Anharmonic oscillator: a perturbative approach

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In this article, the eigenvalues of anharmonic oscillator are investigated with perturbative theory. We use standard perturbative means to expand the lowest 10 eigenvalues to 100th-order, and then use a Páde (50,50) approximant to better approximate each divergent series. The results will then be compared against numerical means. To the author's knowledge, this is the first extensive high order analytical study on the excited states of the x^4 anharmonic oscillator[a].

I. MATHEMATICAL MODEL OF THE UNPERTURBED SYSTEM

The quantum mechanical description, *i.e.*, Schrödinger equation, of harmonic oscillator eventually reduces to this equation

$$-y''(x) + x^2 y(x) = Ey. \tag{1}$$

The solution of which can be found in standard QM text:

$$y^{\mathrm{u}}_m(x) = e^{-x^2/2} H_m(x), \quad m = 0, 1, \ldots, \tag{2}$$
$$E^{\mathrm{u}}_m = 2m + 1, \tag{3}$$

where H_m is the m-th order Hermite polynomial and the superscript u means "unperturbed". The degree of m-th order Hermite polynomial is m.

II. SOLVING THE PERTURBED SYSTEM

The anharmonicity takes many forms, but usually it ends up in a power form, *i.e.*, a term proportional to $x^\alpha (\alpha = 3, 4, \ldots)$. We restrict the discussion to the case of $\alpha = 4$, but the method is applicable to other cases as well.

The equation of interest is

$$-y''(x) + (x^2 - \epsilon x^4) y(x) = Ey. \tag{4}$$

Since we are talking of perturbative approach, there's a coefficient ϵ before the anharmonic term meaning that term is small compared to the harmonic term. Note, however, despite being small, the parameter ϵ should always satisfy $\epsilon <= 0$. Otherwise, the system's Hamiltonian is not bounded from below, leading to instability. For a more analytical proof, please refer to [1].

Given a pair of unperturbed eigenvalue and eigenfunction, the eigenvalue E and eigenfunction y of the perturbed system are expanded in terms of ϵ:

$$E = \sum_{n=0}^{\infty} E_n \epsilon^n, \tag{5}$$
$$y = \sum_{n=0}^{\infty} y_n \epsilon^n, \tag{6}$$

with

$$E_0 = E^{\mathrm{u}}_m, \qquad y_0 = y^{\mathrm{u}}_m, \tag{7}$$

being the unperturbed solution.

It can be shown that the different orders of perturbative solution satisfy a recursion relation [2]:

$$E_n = \frac{\int_{-\infty}^{\infty} y_0(x) \left[W(x) y_{n-1}(x) - \sum_{j=1}^{n-1} E_j y_{n-j}(x) \right] \mathrm{d}x}{\int_{-\infty}^{\infty} \mathrm{d}x \, y_0^2(x)}, \quad n = 1, 2, \ldots \tag{8}$$

and

$$y_n(x) = y_0(x) \int^x \frac{\mathrm{d}t}{y_0^2(t)} \int^t \mathrm{d}s \, y_0(s) \times \left[W(s) y_{n-1}(s) - \sum_{j=1}^{n} E_j y_{n-j}(s) \right], \quad n = 1, 2, \ldots \tag{9}$$

where $W(x) = -x^4$ is the perturbation potential.

A. Update eigenfunction

We try the following *ansatz* for y_n[3]

$$y_n(x) = e^{-x^2/2} \sum_{k=0}^{4n+m} y_{nk} x^k. \tag{10}$$

The form can be justified intuitively as follows. In

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[a] Otherwise, there should be code online and there should be no need for me to implement them by hand.

eq. (9), the integrand for s is

$$\begin{aligned} & y_0(s)\left[W(s)y_{n-1}(s)-\sum_{j=1}^{n}E_j y_{n-j}(s)\right] \\ &= e^{-s^2/2}\sum_{i=0}^{m} y_{ni}s^i\left[-s^4 e^{-s^2/2}\sum_{l=0}^{4n-4+m} y_{n-1,l}s^l\right. \\ &\qquad \left. -\sum_{j=1}^{n}E_j e^{-s^2/2}\sum_{l=0}^{4n-4j+m} y_{n-j,l}s^l\right] \quad (11) \\ &= e^{-s^2}\sum_{i=0}^{4n+2m} b_{ni}s^i = b_n(s), \quad (12) \end{aligned}$$

where in the last line we collected terms in a polynomial of order $4n+2m$. However, eq. (9) is really

$$b_n(s) = \left[(y_n/y_0)' y_0^2\right]' = y_0(s)y_n''(s) - y_n(s)y_0''(s). \quad (13)$$

And the it's not hard to show that the RHS also has a form of

$$e^{-s^2}\sum_{k=0}^{4n+2m+2} c_{nk}s^k, \quad (14)$$

and c_{nk} can be expressed in terms of known $y_{0l}(l = 0,\ldots,m)$ and unknown $y_{nk}(k = 0,\ldots,4n+m)$.

Comparing the coefficients of different powers of s would yield $(4n+2m+3)$ equations for y_{nk}. Note there're more equations than unknowns. But it turns out these relations are not linear independent and in the very least there should be the freedom of choosing integration constant. Solving this set of linear equations completes the update process of eigenfunction.

It can be further shown with induction that either all the odd or all the even terms in y_{nk} vanishes, depending on the parity of $y_0(x)$ of the unperturbed solution.

B. Calculate eigenvalue

Observe eq. (10) and eq. (8) and it's clear that both integrands for the denominator and numerator have the following form:

$$e^{-x^2}\sum_{k=0}^{d/2} e_k x^{2k}, \quad (15)$$

with d being $4n+2m$ for the numerator and $2m$ for the denominator.

The integration of each term is straightforward using Gamma function:

$$\int_{\infty}^{\infty} e^{-x^2}x^{2k}\,\mathrm{d}x = \int_0^{\infty} e^{-t}t^{k/2-1/2}\,\mathrm{d}t = \Gamma(k+1/2). \quad (16)$$

Since the argument of Gamma function is always half-integer and Gamma functions appear in both the numerator and denominator, we could factor out $\Gamma(1/2)$ and the correction would always be a rational number.

III. PÁDE APPROXIMANT

Evidently, the series (5) is divergent due to the singularity at $\epsilon = 0$. To avoid divergence, a Páde (50,50) approximant $P_{50}^{50}(\epsilon)$ is constructed. Formally, this means

$$P_{50}^{50}(\epsilon) = \frac{\sum_{n=0}^{N} A_n\epsilon^n}{\sum_{n=0}^{M} B_n\epsilon^n} \quad (17)$$

and the first $50+50+1$ Taylor expansion coefficients should equal to E_n. The problem again reduces to solve linear equations.

IV. NUMERICAL MEANS

To evaluate the effectiveness of our analytical approximation, numerical approach based on exact diagonalization is used.

