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Binary vectors partially determined by linear equation systems[1]

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Abstract

We address problems of the general form: given a J-dimensional binary (0- or 1-valued) vector a, a system of E of linear equations which a satisfies and a domain $\mathscr{D} \subset \mathbb{R}^J$ which contains a, when is a the unique solution of E in $\mathscr{D}$? More generally, we aim at finding conditions for the invariance of a particular position j, $1 \leqslant j \leqslant J$ (meaning that $b_j = a_j$, for all solutions b of E in $\mathscr{D}$). We investigate two particular choices for $\mathscr{D}$: the set of binary vectors of length J (integral invariance) and the set of vectors in $\mathbb{R}^J$ whose components lie between 0 and 1 (fractional invariance). For each position j, a system of inequalities is produced, whose solvability in the appropriate space indicates variance of the position. A version of Farkas' Lemma is used to specify the alternative system of inequalities, giving rise to a vector using which one can tell for each position whether or not it is fractionally invariant. We show that if the matrix of E is totally unimodular, then integral invariance is equivalent to fractional invariance. Our findings are applied to the problem of reconstruction of two-dimensional binary pictures from their projections (equivalently, $(0, 1)$-matrices from their marginals) and lead to a "structure result" on the arrangement of the invariant positions in the set of all binary pictures which share the same row and column sums and whose values are possibly prescribed at some positions. The relationship of our approach to the problem of reconstruction of higher-dimensional binary pictures is also discussed.

1. Introduction

For precise expression of the ideas intimated in the abstract, we introduce the following conventions and terminology. Our subject matter of concern will be J-dimensional (column) vectors, where J is an arbitrary (but fixed) positive integer. For any set H, we use H^J to denote the set of J-dimensional vectors over H and, for h in H^J, we

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use h_j to denote the jth component of H. We will be particularly concerned with the vectors in

$$\begin{aligned} \mathscr{I} &= \{0,1\}^J, \\ \mathscr{F} &= [0,1]^J, \\ \mathscr{T} &= \mathbb{R}^J, \end{aligned} \tag{1}$$

where $\mathbb{R}$ is the set of real numbers. We use the notation $h > 0$ to abbreviate that all components of a vector h are positive and adopt the corresponding conventions with $=$ and with $\geqslant$.

Let I be another arbitrary (but fixed) positive integer. We will be repeatedly using the letter P to denote an $I \times J$ matrix of real numbers, which we call the *projection matrix* (by analogy to the matrix referred to as the projection matrix [10, p. 100]). We will adopt, usually without explicit definition, the standard terminology and notation regarding matrices that can be found e.g., in [1]. In particular, we use P' to denote the transpose of the matrix P. The columns of P are elements of $\mathbb{R}^I$; we use P_j to denote the vector which is the jth column of P. Note that for any x in $\mathscr{T}$,

$$Px = \sum_{j=1}^{J} x_j P_j. \tag{2}$$

As usual, we define the inner product between vectors y and z in $\mathbb{R}^I$ as

$$\langle y, z \rangle = \sum_{i=1}^{I} y_i z_i. \tag{3}$$

Clearly, $\langle P_j, y \rangle$ is the same as $(P'y)_j$. (By analogy to the matrix used in the "discrete backprojection" of [10, Section 7.3], one may refer to P' as the *backprojection matrix*.)

As an example, consider the element $g = (1,1,1,0,0,1,0,0,0,0,1,0,0,1,1,1)'$ of $\mathscr{I}$. This can be interpreted as the *binary picture*:

1	1	1	0
0	1	0	0
0	0	1	0
0	1	1	1

Then the projection matrix

$$D=\begin{pmatrix}
1&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\
0&1&1&0&0&1&1&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&1&1&0&0&1&1&0&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&0\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1
\end{pmatrix} \tag{4}$$

corresponds to the nine readings that can be achieved by a detector which covers exactly a 2×2 region of contiguous picture positions of the binary picture. In this case, $Dg = (3,3,1,1,2,1,1,3,3)'$ and $Dg = Db$, where b is the 16-dimensional vector that can be interpreted as the (not binary) picture:

−1	1	1	0
2	1	0	0
−2	0	1	0
2	1	1	1

(As explained further below, the "pictures" of this example could have also been referred to as "matrices", but we decided not to use that terminology in order to avoid possible confusion between these pictures and the matrices of projection and backprojection.)

Let a be an arbitrary element of $\mathscr{I}$. For any j $(1 \leqslant j \leqslant J)$, j is said to be an *integrally* (respectively, *fractionally*) *invariant position* of a with respect of P, if for any b in $\mathscr{I}$ (respectively, in $\mathscr{F}$) such that $Pb = Pa$, $b_j = a_j$. If all j $(1 \leqslant j \leqslant J)$ are integrally (respectively, fractionally) invariant positions of a with respect to P, then a is said to be *integrally* (respectively, *fractionally*) *unique* with respect to P. In the discussion that follows, we use the word *variant* to mean "not invariant" and we drop the phrase "with respect to P" whenever doing so does not lead to confusion.

It is obvious that if a in $\mathscr{I}$ is fractionally unique, then it is integrally unique. It follows from the discussion of the example above that Dg does not uniquely determine the vector g, but below we show that g is fractionally (and hence integrally) unique with respect to D.

