Non linearity of a function $f(x)$

$\eta(f)$ – Min distance of $f(x)$ from all n-bit affine functions

$= \min_{g \in A(n)} \{ dist(f, g) \}$

$= 2^{n-1} - \frac{1}{2} \max_{a \in \{0,1\}^n} |W_f(a)|$

Parseval's identity

$$\sum_{a \in \{0,1\}^n} |W_f(a)|^2 = 2^{2n}$$

$U_F(x_1, x_2 \ldots x_n, b)$

$= (x_1, x_2 \ldots x_n, b \oplus F(x))$

Measurement (multi qbit)

$P_r(i) = \langle \psi | M_i^\dagger M_i | \psi \rangle$

outcome state $\rightarrow$ $|\psi'\rangle = \dfrac{M_i |\psi\rangle}{\sqrt{\langle \psi | M_i^\dagger M_i | \psi \rangle}}$

Meas--- operators

M_{00} $|00\rangle\langle 00|$, $|01\rangle\langle 01| = M_{01}$

M_{10} $|10\rangle\langle 10|$, $|11\rangle\langle 11| = M_{11}$

$P_r(00) = \langle \psi | (|00\rangle\langle 00|)^\dagger (|00\rangle\langle 00|) |\psi\rangle$

Walsh spectrum of a Boolean fun $f(x)$

$W_f = [W_f(0), W_f(1) \ldots W_f(2^n - 1)]$

Walsh coeff at point $0^{\otimes n}$

$$W_f(0^{\otimes n}) = \sum_{x \in \{0,1\}^n} (-1)^{f(x)}$$

For
- balanced f $W_f(0^{\otimes n}) = 0$
- constant f $W_f(0^{\otimes n}) = \pm 2^n$

$W_f(0^{\otimes n}) = |\{x : f(x) = 0\}|$

$- |\{x : f(x) = 1\}|$

$= 2^n - 2wt(f)$

$= (2^n - wt(f)) - (wt(f))$

$\quad \downarrow f(x)=0 \qquad \downarrow f(x)=1$

$W_f(a) = | \; | - | \; |$

$= \cancel{2^n - 2wt(f)}$

$= |\{x : f(x) = a \cdot x\}|$

$- |\{x : f(x) \neq a \cdot x\}|$

$= 2^n - 2wt(f(x) \oplus (a \cdot x))$

$\langle \psi | (|00\rangle\langle 00|)^\dagger = (|00\rangle\langle 00| \, |\psi\rangle)^\dagger$

Density Mx form

$P_r(|i\rangle) = tr(|i\rangle\langle i| \rho)$

$\rho' = \dfrac{M_i \rho M_i^\dagger}{tr(|i\rangle\langle i| \rho)}$

$M_0 = |0\rangle\langle 0|$

$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Walsh transform of a func 'f' at point 'a' $\in \{0,1\}^n$

$$W_f(a) = \sum_{x \in \{0,1\}^n} (-1)^{f(x) \oplus a \cdot x}$$

$a \cdot x = a_1 x_1 \oplus a_2 x_2 \ldots a_n x_n$

CNOT & No cloning thm

CNOT $|0\rangle|0\rangle \longrightarrow |0\rangle|0\rangle$

But

CNOT$(\alpha|0\rangle + \beta|1\rangle)|0\rangle$

$\downarrow$

$\alpha|0\rangle|0\rangle + \beta|1\rangle|1\rangle$

$\neq (\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle)$

$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$Y = +iXZ = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$\sqrt{Z} = S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

$\sqrt{S} = T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$

$P(\lambda) = U(0, 0, \lambda)$

$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\lambda} \end{bmatrix}$

$I = P(0)$

$Z = P(\pi)$

$S = P(\pm\pi/2)$

$T = P(\pm\pi/4)$

EPR states / Bell states.

$|\phi^+\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$ $\quad |\psi^+\rangle = \dfrac{|01\rangle + |10\rangle}{\sqrt{2}}$

$|\phi^-\rangle = \dfrac{|00\rangle - |11\rangle}{\sqrt{2}}$ $\quad |\psi^-\rangle = \dfrac{|01\rangle - |10\rangle}{\sqrt{2}}$

$|0\rangle$ — U — •

$|0\rangle$ ——— ⊕

U = H then u get $|\phi^+\rangle$

$|0\rangle \xrightarrow{U} \alpha|0\rangle + \beta|1\rangle$

CNOT gate [2 qbit gate]

$|0x\rangle \xrightarrow{CNOT} |0x\rangle$

$|1x\rangle \xrightarrow{CNOT} |1\bar{x}\rangle$

$|a\rangle|b\rangle \xrightarrow{CNOT} |a\rangle|a \oplus b\rangle$

Matrix representation

$\begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix}_{4 \times 4}$

$$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{matrix}$$

C_H gate [2 qbit

$|a\rangle|b\rangle \xrightarrow{C_H} |a\rangle|a \oplus H|b\rangle\rangle$

$\begin{bmatrix} I & 0 \\ 0 & H \end{bmatrix}_{4 \times 4}$

CCNOT gate

$|a\rangle|b\rangle|c\rangle$

$\downarrow$

$|a\rangle|b\rangle|c \oplus a \cdot b\rangle$

Matrix rep.

