Chaos in light

Robert G. Harrison

Physics Department, Heriot-Watt University, Edinburgh EH14 4AS, UK

Dhruba J. Biswas

MDRS (Physics Group), Bhabha Atomic Research Centre, Bombay 400 085, India

Chaos is an inherent feature of many nonlinear systems. In particular, the transition from a steady to chaotic state occurs independently of the physical properties of the system. Such behaviour occurs in optics, both in lasers and in nonlinear optical devices. Such devices, which are fundamentally simple both in construction and in the mathematics that describes them, provide excellent opportunities for investigating nonlinear phenomena as well as for technological innovation.

MATHEMATICAL discoveries have revolutionized our understanding of nonlinear science. The preconception that physical systems, in general, behave in a predictable manner is seriously in question. Rather than yielding regular and repeatable behaviour, many nonlinear systems also exhibit unstable, even chaotic, solutions. Furthermore, the transition from stable to chaotic behaviour, which may occur when varying a control parameter of the system, follows specific, well-defined routes which are universal in the sense that they are independent of the physical properties of the system they describe. It is these signatures which have been a major impetus to experimentalists in the subsequent search for physical systems that exhibit these phenomena. Such behaviour has now been observed in many branches of science and the number is rapidly growing.

The most recent exciting development is the discovery that such phenomena exist in optics. Here we are concerned, in general, with the nonlinear interaction of light with media contained in optical resonators; the physical properties of the medium, such as absorption and refractive index, being modified by the intensity of the incident radiation.

There are two major areas. The first are lasers or active systems, in which the optical signal is derived from stimulated emission generated within an optical cavity containing a gain medium. The second are passive systems for which the optical signal is but the transmission of an input light signal through an optical cavity containing, for example, an absorptive medium.

Instabilities in laser emission, notably in the form of spontaneously coherent pulsations, have been observed almost since the first demonstration of laser action. However, subsequent theoretical efforts towards understanding these phenomena have been at a modest level, due in part to the wide variety of alternative areas of investigation provided by lasers. It is only with the new mathematical discoveries that these instability phenomena have been investigated to give a deeper insight into the mechanisms of laser action and its deterministic chaotic behaviour. Re-examination of many of these systems show that such effects are quite abundant; the operating window for conventional stable emission in some systems often proving surprisingly small while in some cases, the instabilities are found to prevail just where the lasing emission is optimum.

On the other hand, passive systems which are being increasingly recognized for their potential application as bistable all-optical logic elements, may give rise to similar phenomena. It may, nevertheless, be possible to take advantage of the periodic instabilities that precede chaos in the development of ultra-high frequency all-optical modulators.

Of the variety of physical systems that exhibit deterministic instability phenomena, optical systems, both lasers and passive devices, provide nearly ideal systems for quantitative investigation due to their simplicity both in construction and in the mathematics that describe them, enriched by the possibility of a quantum description. Notable also is the very short timescale (nanosecond to microsecond) over which optical instabilities occur which, in contrast to many other systems, ensures essentially constant environmental conditions during data acquisition. This is particularly important since even small extraneous perturbations, such as noise, may dramatically alter the form of the subsequent temporal evolution of the instability process. These features are fundamental to the rapid establishment of optical systems in this multidisciplinary field.

Universality in chaos

In considering deterministic behaviour, one is tempted into the misconception that such behaviour must be regular since successive states evolve continuously from each other. However, as early as 1892 Poincare showed that particular mechanical systems, where time evolution is governed by hamiltonian equations, could display chaotic behaviour. The subsequent discovery by Lorentz[1] in 1963 that even a simple set of three coupled first-order, nonlinear differential equations can lead to completely chaotic trajectories is recognized as a landmark. This work is fundamental to our understanding of laser instabilities.

At first sight, such behaviour appears alien to our conception of many problems in physical science. This prejudice stems from the dominance of mathematical theory pertaining to linear systems and its subsequent successful application to many fundamental linear problems in the physical sciences. Unfortunately, however, this has led to a narrow vision of the physical world where nonlinear behaviour is the rule and linear behaviour the exception.

Unlike linear systems, nonlinear systems must be treated in their full complexity, and so there is no general analytical approach for solving them. The advances made in understanding many previously intractable nonlinear problems can largely be attributed to the power of contemporary computers, where simulated solutions of nonlinear equations have provided insights into their behaviour and suggested directions for future research.

The temporal evolution in the behaviour of a system can be characterized when presented as a trajectory of a point in the phase space of its dynamical variables. In this representation, consider a familiar dynamical system such as a periodically-forced pendulum in a frictional environment. Such a dynamical system is characterized by the fact that the rate of change of its variable is given as a function of the value of the variable at that time. The space defined by the variables is called the phase space. The pendulum's behaviour can be described by the motion of a point in a two-dimensional phase space whose coordinates are the position and velocity of the pendulum. In

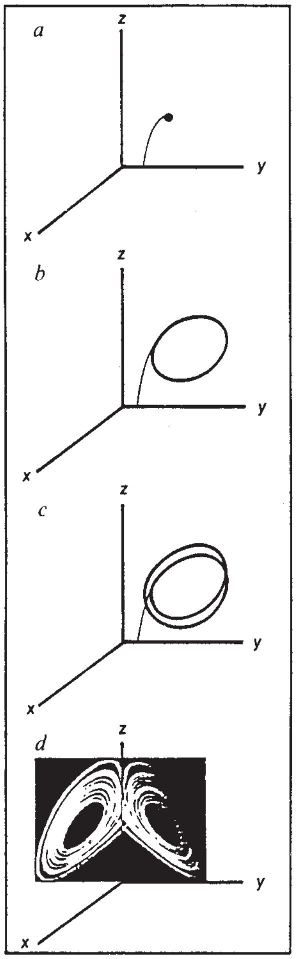


Fig. 1 Phase space portraits of the attractor for the dynamical variables x, y and z. *a*, Stable point corresponding to a steady state in time after initial transients have died out shown as the thin continuous line. *b*, Period-one limit cycle corresponding to a periodic solution in time of single frequency. *c*, Bifurcation to period-two limit cycle; a periodic solution in time with double the period of that in *b*. Successive period doubling bifurcations lead to an eventual chaotic colution. *d* Shows the strange or chaotic attractor for a Lorenz–Haken system describing a single mode laser with a homogeneously broadened two-level gain medium.

more complicated systems, involving many variables, the dimension of the phase space will, however, be considerably larger. If an initial condition of a dissipative dynamical system, such as the pendulum, is allowed to evolve for a long time, the system, after all the transients have died out, will eventually approach a restricted region of the phase space called an attractor. A dynamical system can have more than one attractor in which case different initial conditions lead to different types of long-time behaviour.