We note the obvious fact that if j is an integrally (respectively, fractionally) invariant (respectively, variant) position of a with respect to P and if a b in $\mathscr{I}$ is such that $Pb = Pa$, then j is an integrally (respectively, fractionally) invariant (respectively, variant) position of b with respect to P. This means that fractional and integral invariance and variance of a position j of a can be thought of as a property of a *class* of vectors of which a is only one member. The results that follow in this paper can be interpreted in a similar fashion. We will not provide such interpretations in the body of the paper itself, but emphasize the possibility of doing so, since the approach of referring to properties of classes rather than to properties of individual vectors from those classes is common in the related literature (e.g., [2,12]).

Uniqueness questions have been studied for various special kinds of projection matrices. For example, Fishburn et al. [7] found (using approaches similar in spirit to those presented in this paper) conditions specifying which positions of a vector representing an n-dimensional binary picture are integrally invariant with respect to a projection matrix which describes summations over all maximal $(n-1)$-dimensional subpictures of the binary picture. (It is customary in the discrete mathematics literature to refer to what we call "binary pictures" as "(0,1)-matrices" [2,3,5,8,12]. We chose to use the terminology involving the word "picture" for the purpose of avoiding possible confusion between the picture and the projection matrix, which also happens to be $(0,1)$ if it describes summations; see D in (4).) There are more general approaches than what is taken in our paper, for example, Kellerer [11] considers which sets in a measure space are (up to a null set) uniquely determined by the integrals over them of a family of test functions.

Returning to the classes defined above (more general than those discussed by Fishburn et al. [7], but subsumed under appropriate definitions by the classes of Kellerer [11]), it is found (Theorem 2.3) that for any a in $\mathscr{I}$ there exists a vector $y \in \mathbb{R}^I$ such that $(P'y)_j$ is 0 if, and only if, j is a fractionally variant position of a. If we additionally assume that the matrix P is totally unimodular, then there is a corresponding property for the integrally variant positions of a (Theorem 2.6). These results lead to necessary and sufficient conditions of fractional and integral uniqueness (Theorems 2.4 and 2.7). In Section 3 we apply our results to the problem of reconstruction of two-dimensional binary pictures and show how our findings lead to a "structure result" on the arrangement of the invariant positions in the set of all two-dimensional binary pictures which share the same row and column sums (marginals) and which have prescribed values at some positions. We also outline a method for constructing the structure of such a class of binary pictures. In the final section we discuss the relationship of our approach to some aspects of the literature on reconstructing two- and higher-dimensional binary pictures from various types of projections.

2. Conditions for fractional and integral invariance and uniqueness

We define, for any a in $\mathscr{I}$, the $I \times J$ matrix of real numbers P^a whose jth column, for $1 \leqslant j \leqslant J$, is

$$P_j^a = \begin{cases} P_j & \text{if } a_j = 1, \\ -P_j & \text{if } a_j = 0. \end{cases} \tag{5}$$

Lemma 2.1. *For any j, $1 \leqslant j \leqslant J$, and for any a in $\mathscr{I}$, j is fractionally (respectively, integrally) variant position of a if, and only if, there exists a d in $\mathscr{F}$ (respectively, in $\mathscr{I}$), such that $d_j > 0$ and $P^a d = 0$.*

Proof. Suppose first that j is fractionally (respectively, integrally) variant position of a. Then there exists a b in $\mathscr{F}$ (respectively, in $\mathscr{I}$), such that $Pa = Pb$ and $a_j \neq b_j$. Define d by

$$d_k = \begin{cases} a_k - b_k & \text{if } a_k = 1, \\ b_k - a_k & \text{if } a_k = 0, \end{cases} \tag{6}$$

for $1 \leqslant k \leqslant J$. Clearly, d is in $\mathscr{F}$ (respectively, in $\mathscr{I}$) and $d_j > 0$. Also

$$\begin{aligned} P^a d = \sum_{k=1}^{J} d_k P_k^a &= \sum_{a_k=1} (a_k - b_k) P_k - \sum_{a_k=0} (b_k - a_k) P_k \\ &= \sum_{k=1}^{J} a_k P_k - \sum_{k=1}^{J} b_k P_k = Pa - Pb = 0. \end{aligned} \tag{7}$$

Now suppose that there exists a d in $\mathscr{I}$ (respectively, in $\mathscr{F}$), such that $d_j > 0$ and $P^a d = 0$. Define b by

$$b_k = \begin{cases} a_k - d_k & \text{if } a_k = 1, \\ a_k + d_k & \text{if } a_k = 0, \end{cases} \tag{8}$$

for $1 \leqslant k \leqslant J$. Clearly, b is in $\mathscr{I}$ (respectively, in $\mathscr{F}$) and $a_j \neq b_j$. Furthermore,

$$\begin{aligned} Pa &= Pa - P^a d = \sum_{k=1}^{J} a_k P_k - \sum_{k=1}^{J} d_k P_k^a \\ &= \sum_{a_k=1} (a_k - d_k) P_k + \sum_{a_k=0} (a_k + d_k) P_k = Pb. \end{aligned} \tag{9}$$

It follows that j is a fractionally (respectively, integrally) variant position of a. □

To prove our next theorem, we need the following variant of the classical Farkas' Lemma which can be used to produce the alternative system of linear inequalities [15, p. 201] to the system defining d in Lemma 2.1.