8×8

$\begin{bmatrix} I & 0 \\ 0 & CNOT \end{bmatrix}$

~~BB84 Q~~

$X = U(\pi/2, 0, \pi)$

$Z = P(\pi) = U(0, 0, \pi)$

$Y = U(\pi, \pi/2, \pi/2)$

$H = U(\pi/2, 0, \pi)$

Unitary matrix rep

$U(\theta, \phi, \lambda) = \begin{bmatrix} \cos\theta/2 & \sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{bmatrix}$

$\begin{bmatrix} 1 & (-e^{i\lambda}) \\ e^{i\phi} & e^{i(\lambda+\phi)} \end{bmatrix}$

$0 \leq \theta \leq \pi$

$0 \leq \phi \leq 2\pi$

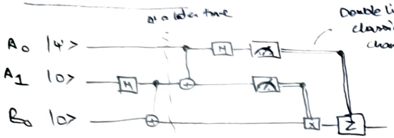
$0 \leq \lambda < 2\pi$

TELEPORTATION

at a later time

Double line means classical channel

NON LOCAL PHENOMENON



Alice Bob

$$\left[\,|\psi\rangle|0\rangle|0\rangle\,\right] \xrightarrow{I\otimes H\otimes I} \left[\,|\psi\rangle\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|0\rangle\,\right]$$

$$\Big\downarrow I_0\otimes CNOT(1,2)$$

PREPARING FOR ENTANGLEMENT

$$\left[\,|\psi\rangle\left[\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right]\right]$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}\alpha|000\rangle+\alpha|011\rangle\\ +\beta|100\rangle+\beta|111\rangle\end{bmatrix}$$

~~Alice measures in 00, 01, 10, 11~~

The state is shared (last two bits first bit is Alice, last is Bob's)

$$\xrightarrow{CNOT_{01}\times I_2} \frac{1}{\sqrt{2}}\begin{bmatrix}\alpha|000\rangle+\alpha|011\rangle\\ +\beta|110\rangle+\beta|101\rangle\end{bmatrix}$$

$$\Big\downarrow H_0\times I_1\times I_2$$

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix}\alpha|000\rangle+\alpha|100\rangle+\alpha|011\rangle+\alpha|111\rangle\\ \beta|010\rangle-\beta|110\rangle+\beta|001\rangle-\beta|101\rangle\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}|00\rangle[\alpha|0\rangle+\beta|1\rangle]+|01\rangle[\alpha|1\rangle+\beta|0\rangle]\\ +|10\rangle[\alpha|0\rangle-\beta|1\rangle]+|11\rangle[\alpha|1\rangle-\beta|0\rangle]\end{bmatrix}$$

State Alice observes	State of Bob's Qbit
$\|00\rangle$	$\alpha\|0\rangle+\beta\|1\rangle$
$\|01\rangle$	$\alpha\|1\rangle+\beta\|0\rangle$
$\|10\rangle$	$\alpha\|0\rangle-\beta\|1\rangle$
$\|11\rangle$	$\alpha\|1\rangle-\beta\|0\rangle$

$$U(\theta,\phi,\lambda) = \begin{bmatrix}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}\,(-e^{i\lambda})\\ e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\,e^{i(\lambda+\phi)}\end{bmatrix}$$

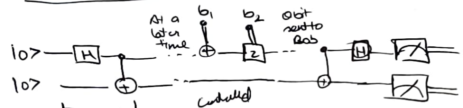
UNITARY GATE REP

$(\theta)\in[0,\pi]$

$\phi\in[0,2\pi)$

$\lambda\in[0,2\pi)$

SUPER DENSE CODING



conditional on $b_0=1$

$$\frac{1}{\sqrt{2}}\left[|00\rangle+|11\rangle\right] \xrightarrow{CX\otimes I} \frac{1}{\sqrt{2}}\left(|b_1\rangle|0\rangle+|\bar{b}_1\rangle|1\rangle\right)$$

$$\Big\downarrow CZ\otimes I$$

$$\frac{\left[(-1)^{b_1\cdot b_2}|b_1\rangle|0\rangle+(-1)^{\bar{b}_1\cdot b_2}|\bar{b}_1\rangle|1\rangle\right]}{\sqrt{2}}$$

