ADVANCED ENGINEERING MATHEMATICS

2130002 – 5TH EDITION

DARSHAN INSTITUTE OF ENGINEERING AND TECHNOLOGY

Name :

Roll No. :

Division :

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UNIT WISE ANALYSIS FROM GTU QUESTION PAPERS

Unit Number ⟼	1	2	3	4	5	6
W – 14	4	28	28	7	28	24
S – 15	-	35	14	14	28	28
W – 15	3	30	25	14	31	16
S – 16	9	28	26	8	31	17
W – 16	3	30	21	14	31	16
S – 17	2	15	38	11	26	27
W – 17	3	13	35	14	26	28
S – 18	-	22	31	14	24	28
Average ⟼	3	25	27	12	28	23
*GTU Weightage ⟼	4	10	20	6	15	15

*Unit weightage out of 70 marks.

Unit No.	Unit Name	Level	GTU Hour
1	Introduction to Some Special Function	Easy	2
2	Fourier Series and Fourier Integral	Medium	5
3	Differential equation and It's Application	Medium	11
4	Series Solution of Differential Equation	Easy	3
5	Laplace Transform and It's Application	Hard	9
6	Partial Differential Equation	Hard	12

LIST OF ASSIGNMENT

Assignment No.	Unit No.	Method No.
1	4	2
	6	6
2	2	1, 2
3	2	3, 4, 5
4	3B	ALL METHODS
5	3A	ALL METHODS
6	5	Proof of Formulae
7	5	GTU asked examples (Method No. 1 to 8)
8	5	GTU asked examples (Method No. 9 to 16)

UNIT 1 – INTRODUCTION TO SOME SPECIAL FUNCTIONS

❖ INTRODUCTION:

✓ Special functions are particular mathematical functions which have some fixed notations due to their importance in mathematics. In this Unit we will study various type of special functions such as Gamma function, Beta function, Error function, Dirac Delta function etc. These functions are useful to solve many mathematical problems in advanced engineering mathematics.

❖ BETA FUNCTION:

✓ If $m > 0$, $n > 0$, then Beta function is defined by the integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ and is denoted by $\beta(m,n)$ OR $B(m,n)$.

$$\mathbf{B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx}$$

✓ Properties:

(1) Beta function is a symmetric function. i.e. $B(m,n) = B(n,m)$, where $m > 0, n > 0$.

(2) $B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$

(3) $\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} \cdot B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

(4) $B(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$

❖ GAMMA FUNCTION:

✓ If $n > 0$, then Gamma function is defined by the integral $\int_0^{\infty} e^{-x}x^{n-1}dx$ and is denoted by $\lceil n$.

$$\mathbf{\lceil n = \int_0^{\infty} e^{-x}\, x^{n-1}\, dx}$$

✓ Properties:

(1) Reduction formula for Gamma Function $\lceil(n+1) = n\lceil n$; where $n > 0$.

(2) If n is a positive integer, then $\Gamma(n+1) = n!$

(3) Second Form of Gamma Function $\int_0^\infty e^{-x^2} x^{2m-1} dx = \frac{1}{2}\Gamma m$

(4) Relation Between Beta and Gamma Function, $B(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$. **W – 15 ; W – 16**

(5) $\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$

(6) $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{n!4^n}$ for $n = 0,1,2,3,\ldots$

Examples: For $n = 0, \Gamma\frac{1}{2} = \sqrt{\pi}$ **W – 16**

For $n = 1, \Gamma\frac{3}{2} = \frac{\sqrt{\pi}}{2}$ For $n = 2, \Gamma\frac{5}{2} = \frac{3\sqrt{\pi}}{4}$

(7) Legendre's duplication formula. **S – 16**

$\Gamma n\, \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}}\,\Gamma(2n)$ OR $\Gamma(n+1)\, \Gamma n+\frac{1}{2} = \frac{\sqrt{\pi}}{2^{2n}}\,\Gamma(2n+1)$

(8) Euler's formula : $\Gamma n\, \Gamma(1-n) = \frac{\pi}{\sin n\pi}$; $0 < n < 1$

METHOD – 1: EXAMPLE ON BETA FUNCTION AND GAMMA FUNCTION

C	1	Find B(4,3). **Answer**: $\frac{1}{60}$	
T	2	Find $B\left(\frac{9}{2},\frac{7}{2}\right)$. **Answer**: $\frac{5\pi}{2048}$	S – 16
H	3	State the relation between Beta and Gamma function.	W – 15 W – 16
H	4	State Duplication (Legendre) formula.	S – 16
C	5	Find $\Gamma\frac{7}{2}$. **Answer**: $\frac{15\sqrt{\pi}}{8}$	S – 16

H	**6**	Find $\overline{\left\lvert \frac{13}{2} \right.}$. **Answer**: $\frac{10395\sqrt{\pi}}{64}$	W – 15
T	**7**	Find $\overline{\left\lvert \frac{5}{4} \right.}\,\overline{\left\lvert \frac{3}{4} \right.}$. **Answer**: $\frac{\pi}{2\sqrt{2}}$	

❖ ERROR FUNCTION AND COMPLEMENTARY ERROR FUNCTION:

✓ The error function of x is defined as below and is denoted by erf(x).

$$\mathbf{erf(x)} = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} dt$$

✓ The complementary error function is denoted by $\mathrm{erf}_c(x)$ and defined as

$$\mathbf{erf_c(x)} = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt$$

✓ Properties:

(1) $\mathrm{erf}(0) = 0$

(2) $\mathrm{erf}_c(0) = 1$

(3) $\mathrm{erf}(\infty) = 1$

(4) $\mathrm{erf}(-x) = -\mathrm{erf}(x)$

(5) $\mathrm{erf}(x) + \mathrm{erf}_c(x) = 1$

❖ UNIT STEP FUNCTION (HEAVISIDE'S FUNCTION): W – 14 ; W – 16

✓ The Unit Step Function is defined by

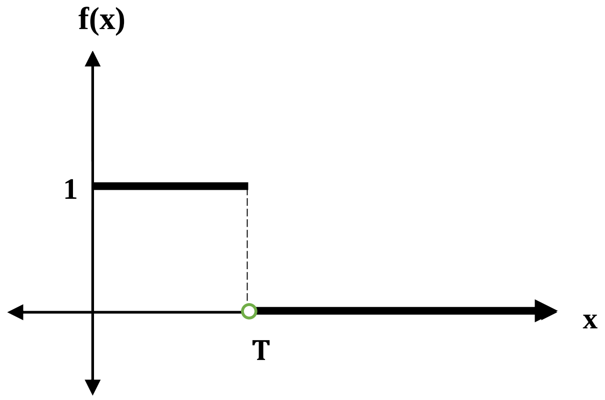
$$u(x-a) = \begin{cases} 1 & ; \quad x \geq a \\ 0 & ; \quad x < a \end{cases}.$$

✓ It is also denoted by $H(x-a)$ or $u_a(x)$.

❖ PULSE OF UNIT HEIGHT:

✓ The pulse of unit height of duration T is defined by

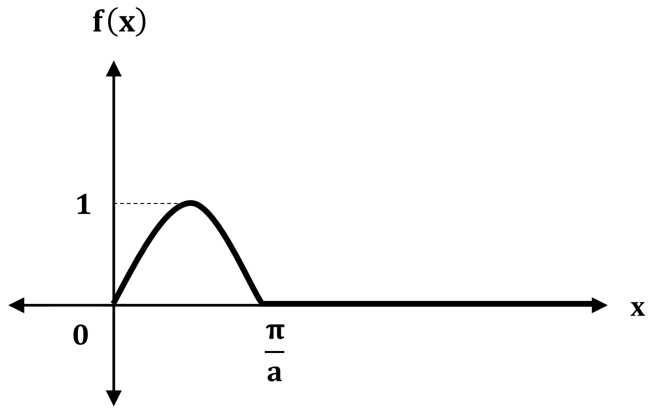
$$f(x)=\begin{cases}1 & ; \quad 0\le x\le T\\ 0 & ; \quad x>T\end{cases}.$$



❖ SINUSOIDAL PULSE FUNCTION:

✓ The sinusoidal pulse function is defined by

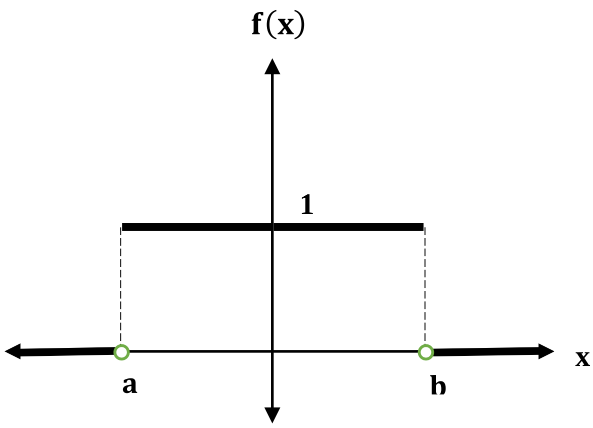
$$f(x)=\begin{cases}\sin ax & ; \quad 0\le x\le \frac{\pi}{a}\\ 0 & ; \quad x>\frac{\pi}{a}\end{cases}$$



❖ RECTANGLE FUNCTION: (W – 17)

✓ A Rectangular function f(x) on ℝ is defined by

$$f(x)=\begin{cases}1 & ; \quad a\le x\le b\\ 0 & ; \quad \text{otherwise}\end{cases}$$



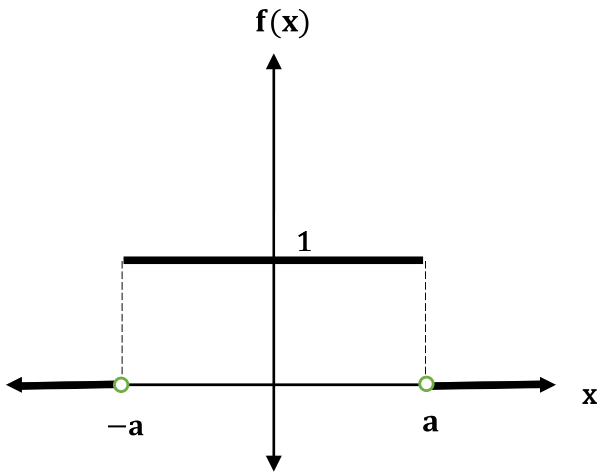
❖ GATE FUNCTION:

✓ A Gate function $f_a(x)$ on ℝ is defined by

$$f_a(x)=\begin{cases}1 & ; \quad |x|\le a\\ 0 & ; \quad |x|>a\end{cases}.$$

✓ Note that gate function is symmetric about axis of co-domain.

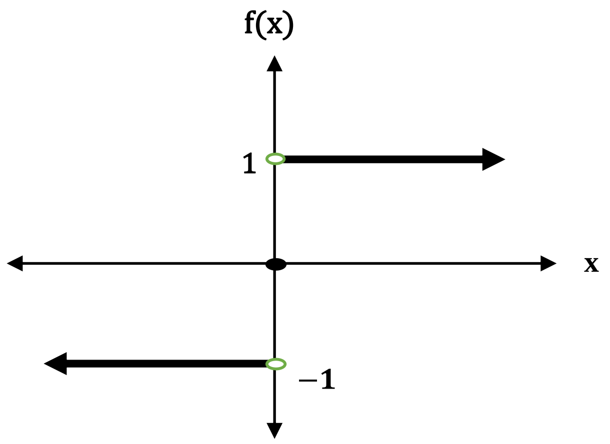
✓ Gate function is also a rectangle function.



❖ SIGNUM FUNCTION:

✓ The Signum function is defined by

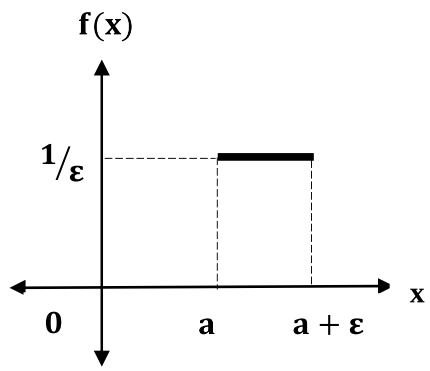
$$f(x) = \begin{cases} -1 & ; \quad x < 0 \\ 0 & ; \quad x = 0 \\ 1 & ; \quad x > 0 \end{cases}.$$



❖ IMPULSE FUNCTION:

✓ An impulse function is defined as below,

$$f(x) = \begin{cases} 0 & ; \quad x < a \\ \frac{1}{\varepsilon} & ; \quad a \le x \le a + \varepsilon \\ 0 & ; \quad x > a + \varepsilon \end{cases}$$



❖ DIRAC DELTA FUNCTION(UNIT IMPULSE FUNCTION): W – 14

✓ A Dirac delta Function $\delta(x - a)$ is defined by $\delta(x - a) = \lim_{\varepsilon \to 0} f(x)$.

Where, $f(x)$ is an impulse function, which is defined as

$$f(x) = \begin{cases} 0 & ; \quad x < a \\ \frac{1}{\varepsilon} & ; \quad a \le x \le a + \varepsilon \\ 0 & ; \quad x > a + \varepsilon \end{cases}.$$

❖ PERIODIC FUNCTION:

✓ A function f is said to be periodic, if $f(x + p) = f(x)$ for all x.

✓ If smallest positive number of set of all such p exists, then that number is called the Fundamental period of $f(x)$.

✓ Note:

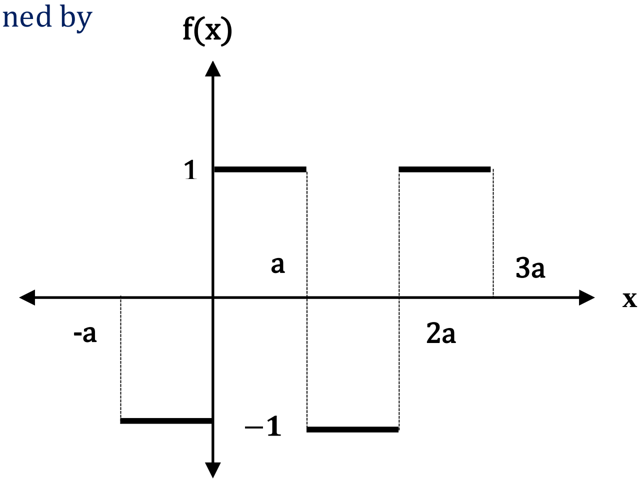
(1) Constant function is periodic without Fundamental period.

(2) Sine and Cosine are Periodic functions with Fundamental period 2π.

❖ SQUARE WAVE FUNCTION:

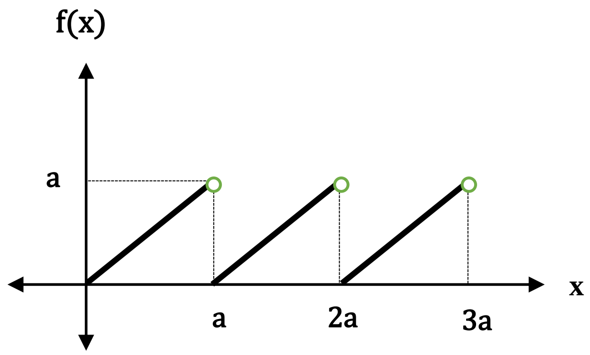
✓ A square wave function f(x) of period "2a" is defined by

$$f(x) = \begin{cases} 1 & ; \quad 0 < x < a \\ -1 & ; \quad a < x < 2a \end{cases}.$$



❖ SAW TOOTH WAVE FUNCTION: (W – 17)

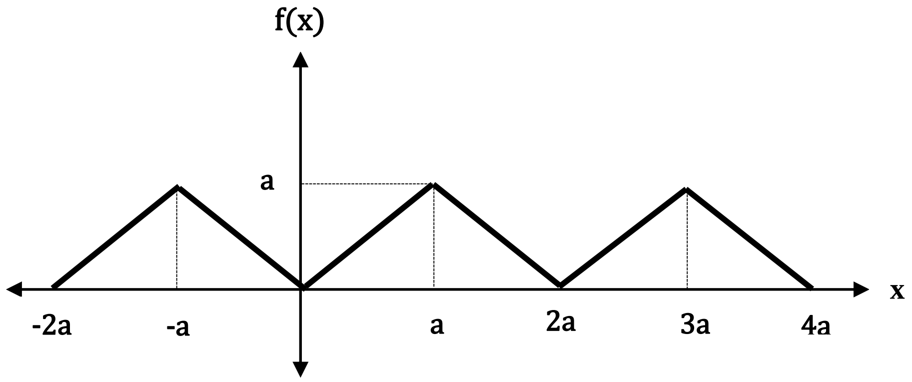
✓ A saw tooth wave function f(x) with period a is defined as $f(x) = x\ ; 0 \le x < a$.



❖ TRIANGULAR WAVE FUNCTION:

✓ A Triangular wave function f(x) having period "2a" is defined by

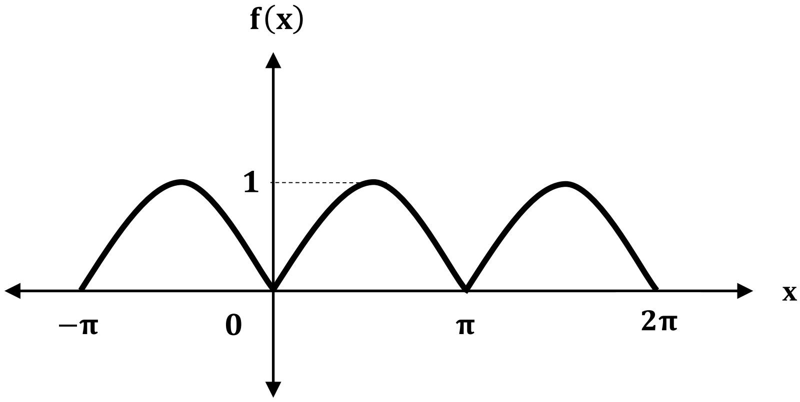
$$f(x) = \begin{cases} x & ; \quad 0 \le x < a \\ 2a - x & ; \quad a \le x < 2a \end{cases}.$$



❖ FULL RECTIFIED SINE WAVE FUNCTION:

✓ A full rectified sine wave function with period "π" is defined as

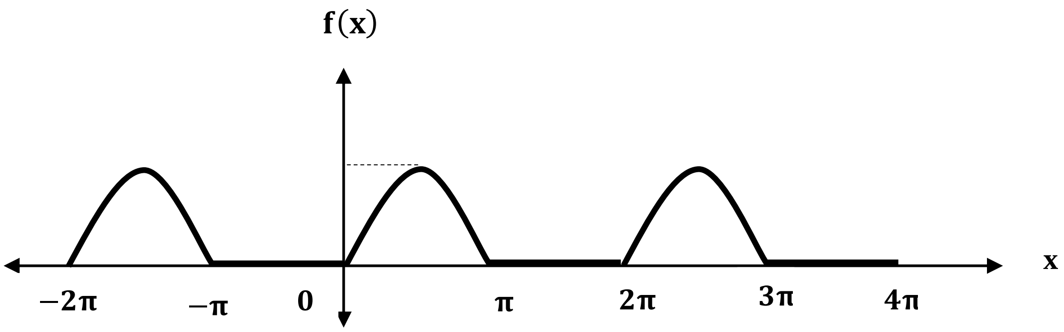
$$f(x) = \sin x \ ; \ 0 \le x < \pi.$$



❖ HALF RECTIFIED SINE WAVE FUNCTION:

✓ A half wave rectified sinusoidal function with period "2π" is defined as

$$f(x) = \begin{cases} \sin x & ; \quad 0 \le x < \pi \\ 0 & ; \quad \pi \le x < 2\pi \end{cases}.$$



❖ BESSEL'S FUNCTION:

✓ A Bessel's function of 1st kind of order n is defined by

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4(2n+2)(2n+4)} - \cdots\right] = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k}$$

METHOD – 2: EXAMPLE ON BESSEL'S FUNCTION

C	**1**	Determine the value $J_{\frac{1}{2}}(x)$. **Answer**: $\sqrt{\frac{2}{\pi x}}\sin x$	S – 16
H	**2**	Determine the value $J_{\left(-\frac{1}{2}\right)}(x)$. **Answer**: $\sqrt{\frac{2}{\pi x}}\cos x$	
C	**3**	Determine the value $J_{\frac{3}{2}}(x)$. **Answer**: $\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$	S – 16
H	**4**	Determine the value $J_{\left(-\frac{3}{2}\right)}(x)$. **Answer**: $\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$	
T	**5**	Using Bessel's function of the first kind, Prove that $J_0(0) = 1$.	

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UNIT-2 » FOURIER SERIES AND FOURIER INTEGRAL

❖ BASIC FORMULAE:

- ✓ Leibnitz's Formula (Take, Given polynomial function as **"u"**)

$$\int \mathbf{u \cdot v \; dx = u\, v_1 - u'\, v_2 + u''\, v_3 - u'''\, v_4 + \cdots}$$

Where, $u', u'', \ldots$ are successive derivatives of u and $v_1, v_2, \ldots$ are successive integrals of v.

- ➢ Choice of u and v is as per LIATE order.

Where,

L means Logarithmic Function

I means Invertible Function

A means Algebraic Function

T means Trigonometric Function

E means Exponential Function

- ✓ When Function is Exponential Function:

$$\int \mathbf{e^{ax} \sin bx \; dx = \frac{e^{ax}}{a^2 + b^2}[a \sin bx - b \cos bx] + c}$$

$$\int \mathbf{e^{ax} \cos bx \; dx = \frac{e^{ax}}{a^2 + b^2}[a \cos bx + b \sin bx] + c}$$

- ✓ When Function is Trigonometric Function:

$$\mathbf{2 \sin a \cos b = \sin(a + b) + \sin(a - b)}$$

$$\mathbf{2 \cos a \sin b = \sin(a + b) - \sin(a - b)}$$

$$\mathbf{2 \cos a \cos b = \cos(a + b) + \cos(a - b)}$$

$$\mathbf{2 \sin a \sin b = -\cos(a + b) + \cos(a - b)}$$

❖ NOTE (FOR EVERY, $n \in \mathbb{Z}$)

- ✓ $\cos n\pi = (-1)^n$
- ✓ $\sin n\pi = 0$
- ✓ $\cos(2n+1)\frac{\pi}{2} = 0$
- ✓ $\cos 2n\pi = (-1)^{2n} = 1$
- ✓ $\sin 2n\pi = 0$
- ✓ $\sin(2n+1)\frac{\pi}{2} = (-1)^n$

❖ INTRODUCTION:

- ✓ We know that Taylor's series representation of functions are valid only for those functions which are continuous and differentiable. But there are many discontinuous periodic functions of practical interest which requires to express in terms of infinite series containing "sine" and "cosine" terms.
- ✓ Fourier series, which is an infinite series representation in term of "sine" and "cosine" terms, is a useful tool here. Thus, Fourier series is, in certain sense, more universal than Taylor's series as it applies to all continuous, periodic functions and discontinuous functions.
- ✓ Fourier series is a very powerful method to solve ordinary and partial differential equations, particularly with periodic functions.
- ✓ Fourier series has many applications in various fields like Approximation Theory, Digital Signal Processing, Heat conduction problems, Wave forms of electrical field, Vibration analysis, etc.
- ✓ Fourier series was developed by Jean Baptiste Joseph Fourier in 1822.

❖ DIRICHLET CONDITION FOR EXISTENCE OF FOURIER SERIES OF $f(x)$:

(1) $f(x)$ is bounded.

(2) $f(x)$ is single valued.

(3) $f(x)$ has finite number of maxima and minima in the interval.

(4) $f(x)$ has finite number of discontinuity in the interval.

❖ NOTE:

- ✓ At a point of discontinuity the sum of the series is equal to average of left and right hand limits of $f(x)$ at the point of discontinuity, say x_0.

$$\text{i.e.}\quad f(x_0) = \frac{f(x_0 - 0) + f(x_0 + 0)}{2}$$

❖ FOURIER SERIES IN THE INTERVAL $(c, c+2L)$:

✓ The Fourier series for the function f(x) in the interval $(c, c+2L)$ is defined by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

Where the constants a_0, a_n and b_n are given by

$$a_0 = \frac{1}{L}\int_c^{c+2L} f(x)\, dx$$

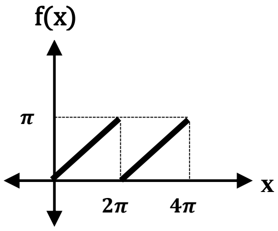
$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

METHOD – 1: EXAMPLE ON FOURIER SERIES IN THE INTERVAL $(C, C+2L)$

C	1	Find the Fourier series for $f(x) = x^2$ in (0,2). **Answer**: $f(x) = \frac{4}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2\pi^2}\cos(n\pi x) - \frac{4}{n\pi}\sin(n\pi x)\right]$	
H	2	Find the Fourier series to represent $f(x) = 2x - x^2$ in (0,3). **Answer**: $f(x) = \sum_{n=1}^{\infty}\left[-\frac{9}{n^2\pi^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{n\pi}\sin\left(\frac{2n\pi x}{3}\right)\right]$	S – 16
C	3	Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2$. **Answer**: $f(x) = \frac{(1-e^{-2})}{2} + \sum_{n=1}^{\infty}\frac{(1-e^{-2})}{n^2\pi^2+1}[\cos n\pi x + (n\pi)\sin n\pi x]$	
T	4	Find the Fourier series of the periodic function $f(x) = \pi\sin\pi x$. Where $0 < x < 1$, $p = 2l = 1$. **Answer**: $f(x) = 2 + \sum_{n=1}^{\infty}\frac{4}{1-4n^2}\cos(2n\pi x)$	
T	5	Find Fourier Series for $f(x) = x^2$; where $0 \le x \le 2\pi$ **Answer**: $f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right]$	

H	6	Show that, $\pi - x = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{\sin 2nx}{n}$, when $0 < x < \pi$.	
C	7	Obtain the Fourier series for $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in interval $0 < x < 2\pi$. Hence prove that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$. **Answer:** $\mathbf{f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx}$	W – 14
H	8	Find Fourier Series for $f(x) = e^{-x}$ where $0 < x < 2\pi$. **Answer:** $\mathbf{f(x) = \frac{1 - e^{-2\pi}}{2\pi} + \sum_{n=1}^{\infty} \frac{1 - e^{-2\pi}}{\pi(n^2 + 1)} [\cos nx + n \sin nx]}$	
H	9	Find Fourier Series for $f(x) = e^{ax}$ in $(0,2\pi); a > 0$ **Answer:** $\mathbf{f(x) = \frac{e^{2a\pi} - 1}{2a\pi} + \sum_{n=1}^{\infty} \frac{e^{2a\pi} - 1}{\pi(n^2 + a^2)} [a \cos nx - n \sin nx]}$	S – 18
H	10	Develop f(x) in a Fourier series in the interval (0,2) if $f(x) = \begin{cases} x, 0 < x < 1 \\ 0, 1 < x < 2. \end{cases}$ **Answer:** $\mathbf{f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2\pi^2} \cos(n\pi x) + \frac{(-1)^{n+1}}{n\pi} \sin(n\pi x) \right]}$	
C	11	For the function $f(x) = \begin{cases} x; & 0 \le x \le 2 \\ 4 - x: 2 \le x \le 4 \end{cases}$, find its Fourier series. Hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$. **Answer:** $\mathbf{f(x) = 1 + \sum_{n=1}^{\infty} \frac{4\,[(-1)^n - 1]}{\pi^2 n^2} \cos\left(\frac{n\pi x}{2}\right)}$	W – 15
T	12	Find the Fourier series for periodic function with period 2 of $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi\,(2 - x), 1 \le x \le 2. \end{cases}$ **Answer:** $\mathbf{f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{\pi n^2} \cos(n\pi x)}$	

H	**13**	Find the Fourier series of $f(x) = \begin{cases} x^2 & ; 0 < x < \pi \\ 0 & ; \pi < x < 2\pi. \end{cases}$ **Answer**: $\mathbf{f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \left[\frac{2(-1)^n \cos nx}{n^2} + \frac{1}{\pi} \left\{ -\frac{\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3} \right\} \sin nx \right]}$	
C	**14**	Find the Fourier Series for the function f(x) given by $f(x) = \begin{cases} -\pi, & 0 < x < \pi \\ x - \pi, & \pi < x < 2\pi \end{cases}$. Hence show that $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$. **Answer**: $\mathbf{f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{(1-(-1)^n)}{n^2\pi} \cos(nx) + \frac{(-1)^n - 2}{n} \sin(nx) \right]}$	W – 16 S – 18
C	**15**	Determine the Fourier series to represent the periodic function as shown in the figure. **Answer**: $\mathbf{f(x) = \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{\sin nx}{n}}$  	

❖ DEFINITION:

- ✓ Odd Function: A function is said to be Odd Function if $\mathbf{f(-x) = -f(x)}$.
- ✓ Even Function: A function is said to be Even Function if $\mathbf{f(-x) = f(x)}$.

❖ NOTE:

- ✓ If f(x) is an even function defined in $(-l, l)$, then $\int_{-l}^{l} f(x)\ dx = 2\int_{0}^{l} f(x)\ dx$.
- ✓ If f(x) is an odd function defined in $(-l, l)$, then $\int_{-l}^{l} f(x)\ dx = 0$.

❖ FOURIER SERIES FOR ODD & EVEN FUNCTION:

- ✓ Let, f(x) be a periodic function defined in $(-L, L)$

f(x) is even, $b_n = 0; n = 1,2,3, \dots$

- ✓ $\mathbf{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)}$

f(x) is odd, $a_0 = 0 = a_n; n = 1,2,3, \dots$

- ✓ $\mathbf{f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)}$

✓ Where,

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

✓ Where,

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Sr. No.	Type of Function	Example
1.	Even Function	✓ $x^2, x^4, x^6, \ldots$ i.e. x^n, where n is even. ✓ Any constant. e.g. $1, 2, \pi \ldots$ ✓ $\cos ax$ ✓ Graph is symmetric about $Y-$ axis. ✓ $\lvert x\rvert, \lvert x^3\rvert, \lvert\cos x\rvert, \ldots$ ✓ $f(-x) = f(x)$
2.	Odd Function	✓ $x, x^3, x^5, \ldots$ i.e. x^m, where m is odd. ✓ $\sin ax$ ✓ Graph is symmetric about Origin. ✓ $f(-x) = -f(x)$
3.	Neither Even nor Odd	✓ e^{ax} ✓ $ax^m + bx^n$; n is even & m is odd number.

