School of Computer Science
University of Birmingham

Adelic Geometry via Geometric Logic

(joint work with Steve Vickers)

Ming Ng
mxn732@bham.ac.uk

November 14, 2021

What this talk is about

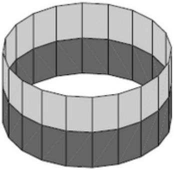
I'm going to discuss two basic themes that cross-cut many different areas of mathematics:

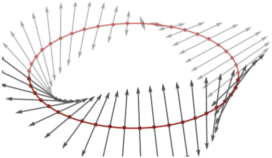
1. What kind of info can homotopical data encode?
2. When can we solve a problem by breaking it into smaller pieces?

I'll then discuss how the research project 'Adelic Geometry via Topos Theory' serves as an interesting test problem for illuminating how these two themes interact with each other.

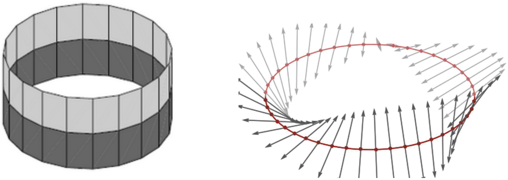
Homotopical Data = Geometric Data

Homotopical Data = Geometric Data



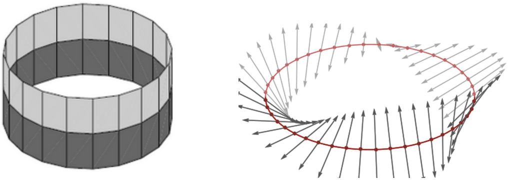


Homotopical Data = Geometric Data



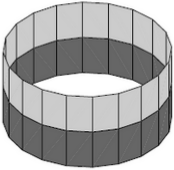
- Question: Are these the only line bundles over S^1 (up to isomorphism)?

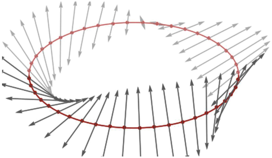
Homotopical Data = Geometric Data



- Question: Are these the only line bundles over S^1 (up to isomorphism)?
- Answer: Yes.

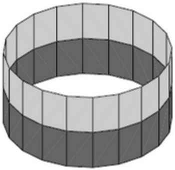
Homotopical Data = Geometric Data

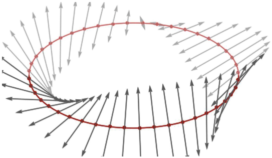




- Question: Are these the only line bundles over S^1 (up to isomorphism)?
- Answer: Yes.
- Why? Exploit the tight relationship between $[S^{k-1}, GL_n(\mathbb{R})]$ and $\mathrm{Vect}_n(S^k)$.

Homotopical Data = Geometric Data





Classification Theorem

Suppose that X is a paracompact space. Let $\mathrm{Vect}_n(X)$ be the set of isomorphism classes of n-dimensional vector bundles over X. Then the map

$$[X, G_n] \longrightarrow \mathrm{Vect}_n(X)$$

given by $f \longmapsto f^*(\gamma_n)$ is a bijection, where γ_n is the universal bundle.

Continuity = Geometricity

A similar attitude occurs in topos theory in regards to geometric logic:

> axioms. This gives rise to a geometric *mathematics*, going beyond the merely logical – technically it is the mathematics that can be conducted in the topos-valid internal mathematics of Grothendieck toposes, and is moreover preserved by the inverse image functors of geometric morphisms. To put it another way, the geometric mathematics has an intrinsic continuity (since geometric morphisms are the continuous maps between toposes).
>
> In this paper I shall survey some of the special features of geometric logic, and a body of established results that combine to support a manifesto *"continuity is geometricity"*. In other words, to "do mathematics continuously" is to work within the geometricity constraints.

