Complex Number:-

$$z = a + ib$$

$$|z| = \sqrt{a^2 + b^2}$$

$$\theta = \arg z = \tan^{-1}(b/a).$$

(a) every complex no. can be represent at a point in a plane which is called complex number plane or Argument plane.

Quadrant	argument	Principle argument
I	α	α
II	$\pi - \alpha$	$\pi - \alpha$
III	$\pi + \alpha$	$-\pi + \alpha$
IV	$2\pi - \alpha$	$-\alpha$

0 1/2 : $\frac{1}{\sqrt{2}}$

1 $\sqrt{3}/2$

$\frac{1}{\sqrt{3}}$ $\frac{1}{\sqrt{2}}$

$\frac{\sqrt{3}}{2}$ $\frac{\sqrt{3}}{1}$

(b) $|z_1 - z_2|$ → distance b/w z_1 & z_2

$\tan\theta$ 60

(c) $|z - z_0| = r$ (complex equation of circle).

60

$\frac{\pi}{2}$ 3 50

(d) $|z - z_0| < r$ represent interior of circle.

-1.5

-1.5

$\arg(R^+) = 0$ | R^- → π

Img^+ → $\pi/2$ | Img^- → $-\pi/2$

Relation b/w circular and hyperbolic functions:-

$\sin ix = i\sinh x$ | $\sinh ix = i\sin x$
$\cos ix = \cosh x$ | $\cosh ix = \cos x$
$\tan ix = i\tanh x$ | $\tanh ix = i\tan x$
$\operatorname{cosec} iz = -i\operatorname{cosech} z$
$\sec iz = \operatorname{sech} z$
$\cot iz = -i\coth z$

→ $\frac{d}{dx}\sinh x = \cosh x$ | $\frac{d}{dx}\cosh x = \sinh x$

$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$ | $\frac{d}{dx}\coth x = -\operatorname{cosech}^2 x$

$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$ | $\frac{d}{dx}\operatorname{cosech} x = -\operatorname{cosech} x \cdot \coth x$.

→ limit of the complex function.

• $\lim_{z \to z_0} f(z)$ exist if it is independent of the path $z \to z_0$.

→ continuity.

$f(z_0) = \lim_{z \to z_0} f(z)$ ✓

→ Differentiability

$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exist. $= f'(z_0)$

analytic function

- function $f(z)$ is analytic at $z=z_0$ if $f(z)$ is differentiable at every point in neighbourhood $|z-z_0| < \delta$ of z_0.

necessary condition for analytic function

- satisfy Cauchy-Rieman equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Proof:- for cauchy R. eq.

let a function $f(z)$ be analytic

$w = f(z) = u + iv$

so $$\frac{dw}{dz} = \lim_{z \to z_0} \text{ or } \lim_{\delta z \to 0} \frac{\delta w}{\delta z} \quad \text{--- (i)}$$

along path $x=0$

$$\frac{dw}{dz} = \lim_{\delta x \to 0} \frac{\delta w}{\delta x} = \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{--- (ii)}$$

along path $y=0$

$$\frac{dw}{dz} = \lim_{i\delta y \to 0} \frac{\delta w}{i\delta y} = \frac{\partial w}{i\partial y} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$= \frac{-i\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{— (ii)}$$

yrom eq (1) & (2).

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{-i\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \& \quad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

(•) Suffiecint condition.

(1) Satisfy C.R.E.
(2) all derivative of C.R.E should be continuous.

(•) Harmonic yunction:-

y(x,y) is called harmonic iy

$$\frac{\partial f^2}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 0 \quad \text{or} \quad \nabla^2 y = 0$$

(1) property :-

if $f(z) = u + iv$ is an an analytic function and u, v are both harmonic function then u & v are called conjugate harmonic of each other.

(1) If $f(z) = u + iv$ be analytic then u and v are harmonic

~~proof~~ Proof — we CR eq

$\nabla^2 f = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ replace

will become = zero.

(2)

for u

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\nabla^2 f \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

i.e $\nabla^2 u = 0$

So u is harmonic.

do similarly for v.

