Solving some instances of the two color problem

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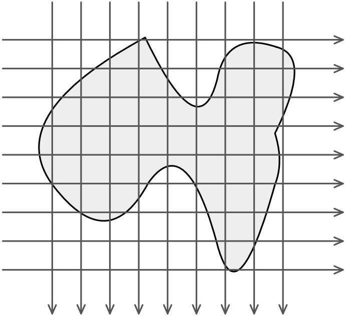
Discrete tomography

- Goal: to reconstruct discrete sets from their projections



Motivations

- Real world inspiration: electron beam techniques - QUANTITEM
- Many other problematics can be recunducted to discrete tomography
  - Es. modeling of problems through discrete matrices

Typical problems

- Typical problems concern
  - Consistency: is there a matrix that satisfies the input constraints ?
  - Reconstruction: rebuilding a matrix that solves the problem
  - Uniqueness: is the solution of the problem unique ?

Constraints

- Often various constraints are imposed on the studied discrete set
  - Connectiveness (Polyominoes): the studied set represents an only connected object
  - Convexity: the searched set is convex in respect to various directions
  - Other: problems modelled with discrete matrices can lead to any arbitrary constraint

The n color problems

- Most basical family of problems defined in discrete tomography
  - Discrete set with cells of n different types (colors, atoms)
  - Projections along parallel lines in various directions
  - Known for each of them the number of cells of the given types
  - No other constraint assumed on the discrete set

Known results

- Known results
  - For more than 2 projections, consistency and reconstruction are proved to be NP-complete even for only one atom
  - Assumed 2 projections, usually ortogonal

Known results: 1 color

- For one atom, complexity determined by Ryser
- Found necessary and sufficient conditions for consistency
- A simple greedy algorithm solves the reconstruction problem
- Usually no guarantee of uniqueness

Known results: more than 2 colors

- For more than two atoms, problem is NP-hard
- Shown at first for 6 atoms (Gardner, Gritzmann, Prangenberg) and in the following for three or more (Chrobak and Dürr)

The two color problem

- The only left undetetermined case in that of the two atoms
- Problem represents a boundary between easy and hard problems
- Determining complexity of an n atom problem gives results also about related problems

Definition of the problem

- **Input** $H = ((h_1^b, h_1^r), \ldots, (h_m^b, h_m^r))$ and $V = ((v_1^b, v_1^r), \ldots, (v_n^b, v_n^r))$ containing non negative integers
- **Task** Find a matrix $A = (a_{i,j})$ such as $\forall 1 \leq i \leq m$

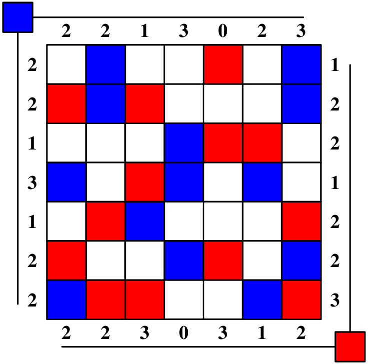
$$h_i^x = |\{a_{i,j} \ : \ a_{i,j} = x\}| \text{ with } 1 \leq j \leq n, \text{ and } x \in \{b, r\},$$

and analogously, $\forall 1 \leq j \leq n$

$$v_i^x = |\{a_{i,j} \ : \ a_{i,j} = x\}| \text{ with } 1 \leq i \leq m, \text{ and } x \in \{b, r\}.$$

A two colored matrix

- An example of a two colored matrix and its projections



Our result

Consistency always satisfied if

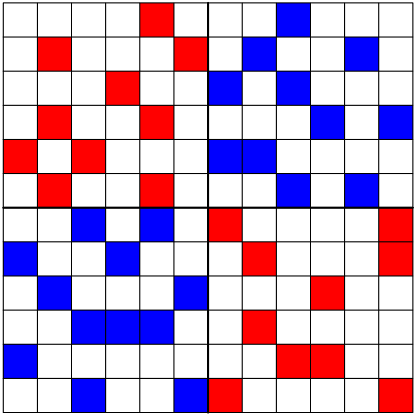
- For all i, j, x we have $0 < h_i^x, v_j^x \leq M$, where:
  - if m and n are both even, $M = \lfloor min\{m, n\}/3 \rfloor$;
  - otherwise, $M = \lfloor min\{m, n\}/4 \rfloor$;
- For two positive integers c_1, c_2 and for all i, j we have