The simplest attractor in phase space is a fixed point. The system is attracted towards this point and stays there. This is the case for a simple pendulum in the presence of friction; regardless of its initial position, the pendulum will eventually come to rest in a vertical position. When the pendulum is under the influence of an external periodic driving force, the system is then fully nonlinear and leads to strikingly different behaviour. Irrespectively of the initial conditions the pendulum always ends up making a periodic motion. The limit or attractor of the motion is a periodic cycle called a limit cycle. However, when the driving force exceeds a certain critical value, the periodic motion of the pendulum breaks down into a more complex chaotic pattern which never repeats itself. This motion represents a third kind of attractor in phase space called a chaotic or strange attractor (see Fig. 1).

A trajectory on a chaotic attractor exhibits most of the properties intuitively associated with random functions, although no randomness is ever explicitly added. The equations of motion are purely deterministic; the random behaviour emerges spontaneously from the nonlinear system. Over short times, the trajectory of each point can be followed, but over longer periods small differences in position are greatly amplified making the predictions of long-term behaviour impossible. As such arbitrarily close initial conditions can lead to trajectories which after a sufficiently long time diverge widely; even for the simplest of systems in which all the parameters are determined exactly, long-term prediction is, therefore, impossible. This behaviour is in marked contrast to that of the fixed point and limits cycle attractors for which, irrespective of starting conditions, the system always settles down to the same solutions.

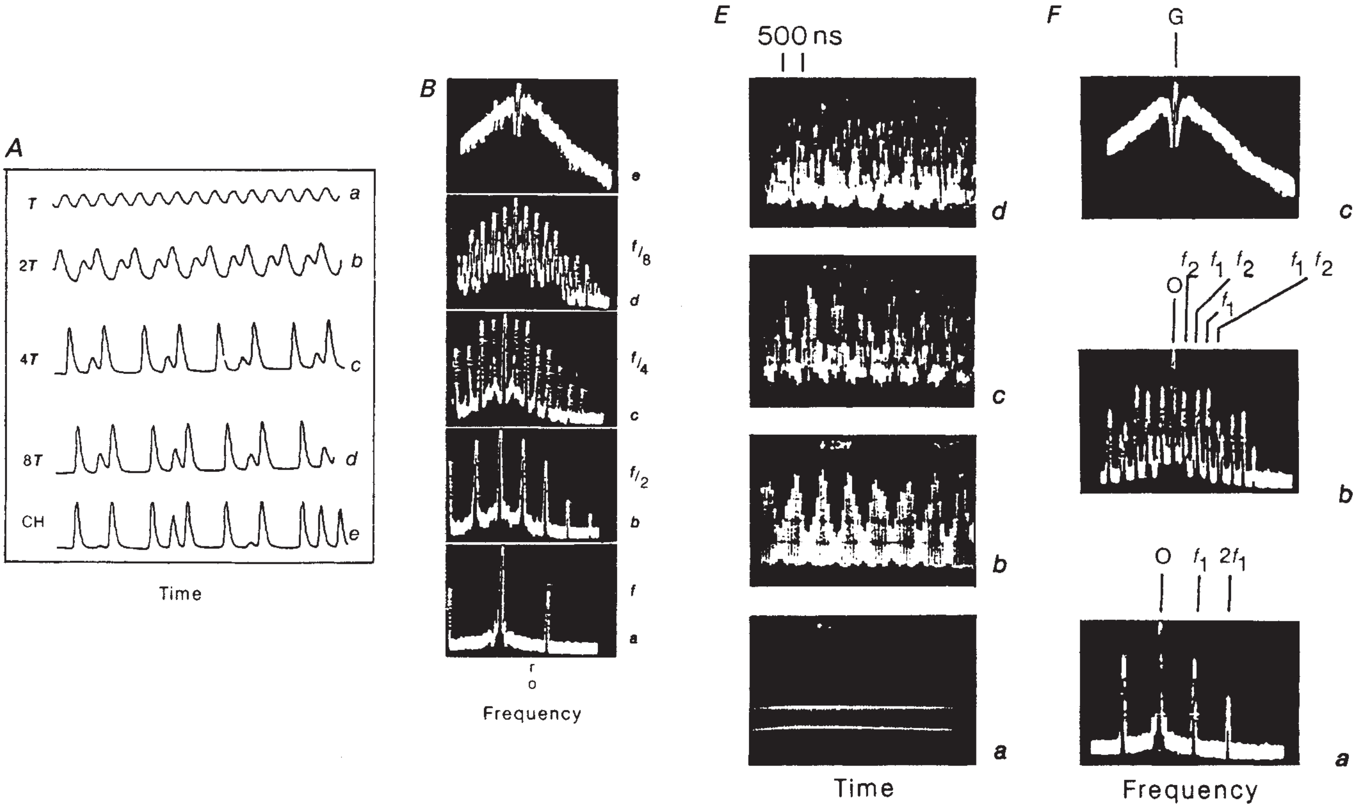
Irratic and aperiodic temporal behaviour of any of the systems' variables implies a corresponding continuous spectrum for its Fourier transform which is, therefore, also a further signature of chaotic motion. However, other factors including noise, can lead to continuous spectra, and distinguishing chaos from noise is one of the major problems of the field. Hence, although time series, power spectra and routes to chaos collectively provide strong evidence of deterministic behaviour further signatures are desirable for its full characterization and in discriminating it from stochastic behaviour. Here analysis of trajectories of a point in the phase space of its dynamical variables is required. However, for a system with, say, N degrees of freedom it seemed that it would be necessary to measure N independent variables; an awesome if not impossible task for complex system. Consequently mathematicians have long tried to develop practical techniques for extracting specific finite dimensional information from the limited output provided by experiment; typically the time record of a specific physical observable; that is, one variable of the system. Here embedding theorems[2] have been recently used to reconstruct phase portraits from which Lyapunov exponents may be determined that measure the average rate of exponential separation or contraction of nearby points on the attractor. These measure intrinsically dynamical properties, unlike power spectra, and provide quantitative measures by which chaotic motion may be distinguished from stochastic behaviour.

