An elementary introduction to partial balayage
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This lecture note on partial balayage is designed for graduate students. Rather than following the historical development, we introduce partial balayage directly through modern approaches, based on some materials in [GS12, GS24, Gus90, KLSS24, KS24], which is consistent to the one in [Gus04, Definition 3.1]. Since the audience consists of graduate students, we will explain the details in a clear and straightforward manner.

For those students who do not familiar with distribution theory (i.e. generalized functions), one can refer to the monograph [FJ98], or my other existing lecture note [Kow22, Kow24] as well, we will not explicitly mention weak/distributional derivatives. We also provided some preliminaries (a version of Hahn-Banach theorem, Sobolev embeddings and integration by parts in weak sense) in Appendix A. Throughout this lecture note, we will only consider real-valued functions unless stated explicitly.

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Contents

CHAPTER 1

Partial balayage of compactly supported bounded measures

1.1. Motivation

The word "balayage" means sweeping in French. Given a compactly supported distribution $\mu \in \mathscr{E}'(\mathbb{R}^n)$, we are interested to a procedure which output a compact supported distribution $\mathrm{Bal}(\mu)$ satisfying

$$\mathrm{Bal}(\mu) = \chi_D + \mu|_{D^\complement} \quad \text{and} \quad \min\{1,\mu\} \leq \mathrm{Bal}(\mu) \leq 1 \tag{1.1.1}$$

for some open set $D \subset \mathbb{R}^n$, where $D^\complement := \mathbb{R}^n \setminus D$ and

$$\chi_D = \begin{cases} 1 & \text{in } D, \\ 0 & \text{in } D^\complement. \end{cases}$$

The open set D in (1.1.1) can be understood as the "region which the measure μ was cleaned", and we see that the measure remain unchanged in $D^\complement$. Therefore, such distribution $\mathrm{Bal}(\mu)$ is called the *balayage* of μ (with respect to Lebesgue measure). It is also possible to discuss the partial balayage on some manifold [**GR18**].

We don't simply choose $D = \{\mu > 1\}$, since our main goal is to choose a suitable D to obtain some "good properties" related to the following object (see Section 1.5 below):

DEFINITION 1.1.1 (quadrature domain). A bounded open set $D \subset \mathbb{R}^n$ (not necessarily connected) is called a quadrature domain (for harmonic functions), corresponding to a distribution $\mu \in \mathscr{E}'(D)$ if

$$\int_D w(x)\,\mathrm{d}x = \langle \mu, w \rangle \quad \text{for all } w \in L^1(D) \text{ with } \Delta w = 0 \text{ in } D. \tag{1.1.2}$$

The notation $\mu \in \mathscr{E}'(D)$ means that μ is a compactly supported distribution satisfying the support condition $\mathrm{supp}(\mu) \subset D$. Since all $L^1(D)$ harmonic functions are in $C^\infty(D)$, thus the distribution pairing in the right-hand-side of (1.1.2) is well-defined. It is interesting to point out that one can use quadrature domain is related to acoustic scattering problem, see e.g. [**KLSS24, KSS24, KS24, SS21**].

In fact, the mean value theorem for harmonic function can be restated as follows:

EXAMPLE 1.1.2 (mean value theorem for harmonic functions). Let $n \geq 2$ be an integer and let $R > 0$ be any constant. If $u \in L^1(B_R(x_0))$ is a solution to $\Delta u = 0$ in $B_R(x_0)$, then $B_R(x_0)$ is a quadrature domain with respect to

$$\mu = |B_R(x_0)|\delta_{x_0},$$

where $|B_R(x_0)|$ is the Lebesgue measure of $B_R(x_0)$ and δ_{x_0} is the Dirac delta at x_0.

EXAMPLE 1.1.3 (A conjecture that has been resolved). We refer a quadrature domain with respect to $\mu \equiv 0$ as *null quadrature domain*. Null quadrature domains are fully characterized in [**EFW25**] for all dimensions $n \geq 2$ (the special case when $n \geq 6$ was done in [**ESW23**]), it must either one of the followings:

(1) complement of a half-space; or
(2) complement of an ellipsoid; or
(3) complement of a cylinder with an ellipsoid base; or
(4) complement of a cylinder with a paraboloid base.

In either case, we see that null quadrature domains are unbounded.

From now on, we always assume that $n \geq 3$. Recall that the function

$$\Phi(x) := (n(n-2)|B_1|)^{-1}|x|^{2-n}$$

is in $L^1_{\rm loc}(\mathbb{R}^n)$, which is a fundamental solution of $-\Delta$, i.e. $-\Delta\Phi = \delta_0$, see e.g. [**GT01**]. Without causing any confusion, we do not explicitly mention the term "almost everywhere (a.e.)" throughout this lecture note. By using the mean value theorem for harmonic functions, one sees for each $w \in L^1(D)$ with $\Delta w = 0$ in D and $\mu \in \mathscr{E}'(D)$ that

$$\begin{aligned}
\left\langle \mu * \frac{1}{|B_r|}\chi_{B_r}, w \right\rangle &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{1}{|B_r|}\chi_{B_r}(x-y)\mu(y)\,\mathrm{d}y \right) w(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \left(\frac{1}{|B_r|} \int_{\mathbb{R}^n} \chi_{B_r}(x-y) w(x)\,\mathrm{d}x \right) \mu(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \left(\frac{1}{|B_r(y)|} \int_{B_r(y)} w(x)\,\mathrm{d}x \right) \mu(y)\,\mathrm{d}y \\
&\overset{\rm MVT}{=} \int_{\mathbb{R}^n} w(y)\mu(y)\,\mathrm{d}y = \langle \mu, w \rangle.
\end{aligned} \tag{1.1.3}$$

This shows that D is a quadrature domain corresponding to μ if and only if D is a quadrature domain corresponding to $\mu * \frac{1}{|B_r|}\chi_{B_r} \in L^\infty_c(\mathbb{R}^n)$ for all $0 < r < \operatorname{dist}(\operatorname{supp}(\mu), \partial D)$. This suggests us to first consider partial balayage of compactly supported bounded measures, as in the title of this chapter.

1.2. From variational problem to obstacle problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected smooth domain. Let $H^1(\Omega)$ be the Hilbert space equipped with the norm

$$\|\cdot\|_{H^1(\Omega)} := \left(\|\cdot\|^2_{L^2(\Omega)} + \|\nabla \cdot\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}, \tag{1.2.1}$$

and let $H_0^1(\Omega)$ be the completion of $C_c^\infty(\Omega)$ with respect to the norm (1.2.1). In fact,

$$H_0^1(\Omega) = \left\{ v \in H^1(\Omega) : v|_{\partial\Omega} = 0 \right\}.$$

Given any $v \in H_0^1(\Omega)$, by using [**Bre11**, Proposition 9.18], one sees that its zero extension

$$\chi_\Omega v \equiv \begin{cases} v & \text{in } \Omega, \\ 0 & \text{in } \Omega^\complement, \end{cases}$$

belongs to $H^1(\mathbb{R}^n)$ and satisfying

$$\nabla(\chi_\Omega v) = \chi_\Omega \nabla v \equiv \begin{cases} \nabla v & \text{in } \Omega, \\ 0 & \text{in } \Omega^\complement. \end{cases} \tag{1.2.2}$$

Let $\lambda_1(\Omega)$ be the *fundamental tone* of Ω defined by

$$\lambda_1(\Omega) := \inf_{0 \not\equiv v \in H_0^1(\Omega)} \frac{\|\nabla v\|_{L^2(\Omega)}^2}{\|v\|_{L^2(\Omega)}^2}.$$

In fact, one has $\lambda_1(\Omega) > 0$ (well-known as *Poincaré inequality*), and $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of $-\Delta$. By using Poincaré inequality, it is easy to see that $H_0^1(\Omega)$ can be equipped with the following equivalent norm

$$\|\cdot\|_{H_0^1(\Omega)} := \|\nabla \cdot\|_{L^2(\Omega)}.$$

Let $A = (a_{ij}) \in (L^\infty(\Omega))_{\rm sym}^{n\times n}$ satisfies the following ellipticity condition:

$$\Lambda^{-1}|\xi|^2 \leq \xi \cdot A(x)\xi \leq \Lambda|\xi|^2 \quad \text{in } \Omega \text{ for all } \xi \in \mathbb{R}^n.$$

Let $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ be a symmetric bilinear form defined by

$$a(v_1, v_2) := \int_\Omega \nabla v_1(x) \cdot A(x) \nabla v_2(x)\,\mathrm{d}x. \tag{1.2.3}$$

It is easy to see that:

(1) $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is continuous:

$$|a(v_1, v_2)| \leq \Lambda \int_\Omega \nabla v_1(x) \cdot \nabla v_2(x)\,\mathrm{d}x \leq \Lambda \|v_1\|_{H_0^1(\Omega)} \|v_2\|_{H_0^1(\Omega)}$$

for all $v_1, v_2 \in H_0^1(\Omega)$.

(2) $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is coercive:

$$|a(v,v)| \geq \Lambda^{-1} \|v\|_{H_0^1(\Omega)}^2 \quad \text{for all } v \in H_0^1(\Omega).$$

Let $\psi \in H_0^1(\Omega)$ and we define

$$\widetilde{\mathbb{K}} := \left\{ v \in H_0^1(\Omega) : v \geq \psi \text{ in } \Omega \right\},$$

which is a non-empty closed convex subset of $H_0^1(\Omega)$. By using Stampacchia's theorem [**Bre11**, Theorem 5.6][1], one reach the following lemma.

LEMMA 1.2.1 ("well-posedness" of a variational problem). *Let $\psi \in H_0^1(\Omega)$, $f \in H^{-1}(\Omega)$ and let $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ be the symmetric continuous coercive bilinear form given in (1.2.3). There exists a unique $u_* \in \widetilde{\mathbb{K}}$ such that*

$$u_* = \operatorname*{argmin}_{u \in \widetilde{\mathbb{K}}} \{a(u,u) - 2\langle f, u\rangle\}$$

and $u_* \in \widetilde{\mathbb{K}}$ can also be characterized by

$$a(u_*, u - u_*) \geq \langle f, u - u_*\rangle \quad \text{for all } u \in \widetilde{\mathbb{K}}. \tag{1.2.4}$$

Here $\langle \cdot, \cdot \rangle$ is the $H^{-1}(\Omega) \times H_0^1(\Omega)$ duality pair.

The following lemma shows that the element u_* in Lemma 1.2.1 also can be characterized as the smallest element of an obstacle problem.

PROPOSITION 1.2.2 (from variational problem to obstacle problem). *Let $\psi \in H_0^1(\Omega)$, $f \in H^{-1}(\Omega)$ and let $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ be the symmetric continuous coercive bilinear form given in (1.2.3). Let $u_* \in \widetilde{\mathbb{K}}$ be the function described in Lemma 1.2.1. If $u \in \widetilde{\mathbb{K}}$ satisfies*

$$-\nabla \cdot (A \nabla u) \geq f \quad \text{in } H^{-1}(\Omega)\text{-sense}, \tag{1.2.5}$$

then $u_* \leq u$ in Ω. In other words, u_* is the smallest element in the collection

$$\mathscr{F}(A,f) := \left\{ u \in H_0^1(\Omega) : \begin{array}{l} -\nabla \cdot (A\nabla u) \geq f \text{ in } H^{-1}(\Omega)\text{-sense} \\ u \geq \psi \text{ in } \Omega \end{array} \right\}.$$

PROOF. Since $\zeta := \min\{u_*, u\} \in \widetilde{\mathbb{K}}$, then from (1.2.4) we have

$$a(u_*, \zeta - u_*) \geq \langle f, \zeta - u_*\rangle. \tag{1.2.6}$$

Since $\zeta - u_* = \min\{u_*, u\} - u_* \leq 0$ in Ω, then from (1.2.5) we have

$$a(u, \zeta - u_*) \leq \langle f, \zeta - u_*\rangle. \tag{1.2.7}$$

We combine (1.2.6) and (1.2.7) to obtain

$$a(u - u_*, \zeta - u_*) \leq 0.$$

[1] Lax-Milgram theorem is a corollary of Stampacchia's theorem, see e.g. [**Bre11**, Corollary 5.8].

By using the definition of ζ, we compute that

$$\begin{aligned}0 &\geq a(u-u_*,\zeta-u_*) = \int_\Omega \nabla(u-u_*)\cdot A\nabla(\zeta-u_*)\,\mathrm{d}x \\ &= \int_{\{\zeta<u_*\}} \nabla(u-u_*)\cdot A\nabla(\zeta-u_*)\,\mathrm{d}x \quad \text{(because } \zeta\leq u_* \text{ in } \Omega\text{, (1.2.2) involved)} \\ &= \int_{\{\zeta<u_*\}} \nabla(\zeta-u_*)\cdot A\nabla(\zeta-u_*)\,\mathrm{d}x \quad \text{(because } \min\{u_*,u\}\equiv\zeta<u_* \text{ implies } \zeta=u\text{)} \\ &= \int_\Omega \nabla(\zeta-u_*)\cdot A\nabla(\zeta-u_*)\,\mathrm{d}x \geq \Lambda^{-1}\|\zeta-u_*\|_{H_0^1(\Omega)}^2,\end{aligned}$$

which implies $u_*=\zeta\equiv\min\{u_*,u\}$ in Ω, which implies our proposition. ☐

1.3. Definition of partial balayage

The main theme of this section is to introduce partial balayage of $\mu\in L_c^\infty(\mathbb{R}^n):=\{\mu\in L^\infty(\mathbb{R}^n):\mu \text{ has compact support}\}\subset\mathscr{E}'(\mathbb{R}^n)$. The Newtonian potential is defined as

$$U^\mu(x):=(\Phi*\mu)(x)=\int_{\mathbb{R}^n}\Phi(x-y)\mu(y)\,\mathrm{d}y.$$

We write $\mu_+:=\max\{\mu,0\}$ and $\mu_-:=-\min\{\mu,0\}$ and see that $\mu=\mu_+-\mu_-$ and $|\mu|=\mu_++\mu_-$. We recall the following mean value theorem for sub-harmonic functions.