METHOD – 2: EXAMPLE ON FOURIER SERIES IN THE INTERVAL $(-L, L)$

C	1	Find the Fourier series of the periodic function $f(x) = 2x$. Where $-1 < x < 1$, $p = 2l = 2$. **Answer**: $f(x) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi}\sin(n\pi x)$	W – 16

H	**2**	Find the Fourier expansion for function $f(x) = x - x^3$ in $-1 < x < 1$. **Answer**: $f(x) = \sum_{n=1}^{\infty} \frac{12(-1)^{n+1}}{n^3\pi^3} \sin n\pi x$	
C	**3**	Find the Fourier series for $f(x) = x^2$ in $-l < x < l$. **Answer**: $f(x) = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4\, l^2(-1)^n}{n^2\pi^2} \cos\left(\frac{n\pi x}{l}\right)$	
H	**4**	Expand $f(x) = x^2 - 2$ in $-2 < x < 2$ the Fourier series. **Answer**: $f(x) = -\frac{2}{3} + \sum_{n=1}^{\infty} \frac{16(-1)^n}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right)$	
C	**5**	Find the Fourier series of $f(x) = x^2 + x$ Where $-2 < x < 2$. **Answer**: $f(x) = \frac{4}{3} + \sum_{n=1}^{\infty} \left[\frac{16(-1)^n}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right) + \frac{4(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\right]$	
T	**6**	Find the Fourier expansion for function $f(x) = x - x^2$ in $-1 < x < 1$. **Answer**: $f(x) = -\frac{1}{3} + \sum_{n=1}^{\infty} \left[\frac{4(-1)^{n+1}}{n^2\pi^2} \cos(n\pi x) + \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)\right]$	
C	**7**	Obtain the Fourier series for $f(x) = x^2$ in the interval $-\pi < x < \pi$ and hence deduce that (i) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. (ii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$. **Answer**: $f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$	S – 16 S – 18
T	**8**	Find the Fourier series expansion of $f(x) = \lvert x\rvert; -\pi < x < \pi$. **Answer**: $f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2\,[(-1)^n - 1]}{\pi n^2} \cos nx$	W – 16
T	**9**	Find the Fourier series of $f(x) = x^3$; $x \in (-\pi, \pi)$. **Answer**: $f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\left[\pi^2 - \frac{6}{n^2}\right] \sin nx$	

H	10	Find the Fourier series of $f(x) = x - x^2; -\pi < x < \pi$. Deduce that: $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$. **Answer**: $f(x) = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4(-1)^{n+1}}{n^2}\cos nx + \frac{2(-1)^{n+1}}{n}\sin nx\right]$	S – 17
H	11	Find the Fourier series of $f(x) = x + x^2; -\pi < x < \pi$. Deduce that: $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$. **Answer**: $f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4(-1)^{n}}{n^2}\cos nx + \frac{2(-1)^{n+1}}{n}\sin nx\right]$	W – 17
C	12	Find the Fourier series of $f(x) = x + \lvert x\rvert; -\pi < x < \pi$. **Answer**: $f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2[(-1)^n - 1]}{\pi n^2}\cos nx + \frac{2(-1)^{n+1}}{n}\sin nx\right]$	W – 14 W – 15
H	13	Find the Fourier series to representation e^x in the the interval $(-\pi, \pi)$. **Answer**: $f(x) = \frac{e^{\pi} - e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty}\frac{(e^{\pi} - e^{-\pi})(-1)^n}{\pi(n^2+1)}[\cos nx + n\sin nx]$	
C	14	Determine the Fourier expansion of $f(x) = \begin{cases} 0, & -2 < x < -1 \\ 1 + x, & -1 < x < 0 \\ 1 - x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$. **Answer**: $f(x) = \frac{1}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1 - \cos\frac{n\pi}{2}\right)\left(\cos\frac{n\pi x}{2}\right)$	
T	15	Find the Fourier series for periodic function f(x) with period 2 Where $f(x) = \begin{cases} -1, -1 < x < 0 \\ 1, \ 0 < x < 1 \end{cases}$. **Answer**: $f(x) = \sum_{n=1}^{\infty}\frac{2 - 2(-1)^n}{\pi n}\sin n\pi x$	

C	**16**	Find Fourier series expansion of the function given by $f(x)=\begin{cases}0, -2<x<0\\ 1,\ \ 0<x<2\end{cases}.$ **Answer**: $\mathbf{f(x)=\frac{1}{2}+\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]}$	
T	**17**	Find the Fourier series for periodic function with period 2, which is given below $f(x)=\begin{cases}0, -1<x<0\\ x,\ \ 0<x<1\end{cases}.$ **Answer**: $\mathbf{f(x)=\frac{1}{4}+\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2\pi^2}\cos n\pi x+\frac{(-1)^{n+1}}{n\pi}\sin n\pi x\right]}$	
C	**18**	Find the Fourier series expansion of the function $f(x)=\begin{cases}-\pi, -\pi\le x\le 0\\ x\ ,\ \ 0\le x\le \pi\end{cases}$. Deduce that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$. **Answer**: $\mathbf{f(x)=-\frac{\pi}{4}+\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{\pi n^2}\cos nx+\frac{1-2(-1)^n}{n}\sin nx\right]}$	S – 16
T	**19**	Obtain the Fourier Series for the function f(x) given by $f(x)=\begin{cases}0\ , -\pi\le x\le 0\\ x^2\ ,\ \ 0\le x\le \pi\end{cases}$. Hence prove $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots=\frac{\pi^2}{12}$. **Answer**: $\mathbf{f(x)=\frac{\pi^2}{6}+\sum_{n=1}^{\infty}\left[\frac{2(-1)^n\cos nx}{n^2}+\frac{1}{\pi}\left\{-\frac{\pi^2(-1)^n}{n}+\frac{2(-1)^n}{n^3}-\frac{2}{n^3}\right\}\sin nx\right]}$	
H	**20**	Find the Fourier Series for the function f(x) given by $f(x)=\begin{cases}-\pi\ , -\pi<x<0\\ x-\pi\ ,\ \ 0<x<\pi\end{cases}.$ **Answer**: $\mathbf{f(x)=-\frac{3\pi}{4}+\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{\pi n^2}\cos nx+\frac{(-1)^{n+1}}{n}\sin nx\right]}$	

T	**21**	Find Fourier series for 2π periodic function $f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}$. Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$. **Answer**: $\mathbf{f(x) = \sum_{n=1}^{\infty} \frac{2k[1-(-1)^n]}{n\pi} \sin nx}$	
C	**22**	If $f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$, $f(x) = f(x + 2\pi)$,for all x then expand f(x) in a Fourier series. **Answer**: $\mathbf{f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}[1-(-1)^n] \cos nx}$	
H	**23**	Find the Fourier Series for the function f(x) given by $f(x) = \begin{cases} 1 + \frac{2x}{\pi} & ; -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi} & ; \ 0 \le x \le \pi \end{cases}$. Hence prove $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$. **Answer**: $\mathbf{f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}[1-(-1)^n] \cos nx}$	

❖ HALF RANGE SERIES:

✓ If a function f(x) is defined only on a half interval $(0, L)$ instead of $(c, c + 2L)$, then it is possible to obtain a Fourier cosine or Fourier sine series.

❖ HALF RANGE COSINE SERIES IN THE INTERVAL $(0, L)$:

$$\mathbf{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)}$$

Where the constants a_0 and a_n are given by

$$a_0 = \frac{2}{L}\int_0^L f(x)\, dx$$

$$a_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

METHOD – 3: EXAMPLE ON HALF COSINE SERIES IN THE INTERVAL $(0, L)$

H	**1**	Find Fourier cosine series for $f(x) = x^2;\ 0 < x \le \pi$. **Answer**: $f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$	S – 15
H	**2**	Find Half-range cosine series for $f(x) = (x-1)^2$ in $0 < x < 1$. **Answer**: $f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \cos(n\pi x)$	S – 15
C	**3**	Find a cosine series for $f(x) = \pi - x$ in the interval $0 < x < \pi$ **Answer**: $f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2[(-1)^{n+1} + 1]}{\pi n^2} \cos nx$	W – 17
H	**4**	Find a cosine series for $f(x) = e^x$ in $0 < x < L$. **Answer**: $f(x) = \frac{e^L - 1}{L} + \sum_{n=1}^{\infty} \frac{2L[e^L(-1)^n - 1]}{n^2\pi^2 + L^2} \cos\left(\frac{n\pi x}{L}\right)$	
C	**5**	Find Half-range cosine series for $f(x) = e^{-x}$ in $0 < x < \pi$. **Answer**: $f(x) = \frac{1 - e^{\pi}}{\pi} + \sum_{n=1}^{\infty} \frac{2[e^{-\pi}(-1)^{n+1} + 1]}{\pi(n^2+1)} \cos nx$	W – 15
C	**6**	Find Half range cosine series for $\sin x$ in $(0, \pi)$ **Answer**: $f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \left[\frac{-(-1)^{1+n} + 1}{1+n} + \frac{-(-1)^{1-n} + 1}{1-n}\right] \cos nx,\ a_1 = 0$	W – 14 S – 18
T	**7**	Find Half-Range cosine series for $f(x) = \begin{cases} kx & ; 0 \le x \le \frac{l}{2} \\ k(l-x); & \frac{l}{2} \le x \le l \end{cases}$. And hence deduce that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$. Answer: $f(x) = \frac{kl}{4} + \frac{2kl}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[2\cos\left(\frac{n\pi}{2}\right) - 1 - (-1)^n\right] \cos\left(\frac{n\pi x}{l}\right)$	S – 16

❖ HALF RANGE SINE SERIES IN THE INTERVAL $(0, L)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

Where the constants a_0 and a_n are given by

$$a_0 = \frac{2}{L}\int_0^L f(x)\, dx$$

$$a_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

METHOD – 4: EXAMPLE ON HALF SINE SERIES IN THE INTERVAL $(0, L)$

C	1	Find the Half range sine series for $f(x) = 2x, 0 < x < 1$. **Answer**: $f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}(-1)^{n+1} \sin n\pi x$	S – 15
H	2	Expand $\pi x - x^2$ in a half-range sine series in the interval $(0, \pi)$ up to first three terms. **Answer**: $f(x) = \sum_{n=1}^{\infty} \frac{4[(-1)^{n+1} + 1]}{n^3\pi} \sin nx$	
T	3	Find half range sine series of $f(x) = x^3$, $0 < x < \pi$. **Answer**: $f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^n \left[\frac{6}{n^2} - \pi^2\right] \sin nx$	S – 17
H	4	Find Half-range sine series for $f(x) = e^x$ in $0 < x < \pi$. **Answer**: $f(x) = \sum_{n=1}^{\infty} \frac{2n[e^{\pi}(-1)^{n+1} + 1]}{\pi(1 + n^2)} \sin nx$	S – 15
C	5	Find the sine series $f(x) = \begin{cases} x & ; 0 < x < \frac{\pi}{2} \\ \pi - x ; & \frac{\pi}{2} < x < \pi \end{cases}$. **Answer**: $f(x) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi} \sin\left(\frac{n\pi}{2}\right) \sin nx$	W – 14

H	6	Find Half range sine series for $\cos 2x$ in $(0, \pi)$. **Answer**: $f(x) = \sum_{\substack{n=1 \\ n\neq 2}}^{\infty} \frac{2n[(-1)^{n+1}+1]}{\pi(n^2-4)} \sin nx,\ b_2 = 0$	W – 15

❖ FOURIER INTEGRALS

✓ Fourier Integral of f(x) is given by

$$f(x) = \int_0^{\infty} [A(\omega)\cos \omega x + B(\omega)\sin \omega x]\, d\omega$$

$$\text{Where, } A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos \omega x\, dx \quad \& \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin \omega x\, dx$$

❖ FOURIER COSINE INTEGRAL

✓ Fourier Cosine Integral of f(x) is given by

$$f(x) = \int_0^{\infty} A(\omega)\cos \omega x\, d\omega$$

$$\text{Where, } A(\omega) = \frac{2}{\pi}\int_0^{\infty} f(x)\cos \omega x\, dx$$

❖ FOURIER SINE INTEGRAL

✓ Fourier Sine Integral of f(x) is given by

$$f(x) = \int_0^{\infty} B(\omega)\sin \omega x\, d\omega$$

$$\text{Where, } B(\omega) = \frac{2}{\pi}\int_0^{\infty} f(x)\sin \omega x\, dx$$

METHOD – 5: EXAMPLE ON FOURIER INTERGRAL

C	1	Using Fourier integral, Prove that $\int_0^\infty \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}\,d\omega = \begin{cases} 0 & ; x < 0 \\ \pi/2 & ; x = 0 \\ \pi e^{-x} & ; x > 0. \end{cases}$	S – 15
C	2	Find the Fourier integral representation of $f(x) = \begin{cases} 1 & ; \lvert x\rvert < 1 \\ 0 & ; \lvert x\rvert > 1 \end{cases}$. Hence calculate the followings. a) $\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$ b) $\int_0^\infty \frac{\sin\omega}{\omega}\,d\omega$. **Answer**: $\mathbf{f(x) = \int_0^\infty \frac{2\sin\omega}{\pi\omega}\cos\omega x\,d\omega}$ **a)** $\begin{cases} \frac{\pi}{2} & ; \lvert x\rvert < 1 \\ 0 & ; \lvert x\rvert > 1 \end{cases}$ **b)** $\frac{\pi}{2}$	W – 14 W – 16
H	3	Find the Fourier integral representation of $f(x) = \begin{cases} 2 & ; \lvert x\rvert < 2 \\ 0 & ; \lvert x\rvert > 2. \end{cases}$ **Answer**: $\mathbf{f(x) = \int_0^\infty \frac{4\sin 2\omega\cos\omega x}{\pi\omega}\,d\omega}$	S – 16 S – 17
C	4	Find the Fourier cosine integral of $f(x) = e^{-kx}$ $(x > 0, k > 0)$. **Answer**: $\mathbf{f(x) = \int_0^\infty \frac{2k}{\pi(k^2+\omega^2)}\cos\omega x\,d\omega}$	W – 16
H	5	Find the Fourier cosine integral of $f(x) = \frac{\pi}{2}e^{-x}, x \geq 0$. **Answer**: $\mathbf{f(x) = \int_0^\infty \frac{1}{(1+\omega^2)}\cos\omega x\,d\omega}$	W – 15
C	6	Using Fourier integral prove that $\int_0^\infty \frac{1-\cos\omega\pi}{\omega}\sin\omega x\,d\omega = \begin{cases} \frac{\pi}{2} & ; 0 < x < \pi \\ 0 & ; x > \pi. \end{cases}$	
H	7	Express $f(x) = \begin{cases} 1 & ; 0 \leq x \leq \pi \\ 0 & ; x > \pi. \end{cases}$ as a Fourier Sine integral and hence evaluate $\int_0^\infty \frac{1-\cos\lambda\pi}{\lambda}\sin\lambda x\,d\lambda$.	W – 17

T	**8**	Find Fourier cosine and sine integral of $f(x) = \begin{cases} \sin x \ ; 0 \le x \le \pi \\ 0 \qquad ; x > \pi. \end{cases}$ **Answer**: a) $\mathbf{f(x) = \int_0^\infty \frac{2(1+\cos\omega\pi)}{\pi\,(1-\omega^2)} \cos\omega x \, d\omega \ ; A(1) = 0}$ b) $\mathbf{f(x) = \int_0^\infty \frac{2\sin\omega\pi}{\pi\,(1-\omega^2)} \sin\omega x \ d\omega \ ; B(1) = 1}$	
T	**9**	Show that $\int_0^\infty \frac{\lambda^3 \sin\lambda x}{\lambda^4+4} d\lambda = \frac{\pi}{2} e^{-x} \cos x, x > 0.$	W – 15

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UNIT-3A » DIFFERENTIAL EQUATION OF FIRST ORDER

❖ INTRODUCTION:

- ✓ A differential equation is a mathematical equation which involves differentials or differential coefficients. Differential equations are very important in engineering problem. Most common differential equations are Newton's Second law of motion, Series RL, RC, and RLC circuits, etc.
- ✓ Mathematical modeling reduces many Natural phenomenon (real world problem) to differential equation(s).
- ✓ In this chapter, we will study, the method of obtaining the solution of ordinary differential equation of first order.

❖ DEFINITION: DIFFERENTIAL EQUATION:

- ✓ An eqn. which involves differential co-efficient is called a Differential Equation.

e.g. $\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = 0$

❖ DEFINITION: ORDINARY DIFFERENTIAL EQUATION:

- ✓ An eqn. which involves function of single variable and ordinary derivatives of that function then it is called an Ordinary Differential Equation.

e.g. $\frac{dy}{dx} + y = 0$

❖ DEFINITION: PARTIAL DIFFERENTIAL EQUATION:

- ✓ An eqn. which involves function of two or more variables and partial derivatives of that function then it is called a Partial Differential Equation.

e.g. $\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = 0$

❖ DEFINITION: ORDER OF DIFFERENTIAL EQUATION:

- ✓ The order of highest derivative which appeared in a differential equation is "Order of D.E".

e.g. $\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} + 5y = 0$ has order 1.

❖ DEFINITION: DEGREE OF DIFFERENTIAL EQUATION:

- When a D.E. is in a polynomial form of derivatives, the highest power of highest order derivative occurring in D.E. is called a "Degree of D.E.".

  e.g. $\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} + 5y = 0$ has degree 2.

NOTE:

- To determine the degree, the D.E has to be expressed in a polynomial form in the derivatives. If the D.E. cannot be expressed in a polynomial form in the derivatives, the degree of D.E. is not defined.

METHOD – 1: EXAMPLE ON ORDER AND DEGREE OF DIFFERENTIAL EQUATION

C	**1**	Find order and degree of $\frac{d^2y}{dx^2} = \left[y + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{4}}$. **Answer: 2, 4**	
H	**2**	Find order and degree of $y = x\frac{dy}{dx} + \frac{x}{\frac{dy}{dx}}$. **Answer: 1, 2**	
C	**3**	Find order and degree of $\left(\frac{d^2y}{dx^2}\right)^3 = \left[x + \sin\left(\frac{dy}{dx}\right)\right]^2$. **Answer: 2, Undefined**	S – 15
H	**4**	Define order and degree of the differential equation. Find order and degree of differential equation $\sqrt{x^2\frac{d^2y}{dx^2} + 2y} = \frac{d^3y}{dx^3}$. **Answer: 3, 2**	S – 17
H	**5**	Find order and degree of differential equation $dy = (y + \sin x)dx$. **Answer: 1, 1**	S – 16

❖ SOLUTION OF A DIFFERENTIAL EQUATION:

✓ A solution or integral or primitive of a differential equation is a relation between the variables which does not involve any derivative(s) and satisfies the given differential equation.

❖ GENERAL SOLUTION (G.S.):

✓ A solution of a differential equation in which the number of arbitrary constants is equal to the order of the differential equation, is called the General solution or complete integral or complete primitive.

❖ PARTICULAR SOLUTION:

✓ The solution obtained from the general solution by giving a particular value to the arbitrary constants is called a particular solution.

❖ SINGULAR SOLUTION:

✓ A solution which cannot be obtained from a general solution is called a singular solution.

❖ LINEAR DIFFERENTIAL EQUATION:

✓ A differential equation is called "LINEAR DIFFERENTIAL EQUATION" if the dependent variable and every derivatives in the equation occurs in the first degree only and they should not be multiplied together.

✓ Examples:

(1) $\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = 0$ is linear.

(2) $\frac{d^2y}{dx^2} + y\frac{dy}{dx} + y = 0$ is non-linear.

(3) $\frac{d^2y}{dx^2} + x^2\left(\frac{dy}{dx}\right)^2 + y = 0$ is non-linear.

✓ A Linear Differential Equation of first order is known as Leibnitz's linear Differential Equation

i.e. $\frac{dy}{dx} + P(x)y = Q(x) + c$ OR $\frac{dx}{dy} + P(y)x = Q(y) + c$

❖ TYPE OF FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATION:

- ✓ Variable Separable Equation
- ✓ Homogeneous Differential Equation
- ✓ Linear(Leibnitz's) Differential Equation
- ✓ Bernoulli's Equation
- ✓ Exact Differential Equation

❖ VARIABLE SEPARABLE EQUATION:

- ✓ If a differential equation of type $\frac{dy}{dx} = f(x, y)$ can be converted into $M(x)dx = N(y)dy$, then it is known as a Variable Separable Equation.
- ✓ The general solution of a Variable Separable Equation is

$$\int \mathbf{M(x)dx} = \int \mathbf{N(y)dy + c}$$

- ✓ Where, c is an arbitrary constant.

❖ NOTE:

- ✓ For convenience, the arbitrary constant can be chosen in any suitable form for the answers. e. g. in the form $\log c, \tan^{-1} c, e^c, \sin c$, etc.

❖ REDUCIBLE TO VARIABLE SEPARABLE EQUATION:

- ✓ If a differential equation of type $\frac{dy}{dx} = f(x, y)$ can be converted into $\frac{dy}{dx} = \varphi\left(\frac{y}{x}\right)$ then it can be converted into variable separable equation by taking $y = vx$ & $\frac{dy}{dx} = x\frac{dv}{dx} + v$.

METHOD – 2: EXAMPLE ON VARIABLE SEPARABLE METHOD

C	**1**	Solve: $9\,y\,y' + 4\,x = 0$. **Answer**: $\mathbf{9y^2 + 4x^2 = c}$	S – 16
H	**2**	Solve $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$ by variable separable method. **Answer**: $\mathbf{e^y = e^x + \frac{x^3}{3} + c}$	S – 18

C	**3**	Solve: $xy' + y = 0\,;\, y(2) = -2$. **Answer**: $\mathbf{x \cdot y = -4}$	
H	**4**	Solve: $L\dfrac{dI}{dt} + RI = 0, I(0) = I_0$. **Answer**: $\mathbf{I = I_0 \cdot e^{-\frac{R}{L}t}}$	
T	**5**	Solve: $(1 + x)ydx + (1 - y)xdy = 0$. **Answer**: $\mathbf{\log(xy) + x - y = c}$	
C	**6**	Solve the following differential equation using variable separable method. $3\, e^x \tan y\, dx + (1 + e^x) \sec^2 y\;\, dy = 0$ **Answer**: $\mathbf{(1 + e^x)^{-3} = c \cdot \tan y}$	W – 17
H	**7**	Solve: $e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$. **Answer**: $\mathbf{(1 - e^x)^{-1} \tan y = c}$	
T	**8**	Solve: $xy' = y^2 + y$. **Answer**: $\mathbf{\dfrac{y}{y+1} = xc}$	
C	**9**	Solve: $xy\dfrac{dy}{dx} = 1 + x + y + xy$. **Answer**: $\mathbf{y - \log(1 + y) = \log x + x + c}$	
H	**10**	Solve: $\tan y\dfrac{dy}{dx} = \sin(x + y) + \sin(x - y)$. **Answer**: $\mathbf{\sec y = -2\cos x + c}$	
H	**11**	Solve: $1 + \dfrac{dy}{dx} = e^{x+y}$. **Answer**: $\mathbf{-(e^{-x-y}) = x + c}$	
T	**12**	Solve: $\dfrac{dy}{dx} = \cos x \cos y - \sin x \sin y$. **Answer**: $\mathbf{\tan\left(\dfrac{x+y}{2}\right) = x + c}$	
H	**13**	Solve: $x\dfrac{dy}{dx} = y + x\, e^{\frac{y}{x}}$. **Answer**: $\mathbf{-\left(e^{-\frac{y}{x}}\right) = \log x + c}$	
C	**14**	Solve: $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\left(\dfrac{y}{x}\right)$. **Answer**: $\mathbf{\sin}(y/x) \mathbf{= x \cdot c}$	

❖ LEIBNITZ'S (LINEAR) DIFFERENTIAL EQUATION:

	Form - 1	Form -2
Form of differential equation	$\frac{dy}{dx} + P(x)y = Q(x)$	$\frac{dx}{dy} + P(y)x = Q(y)$
Integrating factor	$I.F. = e^{\int P(x)\,dx}$	$I.F. = e^{\int P(y)\,dy}$
Solution	$y\,(I.F.) = \int Q(x)\,(I.F.)dx \; + c$	$x\,(I.F.) = \int Q(y)\,(I.F.)dy \; + c$

METHOD – 3: EXAMPLE ON LEIBNITZ'S DIFFERENTIAL EQUATION

C	**1**	Solve: $y' + y\sin x = e^{\cos x}$. **Answer**: $\mathbf{y\,e^{-\cos x} = x + c}$	
H	**2**	Solve: $\frac{dy}{dx} + \frac{1}{x^2}y = 6\,e^{\frac{1}{x}}$. **Answer**: $\mathbf{y\,e^{\frac{-1}{x}} = 6\,x + c}$	
H	**3**	Solve: $y' + 6x^2y = \frac{e^{-2x^3}}{x^2}, y(1) = 0$. **Answer**: $\mathbf{y\,e^{2x^3} = \left(1 - \frac{1}{x}\right)}$	
C	**4**	Solve: $(x+1)\frac{dy}{dx} - y = (x+1)^2e^{3x}$. **Answer**: $\mathbf{\frac{y}{x+1} = \frac{e^{3x}}{3} + c}$	W – 16
H	**5**	Solve: $\frac{dy}{dx} + y = x$. **Answer**: $\mathbf{y\,e^x = e^x\,x - e^x + c}$	
H	**6**	Solve: $x\frac{dy}{dx} + (1+x)y = x^3$. **Answer**: $\mathbf{x\,y\,e^x = x^3e^x - 3\,x^2\,e^x + 6\,x\,e^x - 6\,e^x + c}$	
C	**7**	Solve: $\frac{dy}{dx} + (\cot x)y = 2\cos x$. **Answer**: $\mathbf{y \cdot \sin x = -\frac{\cos 2x}{2} + c}$	S – 16

H	**8**	Solve: $\frac{dy}{dx}+ y\tan x = \sin 2x\,, y(0) = 0$. **Answer**: $\mathbf{y \cdot \sec x = -2\cos x + 2}$	S – 17
T	**9**	Solve: $\frac{dy}{dx}+\frac{4x}{x^2+1}y=\frac{1}{(x^2+1)^3}$. **Answer**: $\mathbf{y \cdot (x^2+1)^2 = \tan^{-1}x + c}$	

❖ BERNOULLI'S DIFFERENTIAL EQUATION:

✓ A differential equation of the form $\frac{dy}{dx}+P(x)y = Q(x)y^n$ OR $\frac{dx}{dy}+P(y)x = Q(y)\,x^n$ is known as Bernoulli's Differential Equation. Where, $n \in \mathbb{R}-\{0,1\}$ such differential equation can be converted into linear differential equation and accordingly can be solved.

❖ EQUATION REDUCIBLE TO LINEAR DIFFERENTIAL EQUATION FORM:

✓ **CASE 1 :** A differential equation of the form $\frac{dy}{dx}+P(x)y = Q(x)\,y^n$...... (1)

Dividing both sides of equation (1) by y^n,

We get, $y^{-n}\frac{dy}{dx}+P(x)y^{1-n} = Q(x)$____(2)

Let, $y^{1-n} = v$

$$\Rightarrow (1-n)y^{-n}\frac{dy}{dx}=\frac{dv}{dx}$$

$$\Rightarrow y^{-n}\frac{dy}{dx}=\frac{1}{(1-n)}\frac{dv}{dx}$$

By Eqⁿ (2), $\frac{1}{(1-n)}\frac{dv}{dx}+P(x)v = Q(x)$

$$\Rightarrow \frac{dv}{dx}+P(x)(1-n)v = Q(x)(1-n)$$

Which is Linear Differential equation and accordingly can be solved.

✓ **CASE 2** : A differential of form $\frac{dy}{dx}+P(x)f(y) = Q(x)\,g(y)$ ......(3)

Dividing both sides of equation (3) by "g(y)" ,

We get, $\frac{1}{g(y)}\frac{dy}{dx}+P(x)\frac{f(y)}{g(y)}=Q(x)$ ......(4)

Let, $\frac{f(y)}{g(y)}=v$

Differentiate with respect to x both the sides,

Eq^n (4) becomes Linear Differential equation and accordingly can be solved.

METHOD – 4: EXAMPLE ON BERNOULLI'S DIFFERENTIAL EQUATION

H	**1**	Solve: $\frac{dy}{dx} + y = -\frac{x}{y}$. **Answer**: $\mathbf{y^2\, e^{2x} = -x\, e^{2x} + \frac{e^{2x}}{2} + c}$	
T	**2**	Solve the following Bernoulli's equation $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$ **Answer**: $\mathbf{\frac{1}{xy} = -\frac{1}{2x^2} + c}$	W – 17
C	**3**	Solve: $\frac{dy}{dx} + \frac{1}{x} y = x^3 y^3$. **Answer**: $\mathbf{\frac{1}{x^2\, y^2} = -x^2 + c}$	W – 15
H	**4**	Solve: $\frac{dy}{dx} + \frac{y}{x} = x^2 y^6$. **Answer**: $\mathbf{y^{-5}\, x^{-5} = \frac{5}{2} x^{-2} + c}$	
H	**5**	Solve: $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$. **Answer**: $\mathbf{\frac{e^{-y}}{x} = \frac{1}{2\, x^2} + c}$	S – 15
C	**6**	Solve: $\frac{dy}{dx} - \frac{\tan y}{1 + x} = (1 + x) e^x \sec y$. **Answer**: $\mathbf{\frac{\sin y}{1 + x} = e^x + c}$	
H	**7**	Solve: $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$. **Answer**: $\mathbf{e^{x^2} \tan y = \frac{(x^2 - 1) e^{x^2}}{2} + c}$	
H	**8**	Solve: $\frac{dy}{dx} - 2\, y \tan x = y^2 \tan^2 x$. **Answer**: $\mathbf{\frac{\sec^2 x}{y} = -\frac{\tan^3 x}{3} + c}$	

C	**9**	Solve: $(x^3 y^2 + x y)dx = dy$. **Answer**: $\mathbf{\frac{e^{\frac{x^2}{2}}}{y} = (2 - x^2)e^{\frac{x^2}{2}} + c}$	
T	**10**	Solve: $x\frac{dy}{dx} + y = y^2 \log x$. **Answer**: $\mathbf{y(\log x + 1) + c\,x\,y = 1}$	
H	**11**	Solve: $e^y \frac{dy}{dx} + e^y = e^x$. **Answer**: $\mathbf{e^{x+y} = \frac{e^{2x}}{2} + c}$	

❖ EXACT DIFFERENTIAL EQUATION:

- ✓ A differential equation of the form $M(x, y)dx + N(x, y)dy = 0$ is said to be Exact Differential Equation if it can be derived from its primitive by direct differential without any further transformation such as elimination etc.
- ✓ The necessary and sufficient condition for differential equation to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
- ✓ Where first order continuous partial derivative of M and N must be exist at all points of $f(x, y)$.
- ✓ The general solution of Exact Differential Equation is

$$\int_{y=\text{constant}} M\,dx + \int (\text{terms of N free from x})dy = c$$

Where, c is an arbitrary constant.

METHOD – 5: EXAMPLE ON EXACT DIFFERENTIAL EQUATION

H	**1**	Solve: $(x^3 + 3xy^2)dx + (y^3 + 3x^2y)dy = 0$. **Answer**: $\mathbf{\frac{x^4}{4} + \frac{3x^2y^2}{2} + \frac{y^4}{4} = c}$	W – 14

C	**2**	Check whether the given differential equation is exact or not $(x^4 - 2xy^2 + y^4)dx - (2x^2y - 4xy^3 + \sin y)dy$ **Answer**: $\mathbf{\frac{x^5}{5} - x^2y^2 + xy^4 + \cos y = c}$	W – 17
H	**3**	Solve: $(x^2 + y^2)dx + 2xydy = 0$. **Answer**: $\mathbf{\frac{x^3}{3} + y^2x = c}$	
H	**4**	Solve: $2\,x\,y\,dx + x^2\,dy = 0$. **Answer**: $\mathbf{x^2y = c}$	
C	**5**	Solve: $\frac{dy}{dx} = \frac{x^2 - x - y^2}{2xy}$. **Answer**: $\mathbf{\frac{x^3}{3} - \frac{x^2}{2} - xy^2 = c}$	W – 15
H	**6**	Solve: $ye^x dx + (2y + e^x)dy = 0.$ **Answer**: $\mathbf{ye^x + y^2 = c}$	S – 15
H	**7**	Solve: $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$. **Answer**: $\mathbf{(e^y + 1)\sin x = c}$	
H	**8**	Test for exactness and solve : $[(x + 1)e^x - e^y]dx - xe^y dy = 0, y(1) = 0$. **Answer**: $\mathbf{x(e^x - e^y) = e - 1}$	
C	**9**	Solve: $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$. **Answer**: $\mathbf{y\sin x + x\sin y + xy = c}$	W – 16
H	**10**	Solve: $\frac{y^2}{x}dx + (1 + 2y\log x)dy = 0, x > 0$. **Answer**: $\mathbf{y^2\log x + y = c}$	
H	**11**	Solve: $(y^2e^{xy^2} + 4x^3)dx + \left(2xye^{xy^2} - 3y^2\right)dy = 0$. **Answer**: $\mathbf{e^{xy^2} + x^4 - y^3 = c}$	

❖ NON-EXACT DIFFERENTIAL EQUATION:

✓ A differential equation which is not exact differential equation is known as Non-Exact Differential Equation. i.e. if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then given equation is Non-Exact Differential Equation.