The discoveries that deterministic chaos proceeds through a limited number of specific routes when a control parameter of the nonlinear system is varied is profoundly significant as such behaviour is not restricted to a particular model description of a particular physical system. Rather, nonlinear physical systems in all branches of science which may be formally described by the same set of mathematical equations will give solutions that evolve identically in time through one or other routes to chaotic motion. The unique effect of such unification between many separate scientific diciplines forms the basis for the foundation of synergetics[3,4]. There are at least three common routes by which a nonlinear system may become chaotic. These are referred to as period doubling, intermittency and two-frequency scenarios.

Period doubling. From considering various difference equations, many of which can be reduced to simple one-dimensional maps, solutions have been found to oscillate between stable values, the period of which successively doubles at distinct values of the external control parameter[5,6,7]. This continues until the number of fixed points becomes infinite at a finite parameter value, where the variation in time of the solutions becomes irregular (see Fig. 1). One example showing such behaviour is the simple logistic map

$$X_{n+1} = rX_n(1 - X_n)$$

Perhaps the most popular application of this map is in describing



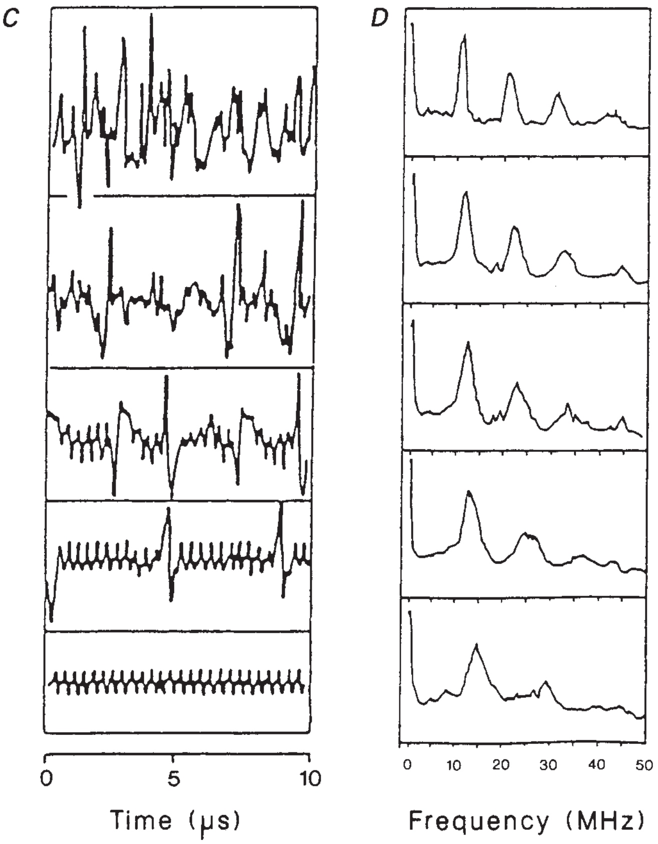


Fig. 2 *A*, *B* Examples of period doubling route to chaos for laser systems. The time recordings shown in *A* are for a CO_2 laser in which cavity length modulation is used as a control parameter for the generation of chaos (after ref. 20) and *B* shows a spectral recording of the lasing emission for an optically pumped far-infrared ammonia laser with cavity length tuning as a control parameter (after ref. 49). The steady-state solution as in Fig. 1*a* are not shown in these recordings. As seen initial oscillation (*Aa* and *Ba*) undergoes successive period doublings (*Ab–d*) or frequency halving (*Bb–d*) before culminating in chaos where in time the output is aperiodic (*Ae*) and in frequency is a broad continuum (*Be*). *C*, *D*, Examples of intermittent route to chaos in lasers. The time recordings shown in *C* are for a He–Ne laser in which cavity mirror tilt is used as the control parameter (after ref. 50, and *D* shows a spectral recording of the lasing emission for a He–Xe laser with laser gain as a control parameter (after ref. 25). Steady-state solutions are not shown in the recordings. The initial instability here is characterized by a narrow spectral peak in frequency and periodic pulsation in time. On varying the control parameter the intermittent bursts of chaos become increasingly frequent in time with a corresponding steady broadening of the peaks in the frequency followed by fully chaotic motion. *E*, *F*, Examples of Ruelle-Takens route to chaos in lasers. The time recordings shown in *E* are for a multimode CO_2 laser in which cavity length tuning is used as a control parameter (after ref. 51) and *F* shows a spectral recording of the lasing emission for a He–Ne laser with cavity mirror tilt as a control parameter (after ref. 50. Steady-state solutions are not shown in the recordings. Initial instability (periodic pulsation, *Ea*; one frequency and its harmonics, *Fa*) develops into quasiperiodic motion with two incommensurate frequencies (*Eb* and *Fb*) as the control parameter is varied before going into eventual chaos (*Ed* and *Fc*).

changes in population ($X_n \to X_{n+1}$) of an organism or species from year to year ($n \to n+1$) where there is no overlap between successive generations. The first term in the brackets describes population growth by birth and the second term, population decline due, for example, to preditorial or other environmental conditions. Dependent on the increase in the value of the control parameter r the population may be steady, oscillatory or chaotic the transition following a period doubling scenario (see refs 8, 9). More generally many complex physical systems, often described by large numbers of coupled differential equations, may be reduced in some conditions to the form of this or similar maps. Period doubling bifurcation to chaos has been experimentally observed in numerous systems (see Fig. 2*A*, *B*).

