FOURIER ANALYSIS (751799001, 701866001, 114-1) - HOMEWORK 4

Return by October 8, 2025 (Wednesday) 23:59

Total marks: 50

Special requirement. All homework must be prepared by using LaTeX.

Exercise 1 (10 points)**.** Prove that for each fixed number $a \in (0,\infty)$ the function $f(x) = e^{-a|x|^2}$ ($x \in \mathbb{R}^n$) belongs to $\mathscr{S}(\mathbb{R}^n)$. Therefore $C_c^\infty(\mathbb{R}^n) \subsetneq \mathscr{S}(\mathbb{R}^n) \subsetneq C^\infty(\mathbb{R}^n)$. (Note: $e^{-|x|}$ is not in Schwartz space since it is not C^∞ near the origin.)

Exercise 2 (10 points)**.** Verify that the function $\mathsf{d}_{\mathscr{S}(\mathbb{R}^n)}$ given in (2.2.5) is a metric. [Hint: If $\|\cdot\|$ is a norm on a vector space, show hat $\frac{\|u+v\|}{1+\|u+v\|} \le \frac{\|u\|}{1+\|u\|} + \frac{\|v\|}{1+\|v\|}$.]

Exercise 3 (10 points)**.** Prove that for each $s \in \mathbb{R}$ the function $f(x) := \langle x \rangle^s$ ($x \in \mathbb{R}^n$) belongs to $\mathscr{O}_{\mathrm{M}}(\mathbb{R}^n)$.

Exercise 4 (10 points)**.** Prove that the function $f(x) := e^{i|x|^2}$ ($x \in \mathbb{R}^n$) belongs to $\mathscr{O}_{\mathrm{M}}(\mathbb{R}^n)$.

Exercise 5 (10 points)**.** Let $\phi_n(x) = e^{-\frac{1}{2}|x|^2}$. Prove that $\hat{\phi}_n = (2\pi)^{\frac{n}{2}}\phi_n$ and $\phi_n(0) = (2\pi)^{-n}\int_{\mathbb{R}^n} \hat{\phi}_n(x)\,dx$.