DIFFERENTIAL EQUATIONS (751873002, 113-2) - HOMEWORK 2

Return by March 13, 2024 (Thursday) 23:59

Total marks: 50

Special requirement. All homeworks must be prepared by using LaTeX.

Exercise 1 (10 points)**.** Show that

$$\|f\|_{L^p(\Omega)} = \sup_{\|g\|_{L^{p'}(\Omega)}=1} \int_\Omega f(\boldsymbol{x})g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

and

$$\|f\|_{L^p(\Omega)} = \sup_{\|g\|_{L^{p'}(\Omega)}=1} \int_\Omega |f(\boldsymbol{x})g(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{x}.$$

Here f is not necessarily in $L^p(\Omega)$.

Exercise 2 (10 points)**.** Let $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ and $h \in L^r(\Omega)$ for some $1 \le p,q,r \le \infty$ with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$. Show that $fgh \in L^1(\Omega)$ and

$$\int_\Omega |f(\boldsymbol{x})g(\boldsymbol{x})h(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{x} \le \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}\|h\|_{L^r(\Omega)}.$$

Exercise 3 (10 points)**.** [1] Show that $\|\cdot\|_{L^p(\Omega)}^p$ defines a norm for each $0<p<1$. In addition, show that $\|\cdot\|_{L^p(\Omega)}$ does not define a norm for each $0<p<1$.

Exercise 4 (10 points)**.** Let $n=2$. Verify that Green's theorem is a special case of divergence theorem, see Exercise 1.0.20 for details.

Exercise 5 (10 points)**.** Let A be a real symmetric matrix. Show that all its eigenvalue are positive if and only if

$$A\boldsymbol{\xi}\cdot\boldsymbol{\xi} \equiv \boldsymbol{\xi}^\intercal A\boldsymbol{\xi} > 0 \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}.$$

[1] It is interesting to compare this exercise with Exercise 3 in Homework 1.