❖ INTEGRATING FACTOR:

✓ A differential equation which is not exact can be made exact by multiplying it by a suitable function of x and y. Such a function is known as Integrating Factor.

❖ SOME STANDARD RULES FOR FINDING I.F.:

(1) If $Mx + Ny \neq 0$ and the given equation is Homogeneous, then $\text{I.F.} = \frac{1}{Mx+Ny}$.

(2) If $Mx - Ny \neq 0$ and the given equation is of the form $f(x, y)\, y\, dx + g(x, y)\, x\, dy = 0$ (OR Non-Homogeneous), then $\text{I.F.} = \frac{1}{Mx-Ny}$.

(3) If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ (i.e. function of only x), then $\text{I.F.} = e^{\int f(x)\, dx}$

(4) If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = g(y)$ (i.e. function of only y), then $\text{I.F.} = e^{\int g(y)\, dy}$

✓ Further, Multiply I.F. to given differential equation. So, It will converted to Exact Differential Equation.

✓ Now, we will get new $M'(x, y)$ and $N'(x, y)$.

Where, $M'(x, y) = M(x, y) \cdot (\text{I.F.})$ & $N'(x, y) = N(x, y) \cdot (\text{I.F.})$

✓ Later on it can be solved same method as used for exact differential equation.

i.e.

$$\int_{y=\text{constant}} M'\, dx + \int (\text{terms of } N' \text{ free from } x)dy = c$$

Where, c is an arbitrary constant.

METHOD – 6: EXAMPLE ON NON-EXACT DIFFERENTIAL EQUATION

H	**1**	Solve $x^2y\, dx - (x^3 + xy^2)dy = 0$. **Answer**: $-\frac{x^2}{2y^2} + \log y = c$	W – 14
C	**2**	Solve: $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$. **Answer**: $\frac{x}{y} - 2\log x + 3\log y = c$	

T	**3**	Solve: $(x^3+y^3)\,dx - x\,y^2dy = 0$. **Answer:** $\log x - \frac{y^3}{3x^3} = c$	
C	**4**	Solve $(x^4 + y^4)dx - xy^3dy = 0$. **Answer:** $\log x - \frac{y^4}{4x^4} = c$	S – 18
H	**5**	Solve: $(x + y)dx + (y - x)dy = 0$. **Answer:** $\frac{1}{2}\log(x^2 + y^2) + \tan^{-1}\left(\frac{x}{y}\right) = c$	
C	**6**	Solve: $(x^2y^2 + 2)ydx + (2 - x^2y^2)xdy = 0$. **Answer:** $\log\frac{x}{y} - \frac{1}{x^2y^2} = 2c$	W – 14
T	**7**	Solve: $y(1 + xy)dx + x(1 + xy + x^2y^2)dy = 0$. **Answer:** $\frac{1}{2x^2y^2} + \frac{1}{xy} - \log y = c$	
H	**8**	Solve: $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$. **Answer:** $-\frac{1}{xy} + \log\frac{x^2}{y} = 3c$	
T	**9**	Solve: $(x^2+y^2 + x)dx + xydy = 0$. **Answer:** $3x^4 + 6x^2y^2 + 4x^3 = 12c$	
C	**10**	Solve: $(x^2+y^2 + 3)dx - 2xydy = 0$. **Answer:** $x^2 - y^2 - 3 = cx$	S – 17
C	**11**	Solve: $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$. **Answer:** $x^3y^2 + \frac{x^2}{y} = c$	

❖ DEFINITION: ORTHOGONAL TRAJECTORY:

✓ A curve which cuts every member of a given family at right angles is a called an Orthogonal Trajectory.

❖ METHODS OF FINDING ORTHOGONAL TRAJECTORY OF $\mathbf{f(x, y, c) = 0}$

(1) Differentiate $f(x, y, c) = 0$ … (1) w.r.t. x.

(2) Eliminate c by using eqn …(1) and its derivative.

(3) Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$. This will give you differential equation of the orthogonal trajectories.

(4) Solve the differential equation to get the equation of the orthogonal trajectories.

❖ METHODS OF FINDING ORTHOGONAL TRAJECTORY OF $\mathbf{f(r, \theta, c) = 0}$

(1) Differentiate $f(r, \theta, c) = 0$ … (1) w.r.t. θ.

(2) Eliminate c by using eqn …(1) and its derivative

(3) Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$. This will give you differential eqn of the orthogonal trajectories.

(4) Solve the differential equation to get the equation of the orthogonal trajectories.

METHOD – 7: EXAMPLE ON ORTHOGONAL TREJECTORY

C	1	Find orthogonal trajectories of $y = x^2 + c$. **Answer:** $\mathbf{\log x + 2y = c}$	
H	2	Find the orthogonal trajectories of the family of circles $x^2 + y^2 = c^2$. **Answer:** $\mathbf{\frac{x}{y} = c}$	S – 16
T	3	Find the orthogonal trajectories of the family of circles $x^2 + y^2 = 2cx$. **Answer:** $\mathbf{x^2 + y^2 = 2ay}$	
C	4	Find Orthogonal trajectories of $r = \mathbf{a\ (1 - \cos\theta)}$. **Answer:** $\mathbf{r = \frac{c}{2}(1 + \cos\theta)}$	W – 17
H	5	Find Orthogonal trajectories of $r^n = a^n \cos n\theta$. **Answer:** $\mathbf{r^n = c^n \sin n\theta}$	

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UNIT-3B » DIFFERENTIAL EQUATION OF HIGHER ORDER

❖ INTRODUCTION:

- ✓ Many engineering problems such as Oscillatory phenomena, Bending of beams, etc. leads to the formulation and solution of Linear Ordinary Differential equations of second and higher order.
- ✓ In this chapter we will study, the method of obtaining the solution of Linear Ordinary Differential equations (homogeneous and nonhomogeneous) of second and higher order.

❖ HIGHER ORDER LINEAR DIFFERENTIAL EQUATION:

- ✓ A linear differential equation with more than one order is known as Higher Order Linear Differential Equation.
- ✓ A general linear differential equation of the nth order is of the form

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_ny = R(x) \ldots\ldots\ldots (A)$$

Where, $P_0, P_1, P_2, \ldots$ are functions of x.

❖ HIGHER ORDER LINEAR DIFFERENTIAL EQUATION WITH CONSTANT CO-EFFICIENT:

- ✓ The n^{th} order linear differential equation with constant co-efficient is

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_ny = R(x) \ldots\ldots\ldots (B)$$

Where, $a_0, a_1, a_2, \ldots$ are constants.

❖ NOTATIONS:

- ✓ Eq. (B) can be written in operator form by taking $D \equiv \frac{d}{dx}$ as below,

$$a_0D^ny + a_1D^{n-1}y + a_2D^{n-2}y + \cdots + a_ny = R(x) \ldots\ldots\ldots (C)$$

OR

$$[f(D)]y = R(x) \ldots\ldots\ldots (D)$$

❖ NOTE:

- ✓ An n^{th} order linear differential equation has n linear independent solution.

❖ AUXILIARY EQUATION:

- ✓ The auxiliary equation for nth order linear differential equation

$$a_0D^ny + a_1D^{n-1}y + a_2D^{n-2}y + \cdots + a_ny = R(x)$$

is derived by replacing D by m and equating with 0.

$$\text{i.e. } a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots + a_n = 0$$

❖ COMPLIMENTARY FUNCTION (C.F. -- y_c):

- ✓ A general solution of $[f(D)]y = 0$ is called complimentary function of $[f(D)]y = R(x)$.

❖ PARTICULAR INTEGRAL (P.I. -- y_p):

- ✓ A particular integral of $[f(D)]y = R(x)$ is $y = \frac{1}{f(D)}R(x)$.

❖ GENERAL SOLUTION [y (x)] OF HIGHER ORDER LINEAR DIFFERENTIAL EQUATION:

- ✓ $\text{G. S.} = \text{C. F} + \text{P. I.}$ i.e. $y(x) = y_c + y_p$

❖ NOTE:

- ✓ In case of higher order homogeneous differential equation, complimentary function is same as general solution.

❖ FORMULA:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2) \qquad a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$(a + b)^3 = a^3 + b^3 + 3ab(a + b) \qquad (a - b)^3 = a^3 - b^3 - 3ab(a - b)$$

❖ METHOD FOR FINDING C.F. OF HIGHER ORDER DIFFERENTIAL EQUATION:

- ✓ Consider, $a_0D^ny + a_1D^{n-1}y + a_2D^{n-2}y + \cdots + a_ny = R(x)$

  The Auxiliary equation is $a_0m^ny + a_1m^{n-1}y + a_2m^{n-2}y + \cdots + a_ny = 0$

- ✓ Let, $m_1, m_2, m_3, \ldots$ be the roots of auxiliary equation.

Case	Nature of the "n" roots	L.I. solutions	General Solutions
1)	$m_1 \neq m_2 \neq m_3 \neq \cdots$	$e^{m_1x}, e^{m_2x}, e^{m_3x}, \ldots$	$y = c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + \cdots$

2)	$m_1 = m_2 = m_3 = m$	$e^{m_1 x}, xe^{m_2 x}, x^2 e^{m_3 x}$	$y = (c_1 + c_2 x + c_3 x^2)e^{mx}$
3)	$m_1 = m_2 = m_3 = m$ $m_4 \neq m_5, \dots$	$e^{mx}, x\, e^{mx}, x^2 e^{mx}$, $e^{m_4 x}, e^{m_5 x}, \dots$	$y = (c_1 + c_2 x + c_3 x^2)e^{mx}$ $+c_4 e^{m_4 x} + c_5 e^{m_5 x} + \cdots$
4)	$m = p \pm iq$	$e^{px}\cos qx, e^{px}\sin qx$,	$y = e^{px}(c_1 \cos qx + c_2 \sin qx)$
5)	$m_1 = m_2 = p \pm iq$	$e^{px}\cos qx, xe^{px}\cos qx$, $e^{px}\sin qx, xe^{px}\sin qx$,	$y = e^{px}[(c_1 + c_2 x)\cos qx$ $+(c_3 + c_4 x)\sin qx]$

METHOD – 1: EXAMPLE ON HOMOGENEOUS DIFFERENTIAL EQUATION

C	**1**	Solve: $y'' + y' - 2y = 0$. **Answer**: $\mathbf{y = c_1 e^{-2x} + c_2 e^{x}}$	
T	**2**	Solve: $y'' - 9y = 0$. **Answer**: $\mathbf{y = c_1 e^{3x} + c_2 e^{-3x}}$	
H	**3**	Solve $y''' - 6y'' + 11y' - 6y = 0$ **Answer**: $\mathbf{y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}}$	S – 18
H	**4**	Solve: $y'' - 6y' + 9y = 0$. **Answer**: $\mathbf{y = (c_1 + c_2 x)\, e^{3x}}$	
C	**5**	Solve $y''' - 3y'' + 3y' - y = 0$ **Answer**: $\mathbf{y = (c_1 + c_2 x + c_3 x^2)\, e^{x}}$	
C	**6**	Solve: $\dfrac{d^4y}{dx^4} - 18\dfrac{d^2y}{dx^2} + 81y = 0$ **Answer**: $\mathbf{y = (c_1 + c_2 x)e^{3x} + (c_3 + c_4 x)e^{-3x}}$	
C	**7**	Solve: $16y'' - 8y' + 5y = 0$. **Answer**: $\mathbf{y = e^{\frac{x}{4}}\left(c_1 \cos\frac{x}{2} + c_2 \sin\frac{x}{2}\right)}$	
H	**8**	Solve: $(D^4 - 1)y = 0$. **Answer**: $\mathbf{y = c_1 e^{-x} + c_2 e^{x} + (c_3 \cos x + c_4 \sin x)}$	
C	**9**	Solve: $y''' - y = 0$. **Answer**: $\mathbf{y = c_1 e^{x} + e^{-\frac{1}{2}x}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)}$	

C	**10**	Solve: $y'' - 5y' + 6y = 0; y(1) = e^2, y'(1) = 3e^2$. **Answer**: $\mathbf{y = e^{3x-1}}$	
H	**11**	Solve: $y'' + 4y' + 4y = 0; y(0) = 1, y'(0) = 1$. **Answer**: $\mathbf{y = (1+3x)e^{-2x}}$	S – 16
T	**12**	Solve: $y'' - 4y' + 4y = 0; y(0) = 3, y'(0) = 1$. **Answer**: $\mathbf{y = (3-5x)e^{2x}}$	W – 14
H	**13**	Solve: $y''' - y'' + 100y' - 100y = 0; y(0) = 4, y'(0) = 11, y''(0) = -299$. **Answer**: $\mathbf{y = e^x + \sin 10x + 3\cos 10x}$	
T	**14**	Solve: $(D^4 + K^4)y = 0$. **Answer**: $\boldsymbol{y = e^{\frac{k}{\sqrt{2}}x}\left\{c_1\cos\frac{k}{\sqrt{2}}x + c_2\sin\frac{k}{\sqrt{2}}x\right\} + e^{-\frac{k}{\sqrt{2}}x}\left\{c_3\cos\frac{k}{\sqrt{2}}x + c_4\sin\frac{k}{\sqrt{2}}x\right\}}$	

❖ METHOD OF FINDING THE PARTICULAR INTEGRAL:

- ✓ There are many methods of finding the particular integral $\frac{1}{f(D)}R(x)$, we shall discuss following four main methods,
  - **(1)** General Methods
  - **(2)** Short-cut Methods involving operators
  - **(3)** Method of Undetermined Co-efficient
  - **(4)** Method of Variation of parameters

❖ GENERAL METHODS:

- ✓ Consider the differential equation

$$a_0D^ny + a_1D^{n-1}y + a_2D^{n-2}y + \cdots + a_ny = R(x)$$

$$\Rightarrow f(D)y = R(x)$$

$$\therefore \text{Particular Integral} = y_p = \frac{1}{f(D)}R(x)$$

- ✓ Particular Integral may be obtained by following two ways:

(1). Method of Factors:

- ✓ The operator $\frac{1}{f(D)}$ may be factorized into n linear factors; then the P.I. will be

$$P.I. = \frac{1}{f(D)} R(x) = \frac{1}{(D-m_1)(D-m_2)\ldots\ldots\ldots\ldots.(D-m_n)} R(x)$$

✓ Now, we know that,

$$\frac{1}{D-m_n} R(x) = e^{m_n x} \int R(x)\, e^{-m_n x} dx$$

✓ On operating with the first symbolic factor, beginning at the right, the particular integral will have form

$$P.I. = \frac{1}{(D-m_1)(D-m_2)\ldots\ldots\ldots\ldots.(D-m_{n-1})} e^{m_n x} \int R(x)\, e^{-m_n x} dx$$

✓ Then, on operating with the second and remaining factors in succession, taking them from right to left, one can find the desired particular integral.

(2). Method of Partial Fractions:

✓ The operator $\frac{1}{f(D)}$ may be factorized into n linear factors; then the P.I. will be

$$P.I. = \frac{1}{f(D)} R(x) = \left(\frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \cdots + \frac{A_n}{D-m_n}\right) R(x)$$

$$= A_1 \frac{1}{D-m_1} R(x) + A_2 \frac{1}{D-m_2} R(x) + \cdots + A_n \frac{1}{D-m_n} R(x)$$

✓ Using $\frac{1}{D-m_n} R(x) = e^{m_n x} \int R(x)\, e^{-m_n x} dx$, we get

$$P.I. = A_1 e^{m_1 x} \int R(x)\, e^{-m_1 x} dx + A_2 e^{m_2 x} \int R(x)\, e^{-m_2 x} dx + \cdots + A_n e^{m_n x} \int R(x)\, e^{-m_n x} dx$$

✓ Out of these two methods, this method is generally preferred.

❖ SHORTCUT METHOD:

✓ $R(x) = e^{ax}$

$$P.I. = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, \text{if } f(a) \neq 0$$

✓ If $f(a) = 0$, $P.I. = \frac{1}{f(D)} e^{ax} = \frac{x}{f'(a)} e^{ax}$, if $f'(a) \neq 0$

✓ In general, If $f^{n-1}(a) = 0$, $P.I. = \frac{1}{f(D)} e^{ax} = \frac{x^n}{f^n(a)} e^{ax}$, if $f^n(a) \neq 0$

✓ $R(x) = \sin(ax + b)$

$$\text{P.I.} = \frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b)\,,\text{if } f(-a^2) \neq 0$$

- ✓ If $f(-a^2) = 0$, $\text{P.I.} = \frac{1}{f(D^2)}\sin(ax+b) = \frac{x}{f'(-a^2)}\sin(ax+b)$, if $f'(-a^2) \neq 0$
- ✓ If $f'(-a^2) = 0$, $\text{P.I.} = \frac{1}{f(D^2)}\sin(ax+b) = \frac{x^2}{f''(-a^2)}\sin(ax+b)$, if $f''(-a^2) \neq 0$ and so on ...
- ✓ $R(x) = \cos(ax+b)$

$$\text{P.I.} = \frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b)\,,\text{if } f(-a^2) \neq 0$$

- ✓ If $f(-a^2) = 0$, $\text{P.I.} = \frac{1}{f(D^2)}\cos(ax+b) = \frac{x}{f'(-a^2)}\cos(ax+b)$, if $f'(-a^2) \neq 0$
- ✓ If $f'(-a^2) = 0$, $\text{P.I.} = \frac{1}{f(D^2)}\cos(ax+b) = \frac{x^2}{f''(-a^2)}\cos(ax+b)$, if $f''(-a^2) \neq 0$ and so on ...
- ✓ $R(x) = x^m; m > 0$
- ✓ In this case convert $f(D)$ in the form of $1+\phi(D)$ or $1-\phi(D)$ form.

$$\text{P.I.} = \frac{1}{f(D)}x^m = \frac{1}{1+\phi(D)}x^m = [\{1-\phi(D)+[\phi(D)]^2-\ \ldots\}]x^m$$

(Using Binomial Theorem)

❖ **NOTE:**

(1) $\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - \cdots$

(2) $\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + \cdots$

(3) $(1+h)^n = 1 + n\cdot\frac{h^1}{1!} + n\cdot(n-1)\cdot\frac{h^2}{2!} + n\cdot(n-1)\cdot(n-2)\cdot\frac{h^3}{3!} + \cdots$

- ✓ $R(x) = e^{ax}\,V(X)$, Where $V(X)$ is a function of x.

$$\text{P.I.} = \frac{1}{f(D)}e^{ax}\,V(x) = e^{ax}\frac{1}{f(D+a)}V(x)$$

METHOD – 2: EXAMPLE ON NON-HOMOGENEOUS DIFFERENTIAL EQUATION

C	1	Solve: $\frac{d^2y}{dx^2}+\frac{dy}{dx}-12y=e^{6x}$. **Answer**: $\mathbf{y=(c_1e^{-4x}+c_2e^{3x})+\frac{1}{30}e^{6x}}$	
H	2	Solve $y''-3y'+2y=e^{3x}$ **Answer**: $\mathbf{y=c_1e^{x}+c_2e^{2x}+\frac{e^{3x}}{2}}$	S – 18
H	3	Solve: $(D^2-2D+1)y=10e^{x}$. **Answer**: $\mathbf{y=(c_1+c_2x)e^{x}+5x^2e^{x}}$	S – 15
H	4	Solve: $\frac{d^2y}{dx^2}+2\frac{dy}{dx}-35y=12e^{5x}$. **Answer**: $\mathbf{y=c_1e^{-7x}+c_2e^{5x}+xe^{5x}}$	
T	5	Solve: $(D^3-7D+6)y=e^{2x}$. **Answer**: $\mathbf{y=c_1e^{x}+c_2e^{2x}+c_3e^{-3x}+\frac{x}{5}e^{2x}}$	
C	6	Solve: $y'''-3y''+3y'-y=4e^{t}$. **Answer**: $\mathbf{y=(c_1+c_2t+c_3t^2)e^{t}+\frac{2}{3}t^3e^{t}}$	W – 14
C	7	Solve: $y''-6y'+9y=6e^{3x}-5\log 2$. **Answer**: $\mathbf{y=(c_1+c_2x)\,e^{3x}+3x^2e^{3x}-\frac{5}{9}\log 2}$	
C	8	Solve: $(D^2-49)y=\sinh 3x$. **Answer**: $\mathbf{y=c_1e^{-7x}+c_2e^{7x}-\frac{1}{40}\sinh 3x}$	
H	9	Find the complete solution of $\frac{d^3y}{dx^3}+8y=\cosh(2x)$. **Answer**: $\mathbf{y=c_1e^{-2x}+e^{x}[c_2\cos(\sqrt{3}x)+c_3\sin(\sqrt{3}x)]+\frac{1}{32}e^{2x}+\frac{x}{24}e^{-2x}}$	W – 15
C	10	Solve: $(D^3-3D^2+9D-27)y=\cos 3x$. **Answer**: $\mathbf{y=c_1e^{3x}+c_2\cos 3x+c_3\sin 3x-\frac{x}{36}(\cos 3x+\sin 3x)}$	W – 16
H	11	Solve $(D^2+9)y=\cos 4x$ **Answer**: $\mathbf{y=c_1\cos 3x+c_2\sin 3x-\frac{1}{7}\cos 4x}$	S – 18

T	**12**	Find the steady state oscillation of the mass-spring system governed by the equation $y'' + 3y' + 2y = 20\cos 2t$. **Answer**: $\mathbf{y = c_1e^{-2t} + c_2e^{-t} + 3\sin 2t - \cos 2t}$	
T	**13**	Solve: $(D^2 + 4)y = \sin 2x$, given that $y = 0$ and $\frac{dy}{dx} = 2$ when $x = 0$. **Answer**: $\mathbf{y = \frac{9}{8}\sin 2x - \frac{x}{4}\cos 2x}$	
C	**14**	Solve: $(D^2 - 4D + 3)y = \sin 3x\cos 2x$. **Answer**: $\mathbf{y = c_1e^{x} + c_2e^{3x} + \frac{\sin x + 2\cos x}{20} + \frac{10\cos 5x - 11\sin 5x}{884}}$	
H	**15**	Solve complementary differential equation $\frac{d^2y}{dx^2} - \frac{6dy}{dx} + 9y = \sin x\cos 2x$. **Answer**: $\mathbf{y = (c_1 + c_2x)e^{3x} + \frac{\cos 3x}{36} - \frac{3\cos x + 4\sin x}{100}}$	W – 15
T	**16**	Solve: $(D^2 + 9)y = \cos 3x + 2\sin 3x$. **Answer**: $\mathbf{y = c_1\cos 3x + c_2\sin 3x - \frac{x}{3}\cos 3x + \frac{x}{6}\sin 3x}$	S – 16
C	**17**	Solve: $y'' + 2y' + 3y = 2x^2$. **Answer**: $\mathbf{y = e^{-x}\left(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x\right) + \left(\frac{2}{3}x^2 - \frac{8}{9}x + \frac{4}{27}\right)}$	W – 14
H	**18**	Solve: $(D^3 - D)y = x^3$. **Answer**: $\mathbf{y = c_1e^{x} + c_2e^{-x} + c_3 - \frac{x^4}{4} - 3x^2}$	W – 16
H	**19**	Solve: $(D^3 - D^2 - 6D)y = x^2 + 1$. **Answer**: $\mathbf{y = c_1 + c_2e^{3x} + c_3e^{-2x} - \frac{x^3}{18} + \frac{x^2}{36} - \frac{25x}{108}}$	
T	**20**	Find the general solution of the following differential equation $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = e^{x}\cos x$ **Answer**: $\mathbf{y = c_1\,e^{-2x} + e^{x}(c_2\cos x + c_3\sin x) + \frac{x\,e^{x}}{20}(3\sin x - \cos x)}$	W – 17
C	**21**	Solve: $(D^3 - D^2 + 3D + 5)y = e^{x}\cos 3x$. **Answer**: $\mathbf{y = c_1e^{-x} + e^{x}(c_2\cos 2x + c_3\sin 2x) - \frac{e^{x}}{65}(3\sin 3x + 2\cos 3x)}$	
H	**22**	Solve: $(D^2 - 5D + 6)y = e^{2x}\sin 2x$. **Answer**: $\mathbf{y = c_1e^{3x} + c_2e^{2x} + e^{2x}\left(-\frac{1}{5}\sin 2x + \frac{1}{10}\cos 2x\right)}$	

C	23	Solve: $(D^2 - 2D + 1)y = x^2 e^{3x}$. **Answer**: $\mathbf{y = (c_1 + c_2 x)e^x + \frac{e^{3x}}{4}\left(x^2 - 2x + \frac{3}{2}\right)}$	
H	24	Solve: $(D^4 - 16)y = e^{2x} + x^4$. **Answer**: $\mathbf{y = c_1 e^{2x} + c_2 e^{-x} + c_3 \cos 2x + c_4 \sin 2x + \frac{x}{32} e^{2x} - \frac{x^4}{16} - \frac{3}{32}}$	S – 17
T	25	Solve: $\frac{d^4 y}{dt^4} - 2\frac{d^2 y}{dt^2} + y = \cos t + e^{2t} + e^t$. **Answer**: $\mathbf{y = (c_1 + c_2 t)e^{-t} + (c_3 + c_4 t)e^t + \left(\frac{\cos t}{4} + \frac{e^{2t}}{9} + \frac{t^2 e^t}{8}\right)}$	
C	26	Solve: $(D^2 + 16)y = x^4 + e^{3x} + \cos 3x$. **Answer**: $\mathbf{y = c_1 \cos 4x + c_2 \sin 4x + \frac{1}{16}\left(x^4 - \frac{3x^2}{4} + \frac{3}{32}\right) + \frac{e^{3x}}{25} + \frac{\cos 3x}{7}}$	
H	27	Solve: $y'' + 4y = 8e^{-2x} + 4x^2 + 2; y(0) = 2, y'(0) = 2$. **Answer**: $\mathbf{y = \cos 2x + 2\sin 2x + e^{-2x} + x^2}$	

❖ METHOD OF UNDETERMINED CO-EFFICIENT:

✓ This method determines P.I. of $f(D)y = R(x)$. In this method we will assume a trial solution containing unknown constants, which will be obtained by substitution in $f(D)y = R(x)$. The trial solution depends upon $R(x)$ (the RHS of the given equation $f(D)y = R(x)$.

✓ Let the given equation be $f(D)y = R(x)$ ...... (A)

$\therefore$ The general solution of (A) is $Y = Y_C + Y_P$

✓ Here we guess the form of Y_P depending on X as per the following table.

	RHS of $\mathbf{f(D)y} = R(x)$	Form of Trial Solution
1.	$R(x) = e^{ax}$	$Y_P = Ae^{ax}$
e.g.	$R(x) = e^{2x}$ $R(x) = e^{2x} - 3e^{-x}$	$Y_P = Ae^{2x}$ $Y_P = Ae^{2x} + Be^{-x}$
2.	$R(x) = \sin ax$ $R(x) = \cos ax$	$Y_P = A \sin ax + B \cos ax$
e.g.	$R(x) = \cos 3x$	$Y_P = A \sin 3x + B \cos 3x$

	$R(x) = 2\sin(4x-5)$	$Y_P = A\sin(4x-5) + B\cos(4x-5)$
3.	$R(x) = a + bx + cx^2 + dx^3$ $R(x) = ax^2 + bx$ $R(x) = ax + b$ $R(x) = c$	$Y_P = A + Bx + Cx^2 + Dx^3$ $Y_P = A + Bx + Cx^2$ $Y_P = A + Bx$ $Y_P = A$
4.	$R(x) = e^{ax}\sin bx$ $R(x) = e^{ax}\cos bx$	$Y_P = e^{ax}(A\sin bx + B\cos bx)$
5.	$R(x) = xe^{ax}$ $R(x) = x^2e^{ax}$	$Y_P = e^{ax}(A + Bx)$ $Y_P = e^{ax}(A + Bx + Cx^2)$
6.	$R(x) = x\sin ax$ $R(x) = x^2\cos ax$	$Y_P = \sin ax\,(A + Bx) + \cos ax\,(C + Dx)$ $Y_P = \sin ax\,(A + Bx + Cx^2) + \cos ax\,(D + Ex + Fx^2)$

METHOD – 3: EXAMPLE ON UNDETERMINED CO-EFFICIENT

C	**1**	Solve: $y'' + 4y = 4\,e^{2x}$ **Answer**: $\mathbf{y = c_1\cos 2x + c_2\sin 2x + \frac{1}{2}e^{2x}}$	
C	**2**	Solve: $y'' + 4y = 2\sin 3x$. **Answer**: $\mathbf{y = c_1\cos 2x + c_2\sin 2x - \frac{2}{5}\sin 3x}$	
C	**3**	Solve: $y'' + 9y = 2x^2$. **Answer**: $\mathbf{y = c_1\cos 3x + c_2\sin 3x + \frac{2}{9}x^2 - \frac{4}{81}}$	S – 17
H	**4**	Solve $y'' - 2y' + 5y = 5x^3 - 6x^2 + 6x$ by method of undetermined coefficients. **Answer**: $\mathbf{y = e^x(c_1\cos 2x + c_2\sin 2x) + x^3}$	S – 18
T	**5**	Solve: $y'' + 4y' = 8x^2$. **Answer**: $\mathbf{y = c_{1+}c_2e^{-4x} + \frac{x}{4} - \frac{x^2}{2} + \frac{2}{3}x^3}$	S – 16
C	**6**	Solve: $y'' - 2y' + y = e^x + x$. **Answer**: $\mathbf{y = (c_1 + x\,c_2)e^x + \frac{x^2e^x}{2} + x + 2}$	

H	7	Solve the following differential equation using the method of undetermined coefficient : $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+4y=2x^2+3e^{-x}$ **Answer**: $\mathbf{y=e^{-x}(c_1\cos\sqrt{3}x+c_2\sin\sqrt{3}x)-\frac{1}{2}x+\frac{1}{2}x^2+e^{-x}}$	W – 17

❖ DEFINITION: WRONSKIAN:

✓ Wronskian of the n function $y_1, y_2, \ldots, y_n$ is defined and denoted by the determinant

$$W(y_1,y_2,\ldots y_n)=\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & y'_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

❖ THEOREM:

✓ Let $y_1, y_2, \ldots y_n$ be differentiable functions defined on some interval I. Then

(1) $y_1, y_2, \ldots y_n$ are linearly independent on I if and only if $W(y_1,y_2,\ldots y_n)\neq 0$ at least one value of $x\in I$.

(2) $y_1, y_2, \ldots y_n$ are linearly dependent on I then $W(y_1,y_2,\ldots y_n)=0$ for all $x\in I$.

METHOD – 4: EXAMPLE ON WRONSKIAN

H	1	Find wronskian of $x, \log x, x(\log x)^2$; $x>0$. **Answer**: **Linear Independent**	
C	2	Find the wronskian e^x, e^{-x} . **Answer**: **Linear Independent**	

❖ METHOD OF VARIATION OF PARAMETERS:

✓ The process of replacing the parameters of an analytic expression by functions is called variation of parameters.

✓ Consider, $y''+p(x)\,y'+q(x)\,y=R(x)$. Where p, q and R(x) are the functions of x.

✓ The general solution of **SECOND** order differential equation by the method of variation of parameters is

$$y(x) = y_c + y_p$$

Where, $y_c = c_1 y_1 + c_2 y_2$

$$y_p(x) = -y_1 \int y_2 \frac{R(x)}{W} dx + y_2 \int y_1 \frac{R(x)}{W} dx$$

- ✓ Note that, y_1 and y_2 are the solutions of $y'' + py' + qy = 0$, $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} \neq 0$.
- ✓ Consider, $y''' + p(x)\, y'' + q(x)\, y' + s(x)\, y = R(x)$. Where p, q, s and $R(x)$ are the functions of x.
- ✓ The general solution of **THIRD** order differential equation by the method of variation of parameters is

$$y(x) = y_c + y_p$$

Where, $y_c = c_1 y_1 + c_2 y_2 + c_3 y_3$ and $y_p(x) = A(x) y_1 + B(x) y_2 + C(x) y_3$

- ✓ Note that,

$$A(x) = \int (y_2 y_3' - y_3 y_2') \frac{R(x)}{W} dx$$

$$B(x) = \int (y_3 y_1' - y_1 y_3') \frac{R(x)}{W} dx$$

$$C(x) = \int (y_1 y_2' - y_2 y_1') \frac{R(x)}{W} dx$$

Also, $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \neq 0$.