Lemma 2.2. *If $1 \leqslant j \leqslant J$ and $z_k \in \mathbb{R}^I$ for $1 \leqslant k \leqslant J$, then one and only one of the following alternatives hold.*

(i) *There exists a y in $\mathbb{R}^I$ such that $\langle z_k, y \rangle \geqslant 0$ for $1 \leqslant k \leqslant J$ and $\langle z_j, y \rangle > 0$.*

(ii) *There exist nonnegative real numbers $\lambda_1, \ldots, \lambda_J$ such that λ_j is not zero and*

$$\sum_{k=1}^{J} \lambda_k z_k = 0. \tag{10}$$

Proof. This is a special case of [15, Theorem 22.2, pp. 198–199]. □

For any a in $\mathscr{I}$, we say that a y in $\boldsymbol{R}^I$ is *a-compatible* if $(P^a)'y \geqslant 0$. Note that, for a given a, the linear combination of a-compatible vectors with nonnegative coefficients is also an a-compatible vector. Clearly, a vector y is a-compatible if, and only if,

$$\langle P_j, y\rangle = \begin{cases} \geqslant 0 & \text{if } a_j = 1, \\ \leqslant 0 & \text{if } a_j = 0. \end{cases} \tag{11}$$

For the case of our example, it is easy to check that $z = (1,2,-1,-2,0,-2,-1,2,1)'$ is g-compatible.

Theorem 2.3. *Let a be any element of $\mathscr{I}$.*

(i) *For $1 \leqslant j \leqslant J$, j is fractionally invariant position of a if, and only if, there exists an a-compatible y such that $\langle P_j, y\rangle \neq 0$.*

(ii) *There exists a (necessarily a-compatible) y in $\boldsymbol{R}^I$ such that, for $1 \leqslant j \leqslant J$,*

$$\langle P_j, y\rangle \begin{cases} > 0 & \textit{if } a_j = 1 \textit{ and } j \textit{ is fractionally invariant position of } a, \\ = 0 & \textit{if } j \textit{ is fractionally variant position of } a, \\ < 0 & \textit{if } a_j = 0 \textit{ and } j \textit{ is fractionally invariant position of } a. \end{cases} \tag{12}$$

Proof. To prove (i), first assume that j is a fractionally variant position of a. Then, by Lemma 2.1, there exists a d in $\mathscr{F}$, such that $d_j > 0$ and $P^a d = 0$. If we now let, for $1 \leqslant k \leqslant J$, $\lambda_k = d_k$ and $z_k = P^a_k$, then we see that (ii) of Lemma 2.2 holds and so (i) of Lemma 2.2 cannot hold. In other words, if y is a-compatible, then $\langle P_j, y\rangle = 0$.

Now assume that there does not exist an a-compatible y in $\boldsymbol{R}^I$ such that $\langle P_j, y\rangle \neq 0$. That means that for every y such that $\langle P^a_k, y\rangle \geqslant 0$ for $1 \leqslant k \leqslant J$ we have that $\langle P^a_j, y\rangle = 0$. That means that (i) of Lemma 2.2 does not hold for $z_k = P^a_k$ and so (ii) of Lemma 2.2 must hold. If we now define, for $1 \leqslant k \leqslant J$, $d_k = \lambda_k / \max\{\lambda_1, \dots, \lambda_J\}$, then we see that d is in $\mathscr{F}$, $d_j > 0$, and $P^a d = 0$. Hence, by Lemma 2.1, j is fractionally variant position of a, which completes the proof of (i).

Regarding (ii), we first note that any y which satisfies (12) is necessarily a-compatible. For every k which is fractionally invariant position of a, let $y^{(k)}$ denote an a-compatible element of $\boldsymbol{R}^I$ such that $\langle P_k, y^{(k)}\rangle \neq 0$. For a k which is a fractionally variant position of a, let $y^{(k)} = 0$. Then

$$y = \sum_{k=1}^{J} y^{(k)} \tag{13}$$

is a-compatible. It follows therefore from (i) that if j is a fractionally variant position of a, then $\langle P_j, y\rangle = 0$ and satisfies (12). On the other hand, if j is a fractionally invariant position of a and $a_j = 1$, then (for all k) $\langle P_j, y^{(k)}\rangle \geqslant 0$ (the $y^{(k)}$ are a-compatible), but $\langle P_j, y^{(j)}\rangle > 0$ and so $\langle P_j, y\rangle > 0$. The argument is similar if j is a fractionally invariant position of a and $a_j = 0$. $\square$

Theorem 2.4. *An a in $\mathscr{I}$ is fractionally unique if, and only if, there exists a y in $\boldsymbol{R}^I$ such that $(P^a)'y > 0$.*

Proof. We observe that if a is fractionally unique, then a y which satisfies (12) will have the required property. On the other hand, if there exists a y such that $(P^a)'y > 0$, then it is a-compatible and, by (i) of Theorem 2.3, j is fractionally invariant position of a for $1 \leqslant j \leqslant J$. $\square$

For our example – D defined by (4), g defined above (4), and z specified just before the statement of the Theorem 2.3 – we see that $(D^g)'z = (1,3,1,1,1,1,1,3,3,1,1,1,1,1,3,1)'$ and so g is fractionally (and hence integrally) unique with respect to D.

