$K_1(\mathcal{V}\mathrm{ar})$ IS GENERATED BY QUASI-AUTOMORPHISMS

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ABSTRACT. This paper provides a complete characterisation of $K_1(\mathcal{V}\mathrm{ar})$, the K_1 group of varieties, solving a problem left open in [Zak17c]. Our approach involves adapting Gillet-Grayson's G-Construction to define a new K-theory spectrum of varieties. There are two levels on which one can read the present paper. On one level, we streamline and extend a series of K-theory results on exact categories to a more general class of categories (including $\mathcal{V}\mathrm{ar}$). On another level, our investigations bring into focus an interesting generalisation of automorphisms ("double exact squares"), which generate K_1. Since our results apply to a wide range of non-additive contexts (e.g. varieties, o-minimal structures, definable sets etc.), this sets up a challenging question: what kind of information do these quasi-automorphisms calibrate?

Our understanding of K-theory is changing. Recent efforts to extend tools from classical algebraic K-theory to non-additive settings have led us to make decisions on what its essential features are. One perspective, influenced by Waldhausen's $S_\bullet$-construction [Wal87], is that K-theory is a framework for analysing the finite assembly and decompositions of objects; non-additive applications of this insight can be found in Campbell's $\widetilde{S}_\bullet$-construction [Cam19] as well as Zakharevich's use of finite disjoint covers in Assemblers [Zak17b]. A related perspective emphasises the view that K-theory breaks an object into two types of pieces. This underpins Campbell-Zakharevich's framework of *CGW categories* [CZ22], which formalises key similarities between exact categories and the category of varieties $\mathcal{V}\mathrm{ar}_k$.

In a different line of work: the study of $K_1(\mathcal{C})$ for arbitrary exact categories began with Gillet-Grayson's G-construction [GG87], which provided an elementary description of its generators. This description was refined by Sherman [She94, She98] and Nenashev [Nen96], before culminating in Nenashev's characterisation of the complete set of relations for K_1 [Nen98b, Nen98a].

The present paper unites the two lines of investigation by extending the K_1 results to a subclass of CGW categories known as *pCGW categories*. These include not only exact categories and varieties, but also finite sets, o-minimal structures, and definable sets. As a result, we provide two alternative, complete characterisations of K_1 applicable to a broad range of non-additive contexts.

Overview. Let us develop the previous remark that K-theory is an abstract framework for breaking an object into two different types of pieces. Consider the following two definitions.

- Let R be a ring. We define $K_0(R)$ as

$$K_0(R) := \left\{ \begin{matrix} \text{free abelian group} \\ \text{fin. gen. proj. } R\text{-modules} \end{matrix} \right\} \Bigg/ \begin{matrix} [M] = [M'], \text{ if } M \cong M' \\ [M] = [M'] + [M''], \text{ if } M' \to M \to M'' \end{matrix}$$

  where $M' \to M \to M''$ is a short exact sequence.
- Let $\mathcal{V}\mathrm{ar}_k$ be the category of k-varieties, i.e. reduced separated schemes of finite type over field k. We define $K_0(\mathcal{V}\mathrm{ar}_k)$ as

$$K_0(\mathcal{V}\mathrm{ar}_k) := \left\{ \begin{matrix} \text{free abelian group} \\ k\text{-varieties} \end{matrix} \right\} \Bigg/ \begin{matrix} [X] = [X'], \text{ if } X \cong X' \\ [X] = [U] + [X \setminus U], \text{ if } U \hookrightarrow X \text{ is a closed immersion} \end{matrix}$$

The analogy is clear. A short exact sequence $M' \to M \to M''$ decomposes the R-module M into two distinct pieces, M' and M'', with $M' \to M$ an admissible monic and $M \to M''$ as an admissible epi. Similarly, $K_0(\mathcal{V}\mathrm{ar}_k)$ decomposes a variety X into U and $X \setminus U$, with $U \hookrightarrow X$ a closed immersion and

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$X \setminus U \hookrightarrow X$ an open immersion. In both cases, an object that is broken into two pieces is viewed as an abstract sum of the two components. But what are the essential features of this decomposition?

