Anharmonic oscillator: a perturbative approach

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In this article, the eigenvalues of anharmonic oscillator are investigated with perturbative theory. We use standard perturbative means to expand the lowest 10 eigenvalues to 100th-order, and then use a Páde (50,50) approximant to better approximate each divergent series. The results will then be compared against numerical means. To the author's knowledge, this is the first extensive high order analytical study on the excited states of the x^4 anharmonic oscillator[a].

I. MATHEMATICAL MODEL OF THE UNPERTURBED SYSTEM

The quantum mechanical description, *i.e.*, Schrödinger equation, of harmonic oscillator eventually reduces to this equation

$$-y''(x) + x^2 y(x) = Ey. \tag{1}$$

The solution of which can be found in standard QM text:

$$y^{\mathrm{u}}_m(x) = e^{-x^2/2} H_m(x), \quad m = 0, 1, \ldots, \tag{2}$$

$$E^{\mathrm{u}}_m = 2m + 1, \tag{3}$$

where H_m is the m-th order Hermite polynomial and the superscript u means "unperturbed". The degree of m-th order Hermite polynomial is m.

II. SOLVING THE PERTURBED SYSTEM

The anharmonicity takes many forms, but usually it ends up in a power form, *i.e.*, a term proportional to $x^\alpha (\alpha = 3, 4, \ldots)$. We restrict the discussion to the case of $\alpha = 4$, but the method is applicable to other cases as well.

The equation of interest is

$$-y''(x) + (x^2 - \epsilon x^4) y(x) = Ey. \tag{4}$$

Since we are talking of perturbative approach, there's a coefficient ϵ before the anharmonic term meaning that term is small compared to the harmonic term. Note, however, despite being small, the parameter ϵ should always satisfy $\epsilon \leq 0$. Otherwise, the system's Hamiltonian is not bounded from below, leading to instability. For a more analytical proof, please refer to [1].

Given a pair of unperturbed eigenvalue and eigenfunction, the eigenvalue E and eigenfunction y of the perturbed system are expanded in terms of ϵ:

$$E = \sum_{n=0}^{\infty} E_n \epsilon^n, \tag{5}$$

$$y = \sum_{n=0}^{\infty} y_n \epsilon^n, \tag{6}$$

with

$$E_0 = E^{\mathrm{u}}_m, \qquad y_0 = y^{\mathrm{u}}_m, \tag{7}$$

being the unperturbed solution.

It can be shown that the different orders of perturbative solution satisfy a recursion relation [2]:

$$E_n = \frac{\int_{-\infty}^{\infty} y_0(x) \left[W(x) y_{n-1}(x) - \sum_{j=1}^{n-1} E_j y_{n-j}(x) \right] \mathrm{d}x}{\int_{-\infty}^{\infty} \mathrm{d}x\, y_0^2(x)}, \quad n = 1, 2, \ldots \tag{8}$$

and

$$y_n(x) = y_0(x) \int^x \frac{\mathrm{d}t}{y_0^2(t)} \int^t \mathrm{d}s\, y_0(s) \times \left[W(s) y_{n-1}(s) - \sum_{j=1}^{n} E_j y_{n-j}(s) \right], \quad n = 1, 2, \ldots \tag{9}$$

where $W(x) = -x^4$ is the perturbation potential.

A. Update eigenfunction

We try the following *ansatz* for y_n[3]

$$y_n(x) = e^{-x^2/2} \sum_{k=0}^{4n+m} y_{nk} x^k. \tag{10}$$

The form can be justified intuitively as follows. In

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[a] Otherwise, there should be code online and there should be no need for me to implement by hand.

eq. (9), the integrand for s is

$$
\begin{aligned}
& y_0(s)\left[W(s)y_{n-1}(s)-\sum_{j=1}^{n}E_j y_{n-j}(s)\right] \\
&= e^{-s^2/2}\sum_{i=0}^{m} y_{ni}s^i\left[-s^4e^{-s^2/2}\sum_{l=0}^{4n-4+m}y_{n-1,l}s^l\right. \\
&\qquad \left. -\sum_{j=1}^{n}E_j e^{-s^2/2}\sum_{l=0}^{4n-4j+m}y_{n-j,l}s^l\right] \qquad (11) \\
&= e^{-s^2}\sum_{i=0}^{4n+2m} b_{ni}s^i = b_n(s), \qquad (12)
\end{aligned}
$$

where in the last line we collected terms in a polynomial of order $4n+2m$. However, eq. (9) is really

$$b_n(s) = \left[(y_n/y_0)'y_0^2\right]' = y_0(s)y_n''(s) - y_n(s)y_0''(s). \quad (13)$$

And the it's not hard to show that the RHS of the above equation also has a form of

$$e^{-s^2}\sum_{k=0}^{4n+2m+2} c_{nk}s^k, \quad (14)$$

and c_{nk} can be expressed in terms of known $y_{0l}(l = 0,\ldots,m)$ and unknown $y_{nk}(k = 0,\ldots,4n+m)$.

Comparing the coefficients of different powers of s would yield $(4n+2m+3)$ equations for y_{nk}. Note there're more equations than unknowns. But it turns out these relations are not linear independent and in the very least there should be the freedom of choosing integration constant. Solving this set of linear equations completes the update process of eigenfunction.

It can be further shown with induction that either all the odd or all the even terms in y_{nk} vanishes, depending on the parity of $y_0(x)$ of the unperturbed solution.

B. Calculate eigenvalue

Observe eq. (10) and eq. (8) and it's clear that both integrands for the denominator and numerator have the following form:

$$e^{-x^2}\sum_{k=0}^{d/2} e_k x^{2k}, \quad (15)$$

with d being $4n+2m$ for the numerator and $2m$ for the denominator.

The integration of each term is straightforward using Gamma function:

$$\int_{\infty}^{\infty} e^{-x^2}x^{2k}\,\mathrm{d}x = \int_0^{\infty} e^{-t}t^{k/2-1/2}\,\mathrm{d}t = \Gamma(k+1/2). \quad (16)$$

Since the argument of Gamma function is always half-integer and Gamma functions appear in both the numerator and denominator, we could factor out $\Gamma(1/2)$ and the correction would always be a rational number.

III. PÁDE APPROXIMANT

Evidently, the series (5) is divergent due to the singularity at $\epsilon = 0$. To avoid divergence, a Páde (50,50) approximant $P_{50}^{50}(\epsilon)$ is constructed. Formally, this means

$$P_{50}^{50}(\epsilon) = \frac{\sum_{n=0}^{50} A_n\epsilon^n}{\sum_{n=0}^{50} B_n\epsilon^n} \quad (17)$$

and $B_0 = 1$ by convention; the lowest $50+50+1$ Taylor expansion coefficients of P_{50}^{50}should equal to E_n. The problem again reduces to solving linear equations.

IV. NUMERICAL MEANS

To evaluate the effectiveness of our analytical approximation, two numerical approachs are used.

