

# Anharmonic oscillator: a perturbative approach

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In this article, the eigenvalues of anharmonic oscillator are investigated with perturbative theory. We use standard perturbative means to expand the lowest 10 eigenvalues to 100th-order, and then use a Páde (50,50) approximant to better approximate each divergent series. The results will then be compared against numerical means. To the author's knowledge, this is the first extensive high order analytical study on the excited states of the  $x^4$  anharmonic oscillator<sup>a</sup>.

## I. MATHEMATICAL MODEL OF THE UNPERTURBED SYSTEM

The quantum mechanical description, *i.e.*, Schrödinger equation, of harmonic oscillator eventually reduces to this equation

$$-y''(x) + x^2 y(x) = Ey. \quad (1)$$

The solution of which can be found in standard QM text:

$$y_m^u(x) = e^{-x^2/2} H_m(x), \quad m = 0, 1, \dots, \quad (2)$$

$$E_m^u = 2m + 1, \quad (3)$$

where  $H_m$  is the  $m$ -th order Hermite polynomial and the superscript u means "unperturbed". The degree of  $m$ -th order Hermite polynomial is  $m$ .

## II. SOLVING THE PERTURBED SYSTEM

The anharmonicity takes many forms, but usually it ends up in a power form, *i.e.*, a term proportional to  $x^\alpha$  ( $\alpha = 3, 4, \dots$ ). We restrict the discussion to the case of  $\alpha = 4$ , but the method is applicable to other cases as well.

The equation of interest is

$$-y''(x) + (x^2 - \epsilon x^4)y(x) = Ey. \quad (4)$$

Since we are talking of perturbative approach, there's a coefficient  $\epsilon$  before the anharmonic term meaning that term is small compared to the harmonic term. Note, however, despite being small, the parameter  $\epsilon$  should always satisfy  $\epsilon \leq 0$ . Otherwise, the system's Hamiltonian is not bounded from below, leading to instability. For a more analytical proof, please refer to [1].

Given a pair of unperturbed eigenvalue and eigenfunction, the eigenvalue  $E$  and eigenfunction  $y$  of the per-

turbed system are expanded in terms of  $\epsilon$ :

$$E = \sum_{n=0}^{\infty} E_n \epsilon^n, \quad (5)$$

$$y = \sum_{n=0}^{\infty} y_n \epsilon^n, \quad (6)$$

with

$$E_0 = E_m^u, \quad y_0 = y_m^u, \quad (7)$$

being the unperturbed solution.

It can be shown that the different orders of perturbative solution satisfy a recursion relation [2]:

$$E_n = \frac{\int_{-\infty}^{\infty} y_0(x) \left[ W(x) y_{n-1}(x) - \sum_{j=1}^{n-1} E_j y_{n-j}(x) \right] dx}{\int_{-\infty}^{\infty} dx y_0^2(x)}, \quad n = 1, 2, \dots \quad (8)$$

and

$$y_n(x) = y_0(x) \int_{-\infty}^x \frac{dt}{y_0^2(t)} \int^t ds y_0(s) \times \left[ W(s) y_{n-1}(s) - \sum_{j=1}^n E_j y_{n-j}(s) \right], \quad n = 1, 2, \dots \quad (9)$$

where  $W(x) = -x^4$  is the perturbation potential.

### A. Update eigenfunction

We try the following *ansatz* for  $y_n$ [3]

$$y_n(x) = e^{-x^2/2} \sum_{k=0}^{4n+m} y_{nk} x^k. \quad (10)$$

The form can be justified intuitively as follows. In

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<sup>a</sup> Otherwise, there should be code online and there should be no need for me to implement them by hand.

eq. (9), the integrand for  $s$  is

$$y_0(s) \left[ W(s)y_{n-1}(s) - \sum_{j=1}^n E_j y_{n-j}(s) \right] \\ = e^{-s^2/2} \sum_{i=0}^m y_{ni} s^i \left[ -s^4 e^{-s^2/2} \sum_{l=0}^{4n-4+m} y_{n-1,l} s^l \right. \\ \left. - \sum_{j=1}^n E_j e^{-s^2/2} \sum_{l=0}^{4n-4j+m} y_{n-j,l} s^l \right] \quad (11)$$

$$= e^{-s^2} \sum_{i=0}^{4n+2m} b_{ni} s^i = b_n(s), \quad (12)$$

where in the last line we collected terms in a polynomial of order  $4n + 2m$ . However, eq. (9) is really

$$b_n(s) = [(y_n/y_0)' y_0^2]' = y_0(s) y_n''(s) - y_n(s) y_0''(s). \quad (13)$$

And it's not hard to show that the RHS also has a form of

$$e^{-s^2} \sum_{k=0}^{4n+2m+2} c_{nk} s^k, \quad (14)$$

and  $c_{nk}$  can be expressed in terms of known  $y_{0l}$  ( $l = 0, \dots, m$ ) and unknown  $y_{nk}$  ( $k = 0, \dots, 4n + m$ ).

Comparing the coefficients of different powers of  $s$  would yield  $(4n+2m+3)$  equations for  $y_{nk}$ . Note there're more equations than unknowns. But it turns out these relations are not linear independent and in the very least there should be the freedom of choosing integration constant. Solving this set of linear equations completes the update process of eigenfunction.

It can be further shown with induction that either all the odd or all the even terms in  $y_{nk}$  vanishes, depending on the parity of  $y_0(x)$  of the unperturbed solution.

### B. Calculate eigenvalue

Observe eq. (10) and eq. (8) and it's clear that both integrands for the denominator and numerator have the following form:

$$e^{-x^2} \sum_{k=0}^{d/2} e_k x^{2k}, \quad (15)$$

with  $d$  being  $4n + 2m$  for the numerator and  $2m$  for the denominator.

The integration of each term is straightforward using Gamma function:

$$\int_0^\infty e^{-x^2} x^{2k} dx = \int_0^\infty e^{-t} t^{k/2-1/2} dt = \Gamma(k+1/2). \quad (16)$$

Since the argument of Gamma function is always half-integer and Gamma functions appear in both the numerator and denominator, we could factor out  $\Gamma(1/2)$  and the correction would always be a rational number.

### III. PÁDE APPROXIMANT

Evidently, the series (5) is divergent due to the singularity at  $\epsilon = 0$ . To avoid divergence, a Páde (50,50) approximant  $P_{50}^{50}(\epsilon)$  is constructed. Formally, this means

$$P_{50}^{50}(\epsilon) = \frac{\sum_{n=0}^N A_n \epsilon^n}{\sum_{n=0}^M B_n \epsilon^n} \quad (17)$$

and the first  $50 + 50 + 1$  Taylor expansion coefficients should equal to  $E_n$ . The problem again reduces to solve linear equations.

### IV. NUMERICAL MEANS

To evaluate the effectiveness of our analytical approximation, numerical approach based on exact diagonalization is used.

