A Proof of the Erdős–Gallai Theorem

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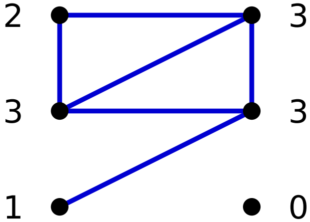
Joint work with Tyler Seacrest

Degree Sequences

Def. The degree sequence π of a graph G is a list of its degrees (with multiplicity), usually listed in decreasing order.



$$\pi = (3, 3, 3, 2, 1, 0)$$

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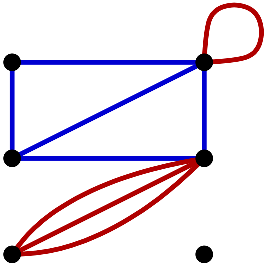

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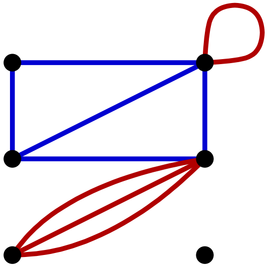

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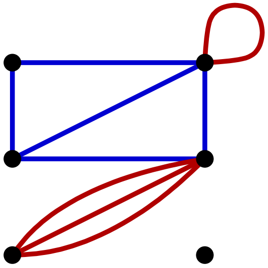

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What lists are graphic?

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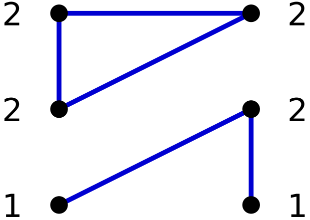

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Hence, the sum of π must be even.

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Ex. Is $\pi = (6, 6, 5, 4, 4, 2, 1)$ graphic? No . . . but why?

The Erdős–Gallai Theorem

Thm. [Erdős–Gallai 1960]

Let $\pi = (d_1 \geq d_2 \geq \ldots \geq d_n)$ be a nonneg int list with even sum.

Then π is graphic if and only if

$$\text{for all } 1 \leq k \leq n, \quad \sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k).$$

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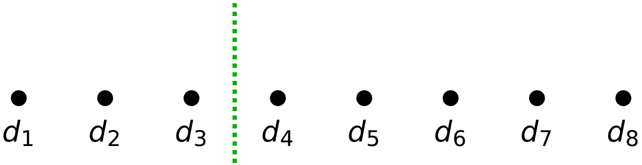

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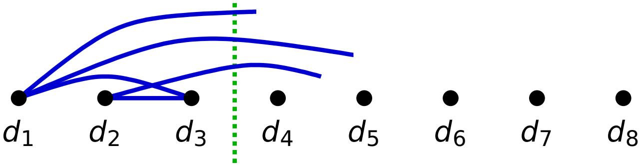

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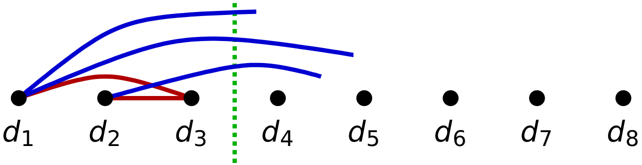

At most $k(k-1)$ can be from edges among the highest k vertices.

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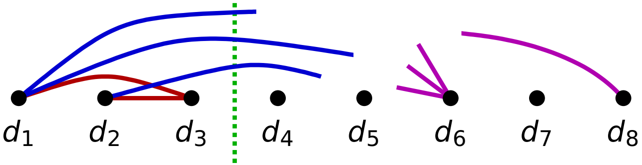

Each remaining vertex can absorb at most k or its degree.

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So π is not graphic!

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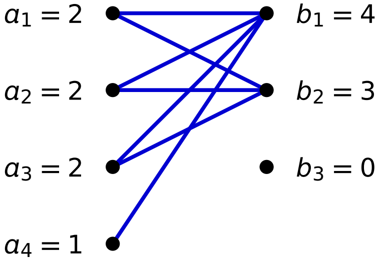

$\tau = (2, 2, 2, 1; 4, 3, 0)$

Note that the a_is are the degrees in one part, and the b_js are the degrees in the other part.