Alice

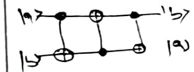
$$\xrightarrow{CX_{1,2}} \frac{(-1)^{b_1\cdot b_2}|b_1\rangle|b_1\rangle+(-1)^{\bar{b}_1\cdot b_2}|\bar{b}_1\rangle|b_1\rangle}{\sqrt{2}}$$

$$\Big\downarrow H\times I$$

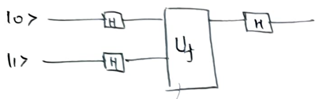
$$|b_2\rangle|b_1\rangle$$

Bob

SWAP GATE USING CNOT



DEUTSCH ALGO



$|x\rangle \to (-1)^{f(x)}|x\rangle$

$$|0\rangle|1\rangle \xrightarrow{H^{\otimes 2}} |+\rangle|-\rangle \xrightarrow{U_f} \left(\frac{(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle}{\sqrt{2}}\right)|-\rangle$$

$$\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|-\rangle$$

$$\downarrow H\otimes I$$

$$\left[\frac{(-1)^{f(0)}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) + (-1)^{f(1)}\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)}{\sqrt{2}}\right]|-\rangle$$

$$= \left[\frac{|0\rangle\left((-1)^{f(0)}+(-1)^{f(1)}\right) + |1\rangle\left((-1)^{f(0)}-(-1)^{f(1)}\right)}{\sqrt{2}\,\sqrt{2}}\right]|-\rangle$$

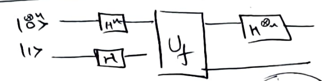
$$Pr(|0\rangle) = \left[\frac{1}{\sqrt{2}}\left[(-1)^{f(0)}+(-1)^{f(1)}\right]\left(\frac{1}{\sqrt{2}}\right)\right]^2$$

$$= \frac{1}{4}\left((-1)^{f(0)}+(-1)^{f(1)}\right)^2$$

$$Pr|1\rangle = \frac{1}{4}\left((-1)^{f(0)}-(-1)^{f(1)}\right)$$

DEUTSCH JOZSA

1 bit o/p function f.



$$|0^{\otimes n}\rangle|1\rangle \xrightarrow{H^{\otimes n}\times H} \frac{1}{\sqrt{2^n}}\left[\sum_{x\in\{0,1\}^n}(-1)^{x\cdot 0}|x\rangle\right]|-\rangle = \frac{1}{\sqrt{2^n}}\left[\sum_{x\in\{0,1\}^n}|x\rangle\right]|-\rangle$$

$$\downarrow U_f$$

$$\frac{1}{\sqrt{2^n}}\left[\sum_{x\in\{0,1\}^n}(-1)^{f(x)}|x\rangle\right]|-\rangle$$

$$\xleftarrow{H^{\otimes n}\times I} \frac{1}{(\sqrt{2^n})^2}\left[\sum_{x\in\{0,1\}^n}(-1)^{f(x)}\sum_{y\in\{0,1\}^n}(-1)^{x\cdot y}|y\rangle\right]|-\rangle$$

$$\frac{1}{(\sqrt{2^n})^2}\left[\sum_{x\in\{0,1\}^n}\left[\sum_{y\in\{0,1\}^n}(-1)^{f(x)\oplus x\cdot y}|y\rangle\right]\right]|-\rangle$$

$$= \frac{1}{2^n}\sum_y\left[\sum_x(-1)^{f(x)\oplus x\cdot y}\right]|y\rangle|-\rangle$$

$$\underset{y=0^n}{Pr(|0^n\rangle)} = \frac{1}{2^{2n}}\left[\sum(-1)^{f(x)}\right]^2$$

For constant f : $Pr(|0^n\rangle) = \frac{1}{2^{2n}}(2^n)^2 = 1$

For balanced f : $Pr(|0^{\otimes n}\rangle) = \frac{1}{2^{2n}}\,0 = 0$

SWAP gate (Not a controlled 2-Q bit gate) XCEPTION)

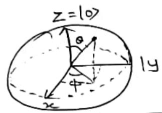
$|00\rangle \to |00\rangle$
$|10\rangle \to |01\rangle$
$|01\rangle \to |10\rangle$
$|11\rangle \to |11\rangle$

$\begin{matrix} 1 & 0 & - & - \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ - & - & - & 1 \end{matrix}$

$U = \begin{matrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{matrix}$

~~QASM SIMULATOR~~

BLOCH SPHERE



$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$= r_1 e^{i\theta_1}|0\rangle + r_2 (e^{i\theta_2})|1\rangle$

$= e^{i\theta_1}\left(r_1|0\rangle + r_2 e^{i(\theta_2-\theta_1)}|1\rangle\right)$