METHOD – 5: EXAMPLE ON VARIATION OF PARAMETERS

C	**1**	Solve: $(D^2 - 4D + 4)y = \frac{e^{2x}}{x^5}$ **Answer**: $\mathbf{y = (c_1 + c_2 x)e^{2x} + \frac{1}{12}\frac{e^{2x}}{x^3}}$	
H	**2**	Solve: $y'' - 3y' + 2y = e^x$ **Answer**: $\mathbf{y = (c_1 e^{2x} + c_2 e^x) - e^x - x e^x}$	W – 16

H	3	Solve: $(D^2-1)y = x\,e^x$ **Answer**: $\mathbf{y = (c_1e^x + c_2e^{-x}) + \frac{e^x}{4}x^2 - \frac{e^x}{8}(1-2x)}$	S – 17
T	4	Use variation of parameter to find general sol^n of $y'' - 4y' + 4y = \frac{e^{2x}}{x}$. **Answer**: $\mathbf{y = e^{2x}(c_1 + c_2x - x + x\log x)}$	S – 17
C	5	Solve: $y'' + 2y' + y = e^{-x}\cos x$ **Answer**: $\mathbf{y = (c_1 + c_2x - \cos x)e^{-x}}$	
T	6	Solve: $(D^2 - 4D + 4)y = \frac{e^{2x}}{1+x^2}$ **Answer**: $\mathbf{y = (c_1 + c_2x)e^{2x} - e^{2x}\frac{1}{2}\log(1+x^2) + xe^{2x}(\tan^{-1}x)}$	
C	7	Find the solution of $y'' + a^2y = \tan ax$ by variation of parameter. **Answer**: $\mathbf{y = c_1\cos ax + c_2\sin ax + \left(\frac{-\cos ax\log(\sec ax + \tan ax)}{a^2}\right)}$	S – 16
H	8	Find solution of $\frac{d^2y}{dx^2} + 9y = \tan 3x$ using variation of parameter . **Answer**: $\mathbf{y = c_1\cos 3x + c_2\sin 3x - \frac{\cos 3x}{9}\log(\sec 3x + \tan 3x)}$	W – 15
T	9	Find the solution of $y'' + 4y = 4\tan 2x$ by variation of parameter. **Answer**: $\mathbf{y = c_1\cos 2x + c_2\sin 2x + -\cos 2x\log(\sec 2x + \tan 2x)}$	S – 18
H	10	Solve: $y'' + y = \sec x$. **Answer**: $\mathbf{y = c_1\cos x + c_2\sin x + x\sin x + \cos x\log(\cos x)}$	
C	11	Solve differential equation using variation of parameter $y'' + 9y = \sec 3x$. **Answer**: $\mathbf{y = c_1\cos 3x + c_2\sin 3x + \frac{1}{3}x\ \sin 3x + \frac{1}{9}\cos 3x\log\cos 3x}$	S – 15
H	12	Solve: $(D^2 + a^2)y = \operatorname{cosec} ax$ **Answer**: $\mathbf{y = c_1\cos ax + c_2\sin ax + \left(\frac{-x\cos ax}{a} + \frac{1}{a^2}\sin ax\log\lvert\sin x\rvert\right)}$	
H	13	Solve the following differential equation $\frac{d^2y}{dx^2} + y = \sin x$ using the method of variation of parameters. **Answer**: $\mathbf{y = c_1\cos x + c_2\sin x - \frac{x\ \cos x}{2} + \frac{\cos x\sin 2x}{4} - \frac{\cos x\cos 2x}{4}}$	W – 17

C	**14**	Solve: $\frac{d^3y}{dx^3}+\frac{dy}{dx}=\text{cosecx}$ **Answer**: $\mathbf{y=c_1+c_2cosax+c_3sinax+\log(\operatorname{cosec}x-\cot x)-\cos x\,(\log\sin x)-x\sin x}$	

❖ CAUCHY – EULER EQUATION

✓ An equation of the form

$$x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-2}}+a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}}+\cdots+a_{n-1}x\frac{dy}{dx}+a_ny=R(x)$$

✓ Where $a_1, a_2, \dots, a_n$ are constants and $R(x)$ is a function of x, is called Cauchy's homogeneous linear equation.

❖ STEPS TO CONVERT CAUCHY-EULER EQ. TO LINEAR DIFFERENTIAL EQ.:

✓ To reduce the above Cauchy – Euler Equation into a linear equation with constant coefficients, we use the transformation $x=e^z$ so that $z=\log x$.

✓ Now, $z=\log x \Longrightarrow \frac{dz}{dx}=\frac{1}{x}$

✓ Now, $\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}$

$$\Longrightarrow \frac{dy}{dx}=\frac{1}{x}\frac{dy}{dz}$$

$$\Longrightarrow x\frac{dy}{dx}=\frac{dy}{dz}=D\,y,\text{where } D=\frac{d}{dz}$$

✓ Similarly, $x^2\frac{d^2y}{dx^2}=D(D-1)y$ & $x^3\frac{d^3y}{dx^3}=D(D-1)(D-2)y$

✓ Using this transformation, the given equation reduces to

$$[D(D-1)(D-2)\dots(D-n+1)+a_1D(D-1)\dots(D-n+2)+\cdots+a_{n-1}D+a_n]y = f(e^z)$$

✓ This is a linear equation with constant coefficients, which can be solved by the methods discussed earlier.

METHOD – 6: EXAMPLE ON CAUCHY EULER EQUATION

C	1	Solve: $(x^2D^2 - 3xD + 4)y = 0; y(1) = 0, y'(1) = 3$ **Answer**: $\mathbf{y = 3\,x^2 \log x}$	
H	2	Solve: $x^2y'' - 2.5xy' - 2y = 0$ **Answer**: $\mathbf{y = c_1x^4 + c_2\frac{1}{\sqrt{x}}}$	
T	3	Solve: $x^2y'' - 4xy' + 6y = 21x^{-4}$ **Answer**: $\mathbf{y = c_1x^2 + c_2x^3 + \frac{1}{2}x^{-4}}$	
H	4	Solve: $(x^2D^2 - 3xD + 4)y = x^2; y(1) = 1, y'(1) = 0$ **Answer**: $\mathbf{y = (1 - 2\log x)x^2 + \frac{1}{2}x^2(\log x)^2}$	
C	5	Solve: $x^3y''' + 2x^2y'' + 2y = 10\left(x + \frac{1}{x}\right)$ **Answer**: $\mathbf{y = c_1\,x^{-1} + x\{c_2\cos(\log x) + c_3\sin(\log x)\} + 5x + 2x^{-1}\log x}$	
H	6	Solve: $(x^2D^2 - 3xD + 3)y = 3\ln x - 4$ **Answer**: $\mathbf{y = c_1x + c_2x^3 + \ln x}$	
C	7	Solve: $x^2D^2y - xDy + y = \sin(\log x)$ **Answer**: $\mathbf{y = (c_1 + c_2\log x)\,e^{\log x} + \frac{1}{2}\cos(\log x)}$	S – 17
T	8	Solve the following Cauchy-Euler equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x \cdot \sin(\log x)$ **Answer**: $\mathbf{y = c_1\cos(\log x) + c_2\sin(\log x) - \frac{(\log x)^2}{4}\cos(\log x) + \frac{\log x}{4}\sin\log x}$	W – 17
H	9	Solve: $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x^2\sin(\log x)$ **Answer**: $\mathbf{y = \left(c_1e^{-2\log x} + c_2e^{-\log x}\right) - \frac{x^2}{170}\left(7\cos(\log x) - 11\sin(\log x)\right)}$	W – 16
T	10	Solve completely the differential equation $x^2\frac{d^2y}{dx^2} - 6x\frac{dy}{dx} + 6y = x^{-3}\log x.$ **Answer**: $\mathbf{y = c_1x^6 + c_2x + \frac{1}{36x^3}\left(\log x + \frac{13}{36}\right)}$	W – 15
C	11	Solve: $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = x\log x$ **Answer**: $\mathbf{y = x[c_1\cos(\log x) + c_2\sin(\log x)] + x\log x}$	

❖ HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION:

- ✓ Reduction of order method for Linear second order O.D.E.
- ✓ **Step-1** Convert given D.E. into $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$ and find $P(x)$ & $Q(x)$.
- ✓ **Step-2** Find U.

$$U = \frac{1}{y_1^2} e^{-\int P\,dx}$$

- ✓ **Step-3** Find V.

$$V = \int U\ dx$$

- ✓ **Step-4** Second solution $y_2 = V \cdot y_1$
- ✓ Finally, General solution is $y = c_1 y_1 + c_2 y_2$.

METHOD – 7: EXAMPLE ON FINDING SECOND SOLUTION

C	1	Find second solution of $x^2y'' - x\,y' + y = 0, y_1 = x$. **Answer**: $\mathbf{y_2 = x \log x}$	
H	2	Find second solution of $x^2y'' - 4\,x\,y' + 6\,y = 0\,, y_1 = x^2\,; x > 0$. **Answer**: $\mathbf{y_2 = x^3}$	
T	3	Find second solution of $x\,y'' + 2\,y' + x\,y = 0, y_1 = \frac{\sin x}{x}$. **Answer**: $\mathbf{y_2 = -\frac{\cos x}{x}}$	

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UNIT-4 » SERIES SOLUTION OF DIFFERENTIAL EQUATION

❖ INTRODUCTION:

- ✓ If homogeneous linear differential equation has constant coefficients, it can be solved by algebraic methods, and its solutions are elementary functions known from calculus ($e^x, \cos x$, etc ...), as we know from unit-3. However, if such an equation has variable coefficients it must usually be solved by other methods (for example Euler-Cauchy equation).
- ✓ There are some linear differential equations which do not come in this category. In such cases we have to find a convergent power series arranged according to powers of the independent variable, which will approximately express the value of the dependent variables.
- ✓ We devote an entire Unit-4 to two standard methods of solution and their applications: Power Series Method and Frobenius Method (an extension of Power Series method).
- ✓ Before actually proceeding to solve linear ordinary differential equations with polynomial coefficient, we will look at some of the basic concepts which require for their study.

❖ DEFINITION: POWER SERIES:

- ✓ An infinite series of the form

$$\sum_{k=0}^{\infty} a_k(x-x_0)^k = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

is called a power series in $(x - x_0)$.

❖ DEFINITION: ANALYTIC FUNCTION:

- ✓ A function is said to be analytic at a point x_0 if it can be expressed in a power series near x_0.

❖ DEFINITION: ORDINARY AND SINGULAR POINT:

- ✓ Let $P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0$ be the given differential equation with variable co-efficient.
- ✓ Dividing by $P_0(x)$,

$$\frac{d^2y}{dx^2} + \frac{P_1(x)}{P_0(x)}\frac{dy}{dx} + \frac{P_2(x)}{P_0(x)}y = 0\,.$$

✓ Let, $P(x) = \frac{P_1(x)}{P_0(x)}$ & $Q(x) = \frac{P_2(x)}{P_0(x)}$

$$\therefore \frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0\,.$$

❖ DEFINITION: ORDINARY POINT AND SINGULAR POINT:

✓ A point x_0 is called an ordinary point of the differential equation if the functions $P(x)$ and $Q(x)$ both are analytic at x_0.

✓ If at least one of the functions $P(x)$ or $Q(x)$ is not analytic at x_0 then x_0 is called a singular point.

❖ DEFINITION: REGULAR SINGULAR POINT AND IRREGULAR SINGULAR POINT:

✓ A singular point x_0 is called regular singular point if both $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are analytic at x_0 otherwise it is called an irregular singular point.

METHOD – 1: EXAMPLE ON SINGULARITY OF DIFFERENTIAL EQUATION

C	**1**	Find singularity of $y'' + (x^2+1)y' + (x^3 + 2x^2 + 3x)y = 0$. **Answer: No singular points**	
H	**2**	Find singularity of $y'' + e^x y' + \sin(x^2)y = 0$. **Answer: No singular points**	
C	**3**	Find singularity of $x^3y'' + 5xy' + 3y = 0$. **Answer: $x = 0$ is an Irregular Singular Point.**	
H	**4**	Find singularity of $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$ **Answer: $x = 1$ & -1 are Regular Singular Point.**	S – 17
H	**5**	Find singularity of $x^3(x-1)y'' + 3(x-1)y' + 7xy = 0$ **Answer:** **$x = 1$ is a Regular Singular Point & $x = 0$ is an Irregular Singular Point.**	S – 17
T	**6**	Find singularity of $(x^2 + 1)y'' + xy' - xy = 0$. **Answer: $x = i, -i$ are Regular Singular Point.**	
C	**7**	Find singularity of $2x(x-2)^2y'' + 3xy' + (x-2)y = 0$. **Answer:** **$x = 0$ is a Regular Singular Point & $x = 2$ is an Irregular Singular Point.**	W – 15

H	**8**	Find singularity of $x(x+1)^2y'' + (2x-1)y' + x^2y = 0$. **Answer**: $\mathbf{x = 0}$ **is a Regular Singular Point** & $\mathbf{x = -1}$ **is an Irregular Singular Point**	
T	**9**	$x = 0$ is a regular singular point of $2x^2y'' + 3xy' + (x^2-4)y = 0$ say true or false. **Answer: True**	S – 16

❖ POWER SERIES SOLUTION AT AN ORDINARY POINT:

- ✓ A power series solution of a differential equation $P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0$ at an ordinary point x_0 can be obtained using the following steps.
- ✓ **Step-1:** Assume that

$$y = \sum_{k=0}^{\infty} a_k(x-x_0)^k = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

is the solution of the given differential equation.

- ✓ Differentiating with respect to x we get,

$$\Rightarrow y' = \frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots$$

$$\Rightarrow y'' = \frac{d^2y}{dx^2} = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots$$

- ✓ **Step-2:** Substitute the expressions of $y, \frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given differential equation.
- ✓ **Step-3:** Equate to zero the co-efficient of various powers of x and find a_2, a_3, a_4 ... etc. in terms of a_0 and a_1.
- ✓ **Step-4:** Substitute the expressions of $a_2, a_3, a_4, \ldots$ in

$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$ which will be the required solution.

METHOD – 2: EXAMPLE ON POWER SERIES METHOD

H	**1**	$y' + 2xy = 0$. **Answer**: $\mathbf{y = a_0 - a_0\,x^2 + \frac{1}{2}a_0x^4 - \frac{1}{6}a_0\,x^6 + \cdots}$	

H	**2**	$y'' + y = 0$. **Answer:** $\mathbf{y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{1}{6} a_1 x^3 + \frac{1}{24} a_0 x^4 + \frac{1}{120} a_1 x^5 + \cdots}$	
C	**3**	$y'' + xy = 0$ in powers of x. **Answer:** $\mathbf{y = a_0 + a_1 x - \frac{1}{6} a_0 x^3 - \frac{1}{12} a_1 x^4 + \frac{1}{180} a_0 x^6 + \cdots}$	W – 16 S – 16
T	**4**	$y'' + x^2 y = 0$. **Answer** $\mathbf{y = a_0 + a_1 x - \frac{1}{12} a_0 x^4 - \frac{1}{20} a_1 x^5 + \frac{1}{672} a_0 x^8 + \frac{1}{1440} a_1 x^9 + \cdots}$	
H	**5**	$y'' = y'$. **Answer:** $\mathbf{y = a_0 + a_1 x + \frac{1}{2} a_1 x^2 + \frac{1}{6} a_1 x^3 + \frac{1}{24} a_1 x^4 + \frac{1}{120} a_1 x^5 + \cdots}$	
C	**6**	Find the power series solution about $x = 0$ of $y'' + xy' + x^2 y = 0$. **Answer:** $\mathbf{y = a_0 \left\{1 - \frac{1}{12} x^4 + \frac{1}{90} x^6 + \cdots\right\} + a_1 \left\{x - \frac{1}{6} x^3 - \frac{1}{40} x^5 + \cdots\right\}}$	
H	**7**	$y'' - 2xy' + 2py = 0$. **Answer:** $\mathbf{y = a_0 + a_1 x - p\, a_0 x^2 + \frac{(1-p)}{3} a_1 x^3 - \frac{p\,(2-p)}{6} a_0 x^4 + \frac{(1-p)\,(3-p)}{30} a_1 x^5 + \cdots}$	
T	**8**	$(1 - x^2) y'' - 2xy' + 2y = 0$. **Answer:** $\mathbf{y = a_0 + a_1 x - a_0 x^2 - \frac{1}{3} a_0 x^4 - \frac{1}{5} a_0 x^6 + \cdots}$	W – 14
C	**9**	$\frac{d^2y}{dx^2}(1 - x^2) - x\frac{dy}{dx} + py = 0$. **Answer:** $\mathbf{y = a_0 + a_1 x - \frac{p}{2} a_0 x^2 + \frac{(1-p)}{6} a_1 x^3 - \frac{p(4-p)}{24} a_0 x^4 + \frac{(9-p)(1-p)}{120} a_1 x^5 + \cdots}$	
C	**10**	$(1 + x^2) y'' + xy' - 9y = 0$. **Answer:** $\mathbf{y = a_0 + a_1 x + \frac{9}{2} a_0 x^2 + \frac{4}{3} a_1 x^3 + \frac{15}{8} a_0 x^4 - \frac{7}{16} a_0 x^6 + \cdots}$	S – 15
H	**11**	$(x^2 + 1) y'' + xy' - xy = 0$ near $x = 0$. **Answer:** $\mathbf{y = a_0 + a_1 x + a_0 \frac{x^3}{6} - a_1 \frac{x^3}{6} + \left(\frac{a_1}{12}\right) x^4 - \left(\frac{3}{40}\right) a_0 x^5 + ..}$	S – 17

H	**12**	$(x-2)y''-x^2y'+9y=0$. **Answer**: $\mathbf{y=a_0\left(1+\frac{9\,x^2}{4}+\frac{9\,x^3}{24}+\frac{90\,x^4}{4}+\cdots\right)+a_1\left(x+\frac{18\,x^3}{24}+\frac{14\,x^4}{4}+\cdots\right)}$	W – 15
H	**13**	$(1-x^2)y''-2xy'+2y=0$. **Answer**: $\mathbf{y=a_0+a_1x-a_0x^2-\frac{a_0}{3}x^4+\cdots}$	W – 17

❖ FROBENIUS METHOD:

- ✓ Frobenius Method is used to find a series solution of a differential equation near regular singular point.
- ✓ **Step-1:** If x_0 is a regular singular point, we assume that the solution is

$$y=\sum_{k=0}^{\infty}a_k(x-x_0)^{m+k}$$

- ✓ Differentiating with respect to x ,we get

$$\Rightarrow\frac{dy}{dx}=\sum_{k=0}^{\infty}(m+k)a_k(x-x_0)^{m+k-1}$$

$$\Rightarrow\frac{d^2y}{dx^2}=\sum_{k=0}^{\infty}(m+k)(m+k-1)a_k(x-x_0)^{m+k-2}$$

- ✓ **Step-2:** Substitute the expressions of y , $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given differential equation.
- ✓ **Step-3:** Equate to zero the co-efficient of least power term in $(x-x_0)$, which gives a quadratic equation in m, called Indicial equation.
- ✓ The format of the series solution depends on the type of roots of the indicial equation.
  - ➢ Here we have the following three cases:
- ✓ **Case-I** Distinct roots not differing by an integer. $(m_1\neq m_2\ \&\ m_1-m_2\notin Z)$
  - ○ When $m_1-m_2\notin Z$, i.e. difference of $m_1$and m_2 is not a positive or negative integer.In this case, the series solution is obtained corresponding to both values of m. Let the solutions be $y=y_1$ and $y=y_2$, then the general solution is $y=c_1y_1+c_2y_2$.

- **Case-II** Equal roots. $(m_1 = m_2)$
  - In this case, we will have only one series solution. i.e. $y = y_1 = \sum_{k=0}^{\infty} a_k (x - x_0)^{m+k}$
  - In terms of a_0 and the variable m. The general solution is $y = c_1 (y_1)_m + c_2 \left(\frac{dy_1}{dm}\right)_m$
- **Case-III** Distinct roots differing by an integer. $(m_1 \neq m_2 \;\&\; m_1 - m_2 \in Z)$
  - When $m_1 - m_2 \in Z$, i.e. difference of m_1 & m_2 is a positive or negative integer.Let the roots of the indicial equation be m_1 & m_2 with $m_1 < m_2$. In this case, the solutions corresponding to the values m_1 & m_2 may or may not be linearly independent.Smaller root must be taken as m_1.
- Here we have the following two possibilities.
  - One of the co-efficient of the series becomes indeterminate for the smaller root m_1 and hence the solution for m_1 contains two arbitrary constants. In this case, we will not find solution corresponding to m_2.
  - Some of the co-efficient of the series becomes infinite for the smaller root m_1 ,then it is required to modify the series by replacing a_0 by $a_0(m + m_1)$. The two linearly independent solutions are obtained by substituting $m = m_1$ in the modified form of the series for y and in $\frac{dy}{dm}$ obtained from this modified form.

METHOD – 3: EXAMPLE ON FROBENIUS METHOD

C	**1**	$4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0.$ **Answer**: $\mathbf{y = c_1 a_0 \left(1 - \frac{x}{2} + \frac{x^2}{24} - \cdots\right) + c_2 a_0 \sqrt{x} \left(1 - \frac{x}{6} + \frac{x^2}{120} - \cdots\right)}$	
T	**2**	$xy'' + y' - y = 0\,.$ **Answer**: $\mathbf{y = c_1 a_0 \left(1 + x + \frac{x^2}{4} + \cdots\right) + c_2 a_0 \log x \left(1 + x + \frac{x^2}{4} + \cdots\right)}$ $\mathbf{-2c_2 a_0 \left(x + \frac{3}{8}x^2 + \cdots\right)}$	
H	**3**	$(x^2 - x)y'' - xy' + y = 0.$ **Answer**: $\mathbf{y = c_1 c_0 x + c_2 c_0 (x \log x + 1 - 3x)}$	

H	**4**	$2x^2y'' + xy' + (x^2-1)y = 0$ **Answer:** $y = c_1x\left(1 - \frac{x^2}{14} + \frac{x^4}{616} - \cdots\right) + c_2\frac{1}{\sqrt{x}}\left(1 + \frac{x^2}{2} + \frac{x^4}{40} + \cdots\right)$	W – 16
C	**5**	$3x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ **Answer:** $y = c_1a_0\left(1 + x + \frac{1}{8}x^2 + \cdots\right) + c_2a_0x^{\frac{1}{3}}\left(1 - \frac{3}{8}x + \frac{3}{112}x^2 - \cdots\right)$	W – 17
T	**6**	$8x^2y'' + 10xy' - (1+x)y = 0$ **Answer:** $y = c_1a_0x^{\frac{1}{4}}\left(1 + \frac{1}{14}x + \frac{1}{616}x^2 + \cdots\right) + \frac{c_2a_0}{x^2}\left(1 + \frac{1}{2}x + \frac{1}{40}x^2 + \cdots\right)$	S – 18

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UNIT-5 » LAPLACE TRANSFORM AND IT'S APPLICATION

❖ INTRODUCTION:

✓ Pierre Simon Marquis De Laplace (1749-1827) was French mathematician.

✓ The word Transform itself indicates about the conversion of one form to another form. The Laplace transforms is a versatile tool for solving differential equations as it transforms differential equations to algebraic equations and algebraic equations are comparatively easier to solve.

✓ By using Laplace transform we can find particular solution of a differential equation without determining the general solution. Laplace useful to finding solution of the problems related with mechanical or electrical driving force which has discontinuities, impulsive or complicated periodic functions. Also we can find solutions of a system of ODE, PDE and integral equations.

✓ In this unit, we will first develop the concept of Laplace transform through definition, properties and theorems. Then we will study about the solution of some differential equation(s).

❖ DEFINITION: LAPLACE TRANSFORM

✓ Let f(t) be a given function defined for all $t \geq 0$, then the Laplace transform of f(t) is denoted by $\mathcal{L}\{f(t)\}$ or $\bar{f}(s)$ or F(S), and is defined as

$$\boldsymbol{\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt}, \qquad \text{provided the integral exist.}$$

❖ PROPERTIES OF LAPLACE TRANSFORMS:

✓ **Linearity property of Laplace transform:** $\boldsymbol{\mathcal{L}\{\alpha \cdot f(t) + \beta \cdot g(t)\} = \alpha \cdot \mathcal{L}\{f(t)\} + \beta \cdot \mathcal{L}\{g(t)\}}$

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$\mathcal{L}\{\alpha \cdot f(t) + \beta \cdot g(t)\} = \int_0^\infty e^{-st}\, [\alpha \cdot f(t) + \beta \cdot g(t)]\, dt$$

$$= \int_0^\infty e^{-st} \cdot \alpha \cdot f(t)\, dt + \int_0^\infty e^{-st} \cdot \beta \cdot g(t)\, dt$$

$$= \alpha \cdot \mathcal{L}\{f(t)\} + \beta \cdot \mathcal{L}\{g(t)\}$$

✓ **Change of Scale of Laplace transform: If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{f(bt)\} = \frac{1}{b}F\left(\frac{s}{b}\right)$.**

Proof: By definition, $\mathcal{L}\{f(bt)\} = \int_0^\infty e^{-st}\, f(bt)\, dt$

Take $bt = u$ then $dt = \frac{du}{b}$ then by above, we have

$$\mathcal{L}\{f(bt)\} = \int_0^\infty e^{-\frac{su}{b}} f(u) \frac{du}{b}$$

$$= \frac{1}{b}\int_0^\infty e^{-\left(\frac{s}{b}\right)u} f(u)\, du$$

$$= \frac{1}{b}F\left(\frac{s}{b}\right)$$

Thus, $\mathcal{L}\{f(bt)\} = \frac{1}{b}F\left(\frac{s}{b}\right)$.

Example: **If $\mathcal{L}\{f(t)\} = \frac{s}{s^2+k^2}$, find $\mathcal{L}\{f(2t)\}$.**

Solution: $\mathcal{L}\{f(2t)\} = \frac{1}{2}\frac{\left(\frac{s}{2}\right)}{\left(\frac{s}{2}\right)^2+k^2} = \frac{s}{s^2+4k^2}$

❖ **LAPLACE TRANSFORM OF SOME STANDARD FUNCTION:**

(1). $\mathcal{L}\{t^n\} = \frac{1}{s^{n+1}}\overline{)n+1}$; **$n > -1$** OR $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ **; n is integer**

Proof: By definition, $L\{f(t)\} = \int_0^\infty e^{-st}\, f(t)\, dt$

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st}\, t^n\, dt$$

Let, $st = x \Longrightarrow sdt = dx$ & hence, When $t \longrightarrow 0 \Rightarrow x \rightarrow 0$ and $t \longrightarrow \infty \Rightarrow x \rightarrow \infty$

$$\Longrightarrow \mathcal{L}\{t^n\} = \int_0^\infty e^{-x}\frac{x^n}{s^n}\frac{dx}{s}$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-x}\, x^n\, dx$$

$\Longrightarrow \mathcal{L}\{t^n\} = \frac{1}{s^{n+1}}\overline{)n+1}$ ($\because$ By definition of Gamma function $\overline{)n} = \int_0^\infty e^{-x}\, x^{n-1}\, dx$)

✓ If **n** is a positive integer, then **n**! $= \overline{)n+1}$

$$\Rightarrow \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

(2). $\mathcal{L}\{1\} = \frac{1}{s}$.

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{0-1}{-s} = \frac{1}{s}$$

(3). $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $s > a$ **(S – 15)**

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \left[\frac{0-1}{-(s-a)}\right] \qquad [\text{When } t \longrightarrow \infty \Longrightarrow e^{-(s-a)t} \longrightarrow 0\ (\because s > a \Rightarrow s - a > 0)]$$

$$\Rightarrow \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

(4). $\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$, $s > -a$ **(W – 14)**

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at}\, dt$$

$$= \int_0^\infty e^{-(s+a)t}\, dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \left[\frac{0-1}{-(s+a)}\right] \qquad [\text{When } t \longrightarrow \infty \Longrightarrow e^{-(s+a)t} \longrightarrow 0(\because s > -a \Longrightarrow s + a > 0)]$$

$$= \frac{1}{s+a}$$

$$\Longrightarrow \mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$

(5). $\boldsymbol{\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}}$ **, $\boldsymbol{s > 0}$** and **a** is a constant.

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at \; dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty$$

$$= 0 - \frac{1}{s^2+a^2}(-a) \qquad [\text{When } t \longrightarrow \infty \Longrightarrow e^{-st} \longrightarrow 0 (\because s > 0)]$$

$$\Longrightarrow \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

(6). $\boldsymbol{\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}}$ **, $\boldsymbol{s > 0}$** and **a** is a constant.