The eigenfunction (2) of the unperturbed system is not normalized

$$\int_{-\infty}^{\infty} y_n^{\mathrm{u}} y_m^{\mathrm{u}}\,\mathrm{d}x = \sqrt{\pi}2^n n!\delta_{nm}. \quad (18)$$

To simplify the discussion, we form normalized eigenfunction

$$y_n^{\mathrm{n}} = \frac{1}{\pi^{1/4}\sqrt{2^n n!}} y_n^{\mathrm{u}} \quad (19)$$

such that

$$\int_{-\infty}^{\infty} y_n^{\mathrm{n}} y_m^{\mathrm{n}}\,\mathrm{d}x = \delta_{nm}. \quad (20)$$

It can be shown

$$\begin{aligned} & \int_{-\infty}^{\infty} y_n^{\mathrm{n}} x^4 y_m^{\mathrm{n}}\,\mathrm{d}x \\ &= \frac{1}{4}(6n^2+6n+3)\delta_{nm} \\ &\quad + (n+3/2)\sqrt{(n+1)(n+2)}\delta_{n,m-2} \\ &\quad + (n-1/2)\sqrt{n(n-1)}\delta_{n,m+2} \\ &\quad + \frac{1}{4}\sqrt{(n+1)(n+2)(n+3)(n+4)}\delta_{n,m-4} \\ &\quad + \frac{1}{4}\sqrt{(n-3)(n-2)(n-1)n}\delta_{n,m+4}. \quad (21) \end{aligned}$$

The full Hamiltonian H to diagonalize is composed of two parts $H = H_0 + \epsilon V$, with

$$H_0 = \mathrm{diag}\,\{1, 3, \ldots, 2N_{\mathrm{cutoff}}+1\}, \quad (22)$$

and the matrix element of V is defined in eq. (21). Choosing an approriate dimension of cutoff N_{cutoff} is a trade-off between cutoff-error and roundoff-error, but N_{cutoff} is simply fixed at 50 in this work. Since the full Hamiltonian H is real and symmetric, the very efficient and stable Jacobi method can be used to determine the eigenvalues.

V. IMPLEMENTATION

Since all coefficients are rational numbers, GMP (GNU Multiple Precision Arithmetic Library) [4] is used to store and manipulate them. And to ensure best performance, Eigen [5] library is used to solve linear equations and calculate eigenvalues. A simple polynomial class, featuring term by term add, multiplication by polynomial, and evaluation, is also implemented by hand. These programs are implemented with the C++ programming language [6].

On average, our method is capable of calculating the coefficients E_n in 168 s (compilation) + 132 s (run) given an unperturbed solution. The long compilation time is due to both Eigen and our hand-written polynomial library relies heavily on template. Although more efficient method based on direct recursion exists, our method is not formidably slow, so no attempt is made to optimize the code[7]. The construction of Páde approximant from E_n and calculating perturbed energy level at 100 different ϵ takes 18 s (compilation) + 3 s (run) on average.

A separate program written in Mathematica performing symbolic integrations according to eq. (8),(9) is implemented to check the correctness of low order. Due to slow symbolic calculations this method applies to n up to 13; and at $n = 13$, each update of E_n and y_n takes around 30 s.

VI. RESULT

A. Graph

In Fig. 1, the lowest 10 eigenvalues in $\epsilon \in [-10, 0]$ are plotted. Despite the series in eq. (5) is inherently divergent[8], Páde approximant converges well. Both solutions capture the trend that the eigenvalue increases as ϵ decreases. But it seems the numerical solution tends to a straight line at large $|\epsilon|$, while the slope of the analytical solution tends to 0. The numerical solution differs visibly from analytical solution below $\epsilon = 3$.

To see the error between the two more clearly, in Fig. 2, we plot the relative error between the two methods against ϵ. Log scale is used to account for the order-of-magnitude difference. Note $E^{\text{numerical}} > E^{\text{analytical}}$ holds for all $\epsilon < 0$ and the relative error is monotonically increasing. Also note there's seemingly crossing at, say, $\epsilon = -3$ between $n = 9$ and $n = 8$ where the slope of error is at minimum.

B. Table

In this section we present the values of $E_n (n = 1, \ldots, 100)$. For each unperturbed energy level, we present exactly solution up to first 20 orders and the approximate solution of all 100 orders. Since $-E_n 2^{3n}$ is a

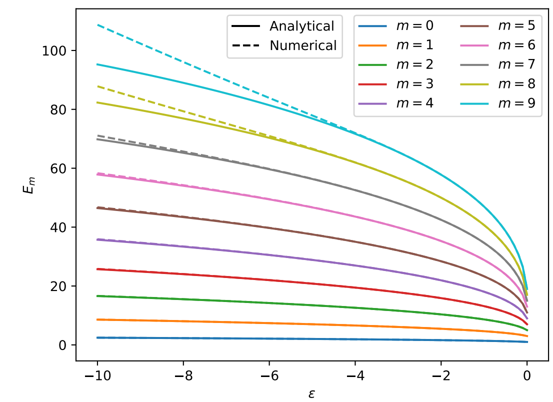

FIG. 1. Eigenvalues of eq. (4) from analytical(solid) and numerical(dashed) means.

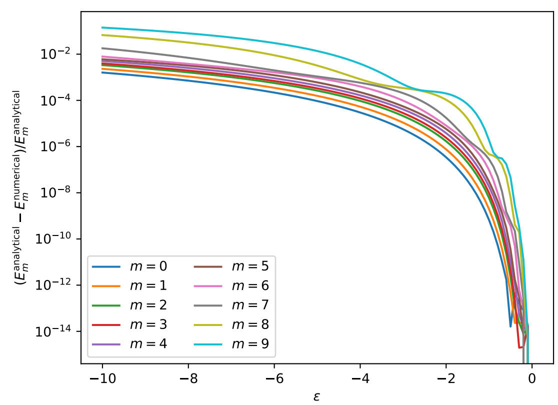

FIG. 2. Relative error between analytical and numerical methods.

positive integer, we present our result in $-E_n 2^{3n}$ instead of E_n.

In reference , the coefficients of ground state correction are given up to $n = 75$. Note the their definition of Hamiltonian differs from ours, and it can be shown that $(-2)^{n-1} E_n$ equals to their A_n.

VII. CONCLUSION

We've completed a high-order perturbative analysis to the x^4 anharmonic oscillator. The convergent behavior is achieved with analytical method at even very large perturbation strength ϵ where numerical method fails to function properly. We would like to comment that the method described is not restricted to quartic potential only, but applicable to other integer exponent $\alpha > 2$ potential as well.

