Every Ideal in an Euclidean Domain is also a principal ideal.

(element with least norm)
Let x be the least element of an Ideal I, [Well ordering principle]
where I is any arbitrary ideal of an ED

so $\langle x \rangle \subseteq I$

And we need to prove $I \subseteq \langle x \rangle$, then we would have $\langle x \rangle = I$, an ideal generated by a single element, thus principal.

To prove $I \subseteq \langle x \rangle$

Let's prove by contradiction,

Let there be an element 'a' in I, $a \in I$
where

$$a \notin \langle x \rangle$$

so $a \neq xr$ [$r \in ED$ $x \in I$ then $rx \in I$]

then $a = xr + d$

$$a - xr = d$$

$a \in I$

$xr \in I$

the $a - xr \in I$

thus $a - xr = d \in I$

but from $a = xr + t$

we must also have the

condition that

i) $\text{Norm}(x) > \text{Norm}(d)$

OR

ii) $\text{Norm}(d) = 0$

and if x is the least element

then there can't be a 'd' with

condition (i), so

$\text{Norm}(d) = 0$

Thus $a = xr \in I$

$a \in \langle x \rangle$

thus $I \subseteq \langle x \rangle$

then $I = \langle x \rangle$

Every ideal in an ED is principal.

~~To prove: PID → UFD~~

~~PID: Every ideal~~

~~To prove PID → UFD~~

UFD

- every non-zero element (not a unit) can be written as product of irreducibles (finite) (not necessarily distinct)
- unique decomposition

unique upto associates

$r = p_1 p_2 \dots p_n$

$r = q_1 q_2 \dots q_m$

$\boxed{m = n}$

R is a PID

Let $r \neq 0$

$r \in R$

r is not a unit

Then if r is reducible, we can write $r = bc$ [b, c are not units]

Now if b, c are irreducible then no need to extend the proof further. But if atleast one of b, c is reducible, let's say $\cancel{b}$ c, then c can be decomposed.

$r = bde \quad c = de$

Now let's take the case when 'd' is irreducible and 'e' is reducible then

$r = bdfg \quad e = fg$

And so on.

$\cancel{r = bdfgh}$ $r = bdfhij \dots$

(We need this extension of reducibility of be finite)

So when we write

$r = bc$

In Ideal notation, we can say

$\langle r \rangle \subset \cancel{\langle b \rangle} \langle c \rangle$ [Proper subset because 'b' is not unit]

$\cancel{\langle c \rangle} \subset \langle f \rangle$

$\cancel{429}$ 28.

291

$\boxed{137}$

5

27.4

291 − 28.

263

$\langle c \rangle \subset \langle e \rangle$ ['d' is not unit,

$\langle r \rangle \subset \langle c \rangle \subset \langle e \rangle \ldots \subset R$

Let $I = \bigcup_{i=1}^{\infty} I_i$

Since R was a PID so any ideal is principal

means: $I = \langle a \rangle$

Since I is sum of all ideals of R then.

$a \in I$, more precisely $a \in I_n$

Then we get $I_n \subseteq I = \langle a \rangle \subseteq I_n$

so $I = I_n$ ('a' an irreducible starts repeating)

Uniqueness

Suppose $r = p_1 p_2 \cdots p_n = q_1 q_2 \cdots q_m$

$m \geq n$

p_i and q_j are irreducibles

Let's say p_i divides some q_j on RHS then we ~~E~~ can have an order coherent.

$p_1 \mid q_1 \Rightarrow q_1 = p_1 u_1$ u is a unit in R since q_1, p_1 are irreducibles.

Cancelling p_1 from both sides in r

$p_2 p_3 \cdots p_n = u q_2 q_3 \cdots q_m$

Similarly we can cancel p_2 from both side $q_2 = p_2 u_2$ $u_2 \to$ unit

~~Hence we got the start~~

Recursively we cancel the irreducibles and at the end, are left with only units.

$m = n$