

10-701

Machine Learning

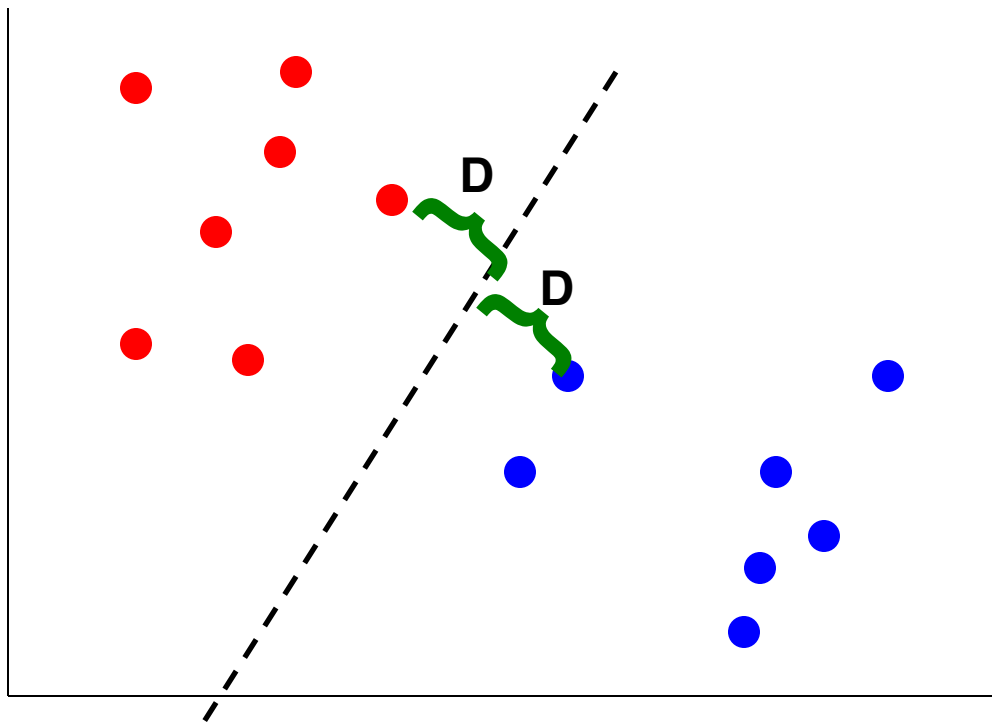
Support Vector Machines

Types of classifiers

- We can divide the large variety of classification approaches into roughly three major types
 1. Instance based classifiers
 - Use observation directly (no models)
 - e.g. K nearest neighbors
 2. Generative:
 - build a generative statistical model
 - e.g., Bayesian networks
 3. Discriminative
 - directly estimate a decision rule/boundary
 - e.g., decision tree

Max margin classifiers

- Instead of fitting all points, focus on boundary points
- Learn a boundary that leads to the largest margin from points on both sides

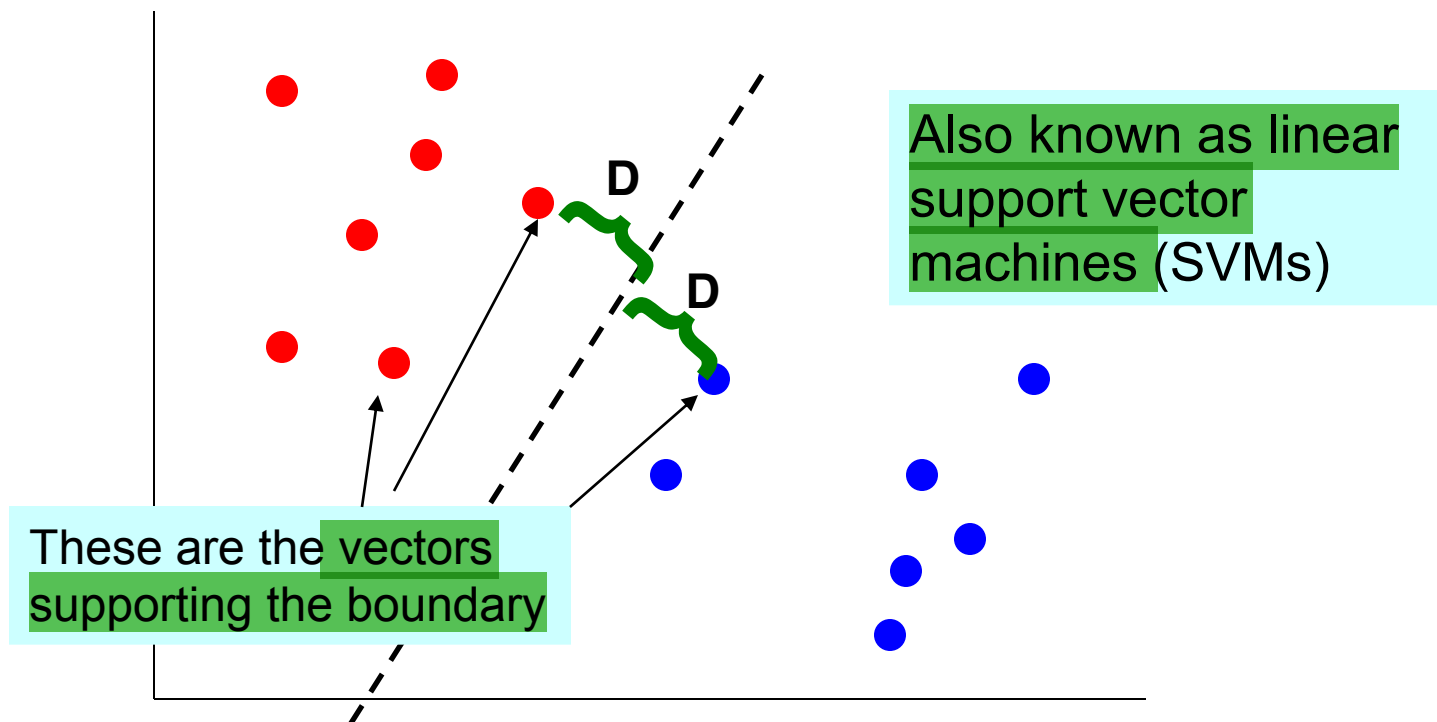


Why?

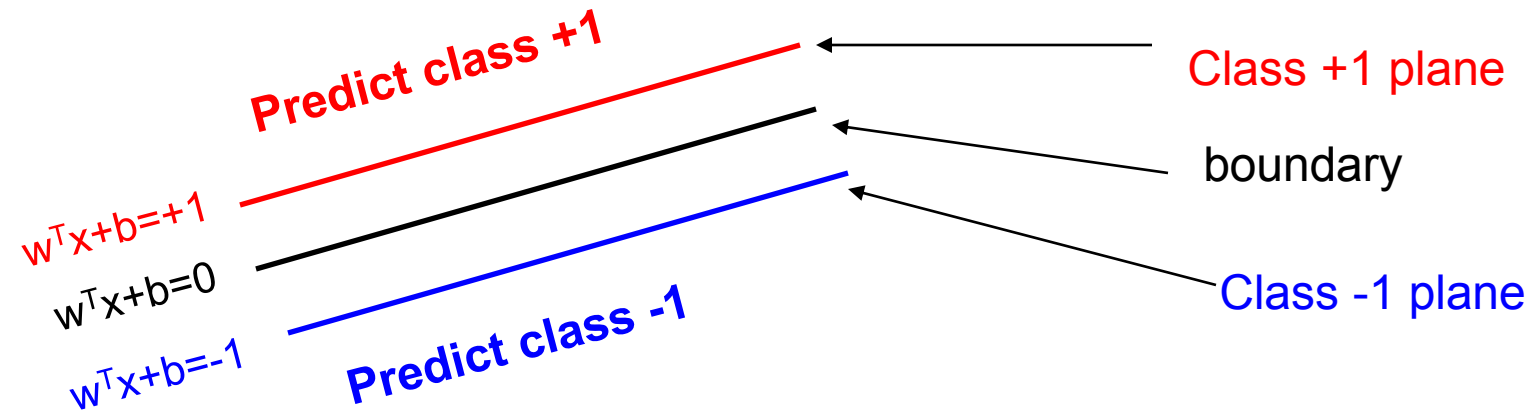
- Intuitive, 'makes sense'
- Some theoretical support
- Works well in practice

Max margin classifiers

- Instead of fitting all points, focus on boundary points
- Learn a boundary that leads to the largest margin from points on both sides