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st} \cos at \; dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right]_0^\infty$$

$$= 0 - \frac{1}{s^2+a^2}(-s) \quad [\text{When } t \longrightarrow \infty \Longrightarrow e^{-st} \longrightarrow 0 \ (\because s > 0)]$$

$$\Longrightarrow \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

(7). $\boldsymbol{\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}}$ **, $\boldsymbol{s^2 > a^2}$ or $\boldsymbol{(s > |a|)}$ (S – 15)**

Proof: $\mathcal{L}\{\sinh at\} = \mathcal{L}\left\{\frac{e^{at} - e^{-at}}{2}\right\}$

$$= \frac{1}{2}[\mathcal{L}\{e^{at}\} - \mathcal{L}\{e^{-at}\}]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$\Longrightarrow \mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$$

(8). $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$, $s^2 > a^2$ or $(s > |a|)$

Proof: $L\{\cosh at\} = \mathcal{L}\left\{\frac{e^{at}+e^{-at}}{2}\right\}$

$$= \frac{1}{2}\left[\mathcal{L}\{e^{at}\} + \mathcal{L}\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right]$$

$$\Rightarrow \mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$$

METHOD – 1: EXAMPLE ON DEFINITION OF LAPLACE TRANSFORM

H	1	Find the Laplace transform of $f(t) = \begin{cases} 0 \ ; & 0 \le t < 3 \\ 4 \ ; & t \ge 3 \end{cases}$. **Answer:** $\frac{4e^{-3s}}{s}$	
C	2	Given that $f(t) = \begin{cases} t+1 \ ; & 0 \le t \le 2 \\ 3 \ ; & t \ge 2 \end{cases}$. Find $\mathcal{L}\{f(t)\}$. **Answer:** $\frac{-e^{-2s}}{s^2} + \frac{1}{s} + \frac{1}{s^2}$	
H	3	Find the Laplace transformation of $f(x) = \begin{cases} e^t & ; 0 < t < 1 \\ 0 & ; \quad t > 1 \end{cases}$. **Answer:** $\frac{e^{1-s}}{1-s} - \frac{1}{1-s}$	
C	4	Find the Laplace transform of $f(t) = \begin{cases} 0 & ; 0 < t < \pi \\ \sin t & ; \quad t > \pi \end{cases}$. **Answer:** $\frac{-e^{-\pi s}}{s^2+1}$	S – 15
H	5	Find the Laplace transform of $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$. **Answer:** $\frac{1}{s^2+1}\left(e^{-\pi s}(s-1) + s\right)$	

❖ SOME IMPORTANT FORMULAE:

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$\cos^3 A = \frac{\cos 3A + 3\cos A}{4} \qquad \cos^2 A = \frac{1+\cos 2A}{2}$$

$$\sin^3 A = \frac{3\sin A - \sin 3A}{4} \qquad \sin^2 A = \frac{1-\cos 2A}{2}$$

$$\cosh at = \frac{e^{at}+e^{-at}}{2} \qquad \cos at = \frac{e^{iat}+e^{-iat}}{2}$$

$$\sinh at = \frac{e^{at}-e^{-at}}{2} \qquad \sin at = \frac{e^{iat}-e^{-iat}}{2i}$$

METHOD – 2: EXAMPLE ON LAPLACE TRANSFORM OF SIMPLE FUNCTIONS

C	**1**	Find the Laplace transform of $t^3 + e^{-3t} + t^{\frac{1}{2}}$. **Answer**: $\frac{3!}{s^4} + \frac{1}{s+3} + \frac{\sqrt{\pi}}{2s^{3/2}}$	
H	**2**	Find the Laplace transform of $\sin\frac{t}{2} + 2^t + t^{\frac{4}{3}}$. **Answer**: $\frac{2}{(4s^2+1)} + \frac{1}{s-\log 2} + \frac{4\overline{)1/3}}{9s^{7/3}}$	
H	**3**	Find the Laplace transform of $t^5 + e^{-100t} + \cos 5t$. **Answer**: $\frac{5!}{s^6} + \frac{1}{s+100} + \frac{s}{s^2+25}$	
C	**4**	Find the Laplace transform of $100^t + 2t^{10} + \sin 10t$. **Answer**: $\frac{1}{s-\log_e 100} + \frac{2\cdot 10!}{s^{11}} + \frac{10}{s^2+100}$	
C	**5**	Find the Laplace transformation of $f(t) = \cosh^2 3t$. **Answer**: $\frac{1}{2s} + \frac{s}{2(s^2-36)}$	

H	**6**	Find $\mathcal{L}[(2t-1)^2]$. **Answer**: $\frac{8}{s^3}-\frac{4}{s^2}+\frac{1}{s}$	
C	**7**	Find the Laplace transformation (i) $\sin(\omega t+\alpha)$ (ii) $\cos(\omega t+b)$ **Answer**: (i) $\cos\alpha\,\frac{\omega}{s^2+\omega^2}+\sin\alpha\,\frac{s}{s^2+\omega^2}$ (ii) $\cos b\,\frac{s}{s^2+\omega^2}-\sin b\,\frac{\omega}{s^2+\omega^2}$	
C	**8**	Find $\mathcal{L}\{\sin 2t\cos 2t\}$. **Answer**: $\frac{2}{s^2+16}$	
H	**9**	Find $\mathcal{L}\{\sin 2t\sin 3t\}$. **Answer**: $\frac{1}{2}\left[\frac{s}{s^2+1}-\frac{s}{s^2+25}\right]$	
C	**10**	Find Laplace transform of $\cos^2(at)$ **Answer**: $\frac{s^2+2a^2}{s(s^2+4a^2)}$	
C	**11**	Find the Laplace transform of $\sin^2 3t$. **Answer**: $\frac{18}{s(s^2+36)}$	W – 14
H	**12**	Find the Laplace transform of $\cos^2 2t$. **Answer**: $\frac{s^2+8}{s(s^2+16)}$	
H	**13**	Find $\mathcal{L}\{\cos^2 t\}$. **Answer**: $\frac{s^2+2}{s(s^2+4)}$	S – 18
H	**14**	Find Laplace transform of $\sin^3(at)$. **Answer**: $\frac{6a^3}{(s^2+a^2)(s^2+9a^2)}$	
C	**15**	Find the Laplace transform of (i) $\sin^3 2t$ (ii) $\cos^3 2t$. **Answer**: (i) $\frac{48}{(s^2+4)(s^2+36)}$ (ii) $\frac{s^3+28s}{(s^2+4)(s^2+36)}$	
H	**16**	Compute $\mathcal{L}\{\cos t\cos 2t\cos 3t\}$. **Answer**: $\frac{1}{4s}+\frac{s}{4(s^2+36)}+\frac{s}{4(s^2+16)}+\frac{s}{4(s^2+4)}$	

❖ THEOREM. FIRST SHIFTING THEOREM:

✓ **Statement:** If $\mathbf{L\{f(t)\} = F(s)}$, then show that $\boldsymbol{\mathcal{L}\{e^{at}f(t)\} = F(s-a)}$.

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

Now, $\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at} f(t)\, dt$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

Since, s and a are constants. $s-a$ is also a constant.

Thus, $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$.

✓ Corollary: If $\boldsymbol{\mathcal{L}\{f(t)\} = F(s)}$, then show that $\mathbf{L\{e^{-at}f(t)\} = F(s+a)}$.

METHOD – 3: EXAMPLE ON FIRST SHIFTING THEOREM

T	**1**	Find $\mathcal{L}\left(e^{-3t}f(t)\right)$, if $\mathcal{L}\left(f(t)\right) = \frac{s}{(s-3)^2}$. **Answer**: $\boldsymbol{\frac{(s+3)}{s^2}}$	
H	**2**	Find $\mathcal{L}\left(e^{-3t}t^{3/2}\right)$. **Answer**: $\boldsymbol{\frac{3\sqrt{\pi}}{4(s+3)^{\frac{5}{2}}}}$	
C	**3**	By using first shifting theorem, obtain the value of $\mathcal{L}\{(t+1)^2e^t\}$. **Answer**: $\boldsymbol{\frac{2}{(s-1)^3} + \frac{2}{(s-1)^2} + \frac{1}{s-1}}$	
C	**4**	Find $\mathcal{L}(e^{2t}\sin 3t)$. **Answer**: $\boldsymbol{\frac{3}{(s-2)^2+9}}$	S – 18
H	**5**	Obtain Laplace transform of $e^{2t}\sin^2 t$ **Answer**: $\boldsymbol{\frac{1}{2}\left[\frac{1}{s-2} - \frac{s-2}{s^2-4s+8}\right]}$	S – 17
C	**6**	Find Laplace transform of $e^{-2t}(\sin 4t + t^2)$. **Answer**: $\boldsymbol{\left(\frac{4}{s^2+4s+20} + \frac{2}{(s+2)^3}\right)}$	W – 14

H	**7**	Find Laplace transform of $e^{-3t}(2\cos 5t - 3\sin 5t)$. **Answer**: $\frac{2s-9}{(s+3)^2+25}$	
T	**8**	Find Laplace transform of $e^{4t}(\sin 2t \cos t)$. **Answer**: $\frac{1}{2}\left[\frac{3}{(s^2-8s+25)}+\frac{1}{(s^2-8s+17)}\right]$	
T	**9**	Find the Laplace transformation of $f(t) = \frac{\cos 2t \sin t}{e^{2t}}$. **Answer**: $\frac{3}{2}\frac{1}{(s^2+4s+13)} - \frac{1}{2}\frac{1}{(s^2+4s+5)}$	
H	**10**	Find $\mathcal{L}\{\cosh 2t \cos 2t\}$. **Answer**: $\frac{1}{2}\left[\frac{s-2}{(s-2)^2+4}+\frac{s+2}{(s+2)^2+4}\right]$	
T	**11**	Find the Laplace transformation of $f(t) = e^{-3t}\cosh 4t \sin 3t$ **Answer**: $\frac{3}{2}\left\{\frac{1}{(s^2+14s+58)}+\frac{1}{(s^2-2s+10)}\right\}$	
C	**12**	Find the Laplace transform of $f(t) = t^2 \cos h\, \pi t$. **Answer**: $\frac{1}{(s-\pi)^3}+\frac{1}{(s+\pi)^3}$	W – 14
H	**13**	Find the Laplace transform of $t^3 \cosh 2t$. **Answer**: $\frac{3}{(s-2)^4}+\frac{3}{(s+2)^4}$	

❖ THEOREM. DIFFERENTIATION OF LAPLACE TRANSFORM:

✓ **Statement:** If $\mathcal{L}\{f(t)\} = F(s)$, then show that $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)\,;\ n = 1,2,3, \ldots$

✓ **Proof:** By definition, $F(s) = \int_0^\infty e^{-st} f(t)\, dt$

$$\Rightarrow \frac{d^n}{ds^n} F(s) = \frac{d^n}{ds^n} \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty \left[\frac{\partial^n}{\partial s^n} e^{-st}\right] f(t)\, dt$$

$$= \int_0^\infty (-1)(t) \frac{\partial^{n-1}}{\partial s^{n-1}} e^{-st} f(t)\, dt$$

$$= \int_0^\infty (-1)^2 (t)^2 \frac{\partial^{n-2}}{\partial s^{n-2}} e^{-st} f(t)\, dt$$

✓ Continuing in this way, we have

$$= (-1)^n \int_0^\infty e^{-st} (t)^n f(t)\, dt$$

$$= (-1)^n \mathcal{L}\{t^n f(t)\}$$

Thus, $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s), n = 1,2,3, \dots$

METHOD – 4: EXAMPLE ON DIFFERENTIATION OF LAPLACE TRANSFORM

C	**1**	Find the value of $\mathcal{L}\{t \sin at\}$. **Answer**: $\frac{\mathbf{2as}}{\mathbf{(s^2+a^2)^2}}$	S – 16
H	**2**	Find the value of $\mathcal{L}\{t \sin 2t\}$. **Answer**: $\frac{\mathbf{4s}}{\mathbf{(s^2+4)^2}}$	S – 15
H	**3**	Find the value of $\mathcal{L}\{t \cosh t\}$. **Answer**: $\frac{\mathbf{1+s^2}}{\mathbf{(s^2-1)^2}}$	
H	**4**	Find the Laplace transform of (i)$t^2 \sin \pi t$ (ii)$t^2 \sin 2t$. **Answer**: (i) $\frac{\mathbf{2\pi(3s^2-\pi^2)}}{\mathbf{(\pi^2+s^2)^3}}$ **(ii)** $\frac{\mathbf{4(3s^2-4)}}{\mathbf{(s^2+4)^3}}$	
C	**5**	Find $\mathcal{L}\{t^2 \sin 4t\}$. **Answer**: $\frac{\mathbf{8(3s^2-16)}}{\mathbf{(s^2+16)^3}}$	
C	**6**	Find $\mathcal{L}\{t \sin 3t \cos 2t\}$ **Answer**: $\frac{\mathbf{5s}}{\mathbf{(s^2+25)^2}} + \frac{\mathbf{s}}{\mathbf{(s^2+1)^2}}$	W – 16
C	**7**	Find $\mathcal{L}(t^2 \cos^2 2t)$. **Answer**: $\frac{\mathbf{1}}{\mathbf{s^3}} + \frac{\mathbf{s(s^2-48)}}{\mathbf{(s^2+16)^3}}$	

H	**8**	Find the Laplace transform of $f(t) = t^2 \cosh 3t$. **Answer**: $\frac{1}{(s-3)^3} + \frac{1}{(s+3)^3}$	S – 16
H	**9**	Find $\mathcal{L}\{t(\sin t - t\cos t)\}$. **Answer**: $\frac{8s}{(s^2+1)^3}$	W – 15
H	**10**	Obtained $\mathcal{L}\{e^{at}\, t\, \sin at\}$. **Answer**: $\frac{2a(s-a)}{[(s-a)^2+a^2]^2}$	
C	**11**	Find $\mathcal{L}(t\, e^{-t} \cos ht)$. **Answer**: $\frac{1}{2}\left[\frac{1}{s^2} + \frac{1}{(s+2)^2}\right]$	
H	**12**	Find the Laplace transform of $t\, e^{4t} \cos 2t$. **Answer**: $\frac{s^2-8s+12}{(s^2-8s+20)^2}$	S – 17

❖ THEOREM. INTEGRATION OF LAPLACE TRANSFORM:

✓ **Statement:** If $\mathbf{L\{f(t)\} = F(s)}$ and if Laplace transform of $\frac{f(t)}{t}$ exists, then $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \mathbf{F(s)ds}$.

Proof: By definition, $F(s) = \int_0^\infty e^{-st} f(t)\, dt$

Integrating both sides with respect to "s" in the range s to ∞.

$$\int_s^\infty F(s)ds = \int_s^\infty \left(\int_0^\infty e^{-st}\, f(t)\, dt\right) ds$$

$$= \int_0^\infty \int_s^\infty e^{-st}\, f(t)\, ds\, dt$$

$$= \int_0^\infty \left(\int_s^\infty e^{-st} ds\right) f(t)\, dt$$

$$= \int_0^\infty \left(\frac{e^{-st}}{-t}\right)_s^\infty f(t)\, dt$$

$$= \int_0^\infty \left(\frac{0 - e^{-st}}{-t}\right) f(t)\, dt$$

$$= \int_0^\infty e^{-st}\left[\frac{f(t)}{t}\right] dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

Thus, $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)ds$.

METHOD – 5: EXAMPLE ON INTEGRATION OF LAPLACE TRANSFORM

C	1	Find $\mathcal{L}\left\{\frac{\sin \omega t}{t}\right\}$. **Answer**: $\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{\omega}\right)$	
H	2	Find $\mathcal{L}\left\{\frac{\sin 2t}{t}\right\}$. **Answer**: $\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)$	
C	3	Find $\mathcal{L}\left\{e^t \frac{\sin t}{t}\right\}$. **Answer**: $(\pi/2) - \tan^{-1}(s-1)$	
H	4	Find $\mathcal{L}\left\{\frac{\cos 3t}{t}\right\}$. **Answer**: ∞	W – 14
C	5	Find $\mathcal{L}\left\{\frac{1-e^t}{t}\right\}$. **Answer**: $\log\left(\frac{s-1}{s}\right)$	
H	6	Find $\mathcal{L}\left(\frac{e^{-bt}-e^{-at}}{t}\right)$. **Answer**: $\log\left(\frac{s+a}{s+b}\right)$	
C	7	Find the Laplace transform of $\frac{1-\cos 2t}{t}$. **Answer**: $\log\left(\frac{\sqrt{s^2+4}}{s}\right)$	W – 17
H	8	Find the Laplace transform of $\frac{1-\cos t}{t}$. **Answer**: $\log\left(\frac{\sqrt{s^2+1}}{s}\right)$	

T	9	Find the Laplace transform of $\frac{\cos at - \cos bt}{t}$. **Answer:** $\log\sqrt{\frac{s^2+b^2}{s^2+a^2}}$	S – 16
T	10	Find the Laplace transformation of $f(t) = \frac{\sin^2 t}{t^2}$. **Answer:** $\frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)$	

❖ THEOREM. LAPLACE TRANSFORM OF INTEGRATION OF A FUNCTION

✓ **Statement:** If $L\{f(t)\} = F(s)$, then $\mathcal{L}\left\{\int_0^t f(t)\ dt\right\} = \frac{F(s)}{s}$.

Proof: By definition $\mathcal{L}\left(\int_0^t f(u)\,du\right) = \int_0^\infty e^{-st}\left\{\int_0^t f(u)\,du\right\}\,dt$,

Suppose that $U = \int_0^t f(u)\,du$ then we have,

$$\mathcal{L}(U) = \int_0^\infty e^{-st}\,U\ dt$$

$$= \left[U\cdot\left(\frac{e^{-st}}{-s}\right) - \int\left\{\frac{dU}{dt}\cdot\left(\frac{e^{-st}}{-s}\right)\right\}dt\right]_0^\infty \qquad \text{(By Integration by parts)}$$

$$= \left[U\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty - \int_0^\infty\left\{\frac{dU}{dt}\left(\frac{e^{-st}}{-s}\right)\right\}dt \ \ldots\ldots\ldots\ldots [1]$$

✓ **In first step**, when $t \to \infty$ then $e^{-st} \to 0$ and $t \to 0$ then $U = \int_0^t f(u)\,du \to 0$.

✓ **In second step**, By Fundamental Theorem of Calculus, $\frac{dU}{dt} = \frac{d}{dt}\left\{\int_0^t f(u)\ du\right\} = f(t)$

Now by [1] we have,

$$\mathcal{L}(U) = 0 - \frac{1}{s}\int_0^\infty e^{-st} f(t)\ \ dt$$

Thus, $\mathcal{L}\left\{\int_0^t f(t)\ dt\right\} = \frac{1}{s}F(s)$.

✓ **Remark:**

Similarly, we can prove that $\int_0^t \int_0^t f(t)\,dt\,dt = \frac{1}{s^2}F(S)$.

METHOD – 6: EXAMPLE ON INTEGRATION OF A FUNCTION

H	**1**	Find $\mathcal{L}\left\{\int_0^t e^{-x}\cos x\,dx\right\}$. **Answer**: $\frac{\mathbf{s+1}}{\mathbf{s[(s+1)^2+1]}}$	
H	**2**	Find $\mathcal{L}\left\{\int_0^t \int_0^t \sin au\,du\,du\right\}$. **Answer**: $\frac{\mathbf{1}}{\mathbf{s^2}}\frac{\mathbf{a}}{\mathbf{(s^2+a^2)}}$	
C	**3**	Find $\mathcal{L}\left\{\int_0^t e^u(u+\sin u)du\right\}$. **Answer**: $\frac{\mathbf{1}}{\mathbf{s}}\left\{\frac{\mathbf{1}}{\mathbf{s^2-2s+1}}+\frac{\mathbf{1}}{\mathbf{s^2-2s+2}}\right\}$	W – 15
C	**4**	Find the Laplace transformation of $f(t) = e^{-3t}\int_0^t t\sin 3t\;dt$. **Answer**: $\frac{\mathbf{6}}{\mathbf{(s^2+6s+18)^2}}$	
H	**5**	Find the Laplace transformation of $\int_0^t e^{-t}\,t\cos t\;dt$. **Answer**: $\frac{\mathbf{(s+1)^2-1}}{\mathbf{s[(s+1)^2+1]^2}}$	
C	**6**	Find the Laplace transformation of $f(t) = \int_0^t \frac{e^t \sin t}{t}\,dt$. **Answer**: $\frac{\mathbf{\cot^{-1}(s-1)}}{\mathbf{s}}$	W – 16

C	7	Evaluate: $\int_0^\infty e^{-2t}\, t\cos t\, dt$. **Answer**: $\frac{3}{25}$	
T	8	Evaluate: $\int_0^\infty e^{-t}\left(\int_0^t \frac{\sin u}{u} du\right) dt$. **Answer**: $\frac{\pi}{4}$	

❖ THEOREM. LAPLACE TRANSFORM OF PERIODIC FUNCTION:

✓ **Statement:** The Laplace transform of a piecewise continuous periodic function $\mathbf{f(t)}$ having period "$\mathbf{p}$" is

$$F(s) = \mathcal{L}\{f(t)\} = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t) dt \text{, Where } s > 0.$$

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$\mathcal{L}\{f(t)\} = \int_0^p e^{-st} f(t)\, dt + \int_p^\infty e^{-st} f(t)\, dt \dots\dots\dots \text{ (A)}$$

Now, $\int_p^\infty e^{-st} f(t)\, dt$

✓ Let, $t = u + p \Longrightarrow dt = du$

✓ When $t \longrightarrow p \Longrightarrow u \longrightarrow 0$

$t \longrightarrow \infty \Longrightarrow u \longrightarrow \infty.$

$$= \int_0^\infty e^{-s(u+p)} f(u+p)\, du$$

Since $f(u)$ is Periodic. i. e. $f(u) = f(u+p)$

$$= \int_0^\infty e^{-s(u+p)} f(u)\, du$$

$$= e^{-sp} \int_0^\infty e^{-su} f(u)\, du = e^{-sp} F(s)$$

✓ By eqn. ...(A)

$$\mathcal{L}\{f(t)\} = \int_0^p e^{-st} f(t)\, dt + \int_p^\infty e^{-st} f(t)\, dt$$

$$\Rightarrow F(s) = \int_0^p e^{-st} f(t)\, dt + e^{-sp} F(s)$$

$$\Rightarrow (1 - e^{-sp}) F(s) = \int_0^p e^{-st} f(t)\, dt$$

$$\Rightarrow F(s) = \frac{1}{1 - e^{-sp}} \int_0^p e^{-st} f(t)\, dt$$

$$\text{Thus, } \mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-sp}} \int_0^p e^{-st} f(t) dt\,;\ \text{Where } s > 0$$

METHOD – 7: EXAMPLE ON L. T. OF PERIODIC FUNCTIONS

H	**1**	Find the Laplace transformation of $f(t) = \begin{cases} 1 & ; 0 \le t < a \\ -1 & ; a < t < 2a \end{cases}$ and f(t) is a period of 2a. **Answer**: $\frac{1}{s} \tanh\left(\frac{as}{2}\right)$	
H	**2**	Find the Laplace transform of the periodic function defined by $f(t) = \frac{t}{2}, 0 < t < 3, f(t+3) = f(t).$ **Answer**: $\frac{1}{2s^2}\left[1 - \frac{3s}{e^{3s} - 1}\right]$	
C	**3**	Find the Laplace transform of the periodic function $f(t) = \begin{cases} 3t & ; \quad 0 \le t < 2 \\ 6 & ; \quad 2 < t < 4 \end{cases}$ **Answer**: $\frac{1}{1 - e^{-4s}}\left[\frac{3}{s^2} - \frac{3e^{-2s}}{s^2} - \frac{6e^{-4s}}{s}\right]$	
C	**4**	Find the Laplace transformation of the periodic function of the waveform $f(t) = \frac{2t}{3}, 0 < t < 3\,; \text{if } f(t) = f(t+3).$ **Answer**: $\frac{2}{3s^2} - \frac{2e^{-3s}}{s(1 - e^{-3s})}$	W – 17

T	5	Find the Laplace transformation of $f(t) = \frac{t}{T}, 0 < t < T$; if $f(t) = f(t+T)$. **Answer**: $\frac{1}{Ts^2} - \frac{e^{-Ts}}{s(1-e^{-Ts})}$	
H	6	Find the Laplace transformation of $f(t) = t^2, 0 < t < 2$ if $f(t) = f(t+2)$. **Answer**: $\frac{1}{(1-e^{-2s})s^3}(2 - 2e^{-2s} - 4se^{-2s} - 4s^2e^{-2s})$	
H	7	Find the Laplace transformation of $f(t) = \begin{cases} t & ; 0 < t < a \\ 2a - t ; & a < t < 2a \end{cases}$ if $f(t) = f(t + 2a)$. **Answer**: $\frac{\tanh\left(\frac{as}{2}\right)}{s^2}$	
C	8	Find the Laplace transform of the half wave rectifier $f(t) = \begin{cases} \sin \omega t, 0 < t < \frac{\pi}{\omega} \\ 0, \quad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$ and $f(t) = f\left(t + \frac{2\pi}{\omega}\right)$. **Answer**: $\frac{\omega}{\left((s^2+\omega^2)\left(1 - e^{-\pi s/\omega}\right)\right)}$	
T	9	Find the Laplace transform of $f(t) = \lvert \sin wt \rvert, t \geq 0$. **Answer**: $\frac{\omega\left(1 + e^{-\pi s/\omega}\right)}{(s^2 + w^2)\left(1 - e^{-\pi s/\omega}\right)}$	

❖ LAPLACE TRANSFORM OF UNIT STEP FUNCTION:

✓ **Statement:** Show that $\mathbf{L\{u(t-a)\} = \frac{e^{-as}}{s}}$

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

Now, $\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\, dt$

$$= \int_0^a e^{-st} u(t-a)\, dt + \int_a^\infty e^{-st} u(t-a)\, dt$$

✓ We know that, $u(t-a) = \begin{cases} 0, & 0 < x < a \\ 1, & x \geq a \end{cases}$

$$= \int_a^\infty e^{-st}\,dt = \left[-\frac{e^{-st}}{s}\right]_a^\infty$$

$$= \left[-\frac{0 - e^{-as}}{s}\right] = \frac{e^{-as}}{s}$$

Thus, $\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$

✓ **Note:** Instead of $\mathbf{u(t-a)}$, we can also write $\mathbf{H(t-a)}$, which is referred as Heaviside's unit step function.

❖ THEOREM. SECOND SHIFTING THEOREM:

✓ **Statement:** If $\boldsymbol{\mathcal{L}\{f(t)\} = F(s)}$, then $\boldsymbol{\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)}$

Proof: By definition, $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

Now, $\mathcal{L}\{f(t-a)u(t-a)\}$

$$= \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt$$

$$= \int_0^a e^{-st} f(t-a)u(t-a)\,dt + \int_a^\infty e^{-st} f(t-a)u(t-a)\,dt$$

✓ **We know that,** $\mathbf{u(t-a)} = \begin{cases} \mathbf{0}, & \mathbf{0 < x < a} \\ \mathbf{1}, & \mathbf{x \geq a} \end{cases}$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

✓ **Let,** $\mathbf{t - a = u \Longrightarrow dt = du}$**. When** $\mathbf{t \longrightarrow a \Longrightarrow u \longrightarrow 0}$ **and** $\mathbf{t \longrightarrow \infty \Longrightarrow u \longrightarrow \infty}$**.**

$$= \int_0^\infty e^{-s(a+u)} f(u)\,du$$

$$= e^{-sa} \int_0^\infty e^{-su} f(u)\,du = e^{-as}F(s)$$

Hence, $\mathcal{L}\{f(t-a) \cdot u(t-a)\} = e^{-as}F(s)$

✓ **NOTE:** $\mathcal{L}\{f(t) \cdot u(t-a)\} = e^{-as}\,\mathcal{L}\{f(t+a)\}$

METHOD – 8: EXAMPLE ON SECOND SHIFTING THEOREM

C	1	Find the Laplace transform of $e^{-3t}\,u(t-2)$. **Answer**: $\frac{e^{-2s}e^{-6}}{s+3}$	W – 17
H	2	Find the Laplace transform of $e^{t}\,u(t-2)$. **Answer**: $\frac{e^{-2s+2}}{s-1}$	
C	3	Find the Laplace transform of $t^2\,u(t-2)$. **Answer**: $e^{-2s}\left(\frac{2!}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right)$	W – 14
H	4	Find the Laplace transform of $(t-1)^2\,u(t-1)$. **Answer**: $\frac{2e^{-s}}{s^3}$	
H	5	Find the Laplace transform of $\cos t\,u(t-\pi)$. **Answer**: $\frac{-se^{-\pi s}}{s^2+1}$	
C	6	Find $\mathcal{L}\left(e^{-t}\sin t\,u(t-\pi)\right)$. **Answer**: $-\frac{e^{-\pi(s+1)}}{s^2+2s+2}$	
C	7	Express the following in terms of unit step function and hence find $F(s)$. $f(t)=\begin{cases} t-1 & ; \quad 1<t<2 \\ 3-t & ; \quad 2<t<3 \end{cases}$. **Answer**: $\frac{e^{-s}}{s^2}-\frac{2e^{-2s}}{s^2}+\frac{e^{-3s}}{s^2}$	
H	8	Express the following in terms of unit step function and hence find $F(s)$. $f(t)=\begin{cases} \sin 2t & ; \quad 2\pi<t<4\pi \\ 0 & ; \quad \text{otherwise} \end{cases}$. **Answer**: $\frac{2(e^{-2\pi s}-e^{-4\pi s})}{s^2+4}$	
T	9	Express the following in terms of unit step function and hence find $F(s)$. $f(t)=\begin{cases} t^2 & ; \quad 0<t<\pi \\ e^{-2t} & ; \quad \pi<t<2\pi \\ \cos 3t & ; \quad t>2\pi \end{cases}$. **Answer**: $\frac{2!}{s^3}-e^{-\pi s}\left(\frac{2!}{s^3}+\frac{2\pi}{s^2}+\frac{\pi^2}{s}\right)-e^{-\pi s}\frac{e^{-2\pi}}{s+2}-e^{-\pi s}\frac{e^{-4\pi}}{s+2}+e^{-2\pi s}\frac{s}{s^2+9}$	

❖ LAPLACE INVERSE TRANSFORM

$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$ **OR** $t^{n-1} \big/ \overline{)n}$

$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

$\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$

$\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$

$\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

METHOD – 9: EXAMPLE ON LAPLACE INVERSE TRANSFORM

C	**1**	Find $\mathcal{L}^{-1}\left\{\frac{6s}{s^2-16}\right\}$. **Answer**: $\mathbf{6\cosh 4t}$	
T	**2**	Find $\mathcal{L}^{-1}\left\{\frac{2s-5}{s^2-4}\right\}$. **Answer**: $\mathbf{2\cosh 2t - \frac{5}{2}\sinh 2t}$	
C	**3**	Find $\mathcal{L}^{-1}\left\{\frac{3s-8}{4s^2+25}\right\}$. **Answer**: $\mathbf{\frac{3}{4}\cos\frac{5}{2}t - \frac{4}{5}\sin\frac{5}{2}t}$	
H	**4**	Find $\mathcal{L}^{-1}\left\{\frac{3(s^2-1)^2}{2s^5}\right\}$. **Answer**: $\mathbf{\frac{3}{2}\left[1 - t^2 + \frac{t^4}{4!}\right]}$	
C	**5**	Find $\mathcal{L}^{-1}\left\{\frac{s^3+2s^2+2}{s^3(s^2+1)}\right\}$. **Answer**: $\mathbf{t^2 + sint}$	
H	**6**	Find $\mathcal{L}^{-1}\left(\frac{4s+15}{16s^2-25}\right)$. **Answer**: $\mathbf{\frac{1}{4}\cos h\frac{5}{4}t + \frac{3}{4}\sin h\frac{5}{4}t}$	

C	7	Find $\mathcal{L}^{-1}\left(\frac{\sqrt{s}-1}{s}\right)^2$. **Answer**: $\mathbf{1+t-4\sqrt{\frac{t}{\pi}}}$	

❖ REMARK: (CHANGE OF SCALE PROPERTY)

✓ **Statement:** If $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then $\mathcal{L}^{-1}\{F(bs)\} = \frac{1}{b} f\left(\frac{t}{b}\right)$.

✓ **Example 1:** Find $\mathcal{L}^{-1}\left(\frac{1}{5s+1}\right)$ using change of scale property.

Solution:

Since, $\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$.

So, $\mathcal{L}^{-1}\left(\frac{1}{5s+1}\right) = \frac{1}{5} e^{-\frac{t}{5}}$

✓ **Example 2:** Find $\mathcal{L}^{-1}\left(\frac{2}{9s^2-4}\right)$ using change of scale property.

Solution:

$$\mathcal{L}^{-1}\left(\frac{2}{9s^2-4}\right) = 2\,\mathcal{L}^{-1}\left(\frac{1}{(3s)^2-4}\right)$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{2}{9s^2-4}\right) = 2\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\sinh 2\left(\frac{t}{3}\right) \qquad \left(\because \mathcal{L}^{-1}\left\{\frac{1}{s^2-4}\right\} = \frac{1}{2}\sinh 2t\right)$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{2}{9s^2-4}\right) = \frac{1}{3}\sinh\frac{2t}{3}.$$

❖ FIRST SHIFTING THEOREM:

✓ **Statement:** If $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then $\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$.

