# Notes for the Australian Mathematics Advanced Stage 6 Course

# Contents

# Part I

# Year 11

# Chapter 1

# Functions

## 1.1 Functions and Relations

A function takes in input and returns exactly one output, whereas a relation can return more than one. A function is a mapping between two sets.

### 1.1.1 Definition of a function

Functions can be thought of as a set of ordered pairs $(x, y)$.

### 1.1.2 Domain, range, independent and dependent variables

**Definition 1.1.1** (Domain and range)**.** The domain of a function is the set of all possible inputs values a function can have. The range of a function is the set of all possible outputs a function can have.

**Definition 1.1.2** (Independent and dependent variables)**.** A function $f(x)$ has $x$ as the independent variable. The output (or $f(x)$ itself) is the dependent variable, as it is dependent on $x$.

Interval notation is used to describe unbroken portions of the real line.

**Definition 1.1.3** (Interval notation)**.** The *open* interval $(a, b)$ represents the set of all real numbers between $a$ and $b$ but not including $a$ or $b$. The *closed* interval $[a, b]$ contains all real numbers between $a$ and $b$ and includes the endpoints $a$ and $b$. The *half-open* interval $[a, b)$ is the set of all real numbers between $a$ and $b$ but only includes the endpoint $a$. Similarly, the half-open interval $(a, b]$ only contains the endpoint $b$.

Interval notation is used with the $\in$ symbol (read as "in") You may use the notation $x \in \mathbb{R}$ to denote $x$ being any real number. Intervals are combined with the union (meaning "or") symbol, such as in

$$x \in (-\infty, 0) \cup (0, \infty)$$

### 1.1.3 Graphs of functions

### 1.1.4 Types of functions and relations

**Definition 1.1.4** (One-to-one)**.** A *one-to-one* function takes in one input and returns exactly one output. It passes both horizontal and vertical line tests. For example,

$$f(x) = x$$

is a one-to-one function.

**Definition 1.1.5** (Many-to-one)**.** A *One-to-many* function has more than one input that produce the same output. It fails the horizontal line test but passes ther vertical. An example is

$$f(x) = x^2$$

where both $x = -1$ and $x = 1$ give the output $1$.

**Definition 1.1.6** (One-to-many)**.** A *one-to-many* relation has an input that outputs more than one number. It passes the horizontal line test but fails the vertical. An example relation is

$$y^2 = x$$

where $x = 1$ has the outputs $-1$ and $+1$.

**Definition 1.1.7** (Many-to-many)**.** A *many-to-many* relation has multiple input which output multiple of the same outputs. It fails both horizontal and vertical line tests. An example is the circle

$$x^2 + y^2 = 1$$

where $x = 1$ or $x = -1$ both produce outputs $1$ and $-1$.

### 1.1.5 Properties of functions

**Definition 1.1.8** (Even function)**.** A function $f$ is even if it satisfies,

$$f(-x) = f(x).$$

The graph of $f$ is symmetric about the $y$-axis.

**Definition 1.1.9** (Odd function)**.** A function $f$ is odd if it satisfies,

$$f(-x) = -f(x).$$

The graph of $f$ has point symmetry about the origin (if you spin the graph $180°$ about the origin, its the same graph).

**Definition 1.1.10** (Algebra of functions)**.** Two functions $f$ and $g$ can be added $(f(x) + g(x))$, subtracted $(f(x) - g(x))$, multiplied $(f(x)g(x))$ together or divided $(f(x)/g(x)$ provided $g(x)$ is never $0)$, forming a new function. The domain of the new function is the intersection of the domain of $f$ and the domain of $g$. The range is more difficult to find.

**Definition 1.1.11** (Function composition)**.** Function composition is another way to combine functions to form new functions. The composition of functions $f$ and $g$ is denoted as

$$f \circ g(x) = f(g(x)).$$

The domain of $f \circ g$ is the domain of $g$, whose outputs must also lie in the domain of $f$. The range of $f \circ g$ is all the outputs that from the range of $g$ as input.

### 1.1.6 Solutions to functions

When we solve the equation

$$f(x) = 0,$$

we are solving for the $x$ values that are sent to $0$ by the function. Graphically, they are the $x$-intercepts. This is because we are finding all ordered pairs whose $y$ (output) value is $0$, corresponding to the $x$-intercepts.

## 1.2 Linear, quadratic and cubic functions

### 1.2.1 Linear functions

A linear function is of the form

$$f(x) = mx + c$$

where $m$ and $c$ are any real numbers. $m$ represents the gradient and $c$ is the $y$-intercept. This produces a straight line graph.