LEMMA 1.3.1 (see e.g. [KLSS24, Appendix A]). *If $w\in L^1(B_R(x_0))$ satisfying $\Delta w\geq 0$ in $B_R(x_0)$, then, provided x_0 is a Lebesgue point of w,*

$$\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} w(x)\,\mathrm{d}x \geq w(x_0).$$

In addition, the mapping

$$r\in\mathbb{R}_{>0}\mapsto\frac{1}{|B_r(x_0)|}\int_{B_r} w(x)\,\mathrm{d}x$$

is monotone increasing, unless there exists an $R'>0$ such that $\Delta w=0$ in $B_{R'}$ in which the case the mapping is constant on $(0,R')$ and increasing on (R',∞).

We first begin with the following lemma.

LEMMA 1.3.2. *Let $\mu\in L_c^\infty(\mathbb{R}^n)$. Then*

$$\mathscr{F}(\mu):=\left\{v\in H^1_{\mathrm{loc}}(\mathbb{R}^n): \begin{array}{l}-\Delta v\leq 1 \text{ and } v\leq U^\mu \text{ in } \mathbb{R}^n\\ v=U^\mu \text{ outside a compact set}\end{array}\right\}\neq\emptyset.$$

PROOF OF LEMMA 1.3.2. We define

$$\tilde{u}:=U^\mu*\phi_r-U^{\mu_-} \quad \text{where} \quad \phi_r:=\frac{1}{|B_r|}\chi_{B_r}. \tag{1.3.1}$$

Using elliptic regularity, we know that $\tilde{u} \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$ for all $1 < p < \infty$. Note that

$$(U^{\mu_+} * \phi_r)(x) = \frac{1}{|B_r|}\int_{B_r} U^{\mu_+}(x-y)\,\mathrm{d}y = \frac{1}{|B_r|}\int_{B_r} U^{\mu_+}(y)\,\mathrm{d}y \quad \text{for all } x \in \mathbb{R}^n.$$

Since $-\Delta U^{\mu_+} = \mu_+$, using mean value theorem for subharmonic functions (Lemma 1.3.1), we see that[2]

$$\begin{aligned} U^{\mu_+} * \phi_r(x) &\le U^{\mu_+}(x) \quad \text{for all } x \in \mathbb{R}^n, \\ U^{\mu_+} * \phi_r(x) &= U^{\mu_+}(x) \quad \text{for all } x \notin \mathrm{supp}\,(\mu) + \overline{B_r}, \end{aligned}$$

which implies

$$\begin{aligned} \tilde{u}(x) &\le U^{\mu}(x) \quad \text{for all } x \in \mathbb{R}^n, \\ \tilde{u}(x) &= U^{\mu}(x) \quad \text{for all } x \notin \mathrm{supp}\,(\mu) + \overline{B_r}. \end{aligned}$$

On the other hand, we see that

$$-\Delta\tilde{u}(x) \le \mu_+ * \phi_r(x) = \frac{1}{|B_r|}\int_{B_r} \mu_+(x)\,\mathrm{d}x \le \frac{1}{|B_r|}\int_{\mathbb{R}^n} \mu_+(x)\,\mathrm{d}x \quad \text{for all } x \in \mathbb{R}^n.$$

We now choose $r > 0$ sufficiently large so that $|B_r| \ge \int_{\mathbb{R}^n} \mu_+(x)\,\mathrm{d}x$, we reach $-\Delta\tilde{u}(x) \le 1$ for all $x \in \mathbb{R}^n$. Thus we conclude that $\tilde{u} \in \mathscr{F}(\mu)$. $\square$

Before we further proceed, lets us introduce the following notion.

DEFINITION 1.3.3 (the term “near”). Let A be *any* set in $\mathbb{R}^n$. We say that a property holds *near* A if there exists an open set $U \supset A$ such that the property holds in U. Sometimes we refer such U an *open neighborhood* of A.

LEMMA 1.3.4. *Let $\mu \in L^\infty_c(\mathbb{R}^n)$. Then there exists a largest element V^μ in $\mathscr{F}(\mu)$, i.e. $V^\mu \ge v$ in $\mathbb{R}^n$ for all $v \in \mathscr{F}(\mu)$. In addition, the element V^μ satisfies*

$$\langle 1 + \Delta V^\mu, V^\mu - U^\mu \rangle = 0, \tag{1.3.2}$$

where $\langle\cdot,\cdot\rangle$ is the $H^{-1}(B_R) \times H^1_0(B_R)$ duality pairing for some suitable chosen $R > 1$.

REMARK 1.3.5. Since $V^\mu \in \mathscr{F}(\mu)$, then one can choose $R > 1$ such that $V^\mu - U^\mu \in H^1_0(B_R)$. Therefore $\Delta(V^\mu - U^\mu) \in H^{-1}(B_R)$. Possibly replacing $R > 1$ by a larger one, one may assume that $B_R \supset \mathrm{supp}\,(\mu)$, and one sees that $\mu \in H^{-1}(B_R)$. Therefore the term ΔV^μ in (1.3.2) can be understood as

$$\Delta V^\mu = -\mu + \Delta(V^\mu - U^\mu) \in H^{-1}(B_R).$$

[2]For two sets A and B in $\mathbb{R}^n$, we define the set $A + B := \{a + b : a \in A, b \in B\}$.

PROOF OF LEMMA 1.3.4. Fixing any $R > 0$ be such that $\tilde{u} = U^{\mu}$ outside B_R, where $\tilde{u}$ is the function given in (1.3.1). Let $\varphi \in H^1(B_R)$ be the unique solution to

$$\begin{cases} -\Delta\varphi = 1 & \text{in } B_R, \\ \varphi = U^{\mu} & \text{on } \partial B_R. \end{cases} \tag{1.3.3}$$

Define

$$\widetilde{\mathscr{F}}(\mu) := \left\{ w \in H_0^1(\Omega) : -\Delta w \geq 0 \text{ and } w \geq \varphi - U^{\mu} \text{ in } B_R \right\}.$$

We claim that there exists a smallest element $u_* \in \widetilde{\mathscr{F}}(\mu)$. If this is the case, then

$$V^{\mu} := \varphi - u_* \tag{1.3.4}$$

is the largest element of $\mathscr{F}(\mu)|_{B_R} := \{v|_{B_R} : v \in \mathscr{F}(\mu)\}$. Since $\tilde{u} = U^{\mu}$ near ∂B_R, then it is necessarily $V^{\mu} = U^{\mu}$ near ∂B_R. Therefore, it we extend V^{μ} by $V^{\mu} := U^{\mu}$ outside B_R, we know that $V^{\mu} \in H^1_{\rm loc}(\mathbb{R}^n)$ is the largest element in $\mathscr{F}(\mu)$.

In particular, the existence of the smallest element in $\tilde{\mathscr{F}}(\mu)$ follows from Proposition 1.2.2 with the bilinear form $a : H_0^1(B_R) \times H_0^1(B_R) \to \mathbb{R}$ defined by

$$a(v_1, v_2) := \int_{B_R} \nabla v_1(x) \cdot \nabla v_2(x) \, \mathrm{d}x \tag{1.3.5}$$

and the observation $\varphi - U^{\mu} \in H_0^1(B_R)$. Indeed, from Proposition 1.2.2, we know that $a(u_*, u - u_*) \geq 0$ for all $u \in H_0^1(B_R)$ with $u \geq \varphi - U^{\mu}$. Choosing $u = \varphi - U^{\mu}$, we have

$$-\langle \Delta u_*, \varphi - U^{\mu} - u_* \rangle = a(u_*, \varphi - U^{\mu} - u_*) \geq 0. \tag{1.3.6}$$

Since $-\Delta u_* \geq 0$ and $u_* \geq \varphi - U^{\mu}$ in B_R, then

$$\langle \Delta u_*, \varphi - U^{\mu} - u_* \rangle = \langle -\Delta u_*, u_* - (\varphi - U^{\mu}) \rangle \geq 0. \tag{1.3.7}$$

Combining (1.3.6) and (1.3.7), we obtain

$$0 = \langle \Delta u_*, \varphi - U^{\mu} - u_* \rangle \overset{(1.3.4)}{=} \langle -\Delta(\varphi - V^{\mu}), V^{\mu} - U^{\mu} \rangle \overset{(1.3.3)}{=} \langle 1 + \Delta V^{\mu}, V^{\mu} - U^{\mu} \rangle,$$

which completes our proof. $\square$

We now ready to define the main object which we are interested.

DEFINITION 1.3.6 (partial balayage for bounded functions). The *partial balayage* $\mathrm{Bal}(\mu)$ of $\mu \in L_c^{\infty}(\mathbb{R}^n)$ (with respect to Lebesgue measure) is defined as

$$\mathrm{Bal}(\mu) := -\Delta V^{\mu} \text{ in } \mathscr{D}'(\mathbb{R}^n),$$

where $\mathscr{D}'(\mathbb{R}^n)$ is the space of distributions on $\mathbb{R}^n$. We also called V^{μ} the *partial reduction* of U^{μ} [**GS09**].

As explained in Remark 1.3.5 above, one can find $R>0$ such that $\mathrm{Bal}(\mu)\in H^{-1}(B_R)$. For each $\mu\in L_c^\infty(\mathbb{R}^n)$, from definition of $\mathscr{F}(\mu)$, it is easy to see that

$$\mathrm{Bal}(\mu)\leq 1 \quad \text{and} \quad V^\mu\leq U^\mu \text{ in } \mathbb{R}^n. \tag{1.3.8}$$

We compute

$$\begin{aligned} U^{\mathrm{Bal}(\mu)}-U^\mu &= \Phi*(\mathrm{Bal}(\mu)-\mu) = -\Phi*(\Delta(V^\mu-U^\mu)) \\ &= -\Delta\Phi*(V^\mu-U^\mu) = \delta_0*(V^\mu-U^\mu) = V^\mu-U^\mu \quad \text{in } \mathscr{D}'(\mathbb{R}^n) \end{aligned}$$

where the convolution is understood as convolution of $\mathscr{D}'(\mathbb{R}^n)$ and $\mathscr{E}'(\mathbb{R}^n)$, which implies the following fundamental equality for partial balayage:

LEMMA 1.3.7. *Let $\mu\in L_c^\infty(\mathbb{R}^n)$, then $U^{\mathrm{Bal}(\mu)}=V^\mu$ in $\mathscr{D}'(\mathbb{R}^n)$.*

Combining Lemma 1.3.4 and Lemma 1.3.7, we reach the following corollary.

COROLLARY 1.3.8. *For each $\mu\in L_c^\infty(\mathbb{R}^n)$, one has $\left\langle 1-\mathrm{Bal}(\mu),U^{\mathrm{Bal}(\mu)-\mu}\right\rangle=0$.*

We *formally* define the bilinear form

$$(\mu_1,\mu_2)_e := \iint_{\mathbb{R}^n\times\mathbb{R}^n}\Phi(x-y)\,\mathrm{d}\mu_1(y)\,\mathrm{d}\mu_2(x) = \langle\mu_2,U^{\mu_1}\rangle,$$

where we write $\mathrm{d}\mu_j(x)=\mu_j(x)\,\mathrm{d}x$ for $j=1,2$, and *formally* denote the "energy"

$$E(\mu_1) := (\mu_1,\mu_1)_e. \tag{1.3.9}$$

Given any bounded smooth domain Ω in $\mathbb{R}^n$, we compute that

$$(\mu_1,\mu_2)_e = \langle\mu_2,U^{\mu_1}\rangle = \langle -\Delta U^{\mu_2},U^{\mu_1}\rangle = -\int_{\partial\Omega}U^{\mu_1}\partial_{\mathbf{n}}U^{\mu_2}\,\mathrm{d}S + \int_\Omega \nabla U^{\mu_1}\cdot\nabla U^{\mu_2}\,\mathrm{d}x. \tag{1.3.10}$$

where $\partial_{\mathbf{n}}$ is the outward normal derivative on $\partial\Omega$. If U^{μ_1} has compact support[3], by choosing $\Omega\supset\mathrm{supp}(\mu_1)$ we see that

$$E(\mu_1) = \|\nabla U^{\mu_1}\|_{L^2(\mathbb{R}^n)}^2 \text{ and in this case we denote } \|\mu_1\|_e := E(\mu_1). \tag{1.3.11}$$

For example, since $U^{\mathrm{Bal}(\mu)-\mu}=V^\mu-U^\mu$ has compact support, then we see that

$$E(\mathrm{Bal}(\mu)-\mu) = \|\nabla(V^\mu-U^\mu)\|_{L^2(\mathbb{R}^n)}^2 \equiv \|\mathrm{Bal}(\mu)-\mu\|_e.$$

We can rewrite Corollary 1.3.8 as

$$(\mathrm{Bal}(\mu)-\mu,1-\mathrm{Bal}(\mu))_e=0.$$

[3]However, in general we do not expect that U^μ has compact support: Let $\Omega\supset\mathrm{supp}(\mu)$ be a bounded smooth domain. By following the ideas in [**KW21**, Theorem 2.5] (see also references therein for more details on non-radiating sources for acoustic waves, electromagnetic waves as well as elastic waves), one can show that U^μ has compact support if and only if $\int_\Omega\mu\cdot w\,\mathrm{d}x=0$ for all $w\in\mathbb{E}(\Omega)$, where $\mathbb{E}(\Omega)$ is the completion of $\left\{w\in H^1(\Omega):\Delta w=0 \text{ in } \Omega\right\}$ in $L^2(\Omega)$.