METHOD – 10: EXAMPLE ON FIRST SHIFTING THEOREM

C	**1**	Find $\mathcal{L}^{-1}\left\{\frac{10}{(s-2)^4}\right\}$. **Answer**: $\frac{5e^{2t}t^3}{3}$	
T	**2**	Find $\mathcal{L}^{-1}\left(\frac{1}{\sqrt{2s+3}}\right)$. **Answer**: $\frac{1}{\sqrt{2\pi}}t^{-\frac{1}{2}}e^{-\frac{3t}{2}}$	
T	**3**	Find $\mathcal{L}^{-1}\left(\frac{3s+1}{(s+1)^4}\right)$. **Answer**: $\frac{3}{2}e^{-t}t^2-\frac{1}{3}e^{-t}t^3$	
H	**4**	Find $\mathcal{L}^{-1}\left(\frac{s}{(s+2)^4}\right)$. **Answer**: $e^{-2t}\left(\frac{t^2}{2}-\frac{t^3}{3}\right)$	
C	**5**	Find $\mathcal{L}^{-1}\left(\frac{s}{(s+2)^2+1}\right)$. **Answer**: $e^{-2t}\cos t-2e^{-2t}\sin t$	
C	**6**	Find $\mathcal{L}^{-1}\left\{\frac{3}{s^2+6s+18}\right\}$. **Answer**: $e^{-3t}\sin 3t$	
H	**7**	Find $\mathcal{L}^{-1}\left(\frac{2s+3}{s^2-4s+13}\right)$. **Answer**: $\frac{1}{3}e^{2t}(6\cos 3t+7\sin 3t)$	
H	**8**	Find $\mathcal{L}^{-1}\left(\frac{1}{s^2+s+1}\right)$. **Answer**: $\sqrt{\frac{2}{3}}\,e^{-\frac{t}{2}}\sin\left(\sqrt{\frac{3}{2}}t\right)$	
H	**9**	Find $\mathcal{L}^{-1}\left(\frac{s}{s^2+s+1}\right)$. **Answer**: $e^{-\frac{t}{2}}\left(\cos\sqrt{\frac{3}{2}}t-\frac{1}{\sqrt{6}}\sin\left(\sqrt{\frac{3}{2}}t\right)\right)$	

H	**10**	Find $\mathcal{L}^{-1}\left(\frac{s+7}{s^2+8s+25}\right)$. **Answer**: $e^{-4t}(\sin 3t+\cos 3t)$	S – 17
C	**11**	Find the inverse Laplace transform of $\frac{6+s}{s^2+6s+13}$, use shifting theorem. **Answer**: $e^{-3t}\cos 2t+\frac{3}{2}e^{-3t}\sin 2t$	

❖ PARTIAL FRACTION METHOD

✓ **Case 1** : If the denominator has non-repeated linear factors $(s-a),(s-b),(s-c)$, then

$$\frac{f(s)}{(s-a)(s-b)(s-c)}=\frac{A}{(s-a)}+\frac{B}{(s-b)}+\frac{C}{(s-c)}$$

✓ **Case 2** : If the denominator has repeated linear factors $(s-a)$, (n times), then

$$\frac{f(s)}{(s-a)^n}=\frac{A_1}{(s-a)}+\frac{A_2}{(s-a)^2}+\frac{A_3}{(s-a)^3}+\cdots+\frac{A_n}{(s-a)^n}$$

✓ **Case 3** : If the denominator has non-repeated quadratic factors (s^2+as+b), (s^2+cs+d) ,

$$\frac{f(s)}{(s^2+as+b)(s^2+cs+d)}=\frac{As+B}{(s^2+as+b)}+\frac{Cs+D}{(s^2+cs+d)}$$

✓ **Case 4** : If the denominator has repeated quadratic factors (s^2+as+b), (n times), then

$$\frac{f(s)}{(s^2+as+b)^n}=\frac{As+B}{(s^2+as+b)}+\frac{Cs+D}{(s^2+as+b)^2}+\cdots\text{(n times)}$$

METHOD – 11: EXAMPLE ON PARTIAL FRACTION METHOD

H	**1**	Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}$. **Answer**: $\mathbf{1-e^{-t}}$	
H	**2**	Find $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$. **Answer**: $\mathbf{\frac{1}{5}[e^{-t}-e^{-2t}]}$	S – 18

H	**3**	Find $\mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s+3)}\right\}$. **Answer:** $\frac{1}{5}[e^{2t}-e^{-3t}]$	S – 15
T	**4**	Find $\mathcal{L}^{-1}\left\{\frac{1}{(s+\sqrt{2})(s-\sqrt{3})}\right\}$. **Answer:** $\frac{e^{\sqrt{3}t}-e^{-\sqrt{2}t}}{\sqrt{3}+\sqrt{2}}$	S – 16
C	**5**	Find $\mathcal{L}^{-1}\left\{-\frac{s+10}{s^2-s-2}\right\}$. **Answer:** $-4e^{2t}+3e^{-t}$	
C	**6**	Find $\mathcal{L}^{-1}\left\{\frac{5s^2+3s-16}{(s-1)(s+3)(s-2)}\right\}$. **Answer:** $2e^{t}+e^{-3t}+2e^{2t}$	
H	**7**	Find $\mathcal{L}^{-1}\left\{\frac{3s^2+2}{(s+1)(s+2)(s+3)}\right\}$. **Answer:** $\frac{5}{2}e^{-t}-14e^{-2t}+\frac{29}{2}e^{-3t}$	
H	**8**	Find $\mathcal{L}^{-1}\left(\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right)$. **Answer:** $\frac{7}{2}e^{3t}-\frac{1}{6}e^{-t}-\frac{4}{3}e^{2t}$	
T	**9**	Find $\mathcal{L}^{-1}\left(\frac{s+2}{s\,(s+1)(s+3)}\right)$. **Answer:** $\frac{2}{3}-\frac{1}{2}e^{-t}-\frac{1}{6}e^{-3t}$	
C	**10**	Find $\mathcal{L}^{-1}\left(\frac{2s+3}{(s+2)\,(s+1)^2}\right)$. **Answer:** $-e^{-2t}+e^{-t}+te^{-t}$	
H	**11**	Find $\mathcal{L}^{-1}\left(\frac{4s+5}{(s-1)^2(s+2)}\right)$. **Answer:** $\frac{1}{3}e^{t}+3te^{t}+\frac{1}{3}e^{-2t}$	
C	**12**	Find $\mathcal{L}^{-1}\left(\frac{3s+1}{(s+1)\,(s^2+2)}\right)$. **Answer:** $-\frac{2}{3}e^{-t}+\frac{2}{3}\cos\sqrt{2}\,t+\frac{7}{3\sqrt{2}}\sin\sqrt{2}\,t$	

C	**13**	Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2-3s+3)}\right\}$. **Answer:** $\frac{1}{3}+e^{\frac{3t}{2}}\left[\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right)-\frac{1}{3}\cos\left(\frac{\sqrt{3}t}{2}\right)\right]$	W – 15
H	**14**	Find the inverse Laplace transform of $\frac{5s+3}{(s^2+2s+5)(s-1)}$. **Answer:** $-e^{-t}\cos 2t+\frac{3}{2}e^{-t}\sin 2t+e^{t}$	
T	**15**	Find $\mathcal{L}^{-1}\left(\frac{s+4}{s\,(s-1)\,(s^2+4)}\right)$. **Answer:** $-1+e^{t}-\frac{1}{2}\sin 2t$	
C	**16**	Find $\mathcal{L}^{-1}\left(\frac{1}{(s^2+2)\,(s^2-3)}\right)$. **Answer:** $\frac{1}{5\sqrt{3}}\sinh\sqrt{3}t-\frac{1}{5\sqrt{2}}\sin\sqrt{2}t$	
H	**17**	Find $\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)\,(s^2+4)}\right)$. **Answer:** $\frac{1}{3}(\cos t-\cos 2t)$	
C	**18**	Find $\mathcal{L}^{-1}\left(\frac{2s^2-1}{(s^2+1)\,(s^2+4)}\right)$. **Answer:** $\frac{3}{2}\sin 2t-\sin t$	W – 16
H	**19**	Find $\mathcal{L}^{-1}\left(\frac{s^2}{(s^2+a^2)\,(s^2+b^2)}\right)$. **Answer:** $\frac{1}{a^2-b^2}(a\sin at-b\sin bt)$	
C	**20**	Find $\mathcal{L}^{-1}\left\{\frac{s^3}{s^4-81}\right\}$. **Answer:** $\frac{\cos 3t+\cosh 3t}{2}$	W – 17
H	**21**	Find $\mathcal{L}^{-1}\left\{\frac{1}{s^4-81}\right\}$. **Answer:** $\frac{\sinh 3t-\sin 3t}{54}$	S – 16

T	22	Find $\mathcal{L}^{-1}\left\{\frac{s}{s^4+64}\right\}$. **Answer:** $\frac{1}{16}[e^{2t}\sin 2t - e^{-2t}\sin 2t]$	

❖ SECOND SHIFTING THEOREM:

✓ **Statement:** If $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\cdot H(t-a)$.

METHOD – 12: EXAMPLE ON SECOND SHIFTING THEOREM

H	1	Find $\mathcal{L}^{-1}\left(\frac{e^{-as}}{s}\right)$. **Answer:** $\mathbf{u(t-a)}$	
C	2	Find the inverse Laplace transform of $\frac{se^{-2s}}{s^2+\pi^2}$. **Answer:** $\mathbf{\cos\pi(t-2)\cdot u(t-2)}$	
H	3	Find $\mathcal{L}^{-1}\left\{\frac{se^{-\pi s}}{s^2-\pi^2}\right\}$. **Answer:** $\mathbf{\cosh(t-\pi)\cdot u(t-\pi)}$	
T	4	Find $\mathcal{L}^{-1}\left(e^{-s}\left\{\frac{\sqrt{s}-1}{s}\right\}^2\right)$. **Answer:** $\left\{\mathbf{1+(t-1)-4\sqrt{\frac{(t-1)}{\pi}}}\right\}\mathbf{u(t-1)}$	
C	5	Find the inverse Laplace transform of $\frac{e^{-4s}(s+2)}{s^2+4s+5}$. **Answer:** $\mathbf{e^{-2(t-4)}\cos(t-4)\cdot u(t-4)}$	
H	6	Find $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{(s+2)(s+3)}\right)$. **Answer:** $\mathbf{\{e^{-2(t-2)}-e^{-3(t-2)}\}\cdot u(t-2)}$	
H	7	Find $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2+8s+25}\right)$. **Answer:** $\mathbf{\frac{e^{-4(t-2)}}{3}\sin 3(t-2)\cdot u(t-2)}$	W – 16

C	**8**	Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{(s^2+2)(s^2-3)}\right\}$. **Answer:** $\mathbf{\dfrac{1}{5}\left[\dfrac{1}{\sqrt{3}}\sinh\sqrt{3}(t-2)-\dfrac{1}{\sqrt{2}}\sin\sqrt{2}(t-2)\right]H(t-2)}$	W – 15

❖ INVERSE LAPLACE TRANSFORM OF DERIVATIVES:

✓ **Statement:** If $\mathcal{L}^{-1}\{F(s)\}=f(t)$, then $\mathcal{L}^{-1}\{F'(s)\}=-t\cdot f(t)$

METHOD – 13: EXAMPLE ON INVERSE LAPLACE TRANSFORM OF DERIVATIVES

H	**1**	Find $\mathcal{L}^{-1}\left\{\log\dfrac{1}{s}\right\}$. **Answer:** $\mathbf{\dfrac{1}{t}}$	
H	**2**	Find $\mathcal{L}^{-1}\left\{\log\left(\dfrac{1+s}{s}\right)\right\}$. **Answer:** $\mathbf{\dfrac{1-e^{-t}}{t}}$	
C	**3**	Find the inverse transform of the function $\ln\left(1+\dfrac{w^2}{s^2}\right)$. **Answer:** $\mathbf{\dfrac{2}{t}(1-\cos wt)}$	
T	**4**	Find $\mathcal{L}^{-1}\left\{\log\dfrac{s+a}{s+b}\right\}$. **Answer:** $\mathbf{\dfrac{e^{-bt}-e^{-at}}{t}}$	
H	**5**	Find $\mathcal{L}^{-1}\left\{\log\dfrac{s+1}{s-1}\right\}$. **Answer:** $\mathbf{\dfrac{e^{t}-e^{-t}}{t}}$	
C	**6**	Find $\mathcal{L}^{-1}\left\{\log\left(\dfrac{s+4}{s+3}\right)\right\}$. **Answer:** $\mathbf{\dfrac{e^{-3t}-e^{-4t}}{t}}$	

H	**7**	Find $\mathcal{L}^{-1}\{\tan^{-1}(s+1)\}$. **Answer**: $-\frac{1}{t}e^{-t}\sin t$	
C	**8**	Find $\mathcal{L}^{-1}\{\tan^{-1}\frac{2}{s}\}$. **Answer**: $\frac{1}{t}\sin 2t$	W – 17
C	**9**	Find $\mathcal{L}^{-1}\left\{\cot^{-1}\left(\frac{s+a}{b}\right)\right\}$. **Answer**: $\frac{be^{-at}\sin bt}{t}$	
H	**10**	Find $\mathcal{L}^{-1}\{\cot^{-1}(\alpha s)\}$. **Answer**: $\frac{1}{t}\sin\frac{t}{a}$	

❖ DEFINITION: CONVOLUTION PRODUCT:

✓ The convolution of f and g is denoted by $f * g$ and is defined as

$$f * g = \int_0^t f(u) \cdot g(t-u)\, du$$

METHOD – 14: EXAMPLE ON CONVOLUTION PRODUCT

H	**1**	Find the value of $1 * 1$. Where " $*$ " denote convolution product. **Answer**: t	
C	**2**	Evaluate $1 * e^{2t}$. **Answer**: $e^{2t} - 1$	
C	**3**	Evaluate $t * e^{t}$. **Answer**: $e^{t} - t - 1$	S – 15
H	**4**	Find the convolution of e^{-t} and $\sin t$. **Answer**: $\frac{e^{-t} + \sin t - \cos t}{2}$	

❖ THEOREM. CONVOLUTION THEOREM

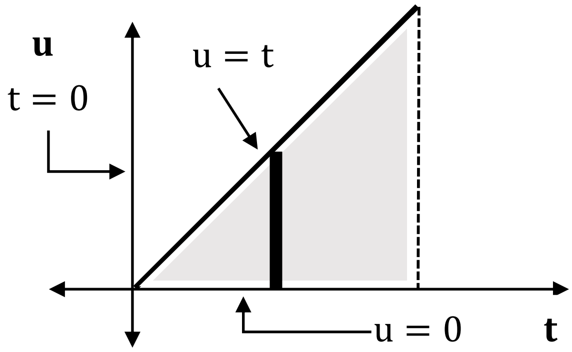
- **Statement:** If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and $\mathcal{L}^{-1}\{G(s)\} = g(t)$, then

$$\mathcal{L}^{-1}\{F(s) \cdot G(s)\} = \int_0^t f(u) \cdot g(t-u)\, du = f * g$$

- **Proof:** Let us suppose $F(t) = \int_0^t f(u) \cdot g(t-u)\, du$

$$\text{Now, } \mathcal{L}\big(F(t)\big) = \int_0^\infty e^{-st}\left(\int_0^t f(u) \cdot g(t-u) du\right) dt = \int_0^\infty \int_0^t f(u) \cdot g(t-u)\, e^{-st} du\, dt$$

- Here, region of integration is entire area lying between the lines $u = 0$ and $u = t$ which is part of the first quadrant.



- Changing the order of integration, we have

$$\mathcal{L}\big(F(t)\big) = \int_0^\infty \int_u^\infty e^{-st} f(u) \cdot g(t-u)\, dt\, du$$

$$= \left(\int_0^\infty e^{-su} f(u) du\right)\left(\int_u^\infty e^{-s(t-u)} g(t-u)\, dt\right)$$

- Let, $t - u = v \Longrightarrow dt = dv$. When $t \longrightarrow u \Longrightarrow v \longrightarrow 0$ and $t \longrightarrow \infty \Longrightarrow v \longrightarrow \infty$.

$$= \left(\int_0^\infty e^{-su} f(u) du\right)\left(\int_0^\infty e^{-sv} g(v)\, dv\right)$$

$$= F(s) \cdot G(s)$$

- Thus, $\mathcal{L}\big(F(t)\big) = F(s) \cdot G(s) \Longrightarrow F(t) = \mathcal{L}^{-1}\{F(s) \cdot G(s)\}$

$$\Longrightarrow \mathcal{L}^{-1}\{F(s) \cdot G(s)\} = \int_0^t f(u) g(t-u) du$$

- Hence, $f * g = \mathcal{L}^{-1}\{F(s) \cdot G(s)\} = \int_0^t f(u) \cdot g(t-u)\, du$ is convolution product of f & g.

METHOD – 15: EXAMPLE ON CONVOLUTION THERORE M

C	**1**	State convolution theorem and using it find $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+3)}\right\}$. **Answer**: $\frac{e^{-t}-e^{-3t}}{2}$	
C	**2**	Using the convolution theorem, obtain the value of $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\}$. **Answer**: $\frac{1-\cos 2t}{4}$	S – 15
H	**3**	Use convolution theorem to find $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\}$. **Answer**: $\frac{1-\cos at}{a^2}$	
T	**4**	State convolution theorem and use it to evaluate $\mathcal{L}^{-1}\left\{\frac{a}{s^2(s^2+a^2)}\right\}$. **Answer**: $\frac{at-\sin at}{a^2}$	
C	**5**	Apply convolution theorem to Evaluate $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$. **Answer**: $\frac{t\sin at}{2a}$	W – 16
H	**6**	Find the inverse Laplace transform of $\frac{s}{(s^2+1)^2}$. **Answer**: $\frac{t\sin t}{2}$	W – 14
C	**7**	State Convolution Theorem and Use to it Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$. **Answer**: $\frac{1}{2a^2}\left(\frac{\sin at}{2a}-t\cos at\right)$	S – 16
H	**8**	Using convolution theorem, obtain $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)^2}\right\}$. **Answer**: $\frac{1}{8}\left(\frac{\sin 2t}{4}-t\cos 2t\right)$	S – 17 S – 18
H	**9**	Using convolution theorem find $\mathcal{L}^{-1}\left(\frac{2}{(s^2+1)(s^2+4)}\right)$. **Answer**: $\frac{2}{3}\left(\sin t-\frac{1}{2}\sin 2t\right)$	

C	**10**	State convolution theorem and using it find $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$. **Answer**: $\frac{a\sin at - b\sin bt}{a^2-b^2}$	W – 17
H	**11**	State convolution theorem and using it to find $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+9)}\right\}$. **Answer**: $\frac{3\sin 3t - 2\sin 2t}{5}$	
T	**12**	Find $\mathcal{L}^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\}$. **Answer**: $\frac{e^{-2t}\, t\sin t}{2}$	
C	**13**	Find the inverse Laplace transform of $\frac{s}{(s+1)\,(s-1)^2}$. **Answer**: $\frac{t\,e^t}{2}+\frac{e^t}{4}-\frac{e^{-t}}{4}$	W – 14
H	**14**	Find $\mathcal{L}^{-1}\left(\frac{1}{(s-2)(s+2)^2}\right)$. **Answer**: $\frac{e^{2t}-e^{-2t}-4t\,e^{-2t}}{16}$	
C	**15**	Find $\mathcal{L}^{-1}\left(\frac{1}{(s-2)^4(s+3)}\right)$. **Answer**: $\frac{e^{-3t}}{625}+\frac{e^{2t}}{6}\left(\frac{t^3}{5}-\frac{3t^2}{25}+\frac{6t}{125}-\frac{6}{625}\right)$	
T	**16**	Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s+a)^3}\right\}$. **Answer**: $\frac{1}{2}\left[-\frac{t^2e^{-at}}{a}-\frac{2te^{-at}}{a^2}-\frac{2e^{-at}}{a^3}+\frac{2}{a^3}\right]$	
C	**17**	State the convolution theorem and verify it for $f(t)=t$ and $g(t)=e^{2t}$. **Answer**: $\frac{1}{s^2(s-2)}$	W – 15
H	**18**	State the convolution theorem and verify it for $f(t)=1$ and $g(t)=\sin t$. **Answer**: $\frac{1}{s(s^2+1)}$	

❖ NOTE:

(1) If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ then $\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\, dt$.

(2) If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ then $\mathcal{L}^{-1}\left\{\frac{F(s)}{s^n}\right\} = \left(\int_0^t \int_0^t \ldots \ldots \text{n times}\right) f(t)\,(dt)^n$; $n \in N$

❖ THEOREM. DERIVATIVE OF LAPLACE TRANSFORM

✓ **Statement: If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$.**

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$\Rightarrow \mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt$$

$$\Rightarrow \mathcal{L}\{f'(t)\} = \left[e^{-st}\int f'(t)dt - \int \left\{\left(\frac{d}{dt}e^{-st}\right)\cdot \int f'(t)dt\right\} dt\right]_0^\infty$$

$$\Rightarrow \mathcal{L}\{f'(t)\} = \left[e^{-st}f(t) - \int (-s)e^{-st}f(t)dt\right]_0^\infty$$

$$\Rightarrow \mathcal{L}\{f'(t)\} = 0 - f(0) + s\int_0^\infty e^{-st}f(t)\, dt = sF(s) - f(0)$$

$$\Rightarrow \mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

❖ NOTE:

✓ $\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0)$.

✓ $\mathcal{L}\{y''(t)\} = s^2\mathcal{L}\{y(t)\} - s\,y(0) - y'(0)$.

✓ $\mathcal{L}\{y'''(t)\} = s^3\mathcal{L}\{y(t)\} - s^2y(0) - sy'(0) - y''(0)$.

METHOD – 16: EXAMPLE ON APPLICATION OF LAPLACE TRANSFORM

T	1	Solve by Laplace transform: $\frac{dy}{dt} - 2y = 4$, given that $t = 0, y = 1$. **Answer**: $\mathbf{y(t) = 3e^{2t} - 2}$	
H	2	Solve by Laplace transform $y'' + 6y = 1, y(0) = 2, y'(0) = 0$. **Answer**: $\mathbf{y(t) = \frac{11}{6}\cos\sqrt{6}t + \frac{1}{6}}$	

C	3	Solve IVP using Laplace transform $y'' + 4y = 0, y(0) = 1, y'(0) = 6.$ **Answer**: $\mathbf{y(t) = \cos 2t + 3\sin 2t}$	
T	4	By using the method of Laplace transform solve the IVP : $y'' + 2y' + y = e^{-t}, y(0) = -1$ and $y'(0) = 1$. **Answer**: $\mathbf{y(t) = \frac{e^{-t}t^2}{2!} - e^{-t}}$	
H	5	By using the method of Laplace transform solve the IVP : $y'' + 4y' + 3y = e^{-t}, y(0) = 1$ and $y'(0) = 1$. **Answer**: $\mathbf{y(t) = \left(\frac{t}{2} + \frac{9}{4}\right)e^{-t} - \frac{5}{4}e^{-3t}}$	
H	6	By using the method of Laplace transform solve the IVP : $y'' + 3y' + 2y = e^{t}, y(0) = 1$ and $y'(0) = 0$. **Answer**: $\mathbf{y(t) = \frac{1}{6}e^{t} - \frac{5}{2}e^{-t} + \frac{4}{3}e^{-2t}}$	S – 15
C	7	Use the Laplace transform to solve the following initial value problem: $y'' - 3y' + 2y = 12e^{-2t}, y(0) = 2$ and $y'(0) = 6$. **Answer**: $\mathbf{y(t) = \frac{1}{6}e^{-2t} + \frac{9}{2}e^{2t} - \frac{8}{3}e^{t}}$	S – 17
C	8	By using the method of Laplace transform solve the IVP : $y'' - 4y' + 3y = 6t - 8, y(0) = 0$ and $y'(0) = 0$. **Answer**: $\mathbf{y(t) = 2t + e^{t} - e^{3t}}$	
C	9	Using Laplace transform solve the IVP $y'' + y = \sin 2t, y(0) = 2, y'(0) = 1.$ **Answer**: $\mathbf{y(t) = \frac{5}{3}\sin t - \frac{1}{3}\sin 2t + 2\cos t}$	W – 14 S – 18
T	10	By Laplace transform solve $y'' + a^2y = K\sin at.$ **Answer**: $\mathbf{y(t) = \left(\frac{k}{2a^2} + \frac{B}{a}\right)\sin a\,t + \left(A - \frac{k}{2a}t\right)\cos at}$	
T	11	Using Laplace transform solve the differential equation $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = e^{-t}\sin t$, where $x(0) = 0, x'(0) = 1$. **Answer**: $\mathbf{x(t) = \frac{e^{-t}(\sin t + \sin 2t)}{3}}$	

T	**12**	Solve the initial value problem: $y'' - 2y' = e^{t}\sin t$, $y(0) = y'(0) = 0$, using Laplace transform. **Answer**: $\mathbf{y(t) = -\frac{1}{4} + \frac{1}{4}e^{2t} - \frac{1}{2}e^{t}\sin t}$	W – 15
C	**13**	Solve the differential equation using Laplace Transformation method $y'' - 3y' + 2y = 4t + e^{3t}$, $y(0) = 1$ and $y'(0) = -1$. **Answer**: $\mathbf{y(t) = 3 + 2t + \frac{1}{2}(e^{3t} - e^{t}) - 2e^{3t}}$	S – 16 W – 16
T	**14**	Solve using Laplace transforms, $y''' + 2y'' - y' - 2y = 0$; where, $y(0) = 1$, $y'(0) = 2$, $y''(0) = 2$. **Answer**: $\mathbf{y(t) = \frac{1}{3}[5e^{t} + e^{-2t}] - e^{-t}}$	W – 17

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UNIT-6 » PARTIAL DIFFERENTIAL EQUATION AND IT'S APPLICATION

❖ INTRODUCTION:

- ✓ A partial differential equation is a mathematical equation involving two or more independent variables, unknown function and its partial derivative with respect to independent variables.
- ✓ Partial differential equations are used to formulate the problems containing functions of several variables, such as propagation of heat or sound, fluid flow, electrodynamics etc.

❖ DEFINITION: PARTIAL DIFFERENTIAL EQUATION:

- ✓ An equation which involves function of two or more variables and partial derivatives of that function then it is called Partial Differential Equation.

  e.g. $\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = 0.$

❖ DEFINITION: ORDER OF DIFFERENTIAL EQUATION:

- ✓ The order of highest derivative which appears in differential equation is "Order of D.E".

  e.g. $\left(\frac{\partial y}{\partial x}\right)^2 + \frac{\partial y}{\partial t} + 5y = 0$ has order 1.

❖ DEFINITION: DEGREE OF DIFFERENTIAL EQUATION:

- ✓ When a D.E. is in a polynomial form of derivatives, the highest power of highest order derivative occurring in D.E. is called a "Degree Of D.E.".

  e.g. $\left(\frac{\partial y}{\partial x}\right)^2 + \frac{\partial y}{\partial t} + 5y = 0$ has degree 2.

❖ NOTATION:

- ✓ Suppose $z = f(x, y)$. For that , we shall use $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$, $\frac{\partial^2 z}{\partial x^2} = r$, $\frac{\partial^2 z}{\partial x \partial y} = s$, $\frac{\partial^2 z}{\partial y^2} = t$.

❖ FORMATION OF PARTIAL DIFFERENTIAL EQUATION:

- ✓ By Eliminating Arbitrary Constants
  - Consider the function $f(x, y, z, a, b) = 0$. Where, a & b are independent arbitrary constants.
  - **Step 1:** $f(x, y, z, a, b) = 0.$ ……(1)

- **Step 2:** $f_x(x,y,z,a,b) = 0$. ……(2) and $f_y(x,y,z,a,b) = 0$. ……(3)
- **Step 3:** Eliminate a & b from eq. (1), eq. (2) & eq. (3).
- We get partial differential equation of the form $F(x,y,z,p,q) = 0$

✓ By Eliminating Arbitrary Functions

- Type 1: Consider, the function $(u,v) = 0$; u and v are functions of x and y
  - **Step 1:** Let, $u = F(v)$.
  - **Step 2:** Find u_x & u_y.
  - **Step 3:** Eliminate the function F from u_x & u_y.
  - **Note:** In such case, for elimination of function, substitution method is used.
- Type 2: Consider, the function $z = f(x,y)$
  - **Step 1:** Find z_x & z_y.
  - **Step 2:** Eliminate the function f from z_x & z_y.
  - **Note:** In such case, for elimination of function, division of z_x & z_y is used.

METHOD – 1: EXAMPLE ON FORMATION OF PARTIAL DIFFERENTIAL EQUATION

H	1	Form the partial differential equation $z = ax + by + ct$. **Answer**: $\mathbf{z = px + qy + t\frac{\partial z}{\partial t}}$	S – 17
C	2	Form the partial differential equation $z = (x-2)^2 + (y-3)^2$. **Answer**: $\mathbf{4z = p^2 + q^2}$	
C	3	Form the partial differential equation for the equation $(x-a)(y-b) - z^2 = x^2 + y^2$. **Answer**: $\mathbf{4(x+pz)(y+qz) - z^2 = x^2 + y^2}$	W – 15
H	4	Eliminate the function f from the relation $f(xy + z^2, x + y + z) = 0$. **Answer**: $\mathbf{\frac{p+1}{q+1} = \frac{y+2zp}{x+2zq}}$	

C	**5**	Form the partial differential equation of $f(x+y+z, x^2+y^2+z^2)=0$. **Answer**: $\dfrac{\mathbf{p+1}}{\mathbf{q+1}}=\dfrac{\mathbf{x+zp}}{\mathbf{y+zq}}$	S – 15
T	**6**	From a partial differential equation by eliminating the arbitrary function $\emptyset$ from $\emptyset(x+y+z, x^2+y^2-z^2)=0$. **Answer**: $\mathbf{(y+z)p-(x+z)q=x-y}$	
H	**7**	Form the partial differential equation $f(x^2-y^2, xyz)=0$. **Answer**: $\dfrac{\mathbf{yz+xyp}}{\mathbf{xz+xyq}}=-\dfrac{\mathbf{x}}{\mathbf{y}}$	W – 14
C	**8**	Form the partial differential equations by eliminating the arbitrary function from $f(x^2+y^2, z-xy)=0$. **Answer**: $\dfrac{\mathbf{x}}{\mathbf{y}}=\dfrac{\mathbf{p-y}}{\mathbf{q-x}}$	W – 17
C	**9**	Form partial differential equation by eliminating the arbitrary function from $xyz=\Phi(x+y+z)$. **Answer**: $\dfrac{\mathbf{p+1}}{\mathbf{q+1}}=\dfrac{\mathbf{yz+xyp}}{\mathbf{xz+xyq}}$	
H	**10**	Form the partial differential equation of $z=f\left(\frac{x}{y}\right)$. **Answer**: $\dfrac{\mathbf{p}}{\mathbf{q}}=-\dfrac{\mathbf{y}}{\mathbf{x}}$	S – 17
H	**11**	Form the partial differential equation by eliminating the arbitrary function from $z=f(x^2-y^2)$. **Answer**: $\dfrac{\mathbf{p}}{\mathbf{q}}=-\dfrac{\mathbf{x}}{\mathbf{y}}$	
T	**12**	Form the partial differential equation of $z=xy+f(x^2+y^2)$. **Answer**: $\dfrac{\mathbf{p-y}}{\mathbf{q-x}}=\dfrac{\mathbf{x}}{\mathbf{y}}$	
C	**13**	Form partial differential equation of $z=f(x+iy)+g(x-iy)$. **Answer**: $\dfrac{\partial^2 \mathbf{z}}{\partial \mathbf{x}^2}+\dfrac{\partial^2 \mathbf{z}}{\partial \mathbf{y}^2}=\mathbf{0}$	

METHOD – 2: EXAMPLE ON DIRECT INTEGRATION

C	1	Solve $\frac{\partial^2 u}{\partial x\,\partial y} = x^3 + y^3$. **Answer**: $\mathbf{u(x,y) = \frac{x^4 y}{4} + \frac{x y^4}{4} + F(y) + g(x)}$	
H	2	Solve $\frac{\partial^2 u}{\partial x\,\partial t} = e^{-t}\cos x$. **Answer**: $\mathbf{u(x,y) = -e^{-t}\sin x + F(t) + g(x)}$	
C	3	Solve $\frac{\partial^3 u}{\partial x^2\,\partial y} = \cos(2x + 3y)$. **Answer**: $\mathbf{u(x,y) = -\frac{\sin(2x+3y)}{12} + xF(y) + G(y) + h(x)}$	
H	4	Solve $\frac{\partial^2 z}{\partial x\,\partial y} = \sin x \sin y$, given that $\frac{\partial z}{\partial y} = -2\sin y$, when $x = 0$ & $z = 0$, when y is an odd multiple of $\frac{\pi}{2}$. **Answer**: $\mathbf{z(x,y) = \cos x\ \cos y + \cos y}$	

❖ LINEAR PDE WITH CONSTANT CO-EFFICIENT:

✓ The n^{th} order linear partial differential equation with constant co-efficient is

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\,\partial y} + \cdots a_n \frac{\partial^n z}{\partial y^n} = F(x,y) \ldots\ldots\ldots (A)$$

Where, $a_0, a_1, \ldots, a_n$ are constants.

