Dynamic Modelling of a Bicycle in Balance

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***Abstract*—In this paper, the dynamic model of a bicycle in its vertical balance state is established using Newton Euler equations. Some special features of bicycle design and operation are explored to simplify the model in the point of view of multi body dynamics. The model can be used for design and stability analysis for bicycles and general mobile robots in a similar structure.**

I. Introduction

As a daily transportation tool, a sport gear or a toy, bicycle has attracted attentions of scientists and engineers since 19th century [1].Even today, the research in bicycle is still active thanks to its relevance to robotics, mechanics and engineering education [2] [3][4][5][6][7]. A suitable dynamic model is the essential step for analyzing the unique behaviours of bicycles like self stabilization, optional design for riding comfort and autonomous control. Depending on the purpose of the modelling, the complexity of the model reported in the literature differs much, ranging from 2 to 20 in terms of order of the system [3]. Newton-Euler equation and Lagrange formulation are two main methods for modelling.

Balance at the vertical position with a reasonable forward velocity is one of the interesting behaviours of bicycles. It is also related to the issue of stability of the bicycle which is cruising on the road. Considering the bicycle's position near the vertical position, some reasonable assumptions on the range of angular displacement of various parts of the bicycle can be made and an elegant and compact second order dynamic model can be established as represented by those reported in [3][8]. That model is set up for the rotation of the bicycle around the single axis pointing to the forward motion, treating the roll angle of the back plane of the bicycle as the output and the steering angle as the input. The stability of the bicycle can then be analysed by examining the gain and the positions of the zeros and the poles of the system.

This paper will extend the models in [3][8] through multi-body dynamic analysis. The couplings among system states, especially between roll angle and steering angle are established. The model can be used for a more comprehensive dynamic analysis for the balance of a bicycle.

The paper is organized as follows. After the introduction in Section 1, kinematical model is described in Section 2. Section 3 is on dynamic model. Conclusion is given in Section 4.

With kind permission of Springer Science+Business Media, the paper is based on Section 4.3 of the author's book "A concise Introduction to Mechanics of Rigid Bodies: Multidisciplinary Engineering", ISBN 978-1-4614-0471-2, Springer, 2012 [9].

II. Kinematic model

Figure 1 schematically shows the side view of a bicycle standing on a horizontal ground. According to [3], the bicycle is divided into the following four main parts: rear wheel, frame (including the rider), front fork and front wheel. The rear wheel and the frame form a plane called *rear plane*, and the front fork and the front wheel form another plane called *front plane*. In the figure, the wheels, the frame and the front fork are all on the same vertical plane. The centres of masses of the frame plane (C), the centres of the wheels (C_1 and C_2) and the distances among them are shown. The point C_2 is treated as the centre of mass of the front plane. Points P_1 and P_2 are the contact points of the rear and the front wheels on the ground respectively. Point P_3 is the intersection between the *steer axis* of the front fork and the ground. The angle (γ) between the steering axis and the ground is called *head angle*, the distance (d) between P_2 and P_3 is called *trail* and the distance (b) between P_2 and P_1 is called *wheel base*. The head angle, the trail and the wheel base are the key parameters for a bicycle. The angular displacement between the front and rear planes is denoted as θ. The angular speed of the rear and front wheels are constant and denoted as ω_1 and ω_2 respectively.

Figure 2 shows the top view of the bicycle in a more general configuration. The universe frame $\{U\} : OXYZ$ is set up at a fixed point on the ground. The Z axis is perpendicular to the ground and is shown as a black dot. The base vectors

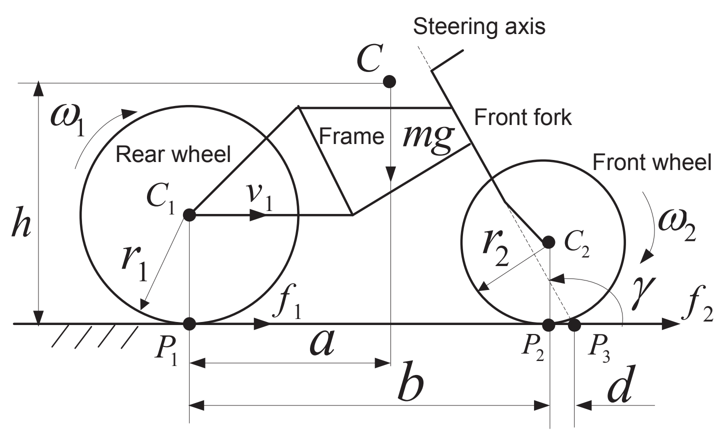

Fig. 1. Bicycle - side view [9]