The eigenfunction (2) of the unperturbed system is not normalized

$$\int_{-\infty}^{\infty} y_n^u y_m^u dx = \sqrt{\pi} 2^n n! \delta_{nm}. \quad (18)$$

To simplify the discussion, we form normalized eigenfunction

$$y_n^n = \frac{1}{\pi^{1/4} \sqrt{2^n n!}} y_n^u \quad (19)$$

such that

$$\int_{-\infty}^{\infty} y_n^n y_m^n dx = \delta_{nm}. \quad (20)$$

It can be shown

$$\int_{-\infty}^{\infty} y_n^n x^4 y_m^n dx \\ = \frac{1}{4} (6n^2 + 6n + 3) \delta_{nm} \\ + (n + 3/2) \sqrt{(n+1)(n+2)} \delta_{n,m-2} \\ + (n - 1/2) \sqrt{n(n-1)} \delta_{n,m+2} \\ + \frac{1}{4} \sqrt{(n+1)(n+2)(n+3)(n+4)} \delta_{n,m-4} \\ + \frac{1}{4} \sqrt{(n-3)(n-2)(n-1)n} \delta_{n,m+4}. \quad (21)$$

The full Hamiltonian  $H$  to diagonalize is composed of two parts  $H = H_0 + \epsilon V$ , with

$$H_0 = \text{diag} \{1, 3, \dots, 2N_{\text{cutoff}} + 1\}, \quad (22)$$

and the matrix element of  $V$  is defined in eq. (21). Choosing an appropriate dimension of cutoff  $N_{\text{cutoff}}$  is a trade-off between cutoff-error and roundoff-error, but  $N_{\text{cutoff}}$  is simply fixed at 50 in this work. Since the full Hamiltonian  $H$  is real and symmetric, the very efficient and stable Jacobi method can be used to determine the eigenvalues.

## V. IMPLEMENTATION

Since all coefficients are rational numbers, GMP (GNU Multiple Precision Arithmetic Library) [4] is used to store and manipulate them. And to ensure best performance, Eigen [5] library is used to solve linear equations and calculate eigenvalues. A simple polynomial class, featuring term by term add, multiplication by polynomial, and evaluation, is also implemented by hand. These programs are implemented with the C++ programming language [6].

On average, our method is capable of calculating the coefficients  $E_n$  in 168s (compilation) + 132s (run) given an unperturbed solution. The long compilation time is due to both Eigen and our hand-written polynomial library relies heavily on template. Although more efficient method based on direct recursion exists, our method is not formidably slow, so no attempt is made to optimize the code[7]. The construction of Páde approximant from  $E_n$  and calculating perturbed energy level at 100 different  $\epsilon$  takes 18s (compilation) + 3s (run) on average.

A separate program written in Mathematica performing symbolic integrations according to eq. (8),(9) is implemented to check the correctness of low order. Due to slow symbolic calculations this method applies to  $n$  up to 13; and at  $n = 13$ , each update of  $E_n$  and  $y_n$  takes around 30s.

## VI. RESULT

### A. Graph

In Fig. 1, the lowest 10 eigenvalues in  $\epsilon \in [-10, 0]$  are plotted. Despite the series in eq. (5) is inherently divergent[8], Páde approximant converges well. Both solutions capture the trend that the eigenvalue increases as  $\epsilon$  decreases. But it seems the numerical solution tends to a straight line at large  $|\epsilon|$ , while the slope of the analytical solution tends to 0. The numerical solution differs visibly from analytical solution below  $\epsilon = 3$ .

To see the error between the two more clearly, in Fig. 2, we plot the relative error between the two methods against  $\epsilon$ . Log scale is used to account for the order-of-magnitude difference. Note  $E^{\text{numerical}} > E^{\text{analytical}}$  holds for all  $\epsilon < 0$  and the relative error is monotonically increasing. Also note there's seemingly crossing at, say,  $\epsilon = -3$  between  $n = 9$  and  $n = 8$  where the slope of error is at minimum.

### B. Table

In this section we present the values of  $E_n$  ( $n = 1, \dots, 100$ ). For each unperturbed energy level, we present exactly solution up to first 20 orders and the ap-

proximate solution of all 100 orders. Since  $-E_n 2^{3n}$  is a

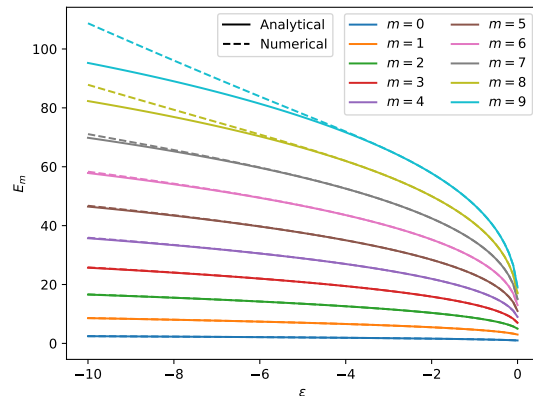


FIG. 1. Eigenvalues of eq. (4) from analytical(solid) and numerical(dashed) means.

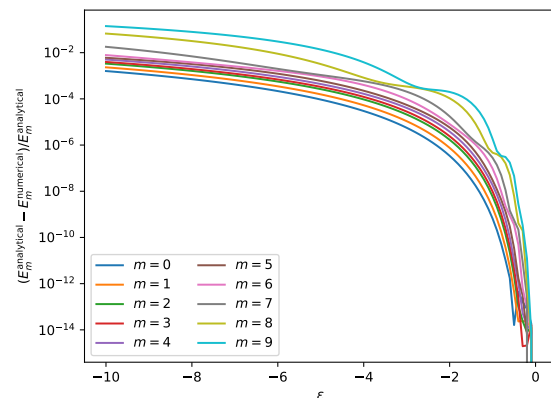


FIG. 2. Relative error between analytical and numerical methods.

positive integer, we present our result in  $-E_n 2^{3n}$  instead of  $E_n$ .

In reference, the coefficients of ground state correction are given up to  $n = 75$ . Note the their definition of Hamiltonian differs from ours, and it can be shown that  $(-2)^{n-1} E_n$  equals to their  $A_n$ .

## VII. CONCLUSION

We've completed a high-order perturbative analysis to the  $x^4$  anharmonic oscillator. The convergent behavior is achieved with analytical method at even very large perturbation strength  $\epsilon$  where numerical method fails to function properly. We would like to comment that the method described is not restricted to quartic potential only, but applicable to other integer exponent  $\alpha > 2$  potential as well.

