Bayes decision rule

x – input feature set
y - label

- If we know the conditional probability p(x | y) and class priors p(y) we can determine the appropriate class by using Bayes rule:

$$P(y=i \mid x) = \frac{P(x \mid y=i)P(y=i)}{P(x)} \stackrel{def}{=} q_i(x)$$

Minimizes our probability of making a mistake

Note that p(x) does not affect our decision

- We can use $q_i(x)$ to select the appropriate class.
- We chose class 0 if $q_0(x) \geq q_1(x)$ and class 1 otherwise
- This is termed the 'Bayes decision rule' and leads to optimal classification.
- However, it is often very hard to compute …

Bayes decision rule

$$P(y=i \mid x) = \frac{P(x \mid y=i)P(y=i)}{P(x)} \stackrel{def}{=} q_i(x)$$

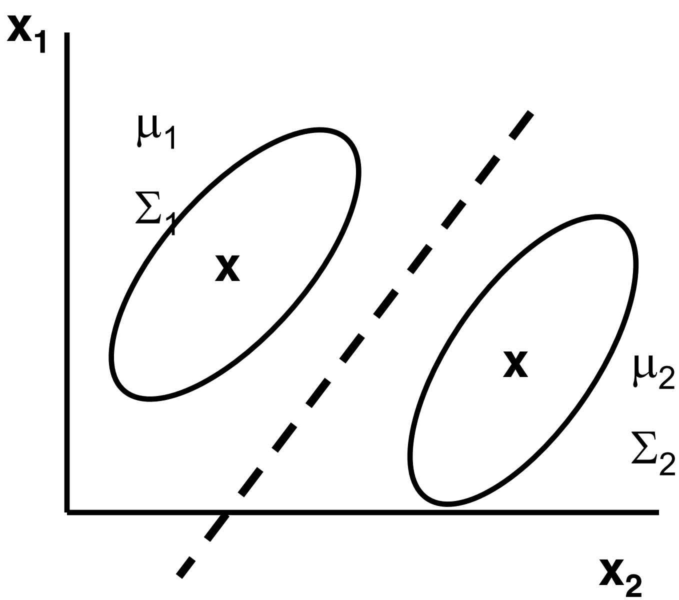
- We can also use the resulting probabilities to determine our **confidence** in the class assignment by looking at the likelihood ratio:

$$L(x) = \frac{q_0(x)}{q_1(x)}$$

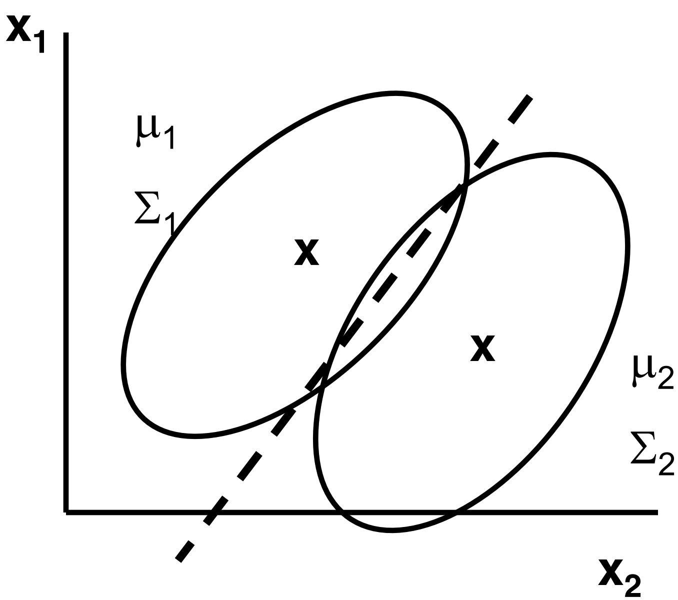
Also known as likelihood ratio, we will talk more about this later