for the axes of the universe frame are denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively. The body frame for the rear plane is $\{B_1\}$: $C\hat{x}_1\hat{y}_1\hat{z}_1$ which has origin at its centre of mass C. Axis $\hat{x}_1$ is in the rear plane and is parallel to the ground. Axis $\hat{y}_1$ is along the axial of the wheel and is perpendicular to the rear plane, and axis $\hat{z}_1 = \hat{x}_1 \times \hat{y}_1$. The body frame for the front plane, $\{B_2\}$: $C_2\hat{x}_2\hat{y}_2\hat{z}_2$, has origin at C_2. Its axes ($\hat{x}_2$, $\hat{y}_2$ and $\hat{z}_2$) are defined in the same way as that for the definition of the axes $\hat{x}_1$, $\hat{y}_1$ and $\hat{z}_1$.

Assume the mass of the rear and front planes are m_1 and m_2 respectively. The axes of their body frames are their principal axes of inertia, and the inertia matrices are ${}^{B_1}I = diag(I_{x_1x_1},\ I_{y_1y_1},\ I_{z_1z_1})$ and ${}^{B_2}I = diag(I_{x_2x_2},\ I_{y_2y_2},\ I_{z_2z_2})$ respectively.

Bicycle is a complex multi-body system if each part of it is modelled accurately. Considering the unique features of the bicycle's structure and motion, the following assumptions are made.

- the bicycle moves on a smooth horizontal ground.
- there is no slippage between the wheels and the ground.
- the steer angle θ is very small, $\theta \to 0$, $\theta \approx \sin\theta$ and $\cos\theta \approx 1$.
- the head angle $\gamma = 90$ degrees and trail $d = 0$.
- the rider is fixed with respect to the frame.
- the masses of wheels are ignored relative to the masses of other parts of the bicycle.
- the front plane is always in an upright position.
- the diameters of the two wheels are the same, $r_1 = r_2$.

The rear plane's linear position is $r_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, where x_1, y_1 and z_1 are the coordinates of the centre of mass C. Its orientation varies with ϕ_1 (*yaw*), the angle between axes $\hat{x}_1$ and X, and δ_1 (*roll*), the angular displacement it has around axis $\hat{x}_1$. The front plane's linear position is $r_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$, where x_2, y_2 and r_2 are the coordinates of point C_2. Its orientation only varies with ϕ_2 (*yaw*), the angle between axes $\hat{x}_2$ and X as it is assumed that it is always kept in an upright position. It is assumed that at the bicycle's initial position as shown in Figure 1, $\phi_1 = \phi_2 = \delta_1 = 0$

The rotation matrices of the rear plane and the front plane respectively are

$$
\begin{aligned}
R_1 &= R_z(\phi_1)R_x(\delta_1) \\
&= \begin{bmatrix} \cos\phi_1 & -\sin\phi_1 & 0 \\ \sin\phi_1 & \cos\phi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\delta_1 & -\sin\delta_1 \\ 0 & \sin\delta_1 & \cos\delta_1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\phi_1 & -\sin\phi_1\cos\delta_1 & \sin\phi_1\sin\delta_1 \\ \sin\phi_1 & \cos\phi_1\cos\delta_1 & -\cos\phi_1\sin\delta_1 \\ 0 & \sin\delta_1 & \cos\delta_1 \end{bmatrix} \quad (1) \\
R_2 &= R_z(\phi_2) = \begin{bmatrix} \cos\phi_2 & -\sin\phi_2 & 0 \\ \sin\phi_2 & \cos\phi_2 & 0 \\ 0 & \sin\delta_2 & 1 \end{bmatrix}
\end{aligned}
$$

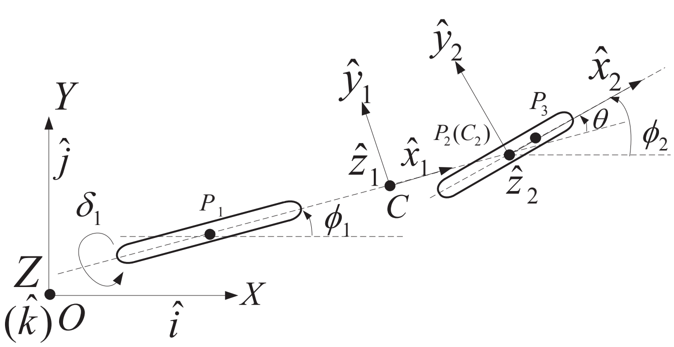

Fig. 2. Bicycle - top view [9]

where $R_z(\bullet)$ and $R_x(\bullet)$ are the basic rotation matrices.

