Interlude - Multivariate Calculus

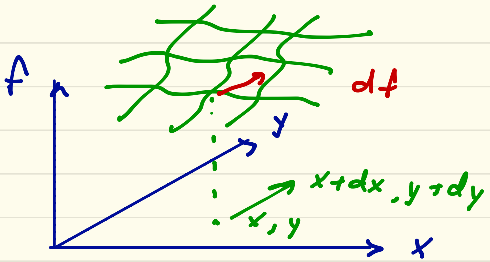
partial derivatives

$$f = f(x,y) \qquad \left(\frac{\partial f}{\partial x}\right)_y = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

ditto $\left(\frac{\partial f}{\partial y}\right)_x$

Total "exact" differential

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$$



$$\oint df = 0$$

Line integral along any closed loop = 0

A thermodynamic state function

$U, S, H, G, \ldots$

Always has this property

How to tell if $A(x,y)\,dx + B(x,y)\,dy$ is exact?

Must exist f such that

$$\left(\frac{\partial f}{\partial x}\right)_y = A \qquad \left(\frac{\partial f}{\partial y}\right)_x = B$$

Because order of differentiation doesn't matter for exact differential, can check

$$\left(\frac{\partial^2 f}{\partial x\,\partial y}\right) = \boxed{\left(\frac{\partial A}{\partial y}\right)_x \overset{?}{=} \left(\frac{\partial B}{\partial x}\right)_y}$$

Called a <u>Maxwell</u> <u>relation</u>

<u>Partial differential relations</u>

Given $df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$

Suppose we want to trace out a path of constant f in x,y space

$$df = 0 = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$$

$$= \left(\frac{\partial f}{\partial x}\right)_y \frac{dx}{dy} + \left(\frac{\partial f}{\partial y}\right)_x$$

$$0 = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_f + \left(\frac{\partial f}{\partial y}\right)_x$$

since $x = x(y,f)$

$$\boxed{\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_f \left(\frac{\partial y}{\partial f}\right)_x = -1}$$

cyclic permutation

Can similarly show

$$0 = \left(\frac{\partial f}{\partial x}\right)_y + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_f$$

$$\boxed{\left(\frac{\partial y}{\partial x}\right)_f = -\frac{\left(\partial f/\partial x\right)_y}{\left(\partial f/\partial y\right)_x} = \frac{1}{\left(\frac{\partial x}{\partial y}\right)_f}}$$

reciprocal rule

Suppose we want to parameterize x & y in terms of a new variable t:

$$x = x(t) \qquad y = y(t)$$

$$dx = \frac{dx}{dt}\,dt \qquad dy = \frac{dy}{dt}\,dt$$

$$df = \left\{ \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{dx}{dt}\right) + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{dy}{dt}\right) \right\} dt$$

Trace $df = 0$ path. Must have

$$\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{dx}{dt}\right) = -\left(\frac{\partial f}{\partial y}\right)_x \left(\frac{dx}{dt}\right)$$

cyclic permutation

$$\frac{(\partial y/\partial t)_f}{(\partial x/\partial t)_f} = -\frac{(\partial f/\partial x)_y}{(\partial f/\partial x)_x} = -\frac{(\partial f/\partial y)_x\,(\partial x/\partial x)_f}{(\partial f/\partial y)_x}$$

$$\boxed{\left(\frac{\partial y}{\partial x}\right)_f = \frac{(\partial y/\partial t)_f}{(\partial x/\partial t)_f}}$$

Homogeneous functions

$f(\lambda x, \lambda y) = \lambda^n f(x,y)$ nth order homogeneous

linear ftns, polynomial of order n, ...

$$\frac{\partial}{\partial \lambda} f(\lambda x, \lambda y) = \left(\frac{\partial f(\lambda x, \lambda y)}{\partial(\lambda x)}\right)_{\lambda y} \left(\frac{\partial \lambda x}{\partial \lambda}\right) + \cdots$$

chain rule

$\lambda x \to x \quad \lambda y \to y$

$$= \left(\frac{\partial f(x,y)}{\partial x}\right)_y \cdot x + \cdots$$

combine

$$\frac{\partial}{\partial \lambda} \lambda^n f(x,y) = n \lambda^{n-1} f(x,y)$$

$$n \lambda^{n-1} f(x,y) = \left(\frac{\partial f(x,y)}{\partial x}\right)_y \cdot x + \cdots$$

$\lambda \to 1 \quad n f(x,y) = \left(\frac{\partial f}{\partial x}\right) x + \left(\frac{\partial f}{\partial y}\right) y$

$n = 1 \quad \boxed{f(x,y) = \left(\frac{\partial f}{\partial x}\right) x + \left(\frac{\partial f}{\partial y}\right) y} \qquad f(\vec{r}) = \vec{r} \cdot \nabla f$

Euler relation

$$f(\lambda x, \lambda y) = \left(\frac{\partial f}{\partial(\lambda x)}\right)(\lambda x) + \left(\frac{\partial f}{\partial(\lambda y)}\right)(\lambda y) = \lambda f(x,y)$$

$$= \lambda \left[\left(\frac{\partial f}{\partial \lambda x}\right) x + \left(\frac{\partial f}{\partial \lambda y}\right) y \right]$$

Thus $\left(\frac{\partial f(\lambda x)}{\partial(\lambda x)}\right) = \left(\frac{\partial f(x)}{\partial x}\right)$

If $f(x,y)$ is nth order homogeneous, $\partial f / \partial x$ must be $n-1$ order homogeneous

$U \to$ 1st order $\left(\frac{\partial U}{\partial S}\right)$ 0th order