❖ NOTATIONS:

✓ Replacing $\frac{\partial}{\partial x} = D$ and $\frac{\partial}{\partial y} = D'$ in Eq. (A) , it can be written in operator form as below,

$$a_0 D^n z + a_1 D^{n-1} D' z + \cdots + a_n D'^n z = F(x,y) \textbf{ OR } [f(D,D')]z = F(x,y)$$

❖ AUXILIARY EQUATION:

✓ The auxiliary equation for n^{th} order PDE $a_0 D^n z + a_1 D^{n-1} D' z + \cdots + a_n D'^n z = F(x,y)$ is derived by replacing D by m , D' by 1 and $F(x,y)$ by 0.

❖ COMPLEMENTARY FUNCTION (C.F.--z_c):

✓ A general solution of $[f(D,D')]z = 0$ is called complementary function of $[f(D,D')]z = F(x,y)$.

❖ PARTICULAR INTEGRAL (P.I.--z_p):

✓ A particular integral of $[f(D, D')]z = F(x, y)$ is $P.I. = \frac{1}{f(D,D')} F(x, y)$.

❖ GENERAL SOLUTION OF PDE:

✓ $G.S. = C.F. + P.I = z_c + z_p$

❖ METHOD FOR FINDING C.F. OF PARTIAL DIFFERENTIAL EQUATION:

✓ Consider, $a_0 D^n z + a_1 D^{n-1} D' z + \cdots + a_n D'^n z = F(x, y)$

✓ The Auxiliary equation is $a_0 m^n z + a_1 m^{n-1} z + \cdots + a_n z = 0$.

✓ Let $m_1, m_2, \dots$ be the roots of auxiliary equation.

Case	Nature of the "n" roots	General Solutions
1.	$m_1 \neq m_2 \neq m_3 \neq m_4 \neq \cdots$	$z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \phi_3(y + m_3 x) + \cdots$
2.	$m_1 = m_2 = m$ $m_3 \neq m_4 \neq \cdots$	$z = \phi_1(y + mx) + x\phi_2(y + mx) + \phi_3(y + m_3 x) + \cdots$
3.	$m_1 = m_2 = m_3 = m$ $m_4 \neq m_5, \dots$	$z = \phi_1(y + mx) + x\phi_2(y + mx)$ $+x^2\phi_3(y + mx) + \phi_4(y + m_4 x) + \cdots$

❖ METHOD FOR FINDING PARTICULAR INTEGRAL:

✓ For partial differential equation the value of Particular integral can be find by following methods.

(1) General Method

(2) Short-cut Method

❖ GENERAL METHOD

- Consider the partial differential equation $f(D, D')z = F(x, y)$
- Particular integral $P.I. = \frac{1}{f(D,D')} F(x, y)$
- Suppose, $f(x, y)$ is factorized into n linear factors.

$$\text{P.I.} = \frac{1}{f(D, D')} F(x, y) = \frac{1}{(D - m_1 D')(D - m_2 D') \dots (D - m_n D')} F(x, y)$$

Which can be evaluated by

$$\frac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\, dx$$

- Where, c is replaced by $y + mx$ after integration.

❖ SHORTCUT METHOD

✓ **Case-1** $F(x, y) = e^{ax+by}$

$$\text{P.I.} = \frac{1}{f(D, D')} e^{ax+by} = \frac{1}{f(a, b)} e^{ax+by}, \text{ if } f(a, b) \neq 0$$

➢ If $f(a, b) = 0$ then $m = \frac{a}{b}$ is a root of auxiliary equation repeated r times.

$$f(D, D') = \left(D - \frac{a}{b} D'\right)^r g(D, D')$$

$$\text{P.I.} = \frac{1}{\left(D - \frac{a}{b} D'\right)^r g(D, D')} e^{ax+by} = \frac{x^r}{r!} \frac{1}{g(a, b)} e^{ax+by}, g(a, b) \neq 0$$

✓ **Case-2** $F(x, y) = \sin(ax + by)$

$$\text{P.I.} = \frac{1}{f(D^2, DD', D'^2)} \sin(ax + by) = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax + by)$$

Where, $f(-a^2, -ab, -b^2) \neq 0$

➢ If $f(-a^2, -ab, -b^2) = 0$, then use general method for finding P.I.

✓ **Case-3** $F(x, y) = \cos(ax + by)$

$$\text{P.I.} = \frac{1}{f(D^2, DD', D'^2)} \cos(ax + by) = \frac{1}{f(-a^2, -ab, -b^2)} \cos(ax + by)$$

Where, $f(-a^2, -ab, -b^2) \neq 0$

➢ If $f(-a^2, -ab, -b^2) = 0$, then use general method for finding P.I.

✓ **Case-4** $F(x, y) = x^m y^n$

$$P.I. = \frac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n$$

- Expand $[f(D, D')]^{-1}$ by using binomial expansion according to the following rules:
  - If $n < m$, expand in power of $\frac{D'}{D}$.
  - If $m < n$, expand in power of $\frac{D}{D'}$.

✓ **Case-5** $f(x, y) = e^{ax+by} V(x, y)$

$$P.I. = \frac{1}{f(D, D')} e^{ax+by} V(x, y) = e^{ax+by} \frac{1}{f(D+a, D'+b)} V(x, y)$$

METHOD – 3: EXAMPLE ON SOLUTION OF HIGHER ORDERED PDE

C	**1**	Solve $\frac{\partial^2 z}{\partial x^2} = z$. **Answer**: $z(x, y) = f(y)e^x + g(y)e^{-x}$	W – 14
H	**2**	Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$, given that when $x = 0, z = e^y$ and $\frac{\partial z}{\partial x} = 1$. **Answer**: $z = e^y \cos x + \sin x$	
C	**3**	Solve $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 2\frac{\partial^3 z}{\partial y^3} = 0$. **Answer**: $\mathbf{z = \phi_1(y + x) + \phi_2[y + (1 + \sqrt{3})x] + \phi_3[y + (1 - \sqrt{3})x]}$	W – 14
H	**4**	Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$. **Answer**: $\mathbf{z = \phi_1(y - 2x) + \phi_2(y + 3x)}$	
H	**5**	Solve $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x}$. **Answer**: $\mathbf{z = \phi_1(y) + x\phi_2(y) + \phi_3(y + 2x) + \frac{1}{4}e^{2x}}$	W – 16

C	**6**	Find complete solution $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial^2 x \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$ **Answer**: $\boldsymbol{\phi_1(y-x) + \phi_2(y+2x) + x\phi_3(y+2x) + \frac{e^{x+2y}}{11}}$	W – 17
H	**7**	Solve $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+3y}$ **Answer**: $\boldsymbol{\phi_1(y+2x) + x\phi_2(y+2x) + \frac{e^{2x+3y}}{16}}$	S – 18
H	**8**	Solve $(D^2 - 3DD' + 2D'^2)z = \cos(x+2y)$. **Answer**: $\mathbf{z = \phi_1(y-2x) + \phi_2(y-x) - \frac{1}{3}\cos(x+2y)}$	
C	**9**	Solve $\frac{\partial^3 z}{\partial^2 x \partial y} = \cos(2x+3y)$. **Answer**: $\mathbf{z = -\frac{1}{12}\sin(2x+3y) + xF(y) + G(y) + \phi(x)}$	S – 18
H	**10**	Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = \sin(2x+y)$. **Answer**: $\mathbf{z = \phi_1(y-2x) + \phi_2(y+3x) + \frac{1}{4}\sin(2x+y)}$	
C	**11**	Solve $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y$. **Answer**: $\mathbf{z = \phi_1(y-2x) + \phi_2(y-x) - \frac{x^3}{3} + \frac{x^2 y}{2}}$	S – 15 W – 17

❖ LAGRANGE'S DIFFERENTIAL EQUATION:

✓ A partial differential equation of the form $Pp + Qq = R$ where P, Q and R are functions of x, y, z, or constant is called lagrange linear equation of the first order.

❖ METHOD FOR OBTAINING GENERAL SOLUTION OF $Pp + Qq = R$:

✓ **Step-1:** From the A.E. $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

✓ **Step-2:** Solve this A.E. by the method of grouping or by the method of multiples or both to get two independent solution $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$.

✓ **Step-3:** The form $F(u,v) = 0$ or $u = f(v)$ & $v = f(u)$ is the general solution $Pp + Qq = R$.

❖ FOLLOWING TWO METHODS WILL BE USED TO SOLVE LANGRAGE'S LINEAR EQUATION

- Grouping Method
  - This method is applicable only if the third variable z is absent in $\frac{dx}{P} = \frac{dy}{Q}$ or it is possible to eliminate z from $\frac{dx}{P} = \frac{dy}{Q}$.
  - Similarly, if the variable x is absent in last two fractions or it is possible to eliminate x from last two fractions $\frac{dy}{Q} = \frac{dz}{R}$, then we can apply grouping method.
- Multipliers Method
  - In this method, we require two sets of multiplier l, m, n and l', m', n'.
  - By appropriate selection multiplier l, m, n (either constants or functions of x, y, z) we may write

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lP + mQ + nR} \text{ Such that, } lP + mQ + nR = 0 \,.$$

  - This implies $ldx + mdy + ndz = 0$
  - Solving it we get $u(x,y,z) = c_1$ ... (1)
  - Again we may find another set of multipliers l', m', n'. So that, $l'P + m'Q + n'R = 0$
  - This gives, $l'dx + m'dy + n'dz = 0$
  - Solving it we get $v(x,y,z) = c_2$ ... (2)
  - From (1) and (2), we get the general solution as $F(u,v) = 0$.

METHOD – 4: EXAMPLE ON LAGRANGE'S DIFFERENTIAL EQUATION

H	1	Solve $x^2p + y^2q = z^2$. **Answer**: $\mathbf{f\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{z} - \frac{1}{y}\right) = 0}$	S – 16
H	2	Solve $y^2p - xyq = x(z - 2y)$. **Answer**: $\mathbf{f(x^2 + y^2, yz - y^2) = 0}$	S – 17

H	**3**	Solve $xp + yq = 3z$. **Answer**: $\mathbf{f\left(\frac{x}{y}, \frac{y^3}{z}\right) = 0}$	S – 18
H	**4**	Find the general solution to the P.D.E. $xp + yq = x - y$. **Answer**: $\mathbf{f\left(\frac{x}{y}, x - y - z\right) = 0}$	W – 15
C	**5**	Solve $(z - y)p + (x - z)q = y - x$. **Answer**: $\mathbf{f(x + y + z, x^2 + y^2 + z^2) = 0}$	S – 15
H	**6**	Solve $x(y - z)p + y(z - x)q = z(x - y)$. **Answer**: $\mathbf{f(x + y + z, xyz) = 0}$	
C	**7**	Solve $(x^2 - y^2 - z^2)p + 2xyq = 2xz$. **Answer**: $\mathbf{f\left(\frac{y}{z}, \frac{x^2 + y^2 + z^2}{z}\right) = 0}$	W – 16
T	**8**	Solve $(y + z)p + (x + z)q = x + y$. **Answer**: $\mathbf{f\left(\frac{x - y}{y - z}, (x + y + z)(x - y)^2\right) = 0}$	
T	**9**	Solve $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$. **Answer**: $\mathbf{f\left(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0}$	
C	**10**	Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$. **Answer**: $\mathbf{f\left(\frac{x - y}{y - z}, \frac{y - z}{z - x}\right) = 0}$	W – 17

❖ NON LINEAR PARTIAL DIFFERENTIAL EQUATION OF FIRST ORDER:

- ✓ A partial differential equation in which p & q occur in more than one order is known as Non Linear Partial Differential Equation.
- ✓ **Type 1:** Equation Of the form $f(p, q) = 0$.
    - **Step 1:** Substitute $p = a$ & $q = b$.
    - **Step 2:** Convert $b = g(a)$.
    - **Step 3:** Complete Solution : $z = ax + by + c \Longrightarrow z = ax + g(a)y + c$
- ✓ **Type 2:** Equation Of the form $f(x, p) = g(y, q)$.
    - **Step 1:** $f(x, p) = g(y, q) = a$

- **Step 2:** Solving equations for p & q. Assume $p = F(x)$ & $q = G(y)$.
- **Step 3:** Complete Solution : $z = \int F(x)\,dx + \int G(y)\,dy + b$.

✓ **Type 3:** Equation Of the form $z = px + qy + f(p, q)$ (Clairaut's form.) W-15

- **Step 1:** Substitute $p = a$ & $q = b$.
- **Step 2:** Complete Solution : $z = ax + by + f(a, b)$.

✓ **Type 4:** Equation Of the form $f(z, p, q) = 0$.

- **Step 1:** Assume $q = ap$
- **Step 2:** Solve the Equation in $dz = p\,dx + q\,dy$

❖ CHARPIT'S METHOD:

✓ Consider, $f(x, y, z, p, q) = 0$.

- **Step 1:** Find value of p & q by using the relation

$$\frac{dx}{\frac{\partial f}{\partial p}} = \frac{dy}{\frac{\partial f}{\partial q}} = \frac{dz}{p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q}} = \frac{dp}{-\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)} = \frac{dq}{-\left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)} \text{ (lagrange} - \text{Charpit eq}^n\text{)}$$

- **Step 2:** Find value of p & q.
- **Step 3:** Complete Solution : $z = \int p\,dx + \int q\,dy + c$.

METHOD – 5: EXAMPLE ON NON-LINEAR PDE

C	**1**	Solve $p^2 + q^2 = 1$. **Answer**: $z = ax \pm \left(\sqrt{1 - a^2}\right) y + c$	S – 18
H	**2**	Solve $\sqrt{p} + \sqrt{q} = 1$. **Answer**: $z = ax + \left(1 - \sqrt{a}\right)^2 y + c$	W – 14
H	**3**	Find the complete integral of $q = pq + p^2$. **Answer**: $z = ax + \frac{a^2}{1 - a} y + c\,;\, a \neq 1$	

C	**4**	Solve $p^2+q^2 = npq$. **Answer**: $z = ax + \frac{na \pm a\sqrt{n^2-4}}{2}y + c$	
T	**5**	Find the complete integral of $p^2 = q + x$. **Answer**: $z = \frac{2}{3}(x+a)^{\frac{3}{2}} + ay + b$	
C	**6**	Solve $p^2 + q^2 = x + y$. **Answer**: $z = \frac{2}{3}(a+x)^{\frac{3}{2}} + \frac{2}{3}(y-a)^{\frac{3}{2}} + b$	
T	**7**	Solve $p^2 - q^2 = x - y$. **Answer**: $z = \frac{2}{3}(a+x)^{\frac{3}{2}} + \frac{2}{3}(a+y)^{\frac{3}{2}} + b$	W – 14 S – 17 S – 18
H	**8**	Solve $p - x^2 = q + y^2$. **Answer**: $z = ax + \frac{x^3}{3} + ay - \frac{y^3}{3} + b$	S – 15
C	**9**	Solve $z = px + qy + p^2q^2$. **Answer**: $z = ax + by + a^2b^2$	
H	**10**	Solve $qz = p(1+q)$. **Answer**: $\log(az-1) = x + ay + b$	
C	**11**	Solve $pq = 4z$. **Answer**: $az = (x + ay + b)^2$	
T	**12**	Using Charpit's method solve $z = px + qy + p^2 + q^2$. **Answer**: $z = ax + by + a^2 + b^2$	
C	**13**	Solve $py = 2\,yx + \log q$. **Answer**: $az = ax^2 + a^2x + e^{ay} + ab$	
H	**14**	Solve by charpit's method $px + qy = pq$. **Answer**: $2az = (ax+y)^2 + b$	S – 16
H	**15**	Solve by charpit's method $p = (z+qy)^2$. **Answer**: $\mathbf{zy = ax + z\sqrt{a}\sqrt{y} + b}$	S – 18

❖ **METHOD OF SEPARATION OF VARIABLES:**

✓ **Step 1:** Let $u(x,y) = X(x) \cdot Y(y)$

- ✓ **Step 2:** Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}$ as requirement and substitute in given Partial Differential Eq^n.
- ✓ **Step 3:** Convert it into Separable Variable equation and equate with constant say k individually.
- ✓ **Step 4:** Solve each Ordinary Differential Equation.
- ✓ **Step 5:** Put value of $X(x)$ & $Y(y)$ in equation $u(x,y) = X(x) \cdot Y(y)$.

METHOD – 6: EXAMPLE ON SEPARATION OF VARIABLES

H	1	Solve the equation by method of separation of variables $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, where $u(0,y) = 8e^{-3y}$. **Answer**: $u(x,y) = 8\,e^{-12x-3y}$	
C	2	Solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ subject to the condition $u(x,0) = 6e^{-3x}$ **Answer**: $u(x,t) = 6\,e^{-3x-2t}$	S – 15 S – 17
H	3	Solve the equation $u_x = 2u_t + u$ given $u(x,0) = 4e^{-4x}$ by the method of separation of variable. **Answer**: $u(x,t) = 4e^{-4x-\frac{5}{2}t}$	S – 16
C	4	Using method of separation of variables solve $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x+y)u$. **Answer**: $u(x,y) = c_1 c_2 e^{x^2+y^2+kx-ky}$	
H	5	Solve $x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$ using method of separation variables. **Answer**: $u(x,y) = c_1\,c_2\,x^k\,y^{\frac{k}{2}}$	W – 16
H	6	Using the method of separation variables, solve the partial differential equation $u_{xx} = 16u_y$. **Answer**: $u(x,y) = \left(c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}\right)\ c_3\ e^{\frac{ky}{16}}$	
H	7	Using method of separation of variables solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u$. **Answer**: $u(x,y) = \left(c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}\right)\ c_3\ e^{(k-2)y}$	

C	**8**	Solve two dimensional Laplace's equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$,using the method separation of variables. **Answer**: $u(x,y)=\left(c_1e^{\sqrt{k}x}+c_2e^{-\sqrt{k}x}\right)\left(c_3\cos\sqrt{k}y+c_4\sin\sqrt{k}y\right)$	
H	**9**	Using the method of separation of variables, solve the partial differential equation $\frac{\partial^2 u}{\partial x^2}=16\frac{\partial^2 u}{\partial y^2}$. **Answer**: $u(x,y)=\left(c_1e^{\sqrt{k}x}+c_2e^{-\sqrt{k}x}\right)\left(c_3e^{\frac{\sqrt{k}}{4}y}+c_4e^{-\frac{\sqrt{k}}{4}y}\right)$	W – 14
H	**10**	Solve the method of separation of variables $\frac{\partial^2 u}{\partial x^2}-4\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$. **Answer**: $u(x,y)=\left(c_1e^{(2+\sqrt{4+k})x}+c_2e^{(2-\sqrt{4+k})x}\right)c_3e^{-ky}$	
C	**11**	Solve $\frac{\partial^2 z}{\partial x^2}-2\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=0$ by the method of separation variable. **Answer**: $z(x,y)=\left(c_1e^{(1+\sqrt{1+k})x}+c_2e^{(1-\sqrt{1+k})x}\right)c_3e^{-ky}$	W – 17

❖ CLASSIFICATION OF SECOND ORDER PARTIAL DIFFERENTIAL EQUATION:

✓ The general form of a non-homogeneous second order P.D.E.

$$A(x,y)\frac{\partial^2 z}{\partial x^2}+B(x,y)\frac{\partial^2 z}{\partial x\,\partial y}+C(x,y)\frac{\partial^2 z}{\partial y^2}+f\left(x,y,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right)=F(x,y)\ \ldots\ldots(1)$$

✓ Equation (1) is said to be

Elliptic, If $B^2-4AC<0$ **Parabolic,** If $B^2-4AC=0$ **Hyperbolic,** If $B^2-4AC>0$

METHOD – 7: EXAMPLE ON CLASSIFICATION OF 2ND ORDER PDE

C	**1**	Classify the 2nd order P.D.E. $t\frac{\partial^2 u}{\partial t^2}+3\frac{\partial^2 u}{\partial x\,\partial t}+x\frac{\partial^2 u}{\partial x^2}+17\frac{\partial u}{\partial x}=0$. **Answer**: **Hyperbolic, if $xt>\frac{9}{4}$; Parabolic, if $xt=\frac{9}{4}$; Elliptical, if $xt<\frac{9}{4}$**	
H	**2**	Classify the 2nd order P.D.E. $4\frac{\partial^2 u}{\partial t^2}-9\frac{\partial^2 u}{\partial t\,\partial x}+5\frac{\partial^2 u}{\partial x^2}=0$. **Answer**: **Hyperbolic**	

H	**3**	Classify the 2nd order P.D.E. $\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x \partial t} + 2\frac{\partial^2 u}{\partial x^2} = 0$. **Answer**: **Elliptic**	

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GUJARAT TECHNOLOGICAL UNIVERSITY

ADVANCED ENGINEERING MATHEMATICS

SUBJECT CODE: 2130002

B.E. 3rd SEMESTER

Type of course: Engineering Mathematics

Prerequisite: The course follows from Calculus, Linear algebra

Rationale: Mathematics is a language of Science and Engineering

Teaching and Examination Scheme:

Teaching Scheme			Credits	Examination Marks						Total Marks
L	T	P	C	Theory Marks			Practical Marks			
				ESE (E)	PA (M)		PA (V)		PA (I)	
					PA	ALA	ESE	OEP		
3	2	0	5	70	20	10	30	0	20	150

Content:

Sr. No.	Topics	Teaching Hrs.	Module Weightage
1	**Introduction to Some Special Functions:** Gamma function, Beta function, Bessel function, Error function and complementary Error function, Heaviside's function, pulse unit height and duration function, Sinusoidal Pulse function, Rectangle function, Gate function, Dirac's Delta function, Signum function, Saw tooth wave function, Triangular wave function, Halfwave rectified sinusoidal function, Full rectified sine wave, Square wave function.	02	4
2	**Fourier Series and Fourier integral:** Periodic function, Trigonometric series, Fourier series, Functions of any period, Even and odd functions, Half-range Expansion, Forced oscillations, Fourier integral	05	10
3	**Ordinary Differential Equations and Applications:** First order differential equations: basic concepts, Geometric meaning of $y' = f(x,y)$ Direction fields, Exact differential equations, Integrating factor, Linear differential equations, Bernoulli equations, Modeling , Orthogonal trajectories of curves.Linear differential equations of second and higher order: Homogeneous linear differential equations of second order, Modeling: Free Oscillations, Euler- Cauchy Equations, Wronskian, Non homogeneous equations, Solution by undetermined coefficients, Solution by variation of parameters, Modeling: free Oscillations resonance and Electric circuits, Higher order linear differential equations, Higher order homogeneous with constant coefficient, Higher order non homogeneous equations. Solution by $[1/f(D)]\, r(x)$ method for finding particular integral.	11	20

4	**Series Solution of Differential Equations:** Power series method, Theory of power series methods, Frobenius method.	03	6
5	**Laplace Transforms and Applications:** Definition of the Laplace transform, Inverse Laplace transform, Linearity, Shifting theorem, Transforms of derivatives and integrals Differential equations, Unit step function Second shifting theorem,	09	15
	Dirac's delta function, Differentiation and integration of transforms, Convolution and integral equations, Partial fraction differential equations, Systems of differential equations		
6	**Partial Differential Equations and Applications:** Formation PDEs, Solution of Partial Differential equations f(x,y,z,p,q) = 0, Nonlinear PDEs first order, Some standard forms of nonlinear PDE, Linear PDEs with constant coefficients,Equations reducible to Homogeneous linear form, Classification of second order linear PDEs.Separation of variables use of Fourier series, D'Alembert's solution of the wave equation,Heat equation: Solution by Fourier series and Fourier integral	12	15

Reference Books:

1. Advanced Engineering Mathematics (8th Edition), by E. Kreyszig, Wiley-India (2007).
2. Engineering Mathematics Vol 2, by Baburam, Pearson
3. W. E. Boyce and R. DiPrima, Elementary Differential Equations (8th Edition), John Wiley (2005)
4. R. V. Churchill and J. W. Brown, Fourier series and boundary value problems (7th Edition), McGraw-Hill (2006).
5. T.M.Apostol, Calculus , Volume-2 (2nd Edition), Wiley Eastern , 1980

Course Outcome:

After learning the course the students should be able to

1. Fourier Series and Fourier Integral
   - Identify functions that are periodic. Determine their periods.
   - Find the Fourier series for a function defined on a closed interval.
   - Find the Fourier series for a periodic function.
   - Recall and apply the convergence theorem for Fourier series.
   - Determine whether a given function is even, odd or neither.
   - Sketch the even and odd extensions of a function defined on the interval [0,L].
   - Find the Fourier sine and cosine series for the function defined on [0,L]

2. Ordinary Differential Equations and Their Applications
   - Model physical processes using differential equations.
   - Solve basic initial value problems, obtain explicit solutions if possible.
   - Characterize the solutions of a differential equation with respect to initial values.
   - Use the solution of an initial value problem to answer questions about a physical system.
   - Determine the order of an ordinary differential equation. Classify an ordinary differential equation as linear or nonlinear.
   - Verify solutions to ordinary differential equations.
   - Identify and solve first order linear equations.
   - Analyze the behavior of solutions.
   - Analyze the models to answer questions about the physical system modeled.

- Recall and apply the existence and uniqueness theorem for first order linear differential equations.
- Identify whether or not a differential equation is exact.
- Use integrating factors to convert a differential equation to an exact equation and then solve.
- Solve second order linear differential equations with constant coefficients that have a characteristic equation with real and distinct roots.
- Describe the behavior of solutions.
- Recall and verify the principal of superposition for solutions of second order linear differential equations.
- Evaluate the Wronskian of two functions.

Seat No.: ________ Enrolment No.__________

GUJARAT TECHNOLOGICAL UNIVERSITY

BE - SEMESTER–III • EXAMINATION – WINTER • 2014

Subject Code: 2130002 **Date: 06-01-2015**
Subject Name: Advanced Engineering Mathematics
Time: 02.30 pm - 05.30 pm **Total Marks: 70**

Instructions:
1. **Attempt all questions.**
2. **Make suitable assumptions wherever necessary.**
3. **Figures to the right indicate full marks.**

Unit-1

Q.1 **(a)** Define the terms: (1) Unit Step Function (2) Dirac Delta Function **04**

Unit-6

(b) Form partial differential equation by eliminating arbitrary function from $F\left(x^2-y^2, xyz\right)=0$ **03**

Unit-4

(c) Find the series solution of $\left(1-x^2\right)y''-2xy'+2y=0$. **07**

Unit-2

Q.2 **(a)** Obtain the Fourier Series of $f(x)=\left(\frac{\pi-x}{2}\right)^2$ in the interval $0<x<2\pi$. Hence deduce that $\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots$. **07**

Unit-2

(b) Find the Fourier Series of $f(x)=x+|x|$ in the interval $-\pi<x<\pi$. **07**

OR

Unit-2

(b) Obtain Half range cosine series of $f(x)=\sin x$ in the interval $0<x<\pi$. Hence deduce that $\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots$. **07**

Unit-5

Q.3 **(a)** (1) Define Laplace Transform. Prove that $L\left(e^{-at}\right)=\frac{1}{s+a}$, $s>-a$ **03**

(2) Find the Laplace Transform of (i) $e^{-2t}(\sin 4t+t^2)$ (ii) $\frac{\cos 3t}{t}$ **04**

Unit-5

(b) Find the inverse Laplace Transform of (i) $\frac{s}{(s+1)(s-1)^2}$ (ii) $\frac{s}{\left(s^2+1\right)^2}$ **07**

OR

Unit-5

Q.3 **(a)** Do as Directed.

(1) Find $L\left(t^2u(t-2)\right)$ **03**

(2) Find the Laplace Transform of (i) $t^2\cosh \pi t$ (ii) $\sin^2 3t$ **04**

Unit-5

(b) Solve the initial value problem by using Laplace Transforms: **07**

$y''+y=\sin 2t$, $y(0)=2$, $y'(0)=1$

Unit-3

Q.4 **(a)** (1) Solve: $\left(x^3+3xy^2\right)dx+\left(3x^2y+y^3\right)dy=0$ **03**

(2) Solve: $x^2y\,dx-\left(x^3+xy^2\right)dy=0$ **04**

Unit-3

(b) Solve: $y''+2y'+3y=2x^2$ **07**

OR

Unit-3 **Q.4** **(a)** (1) Solve the initial value problem $y - 4y + 4y = 0$ $y(0) = 3,\ y'(0) = 1$

(2) Solve: $(x^2y^2 + 2)ydx + (2 - x^2y^2)xdy = 0$ **03**

Unit-3 **(b)** $y''' - 3y'' + 3y \quad y = 4e'$ **04**

Unit-6 **Q.5** **(a)** (i) Solve: $\sqrt{p} + \sqrt{q} = 1$ **07**

(ii) Solve: $\frac{\partial^2 z}{\partial x^2}$ **03**

04

Unit-6 **(b)** Using the method of separation of variables, solve the partial differential equation $\frac{\partial^2 u}{\partial x^2} - 16\frac{\partial^2 u}{\partial y^2}$ **07**

OR

Unit-6 **Q.5** **(a)** (i) Solve: $p^2 - q^2 = x - y$ **03**

(ii) Solve: $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial^2 x \partial y} + 2\frac{\partial^3 z}{\partial y^3} = 0$ **04**

Unit-2 **(b)** Find the Fourier Integral representation of $f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$ **07**

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Seat No.: ________ Enrolment No.___________

GUJARAT TECHNOLOGICAL UNIVERSITY

BE - SEMESTER– III (NEW)EXAMINATION – SUMMER 2015

Subject Code:2130002 **Date: 06/06/2015**

Subject Name:Advanced Engineering Mathematics

Time:02.30pm-05.30pm **Total Marks: 70**

Instructions:

1. **Attempt all questions.**
2. **Make suitable assumptions wherever necessary.**
3. **Figures to the right indicate full marks.**

Unit-3	**Q.1**	**(a)**	(1) Solve the differential equation $\frac{dy}{dx}+\frac{1}{x}=\frac{e^y}{x^2}$.	**04**
			(2) Solve the differential equation $ye^x dx+(2y+e^x)dy=0$.	**03**
Unit-4		**(b)**	Find the series solution of $(1+x^2)y''+xy'-9y=0$.	**07**
Unit-3	**Q.2**	**(a)**	(1)Solve the differential equation using the method of variation of parameter $y''+9y=\sec 3x$.	**04**
			(2) Solve the differential equation $(D^2-2D+1)y=10e^x$.	**03**
Unit-6		**(b)**	Using the method of separation of variables, solve $\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial t}+u$; $u(x,0)=6e^{-3x}$.	**07**
			OR	
Unit-4		**(b)**	Find the series solution of $2x(x-1)y''-(x+1)y'+y=0;\quad x_0=0$	**07**
Unit-2	**Q.3**	**(a)**	Find the Fourier Series for $f(x)=\begin{cases}\pi+x; & -\pi<x<0\\ \pi-x; & 0<x<\pi\end{cases}$	**07**
Unit-2		**(b)**	(1) Find the Half range Cosine Series for $f(x)=(x-1)^2; 0<x<1$.	**04**
			(2) Find the Fourier sine series for $f(x)=e^x; 0<x<\pi$.	**03**
			OR	
Unit-2	**Q.3**	**(a)**	Find the Fourier Series for $f(x)=\begin{cases}-\pi; & -\pi<x<0\\ x-\pi; & 0<x<\pi\end{cases}$.	**07**
Unit-2		**(b)**	(1) Find the Fourier cosine series for $f(x)=x^2; 0<x<\pi$.	**04**
			(2) Find the Fourier sine series for $f(x)=2x; 0<x<1$.	**03**
Unit-5	**Q.4**	**(a)**	(1) Prove that (i) $L(e^{at})=\frac{1}{s-a}; s>a$ (ii) $L(\sinh at)=\frac{a}{s^2-a^2}$.	**04**
			(2) Find the Laplace transform of $t\sin 2t$.	**03**
Unit-5		**(b)**	(1) Using convolution theorem, obtain the value of $L^{-1}\left\{\frac{1}{s(s^2+4)}\right\}$.	**04**
			(2) Find the inverse Laplace transform of $\frac{1}{(s-2)(s+3)}$.	**03**
			OR	
Unit-5	**Q.4**	**(a)**	Solve the initial value problem using Laplace transform: $y''+3y'+2y=e^t,\ y(0)=1,\ y'(0)=0$.	**07**
Unit-5		**(b)**	(1) Find the Laplace transform of $f(t)=\begin{cases}0 \ ; & 0<t<\pi\\ \sin t; & t\ge\pi\end{cases}$.	**04**

(2) Evaluate $t * e^t$. **03**

Unit-2 **Q.5** **(a)** Using Fourier integral representation prove that

$$\int_0^\infty \frac{\cos\lambda x + \lambda \sin\lambda x}{1+\lambda^2} d\lambda = \begin{cases} 0 & if \quad x<0 \\ \frac{\pi}{2} & if \quad x=0 \\ \pi e^{-x} & if \quad x>0 \end{cases}.$$ **07**

Unit-6 **(b)** (1) Form the partial differential equation by eliminating the arbitrary functions from $f\left(x+y+z, x^2+y^2+z^2\right)=0$. **04**

(2) Solve the following partial differential equation $(z-y)p+(x-z)q=y-x$. **03**

OR

Unit-6 **Q.5** **(a)** A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the temperature initially is

$$u(x,0)=\begin{cases} x & ; \quad 0 \le x \le 50 \\ 100-x; & 50 \le x \le 100 \end{cases}$$ **07**

Find the temperature $u(x,t)$ at any time.

