Contents

1 Introduction, Motivation, and Conventions

There have become several notions of a 'vertex algebra' floating around since their introduction by Borcherds in [Bor86]. Two popular definitions are found in [Kac98, §1.3] and [FLM88, §8.10]; several other related concepts can be found in [Kac03]. Ultimately they have been introduced in order to algebraically capture the extremely local behaviour of low-dimensional conformal field theory.

One, seemingly unusual, definition of vertex algebras is given by Beilinson and Drinfeld in [BD04]. What makes this one an outlier is the prevalence of concepts from algebraic geometry. In this sense one can define a vertex operator as a translation-equivariant sheaf of $\mathcal{D}_X$-modules over the affine line $\mathbb{A}^1$, equipped with a certain pairing on stalks called an operator product expansion. Most other definitions are overwhelming algebraic – one could give them to a second year linear algebra student and have them understand.

The definition of vertex algebras is long and involved, but again, is accessible. The motivation behind which, however, is less easily accessed. A typical access path would be a course in quantum field theory or equivalent. To a pure mathematics student, this may be out-of-reach in the short term. And indeed, vertex algebras are a rich source of studies in pure mathematics – the Moonshine phenomenon is one such example. In fact, this is the reason for introduction of these algebras originally. A selection of related papers is [DGM90], [Bor92], [CS99, §30], [MSV99] and [Hel13].

This thesis is accordingly designed as an introduction to the theory of vertex algebras for early graduate students in pure mathematics. Few, if any, results of these algebras will be discussed. Instead, the focus is motivation. Approaching the topic from the viewpoint of [BD04] allows concurrently for a wonderful breadth of topics to be introduced and motivated – fibre bundles, algebraic geometry and sheaf theory being the principle examples.

In a broad sense, the conventional development for quantum field theory is to take classical field theories defined on smooth manifolds and then account for the local effects which appear in quantum theories, by means of Hilbert spaces. An excellent synopsis of this can be found in [Gan06, §4]. The approach offered in this thesis, an interpretation of that done in [BD04], is to do the reverse. That is, one first starts with the language of algebraic geometry, in order to account for the quantum mechanical occurrences. Subsequently the necessary analogue of smoothness and variation is introduced to these objects – in some sense, synthetically.

In Section 2 the focus is mechanics and bundle theory. In such, field theory is developed on fibre bundles $\pi : E \to \Sigma$ of manifolds; variational methods are encoded from the use of the jet bundles $J^q\pi$. It is here that we develop the objects we wish to emulate in the algebraic geometric setting. Several important concepts for differential geometry and for use in Section 3 and Section 4 are introduced.

Section 3 motivates the use of algebraic geometry relevant to physics, and develops the foundation and tools required for later use. The key concept is a variant of Gelfand duality. This section serves as an introduction to algebraic geometry for the new student – an expert may wish to read Section 3.1 and skip ahead. This may be the most interesting section for a physicist, however: it acts as a motivator for why one might want to use the language of algebraic geometry physically.

Section 4 develops the analogue of jet bundles for schemes, and more conveniently the dual notion of $\mathcal{D}$-modules. There, several related topics are presented so as see that $\mathcal{D}$-modules are excellent models for the topics discussed in the prior sections.

Finally, Section 5 exhibits vertex algebras as originating from operator product expansion algebras; as a local algebraic structure on $\mathcal{D}$-modules on the affine line.

Conventions & Prior Knowledge

As stated above, the assumed knowledge base of the reader is to be that of a beginning graduate student of pure math, or otherwise a physicist who is unfamiliar with the mathematical formulation of the topics discussed.

Point-set topology and algebra are central concepts and so assumed to be known, at least to the level of final year undergraduate work. In particular tensor products will be used throughout. The basic notions of category theory are also assumed: for example commutative diagrams, functors, natural transformations and limits. Notions relating to Abelian categories and exact sequences would be helpful, but are not critical.

'Algebra', unless otherwise specified, shall mean 'commutative, unital, associative algebra'. We shall have to deal with non-commutative rings, so commutativity will be specified where possible. All rings are unital. An ideal generated by a subset $S \subseteq R$ of a ring R shall be denoted with angle brackets, as $\langle S \rangle$. A natural transformation between functors F, G shall be denoted with a dot, as $F \dot{\to} G$, following [ML98].

When referring to manifolds and maps between them, the term 'smooth' shall mean 'infinitely differentiable', and all manifolds shall be assumed to be smooth.

In Section 2, when dealing with local coordinates of fibre bundles, we shall use contravariant/covariant notation for indexing, without comment. We shall try to avoid using Einstein summation notation in general.

Acknowledgements

My parents have my eternal gratitude for their unceasing belief in me and my decisions. Nora Ganter, my supervisor, has my deepest thanks for being a wellspring of ideas, an ideal academic role-model, and for possessing limitless patience with me. To the other members of faculty whom I have bothered with miscellanea, including but not exclusive to A. Ram, D. Ridout and G. Zhao; I present here my appreciation. And finally I am thankful to my peers, friends and whetstones M. Treutlein and A. Singh with whom I have discussed and argued countless times; improving my mathematical skills more than I ever could alone.

2 Physical Motivations

2.1 A Brief Re-cap of Classical Mechanics

The starting point and object of emulation for quantum field theory is classical mechanics. Because we assume little to no physical background, we shall impart a brief recap.

The adjective 'classical' refers to the omission of relativistic and quantum considerations to the theory. Practically, this means that the classical theory is only accurate when distances and velocities are neither too small nor too big.

The theory of mechanics is essentially the answer(s) to the following question: **If the state of an object (e.g. velocity, position) is known now, and the forces acting upon it are known, what is the state of the object later?**

'Object' could be interpreted many ways: a density of mass over a volume, some physical system subject to some constraints, or an infinitesimally small point of mass; a particle, described by a position function $\mathbf{r}(t)$ of time.

The breakthrough observation of the classical theory is due to Hamilton and Lagrange, and is based in variational and analytic principals. Let:

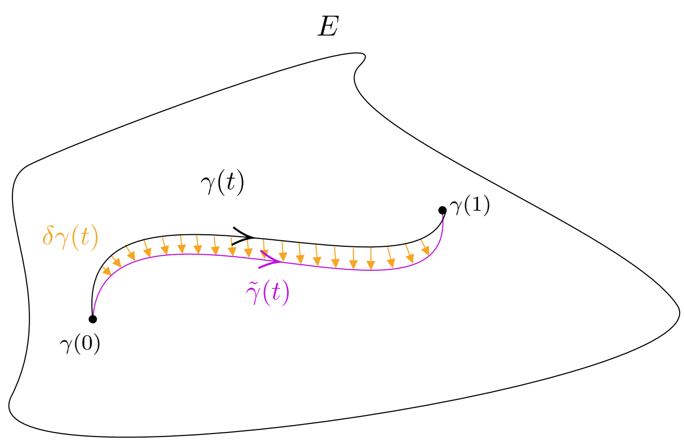
$$\gamma : [t_0, t_1] \to E$$

be a path in a manifold E defined locally by smooth functions $\{q_i(t)\}$, describing the state of our object. At this point we do not assume this to be a description of an actual path in physical space, only a description of the properties of interest depending on time. We fix a function L called the *Lagrangian*, which depends on the local coordinates $\{q_i(t)\}$ and their time derivatives $\{\dot{q}_i(t)\}$. This choice of function is meant to encode the data about the forces acting on our system.

Hamilton's principal is as follows: the actual path taken γ should be minimal (or maximal) among paths with the same endpoints, with respect to the value of the action functional:

$$S[\gamma] = \int_{t_0}^{t_1} L(t, \gamma, \dot{\gamma}) dt.$$

Informally, this says that a small variation $\delta\gamma(t)$ away from the actual path $\gamma(t)$ moves the path away from equilibrium under the forces present. Here is a cartoon of such.



Ultimately, variational calculus says that the actual path can be found by solving the equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}.$$

2.2 Classical Field Theory & Bundles

More sophisticated mechanical theories view the trajectory of a particle not just as embeddings $[t_0, t_1] \to \mathbb{R}^N$, but as *sections of a fibre bundle* $\pi : E \to [t_0, t_1]$. The reason for this is, broadly, that the typical variational methods of Section 2.1 only account for local structure of the mechanics, and not necessarily global structure. Looking instead at mechanics on a fibre bundle allows for concepts which are contemporarily very central; namely *locality* and *gauge*. We shall return soon to these.

In this section we shall make ample use of the language of smooth fibre bundles. We shall introduce the most important concepts, but make no attempt to provide a comprehensive study of the objects – these are primarily to set stage for Section 3 to Section 5. We recommend reading [Ste51] and [Sau89] for a more comprehensive study of bundles, and [GP74] for topics of smooth manifolds like differential forms which we shall utilise without further comment.

We note here that even bundles that are trivial are still interesting under the further constructions. The language of bundles provides a solid framework for fields even when not in interesting cases like when the base is the circle $\mathbb{S}^1$.

We begin with some definitions.

Definition 2.2.1. A *(smooth) fibre bundle*, or *fibred manifold* is a tuple (E, Σ, F, π) comprised of:

I) A manifold E, called the *total space*.

II) A manifold Σ, called the *base space*.

III) A manifold F, called the *fibre*.

IV) A smooth map $\pi : E \to \Sigma$, satisfying:

   a) For each $p \in \Sigma$, the *fibre at p* defined by:

$$E_p := \pi^{-1}(p)$$

   is diffeomorphic to F.

   b) For each $p \in \Sigma$, there is an open neighbourhood $V \subseteq \Sigma$ of p, and a diffeomorphism:

$$\phi : V \times F \to \pi^{-1}(V) \subset E, \tag{2.1}$$

   such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(V) \arrow[dr, "\pi"] & \\ V \times F \arrow[u, "\phi"] \arrow[r, "\mathrm{pr}_V"'] & V \end{tikzcd}$$

That is,

$$\pi \circ \phi(x, f) = x.$$

A *(cross)-section* of the bundle $\pi : E \to \Sigma$ on an open set $U \subseteq \Sigma$ is a smooth map $s : U \to E$ such that:

$$\begin{tikzcd} U \arrow[r, "s"'] \arrow[rr, bend left, "\mathrm{id}_U"] & E \arrow[r, "\pi"'] & U \end{tikzcd}$$

is commutative; that is, $\pi \circ s = \mathrm{id}_U$. When $U = N$, such a map is called a *global section*. Sections are analogous to the notion of 'graph of a function'.

We shall refer to these simply as bundles.

Definition 2.2.2. A *vector bundle* is a fibre bundle (E, Σ, F, π) such that $F \cong \Bbbk^N$ for some $N \in \mathbb{Z}_{\geq 0}$ and $\Bbbk = \mathbb{R}$ or $\mathbb{C}$, and such that for every $p \in \Sigma$, the local trivialisations ϕ can be chosen such that:

$$\phi|_p : \{p\} \times \Bbbk^N \cong \Bbbk^N \to E_p$$

is $\Bbbk$-linear.

Remark 2.2.3. In mathematics literature, the base space Σ is typically referred to as M or X. We shall alternatively use the physics notation Σ, thinking of it as the *worldline* or *worldsheet* of a particle – some intrinsic parameter space for our physical system. The total space E should be thought about as an extrinsic phase space.

Indeed, when Σ is one-dimensional, namely $\Sigma \cong \mathbb{R}$ or $[0, 1]$ or $\mathbb{S}^1$, sections of a bundle $\pi : E \to \Sigma$ are paths in E. When $\dim(\Sigma) > 1$, we call a section σ of π a *field*.

The idea of fields becoming central goes back to Faraday, in order to speak about locality: the idea the that to influence something at a distance, one must propagate a disturbance from us to it. It is perhaps simplest to think of a field simply as a function taking values in Σ, as many do in physics, though this discounts the interesting topology that may come from the bundle structure

Remark 2.2.4. In general, one might consider a more general sort of bundle – with topological spaces in place of manifolds, or a significantly more interesting case of *principal G-bundles*. These are incredibly important for gauge theory in particular, though for the classical case they are largely superfluous. Many of the constructions below extend to principal bundles, however.

Once one has identified the sections of a bundle $\pi : E \to \Sigma$ as the objects of study, the natural course is to discuss how one can study their mechanics. The long and short of it is that the same construction in Section 2.1 can be undertaken geometrically with *jet bundles*.

To introduce some key constructions for later, we shall give a brief overview of a very classical case; namely vector fields and derivations. Such are a natural starting point for a mechanical viewpoint.

We shall not dwell too much on the usual definitions; they are in any differential topology book, for instance [GP74].

Let M be a smooth N-manifold. Derivations are first defined as they are in $\mathbb{R}^N$ – as linear combinations of directional derivatives, induced from $\mathbb{R}^N$ to the manifold by using coordinate charts. Then for any point $p \in M$, one has the notion of tangent space:

$$T_pM = \{\text{Smooth curves } \gamma : (-1, 1) \to M \mid \gamma(0) = p\}_{\sim} \cong \mathbb{R}^m,$$

where two curves γ_1, γ_2 are equivalent under $\sim$ if and only if for some coordinate chart (U, φ) of p,

$$(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

This perspective is natural, coming from doing so in $\mathbb{R}^N$, but it is extraordinarily difficult to calculate anything with this definition. The set

$$TM := \{(x, v) \mid x \in M, v \in T_x M\}$$

has a manifold structure induced from M, and the projection $\tau_M : TM \to M$ is a smooth vector bundle, called the *tangent bundle.* This lets one talk about *vector fields* as sections of the tangent bundle; as smoothly varying assignments of a tangent vectors to each point in M. The set of all vector fields on M we denote by $\mathcal{X}(M)$. There are two equivalent formulations.

Definition 2.2.5. A *derivation at the point* $p \in M$ is a smooth,, $\mathbb{R}$-linear function $\xi : C^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule at p; for any $f, g \in C^\infty(M)$:

$$\xi(fg) = f(p)\xi(g) + g(p)\xi(f). \tag{2.2}$$

That is, ξ locally satisfies the product rule. It happens that this captures the nature of a differential.

The set of all derivations at p is isomorphic to the tangent space $T_x M$.

In this way the notion of vector fields as sections of the tangent bundle becomes interpreted as an $\mathbb{R}$-linear function $P : C^\infty(M) \to C^\infty(M)$ satisfying (2.2) at every point $p \in M$.

The other equivalent way is as follows. Let $\mathfrak{m}_p$ be the maximal ideal of $C^\infty(M)$ at a point $p \in M$. Namely,

$$\mathfrak{m}_p := \{f \in C^\infty(M) \mid f(p) = 0\}.$$

Then $T_p M$ can be described algebraically as the dual bundle to the cotangent bundle, the bundle over M with fibres $\mathfrak{m}_p/\mathfrak{m}_p^2$. This is best seen after the following definition:

Definition 2.2.6. We say the *space of first order jets at* $p \in M$ is the following set:

$$J_p^1 M := {C^\infty(M)} \big/ {\mathfrak{m}_p^2}. \tag{2.3}$$

Remark 2.2.7. Note that there is an injection $\mathfrak{m}_p/\mathfrak{m}_p^2 \hookrightarrow J_p^1 M$. In fact, $\mathfrak{m}_p/\mathfrak{m}_p^2$ is a direct summand of $J_p^1 M$, resulting from the canonical isomorphism:

$$\begin{aligned} C^\infty(M) &\cong \mathbb{R} \oplus \mathfrak{m}_p \\ f &\mapsto (f(p), f(x) - f(p)) \end{aligned}$$

Definition 2.2.6 is best interpreted as smooth functions $f \in C^\infty(M)$, modulo the relation that functions are equivalent if and only if their first order Taylor expansions are the same about p. Indeed, higher order expansions of f will only be different up to elements of $\mathfrak{m}_p^2$.

The summand $\mathfrak{m}_p/\mathfrak{m}_p^2$ then has an interpretation as the cotangent space $T_p^* M$. This is the preferred definition for algebraic geometry, in which maximal ideals are a central object, and curves aren't so well behaved.

The above demonstrates that on smooth manifolds, it can be helpful to consider smooth functions via their local Taylor expansions. This motivates an important definition:

Definition 2.2.8. Given a smooth m-manifold M, a point $x \in M$, and $q \in \mathbb{Z}_{\geq 0}$, the *space of q-jets* is the quotient:

$$J^q_x M := {C^\infty M}\big/{\mathfrak{m}_x^{q+1}}.$$

The equivalence class of f is denoted $[f]^q_x$, and called the *qth jet of f at x*.

That is, $J^q_x M$ is the $\mathbb{R}$-algebra of smooth functions on M under the relation that two smooth functions f and g are equivalent if and only if their Taylor expansions about x agree up to and including order q.

Lemma 2.2.9. For a fixed $q \in \mathbb{Z}_{\geq 0}$, the collection of all such spaces $J^q M := \coprod_{x \in M} J^q_x M$ forms a smooth manifold. Furthermore,

$$\begin{aligned} \sigma_q : J^q M &\to M \\ [f]^q_x &\mapsto x \end{aligned} \tag{2.4}$$

is a smooth vector bundle.

Proof. Refer to [Nes20, §11.11.IX]. □

A quick combinatorial argument shows that there are $1 + \binom{N+q-1}{q-1}$ terms in a qth order Taylor expansion of a function on an N-manifold, for $q \in \mathbb{Z}_{\geq 1}$.

This gives the rank of the bundle $\sigma_q : J^q M \to M$ neatly.

2.3 Jet Bundles

The concept of taking jets and derivations of a smooth function extends to a much more interesting class of functions, namely smooth sections of bundles. Some care needs to be taken to do so, however; in the above case one has the luxury of speaking about field-valued functions with domain the entire manifold. However, there are fibre bundles that have no sections defined on the whole base M.

The classical example to see this is to observe that a global section of the tangent bundle $\tau_{\mathbb{S}^2} : T\mathbb{S}^2 \to \mathbb{S}^2$ is the same as a smooth vector field on the 2-sphere. Thus the fibre bundle obtained from $\tau_{\mathbb{S}^2}$ by removing zero from each fibre has global sections as non-vanishing vector fields on $\mathbb{S}^2$. These cannot exist by the Hairy Ball theorem. So one must speak about jets in terms of local sections.

Definition 2.3.1. Let $\pi : E \to \Sigma$ be a fibre bundle. Following [Sau89], we define the following notation:

I) For some open submanifold W of Σ, the set of local sections of π with domain W shall be called $\Gamma_W(\pi)$. That is,

$$\Gamma_W(\pi) := \{\phi : W \to E \mid \pi \circ \phi = \mathrm{id}_W,\ \phi \text{ smooth}\}.$$

II) We shall denote by $\Gamma_{\mathrm{loc}}(\pi)$ the set of local sections of π regardless of domain.

III) Let $p \in M$. We shall denote by $\Gamma_p(\pi)$ the set of all local sections whose domain contains p.

When talking about vector bundles, it shall become convenient to have local coordinates to work with. Thankfully, there are canonical systems of coordinates.

Let $\pi : E \to \Sigma$ be a rank m vector bundle, with Σ a manifold of dimension N. Let $a \in M$, $U \subseteq M$ a sufficiently small open neighbourhood of $\pi(a)$, and $\phi : U \times \mathbb{R}^m \xrightarrow{\sim} \pi^{-1}(U)$ a local trivialisation as in (2.1). Then given a coordinate system on U, $x : U \xrightarrow{\sim} \mathbb{R}^N$, we can make a coordinate system on $\pi^{-1}(U)$ by:

$$y := (x \circ \mathrm{pr}_U \circ \phi^{-1}, \mathrm{pr}_{\mathbb{R}^m} \circ \phi^{-1}) : \pi^{-1}(U) \to \mathbb{R}^{m+N}.$$

This is called an *adapted coordinate system*. The point is that the first m coordinates are originating from the coordinates on the base, and the last n originate from the fibre. We denote y element-wise by $y = (x^i, u^\alpha)$, for $i \in \{1, \dots, N\}$ and $\alpha \in \{1, \dots, m\}$. In these coordinates, then, a local section $s : U \to E$ takes the form:

$$s : (x^1, \dots, x^N) \mapsto (x^1, \dots, x^N, s^1(x), \dots s^m(x)),$$

where each s^α is the projection of s onto the αth coordinate of $\mathbb{R}^m$

Example 2.3.2. Let Σ be a dimension N manifold. Consider the tangent bundle

$$\tau_\Sigma : T\Sigma \to \Sigma.$$

Fix some $p \in \Sigma$, and a local coordinate system $(x^1, \dots, x^N)$ on Σ about p. Then a basis for the fiber $T_p\Sigma \cong \mathbb{R}^N$ is given by the directional derivatives $\frac{\partial}{\partial x^\ell}$ evaluated at p – a vector field P on Σ is represented locally by:

$$P = \sum_{\ell=1}^{N} \xi^\ell \frac{\partial}{\partial x^\ell}\bigg|_p, \qquad \xi^\ell \in C^\infty(U).$$

The adapted coordinates of τ_Σ are thus $(x^i, \dot{x}^j)$, where for each $j \in \{1, \dots N\}$ we define

$$\dot{x}^j(P) = \xi^j.$$

Having some machinery to describe the local behaviour of bundles, we can now manage a definition of our main objects - jet bundles.

Definition 2.3.3. Let Σ be a dimension N manifold, and let $\pi : E \to \Sigma$ be a bundle.

Let $p \in \Sigma$. Let $s, t \in \Gamma_p(\pi)$ be local sections of π around p. We say that s and t are *equivalent as* 1*-jets* if $s(p) = t(p)$ and in some adapted coordinate system (x^i, u^α) around p:

$$\frac{\partial s^\alpha}{\partial x^i}\bigg|_p = \frac{\partial t^\alpha}{\partial x^i}\bigg|_p, \quad \text{for all } i \in \{1, \dots, N\},\ \alpha \in \{1, \dots, m\}.$$

Observe that this is an equivalence relation. We shall denote the class of s under this relation by $j^1_p s$, the *first order jet of* s at p.

Remark 2.3.4. It is not immediate that this definition is independent of the choice of coordinate system. We shall delegate the proof of such, and related proofs, to [Sau89, Lemmas 4.1.1, 5.1.1 and 6.2.1]. The proof amounts to one or more applications of the chain rule.

Definition 2.3.5. Let Σ be a dimension N manifold, and let $\pi : E \to \Sigma$ be a bundle.

The *first order jet manifold of* π is the set:

$$J^1\pi := \left\{ j^1_p s \mid p \in \Sigma,\, s \in \Gamma_p(\pi) \right\}. \tag{2.5}$$

Compare this to Definition 2.2.6 and Lemma 2.2.9. They are necessarily different, due to the intricacies introduced with bundles that have already been discussed, but they are morally the same - they both parametrize certain smooth functions in terms of their first order Taylor expansions.

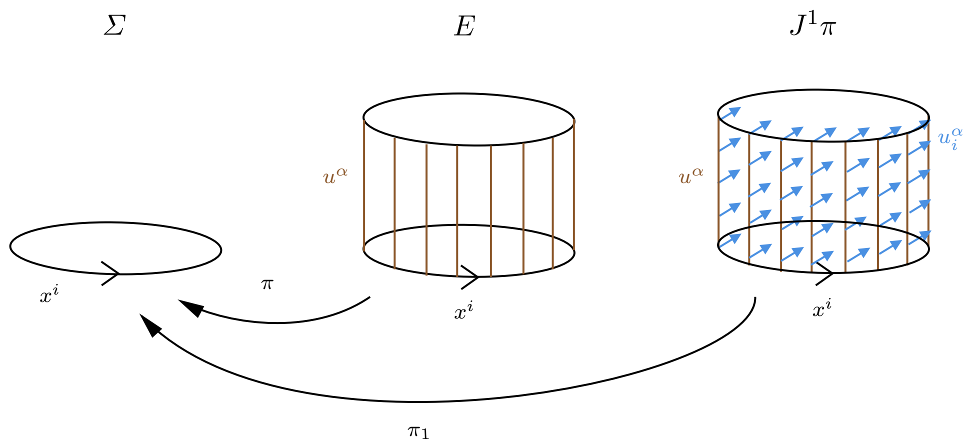
$J^1\pi$ has a preferred set of coordinates on it, induced from the adapted coordinates on π. Namely, if $U \subseteq \Sigma$ is a sufficiently small open set with local coordinates (x^i), we have adapted coordinates (x^i, u^α) on $\pi^{-1}(U)$. From this, we get local coordinates $(x^i, u^\alpha, u^\beta_j)$ on

$$U^1 := \{ j^1_p s \mid s(p) \in U \} \subseteq J^1\pi.$$

Here, $i, j \in \{1, \ldots, N\}$ and $\alpha, \beta \in \{1, \ldots, m\}$, and:

$$\begin{aligned} u^\beta_j &: U^1 \to \mathbb{R} \\ j^1_p s &\mapsto \left. \frac{\partial s^\beta}{\partial x^j} \right|_p . \end{aligned}$$

These coordinates are pictured below, for the simple example of the cylinder bundle $\Sigma = \mathbb{S}^1$ and $E \cong \mathbb{S}^1 \times [0, 1]$.



These coordinates define a smooth atlas for the set $J^1\pi$ – see [Sau89, Proposition 4.1.7]. This atlas induces a second countable, Hausdorff topology on $J^1\pi$, by [Sau89, Corollary 4.1.12]. So $J^1\pi$ is genuinely a manifold.

$J^1\pi$ fits into a commutative diagram:

$$\begin{tikzcd} E \arrow[d, "\pi"'] & J^1\pi \arrow[l, "\pi_{1,0}"'] \arrow[dl, "\pi_1"] \\ \Sigma & \end{tikzcd}$$

where the horizontal and diagonal maps are respectively:

$$\begin{aligned} \pi_{1,0} : J^1\pi &\to E \\ j^1_p s &\mapsto s(p), \\ \pi_1 : J^1\pi &\to \Sigma \\ j^1_p s &\mapsto p. \end{aligned}$$

Both π_1 and $\pi_{1,0}$ define bundles. In fact, the bundle $\pi_{1,0}$ has extra structure – it is an affine bundle: somewhere between a bundle and a vector bundle. This structure is not particularly relevant for us, so we shall not mention affine bundles again, but details can be found in [Sau89, §2.4].

The construction above is much the same for derivatives of arbitrary order, akin to Definition 2.2.8.

Definition 2.3.6. Let $q \in \mathbb{Z}_{>0}$. Let Σ be a dimension N manifold and let $\pi : E \to \Sigma$ be a bundle.

Let $p \in \Sigma$. Let $s, t \in \Gamma_p(\pi)$ be local sections of π around p. We say that s and t are *equivalent as q-jets* if $s(p) = t(p)$ and in some adapted coordinate system (x^i, u^α) around p:

$$\left.\frac{\partial^k s^\alpha}{\partial x^{i_1} \cdots \partial x^{i_k}}\right|_p = \left.\frac{\partial^k t^\alpha}{\partial x^{i_1} \cdots \partial x^{i_k}}\right|_p,$$

for all $k \in \{1, \ldots, q\}$, $\alpha \in \{1, \ldots, m\}$ and any subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N\}$ of size k.

This is again an equivalence relation irrespective of choice of coordinates, and we denote the class of s under this relation by $j^q_p s$, the *qth order jet of s* at p.

Definition 2.3.7. Let $q \in \mathbb{Z}_{>0}$. Let Σ be a manifold and $\pi : E \to \Sigma$ a bundle.

