DIFFERENTIAL EQUATIONS (751873002, 113-2) - HOMEWORK 2

Return by March 13, 2025 (Thursday) 23:59

Total marks: 50 (last update: March 25, 2025)
Special requirement. All homeworks must be prepared by using LaTeX.

Exercise 1 (10 points)**.** Show that

$$\|f\|_{L^p(\Omega)} = \sup_{\|g\|_{L^{p'}(\Omega)}=1} \int_\Omega f(\boldsymbol{x})g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

and

$$\|f\|_{L^p(\Omega)} = \sup_{\|g\|_{L^{p'}(\Omega)}=1} \int_\Omega |f(\boldsymbol{x})g(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{x}.$$

Here f is not necessarily in $L^p(\Omega)$.

Exercise 2 (10 points)**.** Let $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ and $h \in L^r(\Omega)$ for some $1 \leq p,q,r \leq \infty$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Show that $fgh \in L^1(\Omega)$ and

$$\int_\Omega |f(\boldsymbol{x})g(\boldsymbol{x})h(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{x} \leq \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}\|h\|_{L^r(\Omega)}.$$

Exercise 3 (10 points)**.** [1] For each $0 < p < 1$, show that

$$\|f+g\|_{L^p(\Omega)}^p \leq \|f\|_{L^p(\Omega)}^p + \|g\|_{L^p(\Omega)}^p \quad \text{for all } f,g \in L^p(\Omega).$$

In addition, show that $\|\cdot\|_{L^p(\Omega)}$ does not define a norm for each $0 < p < 1$.

Exercise 4 (10 points)**.** Let $n = 2$. Verify that Green's theorem is a special case of divergence theorem, see Exercise 1.0.20 for details.

Exercise 5 (10 points)**.** Let A be a real symmetric matrix. Show that all its eigenvalue are positive if and only if

$$A\boldsymbol{\xi}\cdot\boldsymbol{\xi} \equiv \boldsymbol{\xi}^\intercal A\boldsymbol{\xi} > 0 \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^n \setminus \{\mathbf{0}\}.$$

[1] It is interesting to compare this exercise with Exercise 3 in Homework 1.