Quillen [Qui73] introduced, and later characterised, an exact category as a pair $(\mathcal{C}, \mathcal{S})$, where $\mathcal{C}$ is additive, and $\mathcal{S}$ is a family of sequences $M' \to M \to M''$ satisfying conditions analogous to short exact sequences in abelian categories. This includes the natural condition that admissible monics $M' \to M$ are kernels of admissible epis $M \to M''$, and admissible epis are cokernels of admissible monics. This enabled Quillen to construct a K-theory spectrum $K\mathcal{C}$, recovering $K_0(R)$ on π_0 when $\mathcal{C}$ is the category of finitely-generated R-modules.

Campbell-Zakharevich [CZ22] re-examined Quillen's framework, and made a crucial observation: instead of requiring the two classes of morphisms to compose, it suffices to encode their interaction formally. This added flexibility allows us to extend Quillen's K-theory to non-additive settings like varieties, where sequences like $U \hookrightarrow X \hookleftarrow X \setminus U$ clearly do not compose. In particular, one can adapt Quillen's argument to construct a K-theory spectrum $K\mathcal{V}\mathrm{ar}_k$ whose π_0 recovers the original $K_0(\mathcal{V}\mathrm{ar}_k)$.

We can now define the K-groups $K_n(\mathcal{V}\mathrm{ar}_k) := \pi_n(K\mathcal{V}\mathrm{ar}_k)$ for all n, and ask:

Question 1. What kind of information do the higher K-groups of varieties encode?

This is a challenging question. In classical algebraic K-theory, the coarseness of K_0 as an invariant may be measured by the fact that $K_0(F) = \mathbb{Z}$ for all fields F, whereas $K_1(F) \cong F^\times$. Is there an analogous story in the setting of varieties? More explicitly, how might we measure the loss of information in $K_0(\mathcal{V}\mathrm{ar}_k)$? To what extent can this information be recovered in the higher K-groups? The following summary theorem gives a snapshot of the current landscape.

Summary Theorem 0.1. *Assume the base field k is of characeristic 0, and equip $K_0(\mathcal{V}\mathrm{ar}_k)$ with a ring structure by defining $[X] \cdot [Y] := [(X \times_k Y)_{\mathrm{red}}]$. Two k-varieties X, Y are said to be* piecewise isomorphic *if X and Y admit finite partitions*

$$X_1, \ldots, X_n \qquad \textit{and} \qquad Y_1, \ldots, Y_n$$

into locally closed subvarieties such that $X_i \cong Y_i$ for all n. The following is known:

- *(i) Define $SK_0(\mathcal{V}\mathrm{ar}_k)$ as the freely generated* semiring *on $[X]$ subject to $[X] = [Z] + [X \setminus Z]$. Then, two k-varieties X, Y are piecewise isomorphic iff $[X] = [Y]$ in $SK_0(\mathcal{V}\mathrm{ar}_k)$.*
- *(ii) Let X, Y k-varieties such that* $\dim X \leq 1$. *Then $[X] = [Y]$ in $K_0(\mathcal{V}\mathrm{ar}_k)$ iff they are piecewise isomorphic.*
- *(iii) There exists k-varieties X and Y such that $[X] = [Y]$ in $K_0(\mathcal{V}\mathrm{ar}_k)$ and yet fail to be piecewise isomorphic.*
- *(iv) Let X be a k-variety of any non-negative dimension containing only finitely many rational curves. Then for any k-variety Y, $[X] = [Y]$ in $K_0(\mathcal{V}\mathrm{ar}_k)$ iff they are piecewise isomorphic.*

Proof. (i) appears to be folklore, and is recorded in [Bek17] as well as [CLNS18, Cor. 1.4.9, Chapter 2]. (ii) is [LS10, Props. 5 and 6]. For (iii), various constructions are now known but the first example goes back to [Bor18]. (iv) is [LS10, Theorem 5]. □

Summary Theorem 0.1 sharpens our understanding of what is at stake. Given our high-level characterisation of K-theory as an abstract framework for analysing the finite assembly and decompositions of objects, the following question was natural:

Question 2 ([LL03, Question 1.2])**.** Is it true that two k-varieties are piecewise isomorphic iff they agree in $K_0(\mathcal{V}\mathrm{ar}_k)$?