$$h_i^b = c_1, \; v_j^r = c_2$$

- The projection sums are consistent

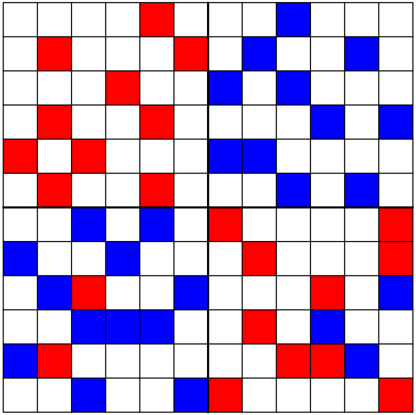
Idea of the algorithm

- Given an efficient recunstruction algorithm
- The idea: decomposing the problem in 4 1-color problems



Idea of the algorithm

- Rarely decomposition is directly appliable
- The algorithm will also merge a limited number of cells in a subproblem relative to the opposite color



Intermediate results

Result is based on three subresults

- Definition of instances of 1-color that always allow a solution
- Theorem that defines some when inserting cells of a color in a partially filled matrix is possible
- Theorem about multiset partitioning

The one color problem

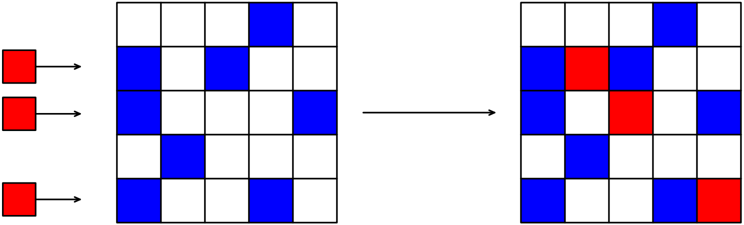
- One color problem studied extensevely by Ryser
- We give an equivalent model confortable to us
- Studied what are the arrays H consistent with a given V

The maximal array of projections

- Defined a poset on the possible projections for a given number of cells
- Array $(k, k, \ldots, k, k-1, k-1, \ldots, k-1)$ and its permutations are the maximal element of the poset
- Consistent with any other feasible projection array
  - Beyond this result, the construction of this poset can be valuable for further studies

Cell insertion problem

- **Input**
  - A $w \times h$ matrix M with some cells of color 1 and others of color 0 (empty cells)
  - In M there are at most $w - c$ cells per row, for a fixed c
  - A set of rows R such that $|R| \leq c$
- **Problem** Can we insert in the empty positions of M a red cell for each row in R such that there is at most a red cell for each column ?
- The given task always has a solution

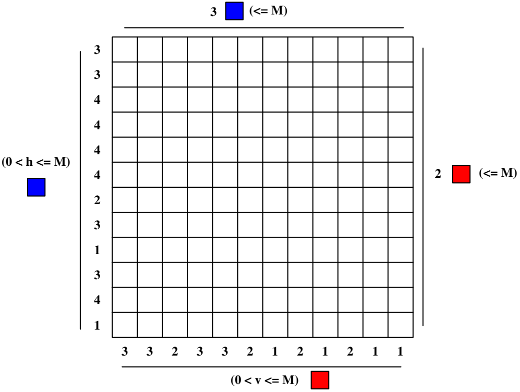


The problem of multiset partitioning

- Multiset S of positive integers with maximal value M
- Goal is to obtain a partition S' and S'' of S such that
  - $|S'| = \lfloor |S|/2 \rfloor$ and $|S''| = \lceil |S|/2 \rceil$
  - The sums of the elements in S' and S'' do not differ of more than M
- We prove that the sets S' and S'' always exists and can be found efficiently