The *qth order jet manifold of* π is the set:

$$J^q\pi := \left\{ j^q_p s \mid p \in \Sigma,\, s \in \Gamma_p(\pi) \right\}.$$

For $q \geq l$, one once more has bundle morphisms:

$$\begin{aligned} \pi_q : J^q\pi &\to \Sigma \\ j^q_p s &\mapsto p \end{aligned}$$

and

$$\begin{aligned} \pi_{q,0} : J^q\pi &\to E \\ j^q_p s &\mapsto s(p) \end{aligned}$$

and

$$\begin{aligned} \pi_{q,l} : J^q\pi &\to J^l\pi \\ j^q_p s &\mapsto j^l_p s. \end{aligned}$$

These fit into a commutative diagram of bundles:

$$\begin{tikzcd} E \arrow[d, "\pi"'] & J^1\pi \arrow[l, "\pi_{1,0}"'] \arrow[dl, "\pi_1"] & J^2\pi \arrow[l, "\pi_{2,1}"'] \arrow[dll, bend left, "\pi_2"] & J^3\pi \arrow[l, "\pi_{3,2}"'] \arrow[dlll, bend left, "\pi_3"] & \cdots \arrow[l] \arrow[dllll, bend left] \\ \Sigma \end{tikzcd}$$

Remark 2.3.8. This is in fact an *inverse limit diagram* of bundles. The inverse limit, denoted

$$J^\infty \pi := \varprojlim J^q \pi,$$

is theoretically a useful tool in many regards. The qth order jet of a section s at a point $p \in \Sigma$ always has a representative described by the qth order Taylor expansion of s at p – the jet of infinite order of s should accordingly be described by the Taylor series of s at p. We know from calculus that these are useful, but considerably more complicated. Besides this it is simply nice to not have to worry about orders of jets in application.

This limit bundle should be an infinite-dimensional manifold – the exact definition of such is not a standard thing. A common method is via Fréchet manifolds. Some description of $J^\infty\pi$ is discussed in [Sau89, §7], [Nes20, §15.20], [Vak98] for example.

We shall manage to largely avoid this discussion, by manner of taking the dual object to jets, namely *rings of differential operators*. The respective infinite order object is significantly easier to work with by using (noncommutative) algebra. This is the focus of Section 4 and introduced briefly in Section 2.6.

As a prospect, we shall note here that our main objects of study of Section 3 and beyond – schemes – have an associated notion of jet bundles. The relevant infinite order object is better behaved than in the manifold case (or at least there is little ambiguity as to how to define it); it is an *ind-scheme*.

To finish off our initial discussion of jet bundles, we note that there are standard notions of coordinates for each $J^q\pi$. Namely, if $U \subseteq \Sigma$ is a sufficiently small open and (x^i, u^α) is an adapted coordinate system for π on U, then one has an induced coordinate system:

$$(x^i, u^\alpha, u^\alpha_{i_1 \cdots i_k}),$$

on the open set:

$$U^q := \{j^q_p s \mid p \in U\}$$

where $k \in \{1, \dots, q\}$ and $\{i_1, \dots, i_k\} \subseteq \{1, \dots, N\}$ is a subset of size k, and

$$\begin{aligned} u^\alpha_{i_1 \cdots i_k}(s) : U^q &\to \mathbb{R} \\ j^q_p s &\mapsto \left.\frac{\partial^k s^\alpha}{\partial x^{i_1} \cdots \partial x^{i_k}}\right|_p \end{aligned}$$

Note that there is an obvious symmetry of coordinates given by interchanging any i_j, i_l, from commutativity of partial derivatives.

2.4 Lifting Properties of Jets and Bundles

So the Jet bundles corresponding to a bundle $\pi : E \to \Sigma$ track the variational behaviour of local sections. In the simplest mechanical case, namely that described in Section 2.1, one has a (possibly local) section of some trivial bundle:

$$\gamma : \mathbb{R} \to M,$$

describing the path of some object. The directional derivatives of this path describe a vector field on a submanifold of M, namely the image of γ.

To invoke the variational principle (see Section 2.1), one needs to be able to 'slightly perturb' the path γ; this needs to also keep track of the pertubation in the vector field induced by γ. It is exactly this which the Jet bundle formalism allows us to do. It remains to discuss how to lift vector fields and paths in fibre bundles to their Jet bundles, in order to see this.

First, note that our formalism utilised a more general bundle $\pi : E \to \Sigma$, to account for the more general field theory, as opposed to just mechanics. To get a trajectory, we should look at paths $[0,1] \to \Sigma$. Accordingly, this path should induce a path in the total space E, which we consider our phase space. Bundles accommodate this lift, but there is choice involved.

To introduce all possible such paths, and a manner in which to work with them, one introduces the following.

Definition 2.4.1. Let Σ be a manifold; denote the group of diffeomorphisms $\Sigma \to \Sigma$ by $\mathrm{Diff}(\Sigma)$.

A *flow* on Σ is a smooth homomorphism of groups $\psi : \mathbb{R} \to \mathrm{Diff}(\Sigma)$. That is, it is a family of diffeomorphisms ψ_t such that $\psi_0 = \mathrm{id}_\Sigma$ and $\psi_t \circ \psi_{t'} = \psi_{t+t'}$ for all $t, t' \in \mathbb{R}$, and such that the map $(t, m) \mapsto \psi_t(m)$ is smooth.

In plain language, a flow designates to each point $p \in \Sigma$ a path

$$t \mapsto \psi_t(p)$$

which evaluates to p at $t = 0$, all done so in a differentiable manner. A flow is typically referred to by ψ_t.

Physically, this is the notion of a 'test particle'. Under a (physical) field, one picks a point in space and pretends as if there were a particle there, and sees which path it takes under the force given by the field. One must use this imaginary particle as opposed to an actual one, because in general having a particle present would alter the field. What path is chosen under a particular field is what we seek to find from mechanics – it is what we shall investigate variationally.

We noted above that to a path in Σ one associates a vector field. Locally one can do the reverse – that is, associate to a vector field a flow ψ with paths whose derivative at each point matches the vector field.

Lemma 2.4.2. Let Σ be a manifold, and let $\Xi \in \mathcal{X}(\Sigma)$ be a vector field.

There exists an open neighbourhood $A = \{(t, x) \mid a_x < t < b_x\} \subseteq \mathbb{R} \times \Sigma$ containing $\{0\} \times \Sigma$, and a smooth map $\psi : A \to \Sigma$ such that:

I) $\psi(0, p) = p$ for all $p \in \Sigma$.

II) $\partial_t \psi(t, p) = \Xi(\psi(t, p))$ for all $p \in \Sigma$.

III) For all $p \in \Sigma$, if a path

$$\gamma : (a, b) \to \Sigma, \qquad \text{where } (a, b) \subseteq (a_x, b_x) \text{ and } \gamma(0) = p$$

satisfies $\dot{\gamma}(t) = \Xi(\gamma(t))$, then $\gamma(t) = \psi(t, p)$ for all $t \in (a, b)$.

Such a ψ is called the (local) flow corresponding to the vector field Ξ.

Hence the question of lifting paths from the base of a bundle to the total space is the same as lifting vector fields from the base to the total space (at least locally, which is all we are concerned with). We next formulate some concepts to capture this.

Definition 2.4.3. Let $\pi : E \to \Sigma$ be a fibre bundle. Let:

$$V\pi := \{\Xi \in TE \mid \pi_* \Xi = 0\} \subseteq TE$$

be the *tangent vectors in E vertical to π.*

$V\pi$ is a submanifold of TE, and is canonically a bundle over E, with structure map

$$\upsilon_\pi := \tau|_{V\pi} : V\pi \to E,$$

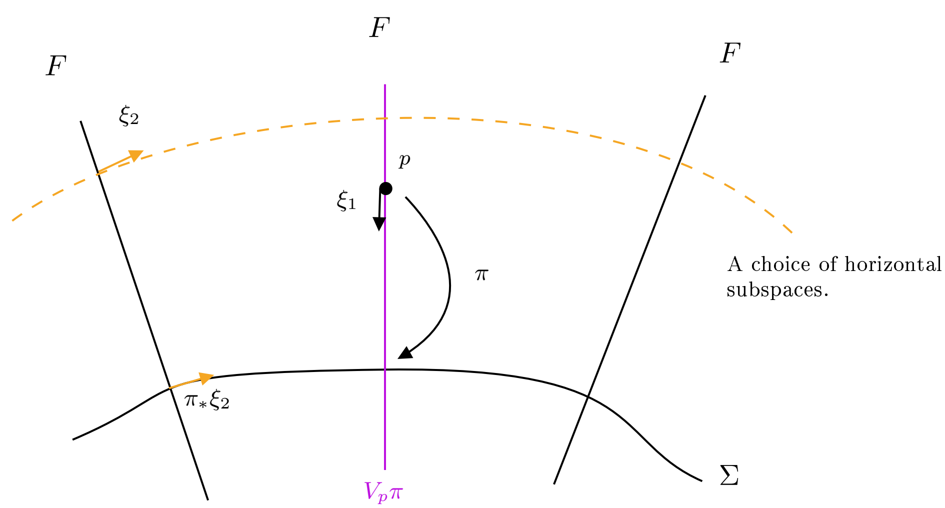
the restriction of the tangent bundle's structure map. It is correspondingly called the *vertical bundle to π.* We denote the space of global sections of $V\pi$ by $\mathcal{V}(\pi)$; the *vertical vector fields on π.* It fits into an exact sequence[1] of bundles on E:

$$0 \longrightarrow V\pi \longrightarrow TE \longrightarrow \pi^* T\Sigma \longrightarrow 0$$

where the first map is inclusion and the second, pushforward by π. There is a corresponding exact sequence of global sections:

$$0 \longrightarrow \mathcal{V}(\pi) \longrightarrow \mathcal{X}(E) \longrightarrow \mathcal{X}(\pi) \longrightarrow 0$$

This rightmost non-zero term is defined as the sections of $\pi^* T\Sigma$, called the *vector fields along π.*



From the figure, one sees that there is an immense amount of choice involved in lifting a path in Σ.

One fixes this via introducing the notion of connection. This is quite a central concept, not just in this paper but in the wider lens of gauge theory, so we shall give a thorough introduction. Informally, a connection is a continuously varying family of subspaces complementary to the fibres $V_p\pi$ of the vertical bundle.

[1] If the reader is unfamiliar with this terminology, one might refer to [ML98, §VIII.3] for the definition. It is an extremely common tool in (homological) algebra, and very convenient.

Definition 2.4.4. Let $\pi : E \to \Sigma$ be a bundle.

An *(Ehresmann) connection* is a subbundle $H\pi$ of the tangent bundle TE, such that $H\pi$ is complementary to $V\pi$. That is,

$$TE = V\pi \oplus H\pi$$

Definition 2.4.5. Let $\pi : E \to \Sigma$ be a bundle. Let $\bigwedge_0^1 \pi$ denote the sections of $\pi^*(T^*E)$; that is, the dual to $\mathcal{X}(\pi) \subseteq \mathcal{X}(E)$. Such sections are correspondingly called *horizontal 1-forms*.

A *connection* or *covariant derivative* on π is a section $\nabla \in \bigwedge_0^1 \pi \otimes \mathcal{X}(E)$, satisfying $\nabla(\sigma) = \sigma$ for all $\sigma \in \bigwedge_0^1 \pi$.

Denote, for any $P \in \mathcal{X}(\Sigma)$, the corresponding vector field by $\nabla_P \in \mathcal{X}(E)$. A connection ∇ is said to be *flat* if these commute; that is:

$$\nabla_{[P,Q]} = \nabla_P \nabla_Q - \nabla_Q \nabla_P \tag{2.6}$$

Definition 2.4.6. Let $\pi : E \to \Sigma$ be a bundle.

A *parallel transport map* PT smoothly associates to each path $\gamma : [0,1] \to \Sigma$, an isomorphism of fibres $PT(\gamma) : E_{\gamma(0)} \to E_{\gamma(1)}$, such that:

I) If c_p denotes the constant path at $p \in \Sigma$, then $PT(c_p) = \mathrm{id}_{E_p}$.

II) PT is invariant under orientation preserving diffeomorphisms of the interval $[0,1]$.

III) If γ_1, γ_2 are two paths in G such that $\gamma_1(1) = \gamma_2(0)$, then

$$PT(\gamma_1 \cdot \gamma_2) = PT(\gamma_2) \circ PT(\gamma_1),$$

where $\gamma_1 \cdot \gamma_2$ is the concatenation of the two paths.

All these invoke the same concept, as we discussed above: creating a preferred lift of a path in a bundle.

Theorem 2.4.7. Let $\pi : E \to \Sigma$ be a bundle. The following are equivalent:

I) An Ehresmann connection on π.

II) A covariant derivative $\nabla \in \bigwedge_0^1 \pi \otimes \mathcal{X}(E)$.

III) A parallel transport map P.

Furthermore, a covariant derivative is flat if and only if the corresponding parallel transport is invariant under endpoint-preserving homotopies of paths.

Proof. A proof of I) being equivalent to II) can be found in [Sau89, §3.5]. A nice, quick proof of the equivalence of II) and III), and the equivalence of flat connections and homotopy invariant parallel transports can be found in [Dum10]. □

As a result of Theorem 2.4.7, one is justified in calling any of the three concepts a connection.

Remark 2.4.8. It is worth remarking here that flat connections being equivalent to homotopy invariant parallel transports is extremely dependant on working with manifolds, simply because manifolds are locally contractible. In particular, in a sufficiently small open subset of Σ, there is a canonical path between any two points. In other contexts (which we shall see in Section 3 and Section 4), one can quite easily define a connection algebraically using Definition 2.4.5, but there is no appropriate notion of path due to the coarse topology.

In the context of principal G-bundles, a similar result holds.

One can use the language of jets to say the notion of connection more succinctly.

Lemma 2.4.9 ([Sau89, §4.6.3])**.** Let $\pi : E \to \Sigma$ be a bundle.

There is a bijective correspondence between the connections of π and the sections of the bundle $\pi_{1,0} : J^1\pi \to E$.

Now we return to the case of lifting vector fields from a bundle to its first jet bundle. We shall only describe here the lifts of vertical vector fields – once one has the notion of connection, tangent vectors in general can be lifted with the same methodology, but with more complex notation.

Let $P \in \mathcal{V}(\pi)$ be a vertical vector field. We have the following diagram:

$$\begin{tikzcd}
 & J^1\upsilon_\pi \arrow[dl, "(\upsilon_\pi)_{1,0}"'] \arrow[d, bend left, "(\upsilon_\pi)_1"] & & \\
V\pi \arrow[r, bend right, "\upsilon_\pi"'] \arrow[dr] & E \arrow[l, bend right, "P"'] \arrow[u, bend left, "j^1P"] \arrow[d, "\pi"] & J^1\pi \arrow[l, "\pi_{1,0}"] \arrow[dl, "\pi_1"] & V\pi_1 \arrow[l, "\upsilon_{\pi_1}"] \arrow[dll, bend left] \\
 & \Sigma & &
\end{tikzcd}$$

The unlabelled arrows above are compositions; here we have taken the first order jet of P,

$$(j^1P)(p) = j^1_p(P), \qquad \text{for all } p \in E,$$

which is a section of $(\upsilon_\pi)_1$, the jet bundle of the vertical bundle of π.

By composing one finds a morphism $j^1P \circ \pi_{1,0} : J^1\pi \to J^1\upsilon_\pi$. And by [Sau89, §4.4.1], there is a canonical isomorphism $i^1 : J^1\upsilon_\pi \to V\pi_1$ between the jets of the vertical bundle vertical bundle of jets.

Definition 2.4.10. Let $P \in \mathcal{V}(\pi)$ be a vertical vector field. Using the notation above, the *first prolongation of* P is the composition

$$J^1\pi \xrightarrow{\pi_{1,0}} E \xrightarrow{j^1P} J^1\upsilon_\pi \xrightarrow{i^1} V\pi_1$$

which we denote by P^1, a section of υ_{π_1}. That is, P^1 is a vertical vector field of the jet bundle π_1, corresponding to P.

Informally, the prolongation P^1 is the vector field P equipped some additional data: information on small variations of sections, under the effects of the vector field – see the picture in section 2.1.

Definition 2.4.11. Let $\pi : E \to \Sigma$ be a bundle. Let $W \subseteq \Sigma$ be an open subset. Let $P \in \mathcal{V}(\pi)$ be a vertical vector field over π, and denote the flow corresponding to P by ψ_t.

If $\phi \in \Gamma_W(\pi)$ is a local section, then the *variation of ϕ induced by X* is the family of local sections

$$\tilde{\psi}_t(\phi) := \psi_t \circ \phi \in \Gamma_W(\pi).$$

The corresponding tangent vector

$$[t \mapsto \tilde{\psi}_t(\phi)] = X \circ \phi$$

is called the corresponding *variation field*.

2.5 Geometrical Variational Calculus

In this section we use the tools developed in Section 2.4 in order to apply the field theory mentioned prior. We include this section primarily to demonstrate how the theory of jets can be used in application to field theory, so that we may apply the same reasoning in later sections to develop notions of field theory in relation to schemes, for instance. For the reader uninformed on differential topology, this section may be skipped – integration is needed and hence differential forms.

Here is where we make some choice on the physics we study: a choice of Lagrangian density.

Definition 2.5.1. Let $\pi : E \to M$ be a bundle on an orientable m-manifold M. Denote a choice of volume form on M by Ω.

A *(first order) Lagrangian density* on π is a choice of smooth function $\mathcal{L} \in C^\infty(J^1\pi)$. The associated *Lagrangian* is the m-form $\mathcal{L}\pi_1^*\Omega$.

Fix a Lagrangian density $\mathcal{L}$. Let $P \in \mathcal{V}(\pi)$ be a vertical vector field with flow ψ_t. Let $\phi \in \Gamma_W(\pi)$ be a local section, thought of as a field. Let $C \subseteq W$ be a compact submanifold of Σ.

For clarity, we have the following diagram:

$$\begin{tikzcd}
E \arrow[d, "\pi"] & & J^1\pi \arrow[ll, "\pi_{1,0}"'] \arrow[dll, "\pi_1"'] \\
M \arrow[u, bend left, "\psi_t\circ\phi"] \arrow[urr, bend right, "j^1(\psi_t\circ\phi)"'] & &
\end{tikzcd}$$

Then the following function is well-defined, and continuous.

$$\begin{aligned}
(-\epsilon, \epsilon) &\to \mathbb{R} \\
t &\mapsto \int_C (j^1(\psi_t \circ \phi))^* L\pi_1^*\Omega
\end{aligned}$$

Definition 2.5.2. With the notation as above, the local section $\phi \in \Gamma_W(\pi)$ is an *extremal of* $\mathcal{L}$ if:

$$\left.\frac{d}{dt}\right|_{t=0} \int_C (j^1(\psi_t \circ \phi))^* L\pi_1^*\Omega = 0$$

for all such vector fields P with flow ψ_t satisfying $X|_{\pi^{-1}(\partial C)} = 0$.

Lemma 2.5.3. The local section ϕ is an extremal of L if and only if:

$$\int_C (j^1\phi)^* d_{P^1} L \pi_1^* \Omega = 0$$

where d_{P^1} is the Lie derivative along P^1, the first prolongation of P.

Proof. Let $p \in C$. We observe that:

$$\begin{aligned}
(j^1\phi)^*(d_{X^1}L)(p) &= X^1_{j^1_p\phi} L \\
&= \left[t \mapsto \psi^1_t(j^1_p\phi)\right](L) \\
&= \left[t \mapsto j^1(\psi_t \circ \phi)(p)\right](L) \\
&= \frac{d}{dt}\bigg|_{t=0} (j^1(\psi_t \circ \phi)(p))^* L(p)
\end{aligned}$$

□

What the above lemma says is that the variation in the traditional sense and the variation described in Section 2.4 via vector fields are one and the same. For brevity we shall not go further in variational calculus using jets, but we shall state a lemma that says that the above recovers the usual Euler-Lagrange equations. Essentially, solving the integral above requires one to extend to the second jet bundle.

Proposition 2.5.4 ([Sau89, §5.5.1])**.** Using the set-up above, suppose that the image of the local section $\phi(C)$ lies within a single coordinate system (x^i, u^α).

Then if ϕ is an extremal for the Lagrangian $\mathcal{L}$, it satisfies the equations:

$$(j^2\phi)^*\left(\frac{\partial \mathcal{L}}{\partial u^\alpha} - \frac{d}{dx^i}\frac{\partial \mathcal{L}}{\partial u^\alpha_i}\right) = 0,$$

where $(x^i, u^\alpha, u^\alpha_i)$ is the coordinate system on the second order jet bundle.

Jet bundles are a wonderful and rich topic by themselves, and worthy of an in-depth treatment that we do not have time for here. Some related texts with application to field theory are [McC94] and [BFR19]. Perhaps the main takeaway is that they are useful tools for encoding dynamics of field theories.

2.6 Sections of Jet Bundles

In this section, we describe the sections of the jet bundles $J^q\Sigma \to \Sigma$, primarily for use in Section 4. Similar results hold for the jets of bundles, but will not be necessary for our needs, as they end up being modules over the sections which we will calculate now.

Let $p \in M$ be a point and (x, U) a coordinate chart about p. Recall from Example 2.3.2 that a vector field P on U has presentation:

$$P = \sum_{i=1}^m \xi^i \frac{\partial}{\partial x_i}, \qquad \xi_i \in C^\infty(U). \tag{2.7}$$

Let $f \in C^\infty(U)$. The map $P(f)$ is then a local section on U of the cotangent bundle. Every section originates in this way. Combined with the identification (2.3) and the bundle projection (2.4), every local section on U of the first order Jet bundle is of the form:

$$P(f) + b, \qquad y \in U.$$

for some $b \in C^\infty(U)$ and some vector field P on U. Since a general first order linear differential operator Q has the form:

$$Q = \sum_{i=1}^{m} (\xi')^i \frac{\partial}{\partial x_i} + \psi', \qquad (\xi')^i, \psi' \in C^\infty(U). \tag{2.8}$$

we are inclined to hypothesize the following:

Proposition 2.6.1. Let $q \in \mathbb{Z}_{\geq 0}$. Let $p \in M$ be a point and (x, U) a coordinate chart about p.

Let $D_q(U)$ denote the vector space of qth order differential operators on U, and $\mathcal{J}^q(U)$ the space of local sections of $J^qM \to M$ on U.

Then both spaces are $C^\infty(U)$-modules, and:

$$D_q(U) \cong \mathrm{Hom}_{C^\infty(U)}(\mathcal{J}^q(U), C^\infty(U)).$$

Before we prove this, we identify $D_q(U)$ in a nicer manner. Consider the operator

$$Q : C^\infty(U) \to C^\infty(U)$$

as in (2.8). Observe that the constant term ψ' can be written coordinate-freely as $Q(1)$. Then one finds that $Q - Q(1)$ has representation in coordinates akin to (2.7), and so is a vector field. Similarly, if P is a vector field on U, then for any $\psi \in C^\infty(U)$, $P + \psi$ is a first order linear differential operator. So we make the following identification:

Lemma 2.6.2. An $\mathbb{R}$-linear map $P : C^\infty(M) \to C^\infty(M)$ is a first order linear differential operator if and only if $P - P(1)$ is a vector field.

Using the identification of vector fields as derivations, as in Definition 2.2.5, we have a nice description of differential operators algebraically.

Let $f, g \in C^\infty(M)$. Then $P - P(1)$ is a derivation if and only if

$$\begin{aligned}(P - P(1))(fg) &= f(P - P(1))(g) + g(P - P(1))(f) \\ &= fP(g) - fP(1)(g) + g\nabla(f) - g\nabla(1)(f) \\ &= fP(g) + gP(f),\end{aligned} \tag{2.9}$$

having made use of the operator $P(1)$ being multiplication by an element of $C^\infty(M)$.

Equation (2.9) is equivalently:

$$P(fg) - fP(g) = gP(f) - fgP(1). \tag{2.10}$$

It becomes convenient to have a notion of Lie bracket on $C^\infty(M)$ in which to express this.

Definition 2.6.3. The commutator on $\mathrm{End}_{\mathbb{R}}(C^\infty(M))$ is the $\mathbb{R}$-bilinear map:

$$\begin{aligned}[\cdot,\cdot] : \mathrm{End}_{\mathbb{R}}(C^\infty(M))^2 &\to \mathrm{End}_{\mathbb{R}}(C^\infty(M))\\ (A,B) &\mapsto [A,B] := A\circ B - B\circ A\end{aligned}$$

Adopting the convention that the operator in $\mathrm{End}_{\mathbb{R}}(C^\infty(M))$ acting by multiplication by an element $f \in C^\infty(M)$ is denoted $\bar{f}$, (2.10) becomes:

$$[P,\bar{f}](g) = g[P,\bar{f}](1).$$

Thus, an $\mathbb{R}$-linear map $P : C^\infty(M) \to C^\infty(M)$ is a first order differential operator if and only if $[P,\bar{f}]$ is a $C^\infty(M)$-module morphism, for all $f \in C^\infty(M)$.

If $T \in \mathrm{End}_{\mathbb{R}}(C^\infty(M))$ is additionally $C^\infty(M)$-linear, then for all $g \in C^\infty(M)$,

$$\begin{aligned}T(g) = gT(1) &= T(1)g,\\ [T,\bar{g}] &= 0.\end{aligned}$$

So T is multiplication by $T(1) \in C^\infty(M)$, and we make the identification:

$$\begin{aligned}\mathrm{Hom}_{C^\infty(M)}(C^\infty(M), C^\infty(M)) &\cong C^\infty(M)\\ &\cong \{T \in \mathrm{End}_{\mathbb{R}}(C^\infty(M)) \mid [T,\bar{f}] = 0 \text{ for all } f \in C^\infty(M)\}.\end{aligned} \tag{2.11}$$

Similarly, one can consider second order differential operators, say ∂_i^2 for some $i \in \{1,\ldots,m\}$. In which case, the commutator $[\partial_i^2, \bar{f}]$ maps $g \in C^\infty(M)$ as

$$g \mapsto \partial_i(f\partial_i g + g\partial_i f) - f\partial_i^2(g) = 2\partial_i f \partial_i g + g\partial_i^2 f.$$

That is, $[\partial_i^2, \bar{f}]$ maps g to a $C^\infty(M)$-linear combination of $\partial_i g$ and g, and so is a first-order linear operator. $\mathbb{R}$-linearity of the bracket then lets us make the following description of linear differential operators:

Proposition 2.6.4. Let $q \in \mathbb{Z}_{\geq 0}$. Let $D_q(M) \subseteq \mathrm{End}_{\mathbb{R}}(C^\infty(M))$ be the (analytically defined) qth order linear differential operators on M. Then each D_q is a $C^\infty(M)$-bimodule with $f \in C^\infty(M)$ acting on the left by

$$f \cdot T = fT$$

and on the right by

$$t \cdot f = T \circ \bar{f}.$$

Furthermore,

$$D_0(M) \cong \{T \in \mathrm{End}_{\mathbb{R}}(C^\infty(M)) \mid [T,\bar{f}] = 0 \text{ for all } f \in C^\infty(M)\} \tag{2.12}$$

and

$$D_q(M) \cong \{T \in \mathrm{End}_{\mathbb{R}}(C^\infty(M)) \mid [T,\bar{f}] \in D_{q-1}(M) \text{ for all } f \in C^\infty(M)\} \tag{2.13}$$

as $C^\infty(M)$-bimodules.

Proof. See [Nes20] Theorem 9.62. $\square$

An identical proof shows that one may replace M in (2.12) and (2.13) by an open subset $U \subseteq M$. Let us now prove Proposition 2.6.1, on the categorical (co)representation of D_q.

