Jacobian Matrix

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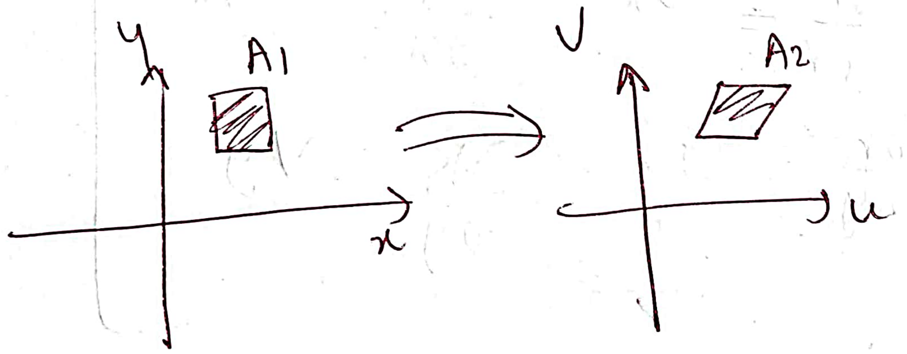
tells us the area/volume change during Linear transformation.

→ Approximation when transformation isn't linear but we take very small dx and dy which are apparently straight.

→ $J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

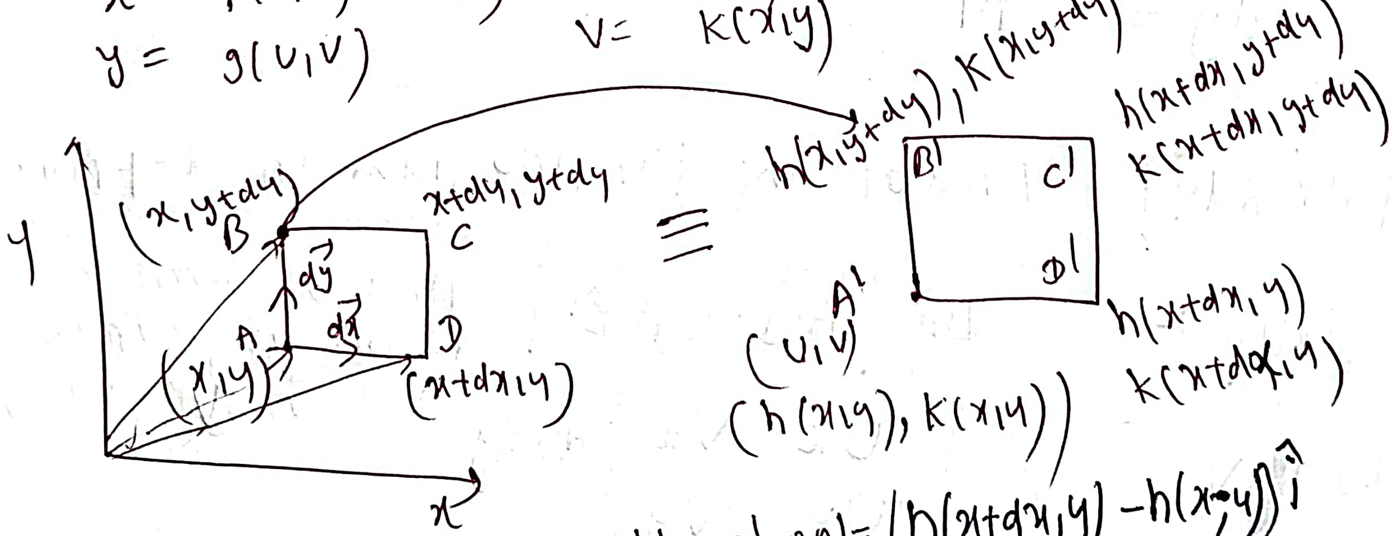


$\boxed{A_2 = |J| A_1}$

Mine

$x = f(u,v)$ ⟹ $u = \cancel{h(u,v)}\ h(x,y)$

$y = g(u,v)$ ⟹ $v = k(x,y)$



$\vec{AD'} = \vec{OD'} - \vec{OA'} = \left(h(x+dx,y) - h(x,y)\right)\hat{i} + \left(k(x+dx,y) - k(x,y)\right)\hat{j}$

$$\vec{A'B'} = \left[h(x, y+dy) - h(x,y)\right]\hat{i} + \left[k[x, y+dy] - k(x,y)\right]\hat{j}$$

$$\vec{A'D'} = \frac{h(x+dx, y) - h(x,y)}{dx}\, dx\,\hat{i} + k\frac{(x+dx, y) - k(x,y)}{dx}\, dx\,\hat{j}$$

$$\boxed{\begin{aligned} \vec{A'D'} &= \frac{\partial h}{\partial x}\, dx\,\hat{i} + \frac{\partial k}{\partial x}\, dx\,\hat{j} \\ \vec{A'B'} &= \frac{\partial h}{\partial y}\, dy\,\hat{i} + \frac{\partial k}{\partial y}\, dy\,\hat{j} \end{aligned}}$$

~~h~~ $\to U$

~~k~~ $\to$

$$\boxed{\begin{aligned} \frac{\partial h}{\partial x} &= U_x & \frac{\partial k}{\partial x} &= V_x \\ \frac{\partial h}{\partial y} &= U_y & \frac{\partial k}{\partial y} &= V_y \end{aligned}}$$

$$\vec{dx} = \vec{AD'}$$

$$\vec{dy} = \vec{AB}$$

$|\vec{dx} \times \vec{dy}|$ = Area of the parallelogram in x-y plane.

$$= |\vec{A'D'} \times \vec{A'B'}|$$ = Area of same region in U-V plane. $= |d\vec{u} \times d\vec{v}| = du\,dV$

$$du\,dV = \begin{vmatrix} U_x & V_x \\ U_y & V_y \end{vmatrix} dx\,dy$$

$$du\,dy = \left| \hat{k} \begin{vmatrix} U_x\,dx & V_x\,dx \\ U_y\,dy & V_y\,dy \end{vmatrix} \right|$$

$$\boxed{du\,dy = \begin{vmatrix} U_x & V_x \\ U_y & V_y \end{vmatrix} dx\,dy}$$

<u>Similarly</u>

$$dx\,dy = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} du\,dv$$

$$\det(A) = \det(A^T)$$

$$\boxed{dx\,dy = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} du\,dv} = J\left(\frac{x, y}{u, v}\right)$$

So

$$\text{Jacobian} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

One more observation

$$J = \begin{bmatrix} \boxed{\begin{matrix} x_u \\ y_u \end{matrix}} & \boxed{\begin{matrix} x_v \\ y_v \end{matrix}} \end{bmatrix}$$

where $\hat{i}$ landed $(1,0)$

where $\hat{j}$ landed $(0,1)$ after linear transformation from

$\boxed{x, y \text{ to } u, v}$