Regression Analysis

Logistic Regression

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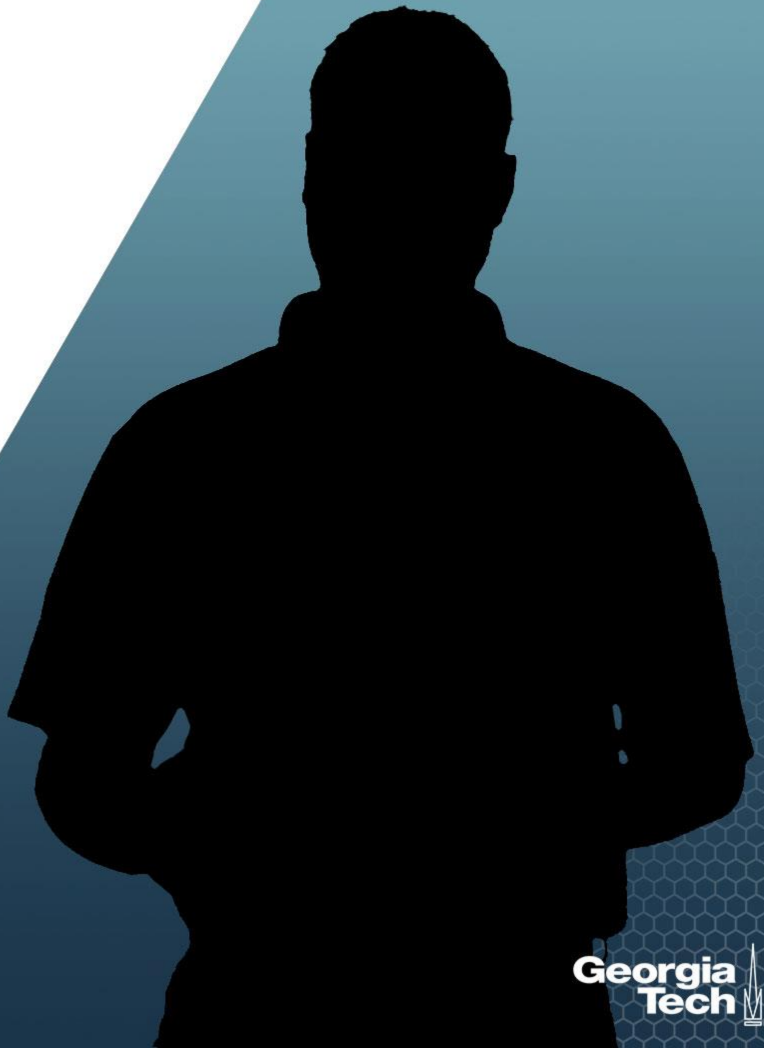

Statistical Inference

About This Lesson

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Model Estimation

Model the probability of success given predictor(s):

$$\text{Logit}\Big(\Pr\big(Y = 1 \mid X_1, \cdots, X_p\big)\Big) = \text{Logit}\Big(\mathrm{p}\big(X_1, \cdots, X_p\big)\Big) = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p$$

Parameters: $\beta_0, \beta_1, \cdots, \beta_p$

Approach: Maximum Likelihood Estimation

$$\max_{\beta_0,\beta_1,\cdots,\beta_p} \mathcal{L}(\beta_0, \beta_1, \cdots, \beta_p) = \prod_{i=1}^{n} \mathrm{p}\big(\boldsymbol{X}_{i,1}, \boldsymbol{X}_{i,2}, \cdots, \boldsymbol{X}_{i,p}\big)^{\boldsymbol{Y}_i} \Big(1 - \mathrm{p}\big(\boldsymbol{X}_{i,1}, \boldsymbol{X}_{i,2}, \cdots, \boldsymbol{X}_{i,p}\big)\Big)^{1-\boldsymbol{Y}_i}$$

or

$$\max_{\beta_0,\beta_1,\cdots,\beta_p} \ell\big(\beta_0, \beta_1, \cdots, \beta_p\big) = \max_{\beta_0,\beta_1,\cdots,\beta_p} \log\big(\mathcal{L}(\beta_0, \beta_1, \cdots, \beta_p)\big)$$

$$= \max_{\beta_0,\beta_1,\cdots,\beta_p} \sum_{i=1}^{n} \left(\boldsymbol{Y}_i \log\left(\frac{\mathrm{e}^{\beta_0+\beta_1 X_1+\cdots+\beta_p X_p}}{1 + \mathrm{e}^{\beta_0+\beta_1 X_1+\cdots+\beta_p X_p}}\right) + (1 - \boldsymbol{Y}_i)\log\left(\frac{1}{1 + \mathrm{e}^{\beta_0+\beta_1 X_1+\cdots+\beta_p X_p}}\right)\right)$$