TABLE I. Exact coefficient of  $E_n$  up to  $n = 20$  for  $E_0^u$ 

$n$	$-E_n 2^{3n}$
1	6
2	84
3	2664
4	123540
5	7333848
6	524147208
7	43572714768
8	4121980396212
9	437015860091640
10	51336059452724760
11	6621407769290178672
12	930758905386276349320
13	141686496049367635656048
14	23229387977224612690619280
15	4081973849686832002155521568
16	765543711410243711551312338420
17	152644886266565584391713812812088
18	32249559868746478308856261490338680
19	7197129672652253378568475159863338160
20	1691928460676486363401186477311314757720

TABLE II. Exact coefficient of  $E_n$  up to  $n = 20$  for  $E_1^u$ 

$n$	$-E_n 2^{3n}$
1	30
2	660
3	31320
4	2081940
5	170435880
6	16215569160
7	1737651694320
8	205795688655540
9	26607913470652680
10	3723933252458267160
11	560699252370983391120
12	90394751875598518863240
13	15544914905835692552987280
14	2842290048673569505197263760
15	551003198488474493684196465120
16	112961391092496352173257140772340
17	24432204681317955463241675314900680
18	5562664656497172471982192662546819960
19	1330383013238816192087544072071900650320
20	333565102755481893703894918020235785930840

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- [1] C. M. Bender and T. T. Wu, Anharmonic oscillator, Phys. Rev. **184**, 1231 (1969).
- [2] C. Bender, S. Orszag, and S. Orszag, *Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory*, Advanced Mathematical Methods for Scientists and Engineers (Springer, 1999).
- [3] In fact, it is shown in [1] that the sum for  $k$  could start from 2 instead of 0 if  $n \geq 1$ .
- [4] T. Granlund and the GMP development team, *GNU MP: The GNU Multiple Precision Arithmetic Library*, 6th ed. (2020), <http://gmplib.org/>.
- [5] G. Guennebaud, B. Jacob, et al., Eigen v3, <http://eigen.tuxfamily.org> (2010).
- [6] Code available at <https://github.com/putian9935/anharmonic-osc>.
- [7] We didn't even turn on compiler optimization flag, e.g. `-O2` or `-Ofast`.
- [8] It can be shown that  $A_n$  scales super-exponentially. Thus the series is divergent.

TABLE III. Exact coefficient of  $E_n$  up to  $n = 20$  for  $E_2^u$ 

$n$	$-E_n 2^{3n}$
1	78
2	2460
3	160632
4	14305020
5	1535988744
6	188110215960
7	25495041543984
8	3754912433232540
9	593920919123280360
10	100088070464024930760
11	17870697976178395841616
12	3367187369124912927538200
13	667538193130776896234040144
14	138928611649281749972298448560
15	30300051071205181574277032253024
16	6915045540402221464641419799181020
17	1649283304963157585066069253330007464
18	410622272668183981562228778917860994280
19	106601227035161318024640742366601345479440
20	28826540363655691102327483978432629561219720

TABLE IV. Exact coefficient of  $E_n$  up to  $n = 20$  for  $E_3^u$ 

$n$	$-E_n 2^{3n}$
1	150
2	6300
3	534600
4	60738300
5	8199819000
6	1247186367000
7	207635609250000
8	37179301503865500
9	7078906156588479000
10	1422022835047769325000
11	299732457805473100590000
12	66028224244090279453695000
13	15157989023243063520623310000
14	3618665345116689592688271630000
15	896948463395568349991868869700000
16	230562265546554690170858715723937500
17	61407665988905530461321136028463975000
18	16934299026103556311924076828956574625000
19	4832527074531557878069819538893268037750000
20	1426359462251378421340254159165455176897125000

TABLE V. Exact coefficient of  $E_n$  up to  $n = 20$  for  $E_4^u$ 

$n$	$-E_n 2^{3n}$
1	246
2	12996
3	1369224
4	190980900
5	3134848888
6	5748774789672
7	1145277005586768
8	243704776908936708
9	54780845374124216280
10	12909166056728713392120
11	3171880841400323898134832
12	809406810491980230872556840
13	213883347293261559096605975088
14	58398225891609806982343388149200
15	16448551333930388750983395578720928
16	4773480671346660668911587492182081220
17	1426049185149588560529569069925225467928
18	438271289668099818802617355671857476165080
19	138502320945473198399637508287150474609963120
20	44991533418599438515152390972648750195089860920

TABLE VI. Exact coefficient of  $E_n$  up to  $n = 20$  for  $E_5^u$ 

$n$	$-E_n 2^{3n}$
1	366
2	23364
3	2952504
4	490546980
5	95310683208
6	20570476322088
7	4797816243861168
8	1189443292455792132
9	310066182169392911400
10	84361702124758951605240
11	23828859146200940320546512
12	6960258294018023836347434280
13	2096209764973891029257110765008
14	649475884761654289683004487505360
15	206667567156572264205032912139079008
16	67452974101130232740910983180011749060
17	22559283605128866560805433773687584796648
18	7725549318398891749701093949956045372628440
19	2707596177960269327080542443377371057342042000
20	970795647566554645481254635712205043163353704760

TABLE VII. Exact coefficient of  $E_n$  up to  $n = 20$  for  $E_6^u$ 

$n$	$-E_n 2^{3n}$
1	510
2	38220
3	5644440
4	1091127180
5	245625725160
6	61178256043320
7	16406098735555440
8	4660093776312281580
9	1387230513223589762760
10	429627345056803841886120
11	137704404716964210720468240
12	45502311953568087819486001080
13	15455286479818544332043410540560
14	5383973555502487536194054696110320
15	1920240325609227756315958323778863840
16	700241809716170442189848657832521355180
17	260810480281117796968229874840754997863560
18	99137146631639354998283926976013719243182920
19	38434121670962451927063055607350960046370924240
20	15190458307547315488739642298798206580805045163880

TABLE VIII. Exact coefficient of  $E_n$  up to  $n = 20$  for  $E_7^u$ 

$n$	$-E_n 2^{3n}$
1	678
2	58380
3	9877032
4	2180850060
5	559129221144
6	158165907341880
7	48044012151997584
8	15418307436741808620
9	5172876008714722280760
10	1801297375941218698298280
11	647659150449728639148581616
12	239524358482197491691323676600
13	90851388119905954404768755292144
14	35262773832368118340351248495421680
15	13981101530066973109178900637361192224
16	5654641079908380097496232729047507718060
17	2330409375544560269525302040720326823910264
18	977795574003170289875384379125260859423232840
19	417406410914987753209614501324937077305410574640
20	181191933667780863275686273200874084523211900557160

TABLE IX. Exact coefficient of  $E_n$  up to  $n = 20$  for  $E_8^u$ 

$n$	$-E_n 2^{3n}$
1	870
2	84660
3	16154280
4	4014543540
5	1156055834520
6	366573079193160
7	124570926823470480
8	44639373314584563540
9	16692295906333920321720
10	6466841110598481273702360
11	2582341143421740531173264880
12	1058832810601068101027028605640
13	444506171440957969399800993108720
14	190630208426068724741346000137682960
15	83368839445052238682099337091426504480
16	37128333972989641369552948304342114024340
17	16819382635460142250188719438936923135862520
18	7743317794767935207526522226320115411624251960
19	3620294854568037645847016611372652073768138719280
20	1717955671666603638721472801287605386511053052911640