In the setting of characteristic 0, item (iii) of the Summary Theorem answers no, signalling a loss of information on the level of K_0. Item (i) tells us the information is lost precisely because $K_0(\mathcal{V}\mathrm{ar}_k)$ involves group completion – akin to an Eilenberg Swindle. Item (ii) tells us that piecewise isomorphism and equivalence in $K_0(\mathcal{V}\mathrm{ar}_k)$ coincide so long as the varieties are of sufficiently low dimension. Put otherwise, the algebraic barriers to geometric information only occur at the higher dimensions. Item (iv) is subtler, and raises interesting questions about how taking piecewise isomorphisms of complex varieties relates to the ampleness of their canonical line bundles (cf. the algebraic hyperbolicity conjecture for surfaces).

In light of this discussion, let us return to Question 1. Some promising initial progress has been made. Using the formalism of Assemblers, Zakharevich constructs a different (but equivalent) K-theory spectrum of varieties, before leveraging its connection with Waldhausen categories to obtain a partial characterisation of $K_1(\mathcal{V}\mathrm{ar}_k)$ [Zak17c, Theorem B]. Inspired by Borisov's work [Bor18], this was later developed in [Zak17a] to illuminate a subtle geometric insight: the failure to extend birational automorphisms of varieties to piecewise isomorphisms is tightly connected to the Lefschetz motive $[\mathbb{A}^1]$ being a zero divisor in $K_0(\mathcal{V}\mathrm{ar}_k)$. In a different vein: [CWZ19] identifies non-trivial elements in $K_n(\mathcal{V}\mathrm{ar}_k)$ by lifting various motivic measures $K_0(\mathcal{V}\mathrm{ar}_k) \to K_0(\mathcal{C})$ to the level of spectra $K\mathcal{V}\mathrm{ar}_k \to K\mathcal{C}$.

Discussion of Main Results. Until recently, a full characterisation of any higher K-group of varieties was not known. In her original paper, Zakharevich [Zak17c, Theorem B] identifies the generators of $K_1(\mathcal{V}\mathrm{ar}_k)$ and some key relations, but does not prove their completeness. Independently from us, an intriguing recent collaboration between algebraic topologists and experts in homological stability has uncovered a homological proof [KLM$^+$24, Prop. 4.1] that Zakharevich's presentation is in fact complete.

We take a different approach. Whereas [KLM$^+$24] utilises homological methods to analyse K_1, the present paper instead relies on techniques from simplicial homotopy theory. Further, whereas [Zak17c] relies on the connection between $\mathcal{V}\mathrm{ar}_k$ and Waldhausen categories, we instead focus on the (tighter) connection between $\mathcal{V}\mathrm{ar}_k$ and exact categories. This sets up the following theorem.

Theorem A (Theorem 2.11)**.** Let $\mathcal{C}$ be a pCGW category, and $\mathcal{SC}$ the simplicial set obtained by applying the $S_\bullet$-construction. Then, there exists a simplicial set $G\mathcal{C}$ such that there is a homotopy equivalence

$$|G\mathcal{C}| \simeq \Omega|\mathcal{SC}|.$$

In particular, $\pi_n|G\mathcal{C}| = K_n\mathcal{C}$ for all n.

In broad strokes: Theorem A extends Gillet-Grayson's G-construction on exact categories [GG87] to a wider class of categories including $\mathcal{V}\mathrm{ar}_k$. The beauty of the G-construction is that it translates a topological problem (i.e. characterising π_1 of a loop space) into a simplicial one, which is more combinatorial and thus easier to work with. To show that this gives us sufficient leverage to characterise K_1 will, of course, take the rest of the paper. Let us also remark that while one can prove Theorem A by adapting the original proof [GG87] to our setting, we provide a more streamlined argument (Theorem 2.8) inspired by Grayson's framework of dominant functors [Gra87].