Specifying a max margin classifier

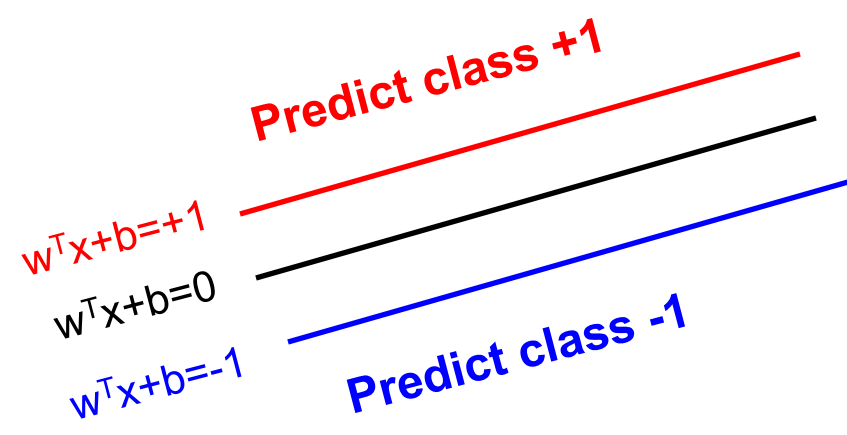


Classify as +1 if $w^T x + b \geq 1$

Classify as -1 if $w^T x + b \leq -1$

Undefined if $-1 < w^T x + b < 1$

Specifying a max margin classifier



Is the linear separation assumption realistic?

We will deal with this shortly, but let's assume it for now

Classify as +1

if

$$w^T x + b \geq 1$$

Classify as -1

if

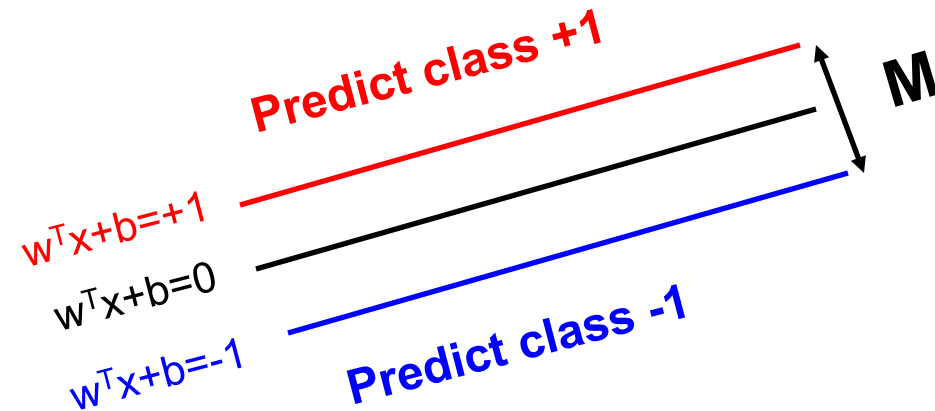
$$w^T x + b \leq -1$$

Undefined

if

$$-1 < w^T x + b < 1$$

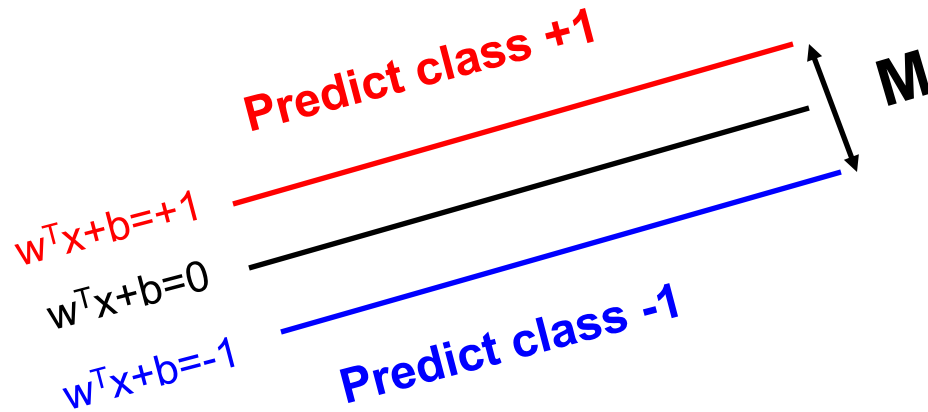
Maximizing the margin



Classify as +1 if $w^T x + b \geq 1$
Classify as -1 if $w^T x + b \leq -1$
Undefined if $-1 < w^T x + b < 1$

- Lets define the width of the margin by M
- How can we encode our goal of maximizing M in terms of our parameters (w and b)?
- Lets start with a few observations

Maximizing the margin



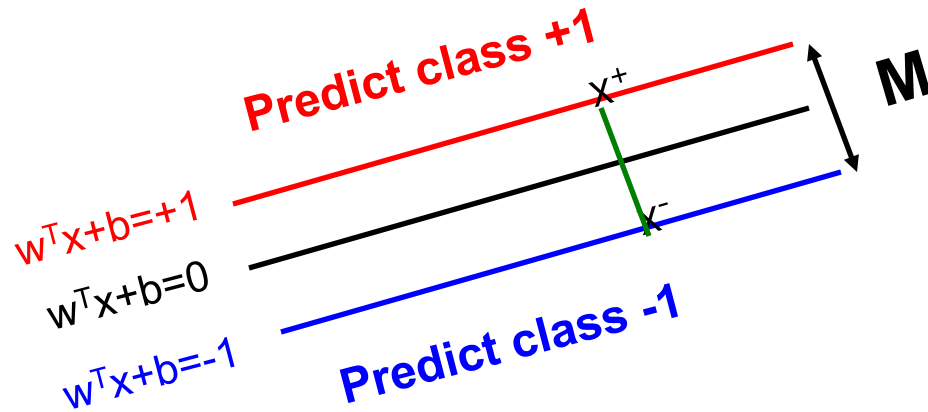
Classify as +1 if $w^T x + b \geq 1$
Classify as -1 if $w^T x + b \leq -1$
Undefined if $-1 < w^T x + b < 1$

- Observation 1: the vector w is orthogonal to the +1 plane
- Why?

Let u and v be two points on the +1 plane,
then for the vector defined by u and v we have
 $w^T(u-v) = 0$

Corollary: the vector w is orthogonal to the -1 plane

Maximizing the margin



Classify as +1 if $w^T x + b \geq 1$

Classify as -1 if $w^T x + b \leq -1$

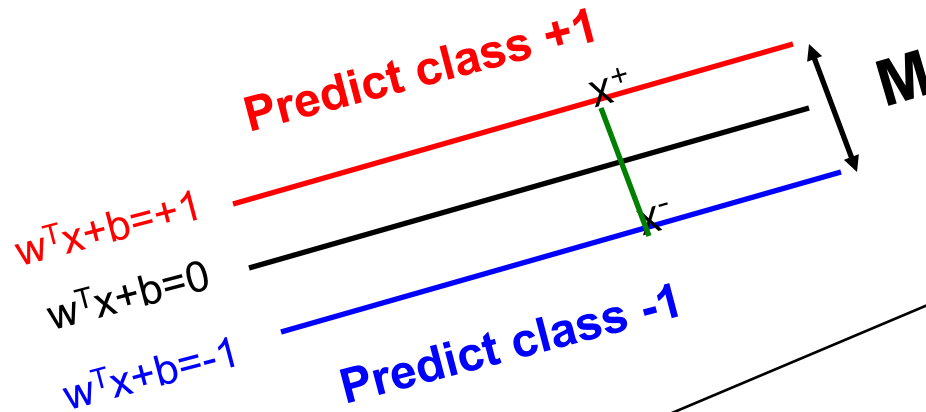
Undefined if $-1 < w^T x + b < 1$

- Observation 1: the vector w is orthogonal to the +1 and -1 planes
- Observation 2: if x^+ is a point on the +1 plane and x^- is the *closest* point to x^+ on the -1 plane then

$$x^+ = \lambda w + x^-$$

Since w is orthogonal to both planes we need to 'travel' some distance along w to get from x^+ to x^-