Accordingly, one sees that the following holds for each $\sigma \in L_c^\infty(\mathbb{R}^n)$ with $\sigma \leq 1$:

$$\begin{aligned}
&(\mathrm{Bal}\,(\mu) - \mu, \sigma - \mathrm{Bal}\,(\mu))_e \\
&\quad = (\mathrm{Bal}\,(\mu) - \mu, \sigma - 1)_e + \overbrace{(\mathrm{Bal}\,(\mu) - \mu, 1 - \mathrm{Bal}\,(\mu))_e}^{=0} = \left\langle \sigma - 1, U^{\mathrm{Bal}\,(\mu) - \mu} \right\rangle \\
&\quad = \langle \sigma - 1, V^\mu - U^\mu \rangle \geq 0 \quad (\text{because } V^\mu \leq U^\mu, \text{ see (1.3.8)}).
\end{aligned}$$

Since

$$\begin{aligned}
&(\mathrm{Bal}\,(\mu) - \mu, \sigma - \mathrm{Bal}\,(\mu))_e \\
&\quad = (\mathrm{Bal}\,(\mu) - \mu, \mu - \mathrm{Bal}\,(\mu))_e + (\mathrm{Bal}\,(\mu) - \mu, \sigma - \mu)_e \\
&\quad = - \|\mathrm{Bal}\,(\mu) - \mu\|_e + (\mathrm{Bal}\,(\mu) - \mu, \sigma - \mu)_e,
\end{aligned}$$

then we now reach

$$\|\mathrm{Bal}\,(\mu) - \mu\|_e \leq (\mathrm{Bal}\,(\mu) - \mu, \sigma - \mu)_e \quad \text{for all } \sigma \in L_c^\infty(\mathbb{R}^n) \text{ with } \sigma \leq 1.$$

For each $\sigma \in L_c^\infty(\mathbb{R}^n)$ with $\sigma \leq 1$ such that $U^{\sigma - \mu}$ has compact support, we see that

$$\begin{aligned}
\|\mathrm{Bal}\,(\mu) - \mu\|_e &\leq (\mathrm{Bal}\,(\mu) - \mu, \sigma - \mu)_e \overset{(1.3.10)}{=} \int_{\mathbb{R}^n} \nabla U^{\mathrm{Bal}\,(\mu) - \mu} \cdot \nabla U^{\sigma - \mu} \, \mathrm{d}x \\
&\leq \left\| \nabla U^{\mathrm{Bal}\,(\mu) - \mu} \right\|_{L^2(\mathbb{R}^n)} \left\| \nabla U^{\sigma - \mu} \right\|_{L^2(\mathbb{R}^n)} \overset{(1.3.11)}{=} \|\mathrm{Bal}\,(\mu) - \mu\|_e^{1/2} \|\sigma - \mu\|_e^{1/2},
\end{aligned}$$

which concludes the following proposition.

PROPOSITION 1.3.9. *If $\mu \in L_c^\infty(\mathbb{R}^n)$, then its partial balayage* $\mathrm{Bal}\,(\mu)$ *minimizes the energy in the following sense:*

$$\|\mathrm{Bal}\,(\mu) - \mu\|_e \leq \|\sigma - \mu\|_e \quad \textit{for all } \sigma \in L_c^\infty(\mathbb{R}^n) \textit{ with } U^\sigma \in \mathscr{F}(\mu).$$

This shows that our definition of partial balayage is consistent to the one in [**Gus04**, Definition 3.1].

1.4. PDE characterization of quadrature domains

Before we discuss the relation between partial balayage and quadrature domain (see Section 1.5), we need to express quadrature domain in terms of PDE, as follows:

THEOREM 1.4.1. *Let D be a bounded open set and let $\mu \in \mathscr{E}'(D)$. The following are equivalent:*

(1) *D is a quadrature domain corresponding to μ;*
(2) *there exists a distribution u satisfying*

$$\begin{cases} \Delta u = \chi_D - \mu & \text{in } \mathbb{R}^n, \\ u = |\nabla u| = 0 & \text{in } D^\complement. \end{cases} \tag{1.4.1}$$

REMARK 1.4.2. Note that even though u is only assumed to be in $\mathscr{D}'(\mathbb{R}^n)$, since $\operatorname{supp}(\mu) \subset D$, one sees that $\Delta u = \chi_D$ near ∂D, and thus Calderón-Zygmund inequality [**GT01**, Theorem 9.11] (or simply referred as "elliptic regularity") and Sobolev embeddings (Appendix A) implies that $u \in C^1$ near ∂D, hence the condition $u = |\nabla u| = 0$ in $D^{\complement}$ is meaningful.

PROOF OF THE IMPLICATION (1) $\Longrightarrow$ (2) IN THEOREM 1.4.1. If D is a quadrature domain corresponding to $\mu \in \mathscr{E}'(D)$, then

$$\int_D \partial^\alpha \Phi(z-x)\,\mathrm{d}x = \langle \mu, \partial^\alpha \Phi(z-\cdot)\rangle \quad \text{for all } z \in D^{\complement} \text{ and } |\alpha| \le 1.$$

Let $u = -\Phi * (\chi_D - \mu)$, which is well-defined since $\chi_D - \mu \in \mathscr{E}'(\mathbb{R}^n)$, and one can verify that u satisfies (1.4.1). $\square$

We now want to prove the implication (2) $\Longrightarrow$ (1). Let u satisfies (1.4.1). For each $w \in L^1(D)$ that solves $\Delta w = 0$ *near* $\overline{D}$, by taking a cutoff function $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\psi = 1$ near $\overline{D}$ we have

$$\int_D w\,\mathrm{d}x - \langle w, \mu\rangle = \langle \chi_D - \mu, \psi w\rangle = \langle \Delta u, \psi w\rangle = \langle u, \Delta(\psi w)\rangle = 0,$$

using that the derivatives of ψ vanish near $\operatorname{supp}(u)$. For general $w \in L^1(D)$ with $\Delta w = 0$ *in* D, we need another argument involving the following Runge approximation result, which can be proved by following the argument in [**KLSS24**, Proposition 2.4], which is basically modified from [**Sak84**, Lemma 5.1], see also [**AH96**, Chapter 11] for related results:

LEMMA 1.4.3. *Let D be a bounded open set. The linear span of*

$$F := \left\{ \partial^\alpha \Phi(z-\cdot)|_D : z \in D^{\complement}, |\alpha| \le 1 \right\}$$

is dense in

$$HL^1(D) := \left\{ w \in L^1(D) : \Delta w = 0 \right\}$$

with respect to the $L^1(D)$ topology.

PROOF. By the Hahn-Banach theorem (Theorem A.1.1), it is enough to show that any bounded linear functional ℓ in $L^1(D)$ that satisfies $\ell|_F = 0$ also satisfies $\ell|_{HL^1(D)} = 0$. Since the dual of $L^1(D)$ is $L^\infty(D)$, there is a function $f \in L^\infty(D)$ with

$$\ell(w) = \int_D f w\,\mathrm{d}x, \quad w \in L^1(D).$$

We extend f by zero to $\mathbb{R}^n$ and consider the function $u = -\Phi * f$ in $\mathbb{R}^n$. By the assumption $\ell|_F = 0$, the function u satisfies

$$\begin{cases} \Delta u = f & \text{in } \mathbb{R}^n, \\ u = |\nabla u| = 0 & \text{in } D^{\complement}. \end{cases}$$

Note that since $f \in L^\infty$, using the Calderón-Zygmund inequality [**GT01**, Theorem 9.11] and Sobolev embeddings (Appendix A) one has $u \in \bigcap_{\alpha<1} C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$. In order to show that $\ell|_{HL^1(D)} = 0$, we take some $w \in HL^1(D)$ and compute

$$\ell(w) = \int_D fw\,\mathrm{d}x = \int_D (\Delta u)w\,\mathrm{d}x.$$

If one can integrate by parts and use the condition $\Delta w = 0$ to conclude that

$$\int_D (\Delta u)w\,\mathrm{d}x = 0. \tag{1.4.2}$$

This implies $\ell|_{HL^1(D)} = 0$ and prove the results. However, the proof of (1.4.2) is somehow delicate due the Calderón-Zygmund inequality [**GT01**, Theorem 9.11] does not hold true when $p = \infty$.

By using [**GT01**, Theorem 3.9], one sees that

$$|\nabla u(x) - \nabla u(y)| \le C|x-y|\log(1/|x-y|) \quad \text{for all } x,y \in \overline{D} \text{ with } |x-y| < e^{-2}.$$

Using the condition $u = |\nabla u| = 0$ in $D^{\complement}$, this implies that uniformly for $x \in D$ near ∂D one has

$$u(x) = O(\delta(x)^2 \log(1/\delta(x))),$$
$$\nabla u(x) = O(\delta(x)\log(1/\delta(x))),$$

where $\delta(x) = \operatorname{dist}(x,\partial D)$. We now introduce the sequence $(\omega_j)_{j=1}^{\infty}$ of Ahlfors-Bers mollifiers [**Ahl64, Ber65**] that satisfy $\omega_j \in C^\infty(\mathbb{R}^n)$, $0 \le \omega_j \le 1$, $\omega_j = 0$ near ∂D, $\omega_j = 1$ outside a neighborhood of ∂D, $\omega_j(x) \to 1$ for $x \notin \partial D$, and

$$|\partial^\alpha \omega_j(x)| \le C_\alpha j^{-1}\delta(x)^{-|\alpha|}(\log 1/\delta(x))^{-1} \quad \text{for } x \notin \partial D,$$

see [**Hed73**, Lemma 4]. One now has

$$\int_D (\Delta u)w\,\mathrm{d}x = \lim_{j\to\infty}\int_D (\Delta u)\omega_j w\,\mathrm{d}x = \lim_{j\to\infty}\int_D \big(\Delta(\omega_j u) - 2\nabla\omega_j\cdot\nabla u - (\Delta\omega_j)u\big)\,w\,\mathrm{d}x.$$

Using the estimates for u and ω_j, the limits corresponding to the last two terms inside the brackets are zero. Moreover, since w is smooth near $\operatorname{supp}(\omega_j)$, we have

$$\int_D (\Delta u)\omega_j w\,\mathrm{d}x = \lim_{j\to\infty}\int_D \omega_j u \Delta w\,\mathrm{d}x = 0,$$

which conclude (1.4.2). □

We now ready to prove the implication (2) $\Longrightarrow$ (1) in Theorem 1.4.1.

Proof of the implication (2) $\Longrightarrow$ (1) in Theorem 1.4.1. Let u satisfies (1.4.1). Since $u \in \mathscr{E}'(\mathbb{R}^n)$, then

$$u = -\Phi * \Delta u = \Phi * (\chi_D - \mu).$$

Using that $u = |\nabla u| = 0$ in $D^{\complement}$, we have

$$\int_D \partial^\alpha \Phi(z-x)\,\mathrm{d}x = \langle u, \partial^\alpha\Phi(z-\cdot)\rangle \quad \text{for all } z \in D^{\complement} \text{ and } |\alpha| \le 1.$$

Now let $w \in L^1(D)$ solves $\Delta w = 0$ in D and use Runge approximation (Lemma 1.4.3) to find a sequence $w_j \in \operatorname{span}\left\{ \partial^\alpha \Phi(z-\cdot)|_D : z \in D^\complement, |\alpha| \leq 1 \right\}$ with $w_j \to w$ in $L^1(D)$. In particular, for any $j \geq 1$ we have

$$\int_D w_j \, \mathrm{d}x = \langle \mu, w_j \rangle. \tag{1.4.3}$$

Since $\mu \in \mathscr{E}'(D)$, by using a deep result on the distribution $\mathscr{E}'(D)$ [**FJ98**], there is a compact set $K \subset D$ and an integer $m \geq 0$ such that

$$|\langle \mu, \varphi \rangle| \leq C \|\varphi\|_{C^m(K)} \quad \text{for all } \varphi \in C^\infty(D).$$

By elliptic regularity and Sobolev embeddings, any $v \in L^1(D)$ with $\Delta v \in H^{s-2}(D)$ satisfies $v \in C^m(K)$ with $s > m + \frac{n}{2}$. By the closed graph theorem, this yields the estimate

$$\|v\|_{C^m(K)} \leq C \left(\|v\|_{L^1(D)} + \|\Delta v\|_{H^{s-2}(D)} \right).$$

Apply this estimate to $v = w_j - w$ gives

$$\|w_j - w\|_{C^m(K)} \leq C \|w_j - w\|_{L^1(D)}.$$

We may take limit $j \to \infty$ in (1.4.3) to conclude that D is a quadrature domain corresponding to μ. □

1.5. Relation between partial balayage and quadrature domain

We now discuss some properties of partial balayage. These properties will explain why it called "partial balayage". Before we further proceed, let us generalize the notion in Definition 1.1.1.

DEFINITION 1.5.1 (quadrature domain for sub-harmonic functions). A bounded open set $D \subset \mathbb{R}^n$ (not necessarily connected) is called a quadrature domain for sub-harmonic functions, corresponding to a *Radon measure* μ *with* $\operatorname{supp}(\mu) \subset D$ if

$$\int_D w(x) \, \mathrm{d}x \geq \int w \, \mathrm{d}\mu \quad \text{for all } w \in L^1(D) \cap L^1(\mathrm{d}\mu) \text{ with } \Delta w \geq 0 \text{ in } D. \tag{1.5.1}$$

REMARK 1.5.2. Note that $\Delta w = 0$ if and only if both $\pm \Delta w \geq 0$. From this, one can easily see that each quadrature domain for sub-harmonic function is also necessary a quadrature domain for harmonic function (Definition 1.1.1) as well.

If $\mu \in L^\infty_c(\mathbb{R}^n)$ with $\operatorname{supp}(D)$, then

$$\int w \, \mathrm{d}\mu = \int_{\mathbb{R}^n} w(x) \mu(x) \, \mathrm{d}x.$$

EXAMPLE 1.5.3. If we write $\mu = |B_R(x_0)| \delta_{x_0}$, the condition $w \in L^1(\mathrm{d}\mu)$ guarantees that x_0 is a Lebesgue point of w. The above mean value theorem for sub-harmonic functions ((Lemma 1.3.1)) show that each ball in $\mathbb{R}^n$ is also a quadrature domain for sub-harmonic functions as well. This example reminds us that don't forget about the assumption $w \in L^1(\mathrm{d}\mu)$ in (1.5.1).

Suppose that D is a quadrature domain for sub-harmonic function with respect to $\mu \in L_c^\infty(\mathbb{R}^n)$. Since the fundamental solution Φ of $-\Delta$ belongs to $L^1_{\rm loc}(\mathbb{R}^n)$, we can choose $w = -\Phi(z-\cdot)$ in (1.1.2) and (1.5.1) to see that

$$U^D(z) \le U^\mu(z) \text{ for all } z \in \mathbb{R}^n, \tag{1.5.2a}$$

$$U^D(z) = U^\mu(z) \text{ for all } z \in D^\complement. \tag{1.5.2b}$$

The following simple observation suggests the strong relation between partial balayage and quadrature domains:

LEMMA 1.5.4. *Let $\mu \in L_c^\infty(\mathbb{R}^n)$. If*

$$\mathrm{Bal}\,(\mu) = \chi_D \quad \textit{for some open set } D, \tag{1.5.3}$$

then (1.5.2a) and (1.5.2b) hold.