TABLE I. Exact coeffcient of E_n up to $n = 20$ for E_0^{u}

n	$-E_n 2^{3n}$
1	6
2	84
3	2664
4	123540
5	7333848
6	524147208
7	43572714768
8	4121980396212
9	437015860091640
10	51336059452724760
11	6621407769290178672
12	930758905386276349320
13	141686496049367635656048
14	23229387977224612690619280
15	4081973849686832002155521568
16	765543711410243711551312338420
17	152644886266565584391713812812088
18	32249559868746478308856261490338680
19	7197129672652253378568475159863338160
20	1691928460676486363401186477311314757720

TABLE II. Exact coeffcient of E_n up to $n = 20$ for E_1^{u}

n	$-E_n 2^{3n}$
1	30
2	660
3	31320
4	2081940
5	170435880
6	16215569160
7	1737651694320
8	205795688655540
9	26607913470652680
10	3723933252458267160
11	560699252370983391120
12	90394751875598518863240
13	15544914905835692552987280
14	2842290048673569505197263760
15	551003198488474493684196465120
16	112961391092496352173257140772340
17	24432204681317955463241675314900680
18	5562664656497172471982192662546819960
19	1330383013238816192087544072071900650320
20	333565102755481893703894918020235785930840

[1] C. M. Bender and T. T. Wu, Anharmonic oscillator, Phys. Rev. **184**, 1231 (1969).

[2] C. Bender, S. Orszag, and S. Orszag, *Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory*, Advanced Mathematical Methods for Scientists and Engineers (Springer, 1999).

[3] In fact, it is shown in [1] that the sum for k could start from 2 instead of 0 if $n \geq 1$.

[4] T. Granlund and the GMP development team, *GNU MP: The GNU Multiple Precision Arithmetic Library*, 6th ed. (2020), `http://gmplib.org/`.

[5] G. Guennebaud, B. Jacob, *et al.*, Eigen v3, http://eigen.tuxfamily.org (2010).

[6] Code available at `https://github.com/putian9935/anharmonic-osc`.

[7] We didn't even turn on compiler optimization flag, e.g. `-O2` or `-Ofast`.

[8] It can be shown that A_n scales super-exponentially. Thus the series is divergent.

TABLE III. Exact coeffcient of E_n up to $n = 20$ for E_2^{u}

n	$-E_n 2^{3n}$
1	78
2	2460
3	160632
4	14305020
5	1535988744
6	188110215960
7	25495041543984
8	3754912433232540
9	593920919123280360
10	100088070464024930760
11	17870697976178395841616
12	3367187369124912927538200
13	667538193130776896234040144
14	138928611649281749972298448560
15	30300051071205181574277032253024
16	6915045540402221464641419799181020
17	1649283304963157585066069253330007464
18	410622272668183981562228778917860994280
19	106601227035161318024640742366601345479440
20	28826540363655691102327483978432629561219720

TABLE IV. Exact coeffcient of E_n up to $n = 20$ for E_3^{u}

n	$-E_n 2^{3n}$
1	150
2	6300
3	534600
4	60738300
5	8199819000
6	1247186367000
7	207635609250000
8	37179301503865500
9	7078906156588479000
10	1422022835047769325000
11	299732457805473100590000
12	66028224244090279453695000
13	15157989023243063520623310000
14	3618665345116689592688271630000
15	896948463395568349991868869700000
16	230562265546554690170858715723937500
17	61407665988905530461321136028463975000
18	16934299026103556311924076828956574625000
19	4832527074531557878069819538893268037750000
20	1426359462251378421340254159165455176897125000

TABLE V. Exact coeffcient of E_n up to $n = 20$ for E_4^{u}

n	$-E_n 2^{3n}$
1	246
2	12996
3	1369224
4	190980900
5	31348488888
6	5748774789672
7	1145277005586768
8	243704776908936708
9	54780845374124216280
10	12909166056728713392120
11	3171880841400323898134832
12	809406810491980230872556840
13	213883347293261559096605975088
14	58398225891609806982343388149200
15	16448551333930388750983395578720928
16	4773480671346660668911587492182081220
17	1426049185149588560529569069925225467928
18	438271289668099818802617355671857476165080
19	138502320945473198399637508287150474609963120
20	44991533418599438515152390972648750195089860920

TABLE VI. Exact coeffcient of E_n up to $n = 20$ for E_5^{u}

n	$-E_n 2^{3n}$
1	366
2	23364
3	2952504
4	490546980
5	95310683208
6	20570476322088
7	4797816243861168
8	1189443292455792132
9	310066182169392911400
10	84361702124758951605240
11	23828859146200940320546512
12	6960258294018023836347434280
13	2096209764973891029257110765008
14	649475884761654289683004487505360
15	206667567156572264205032912139079008
16	67452974101130232740910983180011749060
17	22559283605128866560805433773687584796648
18	7725549318398891749701093949956045372628440
19	2707596177960269327080542443377371057342042000
20	970795647566554645481254635712205043163353704760

TABLE VII. Exact coeffcient of E_n up to $n = 20$ for E_6^{u}

n	$-E_n 2^{3n}$
1	510
2	38220
3	5644440
4	1091127180
5	245625725160
6	61178256043320
7	16406098735555440
8	4660093776312281580
9	1387230513223589762760
10	429627345056803841886120
11	137704404716964210720468240
12	45502311953568087819486001080
13	15455286479818544332043410540560
14	5383973555502487536194054696110320
15	1920240325609227756315958323778863840
16	700241809716170442189848657832521355180
17	260810480281117796968229874840754997863560
18	99137146631639354998283926976013719243182920
19	38434121670962451927063055607350960046370924240
20	15190458307547315488739642298798206580805045163880

TABLE VIII. Exact coeffcient of E_n up to $n = 20$ for E_7^{u}

n	$-E_n 2^{3n}$
1	678
2	58380
3	9877032
4	2180850060
5	559129221144
6	158165907341880
7	48044012151997584
8	15418307436741808620
9	5172876008714722280760
10	1801297375941218698298280
11	647659150449728639148581616
12	239524358482197491691323676600
13	90851388119905954404768755292144
14	35262773832368118340351248495421680
15	13981101530066973109178900637361192224
16	5654641079908380097496232729047507718060
17	2330409375544560269525302040720326823910264
18	977795574003170289875384379125260859423232840
19	417406410914987753209614501324937077305410574640
20	181191933667780863275686273200874084523211900557160

TABLE IX. Exact coeffcient of E_n up to $n = 20$ for E_8^{u}

n	$-E_n 2^{3n}$
1	870
2	84660
3	16154280
4	4014543540
5	1156055834520
6	366573079193160
7	124570926823470480
8	44639373314584563540
9	16692295906333920321720
10	6466841110598481273702360
11	2582341143421740531173264880
12	1058832810601068101027028605640
13	444506171440957969399800993108720
14	190630208426068724741346000137682960
15	83368839445052238682099337091426504480
16	37128333972989641369552948304342114024340
17	16819382635460142250188719438936923135862520
18	7743317794767935207526522226320115411624251960
19	3620294854568037645847016611372652073768138719280
20	1717955671666603638721472801287605386511053052911640