The previous remarks underscore a more fundamental difference. Both [Zak17c] and [KLM$^+$24] are concerned with the K-theory of Assemblers, whereas our paper builds on [CZ22] to develop the K-theory of so-called *pCGW categories*. Precise definitions will be given in due course; for now, it suffices to think of Assemblers and pCGW categories as two distinct yet equivalent ways of defining the K-theory spectrum of varieties.[1] This difference becomes apparent when comparing our respective presentations of $K_1(\mathcal{V}\mathrm{ar}_k)$. In our language:

Theorem B (Theorem 3.17 and Prop. 4.12)**.** Let $\mathcal{C}$ be a pCGW category. Then $K_1(\mathcal{C})$ is generated by *double exact squares*, i.e. by pairs of distinguished squares in $\mathcal{C}$ with identical nodes

$$l := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & C \arrow[d, circ, "g_1"] \\ A \arrow[r, tail, "f_1"'] & B \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & C \arrow[d, circ, "g_2"] \\ A \arrow[r, tail, "f_2"'] & B \end{tikzcd} \right), \tag{1}$$

modulo the following relations

(B1) $$\left\langle \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & A \arrow[d, circ, "1"] \\ O \arrow[r, tail] & A \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & A \arrow[d, circ, "1"] \\ O \arrow[r, tail] & A \end{tikzcd} \right) \right\rangle = 0;$$

[1] For the cautious reader: the weak equivalence of these spectra as spaces is [CZ22, Theorems 7.8 and 9.1].

(B2) $$\left\langle \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, very near start}] \arrow[dr, phantom, "\square"] & O \arrow[d, "\circ"{marking, very near start}] \\ A \arrow[r, tail, "1"'] & A \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, very near start}] \arrow[dr, phantom, "\square"] & O \arrow[d, "\circ"{marking, very near start}] \\ A \arrow[r, tail, "1"'] & A \end{tikzcd} \right) \right\rangle = 0;$$

(B3) Suppose $f_C \colon A \overset{f_A}{\rightarrowtail} B \overset{f_B}{\rightarrowtail} C$ and $f'_C \colon A \overset{f'_A}{\rightarrowtail} B \overset{f'_B}{\rightarrowtail} C$. Under technical conditions (imposed by the 2-simplices of $G\mathcal{C}$), the following splitting relation holds

$$\left\langle \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, very near start}] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[d, "\circ"{marking, very near start}, "g_A"] \\ A \arrow[r, tail, "f_A"'] & B \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, very near start}] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[d, "\circ"{marking, very near start}, "g'_A"] \\ A \arrow[r, tail, "f'_A"'] & B \end{tikzcd} \right) \right\rangle + \left\langle \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, very near start}] \arrow[dr, phantom, "\square"] & \frac{C}{B} \arrow[d, "\circ"{marking, very near start}, "g_B"] \\ B \arrow[r, tail, "f_B"'] & C \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, very near start}] \arrow[dr, phantom, "\square"] & \frac{C}{B} \arrow[d, "\circ"{marking, very near start}, "g'_B"] \\ B \arrow[r, tail, "f'_B"'] & C \end{tikzcd} \right) \right\rangle = \left\langle \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, very near start}] \arrow[dr, phantom, "\square"] & \frac{C}{A} \arrow[d, "\circ"{marking, very near start}, "g_C"] \\ A \arrow[r, tail, "f_C"'] & C \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, very near start}] \arrow[dr, phantom, "\square"] & \frac{C}{A} \arrow[d, "\circ"{marking, very near start}, "g'_C"] \\ A \arrow[r, tail, "f'_C"'] & C \end{tikzcd} \right) \right\rangle .$$

This presentation appears to be new, even in the context of exact categories. How does it compare with K_1 of an Assembler? The following informal discussion may be illuminating.