Proof of Proposition 2.6.1. Let $h \in \mathrm{Hom}_{C^\infty(U)}(\mathcal{J}^q(U), C^\infty(U))$. Consider the morphism:

$$\begin{aligned} j_q : C^\infty(U) &\to \mathcal{J}^q(U) \\ f &\mapsto [f]^q = (x \mapsto [f]^q_x) \end{aligned}$$

and the composition:

$$\begin{tikzcd} C^\infty(U) \arrow[r, "j_q"] \arrow[dr] & \mathcal{J}^q(U) \arrow[d, "h"] \\ & C^\infty(U) \end{tikzcd}$$

We proceed by induction to prove that $h \circ j_q \in D_q$. In the case $q = 0$, j_0 is the identity, so $h \circ j_0 = h$. By assumption, h is a module morphism and so by (2.11) and (2.12), $h \in D_0(U)$.

Assume now that $h \circ j_{q-1} \in D_{q-1}$. Then for any $f, g \in C^\infty(U)$,

$$\begin{aligned} [h \circ j_q, \bar{f}](g) &= h([fg]^q) - fh([g]^q) \\ &= h([fg]^{q-1} + a_q) - fh([g]^{q-1} + b_q), \end{aligned}$$

where a_q and b_q are the sums of the qth order derivatives of fg and g respectively. Thus:

$$\begin{aligned} [h \circ j_q, \bar{f}](g) &= [h \circ j_{q-1}, \bar{f}](g) + h(a_q) - fh(b_q) \\ &= [h \circ j_{q-1}, \bar{f}](g) + a_q h(1) - fh(1)b_q \end{aligned}$$

using the assumption that h is a module morphism. The first term is, by the induction hypothesis, an element of $D_{q-1}(U)$ applied to g, and the other terms are a $C^\infty(U)$-linear combination of derivatives of g of order less than or equal to q, by successive use of the Leibniz rule on fg. The only terms inside a_q that is a qth order derivative of g has coefficient $fh(1)$, cancelling with $fh(1)b_q$. Hence, $[h \circ j_q, \bar{f}] \in D_q(U)$.

Thus we have a morpism:

$$\begin{aligned} \mathrm{Hom}_{C^\infty(U)}(\mathcal{J}^q(U), C^\infty(U)) &\to D_q(U) \\ h &\mapsto h \circ j_q \end{aligned}$$

Conversely, given $P \in D_q(U)$, define a map:

$$\begin{aligned} h_P : \mathcal{J}^q(U) &\to C^\infty(U) \\ [f]^q &\mapsto P(f) \end{aligned}$$

To see this is well defined, take $f, g \in C^\infty(U)$ to be two representatives of $[f]^q$. Recall that this means that for any $y \in U$, there is a neighbourhood V of y such that $(f-g)(x) = a(x)\zeta(x)^{q+1}$ where $a, \zeta \in C^\infty(U)$ and $\zeta(y) = 0$. Then a q-fold application of the Leibniz rule implies that $P(f-g)$ has a factor of ζ, and so $P(f-g)(y) = 0$.

Finally, our maps are inverse to eachother, as:

$$P \mapsto h_P \mapsto h_P \circ j_q : (f \mapsto [f]^q \mapsto P(f))$$

Therefore we have an isomorphism

$$D_q(U) \cong \mathrm{Hom}_{C^\infty(U)}(\mathcal{J}^q(U), C^\infty(U)).$$

□

2.7 Poisson Algebras and Hamiltonians

The following two sections are deliberately detail scarce, but give valuable intuition.

Let us return briefly to the case of classical mechanics, where we are interested in paths $[t_0, t_1] \to E$ into a manifold E. The space of smooth functions $C^\infty(T^*E)$ is naturally an $\mathbb{R}$-algebra, with multiplication and addition defined pointwise. It is also a *Poisson algebra.*

Definition 2.7.1. A *Poisson Algebra* is the following:

1. A module A over a commutative ring R.
2. An associative, commutative multiplication map
$$\bullet : A \otimes_R A \to A.$$
3. A Lie bracket
$$\{\cdot, \cdot\} : A \otimes_R A \to A,$$
such that, for every $f \in A$, $\{f, \cdot\} : A \to A$ is a derivation with respect to the product $\bullet$.

Example 2.7.2. Again, let E be a manifold and let $A = C^\infty(T^*E)$ be the algebra of real-valued functions on E. Recall that if (x^i) are some local coordinates on E, then there are induced coordinates (x^i, p_j) on T^*E. We have a pointwise multiplication map as described above, and a Lie bracket defined in local coordinates as:

$$\{f, g\} := \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x^i} \frac{\partial f}{\partial p_i} \right)$$

which together make A a Poisson algebra.

Here is a question: why care about Poisson algebras? Well, the reason is that it allows one to talk about *Hamiltonians.*

Definition 2.7.3. Let E be a manifold and let $A = C^\infty(T^*E)$ be the Poisson algebra of real-valued functions.

Given a vector field P on T^*E, a *Hamiltonian for P* is a smooth function $H_P \in A$ such that

$$\{H_P, \cdot\} : A \to A$$

is exactly the derivation $P : A \to A$.

So the question becomes: why care about Hamiltonians? Well, if one fixes a Lagrangian $L \in A$, and P is a vector field on T^*E with (local) flow ϕ_t, then for any $f \in A$, one has

$$\{H_P, f\} = \frac{d}{dt} f(\phi_t).$$

So, Hamiltonians govern the time evolution of the system in question – they define the dynamics.

One can do the same for any symplectic manifold (X, ω); this is summarized in [WEK97, Lecture 1].

Thus from a pair (T^*E, L) of a phase space and a Lagrangian, one constructs a pair (A, H) of a Poisson algebra and a Hamiltonian which defines the same dynamics. The second is preferable, because it defines the mechanics intrinsically, unlike the Lagrangian picture. Examples of such are shown in [WEK97].

Furthermore, using the techniques developed in Section 3, one can reconstruct the phase space from the Poisson algebra by taking spectra.

In the field theory case, though, the Poisson structure is less obvious. We shall make the following explicit and formal in Section 4; for now we shall develop.

Recall that we instead focus on the (local) sections of a bundle $\pi : \Sigma \to E$, thinking of Σ as an intrinsic parameter space and E as an extrinsic phase space. For $U \subseteq \Sigma$, let $s, t \in \Gamma_U(\pi)$ be two local sections. The issue that appears is that a local associative multiplication should allow for s and t to be evaluated at different points $x, x' \in \Sigma$. The output of the pairing should of course itself be a field: in order to do so one needs to introduce something acting like a Dirac delta, so that neither coordinate x, x' is preferred under the pairing. The pairing thus should look something like:

$$(s(x), t(x')) \mapsto \phi(x)\delta(x - x') \tag{2.14}$$

Regardless of how to make this appropriately rigorous, a locally defined multiplication map is not much use unless one can define it consistently across the entire bundle. The following are the appropriate formulation with which to approach the topic.

Definition 2.7.4. Let $\pi_1 : E_1 \to \Sigma_1$ and $\pi_2 : E_2 \to \Sigma_2$ be two bundles. Denote by $\mathrm{pr}_i : \Sigma_1 \times \Sigma_2 \to \Sigma_i$ the projection of the Cartesian product onto the ith component. One has the pullbacks:

$$\begin{tikzcd}
 & \mathrm{pr}_1^* E_1 \arrow[dl] \arrow[dr] & & \mathrm{pr}_2^* E_2 \arrow[dl] \arrow[dr] & \\
E_1 \arrow[dr, "\pi_1"'] & & \Sigma_1 \times \Sigma_2 \arrow[dl, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"'] & & E_2 \arrow[dl, "\pi_2"] \\
 & \Sigma_1 & & \Sigma_2 &
\end{tikzcd}$$

The *external tensor product of* π_1 *and* π_2 is the bundle over $\Sigma_1 \times \Sigma_2$ defined by

$$E_1 \boxtimes E_2 := \mathrm{pr}_1^* E_1 \otimes_{\Sigma_1 \times \Sigma_2} \mathrm{pr}_2^* E_2. \tag{2.15}$$

Definition 2.7.5. Let Σ be a manifold. We define the *diagonal embedding* to be

$$\begin{aligned} \Delta : \Sigma &\to \Sigma \times \Sigma \\ x &\mapsto (x, x). \end{aligned} \tag{2.16}$$

Notably, this is a diffeomorphism onto its image.

Morally, the external tensor product allows one to take two bundles (over possibly different bases) and mash them into a single bundle. In practice, though, the sections of $E \boxtimes E$ allow for one to formulate pairs of sections of $E \to \Sigma$ with different variables.

If one denotes the local sections of $\pi : E \to \Sigma$ by $\mathcal{A}$ (really, this is a *sheaf of modules*, but here we shall think of it as a space of functions), then the pairing of fields alluded to in (2.14) becomes a morphism:

$$\mathcal{A} \boxtimes \mathcal{A} \to \Delta_* \mathcal{A} \tag{2.17}$$

One must pushforward by the diagonal both to develop an algebraic notion of δ above, but also so that both sides of (2.17) are defined over the same space $\Sigma \times \Sigma$. A local Lie bracket is defined analogously. [Kac03] (notes courtesy of R. Heluani) calls this notion a *Courant algebroid*. [BD04, §0] refers to them as a *coisson algebra*, short for 'chiral Poisson' or 'compound Poisson'.

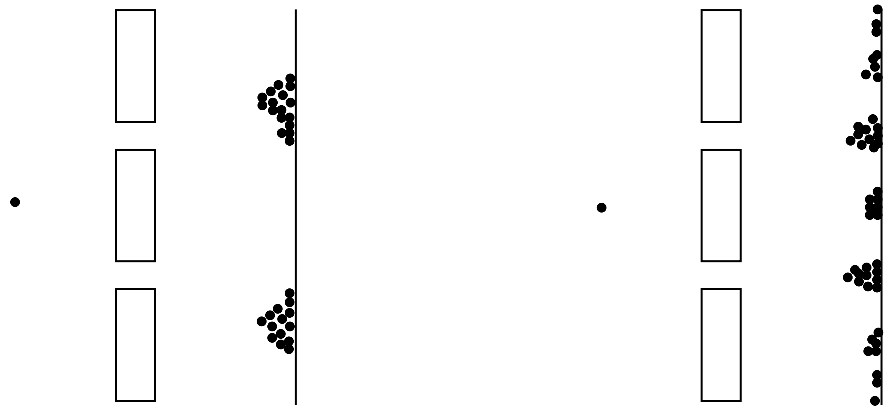
Physically, the sheaf $\mathcal{A}$ takes centre stage once more because in analogy with the case described at the beginning of the section, the dynamics of a field theory are prescribed by a choice of field $\mathcal{H}$; the *Hamiltonian density*. See [Gan06, §4] for a fantastic run-through.

2.8 What is quantization?

Quantization is informally any method by which one can translate the classical notions of physics into a language which describes the extremely short distance behaviour observed in experiment sufficiently well. The 'correct and fully descriptive' method of quantization is likely not known; we do not have the time or will to give a full description here regardless. What will be mentioned here, however, will be a brief description of the most basic behaviour which we desire to model, and which will intuit us into Section 3 and beyond.

One of the most striking behaviours of quantum mechanics is the importance of *observation*. This is best exhibited in the double-slit experiments first performed in the early 20th century. The idea is simple: direct a small enough object (for example a photon or an electron) towards a wall with two sufficiently small slits in it. Classically, the object should hit the wall and be done.

In experiment, however, one observes impacts beyond the initial wall. What's more, first measuring just beyond the wall and first measuring further out exhibit different results. See the cartoon below.



a)   b)

For experts, what this demonstrates is the principle of superposition. Some ([KK14, §1], for instance) describe this as the most fundamental aspect of quantum mechanics. We shall focus on the more obvious phenomenon of observation.

What the above demonstrates is that, there appears some 'non-commutativity' at small distances. A famous notion of the Heisenberg uncertainty relation is developed from this; a tool to study this relation is the *Weyl algebra*:

$$A_N := \frac{\mathbb{C}[x_1, \dots, x_N, y_1, \dots, y_N]}{\langle x_\ell y_\ell - y_\ell x_\ell = 1 \mid 1 \leq \ell \leq N \rangle}, \tag{2.18}$$

thinking of x_ℓ as position and y_ℓ as momentum.

In general, given a Poisson algebra A, a *deformation quantization* of A is a family of non-commutative algebras $A_\hbar$ parametrized by a number $\hbar$, thought of as the 'degree of non-commutativity', along with a family of multiplication maps and brackets $A \times A \to A_\hbar$. This is discussed somewhat in [Kac03], and in [Gan06, pg. 248]. Each $A_\hbar$ is supposed to be seen as a small deformation to the original A.

In Section 2.7, we saw that the algebra A along with a specified element $H \in A$ determines a classical physical system. It is correspondingly called the *algebra of observables*. Such algebras are our starting point for Section 3. After developing some tools to deal with these things, the manner in which we approach quantization in Section 4 is essentially by developing a 'local' action by Weyl algebras onto the relevant object associated to the algebra of observables.

3 Algebraic Geometry from a Physical Perspective

3.1 Measuring Devices and Duality

Observability, as introduced in Section 2.8 becomes a central concept in physics. One thus might consider the same concept for the well-trodden classical physics also.

In 2.1 we described motion of an object by a function $\mathbf{r}(t)$ of time. This is an acceptable description in the classical case in which what is observed and what occurs are equivalent. A more natural description however, once one has identified the criticality of observability, is to describe the position of an object by a system of *measuring devices*, and the associated mathematical objects. Physically, a measuring device is a machine that measures the state of a system and outputs a reading. Mathematically,

Definition 3.1.1. Suppose a system has possible states described by a set of states $\mathcal{S}$; a manifold. A measuring device is a smooth function:

$$f : \mathcal{S} \to \mathbb{R}$$

assigning a state $s \in \mathcal{S}$, a point in a smooth manifold, to its reading $f(s)$.

The assumption that things be smooth is a common one in physics, and very reasonable in the classical case we are in - a small change in state should intuitively result in a small change in reading. An important part of measuring devices is the following; an assumption from physics:

Proposition 3.1.2. Given measuring devices $f, g : \mathcal{S} \to \mathbb{R}$ and a real number $\lambda \in \mathbb{R}$, the functions $f + g, f \cdot g, \lambda f : \mathcal{S} \to \mathbb{R}$, defined pointwise, are all also measuring devices. Furthermore, the constant functions $0 : s \mapsto 0$ and $1 : s \mapsto 1$ are measuring devices.

That is, the set of measuring devices for a system form an $\mathbb{R}$-algebra.

Proposition 3.1.2 is the statement corresponding to the physical idea that if you have two measuring devices, you can construct a device that displays the sum or product of the two measurements, and that the number displayed by one can be scaled.

Proposition 3.1.2, as stated, relies on the knowledge of the possible states of the system $\mathcal{S}$. However, the whole introduction of observability was to reinforce the fact that $\mathcal{S}$ is not a priori a known quantity. This begs the question:

Question 3.1.3. Given an arbitrary $\mathbb{R}$-algebra A; our measuring devices, can we recover the states of our system? That is, given A, does there exist a manifold $\mathcal{S}$ such that

$$A \cong \{f : \mathcal{S} \to \mathbb{R} \mid \text{f is smooth}\}?$$

We shall see that the condition that A be the whole space of functions is too stringent, and too much structure is needed in general for $\mathcal{S}$ to be a manifold. However, we shall see that algebras are indeed a subalgebra of the continuous functions on a Hausdorff space!

A similar and popular result in linear algebra exists for vector spaces, namely that a finite-dimensional vector space V is canonically isomorphic to the dual of its dual space. A similar result holds for $\mathbb{R}$-algebras, with a significantly less stringent condition.

Definition 3.1.4. Let $\Bbbk$ be a field, and A be a $\Bbbk$-algebra. The $\Bbbk$-*algebra dual space*, or $\Bbbk$-*spectrum* of A, is the set:

$$|A| := \{x : A \to \Bbbk \mid x \text{ is a } \Bbbk\text{-algebra morphism}\}.$$

Theorem 3.1.5. Let A be an $\mathbb{R}$-algebra. Consider the set:

$$\tilde{A} := \{\tilde{f} : |A| \to \mathbb{R} \mid f \in A, \text{ for any } x \in |A|,\ \tilde{f}(x) := x(f)\} \tag{3.1}$$

and the map:

$$\begin{aligned} \tau : A &\to \tilde{A} \\ f &\mapsto \tilde{f}. \end{aligned}$$

Then $\tilde{A}$ is a $\Bbbk$-algebra, and $\tau : A \to \tilde{A}$ is a surjective $\mathbb{R}$-algebra morphism. Furthermore, τ is an isomorphism if and only if

$$N_A := \bigcap_{x \in |A|} \ker(x) = 0. \tag{3.2}$$

Proof. The algebra structure on $\tilde{A}$ is given by:

$$\begin{aligned} (\tilde{f} + \tilde{g})(x) &= x(f) + x(g) \\ (\tilde{f}\tilde{g})(x) &= x(f)x(g) \\ (\lambda \tilde{f})(x) &= \lambda x(f) \end{aligned}$$

with units $\tilde{0}, \tilde{1}$. The coherency conditions for vector spaces and rings are inherited from $\Bbbk$.

With this structure, τ is a $\Bbbk$-linear map, and is surjective by definition of $\tilde{A}$.

For $f, g \in A$ and $x \in |A|$, we have

$$\begin{aligned} \tau(fg)(x) = \widetilde{fg}(x) = x(fg) = x(f)x(g) &= (\tilde{f}\tilde{g})(x), \\ \tau(1)(x) = x(1) = 1 &= \tilde{1}(x), \end{aligned}$$

because each x is a morphism, so that $\tau(fg) = \tau(f)\tau(g)$ and $\tau(1) = \tilde{1}$. That is, τ is a ring homomorphism.

It remains to show whether or not τ is injective.

Suppose that τ is injective. Let $f \in N$. Then for all $x \in |A|$,

$$x(f) = 0, \quad \text{so that } \tau(f) = 0.$$

Thus as τ is injective, $f = 0$. So:

$$N_A = \bigcap_{x \in |A|} \ker(x) = 0$$

Suppose now that

$$\bigcap_{x \in |A|} \ker(x) = 0.$$

Let $f \in A$ be such that $\tau(f) = 0$. Then for every $x \in |A|$,

$$\tau(f)(x) = x(f) = 0,$$

and thus $f \in N_A$. Therefore $f = 0$, and so τ is injective.

Therefore τ is an isomorphism of $\Bbbk$-algebras if and only if $N_A = 0$. □

Remark 3.1.6. The notation for the elements $x \in |A|$ are telling – they are to form the point for the set of states corresponding to A.

Remark 3.1.7. In words, the condition for a topological space to be Hausdorff is the imposition that points be distinguishable under the topology. Compare this to the condition (3.2) in 3.1.5 that is equivalent to τ being an isomorphism. This is exactly the imposition that two points in the underlying space be distinguishable by the functions on the space – that there is no pair of distinct points $x, y \in |A|$ such that for all $f \in A$, $f(x) = f(y)$.

Physically, it says that every state of the system is observable.

Corollary 3.1.8. Let A be an $\mathbb{R}$-algebra such that

$$\bigcap_{x \in |A|} \ker(x) = 0. \tag{3.3}$$

Then A is a subalgebra of the real-valued continuous functions on the Hausdorff topological space $|A|$. That is, every element $f \in A$ can be realised as a continuous function on a Hausdorff space.

Proof. From Theorem 3.1.5 and (3.3) we have $A \cong \tilde{A}$, so each $f \in A$ can be identified as a function from $|A|$ to $\mathbb{R}$.

$|A|$ inherits a topology from $\mathbb{R}$. Explicitly, it is the topology generated by the sets $f^{-1}(V)$, where $f \in A$ is a function $|A| \to \mathbb{R}$ and $V \subseteq \mathbb{R}$ is an open set of $\mathbb{R}$. Each f is thus continuous by definition of the topology. It happens that this topology agrees with what is expected in thte case $A = C^\infty(\mathbb{R}^N)$; namely $|A|$ is homeomorphic to $\mathbb{R}^N$. See [Nes20, §3.15-16] for details.

It remains to show that $|A|$ is Hausdorff. Let $x, y \in |A|$ be distinct points of $|A|$. Let $f \in A$ be such that $f(x) \neq f(y)$. Such an f exists by Remark 3.1.7. Then there exist disjoint open neighbourhoods of x and y:

$$f^{-1}\left(\left(-\infty, \frac{f(x)+f(y)}{2}\right)\right) \cap f^{-1}\left(\left(\frac{f(x)+f(y)}{2}, \infty\right)\right) = \emptyset$$

so $|A|$ is Hausdorff. □

So far, we have managed to assign a (nice) space to a (nice) algebra; a space of states to a system of measuring devices. We will defer our requirement that the space be a manifold until later. It happens that our work so far is *natural* in the sense of categories.

Lemma 3.1.9. Let $\mathbb{R}$-Alg denote the category with objects $\mathbb{R}$-algebras and arrows morphisms of $\mathbb{R}$-algebras. Let Haus denote the category with objects Hausdorff spaces, and arrows continuous maps.

For an object A of $\mathbb{R}$-Alg, denote

$$N_A = \bigcap_{x \in |A|} \ker(x).$$

The assignment:

$$\begin{aligned} \nu : \mathbb{R}\text{-Alg} &\to \mathbb{R}\text{-Alg} \\ A &\mapsto {}^{A}\!/\!_{N_A} \end{aligned} \tag{3.4}$$

of quotient maps is a functor.

Proof. Let $\varphi : A \to B$ be a morphism of $\mathbb{R}$-algebras. We define $\nu(\varphi)$ as follows:

$$\begin{tikzcd} A \arrow[r, "\nu_A"] \arrow[d, "\varphi"'] & {}^{A}\!/\!_{N_A} \arrow[d, "\nu(\varphi)"] \arrow[dl, dashed, "\bar{\varphi}"'] \\ B \arrow[r, "\nu_B"'] & {}^{B}\!/\!_{N_B} \end{tikzcd}$$

Here $\bar{\varphi}$ exists uniquely from the universal property of the quotient map, and $\nu(\varphi)$ is defined as the composition $\nu_B \circ \bar{\varphi}$. Thus, for another morphism $\psi : b \to C$, commutativity of the diagram:

$$\begin{tikzcd} A \arrow[r, "\nu_A"] \arrow[d, "\varphi"'] & {}^{A}\!/\!_{N_A} \arrow[d, "\nu(\varphi)"] \arrow[dl, dashed, "\bar{\varphi}"'] \\ B \arrow[r, "\nu_B"'] \arrow[d, "\psi"'] & {}^{B}\!/\!_{N_B} \arrow[d, "\nu(\psi)"] \arrow[dl, dashed, "\bar{\psi}"'] \\ C \arrow[r, "\nu_C"'] & {}^{C}\!/\!_{N_C} \end{tikzcd}$$

and universality imply that $\nu(\psi \circ \varphi) = \nu(\psi) \circ \nu(\varphi)$. It is clear that $\nu(\mathrm{Id}_A) = \mathrm{Id}_{A/N_A}$. Thus, ν is a functor.

□

Corollary 3.1.10. Let A be an $\mathbb{R}$-algebra. Let $B = \nu(A)$.
Then $N_B = 0$.

Corollary 3.1.11 ([Nes20, §3.11])**.** Let A be an $\mathbb{R}$-algebra.
Then, as sets,

$$|A| = |\nu(A)|$$

Note that this is *not* a homeomorphism in general.

Remark 3.1.12. We note that this perspective to algebras is a form of Gelfand-Kolmogorov duality. Typical Gelfand duality uses C^* algebras instead of commutative unital $\mathbb{R}$-algebras. A similar reasoning is occasionally used to motivate quantum mechanics, in terms of C^* algebras. The disadvantage to this is that these are considerably more difficult to work with than algebras in the sense used here.

Instead, we started with a case more suitable for the classical case. The benefit of this is that it quite smoothly transitions into algebraic geometry, as we shall see. What one lacks from not considering C^*-algebras, one makes up for by doing analysis in the setting of $\mathcal{D}$-modules; the topic of Section 4.

3.2 Localization and Sheaves

So, from Section 3.1, we can view an algebra as some continuous functions on a space that is Hausdorff under a mild assumption. An immediate question would be:

Question 3.2.1. Does the topological structure on $|A|$ 'fit' the algebraic structure of A as a space of functions? Since we have a notion of openness on $|A|$, we can restrict functions on $|A|$ to an open subset $U \subseteq |A|$, and get continuous functions on the subspace U. What is the appropriate sense of restricting the algebra A to such subspaces?

We shall relate this question, and many following, to our normal understanding of Hausdorff spaces and manifolds.

Example 3.2.2. Let X be a Hausdorff space, and let $A = C^0(X)$ be the $\mathbb{R}$-algebra of continuous functions from X to $\mathbb{R}$. We note that, given $f, g \in A$, the quotient f/g is well-defined and continuous on X so long as g never vanishes.

For $x \in X$ and $U \subsetneq X$ a proper open neighbourhood of x, one would expect (demand, really) restriction of the algebra A to U to be $C^0(U)$.

Let $f, g \in A$. Suppose $y \in X \setminus U$ is such that

$$\begin{aligned} g(y) &= 0, \\ f(y) &\neq 0 \text{ and} \\ g(x') &\neq 0, \end{aligned}$$

for any $x' \in U$.

Being functions, there is a notion of topological restriction for these functions; associating to $f : X \to \mathbb{R}$ the continuous function $f|_U : U \to \mathbb{R}$. As mentioned above we know $f/g \notin A$, since g vanishes somewhere in X, but as g never vanishes in U,

$$\left.\left(\frac{f}{g}\right)\right|_U := \frac{f|_U}{g|_U} \in C^0(U).$$

So $(f/g)|_U \in C^0(U)$ but, a priori, does not necessarily originate from a globally defined function in A. In Example 3.2.5 below we shall give an example where this is definitively not the case.

The takeaway here is that, in a case in which we have a preferred notion of what the restriction of the whole algebra should be, while the restriction of *functions* will land in the restriction of *algebras*, the latter is not comprised entirely of the former; there are 'more' functions to be had inside of the restriction.

To do this in the case of smooth manifolds we introduce *localization*, and in general, the concept of *sheaf*.

Definition 3.2.3. Let R be a commutative ring with unity.

Let $S \subseteq R$ be a subset that is multiplicatively closed. That is, one has $1 \in S$, and if $s, t \in S$ then $st \in S$.

The *localization of R at S* is the set:

$$S^{-1}R = \left\{ \frac{r}{s} \;\middle|\; r \in R, s \in S \right\} /\!\sim$$

where

$$\frac{r}{s} \sim \frac{r'}{s'} \text{ if and only if there exists an } s'' \in S, \text{ such that } s''(rs' - r's) = 0.$$

Lemma 3.2.4. Let R be a commutative ring with unity, and $S \subseteq R$ a multiplicatively closed subset.

Then the localization $S^{-1}R$ is a ring with operations:

$$\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'}, \qquad \frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'},$$

for all $r, r' \in R$ and $s, s' \in S$.

localization is the tool which allows us to consider formal fractions of ring elements, and formalise the discussions made in Question 3.2.1 and Example 3.2.2.

Example 3.2.5.

I) Let X be a Hausdorff space, and let $A = C^0(X)$ be the $\mathbb{R}$-algebra of continuous functions from X to $\mathbb{R}$. Let $x \in X$ and $U \subsetneq X$ an open neighbourhood of x.

   Set
   $$S_U = \{ f \in A \mid f(x') \neq 0 \text{ for all } x' \in U \}.$$

   Then S is multiplicatively closed, and the localization $S_U^{-1}A$ is a subring of the algebra $C^0(U)$.