In general, we cannot replace in Theorems 2.3 and 2.4 "fractional" by "integral". (Consider, for example, the projection matrix $P = (2,-1)$ and the vector $a = (1,1)'$. In this case, $y = (0)$ is the only a-compatible vector and $\langle P_1, y\rangle = \langle P_2, y\rangle = 0$. Therefore, according to Theorem 2.3(i), both positions are fractionally variant. However, it is clear that they are both integrally invariant.) Nevertheless, we now show that under some special conditions on the projection matrix, results similar to Theorems 2.3 and 2.4 also exist regarding integral invariance and uniqueness.

A matrix is *totally unimodular* if, for all its square submatrices, the absolute value of the determinant of the submatrix is either 0 or 1. It is in general quite laborious to check whether a matrix such as the D of (4) is totally unimodular (see, e.g., [17]). Here we point out the fact, to be used below, that negating a column of a totally unimodular matrix produces another totally unimodular matrix.

Lemma 2.5. *Let P be totally unimodular and $a \in \mathscr{I}$. If j is a fractionally variant position of a, then j is an integrally variant position of a.*

Proof. We first prove the required result for the special case when $a = o$, the J-dimensional zero vector. If j is a fractionally variant position of o, then there exists a p in $\mathscr{F}$ such that $Pp = 0$ and p_j is not 0. By dividing each component of p by p_j, we get a vector q such that $Pq = 0$ and $q_j = 1$ and $q \geqslant 0$. Consider the set

$$C = \{x \mid Px = 0,\ x_j = 1,\ x \geqslant 0\}. \tag{14}$$

This is a nonempty (since q is in it) polyhedral convex set [15]. C contains some extreme points (see [15, Corollary 18.5.3]); let v denote one of them. Then v is the unique solution of a system of equations $Bx = b$ which consists of J linearly independent equations of the larger system of equations

$$\begin{aligned} Px &= 0, \\ x_j &= 1, \\ x_k &= 0, \text{ if } k \neq j. \end{aligned} \tag{15}$$

It is easy to see that B is a totally unimodular matrix and that b has one component which is 1 and all its other components are 0. Hence, by Cramer's rule for solving linear equations in the nonsingular case [1], we see that each component of v is either 1, or 0, or -1. However, the last possibility cannot arise, since v is in C. By the same token, $v_j = 1$ and $Pv = 0 = Po$. Hence, j is an integrally variant position of o.

Now assume that j is a fractionally variant position of any element a of $\mathscr{I}$. By Lemma 2.1 there exists a d in $\mathscr{F}$, such that $d_j > 0$ and $P^a d = 0$. This means that j is a fractionally variant position of o with respect to P^a. The total unimodularity of P implies that of P^a and so, by the previously discussed special case, we have that j is an integrally variant position of o with respect to P^a. That means that there exists a d in $\mathscr{I}$, such that $d_j > 0$ and $P^a d = 0$. By a second application of Lemma 2.1, we get that j is an integrally variant position of a. $\square$

Theorem 2.6. *Let P be totally unimodular and $a \in \mathscr{I}$.*

(i) *For $1 \leqslant j \leqslant J$, j is an integrally invariant position of a if, and only if, there exists an a-compatible y such that $\langle P_j, y\rangle \neq 0$.*

(ii) *There exists a (necessarily a-compatible) y in $\mathbb{R}^I$ such that, for $1 \leqslant j \leqslant J$,*

$$\langle P_j, y\rangle \begin{cases} > 0 & \text{if } a_j = 1 \text{ and } j \text{ is an integrally invariant position of } a, \\ = 0 & \text{if } j \text{ is an integrally variant position of } a, \\ < 0 & \text{if } a_j = 0 \text{ and } j \text{ is an integrally invariant position of } a. \end{cases} \tag{16}$$

Proof. This follows from Theorem 2.3 and Lemma 2.5. $\square$

Theorem 2.7. *Let P be totally unimodular. An a in $\mathscr{I}$ is integrally unique if, and only if, there exists a y in $\mathbb{R}^I$ such that $(P^a)'y > 0$.*

Proof. This follows from Theorem 2.4 and Lemma 2.5. $\square$

3. A structure result for binary pictures with prescribed positions and projections

To illustrate the application of Theorems 2.6 and 2.7, we consider the reconstruction of two-dimensional binary pictures (i.e., (0, 1)-matrices) with prescribed positions (i.e., the value of the picture at these positions is given) from their projections (i.e., row and column sums). We show that it is possible to reorder the rows and the columns of the picture so that, after the reordering, the free positions with invariant value 1 are in the upper-left corner of the picture and the free positions with invariant value 0 are in the lower-right corner, with the variant positions sandwiched between them. Such structure results have been previous obtained under some special assumptions (e.g., in [4] for the case of at most one prescribed position per column), here we show that, even without any assumptions, it is a straightforward consequence of our general theory.