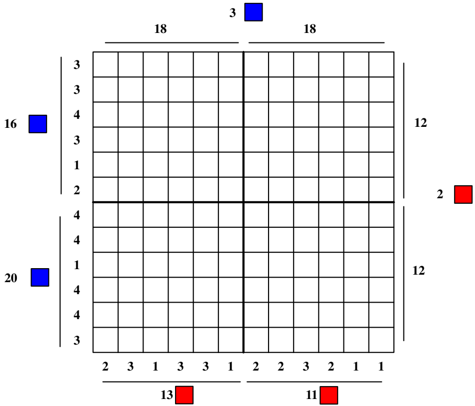
The main algorithm

- The main algorithm



Step 1

- Step 1: partition the arrays of non constant projections in two multisets with sum difference $\leq M$
- Place elements of the first multisets in the first half of the rows (columns)

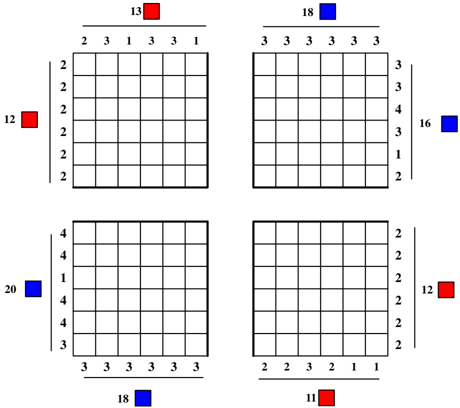


Step 2

- Step 2: part the problem in four subproblems
- Some problem may have an unconsistent cell number for columns and rows
- A number of cells will have to be merged in the neighbour problems
  - Thanks to the partitioning operation, this quantity is limited to $M/2$ if width and height of A is even of to M otherwise.
  - We will assume cells will have to be moved between SW and SE

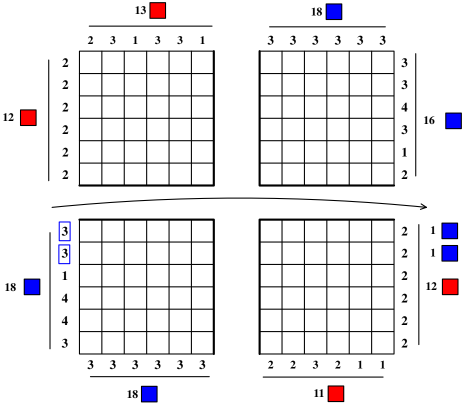
Step 2 (b)

- Execution of step 2



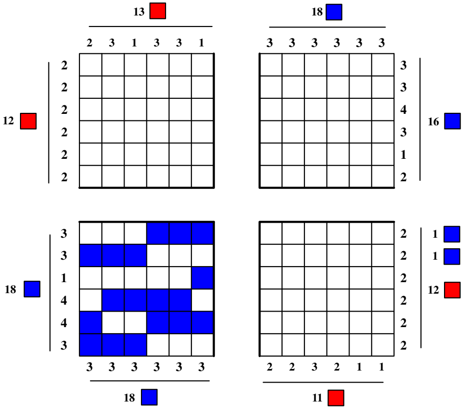
Step 3

- Step 3: move cells from problem *SW* to problem *SE*
- Chose at most a cell for every row



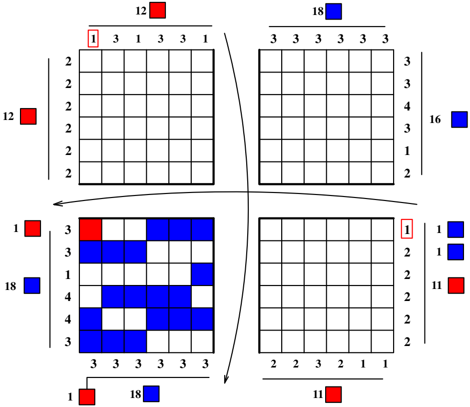
Step 4

- Step 4: solve the subproblem in SW
  - Can use the greedy algorithm
  - As a side has projections $(k, \ldots, k)$, we are sure of a solution



