Scalable ML

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Bochner’s Theorem

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Bochner’s Theorem

Scattered Data Approximation by Holger Wendland
Chapter 6: Positive Definite Functions

Positive Semidefinite Function

Definition: [Positive semidefinite function]

ϕ is a positive semidefinite function if and only if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \phi(t_i - t_j)\xi_i\bar{\xi}_j \geq 0 \qquad \forall\ t_1, \ldots t_n \in \mathbb{R},\ \forall\ \xi_1, \ldots \xi_n \in \mathbb{C}$$

Characteristic Function

Definition: [Characteristic function of a probability distribution]

We say that ϕ is the characteristic function of probability distribution μ if

$$\phi(t) = \int e^{itx}\mu(dx) = \mathbb{E}_{x\sim\mu}\left[e^{itx}\right]$$

$t, x \in \mathbb{R}$, μ is a probability distribution on $\mathbb{R}$

Definition: [Vector-valued generalization of characteristic function]

$$\phi(u) = \int e^{i\langle u,x\rangle}\mu(dx) = \mathbb{E}_{u\sim\mu}\left[e^{i\langle u,x\rangle}\right]$$

$u, x \in \mathbb{R}^d$, μ is a probability distribution on $\mathbb{R}^d$

The Most Beautiful Theorem Ever

Theorem: [Euler's equation]

$$e^{-i\pi} + 1 = 0$$

"like a Shakespearean sonnet that captures the very essence of love, or a painting that brings out the beauty of the human form that is far more than just skin deep, Euler's equation reaches down into the very depths of existence"

The 2nd Most Beautiful Theorem

Theorem: [Bochner]

Part 1

If ϕ is a **characteristic function** of a **probability distribution** on $\mathbb{R}$, then ϕ is a **positive semidefinite function**.

Part 2

If ϕ is a **positive semidefinite function**, **continuous at 0**, $\phi(0) = 1$, then ϕ is a **characteristic function** of a **probability distribution**.

Proofs

We will only prove that 1-dim case, the d-dim case is similar.

Proof of Part 1

We need to prove:

If ϕ is a characteristic function of a probability measure μ on $\mathbb{R}$, then ϕ is a positive semidefinite function, that is

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\phi(t_i - t_j)\xi_i\bar{\xi}_j \geq 0 \qquad \forall\ t_1,\dots t_n \in \mathbb{R},\ \forall\ \xi_1,\dots \xi_n \in \mathbb{C}$$

This part is easy: Since ϕ is a characteristic function $\Rightarrow \phi(t) = \int e^{itx}\mu(dx)$

Therefore,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\underbrace{\phi(t_i - t_j)}_{=\int \underbrace{e^{i(t_i-t_j)x}}_{=e^{it_ix}e^{-it_jx}}\mu(dx)}\xi_i\bar{\xi}_j = \sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\bar{\xi}_j\int e^{it_ix}e^{-it_jx}\mu(dx)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\int\left(\xi_i e^{it_ix}\right)\left(\bar{\xi}_j e^{-it_jx}\right)\mu(dx) = \int\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\xi_i e^{it_ix}\right)\left(\bar{\xi}_j e^{-it_jx}\right)\mu(dx)$$

Proof of Part 1 (Continued)

This is what we know so far:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\phi(t_i-t_j)\xi_i\bar{\xi}_j = \int \sum_{i=1}^{n}\sum_{j=1}^{n} \underbrace{\left(\xi_i e^{it_i x}\right)\left(\bar{\xi}_j e^{-it_j x}\right)}_{=\left|\sum_{i=1}^{n}\xi_i e^{it_i x}\right|^2} \mu(dx)$$

$$= \int \left|\sum_{i=1}^{n}\xi_i e^{it_i x}\right|^2 \geq 0 \mu(dx) \quad \geq 0$$

This is what we wanted to prove for Part 1. QED

Proof of Part 2

The converse (Part 2) is more challenging.

