The 1-Color Problem and the Brylawski Model

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Abstract. In discrete tomography, the *1-color* problem consists in determining the existence of a binary matrix with row and column sums equal to some given input values arranged in two vectors. These two vectors are said to be compatible if the associated *1-color* problem has at least a solution. Here, we start from a vector of projections, and we define an algorithm to compute all the vectors compatible with it, then we show how to arrange them in a partial order structure, and we point out some of its combinatorial properties. Finally, we prove that this poset is a sublattice of the Brylawski lattice too, and we check some common properties.

1 Introduction

The *1-color* problem is a decision problem which asks for the existence of a binary matrix compatible with two integer vectors, called horizontal and vertical projections, whose entries count the number of elements 1 in each row and column of the matrix, respectively. This problem has been approached by Ryser in [9] as a study on combinatorial properties of matrices of zeros and ones. His results are considered to be the milestones in discrete tomography, a discipline that studies geometrical properties of discrete sets of points from projections or from partial information on them.

One of the principal motivations of this discipline is the study of crystalline structures from information obtained through electron beams that cross the material, as described in [8] and [10]; many problems in discrete tomography are also related to network flow, timetabling or data security (see [6], [7] for a survey). Theoretical results on the *1-color* problem are often used in these more complex models.

If there exists a binary matrix whose vectors of projections are H and V, then we say that H and V are compatible. In this work we define a simple procedure to list all the vertical projections which are compatible with a given vector of horizontal projections. The obtained elements can be easily arranged in a partial order structure that can be proved to be a lattice. In order to do this, we go deeper into the strong connections between *1-color* and the Brylawski model (see

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[2]), a structure which is related to a very different context: defined to study the integer partitions, it is often considered the ancestor of many dynamical models used for physical simulations of realistic environments (see i.e. [4]). In fact, the Brylawski model can be considered as a special case of the Ice Pile Model, described in [5].

In [2], the author uses some properties of the model to enumerate the matrices of 0s and 1s with prescribed row and column sums, and he furnishes a lower bound for their number which turns out to be exact in some cases; these studies take into account all the switching components inside a binary matrix, i.e. those elements that can be modified (switched) without altering both the vectors of projections.

Here different aspects of the *1-color* problem have been considered, in particular we study how a vector of projections can be changed in order to maintain its compatibility with a given second one in its orthogonal direction. Finally we show that all the computed vectors still form a lattice that inherits many properties of that of Brylawski.

This approach reveals its usefulness each time the vectors of projections are not given as an input, but they have to be computed through the decomposition of a more complex problem, as in the *n-color* case. For example, in [1] the authors decompose some instances of *2-color* into four *1-color*'s ones that allow a solution to exist.

Exploiting properties of this model, we hope to be able in future works to furnish solving algorithms for special instances of the *n-color* problem by using a *divide et impera* approach. The general idea is to decompose a problem in many *1-color* subproblems, then solve each of them, and finally merge together all the solutions, eventually taking care of some still unplaced elements, maintaining the consistency of the final solution.

This strategy relies on the property of the described lattice of Brylawski to detect elements to move from a subproblem to another without precluding the existence of a solution, in the intent of creating void positions where elements of different colors may find their placement.

It is remarkable that such a simple model can be applied as an invaluable tool to investigate problems in different research fields ranging from discrete tomography to dynamic models and combinatorics, allowing one to consider them in a unified perspective.

2 The *1-Color* Problem

Let us consider a $m \times n$ binary matrix $M = (m_{i,j})$, with $1 \leq i \leq m$ and $1 \leq j \leq n$; we define its horizontal and vertical projections as the vectors $H = (h_1, \ldots, h_m)$ and $V = (v_1, \ldots, v_n)$ of the sums of its elements for each row and each column, respectively, i.e.

$$h_i = \sum_{j=1 \ldots n} m_{i,j} \quad \text{and} \quad v_j = \sum_{i=1 \ldots m} m_{i,j}.$$

Let us consider the following problem:

1-Color (H, V)

Input: two vectors of integers H and V of length m and n respectively, such that for each $1 \leq i \leq m$ and $1 \leq j \leq n$, it holds
- $0 \leq h_i \leq n$;
- $0 \leq v_j \leq m$;
- $\sum h_i = \sum v_j$.