First one is exact diagonalization. The eigenfunction (2) of the unperturbed system is not normalized

$$\int_{-\infty}^{\infty} y_n^{\mathrm{u}}y_m^{\mathrm{u}}\,\mathrm{d}x = \sqrt{\pi}2^n n!\delta_{nm}. \quad (18)$$

To simplify the discussion, we form normalized eigenfunction

$$y_n^{\mathrm{n}} = \frac{1}{\pi^{1/4}\sqrt{2^n n!}}y_n^{\mathrm{u}} \quad (19)$$

such that

$$\int_{-\infty}^{\infty} y_n^{\mathrm{n}}y_m^{\mathrm{n}}\,\mathrm{d}x = \delta_{nm}. \quad (20)$$

It can be shown

$$
\begin{aligned}
&\int_{-\infty}^{\infty} y_n^{\mathrm{n}}x^4y_m^{\mathrm{n}}\,\mathrm{d}x \\
&= \frac{1}{4}(6n^2+6n+3)\delta_{nm} \\
&\quad + (n+3/2)\sqrt{(n+1)(n+2)}\delta_{n,m-2} \\
&\quad + (n-1/2)\sqrt{n(n-1)}\delta_{n,m+2} \\
&\quad + \frac{1}{4}\sqrt{(n+1)(n+2)(n+3)(n+4)}\delta_{n,m-4} \\
&\quad + \frac{1}{4}\sqrt{(n-3)(n-2)(n-1)n}\delta_{n,m+4}. \qquad (21)
\end{aligned}
$$

The full Hamiltonian H to diagonalize is composed of two parts $H = H_0 + \epsilon V$, with

$$H_0 = \mathrm{diag}\,\{1,3,\ldots,2N_{\mathrm{cutoff}}+1\}, \quad (22)$$

and the matrix element V_{nm} is defined in eq. (21). Choosing an approriate dimension of cutoff N_{cutoff} is a trade-off between cutoff-error and roundoff-error, but N_{cutoff} is simply fixed at 50 in this work. Since the full Hamiltonian H is real and symmetric, the very efficient and stable Jacobi method can be used to determine the eigenvalues.

Alternatively, direct integration of eq. (4) can be performed and the shooting method can determine the eigenvalues. Based on the parity of the unperturbed state, we let either $y(0) = 0$ or $y'(0) = 0$. Since region of convergence is usually wedge-shaped, the integration path is chosen to be $\theta = -30^\circ$ to the positive axis. Let $x = re^{i\theta}$ and split y into its real and imaginary parts $y(r) = f(r) + ig(r)$. Given E, we integrate using Runge-Kutta method from $r = 0$ to $r = R$ and check the sign of $f(r)$. In this work, $R = 40$ is used. Since a sign change of $f(R)$ is expected around eigenvalue, bisect search is used to boost performance. The algorithm stops when the bisect window is less than 5×10^{-7}.

V. IMPLEMENTATION

Since all coefficients are rational numbers, GMP (GNU Multiple Precision Arithmetic Library) [4] is used to store and manipulate them. And to ensure best performance, Eigen [5] library is used to solve linear equations and calculate eigenvalues. A simple polynomial class, featuring term by term add, multiplication by polynomial, and evaluation, is also implemented by hand. These programs are implemented with the C++ programming language [6].

On average, our method is capable of calculating the coefficients E_n in 168 s (compilation) + 132 s (run) given an unperturbed solution. The long compilation time is due to both Eigen and our hand-written polynomial library relies heavily on template. Although more efficient method based on direct recursion exists, our method is not formidably slow, so no attempt is made to optimize the code[7]. The construction of Páde approximant from E_n and calculating perturbed energy level at 100 different ϵ takes 18 s (compilation) + 3 s (run) on average.

A separate program written in Mathematica performing symbolic integrations according to eq. (8),(9) is implemented to check the correctness of low order. Due to slow symbolic calculations, this method applies to n up to 13; and at $n = 13$, each update of E_n and y_n takes around 30 s.

VI. RESULT

A. Graph

In Fig. 1, the lowest 10 eigenvalues with $\epsilon = i/10, \quad i = 0, \ldots, 9$ are plotted. The overlapping result from the three methods confirms the correctness of the Implementation.

In Fig. 2, the lowest 10 eigenvalues in $\epsilon \in [-10, 0]$ are plotted. Integration method fails outside $\epsilon > 1$ because the wavefunction is too big to be represented as double-precision floating-point number. Despite the series in eq. (5) is inherently divergent[8], Páde approximant converges well. Both solutions capture the trend that the

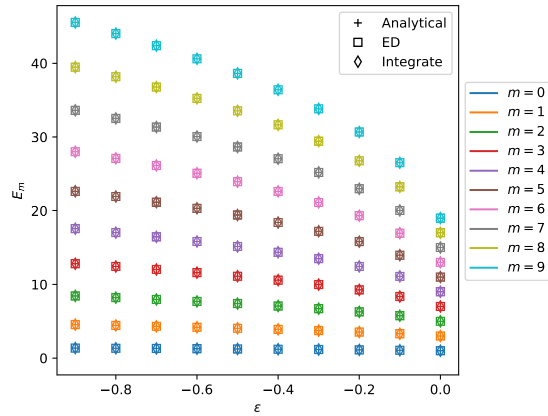

FIG. 1. Eigenvalues of eq. (4) from analytical(plus symbol) and numerical(square and diamond) means around $\epsilon = 0$. The results from different methods matches extremely well.

eigenvalue increases as ϵ decreases. But it seems the numerical solution tends to a straight line at large $|\epsilon|$, while the slope of the analytical solution tends to 0. The numerical solution differs visibly from analytical solution below $\epsilon = 3$.

To see the error between the two more clearly, in Fig. 3, we plot the relative error between the two methods against ϵ. Log scale is used to account for the order-of-magnitude difference. Note $E^{\text{numerical}} > E^{\text{analytical}}$ holds for all $\epsilon < 0$ and the relative error is monotonically increasing. Also note there's seemingly crossing at, say, $\epsilon = -3$ between $n = 9$ and $n = 8$ where the slope of error is at minimum.

B. Table

In this section we present the values of $E_n (n = 1, \ldots, 100)$. For each unperturbed energy level, we present exactly solution up to first 20 orders and the approximate solution of all 100 orders. Since $-E_n 2^{3n}$ is a positive integer, we present our result in $-E_n 2^{3n}$ instead of E_n.

In reference [1], the coefficients of ground state correction are given up to $n = 75$. Note the their definition of Hamiltonian differs from ours, and it can be shown that $(-2)^{n-1} E_n$ equals to their A_n. The result from this work match exactly up to the precision of this reference.

VII. CONCLUSION

We've completed a high-order perturbative analysis to the x^4 anharmonic oscillator. The convergent behavior is achieved with analytical method at even very large perturbation strength ϵ where numerical method fails to function properly. We would like to comment that the method described is not restricted to quartic potential only, but applicable to other integer exponent $\alpha > 2$ potential as well.

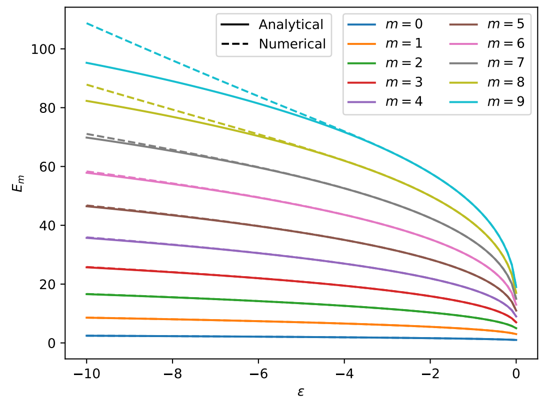

FIG. 2. Eigenvalues of eq. (4) from analytical(solid) and exact diagonalization(dashed). Integration method fails due to convergence issue.