Noting that $\phi_2 = \phi_1 + \theta$ and $\theta \to 0$, the rotation matrix R_2 is rewritten as

$$
R_2 = \begin{bmatrix} \cos\phi_1 - \theta\sin\phi_1 & -\sin\phi_1 - \theta\cos\phi_1 & 0 \\ \sin\phi_1 + \theta\cos\phi_1 & \cos\phi_1 - \theta\sin\phi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

From R_1 and R_2, the expressions of the principal axes of the body frames $\{B_1\}$ and $\{B_2\}$ in the universe frame are obtained,

$$
\begin{aligned}
\hat{x}_1 &= [\cos\phi_1 \quad \sin\phi_1 \quad 0]^T & (2) \\
\hat{y}_1 &= [-\sin\phi_1\cos\delta_1 \quad \cos\phi_1\cos\delta_1 \quad \sin\delta_1]^T \\
\hat{z}_1 &= [\sin\phi_1\sin\delta_1 \quad -\cos\phi_1\sin\delta_1 \quad \cos\delta_1]^T & (3) \\
\hat{x}_2 &= [\cos\phi_1 - \theta\sin\phi_1 \quad \sin\phi_1 + \theta\cos\phi_1 \quad 0]^T & (4) \\
\hat{y}_2 &= [-\sin\phi_1 - \theta\cos\phi_1 \quad \cos\phi_1 - \theta\sin\phi_1 \quad 0]^T \\
\hat{z}_2 &= [0 \ \ 0 \ \ 1]^T
\end{aligned}
$$

The angular velocities of the rear plane, ω_R, is obtained from the rotation matrix R_1,

$$
\omega_R = \dot{\phi}_1\hat{k} + \dot{\delta}_1\hat{x}_1 = \dot{\delta}_1\cos\phi_1\hat{i} + \dot{\delta}_1\sin\phi_1\hat{j} + \dot{\phi}_1\hat{k} \quad (5)
$$

Note the expression of $\hat{x}_1$ in equation (2) is used in the above derivation.

Similarly the angular velocity of the front plane, ω_F, is

$$
\omega_F = \dot{\phi}_2\hat{k}
$$

The angular velocities of two wheels are $\omega_1\hat{y}_1$ and $\omega_2\hat{y}_2$ respectively.

As there are no slippage between the wheels and the ground, the velocities of the points P_1 and P_2 with respect to the universe frame are zero,

$$
v_{P_1} = v_{P_2} = 0
$$

The velocities of C_1 and C_2 are

$$
\begin{aligned}
v_{C_1} &= v_{P_1} + \omega_1\hat{y}_1 \times \overline{P_1C_1} = \omega_1\hat{y}_1 \times r_1\hat{z}_1 = r_1\omega_1\hat{x}_1 \\
v_{C_2} &= v_{P_2} + \omega_2\hat{y}_2 \times \overline{P_2C_2} = \omega_2\hat{y}_2 \times r_2\hat{z}_2 = r_2\omega_2\hat{x}_2
\end{aligned}
$$

As C_1 and C_2 are the points in the rear plane, their velocities are related by

$$
v_{C_2} = v_{C_1} + \omega_R \times \overline{C_1C_2} = r_1\omega_1\hat{x}_1 + \omega_R \times (b\hat{x}_1 + (r_2 - r_1)\hat{z}_1)
$$

Substituting equation (5) into the above equation and considering the assumption that $r_1 = r_2$, we have

$$
v_{C_2} = r_2\omega_2\hat{x}_2 = r_1\omega_1\hat{x}_1 + b\dot{\phi}_1\hat{y}_1 \quad (6)
$$

Because $(r_2\omega_2\hat{x}_2)\times\hat{x}_2=0$,

$$(r_1\omega_1\hat{x}_1+b\dot{\phi}_1\hat{y}_1)\times\hat{x}_2=0$$

and

$$r_1\omega_1\|\hat{x}_1\times\hat{x}_2\|=r_1\omega_1\sin\theta=b\dot{\phi}_1\|\hat{y}_1\times\hat{x}_2\|$$