TABLE X. Exact coeffcient of E_n up to $n = 20$ for E_9^{u}

n	$-E_n 2^{3n}$
1	1086
2	117876
3	25052184
4	6923996340
5	2214159188328
6	778487059956552
7	292907088729453168
8	116045384727244748628
9	47908214409056898325320
10	20463046780979486425277400
11	8996774820508963792598974992
12	4056161451233594608921583680200
13	1869844692748383712711195235962128
14	879402078031756466350628337057112080
15	421206825945193285862265960689062949088
16	205171692865293620863594778860193845459860
17	101522039738075310270837092027362611511548168
18	50982857841264211072697963345552165044153782840
19	25964744959336851103768116305856620858052392828880
20	13402209587883193195141124719169628506488864107995160

TABLE XI. The value of the $-E_n$ with $n = 1, \ldots, 100$ for E_0^{u}. The number following the comma is the power of 10 multipling the decimal.

n	$-E_n$		n	$-E_n$	
1	0.7500000000000000	0	51	0.1774549491325011	75
2	0.1312500000000000	1	52	0.1371585100985822	77
3	0.5203125000000000	1	53	0.1080687553363126	79
4	0.3016113281250000	2	54	0.8676872885291642	80
5	0.2238112792968750	3	55	0.7096770461319298	82
6	0.1999462921142578	4	56	0.5910809250127134	84
7	0.2077708948516846	5	57	0.5011655452280950	86
8	0.2456891772873401	6	58	0.4324420071229820	88
9	0.3256021887746751	7	59	0.3796259964268197	90
10	0.4781043106012490	8	60	0.3389524990800757	92
11	0.7708333164092827	9	61	0.3077189141185188	94
12	0.1354432468922862	11	62	0.2839773024865141	96
13	0.2577262349393415	12	63	0.2663254045275103	98
14	0.5281751322678386	13	64	0.2537640850228387	100
15	0.1160166746583068	15	65	0.2456002702014563	102
16	0.2719757615246876	16	66	0.2413817719864760	104
17	0.6778794692977178	17	67	0.2408551986715980	106
18	0.1790210195015489	19	68	0.2439413849911183	108
19	0.4994011921119655	20	69	0.2507250426950703	110
20	0.1467514010204402	22	70	0.2614570302788745	112
21	0.4531136296684818	23	71	0.2765690405221834	114
22	0.1466652370037318	25	72	0.2967018143757369	116
23	0.4966283069462674	26	73	0.3227493905153705	118
24	0.1755839492534922	28	74	0.3559235776783599	120
25	0.6470221042946597	29	75	0.3978450133890432	122
26	0.2480994545016985	31	76	0.4506701405312378	124
27	0.9884377883559941	32	77	0.5172676031383124	126
28	0.4085842008364257	34	78	0.6014635309853691	128
29	0.1750075894533248	36	79	0.7083838315872806	130
30	0.7757967334354602	37	80	0.8449342603473809	132
31	0.3555183256998041	39	81	0.1020477693952982	135
32	0.1682432154270259	41	82	0.1247795733435630	137
33	0.8213752926846650	42	83	0.1544463180441750	139
34	0.4133016264153365	44	84	0.1934826268889632	141
35	0.2141561571497207	46	85	0.2452869986345015	143
36	0.1141747926177188	48	86	0.3146403879335594	145
37	0.6258125322225467	49	87	0.4083216923637282	147
38	0.3523944748523645	51	88	0.5360193919765655	149
39	0.2037126840152551	53	89	0.7116917856121663	151
40	0.1208147982683254	55	90	0.9556116068446843	153
41	0.7346130817663982	56	91	0.1297462244223361	156
42	0.4576889474357688	58	92	0.1781061513835013	158
43	0.2920146672188311	60	93	0.2471622643123596	160
44	0.1906874900611838	62	94	0.3466999332174321	162
45	0.1273780703868530	64	95	0.4915232511284615	164
46	0.8699690034565581	65	96	0.7042137555250557	166
47	0.6072125057647181	67	97	0.1019500619817492	169
48	0.4329178140381204	69	98	0.1491236212338351	171
49	0.3151420296247126	71	99	0.2203614896688591	173
50	0.2341312637831796	73	100	0.3289353468000507	175

TABLE XII. The value of the $-E_n$ with $n = 1, \ldots, 100$ for E_1^{u}. The number following the comma is the power of 10 multipling the decimal.

n	$-E_n$		n	$-E_n$	
1	0.3750000000000000	1	51	0.1013697382720098	78
2	0.1031250000000000	2	52	0.7999894076405615	79
3	0.6117187500000000	2	53	0.6433065303303194	81
4	0.5082861328125000	3	54	0.5269384387338485	83
5	0.5201290283203125	4	55	0.4395069401405857	85
6	0.6185748733520507	5	56	0.3731610831033226	87
7	0.8285768958663940	6	57	0.3224165529874986	89
8	0.1226637891862035	8	58	0.2833993767834352	91
9	0.1982444038290730	9	59	0.2533468824945158	93
10	0.3468183104375626	10	60	0.2302745907707660	95
11	0.6527398391288978	11	61	0.2127516083919541	97
12	0.1315416766382965	13	62	0.1997479832504679	99
13	0.2827603549273716	14	63	0.1905305321305266	101
14	0.6462619350426325	15	64	0.1845919232030838	103
15	0.1566045280266264	17	65	0.1816031198513451	105
16	0.4013194793106448	18	66	0.1813827886907690	107
17	0.1085007847182149	20	67	0.1838796344801361	109
18	0.3087899189955874	21	68	0.1891652863317557	111
19	0.9231386580424552	22	69	0.1974366106060682	113
20	0.2893216072582752	24	70	0.2090273658972469	115
21	0.9488325779811591	25	71	0.2244300976158352	117
22	0.3250307132935212	27	72	0.2443302288960944	119
23	0.1161079009509754	29	73	0.2696555812966252	121
24	0.4318387635070937	30	74	0.3016462216578736	123
25	0.1669803282377047	32	75	0.3419518043131752	125
26	0.6703372839883187	33	76	0.3927667755460074	127
27	0.2790250074907100	35	77	0.4570183876125177	129
28	0.1202779951451808	37	78	0.5386291110609579	131
29	0.5363224836907425	38	79	0.6428847532700168	133
30	0.2471129926328750	40	80	0.7769539258517119	135
31	0.1175317419895233	42	81	0.9506257966400545	137
32	0.5764879667215923	43	82	0.1177364906865322	140
33	0.2913463211724646	45	83	0.1475829778213682	142
34	0.1515804839338474	47	84	0.1872074696313200	144
35	0.8112255426321887	48	85	0.2402764911690694	146
36	0.4462446620087926	50	86	0.3119905748318578	148
37	0.2521291717902646	52	87	0.4097849326392476	150
38	0.1462154177897528	54	88	0.5443752293033641	152
39	0.8697597331760013	55	89	0.7313299926211286	154
40	0.5303605150557383	57	90	0.9934524761765235	156
41	0.3313227439172527	59	91	0.1364415656868624	159
42	0.2119306535252607	61	92	0.1894351315007845	161
43	0.1387280007164099	63	93	0.2658508397566874	163
44	0.9288338152069379	64	94	0.3770767496381037	165
45	0.6357712219917682	66	95	0.5404894714408601	167
46	0.4446795734592901	68	96	0.7828220531116719	169
47	0.3176727818007192	70	97	0.1145541252717334	172
48	0.2316916636191567	72	98	0.1693498304962881	174
49	0.1724475377449753	74	99	0.2528951783720768	176
50	0.1309319077303230	76	100	0.3814471460671227	178