PROOF. (1.5.2a) is an immediate consequence of (1.3.8), Lemma 1.3.7 and (1.5.3). On the other hand, we combine Corollary 1.3.8 and (1.5.3) to see that

$$0 = \langle 1-\chi_D, U^D - U^\mu \rangle = \int_{D^\complement} (U^D - U^\mu)\,\mathrm{d}x.$$

Since $U^D \overset{(1.5.3)}{=} U^{\mathrm{Bal}\,(\mu)} \overset{\text{Lemma 1.3.7}}{=} V^\mu \overset{(1.3.8)}{\le} U^\mu$, then we conclude (1.5.2b). $\square$

1.6. Structure of partial balayage

The main focus of this section is to prove the following theorem, which is probably the most challenging part of partial balayage theory.

THEOREM 1.6.1. *For each $\mu \in L_c^\infty(\mathbb{R}^n)$, one has*

$$\min\{\mu, 1\} \le \mathrm{Bal}\,(\mu) \le 1 \quad \textit{in } \mathbb{R}^n. \tag{1.6.1}$$

Furthermore, if we define the open sets

$$D(\mu) := (\mathrm{supp}\,(1-\mathrm{Bal}\,(\mu)))^\complement \quad \textit{and} \quad \omega(\mu) := \left\{ x \in \mathbb{R}^n : U^\mu(x) > U^{\mathrm{Bal}\,(\mu)} \right\},$$

then $\omega(\mu) \subset D(\mu)$ and for each measurable set D with $\omega(\mu) \subset D \subset D(\mu)$ we have

$$\mathrm{Bal}\,(\mu) = \chi_D + \chi_{D^\complement}\mu. \tag{1.6.2}$$

REMARK 1.6.2. The set $\omega(\mu)$ is called the *non-contact set* of μ. The set $D(\mu)$ is called the *saturated set* of μ. One sees that $D(\mu)$ is the largest set $\mathscr{O} \subset \mathbb{R}^n$ such that $\mathrm{Bal}\,(\mu)|_{\mathscr{O}} = \chi_{\mathscr{O}}$, therefore $\omega(\mu) \subset D(\mu)$. In fact, if $\mu > 1$ on $\mathrm{supp}\,(\mu)$, then $\mathrm{supp}\,(\mu) \subset \omega(\mu) \subset D(\mu)$, which implies that $\chi_{\omega(\mu)^\complement}\mu = \chi_{D(\mu)^\complement}\mu = 0$ and thus

$$\chi_{\omega(\mu)} = \mathrm{Bal}\,(\mu) = \chi_{D(\mu)},$$

which implies that $|D(\mu)\setminus\omega(\mu)|=0$.

We first state with the following technical lemma before proving our theorem.

LEMMA 1.6.3 (a special case of [**KS00**, Theorem II.6.6]). *Let Ω be any open set in $\mathbb{R}^n$. If $w_1,w_2\in H^1(\Omega)$ satisfy $-\Delta w_j\geq 0$ in $\mathscr{D}'(\Omega)$ for all $j=1,2$, then $-\Delta(\min\{w_1,w_2\})\geq 0$ in $\mathscr{D}'(\Omega)$.*

Now we are ready to prove our theorem.

PROOF OF THEOREM 1.6.1. In order to deliver the ideas clearly, we divide the proof into steps.

Step 1: A minimization problem. Let $R>1$ be the number mentioned in Lemma 1.3.4. Let $\xi\in H_0^1(B_R)$ be the unique solution to

$$-\Delta\xi=(1-\mu)_+\quad\text{in } B_R \tag{1.6.3}$$

and consider the constraint set

$$\widehat{\mathbb{K}}:=\left\{w\in H_0^1(B_R):\omega\geq\xi-u_*\text{ in } B_R\right\},$$

where $u_*\in H_0^1(B_R)$ the function appeared in the proof of Lemma 1.3.4. Note that $\xi-u_*\in\widehat{\mathbb{K}}$, which shows that $\widehat{\mathbb{K}}$ is nonempty. We recall that u_* minimizes the functional $a(\cdot,\cdot)$ among all functions $v\in\widetilde{\mathbb{K}}':=\left\{v\in H_0^1(B_R):v\geq\varphi-U^\mu\text{ in } B_R\right\}$, where $a(\cdot,\cdot)$ is the bilinear form given in (1.3.5). By using Stampacchia's theorem [**Bre11**, Theorem 5.6], there exists a unique $w_*\in\widehat{\mathbb{K}}$ which minimizes the functional $a(\cdot,\cdot)$ in $\widehat{\mathbb{K}}$. Moreover, the minimizer w_* is characterized by the property

$$a(w_*,w-w_*)=\langle-\Delta w_*,w-w_*\rangle\geq 0\quad\text{for all } w\in\widehat{\mathbb{K}}. \tag{1.6.4}$$

Step 2: Complementary formulation. Since $w_*\in\widehat{\mathbb{K}}$, we can restrict (1.6.4) to those satisfying $w\geq w_*$. The definition of the bilinear form $a(\cdot,\cdot)$ implies that

$$-\Delta w_*\geq 0\quad\text{in } B_R. \tag{1.6.5a}$$

Choosing $w=\xi-u_*$ in (1.6.4) gives

$$\langle-\Delta w_*,\xi-u_*-w_*\rangle\geq 0,$$

which along (1.6.5a) and the fact that $w_*\geq\xi-u_*$ implies

$$\langle-\Delta w_*,\xi-u_*-w_*\rangle=0. \tag{1.6.5b}$$

Conversely, if $w_*\in\widehat{\mathbb{K}}$ satisfying (1.6.5a) and (1.6.5b), then for each $w\in\widehat{\mathbb{K}}$ one has

$$\begin{aligned}&\langle-\Delta w_*,w-w_*\rangle\\ &=\Big\langle\overbrace{-\Delta w_*}^{\geq 0\,\because(1.6.5a)},\overbrace{w-(\xi-u_*)}^{\geq 0\,\because w\in\widehat{\mathbb{K}}}\Big\rangle+\overbrace{\langle-\Delta w_*,(\xi-u_*)-w_*\rangle}^{=0\,\because(1.6.5b)}\geq 0.\end{aligned}$$

We conclude that the following are equivalent:

(1) $w_* \in \widehat{\mathbb{K}}$ which minimizes the functional $a(\cdot,\cdot)$ in $\widehat{\mathbb{K}}$;
(2) $w_* \in \widehat{\mathbb{K}}$ satisfies (1.6.4);
(3) $w_* \in \widehat{\mathbb{K}}$ satisfies (1.6.5a) and (1.6.5b).

The advantage of considering the complementary formulation (3) is it does not involving test function $w \in \widehat{\mathbb{K}}$, which allows us to obtain an energy inequality.

Step 3: An energy inequality. We rewrite (1.6.5b) as

$$\langle -\Delta w_*, \xi - w_* \rangle = \langle -\Delta w_*, u_* \rangle . \tag{1.6.6}$$

The inequalities $-\Delta \xi = (1-\mu)_+ \geq 0$ and $w_* \geq \xi - u_*$ (iff $u_* \geq \xi - w_*$) thus imply that

$$\langle -\Delta \xi, \xi - w_* \rangle \leq \langle -\Delta \xi, u_* \rangle ,$$

together with (1.6.6), one finds that

$$\begin{aligned} a(\xi - w_*, \xi - w_*) &= \langle -\Delta(\xi - w_*), \xi - w_* \rangle \\ &\leq \langle -\Delta(\xi - w_*), u_* \rangle = a(\xi - w_*, u_*) \\ &\leq a(\xi - w_*, \xi - w_*)^{1/2} a(u_*, u_*)^{1/2}, \end{aligned}$$

and we reach the following energy inequality

$$a(\xi - w_*, \xi - w_*) \leq a(u_*, u_*). \tag{1.6.7}$$

Step 4: Verifying $w_* = \xi - u_*$. If we can show $\xi - w_* \in \widetilde{\mathbb{K}}'$, since $u_* \in \widetilde{\mathbb{K}}'$ is the minimizer of $a(\cdot,\cdot)$ in $\widetilde{\mathbb{K}}$, then

$$a(u_*, u_*) \leq a(\xi - w_*, \xi - w_*).$$

Together with (1.6.7), we reach

$$a(\xi - w_*, \xi - w_*) = a(u_*, u_*),$$

this means that $\xi - w_* \in \widetilde{\mathbb{K}}'$ is another minimizer of $a(\cdot,\cdot)$ in $\widetilde{\mathbb{K}}$. The uniqueness of minimizers in $\widetilde{\mathbb{K}}'$ implies that $u_* = \xi - w_*$, that is, $w_* = \xi - u_*$.

It remains to show that $\xi - w_* \in \widetilde{\mathbb{K}}'$. Let

$$\phi = \min\{w_*, \xi - (\varphi - U^\mu)\} \text{ in } B_R,$$

where φ is the function given in (1.3.3). By using Lemma 1.6.3, one sees that

$$\phi \leq w_*, \quad \phi \in \widehat{\mathbb{K}}, \quad -\Delta \phi \geq 0 \text{ in } B_R, \tag{1.6.8}$$

now together with (1.6.5a), we have

$$a(\phi,\phi) = \langle -\Delta \phi, \phi \rangle \overset{(1.6.8)}{\leq} \langle -\Delta \phi, w_* \rangle = \langle -\Delta w_*, \phi \rangle \overset{(1.6.5a)(1.6.8)}{\leq} \langle -\Delta w_*, w_* \rangle = a(w_*, w_*).$$

Since $w_* \in \widehat{\mathbb{K}}$ is the unique minimizer of $a(\cdot,\cdot)$ in $\widehat{\mathbb{K}}$, then we conclude $\phi = w_*$ in B_R, which means that $w_* \leq \xi - (\varphi - U^\mu)$. Now we have $\xi - w_* \geq \varphi - U^\mu$, which conclude $\xi - w_* \in \widetilde{\mathbb{K}}'$.

Step 5: Proving (1.6.1). Combining $w_* = \xi - u_*$ and (1.6.5a), we see that

$$-\Delta u_* \leq -\Delta\xi \overset{(1.6.3)}{=} (1-\mu)_+ \quad \text{in } B_R. \tag{1.6.9}$$

Now by (1.3.4) and Lemma 1.3.7 we have $U^{\mathrm{Bal}\,(\mu)} = V^\mu = \varphi - u_*$, that is,

$$u_* = \varphi - U^{\mathrm{Bal}\,(\mu)}.$$

Now from (1.6.9) we reach

$$1 - \mathrm{Bal}\,(\mu) \overset{(1.3.3)}{=} -\Delta(\varphi - U^{\mathrm{Bal}\,(\mu)}) \leq (1-\mu)_+ = \max\{1-\mu, 0\},$$

that is,

$$\mathrm{Bal}\,(\mu) - 1 \geq -\max\{1-\mu,0\} = \min\{\mu - 1, 0\}.$$

Now we have

$$\mathrm{Bal}\,(\mu) \geq 1 + \min\{\mu-1,0\} = \min\{\mu,1\},$$

together with (1.3.8), we conclude (1.6.1).

Step 6: Proving (1.6.2). Now from (1.6.1) we see that $\mathrm{Bal}\,(\mu) \in L^\infty(\mathbb{R}^n)$. By the Calderón-Zygmund inequality, we see that $U^\mu, U^{\mathrm{Bal}\,(\mu)} \in \bigcap_{p<\infty} W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n) \subset \bigcap_{\alpha<1} C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$, which shows that $\omega(\mu)$ is a well-defined open set. From Lemma 1.3.4, it follows that

$$0 \leq \int_{\omega(\mu)} (U^\mu - U^{\mathrm{Bal}\,(\mu)})(1 - \mathrm{Bal}\,(\mu))\,\mathrm{d}x \leq \int_{B_R} (U^\mu - U^{\mathrm{Bal}\,(\mu)})(1 - \mathrm{Bal}\,(\mu))\,\mathrm{d}x = 0,$$

which implies that

$$\int_{\omega(\mu)} \overbrace{(U^\mu - U^{\mathrm{Bal}\,(\mu)})}^{>0} \overbrace{(1 - \mathrm{Bal}\,(\mu))}^{\geq 0}\,\mathrm{d}x = 0,$$

and hence $\mathrm{Bal}\,(\mu)|_{\omega(\mu)} = \chi_{\omega(\mu)}$. Since $\omega(\mu)^\complement = \left\{x \in \mathbb{R}^n : U^{\mathrm{Bal}\,(\mu)} = U^\mu\right\}$, it holds that

$$\mathrm{Bal}\,(\mu) - \mu = -\Delta(U^{\mathrm{Bal}\,(\mu)} - U^\mu) = 0 \quad \text{a.e. in } \omega(\mu)^\complement$$

a.e. in $\omega(\mu)^\complement$, and we reach $\mathrm{Bal}\,(\mu)|_{\omega(\mu)^\complement} = \chi_{\omega(\mu)^\complement}\mu$, and we reach

$$\mathrm{Bal}\,(\mu) = \chi_{\omega(\mu)} + \chi_{\omega(\mu)^\complement}\mu.$$

Consequently, for any measurable set D satisfying $\omega(\mu) \subset D \subset D(\mu)$, by the definition of $D(\mu)$ we have $\mathrm{Bal}\,(\mu)|_{D\setminus\omega(\mu)} = \chi_{D\setminus\omega(\mu)}$ and thus the decomposition

$$\mathrm{Bal}\,(\mu) = \chi_D + \chi_{D^\complement}\mu$$

follows. This complete the proof of Theorem 1.6.1. □

The following lemma also strongly suggests that partial balayage is related to free boundary[4], which is a key lemma in constructing quadrature domains.

LEMMA 1.6.4. *Let $\mu \in L_c^\infty(\mathbb{R}^n)$. Suppose that there exist an open set D satisfying the support condition*

$$\operatorname{supp}(\mu) \subset D \tag{1.6.10}$$

and there exists $u \in \mathscr{E}'(\mathbb{R}^n)$ satisfying

$$\Delta u = \chi_D - \mu \text{ in } \mathbb{R}^n, \quad u > 0 \text{ in } D, \quad u = 0 \text{ in } D^{\complement}, \tag{1.6.11}$$

then $\mathrm{Bal}(\mu) = \chi_D$ *and $D = \omega(\mu)$. In addition, D is a quadrature domain corresponding to μ.*

PROOF. Since $u \in \mathscr{E}'(\mathbb{R}^n)$, then

$$u = \Phi * (-\Delta u) = \Phi * (\mu - \chi_D) = U^\mu - U^D.$$

Since u is non-negative, then then we know that $U^D \in \mathscr{F}(\mu)$. For each $v \in \mathscr{F}(\mu)$, since $u = 0$ in $D^{\complement}$, we see that

$$w := U^D - v = \overbrace{U^D - U^\mu}^{=-u} + U^\mu - v \geq 0 \quad \text{in } D^{\complement}.$$

On the other hand, we have $-\Delta w = 1 + \Delta v \geq 0$ in D. Therefore by using maximum principle [**GT01**, Theorem 8.19], we see that $w \geq 0$ in $\mathbb{R}^n$. This shows that U^D is the largest element in $\mathscr{F}(\mu)$, therefore $V^\mu = U^D$ and then by Definition 1.3.6 we reach

$$\mathrm{Bal}(\mu) = -\Delta U^D = \chi_D.$$

By the above, we see that $D = \{u > 0\} = \{U^\mu - U^D > 0\} = \omega(\mu)$.