Statistical Inference

Maximum Likelihood Estimators (MLEs):

$$\hat{\boldsymbol{\beta}} = (\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_\mathrm{p})$$

Statistical Properties of MLEs:

- Approximate Sampling Distribution: $\hat{\boldsymbol{\beta}} \approx \boldsymbol{N}(\boldsymbol{\beta}, \boldsymbol{V})$
- The normal approximation relies on the assumption of large sample size
- Statistical inference is not reliable for small sample data

1-*α* Approximate Confidence interval $\left\{ \hat{\beta}_j \pm z_{\alpha/2}\sqrt{\mathrm{Var}(\hat{\beta}_j)} \right.$

Statistical Inference (cont'd)

- Hypothesis testing and Confidence Intervals rely on the approximately normal distribution of large sample sizes

- Use the z-test (**Wald test**)
  - Test is for the statistical significance of $\hat{\beta}_j$ *given all other predicting variables in the model*
  - Null hypothesis is that β_j is not significant

    $$\mathrm{H_0}: \beta_j = 0 \;\text{ vs. }\; \mathrm{H_a}: \beta_j \neq 0$$

  - $\mathrm{z{-}value} = \dfrac{\hat{\beta}_j - 0}{\mathrm{se}(\hat{\beta}_j)} = \dfrac{\hat{\beta}_j}{\mathrm{se}(\hat{\beta}_j)}$
  - Reject $\mathrm{H_0}$ if $|\mathrm{z{-}value}|$ is too large
    - Implies that β_j is statistically significant

Statistical Inference (cont'd)

For $\mathbf{H_0}$: $\boldsymbol{\beta_j} = \boldsymbol{b}$ vs. $\mathbf{H_a}$: $\boldsymbol{\beta_j} \neq \boldsymbol{b}$ (to test if the coefficient equals constant b)

- $\text{z−value} = \frac{\hat{\beta}_j - b}{\text{se}(\hat{\beta}_j)}$
- Reject H_0 if $|\text{z−value}| > \text{z}_{\alpha/2}$ for significance level α
- Alternatively, compute P-value
  - P−value = 2Pr$(Z > |\text{z−value}|)$

For $\mathbf{H_0}$: $\boldsymbol{\beta_j} \leq \mathbf{0}$ vs. $\mathbf{H_a}$: $\boldsymbol{\beta_j} > 0$ (to test for a significantly positive coefficient)

  - P−value = Pr$(Z > z - value)$

For $\mathbf{H_0}$: $\boldsymbol{\beta_j} \geq \mathbf{0}$ vs. $\mathbf{H_a}$: $\boldsymbol{\beta_j} < \mathbf{0}$ (to test for a significantly negative coefficient)

  - P−value = Pr$(Z < z - value)$

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Statistical Inference (cont'd)

For $\mathbf{H_0}$: $\boldsymbol{\beta_j} = \boldsymbol{b}$ vs. $\mathbf{H_a}$: $\boldsymbol{\beta_j} \neq \boldsymbol{b}$ (to test if the coefficient equals constant b)

- $\text{z−value} = \frac{\hat{\beta}_j - b}{\text{se}(\hat{\beta}_j)}$
- Reject H_0 if $|\text{z−value}| > \mathrm{z}_{\alpha/2}$ for significance level α
- Alternatively, compute P-value
  - P−value = 2Pr$(Z > |\text{z−value}|)$

For $\mathbf{H_0}$: $\boldsymbol{\beta_j} \leq \mathbf{0}$ vs. $\mathbf{H_a}$: $\boldsymbol{\beta_j} > 0$ (to test for a significantly positive coefficient)

- P−value = Pr$(Z > z - value)$

For $\mathbf{H_0}$: $\boldsymbol{\beta_j} \geq \mathbf{0}$ vs. $\mathbf{H_a}$: $\boldsymbol{\beta_j} < \mathbf{0}$ (to test for a significantly negative coefficient)

- P−value = Pr$(Z < z - value)$

- Because the approximation of the normal distribution relies on large sample size, so do the hypothesis testing procedures.
- What if *n* is small?
  - The hypothesis testing procedure will have a probability of type I error larger than the significance level.
  - In other words, there will likely be more type I errors than expected.