Intermittency. Intermittency[10] means that a signal which behaves regularly in time becomes interrupted by statistically-distributed periods of irregular motion. The average number of these intermittent bursts increases with the external control parameter until the condition becomes completely chaotic (see Fig. 2*C*, *D*).

Two frequency. Turbulence in time was originally considered as a limit of an infinite sequence of instabilities (Hopf bifurcation) evolving from an initial stable solution each of which creates a new basic frequencies[11,12]. However, it has been recently shown[13,14] that after only two or perhaps three instabilities in the third step the trajectory becomes attracted to a bounded region of phase space in which initially closed trajectories separates exponentially; as such the motion becomes chaotic (see 2*E*, *F*).

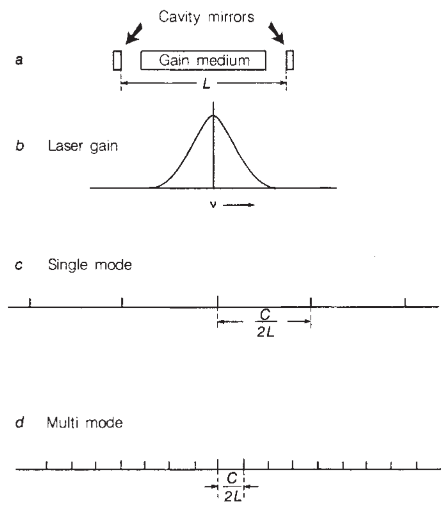


Fig. 3 *a*, A Fabry–Perot laser cavity system with a partially transmitting mirror for coupling out the laser emission; *b*, lorentzian gain profile for a simple two-level homogeneously broadened lasing medium for which each and every atom/molecule emits identically. *c*, Relative position of the cavity modes for a short optical cavity. Here only one mode lies within the gain bandwidth resulting in a single mode emission, *d*, Corresponding position of the modes for a long cavity for which several modes lie within the gain bandwidth.

Chaos in lasers

Chaotic behaviour in lasers may exist in even the simplest of systems: one in which population inversion is established between two discrete energy levels of the medium and where the lasing transition between these two levels is homogeneously broadened. A variety of practical lasers may be controlled to operate in these conditions. A further simplification is that the laser cavity, a Fabry–Perot or ring resonator system surrounding the gain medium, be sufficiently short so that only one resonant frequency of the cavity lies within the bandwidth of the gain medium and that this mode be resonantly tuned to the gain centre frequency. The frequency spacing ($\Delta\nu$) between cavity modes for a Fabry–Perot cavity is given by:

$$\Delta\nu = cn/2L$$

where c is velocity of light, n the refractive index of the lasing medium and L the cavity length. Typical examples of single mode and many mode operation, for short and long cavity lengths respectively, are shown in Fig. 3. In conditions in which the gain or population inversion is maintained at a constant level by, for example, constant electrical or optical excitation, and for the single mode system, lasing occurs with a constant output power at the frequency of the single cavity mode. In this condition, the gain is reduced to a threshold level equal to the cavity losses. Note that even for a multimode system and in the absence of spatial hole burning, lasing is still restricted to a single mode, because the mode with highest gain, here at nor near gain centre, grows at the expense of all other modes. Such behaviour is the accepted operating characteristics of these systems.

However, the discovery[15] that for certain operating conditions emission could be periodic or even chaotic implies that the signal comprises more than one frequency, contrary to the accepted understanding of single mode operation. Prediction of such behaviour were initially identified by Haken through the mathematical equivalence of the equations describing laser action, the Maxwell-Bloch equations, and those derived earlier by Lorenz to describe chaotic motion in fluids. If we consider the trajectory of the Lorenz strange attractor (see Fig. 1*d*) where in the equivalent laser system the dynamic variables x, y and z are the field amplitude (E), polarization of the medium (P), and the population inversion (D), a point (x, y, z) circles in one region for a while, but then suddenly jumps into another region, where it moves for a while until it jumps, seemingly randomly, back into the first region, and so on; the trajectory never intersects. For the laser, such behaviour not only requires a cavity with high transmission but also a gain of at least nine times that required to produce lasing, making the experimental realization of such operation rather impracticable for most lasers of this simple type. A notable exception are optically-pumped far-infrared molecular lasers[16] which will probably become an important research area.

General prerequisites for the onset of deterministic chaos include: that apart from nonlinear interaction there is a sufficiently large phase space; the minimum requirement being that the system possesses at least three degrees of freedom. For example, the simple pendulum possesses only two degrees of freedom and exhibits at most periodic behaviour. In a two-dimensional phase space, described by the variables, velocity and displacement, it is then impossible for the trajectory to be restrained within a basin of attraction without intersecting itself; repetition of trajectory points resulting in only periodic or quasiperiodic solutions. The addition of one further degree of freedom, through the imposition of a modulated external force, leads to the possibility of a non-overlapping trajectory still in a limited phase space but now described by three variables; the so-called strange attractor (see ref. 17).

The Maxwell-Bloch equations described above for the special case of a single-mode laser with field tuned to the centre of the gain line such that both field and polarization are real quantities, satisfy the minimum condition of three independent variable equations, each of which has its own relaxation.