Since $u \in C^1$ attains its minimum in $D^{\complement}$, it holds that $|\nabla u| = 0$ in $D^{\complement}$. Therefore, by (1.6.10), (1.6.11) and then the PDE characterization of quadrature domain (Theorem 1.4.1) to conclude that D is a quadrature domain corresponding to μ. $\square$

1.7. Performing balayage in smaller steps

The main theme of this section is to prove the following theorems.

LEMMA 1.7.1. *If both $\mu_1, \mu_2 \in L_c^\infty(\mathbb{R}^n)$ are non-negative, then* $\mathrm{Bal}(\mu_1 + \mu_2) = \mathrm{Bal}(\mathrm{Bal}(\mu_1) + \mu_2)$.

We will show that similar result also holds true for non-contact sets.

LEMMA 1.7.2. *If $\mu_1, \mu_2 \in L_c^\infty(\mathbb{R}^n)$ such that μ_2 is non-negative, then $\omega(\mu_1 + \mu_2) = \omega(\mu_1) \cup \omega(\mathrm{Bal}(\mu_1) + \mu_2)$.*

[4]If a set Ω takes the form $\Omega = \{v > 0\}$ for some v satisfying some PDE, we sometimes refer such set a "free boundary".

PROOF OF LEMMA 1.7.1. It is suffice to show

$$U^{\mathrm{Bal}(\mu_1+\mu_2)} = U^{\mathrm{Bal}(\mathrm{Bal}(\mu_1)+\mu_2)} \quad \text{in } \mathbb{R}^n. \tag{1.7.1}$$

By using (1.3.8), Lemma 1.3.7 and Definition 1.3.6, we see that

$$U^{\mathrm{Bal}(\mathrm{Bal}(\mu_1)+\mu_2)} \leq U^{\mathrm{Bal}(\mu_1)+\mu_2} = U^{\mathrm{Bal}(\mu_1)} + U^{\mu_2} \leq U^{\mu_1} + U^{\mu_2} \quad \text{in } \mathbb{R}^n \tag{1.7.2}$$

and

$$-\Delta U^{\mathrm{Bal}(\mathrm{Bal}(\mu_1)+\mu_2)} \leq 1 \quad \text{in } \mathbb{R}^n,$$

which shows that $U^{\mathrm{Bal}(\mathrm{Bal}(\mu_1)+\mu_2)} \in \mathscr{F}(\mu_1+\mu_2)$. Since $V^{\mu_1+\mu_2} \overset{\text{Lemma 1.3.7}}{=} U^{\mathrm{Bal}(\mu_1+\mu_2)}$ is the largest element in $\mathscr{F}(\mu_1+\mu_2)$, then we arrive at

$$U^{\mathrm{Bal}(\mathrm{Bal}(\mu_1)+\mu_2)} \leq U^{\mathrm{Bal}(\mu_1+\mu_2)} \quad \text{in } \mathbb{R}^n. \tag{1.7.3}$$

On the other hand, by using (1.3.8), Lemma 1.3.7 and Definition 1.3.6 we observe that

$$U^{\mathrm{Bal}(\mu_1+\mu_2)} - U^{\mu_2} \leq U^{\mu_1+\mu_2} - U^{\mu_2} = U^{\mu_1} \quad \text{in } \mathbb{R}^n$$

and

$$-\Delta\left(U^{\mathrm{Bal}(\mu_1+\mu_2)} - U^{\mu_2}\right) \leq 1-\mu_2 \leq 1 \quad \text{in } \mathbb{R}^n,$$

which shows that $U^{\mathrm{Bal}(\mu_1+\mu_2)} - U^{\mu_2} \in \mathscr{F}(\mu_1)$. Since $V^{\mu_1} \overset{\text{Lemma 1.3.7}}{=} U^{\mathrm{Bal}(\mu_1)}$ is the largest element in $\mathscr{F}(\mu_1)$, then we arrive at

$$U^{\mathrm{Bal}(\mu_1+\mu_2)} - U^{\mu_2} \leq U^{\mathrm{Bal}(\mu_1)} \quad \text{in } \mathbb{R}^n,$$

that is,

$$U^{\mathrm{Bal}(\mu_1+\mu_2)} \leq U^{\mathrm{Bal}(\mu_1)+\mu_2} \quad \text{in } \mathbb{R}^n. \tag{1.7.4}$$

Furthermore, from (1.3.8) one has $-\Delta U^{\mathrm{Bal}(\mu_1+\mu_2)} \leq 1$, together with (1.7.4) we know that $U^{\mathrm{Bal}(\mu_1+\mu_2)} \in \mathscr{F}(\mathrm{Bal}(\mu_1)+\mu_2)$. Since $V^{\mathrm{Bal}(\mu_1)+\mu_2} \overset{\text{Lemma 1.3.7}}{=} U^{\mathrm{Bal}(\mathrm{Bal}(\mu_1)+\mu_2)}$ is the largest element in $\mathscr{F}(\mathrm{Bal}(\mu_1)+\mu_2)$, then we arrive that

$$U^{\mathrm{Bal}(\mu_1+\mu_2)} \leq U^{\mathrm{Bal}(\mathrm{Bal}(\mu_1)+\mu_2)} \quad \text{in } \mathbb{R}^n. \tag{1.7.5}$$

Finally, by combining (1.7.3) and (1.7.5) we reach (1.7.1) and we conclude our lemma. □

PROOF OF LEMMA 1.7.2. We now combine (1.7.1) and (1.7.2) to see that

$$U^{\mathrm{Bal}(\mu_1+\mu_2)} = U^{\mathrm{Bal}(\mathrm{Bal}(\mu_1)+\mu_2)} \leq U^{\mathrm{Bal}(\mu_1)+\mu_2} = U^{\mathrm{Bal}(\mu_1)} + U^{\mu_2} \leq U^{\mu_1} + U^{\mu_2} = U^{\mu_1+\mu_2} \quad \text{in } \mathbb{R}^n.$$

The first inequality is an equality only in $\omega(\mathrm{Bal}(\mu_1)+\mu_2)^{\complement}$ and the second inequality is an equality only in $\omega(\mu_1)^{\complement}$, therefore

$$U^{\mathrm{Bal}(\mu_1+\mu_2)} \leq U^{\mu_1+\mu_2} \quad \text{in } \mathbb{R}^n$$

and the equality holds only in $\omega(\mathrm{Bal}(\mu_1)+\mu_2)^{\complement} \cap \omega(\mu_1)^{\complement} = (\omega(\mathrm{Bal}(\mu_1)+\mu_2) \cup \omega(\mu_1))^{\complement}$. Thus, we reach

$$\omega(\mathrm{Bal}(\mu_1)+\mu_2) \cup \omega(\mu_1) = \left\{ U^{\mathrm{Bal}(\mu_1+\mu_2)} < U^{\mu_1+\mu_2} \right\} \stackrel{\mathrm{def}}{=} \omega(\mu_1+\mu_2),$$

which conclude our lemma. □

1.8. Construction of quadrature domains using partial balayage

By using partial balayage, one can construct quadrature domains as in follows:

THEOREM 1.8.1 ([**KLSS24**, Theorem 7.1]). *Let μ be a positive Radon measure supported in B_ε for some $\varepsilon > 0$. There exists a constant $c_n > 0$ depending only on the dimension such that if*

$$\varepsilon < c_n \mu(\mathbb{R}^n)^{1/n},$$

then there exists an open connected set D with real-analytic boundary which is a quadrature domain corresponding to $\mu \in \mathscr{E}'(D)$. Moreover, for each $w \in L^1(D) \cap L^1(\mathrm{d}\mu)$ satisfying $\Delta w \geq 0$ in D we have

$$\int_D w(x)\,\mathrm{d}x \geq \int w\,\mathrm{d}\mu.$$

REMARK 1.8.2. The proof of analyticity of ∂D involving a free boundary methods called the "moving plane technique". Here μ is not necessarily bounded, this can be done by using the trick in (1.1.3).

It is too difficult to prove the above theorem within a few lectures. We will just prove the following special case in order to discuss the main idea of the construction.

LEMMA 1.8.3. *For each $R > r > 0$, one has*

$$\mathrm{Bal}\left(\frac{R^n}{r^n}\chi_{B_r}\right) = \chi_{B_R} \quad \text{and} \quad \omega\left(\frac{R^n}{r^n}\chi_{B_r}\right) = B_R.$$

In addition, B_R is a quadrature domain corresponding to $\mu = \frac{R^n}{r^n}\chi_{B_r}$.

PROOF. For each $x \in \mathbb{R}^n$, we see that $y \mapsto \Phi(x-y)$ is in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and satisfies $-\Delta\Phi = \delta_x \geq 0$ in $\mathbb{R}^n$. By using the mean value theorem for subharmonic functions (Lemma 1.3.1), we see that

$$\frac{1}{|B_r|}U^{B_r}(x) = \frac{1}{|B_r|}\int_{B_r}\Phi(x-y)\,\mathrm{d}y \geq \frac{1}{|B_R|}\int_{B_R}\Phi(x-y)\,\mathrm{d}y = \frac{1}{|B_R|}U^{B_R}(x) \quad \text{forall } x \in \mathbb{R}^n$$

and the equality holds if and only if $x \in B_R^{\complement}$. In other words, the function $u = \frac{|B_R|}{|B_r|}U^{B_r} - U^{B_R} \in C^1(\mathbb{R}^n)$ satisfies

$$\begin{cases} \Delta u = \chi_{B_R} - \frac{R^n}{r^n}\chi_{B_r} & \text{in } \mathbb{R}^n, \\ u > 0 \text{ in } B_R, \quad u = 0 \text{ in } B_R^{\complement}. \end{cases}$$

The conclusion of the lemma follows by applying Lemma 1.6.4. □

CHAPTER 2

Partial balayage of general unbounded measures

2.1. Motivation

In view of Theorem 1.4.1, we now introduce the concept of the two-phase quadrature domain as in [**KS24**, (1.7)], see also [**EPS11, GS12**].

DEFINITION 2.1.1. Let $D_{\pm}$ be disjoint bounded open subsets of $\mathbb{R}^n$ and let $\mu_{\pm} \in \mathscr{E}'(D_{\pm})$, respectively. If there exists a (compactly supported) distribution u such that

$$\Delta u = (1-\mu_+)\chi_{D_+} - (1-\mu_-)\chi_{D_-} \text{ in } \mathbb{R}^n, \quad u = 0 \text{ in } (D_+ \cup D_-)^{\complement},$$

then we designate such a pair (D_+, D_-) as a *two-phase quadrature domain* (for harmonic functions) corresponding to $(\mu_+, \mu_-) \in \mathscr{E}'(D_+) \times \mathscr{E}'(D_-)$.

EXAMPLE 2.1.2. If $D_{\pm}$ are quadrature domains corresponding to $\mu_{\pm} \in \mathscr{E}'(D_{\pm})$, respectively, and satisfying

$$\overline{D_+} \cap \overline{D_-} = \emptyset, \tag{2.1.1}$$

then by using Theorem 1.4.1 one easily see that (D_+, D_-) is a two-phase quadrature domain corresponding to $(\mu_+, \mu_-) \in \mathscr{E}'(D_+) \times \mathscr{E}'(D_-)$.

REMARK 2.1.3. By using [**GS12**, Theorem 3.1], one sees that such pair (D_+, D_-) has the property that

$$\int_{D_+} h(x)\,\mathrm{d}x - \int_{D_-} h(x)\,\mathrm{d}x = \langle \mu_+, h \rangle - \langle \mu_-, h \rangle \tag{2.1.2}$$

for all $h \in C(\overline{D_+ \cup D_-})$ with $\Delta h = 0$ in $D_+ \cup D_-$. Conversely, if such pair (D_+, D_-) satisfies (2.1.2) with $\mu_{\pm} \in \mathscr{E}'(D_{\pm})$, then there exist "polar sets" Z_+ and Z_- such that $(D_+ \cup Z_+, D_- \cup Z_-)$ is a two-phase quadrature domain corresponding to (μ_+, μ_-). The proof is technical, which involving swept measure, see (2.5.2) below, we will not going to walk through the details there.

One can refer [**EPS11, GS12**] for some other nontrivial examples (i.e. which do not satisfy (2.1.1)). One also may construct two-phase quadrature domains by using a partial balayage procedure. Unlike the one-phase problem above, the "convolution technique" (1.1.3) does not work in this case, therefore one need to introduce the partial balayage of general measures, which is the main theme of this chapter. The framework adopted here largely follows [**GS12, GS24**]. Rather than go through all details, we only highlight some main ideas of partial balayage.

2.2. Maximum principle and δ-subharmonic functions

DEFINITION 2.2.1 ([**Rud87**, Definition 2.8]). Let X be a topological space and consider a function $f : X \to [-\infty, \infty]$. If

$$\{x \in X : f(x) > \alpha\} \text{ is open for each } \alpha \in \mathbb{R},$$

then f is said to be *lower semicontinuous* (*LSC*). If

$$\{x \in X : f(x) < \alpha\} \text{ is open for each } \alpha \in \mathbb{R},$$

then f is said to be *upper semicontinuous* (*USC*).

EXERCISE 2.2.2. If X is compact and $f : X \to (-\infty, \infty)$ is USC, prove that f attains its maximum at some point of X.