— Vickers, 'Continuity and Geometric Logic'

Theories and Models

A *theory* can be viewed as an axiomatic description of mathematical structures (e.g. the theory of groups);

Theories and Models

A *theory* can be viewed as an axiomatic description of mathematical structures (e.g. the theory of groups); a *model* is an object that 'satisfies' these axioms.

Geometric Logic

Theories and Models

A *theory* can be viewed as an axiomatic description of mathematical structures (e.g. the theory of groups); a *model* is an object that 'satisfies' these axioms.

Geometric Theory

A *geometric theory* is a theory whose (formulae featured in its) axioms are built out of certain logical connectives

Geometric Logic

Theories and Models

A *theory* can be viewed as an axiomatic description of mathematical structures (e.g. the theory of groups); a *model* is an object that ‘satisfies’ these axioms.

Geometric Theory

A *geometric theory* is a theory whose (formulae featured in its) axioms are built out of certain logical connectives — i.e. $=$, finite conjunctions $\wedge$, arbitrary (possibly infinite) disjunctions $\bigvee$, and $\exists$.

Example: Theory of Dedekind Reals

As an example, consider the geometric theory of Dedekind reals, which we denote $\mathbb{R}$.

Example: Theory of Dedekind Reals

As an example, consider the geometric theory of Dedekind reals, which we denote $\mathbb{R}$. A model x of $\mathbb{R}$ is a Dedekind real number, which will be represented by two sets of rationals (L, R), whereby:

$$L = \{q \in \mathbb{Q} | q < x\}$$

$$R = \{r \in \mathbb{Q} | x < r\}$$

Otherwise known as the left and right Dedekind sections of the real number.

Points of a Topos

Definition

- A *geometric morphism* $f : \mathcal{F} \to \mathcal{E}$ of toposes is a pair of adjoint functors $f_* : \mathcal{F} \to \mathcal{E}$ and $f^* : \mathcal{E} \to \mathcal{F}$, respectively called the *direct image* and the *inverse image* of f, such that the left adjoint f^* preserves finite limits.

Points of a Topos

Definition

- A *geometric morphism* $f : \mathcal{F} \to \mathcal{E}$ of toposes is a pair of adjoint functors $f_* : \mathcal{F} \to \mathcal{E}$ and $f^* : \mathcal{E} \to \mathcal{F}$, respectively called the *direct image* and the *inverse image* of f, such that the left adjoint f^* preserves finite limits.

Points of a Topos

Definition

- A *geometric morphism* $f : \mathcal{F} \to \mathcal{E}$ of toposes is a pair of adjoint functors $f_* : \mathcal{F} \to \mathcal{E}$ and $f^* : \mathcal{E} \to \mathcal{F}$, respectively called the *direct image* and the *inverse image* of f, such that the left adjoint f^* preserves finite limits.

Definition

1. A *global point* of a topos $\mathcal{E}$ is defined as a geometric morphism $\mathrm{Set} \to \mathcal{E}$.
2. A *generalised point* of a topos $\mathcal{E}$ is a geometric morphism $\mathcal{F} \to \mathcal{E}$.

Topos = Generalised Space

Definition

The classifying topos of a geometric theory $\mathbb{T}$ is a Grothendieck topos $\mathrm{Set}[\mathbb{T}]$ that classifies the models of $\mathbb{T}$ in Grothendieck toposes, i.e. for any Grothendieck topos $\mathcal{E}$, we have an equivalence of categories:

$$\textbf{Geom}(\mathcal{E}, \mathrm{Set}[\mathbb{T}]) \simeq \mathbb{T}\text{-}\mathrm{mod}(\mathcal{E})$$

Topos = Generalised Space

Definition

The classifying topos of a geometric theory $\mathbb{T}$ is a Grothendieck topos $\mathrm{Set}[\mathbb{T}]$ that classifies the models of $\mathbb{T}$ in Grothendieck toposes, i.e. for any Grothendieck topos $\mathcal{E}$, we have an equivalence of categories:

$$\mathbf{Geom}(\mathcal{E}, \mathrm{Set}[\mathbb{T}]) \simeq \mathbb{T}\text{-}\mathrm{mod}(\mathcal{E})$$

Theorem

Every Grothendieck topos is a classifying topos of some geometric theory $\mathbb{T}$, and every geometric theory $\mathbb{T}$ has a classifying topos.