$r_1 = \cos\theta/2 \quad r_2 = \sin\theta/2 \quad \phi = \theta_2 - \theta_1$

$e^{i\theta_1}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right)$

$|0\rangle = |z\rangle$
$|1\rangle = -|z\rangle$
$|+\rangle = |x\rangle$
$|-\rangle = -|x\rangle$
$|i\rangle = |y\rangle$
$|-i\rangle = -|y\rangle$

Density Mx form

$|\psi\rangle\langle\psi| = |\psi'\rangle\langle\psi'| = \cancel{e^{i\theta_1}}\begin{bmatrix}\cos\theta/2 \\ e^{i\phi}\sin\theta/2\end{bmatrix} \cancel{e^{-i\theta_1}}\left[\cos\frac{\theta}{2} \;\; e^{-i\phi}\sin\frac{\theta}{2}\right]$

$= \begin{bmatrix} \cos^2\theta/2 & e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\theta/2 \end{bmatrix}$

$\theta = 2\tan^{-1}\frac{\beta}{\alpha}$

$\phi = \arg\beta - \arg\alpha$

NOT DEPENDENT ON θ_1

COND FOR SAME DENSITY MX

i) $|\alpha| = |\gamma|$
ii) $|\beta| = |\delta|$
iii) $\arg\beta - \arg\alpha = \arg\delta - \arg\gamma$

$|\psi_1\rangle = \alpha|0\rangle + \beta|1\rangle$
$|\psi_2\rangle = \gamma|0\rangle + \delta|1\rangle$

→ Map to same point on Bloch sphere

ALTERNATE CALC

$|0^{\otimes n}\rangle|0^{\otimes n}\rangle$

$\downarrow H^{\otimes n} \otimes I$

$\frac{1}{\sqrt{2^n}}\sum_x |x\rangle|0^{\otimes k}\rangle$

$\downarrow U_f$

$\frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle|f(x)\rangle$

$= \frac{1}{\sqrt{2^n}}\sum_{x\in 1/2\text{ values}}\frac{1}{\sqrt{2}}\left(|x\rangle + |x+s\rangle\right)|f(x)\rangle$

$= \frac{1}{\sqrt{2^{n+1}}}\sum\left(|x\rangle + |x\oplus s\rangle\right)|f(x)\rangle$

$\downarrow H^{\otimes n}$

$= \frac{1}{\sqrt{2^{n+1}}}\sum_x \frac{\left(\sum_y(-1)^{x\cdot y}|y\rangle + \sum_y(-1)^{(x\oplus s)\cdot y}|y\rangle\right)}{\sqrt{2^n}}|f(x)\rangle$

MAPPING A MIXED Q STATE ON BLOCH SPH

SIMON'S CONT.

$P_n(|y\rangle) = \left\|\frac{1}{2^n}\sum_x(-1)^{x\cdot y}|f(x)\rangle\right\|^2$

CASE 1: $s = 0^{\otimes n}$ $\quad y = s\oplus x$

$P_n(|y\rangle) = \left\|\frac{1}{2^n}\sum_x(-1)^{x\cdot y}|x\rangle\right\|^2$

Since all $f(x)$ are distinct ($y\in\{0,1\}^n$ not $\{0,1\}^k$)

$= \frac{1}{2^{2n}}2^n$

$= \frac{1}{2^n}$

CASE 2: $s \neq 0^{\otimes n}$

$P_n(|y\rangle) = \left\|\frac{1}{2^n}\sum_{z\in A}\left((-1)^{x_1\cdot y} + (-1)^{x_2\cdot y}\right)|z\rangle\right\|^2$ (A: Range)

$x_1 = x_2\oplus s$, $f(x_1) = f(x_2) = z$

$= \left\|\frac{1}{2^n}\sum_{z\in A}\left((-1)^{x_1\cdot y} + (-1)^{(x_1\oplus s)\cdot y}\right)|z\rangle\right\|^2$

$= \frac{1}{2^{2n}}\left\|\sum(-1)^{x_1\cdot y}\left(1+(-1)^{s\cdot y}\right)|z\rangle\right\|^2$

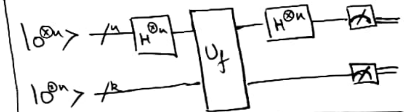
SIMON'S ALGO

PROBLEM STMT : Boolean fn (i) $f(x): \{0,1\}^n \to \{0,1\}^k$
& (ii) $f(x) = f(s\oplus x)$ for FIXED $s\in\{0,1\}^n$
& (iii) $k \geq n/2$

FIND s.