TABLE X. Exact coefficient of  $E_n$  up to  $n = 20$  for  $E_9^u$ 

$n$	$-E_n 2^{3n}$
1	1086
2	117876
3	25052184
4	6923996340
5	2214159188328
6	778487059956552
7	292907088729453168
8	116045384727244748628
9	47908214409056898325320
10	20463046780979486425277400
11	8996774820508963792598974992
12	4056161451233594608921583680200
13	1869844692748383712711195235962128
14	879402078031756466350628337057112080
15	421206825945193285862265960689062949088
16	205171692865293620863594778860193845459860
17	101522039738075310270837092027362611511548168
18	50982857841264211072697963345552165044153782840
19	25964744959336851103768116305856620858052392828880
20	13402209587883193195141124719169628506488864107995160



TABLE XI. The value of the  $-E_n$  with  $n = 1, \dots, 100$  for  $E_0^u$ . The number following the comma is the power of 10 multiplying the decimal.

$n$	$-E_n$	$n$	$-E_n$
1	0.7500000000000000,	0	51 0.1774549491325011,
2	0.1312500000000000,	1	52 0.1371585100985822,
3	0.5203125000000000,	1	53 0.1080687553363126,
4	0.3016113281250000,	2	54 0.8676872885291642,
5	0.2238112792968750,	3	55 0.7096770461319298,
6	0.1999462921142578,	4	56 0.5910809250127134,
7	0.2077708948516846,	5	57 0.5011655452280950,
8	0.2456891772873401,	6	58 0.4324420071229820,
9	0.3256021887746751,	7	59 0.3796259964268197,
10	0.4781043106012490,	8	60 0.3389524990800757,
11	0.7708333164092827,	9	61 0.3077189141185188,
12	0.1354432468922862,	11	62 0.2839773024865141,
13	0.2577262349393415,	12	63 0.2663254045275103,
14	0.5281751322678386,	13	64 0.2537640850228387,
15	0.1160166746583068,	15	65 0.2456002702014563,
16	0.2719757615246876,	16	66 0.2413817719864760,
17	0.6778794692977178,	17	67 0.2408551986715980,
18	0.1790210195015489,	19	68 0.2439413849911183,
19	0.4994011921119655,	20	69 0.2507250426950703,
20	0.1467514010204402,	22	70 0.2614570302788745,
21	0.4531136296684818,	23	71 0.2765690405221834,
22	0.1466652370037318,	25	72 0.2967018143757369,
23	0.4966283069462674,	26	73 0.3227493905153705,
24	0.1755839492534922,	28	74 0.3559235776783599,
25	0.6470221042946597,	29	75 0.3978450133890432,
26	0.2480994545016985,	31	76 0.4506701405312378,
27	0.9884377883559941,	32	77 0.5172676031383124,
28	0.4085842008364257,	34	78 0.6014635309853691,
29	0.1750075894533248,	36	79 0.7083838315872806,
30	0.7757967334354602,	37	80 0.8449342603473809,
31	0.3555183256998041,	39	81 0.1020477693952982,
32	0.1682432154270259,	41	82 0.1247795733435630,
33	0.8213752926846650,	42	83 0.1544463180441750,
34	0.4133016264153365,	44	84 0.1934826268889632,
35	0.2141561571497207,	46	85 0.2452869986345015,
36	0.1141747926177188,	48	86 0.3146403879335594,
37	0.6258125322225467,	49	87 0.4083216923637282,
38	0.3523944748523645,	51	88 0.5360193919765655,
39	0.2037126840152551,	53	89 0.7116917856121663,
40	0.1208147982683254,	55	90 0.9556116068446843,
41	0.7346130817663982,	56	91 0.1297462244223361,
42	0.4576889474357688,	58	92 0.1781061513835013,
43	0.2920146672188311,	60	93 0.2471622643123596,
44	0.1906874900611838,	62	94 0.3466999332174321,
45	0.1273780703868530,	64	95 0.4915232511284615,
46	0.8699690034565581,	65	96 0.7042137555250557,
47	0.6072125057647181,	67	97 0.1019500619817492,
48	0.4329178140381204,	69	98 0.1491236212338351,
49	0.3151420296247126,	71	99 0.2203614896688591,
50	0.2341312637831796,	73	100 0.3289353468000507,

TABLE XII. The value of the  $-E_n$  with  $n = 1, \dots, 100$  for  $E_1^{\text{H}}$ . The number following the comma is the power of 10 multiplying the decimal.

$n$	$-E_n$	$n$	$-E_n$
1	0.3750000000000000,	51	0.1013697382720098,
2	0.1031250000000000,	52	0.7999894076405615,
3	0.6117187500000000,	53	0.6433065303303194,
4	0.5082861328125000,	54	0.5269384387338485,
5	0.5201290283203125,	55	0.4395069401405857,
6	0.6185748733520507,	56	0.3731610831033226,
7	0.8285768958663940,	57	0.3224165529874986,
8	0.1226637891862035,	58	0.2833993767834352,
9	0.1982444038290730,	59	0.2533468824945158,
10	0.3468183104375626,	60	0.2302745907707660,
11	0.6527398391288978,	61	0.2127516083919541,
12	0.1315416766382965,	62	0.1997479832504679,
13	0.2827603549273716,	63	0.1905305321305266,
14	0.6462619350426325,	64	0.1845919232030838,
15	0.1566045280266264,	65	0.1816031198513451,
16	0.4013194793106448,	66	0.1813827886907690,
17	0.1085007847182149,	67	0.1838796344801361,
18	0.3087899189955874,	68	0.1891652863317557,
19	0.9231386580424552,	69	0.1974366106060682,
20	0.2893216072582752,	70	0.2090273658972469,
21	0.9488325779811591,	71	0.2244300976158352,
22	0.3250307132935212,	72	0.2443302288960944,
23	0.1161079009509754,	73	0.2696555812966252,
24	0.4318387635070937,	74	0.3016462216578736,
25	0.1669803282377047,	75	0.3419518043131752,
26	0.6703372839883187,	76	0.3927667755460074,
27	0.2790250074907100,	77	0.4570183876125177,
28	0.1202779951451808,	78	0.5386291110609579,
29	0.5363224836907425,	79	0.6428847532700168,
30	0.2471129926328750,	80	0.7769539258517119,
31	0.1175317419895233,	81	0.9506257966400545,
32	0.5764879667215923,	82	0.1177364906865322,
33	0.2913463211724646,	83	0.1475829778213682,
34	0.1515804839338474,	84	0.1872074696313200,
35	0.8112255426321887,	85	0.2402764911690694,
36	0.4462446620087926,	86	0.3119905748318578,
37	0.2521291717902646,	87	0.4097849326392476,
38	0.1462154177897528,	88	0.5443752293033641,
39	0.8697597331760013,	89	0.7313299926211286,
40	0.5303605150557383,	90	0.9934524761765235,
41	0.3313227439172527,	91	0.1364415656868624,
42	0.2119306535252607,	92	0.1894351315007845,
43	0.1387280007164099,	93	0.2658508397566874,
44	0.9288338152069379,	94	0.3770767496381037,
45	0.6357712219917682,	95	0.5404894714408601,
46	0.4446795734592901,	96	0.7828220531116719,
47	0.3176727818007192,	97	0.1145541252717334,
48	0.2316916636191567,	98	0.1693498304962881,
49	0.1724475377449753,	99	0.2528951783720768,
50	0.1309319077303230,	100	0.3814471460671227,

TABLE XIII. The value of the  $-E_n$  with  $n = 1, \dots, 100$  for  $E_2^u$ . The number following the comma is the power of 10 multiplying the decimal.