Topos = Generalised Space

Definition

The classifying topos of a geometric theory $\mathbb{T}$ is a Grothendieck topos $\mathrm{Set}[\mathbb{T}]$ that classifies the models of $\mathbb{T}$ in Grothendieck toposes, i.e. for any Grothendieck topos $\mathcal{E}$, we have an equivalence of categories:

$$\mathbf{Geom}(\mathcal{E}, \mathrm{Set}[\mathbb{T}]) \simeq \mathbb{T}\text{-}\mathrm{mod}(\mathcal{E})$$

Theorem

Every Grothendieck topos is a classifying topos of some geometric theory $\mathbb{T}$, and every geometric theory $\mathbb{T}$ has a classifying topos.

Slogan

Models = points of a topos. In particular, we can reason in terms of the points of the topos (as a generalised space) as opposed to only reasoning in terms of its objects/sheaves (as a category).

Point-free Topology - A Bird's Eye View

Point-set Topology

- Point = Element of a set
- Space = A set of points, along with a set of opens satisfying some specific axioms.
- Continuous Maps = A function $f : X \to Y$ such that $f^{-1}(U)$ is open for all opens $U \subset Y$

Pointfree Topology

- Point = Model of a geometric theory
- Space = The 'World' in which the point lives with other points (i.e. a Grothendieck topos)
- Continuous Maps = A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ such that $f^* : \mathcal{F} \to \mathcal{E}$ preserves finite limits and small colimits

Generic Model

Fact

There exists a *generic model* $U_{\mathbb{T}}$ living in every classifying topos,

Generic Model

Fact

There exists a *generic model* $U_{\mathbb{T}}$ living in every classifying topos, which possesses the universal property that that any model M in a Grothendieck topos $\mathcal{E}$ can be obtained as $f^*(U_{\mathbb{T}}) \cong M$ via the inverse image functor of some (unique) $f : \mathcal{E} \to \mathrm{Set}[\mathbb{T}]$.

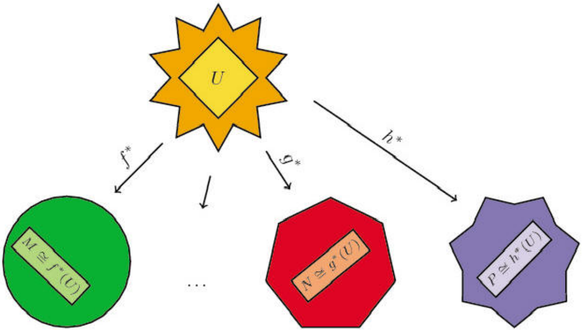
Generic Model

Fact

There exists a *generic model* $U_{\mathbb{T}}$ living in every classifying topos, which possesses the universal property that that any model M in a Grothendieck topos $\mathcal{E}$ can be obtained as $f^*(U_{\mathbb{T}}) \cong M$ via the inverse image functor of some (unique) $f : \mathcal{E} \to \mathrm{Set}[\mathbb{T}]$.

An **important** consequence of this is that any geometric sequent that holds for $U_{\mathbb{T}}$ will hold for all models M of $\mathbb{T}$.

Generic Model



Classifying topos

‘Divide-and-Conquer’

$$X^n + Y^n + Z^n = 0 \qquad (n > 2)$$

‘Divide-and-Conquer’

$$X^n + Y^n + Z^n = 0 \qquad (n > 2)$$

- Question: What are the rational (equiv. integer) solutions to this polynomial?