CLASSICAL SOL^n req $\frac{2^n}{2}+1$ (same as DJ)

QNTM SOL^n reqs. $O(n)$ queries



CALCULATION

$|0^{\otimes n}\rangle|0^{\otimes n}\rangle$

$\downarrow H^{\otimes n}\otimes I$ step 1

$\frac{1}{\sqrt{2^n}}\sum_x|x\rangle|0^{\otimes k}\rangle$

$\downarrow U_f$ step 2

$\frac{1}{\sqrt{2^n}}\sum_x|x\rangle|f(x)\rangle$

$\downarrow H^{\otimes n}\otimes I$ step 3

$\frac{1}{2^n}\sum_x\sum_y(-1)^{x\cdot y}|y\rangle|f(x)\rangle$

$= \sum_y|y\rangle\otimes\left(\frac{1}{2^n}\sum_x(-1)^{x\cdot y}|f(x)\rangle\right)$

SWAP gate (Not a controlled 2-Q bit gate) XCEPTION)

$|00\rangle \to |00\rangle$

$|10\rangle \to |01\rangle$

$|01\rangle \to |10\rangle$

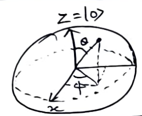
$|11\rangle \to |11\rangle$

$$\begin{matrix} 1 & 0 & & \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ & & & 1 \end{matrix}$$

$$U = \begin{matrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{matrix}$$

~~QASM SIMULATOR~~

BLOCH SPHERE



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$\lvert 0\rangle = \lvert z\rangle$
$\lvert 1\rangle = -\lvert z\rangle$
$\lvert +\rangle = \lvert x\rangle$
$\lvert -\rangle = -\lvert x\rangle$
$\lvert i\rangle = \lvert y\rangle$
$\lvert -i\rangle = -\lvert y\rangle$

Density Mx form

$$|\psi\rangle\langle\psi| = |\psi'\rangle\langle\psi'| = \cancel{e^{i\theta_1}}\begin{bmatrix}\cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2}\end{bmatrix} \cdot \cancel{e^{-i\theta_1}}\begin{bmatrix}\cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2}\end{bmatrix}$$

$$= \begin{bmatrix}\cos^2\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2}\end{bmatrix}$$

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$\theta = 2\tan^{-1}\frac{\beta}{\alpha}$

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$= \frac{1}{\sqrt{2^{n+1}}}\sum\left((|x\rangle) + |x\oplus s\rangle\right)|f(x)\rangle$

$\downarrow H^{\otimes n}$

$$= \frac{1}{\sqrt{2^{n+1}}}\sum_x\left(\frac{\sum_y(-1)^{x\cdot y}|y\rangle + \sum_y(-1)^{(x\oplus s)\cdot y}|y\rangle}{\sqrt{2^n}}\right)|f(x)\rangle$$

MAPPING A MIXED Q STATE ON BLOCH SPH

SIMON'S CONT.

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$= \frac{1}{2^n}$

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$$P_r(|y\rangle) = \left\|\frac{1}{2^n}\sum_{z\in A}\left((-1)^{x_1\cdot y} + (-1)^{x_2\cdot y}\right)|z\rangle\right\|^2$$

A ↓ Range

$x_1 = x_2 \oplus s$, $f(x_1) = f(x_2) = z$

$$= \left\|\frac{1}{2^n}\sum_{z\in A}\left((-1)^{x_1\cdot y} + (-1)^{(x_1\oplus s)\cdot y}\right)|z\rangle\right\|^2$$

$$= \frac{1}{2^{2n}}\left\|\sum(-1)^{x_1\cdot y}\left(1+(-1)^{s\cdot y}\right)|z\rangle\right\|^2$$

SIMON'S ALGO

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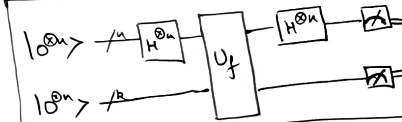
CLASSICAL SOLN req $\frac{2^n}{2}+1$ (same as DJ

QNTM SOLN reqs. $O(n)$ queries



CALCULATION

$|0^{\otimes n}\rangle|0^{\otimes n}\rangle$

$\downarrow H^{\otimes n}\otimes I$ step 1

$\frac{1}{\sqrt{2^n}}\sum_x|x\rangle|0^{\otimes k}\rangle$

$\downarrow U_f$ step 2

$\frac{1}{\sqrt{2^n}}\sum_x|x\rangle|f(x)\rangle$

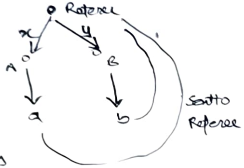
$\downarrow H^{\otimes n}\otimes I$ step 3

$\frac{1}{2^n}\sum_x\sum_y(-1)^{x\cdot y}|y\rangle|f(x)\rangle$

$= \sum_y|y\rangle\otimes\left(\frac{1}{2^n}\sum_x(-1)^{x\cdot y}|f(x)\rangle\right)$