Question: does there exist a binary matrix such that H and V are its horizontal and vertical projections, respectively?

The *1-color* problem is known as *Consistency* problem, as well. Two vectors H and V are *feasible*, if they meet the requirements of the *1-color* problem; furthermore, they are *compatible* if *1-color (H, V)* has a positive answer.

We observe that from each matrix M having projections H and V we can compute a matrix M' having projections H' and V', where H' and V' are obtained from H and V by sorting their elements in decreasing order, respectively, by switching the rows and the columns of M. So, without loss of generality, from now on we will consider only vectors of projections sorted in decreasing order.

Starting from the vector $H = (h_1, \ldots, h_m)$, let us define the vector $\overline{H} = (\overline{h}_1, \ldots, \overline{h}_n)$ as follows:

$$\overline{h}_j = |\{\, h_i \in H \,:\, h_i \geq j \,\}|\,.$$

In [9], Ryser gives a necessary and sufficient condition for a couple of vectors to be compatible, i.e.

Theorem 1. *(Ryser, [9]) Let H and V be a couple of compatible vectors. The problem* 1-color (H, V) *has a positive answer if and only if it holds*

$$\sum_{j=l}^{n} v_j \geq \sum_{j=l}^{n} \overline{h}_j, \qquad \textit{with } 2 \leq l \leq n.$$

Obviously the vectors H and $\overline{H}$ are compatible; as a matter of fact the $m \times n$ matrix whose i-th row has h_i elements 1 followed by $n - h_i$ elements 0 has H and $\overline{H}$ as projections, and it is called *maximal*. Figure 1 shows an example of a matrix which is compatible with a couple of maximal vectors.

In the following we give an alternative characterization of the couples of consistent vectors that allows the construction of a poset with ease.

An immediate check reveals that the class of the matrices whose horizontal and vertical projections are H and V respectively, say $\mathcal{U}(H, V)$, reduces to a singleton if and only if $V = \overline{H}$ by showing that this element admits no switching components. Such a property of the vector $\overline{H}$ allows us to choose it as the minimum of the poset we define in Paragraph 3.

Now we consider the following procedure *Sup* that starts from an integer vector V and gives as output a vector V' equal to V except in two of its elements; we remark that *Sup* acts non deterministically in the choice of a couple of column indexes where the elements of V are modified:

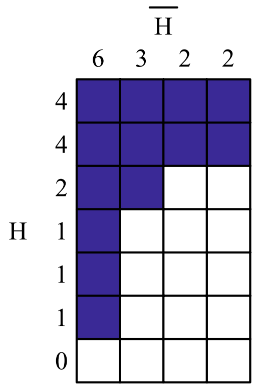

Fig. 1. The computation of the vector $\overline{H}$ and the maximal matrix associated to H and $\overline{H}$

Sup (V)

Step 1: Initialize $V' = V$;
Step 2: *If* there exist two column indexes j, j' such that $v_j - v_{j'} > 1$, and $v_j > v_{j+1}$ and $v_{j'-1} > v_{j'}$
then compute

$$v'_{j'} = v_{j'} + 1 \quad \text{and} \quad v'_j = v_j - 1;$$

and give V' as output
else give V' as output.

The hypothesis $v_j > v_{j+1}$ and $v_{j'-1} > v_{j'}$ maintain the array V' sorted. They impose that if in V there are many consecutive elements with the same value, then *Sup* chooses only one of them. Since we consider only ordered vectors, such a choice can be done without loss of generality.

We will write $Sup(V) \longrightarrow V'$ if V' can be obtained from V through an application of *Sup*.