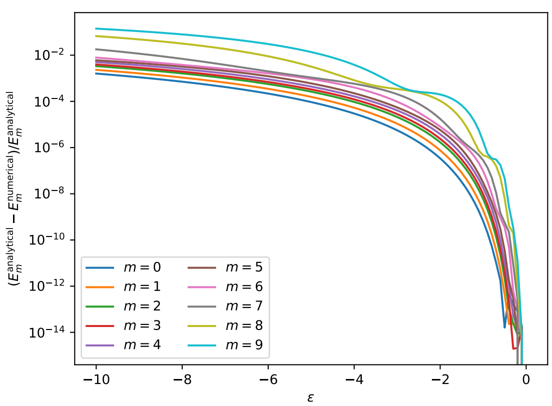

FIG. 3. Relative error between analytical and numerical methods(exact diagonalization only).

[1] C. M. Bender and T. T. Wu, Anharmonic oscillator, Phys. Rev. **184**, 1231 (1969).

[2] C. Bender, S. Orszag, and S. Orszag, *Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory*, Advanced Mathematical Methods for Scientists and Engineers (Springer, 1999).

[3] In fact, it is shown in [1] that the sum for k could start from 2 instead of 0 if $n \geq 1$.

[4] T. Granlund and the GMP development team, *GNU MP: The GNU Multiple Precision Arithmetic Library*, 6th ed. (2020), `http://gmplib.org/`.

[5] G. Guennebaud, B. Jacob, *et al.*, Eigen v3, http://eigen.tuxfamily.org (2010).

[6] Code available at `https://github.com/putian9935/anharmonic-osc`.

[7] We didn't even turn on compiler optimization flag, e.g. `-O2` or `-Ofast`.

[8] It can be shown that A_n scales super-exponentially. Thus the series is divergent.

TABLE I. Exact coeffcient of E_n up to $n = 20$ for E_0^{u}

n	$-E_n 2^{3n}$
1	6
2	84
3	2664
4	123540
5	7333848
6	524147208
7	43572714768
8	4121980396212
9	437015860091640
10	51336059452724760
11	6621407769290178672
12	930758905386276349320
13	141686496049367635656048
14	23229387977224612690619280
15	4081973849686832002155521568
16	765543711410243711551312338420
17	152644886266565584391713812812088
18	32249559868746478308856261490338680
19	7197129672652253378568475159863338160
20	1691928460676486363401186477311314757720

TABLE II. Exact coeffcient of E_n up to $n = 20$ for E_1^{u}

n	$-E_n 2^{3n}$
1	30
2	660
3	31320
4	2081940
5	170435880
6	16215569160
7	1737651694320
8	205795688655540
9	26607913470652680
10	3723933252458267160
11	560699252370983391120
12	90394751875598518863240
13	15544914905835692552987280
14	2842290048673569505197263760
15	551003198488474493684196465120
16	112961391092496352173257140772340
17	24432204681317955463241675314900680
18	5562664656497172471982192662546819960
19	1330383013238816192087544072071900650320
20	333565102755481893703894918020235785930840

TABLE III. Exact coeffcient of E_n up to $n = 20$ for E_2^{u}

n	$-E_n 2^{3n}$
1	78
2	2460
3	160632
4	14305020
5	1535988744
6	188110215960
7	25495041543984
8	3754912433232540
9	593920919123280360
10	100088070464024930760
11	17870697976178395841616
12	3367187369124912927538200
13	667538193130776896234040144
14	138928611649281749972298448560
15	30300051071205181574277032253024
16	6915045540402221464641419799181020
17	1649283304963157585066069253330007464
18	410622272668183981562228778917860994280
19	106601227035161318024640742366601345479440
20	28826540363655691102327483978432629561219720

TABLE IV. Exact coeffcient of E_n up to $n = 20$ for E_3^{u}

n	$-E_n 2^{3n}$
1	150
2	6300
3	534600
4	60738300
5	8199819000
6	1247186367000
7	207635609250000
8	37179301503865500
9	7078906156588479000
10	1422022835047769325000
11	299732457805473100590000
12	66028224244090279453695000
13	15157989023243063520623310000
14	3618665345116689592688271630000
15	896948463395568349991868869700000
16	230562265546554690170858715723937500
17	61407665988905530461321136028463975000
18	16934299026103556311924076828956574625000
19	4832527074531557878069819538893268037750000
20	1426359462251378421340254159165455176897125000

TABLE V. Exact coeffcient of E_n up to $n = 20$ for E_4^{u}

n	$-E_n 2^{3n}$
1	246
2	12996
3	1369224
4	190980900
5	31348488888
6	5748774789672
7	1145277005586768
8	243704776908936708
9	54780845374124216280
10	12909166056728713392120
11	3171880841400323898134832
12	809406810491980230872556840
13	213883347293261559096605975088
14	58398225891609806982343388149200
15	16448551333930388750983395578720928
16	4773480671346660668911587492182081220
17	1426049185149588560529569069925225467928
18	438271289668099818802617355671857476165080
19	138502320945473198399637508287150474609963120
20	44991533418599438515152390972648750195089860920

TABLE VI. Exact coeffcient of E_n up to $n = 20$ for E_5^{u}

n	$-E_n 2^{3n}$
1	366
2	23364
3	2952504
4	490546980
5	95310683208
6	20570476322088
7	4797816243861168
8	1189443292455792132
9	310066182169392911400
10	84361702124758951605240
11	23828859146200940320546512
12	6960258294018023836347434280
13	2096209764973891029257110765008
14	649475884761654289683004487505360
15	206667567156572264205032912139079008
16	67452974101130232740910983180011749060
17	22559283605128866560805433773687584796648
18	7725549318398891749701093949956045372628440
19	2707596177960269327080542443377371057342042000
20	970795647566554645481254635712205043163353704760

TABLE VII. Exact coeffcient of E_n up to $n = 20$ for E_6^{u}

n	$-E_n 2^{3n}$
1	510
2	38220
3	5644440
4	1091127180
5	245625725160
6	61178256043320
7	16406098735555440
8	4660093776312281580
9	1387230513223589762760
10	429627345056803841886120
11	137704404716964210720468240
12	45502311953568087819486001080
13	15455286479818544332043410540560
14	5383973555502487536194054696110320
15	1920240325609227756315958323778863840
16	700241809716170442189848657832521355180
17	260810480281117796968229874840754997863560
18	99137146631639354998283926976013719243182920
19	38434121670962451927063055607350960046370924240
20	15190458307547315488739642298798206580805045163880

TABLE VIII. Exact coeffcient of E_n up to $n = 20$ for E_7^{u}

n	$-E_n 2^{3n}$
1	678
2	58380
3	9877032
4	2180850060
5	559129221144
6	158165907341880
7	48044012151997584
8	15418307436741808620
9	5172876008714722280760
10	1801297375941218698298280
11	647659150449728639148581616
12	239524358482197491691323676600
13	90851388119905954404768755292144
14	35262773832368118340351248495421680
15	13981101530066973109178900637361192224
16	5654641079908380097496232729047507718060
17	2330409375544560269525302040720326823910264
18	977795574003170289875384379125260859423232840
19	417406410914987753209614501324937077305410574640
20	181119193366778086327568627320087408452321190055716O

TABLE IX. Exact coeffcient of E_n up to $n = 20$ for E_8^{u}

n	$-E_n 2^{3n}$
1	870
2	84660
3	16154280
4	4014543540
5	1156055834520
6	366573079193160
7	124570926823470480
8	44639373314584563540
9	16692295906333920321720
10	6466841110598481273702360
11	2582341143421740531173264880
12	1058832810601068101027028605640
13	444506171440957969399800993108720
14	190630208426068724741346000137682960
15	83368839445052238682099337091426504480
16	37128333972989641369552948304342114024340
17	16819382635460142250188719438936923135862520
18	7743317794767935207526522226320115411624251960
19	3620294854568037645847016611372652073768138719280
20	1717955671666603638721472801287605386511053052911640

