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QUANTUM MECHANICS NOTES

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The rotation operator

[1] We can represent $|+z\rangle$ as a vector in a Cartesian coordinate system. If we rotate this vector anticlockwise by 90°, we get $|+x\rangle$. Mathematically, this can be written as,

[1] `rotation_operator.tex`

$$|+x\rangle = \hat{R}\left(\frac{\pi}{2}\mathbf{j}\right)|+z\rangle\,, \tag{1}$$

where $\hat{R}$ is called the *rotation operator*. The terms inside the brackets indicate that the ket is rotated by an angle 90°about the $y-$ axis [2]. The same rotation operator rotates $|-z\rangle$ into $|-x\rangle$, $|-y\rangle$ into $|-x\rangle$, and so on.

[2] Try to be aware of this: we are using the Cartisian coordinate system to represent the spin vectors visually. These spin vectors live in a complex vector space, and are different from the vectors we use to represent classical angular momentum. The Cartesian coordinate system, vector, and the complex vector space are just some of the tools we have developed to reconstruct the physical world in our minds. They do *not* exist in nature, but they are the building blocks of our mental models of the world. We conduct experiments to check if these mental models are indeed in tune with the real world out there. This activity of building mental models and checking if they correctly mimic the reality is called *science*.

Let's now see the effect of this rotation operator on the most general spin-1/2 state given by **??**.

$$\hat{R}\left(\frac{\pi}{2}\mathbf{j}\right)|\psi\rangle = c_+\hat{R}\left(\frac{\pi}{2}\mathbf{j}\right)|+z\rangle + c_-\hat{R}\left(\frac{\pi}{2}\mathbf{j}\right)|-z\rangle \tag{2}$$

Using eq. (1) in eq. (2), we get,

$$\hat{R}\left(\frac{\pi}{2}\mathbf{j}\right)|\psi\rangle = c_+\,|+x\rangle + c_-\,|-x\rangle\,. \tag{3}$$

Yes, the rotation operator has changed the basis in which the state $|\psi\rangle$ was desribed!

The adjoint operator

In the previous section section, we discussed the effect of a rotation operator on a ket vector. What if we are dealing with bra vectors, instead of kets? In other words, what is the bra equation corresponding to eq. (1)? Is it $\langle +x| = \langle +z|\,\hat{R}\left(\frac{\pi}{2}\mathbf{j}\right)$? Let's check.

We know that the amplitude for a state to be in itself is 1. Mathematically, this is written as,

$$\langle +x| + x\rangle = 1$$

Assuming that $\langle +x| = \langle +z|\,\hat{R}\left(\frac{\pi}{2}\mathbf{j}\right)$ is correct, we can write,

$$\langle +z|\,\hat{R}\left(\frac{\pi}{2}\mathbf{j}\right)\hat{R}\left(\frac{\pi}{2}\mathbf{j}\right)|+z\rangle = 1 \tag{4}$$

We have, $\hat{R}\left(\frac{\pi}{2}\mathbf{j}\right)\hat{R}\left(\frac{\pi}{2}\mathbf{j}\right) = |-z\rangle$. This leads to,

$$\langle +z| - z\rangle = 1 \tag{5}$$

This is an absurd result because, the LHS is equal to zero. Therefore, our assumption that $\langle +x| = \langle +z|\,\hat{R}\left(\frac{\pi}{2}\mathbf{j}\right)$ is incorrect. Then, what is the bra equation corresponding to eq. (1)?

Let's now introduce a new operator $\hat{R}^\dagger$, called the *adjoint* of $\hat{R}$. Now the bra equation becomes,

$$\langle +x| = \langle +z| \, \hat{R}^\dagger \left(\frac{\pi}{2}\mathbf{j}\right) \tag{6}$$

Normalisation condition is now satisfied:

$$\begin{aligned}
\langle +x \mid +x \rangle &= \left\langle +z \left| \hat{R}^\dagger \left(\frac{\pi}{2}\mathbf{j}\right) \hat{R} \left(\frac{\pi}{2}\mathbf{j}\right) \right| + z \right\rangle \\
&= \left\langle +z \left| \hat{R}^\dagger \left(\frac{\pi}{2}\mathbf{j}\right) \right| + x \right\rangle \\
&= \langle +z \mid +z \rangle \\
&= 1
\end{aligned}$$

Therefore, in order for the ket to satisfy normalisation condition even after rotation, it is necessary to introduce a new operator whose effect is to negate the action of the rotation operator. Mathematically, the new operator is represented by the adjoint of the rotation operator. Naturally, successive action of rotation operator and its adjoint will leave the ket unchanged. It is useful to interpret this to be equivalent to the action of a new type of operator called *identity operator*. Mathematically, this can be written as,

$$R\hat{R}^\dagger = \hat{I} \tag{7}$$

In summary, a rotation operator $\hat{R}\left(\theta\mathbf{n}\right)$ changes a ket by rotating it by an angle θ around the axis specified by the unit vector $\mathbf{n}$.