‘Divide-and-Conquer’

$$X^n + Y^n + Z^n = 0 \qquad (n > 2)$$

- Question: What are the rational (equiv. integer) solutions to this polynomial? — hard!

'Divide-and-Conquer'

$$X^n + Y^n + Z^n = 0 \qquad (n > 2)$$

- Question: What are the rational (equiv. integer) solutions to this polynomial? — hard!
- Observation #1: Integer solutions imply real and modulo p solutions (in fact p-adic solutions).

‘Divide-and-Conquer’

$$X^n + Y^n + Z^n = 0 \qquad (n > 2)$$

- Question: What are the rational (equiv. integer) solutions to this polynomial? — hard!
- Observation #1: Integer solutions imply real and modulo p solutions (in fact p-adic solutions).
- Observation #2: Real and p-adic solutions are easier to deal with than just integer/rational solutions.

'Divide-and-Conquer'

$$X^n + Y^n + Z^n = 0 \qquad (n > 2)$$

- Question: What are the rational (equiv. integer) solutions to this polynomial? — hard!
- Observation #1: Integer solutions imply real and modulo p solutions (in fact p-adic solutions).
- Observation #2: Real and p-adic solutions are easier to deal with than just integer/rational solutions.

- New Question: Given a polynomial with $\mathbb{Q}$-coefficients, when does knowledge about its $\mathbb{Q}_p$ and $\mathbb{R}$-solutions give us info about its $\mathbb{Q}$-solutions?

Hasse's Local-Global Principle

Local-Global Principle for $\mathbb{Q}$

Some property P is true for $\mathbb{Q}$ iff P is true for all the completions of $\mathbb{Q}$.

Hasse's Local-Global Principle

Local-Global Principle for $\mathbb{Q}$

Some property P is true for $\mathbb{Q}$ iff P is true for all the completions of $\mathbb{Q}$.

Definition of adele ring for $\mathbb{Q}$

The adele ring $\mathbb{A}_{\mathbb{Q}}$ is defined to be the restricted product of all the completions of $\mathbb{Q}$. Morally, the adele ring can be viewed as a device that allows us to reason about all the completions of $\mathbb{Q}$ simultaneously.

Hasse's Local-Global Principle

Local-Global Principle for $\mathbb{Q}$

Some property P is true for $\mathbb{Q}$ iff P is true for all the completions of $\mathbb{Q}$.

Definition of adele ring for $\mathbb{Q}$

The adele ring $\mathbb{A}_{\mathbb{Q}}$ is defined to be the restricted product of all the completions of $\mathbb{Q}$. Morally, the adele ring can be viewed as a device that allows us to reason about all the completions of $\mathbb{Q}$ simultaneously.

Idea

Instead of asking whether a property simultaneously holds for *all completions* of $\mathbb{Q}$ (which forces us to use complicated algebraic constructions like the adele ring $\mathbb{A}_{\mathbb{Q}}$), what if we asked whether a property holds for the *generic completion* of $\mathbb{Q}$?

Generic Completion

"One weakness in the analogy between the collection of $\{K_s\}_{s\in S}$ for a compact Riemann surface S and the collection $\{\mathbb{Q}_p$, for prime numbers p, and $\mathbb{R}\}$ is that [...] no manner of squinting seems to be able to make $\mathbb{R}$ the least bit mistake-able for any of the p-adic fields, nor are the p-adic fields $\mathbb{Q}_p$ isomorphic for distinct p.

***A major theme in the development of Number Theory has been to try to bring $\mathbb{R}$ somewhat more into line with the p-adic fields;** a major mystery is why $\mathbb{R}$ resists this attempt so strenuously."*

— Mazur, 'Passage from Local to Global in Number Theory'

Investigation begins...

Starting point:

For simplicity, let us assume that our base field is $\mathbb{Q}$. Classically, an absolute value of $\mathbb{Q}$ is a function $|\cdot| : \mathbb{Q} \to \mathbb{R}$ such that for all $x, y \in \mathbb{Q}$:

- $|x| \geq 0$, and $|x| = 0$ iff $x = 0$

Investigation begins...