$n$	$-E_n$	$n$	$-E_n$
1	0.9750000000000000,	51	0.2782417452086104,
2	0.3843750000000000,	52	0.2243926705423097,
3	0.3137343750000000,	53	0.1843114693729047,
4	0.3492436523437500,	54	0.1541388682732517,
5	0.4687465649414063,	55	0.1312053996888442,
6	0.7175835264587402,	56	0.1136420900504855,
7	0.1215698315810394,	57	0.1001260937954763,
8	0.2238102217455232,	58	0.8971243625245890,
9	0.4425055676127079,	59	0.8172145118842534,
10	0.9321427947285115,	60	0.7566273573987154,
11	0.2080423056168761,	61	0.7118339167728882,
12	0.4899902515353333,	62	0.6803267422051353,
13	0.1214244899767030,	63	0.6603793566636642,
14	0.3158870905492261,	64	0.6508853760363058,
15	0.8611792472721953,	65	0.6512558308339145,
16	0.2456717688091537,	66	0.6613611911664670,
17	0.7324289197199886,	67	0.6815102684192212,
18	0.2279411507700634,	68	0.7124624839985987,
19	0.7396946044233114,	69	0.7554736396534398,
20	0.2500303815001327,	70	0.8123788246981982,
21	0.8794008610707369,	71	0.8857199300560271,
22	0.3214991286241589,	72	0.9789299135594561,
23	0.1220445451374365,	73	0.1096592092557875,
24	0.4805658842288584,	74	0.1244801155029726,
25	0.1960829257723188,	75	0.1431664447811683,
26	0.8282162685198766,	76	0.1667999123823776,
27	0.3617767501523046,	77	0.1968305436488712,
28	0.1632747597244777,	78	0.2352132650901707,
29	0.7606444488801276,	79	0.2846007436274628,
30	0.3654654557834732,	80	0.3486173960339476,
31	0.1809447926361267,	81	0.4322513647450226,
32	0.9224217810179139,	82	0.5424191420600709,
33	0.4837940481058632,	83	0.6887846448300159,
34	0.2608660912105105,	84	0.8849559365752178,
35	0.1445086323854607,	85	0.1150246398115651,
36	0.8218557382164443,	86	0.1512285530951558,
37	0.4795596708192673,	87	0.2010917780289750,
38	0.2869240584218257,	88	0.2704067944020963,
39	0.1759187901069950,	89	0.3676630823963530,
40	0.1104671689104127,	90	0.5054045194155740,
41	0.7100602266906382,	91	0.7023176023000495,
42	0.4669530670799086,	92	0.9864681638970619,
43	0.3140160142820714,	93	0.1400356119163249,
44	0.2158361158277432,	94	0.2008869139361818,
45	0.1515624921344746,	95	0.2911892666701519,
46	0.1086836097234662,	96	0.4264453968049036,
47	0.7955324132692282,	97	0.6309145973514563,
48	0.5941512847151227,	98	0.9428713232052504,
49	0.4525973463464175,	99	0.1423198554624280,
50	0.3515114836451844,	100	0.2169537385186226,

TABLE XIV. The value of the  $-E_n$  with  $n = 1, \dots, 100$  for  $E_3^u$ . The number following the comma is the power of 10 multiplying the decimal.

$n$	$-E_n$	$n$	$-E_n$
1	0.1875000000000000,	51	0.4885612004700294,
2	0.9843750000000000,	52	0.4030084440980600,
3	0.1044140625000000,	53	0.3384142354430983,
4	0.1482868652343750,	54	0.2891949848658745,
5	0.2502386169433594,	55	0.2514276926770286,
6	0.4757638423919678,	56	0.2223270326163187,
7	0.9900837385654450,	57	0.1998981085532981,
8	0.2216059059135050,	58	0.1827033365495634,
9	0.5274196085772276,	59	0.1697044807960026,
10	0.1324361967898690,	60	0.1601548929725367,
11	0.3489345053740231,	61	0.1535258301809256,
12	0.9608371218795988,	62	0.1494563622142943,
13	0.2757222141234354,	63	0.1477200378497144,
14	0.8227892397182334,	64	0.1482039070653658,
15	0.2549280860067621,	65	0.1508971573331252,
16	0.8191217146224783,	66	0.1558878160669482,
17	0.2727048186774962,	67	0.1633668973744017,
18	0.9400424342333766,	68	0.1736401722068980,
19	0.3353239265793366,	69	0.1871485310101083,
20	0.1237169622174560,	70	0.2044987933248902,
21	0.4718921791943949,	71	0.2265079157071883,
22	0.1859987566907764,	72	0.2542650026252777,
23	0.7572360438408876,	73	0.2892175310325586,
24	0.3182710024931293,	74	0.3332910376991375,
25	0.1380339902233111,	75	0.3890555981217487,
26	0.6173971415705142,	76	0.4599583548793770,
27	0.2846349656013281,	77	0.5506500450477650,
28	0.1351772432879351,	78	0.6674463142908377,
29	0.6609232481268932,	79	0.8189837001951169,
30	0.3324814643971936,	80	0.1017158760593696,
31	0.1719842461887097,	81	0.1278481923542737,
32	0.9142222603788523,	82	0.1626043035804852,
33	0.4991108123193866,	83	0.2092385481623800,
34	0.2796848580421176,	84	0.2723739343176290,
35	0.1607748522707433,	85	0.3586301830404022,
36	0.9475482346438696,	86	0.4775623703747752,
37	0.5722455260453116,	87	0.6430741656094801,
38	0.3539422922647578,	88	0.8755614673669276,
39	0.2240938799070066,	89	0.1205188230737552,
40	0.1451646722308135,	90	0.1676930024905827,
41	0.9616533138168430,	91	0.2358397679412801,
42	0.6511832208865065,	92	0.3352065021320961,
43	0.4505285776740869,	93	0.4814524364444069,
44	0.3183391939767087,	94	0.6987038535366945,
45	0.2296292297680037,	95	0.1024438434023326,
46	0.1690286948136289,	96	0.1517353116560356,
47	0.1269178921453709,	97	0.2270134125956212,
48	0.9717467490483426,	98	0.3430340908766180,
49	0.7583948512077047,	99	0.5234818706418431,
50	0.6031136573533145,	100	0.8066833816806185,

TABLE XV. The value of the  $-E_n$  with  $n = 1, \dots, 100$  for  $E_4^{\text{II}}$ . The number following the comma is the power of 10 multiplying the decimal.