Bigraphic Sequences

A necessary condition:

If a bigraph has degrees $\tau = (a_1 \geq \ldots \geq a_n; b_1 \geq \ldots \geq b_m)$,

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Thm. [Gale 1957, Ryser 1957]

Let $\tau = (a_1 \geq \ldots \geq a_n; b_1 \geq \ldots \geq b_m)$ be a bilist with $\sum a_i = \sum b_j$.

Then τ is bigraphic if and only if

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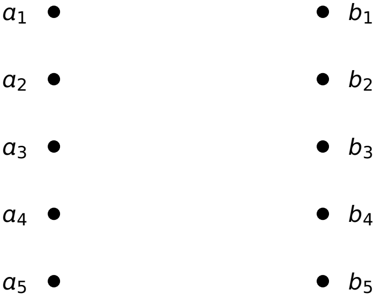

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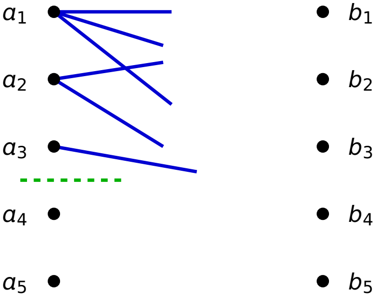

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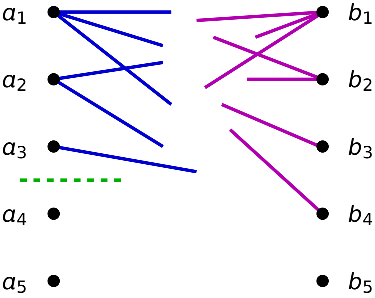

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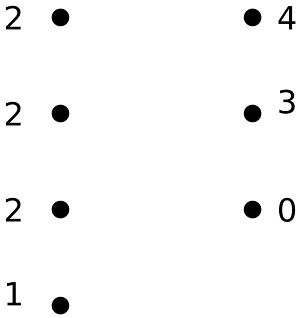
Each vertex in B can absorb at most k or its degree.

The Gale–Ryser Theorem

Many proofs of sufficiency—my favorite uses network flows.



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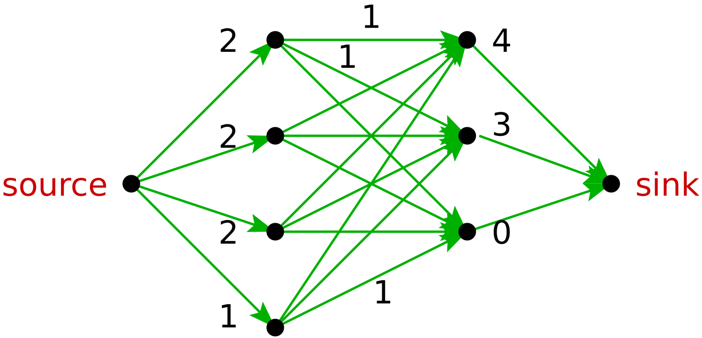

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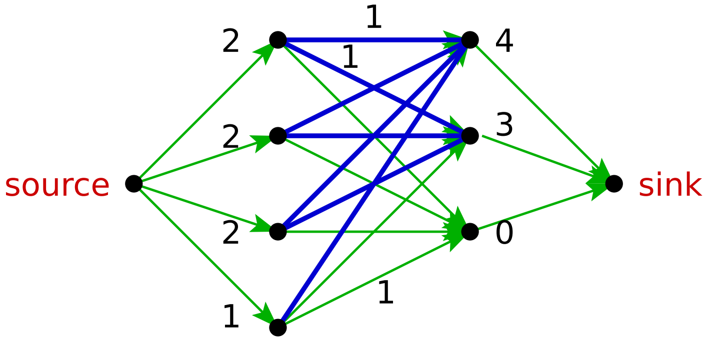

The middle edges with flow 1 realize the bipartite graph.

The Gale–Ryser Theorem

Now apply Max Flow–Min Cut Theorem to the previous network.

Thm. [Ford–Fulkerson 1956]

The maximum flow from the source to the sink is equal
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The checking of the Gale–Ryser conditions is straightforward,
but involves some cases.