Starting point:

For simplicity, let us assume that our base field is $\mathbb{Q}$. Classically, an absolute value of $\mathbb{Q}$ is a function $|\cdot| : \mathbb{Q} \to \mathbb{R}$ such that for all $x, y \in \mathbb{Q}$:

- $|x| \geq 0$, and $|x| = 0$ iff $x = 0$
- $|xy| = |x||y|$

Investigation begins...

Starting point:

For simplicity, let us assume that our base field is $\mathbb{Q}$. Classically, an absolute value of $\mathbb{Q}$ is a function $|\cdot| : \mathbb{Q} \to \mathbb{R}$ such that for all $x, y \in \mathbb{Q}$:

- $|x| \geq 0$, and $|x| = 0$ iff $x = 0$
- $|xy| = |x||y|$
- $|x + y| \leq |x| + |y|$

Investigation begins...

Starting point:

For simplicity, let us assume that our base field is $\mathbb{Q}$. Classically, an absolute value of $\mathbb{Q}$ is a function $|\cdot| : \mathbb{Q} \to \mathbb{R}$ such that for all $x, y \in \mathbb{Q}$:

- $|x| \geq 0$, and $|x| = 0$ iff $x = 0$
- $|xy| = |x||y|$
- $|x + y| \leq |x| + |y|$

We define a *place* as an equivalence class of absolute values whereby $|\cdot|_1 \sim |\cdot|_2$ if there exists some $\alpha \in (0, 1]$ such that $|\cdot|_1 = |\cdot|_2^\alpha$ or $|\cdot|_2 = |\cdot|_1^\alpha$.

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:

  1. $|\cdot|^{\alpha} \sim |\cdot|$

  for any absolute value $|\cdot|$, and $\alpha \in (0, 1]$

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:

  1. $|\cdot|^\alpha \sim |\cdot|$

  for any absolute value $|\cdot|$, and $\alpha \in (0, 1]$

$$[av] \times (0,1] \underset{ex}{\overset{\pi}{\rightrightarrows}} [av]$$

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:
  1. $|\cdot|^\alpha \sim |\cdot|$

  for any absolute value $|\cdot|$, and $\alpha \in (0,1]$

$$[av] \times (0,1] \underset{ex}{\overset{\pi}{\rightrightarrows}} [av]$$

- π is the projection map sending $(|\cdot|, \alpha) \mapsto |\cdot|$
- *ex* is the exponentiation map sending $(|\cdot|, \alpha) \mapsto |\cdot|^\alpha$

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:
  1. $|\cdot|^{\alpha} \sim |\cdot|$

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:
  1. $|\cdot|^{\alpha} \sim |\cdot|$
  2. $|\cdot|^{1} = |\cdot|$
  3. $(|\cdot|^{\alpha})^{\beta} = |\cdot|^{\alpha\cdot\beta}$

  for any absolute value $|\cdot|$, and $\alpha, \beta \in (0, 1]$

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:
  1. $|\cdot|^{\alpha} \sim |\cdot|$
  2. $|\cdot|^{1} = |\cdot|$
  3. $(|\cdot|^{\alpha})^{\beta} = |\cdot|^{\alpha\cdot\beta}$

  for any absolute value $|\cdot|$, and $\alpha, \beta \in (0, 1]$
- In essence, we would like to 'quotient' the topos [*av*] by an algebraic action – two questions:

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:
  1. $|\cdot|^{\alpha} \sim |\cdot|$
  2. $|\cdot|^{1} = |\cdot|$
  3. $(|\cdot|^{\alpha})^{\beta} = |\cdot|^{\alpha\cdot\beta}$

  for any absolute value $|\cdot|$, and $\alpha, \beta \in (0, 1]$
- In essence, we would like to 'quotient' the topos [*av*] by an algebraic action – two questions:

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:
  1. $|\cdot|^{\alpha} \sim |\cdot|$
  2. $|\cdot|^{1} = |\cdot|$
  3. $(|\cdot|^{\alpha})^{\beta} = |\cdot|^{\alpha\cdot\beta}$

  for any absolute value $|\cdot|$, and $\alpha, \beta \in (0, 1]$

- In essence, we would like to 'quotient' the topos [*av*] by an algebraic action – two questions:
  - Is the notion of (real) exponentiation geometric? Ng-Vickers (2021)

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:
  1. $|\cdot|^{\alpha} \sim |\cdot|$
  2. $|\cdot|^{1} = |\cdot|$
  3. $(|\cdot|^{\alpha})^{\beta} = |\cdot|^{\alpha\cdot\beta}$

  for any absolute value $|\cdot|$, and $\alpha, \beta \in (0, 1]$

- In essence, we would like to 'quotient' the topos [*av*] by an algebraic action – two questions:
  - Is the notion of (real) exponentiation geometric? Ng-Vickers (2021)
  - What does it mean to quotient by a monoid action vs. group action?

Global vs. Local Picture

Ostrowski's Theorem for $\mathbb{Q}$

Every absolute value of $\mathbb{Q}$ is equivalent to a (non-Archimedean) p-adic absolute value $|\cdot|_p$ (for some prime p), or the Archimedean absolute value $|\cdot|_\infty$.

Non-Archimedean Place (for fixed prime p)

$$\begin{tikzcd}[av_{NA}] \times (0,\infty) \arrow[r, bend left, "\pi"] \arrow[r, bend right, "ex"'] & {[av_{NA}]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D}\end{tikzcd}$$

Non-Archimedean Place (for fixed prime p)

$$\begin{tikzcd}
{[av_{NA}] \times (0,\infty)} \arrow[r, bend left, "\pi"] \arrow[r, bend right, "ex"'] & {[av_{NA}]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D}
\end{tikzcd}$$

Non-Archimedean Place (for fixed prime p)

$$\begin{tikzcd}[av_{NA}] \times (0,\infty) \arrow[r, bend left, "\pi"] \arrow[r, bend right, "ex"'] & {[av_{NA}]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D}\end{tikzcd}$$

- For any non-Arch. absolute $|\cdot|$, exponentiating $|\cdot|^\alpha$ still yields a non-Arch. absolute value for any $\alpha \in (0,\infty)$ (unlike the Archimedean case).

Non-Archimedean Place (for fixed prime p)

$$\begin{tikzcd} {[av_{NA}] \times (0, \infty)} \arrow[r, bend left, "\pi"] \arrow[r, bend right, "ex"'] & {[av_{NA}]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D} \end{tikzcd}$$

- For any non-Arch. absolute $|\cdot|$, exponentiating $|\cdot|^{\alpha}$ still yields a non-Arch. absolute value for any $\alpha \in (0, \infty)$ (unlike the Archimedean case).
- What is $\mathcal{D}$?

Non-Archimedean Place (for fixed prime p)

$$\begin{tikzcd} {[av_{NA}]} \times (0,\infty) \arrow[r, bend left, "\pi"] \arrow[r, bend right, "ex"'] & {[av_{NA}]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D} \end{tikzcd}$$

- For any non-Arch. absolute $|\cdot|$, exponentiating $|\cdot|^{\alpha}$ still yields a non-Arch. absolute value for any $\alpha \in (0,\infty)$ (unlike the Archimedean case).
- What is $\mathcal{D}$?

Theorem

$\mathcal{D} \simeq \mathrm{Set}$

Archimedean Place

$$\begin{tikzcd}[av_A] \times (0,1] \arrow[r, bend left, "\pi"] \arrow[r, bend right, "\mathfrak{m}"'] & {[av_A]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D}'\end{tikzcd}$$

Archimedean Place

$$
\begin{tikzcd}
{[av_A] \times (0,1]} \arrow[r, bend left, "\pi"] \arrow[r, bend right, "\mathfrak{m}"'] & {[av_A]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D}'
\end{tikzcd}
$$

- Space of Arch. absolute values is acted upon by a **monoid** $(0,1]$-action as opposed to a **group** $(0,\infty)$-action.