TABLE X. Exact coeffcient of E_n up to $n = 20$ for E_9^{u}

n	$-E_n 2^{3n}$
1	1086
2	117876
3	25052184
4	6923996340
5	2214159188328
6	778487059956552
7	292907088729453168
8	116045384727244748628
9	47908214409056898325320
10	20463046780979486425277400
11	8996774820508963792598974992
12	4056161451233594608921583680200
13	1869844692748383712711195235962128
14	879402078031756466350628337057112080
15	421206825945193285862265960689062949088
16	205171692865293620863594778860193845459860
17	101522039738075310270837092027362611511548168
18	50982857841264211072697963345552165044153782840
19	25964744959336851103768116305856620858052392828880
20	13402209587883193195141124719169628506488864107995160

TABLE XI. The value of the $-E_n$ with $n = 1, \ldots, 100$ for E_0^{u}. The number following the comma is the power of 10 multipling the decimal.

n	$-E_n$		n	$-E_n$	
1	0.7500000000000000,	0	51	0.1774549491325011,	75
2	0.1312500000000000,	1	52	0.1371585100985822,	77
3	0.5203125000000000,	1	53	0.1080687553363126,	79
4	0.3016113281250000,	2	54	0.8676872885291642,	80
5	0.2238112792968750,	3	55	0.7096770461319298,	82
6	0.1999462921142578,	4	56	0.5910809250127134,	84
7	0.2077708948516846,	5	57	0.5011655452280950,	86
8	0.2456891772873401,	6	58	0.4324420071229820,	88
9	0.3256021887746751,	7	59	0.3796259964268197,	90
10	0.4781043106012490,	8	60	0.3389524990800757,	92
11	0.7708333164092827,	9	61	0.3077189141185188,	94
12	0.1354432468922862,	11	62	0.2839773024865141,	96
13	0.2577262349393415,	12	63	0.2663254045275103,	98
14	0.5281751322678386,	13	64	0.2537640850228387,	100
15	0.1160166746583068,	15	65	0.2456002702014563,	102
16	0.2719757615246876,	16	66	0.2413817719864760,	104
17	0.6778794692977178,	17	67	0.2408551986715980,	106
18	0.1790210195015489,	19	68	0.2439413849911183,	108
19	0.4994011921119655,	20	69	0.2507250426950703,	110
20	0.1467514010204402,	22	70	0.2614570302788745,	112
21	0.4531136296684818,	23	71	0.2765690405221834,	114
22	0.1466652370037318,	25	72	0.2967018143757369,	116
23	0.4966283069462674,	26	73	0.3227493905153705,	118
24	0.1755839492534922,	28	74	0.3559235776783599,	120
25	0.6470221042946597,	29	75	0.3978450133890432,	122
26	0.2480994545016985,	31	76	0.4506701405312378,	124
27	0.9884377883559941,	32	77	0.5172676031383124,	126
28	0.4085842008364257,	34	78	0.6014635309853691,	128
29	0.1750075894533248,	36	79	0.7083838315872806,	130
30	0.7757967334354602,	37	80	0.8449342603473809,	132
31	0.3555183256998041,	39	81	0.1020477693952982,	135
32	0.1682432154270259,	41	82	0.1247795733435630,	137
33	0.8213752926846650,	42	83	0.1544463180441750,	139
34	0.4133016264153365,	44	84	0.1934826268889632,	141
35	0.2141561571497207,	46	85	0.2452869986345015,	143
36	0.1141747926177188,	48	86	0.3146403879335594,	145
37	0.6258125322225467,	49	87	0.4083216923637282,	147
38	0.3523944748523645,	51	88	0.5360193919765655,	149
39	0.2037126840152551,	53	89	0.7116917856121663,	151
40	0.1208147982683254,	55	90	0.9556116068446843,	153
41	0.7346130817663982,	56	91	0.1297462244223361,	156
42	0.4576889474357688,	58	92	0.1781061513835013,	158
43	0.2920146672188311,	60	93	0.2471622643123596,	160
44	0.1906874900611838,	62	94	0.3466999332174321,	162
45	0.1273780703868530,	64	95	0.4915232511284615,	164
46	0.8699690034565581,	65	96	0.7042137555250557,	166
47	0.6072125057647181,	67	97	0.1019500619817492,	169
48	0.4329178140381204,	69	98	0.1491236212338351,	171
49	0.3151420296247126,	71	99	0.2203614896688591,	173
50	0.2341312637831796,	73	100	0.3289353468000507,	175

TABLE XII. The value of the $-E_n$ with $n = 1, \ldots, 100$ for E_1^{u}. The number following the comma is the power of 10 multipling the decimal.

n	$-E_n$		n	$-E_n$	
1	0.3750000000000000,	1	51	0.1013697382720098,	78
2	0.1031250000000000,	2	52	0.7999894076405615,	79
3	0.6117187500000000,	2	53	0.6433065303303194,	81
4	0.5082861328125000,	3	54	0.5269384387338485,	83
5	0.5201290283203125,	4	55	0.4395069401405857,	85
6	0.6185748733520507,	5	56	0.3731610831033226,	87
7	0.8285768958663940,	6	57	0.3224165529874986,	89
8	0.1226637891862035,	8	58	0.2833993767834352,	91
9	0.1982444038290730,	9	59	0.2533468824945158,	93
10	0.3468183104375626,	10	60	0.2302745907707660,	95
11	0.6527398391288978,	11	61	0.2127516083919541,	97
12	0.1315416766382965,	13	62	0.1997479832504679,	99
13	0.2827603549273716,	14	63	0.1905305321305266,	101
14	0.6462619350426325,	15	64	0.1845919232030838,	103
15	0.1566045280266264,	17	65	0.1816031198513451,	105
16	0.4013194793106448,	18	66	0.1813827886907690,	107
17	0.1085007847182149,	20	67	0.1838796344801361,	109
18	0.3087899189955874,	21	68	0.1891652863317557,	111
19	0.9231386580424552,	22	69	0.1974366106060682,	113
20	0.2893216072582752,	24	70	0.2090273658972469,	115
21	0.9488325779811591,	25	71	0.2244300976158352,	117
22	0.3250307132935212,	27	72	0.2443302288960944,	119
23	0.1161079009509754,	29	73	0.2696555812966252,	121
24	0.4318387635070937,	30	74	0.3016462216578736,	123
25	0.1669803282377047,	32	75	0.3419518043131752,	125
26	0.6703372839883187,	33	76	0.3927667755460074,	127
27	0.2790250074907100,	35	77	0.4570183876125177,	129
28	0.1202779951451808,	37	78	0.5386291110609579,	131
29	0.5363224836907425,	38	79	0.6428847532700168,	133
30	0.2471129926328750,	40	80	0.7769539258517119,	135
31	0.1175317419895233,	42	81	0.9506257966400545,	137
32	0.5764879667215923,	43	82	0.1177364906865322,	140
33	0.2913463211724646,	45	83	0.1475829778213682,	142
34	0.1515804839338474,	47	84	0.1872074696313200,	144
35	0.8112255426321887,	48	85	0.2402764911690694,	146
36	0.4462446620087926,	50	86	0.3119905748318578,	148
37	0.2521291717902646,	52	87	0.4097849326392476,	150
38	0.1462154177897528,	54	88	0.5443752293033641,	152
39	0.8697597331760013,	55	89	0.7313299926211286,	154
40	0.5303605150557383,	57	90	0.9934524761765235,	156
41	0.3313227439172527,	59	91	0.1364415656868624,	159
42	0.2119306535252607,	61	92	0.1894351315007845,	161
43	0.1387280007164099,	63	93	0.2658508397566874,	163
44	0.9288338152069379,	64	94	0.3770767496381037,	165
45	0.6357712219917682,	66	95	0.5404894714408601,	167
46	0.4446795734592901,	68	96	0.7828220531116719,	169
47	0.3176727818007192,	70	97	0.1145541252717334,	172
48	0.2316916636191567,	72	98	0.1693498304962881,	174
49	0.1724475377449753,	74	99	0.2528951783720768,	176
50	0.1309319077303230,	76	100	0.3814471460671227,	178

