DIFFERENTIAL EQUATIONS (751873002, 113-2) - HOMEWORK 1

Return by March 6, 2024 (Thursday) 23:59

Total marks: 50

Special requirement. All homeworks must be prepared by using LaTeX.

Exercise 1 (10 points)**.** For each $1 < p < \infty$ and $\frac{1}{p'} + \frac{1}{p} = 1$, show that the following inequality:

$$ab \le \frac{1}{p} a^p + \frac{1}{p'} b^{p'} \quad \text{for all } a \ge 0 \text{ and } b \ge 0.$$

Exercise 2 (10 points)**.** Let Ω be an open set in $\mathbb{R}^n$. Assume that $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $1 \le p \le \infty$ and $\frac{1}{p'} + \frac{1}{p} = 1$. Show that $fg \in L^1(\Omega)$ and the following Hölder's inequality holds:

$$\int_\Omega |f(\boldsymbol{x})g(\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x} \le \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}$$

and the equality holds if and only if there exists $c \in \mathbb{R}$ such that $|g(\boldsymbol{x})| = c|f(\boldsymbol{x})|^{p-1}$ for a.e. $\boldsymbol{x} \in \Omega$.

Exercise 3 (10 points)**.** Let Ω be an open set in $\mathbb{R}^n$, show that $\|\cdot\|_{L^p(\Omega)}$ defines a norm for each $1 \le p \le \infty$.

Exercise 4 (10 points)**.** Show that

$$\left(\int_{\Omega_2} \left| \int_{\Omega_1} F(\boldsymbol{x}, \boldsymbol{y}) \,\mathrm{d}\boldsymbol{x} \right|^p \mathrm{d}\boldsymbol{y} \right)^{\frac{1}{p}} \le \int_{\Omega_1} \left(\int_{\Omega_2} |F(\boldsymbol{x}, \boldsymbol{y})|^p \,\mathrm{d}\boldsymbol{y} \right)^{\frac{1}{p}} \mathrm{d}\boldsymbol{x}.$$

Exercise 5 (10 points)**.** Deduce that if $f \in L^p(\Omega) \cap L^q(\Omega)$ with $1 \le p \le \infty$ and $1 \le q \le \infty$, then $f \in L^r(\Omega)$ for every r between p and q. More precisely, write

$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q} \quad \text{with } 0 \le \alpha \le 1$$

and prove that

$$\|f\|_{L^r(\Omega)} \le \|f\|_{L^p(\Omega)}^{\alpha} \|f\|_{L^q(\Omega)}^{1-\alpha}.$$