

DIFFERENTIAL EQUATIONS (751873001, 114-1) - HOMEWORK 3

Return by October 10, 2025 (Friday) 23:59

Total marks: 50

Special requirement. All homeworks must be prepared by using L^AT_EX.

Exercise 1 (5+5+5 points). Let $A, B \in \mathbb{C}^{n \times n}$, show that

- (a) $\|A\| = (\operatorname{tr}(A^*A))^{1/2}$, where A^* is the *conjugate transpose* (or *adjoint*) of $A \in \mathbb{C}^{n \times n}$.
- (b) $\|A + B\| \leq \|A\| + \|B\|$,
- (c) $\|AB\| \leq \|A\|\|B\|$.

Exercise 2 (5 points). Show that A_m converges to A if and only if $\lim_{m \rightarrow \infty} \|A_m - A\| = 0$.

Exercise 3 (10 points). For each $A \in \mathbb{C}^{n \times n}$, show that $\det(\exp(A)) = e^{\operatorname{tr}(A)}$. In addition, show that $\operatorname{tr}(A) = \lambda_1 + \cdots + \lambda_n$, where $\lambda_j \in \mathbb{C}$ are eigenvalues (may identical) of A .

Exercise 4 (5+5+5 points). Let

$$A_1 = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

Compute $\exp(A_1)$, $\exp(A_2)$ and $\exp(A_3)$.

Exercise 5 (5 points). Show that for any $a, b, d \in \mathbb{C}$ that

$$\exp \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} e^a & b \frac{e^a - e^d}{a - d} \\ 0 & e^d \end{pmatrix}.$$

Since

$$\lim_{a \rightarrow d} \frac{e^a - e^d}{a - d} = e^a,$$

we simply interpret $\frac{e^a - e^d}{a - d}$ as e^a when $d = a$. (**Hint.** Show that

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^m = \begin{pmatrix} a^m & b \frac{a^m - d^m}{a - d} \\ 0 & d^m \end{pmatrix}$$

for all $m \in \mathbb{N}$ and $a \neq d$.)