We need to prove:

If ϕ is a positive semidefinite function,
continuous at 0,
$\phi(0) = 1$

$\Rightarrow$ then ϕ is a characteristic function of a probability distribution,

that is, there exists μ probability distribution such that

$$\begin{aligned}\phi(t) &= \int e^{itx} \mu(dx) \\ &= \mathbb{E}_{x \sim \mu}\left[e^{itx}\right]\end{aligned}$$

Properties of Positive Semidefinite Functions

We will need 7 lemmas to prove Part 2 and discuss properties of psd functions.

Lemma 1:

If ϕ is a positive semidefinite function,

$a \in \mathbb{R}$

$\psi(t) \doteq \phi(t)e^{ita}$ $\Rightarrow$ then $\psi(t)$ is also a positive semidefinite function.

Proof:

Let $t_1, \ldots, t_n \in \mathbb{R}$ be arbitrary, and $\xi_1, \ldots, \xi_n \in \mathbb{C}$ arbitrary.

$$\Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n} \psi(t_i - t_j)\xi_i\bar{\xi}_j = \sum_{i=1}^{n}\sum_{j=1}^{n} \phi(t_i - t_j)\underbrace{e^{i(t_i-t_j)a}}_{=e^{it_ia}e^{-it_ja}}\xi_i\bar{\xi}_j$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \phi(t_i - t_j)\xi_i e^{it_ia}\bar{\xi}_j e^{-it_ja}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \phi(t_i - t_j)\beta_i\bar{\beta}_j \qquad \text{where } \beta_i = \xi_i e^{it_ia},\ \bar{\beta}_j = \bar{\xi}_j e^{-it_ja}$$

≥ 0, since ϕ is a positive semideifinite function. Q.E.D.

Properties of Positive Semidefinite Functions

Lemma 2:

If $\phi_1, \phi_2, \ldots, \phi_m$ are positive semidefinite functions,

If $\lambda_1, \lambda_2, \ldots, \lambda_m > 0$ positive numbers

$\Rightarrow$ Then $\psi = \sum_{k=1}^{m} \lambda_k \phi_k$ is also a positive semidefinite function.

Proof:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \psi(t_i - t_j)\xi_i\bar{\xi}_j = \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\sum_{k=1}^{m} \lambda_k \phi_k(t_i - t_j) \right] \xi_i\bar{\xi}_j$$

$$= \sum_{k=1}^{m} \lambda_k \underbrace{\sum_{i=1}^{n} \sum_{j=1}^{n} \phi_k(t_i - t_j)\xi_i\bar{\xi}_j}_{\geq 0 \text{ since } \phi_k \text{ is positive semidefinite}} \geq 0$$

This is what we had to prove for Lemma 2. QED

Properties of Positive Semidefinite Functions

Lemma 3:

If ϕ is a positive semidefinite function,

$$\Longrightarrow \begin{cases} \text{(a)} \quad \phi(0) \geq 0,\ \phi(0) \in \mathbb{R} \\ \text{(b)} \quad \phi(t) = \overline{\phi(-t)} \\ \text{(c)} \quad |\phi(t)| \leq |\phi(0)| \ \ \forall t \in \mathbb{R} \end{cases}$$

Remark 1:

From (a) & (c) it follows that $\phi(0) > 0$, otherwise $\phi(t) = 0\ \forall t \in \mathbb{R}$

Proof: ϕ is psd $\Rightarrow \sum_{i=1}^{n} \sum_{j=1}^{n} \phi(t_i - t_j)\xi_i\bar{\xi}_j \geq 0 \quad \forall\, t_1, \ldots t_n \in \mathbb{R},\ \forall\, \xi_1, \ldots \xi_n \in \mathbb{C}$

If $t_1 = 0,\ t_2 = t$, n=2 $\Rightarrow$

$$|\xi_1|^2\phi(t_1 - t_1) + |\xi_2|^2\phi(t_2 - t_2) + \xi_1\bar{\xi}_2\phi(t_1 - t_2) + \bar{\xi}_1\xi_2\phi(t_2 - t_1) \geq 0$$

$$\Rightarrow |\xi_1|^2\phi(0) + |\xi_2|^2\phi(0) + \xi_1\bar{\xi}_2\phi(-t) + \bar{\xi}_1\xi_2\phi(t) \geq 0 \quad \forall\ \xi_1, \xi_2 \in \mathbb{C}$$