   Consider the case where $X = [0,1] \subset \mathbb{R}$ equipped with the subspace topology. Consider the function
   $$h(x) = \sin(1/x),$$
   defined on $U = (0,1]$.

   Suppose that $g \in A$ is a function such that $gh \in A$. If one had $g \in S_U$, then one would have
   $$gh/g = h \in S_U^{-1}A.$$

   h is notably not continuous at $x = 0$, so for gh to be continuous we would require $g(x) = 0$ on an open neighbourhood of 0. Thus $g \notin S_U$, and so f does not arise from the localization of a function. Therefore $S_U^{-1}A \neq C^0(U)$.

Consider the maximal ideal

$$\mathfrak{m}_x = \{f \in A \mid f(x) = 0\}.$$

Then $S = A \setminus \mathfrak{m}_x$ is a multiplicative set. The localization of A at S is called the localization at x, or *the stalk at* x, and denoted C^0_x. We observe here for later that

$$C^0_x = \varinjlim_{U \ni x} S_U^{-1} A = \varinjlim_{U \ni x} C^0(U).$$

II) In the setting of I), if we assume that X is additionally a smooth manifold, and $A = C^\infty(X)$ is the $\mathbb{R}$-algebra of smooth functions, the existence of partitions of unity (see [GP74, §1.8]) in fact guarantees that

$$S_U^{-1} A = C^\infty(U).$$

We introduce some critical commutative algebra notions in the following.

Definition 3.2.6. Let R be a commutative ring with unity.

I) An ideal $\mathfrak{p}$ of R is *prime* if $\mathfrak{p} \neq R$ and, given two elements $s, t \in R$ such that $st \in \mathfrak{p}$, at least one of $s \in \mathfrak{p}$ or $t \in \mathfrak{p}$ holds.

II) An ideal $\mathfrak{m}$ of R is *maximal* if $\mathfrak{m} \neq R$ and if $I \subseteq R$ is an ideal such $\mathfrak{m} \subseteq I$, then $I = R$.

III) A ring R is *local* if it has a unique maximal ideal $\mathfrak{m}$.

IV) A morphism of local rings $(A_1, \mathfrak{m}_1) \to (A_2, \mathfrak{m}_2)$ is a morphism of rings $\varphi : A_1 \to A_2$ such that $\varphi(\mathfrak{m}_1) \subseteq \mathfrak{m}_2$.

Example 3.2.7. Let R be a commutative ring with unity, and $\mathfrak{p}$ a prime ideal of R. Then $S = R \setminus \mathfrak{p}$ is multiplicatively closed, and we denote the localization of R at S by $S^{-1}R = R_\mathfrak{p}$. By an abuse of terminology, we call $R_\mathfrak{p}$ the *localization at* $\mathfrak{p}$. The examples 3.2.5 above are special cases of this, using not just prime ideals but maximal ideals.

It is worth noting for later that $R_\mathfrak{p}$ has exactly one maximal ideal, namely $\mathfrak{p}R_\mathfrak{p}$. So $R_\mathfrak{p}$ is local.

Example 3.2.5 shows that an algebraic approach may not be the best for talking about restriction – defining continuous functions on a subspace is worlds easier than recovering the appropriate functions algebraically. Though, if you have a sufficient notion of smoothness, they end up being the same. It benefits us later equally to have the definition:

Definition 3.2.8. Let X be a topological space, and $\mathcal{C}$ a category.

Define Open(X) to be the category with objects open subsets of X, and with morphisms inclusion maps $U \hookrightarrow V$ of open sets $U \subseteq V \subseteq X$.

Then we define a $\mathcal{C}$*-valued presheaf* to be a contravariant functor

$$\mathcal{F} : \mathrm{Open}(X) \to \mathcal{C}.$$

Unless otherwise specified, we shall take from now on a 'presheaf' to mean a presheaf valued in CRing, the category of commutative unital rings.

Given a morphism $\iota : U \hookrightarrow V$ of Open(X), the image of ι under a presheaf $\mathcal{F}$ is called the *restriction* from V to U, denoted Res^V_U.

This turns out to not be descriptive enough of the local behaviour, so one restricts this to a more easy to manipulate definition as follows.

Definition 3.2.9. Let X be a topological space. Let $\mathcal{F} : \mathrm{Open}(X) \to \mathcal{C}$ be a presheaf valued in an Abelian[2] category $\mathcal{C}$.

One says $\mathcal{F}$ is a *sheaf* if for every open subset $U \subseteq X$, and every open covering $\{U_\alpha\}_{\alpha \in I}$ of U, $\mathcal{F}$ satisfies the following:

Locality) Let $f, g \in \mathcal{F}(U)$. If
$$\mathrm{Res}^U_{U_\alpha} f = \mathrm{Res}^U_{U_\alpha} g$$
for all $\alpha \in I$, then $f = g$.

Gluing) If $f_\alpha \in \mathcal{F}(U_\alpha)$ are a collection of elements such that
$$\mathrm{Res}^{U_\alpha}_{U_\alpha \cap U_\beta} f_\alpha = \mathrm{Res}^{U_\beta}_{U_\alpha \cap U_\beta} f_\beta$$
for all $\alpha, \beta \in I$, then there exists $f \in \mathcal{F}(U)$ such that $\mathrm{Res}^X_{U_\alpha} f = f_\alpha$ for all α.

In plain language, a sheaf is a local assignment of algebraic structures such that if local elements agree locally, then they agree globally; and if some collection of elements defined locally agree on intersections then one can find something globally defined which is locally those elements.

Again, unless stated specifically, we shall assume our sheaves to be valued in CRing.

Definition 3.2.10. Let X be a topological space, and $\mathcal{F}, \mathcal{G}$ two sheaves on X. A *morphism of sheaves* is a natural transformation
$$\tau : \mathcal{F} \stackrel{\cdot}{\longrightarrow} \mathcal{G},$$
recalling that $\mathcal{F}$ and $\mathcal{G}$ are functors $\mathrm{Open}(X) \to \mathrm{CRing}$.

Recall that this means for all open subsets $U \subseteq V \subseteq X$, one has the following commutative diagram:
$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r, "\tau_V"] \arrow[d, "\mathrm{Res}^V_U"'] & \mathcal{G}(V) \arrow[d, "\mathrm{Res}^V_U"] \\
\mathcal{F}(U) \arrow[r, "\tau_U"'] & \mathcal{G}(U)
\end{tikzcd}$$

We now state a rather obvious statement, but a non-trivial and extremely useful one. Full details can be found in [Sta24, Section 009H].

Proposition 3.2.11. Let X be a topological space with basis $\mathcal{U}$, and $\mathcal{F}$ a presheaf on X.

Then it suffices to check the sheaf conditions on unions $\bigcup U_\alpha$ of basis subsets $U_\alpha \in \mathcal{U}$.

[2] If this language is unfamiliar, just take $\mathcal{C}$ to be CRing, Ring or A-Mod for our purposes. See [ML98, §VIII].

Example 3.2.12. Possibly the most important example of a sheaf for our purposes is the following.

Let Σ be a manifold, and let $\pi : E \to \Sigma$ be a rank m vector bundle. Denote by $A = C^\infty(\Sigma)$ the $\mathbb{R}$-algebra of smooth functions on Σ. Consider the presheaf:

$$\Gamma : \mathrm{Open}(\Sigma) \to A\text{-Mod}$$

sending an open $U \subseteq \Sigma$ to $\Gamma_U(\pi)$, the space of sections of π defined on U. We note that each $\Gamma_U(\pi)$ is an A-module. Each section $s \in \Gamma_U(\pi)$ is a smooth function $s : U \to E$, so by general differential topology s is determined exactly by its restriction to open subsets of U. So Γ is an A-Mod valued sheaf.

Example 3.2.13. Let Σ be a manifold, and let $\tau_\Sigma : T\Sigma \to \Sigma$ be its tangent bundle. We denote the sheaf of sections of the tangent bundle by $\mathcal{T}_\Sigma$.

In the context of schemes, we define the tangent sheaf of an affine $\Bbbk$-scheme $\mathrm{Spec}(A)$ to be the sheaf corresponding to the presheaf $U \mapsto \mathrm{Der}_\Bbbk(\mathcal{O}_A(U))$ of derivations, in analogy with the manifold case.

Remark 3.2.14. In reference to physics, the notion of restricting one's algebra of measuring devices is important. There may be global properties one can measure, but ultimately, one will only be able to measure in a compact area, in which the properties may differ. For instance, we assume that energy is universally conserved (in a non-relativistic setting, though a similar notion holds relativistically). However, there are certainly local systems of interest in which energy is not conserved; global behaviour is rarely indicative of local behaviour.

Compare this again to the notion of bundles central to Section 2. Much of the behaviour we are interested in is captured in the sections of bundles, as a sheaf. See Example 3.2.12.

Suppose one has an algebra of measuring devices A, comprised of functions on a space $|A|$, and that A has a sense of restriction. That is, A corresponds to the global sections of a sheaf. Then the sheaf conditions have a nice interpretation.

The presheaf condition imposes that a measuring device agrees with the device measuring the same quantity on a smaller set of possible states. Locality imposes that two measuring devices measuring the same value everywhere must measure the same quantity. Gluing supposes that every quantity measured locally, originates from some universal quantity.

These suit the classical case very well.

Wanting to consider an algebra A as a set of functions on a space $|A|$ with a notion of restriction all together leads to the following definition.

Definition 3.2.15. A *ringed space* is a pair $(X, \mathcal{F})$ comprised of a topological space X and a sheaf on X valued in CRing.

A *locally ringed space* is a ringed space $(X, \mathcal{F})$ such that for every $x \in X$, the *stalk*

$$\mathcal{F}_x := \varinjlim_{U \ni x \text{ open}} \mathcal{F}(U)$$

is a local ring.

Example 3.2.16. Let Σ be a manifold. The assignment

$$C^\infty_\Sigma : \mathrm{Open}(\Sigma) \to \mathrm{CRing}$$

sending an open $U \subseteq \Sigma$ to the ring $C^\infty(U)$ is a sheaf, making $(\Sigma, C^\infty_\Sigma)$ a ringed space.

Furthermore, for any $x \in \Sigma$, the stalk at x is isomorphic to the localization

$$C^\infty_{\Sigma,x} \cong C^\infty(\Sigma)_{\mathfrak{m}_x} \tag{3.5}$$

where

$$\mathfrak{m}_x = \{f \in C^\infty(\Sigma) \mid f(x) = 0\}. \tag{3.6}$$

In particular, C^∞_x is local, so $(\Sigma, C^\infty_\Sigma)$ is a locally ringed space.

Continuous maps between the topological spaces of ringed spaces induce morphisms of sheaves in the following way:

Proposition 3.2.17. Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be ringed spaces. Let $f : X \to Y$ be a continuous map.

I) The map on objects:

$$\begin{aligned} \mathrm{Open}(Y) &\to \mathrm{CRing} \\ V &\mapsto \mathcal{F}(f^{-1}(V)) \end{aligned} \tag{3.7}$$

is a sheaf on Y, called the *direct image sheaf* $f_*\mathcal{F}$, or the *pushforward of f*.

II) The map on objects:

$$\begin{aligned} \mathrm{Open}(X) &\to \mathrm{CRing} \\ U &\mapsto \varinjlim_{W \supseteq f(U)} \mathcal{G}(W) \end{aligned} \tag{3.8}$$

is a presheaf on X. The sheaf corresponding to this presheaf[3] is called the *inverse image sheaf* $f^{-1}\mathcal{G}$, or the *pullback* of f. The colimit over opens W is necessary because $f(U)$ is not necessarily open.

Locally ringed spaces form a category in the following manner.

Definition 3.2.18. A morphism between ringed spaces $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ is a pair $(f, f^\sharp)$, where $f : X \to Y$ is a continuous map, and $f^\sharp : \mathcal{G} \dot\to f_*\mathcal{F}$ is a morphism of sheaves.

A morphism between locally ringed spaces $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ is a morphism of ringed spaces that respects the local ring structure on each stalk. That is, for each $x \in X$, the induced map on stalks $\mathcal{G} \to (f_*\mathcal{F})_x$ is a morphism of local rings.

3.3 Schemes

In Section 3.1, we saw that an $\mathbb{R}$-algebra corresponding to some system of physical measurements is naturally a space of functions on a topological space.

What about measurements giving values in $\mathbb{C}$? Applications in electrical engineering often measure in $\mathbb{C}$. What about $\mathbb{Z}/p\mathbb{Z}$, or $\mathbb{Q}_p$? The work done so far is extremely dependent on the topological properties of $\mathbb{R}$, but there is no good reason why this should be so restrictive. The following example offers a great first step as to abstracting this notion.

[3] To any presheaf one can create a sheaf in a universal manner. We shall not discuss this – see [Sta24, Section 007X]

Example 3.3.1. Let $\Bbbk$ be a field, and A a $\Bbbk$-algebra.

Let $\mathfrak{p}$ be a prime ideal of A. Then the quotient $A/\mathfrak{p}$ is an integral domain, and one has the following composition

$$A \xrightarrow{q} A/\mathfrak{p} \xhookrightarrow{i} \mathrm{Frac}(A/\mathfrak{p})$$

of the quotient map and the inclusion into the fraction field of $A/\mathfrak{p}$. This fraction field is naturally a field extension of $\Bbbk$ – consider the injection of $\lambda \mapsto \lambda \cdot 1$ of $\Bbbk$ into A. In the particular case where $\mathrm{Frac}(A/\mathfrak{p}) \cong \Bbbk$, the composition above is an element of $|A|$ by definition.

Conversely, let K is an integral domain containing $\Bbbk$, and $\varphi : A \to K$ a $\Bbbk$-algebra morphism. Then as $\langle 0 \rangle \subseteq K$ is prime, so too is $\ker(\varphi)$, as the preimage of a prime is prime.

What Example 3.3.1 demonstrates is that the prime ideals of A parametrize (up to an equivalence relation on morphisms $A \to \Bbbk$) the points of the underlying $\Bbbk$-spectrum $|A|$ and more so – they parametrize a more general notion of point, namely a morphism into any ring extension of $\Bbbk$ into an integral domain.

Prime ideals are of course a property only of the ring structure on A. We shall accordingly focus only on rings for the moment.

It happens that a change of base field is incorporated by the notion of a *relative affine scheme*, and our sight is drawn to schemes in general.

Definition 3.3.2. Let R be a commutative ring with unity. The *prime spectrum of* R is the set:

$$\mathrm{Spec}(R) := \{\mathfrak{p} \subsetneq R \mid \mathfrak{p} \text{ is a prime ideal}\}$$

Remark 3.3.3. Prime ideals are important for many reasons more than discussed in Example 3.3.1, such as in number theory, but here they are critical for an algebraic/categorical reason.

Namely, given a morphism of rings $\varphi : R \to T$ and a prime ideal $\mathfrak{q}$ of S, the inverse image $\varphi^{-1}(\mathfrak{q})$ is a prime ideal of R. Thus, the mapping sending $\mathfrak{q} \mapsto \varphi^{-1}(\mathfrak{q})$ is a well-defined function between spectra, denoted

$$\varphi^* : \mathrm{Spec}(T) \to \mathrm{Spec}(R).$$

Moreover, the ability to do localization at primes makes them extremely useful – see Example 3.2.7.

Let R be a commutative unital ring, and fix $r \in R$. Example 3.3.1 inclines us to consider the mapping:

$$\begin{aligned} \mathrm{Spec}(R) &\to \coprod_{\mathfrak{p} \in \mathrm{Spec}(R)} R/\mathfrak{p} \\ \mathfrak{q} &\mapsto r \mod \mathfrak{q}. \end{aligned} \tag{3.9}$$

Furthermore, one has for any prime $\mathfrak{p}$ of R the isomorphism

$$\mathrm{Frac}\left(R/\mathfrak{p}\right) \cong R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}.$$

Composing Equation (3.9) with a product of inclusions into fraction fields, one has a map:

$$\begin{aligned} \bar{r} : \mathrm{Spec}(R) &\to \coprod_{\mathfrak{p} \in \mathrm{Spec}(R)} R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \\ \mathfrak{q} &\mapsto \frac{r}{1} \mod \mathfrak{q}R_{\mathfrak{q}}. \end{aligned}$$

In plain language, Example 3.3.1 tells us that points of a $\Bbbk$-algebra $|A|$ in the sense of $\Bbbk$-spectra correspond to the prime ideals. Correspondingly, the map $\bar{r}$ takes a prime ideal and checks its value in a field corresponding to that prime ideal. What we have done is extend the notion of elements of A being functions on its $\Bbbk$-spectrum, in a manner which doesn't depend on the ring innately being an algebra over a specific field.

Compare this with a common result in analysis, often attributed to Whitney.

Lemma 3.3.4. Let Σ be a manifold. Let $D \subseteq \Sigma$ be a closed subset.

Then there exists a smooth function $f \in C^\infty(\Sigma)$ satisfying

$$f(x) = 0 \text{ if and only if } x \in D.$$

Observe that the function $\bar{r}$ satisfies a similar result; in particular for $D = \{x\}$ a single point: $\bar{r}(\mathfrak{p}) = 0$ if and only if $r \in \mathfrak{p}$. This gives us a notion for closed set on $\operatorname{Spec}(R)$.

Proposition 3.3.5. Let R be a commutative, unital ring. For an ideal I of R, let

$$V(I) = \{\text{Prime ideals } \mathfrak{p} \subsetneq R \mid \mathfrak{p} \supseteq I\}. \tag{3.10}$$

Then the subsets of $\operatorname{Spec}(R)$ of the form $V(I)$ form the closed sets of a topology: the *Zariski Topology.*

Proof. Observe first that

$$V(\langle 0 \rangle) = \operatorname{Spec}(R), \qquad V(R) = \emptyset,$$

and thus $\operatorname{Spec}(R)$ and $\emptyset$ are closed.

Let $\{I_\alpha\}$ be an arbitrary collection of ideals of R. Then

$$J = \sum_\alpha I_\alpha$$

is an ideal of R, and

$$V(J) = \bigcap_\alpha V(I_\alpha),$$

because J is the smallest ideal containing each I_α. So an arbitrary intersection of closed sets is closed.

Finally, if I, I' are two ideals, and $\mathfrak{q}$ is a prime ideal containing the ideal II', then by definition of a prime ideal $\mathfrak{q}$ contains at least one of I or I'. Thus,

$$V(II') = V(I) \cup V(I').$$

So a finite union of closed sets is closed. Thus, we have a topology. $\square$

Alternatively, one can define the Zariski topology in terms of a basis; the *distinguished open sets*:

$$D_R(f) = \operatorname{Spec}(R) \setminus V(\langle f \rangle) = \{\text{Prime ideals } \mathfrak{p} \subsetneq R \mid f \notin \mathfrak{p}\} \tag{3.11}$$

Corollary 3.3.6. Given a morphism of rings $\varphi : R \to T$, the induced map on spectra $\varphi^* : \operatorname{Spec}(T) \to \operatorname{Spec}(R)$ (refer to Remark 3.3.3) is continuous in the Zariski topology.

Proof. Let I be an ideal of R. Observe that

$$\begin{aligned}(\varphi^*)^{-1}(V(I)) &= \{\mathfrak{P} \in \operatorname{Spec}(T) \mid \varphi^{-1}(\mathfrak{P}) \in V(I)\} \\ &= \{\mathfrak{P} \in \operatorname{Spec}(T) \mid I \subseteq \varphi^{-1}(\mathfrak{P})\} \\ &= \{\mathfrak{P} \in \operatorname{Spec}(T) \mid \varphi(I) \subseteq \mathfrak{P}\} \\ &= \{\mathfrak{P} \in \operatorname{Spec}(T) \mid \varphi(I)T \subseteq \mathfrak{P}\}.\end{aligned}$$

The last two equalities hold due to properties of ideals. Thus, $(\varphi^*)^{-1}(V(I)) = V(\varphi(I)T)$. So φ^* takes closed sets to closed sets, and is thus continuous. □

Our work on ringed spaces, and presenting rings as functions on spectra culminates in the following proposition.

Proposition 3.3.7 ([Har77, §II.2.2]). Let R be a commutative, unital ring.

The assignment:

$$\mathcal{O} : U \mapsto \left\{ s : U \to \coprod_{\mathfrak{p} \in U} R_{\mathfrak{p}} \;\middle|\; \begin{array}{l} \text{For each } \mathfrak{p} \in U,\ s(\mathfrak{p}) \in R_{\mathfrak{p}}, \text{ and there exists} \\ \text{an open neighbourhood } \mathfrak{p} \in V \subseteq U \text{ and } f, g \in R \\ \text{such that for each } \mathfrak{q} \in V,\ s(\mathfrak{q}) = f/g, \text{ where } g \notin \mathfrak{q}. \end{array} \right\} \tag{3.12}$$

associating to each Zariski-open set of prime ideals U the ring of functions which are locally quotients of elements of R, is a sheaf.

Furthermore, $\mathcal{O}$ has the properties:

I) For any $\mathfrak{p} \in \operatorname{Spec}(R)$, the stalk at $\mathfrak{p}$,

$$\mathcal{O}_{\mathfrak{p}} = \varinjlim_{U \ni \mathfrak{p} \text{ open}} \mathcal{O}(U)$$

is isomorphic to the localization $R_{\mathfrak{p}}$.

II) $\mathcal{O}(\operatorname{Spec}(R))$ is isomorphic to R.

In particular, $(\operatorname{Spec}(R), \mathcal{O}_R)$ is a locally ringed space.

$\mathcal{O}$ is called the *structure sheaf of* R, or of $\operatorname{Spec}(R)$. When multiple rings are in play, we denote it by $\mathcal{O}_R$.

Definition 3.3.8.

I) A locally ringed space $(X, \mathcal{F})$ is an *affine scheme*, if

$$(X, \mathcal{F}) \cong (\operatorname{Spec}(R), \mathcal{O}_R)$$

for some commutative unital ring R.

II) A locally ringed space $(X, \mathcal{F})$ is a *scheme* if there exists an open cover of X,

$$X = \bigcup_{\alpha \in I} U_\alpha$$

such that for all $\alpha \in I$, $(U_\alpha, i_\alpha^{-1}\mathcal{F})$ is an affine scheme, where U is regarded with the subspace topology and $i_\alpha : U \hookrightarrow X$ is the inclusion map.

Such a scheme $(U_\alpha, i_\alpha^{-1}\mathcal{F})$ is called an *affine patch* of X.

III) A *morphism of schemes* is a morphism of locally ringed spaces.

Remark 3.3.9. Henceforth, we adopt the convention that a scheme is referred to by the underlying topological space X, and the structure sheaf on such is always implicitly denoted by $\mathcal{O}_X$.

Example 3.3.10. Let $\Bbbk$ be a field. By definition of a field, the only prime ideal of $\Bbbk$ is $\langle 0 \rangle$. In particular a field is a local ring. Thus, $\mathrm{Spec}(\Bbbk)$ is a topological space with one point $*$, and the only information in the structure sheaf 3.12 is simply

$$\mathcal{O}(*) = \{* \to \Bbbk_{\langle 0 \rangle}\} = \{* \to \Bbbk\} \cong \Bbbk.$$

In particular $\mathrm{Spec}(\Bbbk)$ is an affine scheme.

Example 3.3.11. Let $\varphi : R \to T$ be a ring morphism. Then as in Remark 3.3.3, there is a continuous map

$$\varphi^* : \mathrm{Spec}(T) \to \mathrm{Spec}(R).$$

Let $\mathfrak{p} \in \mathrm{Spec}(T)$, and $V \subseteq \mathrm{Spec}(R)$ an open neighbourhood of $\varphi^*(\mathfrak{p})$. Consider the map:

$$\begin{gathered}\mathcal{O}_R(V) \to (\varphi^*)_* \mathcal{O}_T(V) = \mathcal{O}_T((\varphi^*)^{-1}(V)) \\ \left(s : V \to \coprod_{\mathfrak{p} \in V} R_{\mathfrak{p}} \right) \mapsto \left(\left(\coprod_{\mathfrak{q} \in (\varphi^*)^{-1}(V)} \varphi_{\mathfrak{q}} \right) \circ s \circ \varphi^* : (\varphi^*)^{-1}(V) \to \coprod_{\mathfrak{q} \in (\varphi^*)^{-1}(V)} T_{\mathfrak{q}} \right)\end{gathered} \tag{3.13}$$

where each

$$\begin{aligned}\varphi_{\mathfrak{q}} : R_{\varphi^{-1}(\mathfrak{q})} &\to T_{\mathfrak{q}} \\ r/s &\mapsto \varphi(r)/\varphi(s)\end{aligned}$$

is the map induced by φ onto localizations. Each $\varphi_{\mathfrak{q}}$ is a morphism of local rings, guaranteeing that the map on stalks is also a morphism of local rings. The map defined in (3.13) is natural with respect to inclusion of open sets in $\mathrm{Spec}(R)$, and so φ^* along with (3.13) define a morphism of schemes induced by φ.

3.4 Sheaves of Modules

In the previous section, we expanded the notion of a point to a prime ideal of a ring, from the prior notion as a morphism from a $\Bbbk$-algebra into $\Bbbk$. In the ringed spaces setting, the former are schemes, while the latter are *schemes over* $\Bbbk$.

Definition 3.4.1. Let S be a scheme. A *scheme over* S, or an S*-scheme*, is a scheme X with a morphism of schemes $p : X \to S$. S is called the *base*, and p the *structure morphism.*

Given two S-schemes $p_X : X \to S$ and $p_Y : Y \to S$, a *morphism of schemes relative to* S is a morphism of schemes $X \to Y$ making the following commute:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "p_X"'] & & Y \arrow[dl, "p_Y"] \\ & S & \end{tikzcd}$$

In the case that $S \cong \mathrm{Spec}(R)$ is an affine scheme, we call a scheme over $\mathrm{Spec}(R)$ simply a *scheme over* R, or an R*-scheme.*

Denote the category of schemes over S by Sch(S), and in particular, the category of schemes over $\mathrm{Spec}(R)$ by Sch(R).

Example 3.4.2. Given a $\Bbbk$-algebra A, we can make an affine scheme over $\Bbbk$ in the following way.

We have an affine scheme $(\mathrm{Spec}(A), \mathcal{O}_A)$ given by the ring structure on A. There is exactly one morphism of $\Bbbk$-algebras $\Bbbk \to A$, namely:

$$\begin{aligned} \Bbbk &\to A \\ \lambda &\mapsto \lambda \cdot 1. \end{aligned}$$

Thus, using the induced map on schemes constructed in Example 3.3.11, we have a morphism of schemes $\mathrm{Spec}(A) \to \mathrm{Spec}(\Bbbk)$, making $\mathrm{Spec}(A)$ a $\Bbbk$-scheme.

Conversely, given a $\Bbbk$-scheme $p : X \to \mathrm{Spec}(\Bbbk)$, one gets the structure of a $\Bbbk$-algebra over each affine patch as follows.

Let $(U, \mathcal{O}|_U)$ be an affine patch of X, with $U \cong \mathrm{Spec}(R)$ for some commutative unital ring R. The morphism p restricts to U, giving a morphism of sheaves

$$p^\sharp : \mathcal{O}_\Bbbk \stackrel{\cdot}{\longrightarrow} p_* \mathcal{O}_R,$$

and a ring morphism on global sections

$$P : \mathcal{O}_\Bbbk(\mathrm{Spec}(\Bbbk)) \cong \Bbbk \to \mathcal{O}_R(p^{-1}(\mathrm{Spec}(\Bbbk))) = \mathcal{O}_R(\mathrm{Spec}(R)) \cong R,$$

using Proposition 3.3.7. Define an action of $\Bbbk$ onto R as follows:

$$\lambda \cdot r := P(\lambda) r. \tag{3.14}$$

Because P is a ring morphism, this is a well-defined $\Bbbk$-action, turning R into a $\Bbbk$-module; that is, a $\Bbbk$-vector space. So R is a $\Bbbk$-algebra.

