FOURIER ANALYSIS (751799001, 701866001, 114-1) - HOMEWORK 1

Return by September 17, 2025 (Wednesday) 23:59

Total marks: 50

Special requirement. All homework must be prepared by using LaTeX.

Exercise 1 (10 points)**.** Define $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ as

$$f(x) = \begin{cases} x + a & \text{if } x > 0, \\ -x & \text{if } x < 0. \end{cases}$$

Determine whether the weak derivative of f exists or not for each $a \in \mathbb{R}$.

Exercise 2 (10 points)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := \operatorname{sgn}(x)\sqrt{|x|}$ for all $x \in \mathbb{R}$, where

$$\operatorname{sgn}(x) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Show that the (order one) weak derivative f' exists, and compute it.

Exercise 3 (10+10 points)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) := H(x) + H(y)$ for all $(x, y) \in \mathbb{R}^2$, where H is the Heaviside function given by

$$H(t) := \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

We denote the multiindices $\alpha = (1, 1)$ and $\beta = (1, 0)$.

(a) Prove that the weak derivatives $\partial^\alpha f$ and $\partial^{\alpha+\beta} f$ exist, and compute them.
(b) Prove that the weak derivative $\partial^\beta f$ does not exist.

Exercise 4 (10 points)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) := H(x) - \operatorname{sgn}(y)$ for all $(x, y) \in \mathbb{R}^2$. We denote the multiindex $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2_{\geq 0}$. Prove that the weak derivative $\partial^\alpha f$ exists if and only if $\alpha_1 \geq 1$ and $\alpha_2 \geq 1$.