Properties of Positive Semidefinite Functions

We already know:

$$|\xi_1|^2\phi(0) + |\xi_2|^2\phi(0) + \xi_1\bar{\xi}_2\phi(-t) + \bar{\xi}_1\xi_2\phi(t) \geq 0 \qquad \forall\ \xi_1, \xi_2 \in \mathbb{C}$$

$$\text{If } \xi_1 = 1,\ \xi_2 = 0 \Rightarrow \quad \phi(0) \geq 0,\ \phi(0) \in \mathbb{R}$$

This proves Lemma 3 (a) QED

$$\text{If } \xi_1 = 1,\ \xi_2 = 1 \Rightarrow \ 2\phi(0) + \phi(-t) + \phi(t) \geq 0$$

$$\Rightarrow \phi(-t) + \phi(t) \in \mathbb{R} \Rightarrow \mathrm{Im}[\phi(-t)] + \mathrm{Im}[\phi(t)] = 0 \Rightarrow \mathrm{Im}[\phi(t)] = -\mathrm{Im}[\phi(-t)]$$

$$\text{If } \xi_1 = 1,\ \xi_2 = i \Rightarrow 2\phi(0) + \bar{i}\phi(-t) + i\phi(t) \geq 0 \Rightarrow 2\phi(0) - i\phi(-t) + i\phi(t) \geq 0$$

$$\Rightarrow i\phi(t) - i\phi(-t) \in \mathbb{R} \quad \Rightarrow \mathrm{Re}[\phi(t) - \phi(-t)] = 0 \quad \Rightarrow \mathrm{Re}[\phi(t)] = \mathrm{Re}[\phi(-t)]$$

$$\Rightarrow \quad \phi(t) = \overline{\phi(-t)}$$

This proves Lemma 3 (b) QED

Properties of Positive Semidefinite Functions

We already know:

$$|\xi_1|^2\phi(0) + |\xi_2|^2\phi(0) + \xi_1\bar{\xi}_2\phi(-t) + \bar{\xi}_1\xi_2\phi(t) \geq 0 \qquad \forall\ \xi_1, \xi_2 \in \mathbb{C}$$

If $\xi_1 = \sqrt{-\phi(t)},\ \bar{\xi}_2 = \sqrt{-\phi(t)} \Rightarrow$

$$|\xi_1|^2 = \left|\sqrt{-\phi(t)}\right|^2 = |\phi(t)|$$

$$|\xi_2|^2 = \left|\sqrt{-\phi(t)}\right|^2 = |\phi(t)|$$

$$\xi_1\bar{\xi}_2 = \sqrt{-\phi(t)}\sqrt{-\phi(t)} = -\phi(t)$$

$$\bar{\xi}_1\xi_2 = \overline{\sqrt{-\phi(t)}\sqrt{-\phi(t)}} = -\overline{\phi(t)}$$

$$\Rightarrow |\phi(t)|\phi(0) + |\phi(t)|\phi(0) - \phi(t)\phi(-t) - \overline{\phi(t)}\phi(t) \geq 0$$

$$\Rightarrow 2|\phi(t)|\phi(0) \geq \phi(t)\underbrace{\phi(-t)}_{=\overline{\phi(t)}\text{ from (b)}} + \overline{\phi(t)}\phi(t)$$

$$\Rightarrow 2|\phi(t)|\phi(0) \geq \phi(t)\overline{\phi(t)} + \overline{\phi(t)}\phi(t) = 2\overline{\phi(t)}\phi(t) = 2|\phi(t)|^2$$

$\Rightarrow \phi(0) \geq |\phi(t)|$ **This proves Lemma 3 (c) QED**

Properties of Positive Semidefinite Functions

Lemma 4:

If ϕ is a positive semidefinite function,
$s, t \in \mathbb{R}$

$$\Rightarrow \; |\phi(t) - \phi(s)|^2 \leq 4\phi(0)\,|\phi(0) - \phi(t-s)|$$

Proof of Lemma 4: [It's a bit tedious…**]**

Since ϕ is a positive semidefinite function,

$$\Rightarrow \; \sum_{i=1}^{n}\sum_{j=1}^{n} \phi(t_i - t_j)\xi_i\bar{\xi}_j \geq 0 \qquad \forall\; t_1, \ldots t_n \in \mathbb{R},\; \forall\; \xi_1, \ldots \xi_n \in \mathbb{C}$$

$$\Rightarrow \; \text{the} \begin{pmatrix} \phi(t_1 - t_1) & \phi(t_1 - t_2) & \phi(t_1 - t_3) \\ \phi(t_2 - t_1) & \phi(t_2 - t_2) & \phi(t_2 - t_3) \\ \phi(t_3 - t_1) & \phi(t_3 - t_2) & \phi(t_3 - t_3) \end{pmatrix} \text{matrix is positive semidefinite}$$

$$\Rightarrow \; \text{the} \begin{pmatrix} \phi(0) & \phi(t_1 - t_2) & \phi(t_1 - t_3) \\ \overline{\phi(t_1 - t_2)} & \phi(0) & \phi(t_2 - t_3) \\ \overline{\phi(t_1 - t_3)} & \overline{\phi(t_2 - t_3)} & \phi(0) \end{pmatrix} \text{matrix is positive semidefinite}$$

Here we used from Lemma 3 that $\phi(t) = \overline{\phi(-t)}$.

Proof of Lemma 4

We already know that the

$$\begin{pmatrix} \phi(0) & \phi(t_1 - t_2) & \phi(t_1 - t_3) \\ \overline{\phi(t_1 - t_2)} & \phi(0) & \phi(t_2 - t_3) \\ \overline{\phi(t_1 - t_3)} & \overline{\phi(t_2 - t_3)} & \phi(0) \end{pmatrix} \text{ matrix is positive semidefinite}$$

Let $t_1 = t,\ t_2 = s,\ t_3 = 0 \Rightarrow$

$$A = \begin{pmatrix} \phi(0) & \phi(t - s) & \phi(t) \\ \overline{\phi(t - s)} & \phi(0) & \phi(s) \\ \overline{\phi(t)} & \overline{\phi(s)} & \phi(0) \end{pmatrix} \text{ matrix is positive semidefinite}$$

$$\Rightarrow \det(A) \geq 0$$

Proof of Lemma 4

We already know that

$$A = \begin{pmatrix} \phi(0) & \phi(t-s) & \phi(t) \\ \overline{\phi(t-s)} & \phi(0) & \phi(s) \\ \overline{\phi(t)} & \overline{\phi(s)} & \phi(0) \end{pmatrix} \text{ matrix is psd,} \quad \det(A) \geq 0$$

$$\begin{aligned} \Rightarrow \quad 0 \leq & \phi(0)[\phi(0)^2 - |\phi(s)|^2] \\ & - \phi(t-s)[\overline{\phi(t-s)}\phi(0) - \phi(s)\overline{\phi(t)}] \\ & + \phi(t)[\overline{\phi(t-s)\phi(s)} - \phi(0)\overline{\phi(t)}] \end{aligned}$$

$$\begin{aligned} = & \phi(0)^3 - \phi(0)|\phi(s)|^2] - |\phi(t-s)|^2\phi(0) \\ & + \phi(t-s)\phi(s)\overline{\phi(t)} + \phi(t)\overline{\phi(t-s)\phi(s)} - \phi(0)|\phi(t)|^2 \end{aligned}$$

$$\begin{aligned} = & \phi(0)^3 - \phi(0)\left[|\phi(s)|^2] + |\phi(t)|^2 + |\phi(t-s)|^2\right] \\ & + \phi(t-s)\phi(s)\overline{\phi(t)} + \phi(t)\overline{\phi(t-s)\phi(s)} \end{aligned}$$