What the above shows, using the particular case of when X is affine, is that a commutative unital ring R is a $\Bbbk$-algebra if and only if $\mathrm{Spec}(R)$ is a scheme over $\Bbbk$.

The above works just as well for when the base scheme $S \cong \mathrm{Spec}(T)$ is the spectrum of a ring and not a field. Then each R above becomes a T-module.

Remark 3.4.3. Let $p : X \to \mathrm{Spec}(\Bbbk)$ be an affine $\Bbbk$-scheme, identifying X with the spectrum of a $\Bbbk$-algebra A. Let $t : \mathrm{Spec}(\Bbbk) \to X$ be a $\Bbbk$-point.

One gets a morphism of rings $T : A \to \Bbbk$ on global sections. Let $\lambda \in \Bbbk$ and $f \in A$. Recalling the definition of the $\Bbbk$-action on A in 3.14, observe:

$$T(\lambda \cdot f) = T(P(\lambda) f) = T(P(\lambda)) T(f).$$

Because $\mathrm{Spec}(\Bbbk)$ is itself a scheme over $\Bbbk$, we have the following diagrams on topological spaces and on global sections:

$$\begin{tikzcd} \mathrm{Spec}(\Bbbk) \arrow[rr, "t"] \arrow[dr] & & X \cong \mathrm{Spec}(A) \arrow[dl, "p"] \\ & \mathrm{Spec}(\Bbbk) & \end{tikzcd} \qquad \begin{tikzcd} \Bbbk & & A \arrow[ll, "T"'] \\ & \Bbbk \arrow[ul] \arrow[ur, "P"'] & \end{tikzcd}$$

where the map $\Bbbk \to \Bbbk$ is the identity. Thus,

$$T(\lambda \cdot f) = \lambda T(f).$$

So each $\Bbbk$-point corresponds to a $\Bbbk$-algebra morphism. Therefore the set $X(\Bbbk)$ is exactly the set $|A|$ from Definition 3.1.4.

Definition 3.4.4. Let X be a scheme over a field $\Bbbk$. Let B be a $\Bbbk$-algebra.

A *B-point* of X is a morphism of schemes over $\Bbbk$, $\mathrm{Spec}(B) \to X$. The *B-points of X* are the set:

$$X(B) := \mathrm{Hom}_{\mathrm{Sch}(\Bbbk)}(\mathrm{Spec}(B), X).$$

When B is algebraically closed, a B-point of X is often referred to as a *geometric point*. This is primarily to differentiate it from the points of the underlying set X.

Example 3.4.5. Here is a simple example of a point of a scheme with no $\mathbb{R}$-points but many $\mathbb{C}$-points. Consider the $\mathbb{R}$-algebra:

$$A = \frac{\mathbb{R}[x, y]}{\langle x^2 + y^2 + 1\rangle}$$

Geometrically, this is the algebra corresponding to a circle of complex radius. Any $\mathbb{R}$-point of $\mathrm{Spec}(A)$ is equivalently an $\mathbb{R}$-algebra morphism $A \to \mathbb{R}$ – it is easy to see that this cannot exist.

We can do something similar without an affine base, motivating the following definition.

Definition 3.4.6. Let $(S, \mathcal{O}_S)$ be a scheme. An *$\mathcal{O}_S$-module* is a sheaf $\mathcal{F}$ on S, such that for each open subset U of S, $\mathcal{F}(U)$ is a $\mathcal{O}_S(U)$-module, and for every inclusion $U \subseteq V$ of open sets of S, the diagram:

$$\begin{tikzcd}
\mathcal{O}_S(V) \times \mathcal{F}(V) \arrow[r] \arrow[d] & \mathcal{F}(V) \arrow[d] \\
\mathcal{O}_S(U) \times \mathcal{F}(U) \arrow[r] & \mathcal{F}(U)
\end{tikzcd}$$

commutes. Above, the horizontal maps are the actions, and the vertical maps are restrictions. The diagram says that the module structures respect restrictions.

A morphism of $\mathcal{O}_S$-modules $\tau : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves such that

$$\tau_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

is a morphism of $\mathcal{O}_S(U)$-modules, for all open subsets $U \subseteq S$.

Denote the category of $\mathcal{O}_S$-modules by $\mathrm{Mod}(\mathcal{O}_S)$.

Example 3.4.7. Let Σ be a manifold, and $\pi : E \to \Sigma$ a rank m vector bundle on Σ. Consider also the locally ringed space $(\Sigma, C^\infty_\Sigma)$. While not a scheme, it is extremely important to our intuition.

Recall the sheaf of sections Γ of Example 3.2.12 – this sheaf takes an open set $U \subseteq \Sigma$ and maps it to the $C^\infty(U)$-module $\Gamma_U(\pi)$.

Fix an open cover $\{U_\alpha\}$ of Σ with a local trivialisation

$$U_\alpha \times \mathbb{R}^m \cong \pi^{-1}(U_\alpha) \subset E.$$

Then as done in Section 2, one can view a local section s on U_α as a map like so:

$$\begin{tikzcd} U_\alpha \times \mathbb{R}^m \arrow[d, bend right, "\pi"'] \\ U_\alpha \arrow[u, bend right, "s"'] \end{tikzcd}$$

That is, one can identify s as an m-tuple of functions in $C^\infty(U_\alpha)$. Thus:

$$\Gamma_{U_\alpha}(\pi) \cong C^\infty(U_\alpha)^{\oplus m}.$$

Example 3.4.7 teaches that the structure sheaf corresponding to a vector bundle of manifolds is locally a free $C^\infty(U_\alpha)$-module.

Definition 3.4.8. Let $(X, \mathcal{O}_X)$ be a scheme. A *vector bundle* or *locally free sheaf* on X is an $\mathcal{O}_X$-module $\mathcal{F}$ such that there exists an open cover $\{U_\alpha\}$ of X such that for each U_α there exists $m \in \mathbb{Z}_{\geq 0}$ such that

$$\mathcal{F}|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}^{\oplus m}.$$

Some definitions allow for the direct sum to be infinite.

Remark 3.4.9. Given this definition of vector bundles on schemes, one can reconstruct a more familiar construction involving open coverings, and a total space. This is [Har77, Ex. §II.5.18].

Alternatively, an equivalent notion of vector bundles is in terms of projective modules of the global sections. [Mor09] examines this in some generality for ringed spaces, quoting [Ser55, §50] and [Swa62]. The same result holds for smooth manifolds – this is [Nes20, §12.32].

The two notions between schemes and manifolds are morally equivalent, but not the same. Notably, vector bundles are not always locally trivialisable in the right sense. Nonetheless they are the right tool in which to approach the constructions in Section 2 in the scheme-theoretic setting.

There is a broad issue with considering only locally free sheaves – the category is not Abelian. That is, the kernel of a morphism of locally free sheaves is rarely itself locally free. This is true even when not considering sheaves, but just modules.

One can state the locally free condition on an $\mathcal{O}_X$-module $\mathcal{F}$ as an exact sequence: for an open cover $\{U_\alpha\}$ of X, one has for each U_α an exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O}_{U_\alpha}^{\oplus m} \longrightarrow \mathcal{F}|_{U_\alpha} \longrightarrow 0.$$

One might weaken this condition so as to allow for infinite direct sums, or to demand that $\mathcal{F}$ is instead locally a quotient of a free module. That is, one demands that $\mathcal{F}$ satisfy the exact sequence:

$$\bigoplus_{i \in I} \mathcal{O}_{U_\alpha} \longrightarrow \mathcal{F}|_{U_\alpha} \longrightarrow 0$$

for all U_α, where I is some indexing set. An $\mathcal{O}_X$-module satsisfying such is called *coherent*.

Unfortunately, unless the scheme X is locally Noetherian, the category of coherent sheaves is not Abelian. However, one can 'add in the kernels' of morphisms between coherent sheaves, to find the desired generalization of vector bundles, characterized by constituting the smallest Abelian sub-category of $\mathcal{O}_X$-modules containing the locally free sheaves.

Definition 3.4.10. Let $(X, \mathcal{O}_X)$ be a scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module.

$\mathcal{F}$ is called *quasi-coherent* if there exists an open cover $\{U_\alpha\}$ of X such that for each U_α, there exists an exact sequence of sheaves:

$$\bigoplus_{j \in J} \mathcal{O}_{U_\alpha} \longrightarrow \bigoplus_{i \in I} \mathcal{O}_{U_\alpha} \longrightarrow \mathcal{F}|_{U_\alpha} \longrightarrow 0.$$

Denote the category of quasi-coherent sheaves on X by $\mathrm{QCoh}(X)$.

Proposition 3.4.11 ([Sta24, Proposition 077P])**.** Let X be a scheme.

Then the category of quasi-coherent sheaves on X, $\mathrm{QCoh}(X)$, is Abelian.

The functors defined in Proposition 3.2.17 were used to relate sheaves on a scheme Y to a scheme X, and vice versa, relative to a morphism $f : X \to Y$. We can do similar for $\mathcal{O}$-modules.

If $\mathcal{F}$ is an $\mathcal{O}_X$-module, then the sheaf-theoretic pushforward $f_*\mathcal{F}$ is easily made to be an $f_*\mathcal{O}_X$-module – for $V \subseteq Y$ open, take the action map to be the action on X:

$$f_*\mathcal{O}_X(V) \times f_*\mathcal{F}(V) = \mathcal{O}_X(f^{-1}(V)) \times \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(f^{-1}(V)) = f_*\mathcal{F}(V).$$

This in turn gives $f_*\mathcal{F}$ a $\mathcal{O}_Y$-module structure, from the morphism $f^\sharp : \mathcal{O}_Y \dot{\to} f_*\mathcal{O}_X$.

The pullback, as defined in Proposition 3.2.17, is a $f^{-1}\mathcal{O}_Y$-module in a similar way. However, there is no immediate action by $\mathcal{O}_X$ and so requires a special definition.

Definition 3.4.12. Let $f : X \to Y$ be a morphism of schemes, and $\mathcal{G}$ an $\mathcal{O}_Y$-module. The *pullback of* $\mathcal{G}$ *by* f is the sheaf $f^*\mathcal{G}$ defined by

$$f^*\mathcal{G} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G},$$

$f^*\mathcal{G}$ is an $\mathcal{O}_X$-module, with local $\mathcal{O}_X$ action given by multiplication on the left.

We finish off this subsection with a series of useful results.

Lemma 3.4.13 ([Sta24, Lemma 0096])**.** Let $f : X \to Y$ be a morphism of schemes.

The functors

$$f_* : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_Y),$$
$$f^* : \mathrm{Mod}(\mathcal{O}_Y) \to \mathrm{Mod}(\mathcal{O}_X)$$

are adjoint. That is, for any $\mathcal{O}_X$-module $\mathcal{F}$ and any $\mathcal{O}_Y$-module $\mathcal{G}$, there is a natural bijection:

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_X)}(f^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_Y)}(\mathcal{G}, f_*\mathcal{F}).$$

Corollary 3.4.14. Let $f : X \to Y$ be a morphism of schemes.

The functor f_* is right exact. That is, if

$$\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is an exact sequence in $\mathrm{Mod}(\mathcal{O}_X)$, then

$$f_*\mathcal{F} \longrightarrow f_*\mathcal{G} \longrightarrow f_*\mathcal{H} \longrightarrow 0$$

is an exact sequence in $\mathrm{Mod}(\mathcal{O}_Y)$.

Proof. If a functor is a left-adjoint, as f_* is by Lemma 3.4.13, then it is right exact by an easy manipulation of definitions. □

Corollary 3.4.15. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.

Then $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module. Hence f_* restricts to a functor

$$f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y).$$

Proof. Apply Corollary 3.4.14 to Definition 3.4.10. □

Lemma 3.4.16 ([Har77, §II.5.3])**.** Let $X \cong \mathrm{Spec}(R)$ be an affine scheme. Let $f \in R$, and consider the corresponding distinguished open set $D_R(f)$ (see (3.11)). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module.

Let $t \in \mathcal{F}(D_R(f))$ be a local section. Then $\mathcal{F}$ is a quasi-coherent sheaf if and only if there exists $n \in \mathbb{Z}_{>0}$ and a global section $s \in \mathcal{F}(X)$ such that

$$\mathrm{Res}^X_{D_R(f)}\, s = f^n t.$$

That is, sections defined locally about f extend globally to a section on X, up to a power of f.

Equivalently, $\mathcal{F}$ is quasi-coherent if and only if

$$\mathcal{F}(D_R(f)) \cong R_f \otimes_R \mathcal{F}(\mathrm{Spec}(R)).$$

Proposition 3.4.17 ([Har77, §II.5.5])**.** Let R be a ring, and let $X = \mathrm{Spec}(R)$ be the resulting affine scheme.

The functor

$$\mathrm{QCoh}(X) \to R\text{-Mod}$$

which sends objects as

$$\mathcal{F} \mapsto \mathcal{F}(X)$$

is an equivalence of categories.

3.5 Other Useful Constructions on Schemes

In this section we describe a selection of other objects that will be required of us in Section 4.

Definition 3.5.1. Let $(X, \mathcal{O}_X)$ be a scheme. A subsheaf $\mathcal{J}$ of $\mathcal{O}_X$ that is an $\mathcal{O}_X$-module is called an *sheaf of ideals*.

Recall that this means that there exists a monomorphism of $\mathcal{O}_X$-modules

$$\mathcal{J} \dot{\hookrightarrow} \mathcal{O}_X.$$

The *scheme associated to the sheaf of ideals* $\mathcal{J}$ is the scheme with underlying set

$$V(\mathcal{J}) := \left\{ z \in X \;\middle|\; \left({}^{\mathcal{O}_X}\!/_{\mathcal{J}} \right)_z = 0 \right\}, \tag{3.15}$$

which is given the subspace topology, and has structure sheaf given by the pullback of the inclusion $i : V(\mathcal{J}) \hookrightarrow X$ of topological spaces:

$$\mathcal{O}_{V(\mathcal{J})} := i^{-1} \left({}^{\mathcal{O}_X}\!/_{\mathcal{J}} \right).$$

Example 3.5.2. An important class of ideal sheaves originate from *closed immersions*; morphisms $f : Z \to X$ of schemes such that the image of f is a closed subset of X, and the induced morphism

$$f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$$

is an epimorphism. The kernel of $f^\sharp$ is a quasi-coherent $\mathcal{O}_X$-module, because $\mathcal{O}_X$ and $f_*\mathcal{O}_Y$ are. This kernel is *the sheaf of ideals corresponding to Z.*

Conversely, given a sheaf of ideals $\mathcal{J}$ of X, the associated inclusion $i : V(\mathcal{J}) \hookrightarrow X$ is a closed immersion. When X is affine $V(\mathcal{J})$ is exactly the vanishing set of an ideal, by Proposition 3.4.17, and so closed. Hence $V(\mathcal{J})$ is in general a finite union of closed sets, and thus closed. And, the induced monomorphism $\mathcal{O}_{V(\mathcal{J})} \overset{\cdot}{\hookrightarrow} \mathcal{O}_X$ is equivalent to an epimorphism $\mathcal{O}_Z \overset{\cdot}{\twoheadrightarrow} i_*\mathcal{O}_X$ by the adjoint property of pullbacks.

Definition 3.5.3. Fix a scheme S. Let X and Y be schemes over S. Then the *fibre product of X and Y*, $X \times_S Y$, is the following pullback in the category of schemes:

$$\begin{tikzcd}
W \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_S Y \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & X \arrow[d] \\
& Y \arrow[r] & S
\end{tikzcd}$$

That is, $X \times_S Y$ together with the projections (p_1, p_2) is the scheme making the above square commute, such that for any other such scheme W making the diagram commute, there is a unique morphism $W \to X \times_S Y$.

[Har77, Theorem 3.3] shows that such a fibre product always exists, and is unique, up to unique isomorphism.

An important special case is when one has two fixed base schemes of interest S, S', and a scheme morphism $S' \to S$. In this case, given a scheme X over S, one can construct a scheme over S', namely $X \times_S S'$. Being a pullback, it is universal, and so encodes as much data about the S-scheme structure of X as possible while extending to S'. Example 3.4.2 shows that this amounts to extending the $\mathcal{O}_S$-module structure on X to that of S'; an extension of scalars, when $S \subseteq S'$ is a field extension. Thus this case is called a *base change.*

Remark 3.5.4. The fibre product of affine schemes is dual to the tensor product. Namely, let $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ be $\mathrm{Spec}(C)$-schemes. Then $X \times_{\mathrm{Spec}(C)} Y \cong \mathrm{Spec}(A \otimes_C B)$.

Base change is particularly valuable in our setting of algebras of observables; given a $\Bbbk$-algebra A and a morphism of rings $\Bbbk \to K$, the base change $A \otimes_\Bbbk K$ is the extension of readings into a 'larger' number system K.

Example 3.5.5. Let S be a scheme, and $t : X \to S$ a scheme over S.

Then one always has a morphism $\Delta_S : X \to X \times_S X$; the *diagonal morphism*, defined by the universal property of the fibre product:

$$\begin{tikzcd}
X \arrow[dr, dotted, "\Delta_S"] \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\
& X \times_S X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & X \arrow[d, "t"] \\
& X \arrow[r, "t"'] & S
\end{tikzcd}$$

From here we can make a somewhat odd definition, but from the perspective of 'algebra first', it is perhaps the natural one.

Definition 3.5.6. Let $A = C^\infty(\mathbb{R}^n)$ be the $\mathbb{R}$-algebra of smooth functions on Euclidean space. Write $S = \mathrm{Spec}(A)$ for the resulting scheme. The real points $X(\mathbb{R})$ of an affine scheme $X = \mathrm{Spec}(B)$ over $\mathbb{R}$ is a *smooth manifold*, seen as a locally ringed space, if:

I) B satisfies (3.2), so that $X(\mathbb{R})$ is Haussdorff, and

II) There is a morphism of schemes $\varphi : X \to S$ over $\mathbb{R}$ such that there is a countable open covering $\{U_\alpha\}$ of X, such that φ restricts to an isomorphism on each U_α.

That is, $X(\mathbb{R})$ is a manifold if it is second countable, Hausdorff, and B is locally the smooth functions on Euclidean space.

This is the definition given in [Nes20, §4.1] (we have baked in the 'complete' condition by our definition of the structure sheaf). The reader does not lose anything by continuing to think of manifolds in the usual sense - this definition is primarily conceptual; precognisance of differential structure is not required. The two definitions are equivalent by [Nes20, §5].

One can define another object in analogy, perhaps more natural for algebraic geometry or commutative algebra:

Definition 3.5.7. A *variety* is a scheme X over a field $\Bbbk$ such that:

I) The structure morphism $X \to \Bbbk$ is separated. That is, the corresponding diagonal morphism

$$\Delta := \Delta_{\mathrm{Spec}(\Bbbk)} : X \to X \times_{\Bbbk} X$$

is a closed immersion.

II) The structure morphism $X \to \Bbbk$ is of finite type. That is, X has a finite covering by open affine subsets $\{U_\alpha \cong \mathrm{Spec}(R_\alpha)\}$ such that each R_α is a finite dimensional $\Bbbk$-algebra.

III) For every non-empty affine patch $\mathrm{Spec}(R) \cong U \subseteq X$, R is an integral domain.

This definition is quite different from that of a manifold, but morally the same. Condition I) is analogous to the Hausdorff condition; compare it with the common result in topology that a space Y is Hausdorff if and only if the diagonal $\Delta(Y) \subseteq Y \times Y$ is a closed subset. Condition II) models X as locally a quotient of $\Bbbk[x_1, \ldots, x_n]$, mimicking $C^\infty(\mathbb{R}^n)$. Condition III) says that there are no nilpotent elements locally – this condition is occasionally not invoked when considering $\mathcal{D}$-modules and related structures as in [BB93].

The most important condition for Section 4 is separatedness – the diagonal being a closed immersion is a critical assumption for working with $\mathcal{D}$-modules.

We introduce a final notion here; of sufficiently simple varieties to work with.

Definition 3.5.8. Let R be a commutative unital ring.

R is said to have *Krull dimension* $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ if n is the supremum of integers m, such that there exists a chain of distinct prime ideals of R:

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_m.$$

One writes $\dim(R) = n$.

Definition 3.5.9. Let $\Bbbk$ be a field. An affine scheme X locally of finite type over $\Bbbk$ is *smooth* if it is of finite type, and for each $x \in X$,

$$\dim_{\Bbbk}\left(\mathfrak{m}_x/\mathfrak{m}_x^2\right) = \dim(\mathcal{O}_{X,x}) \tag{3.16}$$

where $\mathfrak{m}_x$ is the maximal ideal of $\dim(\mathcal{O}_{X,x})$. That is, the cotangent space (remember Remark 2.2.7!) is of equal to the dimension to the stalk; in some way, the number of directions tangent to a point is as small as possible.

A (possibly non-affine) scheme X over $\Bbbk$ is smooth if for any $x \in X$, there exists an affine neighbourhood U such that U is smooth.

Definition 3.5.10. Let $\Bbbk$ be a field, and X a variety over $\Bbbk$. We call X a curve over $\Bbbk$ if:

I) X is dimension 1. That is, if Y_1 and Y_2 are two closed irreducible subsets of X such that $Y_1 \subsetneq Y_2$, then $Y_2 = X$.

II) X is smooth.

One can show that a smooth $\mathbb{R}$-variety satisfies the assumptions of Definition 3.5.6: the (possibly empty) set of $\mathbb{R}$-points is a manifold.

Example 3.5.11. The quintessential example of a variety is *affine space*; for a field $\Bbbk$ set

$$\mathbb{A}^n := \operatorname{Spec}(\Bbbk[x_1, \ldots, x_n]).$$

Note that

$$\mathbb{A}^n \times_{\Bbbk} \mathbb{A}^m = \operatorname{Spec}(\Bbbk[x_1, \ldots, x_n, y_1, \ldots, y_m]) = \mathbb{A}^{n+m}.$$

4 Synthetic Dynamics in Algebraic Geometry

4.1 The Sheaf of Differential Operators, $\mathcal{D}_X$

We return to the case discussed in Section 3, where our main object of studies is an algebra of observables and the associated affine scheme. A ring, by itself, has no notion of differentiability. Thankfully, the objects discussed in Section 2.2 have exquisite representative properties, that we shall be able to use to transport differentiability to general schemes.

We begin with the notion of the sheaf of differential operators of a scheme. Recall Proposition 2.6.4. This says that linear differential operators on a manifold M can be expressed purely algebraically, when viewed as endomorphisms of the ring of smooth functions. This lets us translate differential operators to arbitrary algebras.

Definition 4.1.1. Let R be a commutative unital ring, and B an R-algebra. Then the set of qth order differential operators on R is the set:

$$F_qD_R(B) := \left\{P \in \mathrm{End}_R(B) \mid [P, \bar{f}] \in F_{q-1}D_R \text{ for all } f \in B\right\}$$

and

$$F_0D_R(B) := \left\{P \in \mathrm{End}_R(B) \mid [P, \bar{f}] = 0 \text{ for all } f \in B\right\} \cong B.$$

As before the operator $\bar{f} \in \mathrm{End}_R(B)$ is multiplication by $f \in B$, and the commutator is defined analogously to Definition 2.6.3.

The 'F' here stands for filtration – from the definition we immediately see that:

$$B \cong F_0D_R(B) \subseteq F_1D_R(B) \subseteq F_2D_R(B) \subseteq F_3D_R(B) \subseteq \dots$$

Each $F_qD_R(B)$ is an R-bimodule, with left and right actions given by:

$$\begin{aligned} f \cdot P &:= fP, \\ P \cdot f &:= P \circ \bar{f}, \end{aligned} \tag{4.1}$$

where $f \in R$ and $P \in F_qD_R(B)$.

Lemma 4.1.2. The union of these bimodules,

$$D_R(B) := \bigcup_{q \in \mathbb{Z}_{\geq 0}} F_qD_R(B),$$

is a unital ring, with multiplication given by composition.

Proof. To see this multiplication is well-defined, let $P \in F_sD_R(B)$ and $Q \in F_tD_R(B)$ for some $s, t \in \mathbb{Z}_{\geq 0}$. Then for any $f, g \in R$:

$$\begin{aligned} [P \circ Q, \bar{f}](g) &= P \circ Q \circ \bar{f}(g) - fP \circ Q(g) \\ &= P(Q(fg)) - fP(Q(g)) \\ &= P(Q(fg)) - P(fQ(g)) + P(fQ(g)) - fP(Q(g)) \\ &= (P \circ [Q, \bar{f}])(g) + [P, \bar{f}](Q(g)). \end{aligned}$$

Observe that, by definition,

$$[P, \bar{f}](Q(g)) \in F_{s-1}D_R(B).$$

We claim that $[P \circ Q, \bar{f}]$ is an element of $F_{t+s-1}D_R(B)$ for any such P, Q. We proceed by induction on t, for fixed $s \in \mathbb{Z}_{\geq 0}$.

If $t = 0$, then

$$[P \circ Q, \bar{f}](g) = [P, \bar{f}](Q(g)) \in F_{s-1}D_R(B).$$

Supposing that the result is true for $t - 1$, we notice that as $[Q, \bar{f}] \in F_{t-1}D_R$, we have

$$P \circ [Q, \bar{f}] \in F_{s+t-2}D_R(B),$$

by our induction hypothesis. Thus

$$[P \circ Q, \bar{f}] \in F_{\max\{s+t-2, s-1\}}D_R(B) \subseteq F_{s+t-1}D_R(B).$$

So $P \circ Q \in F_{s+t}D_R(B)$, and hence multiplication in $D_R(B)$ is well-defined.

$D_R(B)$ inherits the remaining conditions to be a ring from $\mathrm{End}_R(B)$; its unit is $\bar{1}$. $\square$

Note however that $D_R(B)$ is *not* commutative, even in the simplest cases.

Example 4.1.3. Consider the case where $R = \Bbbk$ is a field of characteristic 0, and $B = \Bbbk[x_1, \ldots, x_N]$ is the free $\Bbbk$-algebra on $N \in \mathbb{Z}_{>0}$ variables. Then:

$$\begin{aligned} D_\Bbbk(B) &\cong \Bbbk[x_1, \ldots, x_N, \partial_1, \ldots, \partial_N] \\ &:= \frac{\Bbbk[x_1, \ldots, x_N, y_1, \ldots, y_N]}{\langle x_\ell y_\ell - y_\ell x_\ell = 1\rangle}, \end{aligned}$$

where each ∂_ℓ is defined to be the image of y_ℓ under the quotient; each acts formally as the usual partial derivative on polynomials. This ring is referred to as the *Weyl algebra* $A_N(\Bbbk)$. For a proof, see [MR00, §3.6, §15]. Once again, the multiplication can be seen as composition of operators.

Importantly, $\partial_\ell \bar{x}_\ell = x_\ell \partial_\ell + 1$ as operators, so the ring is not commutative.

Example 4.1.4. Let $\Bbbk$ be a characteristic 0 field, let $B = \Bbbk[x_1, \ldots, x_N]$, and let $I \subseteq B$ be an ideal. Consider the affine variety $C = B/I$. Then we have a nice description of the operators on C, namely:

$$D_\Bbbk(C) = \frac{\{P \in A_N(\Bbbk) \mid P(I) \subseteq I\}}{IA_N(\Bbbk)}.$$

Plainly, the differential operators on an affine variety B/I are those operators coming from B whose action upon I remains in I.