Archimedean Place

$$
\begin{tikzcd}
{[av_A] \times (0,1]} \arrow[r, bend left, "\pi"] \arrow[r, bend right, "\mathfrak{m}"'] & {[av_A]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D}'
\end{tikzcd}
$$

- Space of Arch. absolute values is acted upon by a **monoid** $(0,1]$-action as opposed to a **group** $(0,\infty)$-action.
- Can we play the same game as we did in the Non-Archimedean case?

Archimedean Place

$$
\begin{tikzcd}
{[av_A] \times (0,1]} \arrow[r, bend left, "\pi"] \arrow[r, bend right, "\mathfrak{m}"'] & {[av_A]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D}'
\end{tikzcd}
$$

- Space of Arch. absolute values is acted upon by a **monoid** $(0,1]$-action as opposed to a **group** $(0,\infty)$-action.
- Can we play the same game as we did in the Non-Archimedean case? Answer: No!

Archimedean Place

$$
\begin{tikzcd}
{[av_A] \times (0,1]} \arrow[r, bend left, "\pi"] \arrow[r, bend right, "\mathfrak{m}"'] & {[av_A]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D}'
\end{tikzcd}
$$

- Space of Arch. absolute values is acted upon by a **monoid** $(0, 1]$-action as opposed to a **group** $(0, \infty)$-action.
- Can we play the same game as we did in the Non-Archimedean case? Answer: No! (The topos $\mathcal{D}'$ has non-trivial forking in its sheaves)

Archimedean Place

$$\begin{tikzcd}[avA] \times (0,1] \arrow[r, bend left, "\pi"] \arrow[r, bend right, "\mathfrak{m}"'] & [avA] \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D}'\end{tikzcd}$$

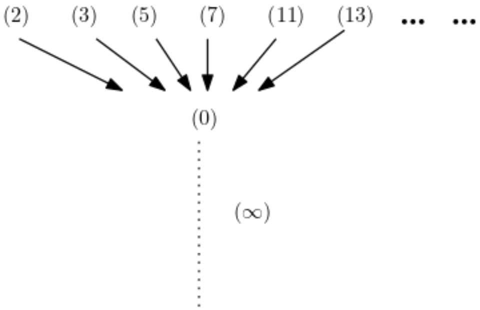
- Space of Arch. absolute values is acted upon by a **monoid** $(0, 1]$-action as opposed to a **group** $(0, \infty)$-action.
- Can we play the same game as we did in the Non-Archimedean case? Answer: No! (The topos $\mathcal{D}'$ has non-trivial forking in its sheaves)
- So what is $\mathcal{D}'$?

Candidate Picture

$\mathcal{D}' \simeq \overleftarrow{[0,1]}$

Candidate Picture

$\mathcal{D}' \simeq \overleftarrow{[0,1]}$ (the space of 'upper reals' between 0 and 1)



Candidate Picture

$\mathcal{D}' \simeq \overleftarrow{[0,1]}$ (the space of 'upper reals' between 0 and 1)

- The Arakelov compactification of $\mathrm{Spec}(\mathbb{Z})$ suggests that we add a single point at infinity to $\mathrm{Spec}(\mathbb{Z})$ corresponding to the 'Archimedean prime' ...

Candidate Picture

$\mathcal{D}' \simeq \overleftarrow{[0,1]}$ (the space of 'upper reals' between 0 and 1)

- The Arakelov compactification of $\mathrm{Spec}(\mathbb{Z})$ suggests that we add a single point at infinity to $\mathrm{Spec}(\mathbb{Z})$ corresponding to the 'Archimedean prime' . . . our candidate picture suggests that there is some blurring going on at infinity, and that infinity is not just a classical point with no intrinsic structure.