Proof of Lemma 4

We already know that

$$0 \leq \phi(0)^3 - \phi(0)\left[\overbrace{|\phi(s)|^2 + |\phi(t)|^2}^{|\phi(t)-\phi(s)|^2 + \phi(t)\overline{\phi(s)} + \phi(s)\overline{\phi(t)}} + |\phi(t-s)|^2\right]$$
$$+ \phi(t-s)\phi(s)\overline{\phi(t)} + \phi(t)\overline{\phi(t-s)\phi(s)}$$

$$\Rightarrow\ 0 \leq \phi(0)^3 - \phi(0)\left[|\phi(t)-\phi(s)|^2 + |\phi(t-s)|^2\right]$$
$$\left.\begin{aligned} & - \phi(0)[\phi(t)\overline{\phi(s)} + \phi(s)\overline{\phi(t)}] \\ & + \phi(t-s)\phi(s)\overline{\phi(t)} + \phi(t)\overline{\phi(t-s)\phi(s)} \end{aligned}\right\}$$

This is real, since the whole thing is real
(*) = Re(*)

$$\begin{aligned} \Rightarrow\ 0 \leq\ & \phi(0)^3 - \phi(0)\left[|\phi(t)-\phi(s)|^2 + |\phi(t-s)|^2\right] \\ & - \mathrm{Re}[\phi(0)\phi(t)\overline{\phi(s)}] \\ & - \mathrm{Re}[\phi(0)\phi(s)\overline{\phi(t)}] \\ & + \mathrm{Re}[\phi(t-s)\phi(s)\overline{\phi(t)}] \\ & + \mathrm{Re}[\phi(t)\overline{\phi(t-s)\phi(s)}] \end{aligned}$$

Proof of Lemma 4

We already know that

$$
\begin{aligned}
0 \le & \phi(0)^3 - \phi(0)\left[|\phi(t)-\phi(s)|^2 + |\phi(t-s)|^2\right] \\
& - \mathrm{Re}[\phi(0)\phi(t)\overline{\phi(s)}] \\
& - \mathrm{Re}[\phi(0)\phi(s)\overline{\phi(t)}] \\
& + \mathrm{Re}[\phi(t-s)\phi(s)\overline{\phi(t)}] \\
& + \mathrm{Re}[\phi(t)\overline{\phi(t-s)\phi(s)}]
\end{aligned}
$$

Let us use the fact that $\mathrm{Re}[A+B] = \mathrm{Re}[A] + \mathrm{Re}[B] = \mathrm{Re}[A+\bar{B}]$

Therefore,

$$
\begin{aligned}
& - \mathrm{Re}[\phi(0)\phi(t)\overline{\phi(s)}] \\
& - \mathrm{Re}[\phi(0)\phi(s)\overline{\phi(t)}] \\
& + \mathrm{Re}[\phi(t-s)\phi(s)\overline{\phi(t)}] \\
& + \mathrm{Re}[\phi(t)\overline{\phi(t-s)\phi(s)}]
\end{aligned}
=
\begin{aligned}
& - \mathrm{Re}[\phi(0)\phi(s)\overline{\phi(t)}] \\
& - \mathrm{Re}[\phi(0)\phi(s)\overline{\phi(t)}] \\
& + \mathrm{Re}[\phi(t-s)\phi(s)\overline{\phi(t)}] \\
& + \mathrm{Re}[\overline{\phi(t)}\phi(t-s)\phi(s)]
\end{aligned}
$$

$$= -2\mathrm{Re}[\phi(s)\overline{\phi(t)}(\phi(0)-\phi(t-s))]$$

Therefore,

$$0 \le \phi(0)^3 - \phi(0)\left[|\phi(t)-\phi(s)|^2 + |\phi(t-s)|^2\right] - 2\mathrm{Re}[\phi(s)\overline{\phi(t)}(\phi(0)-\phi(t-s))]$$