**Definition 1.2.1** (Point-gradient form)**.** The unique line that passes through the point $(x_1, y_1)$ and has a gradient of $m$ has the equation

$$y - y_1 = m(x - x_1)$$

If you had only two points and no gradient, first calculate the gradient from the two points, and use the above equation. The equation for gradient between two points $(x_1, y_1)$ and $(x_2, y_2)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

**Definition 1.2.2** (Parallel and perpendicular lines)**.** Two lines a parallel if and only if their gradients are equal. If their $y$-intercepts are also equal then they are they exact same line. Two lines are perpendicular if and only if their gradients are negative recipricols of each other. That is, if two gradients are $m_1$ and $m_2$, then

$$m_1 m_2 = -1.$$

### 1.2.2 Quadratic functions

A quadratic function is of the form

$$f(x) = ax^2 + bx + c$$

for some real numbers $a, b$ and $c$.

**Definition 1.2.3** (Vertex)**.** The *vertex* (or *turning point*) of a parabola is where the curve of the parabola changes direction. The $x$-value of the vertex is

$$x = \frac{-b}{2a}.$$

The parabola is symmetric across the line containing the vertex.

Parabolas may have zero, one or two vertices, which also means the equation $f(x) = 0$ has zero, one or two solutions.

**Definition 1.2.4** (Completing the square)**.** Given a parabola of the form

$$f(x) = ax^2 + bx + c,$$

completing the square allows us to form a square term plus a constant.

Steps to completing the square:

i) Factor out the coefficient of $x^2$.

$$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

ii) Add and subtract the square of the half of the coefficient of $x$ in the bracket.

$$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right)$$

iii) The first three terms above form a perfect square

$$f(x) = a\left(\left(x - \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right)$$

iv) Expand the brackets.

$$f(x) = a\left(x - \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

We also call this form *vertex form*. If

$$f(x) = a(x-h)^2 + k$$

then the vertex is at $(h, k)$

Factorising quadratics may be done in many ways. I propose only one.

**Example 1.2.5** (Factorising quadratics)**.** Suppose we have to factorise

$$f(x) = 6x^2 - 17x + 12.$$

We need to find two numbers $a$ and $b$ that

- MULTIPLY to $72$,
- SUM to $-17$.

You start listing pairs of positive numbers that multiple to $72$ in your head (or on paper). Whilst you scan for pairs, think about which pair could possibly combined (with addition and subtraction) to make $-17$. That pair is $-9$ and $-8$. Check that the sign of the product is positive. Split the middle term into $-9x$ and $-8x$ to get

$$f(x) = (6x^2 - 9x) + (-8x + 12),$$

group the terms as such and factorise.

**Definition 1.2.6** (The quadratic formula)**.** Given a quadratic

$$f(x) = ax^2 + bx + c,$$

the solutions to $f(x) = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We call $b^2 - 4ac$ the *discriminant* of the parabola.

**Definition 1.2.7** (Discriminant)**.** The discriminant of a quadratic function $f(x) = ax^2 + bx + c$ is

$$\Delta = b^2 - 4ac.$$

1. If $\Delta > 0$, there are two solutions. The graph does not touch the $x$-axis.
2. If $\Delta = 0$, there is one solution. The vertex is the only point that touches the $x$-axis.
3. If $\Delta < 0$, there is no solutions. The graph passes through the $x$-axis at different points.

### 1.2.3 Simultaneous equations

### 1.2.4 The equation $f(x) = k$

When we algebraically solve the equation $f(x) = k$, graphically, we are finding the point of intersection of the graph of $f$ with the horizontal line $y = k$.

### 1.2.5 Application: break-even point

### 1.2.6 Cubic functions

The cubic functions we will study here are of the form

$$\begin{aligned} f(x) &= kx^3 && (1.1)\\ \text{or } f(x) &= k(x-b)^3 + c && (1.2)\\ \text{or } f(x) &= k(x-a)(x-b)(x-c) && (1.3) \end{aligned}$$

where $a, b, c, k$ are real numbers.

- For Function 1.1, its only turning point is at $(0,0)$ and increasing $|k|$ makes it more narrow. The sign of $k$ determines the sign of the function as $x \to \pm\infty$.
- For Function 1.2, its only turning point is at $(b,c)$, and is only a shifted Function 1.1.
- For Function 1.3, it has three $x$-intercepts at $a$, $b$ and $c$. The sign of $k$ determines where the function starts and ends.

## 1.3 Further functions and relations

### 1.3.1 Polynomials and the polynomial functions

**Definition 1.3.1** (Polynomials and polynomial function)**.** A polynomial is an expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $n$ is a nonnegative ($\geq 0$) integer, $a_1, \ldots, a_n$ are real numbers with $a_n \neq 0$. A polynomial function is a function built from the polynomial expression, that is

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

**Definition 1.3.2** (Polynomial terms)**.** Let $p$ be a polynomial function, that is

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

We call $n$ the *degree* of the polynomial. The coefficient $a_n$ is called the *leading coefficient*.