As it is assumed that $\theta\to 0$, $\sin\theta\approx\theta$, then $\hat{y}_1$ is nearly perpendicular to $\hat{x}_2$, $\|\hat{y}_1\times\hat{x}_2\|\approx 1$. As a result,

$$\begin{aligned} r_1\omega_1\theta &= b\dot{\phi}_1 \\ \dot{\phi}_1 &= \frac{r_1\omega_1}{b}\theta \end{aligned} \tag{7}$$

Then the angular velocity of the rear plane can be expressed as a function of ω_1, θ and $\dot{\delta}_1$,

$$\omega_R=\dot{\delta}_1\cos\phi_1\hat{i}+\dot{\delta}_1\sin\phi_1\hat{j}+\frac{r_1\omega_1}{b}\theta\hat{k} \tag{8}$$

The relation between ω_1 and ω_2 can be obtained from equation (6). Taking inner products of the both sides of the equation with $\hat{x}_1$, we have

$$r_2\omega_2\hat{x}_1^T\hat{x}_2=r_1\omega_1+b\dot{\phi}_1\hat{x}_1^T\hat{y}_1=r_1\omega_1$$

As $\hat{x}_1^T\hat{x}_2=\cos\theta$, then

$$\omega_1=\frac{r_2}{r_1}\omega_2\cos\theta$$

The linear velocities of the rear plane and the front plane respectively are,

$$\begin{aligned} v_R &= v_{C_1}=v_1\hat{x}_1=v_1(\cos\phi_1\hat{i}+\sin\phi_1\hat{j}) \\ v_F &= v_{C_2}=v_2\hat{x}_2=v_2(\cos\phi_2\hat{i}+\sin\phi_2\hat{j}) \end{aligned}$$

where $v_1=r_1\omega_1$ and $v_2=r_2\omega_2$ are the speeds of the rear and front wheels respectively.

III. Dynamic model

To have the dynamic model, the accelerations of the various parts of the bicycle are studied first. Assume that ω_1 is a constant, the angular acceleration of the rear plane is

$$\alpha_R=\dot{\omega}_R=\frac{v_1}{b}\dot{\theta}\hat{k}+\ddot{\delta}_1\hat{x}_1+\dot{\delta}_1\dot{\hat{x}}_1 \tag{9}$$

where

$$\dot{\hat{x}}_1=\omega_R\times\hat{x}_1$$

Substituting equations (8) and equation (2) into the above equation, we have

$$\dot{\hat{x}}_1=\frac{v_1}{b}\theta(-\sin\phi_1\hat{i}+\cos\phi_1\hat{j}) \tag{10}$$

Substituting it into equation (9) and considering equation (2), we have

$$\alpha_R=\alpha_x\hat{i}+\alpha_y\hat{j}+\alpha_z\hat{k} \tag{11}$$

where

$$\begin{aligned} \alpha_x &= \ddot{\delta}_1\cos\phi_1-\frac{v_1}{b}\dot{\delta}_1\theta\sin\phi_1 \\ \alpha_y &= \ddot{\delta}_1\sin\phi_1+\frac{v_1}{b}\dot{\delta}_1\theta\cos\phi_1 \\ \alpha_z &= \frac{v_1}{b}\dot{\theta} \end{aligned}$$

The linear acceleration of the rear plane is

$$a_R=\dot{v}_1\hat{x}_1+v_1\dot{\hat{x}}_1 \tag{12}$$

Considering that v_1 (the linear speed of the rear wheel) is constant and substituting equation (10) into the above equation, we have

$$a_R=\frac{v_1^2}{b}\theta(-\sin\phi_1\hat{i}+\cos\phi_1\hat{j}) \tag{13}$$

This is the centripetal acceleration due to the instantaneous circular motion of C_1.

Assume that the rider keeps the bicycle in balance by adjusting steer angle θ only. Ignoring the reaction force from the front plane, the rear plane is subject to the following forces.

- gravitational force: $-mg\hat{k}$, at the centre of mass C, where $g=9.8\ m/s^2$ is the magnitude of gravitational acceleration.
- static friction forces $f_1\hat{x}_1$ at the contact point P_1 between the wheel and the ground.
- reaction force from the ground $f_2\hat{k}$ at the at the contact point P_1 between the wheel and the ground.

