A Project Report on

A BRIEF STUDY ON WAVELET THEORY AND ITS APPLICATIONS

Submitted by

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CERTIFICATE

This is to certify that this dissertation entitled, **"A Brief Note on Wavelet Theory and its Applications"** submitted by **Ms. Pushpa B.,** in partial fulfilment of the requirements for the Degree of **Master of Science in Mathematics** to Bangalore University is based on the work carried out under my guidance and supervision. It is also certified that this dissertation or any part thereof has not been submitted elsewhere for any other degree.

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Pushpa B.

DECLARATION

I, Pushpa B., student of IV Semester M. Sc Mathematics hereby declare that the project work entitled **"A Brief Note on Wavelet Theory and its Applications"** submitted to the Bangalore University during the academic year 2021-22 under the supervision of **Mr. Nagarjun V.,** Assistant Professor, Department of Mathematics, The Oxford College of Science is a bonafide work done by me. This project work is submitted in partial fulfilment of the requirements for the award of the degree **"Master of Science"** in Mathematics. We further declare that the results of this work have not been submitted previously to this or any other university or institution for the award of any degree.

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TABLE OF CONTENTS

PREFACE

This project is organized as follows. The Fourier transform will be shortly addressed in Chapter 2. The chapter discusses the continuous, discrete, fast and short time Fourier transforms. From the short time Fourier transform the link to the continuous wavelet transform will be made in Chapter 3. The wavelet functions and the continuous wavelet analysis method will be explained together with a discretized version of the continuous wavelet transform. The true discrete wavelet transform uses filter banks for the analysis and reconstruction of the time signal. Filter banks and the discrete wavelet transform are the subject of Chapter 4. Wavelet analysis can be applied for many different purposes. It is not possible to mention all different applications, the most important application fields will be presented in Chapter 5. Finally conclusions will be drawn in Chapter 6.

CHAPTER 1

INTRODUCTION AND BASIC CONCEPTS

1.1. Historical Overview

Most signals are represented in the time domain. More information and concepts about the time signals can be obtained by applying signal analysis, i. e., the time signals are transformed using an analysis function. The Fourier transform is the most commonly known method to analyze a time signal for its frequency content. A new analysis method is the wavelet analysis. The wavelet analysis differs from the Fourier analysis by using short wavelets instead of long waves for the analysis function. The wavelet analysis has some major advantages over Fourier transform which makes it an interesting alternative for many applications. The use and fields of application of wavelet analysis have grown rapidly in the last years. In 1807, Joseph Fourier developed a method for representing a signal with a series of coefficients based on an analysis function. He laid the mathematical basis from which the wavelet theory is developed. The first to mention wavelets was Alfred Haar in 1909 in his Ph. D. thesis. In the 1930s, Paul Levy found the scale-varying Haar basis function superior to Fourier basis functions. The transformation method of decomposing a signal into wavelet coefficients and reconstructing the original signal again is derived in 1981 by Jean Morlet and Alex Grossman. In 1986, Stephane Mallat and Yves Meyer developed a multiresolution analysis using wavelets. They mentioned the scaling function of wavelets for the first time, it allowed researchers and mathematicians to construct their own family of wavelets using the derived criteria. Around 1998, Ingrid Daubechies used the theory of multiresolution wavelet analysis to construct her own family of wavelets. Haar set of wavelet orthonormal basis functions have become the cornerstone of wavelet applications today. With her work the theoretical treatment of wavelet analysis is as much as covered.

1.2. Objective

The Fourier transform (FT) only retrieves the global frequency content of a signal. Hence, the Fourier transform (FT) is only useful for stationary and pseudo-stationary signals. The Fourier transform (FT) doesn't give satisfactory results for signals that are highly non-stationary, noisy, a-periodic etc. These types of signals can be analyzed using local analysis methods. These methods include the short time Fourier transform and the wavelet analysis. All analysis methods are based on the principle of computing the correlation between the signal and an analysis function. Since the wavelet transform is a new technique, the principles and analysis methods are not widely known. This report presents an overview of the theory and applications of the wavelet transform. It is invoked by the following problem definition: Perform a literature study to gain more insight in the wavelet analysis and its properties and give an overview of the fields of application.

1.3. Approach

The Fourier transform (FT) is probably the most widely used signal analysis method. Understanding the Fourier transform is necessary to understand the wavelet transform. The transition from the Fourier transform to the wavelet transform is best explained through the short time Fourier transform (STFT). The STFT calculates the Fourier transform of a windowed part of the signal and shifts the window over the signal. Wavelet analysis can be performed in several ways, a continuous wavelet transform, a discretized continuous wavelet transform and a true discrete wavelet transform. The application of wavelet analysis becomes more widely spread as the analysis technique becomes more generally known. The fields of application vary from science, engineering, medicine to finance. This report gives an introduction into wavelet analysis. The basics of the wavelet theory are treated, making it easier to understand the available literature. More detailed information about wavelet analysis can be obtained using the references mentioned in this report. The applications described are thought to be of most interest to mechanical engineering. The various analysis methods presented in this report will be compared using the time signal $x(t)$, shown in Fig.1. From 0.1 s up to 0.3 s the signal consists of a sine with a frequency of 45 Hz at 0.2 s the signal has a pulse. At 0.4 s the signal shows a sinusoid with a frequency of 250 Hz which changes to 75 Hz at 0.5 s. The time interval from 0.7 s up to 0.9 s consists of two superposed sinusoids with frequencies of 30 Hz and 110 Hz. The signal is sampled at a frequency of 1 kHz.

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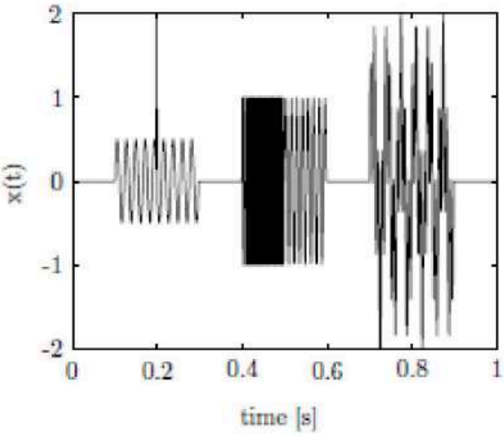

FIGURE 1

CHAPTER 2
FOURIER ANALYSIS

2.1. Fourier Analysis

The Fourier transform (FT) is probably the most widely used signal analysis method. In 1807, a French mathematician Joseph Fourier discovered that a periodic function can be represented by an infinite sum of complex exponentials. Many years later his idea was extended to non-periodic functions and then to discrete time signals. In 1965 the FT became even more popular by the development of the Fast Fourier transform (FFT). The Fourier transform retrieves the global information of the frequency content of a signal and will be discussed in Section 2.2. A computationally more effective method is the fast Fourier transform (FFT) which is the subject of Section 2.3. For stationary and pseudo-stationary signals the Fourier transform gives a good description. However, for highly non-stationary signals some limitations occur. These limitations are overcome by the short time Fourier transform (STFT), presented in Section 2.4. The STFT is a time frequency analysis method which is able to reveil the local frequency information of a signal.

2.2. Fourier Transform

The Fourier transform decomposes a signal into orthogonal trigonometric basis functions. The FT of a continuous signal $x(t)$ is defined in (1). The Fourier transformed signal $X_{FT}(f)$ gives the global frequency distribution of the time signal $x(t)$ [8], [16]. The original signal can be reconstructed using the inverse Fourier transform (2).

(1) $$X_{FT}(f) = \int_{-\infty}^{\infty} x(t)\ e^{-j2\pi ft}\ dt.$$

(2) $$x(t) = \int_{-\infty}^{\infty} X_F T(f)\ e^{j2\pi ft}\ df.$$

Using these equations, a signal $x(t)$ can be transformed into the frequency domain and back again. The Fourier transform and reconstruction are possible if the below mentioned following Dirichlet conditions are fulfilled [8], [15].

(1) The integral $\int_{-\infty}^{\infty} |x(t)| \, dt$ must exist i. e., the Fourier transform $X_{FT} \longrightarrow 0$ as $|f| \longrightarrow \infty$.

(2) The time signal $x(t)$ and its Fourier transform $X_{FT}(f)$ are single-valued, i. e., no two values occur at equal time instant t or frequency f.

(3) The time signal $x(t)$ and its Fourier transform $X_{FT}(f)$ are piece-wise continuous. piece-wise continuous functions must have a value at the point of the discontinuity which equals the mean of the surrounding points. Furthermore of the discontinuity must be of finite size and the number of discontinuities must not increase without limit in a finite time interval [16].

(4) A sufficient, but not necessary condition is that the functions $x(t)$ and $X_{FT}(f)$ have upper and lower bounds. The Dirac delta function for example disobeys this condition.