The generator of rotations

The rotation of a vector by an infinitesimally small angle $d\phi$ about the $z-$ axis can be presented by the operator:

$$\hat{R}(d\phi\mathbf{k}) = \hat{I} - \frac{i}{\hbar}\hat{J}_z d\phi \tag{8}$$

Here, $\hat{J}_z$ is called the *generator of rotations*. We can rotate a vector by a finite angle ϕ by repeatedly carrying out inifinitesimally small rotations of magnitude $d\,\phi$. [3]

[3] In Eq 14, since the LHS has the dimensions of angle, the RHS should also have the same dimensions. That means, the dimensions of $\hbar$ and $\hat{J}_z$ should cancel. Since, $\hbar$ has the dimensions of angular momentum, $\hat{J}_z$ should also have the dimensions of angular momentum (otherwise they can't cancel each other).

The adjoint of the above rotation operator is obtained by:

1. Replacing the complex numbers by their respective conjugates
2. Replacing the generator of rotations operator by its adjoint

Therefore,

$$\hat{R}^\dagger(d\phi\mathbf{k}) = \hat{I} + \frac{i}{\hbar}\hat{J}_z^\dagger d\phi \tag{9}$$

Since, $\hat{R}\hat{R}^\dagger = \hat{I}$, we have,

$$\left(\hat{I} - \frac{i}{\hbar}\hat{J}_z d\phi\right)\left(\hat{I} + \frac{i}{\hbar}\hat{J}_z^\dagger d\phi\right) = \hat{I}$$
$$\hat{I} + \frac{i}{\hbar}\hat{J}_z^\dagger d\phi - \frac{i}{\hbar}\hat{J}_z d\phi + \frac{1}{\hbar^2}\hat{J}_z\hat{J}_z^\dagger (d\phi)^2 = \hat{I}$$

Since the angle $d\phi$ is infinitesimally small, the last term on the LHS can be neglected. This leads to,

$$\hat{I} + \frac{i}{\hbar}\left(\hat{J}_z^\dagger - \hat{J}_z\right) d\phi = \hat{I}, \tag{10}$$

which can be true only if $\hat{J}_z^\dagger = \hat{J}_z$. Operators that are equal to their own adjoints are said to be *self-adjoint* or *Hermitian*. [4]

[4] The generaot of rotations is a Hermitian operator, but the rotation operator is not.

There is one more way to discover that $\hat{J}_z$ is Hermitian. If we rotate the ket in clockwise direction, the rotation operator becomes:

$$\begin{aligned} \hat{R}(-\phi\mathbf{k}) &= \hat{I} - \frac{i}{\hbar}\hat{J}_z(-d\phi) \\ &= \hat{I} + \frac{i}{\hbar}\hat{J}_z d\phi \end{aligned} \tag{11}$$

By the definition of adjoint of rotation operator, we have,

$$\begin{aligned} \hat{R}(-d\phi\mathbf{k}) &= \hat{R}^\dagger(d\phi\mathbf{k}) \\ \hat{I} - \frac{i}{\hbar}\hat{J}_z(-d\phi) &= \hat{I} + \frac{i}{\hbar}\hat{J}_z^\dagger d\phi \end{aligned} \tag{12}$$

This can be true only if $\hat{J}_z = \hat{J}_z^\dagger$.

Identity operators

[5] The general state $|\psi\rangle$ of a spin-$\frac{1}{2}$ particle is:

[5] identity_operator.tex

$$|\psi\rangle = c_+ \, |{+z}\rangle + c_- \, |{-z}\rangle$$

Multiplying by by $|{+z}\rangle$, we get,

$$\langle +z|\psi\rangle = c_+ \, \langle +z| + z\rangle$$

or,

$$c_+ = \langle +z|\psi\rangle$$

Similarly, by multiplying by by $|{-z}\rangle$, we get,

$$c_- = \langle -z|\psi\rangle$$

Now, $|\psi\rangle$ can be written as,

$$\begin{aligned} |\psi\rangle &= \langle +z|\psi\rangle \, |{+z}\rangle + \langle -z|\psi\rangle \, |{-z}\rangle \\ &= |{+z}\rangle \, \langle +z|\psi\rangle + |{-z}\rangle \, \langle -z|\psi\rangle \\ &= |{+z}\rangle \, \langle +z| \, |\psi\rangle + |{+z}\rangle \, \langle -z| \, |\psi\rangle \end{aligned}$$

Or,

$$\boxed{|\psi\rangle = [|{+z}\rangle \, \langle +z| + |{-z}\rangle \, \langle -z|] \, |\psi\rangle}$$

On the right hand side, the term in the square bracket can be considered as an operator that acts on the state $|\psi\rangle$, but doesn't change it at all. It is example of an *identity operator*.