Unit-6 **(b)** (1) Solve $\frac{\partial^2 z}{\partial x^2}+3\frac{\partial^2 z}{\partial x \partial y}+2\frac{\partial^2 z}{\partial y^2}=x+y$. **04**

(2) Solve $p-x^2=q+y^2$. **03**

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Seat No.: ________ Enrolment No.___________

GUJARAT TECHNOLOGICAL UNIVERSITY

BE - SEMESTER–III (New) EXAMINATION – WINTER 2015

Subject Code:2130002 Date:31/12/2015

Subject Name: Advanced Engineering Mathematics

Time: 2:30pm to 5:30pm Total Marks: 70

Instructions:

1. **Attempt all questions.**
2. **Make suitable assumptions wherever necessary.**
3. **Figures to the right indicate full marks.**

Unit-1	**Q.1**		**Answer the following one mark each questions:**	**14**
		1	Find $\Gamma\left(\frac{13}{2}\right)$	
Unit-1		**2**	State relationship between beta and gamma functions.	
Unit-1		**3**	Represent graphically the given saw-tooth function $f(x) = 2x, \quad 0 \le x < 2$ and $f(x+2) = f(x)$ for all x.	
Unit-5		**4**	For a periodic function f with fundamental period p, state the formula to find Laplace transform of f.	
Unit-5		**5**	Find $L\left(e^{-3t}f(t)\right)$, if $L\left(f(t)\right) = \frac{s}{(s-3)^2}$.	
Unit-5		**6**	Find $L[(2t-1)^2]$.	
Unit-2		**7**	Find the extension of the function $f(x) = x + 1$, define over $(0,1]$ to $[-1,1] - \{0\}$ which is an odd function.	
Unit-2		**8**	Is the function $f(x) = \begin{cases} x, & 0 \le x \le 2 \\ x^2, & 2 < x \le 4 \end{cases}$; continuous on $[0,4]$? Give reason.	
Unit-3		**9**	Is the differential equation $\frac{dy}{dx} = \frac{y}{x}$ exact? Give reason.	
Unit-3		**10**	Give the differential equation of the orthogonal trajectory to the equation $y = cx^2$.	
Unit-3		**11**	If $y = c_1y_1 + c_2y_2 = e^x(c_1\cos x + c_2\sin x)$ is a complementary function of a second order differential equation, find the Wronskian $W(y_1, y_2)$.	
Unit-3		**12**	Solve $(D^2 + D + 1)y = 0$; where $D = \frac{d}{dt}$.	
Unit-6		**13**	Is $u(t,x) = 50e^{(t-x)/2}$, a solution to $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + u$?	
Unit-6		**14**	Give an example of a first order partial differential equation of Clairaut's form.	
Unit-3	**Q.2**	**(a)**	Solve: $\frac{dy}{dx} = \frac{x^2 - x - y^2}{2xy}$.	**03**

Unit-3		(b)	Solve: $\frac{dy}{dx}+\frac{1}{x}y=x^3y^3$.	04
Unit-4		(c)	Find the series solution of $(x-2)\frac{d^2y}{dx^2}-x^2\frac{dy}{dx}+9y=0$ about $x_0=0$.	07
			OR	
Unit-3		(c)	Explain regular-singular point of a second order differential equation and find the roots of the indicial equation to $x^2y''+xy'-(2-x)y=0$.	07
Unit-3	**Q.3**	(a)	Find the complete solution of $\frac{d^3y}{dx^3}+8y=\cosh(2x)$.	03
Unit-3		(b)	Find solution of $\frac{d^2y}{dx^2}+9y=\tan 3x$, using the method of variation of parameters.	04
Unit-6		(c)	Using separable variable technique find the acceptable general solution to the one-dimensional heat equation $\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$ and find the solution satisfying the conditions $u(0,t)=u(\pi,t)=0$ for $t>0$ and $u(x,0)=\pi-x,\quad 0<x<\pi$.	07
			OR	
Unit-3	**Q.3**	(a)	Solve completely, the differential equation $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=\cos(2x)\sin x$.	03
Unit-3		(b)	Solve completely the differential equation $x^2\frac{d^2y}{dx^2}-6x\frac{dy}{dx}+6y=x^{-3}\log x$.	04
Unit-6		(c)	(i) Form the partial differential equation for the equation $(x-a)(y-b)-z^2=x^2+y^2$. (ii) Find the general solution to the partial differential equation $xp+yq=x-y$.	07
Unit-2	**Q.4**	(a)	Find the Fourier cosine integral of $f(x)=\frac{\pi}{2}e^{-x},\ x\geq 0$.	03
Unit-2		(b)	For the function $f(x)=\cos 2x$, find its Fourier sine series over $[0,\pi]$.	04
Unit-2		(c)	For the function $f(x)=\begin{cases} x; & 0\leq x\leq 2 \\ 4-x; & 2\leq x\leq 4\end{cases}$, find its Fourier series. Hence show that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{16}$.	07
			OR	
Unit-2	**Q.4**	(a)	Find the Fourier cosine series of $f(x)=e^{-x}$, where $0\leq x\leq\pi$.	03
Unit-2		(b)	Show that $\int_0^\infty \frac{\lambda^3\sin\lambda x}{\lambda^4+4}d\lambda=\frac{\pi}{2}e^{-x}\cos x,\ x>0$.	04
Unit-2		(c)	Is the function $f(x)=x+\lvert x\rvert$, $-\pi\leq x\leq\pi$ even or odd? Find its Fourier series over the interval mentioned.	07
Unit-5	**Q.5**	(a)	Find $L\left\{\int_0^t e^u(u+\sin u)du\right\}$.	03
Unit-5		(b)	Find $L^{-1}\left\{\frac{1}{s(s^2-3s+3)}\right\}$.	04
Unit-5		(c)	Solve the initial value problem: $y''-2y'=e^t\sin t,\ y(0)=y'(0)=0$, using Laplace transform.	07
			OR	
Unit-5	**Q.5**	(a)	Find $L\{t(\sin t-t\cos t)\}$.	03
Unit-5		(b)	Find $L^{-1}\left\{\frac{e^{-2s}}{(s^2+2)(s^2-3)}\right\}$.	04
Unit-5		(c)	State the convolution theorem and verify it for $f(t)=t$ and $g(t)=e^{2t}$.	07

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Seat No.: ________ Enrolment No.___________

GUJARAT TECHNOLOGICAL UNIVERSITY

BE - SEMESTER–III(New) EXAMINATION – SUMMER 2016

Subject Code:2130002 **Date:07/06/2016**

Subject Name:Advanced Engineering Mathematics

Time:10:30 AM to 01:30 PM **Total Marks: 70**

Instructions:

1. **Attempt all questions.**
2. **Make suitable assumptions wherever necessary.**
3. **Figures to the right indicate full marks.**

Q.1 **Answer the following one mark each questions :** **14**

Unit-3	1	Integreating factor of the differential equation $\frac{dx}{dy}+\frac{3x}{y}=\frac{1}{y^2}$ is _________	
Unit-3	2	The general solution of the differential equation $\frac{dy}{dx}+\frac{y}{x}$ =tan2x_____________.	
Unit-3	3	The orthogonal trajectory of the family of curve $x^2+y^2=c^2$ is __________	
Unit-3	4	Particular integral of $(D^2+4)y=\cos 2x$ is _________	
Unit-4	5	X=0 is a regular singular point of $2x^2y''+3xy'(x^2-4)y=0$ say true or false.	
Unit-6	6	The solution of $(y-z)p+(z-x)q=x-\ y\ is$_________	
Unit-3	7	State the type ,order and degree of differential equation $(\frac{dx}{dy})^2+5y^{\frac{1}{3}}$ = x is __________	
Unit-6	8	Solve (D+D')z= cos x	
Unit-6	9	Is the partial differential equation $2\frac{\partial^2u}{\partial x^2}+4\frac{\partial^2u}{\partial x\partial y}+\ 3\ \frac{\partial^2u}{\partial y^2}=6$ elliptic?	
Unit-5	10	$L^{-1}\left(\frac{1}{(s+a)^2}\right)=$ ________	
Unit-5	11	If f(t) is a periodic function with period t then L$[f(t)]=$ _______	
Unit-5	12	Laplace transform of f(t) is defined for +ve and –ve values of t. Say true or false.	
Unit-1	13	State Duplication (Legendre) formula.	
Unit-1	14	Find B $(\frac{9}{2},\frac{7}{2})$	

Unit-3 **Q.2** **(a)** Solve : $9y\,y^{'}+4x=0$ **03**

Unit-3 (b) Solve : $\frac{dy}{dx} + y\cot x = 2\cos x$ **04**

Unit-4 (c) Find series solution of $y'' + xy = 0$ **07**

OR

Unit-1 (c) Determine the value of (a) $J_{\frac{1}{2}}(x)$ (b) $J_{\frac{3}{2}}(x)$ **07**

Unit-3 **Q.3** **(a)** Solve $(D^2 + 9)y = 2\sin 3x + \cos 3x$ **03**

Unit-3 **(b)** Solve $y'' + 4y' = 8x^2$ by the method of undetermined coefficients. **04**

Unit-6 **(c)** (i) Solve $x^2 p + y^2 q = z^2$ **07**

(ii) Solve by charpit's method px+qy = pq

OR

Unit-3 **Q.3** **(a)** Solve $y'' + 4y' + 4 = 0$, $y(0) = 1$, $y'(0) = 1$ **03**

Unit-3 **(b)** Find the solution of $y'' + a^2 y' = \tan ax$, by the method of variation of parameters. **04**

Unit-6 **(c)** Solve the equation $u_x = 2u_t + u$ given u(x,0)= $4e^{-4x}$ by the method of separation of variable. **07**

Unit-2 **Q.4** **(a)** Find the fourier transform of the function f(x) = e^{-ax^2} **03**

Unit-2 **(b)** Obtain fourier series to represent f(x) =x^2 in the interval $-\pi < x < \pi$ .Deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ **04**

Unit-2 **(c)** Find Half-Range cosine series for **07**

F(x) = kx , $0 \leq x \leq \frac{l}{2}$

= k(l-x) , $\frac{l}{2} \leq x \leq l$

Also prove that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

OR

Unit-2 **Q.4** **(a)** Expres the function **03**

F(x)= 2 , |x| < 2

= 0 , |x| > 2 as Fourier integral.

Unit-2 **(b)** Find the fourier series expansion of the function **04**

F(x) = $-\pi$ $\quad -\pi < x < 0$

= x $\quad 0 < x < \pi$

Unit-2 **(c)** Find fourier series to represent the function **07**

F(x) = 2x-x^2 in $0 < x < 3$

Unit-5 **Q.5** **(a)** $Find\ \ L^{-1}\left\{\frac{1}{(s+\sqrt{2})(s-\sqrt{3})}\right\}$ **03**

Unit-5 **(b)** Find the laplace transform of **04**

(i) $\frac{cosat - cosbt}{t}$

(ii) tsinat

Unit-5 **(c)** State convolution theorem and use to it evaluate $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$ **07**

OR

Unit-5	**Q.5**	**(a)**	L $\{t^2 \cosh 3t\}$	**03**
Unit-5		**(b)**	Find $L^{-1}\left\{\frac{1}{s^4-81}\right\}$	**04**
Unit-5		**(c)**	Solve the equation $y'' - 3y' + 2y = 4t + e^{3t}$,when y(0)=1 , $y'(0) = -1$	**07**

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Seat No.: ________ Enrolment No.___________

GUJARAT TECHNOLOGICAL UNIVERSITY

BE - SEMESTER–III(New) • EXAMINATION – WINTER 2016

Subject Code:2130002 Date:30/12/2016

Subject Name:Advanced Engineering Mathematics

Time:10:30 AM to 01:30 PM Total Marks: 70

Instructions:

1. **Attempt all questions.**
2. **Make suitable assumptions wherever necessary.**
3. **Figures to the right indicate full marks.**

MARKS

Q.1 Answer the following one mark questions **14**

Unit-1 1 Find $\Gamma\left(\frac{1}{2}\right)$.

2 State relation between beta and gamma function. Unit-1

Unit-1 3 Define Heaviside's unit step function.

4 Define Laplace transform of f (t), t ≥ 0. Unit-5

Unit-5 5 Find Laplace transform of $t^{\frac{-1}{2}}$.

6 Find L { $\frac{sinat}{t}$ }, given that L { $\frac{sint}{t}$ } = $tan^{-1}\{\frac{1}{s}\}$. Unit-5

Unit-2 7 Find the continuous extension of the function f(x) = $\frac{x^2+x-2}{x^2-1}$ to x = 1

Unit-2 8 Is the function f(x) = $\frac{1}{x}$ continuous on [-1, 1] ? Give reason.

Unit-3 9 Solve $\frac{dy}{dx} = e^{3x-2y} + x^2e^{-2y}$.

Unit-3 10 Give the differential equation of the orthogonal trajectory of the family of circles $x^2 + y^2 = a^2$.

11 Find the Wronskian of the two function sin2x and cos2x . Unit-3

Unit-3 12 Solve ($D^2 + 6D + 9$) x = 0; D = $\frac{d}{dt}$.

Unit-6 13 To solve heat equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ how many initial and boundary conditions are required.

Unit-6 14 Form the partial differential equations from z = f(x + at) + g(x –at).

Unit-3 **Q.2** **(a)** Solve : (x+1)$\frac{dy}{dx} - y = e^{3x}(x+1)^2$. **03**

(b) Solve : $\frac{dy}{dx} + \frac{ycosx + siny + y}{sinx + xcosy + x} = 0$ Unit-3 **04**

Unit-4 **(c)** Find the series solution of $\frac{d^2y}{dx^2} + xy = 0$. **07**

OR

Unit-4 **(c)** Find the general solution of $2x^2y'' + xy' + (x^2 - 1)y = 0$ by using frobenius method. **07**

Unit-3 **Q.3** **(a)** Solve : $(D^3 - 3D^2 + 9D - 27)$y = cos3x. **03**

Unit-3 **(b)** Solve : $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x^2\sin(lnx)$. **04**

Unit-6 **(c)** (i) Solve : $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x}$. **03**

04

(ii) find the general solution to the partial differential equation

$(x^2 - y^2 - z^2)p + 2xyq = 2xz.$

OR

Unit-3	**Q.3**	**(a)**	Solve : $(D^3 - D)y = x^3$.	**03**
Unit-3		**(b)**	Find the solution of $y'' - 3y' + 2y = e^x$, using the method of variation of parameters.	**04**
Unit-6		**(c)**	Solve $x\frac{\partial u}{\partial x}$ - 2y $\frac{\partial u}{\partial y} = 0$ using method of separation of variables.	**07**
Unit-2	**Q.4**	**(a)**	Find the Fourier cosine integral of $f(x) = e^{-kx}, x > 0, k > 0$	**03**
Unit-2		**(b)**	Express f(x) = |x|, $-\pi < x < \pi$ as fouries series.	**04**
Unit-2		**(c)**	Find Fourier Series for the function f(x) given by $f(x) = \begin{cases} 1 + \frac{2x}{\pi}; & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi} & ; 0 \le x \le \pi \end{cases}$ Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots. = \frac{\pi^2}{8}$.	**07**

OR

Unit-2	**Q.4**	**(a)**	Obtain the Fourier Series of periodic function function f(x) =2x, -1 < x < 1, p = 2L =2	**03**
Unit-2		**(b)**	Show that $\int_0^\infty \frac{\sin\lambda\cos\lambda}{\lambda} d\lambda = 0$, if x > 1.	**04**
Unit-2		**(c)**	Expand f(x) in Fourier series in the interval (0, 2π) if $f(x) = \begin{cases} -\pi & ; 0 < x < \pi \\ x - \pi & ; \pi < x < 2\pi \end{cases}$ and hence show that $\sum_{r=0}^{\infty} \frac{1}{(2r+1)^2} = \frac{\pi^2}{8}$.	**07**
Unit-5	**Q.5**	**(a)**	Find L$\{\int_0^t e^t \frac{\sin t}{t} dt\}$.	**03**
Unit-5		**(b)**	Find $L^{-1}\{\frac{2s^2-1}{(s^2+1)(s^2+4)}\}$.	**04**
Unit-5		**(c)**	Solve initial value problem : $y'' - 3y' + 2y = 4t + e^{3t}$,y(0) = 1 and $y'(0) = -1$,using Laplace transform.	**07**

OR

Unit-5	**Q.5**	**(a)**	Find L {tsin3tcos2t}.	**03**
Unit-5		**(b)**	Find $L^{-1}\{\frac{e^{-3s}}{s^2+8s+25}\}$.	**04**
Unit-5		**(c)**	State the convolution theorem and apply it to evaluate $L^{-1}\{\frac{s}{(s^2+a^2)^2}\}$.	**07**

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Seat No.: ________ Enrolment No.___________

GUJARAT TECHNOLOGICAL UNIVERSITY

BE - SEMESTER–III (NEW) - EXAMINATION – SUMMER 2017

Subject Code: 2130002 **Date: 25/05/2017**

Subject Name: Advanced Engineering Mathematics

Time: 10:30 AM to 01:30 PM **Total Marks: 70**

Instructions:

1. **Attempt all questions.**
2. **Make suitable assumptions wherever necessary.**
3. **Figures to the right indicate full marks.**

				MARKS
	Q.1		**Short Questions**	**14**
Unit-3		1	What are the order and the degree of the differential equation $y''+3y^2 = 3\cos x$.	
Unit-3		2	What is the integrating factor of the linear differential equation: $y'-(1/x)y = x^2$	
Unit-3		3	Is the differential equation $ye^x dx + (2y + e^x)dy = 0$ is exact? Justify.	
Unit-3		4	Solve: $y''+11y'+10y = 0$.	
Unit-3		5	Find particular integral of : $y'''+y' = e^{2x}$	
Unit-3		6	If $y = (c_1 + c_2 x)e^x$ is a complementary function of a second order differential equation, find the Wronskian $W(y_1, y_2)$.	
Unit-1		7	Find the value of $\Gamma\left(\frac{7}{2}\right)$	
Unit-2		8	What is the value of the Fourier coefficients a_0 and b_n for $f(x) = x^2, -1 < x < 1$.	
Unit-5		9	Find $L\{e^{3t+3}\}$	
Unit-5		10	Find $L^{-1}\left(\frac{4}{s^2} - \frac{1}{(s^2+9)}\right)$	
Unit-4		11	Find the singular point of the differential equation $(1-x^2)y''-2xy'+n(n+1)y = 0$	
Unit-6		12	Obtain the general integral of $\frac{\partial^3 z}{\partial x^3} = 0$	
Unit-6		13	Obtain the general integral of $p + q = z$	
		14	State the relationship between beta and gamma function.	Unit-1
Unit-3	**Q.2**	(a)	Solve: $(x^2 + y^2 + 3)dx - 2xydy = 0$	**03**
Unit-3		(b)	Solve: $\frac{dy}{dx} + (\tan x)y = \sin 2x, y(0) = 0$	**04**
Unit-3		(c)	$(D^4 - 16)y = e^{2x} + x^4$, *where* $D \equiv d/dx$	**07**
			OR	
Unit-3		(c)	Use the method of variation of parameters to find the	**07**

general solution of $y''-4y'+4y = \frac{e^{2x}}{x}$

Unit-2	Q.3	(a)	Find half range sine series of $f(x) = x^3, 0 \le x \le \pi$	03
Unit-2		(b)	Find the Fourier integral representation of the function $f(x) = \begin{cases} 2, & \lvert x\rvert < 2 \\ 0, & \lvert x\rvert > 2 \end{cases}$	04
Unit-2		(c)	Find the Fourier series expansion for the 2π- periodic function $f(x) = x - x^2$ in the interval $-\pi \le x \le \pi$ and show that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$	07
			OR	
Unit-4	Q.3	(a)	Discuss about ordinary point, singular point, regular singular point and irregular singular point for the differential equation: $x^3(x-1)y''+3(x-1)y'+7xy = 0$	03
Unit-3		(b)	Use the method of undetermined coefficients to solve the differential equation $y''+9y = 2x^2$	04
Unit-4		(c)	Find the series solution of $(x^2+1)y''+xy'-xy = 0$ about $x_0 = 0$.	07
Unit-3	Q.4	(a)	Solve: $(D^2-1)y = xe^x$, *where* $D \equiv d/dx$	03
Unit-3		(b)	Solve: $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \sin(\ln x)$	04
Unit-5		(c)	Use Laplace Transform to solve the following initial value problem: $y''-3y'+2y = 12e^{-2t}, y(0) = 2, y'(0) = 6$	07
			OR	
Unit-5	Q.4	(a)	Obtain $L\left\{e^{2t}\sin^2 t\right\}$	03
Unit-5		(b)	Find $L^{-1}\left[\frac{s+7}{s^2+8s+25}\right]$	04
Unit-5		(c)	Using Convolution theorem, obtain $L^{-1}\left[\frac{1}{(s^2+4)^2}\right]$	07
Unit-5	Q.5	(a)	Find the Laplace Transform of $te^{4t}\cos 2t$	03
Unit-6		(b)	Form the partial differential equation from the following: 1) $z = ax + by + ct$ 2) $z = f\left(\frac{x}{y}\right)$	04
Unit-6		(c)	Using the method of separation of variables solve, $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ where $u(x,0) = 6e^{-3x}$	07
			OR	
Unit-6	Q.5	(a)	Obtain the solution of the partial differential equation: $p^2 - q^2 = x - y$, where $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$	03
Unit-6		(b)	Solve: $y^2p - xyq = x(z-2y)$, where $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$	04
Unit-6		(c)	Find the solution of the wave equation	07

$u_{tt} = c^2 u_{xx}, \ 0 \le x \le L$ satisfying the conditions:

$u(0,t) = u(L,t) = 0, \ u_t(x,0) = 0, \ u(x,0) = \frac{\pi x}{L}, 0 \le x \le L$

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Seat No.: ________ Enrolment No.___________

GUJARAT TECHNOLOGICAL UNIVERSITY

BE - SEMESTER–III (NEW) EXAMINATION – WINTER 2017

Subject Code: 2130002 **Date:06/11/2017**

Subject Name: Advanced Engineering Mathematics

Time: 10:30 AM to 01:30 PM **Total Marks: 70**

Instructions:

1. **Attempt all questions.**
2. **Make suitable assumptions wherever necessary.**
3. **Figures to the right indicate full marks.**

Unit-3	**Q.1**	**(a)**	Solve the following differential equation using variable separable method $3e^x \tan y\, dx + (1+e^x)\sec^2 y\, dy = 0$	**03**
Unit-5		**(b)**	Find the Laplace transform of $t\sin^2 3t$.	**04**
Unit-2		**(c)**	Given that $f(x) = x + x^2$ for $-\pi < x < \pi$, find the Fourier expression of $f(x)$. Deduce that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots.$	**07**
Unit-1	**Q.2**	**(a)**	Define rectangle function and saw-tooth wave function. Also sketch the graphs.	**03**
Unit-3		**(b)**	Find the general solution of the following differential equation : $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = e^x \cos x$	**04**
Unit-4		**(c)**	Find the power series solution of $(1-x^2)y'' - 2xy' + 2y = 0$ about the ordinary point $x = 0$	**07**
			OR	
Unit-4		**(c)**	Find the power series solution of $3x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ about the point $x = 0$, using Frobenius method.	**07**
Unit-2	**Q.3**	**(a)**	Express $f(x) = \begin{cases} 1, & for\ 0 \le x \le \pi \\ 0, & for\ x > \pi \end{cases}$ As a Fourier sine integral and hence evaluate $\int_0^\infty \frac{1-\cos(\pi\lambda)}{\lambda} \sin(x\lambda)\, d\lambda$	**03**
Unit-3		**(b)**	Check whether the given differential equations is exact or not $(x^4 - 2xy^2 + y^4)dx - (2x^2y - 4xy^3 + \sin y)dy = 0$ Hence find the general solution.	**04**
Unit-3		**(c)**	Solve the following differential equation using the method of undetermined coefficient : $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 2x^2 + 3e^{-x}$	**07**
			OR	
Unit-2	**Q.3**	**(a)**	Find the cosine series for $f(x) = \pi - x$ in the interval $0, < x < \pi$.	**03**
Unit-3		**(b)**	Solve the following differential equation $\frac{d^2y}{dx^2} + y = \sin x$ using the method of variation of parameters.	**04**
Unit-3		**(c)**	Solve the following Cauchy-Euler equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x . \sin(\log x)$	**07**

Unit-3	Q.4	(a)	Find the orthogonal trajectory of the cardioids $r = a(1-\cos\theta)$	03
Unit-5		(b)	Find the Laplace transforms of : (i) $e^{-3t}u(t-2)$ (ii) $\dfrac{1-\cos 2t}{t}$	04
Unit-6		(c)	Solve the equation $\dfrac{\partial^2 z}{\partial x^2}-2\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}=0$ by the method of separation of variables.	07

OR

Unit-3	Q.4	(a)	Solve the following Bernoulli's equation: $\dfrac{dy}{dx}+\dfrac{y}{x}=\dfrac{y^2}{x^2}$	03
Unit-5		(b)	Find the inverse Laplace transforms of : (i) $\tan^{-1}\left(\dfrac{2}{s}\right)$ (ii) $\dfrac{s^3}{s^4-a^4}$	04
Unit-6		(c)	Find the complete solution of the following partial differential equations: (i) $\dfrac{\partial^3 z}{\partial x^3}-3\dfrac{\partial^3 z}{\partial x^2\partial y}+4\dfrac{\partial^3 z}{\partial y^3}=e^{x+2y}$ (ii) $\dfrac{\partial^2 z}{\partial x^2}+3\dfrac{\partial^2 z}{\partial x\partial y}+2\dfrac{\partial^2 z}{\partial y^2}=x+y$	07
Unit-6	Q.5	(a)	Form the partial differential equations by eliminating the arbitrary function from $f(x^2+y^2, z-xy)=0$	03
Unit-6		(b)	Solve the following Lagrange's linear differential equation: $(x^2-yz)p+(y^2-zx)q=z^2-xy$	04
Unit-5		(c)	Solve the following initial value problem using the method of Laplace transforms $y'''+2y''-y'-2y=0$ given that $y(0)=1,\ y'(0)=2,\ y''(0)=2$	07

OR

Unit-5	Q.5	(a)	Find the Laplace transform of the periodic function of the waveform $f(t)=\dfrac{2t}{3},\ \ 0\le t\le 3,\ \ f(t+3)=f(t)$	03
Unit-5		(b)	Using the convolution theorem, find $L^{-1}\left\{\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right\},\ a\ne b$	04
Unit-6		(c)	A tightly stretched string of length l with fixed ends is initially in equilibrium position. It is set vibrating by giving each point a velocity $v_0\sin^3\dfrac{\pi x}{l}$ .find the displacement $y(x,t)$.	07

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Seat No.: ________ **Enrolment No.___________**

GUJARAT TECHNOLOGICAL UNIVERSITY

BE - SEMESTER–III (NEW) - EXAMINATION – SUMMER 2018

Subject Code:2130002 **Date:16/05/2018**

Subject Name:Advanced Engineering Mathematics

Time:10:30 AM to 01:30 PM **Total Marks: 70**

Instructions:

1. **Attempt all questions.**
2. **Make suitable assumptions wherever necessary.**
3. **Figures to the right indicate full marks.**

Unit-3	**Q.1**	**(a)**	Solve $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$ by variable separable method.	**03**
Unit-3		**(b)**	Solve $\frac{dy}{dx} + y\sin x = e^{\cos x}$	**04**
Unit-5		**(c)**	State convolution theorem and hence find $L^{-1}\left[\frac{1}{(s^2+4)^2}\right]$	**07**
Unit-3	**Q.2**	**(a)**	Solve $y''-3y'+2y = e^{3x}$	**03**
Unit-2		**(b)**	Find Fourier series for $f(x) = x^2;\ -\pi \le x \le \pi$	**04**
Unit-2		**(c)**	Find Fourier series in the interval $(0,2\pi)$ if $f(x) = \begin{cases} -\pi & 0 < x < \pi \\ x-\pi & \pi < x < 2\pi \end{cases}$ and hence show that $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$	**07**
			OR	
Unit-4		**(c)**	Find the series solution of $y''+x^2y=0$ about an ordinary point $x=0$	**07**
Unit-3	**Q.3**	**(a)**	Solve $y'''-6y''+11y'-6y=0$	**03**
Unit-3		**(b)**	Solve $(D^2+9)y = \cos 4x$	**04**
Unit-3		**(c)**	Solve $y''+4y = 4\tan 2x$ by method of variation parameter.	**07**
			OR	
Unit-5	**Q.3**	**(a)**	Find $L^{-1}\left[\frac{1}{(s+1)(s+2)}\right]$	**03**
Unit-3		**(b)**	Solve $y''-2y'+5y = 5x^3 - 6x^2 + 6x$ by method of undetermined coefficients.	**04**
Unit-4		**(c)**	Find the series solution of $8x^2y''+10xy'-(1+x)y=0$	**07**
Unit-3	**Q.4**	**(a)**	Solve $(x^4+y^4)dx - xy^3dy = 0$	**03**
Unit-2		**(b)**	Express sinx as cosine series in $0 < x < \pi$	**04**
Unit-2		**(c)**	Find Fourier series for $f(x) = e^{ax}$ in $(0,2\pi);\ a > 0$	**07**
			OR	
Unit-5	**Q.4**	**(a)**	Find $L[\cos^2 t]$	**03**

Unit-5		**(b)**	Find $L\left[e^{2t}\sin 3t\right]$	**04**
Unit-5		**(c)**	Solve $y''+y=\sin 2t;$ with $y(0)=2,\ y'(0)=1$ by using Laplace transform.	**07**
Unit-6	**Q.5**	**(a)**	Solve $p^2+q^2=1$	**03**
		(b)	Solve $xp+yq=3z$ Unit-6	**04**
Unit-6		**(c)**	Solve $\frac{\partial^2 z}{\partial x^2}-4\frac{\partial z}{\partial x\partial y}+4\frac{\partial^2 z}{\partial y^2}=e^{2x+3y}$	**07**
			OR	
Unit-6	**Q.5**	**(a)**	Solve $p^2-q^2=x-y$	**03**
Unit-6		**(b)**	Solve $\frac{\partial^3 z}{\partial x^2\partial y}=\cos(2x+3y)$	**04**
Unit-6		**(c)**	Solve by Charpit's method $p=(z+qy)^2$	**07**

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