Many signals, especially periodic signals, do not fulfill the Dirichlet conditions, so the continuous Fourier transform of (1) cannot be applied. Most experimentally obtained signals are not continuous in time, but sampled as discrete time intervals ΔT. Furthermore they are of finite length with a total measurement time T, divided into $N = \frac{T}{\Delta T}$ intervals. These kind of signals can be analyzed in the frequency domain by using the discrete Fourier transform (DFT), defined in (3). Due to the sampling of the signal, the frequency spectrum becomes periodic, so the frequencies that can be analyzed are finite [24]. The DFT is evaluated at discrete frequencies $f_n = \frac{n}{T}$, $n = 0, 1, 2, \ldots, N-1$. The inverse DFT reconstructs the original discrete time signal and is given in (4).

$$X_{DFT}(f_n) = \frac{1}{N} \sum_{K=0}^{N-1} x(k) \; e^{-j2\pi k \Delta T}. \tag{3}$$

$$x(k) = \frac{1}{\Delta T} \sum_{f_n=0}^{\frac{N-1}{T}} X_{DFT}(f_n) \; e^{j2\pi f_n k \Delta T}. \tag{4}$$

2.3. Fast Fourier Transform

The calculation of the DFT can become very time-consuming for large signals (large N). The fast Fourier transform (FFT) algorithm does not take an arbitrary number of intervals N, but only the intervals $N = 2^m$, $m \in N$. The reduction in the number of intervals makes the FFT very fast, as the name implies. A drawback compared to the ordinary DFT is that the signal must have 2^m samples, this is however in general no problem. In practice the calculation of the FFT can suffer from two problems. First since only a small part of the signal $x(t)$ on the interval $0 \leqslant t \leqslant T$ is used, leakage can occur. Leakage is caused by the discontinuities introduced by periodically extending the signal. Leakage causes energy of fundamental frequencies to leak out to neighbouring frequencies. A solution to prevent signal leakage is by applying a window to the signal which makes the signal more periodic in the time interval. A disadvantage is that the window itself has a contribution in the frequency spectrum. The second problem is the limited number of discrete signal values, this can lead to aliasing. Aliasing causes fundamental frequencies to appear as different frequencies in the frequency spectrum and is closely related to the sampling rate of the original signal. Aliasing can be prevented if the sampling theorem of Shannon is fulfilled. The theorem of Shannon states that no information is lost by the discretization if the sample time ΔT equals or is smaller than $\Delta T = \frac{2}{f_{max}}$. For more detailed information regarding both problems the reader is referred to [8], [16]. The FFT transform of the time signal $x(t)$ of Fig. 1 is shown in Fig. 2. The figures shows five major peaks at 30, 45, 75, 110 and 250 Hz. The noise in between the peaks indicates the existence of other frequencies in the time signal. These frequencies are present because of the sudden changes in the time signal and the pulse at 0.4 s. The FFT of Fig. 2 shows peaks at the correct frequencies, however the time structure of the original signal cannot be seen in the figure. The FFT is not very useful for analyzing non-stationary signals since it does not describe the frequency content of a signal at specific times.

2.4. Short Time Fourier Transform

The limitation of the Fourier transform, i. e., it gives only the global frequency content of a signal, is overcome by the short time Fourier transform (STFT). The STFT is able to retrieve both frequency and time information from a signal. The STFT calculates the Fourier transform of a windowed part of the original signal, where the window shifts along the time axis. In other words, a signal $x(t)$ is windowed by a window $g(t)$ of limited extend, centered at time T. Of the windowed signal a FT is taken, giving the frequency content of the signal in the windowed time interval. The STFT is defined in (2). Here,

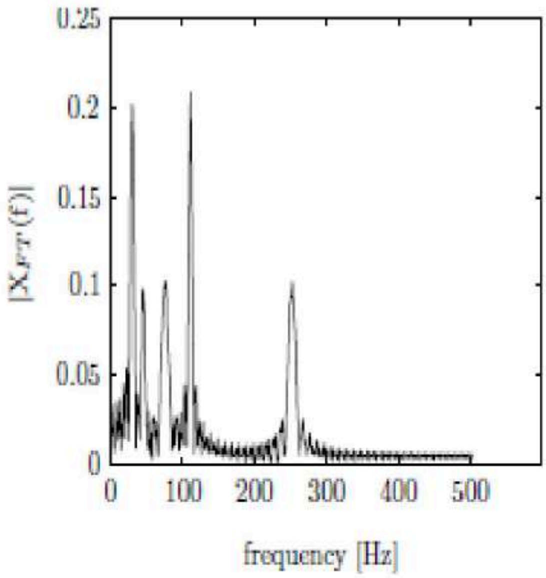

FIGURE 2

the $*$ denotes the complex conjugated [24], [22]. It can be seen that the STFT is nothing more than the FT of the signal $x(t)$ multiplied by a window $g(t)$.

$$X_{STFT}(\tau, f) = \int_{-\infty}^{\infty} x(t)g^*(t-\tau)e^{-j2\pi ft}\,dt. \tag{5}$$

The performance of the STFT analysis depends critically on the chosen window $g(t)$ [22]. A short window gives a good time resolution, but different frequencies are not identified very well. This can be seen in Fig. 3 for a window length of 0.03 s. A long window gives an inferior time resolution, but a better frequency resolution, as shown in Fig.3 for a window length of 0.6 s. It is not possible to get both a good time resolution and a good frequency resolution. This is known as the Heisenberg inequality [22]. The Heisenberg inequality states that the product of time resolution Δt and frequency resolution Δf (bandwidth-time product) is constant, i. e., the time-frequency plane is divided into blocks with an equal area. The bandwidth-time product is lower bounded (minimum block size) as

$$\Delta t\ \Delta f \geqslant \frac{1}{4\pi}. \tag{6}$$

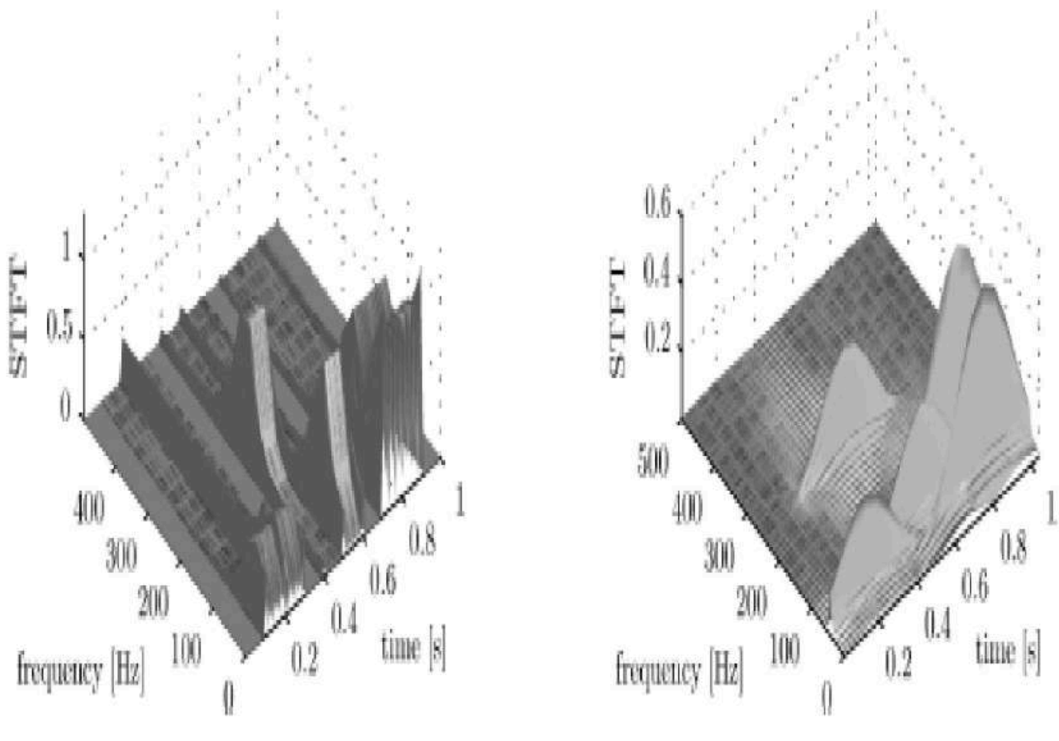

FIGURE 3

A compromise between the time and frequency resolution is shown in Fig.3 for a window length of 0.15 s. With this window length the STFT shows both a reasonable time and frequency resolution. Note that the sharp peak at 0.2 s cannot be distinguished clearly. The example shows that finding a proper window length is critical for the quality of the STFT. The STFT uses a fixed window length, so Δt and Δf are constant. With a constant Δt and Δf T the time-frequency plane is divided into blocks of equal size as shown

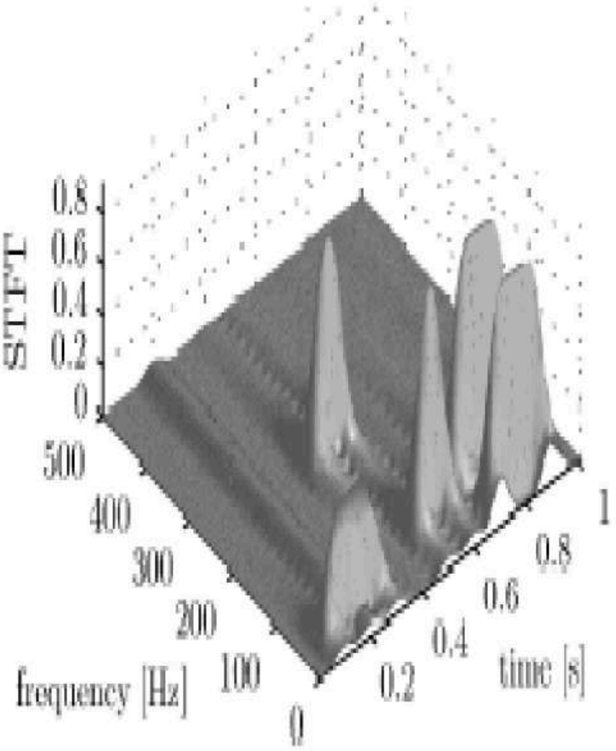

FIGURE 4

in Fig. 4. This resolution is not satisfactory. Low frequency components often last a long period of time, so a high frequency resolution is required. High frequency components often appear as short bursts, invoking the need for a higher time resolution. The basic