A function s that is upper semicontinuous (USC) and satisfies $\Delta s \geq 0$ (in the sense of distributions) will be referred to as *subharmonic*. Similarly, s will be termined *superharmonic* if $-s$ is subharmonic. In addition, we use the term *harmonic* when s is both subharmonic and superharmonic. Here we remind the readers that the terminology "harmonic" introducing here is slightly different with the one in Chapter 1: s is harmonic if and only if $\Delta s = 0$ (in the sense of distributions) and s is *continuous*. The partial balayage heavily relies on the following concept:

LEMMA 2.2.3 ([**GS24**, Proposition 2.8]). *Let Ω be any domain (i.e. open and connected) in $\mathbb{R}^n$. The maximum principle holds on any domain (i.e. open and connected) Ω in $\mathbb{R}^n$, that is, every subharmonic function s which is bounded from above and satisfies*

$$\limsup_{x \to z} s(z) \leq 0 \text{ for all } z \in \partial\Omega$$

must also satisfy $s \leq 0$ in Ω.

Here we remind the readers that we do not impose any assumptions on the boundary $\partial\Omega$ in Lemma 2.2.3. As mentioned in [**GS12**], by a δ-*subharmonic* function on an open set Ω we mean a function $w = s_1 - s_2$ for some subharmonic functions s_1 and s_2 on Ω. However, such function is defined only quasi-everywhere on Ω, i.e. outside the *polar set* where $s_1 = s_2 = -\infty$. One may define w on such polar sets by using some suitable fine topology [**GS12**, section 2.2], here we skip those technical details.

2.3. Definition and some properties of partial balayage

Given an open set $D \subset \mathbb{R}^n$ and a positive measure μ with compact support on $\mathbb{R}^n$, we define

$$\mathscr{F}_D(\mu) := \left\{ v \in \mathscr{D}'(\mathbb{R}^n) : \begin{array}{l} -\Delta v \leq 1 \text{ in } D, \quad v \leq U^\mu \text{ in } \mathbb{R}^n \\ \{v < U^\mu\} \text{ is bounded} \end{array} \right\}.$$

We denote $\mathscr{F}(\mu) := \mathscr{F}_{\mathbb{R}^n}(\mu)$, one will later see that this is exactly same as the one in Lemma 1.3.2 above for the case when μ is bounded. Obviously, $\mathscr{F}(\mu) \subset \mathscr{F}_D(\mu)$, and by [**GS24**, Proposition 3.3][1] one can guarantee $\mathscr{F}(\mu) \neq \emptyset$, and so is $\mathscr{F}_D(\mu)$.

We will need the following technical lemma [**BP04**] (see also [**GS12**, Corollary 2.3] for a short algernative proof):

LEMMA 2.3.1 (Kato's inequality). *If w is a δ-subharmonic function on an open set, then*

$$-\Delta \min\{w,0\} \geq (-\Delta w)\chi_{\{w\leq 0\}}.$$

If u and v are subharmonic functions on an open set Ω, then $v-u$ is δ-subharmonic on Ω. By using the Kato's inequality, one sees that

$$\begin{aligned} \Delta \max\{u,v\} &= -\Delta \min\{-u,-v\} = -\Delta\left(\min\{v-u,0\}-v\right) \\ &\geq (-\Delta(v-u))\chi_{\{v-u\leq 0\}} + \Delta v = (\Delta u)\chi_{\{v\leq u\}} - (\Delta v)\chi_{\{v\leq u\}} + \Delta v \\ &= (\Delta u)\chi_{\{u\geq v\}} + (\Delta v)\chi_{\{v>u\}}, \end{aligned} \tag{2.3.1}$$

and the following corollary (a generalization of Lemma 1.6.3 above) follows:

COROLLARY 2.3.2. *If u and v are subharmonic functions on Ω, then so also is* $\max\{u,v\}$.

Let $u,v \in \mathscr{F}_D(\mu)$. Note that $-\Delta(v-U^1) \leq 0$ and $-\Delta(w-U^1) \leq 0$ in D, then by using Corollary 2.3.2 one sees that

$$0 \geq -\Delta \max\{v-U^1, w-U^1\} = -\Delta\left(\max\{v,w\}-U^1\right) = -\Delta \max\{v,w\} - 1 \text{ in } D.$$

On the other hand, one also sees that $\max\{u,v\} \leq U^\mu$ in $\mathbb{R}^n$ and $\{\max\{u,v\} < U^\mu\} \subset \{v < U^\mu\}$ is bounded, therefore we conclude that

$$\max\{u,v\} \in \mathscr{F}_D(\mu) \quad \text{for all } u,v \in \mathscr{F}_D(\mu).$$

Now, similar to [**GS24**, Section 3], by using standard potential theoretic arguments [**AG01**, Section 3.7] show that $\mathscr{F}_D(\mu)$ has a largest element V_D^μ, which has a USC representative. Again, we also called V_D^μ the *partial reduction* of U^μ [**GS09**]. Accordingly, we can define the non-contact set by

$$\omega_D(\mu) := \left\{V_D^\mu < U^\mu\right\}, \quad \omega(\mu) := \omega_{\mathbb{R}^n}(\mu),$$

and the *partial balayage* is defined by

$$\mathrm{Bal}_D(\mu) := -\Delta V_D^\mu \text{ in } \mathscr{D}'(\mathbb{R}^n).$$

Obviously, one has $V^\mu \leq V_D^\mu$ and $\omega_D(\mu) \subset \omega(\mu) \cap D$ for any open set D. By using Lemma 2.2.3, one sees that the maximum principle holds on $\omega_D(\mu)$. By using the fact $V_D^\mu = U^\mu$ in $D^\complement$, one can

[1]This proposition is due to Simon Larson.

easily verify that (by using the ideas in the proof of Lemma 1.3.7)

$$V^{\mu} = U^{\mathrm{Bal}_D(\mu)}.$$

In fact, the following structure theorem holds:

$$\mathrm{Bal}_D(\mu) = \chi_{\omega_D(\mu)} + \mu\chi_{\omega_D(\mu)^{\complement}} + \nu \tag{2.3.2}$$

for some measure $\nu \geq 0$ which is supported on $\partial D \cap \partial\omega_D(\mu)$.

2.4. Construction of two-phase quadrature domains

Given a signed measure $\mu = \mu_+ - \mu_-$ with compact support and a Borel function $u : \mathbb{R}^n \to [-\infty, +\infty]$, we define the signed measure

$$\eta(u,\mu) := \left((\mu_+ - 1)_+ - (\mu_+ - 1)_-\chi_{\{u>0\}}\right) - \left((\mu_- - 1)_+ - (\mu_- - 1)_-\chi_{\{u<0\}}\right).$$

We first prove some properties of η:

LEMMA 2.4.1 ([**GS12**, Lemma 4.1]). *Let $u, u_1, u_2 : \mathbb{R}^n \to [-\infty,\infty]$ be Borel measurable functions, μ, μ_1, μ_2 be signed measures with compact supports, and $A \subset \mathbb{R}^n$ be Borel sets. Then*

(a) $\eta(-u,-\mu) = -\eta(u,\mu)$;
(b) $\mu - 1 \leq \eta(u,\mu) \leq \mu + 1$; *and*
(c) $u_1\chi_A \leq u_2\chi_A$ *and* $\mu_1\chi_A \geq \mu_2\chi_A$ *imply that* $\eta(u_1,\mu_1)\chi_A \geq \eta(u_2,\mu_2)\chi_A$.

Part (a) is obvious (left as exercise).

PROOF OF LEMMA 2.4.1(B). Note that

$$\begin{aligned}
\mu - 1 &= (\mu_+ - 1) - \mu_- = ((\mu_+ - 1)_+ - (\mu_+ - 1)_-) - ((\mu_- - 1)_+ - (\mu_- - 1)_- + 1) \\
&= \left((\mu_+ - 1)_+ - (\mu_+ - 1)_-\chi_{\{u<0\}}\right) - \overbrace{(\mu_+ - 1)_-\chi_{\{u\geq 0\}}}^{\geq 0} \\
&\quad - \left((\mu_- - 1)_+ - (\mu_- - 1)_-\chi_{\{u<0\}}\right) + \overbrace{(\mu_- - 1)_-\chi_{\{u\geq 0\}} - 1}^{\leq 0} \\
&\leq \left((\mu_+ - 1)_+ - (\mu_+ - 1)_-\chi_{\{u>0\}}\right) - \left((\mu_- - 1)_+ - (\mu_- - 1)_-\chi_{\{u<0\}}\right) = \eta(u,\mu).
\end{aligned}$$

Using similar computations, one can show that (left as exercise)

$$\mu + 1 = \mu_+ - (\mu_- - 1) \geq \eta(u,\mu),$$

and (b) follows. ☐

PROOF OF LEMMA 2.4.1(C). From $\mu_1\chi_A \geq \mu_2\chi_A$ we know that

$$(\mu_1)_+\chi_A = \max\{\mu_1, 0\}\chi_A = \max\{\mu_1\chi_A, 0\} \geq \max\{\mu_2\chi_A, 0\} = (\mu_2)_+\chi_A$$

and thus

$$\begin{aligned} ((\mu_1)_+ - 1)_+ \chi_A &= \max\{((\mu_1)_+ - 1), 0\}\chi_A = \max\{((\mu_1)_+\chi_A - \chi_A), 0\} \\ &\geq \max\{((\mu_2)_+\chi_A - \chi_A), 0\} = \max\{((\mu_2)_+ - 1), 0\}\chi_A = ((\mu_2)_+ - 1)_+ \chi_A. \end{aligned} \tag{2.4.1}$$

Similarly (left as exercise), one can show that $(\mu_1)_-\chi_A \leq (\mu_2)_-\chi_A$ and hence

$$((\mu_1)_- - 1)_+ \chi_A \leq ((\mu_1)_- - 1)_+ \chi_A. \tag{2.4.2}$$

From $u_1\chi_A \leq u_2\chi_A$ we know that

$$\{u_1 > 0\} \cap A \subset \{u_2 > 0\} \cap A$$

and thus from $(\mu_1)_+\chi_A \geq (\mu_2)_+\chi_A$ we now see that

$$\begin{aligned} ((\mu_1)_+ - 1)_- \chi_{\{u_1>0\}\cap A} &= \max\{-((\mu_1)_+ - 1), 0\}\chi_{\{u_1>0\}\cap A} \\ &= \max\left\{-(\mu_1)_+\chi_A\chi_{\{u_1>0\}} + \chi_{\{u_1>0\}\cap A}, 0\right\} \\ &\leq \max\left\{-(\mu_2)_+\chi_A\chi_{\{u_1>0\}} + \chi_{\{u_1>0\}\cap A}, 0\right\} \\ &= \max\{-((\mu_2)_+ - 1), 0\}\chi_{\{u_1>0\}\cap A} = ((\mu_2)_+ - 1)_- \chi_{\{u_1>0\}\cap A} \\ &\leq ((\mu_2)_+ - 1)_- \chi_{\{u_2>0\}\cap A} \end{aligned} \tag{2.4.3}$$

Similarly (left as exercise), one can show that $\{u_1 < 0\} \cap A \supset \{u_2 < 0\} \cap A$ and thus

$$((\mu_1)_- - 1)_- \chi_{\{u_1<0\}\cap A} \geq ((\mu_2)_- - 1)_- \chi_{\{u_2<0\}\cap A}. \tag{2.4.4}$$

We finally combine (2.4.1), (2.4.2), (2.4.3) and (2.4.4) to conclude (c). □

It is convenient to define $W_D^\mu := U^\mu - V_D^\mu$, which has a LSC representation, and we also denote $W^\mu := W_{\mathbb{R}^n}^\mu$. We now define

$$\tau_\mu := \{w : w \text{ is subharmonic}, -\Delta w \geq \eta(w,\mu) \text{ and } w \geq -W^{\mu_-} \text{ in } \mathbb{R}^n\}.$$

Fix any $\varphi \in C^\infty(\mathbb{R}^n)$ with $\Delta\varphi = 1$, for example, $\varphi(x) = \frac{|x|^2}{2n}$ for all $x \in \mathbb{R}^n$, we now consider the collection

$$\tau'_\mu := \left\{w + U^{\mu_-} - \varphi : w \in \tau_\mu\right\}.$$

By using Lemma 2.4.1(b), one sees that

$$-\Delta(w + U^{\mu_-} - \varphi) = -\Delta w + \mu_- + 1 \geq \eta(w,\mu) + \mu_- + 1 \geq \mu + \mu_- = \mu_+ \geq 0.$$

However, we also see that $w + U^{\mu_-} - \varphi = w - (-U^{\mu_-} + \varphi)$ and sees that $\Delta(-U^{\mu_-} + \varphi) = \mu_- + 1 \geq 0$, which shows that the elements of τ'_μ are δ-subharmonic functions, therefore in general such functions are defined only quasi-everywhere on $\mathbb{R}^n$, i.e. outside the *polar set* where $w = -U^{\mu_-} + \varphi = -\infty$, as mentioned above. In fact, one can suitably refined each element of τ'_μ on a polar set to make them superharmonic (and we skip these technical details here). In the special case when μ_- is bounded, by using the Calderón-Zygmund inequality [**GT01**, Theorem 9.11] and

Sobolev embeddings (Appendix A), one sees that $-U^{\mu_-}+\varphi \in C^1$. In this case, the polar set is empty and $w+U^{\mu_-}-\varphi$ has a USC representation.

We now prove a fundamental property of τ_μ'.

LEMMA 2.4.2 ([**GS12**, Lemma 4.2]). *If* $v_1,v_2 \in \tau_\mu'$, *then* $\min\{v_1,v_2\} \in \tau_\mu'$.

PROOF. Let $v_1,v_2 \in \tau_\mu'$ and write $v_i = w_i + U^{\mu_-} - \varphi$ where $w_i \in \tau_\mu$. By observing that

$$\min\{v_1,v_2\} = \min\{w_1,w_2\} + U^{\mu_-} - \varphi,$$

by using Corollary 2.3.2 one can show that $\min\{v_1,v_2\}$ is δ-subharmonic function and $\min\{w_1,w_2\} \geq -W^{\mu_-}$ in $\mathbb{R}^n$, as well as

$$\eta(\min\{w_1,w_2\},\mu) = \eta(w_1,\mu)\chi_{\{w_1-w_2\leq 0\}} + \eta(w_2,\eta)\chi_{\{w_1-w_2>0\}}.$$

By using (2.3.1), one further computes that

$$\eta(\min\{w_1,w_2\},\mu) \leq -(\Delta w_1)\chi_{\{w_1-w_2\leq 0\}} - (\Delta w_2)\chi_{\{w_1-w_2>0\}} \leq -\Delta \min\{w_1,w_2\},$$

which conclude our lemma. □

The following two technical lemmas, regarding some monotonicity properties, can be found in [**GS12**, Theorem 4.3].