TABLE XIII. The value of the $-E_n$ with $n = 1, \ldots, 100$ for E_2^{u}. The number following the comma is the power of 10 multipling the decimal.

n	$-E_n$		n	$-E_n$	
1	0.9750000000000000	1	51	0.2782417452086104	80
2	0.3843750000000000	2	52	0.2243926705423097	82
3	0.3137343750000000	3	53	0.1843114693729047	84
4	0.3492436523437500	4	54	0.1541388682732517	86
5	0.4687465649414063	5	55	0.1312053996888442	88
6	0.7175835264587402	6	56	0.1136420900504855	90
7	0.1215698315810394	8	57	0.1001260937954763	92
8	0.2238102217455232	9	58	0.8971243625245890	93
9	0.4425055676127079	10	59	0.8172145118842534	95
10	0.9321427947285115	11	60	0.7566273573987154	97
11	0.2080423056168761	13	61	0.7118339167728882	99
12	0.4899902515353333	14	62	0.6803267422051353	101
13	0.1214244899767030	16	63	0.6603793566636642	103
14	0.3158870905492261	17	64	0.6508853760363058	105
15	0.8611792472721953	18	65	0.6512558308339145	107
16	0.2456717688091537	20	66	0.6613611911664670	109
17	0.7324289197199886	21	67	0.6815102684192212	111
18	0.2279411507700634	23	68	0.7124624839985987	113
19	0.7396946044233114	24	69	0.7554736396534398	115
20	0.2500303815001327	26	70	0.8123788246981982	117
21	0.8794008610707369	27	71	0.8857199300560271	119
22	0.3214991286241589	29	72	0.9789299135594561	121
23	0.1220445451374365	31	73	0.1096592092557875	124
24	0.4805658842288584	32	74	0.1244801155029726	126
25	0.1960829257723188	34	75	0.1431664447811683	128
26	0.8282162685198766	35	76	0.1667999123823776	130
27	0.3617767501523046	37	77	0.1968305436488712	132
28	0.1632747597244777	39	78	0.2352132650901707	134
29	0.7606444488801276	40	79	0.2846007436274628	136
30	0.3654654557834732	42	80	0.3486173960339476	138
31	0.1809447926361267	44	81	0.4322513647450226	140
32	0.9224217810179139	45	82	0.5424191420600709	142
33	0.4837940481058632	47	83	0.6887846448300159	144
34	0.2608660912105105	49	84	0.8849559365752178	146
35	0.1445086323854607	51	85	0.1150246398115651	149
36	0.8218557382164443	52	86	0.1512285530951558	151
37	0.4795596708192673	54	87	0.2010917780289750	153
38	0.2869240584218257	56	88	0.2704067944020963	155
39	0.1759187901069950	58	89	0.3676630823963530	157
40	0.1104671689104127	60	90	0.5054045194155740	159
41	0.7100602266906382	61	91	0.7023176023000495	161
42	0.4669530670799086	63	92	0.9864681638970619	163
43	0.3140160142820714	65	93	0.1400356119163249	166
44	0.2158361158277432	67	94	0.2008869139361818	168
45	0.1515624921344746	69	95	0.2911892666701519	170
46	0.1086836097234662	71	96	0.4264453968049036	172
47	0.7955324132692282	72	97	0.6309145973514563	174
48	0.5941512847151227	74	98	0.9428713232052504	176
49	0.4525973463464175	76	99	0.1423198554624280	179
50	0.3515114836451844	78	100	0.2169537385186226	181

TABLE XIV. The value of the $-E_n$ with $n = 1, \ldots, 100$ for E_3^{u}. The number following the comma is the power of 10 multipling the decimal.

n	$-E_n$		n	$-E_n$	
1	0.1875000000000000	2	51	0.4885612004700294	82
2	0.9843750000000000	2	52	0.4030084440980600	84
3	0.1044140625000000	4	53	0.3384142354430983	86
4	0.1482868652343750	5	54	0.2891949848658745	88
5	0.2502386169433594	6	55	0.2514276926770286	90
6	0.4757638423919678	7	56	0.2223270326163187	92
7	0.9900837385654450	8	57	0.1998981085532981	94
8	0.2216059059135050	10	58	0.1827033365495634	96
9	0.5274196085772276	11	59	0.1697044807960026	98
10	0.1324361967898690	13	60	0.1601548929725367	100
11	0.3489345053740231	14	61	0.1535258301809256	102
12	0.9608371218795988	15	62	0.1494563622142943	104
13	0.2757222141234354	17	63	0.1477200378497144	106
14	0.8227892397182334	18	64	0.1482039070653658	108
15	0.2549280860067621	20	65	0.1508971573331252	110
16	0.8191217146224783	21	66	0.1558878160669482	112
17	0.2727048186774962	23	67	0.1633668973744017	114
18	0.9400424342333766	24	68	0.1736401722068980	116
19	0.3353239265793366	26	69	0.1871485310101083	118
20	0.1237169622174560	28	70	0.2044987933248902	120
21	0.4718921791943949	29	71	0.2265079157071883	122
22	0.1859987566907764	31	72	0.2542650026252777	124
23	0.7572360438408876	32	73	0.2892175310325586	126
24	0.3182710024931293	34	74	0.3332910376991375	128
25	0.1380339902233111	36	75	0.3890555981217487	130
26	0.6173971415705142	37	76	0.4599583548793770	132
27	0.2846349656013281	39	77	0.5506500450477650	134
28	0.1351772432879351	41	78	0.6674463142908377	136
29	0.6609232481268932	42	79	0.8189837001951169	138
30	0.3324814643971936	44	80	0.1017158760593696	141
31	0.1719842461887097	46	81	0.1278481923542737	143
32	0.9142222603788523	47	82	0.1626043035804852	145
33	0.4991108123193866	49	83	0.2092385481623800	147
34	0.2796848580421176	51	84	0.2723739343176290	149
35	0.1607748522707433	53	85	0.3586301830404022	151
36	0.9475482346438696	54	86	0.4775623703747752	153
37	0.5722455260453116	56	87	0.6430741656094801	155
38	0.3539422922647578	58	88	0.8755614673669276	157
39	0.2240938799070066	60	89	0.1205188230737552	160
40	0.1451646722308135	62	90	0.1676930024905827	162
41	0.9616533138168430	63	91	0.2358397679412801	164
42	0.6511832208865065	65	92	0.3352065021320961	166
43	0.4505285776740869	67	93	0.4814524364444069	168
44	0.3183391939767087	69	94	0.6987038535366945	170
45	0.2296292297680037	71	95	0.1024438434023326	173
46	0.1690286948136289	73	96	0.1517353116560356	175
47	0.1269178921453709	75	97	0.2270134125956212	177
48	0.9717467490483426	76	98	0.3430340908766180	179
49	0.7583948512077047	78	99	0.5234818706418431	181
50	0.6031136573533145	80	100	0.8066833816806185	183