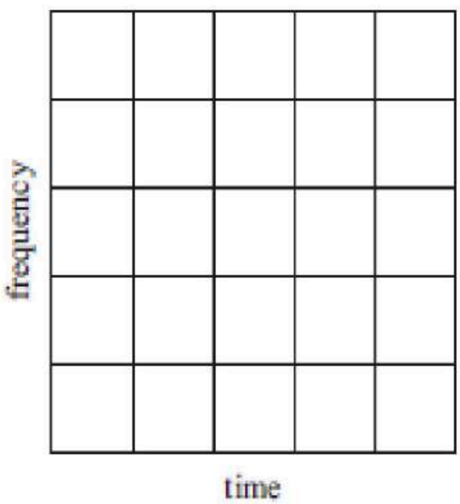

FIGURE 5

differences between the Wavelet Transform (WT) and the STFT are first, the window width can be changed in the WT as a function of the analyzing frequency. Next, the analysis function of the WT can be chosen with more freedom. The wavelet transform is the subject of the next chapter.

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CHAPTER 3
WAVELET ANALYSIS

The analysis of a non-stationary signal using the Fourier Transform or the STFT does not give satisfactory results. Better results can be obtained using wavelet analysis. One of the main advantage of wavelet analysis is the ability to perform local analysis [17]. Wavelet analysis is able to reveal signal aspects that other analysis techniques miss such as trends, breakdown points, discontinuities, etc. In comparison to the STFT, wavelet analysis makes it possible to perform a multiresolution analysis. The general idea of multiresolution analysis will be discussed in Section Multiresolution analysis. The wavelet functions and their properties are the subject of Section Wavelets. The continuous wavelet transform (CWT) will be treated in Section Continious wavelet Transform together with the discretized version of the CWT.

3.1. Multiresolution Analysis

The time-frequency resolution problem is caused by the Heisenberg uncertainty principle and exists regardless of the used analysis technique. For the STFT, a fixed time-frequency resolution is used. By using an approach called multiresolution analysis (MRA) it is possible to analyze a signal at different frequencies with different resolutions. The change in resolution is schematically displayed in below figure. For the resolution of above figure

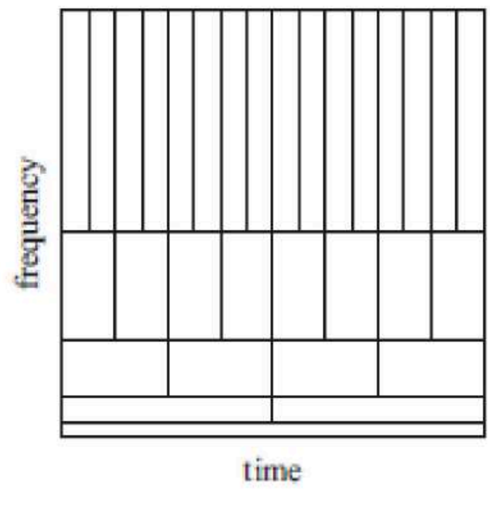


FIGURE 6

it is assumed that low frequencies last for the entire duration of the signal, whereas high frequencies appear from time to time as short burst. This is often the case in practical applications. The wavelet analysis calculates the correlation between the signal under consideration and a wavelet function $\Psi(t)$. The similarity between the signal and the analyzing wavelet function is computed separately for different time intervals, resulting in a two dimensional representation. The analyzing wavelet function $\Psi(t)$ is also referred to as the mother wavelet.

3.2. Wavelets

In comparison to the Fourier transform, the analyzing function of wavelet transform can be chosen with more freedom, without the need of using sine-forms. A wavelet function $\Psi(t)$ is a small wave, which must be oscillatory in some way to differentiate between the different frequencies [24]. The wavelet contains both the analyzing shape and the window. This figure shows an example of a possible wavelet, known as the Morlet wavelet. For the CWT several kind of wavelet functions are developed which all have specific properties [24]. An analyzing function $\Psi(t)$ is classified as a wavelet if the following mathematical criteria are satisfied [1].

1. A wavelet must have finite energy

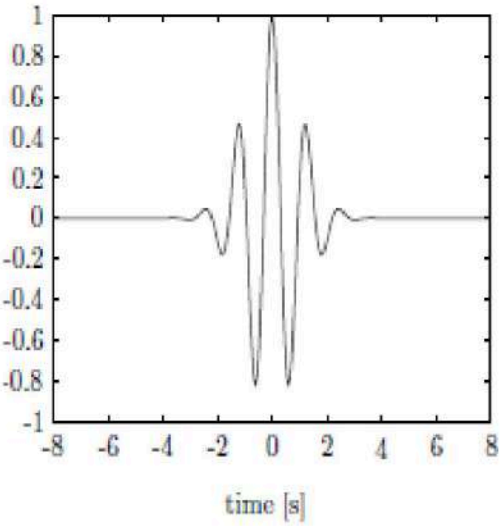


FIGURE 7

$$E = \int_{-\infty}^{\infty} |\psi(t)|^2 \, dt < \infty. \tag{7}$$

The energy E equals the integrated squared magnitude of the analyzing function $\Psi(t)$ and must be less than infinity.

2. If $\Psi(f)$ is the Fourier transform of the wavelet $\Psi(t)$, the following condition must hold

$$C_{\psi} = \int_{-\infty}^{\infty} \frac{|\psi(f)|^2}{f}\, df < \infty. \tag{8}$$

This condition implies that the wavelet has no zero frequency component ($\Psi(0) = 0$), i. e., the mean of the wavelet $\Psi(t)$ must equal zero. This condition is known as the admissibility constant. The value of C_{Ψ} depends on the chosen wavelet.

3. For complex wavelets the Fourier transform $\Psi(f)$ must be both real and vanish for negative frequencies.

3.3. Continuous Wavelet Transform

The continuous wavelet transform is defined as [1], [24], [19]

$$X_{WT}(\tau, s) = \frac{1}{\sqrt{|s|}} \int_{-\infty}^{\infty} x(t)\ \psi^{*}\left(\frac{t-r}{s}\right) dt. \tag{9}$$

The transformed signal $X_{WT(\tau,s)}$ is a function of the translation parameter τ and the scale parameter s. The mother wavelet is denoted by Ψ, the * indicates that the complex conjugate is used in case of a complex wavelet. The signal energy is normalized at every scale by dividing the wavelet coefficients by $\frac{1}{\sqrt{|s|}}$. This ensures that the wavelets have the same energy at every scale. The mother wavelet is contracted and dilated by changing the scale parameter s. The variation in scale s changes not only the central frequency f_c of the wavelet, but also the window length. Therefore the scale s is used instead of the frequency for representing the results of the wavelet analysis. The translation parameter τ specifies the location of the wavelet in time, by changing τ the wavelet can be shifted over the signal. For constant scale s and varying translation τ the rows of the time-scale plane are filled, varying the scale s and keeping the translation τ constant fills the columns of the time-scale plane. The elements in $X_{WT(\tau,s)}$ are called wavelet coefficients, each wavelet coefficient is associated to a scale (frequency) and a point in the time domain.

