Constructing $(0,1)$-matrices with given line sums and certain fixed zeros

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Outline

- Motivation (discrete tomography)
- Recall some facts (old and recent)
- New extension
- A generalized Ryser algorithm
- Application

Motivation

Feasible set in the transportation problem:

$$\begin{array}{ll} \sum_{j=1}^{n} x_{i,j} = r_i & (i \leq m) \\ \sum_{i=1}^{m} x_{i,j} = s_j & (j \leq n) \\ x_{i,j} \geq 0 & (i \leq m, j \leq n). \end{array}$$

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More complicated versions:

- $0 \leq x_{ij} \leq 1$ or other simple bounds.
- $x_{ij} \in \{0, 1\}$
- some x_{ij}'s are fixed to zero (in a certain pattern)

Example

$(0, 1)$-matrix with given line sums and a zero block

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & O \end{bmatrix}$$

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$R = (5, 5, 5, 4, 4, 3, 2, 2)$, $S = (5, 5, 5, 3, 4, 3, 3, 2)$, A_1: 5×4.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Motivation/previous work: discrete tomography

- Discrete tomography: binary images with horizontal and vertical projections.
- Fixing zeros (or ones) correspond to known properties/geometry.
- Herman, G.T., Kuba, A. (Eds.): *Discrete Tomography. Foundations, Algorithms, and Applications*, 1999.
- Del Lungo, A. et al. (Eds.). *Special issue on discrete tomography*. *LAA*, **339** (2001).
- Chen, W.Y.C.: Integral matrices with given row and column sums. *Journ. of Comb. Theory, Ser. A*, **61**, 153–172 (1992).
- Brualdi, Anstee (1982, two papers), Fulkerson (1959), ...

Some known results for $\mathcal{A}(R, S) = \dots$

- Assume R and S nonincreasing (monotone).

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Theorem. *Equivalent:*
(i) $\mathcal{A}(R,S)$ is nonempty.
(ii) $S \prec R^*$ (Gale-Ryser theorem), i.e.,

$$\begin{array}{lll} \sum_{j=1}^{k} s_j & \le \sum_{j=1}^{k} r_j^* & \qquad (k \le n-1) \\ \sum_{j=1}^{n} s_j & = \sum_{j=1}^{n} r_j^*. & \end{array}$$

(iii) The structure matrix T is nonnegative.

Conjugate vector: $r_j = (R^*)_j = |\{i : r_i \ge k\}|$.
Ryser's structure matrix $T = [t_{kl}]$: for $(0 \le k \le m,\ 0 \le l \le n)$

$$t_{k,l} = kl + \sum_{i=k+1}^{m} r_i - \sum_{j=1}^{l} s_j$$

Extension: zero block; class $\mathcal{A}_{p,q}(R,S)$

- Brualdi and Dahl, LAA, Vol. 371 (2003), 191–207.
- New structure matrix T: For $0 \le k \le p$ and $0 \le l \le q$ let

$$
\begin{aligned}
t_{k,l} = \quad & \tau + kl - \textstyle\sum_{i=1}^{k} r_i - \sum_{j=1}^{l} s_j \\
& - \textstyle\sum_{i=p+1}^{m} (r_i - l)^+ - \sum_{j=q+1}^{n} (s_j - k)^+ .
\end{aligned}
$$

  Here $\tau = \sum_i r_i = \sum_j s_j$. T has size $(p+1) \times (q+1)$.
- Specializes into Ryser's matrix when $p = m$, $q = n$.

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- May be derived from the maxflow-mincut theorem (bipartite graph) and a "cut reduction".
- T has interesting properties ...

Example

Let $m = n = 10,\ p = q = 7,\ R = (9, 8, 6, 6, 5, 5, 4, 3, 3, 3)$, and $S = (7, 6, 6, 6, 6, 6, 6, 4, 3, 2)$.

$$T = \begin{bmatrix} 34 & 30 & 27 & 24 & 18 & 12 & 6 & 0 \\ 28 & 25 & 23 & 21 & 16 & 11 & 6 & 1 \\ 23 & 21 & 20 & 19 & 15 & 11 & 7 & 3 \\ 19 & 18 & 18 & 18 & 15 & 12 & 9 & 6 \\ 14 & 14 & 15 & 16 & 14 & 12 & 10 & 8 \\ 9 & 10 & 12 & 14 & 13 & 12 & 11 & 10 \\ 4 & 6 & 9 & 12 & 12 & 12 & 12 & 12 \\ 0 & 3 & 7 & 11 & 12 & 13 & 14 & 15 \end{bmatrix}.$$

Here $T \geq O$ so $\mathcal{A}_{7,7}(R, S)$ is nonempty.