(Q). If $f(z) = u + iv$ be analytic function then u = constant, v = constant for family of orthogonal curve.

proof. If u = const

$$du = 0$$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} \cdot dy = 0$$

$$\frac{\left(\frac{\partial u}{\partial x}\right)}{\frac{\partial u}{\partial y}} \cdot dx + dy = 0$$

$$m_1 = \frac{dy}{dx} = -\frac{(\partial u/\partial x)}{(\partial u/\partial y)} \quad \text{— (1)}$$

for v

$$dv = \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy = 0$$

$$m_2 = \frac{dy}{dx} = -\frac{(\partial v/\partial x)}{(\partial v/\partial y)} \quad \text{— (2)}$$

$$m_1 \cdot m_2 = \frac{\frac{\partial u}{\partial x} \times \frac{\partial v}{\partial x}}{\frac{\partial u}{\partial y} \times \frac{\partial v}{\partial y}} \Rightarrow \frac{\frac{\partial v}{\partial y} \times -\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial y} \times \frac{\partial v}{\partial y}}$$

$$\boxed{m_1 \cdot m_2 = -1}$$

(8). If $f(z) = u + iv$ be analytic function with constant modulus then function $f(z)$ is constant.

CRE — (1)

proof! -

$$|f(z)| = \text{i.e. const}$$

$$\sqrt{x^2 + y^2} = c$$

$$u^2 + v^2 = c \quad \text{— (2)}$$

$$\frac{\partial}{\partial x}(2) \Rightarrow 2u \cdot \frac{\partial u}{\partial x} + 2v \cdot \frac{\partial v}{\partial x} = 0 \quad \text{— (3)}$$

$$\frac{\partial}{\partial y}(2) \Rightarrow 2u \cdot \frac{\partial u}{\partial y} + 2v \cdot \frac{\partial v}{\partial y} \quad \text{— (4)}$$

squaring and adding eq. (3) & (4).

$$\left(u \cdot \frac{\partial u}{\partial x}\right)^2 + \left(v \cdot \frac{\partial v}{\partial x}\right)^2 + \left(u \cdot \frac{\partial u}{\partial y}\right)^2 + \left(v \cdot \frac{\partial v}{\partial y}\right)^2$$

$$\cancel{u^2 (u_x^2 + v_x^2)}$$

$$\left(u \cdot \frac{\partial u}{\partial x}\right)^2 + \left(u \cdot \frac{\partial u}{\partial y}\right)^2 + \left(v \cdot \frac{\partial v}{\partial x}\right)^2 + \left(v \frac{\partial v}{\partial y}\right)^2$$

$$\cancel{u^2 \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right) + v^2 [}$$

$$u^2\left[\cancel{u_x^2+u_y^2}\right]+v^2\left[v_x^2+v_y^2\right] \quad (2)$$

$$u^2\left[u_x^2+u_x^2\right]+\cancel{v^2}$$

$\left|\dfrac{d}{dz} f(z)\right| = \left|\dfrac{\partial}{\partial x}(u+iv)\right|$

$= \left|\dfrac{\partial}{\partial y}(u+iv)\right|$

$= 0.$

from eq. (4)

$$2u\cdot\left(-\frac{\partial v}{\partial x}\right)+2v\cdot\left(\frac{\partial u}{\partial x}\right)=0 \quad (5)$$

$$-u\cdot v_x + v\cdot u_x = 0 \quad (5)$$

eq. (3)

$\hookrightarrow \quad u\cdot u_x + v\cdot v_x = 0$

$$(\text{eq. } 3)^2 + (\text{eq. } 5)^2$$

$$u^2 u_x^2 + v^2\cdot v_x^2 + \cancel{2u\cdot u_x\cdot v\cdot v_x}$$

$$u^2\cdot v_x^2 + v^2\cdot u_x^2 - \cancel{2u\cdot v_x\cdot v\cdot u_x}$$

$$\cancel{u^2\left(u_x^2+v_x^2\right)}$$

$$(u^2+v^2)\,u_x^2 + (u^2+v^2)\cdot v_x^2 = 0$$

$$(u^2+v^2)\left(u_x^2+v_x^2\right) = 0$$

$$|f'(z)| = 0$$

$$f'(z) = 0$$

$$f(z) = \text{constant}$$

① Construction of analytic function:
(Milne thomson method).

① when u (real part) is given.