Proof of Lemma 4

We already know that

$$0 \leq \phi(0)^3 - \phi(0)\left[|\phi(t)-\phi(s)|^2 + |\phi(t-s)|^2\right] - 2\mathrm{Re}[\phi(s)\overline{\phi(t)}(\phi(0)-\phi(t-s))]$$

Since $|\mathrm{Re}[\phi(s)]| \leq |\phi(s)| \leq \phi(0)$

and $|\mathrm{Re}[\overline{\phi(t)}]| \leq |\overline{\phi(t)}| = |\phi(t)| \leq \phi(0)$

we have that

$$|\mathrm{Re}[\phi(s)\overline{\phi(t)}(\phi(0)-\phi(t-s))]| \leq \phi(0)\phi(0)|\phi(0)-\phi(t-s)|$$

Therefore,

$$0 \leq \phi(0)^3 - \phi(0)\left[|\phi(t)-\phi(s)|^2 + |\phi(t-s)|^2\right] + 2\phi(0)^2|\phi(0)-\phi(t-s)|$$

$$\Rightarrow \ 0 \leq \phi(0)^2 - \left[|\phi(t)-\phi(s)|^2 + |\phi(t-s)|^2\right] + 2\phi(0)|\phi(0)-\phi(t-s)|$$

Since from Remark 1 we can assume that $\phi(0) > 0$, otherwise $\phi(t) = 0 \ \forall t \in \mathbb{R}$.

Proof of Lemma 4

We already know that

$$\Rightarrow \; 0 \leq \phi(0)^2 - \left[|\phi(t) - \phi(s)|^2 + |\phi(t-s)|^2\right] + 2\phi(0)|\phi(0) - \phi(t-s)|$$

$$\Rightarrow \; |\phi(t) - \phi(s)|^2 \leq \underbrace{\phi(0)^2 - |\phi(t-s)|^2}_{= (\phi(0) + \underbrace{|\phi(t-s)|}_{\leq \phi(0)})\underbrace{(\phi(0) - |\phi(t-s)|)}_{\leq |\phi(0) - \phi(t-s)|}} + 2\phi(0)|\phi(0) - \phi(t-s)|$$

Since $|A| - |B| \leq |A - B|$

$$\Rightarrow \; |\phi(t) - \phi(s)|^2 \leq 4\phi(0)|\phi(0) - \phi(t-s)|$$

This is what we had to prove for Lemma 4. QED

Uniform Continuity of PSD Functions

Definition [Uniformly Continuous Function]

We sat that $f : \mathcal{X} \to \mathbb{R}$ is uniformly continuous on $\mathcal{X}$

If for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\text{If } |x - y| < \delta, \text{ then } |f(x) - f(y)| < \epsilon.$$

Lemma 5 [Uniform continuity of PSD Functions]:

If ϕ is a positive semidefinite function,

ϕ is continuous at 0,

$$\Rightarrow \text{ then } \phi \text{ is uniformly continuous on } \mathbb{R}.$$

Proof of Lemma 5:

It follows very easily from Lemma 4:

$$|\phi(t) - \phi(s)|^2 \leq 4\phi(0)\,|\phi(0) - \phi(t-s)|$$

Therefore since ϕ is continuous at 0: If $|t - s| < \delta$, then $|\phi(0) - \phi(t-s)| < \epsilon_1$.

$\Rightarrow |\phi(t) - \phi(s)| \leq \sqrt{4\phi(0)\epsilon_1} \leq \epsilon_2$ This finishes to proof of Lemma 5. QED

Bochner's Theorem

Assumption 1: $\int |\phi(t)|dt < \infty$

In what follows we will assume that Assumption 1 holds.

Technically we don't need this, but it simplifies the proofs.