Putting it together



- $w^T x^+ + b = +1$

- $w^T x^- + b = -1$

- $x^+ = \lambda w + x^-$

- $|x^+ - x^-| = M$

$$w^T x^+ + b = +1$$

\Rightarrow

$$w^T (\lambda w + x^-) + b = +1$$

\Rightarrow

$$w^T x^- + b + \lambda w^T w = +1$$

\Rightarrow

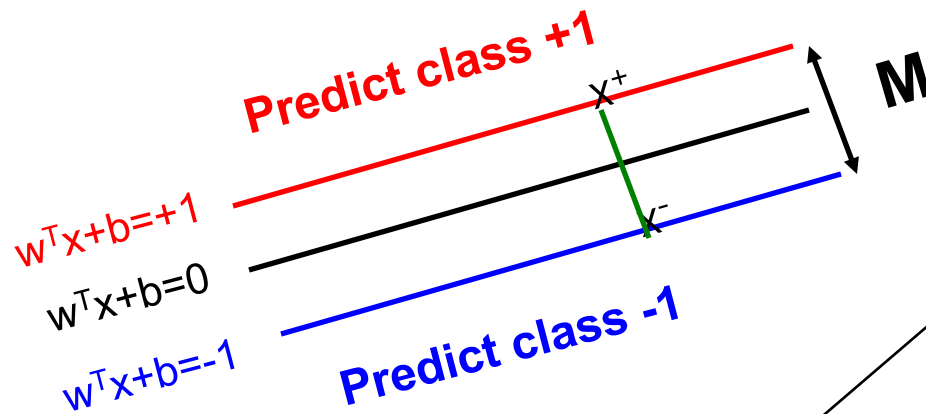
$$-1 + \lambda w^T w = +1$$

\Rightarrow

$$\lambda = 2/w^T w$$

We can now define M in terms of w and b

Putting it together



- $w^T x^+ + b = +1$
- $w^T x^- + b = -1$
- $x^+ = \lambda w + x^-$
- $|x^+ - x^-| = M$
- $\lambda = 2/w^T w$

$$M = |x^+ - x^-|$$

\Rightarrow

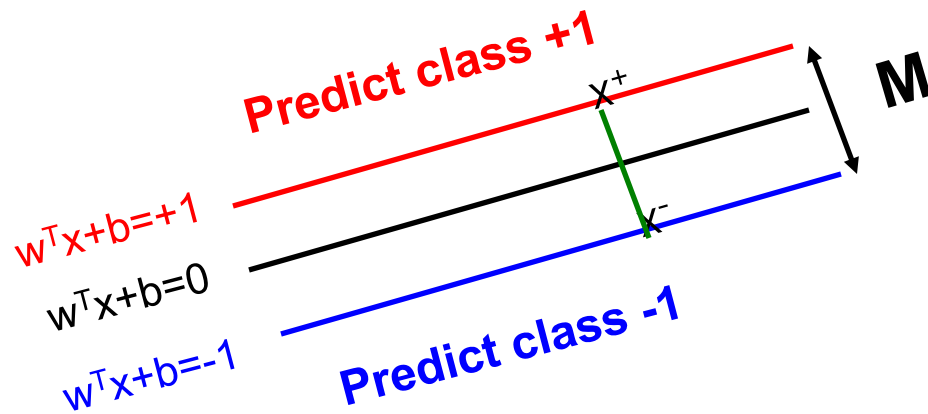
$$M = |\lambda w| = \lambda |w| = \lambda \sqrt{w^T w}$$

\Rightarrow

$$M = 2 \frac{\sqrt{w^T w}}{w^T w} = \frac{2}{\sqrt{w^T w}}$$

We can now define M in terms of w and b

Finding the optimal parameters



$$M = \frac{2}{\sqrt{w^T w}}$$

We can now search for the optimal parameters by finding a solution that:

1. Correctly classifies all points
2. Maximizes the margin (or equivalently minimizes $w^T w$)

Several optimization methods can be used:
Gradient descent, simulated annealing, EM
etc.

Quadratic programming (QP)

Quadratic programming solves optimization problems of the following form:

$$\min_U \frac{u^T R u}{2} + d^T u + c$$

- u -vector (unknown)
- R – squared matrix
- d – vector
- c - scalar

subject to n inequality constraints:

$$\begin{array}{ccc} a_{11}u_1 + a_{12}u_2 + \dots & \leq & b_1 \\ \vdots & & \vdots \end{array}$$

Quadratic term

$$a_{n1}u_1 + a_{n2}u_2 + \dots \leq b_n$$

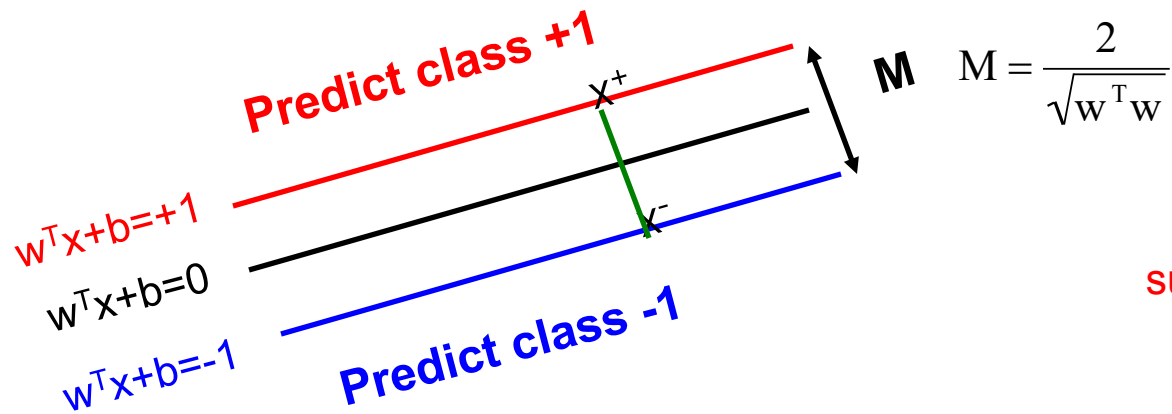
When a problem can be specified as a QP problem we can use solvers that are better than gradient descent or simulated annealing

and k equivalency constraints:

$$\begin{matrix} a_{n+1,1}u_1 + a_{n+1,2}u_2 + \dots = b_{n+1} \\ \vdots \\ \vdots \\ \vdots \end{matrix}$$

$$a_{n+k,1}u_1 + a_{n+k,2}u_2 + \dots = b_{n+k}$$

SVM as a QP problem



$$\text{Min } (w^T w)/2$$

subject to the following inequality constraints:

For all x in class + 1

$$w^T x + b \geq 1$$

For all x in class - 1

$$w^T x + b \leq -1$$

A total of n constraints if we have n input samples

$$\min_U \frac{u^T R u}{2} + d^T u + c$$

subject to n inequality constraints:

$$a_{11}u_1 + a_{12}u_2 + \dots \leq b_1$$

$$\vdots \quad \quad \quad \vdots$$

$$a_{n1}u_1 + a_{n2}u_2 + \dots \leq b_n$$

and k equality constraints:

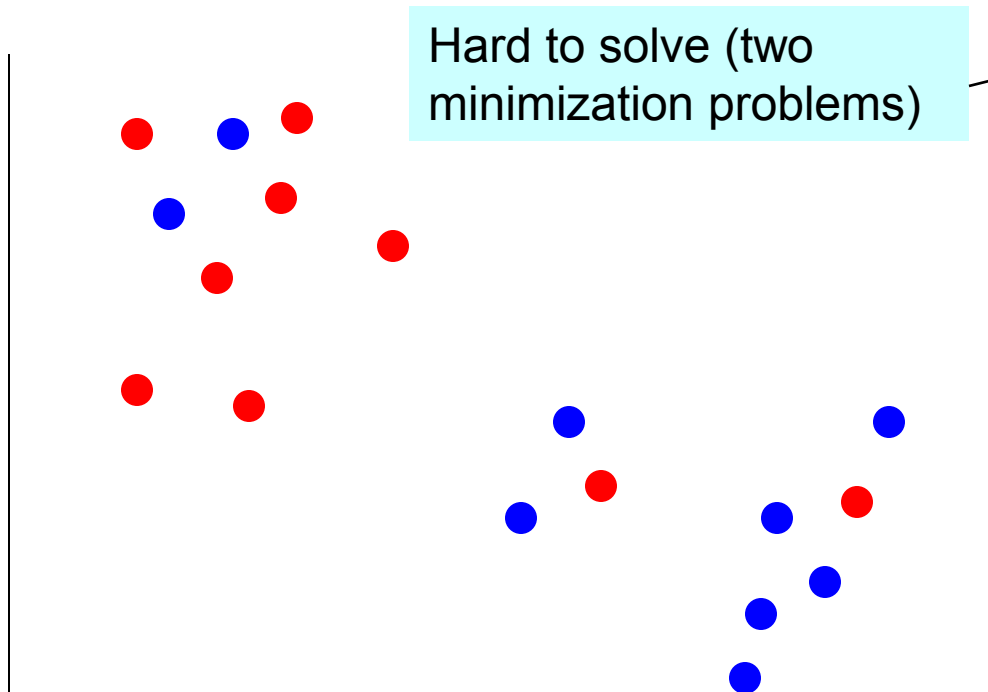
$$a_{n+1,1}u_1 + a_{n+1,2}u_2 + \dots = b_{n+1}$$

$$\vdots \quad \quad \quad \vdots$$

$$a_{n+k,1}u_1 + a_{n+k,2}u_2 + \dots = b_{n+k}$$

Non linearly separable case

- So far we assumed that a linear plane can perfectly separate the points
- But this is not usually the case
 - noise, outliers



How can we convert this to a QP problem?