$n$	$-E_n$	$n$	$-E_n$
1	0.3075000000000000,	51	0.6163890296662139,
2	0.2030625000000000,	52	0.5205828964253396,
3	0.2674265625000000,	53	0.4473282611276090,
4	0.4662619628906250,	54	0.3909700641847076,
5	0.9566799587402344,	55	0.3474754652108244,
6	0.2192983547085571,	56	0.3139455596562416,
7	0.5461106326993790,	57	0.2882859171524893,
8	0.1452593665772299,	58	0.2689823818628982,
9	0.4081491036275343,	59	0.2549476536179022,
10	0.1202259776809133,	60	0.2454162609110759,
11	0.3692555289483075,	61	0.2398733138169726,
12	0.1177841929154209,	62	0.2380075146027955,
13	0.3890515423213611,	63	0.2396823138132173,
14	0.1327821926033944,	64	0.2449214656602231,
15	0.4674959465640538,	65	0.2539069713797716,
16	0.1695881007658308,	66	0.2669887836591426,
17	0.6332930558404064,	67	0.2847068861755240,
18	0.2432894384108430,	68	0.3078276320979706,
19	0.9610529104855464,	69	0.3373976922131465,
20	0.3902393462071975,	70	0.3748208185688765,
21	0.1628430471253280,	71	0.4219651182781612,
22	0.6982043824343618,	72	0.4813119919841363,
23	0.3075431760213506,	73	0.5561628070675839,
24	0.1391487802899673,	74	0.6509264589911897,
25	0.6466052892687969,	75	0.7715212862050487,
26	0.3085418281762928,	76	0.9259399413703431,
27	0.1511541839149337,	77	0.1125048205688947,
28	0.7600810934681406,	78	0.1383722061648816,
29	0.3922119933709182,	79	0.1722477296496474,
30	0.2076233605082946,	80	0.2169821287079190,
31	0.1127161399479063,	81	0.2765671104682771,
32	0.6273347955726312,	82	0.3566357120502071,
33	0.3578125893995756,	83	0.4652000697000462,
34	0.2090678897460641,	84	0.6137471959285830,
35	0.1250898401633373,	85	0.8188784708482159,
36	0.7660976563471842,	86	0.1104780791939786,
37	0.4800612110758221,	87	0.1506978960711500,
38	0.3076681030653535,	88	0.2078076161666302,
39	0.2015885430385612,	89	0.2896601701140670,
40	0.1349810924305350,	90	0.4080750475919190,
41	0.9232785031356141,	91	0.5809879549137958,
42	0.6448778478380773,	92	0.8358395456539431,
43	0.4597725644878483,	93	0.1214957851350285,
44	0.3344793519294535,	94	0.1784172131980461,
45	0.2481990399771389,	95	0.2646702682904439,
46	0.1877947242762296,	96	0.3965727965713836,
47	0.1448342619777139,	97	0.6001328088419278,
48	0.1138204024656286,	98	0.9171421587277599,
49	0.9111505158330104,	99	0.1415303974744693,
50	0.7427551103312338,	100	0.2205191421700687,

TABLE XVI. The value of the  $-E_n$  with  $n = 1, \dots, 100$  for  $E_5^u$ . The number following the comma is the power of 10 multiplying the decimal.

$n$	$-E_n$	$n$	$-E_n$
1	0.4575000000000000,	51	0.5949903137129013,
2	0.3650625000000000,	52	0.5150543709294194,
3	0.5766609375000000,	53	0.4533536030668839,
4	0.1197624462890625,	54	0.4056521609313773,
5	0.2908651220947266,	55	0.3688906874652121,
6	0.7847013977847290,	56	0.3408519051758309,
7	0.2287777063303550,	57	0.3199317458118899,
8	0.7089634492729855,	58	0.3049821345279095,
9	0.2310173080633528,	59	0.2952032842876264,
10	0.7856795762177460,	60	0.2900709899981089,
11	0.2774044306273681,	61	0.2892894689060787,
12	0.1012850886620876,	62	0.2927637345056157,
13	0.3812983350096858,	63	0.3005879304519489,
14	0.1476737190300024,	64	0.3130478959567594,
15	0.5873845542412603,	65	0.3306377704238689,
16	0.2396411037648613,	66	0.3540918928259432,
17	0.1001833443098515,	67	0.3844348052986635,
18	0.4288541365581818,	68	0.4230540328251561,
19	0.1878772261349102,	69	0.4718027235795924,
20	0.8420310001049046,	70	0.5331425170104178,
21	0.3859793789182782,	71	0.6103416068235188,
22	0.1809323690505331,	72	0.7077495322359895,
23	0.8672601782915448,	73	0.8311797207120296,
24	0.4250596062893444,	74	0.9884446497235448,
25	0.2130184877856855,	75	0.1190108851207628,
26	0.1091585936607595,	76	0.1450555127016756,
27	0.5719766897836153,	77	0.1789504287252610,
28	0.3064648239397741,	78	0.2234196177066477,
29	0.1679030921924805,	79	0.2822541664058013,
30	0.9405790071019297,	80	0.3607710231271199,
31	0.5387165032541003,	81	0.4664855053586822,
32	0.3154349474319097,	82	0.6101043006331455,
33	0.1887931017805696,	83	0.8070024156309697,
34	0.1154830323430693,	84	0.1079436092238762,
35	0.7218096555603444,	85	0.1459882951328496,
36	0.4608987114759870,	86	0.1996121028100609,
37	0.3005821358908226,	87	0.2759009607137340,
38	0.2001633853064307,	88	0.3854498346078284,
39	0.1360670047142689,	89	0.5442301651905933,
40	0.9439438057156202,	90	0.7765156269280664,
41	0.6680975245343814,	91	0.1119500564970991,
42	0.4822889626142365,	92	0.1630645481917304,
43	0.3549939312506358,	93	0.2399448186618568,
44	0.2663501848933499,	94	0.3566449856640564,
45	0.2036462741583659,	95	0.5354155187071251,
46	0.1586227213128151,	96	0.8117715941129232,
47	0.1258332524233368,	97	0.1242863901825221,
48	0.1016351826172831,	98	0.1921408294717308,
49	0.8355856995061861,	99	0.2999040923239406,
50	0.6990659553162657,	100	0.4725772114952314,

TABLE XVII. The value of the  $-E_n$  with  $n = 1, \dots, 100$  for  $E_6^u$ . The number following the comma is the power of 10 multiplying the decimal.

