FOURIER ANALYSIS (751799001, 701866001, 114-1) - HOMEWORK 7

Return by November 12, 2025 (Wednesday) 23:59

Total marks: 50

Special requirement. All homework must be prepared by using LaTeX.

Exercise 1 (10+10 points)**.** For a fixed $x \in \mathbb{R}$, we see that $t \mapsto e^{\mathbf{i}x\sin t}$ is a 2π-period function, therefore one sees that its Fourier series on $(0, 2\pi)$ is well-defined, and converges uniformly on $[0, 2\pi]$. Write

$$e^{\mathbf{i}x\sin t} = \sum_{n\in\mathbb{Z}} J_n(x)e^{\mathbf{i}nt} \quad \text{for all } t \in \mathbb{R}.$$

The function J_n is called the *Bessel function of the first kind* of order $n \in \mathbb{Z}$.

(a) Compute $\sum_{n\in\mathbb{Z}} |J_n(x)|^2$ for all $x \in \mathbb{R}$.

(b) For any $x \neq 0$, express $x^2 J_n''(x)$ in terms of $J_n'(x)$ and $J_n(x)$, where $J_n'(x) = \frac{\mathrm{d}}{\mathrm{d}x}J(x)$ and $J_n''(x) = \frac{\mathrm{d}}{\mathrm{d}x}J'(x)$.

Exercise 2 (10+10+10 points)**.** We now consider the normalized Fourier transform as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-\mathbf{i}2\pi x\cdot\xi}\,\mathrm{d}x \quad \text{for all } f \in \mathscr{S}(\mathbb{R}^n),$$

which is often used in signal processing (see, e.g., Python's Numpy package). In this case, the inverse Fourier transform is given by

$$\check{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{\mathbf{i}2\pi x\cdot\xi}\,\mathrm{d}x \quad \text{for all } f \in \mathscr{S}(\mathbb{R}^n).$$

Now we fix $n = 1$.

(a) Show the Poisson summation formula

$$\sum_{k\in\mathbb{Z}} \hat{f}(k) = \sum_{n\in\mathbb{Z}} f(n) \quad \text{for all } f \in \mathscr{S}(\mathbb{R}).$$

[**Hint.** Compute the Fourier series $\Phi(x) = \sum_{n\in\mathbb{Z}} f(n+x)$ and find that value $\Phi(0)$]

(b) Suppose that $g \in \mathscr{S}(\mathbb{R})$. Show that

$$\sum_{k\in\mathbb{Z}} \hat{g}\left(\xi - \frac{k}{T}\right) = T\sum_{n\in\mathbb{Z}} g(nT)e^{-\mathbf{i}2\pi nT\xi} \quad \text{for all } \xi \in \mathbb{R}.$$

In particular, if $g \in \mathscr{S}(\mathbb{R})$ and $\operatorname{supp}(\hat{g}) \subset [-\frac{1}{T}, \frac{1}{T}]$,

$$\hat{g}(\xi) = T\sum_{n\in\mathbb{Z}} g(nT)e^{-\mathbf{i}2\pi nT\xi} \quad \text{for all } \xi \in \left[-\frac{1}{T}, \frac{1}{T}\right].$$

This implies that g can be reconstructed based on partial knowledge of g, namely $\{g(nT)\}_{n\in\mathbb{Z}}$. [**Hint.** Choose $f(x) = g(Tx)e^{-\mathbf{i}2\pi Tx\xi}$ in (a)]

(c) Show that if $g \in \mathscr{S}(\mathbb{R})$ satisfies $\operatorname{supp}(\hat{g}) \subset [-B, B]$, then

$$g(x) = \sum_{k\in\mathbb{Z}} g\left(\frac{k}{2B}\right) \operatorname{sinc}(2Bx - k) \quad \text{for all } x \in \mathbb{R},$$

where sinc is the normalized sinc function given by

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

This result is called the *Shannon interpolation theorem*.

Remark. In mathematics, audio signals are modeled by a continuous function g. However, in practice, it is not possible to recorded the full audio signal g, instead, the signals are typically recorded at a *sample rate* of f_s, i.e., f_s samples per second, with each sample represented as a floating-point number[1] between -1 and 1. If we want to capture the signal contains no frequency components above B Hz (called the *bandwidth*), i.e. $\operatorname{supp}(\hat{f}) \subset [-B, B]$, in view of Exercise 2(b)–(c) with $T = 1/B$, we see that the audio signal g can be reconstructed based on the discrete point $\{g\left(\frac{k}{2B}\right)\}_{k\in\mathbb{Z}}$, which means that a nearly perfect reconstruction is guaranteed provided that $B < f_s/2$. For example, the sample rate $f_s = 16\,\text{kHz}$ captures frequencies up to 8 kHz, which is sufficient for speech recording (https://arxiv.org/abs/2501.01650). Human speech frequency is close to the piano's middle C ($\approx 262\,\text{Hz}$) based on the modern A440 pitch standard[2]. In the standard piano with 88 key, the first key is tuned to be 27.5 Hz and the last key is tuned to be 4186 Hz. The standard rate $f_s = 44.1\,\text{kHz}$ captures frequencies up to 22050 Hz, which covers the human hearing range of approximately 20 Hz to 20 kHz.

[1]For storage efficiency, the audio is usually stored in 16-bit integer format (16-bit PCM) by multiplying each sample by 32767 and rounding to the nearest integer. There are exactly $2^{16} = 65,536$ integers ranging from -32767 to 32767, which explains the term "16-bit".

[2]https://en.wikipedia.org/wiki/Piano_key_frequencies