The moment of force with respect to point P_1 is thus

$$M_{P_1}=\overline{P_1C_1}\times(-mg\hat{k})$$

Noting that $\overline{P_1C_1}=a\hat{x}_1+h\hat{z}_1$ and expressions of $\hat{x}_1$ and $\hat{z}_1$ given by equations (2) and (3) respectively, we have

$$M_{P_1}=mg[-(a\sin\phi_1-h\cos\phi_1\sin\delta_1)\hat{i}+(a\cos\phi_1+h\sin\phi_1\sin\delta_1)\hat{j}] \tag{14}$$

The angular momentum of the rear plane with respect to C_1 is,

$$H_{C_1}=R_1\,{}^{B_1}IR_1^T\omega_R \tag{15}$$

Applying Newton-Euler formulation, we have the following equation of motion,

$$M_{P_1}=R_1\,{}^{B_1}IR_1^T\alpha_R+\overline{P_1C_1}\times(ma_R)+\omega_R\times H_{C_1} \tag{16}$$

The equation will become very complex after being expanded. It can be simplified by considering that in a normal situation, angle δ_1 is usually very small. It is reasonable to assume that $\delta_1\approx\sin\delta_1$ and $\cos\delta_1\approx 1$. Furthermore, as we are only concerned with the balance of the bicycle, the X axis of the universe frame can be chosen to be aligned with $\hat{x}_1$ at the instant of interest and as such $\phi_1=0$ and $\hat{i}=\hat{x}_1$. With these assumptions, the terms R_1 (equation (1)), ω_R (equation (8)), α_R (equation (11)), a_R (equation (13)) and M_{P_1} (equation (14))), the terms used in the equation can be simplified such that

$$\begin{aligned} R_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\delta_1 \\ 0 & \delta_1 & 1 \end{bmatrix} \\ \omega_R &= \dot{\delta}_1\hat{i}+\frac{v_1}{b}\theta\hat{k} \\ \alpha_R &= \ddot{\delta}_1\hat{i}+\frac{v_1}{b}\dot{\delta}_1\theta\hat{j}+\frac{v_1}{b}\dot{\theta}\hat{k} \\ a_R &= \frac{v_1^2}{b}\theta\hat{j} \\ M_{P_1} &= mg(h\delta_1\hat{i}+a\hat{j}) \end{aligned}$$

and

$$\begin{aligned}\hat{x}_1 &= = [\cos\phi_1 \quad \sin\phi_1 \quad 0]^T = [1 \quad 0 \quad 0]^T \\ \hat{z}_1 &= [\sin\phi_1 \sin\delta_1 \quad -\cos\phi_1 \sin\delta_1 \quad \cos\delta_1]^T \approx [0 \quad -\delta_1 \quad 1]^T \\ \overline{P_1C_1} &= a\hat{x}_1 + h\hat{z}_1 = [a \quad -h\delta_1 \quad h]^T\end{aligned}$$

Substituting the above terms into equation (16) and noting the definitions of ${}^{B_1}I$ and H_{C_1}, and the fact that $I_{y_1y_1} = I_{z_1z_1}$, we have

$$I_{x_1x_1}\ddot{\delta}_1 - mgh\delta_1 - \frac{mv_1^2h}{b}\theta = 0 \tag{17}$$

$$I_{x_1x_1}\frac{v_1}{b}\theta\dot{\delta}_1 - mga = 0 \tag{18}$$

$$(1+\delta_1^2)I_{y_1y_1}\dot{\theta} + mv_1a\theta = 0 \tag{19}$$

The above equations of motion for the bicycle near its balance state are valid given the assumptions made. They will be much more complex without those assumptions. From those equations, the steer angel θ and the rear wheel's speed v can be determined to make δ_1 close to the value needed for the balance of the bicycle. The effects of the design parameters (a, b, h, m_1 and $I_{x_1x_1}$ etc.) on the dynamic behaviour of the bicycle can also be studied.

IV. Conclusion

A comprehensive dynamic model of a bicycle near its vertical balance position is established through multi-body dynamic analysis. It uncovers the dynamic relation among the key system states and its relevance to the geometrical parameters of the bicycle. The model can be used for not only the stability analysis, but also the determination of the design parameters of the bicycle. The future work will be on making the model more comprehensive by relaxing the assumptions made in the paper.

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