Bayes decision rule: Example

Normal Gaussians

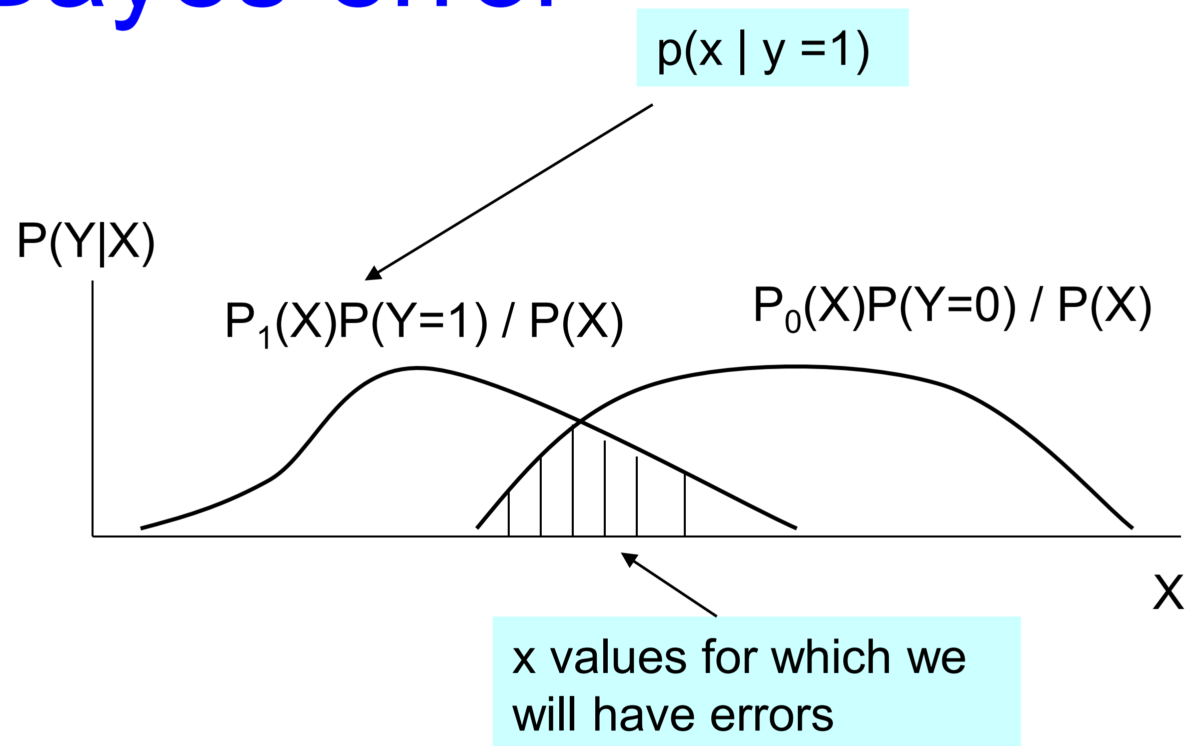


Normal Gaussians



Bayes error

- For the Bayes decision rule we can calculate the probability of an error
- This is the probability that we assign a sample to the wrong class, also known as the **risk**

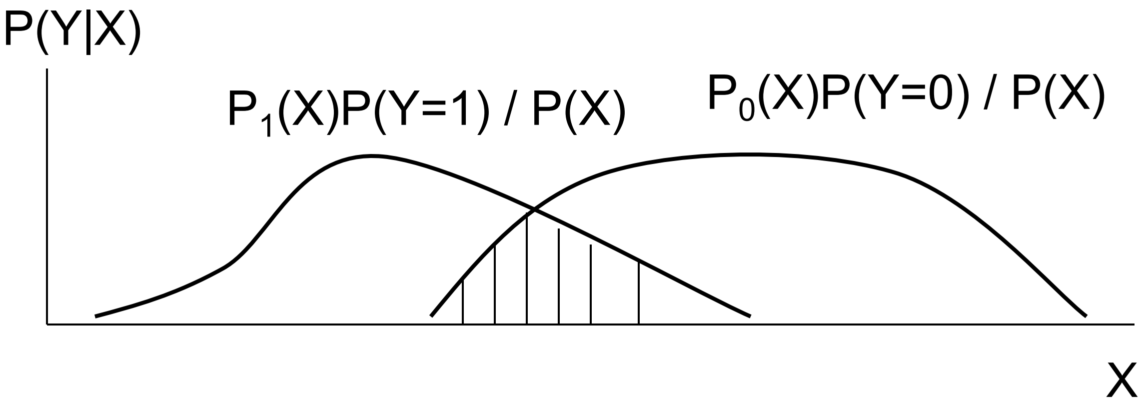


- The risk for sample x is:

$$R(x) = \min\{P_1(x)P(y=1),\ P_0(x)P(y=0)\} \,/\, P(x)$$

Risk can be used to determine a ‘reject’ region

Bayes error



- The probability that we assign a sample to the wrong class, is known as the **risk**

- The risk for sample x is:

$R(x) = \min\{P_1(x)P(y=1), P_0(x)P(y=0)\} / P(x)$

- We can also compute the expected risk (the risk for the entire range of values of x):

$$
\begin{aligned}
E[r(x)] &= \int_x r(x)p(x)dx \\
&= \int_x \min\{p_1(x)p(y=1), p_0(x)p(y=0)\}dx \\
&= p(y=0)\int_{L_1} p_0(x)dx + p(y=1)\int_{L_0} p_1(x)dx
\end{aligned}
$$

L_1 is the region where we assign instances to class 1

Loss function

- The risk value we computed assumes that both errors (assigning instances of class 1 to class 0 and vice versa) are equally harmful.
- However, this is not always the case.
- Why?
- In general our goal is to minimize loss, often defined by a loss function: $L_{0,1}(x)$ which is the penalty we pay when assigning instances of class 0 to class 1

$$E[L] = L_{0,1} p(y=0) \int_{L_1} p_0(x) dx + L_{1,0} p(y=1) \int_{L_0} p_1(x) dx$$