Nonbipartite?

How to relate the nonbipartite world to bipartite land?

Durfee square

Def. Given a partition $d_1 \geq d_2 \geq \ldots \geq d_n$, the Durfee square number m is the largest i such that $d_i \geq i$.

$d_1 = 4$ ● ● ● ●

$d_2 = 2$ ● ●

$d_3 = 2$ ● ●

$d_4 = 1$ ●

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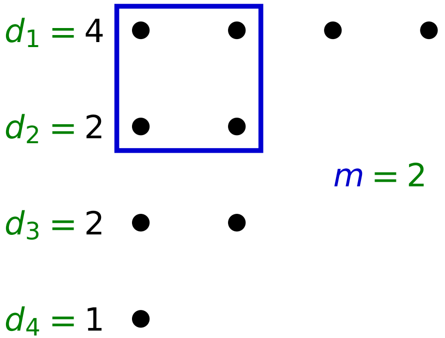

Ryser Condition

Thm. [Ryser 1957?]

Let $\pi = (d_1 \geq d_2 \geq \ldots \geq d_n)$ be a list with Durfee square m.

Let $\tau = (\tilde{d}_1, \ldots, \tilde{d}_n; \tilde{d}_1, \ldots, \tilde{d}_n)$ where $\tilde{d}_i = \begin{cases} d_i + 1 & \text{if } d_i \leq m, \\ d_i & \text{if } d_i > m. \end{cases}$

Then π is graphic if and only if τ is bigraphic.

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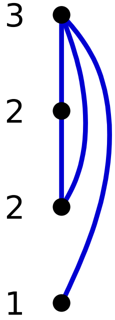

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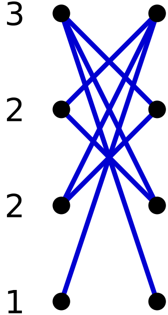

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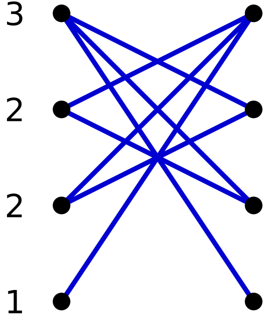

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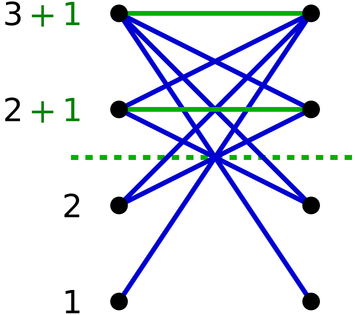

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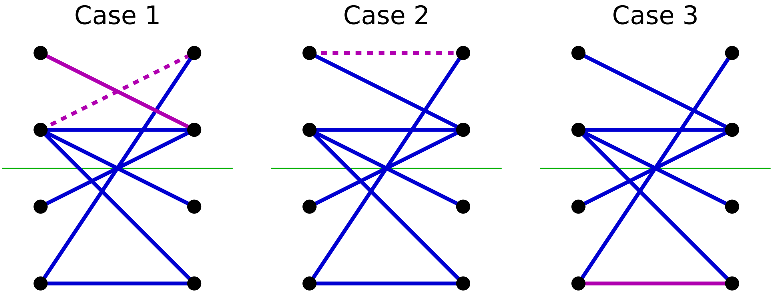

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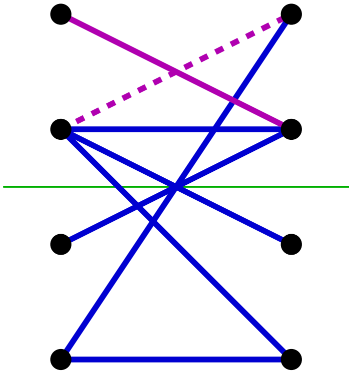

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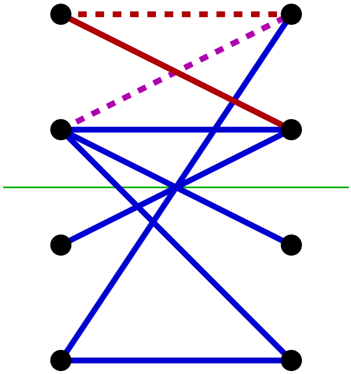