Testing for Subsets of Coefficients

Full model:

$$\text{Logit}\left(\mathrm{p}(X_1, \cdots, X_p, Z_1, \cdots, Z_q)\right)$$
$$= \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p + \alpha_1 Z_1 + \cdots + \alpha_q Z_q$$

Reduced model:

$$\text{Logit}\left(\mathrm{p}(X_1, \cdots, X_p)\right) = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p$$

The hypothesis test:

$$\mathbf{H_0}: \boldsymbol{\alpha_1 = \alpha_2 = \cdots = \alpha_q = 0}$$

vs.

$$\mathbf{H_a}: \boldsymbol{\alpha_i \neq 0 \text{ for at least one } \alpha_i, \mathrm{i} = 1, \cdots, q}$$

- Maximize the likelihood function under reduced model: $\mathcal{L}\,(\bar{\beta}_0, \bar{\beta}_1, \ldots, \bar{\beta}_p)$
- Maximize the likelihood function under full model: $\mathcal{L}(\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_p, \hat{\alpha}_1, \ldots, \hat{\alpha}_q)$
- Test Statistics
  - $\text{Deviance} = \log\left(\mathcal{L}(\bar{\beta}_0, \bar{\beta}_1, \ldots, \bar{\beta}_p)\right) - \log\left(\mathcal{L}(\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_p, \hat{\alpha}_1, \ldots, \hat{\alpha}_q)\right) \approx \chi_q^2$
  - P-value = Pr($\chi_q^2 >$ Deviance)

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Testing for Subsets of Coefficients

Full model:

$$\text{Logit}\left(\text{p}(X_1, \cdots, X_p, Z_1, \cdots, Z_q)\right)$$
$$= \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p + \alpha_1 Z_1 + \cdots + \alpha_q Z_q$$

Reduced model:

$$\text{Logit}\left(\text{p}(X_1, \cdots, X_p\ )\right) = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p$$

The hypothesis test:

$$\mathbf{H_0}:\ \boldsymbol{\alpha_1 = \alpha_2 = \cdots = \alpha_q = 0}$$

vs.

$$\mathbf{H_a}:\ \boldsymbol{\alpha_i \neq 0 \text{ for at least one } \alpha_i, \mathrm{i} = 1, \cdots, q}$$

- The hypothesis test for subsets of coefficients is approximate
  - It relies on large sample size
- This is not a test for goodness of fit!
  - It only compares two models

- Maximize the likelihood function under reduced model: $\mathcal{L}(\bar{\beta}_0, \bar{\beta}_1,\ldots, \bar{\beta}_p))$
- Maximize the likelihood function under full model: $\mathcal{L}(\hat{\beta}_0, \hat{\beta}_1,\ldots, \hat{\beta}_p, \hat{\alpha}_1,\ldots, \hat{\alpha}_q)$
- Test Statistics
  - $\text{Deviance} = \log\left(\mathcal{L}(\bar{\beta}_0, \bar{\beta}_1,\ldots, \bar{\beta}_p)\right) - \log\left(\mathcal{L}(\hat{\beta}_0, \hat{\beta}_1,\ldots, \hat{\beta}_p, \hat{\alpha}_1,\ldots, \hat{\alpha}_q)\right) \approx \chi^2_q$
  - P-value = $\Pr(\chi^2_q > \text{Deviance})$

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Testing for Overall Regression

Full model:

Logit $\left(\mathrm{p}(X_1, \cdots, X_p)\right) = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p$

Reduced model:

Logit $\left(\mathrm{p}(X_1, \cdots, X_p)\right) = \beta_0$

The hypothesis test:

$\mathbf{H_0}$: $\boldsymbol{\beta_1 = \beta_2 = \cdots = \beta_p = 0}$

vs.

$\mathbf{H_a}$: $\boldsymbol{\beta_i \neq 0}$ **for at least one** $\boldsymbol{\beta_i}$, $\mathbf{i = 1, \cdots, \boldsymbol{p}}$

- Maximize the likelihood function under reduced model: $\mathcal{L}\,(\bar{\beta}_0)$
- Maximize the likelihood function under full model: $\mathcal{L}(\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_p)$
- Test Statistics
  - $\text{Deviance} = \log\left(\mathcal{L}\,(\bar{\beta}_0)\,\right) - \log(\mathcal{L}(\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_p)) \approx \chi_p^2$
  - P-value = Pr($\chi_p^2 >$ Deviance)

Summary