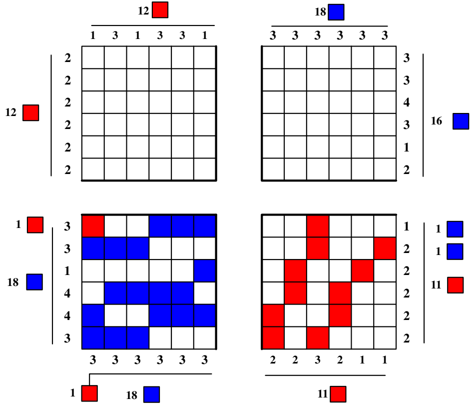
Step 5

- Step 5: insert cells in *SW* to balance projections in *SE*, at most one per column and one per row
  - The number of cells will be limited thanks to the partitioning operation
  - Thanks to the theorem about cell insertion, again we guarantee a solution



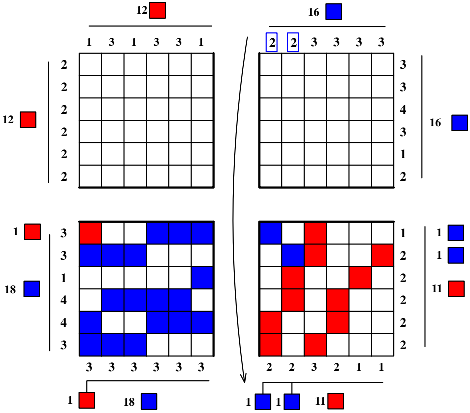
Step 6

- Step 6: solve *SE*



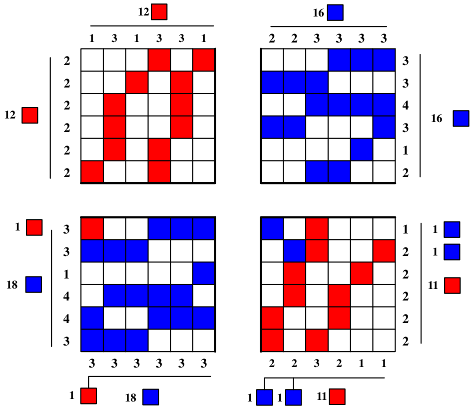
Step 7

- Step 7: place cells of color 1 (blue) in SE, at most one per column



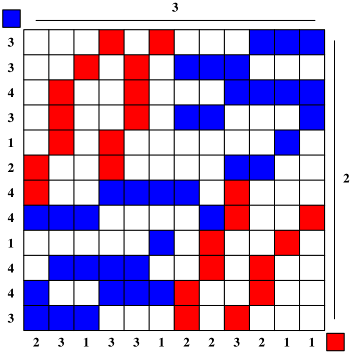
Step 8

- Step 8: solve *NE* and *NW*
  - Since we have subtracted at most a cell per column from the vertical projections, we know that these problems must have a solution



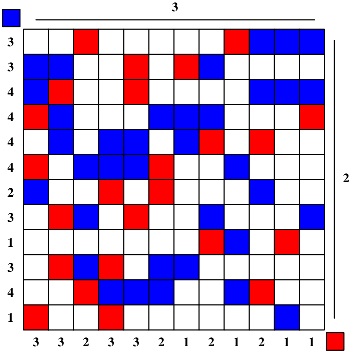
Solution

- Obtained an instance satisfying the projections



Solution

- The permutation in step 1 can now easily be reversed to obtain a solution for the *original* projections

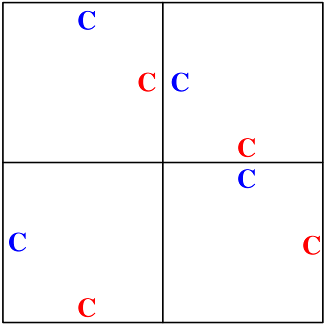


Conclusions

- The result defines some instances that are easy to solve
- Constant or limited projections seem to be a measure of 'solvability' of a problem
- Objective: reduce to a minimal number the instances for which we do not have an efficient algorithm
  - Reducing this number to 0 would prove that the problem is in P
  - Otherwise, having a reduced number of 'hard' instances would give noticeable help in finding an NP-completeness reduction

Further research

- Maybe the result could be expanded or used in other solving algorithms thanks to composition ?



C - constant projections

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