The WT also has an inverse transformation, as was the case for the FT and the STFT. The inverse continuous wavelet transformation (ICWT) is defined by

$$x(t) = \frac{1}{C_\psi^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_{WT}(\tau, s)\frac{1}{s^2}\psi\left(\frac{t-\tau}{s}\right)d\tau\ ds. \tag{10}$$

Note that the admissibility constant C_Ψ must satisfy the second wavelet condition. A wavelet function has its own central frequency f_c at each scale, the scale s is inversely proportional to that frequency. A large scale corresponds to a low frequency, giving global information of the signal. Small scales correspond to high frequencies, providing detail signal information. For the WT, the Heisenberg inequality still holds, the bandwidth time product $\Delta t\ \Delta f$ is constant and lower bounded. Decreasing the scale s, i. e. a shorter window, will increase the time resolution Δt, resulting in a decreasing frequency resolution Δf. This implies that the frequency resolution Δf is proportional to the frequency f, i. e., wavelet analysis has a constant relative frequency resolution [24]. The Morlet wavelet is obtained using a Gaussian window, where f_c is the center frequency and f_b is the bandwidth parameter.

$$\psi(t) = g(t)e^{-j2\pi f_c t}, \quad g(t)\sqrt{\pi f_b}e^{t^2/f_b}. \tag{11}$$

The center frequency f_c and the bandwidth parameter f_b of the wavelet are the tuning parameters. For the Morlet wavelet, scale and frequency are coupled as

$$f = \frac{f_c}{s}. \tag{12}$$

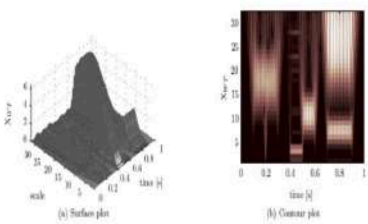
The calculation of the continuous wavelet transform is usually performed by taking the discrete values for the scaling parameter s and translation parameter τ. The resulting wavelet coefficients are called wavelet series. For analysis purposes only, the discretization can be done arbitrarily, however if reconstruction is required, the wavelet restrictions become important. The constant relative frequency resolution of the wavelet analysis is also known as the constant Q property. Q is the quality factor of the filter, defined as the center-frequency f_c divided by the bandwidth f_b [24]. For a constant Q analysis (constant relative frequency resolution), a dyadic sample-grid for the scaling seems suitable. A dyadic grid is also found in the human hearing and music. A dyadic grid discretizes the scale parameter on a logarithmic scale. The time parameter is discretized with respect to the scale parameter. The dyadic grid is one of the most simple and efficient discretization methods for practical purposes and leads to the construction of an orthonormal wavelet basis [1]. Wavelet series can be calculated as

$$X_{WT_{m,n}} = \int_{-\infty}^{\infty} x(t)\psi_{m,n}(t) \quad with \quad \psi_{m,n}(t) = s_0^{-m/2}\psi(s_0^{-m}t - n\tau_0). \tag{13}$$

The integers m and n control the wavelet dilatation and translation. For a dyadic grid, $s_0 = 0$ and $\tau_0 = 1$. Discrete dyadic grid wavelets are chosen to be orthonormal, i. e., they are orthogonal to each other and normalized to have unit energy [1]. This choice allows the reconstruction of the original signal by

$$x(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} X_{WT_{m,n}}\psi_{m,n}(t). \tag{14}$$

The discretized CWT of the signal of Fig. 1, analyzed with the Morlet wavelet, is shown in Fig. 3. Both a surface and a contour plot of the wavelet coefficients are shown. In literature, most of the time the contour plot is used for representing the results of a CWT. Note that in both figures large scales correspond to low frequencies and small scales to high frequencies. The CWT in figure gives a good frequency resolution for high frequencies (small scales) and a good time resolution for low frequencies (large scales). The different frequencies are detected at the correct time instants, the sharp peak at 0.2 s is detected well as can be seen by the existence of the peak at small scales at 0.2 s. The



(a) Surface plot (b) Contour plot

FIGURE 8

true discrete wavelet transform makes use of filter banks for the analysis and synthesis of the signal and will be discussed in the next chapter.

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CHAPTER 4
DISCRETE WAVELET TRANSFORMS

The CWT performs a multiresolution analysis by contraction and dilatation of the wavelet functions. The discrete wavelet transform (DWT) uses filter banks for the construction of the multiresolution time-frequency plane. Filter banks will be introduced in next Section. The DWT uses multiresolution filter banks and special wavelet filters for the analysis and reconstruction of signals. The DWT will be discussed.

4.1. Filter Banks

A filter bank consists of filters which separate a signal into frequency bands [27]. An example of a two channel filter bank is shown in Fig. 4. A discrete time signal $x(k)$ enters the analysis bank and is filtered by the filters $L(z)$ and $H(z)$ which separate the frequency content of the input signal in frequency bands of equal width. The filters $L(z)$ and $H(z)$ are therefore respectively a low-pass and a high-pass filter. The output of the filters each contain half the frequency content, but an equal amount of samples as the input signal. The two outputs together contain the same frequency content as the input signal, however the amount of data is doubled. Therefore downsampling by a factor two, denoted by $(\downarrow 2)$, is applied to the outputs of the filters in the analysis bank. Reconstruction of the original signal is possible using the synthesis filter bank [27], [24]. In the synthesis bank the signals are upsampled $(\uparrow 2)$ and passed through the filters $L'(z)$ and $H'(z)$. The filters in the synthesis bank are based on the filters in the analysis bank. The outputs of the filters in the synthesis bank are summed, leading to the reconstructed signal $y(k)$. The different output signals of the analysis filter bank are called subbands, the filter-bank technique is also called subband coding [24].

4.2. Down and Upsampling

The low-pass and high-pass filters $L(z)$ and $H(z)$ split the frequency content of the signal in half. It therefore seems logical to perform a down sampling with a factor two to avoid redundancy. If half of the samples of the filtered signals $cl(k)$ and $ch(k)$ are reduced, it is still possible to reconstruct the signal $x(k)$ [27]. The downsampling operation $(\downarrow 2)$

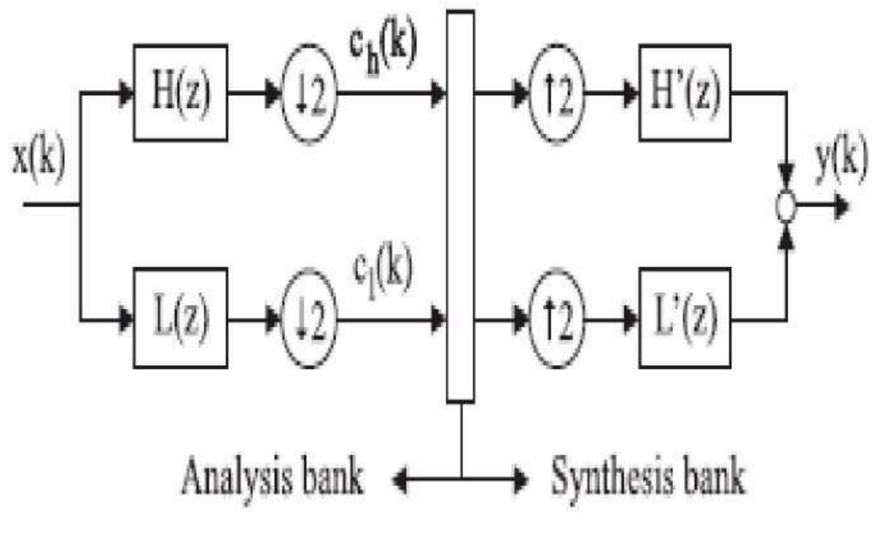

FIGURE 9

saves only the even-numbered components of the filter output, hence it is not invertible. In the frequency domain, the effect of discarding information is called aliasing. If the Shannon sampling theorem is met, no loss of information occurs [8], [27]. The sampling theorem of Shannon states that downsampling a sampled signal by a factor M produces a signal whose spectrum can be calculated by partitioning the original spectrum into M equal bands and summing these bands [24]. In the synthesis bank the signals are first upsampled before filtering. The upsampling by a factor two $(\uparrow 2)$ is performed by adding zeros in between the samples of the original signal. Note that first downsampling a signal and than upsampling it again will not return the original signal.

$$x = \begin{vmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \end{vmatrix} \quad (\downarrow 2)x = \begin{vmatrix} x(0) \\ x(2) \\ x(4) \end{vmatrix} \quad (\uparrow 2)(\downarrow 2)x = \begin{vmatrix} x(0) \\ 0 \\ x(2) \\ 0 \\ x(4) \end{vmatrix} \tag{15}$$

The transpose of $(\downarrow 2)$ is $(\uparrow 2)$. Since transposes come in reverse order, synthesis can be preformed as the transpose of the analysis. Furthermore $(\downarrow 2)(\uparrow 2) = I$, since $(\uparrow 2)$ is the right-inverse of ($\downarrow 2$) [27]. This indicates that it is possible to obtain the original signal again with up- and downsampling. By first inserting zeros and then removing them, the original signal is obtained again.

4.3. Perfect Reconstruction

For perfect reconstruction to be possible, the filter bank should be biorthogonal. Now, furthermore some design criteria for both the analysis and synthesis filters should be met to prevent aliasing and distortion and to guarantee a perfect reconstruction [27]. In the two channel filter bank of Fig. 4, the filters $L(z)$ and $H(z)$ split the signal into two frequency bands, i. e., the filters are respectively a low-pass and a high-pass filter. If the filters were perfect brick-wall filters, the downsampling would not lead to loss of information. However ideal filters cannot be realized in practice, so a transition band exists. Besides aliasing, this leads to an amplitude and phase distortion in each of the channels of the filter band [24]. For the two channel filter bank of aliasing can be prevented by designing the filters of the synthesis filter bank as [27].

$$L^{'}(z) = H(-z). \tag{16}$$

$$H^{'}(z) = -L(-z). \tag{17}$$

To eliminate distortion, a product filter $P_O(z) = L^{'}(z)L(z)$ is defined. Distortion can be avoided if

$$P_0(z) - P_0(-z) = 2z^{-N}. \tag{18}$$

where N is the overall delay in the filter bank. Generally an N^{th} order filter produces a delay of N samples [24]. The perfect reconstruction filter bank can be designed in two steps [27]:

1. Design a low-pass filter PO satisfying (18).
2. Factor PO into $L^{'}(z)L(z)$ and use (16) and (17) to calculate$H(z)$ and $H^{'}(z)$.