←

CHSH game



AB wins if

$x \cdot y = a \oplus b$

- In classical there can be 4 strategies
  - Send O/P 0 always
  - O/P 1 always
  - O/P x
  - O/P $\bar{x}$
- Winning probability = 3/4 [any case]
- QNTM - SHARE ENTANGLED STATE BEFORE GAME

$$\text{Prob of win} = P_r(|00\rangle) + P_r(|11\rangle) = \frac{1}{2}\left[\cos^2\frac{\pi}{8} + \cos^2\frac{\pi}{8}\right] = \cos^2\frac{\pi}{8} \approx 0.854$$

Cas 1

Cas 2 : $P_r(|00\rangle) + P_r(|11\rangle) = 0.854$

Cas 3 : $P_r(|00\rangle) + P_r(|11\rangle) = 0.854$

Cas 4 : $P_r(|01\rangle) + P_r[|10\rangle] = \frac{1}{2}\left(1 + \sin\frac{\pi}{4}\right) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) = 0.854$

00	Cas 1 : $\lvert\phi^+\rangle$ ~~$I \otimes U$~~	$I \otimes U_3(-\pi/4, 0, 0)$
01	Cas 2	$I \otimes U_3(\pi/4, 0, 0)$
10	Cas 3	$H \otimes U_3(-\pi/4, 0, 0)$
11	Cas 4	$H \otimes U_3(\pi/4, 0, 0)$

Cas 3

$$U(\theta, \phi, \lambda) = \begin{bmatrix} \cos\theta/2 & -e^{i\lambda}\sin\theta/2 \\ e^{i\phi}\sin\theta/2 & e^{i(\lambda+\phi)}\cos\theta/2 \end{bmatrix}$$

$$U(-\pi/4, 0, 0) = \begin{bmatrix} \cos\pi/8 & +\sin\pi/8 \\ -\sin\pi/8 & \cos\pi/8 \end{bmatrix}$$

- $|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

- Case 1 : $xy = 00$

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} \xrightarrow{I \otimes U_3(-\pi/4, 0, 0)} \frac{1}{\sqrt{2}}\left[|0\rangle\left(\cos\frac{\pi}{8}|0\rangle - \sin\frac{\pi}{8}|1\rangle\right) + |1\rangle\left(\sin\frac{\pi}{8}|0\rangle + \cos\frac{\pi}{8}|1\rangle\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\cos\frac{\pi}{8}|00\rangle - \sin\frac{\pi}{8}|01\rangle + \sin\frac{\pi}{8}|10\rangle + \cos\frac{\pi}{8}|11\rangle\right]$$

- Case 2 : $xy = 01$

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} \xrightarrow{I \otimes U_3(\pi/4, 0, 0)} \frac{1}{\sqrt{2}}\left[\cos\frac{\pi}{8}|00\rangle + \sin\frac{\pi}{8}|01\rangle - \sin\frac{\pi}{8}|10\rangle + \cos\frac{\pi}{8}|11\rangle\right]$$

~~$P_r(\text{win}) = P_r$~~ [

$$U(\pi/2, 0, \pi) = \begin{bmatrix} \cos\pi/4 & \sin\pi/4 \\ \sin\pi/4 & -\cos\pi/4 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = H$$

- Cas 3 : $xy = 10$

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} \xrightarrow{\overset{H}{U(\pi/2, 0, \pi)} \otimes U_3(-\pi/4, 0, 0)} = \frac{1}{2}\left[(|0\rangle + |1\rangle)\left(\cos\frac{\pi}{8}|0\rangle - \sin\frac{\pi}{8}|1\rangle\right) + (|0\rangle - |1\rangle)\left(\sin\frac{\pi}{8}|0\rangle + \cos\frac{\pi}{8}|1\rangle\right)\right]$$

$$= \frac{1}{2}\left[\left(\cos\frac{\pi}{8} + \sin\frac{\pi}{8}\right)|00\rangle + \left(-\sin\frac{\pi}{8} + \cos\frac{\pi}{8}\right)|01\rangle + \left(\cos\frac{\pi}{8} - \sin\frac{\pi}{8}\right)|10\rangle + \left(-\sin\frac{\pi}{8} - \cos\frac{\pi}{8}\right)|11\rangle\right]$$

- Cas 4 : $xy = 11$

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} \xrightarrow{\overset{H}{U(\pi/2, 0, \pi)} \otimes U(\pi/4, 0, 0)}$$

$$= \frac{1}{2}\left[\left(\cos\frac{\pi}{8} - \sin\frac{\pi}{8}\right)|00\rangle + \left(\sin\frac{\pi}{8} + \cos\frac{\pi}{8}\right)|01\rangle + \left(\cos\frac{\pi}{8} + \sin\frac{\pi}{8}\right)|10\rangle + \left(\sin\frac{\pi}{8} - \cos\frac{\pi}{8}\right)|11\rangle\right]$$