On Generators. Double exact squares describe how to break an object into two distinct parts – for instance, Equation (1) shows B being broken into A and C. Interestingly, these squares generalise the usual notion of an automorphism – see Example 3.16. By contrast, [Zak17c] shows that K_1 of an Assembler is generated by *piecewise automorphisms*, which break an object into n many pieces simultaneously. This difference reflects a trade-off between simplicity vs. flexibility. Our Theorem B presents a simpler set of generators for $K_1(\mathcal{V}\mathrm{ar}_k)$ than [Zak17c], which can be advantageous when e.g. constructing derived motivic measures, as done in [CWZ19].[2] On the other hand, the generality of piecewise automorphisms makes the Assemblers formalism better suited for investigating e.g. scissors congruence of convex polytopes, as done in [KLM+24], where simultaneous decomposition is essential.[3]

On Relations. There is an interesting discrepancy regarding the relations of [Zak17c] and Theorem B. In Zakharevich's presentation, the composition of piecewise automorphisms always split in K_1. More precisely:

$$\left\langle A \underset{f_2}{\overset{f_1}{\rightrightarrows}} B \right\rangle + \left\langle B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C \right\rangle = \left\langle A \underset{g_2 f_2}{\overset{g_1 f_1}{\rightrightarrows}} C \right\rangle \qquad \text{in } K_1,$$

where f_i, g_i are piecewise automorphisms. Figure 1 gives an informal illustration.

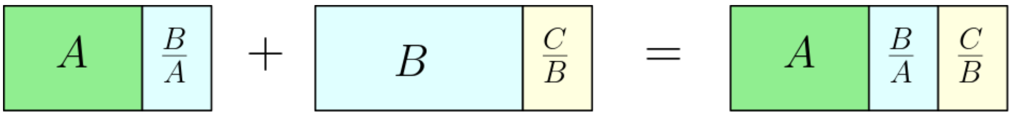

FIGURE 1. LHS: the piecewise automorphisms induced by closed immersions $A \overset{f}{\rightarrowtail} B$ and $B \overset{g}{\rightarrowtail} C$. RHS: the piecewise automorphism induced by their composition $A \overset{gf}{\rightarrowtail} C$.

By contrast, Relation (B3) of Theorem B asserts that composition splits in K_1 only when a technical condition is satisfied – see Warning 4.2 for details. Proposition 4.12 gives evidence that this condition is both non-trivial and necessary in general. Our analysis relies on Theorem C, an alternative presentation of K_1 inspired by Nenashev [Nen98a], and is the final main result of our paper.

Theorem C (Corollary 4.14)**.** Let $\mathcal{C}$ be a pCGW category. Then $K_1(\mathcal{C})$ is generated by double exact squares subject to the following relations:

[2]Technically, [CWZ19] views $\mathcal{V}\mathrm{ar}_k$ as a *subtractive category* before applying the $\widetilde{S}_\bullet$*-construction* as defined in [Cam19], but this is equivalent to viewing $\mathcal{V}\mathrm{ar}_k$ as a CGW category and applying the $S_\bullet$-construction; see [CZ22, Example 7.4].

[3]*Details.* Define two polytopes P and Q to be scissors congruent if: (i) $P = \bigcup_{i=1}^m P_i$ and $Q = \bigcup_{i=1}^m Q_i$ such that $P_i \cong Q_i$, and (ii) $P_i \cap P_j = Q_i \cap Q_j = \emptyset$ for $i \neq j$. The key hypothesis here is convexity. In particular, pairwise unions like $P_j \cup P_k$ may not form a convex polytope, so decomposition and reassembly must be done simultaneously.

(N1) $\langle l \rangle = 0$ if l is a pair of identical squares, say

$$l = \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & C \arrow[d, o->, "g"] \\ A \arrow[r, tail, "f"'] & B \end{tikzcd} , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & C \arrow[d, o->, "g"] \\ A \arrow[r, tail, "f"'] & B \end{tikzcd} \right).$$