TABLE XIII. The value of the $-E_n$ with $n = 1, \ldots, 100$ for E_2^{u}. The number following the comma is the power of 10 multipling the decimal.

n	$-E_n$	n	$-E_n$
1	0.9750000000000000, 1	51	0.2782417452086104, 80
2	0.3843750000000000, 2	52	0.2243926705423097, 82
3	0.3137343750000000, 3	53	0.1843114693729047, 84
4	0.3492436523437500, 4	54	0.1541388682732517, 86
5	0.4687465649414063, 5	55	0.1312053996888442, 88
6	0.7175835264587402, 6	56	0.1136420900504855, 90
7	0.1215698315810394, 8	57	0.1001260937954763, 92
8	0.2238102217455232, 9	58	0.8971243625245890, 93
9	0.4425055676127079, 10	59	0.8172145118842534, 95
10	0.9321427947285115, 11	60	0.7566273573987154, 97
11	0.2080423056168761, 13	61	0.7118339167728882, 99
12	0.4899902515353333, 14	62	0.6803267422051353, 101
13	0.1214244899767030, 16	63	0.6603793566636642, 103
14	0.3158870905492261, 17	64	0.6508853760363058, 105
15	0.8611792472721953, 18	65	0.6512558308339145, 107
16	0.2456717688091537, 20	66	0.6613611911664670, 109
17	0.7324289197199886, 21	67	0.6815102684192212, 111
18	0.2279411507700634, 23	68	0.7124624839985987, 113
19	0.7396946044233114, 24	69	0.7554736396534398, 115
20	0.2500303815001327, 26	70	0.8123788246981982, 117
21	0.8794008610707369, 27	71	0.8857199300560271, 119
22	0.3214991286241589, 29	72	0.9789299135594561, 121
23	0.1220445451374365, 31	73	0.1096592092557875, 124
24	0.4805658842288584, 32	74	0.1244801155029726, 126
25	0.1960829257723188, 34	75	0.1431664447811683, 128
26	0.8282162685198766, 35	76	0.1667999123823776, 130
27	0.3617767501523046, 37	77	0.1968305436488712, 132
28	0.1632747597244777, 39	78	0.2352132650901707, 134
29	0.7606444488801276, 40	79	0.2846007436274628, 136
30	0.3654654557834732, 42	80	0.3486173960339476, 138
31	0.1809447926361267, 44	81	0.4322513647450226, 140
32	0.9224217810179139, 45	82	0.5424191420600709, 142
33	0.4837940481058632, 47	83	0.6887846448300159, 144
34	0.2608660912105105, 49	84	0.8849559365752178, 146
35	0.1445086323854607, 51	85	0.1150246398115651, 149
36	0.8218557382164443, 52	86	0.1512285530951558, 151
37	0.4795596708192673, 54	87	0.2010917780289750, 153
38	0.2869240584218257, 56	88	0.2704067944020963, 155
39	0.1759187901069950, 58	89	0.3676630823963530, 157
40	0.1104671689104127, 60	90	0.5054045194155740, 159
41	0.7100602266906382, 61	91	0.7023176023000495, 161
42	0.4669530670799086, 63	92	0.9864681638970619, 163
43	0.3140160142820714, 65	93	0.1400356119163249, 166
44	0.2158361158277432, 67	94	0.2008869139361818, 168
45	0.1515624921344746, 69	95	0.2911892666701519, 170
46	0.1086836097234662, 71	96	0.4264453968049036, 172
47	0.7955324132692282, 72	97	0.6309145973514563, 174
48	0.5941512847151227, 74	98	0.9428713232052504, 176
49	0.4525973463464175, 76	99	0.1423198554624280, 179
50	0.3515114836451844, 78	100	0.2169537385186226, 181

TABLE XIV. The value of the $-E_n$ with $n = 1, \ldots, 100$ for E_3^{u}. The number following the comma is the power of 10 multipling the decimal.

n	$-E_n$		n	$-E_n$	
1	0.1875000000000000,	2	51	0.4885612004700294,	82
2	0.9843750000000000,	2	52	0.4030084440980600,	84
3	0.1044140625000000,	4	53	0.3384142354430983,	86
4	0.1482868652343750,	5	54	0.2891949848658745,	88
5	0.2502386169433594,	6	55	0.2514276926770286,	90
6	0.4757638423919678,	7	56	0.2223270326163187,	92
7	0.9900837385654450,	8	57	0.1998981085532981,	94
8	0.2216059059135050,	10	58	0.1827033365495634,	96
9	0.5274196085772276,	11	59	0.1697044807960026,	98
10	0.1324361967898690,	13	60	0.1601548929725367,	100
11	0.3489345053740231,	14	61	0.1535258301809256,	102
12	0.9608371218795988,	15	62	0.1494563622142943,	104
13	0.2757222141234354,	17	63	0.1477200378497144,	106
14	0.8227892397182334,	18	64	0.1482039070653658,	108
15	0.2549280860067621,	20	65	0.1508971573331252,	110
16	0.8191217146224783,	21	66	0.1558878160669482,	112
17	0.2727048186774962,	23	67	0.1633668973744017,	114
18	0.9400424342333766,	24	68	0.1736401722068980,	116
19	0.3353239265793366,	26	69	0.1871485310101083,	118
20	0.1237169622174560,	28	70	0.2044987933248902,	120
21	0.4718921791943949,	29	71	0.2265079157071883,	122
22	0.1859987566907764,	31	72	0.2542650026252777,	124
23	0.7572360438408876,	32	73	0.2892175310325586,	126
24	0.3182710024931293,	34	74	0.3332910376991375,	128
25	0.1380339902233111,	36	75	0.3890555981217487,	130
26	0.6173971415705142,	37	76	0.4599583548793770,	132
27	0.2846349656013281,	39	77	0.5506500450477650,	134
28	0.1351772432879351,	41	78	0.6674463142908377,	136
29	0.6609232481268932,	42	79	0.8189837001951169,	138
30	0.3324814643971936,	44	80	0.1017158760593696,	141
31	0.1719842461887097,	46	81	0.1278481923542737,	143
32	0.9142222603788523,	47	82	0.1626043035804852,	145
33	0.4991108123193866,	49	83	0.2092385481623800,	147
34	0.2796848580421176,	51	84	0.2723739343176290,	149
35	0.1607748522707433,	53	85	0.3586301830404022,	151
36	0.9475482346438696,	54	86	0.4775623703747752,	153
37	0.5722455260453116,	56	87	0.6430741656094801,	155
38	0.3539422922647578,	58	88	0.8755614673669276,	157
39	0.2240938799070066,	60	89	0.1205188230737552,	160
40	0.1451646722308135,	62	90	0.1676930024905827,	162
41	0.9616533138168430,	63	91	0.2358397679412801,	164
42	0.6511832208865065,	65	92	0.3352065021320961,	166
43	0.4505285776740869,	67	93	0.4814524364444069,	168
44	0.3183391939767087,	69	94	0.6987038535366945,	170
45	0.2296292297680037,	71	95	0.1024438434023326,	173
46	0.1690286948136289,	73	96	0.1517353116560356,	175
47	0.1269178921453709,	75	97	0.2270134125956212,	177
48	0.9717467490483426,	76	98	0.3430340908766180,	179
49	0.7583948512077047,	78	99	0.5234818706418431,	181
50	0.6031136573533145,	80	100	0.8066833816806185,	183