In order for our theory to be applicable, a picture with prescribed has to be represented by a vector according to some fixed rule of mapping the free positions in the picture into positions in the vector. We now make precise the associated concepts.

Let M and N be positive integers and let $\boldsymbol{L} = \{(m,n) \mid 1 \leqslant m \leqslant M,\ 1 \leqslant n \leqslant N\}$. An $M \times N$ *binary picture* A is a mapping of $\boldsymbol{L}$ into $\{0,1\}$. Elements of $\boldsymbol{L}$ will be referred to as *positions* of A and the value of A for position (m,n) will be denoted by $A_{(m,n)}$. Let $\boldsymbol{K}$ be an arbitrary subset of $\boldsymbol{L}$ (possibly empty); we will refer to elements of $\boldsymbol{K}$ as the *prescribed positions* of A and to the other elements of $\boldsymbol{L}$ as the *free* positions of A.

With every $M \times N$ binary picture A and every set $\boldsymbol{K}$ of prescribed positions of A, we associate a binary vector a and a projection matrix P as follows. Let K be the number of elements in $\boldsymbol{K}$. The vector a has $J = M \times N - K$ positions, one for each free position of the picture A. We find it convenient to denote by $a_{[m,n]}$ the component of a which corresponds to the (free) position (m,n) of A and for any such position we define $a_{[m,n]} = A_{(m,n)}$. (Thus, for $1 \leqslant j \leqslant J$, there is a unique free position (m,n) of A such that $[m,n] = j$.) We define $I = M + N$ and so the projection matrix P has one row for each row of the picture A and one row for each column of the picture A. The columns $P_{[m,n]}$ of P correspond to the free positions of A. For every free position (m,n) of A, all components of $P_{[m,n]}$ are 0 except for the two components which correspond to the mth row and the nth column of A, respectively, and these two components are 1. A free position (m,n) of A is said to be *invariant* (with respect to row and column sums and the given set of prescribed positions) if $[m,n]$ is an integrally invariant position of a (with respect to P), otherwise (m,n) is said to be a *variant* position of A. (So A has three kinds of positions: prescribed, invariant and variant.) We say that A is *unique* if all its free positions are invariant.

To tie our formal definition to the problem of reconstruction of two-dimensional binary pictures with prescribed positions from their row and column sums, we state the following easily seen fact. For every $M \times N$ binary picture A, every set of prescribed positions of A and every free position (m,n) of A, (m,n) is an invariant position of A if, and only if, for every $M \times N$ binary picture B which has the same row and column sums as A and whose values for the prescribed positions are the same as those of A for those positions, it is also the case that $B_{(m,n)} = A_{(m,n)}$. (We could have used an alternative way of tying the picture reconstruction problem into our theory, by essentially letting $J = M \times N$, $a = A$, and introducing for each prescribed position an extra row in P with a single 1 in it. Our chosen approach of using a smaller projection matrix appeared to us more convenient.)

Lemma 3.1. *For every $M \times N$ binary picture A and every set of prescribed positions of A, the associated projection matrix P is totally unimodular.*

Proof. We first observe that the entries of P are either 0 or 1. From this (and from the obvious fact that the transpose of a totally unimodular matrix is totally unimodular) it follows from [17, Theorem 19.3, p. 269] that P is totally unimodular provided that each collection of the rows of P can be split in two parts so that the sum of the rows in one part minus the sum of the rows in the other part is a row with entries only 0, +1, and −1. To see that this condition is satisfied, for any collection of the rows of P let the one part consist of those rows of P which correspond to the rows of the picture A and the other part consist of those rows of P which correspond to the columns of the picture A. Clearly, the sum of the rows in each of the parts has entries which are either 0 or 1, from which the required condition follows. $\square$

For the proof of our next theorem it will be convenient to introduce some additional notation. Let P be the projection matrix which is associated with an $M \times N$ binary picture A (and a set of prescribed positions of A). We can index each of the I rows of P by either r_m (if that row of P corresponds to the mth row of A) or by c_n (if that row of P corresponds of the nth column of A). In a similar fashion, for an arbitrary element y in $\mathbb{R}^I$, each component of y can be identified as either y_{r_m} (for some $m, 1 \leqslant m \leqslant M$) or y_{c_n} (for some $n, 1 \leqslant n \leqslant N$). Specifically, we define $r_m = m$ (for $1 \leqslant m \leqslant M$) and $c_n = M + n$ (for $1 \leqslant n \leqslant N$). Using this notation, we see that

$$\langle P_{[m,n]}, y \rangle = y_{r_m} + y_{c_n}. \tag{17}$$

Theorem 3.2. *For every $M \times N$ binary picture A and every set of prescribed positions of A, there exists a finite set $\boldsymbol{T}$ of real numbers and, for any t in $\boldsymbol{T}$, two sets $\boldsymbol{R}_t$ and $\boldsymbol{C}_t$ of integers such that*

$$\bigcup_{t \in \boldsymbol{T}} \boldsymbol{R}_t = \{r_1, \ldots, r_M\} \quad \textit{and} \quad \bigcup_{t \in \boldsymbol{T}} \boldsymbol{C}_t = \{c_1, \ldots, c_N\} \tag{18}$$

