FOURIER ANALYSIS (751799001, 701866001, 114-1) - HOMEWORK 6

Return by November 5, 2025 (Wednesday) 23:59

Total marks: 50

Special requirement. All homework must be prepared by using LaTeX.

Exercise 1 (10+10+10 points)**.** Let $n = 1$.

(a) Show that

$$T_1(\varphi) := \lim_{\epsilon \to 0_+} \int_{|x|>\epsilon} \frac{\varphi(x)}{x} \, \mathrm{d}x \quad \text{for all } \varphi \in \mathscr{S}(\mathbb{R}^1)$$

defines a tempered distribution (i.e., an element in $\mathscr{S}'(\mathbb{R}^1)$).

(b) How about

$$T_2(\varphi) := \lim_{\epsilon \to 0_+} \left(\int_{x<-\epsilon} \frac{\varphi(x)}{x} \, \mathrm{d}x + \int_{x>2\epsilon} \frac{\varphi(x)}{x} \, \mathrm{d}x \right) \quad \text{for all } \varphi \in \mathscr{S}(\mathbb{R}^1)?$$

(c) Since $\ln |x|$ is locally integrable, thus it defines a tempered distribution (i.e., an element in $\mathscr{S}'(\mathbb{R}^1)$). Compute its distributional derivative.

Remark. In fact, the tempered distribution $T_1 \in \mathscr{S}'(\mathbb{R}^1)$ in (a) is called the *principle-value* $\frac{1}{x}$, denoted by $\mathrm{pv}\frac{1}{x} \in \mathscr{S}'(\mathbb{R}^n)$. After studying T_2 in (b), you will know why term "principle-value" cannot be omitted.

Exercise 2 (10+10 points)**.** Let $n = 1$.

(a) Let γ be the Euler-Mescheroni constant given by

$$\gamma := \int_0^1 \frac{1-\cos t}{t} \, \mathrm{d}t - \overbrace{\lim_{M \to +\infty} \int_1^M \frac{\cos t}{t} \, \mathrm{d}t}^{\text{improper integral}}.$$

Show that

$$\gamma + \ln |x| = \lim_{M \to +\infty} \int_0^M \frac{\chi_{[0,1]}(t) - \cos(xt)}{t} \, \mathrm{d}t \quad \text{for all } x \in \mathbb{R}^1 \setminus \{0\}.$$

[**Hint.** Applying the fundamental theorem of calculus on the function $F(x) := \int_0^x \frac{1-\cos t}{t} \, \mathrm{d}t - \lim_{M \to +\infty} \int_x^M \frac{\cos t}{t} \, \mathrm{d}t$ for all $x > 0$]

(b) Compute the $\mathscr{S}'(\mathbb{R}^n)$-Fourier transform of $f(x) := \ln |x|$ for all $x \in \mathbb{R}^1 \setminus \{0\}$.
[**Note.** The answer consists of a distribution called the *principle-value* $\frac{1}{|x|}$, denoted as $\mathrm{pv}\frac{1}{|x|} \in \mathscr{S}'(\mathbb{R}^n)$, which is defined as the distributional derivative of $\mathrm{sgn}\,(x) \ln |x| \in \mathscr{S}'(\mathbb{R}^n)$.]