- Minimize training errors?

$$\min w^T w$$

$$\min \# \text{errors}$$

- Penalize training errors:

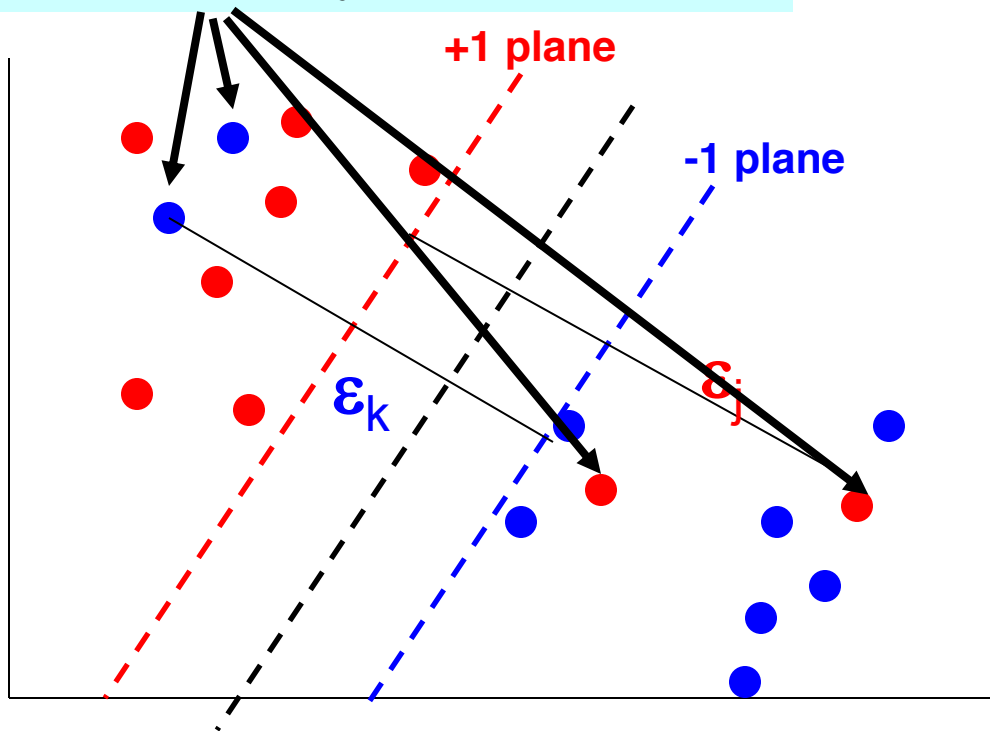
$$\min w^T w + C * (\# \text{errors})$$

Hard to encode in a QP problem

Non linearly separable case

- Instead of minimizing the number of misclassified points we can minimize the *distance* between these points and their correct plane

These are also support vectors since they impact the parameters of the decision boundary



The new optimization problem is:

$$\min_w \frac{w^T w}{2} + \sum_{i=1}^n C \varepsilon_i$$

subject to the following inequality constraints:

For all x_i in class + 1

$$w^T x + b \geq 1 - \varepsilon_i$$

For all x_i in class - 1

$$w^T x + b \leq -1 + \varepsilon_i$$

Wait. Are we missing something?

Final optimization for non linearly separable case

The new optimization problem is:

$$\min_w \frac{w^T w}{2} + \sum_{i=1}^n C \varepsilon_i$$

subject to the following inequality constraints:

For all x_i in class + 1

$$w^T x + b \geq 1 - \varepsilon_i$$

For all x_i in class - 1

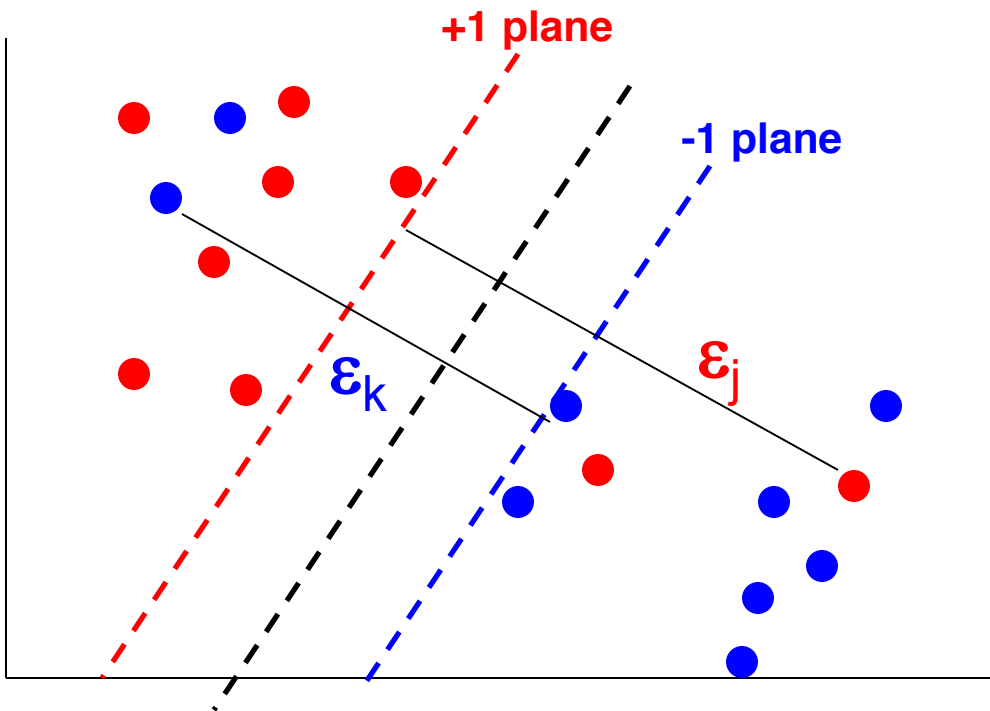
$$w^T x + b \leq -1 + \varepsilon_i$$

For all i

$$\varepsilon_i \geq 0$$

A total of n constraints

Another n constraints



Where we are

Two optimization problems: For the separable and non separable cases

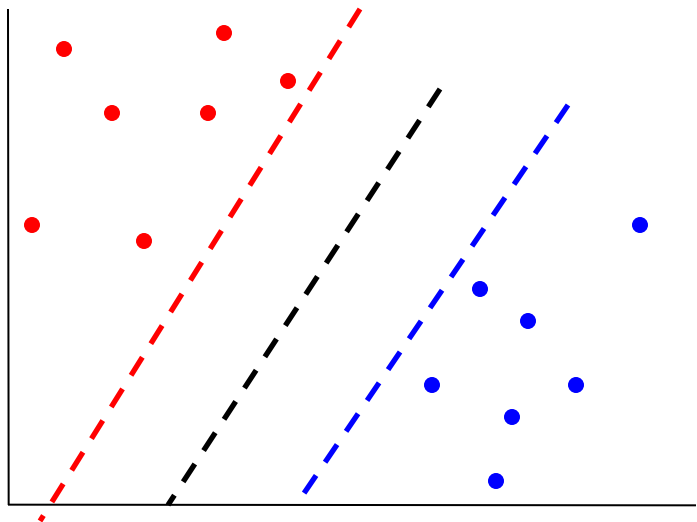
$$\min_w \frac{w^T w}{2}$$

For all x in class + 1

$$w^T x + b \geq 1$$

For all x in class - 1

$$w^T x + b \leq -1$$



$$\min_w \frac{w^T w}{2} + \sum_{i=1}^n C \varepsilon_i$$

For all x_i in class + 1

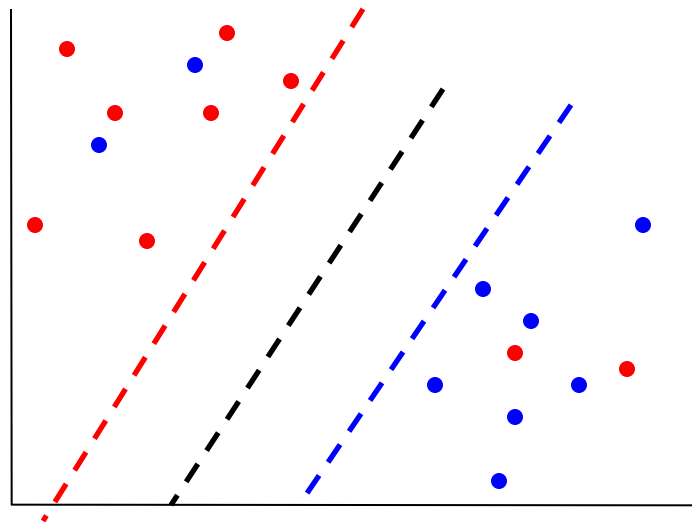
$$w^T x + b \geq 1 - \varepsilon_i$$

For all x_i in class - 1

$$w^T x + b \leq -1 + \varepsilon_i$$

For all i

$$\varepsilon_i \geq 0$$



Where we are

Two optimization problems: For the separable and non separable cases

$$\text{Min } (w^T w)/2$$

For all x in class + 1

$$w^T x + b \geq 1$$

For all x in class - 1

$$w^T x + b \leq -1$$

$$\min_w \frac{w^T w}{2} + \sum_{i=1}^n C \varepsilon_i$$

For all x_i in class + 1

$$w^T x + b \geq 1 - \varepsilon_i$$

For all x_i in class - 1

$$w^T x + b \leq -1 + \varepsilon_i$$

For all i

$$\varepsilon_i \geq 0$$

- Instead of solving these QPs directly we will solve a dual formulation of the SVM optimization problem
- The main reason for switching to this type of representation is that it would allow us to use a neat trick that will make our lives easier (and the run time faster)

An alternative (dual) representation of the SVM QP

- We will start with the linearly separable case
- Instead of encoding the correct classification rule and constraint we will use LaGrange multipliers to encode it as part of the our minimization problem

$$\text{Min } (w^T w)/2$$

For all x in class +1

$$w^T x + b \geq 1$$

For all x in class -1

$$w^T x + b \leq -1$$

Why?



$$\text{Min } (w^T w)/2$$

$$(w^T x_i + b) y_i \geq 1$$

An alternative (dual) representation of the SVM QP

$$\text{Min } (w^T w)/2$$

$$(w^T x_i + b)y_i \geq 1$$

- We will start with the linearly separable case
- Instead of encoding the correct classification rule a constraint we will use Lagrange multipliers to encode it as part of the our minimization problem

Recall that Lagrange multipliers can be applied to turn the following problem:

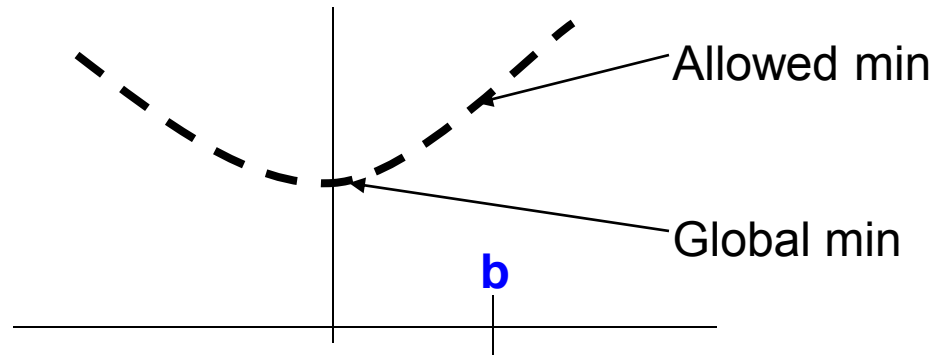
$$\min_x x^2$$

$$\text{s.t. } x \geq b$$

To

$$\min_x \max_{\alpha} x^2 - \alpha(x - b)$$

$$\text{s.t. } \alpha \geq 0$$



Lagrange multiplier for SVMs

Dual formulation

$$\min_{w,b} \max_{\alpha} \frac{w^T w}{2} - \sum_i \alpha_i [(w^T x_i + b)y_i - 1]$$

$$\alpha_i \geq 0 \quad \forall i$$

Original formulation

$$\text{Min } (w^T w)/2$$

$$(w^T x_i + b)y_i \geq 1$$

Using this new formulation we can derive w and b by taking the derivative w.r.t. w and α leading to:

$$w = \sum_i \alpha_i x_i y_i$$

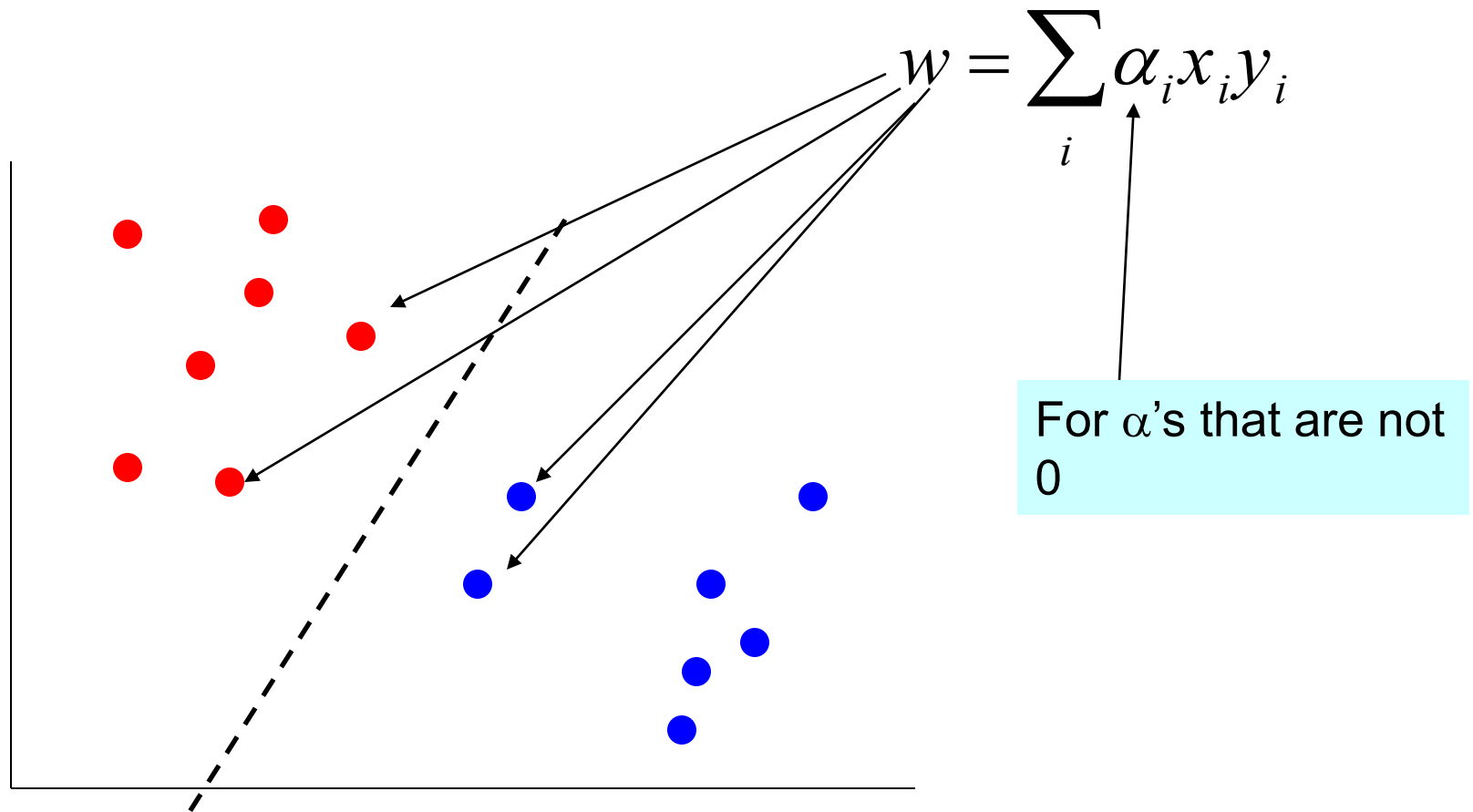
$$b = y_i - w^T x_i$$

$$\text{for } i \text{ s.t. } \alpha_i > 0$$

Finally, taking the derivative w.r.t. b we get:

$$\sum_i \alpha_i y_i = 0$$

Dual SVM - interpretation



Dual SVM for linearly separable case

Substituting w into our target function and using the additional constraint we get:

Dual formulation

$$\max_{\alpha} \sum_i \alpha_i - \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j$$

$$\sum_i \alpha_i y_i = 0$$

$$\alpha_i \geq 0 \quad \forall i$$

$$\min_{w,b} \max_{\alpha} \frac{w^T w}{2} - \sum_i \alpha_i [(w^T x_i + b) y_i - 1]$$

$$\alpha_i \geq 0 \quad \forall i$$

$$w = \sum_i \alpha_i x_i y_i$$

$$b = y_i - w^T x_i$$

$$\text{for } i \text{ s.t. } \alpha_i > 0$$

$$\sum_i \alpha_i y_i = 0$$

Dual SVM for linearly separable case

Our dual target function: $\max_{\alpha} \sum_i \alpha_i - \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j$

$$\sum_i \alpha_i y_i = 0$$

$$\alpha_i \geq 0 \quad \forall i$$

Dot product for all training samples

Dot product with training samples

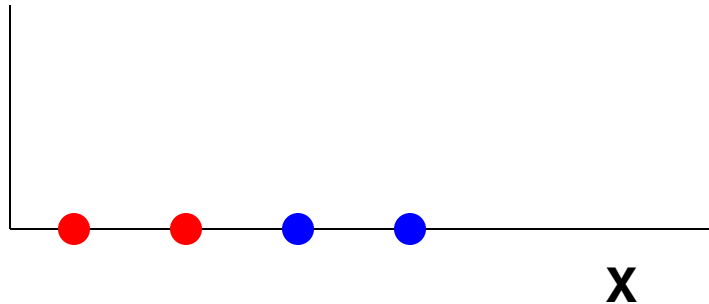
To evaluate a new sample \mathbf{x}_k
we need to compute:

$$\mathbf{w}^T \mathbf{x}_k + b = \sum_i \alpha_i y_i \mathbf{x}_i^T \mathbf{x}_k + b$$

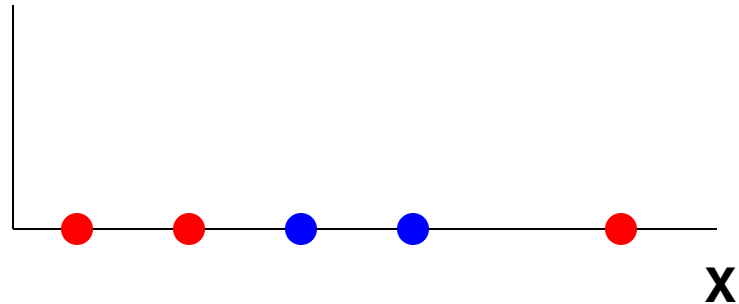
Is this too much computational work (for example when using transformation of the data)?

Classifying in 1-d

Can an SVM correctly
classify this data?

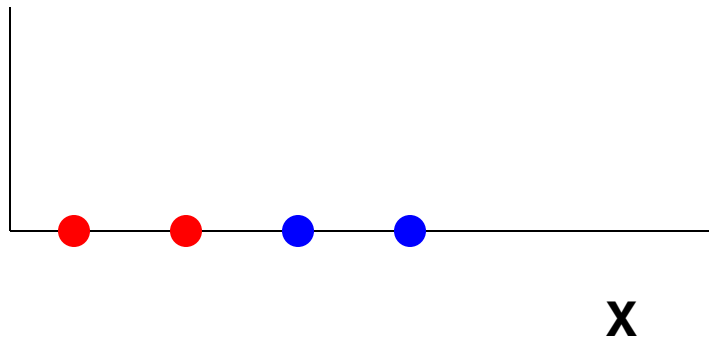


What about this?

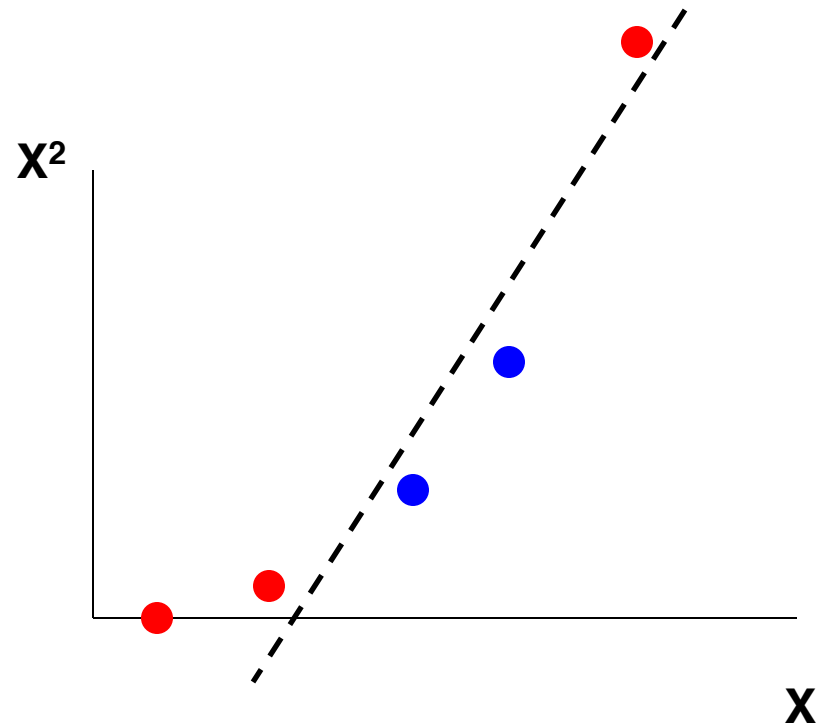


Classifying in 1-d

Can an SVM correctly
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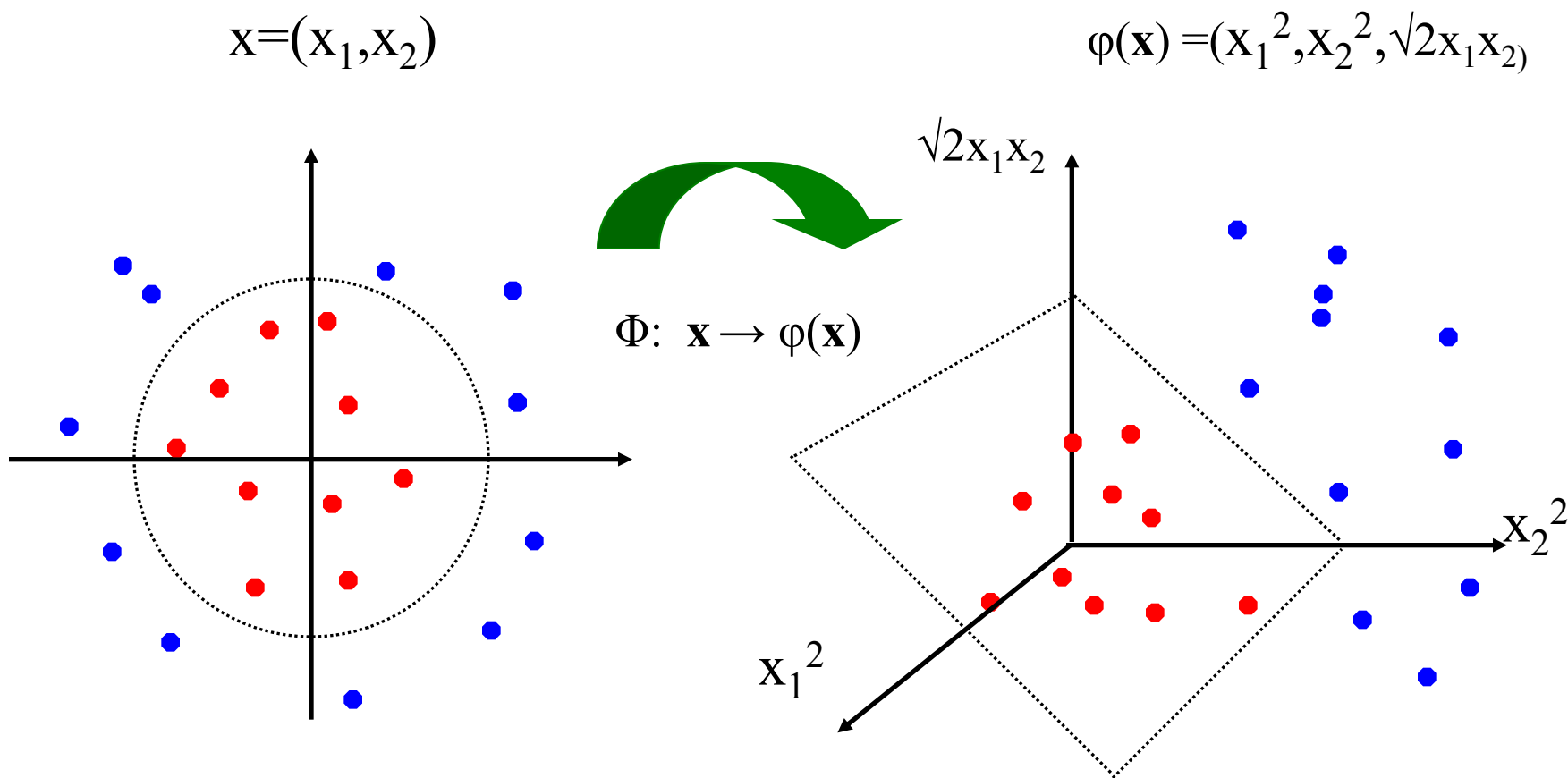