Theorem 2. *Consider two compatible vectors H and V, and let $Sup(V) \longrightarrow V'$. The vectors H and V' are compatible.*

Proof. If $V' = V$ then the result is trivial, otherwise starting from a matrix M which is compatible with H and V, we construct a new matrix M' compatible with H and V' as follows: let j and j' be the column indexes modified by the procedure *Sup*. Since $v_j > v_{j'} + 1$, then there exist two elements $m_{i,j} = 1$, and $m_{i,j'} = 0$. The matrix M' is equal to M except for the two elements $m'_{i,j} = 0$, and $m'_{i,j'} = 1$. In this way the horizontal projections are the same in the two matrices M and M', while this latter has vertical projections equal to V'. $\square$

Corollary 1. *Let V and V' be two vectors compatible with H. If $Sup(V) \longrightarrow V'$, then $|\mathcal{U}(H, V)| \leq |\mathcal{U}(H, V')|$.*

We start by showing that each vector V compatible with H can be obtained by iterating the procedure Sup starting from $\overline{H}$.

Consider a matrix M that is compatible with H and V as in Figure 2 (left). The sliding of all the elements of the matrix till the leftmost available positions in the same row, produces a matrix M' with projections H and $\overline{H}$, as shown in Figure 2 (right). Each element that is moved from a column j to a column $j' < j$ to obtain M' from M produces a change in V which can be considered as the application of the Sup operation to it, consequently all the moves of the elements which transform M into M' are a sequence of applications of Sup to derive $\overline{H}$ from V, and viceversa.

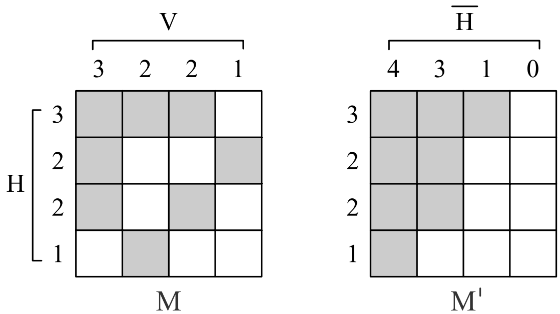

Fig. 2. Sliding all of the elements of a matrix M with projections H and V (left) to the leftmost possible positions, we obtain a matrix M' with projections H and $\overline{H}$ (right). Analyzing the differences between the two matrices, one can obtain all the applications of Sup needed to obtain V from $\overline{H}$.

Let us define $\mathcal{U}_n(H, \star)$ the class of the vectors which have dimension n, and which are compatible with H. Given two elements V and V' of $\mathcal{U}_n(H, \star)$, we say that V' is greater than or equal to V, say $V \leq_S V'$, if V' can be reached from V with a series (eventually void) of successive applications of Sup. From Theorem 2, it holds that V' is greater than V if the vectors which are compatible with V are a subset of those compatible with V'.

We observe that $\overline{H}$ is the minimum of the class $\mathcal{U}_n(H, \star)$; similarly, we can also define the maximum element of $\mathcal{U}_n(H, \star)$ as follows:

Theorem 3. *Let H be an integer vector. If*

$$V_M = (k, k, \ldots, k, k-1, \ldots, k-1)$$

is a vector feasible with H, then it is also compatible with H. Furthermore, if V is a vector compatible with H of the same dimension as V_M, then it holds $V \leq_S V_M$.

The theorem can be easily verified by observing that V_M can be obtained from any vector feasible with H through a finite number of applications of Sup: an array in this form is the only fixed point of the Sup operation. This can be easily understood acting symmetrically to what already done in the proof of Corollary 1.

This result was also proved by induction in [1] with the purpose to furnish a property that allows the decomposition of a two color problem of a certain family in several one color problems that always admit a solution.

3 The Brylawski Model

Let us recall the following two standard definitions: a *partially ordered set*, say *poset*, is a set P together with a binary relation $\leq$ on it which is reflexive, transitive and anti-symmetric. A poset in which any two elements have a unique *sup* and a unique *inf* element is said to be a *lattice*. An immediate check reveals that the couple $(\mathcal{U}_n(H,\star), \leq_S)$ is a poset.

Let H be a given integer vector, and let V be an integer vector whose elements sum to s, and which is compatible with H. Since by hypothesis the elements of V are ordered in decreasing order, we may regard V as an integer partition.