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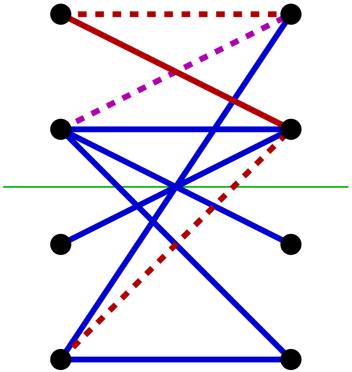

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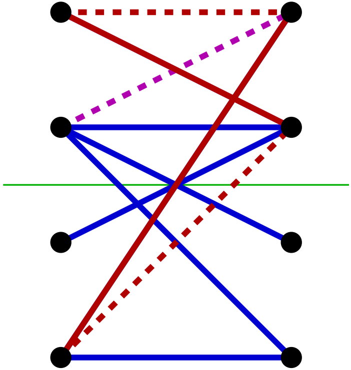

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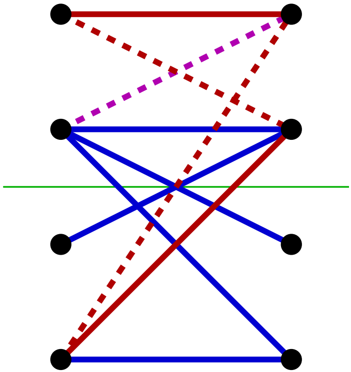

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Proof that τ bigraphic implies π is graphic:

Case 1

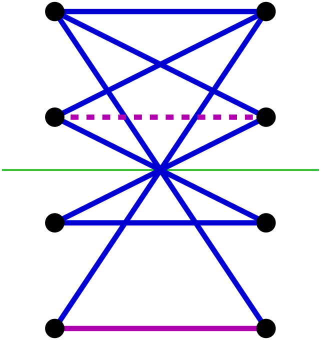
All the instances of Case 1 can be handled similarly.

All instances of Case 1 are handled, and then Cases 2 and 3.

Ryser Condition

Cases 2 and 3: all instances of Case 1 handled.

By parity, there are an even number of bad edges.



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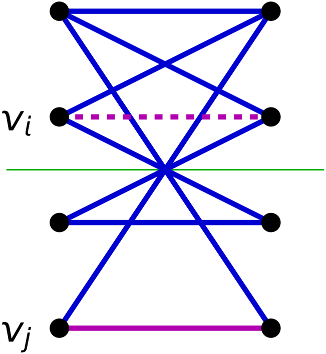

Case A: $i \leq m$ and $j > m$

We can find v_k so that
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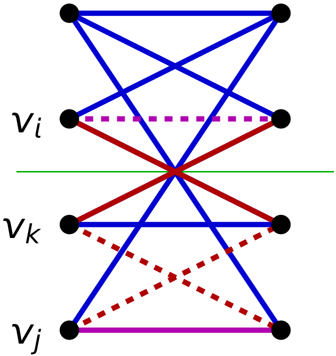

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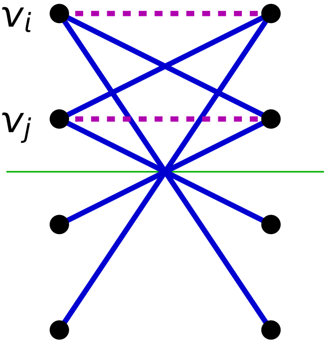

Case B: $i, j \leq m$

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Now proceed with Case A.