And now?



Non-linear SVMs: 2D

- The original input space (\mathbf{x}) can be mapped to some higher-dimensional feature space ($\phi(\mathbf{x})$) where the training set is separable:

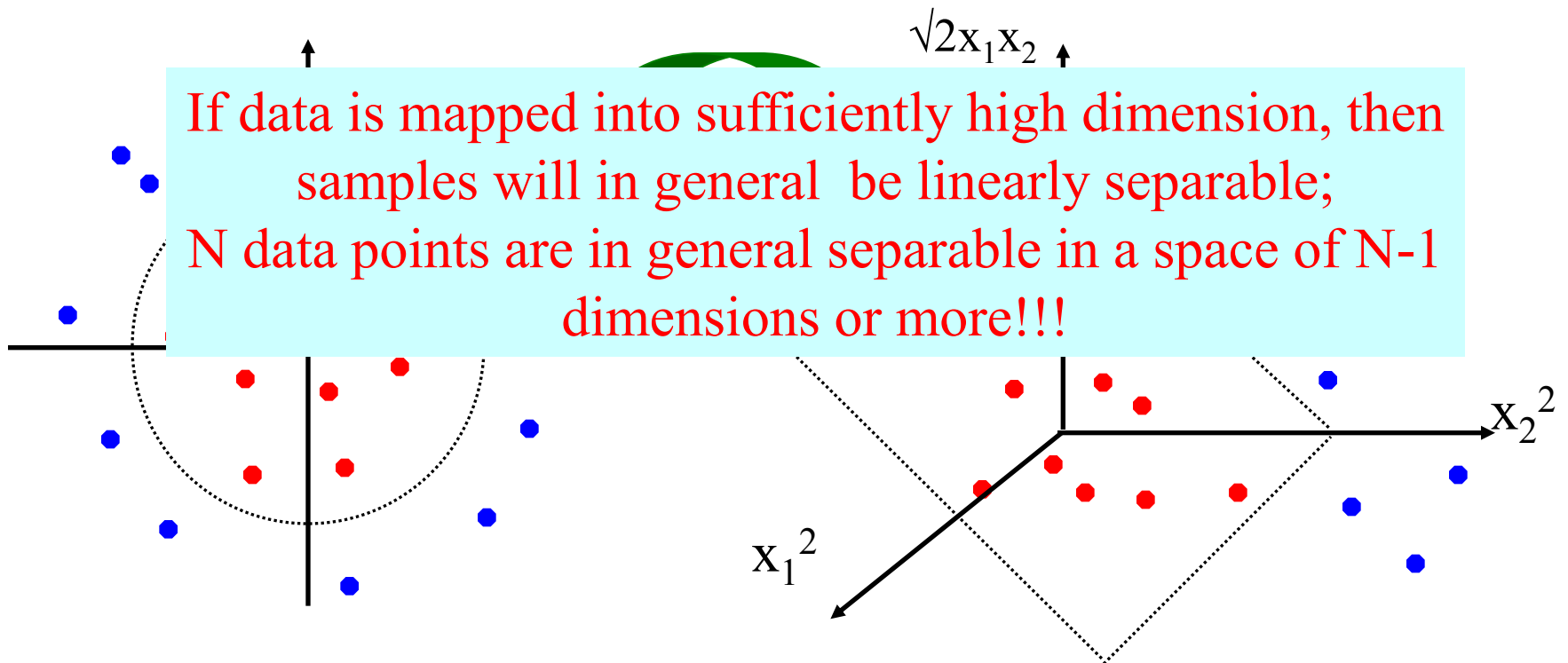


Non-linear SVMs: 2D

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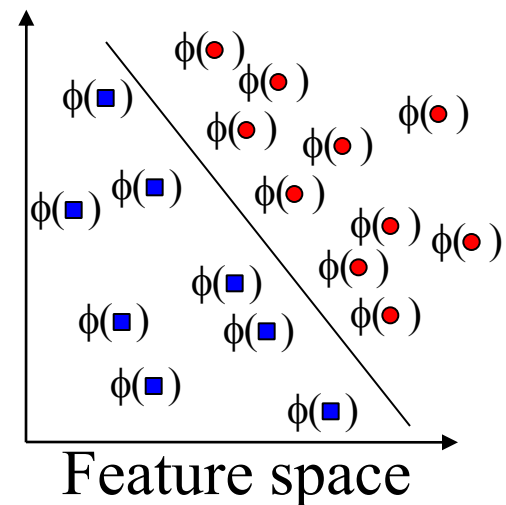
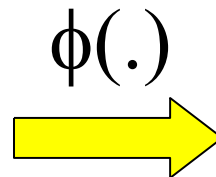
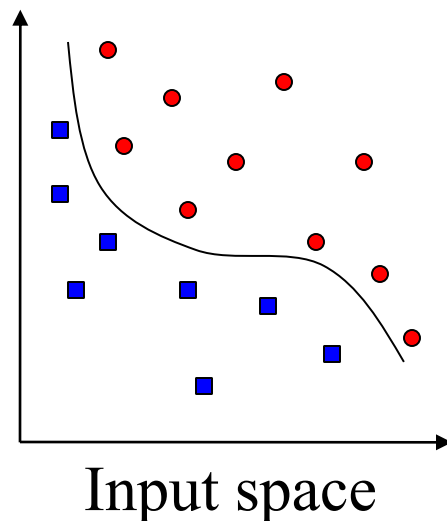
$$\mathbf{x} = (x_1, x_2)$$

$$\phi(\mathbf{x}) = (x_1^2, x_2^2, \sqrt{2}x_1x_2)$$



Transformation of Inputs

- Possible problems
 - High computation burden due to high-dimensionality
 - Many more parameters
- SVM solves these two issues simultaneously
 - “Kernel tricks” for efficient computation
 - Dual formulation only assigns parameters to samples, not features



Quadratic kernels

- While working in higher dimensions is beneficial, it also increases our running time because of the dot product computation
- However, there is a neat trick we can use
- consider all quadratic terms for $x^1, x^2 \dots x^m$

$$\max_{\alpha} \sum_i \alpha_i - \sum_{i,j} \alpha_i \alpha_j y_i y_j \Phi(\mathbf{x}_i) \Phi(\mathbf{x}_j)$$

$$\sum_i \alpha_i y_i = 0$$

$$\alpha_i \geq 0 \quad \forall i$$

m is the number of features in each vector

The $\sqrt{2}$ term will become clear in the next slide

$$\Phi(x) = \begin{pmatrix} \frac{1}{\sqrt{2}}x^1 \\ \vdots \\ \frac{1}{\sqrt{2}}x^m \\ (x^1)^2 \\ \vdots \\ (x^m)^2 \\ \sqrt{2}x^1x^2 \\ \vdots \\ \sqrt{2}x^{m-1}x^m \end{pmatrix}$$

$m+1$ linear terms

 m quadratic terms

 $m(m-1)/2$ pairwise terms

Dot product for quadratic kernels

How many operations do we need for the dot product?