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\kappa E + \kappa P$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \gamma_\perp ED - \gamma_\perp P$$

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \gamma_\parallel(\lambda+1) - \gamma_\parallel D - \gamma_\parallel \lambda EP$$

where κ is the cavity decay rate, $\gamma_\perp$ is the decay rate of atomic

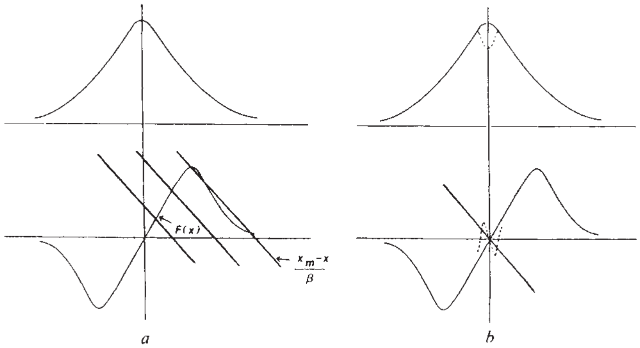


Fig. 4 Graphical solution of the equation of active cavity mode: $F(x)=(x-x_m)/\beta$. Top traces represent gain profile and the corresponding dispersion features associated with these is shown in the bottom traces. *a*, Graphical solution is illustrated for different detunings of the cavity mode. As seen only when the mode is tuned sufficiently away from the line centre, intersection on more than one point is possible resulting in the splitting of the mode into different frequencies all of which fill the same number of half-wavelengths between the cavity mirrors; spontaneous or passive mode splitting. *b*, When the oscillating cavity mode burns a spectral hole into the gain, the associated dispersion takes just the shape required for giving rise to more than one intersection point. As seen, the gain at the split frequencies are indeed higher than that at the original frequency; induced or active mode splitting.

polarization, $\gamma_\parallel$ is the decay rate of population inversion, λ is the pumping parameter, E is the field inside the cavity, D is the population inversion, and P is the atomic polarization.

However, if one variable relaxes faster than the others the stationary solution for that variable may be taken, so resulting in a reduced number of coupled differential equations; commonly termed adiabatic elimination of the fast variables[3,4] In many systems, polarization and population inversion have relaxation times much shorter than the cavity lifetime and both variables can be adiabatically eliminated. With just one variable describing the dynamics, the laser must show a stable behaviour (fixed point in phase space; see Fig. 1*a*). This group of lasers, comprises many common systems such as He–Ne, Ar^+, Dye and lasers. In some cases, only polarization is fast and hence two variables describe the dynamics. In this class, we find ruby, Nd and CO_2 lasers which exhibit oscillating behaviour in some conditions, although ringing is always damped.

Since many lasers are not described by the full set of Maxwell-Bloch equations normally chaotic behaviour from these systems cannot be obtained. For these systems with less than three variables the addition of independent external control parameters to the system, as for the pendulum considered above, have been extensively considered[18] as a means to provide the extra degrees of freedom. Active modulation of a parameter such as population inversion, field, or cavity length as well as injection of a constant field detuned from the cavity resonance and also the use of intracavity saturable absorbers have all been considered[19–21]. For multimode rather than single-mode lasers intrinsic modulation of inversion (or photon flux) by multimode parametric interaction ensures additional degrees of freedom[18]. When the field is detuned from gain centre the field amplitude, polarization and population inversion are complex, providing (in the absence of adiabatic elimination) five rather than three nonlinear equations for single mode systems which is more than sufficient to yield deterministic chaos for suitable parameter values[22]. Also of significance is the remarkably low threshold found for the generation of instabilities and chaos in single mode inhomogeneously broadened laser systems[23–25]. Compared with homogeneously-broadened systems this is attributed to the increased number of independent gain packets available in inhomogeneous systems. Pulsating instabilities and routes to chaos have also been reported for Raman lasers[26] where the instability threshold is again found to be reduced. Significantly, it is also found that instabilities are greatest in conditions for which the laser produces maximum output.

Physical mechanism

A transition from a steady laser output to an oscillatory and subsequently chaotic emission implies the generation of further oscillating frequencies to that of the original stable emission. That this should occur even for a so-called single frequency (single mode) laser seems to be a contradiction in terms. The explanation is in the phenomena of mode splitting[27,28]. This occurs in a region of rapidly varying dispersion when the oscillating cavity mode splits into more than one frequency all of which fill the same number of half-wavelengths between the cavity mirrors. Coupling between these several frequencies having common mode index is a prerequisite for single-mode pulsating instabilities. Dispersion changes its value rapidly near the wings of the gain curve and the splitting that occurs in this region is known as passive mode splitting. An oscillating cavity mode can also induce large variation in dispersion if it can locally saturate the gain curve, hence this splitting is termed induced mode splitting.

To understand spontaneous mode splitting we consider the equation of the oscillating active cavity mode

$$\frac{mc}{2L}=\nu n(\nu) \tag{1}$$

where, m is the mode index and $n(\nu)$ is the index of refraction at the laser oscillation frequency ν. In terms of the empty resonator mode frequencies, $\nu_m = mc/2L$ obtained by putting $n(\nu)=1$ the above equation may be re-written as,

$$\nu_m-\nu=\nu[n(\nu)-1] \tag{2}$$

When the value of $n(\nu)$, either for a lorentzian or gaussian gain profile which are common to most lasers, is substituted this equation may be re-expressed as,

$$x_m-x=\beta F(x) \tag{3}$$

where x and x_m are respectively the laser oscillation frequency and empty cavity resonant frequency normalized as a detuning from the atomic resonance. $F(x)$ has the same functional dependence with x as $n(\nu)$ has with ν and it depends on whether the gain is gaussian or lorentzian. The mode splitting factor, β, is dimensionless and is given by

$$\beta=K\frac{cg}{(\Delta\nu)} \tag{4}$$

where g is the peak small-signal incremental gain, $\Delta\nu$ is the FWHM (full width at half maximum) value of gain-width, c is the velocity of light in vacuum and K is a numerical factor equal to $\pi^{-3/2}\sqrt{\ln 2}$ for gaussian gain profile and π^{-1} for lorentzian gain profile. Equation (3) has been graphically solved for different detunings for a gaussian gain profile in Fig. 4*a*. Under certain detuning (near the wings of the gain) the equation is simultaneously satisfied by more than one value of x. Physically this means that the cavity mode is split into more than one frequency and all of which correspond to the same number of half-wavelengths within the resonator cavity. Such effects occur spontaneously at the wings of the gain curve and hence the name spontaneous mode splitting.