The design of the product filter PO of the first step and the factorization of the second step can be done in several ways. More information about the wavelet filter design can be found in [27].

4.4. Multiresolution Filter Banks

The CWT of Chapter 3 performs a multiresolution analysis which makes it possible to analyze a signal at different frequencies with different resolutions. For high frequencies (low scales), which last a short period of time, a good time resolution is desired. For low frequencies (high scales) a good frequency resolution is more important. The CWT has a time-frequency resolution as shown in Fig. 3. This multiresolution can also be obtained using filter banks, resulting in the discrete wavelet transform (DWT). Note that the discretized version of the CWT is not equal to the DWT, the DWT uses filter banks, whereas the discretized CWT uses discretized versions of the scale and dilatation axes. The low-pass and high-pass filtering branches of the filter bank retrieve respectively the approximations and details of the signal $x(k)$. In Fig. 4, a three level filter bank is shown. The filter bank can be expanded to an arbitrary level, depending on the desired resolution. The coefficients $c_l(k)$ (see Fig. 4) represent the lowest half of the frequencies in $x(k)$, downsampling doubles the frequency resolution. The time resolution is halved, i. e., only half the number of samples are present in $c_l(k)$. In the second level, the outputs of $L(z)$ and $H(z)$ double the time resolution and decrease the frequency content, i. e., the width of the window is increased. After each level, the output of the high-pass filter represents the highest half of the frequency content of the low-pass filter of the previous level, this leads to a pass-band. The time-frequency resolution of the analysis bank of Fig. 4 is similar to the resolution shown in Fig. 3 For a special set of filters $L(z)$ and $H(z)$ this structure is called the DWT, the filters are called wavelet filters.

4.5. Wavelet Filters

The relationship between the CWT and the DWT is not very obvious. The wavelets in the CWT have a center frequency and act as a band-pass filter in the convolution of

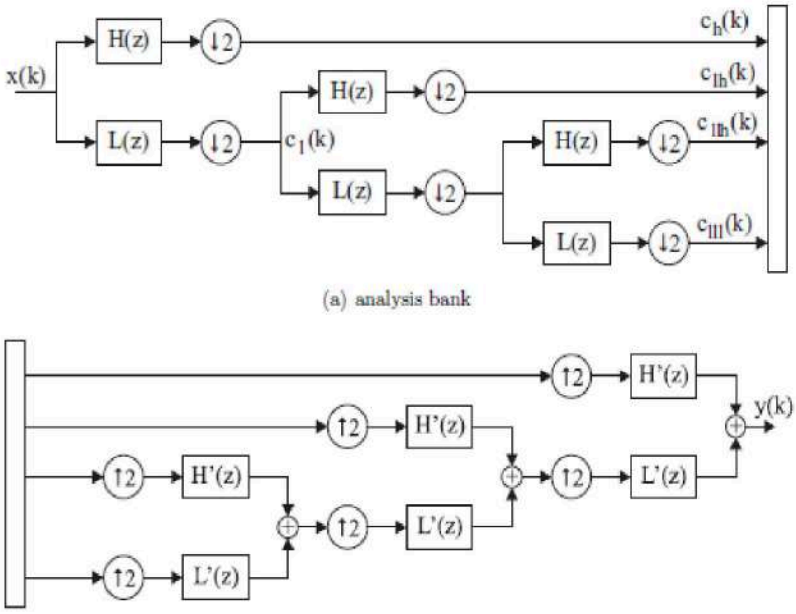

FIGURE 10

the wavelet function with the signal $x(t)$. The sequence of low-pass filter, downsampling and high-pass filter also acts as a band-pass filter. In order to facilitate the comparison between the DWT and CWT, the filter bank of Fig. 4 is rewritten to Fig. 3 [24]. An increase in downsampling rate leads to a larger time grid for lower frequencies (higher scales). The filters can be interpreted as the wavelet functions at different scales. However they are not exact scaled versions of each other, if the number of levels is increased and the impulse responses of the equivalent filters converge to a stable waveform, the filters $L(z)$ and $H(z)$ are wavelet filters [24], [6]. The subsequent filters then become scaled versions of each other. The wavelet filters represent the frequency content of a wavelet function at a specific scale. The wavelet filters can be classified into two classes, orthogonal and biorthogonal wavelets. Several wavelet families, designed for the DWT are discussed with their properties. The limit wavelet functions, i. e., the stable waveforms, can be constructed the easiest from the synthesis bank. For the lower branch of Fig. 4, consisting of only low-pass filters and upsampling, the impulse response converges to a final function $l(n)$ for which the following difference equation holds [1], [24]

$$\phi(t) = \sum_{n=0}^{N} l(n)\phi(2t-n). \tag{19}$$

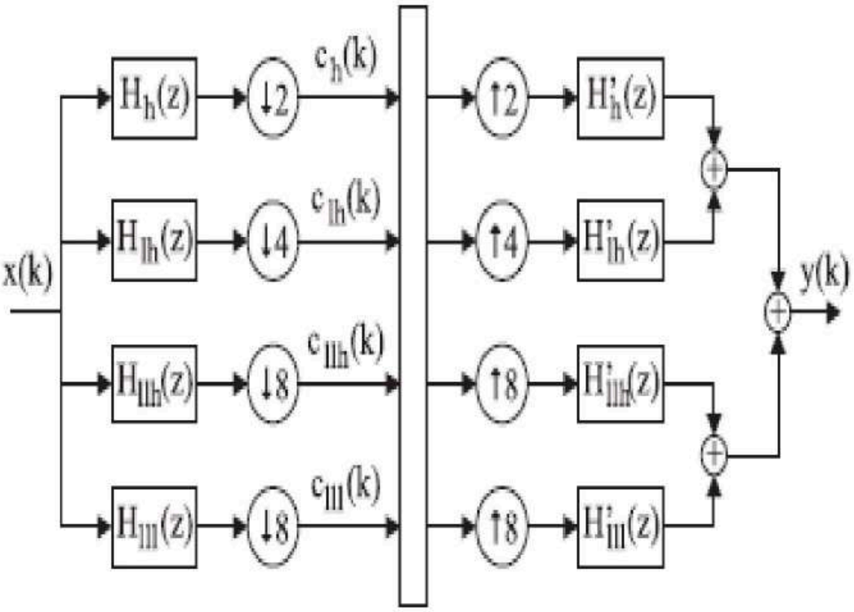

FIGURE 11

This function is known as the scaling function of the wavelet. For the band-pass sequences the impulse responses converge holding the final difference equation which can be calculated as

$$f_t = \frac{f_s}{2^{p+1}} \tag{20}$$

The final equation of the band-pass sequences $h(n)$ is the wavelet function ψ (t). The subband with wavelet coefficients c_{lll} is called the approximation subband cA and contains the lowest frequencies. The other subbands are called detail subbands cD and give the detail information of the signal. The wavelet coefficients represent the signal content in the various frequency bands.For a $p-level$ decomposition, the highest frequency observed in the approximation wavelet coefficients c_{lll} can be calculated as a function of the sample frequency f_s as

$$f_l = \frac{f_s}{2^{p+1}} \tag{21}$$

The frequency content of the approximation frequency band cA and detail frequency bands cD can be calculated as

$$f_{cA} = [0, 2^{p-1} f_s]. \tag{22}$$

$$f_{c_{D_p}} = [2^{-p-1} f_s \quad , \quad 2^{-p} f_s]. \tag{23}$$

The success of a certain decomposition depends strongly on the chosen wavelet filters, depending on the signal properties [24]. This is not the case with the STFT. Further more it is not possible to determine a mean value of a signal using the WT. The DWT of the signal of Fig. 1. with a three level filter bank and a db4 wavelet function is shown in Fig. 4. There exists a trade-off between the order of the wavelets and the computation time. Higher order wavelets are smoother and are better able to distinguish between the various frequencies, but require more computation time. The DWT of Fig. 4 shows the original signal in the top figure. The other figures contain the wavelet coefficients of the various sub bands. The frequencies ranges of the various sub bands are given in the below table.

subband	$f_{tow} Hz$	$f_{high} Hz$
C_{lll}	0	62.5
C_{llh}	62.5	125
C_{lh}	125	250
C_h	250	500