and for all u and v in $\boldsymbol{T}$,

$$\text{if } u \neq v, \quad \text{then} \quad \boldsymbol{R}_u \cap \boldsymbol{R}_v = \emptyset \quad \text{and} \quad \boldsymbol{C}_u \cap \boldsymbol{C}_v = \emptyset \tag{19}$$

and whenever (m,n) is a free position of A such that $r_m \in \boldsymbol{R}_u$ and $c_n \in \boldsymbol{C}_v$, then

$$v - u \quad \begin{cases} > 0 & \textit{if, and only if, } A_{(m,n)} = 1 \textit{ and } (m,n) \textit{ is an invariant} \\ & \textit{position of } A, \\ = 0 & \textit{if, and only if, } (m,n) \textit{ is a variant position of } A, \\ < 0 & \textit{if, and only if, } A_{(m,n)} = 0 \textit{ and } (m,n) \textit{ is an invariant} \\ & \textit{position of } A. \end{cases} \tag{20}$$

Proof. Let a be the binary vector and P be the projection matrix associated with the picture A and the given set of prescribed positions. By Lemma 3.1, P is totally unimodular and so by Theorem 2.6(ii), there exists a y in $\mathbb{R}^I$ such that, for every position $[m,n]$ of a,

$$\langle P_{[m,n]}, y \rangle \quad \begin{cases} > 0 & \text{if } a_{[m,n]} = 1 \text{ and } [m,n] \text{ is an integrally invariant} \\ & \text{position of } a, \\ = 0 & \text{if } [m,n] \text{ is an integrally variant position of } a, \\ < 0 & \text{if } a_{[m,n]} = 0 \text{ and } [m,n] \text{ is an integrally invariant} \\ & \text{position of } a. \end{cases} \tag{21}$$

In view of (17) and the definition of a as well as that of invariance for the free positions of A, this can be rewritten as: there exists a y in $\mathbb{R}^I$ such that, for every free position (m,n) of A,

$$y_{r_m} + y_{c_n} \begin{cases} > 0 & \text{if } A_{(m,n)} = 1 \text{ and } (m,n) \text{ is an invariant position of } A, \\ = 0 & \text{if } (m,n) \text{ is a variant position of } A, \\ < 0 & \text{if } A_{(m,n)} = 0 \text{ and } (m,n) \text{ is an invariant position of } A. \end{cases} \tag{22}$$

Now define $\boldsymbol{T}$ by

$$\boldsymbol{T} = \{-y_{r_1}, \ldots, -y_{r_M}, y_{c_1}, \ldots, y_{c_N}\} \tag{23}$$

and, for any t in $\boldsymbol{T}, \boldsymbol{R}_t$ and $\boldsymbol{C}_t$ by

$$\begin{aligned} \boldsymbol{R}_t &= \{j \mid 1 \leqslant j \leqslant M \text{ and } y_j = -t\}, \\ \boldsymbol{C}_t &= \{j \mid M+1 \leqslant j \leqslant N \text{ and } y_j = t\}. \end{aligned} \tag{24}$$

It is easy to see that these sets satisfy (18) and (19). Also, if (m,n) is a free position of A such that $r_m \in \boldsymbol{R}_u$ and $c_n \in \boldsymbol{C}_v$, then $y_{r_m} + y_{c_n} = v - u$ and so the 'if' part of (20) follows immediately from (22). The 'only if' part also follows, since the mutually exclusive conditions listed on the right-hand side of (22) exhaust all possibilities for a free position of A. □

Corollary 3.3. *For every $M \times N$ binary picture A and every set of prescribed positions of A, there is a reordering of the rows and columns of A such that for the resulting $M \times N$ binary picture B the following is the case.*

(i) *For $1 \leqslant m \leqslant M$ and $1 \leqslant n, n', n'' \leqslant N$,*

a. *if $B_{(m,n)} = 1$, (m,n) is an invariant position of B and (m,n') is a variant position of B, then $n < n'$; and*

b. *if (m,n') is a variant position of $B, B_{(m,n'')} = 0$ and (m,n'') is an invariant position of B, then $n' < n''$.*

(ii) *For $1 \leqslant m, m', m'' \leqslant M$ and $1 \leqslant n \leqslant N$,*

a. *if $B_{(m,n)} = 1$, (m,n) is an invariant position of B and (m',n) is a variant position of B, then $m < m'$; and*

b. *if (m',n) is a variant position of $B, B_{(m'',n)} = 0$ and (m'',n) is an invariant position of B, then $m' < m''$.*

Proof. Considering the statement of Theorem 3.2, reorder the rows and columns of A in such a way that for the positions in the resulting matrix B the following are true:

$$\text{if } m \in \boldsymbol{R}_u,\ m' \in \boldsymbol{R}_{u'} \quad \text{and} \quad u < u', \quad \text{then} \quad m < m', \tag{25}$$

$$\text{if } n \in \boldsymbol{C}_v,\ n' \in \boldsymbol{C}_{v'} \quad \text{and} \quad v > v', \quad \text{then} \quad n < n'. \tag{26}$$

Now we prove Corollary 3.3(i)a, the other cases can be proved similarly. Suppose that $B_{(m,n)} = 1$, (m,n) is an invariant position of B and (m,n') is a variant position of B. Let u, v and v' be such that $m \in \boldsymbol{R}_u$, $n \in \boldsymbol{C}_v$ and $n' \in \boldsymbol{C}_{v'}$. According to (20), $v-u>0$ and $v'-u=0$. It follows that $v>v'$ and so, by (26), that $n<n'$. $\square$

We now demonstrate Theorem 3.2 and Corollary 3.3 by an example. Consider the 6×8 binary picture A described by the contents of the large rectangle below.