LEMMA 2.4.3. *Let* u_1,u_2 *be* δ*-subharmonic functions with compact supports. If* $-\Delta u_1 \geq \eta(u_1,\mu)$ *and* $-\Delta u_2 \leq \eta(u_2,\mu)$*, then* $u_2 \leq u_1$.

PROOF. One computes that the function $v = u_2 - u_1$ satisfies

$$\begin{aligned} -\Delta v &\leq \eta(u_2,\mu) - \eta(u_1,\mu) \\ &= (\mu_+ - 1)_-\chi_{\{u_1>0\}} - (\mu_+ - 1)_-\chi_{\{u_2>0\}} + (\mu_- - 1)_-\chi_{\{u_2<0\}} - (\mu_- - 1)\chi_{\{u_1<0\}}, \end{aligned}$$

so $-(\Delta v)\chi_{\{v\geq 0\}} \leq 0$. By using the Kato's inequality (Lemma 2.3.1), one sees that

$$\Delta v_+ \geq (\Delta v)\chi_{\{v\geq 0\}} \geq 0.$$

Thus v_+. when suitably redefined on a polar set, is subharmonic. Since v has compact support, the maximum principle (Lemma 2.2.3) shows that $v_+ \equiv 0$, which implies our lemma. □

LEMMA 2.4.4. *Let* u *be a* δ*-subharmonic function. Then the following hold:*

(1) *If* $-\Delta u \leq \eta(u,\mu)$*, then* $u \leq W^{\mu_+}$.
(2) *If* $-\Delta u \geq \eta(u,\mu)$*, then* $u \geq -W^{\mu_-}$ *and so* $u \in \tau_\mu$.

PROOF. First of all, we remind the readers that W^{μ_+} is non-negative, δ-subharmonic and has compact support. Since $\mathrm{Bal}(\mu_+) \leq 1$ in $\mathbb{R}^n$, by the structure of partial balayage (2.3.2) we see that

$$\mu_+\chi_{\{W^{\mu_+}=0\}} = \mu_+\chi_{\omega(\mu_+)^\complement} \leq 1. \tag{2.4.5}$$

Consequently, together with Lemma 2.4.1(c) we compute that

$$\begin{aligned} -\Delta W^{\mu_+} &= \mu_+ - \mathrm{Bal}\,(\mu_+) = \mu_+ - \chi_{\{W^{\mu_+}>0\}} - \mu_+\chi_{\{W^{\mu_+}=0\}} \\ &= (\mu_+ - 1)\chi_{\{W^{\mu_+}>0\}} \overset{(2.4.5)}{=} (\mu_+ - 1)_+ - (\mu_+ - 1)_-\chi_{\{W^{\mu_+}>0\}} = \eta(W^{\mu_+},\mu_+) \\ &\overset{\text{Lemma 2.4.1(c)}}{\geq} \eta(W^{\mu_+},\mu). \end{aligned}$$

Now we choose $u_1 = W^{\mu_+}$ and $u_2 = u$ in Lemma 2.4.3 to conclude $u \leq W^{\mu_+}$, which complete the proof of (1).

We now replacing μ by $-\mu$ to obtain

$$\begin{aligned} -\Delta(-W^{\mu_-}) &= \Delta W^{(-\mu)_+} = \eta(W^{(-\mu)_+},(-\mu)_+) \\ &\leq -\eta(W^{\mu_-},-\mu) \overset{\text{Lemma 2.4.1(a)}}{\geq} \eta(-W^{\mu_-},\mu). \end{aligned}$$

Now we choose $u_1 = u$ and $u_2 = -W^{\mu_-}$ in Lemma 2.4.3 to conclude $-W^{\mu_-} \leq u$, which complete the proof of (2). $\square$

We now follow the arguments in [**GS12**, Theorem 4.4, Theorem 4.5, Corollary 4.6 and Remark 1] to establish the following lemma (the proof is technical, which involving swept measure, see (2.5.2) below, we will not going to walk through the details there):

LEMMA 2.4.5. *Let $\mu_\pm$ be positive measures with disjoint compact supports in $\mathbb{R}^n$ and let $\mu = \mu_+ - \mu_-$. Then the set τ_μ contains a least element $\overline{W}^\mu$. If the following support conditions hold:*

$$\mathrm{supp}\,(\mu_\pm) \subset D_\pm := \left\{\pm\overline{W}^\mu > 0\right\}, \tag{2.4.6}$$

then both $D_\pm$ are open sets in $\mathbb{R}^n$ and the pair of domains (D_+,D_-) is a two-phase quadrature domain in the sense of Definition 2.1.1.

We now ready to prove the following theorem.

THEOREM 2.4.6. *Let $\mu_\pm$ be positive measures with disjoint compact supports in $\mathbb{R}^n$ and let $\mu = \mu_+ - \mu_-$. If*

$$\mathrm{supp}\,(\mu_+) \subset \omega_{\overline{\omega(\mu_-)}^\complement}(\mu_+), \quad \mathrm{supp}\,(\mu_-) \subset \omega_{\overline{\omega(\mu_+)}^\complement}(\mu_-), \tag{2.4.7}$$

then there exist two disjoint open bounded sets $D_\pm$ such that (D_+,D_-) is a two-phase quadrature domain in the sense of Definition 2.1.1.

REMARK. Since $\omega_D(\mu) \subset \omega(\mu) \cap D$ for any open set D, then

$$\omega_{\overline{\omega(\mu_-)}^\complement}(\mu_+) \subset \overline{\omega(\mu_-)}^\complement, \quad \omega_{\overline{\omega(\mu_+)}^\complement}(\mu_-) \subset \overline{\omega(\mu_+)}^\complement,$$

therefore the condition (2.4.7) implies

$$\mathrm{supp}\,(\mu_+) \cap \overline{\omega(\mu_-)} = \emptyset, \quad \mathrm{supp}\,(\mu_-) \cap \overline{\omega(\mu_+)} = \emptyset. \tag{2.4.8}$$

This means that $\operatorname{supp}(\mu_+)$ and $\operatorname{supp}(\mu_-)$ cannot "too close to each others".

REMARK. In fact, (2.4.7) can be guaranteed when

$$\limsup_{r\to 0_+} \frac{\mu_+(B_r(x))}{r^n} > \frac{1}{c_n} \quad \text{for all } x \in \operatorname{supp}(\mu_+),$$

$$\limsup_{r\to 0_+} \frac{\mu_-(B_r(y))}{r^n} > \frac{1}{c_n} \quad \text{for all } y \in \operatorname{supp}(\mu_-),$$

for some positive constant c_n depending only on dimension n, see the proof of [**KS24**, Theorem 3.2].

PROOF OF THEOREM 2.4.6. We define

$$u = W^{\mu_+} - W^{\mu_-}_{\overline{\omega(\mu_+)}^\complement}, \quad v := W^{\mu_+}_{\overline{\omega(\mu_-)}^\complement} - W^{\mu_-},$$

and using the disjoint condition (2.4.8), we observe that

$$\{u<0\} = \omega_{\overline{\omega(\mu_+)}^\complement}(\mu_-), \quad \{u>0\} = \omega(\mu_+),$$

$$\{v>0\} = \omega_{\overline{\omega(\mu_-)}^\complement}(\mu_+), \quad \{v<0\} = \omega(\mu_-).$$

Now from (2.4.7) we see that

$$\operatorname{supp}(\mu_+) \subset \{v>0\}, \quad \operatorname{supp}(\mu_-) \subset \{u<0\}.$$

On the other hand, by using the structure of partial balayage (2.3.2), one sees that

$$\begin{aligned}
-\Delta u &= \mu_+ - \operatorname{Bal}(\mu_+) - \mu_- + \operatorname{Bal}_{\overline{\omega(\mu_+)}^\complement}(\mu_-) \\
&= (\mu_+ - 1)\chi_{\omega(\mu_+)} - (\mu_- - 1)\chi_{\overline{\omega(\mu_+)}^\complement} + \nu \\
&\geq (\mu_+ - 1)\chi_{\{u>0\}} - (\mu_- - 1)\chi_{\{u<0\}}.
\end{aligned}$$

In view of the structure of partial balayage (2.3.2) (with $D = \mathbb{R}^n$), one observes that $\mu_+ \leq 1$ outside $\omega(\mu_+) = \{u>0\}$, hence one sees that

$$-\Delta u \geq (\mu_+ - 1)\chi_{\{u>0\}} - (\mu_- - 1)\chi_{\{u<0\}} = \eta(u,\mu).$$

Now using Lemma 2.4.4 we see that $u \in \tau_\mu$, and we reach $u \geq \overline{W}^\mu$. Consequently, we confirm the support condition

$$\operatorname{supp}(\mu_-) \subset \{u<0\} \subset \{\overline{W}^\mu < 0\}.$$

One can similar show that

$$\operatorname{supp}(\mu_+) \subset \{v>0\} \subset \{\overline{W}^\mu > 0\},$$

and now the condition (2.4.6) is satisfied. Finally, we use Lemma 2.4.5 to conclude our theorem with $D_\pm = \{\pm W^\mu > 0\}$. $\square$

2.5. Classical balayage as a special case of partial balayage

Some parts in Section 2.3 can be slightly generalized. Let λ be a positive measure, and the discussions in Section 2.3 corresponds to the special case $\lambda = 1$.

Given any open set $D \subset \mathbb{R}^n$ and a positive measure μ with compact support on $\mathbb{R}^n$, we define

$$\mathscr{F}_D^\lambda(\mu) := \left\{ v \in \mathscr{D}'(\mathbb{R}^n) : \begin{array}{l} -\Delta v \le \lambda \text{ in } D, \quad v \le U^\mu \text{ in } \mathbb{R}^n \\ \{v < U^\mu\} \text{ is bounded} \end{array} \right\}.$$

We denote $\mathscr{F}^\lambda(\mu) := \mathscr{F}_{\mathbb{R}^n}^\lambda(\mu)$.

If we assume that $\mathscr{F}_D^\lambda(\mu) \neq \emptyset$, similarly, one can show that $\mathscr{F}_D^\lambda(\mu)$ has a largest element $V_D^{\mu,\lambda}$ element, which is called the *partial reduction of U^μ with respect to λ*. Accordingly, we can define the non-contact set by

$$\omega_D^\lambda(\mu) := \left\{ V_D^{\mu,\lambda} < U^\mu \right\}, \quad \omega^\lambda(\mu) := \omega_{\mathbb{R}^n}^\lambda(\mu),$$

and the *partial balayage with respect to λ* is defined by

$$\mathrm{Bal}_D^\lambda(\mu) := -\Delta V_D^{\mu,\lambda} \text{ in } \mathscr{D}'(\mathbb{R}^n).$$

In fact, the following structure theorem holds:

$$\mathrm{Bal}_D^\lambda(\mu) = \lambda \chi_{\omega_D^\lambda(\mu)} + \mu \chi_{\omega_D^\lambda(\mu)^\complement} + \nu$$

for some measure $\nu \ge 0$ which is supported on $\partial D \cap \partial \omega_D^\lambda(\mu)$.

Let $D = \mathbb{R}^n$, let Ω be any *bounded* open set in $\mathbb{R}^n$ and let

$$\lambda = \begin{cases} 0 & \text{in } \Omega, \\ +\infty & \text{in } \Omega^\complement. \end{cases}$$

Now we see that

$$\mathscr{F}^\lambda(\mu) = \left\{ v \in \mathscr{D}'(\mathbb{R}^n) : \begin{array}{l} -\Delta v \le 0 \text{ in } \Omega, \quad v \le U^\mu \text{ in } \mathbb{R}^n \\ \{v < U^\mu\} \text{ is bounded} \end{array} \right\}.$$

By using the *Perron's method of subharmonic functions* [**GT01**, Section 2.8], which involving maximum principle (Lemma 2.2.3), one sees that the largest element $V^{\mu,\lambda}$ in $\mathscr{F}^\lambda(\mu)$ satisfies

$$-\Delta V^{\mu,\lambda} = 0 \text{ in } \Omega, \quad V^{\mu,\lambda} = U^\mu \text{ in } \Omega^\complement, \tag{2.5.1}$$

and we see that

$$\mathrm{Bal}^\lambda(\mu) = -\Delta(V^{\mu,\lambda} - U^\mu) + \mu \text{ in } \mathscr{D}'(\mathbb{R}^n).$$

In this case, we also denote

$$V^{\mu,\lambda} = \hat{R}_{U^\mu}^{\Omega^\complement}$$

called the *regularized reduction of the superharmonic function* U^{μ} *relative to* $\Omega^{\complement}$, and we also denote

$$\mu^{\Omega^{\complement}} = \mathrm{Bal}^{\lambda}(\mu) = -\Delta \hat{R}_{U^{\mu}}^{\Omega^{\complement}} \tag{2.5.2}$$

called the *swept measur*e, see [**GS09, GS12**].

It is more convenient to write $u := U^{\mu} - V_D^{\mu,\lambda} \in \mathscr{E}'(\mathbb{R}^n)$, which satisfies

$$\Delta u = \mu \text{ in } \Omega, \quad u = 0 \text{ in } \Omega^{\complement}, \tag{2.5.3}$$

and now the partial balayage can be represented as

$$\mathrm{Bal}^{\lambda}(\mu) = -\Delta u + \mu \text{ in } \mathscr{D}'(\mathbb{R}^n). \tag{2.5.4}$$

In other words, one may solve the Dirichlet problem by using the partial balayage (2.5.4). This method is invented by Henri Poincaré [**Poi90, Poi99**]. Note that

$$\mathrm{supp}\left(\mathrm{Bal}^{\lambda}(\mu)\right) \cap \Omega = \emptyset,$$

which means that this process *completely swept out* the measure μ in the region Ω. Therefore, we usually denote

$$\mathrm{Bal}(\mu, \Omega^{\complement}) := \mathrm{Bal}^{\lambda}(\mu)$$

and called it the *classical balayage*, which is exactly same as the one in [**Gus04**, Section 2]. In other words, the classical balayage can be viewed as a special case of partial balayage.