→ find $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$

then find

$$\phi_1(z,0) = \frac{\partial u}{\partial x}\Big|_{(z,0)}$$

$$\phi_2(z,0) = \frac{\partial u}{\partial y}\Big|_{(z,0)}$$

sol ⇒ $f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz + c$

② when v (imag. part) is given.

$$\phi_1(z,0) = \frac{\partial v}{\partial x}\Big|_{(z,0)}$$

$$\phi_2(z,0) = \frac{\partial v}{\partial y}\Big|_{(z,0)}$$

$$f(z) = \int [\phi_2(z,0) + i\phi_1(z,0)]\, dz + c$$

③ when $u - v$ is given.
we have

$$f(z) = u + iv \quad \text{—①}$$
$$if(z) = iu - v \quad \text{—②}$$

$$(1+i) f(z) = (u - v) + i(u + v)$$

$F(z) = U + iV$

$$\boxed{f(z) = \frac{F(z)}{1+i}} \quad — (A)$$

now further procede case-I as U is given and find $F(z)$ then use equation — (A) to find $f(z)$

⊙ polar form of CR equations

$$f(z) = u + iv = r\cos\theta + i r\sin\theta$$

$$u + iv = f(re^{i\theta}) \quad re^{i\theta} \quad — (A)$$

$$\frac{\partial}{\partial r}(A) \Rightarrow \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}) \cdot e^{i\theta}$$

$$\frac{\partial}{\partial \theta}(A) \Rightarrow \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\, r \cdot e^{i\theta} \cdot i$$

$$\Downarrow$$

$$\frac{1}{ri}\left[\frac{\partial u}{\partial \theta} + \frac{i\,\partial v}{\partial \theta}\right] = f'(re^{i\theta})\, e^{i\theta}$$

$$-\frac{i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\, e^{i\theta} \quad — (2)$$

$\frac{\partial y}{\partial \theta}$ from eq. (1) & (2)

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = -\frac{i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

:- Complex Integration :-

cauchy

→ Cauchy Integral theoram:-

if $f(z)$ is an analytic function within and on a close curve c then

$$\int_C f(z)\,dz = 0$$

Proof:- let $f(z) = u + iv$ be analytic within and on closed curve C

we have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ — (1)

now $$\int_C f(z)\cdot dz = \int (u+iv)\,(dx+i\,dy)$$

$$= \int u\,dx + ui\,dy + iv\cdot dx - v\cdot dy$$

$$= \int u\,dx - v\,dy + i(u\,dy + v\,dx)$$

→ [by green theoram

$$\oint P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + \iint i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

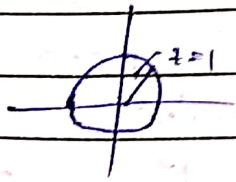
→ By C.R.E.

$$= \iint \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= 0 + i0 = \underline{0}$$

proved

Q. $\int_c e^z \, dz$ where c is $|z|=1$



By C.T.T, $\int_c e^z \, dz = 0$

Extension of cauchy theorem:-

let $f(z)$ be analytic in the region bounded by two close curve C_1 and C_2 then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Cauchy integral formula:-

let $f(z)$ be analytic whithin and on closed curve c and a is any point within c. then

$$f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-a)} \, dz$$

Proof:-

→ let $f(z)$ be analytic within and on closed curve c and a is any point within c

→ cosider a function $\phi z = \frac{f(z)}{z-a}$

describe a circle C_1 with center 'a' of small radius whose equation is $|z-a| = r$ now the function $\phi(z)$ is analytic in a region bounded by closed curve C & C_1 then by extension of cauchy's theorem we have

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{(z-a)}\,dz. \quad \text{——— (1)}$$

since, eq^n of C_1 is $|z-a| = r$

$z - a = re^{i\theta}$

putting these values in equation 1 R.H.S=1

$z = a + re^{i\theta}$

& $dz = rie^{i\theta}\,d\theta.$

$$\int_C \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} \frac{f(z)}{re^{i\theta}}\, rie^{i\theta}.\,d\theta$$

$$= \int_0^{2\pi} \frac{f(a+re^{i\theta})\cdot \cancel{r}i.\cancel{e^{i\theta}}}{\cancel{re^{i\theta}}}\,.d\theta$$

$$= i\int_0^{2\pi} f(a+re^{i\theta})\cdot d\theta$$

$\because$ C_1 is very small so making $r \to 0$

$$= i\int_0^{2\pi} f(a)\cdot d\theta$$

$$= i2\pi\, f(a)$$

$$\boxed{f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} \cdot dz}$$

proved,

11/5/

Date ______
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:- Cauchy's Inequality !-