	C_4	‖	C_3	‖	C_2 ‖	C_1		‖
$\boldsymbol{R}_1$	×	1	1	×	×	1	0	1
	1	1	×	×	×	0	1	0
$\boldsymbol{R}_3$	×	1	1	0	×	0	0	×
	1	1	0	1	×	×	0	×
$\boldsymbol{R}_4$	1	0	×	0	0	0	0	×
	0	1	×	0	×	×	0	0

In this picture × indicates a prescribed position, the other 31 positions are free. Let a and P denote the associated (31-dimensional) binary vector and (14×31) projection matrix. It is obvious that the positions of A inside the three boxes drawn by heavy lines are variant. That the other free positions are invariant follows from Theorem 2.6(i), by observing that the vector $y=(-1,-1,-3,-3,-4,-4,4,4,3,3,2,1,1,1)'$ is a-compatible and that $\langle P_{[m,n]}, y\rangle \neq 0$ for any free position (m,n) outside the heavy boxes. Furthermore, it is easily checked that the same y satisfies (16) and so can be used as the y for the construction in the proof of Theorem 3.2. This gives rise to $\boldsymbol{T}=\{1,2,3,4\}$ and

$$\begin{array}{ll}
\boldsymbol{R}_1=\{1,2\}, & \boldsymbol{C}_1=\{12,13,14\},\\
\boldsymbol{R}_2=\{\}, & \boldsymbol{C}_2=\{11\},\\
\boldsymbol{R}_3=\{3,4\}, & \boldsymbol{C}_3=\{9,10\},\\
\boldsymbol{R}_4=\{5,6\}, & \boldsymbol{C}_4=\{7,8\}.
\end{array}$$

(Note that $\boldsymbol{R}_2$ is empty.) We see that, in this case, (25) and (26) are satisfied and so Corollary 3.3 is valid for A as it is (i.e., without a reordering of its rows and columns).

A problem with Theorem 3.2 and Corollary 3.3 is that they are not constructive. The construction in them (which provides us with the sets $\boldsymbol{T}, \boldsymbol{R}_t$ and $\boldsymbol{C}_t$) depends on having available a vector y of certain properties. The existence of such a y is guaranteed by Theorem 2.6(ii), but so far we have not indicated any methodology by which such a y may be produced. In the rest of this section we remedy this situation.

We first note that once we have the finite sets $\boldsymbol{T}, \boldsymbol{R}_t$ and $\boldsymbol{C}_t$ whose existence is postulated in Theorem 3.2, a y which satisfies (21) can easily be produced by setting $y_{r_m}=-t$ if $r_m\in \boldsymbol{R}_t$ and $y_{c_n}=t$ if $c_n\in \boldsymbol{C}_t$. We therefore concentrate on giving a construction of $\boldsymbol{T}, \boldsymbol{R}_t$ and $\boldsymbol{C}_t$. Since this construction is not making an essential use of the main results of this paper and since detailed proofs of facts concerning it would be very similar in spirit to material presented in [6], we forgo giving such proofs and simply present the construction together with the fundamental facts regarding it.

With every $M \times N$ binary picture A and every set of prescribed positions of A, we associate a digraph D. (This digraph is very closely related to the incidence graph of [6].) The nodes of D are the first $M+N$ positive integers: i.e., the set $\{r_1,\dots,r_M\} \cup \{c_1,\dots,c_N\}$. There is an arc in the digraph corresponding to every free position (m,n) of the picture A: if $A_{(m,n)} = 1$, then the corresponding arc is from r_m to c_n and if $A_{(m,n)} = 0$, then the corresponding arc is from c_n to r_m. The most essential fact (for our construction to achieve its aim) is that a free position (m,n) of the picture A is variant if, and only if, the corresponding arc is in a strong component of D.

Suppose that D has T strong components. It is possible to label the sets of nodes in the various strong components of D as $\boldsymbol{N}_1,\dots,\boldsymbol{N}_T$ in such a way that if $v > u$, then there is no arc in D from a node in $\boldsymbol{N}_v$ to a node in $\boldsymbol{N}_u$. (This is because the condensation of a digraph is acyclic; see, e.g., [9].) If we now define $\boldsymbol{T} = \{1,\dots,T\}$ and, for any $t \in \boldsymbol{T}, \boldsymbol{R}_t = \boldsymbol{N}_t \cap \{r_1,\dots,r_M\}$ and $\boldsymbol{C}_t = \boldsymbol{N}_t \cap \{c_1,\dots,c_N\}$, then we obtain sets with the properties required in the statement of Theorem 3.2.