You will often come across polynomials in their factored form, and asked to sketch. That is, to sketch something of the form

$$p(x) = k(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n).$$

First, all the roots are also the $x$-intercepts. Then find the sign of the leading coefficient (sign of $k$ above) as that tells you where the function starts and ends.

### 1.3.2 The hyperbolic function

The hyperbolic function is important to study because they represent quantities that decrease rapidly as its dependent variable increases.

**Definition 1.3.3** (The hyperbolic function)**.** The hyperbolic function we will study here is of the form

$$f(x) = \frac{k}{x}.$$

It has asymptotes $x = 0$ and $y = 0$.

### 1.3.3 The absolute value function

The use of the absolute value function is to find the "size" of real numbers. That is the size of $-2$ is $2$ because it is a distance of $2$ away from $0$ on the numberline.

**Definition 1.3.4** (The absolute value function)**.** The basic absolute value function is

$$f(x) = |x|$$

and is defined by the piecewise function

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

We can generalise the above function further to absolute functions of the form

$$f(x) = |ax + b|$$

where $a$ and $b$ are real numbers. The piecewise function that defines these are

$$f(x) = \begin{cases} ax + b & \text{if } x \geq \frac{-b}{a} \\ \\ -ax - b & \text{if } x < \frac{-b}{a} \end{cases}.$$

How to sketch $f(x) = |ax + b|$.

1. Find the $x$ intercept by solving $ax + b = 0$. Then sketch each seperate line according to the cases above.

How to solve absolute value equations. Let us try to solve $|ax + b| = k$.

1. Split the equation into two equations. That is

$$\begin{cases} ax + b = k & \text{if } x \geq \frac{-b}{a} \\ \\ -ax - b = k & \text{if } x < \frac{-b}{a} \end{cases}.$$

2. Solve each case seperately. Check that your solution for $x$ lies in the corresponding restriction $x$.

Graphically, we are finding the point of intersection between $|ax + b|$ and $y = k$.

### 1.3.4 Function transormations

**Theorem 1.3.5.** *Given a function $f(x)$, the function $-f(x)$ is the function $f$ reflected across the $x$-axis.*

**Theorem 1.3.6.** *Given a function $f(x)$, the function $f(-x)$ is the function $f$ reflected across the $y$-axis.*

**Theorem 1.3.7.** *Given a function $f(x)$, the function $-f(-x)$ is the function $f$ reflected across the $x$-axis, then reflected across the $y$ axis.*

### 1.3.5 Circles

A circle with radius $r$ and centre at the origin has the equation

$$x^2 + y^2 = r^2.$$

Note this is not a function. A general circle with radius $r$ and centre at $(a, b)$ has the equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

The equation is derived with the Pythagoras' theorem. If the question gave $x^2 + y^2 + ax + by + c = 0$, complete the square to get the equation into the form $(x - a)^2 + (y - b)^2 = r^2$.

### 1.3.6 Semicircles

A positive semicircle with radius $r$ and centre at the origin has the equation

$$y = \sqrt{r^2 - x^2}.$$

A negative semicircle has the equation

$$y = -\sqrt{r^2 - x^2}.$$

Both of these semicircles are functions.

# Chapter 2

# Trigonometric Functions

## 2.1 Trigonometry and measure of angles

### 2.1.1 Introduction to trigonometry

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

This can be remembered with the handy mnemonic SOH CAH TOA. Pronounced (SO-KAH-TOWA).

### 2.1.2 Special formulae

**Theorem 2.1.1** (Sine rule)**.** *Let a triangle have angles $A$, $B$, $C$, and side lengths $a$, $b$, $c$ opposite their respective angles. The sine rule states that*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

*Consequently, we have that*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

**Theorem 2.1.2** (Cosine rule)**.** *Let a triangle have angles $A$, $B$, $C$, and side lengths $a$, $b$, $c$ opposite their respective angles. The cosine rule states that*

$$c^2 = a^2 + b^2 - 2ab\cos C$$

*Similarly, $a^2 = b^2 + c^2 - 2bc\cos A$ and $b^2 = a^2 + c^2 - 2ac\cos B$. There are also the formulas for the angle at a vertex*

$$C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right).$$

*The angle formulas are similar for the other angles $B$ and $A$.*

**Theorem 2.1.3** (Area of a triangle)**.** *Let a triangle have two sides $a$ and $b$, and an included angle (angle between the sides) $C$. Then the area of the triangle is given by*

$$A = \frac{1}{2} ab \sin C.$$

DON'T FORGET THE AMBIGUOUS CASE.

**Definition 2.1.4** (Angle of elevation and depression)**.** The angle of elevation an object makes with an observer, is the angle the object makes with the horizontal. The observer has to "look" up for it to be called an angle of elevation. Similarly, the angle of elevation is the angle the object makes with the horizontal, but the observer has to "look" down.

# Part II

# Year 12