if $f(z)$ is analytic within an on a circle c defined by

$|z-z_0| = r$ and if $f(z)$ is bounded

$|f(z)| \leq M \ \forall z$ on c

then proove that

$$|f(a)| \leq \frac{Mn!}{r^n}$$

proof :- Since $f(z)$ is analytic within an on circle c given by $|z-z_0| = r$ —— (i)

then by cauchy integral formula by higher derivative

$$|f^{(n)}(z_0)| \leq \frac{Mn!}{r^n}$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$$\Rightarrow |f^{(n)}(z_0)| = \left|\frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right|$$

ta

$$\leq \frac{n!}{|2\pi i|} \int_C \frac{|f(z)|}{|z-z_0|^{n+1}} |dz|$$

$$\leq \frac{n!}{2\pi} \int_C \frac{M}{r^{n+1}} |dz|$$

$$= \frac{n!\,M}{2\pi r^{n+1}} \int_C |dz|$$

$\int_C |dz|$ = perimeter of C

$$= \frac{n!\,M}{2\pi r^{n+1}} 2\pi r$$

$$\leq \frac{n!\,M}{r^n}$$

=) Morera's Theorem:-

If function $f(z)$ is continious in a domain D and let for every close curve C in the domain D

$$\oint f(z)\,dz = 0$$

then $f(z)$ is analytic in D it is converse of the cauchy theorem.

all are same $\Rightarrow$ $\left\{\begin{array}{l}\text{analytic}\\ \text{Regular}\\ \text{holomorphic}\end{array}\right\}$

[but a little difference]

⑥ Entire function (or Integral function).

A function $f(z)$ is said to be entire function if it is analytic in every finite region of z plane.

⑥ Liouville's theorem :-
If $f(z)$ is an entire function and it is bounded for all values of z then $f(z)$ is constant

proof :-

let $f(z)$ is an entire function i.e it is analytic in every finite region of z, and let $f(z)$ is bounded it means there exist a +ve real no. M such that $|f(z)| \le M \ \forall z$

→ to prove that $f(z)$ is constant.

let z_1 and z_2 are two distinct point in the z-plane.

Draw a circle C with centre at z_1 and of Radius R such that z_2 lies within C

So it's equation is

since $|z-z_1| = R$ —— (2)

Since $f(z)$ is an entire function so that it is analytic within and on circle C and z_1, z_2 are points within C then by cauchy integral formula

$$f(z_1) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_1)}\,dz$$

and

$$f(z_2) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_2)}\,dz$$

$$\Rightarrow f(z_1) - f(z_2) = \frac{1}{2\pi i}\oint_C f(z)\left\{\frac{1}{z-z_1} - \frac{1}{z-z_2}\right\} dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)\,(z_1-z_2)}{(z-z_1)(z-z_2)}\,dz$$

$$|f(z_1) - f(z_2)| = \left|\frac{z_1-z_2}{2\pi i}\int_C \frac{f(z)}{(z-z_1)\,(z-z_2)}\,dz\right|$$

$$\leq \frac{|z_1-z_2|}{|2\pi i|}\int_C \frac{|f(z)|}{|z-z_1|\,|z-z_2|}\,|dz|$$

$$\leq \frac{|z_1-z_2|}{2\pi}\int_C \frac{M}{R\,|z-z_2|}\,|dz| \quad \text{using eq (1) \& (2)}$$

$$\leq \frac{|z_1 - z_2|}{2\pi} \frac{M}{R} \int_C \frac{1}{|z - z_2|} \cdot |dz|$$

since ye thi R se hi
chota hoga

$$\leq \frac{|z_1 - z_2|}{2\pi} \frac{M}{R^2} \underbrace{\int_C |dz|}_{l}$$

$$\leq \frac{|z_1 - z_2| \cdot M}{2\pi R^2} \times 2\pi R = \frac{|z_1 - z_2| M}{R}$$

$$\leq \cancel{\lim_{R \to \infty} \frac{|z_1 - z_2| M}{R}}$$

≤ 0 applying $\lim_{R \to \infty}$

$$|f(z_1) - f(z_2)| = 0$$

$$\boxed{f(z_1) = f(z_2)}$$

Hence $f(z)$ is constant.