(N2) Given a good 3×3 diagram

$$\left(\begin{tikzcd} X_{00} \arrow[r, tail, "f_0"] \arrow[d, tail, "h_0"'] \arrow[dr, phantom, "\circlearrowleft"] & X_{01} \arrow[d, tail, "h_1"] & X_{02} \arrow[l, o->, "g_0"'] \arrow[d, tail, "h_2"] \\ X_{10} \arrow[r, tail, "f_1"] & X_{11} \arrow[dr, phantom, "\circlearrowleft"] & X_{12} \arrow[l, o->, "g_1"'] \\ X_{20} \arrow[r, tail, "f_2"] \arrow[u, o->, "j_0"] & X_{21} \arrow[u, o->, "j_1"] & X_{22} \arrow[l, o->, "g_2"'] \arrow[u, o->, "j_2"'] \end{tikzcd} , \quad \begin{tikzcd} X_{00} \arrow[r, tail, "f'_0"] \arrow[d, tail, "h'_0"'] \arrow[dr, phantom, "\circlearrowleft"] & X_{01} \arrow[d, tail, "h'_1"] & X_{02} \arrow[l, o->, "g'_0"'] \arrow[d, tail, "h'_2"] \\ X_{10} \arrow[r, tail, "f'_1"] & X_{11} \arrow[dr, phantom, "\circlearrowleft"] & X_{12} \arrow[l, o->, "g'_1"'] \\ X_{20} \arrow[r, tail, "f'_2"] \arrow[u, o->, "j'_0"] & X_{21} \arrow[u, o->, "j'_1"] & X_{22} \arrow[l, o->, "g'_2"'] \arrow[u, o->, "j'_2"'] \end{tikzcd} \right)$$

defined by the following 6 double exact squares

$$l_i := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & X_{i2} \arrow[d, o->, "g_i"] \\ X_{i0} \arrow[r, tail, "f_i"] & X_{i1} \end{tikzcd} , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & X_{i2} \arrow[d, o->, "g'_i"] \\ X_{i0} \arrow[r, tail, "f'_i"] & X_{i1} \end{tikzcd} \right) \qquad \text{for all } i \in \{0,1,2\}$$

$$l^i := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & X_{2i} \arrow[d, o->, "j_i"] \\ X_{0i} \arrow[r, tail, "h_i"] & X_{1i} \end{tikzcd} , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & X_{2i} \arrow[d, o->, "j'_i"] \\ X_{0i} \arrow[r, tail, "h'_i"] & X_{1i} \end{tikzcd} \right) \qquad \text{for all } i \in \{0,1,2\},$$

the following 6-term relation holds

$$\langle l_0 \rangle + \langle l_2 \rangle - \langle l_1 \rangle = \langle l^0 \rangle + \langle l^2 \rangle - \langle l^1 \rangle$$

Does Theorem C imply a mistake in [Zak17c]? Not necessarily. A key distinction lies in the different choices of weak equivalences. Zakharevich's framework defines weak equivalences in a way that effectively hardcodes the splitting of sequences at the categorical level (see [Zak17c, Theorem 2.1]). This draws parallels with the classical Additivity Theorem [on the level of spectra], but contrasts with the pCGW setting, where weak equivalences are isomorphisms. To illustrate, consider the projective line in the sequence

$$* \hookrightarrow \mathbb{P}^1 \hookleftarrow \mathbb{A}^1.$$

In the pCGW setting, this sequence clearly fails to split since $\mathbb{P}^1 \ncong \mathbb{A}^1 \coprod \{*\}$. By contrast, this sequence splits up to weak equivalence in Zakharevich's framework since

$$\mathbb{P}^1 \dashleftarrow \mathbb{A}^1 \coprod \{*\} \xrightarrow{\cong} \mathbb{A}^1 \coprod \{*\}$$

defines a weak equivalence in the sense of [Zak17c], Definition 1.7. We discuss the implications for the K-theory presentation more fully in Section 5.1.

Finally, we remark that our results extend to settings not modelled by the Assemblers framework. One obvious example is exact categories. More interestingly, there is suggestive evidence that they also apply to matroids (Example 1.14), although some details remain to be worked out (see Section 5.4).