BLOCH SPH

z arbitrary complex number

$= x + iy$

$= r e^{i\theta}$

$|\psi\rangle = r_1 e^{i\theta_1}|0\rangle + r_2 e^{i\theta_2}|1\rangle$

$= e^{i\theta_1}[r_1|0\rangle + \underbrace{r_2 e^{i\phi}}|1\rangle]$ $\quad \phi = \theta_2 - \theta_1$

$|\psi'\rangle = \underbrace{[z|0\rangle + (x+iy)|1\rangle]}$

Multiplying $e^{-i\theta_1}$ won't affect the Pr of each state

$|z|^2 + |x+iy|^2 = 1$

$z^2 + x^2 + y^2 = 1$

Satisfy Bloch sph surface

Now we can represent

$z = r\cos\theta/2$

$x = r\sin\theta/2 \cos\phi$

$y = r\sin\theta/2 \sin\phi$

$r = 1 \mid 0 \le \theta \le \pi \mid 0 \le \phi < 2\pi$

$|\psi'\rangle = \cancel{r}\cos\theta/2 |0\rangle + \cancel{r}\sin\theta/2(\cos\phi + i\sin\phi)|1\rangle \quad r = 1$

$= \cos\theta/2 |0\rangle + \sin\theta/2\, e^{i\phi}|1\rangle$

$$A = \begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+\cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & 1-\cos\theta \end{bmatrix}$$

$$= \frac{1}{2}\left[I + x\sigma_x + y\sigma_y + z\sigma_z\right]$$

$\sigma_{x,y,z}$ = Pauli matrices

$x, y, z \in \mathbb{R}$

$x = \sin\theta\cos\phi$

$y = \sin\theta\sin\phi$

$z = \cos\phi$

Pure state

$x = \sum p_i \sin\theta_i \cos\phi_i$

$y = \sum p_i \sin\theta_i \sin\phi_i$

$z = \sum p_i \cos\theta_i$

Mixed state

$$Pr(|0\rangle\langle 0|) = \text{trace}\left[|0\rangle\langle 0|\rho\right]$$

$$= \text{trace}\left[\begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}\rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22}\end{bmatrix}\right]$$

$$= \text{tr}\begin{bmatrix}\rho_{11} & \rho_{12} \\ 0 & 0\end{bmatrix}$$

$$= \rho_{11}$$

$$Pr(i) = \langle\psi|M_i^\dagger M_i|\psi\rangle$$

$M_i \rightarrow$ Measurement operators

$$= \text{tr}(|i\rangle\langle i|\rho_\psi)$$

$$\rho' = \frac{M_i \rho M_i^\dagger}{\text{tr}(|i\rangle\langle i|\rho)}$$

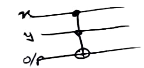
Born Rule

$|\alpha|^2 = Pr(x)$

ORACLE

1) $xy = f(x,y)$

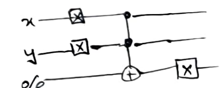
orcl. ccx(x, y, o/p)



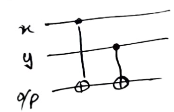
2) $x + y = f(x,y)$ (OR)

orcl. cx(x, o/p)

orcl. cx(y, o/p)



3) $x \oplus y = f(x,y)$ XOR



GROVER'S ALGO

- SQ ROOT IMPROVEMENT ON SEARCHING in an unsorted array

- GROVER'S ITERATE

$G = H^{\otimes n} U_0 H^{\otimes n} U_f$

U_0: Operator that flips the phase of all the basis states except the $|0^{\otimes n}\rangle$ state

U_f: Oracle that marks the elements of interest

$$U_0|x\rangle = \begin{cases} -|x\rangle & x \neq 0^{\otimes n} \\ |x\rangle & x = 0^{\otimes n} \end{cases} = 2|0^{\otimes n}\rangle\langle 0^{\otimes n}| - I$$

- Expanding G

$G = H^{\otimes n}\left[2|0^{\otimes n}\rangle\langle 0^{\otimes n}| - I\right] H^{\otimes n} U_f$

$= \left[2 H^{\otimes n}|0^{\otimes n}\rangle\langle 0^{\otimes n}|H^{\otimes n} - H^{\otimes n} I H^{\otimes n}\right] U_f$

$= (2|s\rangle\langle s| - I)U_f$

$|s\rangle = \frac{1}{\sqrt{2^n}}\sum |x\rangle$, $x \in \{0,1\}^n$

U_f marks the good state

$U_f(\alpha|\text{bad}\rangle + \beta|\text{good}\rangle) \rightarrow \alpha|\text{bad}\rangle - \beta|\text{good}\rangle$

Reflection about the bad state

$|0^n\rangle$ — $\boxed{H^{\otimes n}}$ — $\boxed{G}$ — $\boxed{}$

$O(\sqrt{2^n})$

$G = U_f H^{\otimes n} U_{0^t} H$

MOSCA

U_f = ...

