COMPLEX ANALYSIS (701026001, 112-1) - HOMEWORK 1

Return to TA by: September 26, 2023 (Tuesday) 16:00

Total marks: 50 (with 6 bonus marks)

Exercise 1. (10 points) Show that $|z| \leq |\mathfrak{Re} z| + |\mathfrak{Im} z|$ for all $z \in \mathbb{C}$. When is equality possible?

Exercise 2. (10 points) Determine whether the series $\sum_{k=1}^{\infty} \frac{\mathbf{i}}{k^2+\mathbf{i}}$ and $\sum_{k=1}^{\infty} \frac{1}{k+\mathbf{i}}$ converges or not.

Exercise 3.

(a) (5 points) Show that the n^{th} roots of 1, aside from 1, satisfy the "cyclotomic" equation $z^{n-1} + z^{n-2} + \cdots + z + 1 = 0$.

(b) (5 points) Let $P(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$. Show that all the zeros of $P(z)$ are inside the closed unit disc.

Exercise 4. (10 points) Let $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^{n-1}$ with all $a_i \in \mathbb{R}$ and $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$. Show that all the zeros of $P(z)$ are inside the closed unit disc.

Exercise 5. (10 points) Let $S = \{x + \mathbf{i}y : x = 0\} \cup \left\{x + \mathbf{i}y : x > 0, y = \sin \frac{1}{x}\right\}$. Show that S is topologically connected.

Exercise 6 (Bonus)**.** Use MATLAB (or other software) to plot the region $\{\varphi(z) : |z| < 1\}$ when

(a) (2 points) $\varphi(z) = \frac{1}{m} z^m$ (choose any two integers $m \geq 2$ to plot)

(b) (2 points) $\varphi(z) = z - \frac{2\sqrt{2}}{3} z^2 + \frac{1}{3} z^3$ (zoom in near the point $0.4 + \mathbf{i}0$)

(c) (2 points) $\varphi(z) = (z-1)^2 - (1 - \frac{\mathbf{i}}{2})(z-1)^3$ (zoom in near the origin 0)