TABLE XV. The value of the $-E_n$ with $n = 1, \ldots, 100$ for E_4^{u}. The number following the comma is the power of 10 multipling the decimal.

n	$-E_n$		n	$-E_n$	
1	0.3075000000000000	2	51	0.6163890296662139	84
2	0.2030625000000000	3	52	0.5205828964253396	86
3	0.2674265625000000	4	53	0.4473282611276090	88
4	0.4662619628906250	5	54	0.3909700641847076	90
5	0.9566799587402344	6	55	0.3474754652108244	92
6	0.2192983547085571	8	56	0.3139455596562416	94
7	0.5461106326993790	9	57	0.2882859171524893	96
8	0.1452593665772299	11	58	0.2689823818628982	98
9	0.4081491036275343	12	59	0.2549476536179022	100
10	0.1202259776809133	14	60	0.2454162609110759	102
11	0.3692555289483075	15	61	0.2398733138169726	104
12	0.1177841929154209	17	62	0.2380075146027955	106
13	0.3890515423213611	18	63	0.2396823138132173	108
14	0.1327821926033944	20	64	0.2449214656602231	110
15	0.4674959465640538	21	65	0.2539069713797716	112
16	0.1695881007658308	23	66	0.2669887836591426	114
17	0.6332930558404064	24	67	0.2847068861755240	116
18	0.2432894384108430	26	68	0.3078276320979706	118
19	0.9610529104855464	27	69	0.3373976922131465	120
20	0.3902393462071975	29	70	0.3748208185688765	122
21	0.1628430471253280	31	71	0.4219651182781612	124
22	0.6982043824343618	32	72	0.4813119919841363	126
23	0.3075431760213506	34	73	0.5561628070675839	128
24	0.1391487802899673	36	74	0.6509264589911897	130
25	0.6466052892687969	37	75	0.7715212862050487	132
26	0.3085418281762928	39	76	0.9259399413703431	134
27	0.1511541839149337	41	77	0.1125048205688947	137
28	0.7600810934681406	42	78	0.1383722061648816	139
29	0.3922119933709182	44	79	0.1722477296496474	141
30	0.2076233605082946	46	80	0.2169821287079190	143
31	0.1127161399479063	48	81	0.2765671104682771	145
32	0.6273347955726312	49	82	0.3566357120502071	147
33	0.3578125893995756	51	83	0.4652000697000462	149
34	0.2090678897460641	53	84	0.6137471959285830	151
35	0.1250898401633373	55	85	0.8188784708482159	153
36	0.7660976563471842	56	86	0.1104780791939786	156
37	0.4800612110758221	58	87	0.1506978960711500	158
38	0.3076681030653535	60	88	0.2078076161666302	160
39	0.2015885430385612	62	89	0.2896601701140670	162
40	0.1349810924305350	64	90	0.4080750475919190	164
41	0.9232785031356141	65	91	0.5809879549137958	166
42	0.6448778478380773	67	92	0.8358395456539431	168
43	0.4597725644878483	69	93	0.1214957851350285	171
44	0.3344793519294535	71	94	0.1784172131980461	173
45	0.2481990399771389	73	95	0.2646702682904439	175
46	0.1877947242762296	75	96	0.3965727965713836	177
47	0.1448342619777139	77	97	0.6001328088419278	179
48	0.1138204024656286	79	98	0.9171421587277599	181
49	0.9111505158330104	80	99	0.1415303974744693	184
50	0.7427551103312338	82	100	0.2205191421700687	186

TABLE XVI. The value of the $-E_n$ with $n = 1, \ldots, 100$ for E_5^{u}. The number following the comma is the power of 10 multipling the decimal.

n	$-E_n$		n	$-E_n$	
1	0.4575000000000000	2	51	0.5949903137129013	86
2	0.3650625000000000	3	52	0.5150543709294194	88
3	0.5766609375000000	4	53	0.4533536030668839	90
4	0.1197624462890625	6	54	0.4056521609313773	92
5	0.2908651220947266	7	55	0.3688906874652121	94
6	0.7847013977847290	8	56	0.3408519051758309	96
7	0.2287777063303550	10	57	0.3199317458118899	98
8	0.7089634492729855	11	58	0.3049821345279095	100
9	0.2310173080633528	13	59	0.2952032842876264	102
10	0.7856795762177460	14	60	0.2900709899981089	104
11	0.2774044306273681	16	61	0.2892894689060787	106
12	0.1012850886620876	18	62	0.2927637345056157	108
13	0.3812983350096858	19	63	0.3005879304519489	110
14	0.1476737190300024	21	64	0.3130478959567594	112
15	0.5873845542412603	22	65	0.3306377704238689	114
16	0.2396411037648613	24	66	0.3540918928259432	116
17	0.1001833443098515	26	67	0.3844348052986635	118
18	0.4288541365581818	27	68	0.4230540328251561	120
19	0.1878772261349102	29	69	0.4718027235795924	122
20	0.8420310001049046	30	70	0.5331425170104178	124
21	0.3859793789182782	32	71	0.6103416068235188	126
22	0.1809323690505331	34	72	0.7077495322359895	128
23	0.8672601782915448	35	73	0.8311797207120296	130
24	0.4250596062893444	37	74	0.9884446497235448	132
25	0.2130184877856855	39	75	0.1190108851207628	135
26	0.1091585936607595	41	76	0.1450555127016756	137
27	0.5719766897836153	42	77	0.1789504287252610	139
28	0.3064648239397741	44	78	0.2234196177066477	141
29	0.1679030921924805	46	79	0.2822541664058013	143
30	0.9405790071019297	47	80	0.3607710231271199	145
31	0.5387165032541003	49	81	0.4664855053586822	147
32	0.3154349474319097	51	82	0.6101043006331455	149
33	0.1887931017805696	53	83	0.8070024156309697	151
34	0.1154830323430693	55	84	0.1079436092238762	154
35	0.7218096555603444	56	85	0.1459882951328496	156
36	0.4608987114759870	58	86	0.1996121028100609	158
37	0.3005821358908226	60	87	0.2759009607137340	160
38	0.2001633853064307	62	88	0.3854498346078284	162
39	0.1360670047142689	64	89	0.5442301651905933	164
40	0.9439438057156202	65	90	0.7765156269280664	166
41	0.6680975245343814	67	91	0.1119500564970991	169
42	0.4822889626142365	69	92	0.1630645481917304	171
43	0.3549939312506358	71	93	0.2399448186618568	173
44	0.2663501848933499	73	94	0.3566449856640564	175
45	0.2036462741583659	75	95	0.5354155187071251	177
46	0.1586227213128151	77	96	0.8117715941129232	179
47	0.1258332524233368	79	97	0.1242863901825221	182
48	0.1016351826172831	81	98	0.1921408294717308	184
49	0.8355856995061861	82	99	0.2999040923239406	186
50	0.6990659553162657	84	100	0.4725772114952314	188