$n$	$-E_n$	$n$	$-E_n$
1	0.6375000000000000,	51	0.4568773167774281,
2	0.5971875000000000,	52	0.4058480280997044,
3	0.1102429687500000,	53	0.3663369447696654,
4	0.2663884716796875,	54	0.3359372872406965,
5	0.7495902257080078,	55	0.3128988957740627,
6	0.2333765260441589,	56	0.2959564057316568,
7	0.7823037498262139,	57	0.2842107187889343,
8	0.2777632341571022,	58	0.2770478914035133,
9	0.1033567274528436,	59	0.2740849918827631,
10	0.4001216451235151,	60	0.2751361440975752,
11	0.1603090259211187,	61	0.2801945334204003,
12	0.6621457862430264,	62	0.2894280278283636,
13	0.2811300233555548,	63	0.3031875582244702,
14	0.1224173855803767,	64	0.3220287237215521,
15	0.5457651257100985,	65	0.3467484194847602,
16	0.2487758655846656,	66	0.3784397996724606,
17	0.1158231201086569,	67	0.4185707800567203,
18	0.5503217139304315,	68	0.4690937992900739,
19	0.2666902925646700,	69	0.5325980291730832,
20	0.1317562231847462,	70	0.6125201298321929,
21	0.6633910695713141,	71	0.7134366828886839,
22	0.3403381267855611,	72	0.8414716392949145,
23	0.1778853310085093,	73	0.1004867039649360,
24	0.9471850572326485,	74	0.1214787247171807,
25	0.5138013091659210,	75	0.1486459543198272,
26	0.2839484947921437,	76	0.1840802633620146,
27	0.1598816405598488,	77	0.2306767820057316,
28	0.9172976703971315,	78	0.2924728358062778,
29	0.5363095343518711,	79	0.3751421234489202,
30	0.3195611934325641,	80	0.4867204254156147,
31	0.1940712024665472,	81	0.6386790522181790,
32	0.1201336494738406,	82	0.8475242614712964,
33	0.7580245450301205,	83	0.1137197877044484,
34	0.4875559056225766,	84	0.1542707085981497,
35	0.3196566613683530,	85	0.2115653525031628,
36	0.2136213513028494,	86	0.2932718225313381,
37	0.1455051551744951,	87	0.4108779986877816,
38	0.1010054079152379,	88	0.5817349482635982,
39	0.7144838221827710,	89	0.8322638105298241,
40	0.5149452845492426,	90	0.1203026514128130,
41	0.3780782260658095,	91	0.1756804049813636,
42	0.2827339619095333,	92	0.2591563307335460,
43	0.2153118622155866,	93	0.3861428026211881,
44	0.1669416933191457,	94	0.5810850892211912,
45	0.1317583433359109,	95	0.8830717598022435,
46	0.1058314620713422,	96	0.1355115470829480,
47	0.8649278960532462,	97	0.2099621198718526,
48	0.7190803560053887,	98	0.3284361426922829,
49	0.6080101521931790,	99	0.5186423675161670,
50	0.5227374268359755,	100	0.8267124484291338,

TABLE XVIII. The value of the  $-E_n$  with  $n = 1, \dots, 100$  for  $E_7^u$ . The number following the comma is the power of 10 multiplying the decimal.

$n$	$-E_n$	$n$	$-E_n$
1	0.8474999999999999, 90	51	0.2864576956403406, 90
2	0.9121874999999999, 92	52	0.2614641871216309, 92
3	0.1929107812500000, 94	53	0.2423246400873491, 94
4	0.5324340966796874, 96	54	0.2280033595909620, 96
5	0.1706326968823242, 98	55	0.2177547522946587, 98
6	0.6033550542521667, 100	56	0.2110577211404799, 100
7	0.2290917022323493, 102	57	0.2075710786909981, 102
8	0.9190027378047591, 104	58	0.2071045913077593, 104
9	0.3854092962082268, 106	59	0.2096022085035395, 106
10	0.1677588909809681, 108	60	0.2151354456780130, 108
11	0.7539744843376435, 110	61	0.2239059894867257, 110
12	0.3485538159761891, 112	62	0.2362575403383597, 112
13	0.1652577122875409, 114	63	0.2526978225100913, 114
14	0.8017826492588964, 116	64	0.2739326962375532, 116
15	0.3973668052045404, 118	65	0.3009155240041969, 118
16	0.2008932071329774, 120	66	0.3349165295142665, 120
17	0.1034909658212762, 122	67	0.3776190486735576, 122
18	0.5427855798174454, 124	68	0.4312525973284164, 124
19	0.2896338800149804, 126	69	0.4987769906065196, 126
20	0.1571589504955659, 128	70	0.5841379626377103, 128
21	0.8668158443167112, 130	71	0.6926237763100751, 130
22	0.4858348021009420, 132	72	0.8313655685678594, 132
23	0.2766559751780892, 134	73	0.1010043749858392, 134
24	0.1600405535855561, 136	74	0.1241891871912073, 136
25	0.9404494313735363, 138	75	0.1545132914178496, 138
26	0.5613791718166484, 140	76	0.1945048509737747, 140
27	0.3404171633237804, 142	77	0.2476981035119704, 142
28	0.2097174528624448, 144	78	0.3190720172482583, 144
29	0.1312714792118446, 146	79	0.4156958541019060, 146
30	0.8349703805915372, 148	80	0.5476861241227151, 148
31	0.5397504913741538, 150	81	0.7296354882997953, 150
32	0.3546450009157600, 152	82	0.9827620292982525, 152
33	0.2368805207257806, 154	83	0.1338165911268127, 154
34	0.1608612217740305, 156	84	0.1841800569550901, 156
35	0.1110722312995723, 158	85	0.2562117525672430, 158
36	0.7798830531197261, 160	86	0.3601910503678910, 160
37	0.5568675336700459, 162	87	0.5116802842985579, 162
38	0.4043815157827464, 164	88	0.7344320819312913, 164
39	0.2986453431319782, 166	89	0.1064995789086632, 166
40	0.2243069618967584, 168	90	0.1560070875732223, 168
41	0.1713323927076954, 170	91	0.2308333850685106, 170
42	0.1330832084438063, 172	92	0.3449603516834628, 172
43	0.1051144242910826, 174	93	0.5206131913463705, 174
44	0.8441441035534458, 176	94	0.7934077747406583, 176
45	0.6891895997855279, 178	95	0.1220881768523695, 178
46	0.5719722580299542, 180	96	0.1896740418209394, 180
47	0.4824654662889382, 182	97	0.2974823844496829, 182
48	0.4135699235496021, 184	98	0.4709732693228118, 184
49	0.3602101357368651, 186	99	0.7526206640641758, 186
50	0.3187257591313054, 188	100	0.1213849211210191, 188



TABLE XIX. The value of the  $-E_n$  with  $n = 1, \dots, 100$  for  $E_8^u$ . The number following the comma is the power of 10 multiplying the decimal.

