GEOMETRY (701939001, 751764001, 113-2) - HOMEWORK 3

Return to TA by: March 25, 2025 (Tuesday) 16:00

Total marks: 50

Exercise 1 (10 points)**.** Let $\{\mathcal{K}^{(k)}\}_{k\in\mathbb{N}}$ be a sequence of nonempty compact sets in $\mathbb{R}^n$ such that $\mathcal{K}^{(1)} \supset \mathcal{K}^{(2)} \supset \mathcal{K}^{(3)} \cdots$. Show that $\bigcap_{k\in\mathbb{N}} \mathcal{K}^{(k)} \neq \emptyset$.

Exercise 2 (10 points)**.** Let $\{\mathcal{K}^{(t)}\}_{t\in(0,1)}$ be a collection of nonempty compact sets in $\mathbb{R}^n$ such that $\mathcal{K}^{(t_1)} \subset \mathcal{K}^{(t_2)}$ for all $0 < t_1 < t_2 < 1$. Show that $\bigcap_{t\in(0,1)} \mathcal{K}^{(t)} \neq \emptyset$.

Exercise 3 (10 points)**.** Given any collection of sets $\{A_\alpha\}_{\alpha\in\Lambda}$ in $\mathbb{R}^n$. Show that

$$\overline{\bigcup_{\alpha\in\Lambda} A_\alpha} = \bigcup_{\alpha\in\Lambda} \overline{A_\alpha} \quad \text{and} \quad \overline{\bigcap_{\alpha\in\Lambda} A_\alpha} \overset{(1)}{\subset} \bigcap_{\alpha\in\Lambda} \overline{A_\alpha}.$$

Is the equality holds in (1)? Prove or disprove it.

Exercise 4 (10 points)**.** Given any collection of sets $\{A_\alpha\}_{\alpha\in\Lambda}$ in $\mathbb{R}^n$. Show that

$$\bigcup_{\alpha\in\Lambda} \operatorname{int}(A_\alpha) \overset{(2)}{\subset} \operatorname{int}\left(\bigcup_{\alpha\in\Lambda} A_\alpha\right) \quad \text{and} \quad \operatorname{int}\left(\bigcap_{\alpha\in\Lambda} A_\alpha\right) \overset{(3)}{\subset} \bigcap_{\alpha\in\Lambda} \operatorname{int}(A_\alpha).$$

Is the equality holds in (2) and (3)? Prove or disprove them.

Exercise 5 (10 points)**.** Let A and B be compact sets in $\mathbb{R}^n$. Show that $A + B := \{a + b : a \in A, b \in B\}$ is also a compact set in $\mathbb{R}^n$.