Induced mode splitting may be explained in terms of distortions in the dispersion caused by hole burning in the gain profile by a single mode operating above lasing threshold (see Fig. 4*b*). The dispersion may be so distorted that several new frequencies satisfy the boundary conditions. The onset of these sideband frequencies (the gain at which can be more than that at the parent oscillating frequency; see Fig. 4*b*) gives rise to pulsations

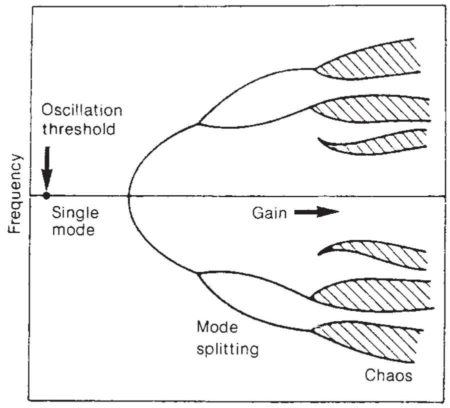


Fig. 5 Mode splitting sequence in which an initially single mode (single frequency lasing oscillation) successively bifurcates on varying gain to generate emission with increasing complex frequency context culminating in a broad band (chaotic) spectrum (after ref. 52).

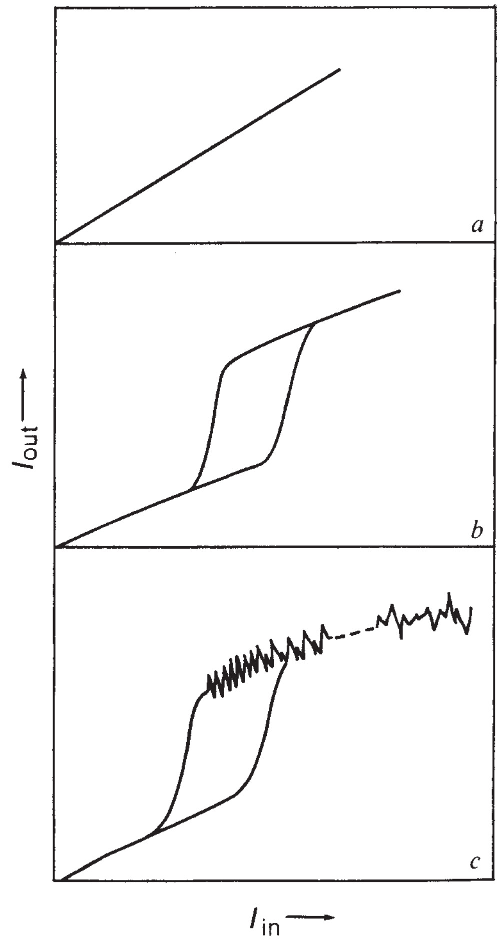


Fig. 6 Transmitted intensity as a function of input intensity for a simple optical resonator, for example, Fabry-Perot or ring cavity. *a*, An empty cavity showing characteristic linear behaviour; *b*, a characteristic optical bistable action and associated hysteresis; *c*, appearance of periodic and chaotic behaviour in the transmitted signal, on increasing the input signal intensity, that may occur in the form of trace *b* for different parameter conditions of the nonlinear cavity.

in the intensity output of the laser. Because the dispersive effects are caused by the oscillating mode itself, this splitting is termed 'induced mode splitting'. This effect should be strongest at line centre and probably exists over much of the lasing tuning range. In view of these effects we may now consider two line broadening situations common to most lasers—homogeneous and inhomogeneous broadening.

The saturating characteristics of an inhomogeneously broadened gain medium is ideal for the realization of mode splitting. In such a medium, unlike the homogeneously broadened case more than one frequency may naturally oscillate since independent sets of atoms/molecules are responsible for lasing at different frequencies across the gain-bandwidth and thus passive mode splitting may be readily observed. The oscillating frequency may also burn a spectral hole into the gain medium thus favouring induced mode splitting. Furthermore, the side bands split again as they again burn spectral holes. This process will continue and may eventually lead to chaos (see Fig. 5). On the other hand, when a homogeneously-broadened gain medium saturates, all the atoms or molecules actually contribute at the oscillating frequency thereby making the possibility of spectral hole burning almost impossible. In such a system, therefore, only the frequency with highest gain would eventually grow. However, for the Haken-Lorentz system, in an extremely high-gain medium survival of more than one frequency is possible.

Chaos in nonlinear optical devices

In parallel with work on laser instabilities, Ikeda predicted oscillation and chaos in passive optical systems[29]. In these systems, the cavity contains a medium whose refractive index is modified by the intensity of the input light signal. This may be expressed as

$$n(I) = n_0 + n_2 I \tag{5}$$

where n_0 and n_2 are respectively the ordinary and nonlinear contribution towards the refractive index and I is the light intensity.