Sullivan's Arithmetic Square (a.k.a. 'The Hasse Square'):

$$
\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & \prod_p \widehat{\mathbb{Z}}_p \arrow[d] \\
\mathbb{Q} \arrow[r] & \mathbb{Q} \otimes_{\mathbb{Z}} \prod_p \widehat{\mathbb{Z}}_p = \mathbb{A}^f_{\mathbb{Q}}
\end{tikzcd}
$$

By way of conclusion...

By way of conclusion...

- Theme #1: Viewing toposes as a framework uniting logic and topology
- Theme #2: Local-Global issues, and its connections to Theme #1 via generic reasoning

By way of conclusion...

- Theme #1: Viewing toposes as a framework uniting logic and topology
- Theme #2: Local-Global issues, and its connections to Theme #1 via generic reasoning

- Pulling away from the set theory reveals key insights into the deep nerve connecting topology and algebra.

By way of conclusion...

- Theme #1: Viewing toposes as a framework uniting logic and topology
- Theme #2: Local-Global issues, and its connections to Theme #1 via generic reasoning

- Pulling away from the set theory reveals key insights into the deep nerve connecting topology and algebra.
- Some very interesting indications that there is some blurring at infinity in our picture of $\overline{\mathrm{Spec}(\mathbb{Z})}$ — interesting to explore the precise implications of this.

References I

[1] Boileau, A., Joyal, A. *La Logique des Topos*, The Journal of Symbolic Logic, Vol. 46, No. 1, pp. 6-16, (1981).

[2] Caramello, O., *Theories, Sites, Toposes: Relating and studying mathematical theories through topos-theoretic 'bridges'*, Oxford University Press (2017).

[3] Connes, A., Consani, C., *Absolute Algebra and Segal's Γ-rings*, Journal of Number Theory Volume 162, pp. 518-551, May 2016.

[4] Egan, G., *Geometry and Waves [Extra]*, `http://www.gregegan.net/ORTHOGONAL/03/WavesExtra.html`

[5] Hatcher, A.: *Vector Bundles and K-theory*, https://www. math.cornell.edu/ hatcher/VBKT/VBpage.html

[6] Johnstone, P.T.: *Topos Theory*, Dover Publications Inc., (1977)

References II

[7] Johnstone, P.T.: *Sketches of an Elephant: A Topos Theory Compendium, Vol. 1*, Clarendon Press, (2002)

[8] Johnstone, P.T.: *Sketches of an Elephant: A Topos Theory Compendium, Vol. 2*, Clarendon Press, (2002)

[9] Mazur, B., *On the Passage from Local to Global in Number Theory*, Bulletin of the AMS, 29 No. 1, (1993).

[10] Moerdijk, I. *The classifying topos of a continuous groupoid, I.*, Transactions of the American Mathematical Society Volume 310, Number 2, pp. 629-668, 1988.

[11] Nlab, *Vector Bundles*, `https://ncatlab.org/nlab/show/vector+bundle`.

[12] Ng, M., and Vickers, S. *Point-free Construction of Real Exponentiation*, to appear in LMCS.

References III

[13] Sullivan, D., *Geometric Topology - Localization, Periodicity, and Galois Symmetry (The 1970 MIT notes)*, `https://www.maths.ed.ac.uk/~v1ranick/books/gtop.pdf`

[14] Vickers, S., *Localic Completion of Generalized Metric Spaces*, preprint.

[15] Vickers, S., *Continuity and Geometric Logic*, J. Applied Logic (12), pp. 14-27, (2014).

[16] Zakharevich, I., *Atttiudes of K-theory*, *NOTICES OF THE AMERICAN MATHEMATICAL SOCIETY*, Vol. 66, No. 7, pp. 1034-1044.