Remark:

Since $\int |\phi(t)|dt < \infty$, therefore its inverse Fourier transform exists:

$$f(x) \doteq \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \phi(t) dt$$

Lemma 6:

$$(a) \quad f(x) \geq 0, \ \forall x$$

$$(b) \quad \int_{\mathbb{R}} f(x) dx = 1,$$ i.e. f is a density function of a distribution

Observation: From Lemma 6 the Bochner's theorem [Part 2] follows since according to this lemma φ is the Fourier transform of a density function *f*, that is φ is indeed a characteristic function of a distribution. Q.E.D.

Lebesgue's Dominated Convergence Theorem

We will need Lebesgue's Dominated Convergence Theorem:

Theorem [Lebesgue's Dominated Convergence]:

$$\left.\begin{array}{l} \text{If } \lim_{n\to\infty} f_n(x) = f(x), \ \forall x \\ \quad |f_n(x)| \le g(x), \ \forall x \\ \quad \Omega \subset \mathbb{R}^d \\ \quad \int_\Omega g(x)dx < \infty \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \lim_{n\to\infty} \int_\Omega f_n(x)dx = \int_\Omega \underbrace{\lim_{n\to\infty} f_n(x)}_{= f(x)} dx \\ \\ \lim_{n\to\infty} \int_\Omega |f_n(x) - f(x)| dx = 0 \end{array}\right.$$

Proof of Lemma 6

$$f(x) \doteq \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\phi(t)dt$$

First we will prove that $f(x) \geq 0 \ \forall x$.

Since ϕ is psd function, we know that

$$0 \leq \int_0^T \int_0^T e^{-itx}e^{isx}\phi(t-s)dtds, \quad \forall T$$

$$\Rightarrow 0 \leq \lim_{T\to\infty} \frac{1}{2\pi T}\int_0^T \int_0^T e^{-itx}e^{isx}\phi(t-s)dtds$$

Using integral transform

$$= \lim_{T\to\infty} \frac{1}{2\pi T}\int_{-T}^T e^{-iux}\phi(u)\,(T-|u|)\,du$$

$$= \frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^T e^{-iux}\phi(u)\left(1-\frac{|u|}{T}\right)du$$

$$= \frac{1}{2\pi}\lim_{T\to\infty}\int_{-\infty}^{\infty} e^{-iux}\phi(u)\left(1-\frac{|u|}{T}\right)1_{[-T,T]}(u)du$$

Proof of Lemma 6

So far we know that

$$0 \leq \frac{1}{2\pi} \lim_{T\to\infty} \int_{-\infty}^{\infty} \underbrace{e^{-iux}\phi(u)\left(1 - \frac{|u|}{T}\right) 1_{[-T,T]}(u)du}_{= h_T(u) \leq |\phi(u)|}$$

Since $h_T(u) \doteq e^{-iux}\phi(u)\left(1 - \frac{|u|}{T}\right) 1_{[-T,T]}(u) \leq |\phi(u)|$,

and we assumed that $\int |\phi(u)|du < \infty$,

$\Rightarrow$ dominated convergence can be applied.

$$\Rightarrow 0 \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\lim_{T\to\infty} e^{-iux}\phi(u)\left(1 - \frac{|u|}{T}\right) 1_{[-T,T]}(u)du}_{= e^{-iux}\phi(u)}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux}\phi(u)du \doteq f(x)$$

$\Rightarrow 0 \leq f(x)\ \forall x$. This is what we wanted to prove. QED.

Proof of Lemma 6

The next step is to prove that $\int_{\mathbb{R}} f(x)dx = 1$, i.e. f is density

where $f(x) \doteq \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\phi(t)dt$

Lemma 7:

$$\phi(t) = \int_{\mathbb{R}} e^{itx} f(x)dx$$

Proof of Lemma 7: The correct proof is tedious but the main idea is that the Fourier transform of the inverse Fourier transform is the identity.

Remark: From Lemma 7 we have that

$$1 = \phi(0) = \int_{\mathbb{R}} e^{i0x} f(x)dx = \int_{\mathbb{R}} f(x)dx$$

Assumption of Lemma 6

This is what we wanted to prove. QED.

Thanks for your Attention! ☺