Proof. This is a straightforward corollary to [MR00, §15.5.13]. $\square$

Remark 4.1.5. The perspective taken in [Ber] is to extend locally the notion of an action of a Weyl algebra. The end result is $\mathcal{D}$-modules, which we shall soon see. One reason to desire this was mentioned in Section 2.8 – Weyl algebras model the uncertainty relations that pop up in quantum mechanics. In this way, considering sheaves of modules of Weyl algebras is implicitly account for uncertainty principals. It is for this reason that we consider $\mathcal{D}$-modules in relation to physics, for our further discussion.

Here is a technical lemma for later.

Lemma 4.1.6. Let $\Bbbk$ be a field of characteristic 0. Let B be a finitely generated $\Bbbk$-algebra. That is, there exists $N \in \mathbb{Z}_{\geq 0}$ such that B is a quotient of $\Bbbk[x_1, \dots, x_N]$.

For any $f \in B$, denote by $B_f = B[f^{-1}]$ the localization of B at f. Then

$$D_{\Bbbk}(B_f) \cong B_f \otimes_B D_{\Bbbk}(B)$$

Proof. This follows directly from Example 4.1.4 and base change formulae. □

Recall Proposition 2.6.1 – this related the sections of jet bundles (on open neighbourhoods in smooth manifolds) to modules of finite order differential operators. Can we say something similar for a general algebra?

Proposition 4.1.7. Let R be a commutative unital ring, and B an R-algebra. Then the module of qth order differentials is (co)represented as:

$$F_q D_R(B) \cong \mathrm{Hom}_B(\mathcal{P}^q_{B/R}, B), \tag{4.2}$$

where $\mathcal{P}^q_{B/R}$ is the *module of principal parts of order q*, defined by:

$$\mathcal{P}^q_{B/R} := \frac{B \otimes_R B}{\mathcal{J}^{q+1}}, \tag{4.3}$$

and $\mathcal{J}$ is the ideal defined by

$$\mathcal{J} := \ker(a \otimes b \mapsto ab) = \langle 1 \otimes a - a \otimes 1 \rangle.$$

All modules here use their left B-module structure.

The name is originally due to Grothendieck [Gro67, §16.3].

Proof. By tensor-Hom adjunction, we have isomorphisms of B-bimodules:

$$\begin{gathered} \mathrm{Hom}_B(B \otimes_R B, B) \cong \mathrm{Hom}_R(B, \mathrm{Hom}_B(B, B)) \cong \mathrm{Hom}_R(B, B) \\ \varphi(-) \mapsto (b \mapsto \varphi(b \otimes -)) \mapsto \varphi(1 \otimes -) \end{gathered} \tag{4.4}$$

The $B \otimes_R B$ module structure of $\mathrm{Hom}_R(B, B)$ is, like (4.1),

$$((f \otimes g) \cdot P)(h) = f P(gh), \qquad f, g, h \in B \text{ and } P \in \mathrm{Hom}_R(B, B).$$

Thus we may write

$$[P, \bar{f}] = (1 \otimes f - f \otimes 1) \cdot P.$$

Note that Definition 4.1.1 is equivalently:

$$F_q D_R(B) = \left\{ P \in \mathrm{End}_R(B) \mid [\dots[P, \bar{f}_0], \bar{f}_1], \dots], \bar{f}_q] = 0 \text{ for all } f_0, \dots f_q \in B \right\},$$

so $P \in F_q D_R(B)$ if and only if

$$(1 \otimes f_q - f_q \otimes 1) \dots (1 \otimes f_0 - f_0 \otimes 1) \cdot P = 0, \qquad \text{for all } f_0, \dots f_q \in B.$$

That is, $F_q D_R(B)$ is the submodule of $\mathrm{Hom}_R(B, B)$ annihilated by the action of the ideal $\mathcal{J}^{q+1} \subset B \otimes_R B$.

Since the action of $B \otimes_R B$ on $\mathrm{Hom}_B(B \otimes_R B, B)$ is simply

$$(f \otimes g \cdot \varphi)(h) = \varphi((f \otimes g)h),$$

this submodule $F_q D_R(B)$ is isomorphic to $\mathrm{Hom}_B(B \otimes_R B / \mathcal{J}^{n+1}, B)$, by (4.4). □

So even in the purely algebraic case, we have an appropriate notion of sections of finite order jet bundles, namely $\mathcal{P}^q_{B/R}$. It is easiest to extend this to scheme in the case where $R = \Bbbk$ is a field of characteristic 0.

Definition 4.1.8 ([BD04, §2.3.3])**.** Let $\Bbbk$ be a field of characteristic 0, and X a $\Bbbk$-scheme. Let $q \in \mathbb{Z}_{\geq 0}$.

Let $\mathcal{J}$ be the sheaf of ideals corresponding to the diagonal morphism[4] $\Delta : X \to X \times_{\Bbbk} X$. Then the *qth Jet scheme* of X is defined to be the scheme corresponding to $\mathcal{J}^n$, denoted $X^{(q)}$.

Note that when X is affine one recovers the scheme dual to the module of principal parts (4.3).

Remark 4.1.9. Let X be a $\Bbbk$-scheme once again. The limit of Jet schemes

$$(X \times X)^{\vee} := \varinjlim X^{(q)}$$

is in general not a scheme in the regular sense. It is instead an inductive limit of schemes (often the schemes are described as representable presheaves of sets in this setting). Hence, the term *ind-scheme*. This is essentially the same issue that was discussed constructing the infinite order jet bundle discussed in Remark 2.3.8. In practice the distinction will not be so relevant for us – a discussion of formal schemes can be found in [Har77, §2.9] or [Gro60, §10]. In essence, one works with the limit as a locally ringed space, not necessarily as a scheme.

More explicitly, $(X \times X)^{\vee}$ is the diagonal $\Delta(X) \cong X$ as a topological space, and has structure sheaf

$$\mathcal{O}_{(X\times X)^{\vee}} = \varprojlim_{q} \left(\mathcal{O}_{X\times X} \big/ \mathcal{J}^q \right), \tag{4.5}$$

restricted to $\Delta(X)$. We also call this the *formal neighbourhood of the diagonal*.

Having the description allows us to describe the differential operators on affine schemes over a ring, and by gluing, on non-affine ones.

Definition 4.1.10. Let X be a smooth scheme over a field $\Bbbk$. For an affine patch U of X, denote the $\Bbbk$-algebra of sections on U by $B = \mathcal{O}_X(U)$.

Then the *sheaf of differential operators on* X is the sheaf $\mathcal{D}_X$ defined on affine patches U of X by:

$$\mathcal{D}_X(U) := D_{\Bbbk}(B) = \bigcup_{q \in \mathbb{Z}_{\geq 0}} F_q D_{\Bbbk}(B).$$

Denote by $\mathcal{D}^{\leq q}_X$ the sheaf of finite order operators:

$$U \mapsto F_q D_{\Bbbk}(B).$$

This definition is also originally due to Grothendieck. See [Gro67, §16.7-8]. To see that this definition makes sense (that is, defining a sheaf by its value on affine patches is well-defined) refer to Proposition 3.2.11 and apply the coherency condition of Lemma 4.1.6. Recall also from (4.1) that each filtration of $F_q D_{\Bbbk}(B)$ is a B-module (taking the left action here) and hence so is $D_{\Bbbk}(B)$.

These show that, on a smooth $\Bbbk$-scheme, $\mathcal{D}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules.

[4] Otherwise, as the diagonal is locally a closed immersion ([Sta24, Lemma 01KJ]), the diagonal inclusion $X \to X \times_{\Bbbk} X \setminus \partial(\Delta(X))$ is a closed immersion. Take then the sheaf $\mathcal{J}$ to be the sheaf of ideals corresponding to this inclusion. Alternatively, as in [EH00, pg. 93], one can defined the corresponding 'diagonal subscheme' by taking affine patches, taking the quotient by the diagonal there, and gluing them back together.

Remark 4.1.11. Note that Definition 4.1.10 requires smoothness, while our more general definition Definition 4.1.1 requires no condition on the algebras. The reason for this is that smooth schemes locally have their differential operators as elements of Weyl algebras (see Example 4.1.3) allowing one to guarantee that $\mathcal{D}_X$ is quasi-coherent, by applying Lemma 4.1.6.

Ultimately the key property that smooth schemes possess is that the ring of differential operators are generated locally by tangent vectors as an algebra (this is [Ber, §9.2]). Compare this with the definition of smoothness Equation (3.16), which says informally that the (co)tangent vectors at a stalk contain all the information of the stalk.

One can define the sheaf $\mathcal{D}_X$ in an analogous way for non-smooth schemes, but in application one rarely wants to. Besides the sheaf-theoretic problems that occur, making defining the critically important *$\mathcal{D}_X$-modules* difficult, differential operators on algebras are generally poorly behaved without smoothness. The typical example of this is found in [BGG72], which among other things is not Noetherian.

One may want for physical applications to look at non-smooth schemes and $\mathcal{D}$-modules (which we have not defined yet) on them, and one can accommodate for this via a result known as *Kashiwara's lemma*. We shall introduce $\mathcal{D}$-modules in way in which this lemma is quite natural.

4.2 Crystals & $\mathcal{D}$-Modules

We briefly recall the set-up discussed in Section 3.1. One begins with an arbitrary algebra A over a field $\Bbbk$, and views it as an algebra of continuous functions over a space:

$$|A| = \mathrm{Hom}_{\Bbbk}(A, \Bbbk).$$

Theorem 3.1.5, Remark 3.1.7 and Corollary 3.1.8 together say that points $x \in |A|$ are distinguishable only up to elements of the ideal

$$N_A = \bigcap_{x \in |A|} \ker(x).$$

But Example 3.3.1 and the rest of our discussion on base change led us to prefer talking about $\mathrm{Spec}(A)$ for a more universal construction. We saw that (up to a certain equivalence) primes correspond to the kernels of $\Bbbk$-algebra morphisms from A into an extension of $\Bbbk$. Correspondingly, the appropriate analogue to N_A is

$$\mathfrak{N}_A := \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p}.$$

$\mathfrak{N}_A$ is called the *nilradical* of A. One can show that this is equal to the set of elements $f \in A$ such that there exists $n \in \mathbb{Z}_{\geq 0}$ such that $f^n = 0$, hence the name.

One will find that the nilradical invokes a similar distinguishability result.

Definition 4.2.1 ([Lur]). Let $\Bbbk$ be a field of characteristic 0. Let X be a scheme over $\Bbbk$, and A a $\Bbbk$-algebra. Let $\mathfrak{N}_A$ denote the nilradical of A, and $q : A \twoheadrightarrow A/\mathfrak{N}_A$ the corresponding quotient map.

We say that two A-points of $x, y : \mathrm{Spec}(A) \to X$ are *infinitesimally close* if they have the same image under the map:

$$X(A) \to X\left({}^{A}\!\big/_{\mathfrak{N}_A}\right)$$
$$(z : \mathrm{Spec}(A) \to X) \mapsto \left(z \circ q^* : \mathrm{Spec}\left({}^{A}\!\big/_{\mathfrak{N}_A}\right) \to X\right).$$

Diagrammatically, x and y are infinitesimally close if the following diagram is commutative:

$$\begin{tikzcd} & \mathrm{Spec}(A) \arrow[dr, "x"] & \\ \mathrm{Spec}\left({}^{A}\!\big/_{\mathfrak{N}_A}\right) \arrow[ur, "q^*"] \arrow[dr, "q^*"'] & & X \\ & \mathrm{Spec}(A) \arrow[ur, "y"'] & \end{tikzcd}$$

Lemma 4.2.2. For an arbitrary ideal I, the quotient $A \twoheadrightarrow A/I$ induces a bijection between the set of prime ideals in A containing I and the set of prime ideals of A/I. Then as the nilradical is the intersection of all primes in A, the quotient q induces a bijection $\mathrm{Spec}(A/\mathfrak{N}) \to \mathrm{Spec}(A)$. It is straightforward to verify that this is also a homeomorphism.

Thus, if two A-points x, y of a scheme X are infinitesimally close, and $\mathfrak{p} \in \mathrm{Spec}(A)$, then by tracking $\mathfrak{p}$ around the above commutative diagram using the homeomorphism q^* one finds that

$$x(\mathfrak{p}) = y(\mathfrak{p}).$$

So infinitesimally close points are identical as maps on topological spaces.

The field theory performed in Section 2.2-2.4 required the existence of a flat connection along a bundle. There is a purely algebraic notion of a flat connection, but because of the odd topology of schemes, the notion of 'connection' is not immediately equivalent to the notion of 'parallel transport', unlike in the case of manifolds. In some ways, the parallel transport notion is the conceptually more important concept.

More explicitly, take for example the quite simple variety $\mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[t])$. The prime ideals of $\mathbb{C}[t]$ are just $\langle 0 \rangle$ and ideals $\langle t - a \rangle$ generated by the degree 1 polynomials, for any $a \in \mathbb{C}$. Since $\mathbb{C}[t]$ is a PID, the vanishing set of any ideal (recall (3.10)) is necessarily contained in a finite number of prime ideals; those degree 1 polynomials which are the irreducible components of the generator for the ideal. So any open set in $\mathbb{A}^1_{\mathbb{C}}$ is necessarily the complement of a finite set. In particular, this is almost always not contractible; there are no preferred paths on which to do parallel transport, even locally.

For a discussion of this in the context of fundamental groups of schemes, see [Mil80, §1].

Thankfully, the notion of infinitesimally close points allows for this to be remedied. Recall here that, as discussed in Section 3.4, our preferred analogue of bundles is quasi-coherent sheaves.

Definition 4.2.3 ([Lur] and [GGK+68, §IX.4])**.**
Let X be a scheme over a field $\Bbbk$ of characteristic 0.

A *crystal of quasi-coherent sheaves* is the following data:

I) A quasi-coherent sheaf $\mathcal{F}$ on X.

II) For every $\Bbbk$-algebra A, a collection of quasi-coherent sheaves on $\mathrm{Spec}(A)$:

$$\{\mathcal{F}(x) \mid x \in X(A)\},$$

where for any A-point $x : \mathrm{Spec}(A) \to X$, $\mathcal{F}(x)$ is the pullback $x^*\mathcal{F}$.

which satisfy the following:

III) Closeness induces natural isomorphisms on pullbacks.

That is, for every pair of infinitesimally close A-points $x, y : \mathrm{Spec}(A) \to X$, there exists an isomorphism of quasi-coherent sheaves

$$\eta_{x,y} : \mathcal{F}(x) \dot{\to} \mathcal{F}(y), \tag{4.6}$$

compatible with base-change. That is, given a morphism of rings $\varphi : A \to B$, the isomorphism $\eta_{x',y'}$ corresponding to the B-points $x' := x \circ \varphi^*, y' := y \circ \varphi^*$ is given by

$$\eta_{x',y'} = \eta_{x,y} \otimes_A B.$$

IV) Closeness is a transitive property on pullbacks.

That is, if $x, y, z : \mathrm{Spec}(A) \to X$ are three A-points such that x is infinitesimally close to y and y is infinitesimally close to z, then by Definition 4.2.1 x is infinitesimally close to z. We require that the isomorphisms in (4.6) match this relation; that:

$$\eta_{x,z} \cong \eta_{y,z} \circ \eta_{x,y}.$$

A morphism of crystals of quasi-coherent sheaves on X is a morphism of sheaves $\tau : \mathcal{F} \dot{\to} \mathcal{F}'$ and a mapping $\eta_{x,y} \mapsto \eta'_{x,y}$ between the given isomorphisms compatible with the induced map of τ on pullbacks.

This is our notion of parallel transport for schemes. Conceptually, it is quite nice, though computationally it is quite bad. Thankfully, one can unpack this definition to be much more hands-on. This lemma gives a direct connection to jets discussed in Section 4.1.

Lemma 4.2.4. Let $\Bbbk$ be a field of characteristic 0, A a $\Bbbk$-algebra, and X a smooth separated $\Bbbk$-scheme.

Two A-points $x, y : \mathrm{Spec}(A) \to X$ are infinitesimally close if and only if

$$(x, y) : \mathrm{Spec}(A) \to X \times_{\Bbbk} X$$

factors through the formal neighbourhood of the diagonal $(X \times X)^\vee$. See Remark 4.1.9 and (4.5) for the definition.

Proof. The assumption that x and y be infinitesimally close is equivalent to there existing a commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}\left(A/\mathfrak{N}_A\right) \arrow[r, "q^*"] \arrow[dr, dashed] & \mathrm{Spec}(A) \arrow[r, "{(x,y)}"] & X \times_{\Bbbk} X \\
& \Delta(X) \arrow[ur, hook] &
\end{tikzcd}$$

Here we regard the pair (x, y) as an A-point of $X \times_{\Bbbk} X$; the statement above is true by using the definition and applying projections.

In the following, we make use of Proposition 3.4.17 to equate quasi-coherent sheaves on an affine scheme with their global sections.

A morphism of schemes factors through a closed subset if and only if the pullback of the corresponding sheaf of ideals is zero, essentially by definition. In our case, this means that the ideal $((x, y) \circ q^*)^* \mathcal{J} \cdot \mathcal{O}_{A/\mathfrak{N}_A} = 0$, where $\mathcal{J}$ is the sheaf of ideals corresponding to the diagonal.

Then by the diagram above, the ideal $(x, y)^* \mathcal{J} \cdot \mathcal{O}_A$ must be contained in the nilradical $\mathfrak{N}_A$. X is smooth over $\Bbbk$, and so locally Noetherian. Thus, by [Sta24, Lemma 00IM], one must have $(x, y)^* \mathcal{J}^n = 0$ for some $n \in \mathbb{Z}_{\geq 0}$.

Thus (x, y) must factor through the closed subscheme corresponding to $\mathcal{J}^n$. By the definition of $(X \times X)^\vee$ and limits, this corresponds to the following commutative diagram:

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[rr, "{(x,y)}"] \arrow[dr, dashed] & & X \times_{\Bbbk} X \\ & (X \times X)^\vee \arrow[ur] & \end{tikzcd}$$

This proves the forward implication. The reverse implication is the same argument, in reverse. □

Note that the above only needed X to be locally Noetherian, not smooth. We stated it as such for a result later.

Corollary 4.2.5. Let $\Bbbk$ be a field of characteristic 0, A a $\Bbbk$-algebra, and X a smooth separated $\Bbbk$-scheme.

Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Consider the projection morphisms from the formal neighbourhood of the diagonal:

$$\begin{tikzcd} & (X \times X)^\vee \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & X \end{tikzcd}$$

Then an isomorphism of locally ringed spaces

$$\pi_1^* \mathcal{F} \cong \pi_2^* \mathcal{F} \tag{4.7}$$

induces an isomorphism of quasi-coherent sheaves:

$$\eta_{x,y} : \mathcal{F}(x) \dot{\to} \mathcal{F}(y),$$

for all infinitesimally close A-points x, y.

Proof. By Lemma 4.2.4, for any pair of infinitesimally close A points x, y, one has a commutative diagram of locally ringed spaces:

$$\begin{tikzcd} & \mathrm{Spec}(A) \arrow[d] & \\ & (X \times X)^\vee \arrow[dl, "\pi_1"'] \arrow[d] \arrow[dr, "\pi_2"] & \\ X & X \times_{\Bbbk} X \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & X \end{tikzcd}$$

where the vertical composition is the A-point (x, y). Commutativity of the diagram along with the isomorphism $\pi_1^*\mathcal{F} \cong \pi_2^*\mathcal{F}$ prove the corollary. $\square$

Lemma 4.2.6. Let $\Bbbk$ be a field of characteristic 0, X a smooth separated $\Bbbk$-scheme, and $\mathcal{F}$ a quasi-coherent sheaf on X. Let π_1 and π_2 be the projections of the formal neighbourhood of the diagonal.

Then an isomorphism $\pi_1^*\mathcal{F} \cong \pi_2^*\mathcal{F}$ is equivalent to a morphism of quasi-coherent sheaves:

$$\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} \dot{\to} \mathcal{F}.$$

Proof. Let $\pi_i^{(q)}$ denote the ith projection of the qth order neighbourhood $X^{(q)}$ to X. By definition of inductive limits, an isomorphism $\pi_1^*\mathcal{F} \cong \pi_2^*\mathcal{F}$ is equivalent to an inductive system of morphisms $(\pi_1^{(q)})^*\mathcal{F} \to (\pi_2^{(q)})^*\mathcal{F}$.

By adjunction of the pullback, our induced morphism $(\pi_1^{(q)})^*\mathcal{F} \to (\pi_2^{(q)})^*\mathcal{F}$ is equivalently a morphism of quasi-coherent sheaves:

$$\mathcal{F} \dot{\to} (\pi_1^{(q)})_*(\pi_2^{(q)})^*\mathcal{F} \cong \mathcal{O}_{X^{(q)}} \otimes_{\mathcal{O}_X} \mathcal{F} \cong (\mathcal{O}_X \otimes_{\Bbbk} \mathcal{O}_X)/\mathcal{J}^q \otimes_{\mathcal{O}_X} \mathcal{F} \tag{4.8}$$

On an affine patch U, the righthand side gives:

$$((\mathcal{O}_X \otimes_{\Bbbk} \mathcal{O}_X)/\mathcal{J}^q \otimes_{\mathcal{O}_X} \mathcal{F})(U) = (B \otimes_{\Bbbk} B)/\langle 1 \otimes b - b \otimes 1 \rangle \otimes_B \mathcal{F}(U)$$

where $B = \mathcal{O}_X(U)$. Recall that this left term inside the tensor is the module of principal parts $\mathcal{P}^q_{B|\Bbbk}$ of Proposition 4.1.7! Working over a field, $\mathcal{P}^q_{B|\Bbbk}$ is free and finitely generated, and so the isomorphism (4.2) extends to an isomorphism:

$$\mathcal{P}^q_{B|\Bbbk} \cong \mathrm{Hom}_B(F_q D_R(B), B)$$

Thus our map on local sections $\mathcal{F}(U) \to \mathcal{P}^q_{B|\Bbbk} \otimes_B \mathcal{F}(U)$ is equivalently a morphism:

$$F_q D_R(B) \otimes_B \mathcal{F}(U) \to \mathcal{F}(U)$$

So by smoothness of X, (4.8) is equivalent to a morphism of $\mathcal{O}_X$-modules:

$$\mathcal{D}_X^{\leq q} \otimes_{\mathcal{O}_X} \mathcal{F} \dot{\to} \mathcal{F}$$

All together the inductive system of morphisms $(\pi_1^{(q)})^*\mathcal{F} \to (\pi_2^{(q)})^*\mathcal{F}$ dualizes to a projective system $\mathcal{D}_X^{\leq q} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$, and hence a morphism:

$$\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} \dot{\to} \mathcal{F}$$

Each of the above steps is an equivalence, so giving such an action also gives isomorphisms $\pi_1^*\mathcal{F} \cong \pi_2^*\mathcal{F}$. $\square$

We take this time to introduce the (perhaps overdue) concept of $\mathcal{D}_X$-modules, which comes out quite naturally as an attempt to develop computational tools for crystals; as a desire to develop parallel transport on schemes.

Definition 4.2.7. Let X be a smooth scheme over a field $\Bbbk$ of characteristic 0. A *left (resp. right) $\mathcal{D}_X$-module* is a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, such that for every affine patch U of X, $\mathcal{F}(U)$ is a left (resp. right) $\mathcal{D}_X(U)$-module.

In the circumstance in which we wish to consider possibly non-quasi-coherent $\mathcal{O}_X$-modules, we shall call the corresponding objects *$\mathcal{D}_X$-sheaves*, following [BD04, §3.5.1][5].

In this paper what we shall refer to as a $\mathcal{D}_X$-module shall mean a **left** $\mathcal{D}_X$-module.

Example 4.2.8.

I) Let X be a smooth scheme over a field $\Bbbk$. Then the structure sheaf $\mathcal{O}_X$ is a left $\mathcal{D}_X$-module – every local section of $\mathcal{D}_X(U)$ is definitionally an endomorphism on $\mathcal{O}_X(U)$.

II) Let X be a smooth scheme over a field $\Bbbk$. Then $\mathcal{D}_X$ is a left and right module over itself, with action given by multiplication.

III) Let $X = \mathbb{A}^1_{\mathbb{C}}$. By Proposition 3.4.17 and Example 4.1.3 a $\mathcal{D}_X$-module is just an A_1-module, where $A_1 = \mathbb{C}[t, \partial_t]$ is the first order Weyl algebra. Then the ring of Laurent polynomials $\mathbb{C}[t, t^{-1}]$ is an A_1-module in the obvious way. Furthermore it is generated as an A_1-module by t^{-1}. One has $(\partial_t \bar{t})(t^{-1}) = 0$, so in fact

$$\mathbb{C}[t, t^{-1}] \cong A_1[t^{-1}] \cong \frac{A_1}{A_1(\partial_t \bar{t})}.$$

What this says is that one can view $\mathbb{C}[t, t^{-1}]$ as the module which recovers the solution space to the differential equation:

$$0 = \partial_t \bar{t} = 1 + t\partial_t,$$

in any $\mathbb{C}$-algebra B which is also an A_1-module, by computing

$$\mathrm{Hom}_{A_1}(\mathbb{C}[t, t^{-1}], B).$$

This is an extremely important notion, and the original reason for introducing $\mathcal{D}$-modules; correspoding to Mikio Sato's work on microlocal analysis. We will not have time to discuss this, but it may be wise to have this concept in the back of one's mind.

IV) Again, let $X = \mathbb{A}^1_{\mathbb{C}}$. One has a short exact sequence:

$$0 \longrightarrow \mathbb{C}[t] \hookrightarrow \mathbb{C}[t, t^{-1}] \twoheadrightarrow \tfrac{\mathbb{C}[t,t^{-1}]}{\mathbb{C}[t]} \longrightarrow 0$$

Denote by δ the image of t^{-1} under the quotient on the right. Similar to above one then has:

$$\frac{\mathbb{C}[t, t^{-1}]}{\mathbb{C}[t]} \cong A_1\delta \cong \frac{A_1}{A_1(t)}$$

From this, one might interpret δ as a type of Dirac function. A nice bonus to considering $\mathcal{D}$-modules is that one has access to algebraically defined notions of distributions. In fact, right $\mathcal{D}$-modules can be seen as acting as distributional derivates (see for example [Ber] or [Eti17] for this perspective).

[5] Strictly speaking, these are defined as sheaves on the *étale topology* of X, not the Zariski topology. For our purposes, however, this does not impact us much. A Zariski cover is an open immersion of schemes and so also an étale cover - so every étale sheaf descends to a Zariski sheaf. The reason for considering étale topology is that the operations constructed in [BD04, §3] are local with respect to this topology.

What Lemma 4.2.4, Corollary 4.2.5 and Lemma 4.2.6 lead us to is the following.

Theorem 4.2.9 ([Lur]). Let X be a smooth separated scheme over a field $\Bbbk$ of characteristic 0.

Then the category of crystals of quasi-coherent sheaves on X is categorically equivalent to the category of $\mathcal{D}_X$-modules.