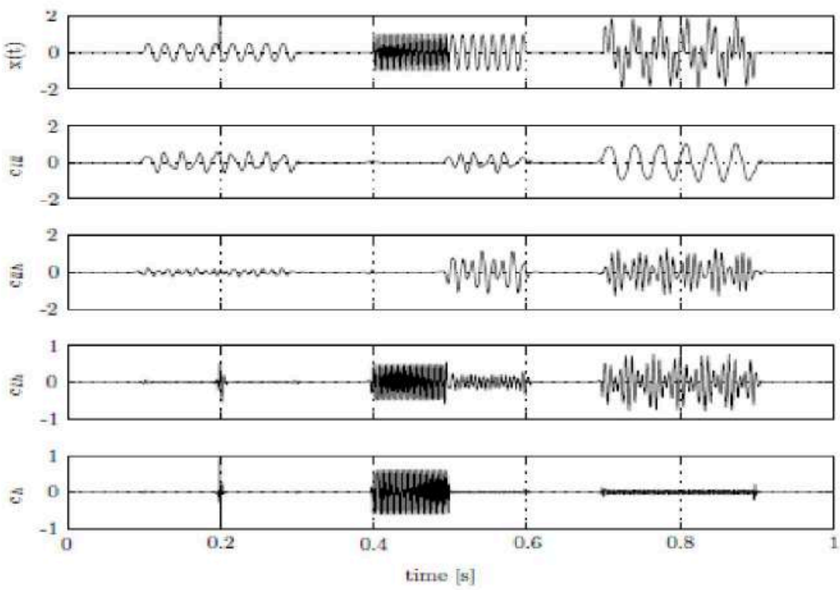

FIGURE 12

From figure, one can see that the frequency content of the signal $x(t)$ is decomposed to the correct frequency bands. The discontinuities at the start and beginning of the various parts are visible in all frequency bands. The sine with a frequency of 250 Hz is visible in both clh and ch since this frequency is located exactly at the edge of both bands. Signal components with specific frequencies appear also in surrounding subbands, however with lower amplitudes. This is because the low-pass and high-pass filters are not perfect brick wall filters. The order of the used wavelet also has an effect on this, a higher order wavelet will produce less undesired frequency content in the surrounding subbands. The peak at 0.2 s appears in the subbands with the highest frequencies. The wavelet coefficients in the different frequency bands of the DWT can be processed in several ways. By adjusting the wavelet coefficients the reconstructed signal of the synthesis filter bank can be changed in comparison to the original signal. This gives the DWT some attractive properties over linear filtering. Compared to the CWT, the DWT is easier to compute and the wavelet coefficients are easier to interpret since no conversion from scale to frequency has to be made.

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CHAPTER 5
APPLICATIONS

Wavelet analysis can be applied for many different purposes. The areas of application differ from science to medicine and finance. It is not possible to mention all different application fields. In this chapter, applications are presented which have some analogy with mechanical engineering and in particular dynamics and control, in which scope this report is written. The wavelet applications found in literature will be shortly summarized in this chapter, more detailed information of the various applications can be found in the corresponding references. The application fields include numerical analysis, signal analysis, control applications and audio applications.

5.1. Numerical Analysis

Wavelet analysis is a powerful tool for numerical analysis and can be very computational efficient. This section deals with the use of wavelets for solving the two types of differential equations, Ordinary Differential Equations (ODE's) and Partial Differential Euations (PDE's). Another useful numerical application for wavelets is the compression of one or two dimensional data, these subjects will be further addressed in respectively Section Audio compression and Section Image and Video compression.

5.2. Ordinary Differential Equations

The solution of ordinary differential equations using wavelets is presented in [22]. The ODEs considered are of the form

$$L_u(x) \;=\; f(x)\ for\ \ x \in [0,1]\ where\ \ L \;= \sum_{j=0}^{m} a_j(x)D^j, \tag{24}$$

with boundary conditions.

$$(B_0u)(0) \;=\; g_0\ and\ (B_1u)(1) \;=\; g_1, \tag{25}$$

where

$$B_i = \sum_{j=0}^{m} b_{i,j}D^j \tag{26}$$

For ODEs with coefficients $a_j(x)$ independent of x, the solution of the ODE can be found with use of the Fourier transform [22]. Wavelet transformation is a possible substitute for the Fourier transform and thus a possible tool for finding the solution of ODEs. The solution is found by first calculating the Fourier transform of the right-hand side of the ODE, then dividing the coefficients of each basis function by the corresponding eigenvalue and finally taking the inverse Fourier transform. The solution of ODEs can also be found using the DWT with compactly supported wavelet filters. The wavelet analysis replaces the Fourier transform. The resulting algorithm is faster than the algorithm which uses the Fourier transform because of the subsampling in the wavelet transform, especially in the coarser levels.

5.3. Partial Differential Equations

Partial Differential Equations (PDEs) can be used to describe physical phenomena. They are often difficult to solve analytically, hence numerical methods have to be used. Partial differential equations that encounter either singularities or steep changes require non-uniform time-spatial grids or moving elements. Wavelet analysis is an efficient method for solving these PDEs [5]. Wavelet transform can track the position of a moving steep front and increase the local resolution of the grid by adding higher resolution wavelets. In the smoother regions, a lower resolution can be used. The resolution grid is adapted dynamically in [5] using an adaptive wavelet collocation method. The minimum and maximum wavelet resolutions are determined in advance. Grid points are removed or added according to the magnitude of the corresponding wavelet coefficient. If the wavelet coefficient is located below a predefined threshold, grid point can be removed. Transient changes in the solution are accounted for by adding extra grid points at the same and at lower levels. The adaptation procedure can be described as:

(1) Compute the interpolation for each odd grid point from the lower level of the grid [5].
(2) Compute the corresponding wavelet coefficient d_k^j and apply the grid reduction/extension according to

(i) if $d_k^j < \epsilon$, remove the grid point.

(ii) if $d_k^j \geq \epsilon$, maintains the grid point. The number of allocation points is optimised without affecting the accuracy of the solution [5].

5.4. Signal Analysis

Signals are always the input for a wavelet analysis. The resulting wavelet coefficients can be manipulated in many ways to achieve certain results, these include denoising, compression, feature detection, etc. This section discusses various applications of wavelets in signal analysis.

5.5. Audio Compression

The DWT is a very useful analysis tool for signal compression. The filter banks of the DWT are not regular, but close to regular in the first few octaves of the subband decomposition [22]. Effective speech and audio compression algorithms use knowledge of the human hearing. Human hearing is associated with critical bands. Around a frequency fm there is masking. A neighboring frequency with magnitude below $T(fm, f)$ is masked by fm and is not audible [27]. The magnitude $T(fm, f)$ can be calculated as

$$T(f_m, f) = \begin{cases} M(f_m)(\frac{f}{f_m})^{28}, & f \leq f_m \\ M(f_m)(\frac{f}{f_m})^{-10}, & f > f_m \end{cases} \tag{27}$$

where $M(fm)$ is called as the masking threshold, independent of the signal. For the audio compression the frequency allocation of the DWT approximates the critical bands of the human ear. The frequencies fm with large power are detected and the masking envelopes $T(fm, f)$ are computed. From this the masking curve is constructed which is used to determinate which wavelet coefficients are kept and which are removed in order to compress the signal. Speech compression can be used e. g. in mobile communications to reduce the transmission time. Speech signals can be divided into voiced and unvoiced sounds. Voiced sound have mainly low-frequency content, whereas unvoiced sounds (e.g.

a hiss) have energy in all frequency bands. The human hearing is associated to non uniform critical bands, which can be approximated using a four-level dyadic filter bank.

5.6. Image and Video Compression

Images are analyzed and synthesized by 2D filter banks. In images and videos the low frequencies, extracted by high scale wavelet functions, represent flat backgrounds. The high frequencies (low scale wavelet functions) represent regions with texture [27]. The compression is performed using a $p - level$ filter bank. The low-pass subband gives an approximation of the original image, the other bands contain detail information. Bit allocation is now crucial, subimages with low energy levels should have fewer bits [27]. Some important properties of the filter bank used for image and video compression are:

(1) The synthesis scaling function should be smooth and symmetric.
(2) The high-pass analysis filter should have no DC leakage. The low-pass subband should contain all the DC energy.
(3) The analysis filters should be chosen to maximize the coding gain.
(4) The high-pass filter should have a good stopband attenuation to minimize leakage of quantization noise into low frequencies.

In [21] a multiwavelet transform based on two scaling functions and two wavelet functions is used for image compression. By using two scaling functions and two wavelets functions, the properties regularity, orthogonality and symmetry are ensured simultaneously. For the DWT, this corresponds to using two low-pass filters and two high-pass filters at each level of the filter bank. An image, decomposed in wavelet coefficients, can be compressed by ignoring all coefficients below some threshold value [13]. The threshold value is also determined based on a quality number calculated using the signal to noise ratio. The tradeoff between compression level and image quality depends on the choice of wavelet function, the filter order, filter length and the decomposition level. The optimal choice depends on the original image quality and computational complexity. In video compression, a new dimension is added, namely time. This expands the wavelet analysis from 2D to 3D. The compression can be performed analogue to the image compression described in [27]. Another video compression method is by motion estimation. This approach is based on compression of the separate images in combination with a motion vector. In the synthesis bank the separate images are reconstructed and the motion vector is used to form the subsequent frames [27].

5.7. JPEG 2000

The continual expansion of multimedia and internet applications leads to the need of a new image compression standard which meets the needs and requirements of the new technologies. The new proposed standard by the JPEG (Joint Photographic Experts Group) committee is called the JPEG 2000 standard [**?**], [26]. The old JPEG standard used the discrete cosine transform (DCT) for the compression of images. The new JPEG 2000 standard is based on the wavelet transform. The new standard provides a low bit-rate operation on still images with a performance superior compared to existing standards [**?**], [26]. The compression engine of the JPEG 2000 standard can be split into three parts, the preprocessing, the core processing and the bit-stream formation part. The preprocessing part tiles the image into rectangular blocks, which are compressed independently. The tiling reduces memory requirements and allows decoding of specific parts of the image. All samples of the separate tiles are DC level shifted and component transformations are included to improve compression and allow for visually relevant quantization. The core processing includes the discrete wavelet transform to decompose the tile components into different levels. The wavelet coefficients of the decomposition levels describe the horizontal and vertical spatial frequency characteristics of the tile components. The wavelet transform uses a one-dimensional subband decomposition. The low-pass samples represent a low-resolution version of the original set, the high-pass samples represent a down-sampled residual version. Images are transformed by performing the DWT in vertical and horizontal directions. After the DWT, all coefficients are quantized. The JPEG 2000 standard supports separate quantization step-sizes for each subband. Finally the core processing part performs entropy coding.