$n$	$-E_n$	$n$	$-E_n$
1	0.1087500000000000,	3	0.1493602943342593,
2	0.1322812500000000,	4	0.1402848579118311,
3	0.3155132812500000,	5	0.1336782351683650,
4	0.9801131689453125,	6	0.1292204241552506,
5	0.3528002424682617,	8	0.1266969827142530,
6	0.1398365322849884,	10	0.1259819834087036,
7	0.5940004674123310,	11	0.1270278110698450,
8	0.2660713989411864,	13	0.1298606705680488,
9	0.1243672959978425,	15	0.1345812599088528,
10	0.6022715112751797,	16	0.1413705707262591,
11	0.3006240752783768,	18	0.1505012687891305,
12	0.1540804530088016,	20	0.1623556412505664,
13	0.8085520156618402,	21	0.1774517398964715,
14	0.4334429114034464,	23	0.1964801797830896,
15	0.2369484930257279,	25	0.2203551756675389,
16	0.1319063399769136,	27	0.2502849618428871,
17	0.7469306344747358,	28	0.2878689567436593,
18	0.4298404851370526,	30	0.3352322135789543,
19	0.2512084189670886,	32	0.3952123048386201,
20	0.1490089017163780,	34	0.4716205144736538,
21	0.8966976597147372,	35	0.5696091017151410,
22	0.5472494482587509,	37	0.6961910414294646,
23	0.3386243432997396,	39	0.8609804621318391,
24	0.2124054335616395,	41	0.1077254722562530,
25	0.1350445833494651,	43	0.1363488459889685,
26	0.8702155722485635,	44	0.1745584989036700,
27	0.5683420456847837,	46	0.2260145205477251,
28	0.3762175538015916,	48	0.2959290828911737,
29	0.2524310641116154,	50	0.3917832621305500,
30	0.1716965141516950,	52	0.5244001318990277,
31	0.1183982450651360,	54	0.7095629736461352,
32	0.8278497411302496,	55	0.9704734857079413,
33	0.5870057523357540,	57	0.1341513638708255,
34	0.4221651274430417,	59	0.1874045251737290,
35	0.3079898095692558,	61	0.2645417634192801,
36	0.2279645938457261,	63	0.3773062256530930,
37	0.1712123101854065,	65	0.5436712264083174,
38	0.1304948529140530,	67	0.7913692779452063,
39	0.1009462787486133,	69	0.1163539138578061,
40	0.7926215977618217,	70	0.1727829885823601,
41	0.6317596487979069,	72	0.2591188584068681,
42	0.5111767937035336,	74	0.3924060196262301,
43	0.4198918451515982,	76	0.6000287996722207,
44	0.3501511788598771,	78	0.9263394535820547,
45	0.2964296299580445,	80	0.1443751072038404,
46	0.2547552946441054,	82	0.2271441972538752,
47	0.2222493104691814,	84	0.3607130208737314,
48	0.1968095661390483,	86	0.5781454417161856,
49	0.1768916211517597,	88	0.9351738409390382,
50	0.1613558723509342,	90	0.1526486481149450,

TABLE XX. The value of the  $-E_n$  with  $n = 1, \dots, 100$  for  $E_g^{\text{II}}$ . The number following the comma is the power of 10 multiplying the decimal.

$n$	$-E_n$	$n$	$-E_n$
1	0.1357500000000000, 93	51	0.6561748446493809, 93
2	0.1841812500000000, 95	52	0.6352337867989922, 95
3	0.4893004687500001, 97	53	0.6233287000878622, 97
4	0.1690428793945312, 99	54	0.6199283943382454, 99
5	0.6757077601098633, 101	55	0.6248474476077912, 101
6	0.2969692458940704, 103	56	0.6382313884154033, 103
7	0.1396689838073030, 105	57	0.6605635306566455, 105
8	0.6916843934490964, 107	58	0.6926938847010161, 107
9	0.3569440127093859, 109	59	0.7358925554046821, 109
10	0.1905769741253880, 111	60	0.7919322824353243, 111
11	0.1047362436134015, 113	61	0.8632075404761096, 113
12	0.5902491759092073, 115	62	0.9529012263671206, 115
13	0.3401227682385768, 118	63	0.1065214863905161, 118
14	0.1999528826744964, 120	64	0.1205685084963463, 120
15	0.1197141801711704, 122	65	0.1381618807698945, 122
16	0.7289162797451838, 124	66	0.1602693362496664, 124
17	0.4508484240964853, 126	67	0.1881787788922953, 126
18	0.2830117131828137, 128	68	0.2236140590804166, 128
19	0.1801666105148484, 130	69	0.2688971824395599, 130
20	0.1162456380103121, 132	70	0.3271770214111639, 132
21	0.7597992208516458, 134	71	0.4027539255226754, 134
22	0.5028878477189503, 136	72	0.5015435622268172, 136
23	0.3369461470409486, 138	73	0.6317442780006320, 138
24	0.2284890771078816, 140	74	0.8048039911649113, 140
25	0.1567877095401951, 143	75	0.1036830954165135, 143
26	0.1088554808875163, 145	76	0.1350666826632063, 145
27	0.7646303397663590, 147	77	0.1778954878002399, 147
28	0.5433823195470802, 149	78	0.2368713844663874, 149
29	0.3906774398998381, 151	79	0.3188205899542757, 151
30	0.2841909876762504, 153	80	0.4337324781345921, 153
31	0.2091769834871307, 155	81	0.5963423708531865, 155
32	0.1558015976248495, 157	82	0.8285609422100905, 157
33	0.1174452834502065, 160	83	0.1163230658697405, 160
34	0.8961130650889597, 162	84	0.1649977222148192, 162
35	0.6921757076557509, 164	85	0.2364392199638066, 164
36	0.5413300026115089, 166	86	0.3422550634693977, 166
37	0.4287136988507646, 168	87	0.5004134476548942, 168
38	0.3438739433319517, 170	88	0.7389538402038722, 170
39	0.2793979594069452, 173	89	0.1101985819230065, 173
40	0.2299860674164497, 175	90	0.1659459842088674, 175
41	0.1918191948231558, 177	91	0.2523200701827696, 177
42	0.1621239891358860, 179	92	0.3873414585128961, 179
43	0.1388715475063974, 181	93	0.6002839001121584, 181
44	0.1205671836007381, 183	94	0.9390827699806588, 183
45	0.1061026879418339, 186	95	0.1482859476174139, 186
46	0.9465180993569096, 188	96	0.2363252291079442, 188
47	0.8559586041722360, 190	97	0.3801015931866250, 190
48	0.7847047047468808, 192	98	0.6169285488933802, 192
49	0.7292733651573322, 195	99	0.1010374079926928, 195
50	0.6870668355008321, 197	100	0.1669585832804385, 197