TABLE XVII. The value of the $-E_n$ with $n = 1, \ldots, 100$ for E_6^{u}. The number following the comma is the power of 10 multipling the decimal.

n	$-E_n$		n	$-E_n$	
1	0.6375000000000000	2	51	0.4568773167774281	88
2	0.5971875000000000	3	52	0.4058480280997044	90
3	0.1102429687500000	5	53	0.3663369447696654	92
4	0.2663884716796875	6	54	0.3359372872406965	94
5	0.7495902257080078	7	55	0.3128988957740627	96
6	0.2333765260441589	9	56	0.2959564057316568	98
7	0.7823037498262139	10	57	0.2842107187889343	100
8	0.2777632341571022	12	58	0.2770478914035133	102
9	0.1033567274528436	14	59	0.2740849918827631	104
10	0.4001216451235151	15	60	0.2751361440975752	106
11	0.1603090259211187	17	61	0.2801945334204003	108
12	0.6621457862430264	18	62	0.2894280278283636	110
13	0.2811300233555548	20	63	0.3031875582244702	112
14	0.1224173855803767	22	64	0.3220287237215521	114
15	0.5457651257100985	23	65	0.3467484194847602	116
16	0.2487758655846656	25	66	0.3784397996724606	118
17	0.1158231201086569	27	67	0.4185707800567203	120
18	0.5503217139304315	28	68	0.4690937992900739	122
19	0.2666902925646700	30	69	0.5325980291730832	124
20	0.1317562231847462	32	70	0.6125201298321929	126
21	0.6633910695713141	33	71	0.7134366828886839	128
22	0.3403381267855611	35	72	0.8414716392949145	130
23	0.1778853310085093	37	73	0.1004867039649360	133
24	0.9471850572326485	38	74	0.1214787247171807	135
25	0.5138013091659210	40	75	0.1486459543198272	137
26	0.2839484947921437	42	76	0.1840802633620146	139
27	0.1598816405598488	44	77	0.2306767820057316	141
28	0.9172976703971315	45	78	0.2924728358062778	143
29	0.5363095343518711	47	79	0.3751421234489202	145
30	0.3195611934325641	49	80	0.4867204254156147	147
31	0.1940712024665472	51	81	0.6386790522181790	149
32	0.1201336494738406	53	82	0.8475242614712964	151
33	0.7580245450301205	54	83	0.1137197877044484	154
34	0.4875559056225766	56	84	0.1542707085981497	156
35	0.3196566613683530	58	85	0.2115653525031628	158
36	0.2136213513028494	60	86	0.2932718225313381	160
37	0.1455051551744951	62	87	0.4108779986877816	162
38	0.1010054079152379	64	88	0.5817349482635982	164
39	0.7144838221827710	65	89	0.8322638105298241	166
40	0.5149452845492426	67	90	0.1203026514128130	169
41	0.3780782260658095	69	91	0.1756804049813636	171
42	0.2827339619095333	71	92	0.2591563307335460	173
43	0.2153118622155866	73	93	0.3861428026211881	175
44	0.1669416933191457	75	94	0.5810850892211912	177
45	0.1317583433359109	77	95	0.8830717598022435	179
46	0.1058314620713422	79	96	0.1355115470829480	182
47	0.8649278960532462	80	97	0.2099621198718526	184
48	0.7190803560053887	82	98	0.3284361426922829	186
49	0.6080101521931790	84	99	0.5186423675161670	188
50	0.5227374268359755	86	100	0.8267124484291338	190

TABLE XVIII. The value of the $-E_n$ with $n = 1, \ldots, 100$ for E_7^{u}. The number following the comma is the power of 10 multipling the decimal.

n	$-E_n$		n	$-E_n$	
1	0.8474999999999999	2	51	0.2864576956403406	90
2	0.9121874999999999	3	52	0.2614641871216309	92
3	0.1929107812500000	5	53	0.2423246400873491	94
4	0.5324340966796874	6	54	0.2280033595909620	96
5	0.1706326968823242	8	55	0.2177547522946587	98
6	0.6033550542521667	9	56	0.2110577211404799	100
7	0.2290917022323493	11	57	0.2075710786909981	102
8	0.9190027378047591	12	58	0.2071045913077593	104
9	0.3854092962082268	14	59	0.2096022085035395	106
10	0.1677588909809681	16	60	0.2151354456780130	108
11	0.7539744843376435	17	61	0.2239059894867257	110
12	0.3485538159761891	19	62	0.2362575403383597	112
13	0.1652577122875409	21	63	0.2526978225100913	114
14	0.8017826492588964	22	64	0.2739326962375532	116
15	0.3973668052045404	24	65	0.3009155240041969	118
16	0.2008932071329774	26	66	0.3349165295142665	120
17	0.1034909658212762	28	67	0.3776190486735576	122
18	0.5427855798174454	29	68	0.4312525973284164	124
19	0.2896338800149804	31	69	0.4987769906065196	126
20	0.1571589504955659	33	70	0.5841379626377103	128
21	0.8668158443167112	34	71	0.6926237763100751	130
22	0.4858348021009420	36	72	0.8313655685678594	132
23	0.2766559751780892	38	73	0.1010043749858392	135
24	0.1600405535855561	40	74	0.1241891871912073	137
25	0.9404494313735363	41	75	0.1545132914178496	139
26	0.5613791718166484	43	76	0.1945048509737747	141
27	0.3404171633237804	45	77	0.2476981035119704	143
28	0.2097174528624448	47	78	0.3190720172482583	145
29	0.1312714792118446	49	79	0.4156958541019060	147
30	0.8349703805915372	50	80	0.5476861241227151	149
31	0.5397504913741538	52	81	0.7296354882997953	151
32	0.3546450009157600	54	82	0.9827620292982525	153
33	0.2368805207257806	56	83	0.1338165911268127	156
34	0.1608612217740305	58	84	0.1841800569550901	158
35	0.1110722312995723	60	85	0.2562117525672430	160
36	0.7798830531197261	61	86	0.3601910503678910	162
37	0.5568675336700459	63	87	0.5116802842985579	164
38	0.4043815157827464	65	88	0.7344320819312913	166
39	0.2986453431319782	67	89	0.1064995789086632	169
40	0.2243069618967584	69	90	0.1560070875732223	171
41	0.1713323927076954	71	91	0.2308333850685106	173
42	0.1330832084438063	73	92	0.3449603516834628	175
43	0.1051144242910826	75	93	0.5206131913463705	177
44	0.8441441035534458	76	94	0.7934077747406583	179
45	0.6891895997855279	78	95	0.1220881768523695	182
46	0.5719722580299542	80	96	0.1896740418209394	184
47	0.4824654662889382	82	97	0.2974823844496829	186
48	0.4135699235496021	84	98	0.4709732693228118	188
49	0.3602101357368651	86	99	0.7526206640641758	190
50	0.3187257591313054	88	100	0.1213849211210191	193