TABLE XV. The value of the $-E_n$ with $n = 1, \ldots, 100$ for E_4^{u}. The number following the comma is the power of 10 multipling the decimal.

n	$-E_n$		n	$-E_n$	
1	0.3075000000000000,	2	51	0.6163890296662139,	84
2	0.2030625000000000,	3	52	0.5205828964253396,	86
3	0.2674265625000000,	4	53	0.4473282611276090,	88
4	0.4662619628906250,	5	54	0.3909700641847076,	90
5	0.9566799587402344,	6	55	0.3474754652108244,	92
6	0.2192983547085571,	8	56	0.3139455596562416,	94
7	0.5461106326993790,	9	57	0.2882859171524893,	96
8	0.1452593665772299,	11	58	0.2689823818628982,	98
9	0.4081491036275343,	12	59	0.2549476536179022,	100
10	0.1202259776809133,	14	60	0.2454162609110759,	102
11	0.3692555289483075,	15	61	0.2398733138169726,	104
12	0.1177841929154209,	17	62	0.2380075146027955,	106
13	0.3890515423213611,	18	63	0.2396823138132173,	108
14	0.1327821926033944,	20	64	0.2449214656602231,	110
15	0.4674959465640538,	21	65	0.2539069713797716,	112
16	0.1695881007658308,	23	66	0.2669887836591426,	114
17	0.6332930558404064,	24	67	0.2847068861755240,	116
18	0.2432894384108430,	26	68	0.3078276320979706,	118
19	0.9610529104855464,	27	69	0.3373976922131465,	120
20	0.3902393462071975,	29	70	0.3748208185688765,	122
21	0.1628430471253280,	31	71	0.4219651182781612,	124
22	0.6982043824343618,	32	72	0.4813119919841363,	126
23	0.3075431760213506,	34	73	0.5561628070675839,	128
24	0.1391487802899673,	36	74	0.6509264589911897,	130
25	0.6466052892687969,	37	75	0.7715212862050487,	132
26	0.3085418281762928,	39	76	0.9259399413703431,	134
27	0.1511541839149337,	41	77	0.1125048205688947,	137
28	0.7600810934681406,	42	78	0.1383722061648816,	139
29	0.3922119933709182,	44	79	0.1722477296496474,	141
30	0.2076233605082946,	46	80	0.2169821287079190,	143
31	0.1127161399479063,	48	81	0.2765671104682771,	145
32	0.6273347955726312,	49	82	0.3566357120502071,	147
33	0.3578125893995756,	51	83	0.4652000697000462,	149
34	0.2090678897460641,	53	84	0.6137471959285830,	151
35	0.1250898401633373,	55	85	0.8188784708482159,	153
36	0.7660976563471842,	56	86	0.1104780791939786,	156
37	0.4800612110758221,	58	87	0.1506978960711500,	158
38	0.3076681030653535,	60	88	0.2078076161666302,	160
39	0.2015885430385612,	62	89	0.2896601701140670,	162
40	0.1349810924305350,	64	90	0.4080750475919190,	164
41	0.9232785031356141,	65	91	0.5809879549137958,	166
42	0.6448778478380773,	67	92	0.8358395456539431,	168
43	0.4597725644878483,	69	93	0.1214957851350285,	171
44	0.3344793519294535,	71	94	0.1784172131980461,	173
45	0.2481990399771389,	73	95	0.2646702682904439,	175
46	0.1877947242762296,	75	96	0.3965727965713836,	177
47	0.1448342619777139,	77	97	0.6001328088419278,	179
48	0.1138204024656286,	79	98	0.9171421587277599,	181
49	0.9111505158330104,	80	99	0.1415303974744693,	184
50	0.7427551103312338,	82	100	0.2205191421700687,	186

TABLE XVI. The value of the $-E_n$ with $n = 1, \ldots, 100$ for E_5^{u}. The number following the comma is the power of 10 multipling the decimal.

n	$-E_n$	n	$-E_n$
1	0.4575000000000000, 2	51	0.5949903137129013, 86
2	0.3650625000000000, 3	52	0.5150543709294194, 88
3	0.5766609375000000, 4	53	0.4533536030668839, 90
4	0.1197624462890625, 6	54	0.4056521609313773, 92
5	0.2908651220947266, 7	55	0.3688906874652121, 94
6	0.7847013977847290, 8	56	0.3408519051758309, 96
7	0.2287777063303550, 10	57	0.3199317458118899, 98
8	0.7089634492729855, 11	58	0.3049821345279095, 100
9	0.2310173080633528, 13	59	0.2952032842876264, 102
10	0.7856795762177460, 14	60	0.2900709899981089, 104
11	0.2774044306273681, 16	61	0.2892894689060787, 106
12	0.1012850886620876, 18	62	0.2927637345056157, 108
13	0.3812983350096858, 19	63	0.3005879304519489, 110
14	0.1476737190300024, 21	64	0.3130478959567594, 112
15	0.5873845542412603, 22	65	0.3306377704238689, 114
16	0.2396411037648613, 24	66	0.3540918928259432, 116
17	0.1001833443098515, 26	67	0.3844348052986635, 118
18	0.4288541365581818, 27	68	0.4230540328251561, 120
19	0.1878772261349102, 29	69	0.4718027235795924, 122
20	0.8420310001049046, 30	70	0.5331425170104178, 124
21	0.3859793789182782, 32	71	0.6103416068235188, 126
22	0.1809323690505331, 34	72	0.7077495322359895, 128
23	0.8672601782915448, 35	73	0.8311797207120296, 130
24	0.4250596062893444, 37	74	0.9884446497235448, 132
25	0.2130184877856855, 39	75	0.1190108851207628, 135
26	0.1091585936607595, 41	76	0.1450555127016756, 137
27	0.5719766897836153, 42	77	0.1789504287252610, 139
28	0.3064648239397741, 44	78	0.2234196177066477, 141
29	0.1679030921924805, 46	79	0.2822541664058013, 143
30	0.9405790071019297, 47	80	0.3607710231271199, 145
31	0.5387165032541003, 49	81	0.4664855053586822, 147
32	0.3154349474319097, 51	82	0.6101043006331455, 149
33	0.1887931017805696, 53	83	0.8070024156309697, 151
34	0.1154830323430693, 55	84	0.1079436092238762, 154
35	0.7218096555603444, 56	85	0.1459882951328496, 156
36	0.4608987114759870, 58	86	0.1996121028100609, 158
37	0.3005821358908226, 60	87	0.2759009607137340, 160
38	0.2001633853064307, 62	88	0.3854498346078284, 162
39	0.1360670047142689, 64	89	0.5442301651905933, 164
40	0.9439438057156202, 65	90	0.7765156269280664, 166
41	0.6680975245343814, 67	91	0.1119500564970991, 169
42	0.4822889626142365, 69	92	0.1630645481917304, 171
43	0.3549939312506358, 71	93	0.2399448186618568, 173
44	0.2663501848933499, 73	94	0.3566449856640564, 175
45	0.2036462741583659, 75	95	0.5354155187071251, 177
46	0.1586227213128151, 77	96	0.8117715941129232, 179
47	0.1258332524233368, 79	97	0.1242863901825221, 182
48	0.1016351826172831, 81	98	0.1921408294717308, 184
49	0.8355856995061861, 82	99	0.2999040923239406, 186
50	0.6990659553162657, 84	100	0.4725772114952314, 188