REMARK 2.5.1. In the special case when Ω is a bounded Lipschitz domain and $\mu \in L^2(\Omega) = \left\{\mu \in L^2(\mathbb{R}^n) : \mu = 0 \text{ in } \Omega^{\complement}\right\}$, there exists a unique solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$ of the Dirichlet problem (2.5.3), see also (1.2.2). In this case, its outward normal derivative $\partial_{\mathbf{n}} u$ on $\partial\Omega$ can be well-defined in the sense of $H^{-1/2}(\partial\Omega)$. The precise definition of (2.5.4) is

$$\left\langle \mathrm{Bal}(\mu, \Omega^{\complement}), \tilde{\phi} \right\rangle = \int_{\mathbb{R}^n} u \Delta\tilde{\phi} \, \mathrm{d}x + \int_{\mathbb{R}^n} \mu \tilde{\phi} \, \mathrm{d}x \quad \text{for any } \tilde{\phi} \in C_c^{\infty}(\mathbb{R}^n).$$

Since $u \in H_0^1(\Omega)$ and $\mu = 0$ in $\Omega^{\complement}$, by writing $\phi = \tilde{\phi}|_{\partial\Omega}$, one sees that

$$\begin{aligned} \left\langle \mathrm{Bal}(\mu, \Omega^{\complement}), \tilde{\phi} \right\rangle &= \int_{\Omega} u \Delta\tilde{\phi} \, \mathrm{d}x + \int_{\Omega} \mu \tilde{\phi} \, \mathrm{d}x = -\int_{\Omega} \nabla u \cdot \nabla\tilde{\phi} \, \mathrm{d}x + \int_{\Omega} \mu \tilde{\phi} \, \mathrm{d}x \\ &= -\int_{\Omega} \partial_{\mathbf{n}} u \phi \, \mathrm{d}S + \overbrace{\int_{\Omega} (\Delta u + \mu)\tilde{\phi} \, \mathrm{d}x}^{=0 \text{ by } (2.5.3)}, \end{aligned}$$

where $\partial_{\mathbf{n}}$ is the outward normal derivative on $\partial\Omega$. If we denote $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ and its distributional adjoint $\gamma^* : H^{-1/2}(\partial\Omega) \to H^{-1}(\Omega)$ [**McL00**], by slightly abuse the notation, the classical balayage $\mathrm{Bal}(\mu, \Omega^{\complement})$ can be understood in the sense of $H^{-1/2}(\partial\Omega)$:

$$\mathrm{Bal}(\mu, \Omega^{\complement}) = -\partial_{\mathbf{n}} u.$$

REMARK 2.5.2. In the case when $\mu = \delta_x$ for some $x \in \Omega$, one sees that $u = G_\Omega(\cdot,x)$ is the Green function of Ω with pole at x, and the corresponding classical balayage $\mathrm{Bal}\,(\mu,\Omega^\complement)$ is exactly the harmonic measure ω_x^Ω. Suggested by previous remark, we sometimes abuse the notation by writing

$$\mathrm{d}\omega_x^\Omega = -\partial_{\mathbf{n}} G_\Omega(\cdot,x)\,\mathrm{d}S.$$

2.6. Pompeiu conjecture*

In fact, the method of partial balayage can be extended for Helmholtz operator $\Delta + k^2$, but the extension is highly nontrivial, see [**GS24, KLSS24, KS24**]. One may consider a definition generalizing Definition 1.1.1:

DEFINITION 2.6.1 ([**KLSS24**, Definition 1.1]). Let $k > 0$. A bounded open set $D \subset \mathbb{R}^n$ (not necessarily connected) is called a quadrature domain for $(\Delta + k^2)$, or a *k-quadrature domain*, corresponding to a distribution $\mu \in \mathscr{E}'(D)$, if

$$\int_D w(x)\,\mathrm{d}x = \langle \mu, w\rangle$$

for all $w \in L^1(D)$ satisfying $(\Delta + k^2)w = 0$ in D.

The first question is whether k-quadrature domains even exist for $k > 0$. This is indeed the case. In fact, balls are always k-quadrature domains. This is a consequence of a mean value theorem for the Helmholtz equation which goes back to H. Weber [**Web68, Web69**], see the arXiv version of [**KLSS24**] for a detailed proof, see also [**CH89**, page 289]. The mean value theorem takes the form

$$\int_{B_r(a)} w(x)\,\mathrm{d}x = c_{n,k,r}^{\mathrm{MVT}} w(a)$$

whenever $w \in L^1(B_r(a))$ and $(\Delta + k^2)w = 0$ in $B_r(a)$. However, unlike for harmonic functions, the constant $c_{n,k,r}^{\mathrm{MVT}}$ has varying sign depending on k, r. In particuilar, the constant vanishes when $J_{n/2}(kr) = 0$ where J_α denotes the Bessel function of the first kind. More details are given in [**KLSS24**, Appendix A], and detailed proofs also provided in arXiv version of the paper. It is also important to mention that D is a k-quadrature domain corresponding to $\mu \in \mathscr{E}'(D)$ if and only if there is a distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ satisfying

$$\begin{cases} (\Delta + k^2)u = \chi_D - \mu & \text{in } \mathbb{R}^n, \\ u = |\nabla u| = 0 & \text{in } D^\complement, \end{cases}$$

see [**KLSS24**, Proposition 2.1].

We are interested in the case when $\mu \equiv 0$:

DEFINITION 2.6.2 ([**KLSS24**, Definition 1.1]). Let $k > 0$. A bounded open set $D \subset \mathbb{R}^n$ (not necessarily connected) is called a null quadrature domain for $(\Delta + k^2)$, or a *null k-quadrature* *domain* if

$$\int_D w(x)\,\mathrm{d}x = 0$$

for all $w \in L^1(D)$ satisfying $(\Delta + k^2)w = 0$ in D.

One sees that D is a null k-quadrature domain if and only if there is a distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ satisfying

$$\begin{cases} (\Delta + k^2)u = \chi_D & \text{in } \mathbb{R}^n, \\ u = |\nabla u| = 0 & \text{in } D^{\complement}. \end{cases} \tag{2.6.1}$$

EXAMPLE 2.6.3. Write $j_{\alpha,m}$ be the m^{th} positive zero of J_α. By using mean value theorem, one sees that each ball with radius R satisfying $J_{n/2}(kR) = 0$, i.e. $R = k^{-1} j_{\frac{n}{2},m}$ for some $m \in \mathbb{N}$, is a null-quadrature domain. In [KS24, Example A.2], we also show that each ball with radius $R = k^{-1} j_{\frac{n}{2},m}$ is a null-quadrature domain by showing that

$$\tilde{u}_m(x) := \begin{cases} \dfrac{(k^{-1} j_{\frac{n}{2},m})^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(j_{\frac{n}{2},m}) - |x|^{\frac{2-n}{2}} (k|x|)}{k^2 (k^{-1} j_{\frac{n}{2},m})^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(j_{\frac{n}{2},m})} & \text{for all } |x| < k^{-1} j_{\frac{n}{2},m}, \\ 0 & \text{otherwise} \end{cases}$$

is in $C^{1,1}(\mathbb{R}^n)$ and satisfies (2.6.1) with $D = B_R$ provided $R = k^{-1} j_{\frac{n}{2},m}$.

We now assume that

$$\begin{cases} D \text{ is a null } k\text{-quadrature domain which is bounded} \\ \text{such that } \partial D \text{ is homeomorphic to a sphere} \end{cases} \tag{2.6.2}$$

By using maximum principle and the fact that the first Dirichlet eigenfunction is positive, one sees that k is strictly larger than the first Dirichlet eigenvalue of D, and thus maximuim principle does not hold on D in the case. Therefore it is not possible to construct null k-quadrature domain by using partial balayage at the moment.

In particular, by using [Wil81, Theorem 1] and [Wil76] the assumptions in (2.6.2) is equivalent to the assumptions in the Pompeiu conjecture [Pom29, Zal92, Zal01], which is still open until today. It is worth to mention that if D satisfies assumption (2.6.2) of Pompeiu conjecture holds, then its boundary ∂D must analytic [Wil81]. See also [Avi86, BST73, BK82, GS93] for some related results. The following conjecture is still remain unanswered:

CONJECTURE 2.6.4 (Pompeiu conjecture [Yau82, Problem 80]). *If D satisfies (2.6.2), then D has to be a ball.*

It is easy to see that $k > 0$ is also a Neumann eigenvalue of D with eigenfunction $v := u - k^{-2}$, where u is given in (2.6.1), which satisfies

$$v|_{\partial D} = -k^{-2}.$$

The main difficulty is the knowledge of $v|_{\partial D}$ does not explicitly contained in the Courant minimax characterization of Neumann eigenvalues. Therefore we also believe that the Courant minimax principle is not helpful in the study of Pompeiu conjecture.

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APPENDIX A

Preliminaries

A.1. A version of Hahn-Banach theorem

There are several versions of Hahn-Banach theorems. Here we exhibit a version which is very useful in the context of PDE:

THEOREM A.1.1 (Hahn-Banach [**Bre11**, Corollary 1.8]). *Let $F \subset E$ be a linear subspace. If*

$$\langle f, x\rangle = 0 \text{ for all } x \in F \implies f \equiv 0,$$

then $\overline{F} = E$.

A.2. Sobolev embeddings

Before introducing the Sobolev embeddings, we first introducing the following concept:

DEFINITION A.2.1. Let X and Y be two Banach spaces. We say that the space X is *continuous embedded* in Y if

$$\|v\|_Y \le c\|v\|_X \quad \text{for all } v \in X. \tag{A.2.1}$$

We say that the space X is *compactly embedded* in Y if (A.2.1) holds and each bounded sequence in X has a convergent subsequence in Y.

Many authors (including myself) simply denote $X \subset Y$ if the Banach space X is continuous embedded in another Banach space Y, despite that X is not necessarily a subset of Y. We will also denote $X \Subset Y$ if X is compactly embedded in Y. Here and after (including the next theorem), we will use these notations without mentioning explicitly. Let $\lfloor x \rfloor$ denotes the integer part of x, and we have the following theorem:

THEOREM A.2.2 (Sobolev embedding theorems [**AH09**, Theorem 7.3.7 and Theorem 7.3.8]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then the following statements are valid:*

(a) *If $k < \frac{n}{p}$, then $W^{k,p}(\Omega) \Subset L^q(\Omega)$ for any $q < p^*$ and $W^{k,p}(\Omega) \subset L^q(\Omega)$ when $q \le p^*$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{k}{n}$.*
(b) *If $k = \frac{n}{p}$, then $W^{k,p}(\Omega) \Subset L^q(\Omega)$ for any $q < \infty$.*
(c) *If $k > \frac{n}{p}$, then*

$$W^{k,p}(\Omega) \Subset C^{k-\lfloor \frac{n}{p} \rfloor - 1,\beta}(\Omega) \quad \text{for all } \beta \in \left[0, \left\lfloor \frac{n}{p} \right\rfloor + 1 - \frac{n}{p}\right)$$

$$W^{k,p}(\Omega) \subset C^{k-\lfloor \frac{n}{p} \rfloor - 1,\beta}(\Omega) \quad \text{with } \beta = \begin{cases} \lfloor \frac{n}{p} \rfloor + 1 - \frac{n}{p} & \text{if } \frac{n}{p} \notin \mathbb{Z}, \\ \text{any positive number } < 1 & \text{if } \frac{n}{p} \in \mathbb{Z}. \end{cases}$$

REMARK A.2.3. Theorem A.2.2 is also valid for $W^{k,p}$-spaces with $k \in \mathbb{R}$, see e.g. [**AH09, McL00**] for precise definitions. Here we will cover these topics in this lecture note. Part (c) of Theorem A.2.2 in particular gives some sufficient condition in terms of weak derivatives to guarantee the well-definedness of the strong/classical derivatives.

It is important to mention that the proof of Theorem A.2.2 is based on the existence of the bounded linear extension operator

$$E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$$

for any nonnegative integer k and for any $1 \leq p \leq \infty$. In fact, the operator norm of the extension operator can be eplicitly given:

THEOREM A.2.4 ([**Bur99**, Theorem 3.4]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. There exists a constant $C = C(\Omega) > 1$ such that*

$$(C^{-1}k)^k \leq \inf_E \|E\|_{W^{k,p}(\Omega)\to W^{k,p}(\mathbb{R}^n)} \leq (Ck)^k$$

for all $k \in \mathbb{N}$ and for all $1 \leq p \leq \infty$, where the infimum is taken over all extension operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$.

REMARK A.2.5. Here we emphasize that the constant C in Theorem A.2.4 is independent of both k and p.

A.3. Integration by parts

The following version of integration by parts is widely-used in the context of PDE:

THEOREM A.3.1 (Integration by parts [**EG15**, Theorem 1 in Section 4.3]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and given $1 \leq p < \infty$. The mapping*

$$\mathrm{Tr} : C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega), \quad \mathrm{Tr}\,(f) = f|_{\partial\Omega} \tag{A.3.1}$$

can be uniquely extended to a bounded surjective linear operator $W^{1,p}(\Omega) \to \mathrm{Tr}\,(W^{1,p}(\Omega)) \subset L^p(\partial\Omega)$. Furthermore, for all $\boldsymbol{\varphi} \in (C^1(\mathbb{R}^n))^n$ and $f \in W^{1,p}(\Omega)$, we have

$$\int_\Omega f \,\mathrm{div}\,(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} = -\int_\Omega \nabla f \cdot \boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} (\boldsymbol{\nu}\cdot\boldsymbol{\varphi})\mathrm{Tr}\,(f)\,\mathrm{d}\mathscr{H}^{n-1}, \tag{A.3.2}$$

where $\boldsymbol{\nu}$ is the unit outer normal to $\partial\Omega$.

REMARK A.3.2. Here we refer the advance monograph [**EG15**] for the precise meaning of $\boldsymbol{\nu}$, which is well-defined for $\mathscr{H}^{n-1}$-a.e. on $\partial\Omega$. The function $\mathrm{Tr}(f)$ given in (A.3.1) is called the *trace* of f on $\partial\Omega$. We usually still denote $\mathrm{d}\mathscr{H}^{n-1}$ by $\mathrm{d}S_{\boldsymbol{x}}$. If there is no ambituity, we sometime omit the notation the trace operator (A.3.1) and simply write (A.3.2) as

$$\int_\Omega f \,\mathrm{div}\,(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} = -\int_\Omega \nabla f \cdot \boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} (\boldsymbol{\nu}\cdot\boldsymbol{\varphi}) f\,\mathrm{d}S_{\boldsymbol{x}}.$$