5.8. Wavelet Coding

The one dimensional (1D) wavelet transform can be extended to a two dimensional wavelet transform using separable wavelet filters. The orthogonal wavelet transform is not ideal for a coding system, the number of wavelet coefficients exceeds the number of input coefficients. The aim of coding systems is to reduce the amount of information, not to increase it. In order to eliminate border effects, the wavelets should have linear phase, this is not possible with orthogonal filters, except for the trivial Haar filters. The biorthogonal wavelet transform can use linear phase filters and permits the use of symmetric filters. The symmetric wavelet transform (SWT) solves the problems of coefficient expansion and border discontinuities and improves the performance of image coding. The disadvantage of biorthogonal filters is that they are not energy preserving. This is however not a big problem since there are linear phase filter coefficients which are close to being orthogonal [27].

5.9. Texture Classification

Another application of the Discrete Wavelet Transform is to perform texture classification [22]. For interpretation and analysis, e.g. of images, the human visual system relies for a great extend on texture perception. Many texture classification algorithms make use of the Gabor transform, which is another time-frequency analysis technique. In [22], a feature extraction algorithm based on the wavelet transform is proposed. The wavelet frame analysis performs upsampling on the filters rather than downsampling the image. The advantage of the wavelet transform over the Gabor transform is that the wavelet transform is computationally less demanding and covers the complete time frequency plane. The zero crossings of the wavelet transform correspond to edges in the image. The proposed algorithm of [22]. Includes information from varying length scales and makes it able to distinguish micro and macrotextures. The zero crossing information can be derived from the wavelet detail subbands. Each level of the DWT has three subbands, for respectively the horizontal, vertical and diagonal directions. From the transform a feature vector is extracted which represents the texture. Using the number of zero crossings of the various subbands, an average number of zero crossings per pixel is determined. It is assumed that the texture image is homogenous. Since only the texture classification is required, this assumption poses no problem [22]. For the texture classification, symmetric wavelet filters are used because of the linear phase which minimizes distortion effects. Fewer decomposition levels, leading to larger vector lengths, improve the classification.

5.10. Denoising

The denoising and feature detection of signals using the wavelet transform is done by representing the signal by a small number of coefficients [27]. This wavelet shrinkage is based on thresholding, as developed by Donoho and Johnstone [9]. The signal is composed into L levels before thresholding is applied. There are two types of thresholding, hard

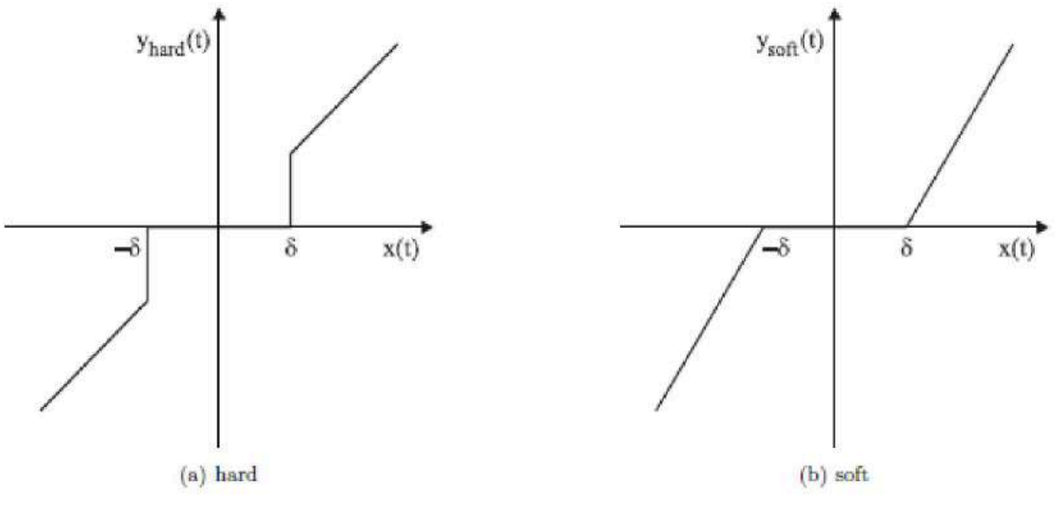

FIGURE 13

and soft thresholding with threshold δ. Hard thresholding zeros out small coefficients, resulting in an efficient representation. Soft thresholding softens the coefficients exceeding the threshold by lowering them by the threshold value. When thresholding is applied, no perfect reconstruction of the original signal is possible. Soft thresholding gives better compression performance [26]. The outputs of soft and hard thresholding can be written as

$$y_{hard} = \begin{cases} x(t), & |x(t)| > \delta \\ 0, & |x(t)| \leq \delta. \end{cases} \tag{28}$$

$$y_{soft} = \begin{cases} sign(x(t))\ (|x(t)| - \delta), & |x(t)| > \delta \\ 0, & |x(t)| \leq \delta. \end{cases} \tag{29}$$

Only the large coefficients are used for the reconstruction of the image. The denoising is not limited to a special kind of noise, different kinds of disturbances can be filtered out of the images. Thresholding generally gives a low-pass version of the original signal. An appropriate threshold δ can suppress noise present in a signal. For denoising applications, generally soft thresholding is used. It is assumed that the noise power is smaller than the signal power. If this is not the case, the denoising by thresholding removes either besides the noise a large part of the signal or leaves a larger part of the noise in the signal. Some of the signals power is removed with the noise, it is generally not possible to filter out all the noise without affecting the original signal.

5.11. Fingerprints

The FBI has millions of fingerprints which they have to digitize to improve search capabilities. The FBI uses a wavelet/scalar quantization (WSQ) algorithm for the compression of the gray scale fingerprint images [27], [2]. The compression algorithm consists of three main steps: a DWT, scalar quantization and entropy coding. For the DWT a two channel perfect reconstruction linear phase filter bank is used since it is symmetric and since it prevents image content to shift between the various subbands. The quantization of the DWT coefficients is done according to uniform scalar quantization characteristics. Finally the quantized indices are entropy-encoded using Huffman coding. Huffman coding is an entropy encoding algorithm used for data compression that finds the optimal system of encoding bits based on the relative frequency of each coefficients. For the reconstruction, first the entropy coding is reversed, then the quantization is done and finally an inverse

DWT is performed [27], [2]. The wavelet analysis is preferred over the JPEG standard, since the JPEG standard merges ridges in the true image during compression. Each of the fingerprint card uses 10 MB of data, this is compressed around 20:1. In [26] an image based method for fingerprint recognition is proposed. Note that this method is not known to be used by the FBI. The proposed method matches fingerprints based on the features extracted by a wavelet transform. For this a j-level 2D dyadic grid wavelet decomposition is performed, representing the image in $3j+1$ subimages; one approximation and $3j$ detail subimages. The wavelet detail coefficients correspond to edges and high frequencies. The dominant frequencies of fingerprint images are located in the middle frequency channels [27]. The wavelet transform detects the ridges in the fingerprint and the distance between the ridges very well. The wavelet coefficients are used to calculate a normalized l2-norm feature vector F of each subimage d_j^k which approximates the energy distribution of the image on different scales (2^j) and orientations (k) as

$$F = \left[(e_j^1 \; e_j^2 \; e_j^3)_{j=1,2,\ldots J} \right]. \tag{30}$$

$$e_j^k = \frac{||d_j^k||_2}{\displaystyle\sum_{i=1}^{J} \sum_{l=1}^{3} ||d_i^l||_2}. \tag{31}$$

A measure for the similarity between different feature vectors is obtained by using the intersection operator [19]. The best results are obtained using Daubechies and Symlet orthonormal wavelet filters.

5.12. Control Applications

Wavelet analysis can be used for the modeling and control of the dynamical behavior of systems and the partitioning and decoupling of system responses [1]. The Morlet wavelet is very useful for the detection of system nonlinearities because of its good support in both frequency and time domain. The separation of modes that are close in frequency can also be done using Morlet wavelets. Furthermore, natural frequencies and damping ratios of multi-degree-of-freedom (MDOF) systems can be identified. Research has also been done on the determination of modal parameters through a wavelet estimation of the technique and on the analysis of impulse responses.This section deals with several control applications where wavelets are used, these include among others motion detection and tracking, nonlinear adaptive control, encoder-quantization denoising, repetitive control, timefrequency adaptive ILC and system identification.

