FOURIER ANALYSIS (751799001, 701866001, 114-1) - HOMEWORK 3

Return by October 1, 2025 (Wednesday) 23:59

Total marks: 50

Special requirement. All homework must be prepared by using LaTeX.

Exercise 1 (10 points)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) := x|y|$ for each $(x,y) \in \mathbb{R}^2$. Prove that the weak derivative $\partial_1^2 \partial_2 f$ exists, while the weak derivative $\partial_1 \partial_2^2 f$ does not.

Exercise 2 (10 points)**.** Let $\epsilon \in (0,1)$ and consider the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} |x|^{-\epsilon} & \text{if } x \in \mathbb{R}^n \setminus \{0\}, \\ 1 & \text{if } x = 0. \end{cases}$$

Prove that $\partial_j f$ exists in the weak sense for each $j \in \{1, \cdots, n\}$ if and only if $n \geq 2$. Also compute the weak derivatives $\partial_j f$ for all $j = 1, \cdots, n$ in the case when $n \geq 2$.

Exercise 3 (10 points)**.** Prove Theorem 1.4.2 for the special case when $f \in C^1(\mathbb{R})$. [Hint: compute the Fourier coefficients of f']

Exercise 4 (10 points)**.** Verify that the Fejér kernel is an approximate identity in the sense of Definition 1.3.6.

Exercise 5 (10 points)**.** Let $T > 0$ and let $f : \mathbb{R}^n \to \mathbb{C}$ be a function with period $2T$ on each variable. Show that the Fourier series of f is given by

$$f(x) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k) e^{i \frac{\pi}{T} k \cdot x} \quad \text{with} \quad \hat{f}(x) = \fint_{[-T,T]^n} f(y) e^{-i \frac{\pi}{T} k \cdot y} \, \mathrm{d}y.$$