Ryser's algorithm: constructs $A \in \mathcal{A}(R, S)$

- Starts with the "maximal matrix" $\overline{A}$ with row sum R and ones left justified, i.e., column sum is R^*.
- Shift the last 1 in certain rows of $\overline{A}$ to column n in order to achieve the sum s_n.
- The 1's in column n are to appear in those rows in which $\overline{A}$ has the largest row sums, giving preference to the bottommost positions in case of ties.
- Proceed inductively to construct columns $n-1, \dots, 2, 1$.

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- may be extended to "zero block": construct $A \in \mathcal{A}_{p,q}(R, S)$
- constructs a canonical matrix, 3 times (modified) Ryser

Main question

- Find other patterns P for the fixed zeros such that simple algorithms may be used to construct a matrix in that class.

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- Note that network flow algorithms (maxflow) may be used for *any* pattern.
- So: simpler and faster than these, i.e., avoid augmenting on arbitrary paths.

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- Special case (*Young-pattern*): P is specified by a vector $K = (k_1, k_2, \ldots, k_n)$ satisfying $k_1 \geq k_2 \geq \cdots \geq k_n$ and for each $j \leq n$

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$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 1 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Our approach

1. Show the existence of a canonical matrix $\tilde{A}$ in the class $\mathcal{A}_P(R, S)$.
2. Use the special properties of this canonical matrix $\tilde{A}$ to set up an algorithm for finding $\tilde{A}$ column by column.

Some results

Lemma. *$\mathcal{A}_P(R,S)$ contains a matrix $\widehat{A}$ where the 1's in column n appear in those rows in P_n in which r_i is largest, giving preference to the bottommost positions in case of ties.*

- Interchange:

$$T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- ... gives "last-column canonical matrix"

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Theorem. *$\mathcal{A}_P(R,S)$ contains a unique matrix $\widetilde{A}$ such that for each $k \le n$ the submatrix A_k consisting of the first k columns of A is a last-column canonical matrix in its class.*

Connectivity

Theorem. *Let A and B be two given matrices in $\mathcal{A}_P(R, S)$. Then there is a sequence of interchanges that transforms A to B with every intermediary matrix in $\mathcal{A}_P(R, S)$.*

Generalized Ryser algorithm:

1. (Initialize) *Let* $k = n$ *and let* $\hat{R} = R$.
2. (Determine column k) *Find the indices in* P_k *corresponding to the* s_k *largest positive components of* $\hat{R}$ *where we prefer largest indices in case of ties. Let the* k*'th column of* $\widetilde{A}$ *have ones in the* s_k *positions just found.*
3. (Update row sum) *Let* $\hat{R} = R - \widetilde{A}(:, k)$ *(the row sum vector after the last column has been deleted). If* $k > 1$*, reduce* k *by 1 and go to Step 2.*

Example

Let $m = 5$, $n = 8$, $R = (5, 5, 4, 4, 3)$, $S = (3, 3, 3, 3, 3, 2, 2, 2)$, and let P correspond to the Young-pattern given by $K = (5, 5, 5, 5, 4, 4, 3, 3)$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & \mathbf{0} & \mathbf{0} \\ 1 & 1 & 1 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & \mathbf{0} & \mathbf{0} \\ 1 & 1 & 1 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \rightarrow$$

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Application: reconstruction (in Discrete Tomography)

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Theorem Let $A \in \mathcal{A}_P(R, S)$. Equivalent:

- A is the unique matrix in $\mathcal{A}_P(R, S)$.
- a certain graph G_A is acyclic.
- Starting with A and recursively deleting rows and columns of all 0's in positions of P, or rows and columns of all 1's in positions in P one eventually obtains the empty matrix.

Reconstruction algorithm:

1. Initialize A by setting $a_{ij} = 0$ for all $(i,j) \notin P$.
2. Find a line in the matrix such that its line sum, together with the fixing outside P and the previously determined entries in the algorithm, forces all remaining entries to be all one or all zero.
3. Set these entries accordingly, and return to Step 2 until A is completely specified.