5.13. Motion Detection and Tracking

Motion detection of objects can be done using an algorithm based on wavelet transform [?]. The algorithm is part of a vehicle tracking system and uses the Gabor and Mallat wavelet transforms to improve the accuracy and the speed of the vehicle detection. The Gabor wavelet analysis estimates image flow vectors, object detection is then done using the Mallat wavelet transform. The first stage of the algorithm is the computation of an image flow field based on a convolution with a Gabor wavelet transform, which for an image $I(x)$ is defined as

$$J_j(x) = \int I(x')\ \psi_j(x - x')\ d^2x'. \tag{32}$$

The Gabor wavelets can be written in the shape of plane waves with wave vector k_j, restricted by a Gaussian envelope function, as

$$\psi_j(x) = \frac{k_j^2}{\sigma^2}\ e^{\left(-\frac{k_j^2 x^2}{2\sigma^2}\right)} e^{(ikjx)}. \tag{33}$$

The width of the Gaussian envelope can be controlled by the parameter given σ. The second stage of the algorithm performs motion hypothesis from the image flow field by extracting local maxima in the image flow histogram (a histogram over the flow field vectors). A low-pass filter avoids the detection of too many irrelevant maxima. The third stage of the algorithm uses the Mallat wavelet transform (a two channel DWT subband coder) to find the matching edges between two frames using the image flow field. The resulting accordance maps are integrated over a sequence of frames in order to improve the quality. Finally, the last stage of the algorithm performs the segmentation. The algorithm generates motion hypotheses on a coarse level, but segments them on a single pixel level, allowing to segment small, disconnected or openworked objects. Another advantage of the algorithm is that the motion of the objects does not need to be continuous. A drawback is that when vehicles move close to each other with similar speed they are identified as one object. However, the segmentation still shows multiple vehicles.

5.14. Robot Positioning

The grasping skill of a robot manipulator can be done using a camera, the images are analyzed using Gabor wavelets [26]. Gabor wavelets are useful for object recognition. For the grasping task the inverse procedure is performed, the object is known, the robot position is to be determined. For the pre-grasp face it is assumed that the object of interest is within the range of the camera, the type and vertical position of the object are known. The images of the camera are analyzed using a 2D Gabor bell function

$$\psi(x,y) = e^{\left(-\frac{x^2+\alpha^2y^2}{2\sigma^2}\right)}\left[e^{\left(-2\pi i\frac{x}{\lambda}\right)} - e^{\left(-\frac{\sigma^2}{2\alpha}\right)}\right]. \tag{34}$$

The period length is defined by the symbol λ, σ and $\frac{\alpha}{\sigma}$ are respectively the longitudinal and transversal width. The last term of the function removes the DC component, making the filter invariant to shifts in gray-level. The camera looks downward vertically to avoid distortion of the image shape, this at the cost of a robot transfer delay and extra kinematic restrictions on the workspace. The positioning of the robot is divided into a coarse, fast translational part and a rotational and translational part. The image of the camera is preprocessed and the objects center of gravity is mapped. The rotational positioning is obtained using a region of interest (ROI). The positioning of the manipulator robot using Gabor filters is stable up to extremely poor illumination conditions and performs well even for partly covered objects or textured backgrounds [21]. The proposed method is calibration free, direct, fast and robust.

5.15. Nonlinear Adaptive Wavelet Control

Using constructive wavelet networks, a nonlinear Adaptive Wavelet Controller (AWC) can be constructed [26]. The orthonormality and multi-resolution properties of wavelet networks make it possible to adjust the structure of nonlinear adaptive wavelet controller on-line. The adaptive wavelet controller is first constructed by a simple structure. If the tracking error does not converge in the specified adaptation period using the current wavelet structure, a new wavelet resolution is considered to be necessary and added. This allows the construction and tuning of the wavelet controller from a coarse level to a finer level while retaining the closed-loop stability [22]. Through adaptation, the desired control performance can be achieved asymptotically. A class of SISO nonlinear dynamic systems with state vector x $\in R^n$ and control input $u \in R$ equals

$$\begin{cases} x_i = x_{i+1}, \; i = 1, 2, \ldots, n-1 \\ x_n = f(x, u), \\ y = x_1. \end{cases} \tag{35}$$

The control objective is to find an appropriate control input $u(t)$ for the nonlinear system such that it tracks the desired trajectory. The structure of wavelet networks cannot be infinitely large for control purposes. The adaptive controller can be written as

$$u(z) = \hat{w}_J^T \phi(z) + \sum_{j=J}^{J_p} \frac{sign\Big(\frac{e_q, (j-J)}{\epsilon_0}\Big) + 1}{2} \hat{v}_j^T \phi(z). \tag{36}$$

The father wavelet (scaling function) is denoted by ϕ and the mother wavelet (wavelet function) by ψ, $e_{q,j}$ is the performance evaluation. The tuning parameters are w and v. When a new resolution is added, the tracking error will keep decreasing till the desired control performance is reached. The stability of each resolution can be guaranteed by using Lyapunovs direct method [21].

5.16. Encoder-Quantization Denoising

Quantization noise can in some cases be filtered out using a low-pass filter, however for slow changing signals this approach fails. Since the DWT can filter out noise at all the frequencies it is a good alternative for denoising encoder signals [24]. The noise is filtered out by determining thresholds for the various subbands, since the amplitude of all the quantization errors is always the same, the same threshold can be used for all subbands. For denoising purposes the choice of waveform is critical [24]. However the DWT is not able to cancel quantization errors for very low signal speeds, for this only dithering or the use of raw encoder signals helps. The denoising of an encoder signal can be summarized as

(1) Determine the lowest signal speed and determine the lowest frequency in the quantization error as function of the speed x and the encoder resolution δ, $f_q = \frac{x}{\Delta}$.
(2) Adjust the threshold levels of the various subbands to $\delta = \Delta$.
(3) Use the bior5.5-wavelet function for the signal decomposition and apply soft thresholding before reconstructing the signal [24].

5.17. Real-Time Feature Detection

Feature detection is based on distinguishing signal parts with different frequency content. The wavelet transform enables the possibility to distinguish between various frequencies in time. Most feature detection algorithms are processed off-line or in a delayed loop. In [24], an on-line approach is followed using a real-time implementation of the DWT. The real-time detection must be fast and the number of false should be minimal. The controller adaption must be done as fast as possible to minimize the decrease in performance. There exists a trade-off between detection speed and accuracy. The detection is successful if the coefficients of the wavelet transform arise a threshold value. The detection speed of the wavelet filter is faster than a simple threshold-based detection. A proposed application of the real-time feature detection is the adaptation of the controller of a CD-player [24]. The controller can be adapted based on detected disc defects. The Haar wavelet has the shortest delay time, but is not able to separate the different disturbances on the CD-player (shocks and disc defects). Even though good results were obtained using the Daubechies wavelet family (Appendix A), for shock detection a new, optimized, waveform with zero mean and normalized energy is derived [24]. Since the optimal waveform is dependent on the actual shock, an adaptive analyzing filter would increase the detection speed. Disc-scratches are best isolated in the first decomposition level, whereas shocks are best detected in higher levels.

5.18. Repetitive Control

In [3], The DWT is used to reduce the memory size of a repetitive controller. Repetitive controllers usually contain a memory for the error signal and a low-pass filter to ensure stability. The size of the memory is determined by the sampling time. The Discrete WT decomposes the error signal into high-pass (detail) and low-pass (approximation) coefficients. If only the levels containing the low-frequent approximation coefficients are used and memorized, the error signal is compressed. The synthesized signal of the inverse discrete wavelet transform (IDWT) is used as input signal for the low-pass filter. The wavelet shape, scale and the decomposition tree level determine the system performance and required memory size. The DWT can be seen as an extra filter added to the repetitive controller and uses little CPU time during analysis and synthesis of the error signal.

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CHAPTER 6
CONCLUSION

Signals are represented in the time domain can be evaluated for their properties in the frequency domain by applying signal analysis. The most commonly known method to analyze a time signal for its frequency content is the Fourier transform. The wavelet transform is a relatively new technique which has some attractive characteristics. This report discusses the main issues regarding the wavelet transform and provides a general introduction of the wavelet theory. The various wavelet analysis methods are described in comparison to the widely known Fourier transform. The Fourier transform only retrieves the global frequency content of a signal, all time information is lost. To overcome this problem the short time Fourier transform is developed, however this method suffers from a limitation due to a fixed resolution in both time and frequency. The multiresolution analysis of the local frequency content of a signal is made possible by wavelet analysis. Two different kinds of wavelet transform can be distinguished, a continuous and a discrete wavelet transform. The continuous wavelet transform is calculated by the convolution of the signal and a wavelet function. A wavelet function is a small oscillatory wave which contains both the analysis and the window function. The discrete wavelet transform uses filter banks for the analysis and synthesis of a signal. The filter banks contain wavelet filters and extract the frequency content of the signal in various subbands. A Wavelet analysis has a wide range of applications. In this project the applications which are of most interest for mechanical engineers have been mentioned. Wavelet analysis can be applied for numerical analysis, i. e., solving ordinary and partial differential equations. Further more the wavelet transform is used in signal analysis, e.g. for compression, denoising and feature extraction. For control applications wavelets are used in motion tracking, robot positioning, identification and both linear and nonlinear control purposes. Finally, wavelets are a powerful tool for the analysis and adjustment of audio signals.

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