To demonstrate this on the 6×8 binary picture A of the example given above, we see that the associated digraph D has four strong components and if we define $\boldsymbol{N}_1 = \{r_1,r_2,c_6,c_7,c_8\}$, $\boldsymbol{N}_2 = \{c_5\}$, $\boldsymbol{N}_3 = \{r_3,r_4,c_3,c_4\}$ and $\boldsymbol{N}_4 = \{r_5,r_6,c_1,c_2\}$, then these are sets of nodes in the strong components of D with the property required in the construction above. Furthermore, the $\boldsymbol{T}, \boldsymbol{R}_t$ and $\boldsymbol{C}_t$ which are defined by the construction will be exactly the ones specified earlier in the example.

4. Discussion

The problem of reconstruction of binary pictures from their projections has an extensive literature. For survey see, e.g., [5]. For results connected to variant/invariant positions see, e.g., [12,14,16]. A method for reconstruction of unique binary pictures with prescribed positions has been presented in [13]. The study of uniqueness is extended to higher-dimensional binary pictures in [7].

We consider it one of the most interesting insights that has come out of the work for this paper that it now appears that many of the important concepts associated with the reconstruction of binary pictures are not really concepts restricted to that topic, but in fact are just special instances of concepts associated with general binary vectors satisfying linear equation systems. As a demonstrative example, we discuss the concept of additivity introduced in [7]. (Other concepts in [7] which are similarly closely related to ones introduced in this paper are those of bad configurations and weakly bad configurations.)

We first consider two-dimensional binary pictures. An $M \times N$ binary picture A is said to be *additive* if there exists an element x of $\mathbb{R}^{M+N}$ such that, for $1 \leqslant m \leqslant M$ and $1 \leqslant n \leqslant N$,

$$A_{(m,n)} = 1 \quad \text{if, and only if, } x_{r_m} + x_{c_n} \geqslant 0. \tag{27}$$

We now show how a result which is part of Theorem 3 in [7] follows from our general theory.

Corollary 4.1. *An $M \times N$ binary picture A (with no prescribed positions) is unique if, and only if, it is additive.*

Proof. By repeating the argument at the beginning of the proof of Theorem 3.2 we can derive that there exists a y in $\mathbb{R}^I$ such that, for every position (m,n) of A, (22) holds. It follows immediately that if A is unique, then it is additive (simply set $x = y$).

Suppose now that A is additive and let x be the element which satisfies (27). Let

$$\delta = \max\{x_{r_m} + x_{c_n} \mid 1 \leqslant m \leqslant M,\ 1 \leqslant n \leqslant N,\ x_{r_m} + x_{c_n} < 0\}. \tag{28}$$

If the set on the right-hand side of (28) is empty, we set $\delta = -1$. For $1 \leqslant j \leqslant M+N$, set $y_j = x_j - \delta/3$. For this y in $\mathbb{R}^I$ it is the case that if $A_{(m,n)} = 1$, then $y_{r_m} + y_{c_n} > 0$ and if $A_{(m,n)} = 0$, then $y_{r_m} + y_{c_n} < 0$. It follows from Theorem 2.7 that A is unique. □

The notion of additiveness can be generalized to the problem of reconstruction of higher-dimensional binary pictures from projections on axes. In this problem the associated projection matrix P is one which describes the taking of sums in all possible hyperplanes. For example, if A is a $2 \times 2 \times 2$ binary picture, then the projection matrix P can be chosen as

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}, \tag{29}$$

We note that for a three-dimensional binary picture each column of the projection matrix P which describes projections on axes contains exactly three 1's (corresponding to the three planes containing the picture position giving rise to that column of P).

If we now generalize the definition of additivity in the obvious way (in the three-dimensional case three components of x should be added together in (27)), then we see that the second part of the proof of Corollary 4.1 is still valid and so additivity implies uniqueness in this general case as well. However, it is shown in [7] that the converse is not valid. The reason why we cannot repeat the first part of the proof of Corollary 4.1 is that the P of (29) is not totally unimodular, as can be seen considering the submatrix consisting of the rows 1, 3, 5 and columns 2, 3, 5:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \tag{30}$$

The determinant of this submatrix is -2.

A similar situation exists when we consider the reconstruction of higher-dimensional binary pictures from sums along lines (i.e., the position changes parallel to one of the axes and is kept constant with respect to all the other ones). For a $2 \times 2 \times 2$ binary picture, the corresponding projection matrix can be chosen to be

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{31}$$

which is totally unimodular, but if we consider a $3 \times 3 \times 3$ binary picture, the projection matrix associated with taking sums along lines is a 27×27 matrix which has the submatrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \tag{32}$$

and so is not totally unimodular.

In conclusion, we have presented a general theory for determining the invariance of positions of binary vectors satisfying certain equation systems. The applicability of the theory to two-dimensional binary pictures sharing the same row and column sums was possible due to the total unimodularity of the projection matrix describing the taking of row and column sums. The corresponding matrices for higher dimensions are typically not totally unimodular and so alternative paths will have to be sought to make the general theory applicable to the reconstruction of such higher-dimensional binary pictures from various types of projections.

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