So, we relate the above defined order on $\mathcal{U}(H,\star)$ to the *dominance order* on the integer partition of s, defined by Brylawski in [2] as follows: given two integer partitions V and V'

$$V \leq_D V' \quad \text{if and only if} \quad \sum_{j=1}^{k} v_j \geq \sum_{j=1}^{k} v'_j$$

for each $1 \leq k \leq n$, with n being the common dimension of V and V', as usual.

The Brylawski model is very popular in combinatorics, and it can be viewed as an extension of the well-known sandpile model (see [3],[4]), where, under some assumptions the grains of sand are allowed to slide.

We immediately realize that the dominance order bases on the condition given by Ryser in Theorem 1; hence we can express the consistency of two vectors H and V using it:

Lemma 1. *H is compatible with V if and only if $\overline{H} \leq_D V$.*

It is immediate to verify that if $V \leq_S V'$, then it holds $V \leq_D V'$; such a correspondence allows us to use some results of Brylawski in the discrete tomography environment.

In particular, we notice that the procedure *Sup* applied to one element V in $\mathcal{U}_n(H,\star)$ does not compute the elements of the same class which lie immediately below it in the poset $(\mathcal{U}_n(H,\star), \leq_S)$, as shown in Fig. 3: let us consider the vector $\overline{H} = (3,1,0)$ which is the maximum element of the class $\mathcal{U}_n(H,\star)$, with $H = (2,1,1)$ (Fig. 3,(a)). The computation of *Sup* on $\overline{H}$ produces $(3,1,0) \longrightarrow (2,2,0)$ and $(3,1,0) \longrightarrow (2,1,1)$ (Fig. 3,(b)). Unfortunately, the obtained poset structure needs to be refined since $(2,1,1)$ is not an immediate successor of $\overline{H}$, as pointed out by the application of *Sup* to $(2,2,0)$ (Fig. 3,(c)). This makes the partial order quite difficult to analyze.

Brylawski defines two simple rules to avoid the problem, and he uses them to directly arrange the integer partition of s in a structure that turns out to be a lattice: let V be an integer partition of s, we construct an integer partition V' as follows

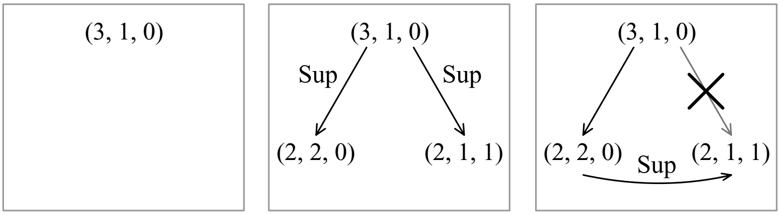

Fig. 3. The steps to determine the poset structure of $\mathcal{U}(H,\star)$, with $H=(2,1,1)$. The minimum and the maximum elements of the poset are $\overline{H}=(3,1,0)$ and $H_M=(2,1,1)$, respectively.

Vertical rule: if $v_i \geq v_{i+1}+2$, then $v'_i = v_i - 1, v'_{i+1} = v_{i+1}+1$, with $1 \leq i \leq n$. The other elements of V and V' coincide.

Horizontal rule: if there exist two indexes i,k such that
- $v_i - v_{i+k} = 2$;
- $v_{i+j} = v_i - 1$ for each $0 < j < k$;
- $v_{i+k} = v_i - 2$,

then $v'_i = v_i - 1, v'_{i+k} = v_{i+k}+1$. The other elements of V and V' coincide.

Brylawski proves that the order $\leq_B$ such that $V \leq_B V'$ if and only if V' can be derived from V by applying one of these two rules, is equivalent to the order $\leq_D$. Consequently it also holds that if $V \leq_S V'$, then $V \leq_B V'$.

This can be nicely rephrased saying that the procedure *Sup* intrinsically contains both these rules, but sometimes they act together in a single appliance of *Sup*.

Remark 1:

The procedure *Sup* preserves the length of its input vector, while if V and V' are respectively the input and the output partitions of the two rules of Brylawski, then it holds $0 \leq |V'|-|V| \leq 1$, with $|V|$ and $|V'|$ being the number of elements of V and V', respectively. This depends from the fact that *Sup* acts on two elements of an array by modifying only their values, and not their number, while the rules of Brylawsky are defined on integer partitions, without any constraints on the number of their elements except those due to the cardinality of the partition sum, i.e. a partition of n elements could generate a partition with $n+1$ elements by adding a 1 in the $(n+1)$th position; as an example the two element partition $(4,3)$ could generate the three element one $(4,2,1)$.