TABLE XIX. The value of the $-E_n$ with $n = 1, \ldots, 100$ for E_8^{u}. The number following the comma is the power of 10 multipling the decimal.

n	$-E_n$		n	$-E_n$	
1	0.1087500000000000	3	51	0.1493602943342593	92
2	0.1322812500000000	4	52	0.1402848579118311	94
3	0.3155132812500000	5	53	0.1336782351683650	96
4	0.9801131689453125	6	54	0.1292204241552506	98
5	0.3528002424682617	8	55	0.1266969827142530	100
6	0.1398365322849884	10	56	0.1259819834087036	102
7	0.5940004674123310	11	57	0.1270278110698450	104
8	0.2660713989411864	13	58	0.1298606705680488	106
9	0.1243672959978425	15	59	0.1345812599088528	108
10	0.6022715112751797	16	60	0.1413705707262591	110
11	0.3006240752783768	18	61	0.1505012687891305	112
12	0.1540804530088016	20	62	0.1623556412505664	114
13	0.8085520156618402	21	63	0.1774517398964715	116
14	0.4334429114034464	23	64	0.1964801797830896	118
15	0.2369484930257279	25	65	0.2203551756675389	120
16	0.1319063399769136	27	66	0.2502849618428871	122
17	0.7469306344747358	28	67	0.2878689567436593	124
18	0.4298404851370526	30	68	0.3352322135789543	126
19	0.2512084189670886	32	69	0.3952123048386201	128
20	0.1490089017163780	34	70	0.4716205144736538	130
21	0.8966976597147372	35	71	0.5696091017151410	132
22	0.5472494482587509	37	72	0.6961910414294646	134
23	0.3386243432997396	39	73	0.8609804621318391	136
24	0.2124054335616395	41	74	0.1077254722562530	139
25	0.1350445833494651	43	75	0.1363488459889685	141
26	0.8702155722485635	44	76	0.1745584989036700	143
27	0.5683420456847837	46	77	0.2260145205477251	145
28	0.3762175538015916	48	78	0.2959290828911737	147
29	0.2524310641116154	50	79	0.3917832621305500	149
30	0.1716965141516950	52	80	0.5244001318990277	151
31	0.1183982450651360	54	81	0.7095629736461352	153
32	0.8278497411302496	55	82	0.9704734857079413	155
33	0.5870057523357540	57	83	0.1341513638708255	158
34	0.4221651274430417	59	84	0.1874045251737290	160
35	0.3079898095692558	61	85	0.2645417634192801	162
36	0.2279645938457261	63	86	0.3773062256530930	164
37	0.1712123101854065	65	87	0.5436712264083174	166
38	0.1304948529140530	67	88	0.7913692779452063	168
39	0.1009462787486133	69	89	0.1163539138578061	171
40	0.7926215977618217	70	90	0.1727829885823601	173
41	0.6317596487979069	72	91	0.2591188584068681	175
42	0.5111767937035336	74	92	0.3924060196262301	177
43	0.4198918451515982	76	93	0.6000287996722207	179
44	0.3501511788598771	78	94	0.9263394535820547	181
45	0.2964296299580445	80	95	0.1443751072038404	184
46	0.2547552946441054	82	96	0.2271441972538752	186
47	0.2222493104691814	84	97	0.3607130208737314	188
48	0.1968095661390483	86	98	0.5781454417161856	190
49	0.1768916211517597	88	99	0.9351738409390382	192
50	0.1613558723509342	90	100	0.1526486481149450	195

TABLE XX. The value of the $-E_n$ with $n = 1, \ldots, 100$ for E_9^{u}. The number following the comma is the power of 10 multipling the decimal.

n	$-E_n$		n	$-E_n$	
1	0.1357500000000000	3	51	0.6561748446493809	93
2	0.1841812500000000	4	52	0.6352337867989922	95
3	0.4893004687500001	5	53	0.6233287000878622	97
4	0.1690428793945312	7	54	0.6199283943382454	99
5	0.6757077601098633	8	55	0.6248474476077912	101
6	0.2969692458940704	10	56	0.6382313884154033	103
7	0.1396689838073030	12	57	0.6605635306566455	105
8	0.6916843934490964	13	58	0.6926938847010161	107
9	0.3569440127093859	15	59	0.7358925554046821	109
10	0.1905769741253880	17	60	0.7919322824353243	111
11	0.1047362436134015	19	61	0.8632075404761096	113
12	0.5902491759092073	20	62	0.9529012263671206	115
13	0.3401227682385768	22	63	0.1065214863905161	118
14	0.1999528826744964	24	64	0.1205685084963463	120
15	0.1197141801711704	26	65	0.1381618807698945	122
16	0.7289162797451838	27	66	0.1602693362496664	124
17	0.4508484240964853	29	67	0.1881787788922953	126
18	0.2830117131828137	31	68	0.2236140590804166	128
19	0.1801666105148484	33	69	0.2688971824395599	130
20	0.1162456380103121	35	70	0.3271770214111639	132
21	0.7597992208516458	36	71	0.4027539255226754	134
22	0.5028878477189503	38	72	0.5015435622268172	136
23	0.3369461470409486	40	73	0.6317442780006320	138
24	0.2284890771078816	42	74	0.8048039911649113	140
25	0.1567877095401951	44	75	0.1036830954165135	143
26	0.1088554808875163	46	76	0.1350666826632063	145
27	0.7646303397663590	47	77	0.1778954878002399	147
28	0.5433823195470802	49	78	0.2368713844663874	149
29	0.3906774398998381	51	79	0.3188205899542757	151
30	0.2841909876762504	53	80	0.4337324781345921	153
31	0.2091769834871307	55	81	0.5963423708531865	155
32	0.1558015976248495	57	82	0.8285609422100905	157
33	0.1174452834502065	59	83	0.1163230658697405	160
34	0.8961130650889597	60	84	0.1649977222148192	162
35	0.6921757076557509	62	85	0.2364392199638066	164
36	0.5413300026115089	64	86	0.3422550634693977	166
37	0.4287136988507646	66	87	0.5004134476548942	168
38	0.3438739433319517	68	88	0.7389538402038722	170
39	0.2793979594069452	70	89	0.1101985819230065	173
40	0.2299860674164497	72	90	0.1659459842088674	175
41	0.1918191948231558	74	91	0.2523200701827696	177
42	0.1621239891358860	76	92	0.3873414585128961	179
43	0.1388715475063974	78	93	0.6002839001121584	181
44	0.1205671836007381	80	94	0.9390827699806588	183
45	0.1061026879418339	82	95	0.1482859476174139	186
46	0.9465180993569096	83	96	0.2363252291079442	188
47	0.8559586041722360	85	97	0.3801015931866250	190
48	0.7847047047468808	87	98	0.6169285488933802	192
49	0.7292733651573322	89	99	0.1010374079926928	195
50	0.6870668355008321	91	100	0.1669585832804385	197