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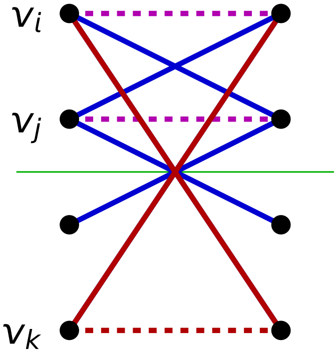

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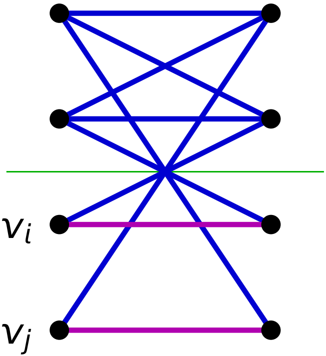

Case C: $i, j > m$

We can find $k < m$

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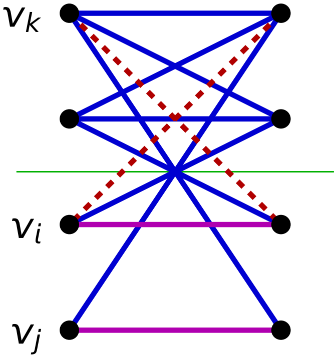
Cases 2 and 3: all instances of Case 1 handled.

By parity, there are an even number of bad edges.



Case C: $i, j > m$

We can find $k < m$
so that $v_i \not\leftrightarrow v_k$.

Now proceed with Case A.

Strengthened Erdős–Gallai

Thm. [Zverovich–Zverovich 1992]

It is sufficient to check the first m EG inequalities,
where m is the Durfee square number.

That is,

Let $\pi = (d_1 \geq d_2 \geq \ldots \geq d_n)$ be a nonneg int list with even sum.

Then π is graphic if and only if

$$\text{for all } 1 \leq k \leq m, \quad \sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k).$$

Key point: for $k > m$, we have $\min(d_i, k) = d_i$.

Equivalent Gale–Ryser

Lem. Let $\tau = (\tilde{d}_1, \ldots, \tilde{d}_n; \tilde{d}_1, \ldots, \tilde{d}_n)$. Then

$$\sum_{i=1}^{k} \tilde{d}_i \leq \sum_{i=1}^{n} \min(\tilde{d}_i, k) \text{ iff } \sum_{i=1}^{k} \max(k, \tilde{d}_i) \leq k^2 + \sum_{i=k+1}^{n} \min(\tilde{d}_i, k).$$

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Proof.

Subtract first k terms from right side:

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Strengthened Gale–Ryser Inequalities

Lem.

It is sufficient to check the first m GR inequalities for τ,
where m is the Durfee square number for π.

That is,
Let $\tau = (\tilde{d}_1, \ldots, \tilde{d}_n; \tilde{d}_1, \ldots, \tilde{d}_n)$.
Then τ is bigraphic if and only if

$$\text{for all } 1 \leq k \leq m, \quad \sum_{i=1}^{k} \tilde{d}_i \leq \sum_{i=k+1}^{n} \min(\tilde{d}_i, k).$$

This is more complicated than for the ER inequalities.

Strengthened Gale–Ryser

Lem. Let $\tau = (\tilde{d}_1, \ldots, \tilde{d}_n; \tilde{d}_1, \ldots, \tilde{d}_n)$.

Then τ bigraphic iff $\forall\ 1 \leq k \leq m$, $\displaystyle\sum_{i=1}^{k} \tilde{d}_i \leq \sum_{i=k+1}^{n} \min(\tilde{d}_i, k)$.

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&= k^2 - k\ell + \sum_{i=1}^{k} \min(\tilde{d}_i, \ell) + \sum_{i=k+1}^{n} \min(\tilde{d}_i, \ell)
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&\le k^2 + \sum_{i=k+1}^{n} \min(\tilde{d}_i, \ell) \le k^2 + \sum_{i=k+1}^{n} \min(\tilde{d}_i, k) \qquad \blacksquare
\end{aligned}$$

Proving Erdős–Gallai

Sufficiency: We need to verify the first m GR ineq for τ.

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&\leq \sum_{i=1}^{k} k + \sum_{i=k+1}^{n} \min(d_i, k) \\
&\leq \sum_{i=1}^{n} \min(d_i, k) \leq \sum_{i=1}^{n} \min(\tilde{d}_i, k).
\end{aligned}$$

This completes the proof of the Erdős–Gallai Theorem! $\blacksquare$

A Proof of the Erdős–Gallai Theorem

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Joint work with Tyler Seacrest