The following result holds:

Theorem 4. *The structure* $(\mathcal{U}_n(H,\star),\leq_S)$ *is a lattice.*

Proof. Since $\mathcal{U}_n(H,\star)$ has a maximum element $\overline{H}$, then each of its subsets has at least a *sup*. Now we prove that the *sup* of two of its elements, say V and V', is unique. Let assume $\hat{V} = sup\{V,V'\}$ in the Brylawski lattice. Since $|\overline{H}| = |\hat{V}|$ and $|\overline{H}| \leq |\hat{V}| \leq |V|$, then $|\hat{V}| = |V|$ by Remark 1. So there exists a sequence of applications of the Brylawski rules starting from $\overline{H}$ and leading to $\hat{V}$. Each of this application can be performed by an application of the procedure Sup, so $\hat{V} \in \mathcal{U}_n(H, \star)$ as desired. A similar argument holds to define the minimum of two elements of $\mathcal{U}_n(H, \star)$. $\square$

So, just to make a comparison between the Brylawski lattice of the partition of an integer s, and the correspondent lattice $\mathcal{U}(H, \star)$, we may note that the latter contains only the elements having with limited dimensions. Obviously the two lattices collapse if we assume $n \geq s$.

Final Remarks:

Many new questions arise from the study of the lattice $(\mathcal{U}(H, \star), \leq_S)$, and interesting results on the Brylawski model naturally reflects as new properties on it. As an example, since it holds that $\overline{\overline{H}} = H$, then there exists a strong duality between the classes $(\mathcal{U}(H, \star), \leq_S)$ and $(\mathcal{U}(\star, \overline{H}), \leq_S)$, i.e. the class of all the horizontal projections consistent with $\overline{H}$.

In particular we have that if $H \leq H'$, then $\overline{H'} \leq \overline{H}$, so the procedure Sup inverts the order $\leq_S$. As an example, in Fig. 4 there are depicted two lattices: the one on the left is $(\mathcal{U}(H, \star), \leq_S)$, with $H = (2, 2, 1, 1, 1)$, while the one on the right is the upside-down version of its dual. The elements are linked by the *overline* operator.

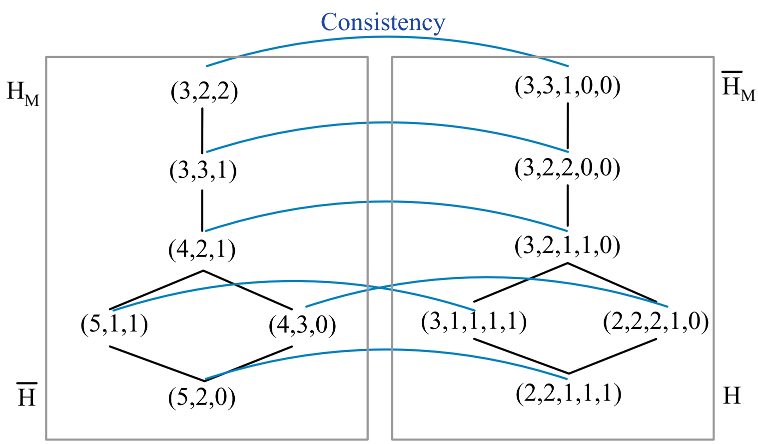

Fig. 4. Integer partitions of the number 7. To the left: poset of partitions with 3 elements, and no element greater than 5. To the right: poset of partitions with 5 elements, and no element greater than 3.

Obviously both the two lattices can be found embedded in the Brylawski lattice of the partition of the number seven. This implies that each Brylawski lattice is composed by many reverse symmetrical parts.

Finally, as an immediate consequence of the above observation, it holds the following combinatorial result:

Theorem 5. *The number of integer partitions of s having at most length m, and whose maximum value is n is equal to the number of integer partitions of s having at most length n, and whose maximum value is m.*

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