To understand this phenomena we must consider how the light field within the cavity changes, according to the intensity dependent nonlinear phase shift produced by this refractive index, with successive round trips. The simplest case is one in which the intensity-dependent refractive index immediately responds to changes in intensity and where the cavity is of low finesse. The intensity of the signal inside the cavity, here a ring resonator, for successive round trips then obey the one-dimensional mapping rule[30]

$$\begin{aligned} I_{n+1} &= I_{in}\left\{1 + C\cos\left[\frac{2\pi}{\lambda}(n_2 I_n)L + \phi_0\right]\right\} \\ &= F(I_{\text{in}}, C, I_n) \end{aligned} \tag{6}$$

where $\phi_0 = (2\pi/\lambda)n_0 L$, is the normal round trip optical phase shift experienced by the input signal in the cavity in the absence of nonlinearity, C is a constant for the particular cavity and we assume that the nonlinear medium fills the whole cavity of length L. The stationary solution of this mapping, denoted by I_s, occurs when $I_{n+1} = I_n$ and yields the relation

$$I_s = I_{\text{in}}\left\{1 + C\cos\left[\frac{2\pi L}{\lambda}(n_2 I_s) + \phi_0\right]\right\} \tag{7}$$

where the transmitted signal (I_T) is then simply given by $I_T = I_s T$, where T is the transmission of one of the cavity mirrors.

In the limit of low intensity, the nonlinear term is negligible and the transmitted signal of the ring cavity reduces to the standard expression

$$I_T = I_{\text{in}}\{1 + C\cos\phi_0\}T \tag{8}$$

for this optical cavity. Here the intensity of transmitted signal scales linearly with that of the input signal (see Fig. 6*a*) its magnitude depending on the degree to which the input signal frequency is resonant with the cavity. Tuning the length of the cavity, thereby varying the phase ϕ_0, leads to the familiar resonant transmission peaks characteristic of optical cavities. Returning to the nonlinear equation a corresponding plot of transmitted intensity as a function of input intensity is shown in Fig. 6*b*. The nonlinear form of this curve is the signature of optical bistability, of interest because of its application to all optical logic elements. For example, this particular curve shows switching action and memory in which at a critical input intensity the output signal switches to a higher value but on the return cycle switches down for a reduced input intensity due to optical hysteresis. A stable and reproducible operation is required for such application. However, there are parameter conditions, for the input intensity, and cavity finesse for these systems for which the transmitted signal exhibits a behaviour far from this ideal, manifesting not only pulsating instabilities but also full chaotic characteristics. Such regions are schematically indicated on the bistability curve (Fig. 6*c*).

To understand this behaviour we need to examine more closely the mapping behaviour of equation (6) as shown in Fig. 7*a*. Intersection of this curve with the line $I_{n+1} = I_n$ gives the stationary solution to the mapping. Starting from an arbitrary value of I_n, the corresponding value of I_{n+1} (vertical intersection with $F(I_{in}; C; I_n)$) is then the new value of I_n (horizontal intersection with line $I_{n+1} = I_n$). Successive iteration generates a spiral of successively decreasing steps which converge to a single point intersection shown. This is then a stable single point solution. In time, the iteration steps are then transients which die out to this steady-state solution. Similar constructions are shown in Fig. 7*b* and *c* for two higher values of C; the input intensity could, of course, be used as an alternative control parameter. As seen in equation (6) this is manifested as an increase in amplitude of the map. In Fig. 7*b*, successive iterations, do not converge to a single point but stabilize to a two-point solution; shown by the intersection of the iterations with the mapping function $F(I_{in}; C; I_n)$. The corresponding signal in time, therefore, shows that after transients have died out the transmitted signal exhibits periodic oscillation or limit cycle behaviour. For further increase in C, the oscillatory signal successively bifurcates to yield oscillations which are each double the period of the preceeding ones; the so-called period doubling sequence, as shown in Fig. 2 for laser systems. Ultimately for sufficiently high C a chaotic solution is obtained for which, as seen in Fig. 7*d*, successive bifurcations never converge to fixed points. This figure also shows that a small change in the starting conditions, value of I_n, will soon result in wildly different iterative steps; unlike the stable and periodic solutions such behaviour which characterizes deterministic chaos is often referred to as being sensitive to initial conditions. As such a small difference in the initial condition is amplified by many operations. Mathematically, this is quantified by the Lyapunov exponent λ which for this sytem is

$$\lambda = \log \frac{I_{in}C}{2} \quad (9)$$

In general, a positive Lyapunov exponent implies exponential separation of nearby trajectories, the signature of deterministic chaos, while a negative exponent implies exponential contraction, the signature of an attracting fixed point. For chaotic behaviour one, therefore, requires $I_{in}C/2 > 1$, whereas bistable action without the presence of instability requires $I_{in}C/2 < 1$. Evidently a transition to chaos may then occur on increasing the input intensity or the value of the parameter C. For convenience, we have assumed that the medium responds instantaneously to the field or more precisely the relaxation time of the medium is much shorter than the cavity round trip time of the field. For systems in which the converse situation applies bistable operation is favoured although even here sideband instabilities may arise.