TABLE XVII. The value of the $-E_n$ with $n = 1, \ldots, 100$ for E_6^{u}. The number following the comma is the power of 10 multipling the decimal.

n	$-E_n$		n	$-E_n$	
1	0.6375000000000000,	2	51	0.4568773167774281,	88
2	0.5971875000000000,	3	52	0.4058480280997044,	90
3	0.1102429687500000,	5	53	0.3663369447696654,	92
4	0.2663884716796875,	6	54	0.3359372872406965,	94
5	0.7495902257080078,	7	55	0.3128988957740627,	96
6	0.2333765260441589,	9	56	0.2959564057316568,	98
7	0.7823037498262139,	10	57	0.2842107187889343,	100
8	0.2777632341571022,	12	58	0.2770478914035133,	102
9	0.1033567274528436,	14	59	0.2740849918827631,	104
10	0.4001216451235151,	15	60	0.2751361440975752,	106
11	0.1603090259211187,	17	61	0.2801945334204003,	108
12	0.6621457862430264,	18	62	0.2894280278283636,	110
13	0.2811300233555548,	20	63	0.3031875582244702,	112
14	0.1224173855803767,	22	64	0.3220287237215521,	114
15	0.5457651257100985,	23	65	0.3467484194847602,	116
16	0.2487758655846656,	25	66	0.3784397996724606,	118
17	0.1158231201086569,	27	67	0.4185707800567203,	120
18	0.5503217139304315,	28	68	0.4690937992900739,	122
19	0.2666902925646700,	30	69	0.5325980291730832,	124
20	0.1317562231847462,	32	70	0.6125201298321929,	126
21	0.6633910695713141,	33	71	0.7134366828886839,	128
22	0.3403381267855611,	35	72	0.8414716392949145,	130
23	0.1778853310085093,	37	73	0.1004867039649360,	133
24	0.9471850572326485,	38	74	0.1214787247171807,	135
25	0.5138013091659210,	40	75	0.1486459543198272,	137
26	0.2839484947921437,	42	76	0.1840802633620146,	139
27	0.1598816405598488,	44	77	0.2306767820057316,	141
28	0.9172976703971315,	45	78	0.2924728358062778,	143
29	0.5363095343518711,	47	79	0.3751421234489202,	145
30	0.3195611934325641,	49	80	0.4867204254156147,	147
31	0.1940712024665472,	51	81	0.6386790522181790,	149
32	0.1201336494738406,	53	82	0.8475242614712964,	151
33	0.7580245450301205,	54	83	0.1137197877044484,	154
34	0.4875559056225766,	56	84	0.1542707085981497,	156
35	0.3196566613683530,	58	85	0.2115653525031628,	158
36	0.2136213513028494,	60	86	0.2932718225313381,	160
37	0.1455051551744951,	62	87	0.4108779986877816,	162
38	0.1010054079152379,	64	88	0.5817349482635982,	164
39	0.7144838221827710,	65	89	0.8322638105298241,	166
40	0.5149452845492426,	67	90	0.1203026514128130,	169
41	0.3780782260658095,	69	91	0.1756804049813636,	171
42	0.2827339619095333,	71	92	0.2591563307335460,	173
43	0.2153118622155866,	73	93	0.3861428026211881,	175
44	0.1669416933191457,	75	94	0.5810850892211912,	177
45	0.1317583433359109,	77	95	0.8830717598022435,	179
46	0.1058314620713422,	79	96	0.1355115470829480,	182
47	0.8649278960532462,	80	97	0.2099621198718526,	184
48	0.7190803560053887,	82	98	0.3284361426922829,	186
49	0.6080101521931790,	84	99	0.5186423675161670,	188
50	0.5227374268359755,	86	100	0.8267124484291338,	190

TABLE XVIII. The value of the $-E_n$ with $n = 1, \ldots, 100$ for E_7^{u}. The number following the comma is the power of 10 multipling the decimal.

n	$-E_n$	n	$-E_n$
1	0.8474999999999999, 2	51	0.2864576956403406, 90
2	0.9121874999999999, 3	52	0.2614641871216309, 92
3	0.1929107812500000, 5	53	0.2423246400873491, 94
4	0.5324340966796874, 6	54	0.2280033595909620, 96
5	0.1706326968823242, 8	55	0.2177547522946587, 98
6	0.6033550542521667, 9	56	0.2110577211404799, 100
7	0.2290917022323493, 11	57	0.2075710786909981, 102
8	0.9190027378047591, 12	58	0.2071045913077593, 104
9	0.3854092962082268, 14	59	0.2096022085035395, 106
10	0.1677588909809681, 16	60	0.2151354456780130, 108
11	0.7539744843376435, 17	61	0.2239059894867257, 110
12	0.3485538159761891, 19	62	0.2362575403383597, 112
13	0.1652577122875409, 21	63	0.2526978225100913, 114
14	0.8017826492588964, 22	64	0.2739326962375532, 116
15	0.3973668052045404, 24	65	0.3009155240041969, 118
16	0.2008932071329774, 26	66	0.3349165295142665, 120
17	0.1034909658212762, 28	67	0.3776190486735576, 122
18	0.5427855798174454, 29	68	0.4312525973284164, 124
19	0.2896338800149804, 31	69	0.4987769906065196, 126
20	0.1571589504955659, 33	70	0.5841379626377103, 128
21	0.8668158443167112, 34	71	0.6926237763100751, 130
22	0.4858348021009420, 36	72	0.8313655685678594, 132
23	0.2766559751780892, 38	73	0.1010043749858392, 135
24	0.1600405535855561, 40	74	0.1241891871912073, 137
25	0.9404494313735363, 41	75	0.1545132914178496, 139
26	0.5613791718166484, 43	76	0.1945048509737747, 141
27	0.3404171633237804, 45	77	0.2476981035119704, 143
28	0.2097174528624448, 47	78	0.3190720172482583, 145
29	0.1312714792118446, 49	79	0.4156958541019060, 147
30	0.8349703805915372, 50	80	0.5476861241227151, 149
31	0.5397504913741538, 52	81	0.7296354882997953, 151
32	0.3546450009157600, 54	82	0.9827620292982525, 153
33	0.2368805207257806, 56	83	0.1338165911268127, 156
34	0.1608612217740305, 58	84	0.1841800569550901, 158
35	0.1110722312995723, 60	85	0.2562117525672430, 160
36	0.7798830531197261, 61	86	0.3601910503678910, 162
37	0.5568675336700459, 63	87	0.5116802842985579, 164
38	0.4043815157827464, 65	88	0.7344320819312913, 166
39	0.2986453431319782, 67	89	0.1064995789086632, 169
40	0.2243069618967584, 69	90	0.1560070875732223, 171
41	0.1713323927076954, 71	91	0.2308333850685106, 173
42	0.1330832084438063, 73	92	0.3449603516834628, 175
43	0.1051144242910826, 75	93	0.5206131913463705, 177
44	0.8441441035534458, 76	94	0.7934077747406583, 179
45	0.6891895997855279, 78	95	0.1220881768523695, 182
46	0.5719722580299542, 80	96	0.1896740418209394, 184
47	0.4824654662889382, 82	97	0.2974823844496829, 186
48	0.4135699235496021, 84	98	0.4709732693228118, 188
49	0.3602101357368651, 86	99	0.7526206640641758, 190
50	0.3187257591313054, 88	100	0.1213849211210191, 193