Proof. The aforementioned results have already proven that the first three conditions of Definition 4.2.3 are equivalent to a morphism

$$A : \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} \dot{\to} \mathcal{F}. \tag{4.9}$$

It remains to show the transitivity condition is equivalent to (4.9) defining a $\mathcal{D}_X$-module action. This follows by dualizing the diagram:

$$\begin{tikzcd}
(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X) \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r, "m \otimes \mathrm{id}_{\mathcal{F}}"] \arrow[dd, leftrightarrow, "\cong"'] & \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[dr, "A"] & \\
& & \mathcal{F} \\
\mathcal{D}_X \otimes_{\mathcal{O}_X} (\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}) \arrow[r, "\mathrm{id}_{\mathcal{D}_X} \otimes A"'] & \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[ur, "A"'] &
\end{tikzcd}$$

using the methods in Lemma 4.2.6. Here, m is the (local) multiplication map. This diagram commutes exactly when A defines a $\mathcal{D}_X$-module structure. □

Remark 4.2.10. The language of crystals allows for a nice definition of $\mathcal{D}_X$-modules on schemes that are singular (i.e. not smooth).

Definition 4.2.3 makes no use of the notion of smoothness, so one can define the category of crystals of quasi-coherent sheaves $\mathrm{Crys}^l(Z)$ for a possibly non-singular $\Bbbk$-scheme Z that is only separated and locally finite type (and hence Noetherian). By Definition 3.5.1 and Example 3.5.2, a closed embedding $i : Z \to X$ of Z into a smooth $\Bbbk$-scheme X induces an equivalence between quasi-coherent sheaves on Z and quasi-coherent sheaves with support contained in Z; that is the sheaves whose stalks vanish outside of Z.

The rest is straightforward – one can realise $\mathrm{Crys}^l(Z)$ as a full subcategory of $\mathrm{Crys}^l(X)$. Then by Theorem 4.2.9 one should define a $\mathcal{D}_Z$-module to be a $\mathcal{D}_X$-module whose support is contained in X.

This result is broadly known as *Kashiwara's lemma*. The use for us is conceptual: we are not prohibited to solely considering smooth schemes in our application.

[GR14] presents this argument in higher generality, in the context of stacks.

Remark 4.2.11. With the terminology defined this section, one might view the functor $\nu : \mathbb{R}\text{-Alg} \to \mathbb{R}\text{-Alg}$ defined in (3.4) as an operation which identifies all infinitesimally close points, in the specific case of $\mathbb{R}$-algebras. The corresponding functor υ under the equivalence Example 3.3.1, sending a ring R to the quotient $R/\mathfrak{N}_R$ is more useful in a broader context. They are used to define the *de Rham space*.

Given a Set-valued presheaf X on $\mathrm{CRing}^{\mathrm{op}}$ (a common broadened notion of scheme), the de Rham space X_{dR} of X is the pre-composition of X with υ. On schemes it is easier to describe; it is the functor on spaces which sends A-points of X to $\upsilon(A)$-points. In the more general notion of presheaves, a crystal is more succinctly a quasi-coherent sheaf over X_{dR}.

4.3 Algebraic Connections

The previous section showed, among other things, that $\mathcal{D}$-modules are categorically equivalent to a notion of parallel transport. This short section demonstrates that giving a quasi-coherent sheaf $\mathcal{D}$-module structure is also equivalent to giving the sheaf a flat connection, algebraically.

Let X be a smooth scheme over a field $\Bbbk$, and $\mathcal{F}$ a left $\mathcal{D}_X$-module. For a local section $P \in \mathcal{D}_X(U)$, denote the corresponding endomorphism of $\mathcal{F}(U)$ given by the action of P by ∇_P.

In particular, pick $P, Q \in F_1 D_{\Bbbk}(\mathcal{O}(U))$ such that $P(1) = 0$. By construction, this means that P and Q are derivations, and satisfy the Leibniz rule. For local sections $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{F}(U)$ we then have the following:

$$\nabla_P(fs) = P(f)s + f\nabla_P(s).$$

And by definition of an action,

$$\begin{aligned} \nabla_{fP}(s) &= f\nabla_P(s), \\ \nabla_{[P,Q]}(s) &= [\nabla_P, \nabla_Q](s). \end{aligned}$$

So the $\mathcal{D}_X$-module structure determines a flat connection on $\mathcal{F}$.

The converse also holds, due to the assumption that X is smooth. In this case, the differential operators are locally generated by the derivations as a (non-commutative) $\mathcal{O}_X(U)$-algebra, as discussed in Section 4.1. Hence the above formulae define a $\mathcal{D}_X$-action, by extending the action from the derivations.

Thus to give a $\mathcal{D}_X$-action on a sheaf $\mathcal{F}$ is equivalent to giving a flat connection on $\mathcal{F}$. Importantly, this gives a useful decomposition of the space of local sections of any $\mathcal{D}$-module, in analogy with the bundle case.

Definition 4.3.1. Let X be a smooth scheme over a field $\Bbbk$ of characteristic 0, and $\mathcal{F}$ a $\mathcal{D}_X$-module.

A local section $s \in \mathcal{F}(U)$ is called *horizontal* if for any local operator $P \in \mathcal{D}_X(U)$,

$$\nabla_P(s) = 0. \tag{4.10}$$

Otherwise, it is called *vertical*. The zero section is both vertical and horizontal by convention.

4.4 Coherence and Functoriality

We will now develop some of the coherency conditions we will need to relate $\mathcal{D}$-modules to each other, and to our original physical picture. Unlike with $\mathcal{O}$-modules, the pullback/pushforward functors for $\mathcal{D}$-modules are a bit finicky. The natural setting for these is *derived categories*; the related derived functors happen to smooth out much of the issues the non-derived functors have. See [Ber, Lectures 17-18] for an excellent summary, and [BD04, §2.1] for some results. For a quick introduction to derived categories in a more general sense, see [Nee21, §2.2].

We shall ignore the derived picture here for the sake of brevity. From here on out we shall use varieties, for computational reasons.

We first begin by defining a way to translate vector fields to the pull back of vector fields along a map. Let X and Y be smooth varieties over a field $\Bbbk$ of characteristic 0. Let $f : X \to Y$

be a morphism of schemes, and let TX and TY the tangent bundles[6] to X and Y. One has the following diagram of schemes, inducing a map δ_f from the universal properties of pullbacks.

$$\begin{tikzcd}
TX \arrow[dr, dashed, "\delta_f"'] \arrow[drr, bend left, "f_*"] \arrow[ddr, bend right, "\tau_X"'] & & \\
& f^*TY \arrow[r] \arrow[d] & TY \arrow[d, "\tau_Y"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

This induces a morphism on sheaves of sections, which we also denote by δ_f:

$$\delta_f : \mathcal{T}_X \to f^*\mathcal{T}_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{T}_Y \tag{4.11}$$

Definition 4.4.1. Let $\Bbbk$ be a field of characteristic 0. Let X and Y be smooth varieties over $\Bbbk$, and $f : X \to Y$ a morphism of schemes. Let $\mathcal{G}$ be a left $\mathcal{D}_Y$-module. Then the *inverse image functor* f^* is defined to be the standard pullback on sheaves of $\mathcal{O}_Y$-modules:

$$f^*\mathcal{G} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}.$$

A priori, this is only an $\mathcal{O}_X$-module. It is additionally a left $\mathcal{D}_X$-sheaf, from the following.

Since X is smooth, recall that for any open affine $U \subseteq X$, the local sections $\mathcal{D}_X(U)$ are generated by the vector fields on U. Let $P \in F_1D_{\Bbbk}(\mathcal{O}_X(U))$ be a vector field, and let $g \in \mathcal{O}_X(U)$ and $s \in f^{-1}\mathcal{G}(U)$ be local sections. Then we define:

$$P \cdot (g \otimes s) = P(g) \otimes s + g\delta_f(P) \cdot (1 \otimes s).$$

From our discussion in Section 4.3, this extends to a $\mathcal{D}_X$-action on $f^*\mathcal{G}$.

Remark 4.4.2. The inverse image of $\mathcal{D}_Y$ is commonly denoted $\mathcal{D}_{Y\to X}$ and called the *transfer module* in the literature.

Example 4.4.3. Let X be a smooth variety on a field $\Bbbk$ of characteristic 0. Denote by pr_i the projection of $X \times X$ onto the ith copy of X.

Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{D}_X$-modules. Then one has the $\mathcal{D}_{X\times X}$-modules $\mathrm{pr}_1^* \mathcal{F}$ and $\mathrm{pr}_2^* \mathcal{G}$. One can then define the sheaf on $X \times X$:

$$\mathcal{F} \boxtimes \mathcal{G} := \mathrm{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X\times X}} \mathrm{pr}_2^* \mathcal{G},$$

called the *exterior tensor product* of $\mathcal{F}$ and $\mathcal{G}$. This is an $\mathcal{O}_{X\times X}$-module in the usual way, but also inherits a $\mathcal{D}_{X\times X}$-module structure in a simple manner. Let $V \subseteq X \times X$ be an affine patch. Then $P \in \mathcal{D}_{X\times X}(V)$ acts on $(\mathcal{F} \boxtimes \mathcal{G})(V)$ as a derivation:

$$P \cdot (f \otimes g) := (P \cdot f) \otimes g + f \otimes (P \cdot g), \qquad f \in \mathrm{pr}_1^* \mathcal{F}(V),\ g \in \mathrm{pr}_2^* \mathcal{G}(V). \tag{4.12}$$

[6] We haven't defined these bundles explicitly. The exact definition is not important – the diagram below is primarily to put the morphisms defined in perspective. The tangent sheaf, however, was defined in Example 3.2.13, or alternatively is the subsheaf of $\mathcal{D}$ corresponding to the filtration component $F_1D_{\Bbbk}$.

Remark 4.4.4. One can obviously extend this notion of external tensor product to sheaves on two distinct non-singular varieties X and Y. We shall leave it at the above case – it is all we shall need – though the operation on different bases is also very interesting. The related operation for vector bundles as in (2.15), for example, defines a monoidal (or tensor, for physicists) structure on the category of vector bundles. Monoidal categories are a major theme of [BD04].

Example 4.4.5. We calculate here, for use in Section 5, the explicit action in Equation (4.12) in the case of the affine line.

Let $\Bbbk$ be a field of characteristic 0, and $X = \mathbb{A}^1_{\Bbbk} = \operatorname{Spec}(\Bbbk[t])$ the affine line over $\Bbbk$. Since X is affine, when considering quasi-coherent sheaves it will suffice to consider their global sections, by Proposition 3.4.17.

The fibre product $X \times X$ is then

$$\mathbb{A}^2 = \operatorname{Spec}(\Bbbk[t] \otimes \Bbbk[t]) \cong \operatorname{Spec}(\Bbbk[t \otimes 1, 1 \otimes t]).$$

It becomes convenient to occasionally write $t \otimes 1 = t_1$ and $1 \otimes t = t_2$. The projections pr_i are then dual to the inclusions

$$\iota_i : t \mapsto t_i, \qquad i \in \{1, 2\},$$

and we have (global sections of) the sheaf of differential operators being the Weyl algebra (recall Example 4.1.3):

$$\mathcal{D}_{X\times X}(X \times X) = A_2(\Bbbk) = \Bbbk[t_1, t_2, \partial_1, \partial_2].$$

A quick computation on global sections shows that the associated morphism δ_{pr_i} (recall (4.11)) for $i \in \{1, 2\}$ is:

$$\begin{aligned} \delta_{\mathrm{pr}_i} : \mathcal{T}_{\mathbb{A}^2}(\mathbb{A}^2) = \Bbbk[\partial_1, \partial_2] &\to \mathrm{pr}_i^* \mathcal{T}_{\mathbb{A}^1}(\mathbb{A}^2) = \Bbbk[t_1, t_2] \otimes_{\Bbbk[t_i]} \Bbbk[\partial_i] \\ \partial_i &\mapsto 1 \otimes \partial_i \\ \partial_j &\mapsto 0, \qquad\qquad j \neq i. \end{aligned} \tag{4.13}$$

Let $\mathcal{F}$ and $\mathcal{G}$ be two $\mathcal{D}_X$-modules and $f \in \mathcal{F}(X)$ and $g \in \mathcal{G}(X)$ be two global sections. Denote by $\tilde{f}$ and $\tilde{g}$ their image in $\mathrm{pr}_1^* \mathcal{F}$ and $\mathrm{pr}_2^* \mathcal{G}$.

Then $\partial_1, \partial_2 \in \mathcal{D}_{X\times X}(X \times X)$ acts on $(\mathcal{F} \boxtimes \mathcal{G})(X \times X)$ simply as the derivative on different factors:

$$\begin{aligned} \partial_1 \cdot (\tilde{f} \otimes \tilde{g}) &= (\partial_1 \cdot \tilde{f}) \otimes \tilde{g} + \tilde{f} \otimes (\partial_1 \cdot \tilde{g}) = (\partial_1 \cdot \tilde{f}) \otimes \tilde{g} \\ \partial_2 \cdot (\tilde{f} \otimes \tilde{g}) &= (\partial_2 \cdot \tilde{f}) \otimes \tilde{g} + \tilde{f} \otimes (\partial_2 \cdot \tilde{g}) = \tilde{f} \otimes (\partial_2 \cdot \tilde{g}) \end{aligned}$$

Remark 4.4.6. The notion of exterior tensor product gives one half of the desired notion of 'Poisson algebra' discussed in Section 2.7, in the context of schemes. For a smooth variety X over a field of characteristic 0, a naïve notion of the associative product on a $\mathcal{D}_X$-module $\mathcal{F}$ would be a morphism like

$$\mathcal{F} \boxtimes \mathcal{F} \dashrightarrow \mathcal{F}. \tag{4.14}$$

Of course, the left-hand-side is a $\mathcal{D}_{X\times_{\Bbbk} X}$-module while the right-hand-side is a $\mathcal{D}_X$-module. The natural solution to this would be to pushforward along the diagonal closed embedding

$$\Delta : X \to X \times_{\Bbbk} X,$$

which identifies X with the closed subset $\Delta(X) \subseteq X \times_{\Bbbk} X$. The problem is: 'What is the appropriate notion of pushforward for $\mathcal{D}_X$-module?'

The issue is the difference between left and right $\mathcal{D}$-modules. The two categories are equivalent, but in practice the natural functors like the pullback f^* or pushforward f_* of $\mathcal{O}$-modules are only defined between left and right $\mathcal{D}$-modules respectively. This is mirrored in the difference between differentiation of functions and differentiation of distributions, discussed in [Ber] and briefly alluded to in Example 4.2.8.

The typical approach to fixing this is by going through derived categories, as mentioned above. This is difficult to mesh with an appropriate notion of (4.14). Instead we shall at the adjoint functors to the pullback in a category containing the category of $\mathcal{D}_X$-modules. We note here that the notation is potentially confusing: the functor we shall describe is *not* the usual pushforward.

One can alternatively define the appropriate adjoint organically, with some intuition developed in the prior chapters. We shall do so only in the case we are interested in, namely the case of the diagonal closed immersion.

Definition 4.4.7. Let X be a smooth variety over a field $\Bbbk$ of characteristic 0. Let $\mathcal{F}$ be a $\mathcal{D}_X$-module. Denote by pr_1 the projection of $X \times_{\Bbbk} X$ onto the 1st copy of X.

The *completion of $\mathcal{F}$ along the diagonal* is the sheaf of modules on $X \times_{\Bbbk} X$:

$$\widehat{\Delta}_* \mathcal{F} := \varprojlim_n \left(\mathcal{O}_{X\times X} / \mathcal{J}^n \right) \otimes_{\mathcal{O}_{X\times X}} \mathrm{pr}_1^* \mathcal{F}, \tag{4.15}$$

where $\mathcal{J}$ is the sheaf of ideals corresponding to the closed immersion $\Delta : X \to X \times_{\Bbbk} X$. There is an action of $\mathcal{D}_{X\times X}$ upon $\widehat{\Delta}_* \mathcal{F}$, by acting as a derivation; using the module structure of each term in the tensor individually.

This should be familiar – it is the limit of the right-hand-side of (4.8)!

Here is an interpretation for this sheaf. One wants a sheaf on $X \times_{\Bbbk} X$ which is morally the same as $\mathcal{F}$, defined instead on the closed subscheme $\Delta(X)$ which is topologically the same as X. So, one forces the sheaf $\mathcal{F}$ into a sheaf on the diagonal by pulling back the projection and tensoring by the structure sheaf of the diagonal subscheme. This doesn't convey enough data to be have a well-defined $\mathcal{D}$-action – see our proof of Lemma 4.2.6! So one must take the completion; the limit in (4.15).

As noted in Definition 3.5.1 and Remark 4.2.10, the structure sheaf of the diagonal subscheme has support contained in $\Delta(X)$; its stalks vanish elsewhere. Then similar to our motivation in Section 3.2, we need to 'add more functions' to account for this.

Definition 4.4.8. Let X be a smooth variety over a field $\Bbbk$ of characteristic 0. Let $\mathcal{F}$ be a $\mathcal{D}_X$-module. Let $U = (X \times_{\Bbbk} X) \setminus \Delta(X)$ be the complementary subset to the diagonal $\Delta(X)$, and:

$$j : U \to X \times_{\Bbbk} X$$

be the corresponding open embedding.

The *localized completion of $\mathcal{F}$* is the sheaf:

$$\widetilde{\Delta}_* \mathcal{F} := \widehat{\Delta}_* \mathcal{F} \otimes_{\mathcal{O}_{X\times X}} j_* \mathcal{O}_U \tag{4.16}$$

where, for our use, $j_* \mathcal{O}_U$ is the subsheaf of $\mathcal{O}_{X\times X}$ uniquely determined by the property that each of its local sections are killed by some power of $\mathcal{J}$. The localized completion again has an action by $\mathcal{D}_{X\times X}$ as derivations.

The existence of $j_*\mathcal{O}_U$ is shown in [BD04, §3.5.2]. One might think of it as the pushforward in the special case of an open immersion, hence the notation.

The unfortunate thing about the completion and its localization is that they are not quasi-coherent $\mathcal{O}_{X\times X}$-modules in general; they are not $\mathcal{D}_{X\times X}$-modules but $\mathcal{D}_{X\times X}$-sheaves, as we called them in Definition 4.2.7. One way to see this that these sheaves act similar to extension of a sheaf by zero (think excision in algebraic topology). Resources that exhibit this are [Har77, §5.2.3] and [Gro66, §1-2]. Alternatively, a good analogy to think of these things is as an extension of the notion of meromorphic functions. See [Gro60, §20.1-2].

The sheaves as we have defined them may seem slightly arbitrary. They are, however, the 'correct' notions, primarily due to the functor $\mathcal{F} \mapsto \widehat{\Delta}_*\mathcal{F}$ being right adjoint to the normal pullback functor Δ^* between categories of $\mathcal{D}$-sheaves. See again [BD04, §3.5.2].

Our final descriptor for 'nice' $\mathcal{D}$-modules is *equivariance*. If one recalls that quasi-coherent sheaves are the appropriate generalization for bundles on schemes, then equivariant quasi-coherent sheaves are the same notion as the (perhaps more familiar case) of equivariant bundles – that is, bundles whose fibres are identifiable under a group action. An equivariant $\mathcal{D}$-module can be thought to be thought as a prolongation of equivariance to jet bundles, using the picture of Section 2 and the translation of such in Section 4. We won't need the full description, only *translation equivariance*.

We note here that the affine line is a group object in the category of schemes. In more concrete terms, $\mathbb{A}^1$ is a group whose multiplication and inversion maps are morphisms of schemes (much like a Lie group). We denote the multiplication map by:

$$m : \mathbb{A}^1 \times_{\mathbb{k}} \mathbb{A}^1 \to \mathbb{A}^1,$$

which on global sections is the map

$$\begin{aligned} \mathbb{k}[t_1] \otimes_{\mathbb{k}} \mathbb{k}[t_2] &\leftarrow \mathbb{k}[t] \\ t_1 + t_2 &\leftarrow\!\shortmid t. \end{aligned}$$

Recall also the projection $\mathrm{pr}_2 : \mathbb{A}^1 \times_{\mathbb{k}} \mathbb{A}^1 \to \mathbb{A}^1$ onto the second component.

In the wider context of equivariance it may be worth viewing m as $\mathbb{A}^1$ acting upon itself by translation.

Definition 4.4.9. Let $\mathbb{k}$ be a field of characteristic 0, and consider the affine line $\mathbb{A}^1$ over $\mathbb{k}$. Denote by

$$\mathrm{pr}_{23} : \mathbb{A}^1 \times_{\mathbb{k}} \mathbb{A}^1 \times_{\mathbb{k}} \mathbb{A}^1 \to \mathbb{A}^1 \times_{\mathbb{k}} \mathbb{A}^1;$$

the projection onto the second and third components.

A $\mathcal{D}_{\mathbb{A}^1}$-module $\mathcal{F}$ is *(weakly) translation equivariant* if it is equipped with an isomorphism of $\mathcal{D}_{\mathbb{A}^2}$-modules

$$\mathbf{\Phi} : m^*\mathcal{F} \xrightarrow{\sim} \mathrm{pr}_2^*\,\mathcal{F}, \tag{4.17}$$

satisfying the associativity condition

$$\mathrm{pr}_{23}^*\,\mathbf{\Phi} \circ (1 \times m)^*\mathbf{\Phi} = (m \times 1)^*\mathbf{\Phi}.$$

A morphism of translation equivariant $\mathcal{D}_{\mathbb{A}^1}$-modules $(\mathcal{F}, \mathbf{\Phi})$ and $(\mathcal{G}, \mathbf{\Psi})$ is a morphism $\varphi : \mathcal{F} \xrightarrow{\sim} \mathcal{G}$ satisfying:

$$\mathrm{pr}_2^*\,\varphi \circ \mathbf{\Phi} = \mathbf{\Psi} \circ m^*\varphi.$$

Denote the category of translation equivariant $\mathcal{D}_{\mathbb{A}^1}$-modules by $\mathcal{M}_T^{\ell}(\mathbb{A}^1)$.

5 Operator Product Expansion

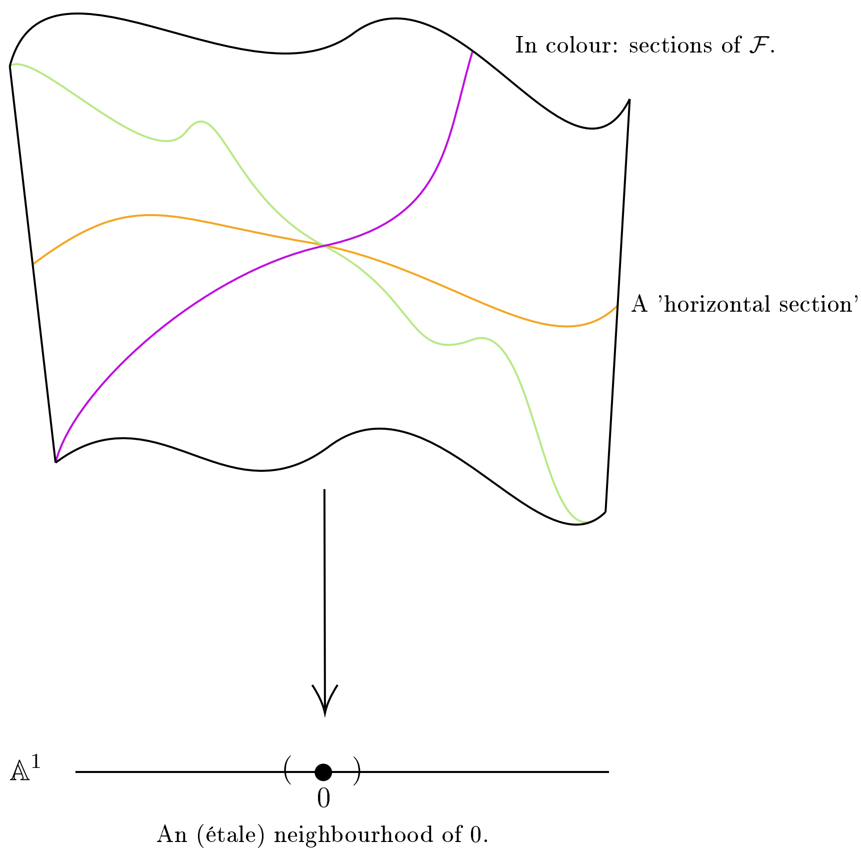
We now return to the circumstance discussed in Section 2.2 to Section 2.7. In such we considered a fiber bundle $\pi : E \to M$ equipped with a flat connection, whose sections were considered as states of a physical system. In the case $M = \mathbb{R}$, these are exactly paths of a particle. The importance of Poisson algebras were stressed - ultimately these were morphisms from sections of the exterior product of π with itself to the sections of π.

Our current goal is to construct an analogue of Poisson algebras in the context of $\mathcal{D}$-modules. This section develops our analogue of the associative structure. In Section 3, the use of algebraic geometry as a physical tool was introduced. In short, schemes acted as a way in which to encode measurement data. The analogue to manifolds became smooth separated schemes and the analogue to bundles became quasi-coherent sheaves on such schemes.

Henceforth fix be a field of characteristic 0, $\Bbbk$. Let X be a curve over $\Bbbk$. We will take for most of this section $X = \mathbb{A}^1$. This in particular covers the one and two dimensional physical cases (when $\Bbbk = \mathbb{Q}$ or $\mathbb{R}$ and $\mathbb{C}$ respectively). Take a $\mathcal{D}_X$-module $\mathcal{F}$.

From Section 4, giving this sheaf $\mathcal{F}$ the structure of a $\mathcal{D}_X$-module is equivalent to specifying a flat connection on $\mathcal{F}$, and allows us to invoke some dynamical information.

Here is a cartoon depicting the set-up, where the quasi-coherent sheaf is pictured as a vector bundle.



Definition 5.0.1 ([BD04, §3.5.8])**.** Let X be a curve over a field of characteristic 0. Let $\mathcal{F}$ be a left $\mathcal{D}_X$-module.

The *space of binary operator product expansions (opes)* associated to $\mathcal{F}$ is the following set of morphisms of $\mathcal{D}_X$-sheaves:

$$O_2(\mathcal{F}) := \mathrm{Hom}_{\mathcal{D}_{X\times X}}(\mathcal{F}\boxtimes\mathcal{F}, \widetilde{\Delta}_*\mathcal{F})$$

A binary ope is an element $\bullet \in O_2(\mathcal{F})$; that is, a morphism $\mathcal{F}\boxtimes\mathcal{F} \to \widetilde{\Delta}_*\mathcal{F}$ respecting the $\mathcal{D}_X$-action.

When clear we shall shorten 'binary ope' to simply 'ope'.

Example 5.0.2. Consider the case $X = \mathrm{Spec}(\Bbbk[t]) = \mathbb{A}^1$. Let's first calculate explicitly the sheaves involved in Definition 5.0.1, using Section 4.4.

We compute, first of all, the completion of $\mathcal{F}$ along the diagonal:

$$\widehat{\Delta}_*\mathcal{F} = \varprojlim_n \left(\mathcal{O}_{X\times X}/\mathcal{J}^n\right) \otimes_{\mathcal{O}_{X\times X}} \mathrm{pr}_1^*\mathcal{F}.$$

Explicitly, we have $\Bbbk$-algebra isomorphisms

$$\mathcal{O}_{X\times X}(X\times X) \cong \Bbbk[t]\otimes_{\Bbbk}\Bbbk[t] \cong \Bbbk[t_1,t_2],$$

or as an algebra over $\mathcal{O}_X(X)\cong\Bbbk[t_1]$,

$$\mathcal{O}_{X\times X}(X\times X) \cong \Bbbk[t_1][t_2] = \Bbbk[t_1][t_1-t_2],$$

having written once more $t_1 = t\otimes 1$ and $t_2 = 1\otimes t$. The ideal of the diagonal is then simply

$$\mathcal{J} = \langle t_1 - t_2\rangle.$$

Having done this, for any $n\in\mathbb{Z}_{\geq 0}$ the global sections of $\mathcal{O}_{X\times X}/\mathcal{J}^n$ are identified with polynomials in (t_1-t_2) of degree strictly less than n, with coefficients in $\Bbbk[t_1]$. One could just as well have taken coefficients in $\Bbbk[t_2]$ – the difference being which projection one takes $X\times_{\Bbbk} X\to X$.