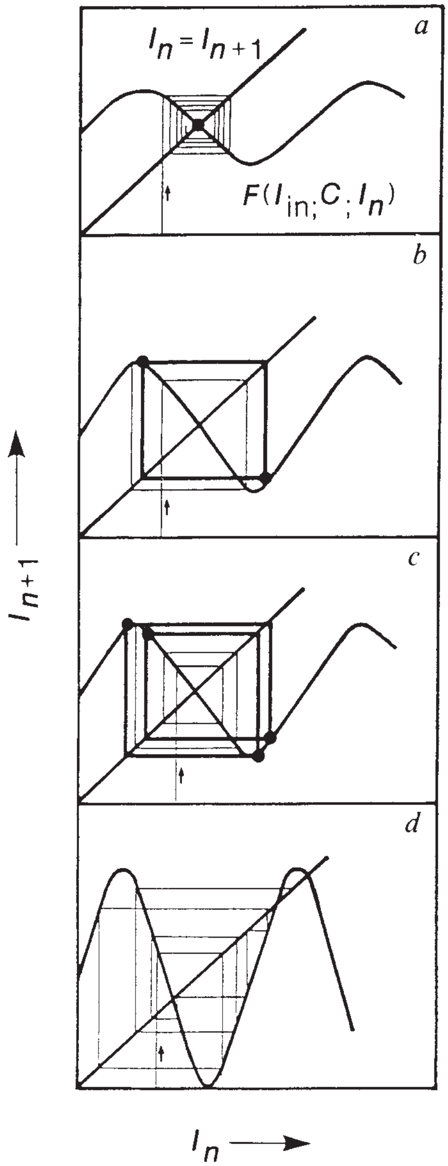


Fig. 7 Graphical solutions of one dimensional map $I_{n+1} = F(I_{in}, C, I_n)$ for various values of control parameter C. The line $I_n = I_{n+1}$ is the condition when the transmitted signal does not change with successive iterations. *a*, A fixed point or steady-state solution I_s for the field in the cavity since successive iterations converge to the single intersection point of the two curves. *b*, An increased value of C, shows the iterations to stabilize on a two-point solution corresponding to a periodic output in time. For further increase in the value of C, a four-point solution is obtained (*c*); that is iterations here have to cycle twice before repeating themselves so corresponding to a periodic oscillation of twice that for the trace *b*. This period doubling behaviour repeats as the control parameter C is further increased until a value is reached when iterations no longer repeat themselves. This is shown in the trace *d* which characterizes an example of chaotic behaviour. We note that the stable and periodic solutions (traces *a–c*) are invariant with the starting value of I_n. In contrast, the chaotic trajectory of trace *d* is very sensitive to this initial condition; following a very different path for even the smallest change. The time series of Fig. 2*A* are representative of the behaviour described here.

Observations of these phenomena, although still limited, have been made in various optical systems such as hybrid bistable devices, and all optical bistable systems. In the hybrid system, nonlinearity in a transparent dielectric medium is caused by an externally-applied voltage rather than by the laser signal itself. The impressed voltage is coupled to the transmitted light signal through a controllable electronic delay with variable gain which serves as the control parameter of the system. On increasing the feedback gain the transmitted optical signal undergoes a period doubling bifurcation though only up to period eight before becoming chaotic. Here[31] as in many other systems, both passive and active, the effect of external noise[32] interrupts the otherwise infinite series of period doublings to create what is termed a bifurcation gap.

The first observation of bifurcation to periodic and chaotic states in an all optical bistable system was made with a single mode optical fibre as the transparent nonlinear medium in a ring cavity[33], the refractive index here being directly modified by the input intensity through the Kerr effect. Here, the high intensities required to induce sufficient nonlinearity precluded the use of a continuous wave (cw) input source and instead a mode-locked train of intense short pulses, separated in time by an amount equal to the round trip time of the ring cavity, was used. Each pulse is, in effect, an iterative step in which the signal of one pulse in circulating the ring cavity adds to the signal of the next incoming pulse, the process repeating for the whole train. Since bifurcations are caused by the interference between the incident field and cavity field which, suffers a nonlinear phase shift in each round trip of the cavity (see equation (6)), the transmitted signal is evidently modified by the intensity of the pulses. With this system period doubling followed by chaos was observed.

The most fundamental of all nonlinear optical systems is nevertheless, one containing a two-level medium as originally analysed by Ikeda for a ring cavity[33]. Near resonant excitation of the medium will, for a sufficiently intense incident signal, cause saturation of the absorption which thereby modifies the associated anomalous dispersion feature of the two-level transition. The mechanism is, in many respects, equivalent to that which gives rise to mode splitting in lasers described earlier for a two-level gain rather than absorptive medium. As for the laser, the intensity dependent change in refraction (dispersion) is resonantly enhanced resulting in considerably reduced power requirements to those used in the fibre experiments. Both atomic and molecular gases are potentially useful media for investigation here, providing discrete levels many of which are in near resonance with available laser lines and approximate reasonably closely to two-level systems. The first corroboration of period doubling routes to chaos in such systems were made in both ring and Fabry-Perot cavities containing ammonia gas near resonantly-excited by pulsed CO_2 radiation[34,35] Recently, experiments have been extended to cw pump laser conditions[36] in which sodium vapour was used as the nonlinear medium. Preliminary results show evidence of period doubling with additional instabilities yet to be fully characterized.

Recognizing the problems that chaotic behaviour may pose for fast all-optical bistable elements, yet, operated in the period doubling window and, in particular, period two oscillation, for which the operational parameter window is largest, they offer the attractive prospect of high-frequency all-optical modulators. But such an operation requires that the relaxation time of the nonlinear medium be much faster than the cavity round trip time, then for small devices (<1 mm in length) the frequency of modulation may, in principle, be in the THz range provided that nonlinear materials may be found with sufficiently short relaxation times. The search for fast large nonlinearities is one of the main efforts in this area.

Conclusions

Nonlinear optics is proving valuable to the fields of nonlinear dynamics and deterministic chaos. On one hand, basically simple optical systems can be constructed exhibiting the most interesting classes of chaotic behaviour enriched by the possibility of a quantum description. On the other hand, lasers and related nonlinear otical devices have a large and growing technical application, and the understanding, control and possible exploitation of sources of instability in these systems has considerable practical importance.

Experimental findings are, in general, not yet sufficiently comprehensive to permit the quantitative analysis necessary to fully test the theoretical models. More carefully controlled experiments are forthcoming from which, along with time series, power spectra and identification of routes to chaos, embedding procedures may be implemented to the attractors describing the dynamical behaviour of these systems.

More detailed discussions of points in this review can be found elsewhere. For laser instabilities see refs 37-39 and also refs 40, 41 which cover both active and passive systems. For passive systems see ref. 42. Comprehensive treatments on the more general principles of deterministic chaos can be found in refs 3, 4. See the recent text on deterministic chaos by Schuster[43]. See refs 8, 9, 44-46 for articles on the general aspects of deterministic chaos, and refs 47 and 48 for discussions on the device application of optical bistable systems.

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