$$\begin{aligned}
 \Phi(x)\Phi(z) = & \begin{pmatrix} 1 \\ \sqrt{2}x^1 \\ \vdots \\ \sqrt{2}x^1 \\ (x^1)^2 \\ \vdots \\ (x^m)^2 \\ \sqrt{2}x^1x^2 \\ \vdots \\ \sqrt{2}x^{m-1}x^m \end{pmatrix} \bullet \begin{pmatrix} 1 \\ \sqrt{2}z^1 \\ \vdots \\ \sqrt{2}z^2 \\ (z^1)^2 \\ \vdots \\ (z^m)^2 \\ \sqrt{2}z^1z^2 \\ \vdots \\ \sqrt{2}z^{m-1}z^m \end{pmatrix} \\
 = & \sum_i 2x^i z^i + \sum_i (x^i)^2 (z^i)^2 + \sum_i \sum_{j=i+1} 2x^i x^j z^i z^j + 1 \\
 & \qquad \qquad \qquad m \qquad \qquad \qquad m \qquad \qquad \qquad m(m-1)/2 \qquad \qquad \qquad \text{=} \sim m^2
 \end{aligned}$$

The kernel trick

How many operations do we need for the dot product?

$$= \sum_i 2x^i z^i + \sum_i (x^i)^2 (z^i)^2 + \sum_i \sum_{j=i+1} 2x^i x^j z^i z^j + 1$$

m
 m
 $m(m-1)/2$
 $\approx m^2$

However, we can obtain dramatic savings by noting that

dot product

$$\begin{aligned}
 (x.z + 1)^2 &= (x.z)^2 + 2(x.z) + 1 \\
 &= \left(\sum_i x^i z^i\right)^2 + \sum_i 2x^i z^i + 1 \\
 &= \sum_i 2x^i z^i + \sum_i (x^i)^2 (z^i)^2 + \sum_i \sum_{j=i+1} 2x^i x^j z^i z^j + 1
 \end{aligned}$$

We only need m operations!

Note that to evaluate a new sample we are also using dot products so we save there as well

Where we are

Our dual target function:

$$\begin{aligned} \max_{\alpha} \quad & \sum_i \alpha_i - \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j \\ & \sum_i \alpha_i y_i = 0 \\ & \alpha_i \geq 0 \quad \forall i \end{aligned}$$

mn^2 operations at each iteration

To evaluate a new sample x_j we need to compute:

$$\mathbf{w}^T x_j + b = \sum_i \alpha_i y_i \mathbf{x}_i^T \mathbf{x}_j + b$$

mr operations where r are the number of support vectors ($\alpha_i > 0$)

Other kernels

- The kernel trick works for higher order polynomials as well.
- For example, a polynomial of degree 4 can be computed using $(x.z+1)^4$ and, for a polynomial of degree d $(x.z+1)^d$
- Beyond polynomials there are other very high dimensional basis functions that can be made practical by finding the right Kernel Function

-Radial-Basis-style Kernel Function: $K(x,z) = \exp\left(-\frac{(x-z)^2}{2\sigma^2}\right)$

- Neural-net-style Kernel Function: $K(x,z) = \tanh(\kappa x.z - \delta)$

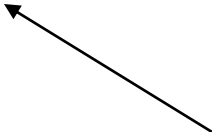
Dual formulation for non linearly separable case

Dual target function:

$$\max_{\alpha} \sum_i \alpha_i - \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j y_i y_j \mathbf{x}_i \mathbf{x}_j$$

$$\sum_i \alpha_i y_i = 0$$

$$C > \alpha_i \geq 0 \quad \forall i$$



The only difference is
that the α_i 's are now
bounded

To evaluate a new sample x_j
we need to compute:

$$\mathbf{w}^T x_j + b = \sum_i \alpha_i y_i \mathbf{x}_i \mathbf{x}_j + b$$

Why do SVMs work?

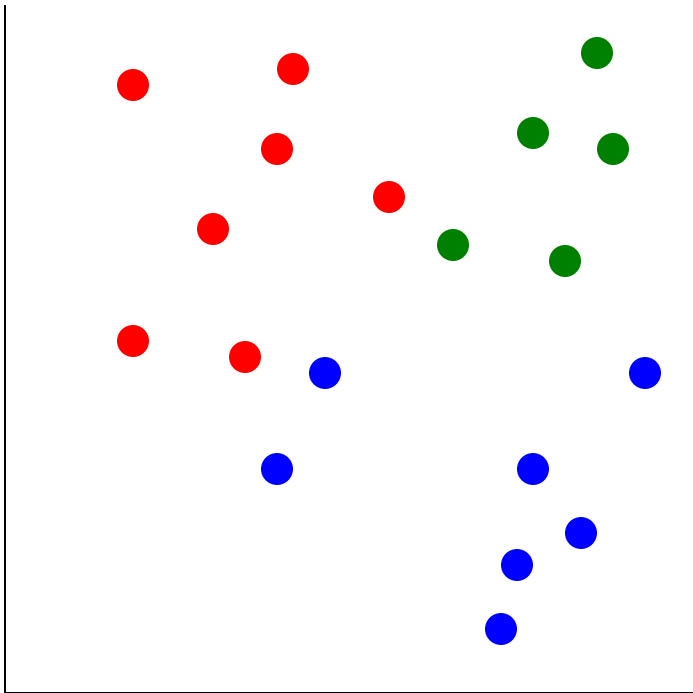
- If we are using huge features spaces (with kernels) how come we are not overfitting the data?
 - Number of parameters remains the same (and most are set to 0)
 - While we have a lot of input values, at the end we only care about the support vectors and these are usually a small group of samples
 - The minimization (or the maximizing of the margin) function acts as a sort of regularization term leading to reduced overfitting

Software

- A list of SVM implementation can be found at <http://www.kernel-machines.org/software.html>
- Some implementation (such as LIBSVM) can handle multi-class classification
- SVMLight is among one of the earliest implementation of SVM
- Several Matlab toolboxes for SVM are also available

Multi-class classification with SVMs

What if we have data from more than two classes?



- Most common solution: One vs. all
 - create a classifier for each class against all other data
 - for a new point use all classifiers and compare the margin for all selected classes

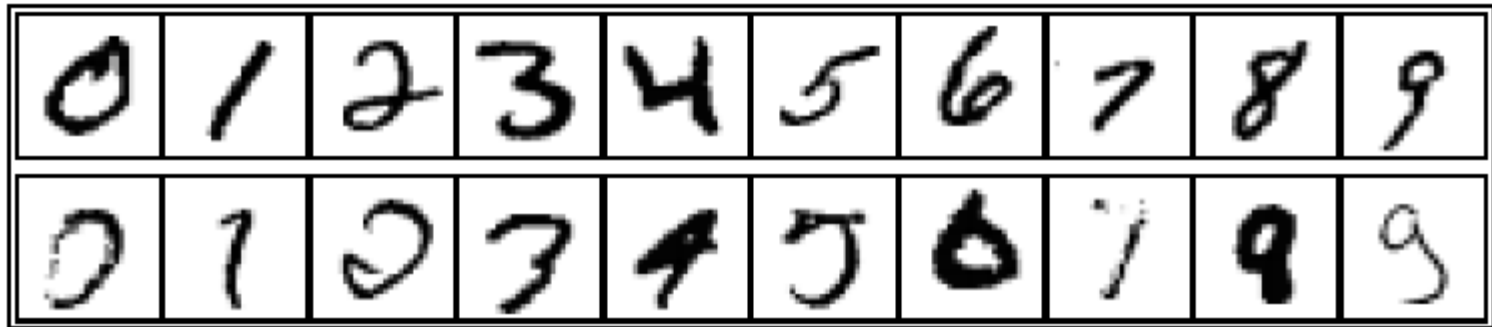
Note that this is not necessarily valid since this is not what we trained the SVM for, but often works well in practice

Applications of SVMs

- Bioinformatics
- Machine Vision
- Text Categorization
- Ranking (e.g., Google searches)
- Handwritten Character Recognition
- Time series analysis

→ Lots of very successful applications!!!

Handwritten digit recognition



3-nearest-neighbor = 2.4% error

400-300-10 unit MLP = 1.6% error

LeNet: 768-192-30-10 unit MLP = 0.9% error

Current best (kernel machines, vision algorithms) \approx 0.6% error

Important points

- Difference between regression classifiers and SVMs'
- Maximum margin principle
- Target function for SVMs
- Linearly separable and non separable cases
- Dual formulation of SVMs
- Kernel trick and computational complexity