$$U_{0^t} = \begin{cases} |x\rangle \rightarrow -|x\rangle & x \neq 0 \\ |0\rangle \rightarrow |0\rangle \end{cases}$$

$|x\rangle|0\rangle \xrightarrow{U_f} |x\rangle|f(x)\rangle$

$|000\rangle = \frac{1}{\sqrt{2^n}} H\left(\sum_y (-1)^{111\cdot y}|111\rangle\right)$

$H|0^n\rangle = \sum_y (-1)^{111\cdot y}$

$f(x_1, x_2, x_3)$, $x_1 x_2 x_3$

0	0	000
1	1	001
1	1	010
2	2	011
1	1	100
2	2	101
2	2	110
3	3	111

$\frac{1}{2\sqrt{2}}\left(|000\rangle \ldots |111\rangle\right)$

$\downarrow U_f$

$\frac{1}{\sqrt{2^3}}\left((-1)^0|000\rangle + (-1)^0|001\rangle \ldots (-1)^1|111\rangle\right)$

$\left(|000\rangle \ldots |110\rangle - |111\rangle\right)$

$\downarrow H^{\otimes n}$

$= \sum_y (-1)^{000\cdot y}|000\rangle + \sum_y (-1)^{001\cdot y}|001\rangle \ldots$

$\sum_y (-1)^{110\cdot y}|110\rangle - \sum_y (-1)^{111\cdot y}|111\rangle$

$= \sum_y (-1)^y |000\rangle$

$|000\rangle \quad |111\rangle \rightarrow |111\rangle$

$\sqrt{2} \quad \sqrt{2}$

$|000\rangle = \frac{1}{\sqrt{2^3}} \sum_y (-1)^{111\cdot y}|111\rangle$

$|000\rangle = \frac{1}{\sqrt{2^3}} \sum_y$

+ $R_k \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i2\pi}{2^k}} \end{bmatrix}$ $\quad R_k^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{i2\pi}{2^k}} \end{bmatrix}$

$0.00..1$
$\uparrow$
k^{th} position.

+ It carries out only 1 bit at k^{th} position.

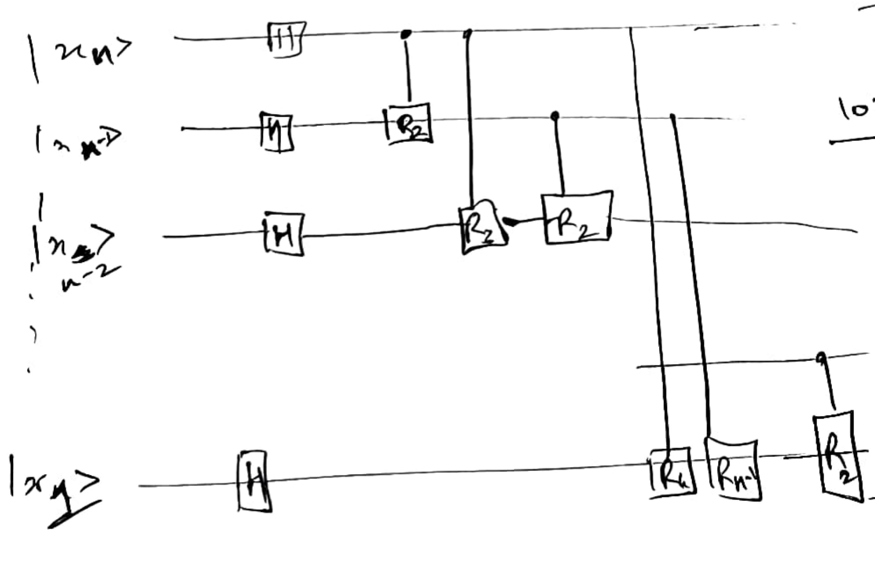
+ Phase estimation.

$$\frac{1}{\sqrt{2}} \sum_y e^{2\pi i \frac{x y}{2^n}} |y\rangle \longrightarrow |x\rangle \qquad x = x_1 x_2 x_3 \cdots x_n$$

we can't predict x beyond ~~..~~ the size of y bits.

$$+ \quad |x\rangle \longrightarrow \underbrace{\frac{1}{\sqrt{2^n}} \sum_y e^{2\pi i \frac{x}{2^n} y} |y\rangle}_{\text{QFT} = \text{inverse (phase estimate)}} \Bigg] \text{ form of DFT}$$

QFT



$$\frac{|0\rangle + e^{2\pi i (0.x_n)}|1\rangle}{\sqrt{2}}$$

$$\frac{|0\rangle + e^{2\pi i (0.x_{n-1} x_n)}|1\rangle}{\sqrt{2}}$$

$$\frac{|0\rangle + e^{2\pi i (0.x_1 x_2 \cdots x_n)}|1\rangle}{\sqrt{2}}$$