TABLE XIX. The value of the $-E_n$ with $n = 1, \ldots, 100$ for E_8^{u}. The number following the comma is the power of 10 multipling the decimal.

n	$-E_n$		n	$-E_n$	
1	0.1087500000000000,	3	51	0.1493602943342593,	92
2	0.1322812500000000,	4	52	0.1402848579118311,	94
3	0.3155132812500000,	5	53	0.1336782351683650,	96
4	0.9801131689453125,	6	54	0.1292204241552506,	98
5	0.3528002424682617,	8	55	0.1266969827142530,	100
6	0.1398365322849884,	10	56	0.1259819834087036,	102
7	0.5940004674123310,	11	57	0.1270278110698450,	104
8	0.2660713989411864,	13	58	0.1298606705680488,	106
9	0.1243672959978425,	15	59	0.1345812599088528,	108
10	0.6022715112751797,	16	60	0.1413705707262591,	110
11	0.3006240752783768,	18	61	0.1505012687891305,	112
12	0.1540804530088016,	20	62	0.1623556412505664,	114
13	0.8085520156618402,	21	63	0.1774517398964715,	116
14	0.4334429114034464,	23	64	0.1964801797830896,	118
15	0.2369484930257279,	25	65	0.2203551756675389,	120
16	0.1319063399769136,	27	66	0.2502849618428871,	122
17	0.7469306344747358,	28	67	0.2878689567436593,	124
18	0.4298404851370526,	30	68	0.3352322135789543,	126
19	0.2512084189670886,	32	69	0.3952123048386201,	128
20	0.1490089017163780,	34	70	0.4716205144736538,	130
21	0.8966976597147372,	35	71	0.5696091017151410,	132
22	0.5472494482587509,	37	72	0.6961910414294646,	134
23	0.3386243432997396,	39	73	0.8609804621318391,	136
24	0.2124054335616395,	41	74	0.1077254722562530,	139
25	0.1350445833494651,	43	75	0.1363488459889685,	141
26	0.8702155722485635,	44	76	0.1745584989036700,	143
27	0.5683420456847837,	46	77	0.2260145205477251,	145
28	0.3762175538015916,	48	78	0.2959290828911737,	147
29	0.2524310641116154,	50	79	0.3917832621305500,	149
30	0.1716965141516950,	52	80	0.5244001318990277,	151
31	0.1183982450651360,	54	81	0.7095629736461352,	153
32	0.8278497411302496,	55	82	0.9704734857079413,	155
33	0.5870057523357540,	57	83	0.1341513638708255,	158
34	0.4221651274430417,	59	84	0.1874045251737290,	160
35	0.3079898095692558,	61	85	0.2645417634192801,	162
36	0.2279645938457261,	63	86	0.3773062256530930,	164
37	0.1712123101854065,	65	87	0.5436712264083174,	166
38	0.1304948529140530,	67	88	0.7913692779452063,	168
39	0.1009462787486133,	69	89	0.1163539138578061,	171
40	0.7926215977618217,	70	90	0.1727829885823601,	173
41	0.6317596487979069,	72	91	0.2591188584068681,	175
42	0.5111767937035336,	74	92	0.3924060196262301,	177
43	0.4198918451515982,	76	93	0.6000287996722207,	179
44	0.3501511788598771,	78	94	0.9263394535820547,	181
45	0.2964296299580445,	80	95	0.1443751072038404,	184
46	0.2547552946441054,	82	96	0.2271441972538752,	186
47	0.2222493104691814,	84	97	0.3607130208737314,	188
48	0.1968095661390483,	86	98	0.5781454417161856,	190
49	0.1768916211517597,	88	99	0.9351738409390382,	192
50	0.1613558723509342,	90	100	0.1526486481149450,	195

TABLE XX. The value of the $-E_n$ with $n = 1, \ldots, 100$ for E_9^{u}. The number following the comma is the power of 10 multipling the decimal.

n	$-E_n$	n	$-E_n$
1	0.1357500000000000, 3	51	0.6561748446493809, 93
2	0.1841812500000000, 4	52	0.6352337867989922, 95
3	0.4893004687500001, 5	53	0.6233287000878622, 97
4	0.1690428793945312, 7	54	0.6199283943382454, 99
5	0.6757077601098633, 8	55	0.6248474476077912, 101
6	0.2969692458940704, 10	56	0.6382313884154033, 103
7	0.1396689838073030, 12	57	0.6605635306566455, 105
8	0.6916843934490964, 13	58	0.6926938847010161, 107
9	0.3569440127093859, 15	59	0.7358925554046821, 109
10	0.1905769741253880, 17	60	0.7919322824353243, 111
11	0.1047362436134015, 19	61	0.8632075404761096, 113
12	0.5902491759092073, 20	62	0.9529012263671206, 115
13	0.3401227682385768, 22	63	0.1065214863905161, 118
14	0.1999528826744964, 24	64	0.1205685084963463, 120
15	0.1197141801711704, 26	65	0.1381618807698945, 122
16	0.7289162797451838, 27	66	0.1602693362496664, 124
17	0.4508484240964853, 29	67	0.1881787788922953, 126
18	0.2830117131828137, 31	68	0.2236140590804166, 128
19	0.1801666105148484, 33	69	0.2688971824395599, 130
20	0.1162456380103121, 35	70	0.3271770214111639, 132
21	0.7597992208516458, 36	71	0.4027539255226754, 134
22	0.5028878477189503, 38	72	0.5015435622268172, 136
23	0.3369461470409486, 40	73	0.6317442780006320, 138
24	0.2284890771078816, 42	74	0.8048039911649113, 140
25	0.1567877095401951, 44	75	0.1036830954165135, 143
26	0.1088554808875163, 46	76	0.1350666826632063, 145
27	0.7646303397663590, 47	77	0.1778954878002399, 147
28	0.5433823195470802, 49	78	0.2368713844663874, 149
29	0.3906774398998381, 51	79	0.3188205899542757, 151
30	0.2841909876762504, 53	80	0.4337324781345921, 153
31	0.2091769834871307, 55	81	0.5963423708531865, 155
32	0.1558015976248495, 57	82	0.8285609422100905, 157
33	0.1174452834502065, 59	83	0.1163230658697405, 160
34	0.8961130650889597, 60	84	0.1649977222148192, 162
35	0.6921757076557509, 62	85	0.2364392199638066, 164
36	0.5413300026115089, 64	86	0.3422550634693977, 166
37	0.4287136988507646, 66	87	0.5004134476548942, 168
38	0.3438739433319517, 68	88	0.7389538402038722, 170
39	0.2793979594069452, 70	89	0.1101985819230065, 173
40	0.2299860674164497, 72	90	0.1659459842088674, 175
41	0.1918191948231558, 74	91	0.2523200701827696, 177
42	0.1621239891358860, 76	92	0.3873414585128961, 179
43	0.1388715475063974, 78	93	0.6002839001121584, 181
44	0.1205671836007381, 80	94	0.9390827699806588, 183
45	0.1061026879418339, 82	95	0.1482859476174139, 186
46	0.9465180993569096, 83	96	0.2363252291079442, 188
47	0.8559586041722360, 85	97	0.3801015931866250, 190
48	0.7847047047468808, 87	98	0.6169285488933802, 192
49	0.7292733651573322, 89	99	0.1010374079926928, 195
50	0.6870668355008321, 91	100	0.1669585832804385, 197