Thus the limit is the associated power series:

$$\varprojlim_n \left(\mathcal{O}_{X\times X}/\mathcal{J}^n\right)(X\times X) \cong \Bbbk[t_1][[t_1-t_2]].$$

Hence, the global sections of the completion of $\mathcal{F}$ are interpretted as formal power series in (t_1-t_2), with coefficients valued in $\mathcal{F}$:

$$\widehat{\Delta}_*\mathcal{F}(X\times X) = \mathrm{pr}_1^*\mathcal{F}(X\times X)[[t_1-t_2]] = \mathcal{F}(\Bbbk[t_1])[[t_1-t_2]].$$

So, upon localizing, the global sections of our sheaf are formal Laurent series with coefficients valued in $\mathcal{F}$:

$$\widetilde{\Delta}_*\mathcal{F}(X\times X) = \mathcal{F}(\Bbbk[t_1])((t_1-t_2)).$$

Similar holds for any (étale) open U of $X\times X$. We write:

$$\widetilde{\Delta}_*\mathcal{F} = \mathcal{F}((t_1-t_2)), \tag{5.1}$$

where $\mathcal{F}$ is viewed as taking values on the first (or equivalently, the second) component of $X\times X$.

Example 5.0.3. In fact, a similar result is true for any $\mathcal{D}$-module $\mathcal{F}$ over a smooth curve on a characteristic 0 field $\Bbbk$. In this circumstance, the variables t_1, t_2 should instead be seen as local coordinates on the curve. Once more, Equation (5.1) holds. See [BD04, §3.5.5].

Now let $\bullet \in O_2(\mathcal{F})$ be a binary ope. Let U be an open subset of $X \times_{\Bbbk} X$, and V the projection of U onto X. Then our ope locally takes the form:

$$\begin{aligned} \bullet : (\mathcal{F} \boxtimes \mathcal{F})(U) &\to \mathcal{F}(V)((t_1 - t_2)) \\ (f, g) &\mapsto \sum_{\ell \geq N} (t_1 - t_2)^{\ell} (f \bullet_{\ell} g) \end{aligned}$$

where $(f \bullet_{\ell} g) \in \mathcal{F}(U)$ are local sections and the coefficients of the Laurent series.

Composition with opes is a little tricky. The most notable issue being that the codomain $\widetilde{\Delta}_* \mathcal{F}$ is not the same $\mathcal{D}_X$-sheaf as either input domain $\mathcal{F}$. The manner of fixing this is to change where one composes. To see this explicitly, observe the following diagram of $\mathcal{D}_{X^3}$-sheaves:

$$\begin{tikzcd} & \widetilde{\Delta}_*^{\{2,3\}}(\mathcal{F} \boxtimes \mathcal{F}) \arrow[r, "\widetilde{\Delta}_*^{\{2,3\}}(\bullet)"] & \widetilde{\Delta}_*^{\{2,3\}} \widetilde{\Delta}_* \mathcal{F} \\ \mathcal{F} \boxtimes \mathcal{F} \boxtimes \mathcal{F} \arrow[ur, "\mathrm{id}_{\mathcal{F}} \boxtimes \bullet"] \arrow[dr, "\bullet \boxtimes \mathrm{id}_{\mathcal{F}}"'] \arrow[rr, dashed, "\square"] & & \widetilde{\Delta}_*^{\{1,2,3\}} \mathcal{F} \arrow[u, hook] \arrow[d, hook] \\ & \widetilde{\Delta}_*^{\{1,2\}}(\mathcal{F} \boxtimes \mathcal{F}) \arrow[r, "\widetilde{\Delta}_*^{\{1,2\}}(\bullet)"'] & \widetilde{\Delta}_*^{\{1,2\}} \widetilde{\Delta}_* \mathcal{F} \end{tikzcd} \tag{5.2}$$

The new sheaves introduced here are not scary - they are merely (4.16), but performed along slightly different closed embeddings. The morphisms in question are:

$$\Delta^{\{2,3\}} := \mathrm{id}_X \times_{\Bbbk} \Delta : X^2 \to X^3$$

and

$$\Delta^{\{1,2\}} := \Delta \times_{\Bbbk} \mathrm{id}_X : X^2 \to X^3$$

and $\Delta^{\{1,2,3\}}$ is the induced map in the fibre diagram:

$$\begin{tikzcd} X \arrow[dr, dashed, "\Delta^{\{1,2,3\}}"] \arrow[drr, bend left, "\Delta^{\{1,2\}}"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\ & X \times_{\Bbbk} X \times_{\Bbbk} X \arrow[r, "\mathrm{pr}_{23}"] \arrow[d, "\mathrm{pr}_1"'] & X \times_{\Bbbk} X \arrow[d] \\ & X \arrow[r] & \mathrm{Spec}(\Bbbk) \end{tikzcd}$$

However it may be easiest to think of it in our preferred setting $X = \mathbb{A}^1$ in which the map is simply:

$$\begin{aligned} \Delta^{\{1,2,3\}} : \mathbb{A}^1 &\to \mathbb{A}^3 \\ x &\mapsto (x, x, x) \end{aligned}$$

A composition of opes is the composition of morphisms given by the top or bottom rows of Equation (5.2). The right-most horizontal morphisms are:

$$\begin{aligned}
\widetilde{\Delta}_*^{\{2,3\}}(\bullet) : \left(f, \sum_{m \geq M} (t_2 - t_3)^m h_m \right) &\mapsto \sum_{m \geq M} (t_2 - t_3)^m (f \bullet h_m) \\
&= \sum_{m \geq M} (t_2 - t_3)^m \sum_{\ell \geq N} (t_1 - t_3)^\ell (f \bullet_\ell h_m) \\
\widetilde{\Delta}_*^{\{1,2\}}(\bullet) : \left(\sum_{\ell \geq N'} (t_1 - t_2)^\ell f_\ell, g \right) &\mapsto \sum_{\ell \geq N'} (t_1 - t_2)^\ell (f_\ell \bullet g) \\
&= \sum_{\ell \geq N'} (t_1 - t_2)^\ell \sum_{m \geq M'} (t_2 - t_3)^m (f_\ell \bullet_m g)
\end{aligned}$$

There is an issue with this approach – namely that the two compositions are not naturally in $\mathrm{Hom}_{\mathcal{D}_{X^3}}(\mathcal{F}^{\boxtimes 3}, \widetilde{\Delta}_*^{\{1,2,3\}}\mathcal{F})$, as one would expect from a composition, but rather in $\mathrm{Hom}_{\mathcal{D}_{X^3}}(\mathcal{F}^{\boxtimes 3}, \widetilde{\Delta}_*^{\{i,j\}}\widetilde{\Delta}_*\mathcal{F})$. The reason for this behaviour is that pushing forward the diagonal $\Delta^{\{1,2,3\}} : X \hookrightarrow X^3$ on $\mathcal{F}$ forces support on the diagonal in X^3, while doing so with $\Delta^{\{i,j\}} : X^2 \hookrightarrow X^3$ does not. There is a physical reason that one must have this, discussed later in Remark 5.0.10.

Definition 5.0.4. Let X be a curve over a field of characteristic 0. Let $\mathcal{F}$ be a $\mathcal{D}_X$-module, and $\bullet \in O_2(\mathcal{F})$ a binary ope.

I) We say that $\bullet$ is *associative* if the images of the aforementioned compositions lie in $\mathrm{Hom}_{\mathcal{D}_{X^3}}(\mathcal{F}^{\boxtimes 3}, \widetilde{\Delta}_*^{\{1,2,3\}}\mathcal{F})$, and the two compositions coincide. To be more precise, $\bullet$ is associative if there exists a $\mathcal{D}_X$-sheaf morphism

$$\square : \mathcal{F} \boxtimes \mathcal{F} \boxtimes \mathcal{F} \to \widetilde{\Delta}^{\{1,2,3\}},$$

making the diagram (5.2) commute.

II) A *unit* for the binary ope $\bullet$ is a horizontal section[7] of $\mathcal{D}_X$-modules $\mathbb{1}$, such that for every local section f of $\mathcal{F}(V)$,

$$f \bullet \mathbb{1} \text{ and } \mathbb{1} \bullet f \in \widehat{\Delta}_*\mathcal{F}(U) \subseteq \widetilde{\Delta}_*\mathcal{F}(U), \tag{5.3}$$

and modulo the sheaf of ideals $\mathcal{J}\widehat{\Delta}_*$, one has $f \bullet \mathbb{1} = \mathbb{1} \bullet f = f$.

III) The ope $\bullet$ is said to be *commutative* if for any two local sections f, g of $\mathcal{F}$,

$$f \bullet g = g \bullet f.$$

[7]For an affine X, this is easy to define: pick a global section satisfying (4.10). When X is not affine, the definition takes more finesse. A horizontal section is a $\mathcal{D}_X$-module morphism $\omega_X \to \mathcal{F}$ from the canonical line bundle. All this is doing is relating local coordinates on X to local sections of $\mathcal{F}$; letting one extend the notion of horizontal section globally. One can check that these notions agree.

Definition 5.0.5 ([BD04, §3.5.9])**.** Let X be a curve over a field of characteristic 0.

An *ope algebra* is a pair $(\mathcal{F}, \bullet)$ consisting of a $\mathcal{D}_X$-module $\mathcal{F}$ and a choice of binary ope $\bullet \in O_2(\mathcal{F})$, with a unit $\mathbb{1}$. In this paper we shall assume that $\bullet$ is associative and commutative.

Definition 5.0.6. An ope algebra $(\mathcal{F}, \bullet)$ over the affine line $X = \mathbb{A}^1$ is said to be translation equivariant if the underlying $\mathcal{D}_X$-module $\mathcal{F}$ is translation equivariant (recall Definition 4.4.9), and the ope is invariant under the supplied isomorphism $\mathbf{\Phi}$ of (4.17).

We denote a translation equivariant ope algebra by the tuple $(\mathcal{F}, \mathbf{\Phi}, \bullet)$.

Suppose we are given a translation equivariant ope algebra $(\mathcal{F}, \mathbf{\Phi}, \bullet)$ over the affine line $\mathbb{A}^1$. Fix a point $t \in \mathbb{A}^1$. Then the isomorphism:

$$\mathbf{\Phi} : m^*\mathcal{F} \dot{\to} \mathrm{pr}_2^* \mathcal{F}$$

induces an isomorphism on stalks at the point $(t, 0) \in \mathbb{A}^2$:

$$\mathbf{\Phi}_{(t,0)} : (m^*\mathcal{F})_{(t,0)} \to (\mathrm{pr}_2^* \mathcal{F})_{(t,0)}.$$

By definition of m and pr_2, this gives an isomorphism:

$$\mathcal{F}_t \to \mathcal{F}_0.$$

So, we can identify the stalks of $\mathcal{F}$ with one another!

Then we have the following data:

I) A $\mathbb{Z}_2$-graded vector space $V := \mathcal{F}(\mathbb{A}^1)$, with gradation given by the horizontal and vertical sections. We shall call this vector field the *space of states*.

II) A unit $\mathbb{1}$ with respect to $\bullet$. We shall relabel this to $|0\rangle$ and call it the *vacuum state*.

III) A *state-field correspondence*. Namely, the map:

$$\begin{aligned} V &\to \mathrm{End}_{\mathbb{k}}(V)((t)) \\ f &\mapsto Y(f,t) := (f \bullet -) = \sum_{\ell \geq N} t^\ell (f \bullet_\ell -) \end{aligned} \tag{5.4}$$

Here we have used translation invariance of $\bullet$ to take $t_2 = 0$ without loss of generality, and relabelled $t_1 = t$. Essentially we are looking at local behaviour about 0.

The terms $(f \bullet_\ell -)$ are called the *modes of the state f*.

It is common practice to choose a different indexing convention for (5.4). We define the *kth mode of f* to be $f_{(k)} := (f \bullet_{-k-1} -) \in \mathrm{End}_{\mathbb{k}}(V)$, so that:

$$Y(f,z) = \sum_{k \leq -N-1} z^{-k-1} f_{(k)}$$

IV) A *translation operator* $T \in \mathrm{End}_{\mathbb{k}}(V)$ defined by

$$Tf := \partial_t Y(f,t)|0\rangle|_{t=0}, \qquad f \in V, \tag{5.5}$$

where we make use of (5.3) to meaningfully evaluate at $t = 0$. Explicitly:

$$\partial_t Y(f,t)|0\rangle = \sum_{\ell \geq 0} \ell t^{\ell-1}(f \bullet_\ell |0\rangle),$$

so that:

$$Tf = \partial_t Y(f,t)|0\rangle|_{t=0} = (f \bullet_1 |0\rangle) = f_{(-2)}|0\rangle. \tag{5.6}$$

V) By definition of the vacuum state, one has

$$Y(|0\rangle, t) = \mathrm{id}_V + \sum_{\ell \geq 1} t^\ell (|0\rangle \bullet_\ell -).$$

Since the vacuum state $|0\rangle$ is a horizontal section, we have:

$$T|0\rangle = 0 = |0\rangle_{(-2)}|0\rangle,$$

implying $|0\rangle_{(-2)} = 0$, and by definition of T

$$[T, |0\rangle_{(n)}] = -n|0\rangle_{(n-1)}.$$

So by induction on n, $|0\rangle_{(-n)} = 0$ for $n \in \mathbb{Z}_{\geq 2}$. That is,

$$Y(|0\rangle, t) = \mathrm{id}_V. \tag{5.7}$$

And for any local section f of $\mathcal{F}$,

$$Y(f,z)|0\rangle = \sum_{\ell \geq 0} z^\ell (f \bullet_\ell |0\rangle).$$

Since $\ell \geq 0$, we can meaningfully evaluate at $z = 0$ to see:

$$Y(f,z)|0\rangle|_{z=0} = f \bullet_0 |0\rangle = f. \tag{5.8}$$

VI) Finally, associativity and commutativity conditions show that there is a natural action of the symmetric group Σ_3 on the elements of the composition $(- \bullet - \bullet -)$. To be more precise, for sections $f, g, h \in V$, we can identify the following expressions:

$$\begin{aligned}
(f \bullet (g \bullet -)) &= Y(f,t_1)Y(g,t_2) \in \mathrm{End}_{\Bbbk}(V)((t_2))((t_1)), \\
((f \bullet g) \bullet -) &= Y(Y(f,t_1)g, t_2) \in \mathrm{End}_{\Bbbk}(V)((t_1))((t_1 - t_2)), \\
(g \bullet (f \bullet -)) &= Y(g,t_2)Y(f,t_1) \in \mathrm{End}_{\Bbbk}(V)((t_1))((t_2)),
\end{aligned}$$

by applying the associativity condition (5.2) and symmetry several times. In other words, there exists $n \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}/2\mathbb{Z}$ such that

$$(t_1 - t_2)^n Y(f,t_1)Y(g,t_2) = (-1)^m (t_1 - t_2)^n Y(g,t_2)Y(f,t_1). \tag{5.9}$$

If the condition (5.9) holds for all f, g, then all fields are said to be *mutually local*.

This is exactly[8] the definition of a *vertex algebra*, á la Kac [Kac98, Definition 1.3]!

[8] Except here we have not demanded V be finite-dimensional.

On the reverse, if one is given a $\mathbb{Z}/2\mathbb{Z}$-graded $\Bbbk$-vector space V, a linear map

$$Y : V \to \mathrm{End}_{\Bbbk}(V)((t))$$

satisfying (5.9), an endomorphism $T \in \mathrm{End}_{\Bbbk}(V)$ satisfying (5.6), and a vector $|0\rangle \in V$ satisfying (5.7) and (5.8), then one can construct a translation equivariant ope algebra over $\mathbb{A}^1$ as follows.

I) Consider the $\Bbbk[t]$-module $V[t]$. By Proposition 3.4.17 this defines a quasi-coherent sheaf $\mathcal{V}$ on $\mathbb{A}^1$, with $V[t]$ the global sections of $\mathcal{V}$. It is straightforward to check that this is translation equivariant.

II) Let $f \in V[t]$. $\mathcal{V}$ becomes a $\mathcal{D}^1_{\mathbb{A}}$-module from the following action by the Weyl algebra A_1 on global sections:

$$\begin{aligned} t \cdot f &:= tf \\ \partial_t \cdot f &:= \partial_t f - Tf \end{aligned} \tag{5.10}$$

III) $\mathcal{V}$ is equipped with a translation invariant binary ope given by (5.4), using the map Y.

IV) The vacuum state $|0\rangle$ is a horizontal (global) section of $\mathcal{V}$ by (5.10), and acts as a unit for the ope by construction.

V) Applying (5.9) along with the vacuum axioms implies that the ope is associative and commutative.

What we have done is summarized below.

Theorem 5.0.7. *Let $\Bbbk$ be a field of characteristic* 0.

The category of translation equivariant ope algebras over the affine line $\mathbb{A}^1_{\Bbbk}$ is equivalent to the category of vertex algebras.

The remainder of this paper is several remarks on generalisations, physical perspective, and other thoughts, and may be comfortably ignored.

Remark 5.0.8. A related but widely used construction is a *vertex operator algebra*. Briefly, these are vertex algebras with the additional structure of a stress energy vector ω, whose modes satisfy the commutation relations of the *Virasoro algebra*. This has a nice interpretation here.

The Witt algebra $\mathfrak{Witt}$ is the Lie algebra defined abstractly by the vector space $\Bbbk[L_m \mid m \in \mathbb{Z}]$ with Lie bracket satisfying relations $[L_m, L_n] = (m-n)L_{m+n}$, or more concretely generated by $L_m = -t^{m+1}\partial_t$. The positive order terms of such are present in the first order Weyl algebra.

The Virasoro algebra $\mathfrak{Vir}$ is the central extension of the Witt algebra: it fits into an exact sequence

$$0 \longrightarrow \Bbbk C \longrightarrow \mathfrak{Vir} \longrightarrow \mathfrak{Witt} \longrightarrow 0,$$

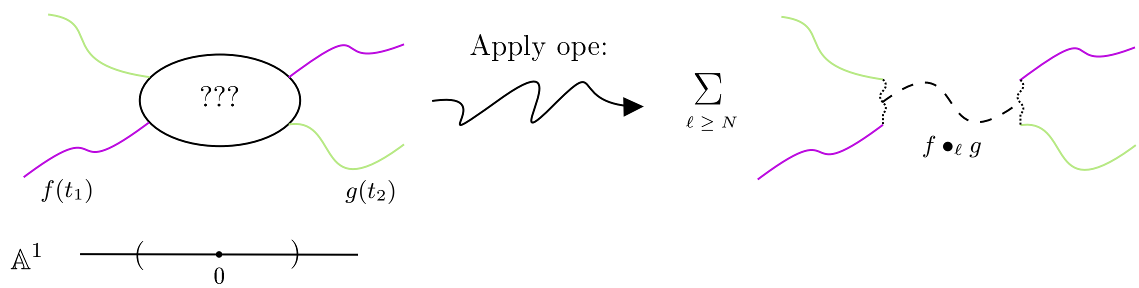
so that the commutation relations become $[L_m, L_n] = (m-n)L_{m+n} + (1/12)(m^3 - m)\delta_{m,-n}C$; it is these relations that the vector ω must satisfy inside the vertex operator algebra.

Recall from Section 4.3 that introducing $\mathcal{D}$-module structure is equivalent to a flat connection. From the work in this section a vertex algebra can be interpreted as possessing a flat connection – adding in the structure of the Virasoro algebra might be interpreted as artificially introducing curvature, here represented by C. This perpective is discussed briefly in [Gan06, §3].

Remark 5.0.9. Operator product expansions were used here to model the associative product of local Poisson algebras. The alternate view, which is categorically much more pleasant, is *chiral algebras*, which model locally the Lie bracket of Poisson algebras. These algebras are the main point of study in [BD04]. However chiral products and associative, commutative ope products are in bijection – this is [BD04, §3.5.10].

Remark 5.0.10. One might consider an ope algebra on $\mathbb{A}^1$ geometrically as follows. We remarked prior that much of the intricacy of defining a local Poisson structure comes from being unable to multiply two sections/fields at the same point. Physically, evaluating two fields at the same point would be similar to attempting to place two point charges on top of one another; a process which would involve a lot of singularities.

In this sense an ope takes the unknown behaviour at and around these singular points and approximates them with a series of other fields, weighted by distance $(t_1 - t_2)$: see the picture below, which one might think of as a zoomed-in version of the picture above.



One can then interpret translation equivariance physically – fields always pair in the same way, regardless of what point one is performing this local approximation about. If there were a special point that did not obey the behaviour elsewhere, then this would be a preferred point. Relativity does not like preferred points, so such a point would be bad.

6 References

[BB93] A. Beilinson and J. Bernstein. A Proof of Jantzen Conjectures. *Advances in Soviet Mathematics*, 16(1):1–50, 1993.

[BD04] A. Beilinson and V. Drinfeld. *Chiral Algebras*. Colloquium Publishing, 1st edition, 2004.

[Ber] J. Bernstein. $\mathcal{D}$-Modules.

[BFR19] R. Brunetti, K. Fredenhagen, and P. L. Ribeiro. Algebraic Structure of Classical Field Theory: Kinematics and Linearized Dynamics for Real Scalar Fields. *Communications in Mathematical Physics*, 368(2):519–584, May 2019.

[BGG72] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand. Differential Operators on a Cubic Cone. *Russian Mathematical Surveys*, 27(1):169, Feb 1972.

[Bjö93] J.-E. Björk. *Analytic D-Modules and Applications*. Springer Dordrecht, 1st edition, 1993.

[Bor86] R. E. Borcherds. Vertex algebras, Kac-Moody algebras, and the Monster. *Proc Natl. Acad. Sci. USA*, 83:3068–3071, 1986.

[Bor88] R. E. Borcherds. Generalised Kac-Moody Algebras. *Journal of Algebra*, 115:501–512, 1988.

[Bor92] R. E. Borcherds. Monstrous moonshine and monstrous Lie superalgebras. *Inventiones mathematicae*, 109:405–444, 1992.

[CS99] J. H. Conway and N. J. A. Sloane. *Sphere Packings, Lattices and Groups*. Springer, 3rd edition, 1999.

[DGM90] L. Dolan, P. Goddard, and P. Montague. Conformal Field Theory, Triality, and the Monster Group. *Physics Letters B*, 236(2):165–172, 1990.

[DP83] A. Dold and D. Puppe. Duality, Trace and Transfer. *Proceedings of the Steklov Institute*, 154:81–97, 1983.

[Dum10] F. Dumitrescu. Connections and Parallel Transport. *Journal of Homotopy and Related Structures*, 5(1):171–175, 2010.

[EH00] D. Eisenbud and J. Harris. *The Geometry of Schemes*. Springer New York, 1st edition, 2000.

[Eti17] P. Etingof. Introduction to Algebraic D-Modules, 2017.

[FLM88] I. Frenkel, J. Lepowsky, and A. Meurman. *Vertex Operator Algebras and the Monster*. Academic Press, 1st edition, 1988.

[Gan06] T. Gannon. *Moonshine Beyond the Monster*. Cambridge University Press, 2006.

[GGK+68] J. Giraud, A. Grothendieck, S. L. Kleiman, M. Raynaud, and J. Tate. *Dix Exposés Sur La Cohomologie Des Schémas*, volume 3. North-Holland Publishing, 1968.

[GP74] V. Guillemin and A. Pollack. *Differential Topology*. American Mathematical Soceity Chelsea Publishing, 1974.

[GR14] D. Gaitsgory and N. Rozenblyum. Crystals and $\mathcal{D}$-modules, 2014.

[Gro60] A. Grothendieck. Éléments de Géométrie Algébrique: I. Le Langage des Schéma. *Publications Mathématiques de l'IHÉS*, 4:5–228, 1960.

[Gro66] A. Grothendieck. Cohomologie Locale des Faisceaux Cohérents et Théorèmes de Lefschetz Locaux et Globaux (SGA 2), 1966.

[Gro67] A. Grothendieck. Éléments de Géométrie Algébrique: Étude Locale des Schémas et des Morphismes de Schéma, Quatrième Partie. *Publications Mathématiques de l'IHÉS*, 32:5–361, 1967.

[Har77] R. Hartshorne. *Algebraic Geometry*. Springer, 1977.

[Hel13] R. Heluani. Sheaves of Vertex Algebras. Lecture Notes, 2013.

[Kac98] V. G. Kac. *Vertex Algebras for Beginners*. American Mathematical Society, 2nd edition, 1998.

[Kac03] V. G. Kac. Vertex Algebras. Lecture Notes, 2003.

[KK14] M. Kashiwara and T. Kawai. *Algebraic Analysis: Papers Dedicated to Professor Mikio Sato on the Occasion of His Sixtieth Birthday*, volume 2. Elsevier Science, 2014.

[Kli82] W. Klingenberg. *Riemannian Gemetry*, volume 1 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., 1982.

[KV11] J. Krasilshchik and A. Verbovetsky. Geometry of Jet Spaces and Integrable Systems. *Journal of Geometry and Physics*, 61(9):1633–1674, 2011.

[Lur] J. Lurie. Notes on Crystals and Algebraic $\mathcal{D}$-Modules.

[McC94] P. McCloud. Jet Bundles in Quantum Field Theory: the BRST–BV Method. *Classical and Quantum Gravity*, 11(3):567–587, March 1994.

[Mil80] J. S. Milne. *Étale Cohomology*. Princeton University Press, 1980.

[ML98] S. Mac Lane. *Categories for the Working Mathematician*. Springer New York, 2nd edition, 1998.

[Mor09] A. S. Morye. Note on the Serre-Swan Theorem. *Mathematische Nachrichten*, 286(2–3):272–278, 2009.

[MR00] J. C. McConnell and J. C. Robson. *Noncommutative Noetherian Rings*. Graduate Studies in Mathematics. American Mathematical Society, 2000.

[MSV99] F. Malikov, V. Schechtman, and A. Vaintrob. Chiral de Rham Complex. *Communications in Mathematical Physics*, 204(2):439–473, 1999.

[MT10] I. Moerdijk and B. Toën. *Simplicial Methods for Operads and Algebraic Geometry*. Birkhäuser Basel, 2010.

[Nee21] A. Neeman. Grothendieck Duality Made Simple, 2021.

[Nes20] J. Nestruev. *Smooth Manifolds and Observables*. Springer, 2nd edition, 2020.

[Nis87] K. Nishijima. *Quantum Field Theory*. Springer Nature, 1st edition, 1987.

[Sau89] D. J. Saunders. *The Geometry of Jet Bundles*. Cambridge University Press, 1st edition, 1989.

[Ser55] J. P. Serre. Faisceaux Algébriques Cohérents. *Annals of Mathematics*, 61(2):197–278, 1955.

[Sta24] The Stacks Project authors. The Stacks Project, 2024.

[Ste51] N. Steenrod. *The Topology of Fibre Bundles*. Princeton University Press, 1951.

[Swa62] R. G. Swan. Vector Bundles and Projective Modules. *Transactions of the American Mathematical Society*, 105(2):264–277, 1962.

[Vak98] R. Vakil. A Beginner's Guide to Jet Bundles from the Point of View of Algebraic Geometry, 1998.

[WEK97] E. Witten, P. Etingof, and D. Kazhdan. Field Theory, 1997.

[Yek01] A. Yekutieli. The Continuous Hochschild Cochain Complex of a Scheme, 2001.