Implications for Characterising K_n. Many of the results of the present paper are inspired by Nenashev's work [Nen96] characterising $K_1(\mathcal{C})$ for an exact category $\mathcal{C}$. Grayson [Gra12] later extended this to characterise $K_n(\mathcal{C})$ for all n, and we expect our approach to generalise similarly to the higher K-groups of varieties (or, more generally, the higher K-groups of pCGW categories). It is currently unclear how one might analogously extend the methods from [Zak17c] or [KLM$^+$24].

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CONTENTS

1. PRELIMINARIES

1.1. **CGW Categories.** The key definition in [CZ22] is the *CGW category*. This is essentially a category equipped with two subclasses of maps, $\mathcal{M}$ and $\mathcal{E}$ (analogous to admissible monics and epis in exact categories), along with a collection of square diagrams ("distinguished squares") that encode how $\mathcal{M}$ and $\mathcal{E}$-morphisms interact.

This is presented using the language of double categories. Recall that a *double category* $\mathcal{C}$ is an internal category in Cat. For the present paper, we will require the following refinement.

Definition 1.1. A *good double category* is a triple of categories $(\mathcal{C}, \mathcal{M}, \mathcal{E})$ presented by the data:

- *Objects.* All three categories have the same objects: $\mathrm{ob}(\mathcal{E}) = \mathrm{ob}(\mathcal{M}) = \mathrm{ob}(\mathcal{C})$.
- *Morphisms.*
  $\mathcal{M}$**-morphisms:** $\mathcal{M}$ is a subcategory of $\mathcal{C}$. Its morphisms are denoted $\rightarrowtail$.
  $\mathcal{E}$**-morphisms:** Either $\mathcal{E}$ or $\mathcal{E}^{\mathrm{op}}$ is a subcategory of $\mathcal{C}$. Its morphisms are denoted $\circ\!\!\rightarrow$.
- *Distinguished Squares.* A collection of square diagrams, denoted

$$\begin{tikzcd} A \arrow[r, tail, "f'"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\ C \arrow[r, tail, "f"] \arrow[ur, phantom, "\square"] & D \end{tikzcd}$$

  where $f, f' \in \mathcal{M}$ and $g, g' \in \mathcal{E}$. These squares closed under horizontal and vertical composition. They are also required to *interact well with isomorphisms* in the following sense: if

$$\begin{tikzcd} A \arrow[r, tail, "f'"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\ C \arrow[r, tail, "f"] & D \end{tikzcd}$$

  defines a commutative diagram in $\mathcal{C}$, and either both $\mathcal{M}$-morphisms or both $\mathcal{E}$-morphisms are isomorphisms, then the square is distinguished.

Convention 1.2. We typically denote a good double category as $\mathcal{C} = (\mathcal{M}, \mathcal{E})$. When the context is clear, we simply write $\mathcal{C}$. When we wish to regard $\mathcal{C}$ as an ordinary 1-category, we refer to $\mathcal{C}$ as the *ambient category* – since it contains $\mathcal{M}$ and $\mathcal{E}$ (or $\mathcal{E}^{\mathrm{op}}$) as subcategories.

We now introduce a couple of helper definitions, before defining a CGW category.

Definition 1.3. Let $C = (\mathcal{E}, \mathcal{M})$ be a good double category, and $\mathcal{D}$ be any (ordinary) category.

(1) Define $\mathrm{Ar}_\square \mathcal{E}$
  - Objects: Morphisms $A \circ\!\!\rightarrow B$ in $\mathcal{E}$.
  - Morphisms: $\mathrm{Hom}_{\mathrm{Ar}_\square \mathcal{E}}(A \overset{g}{\circ\!\!\rightarrow} B, A' \overset{g'}{\circ\!\!\rightarrow} B') = \left\{ \text{distinguished squares} \begin{tikzcd} A \arrow[r, tail] \arrow[d, "g"'] & A' \arrow[d, "g'"] \\ B \arrow[r, tail] \arrow[ur, phantom, "\square"] & B' \end{tikzcd} \right\}$.

$\mathrm{Ar}_\Box\mathcal{M}$ is defined analogously.

(2) Define $\mathrm{Ar}_\triangle\mathcal{D}$
- Objects: Morphisms $A \to B$ in $\mathcal{D}$.
- Morphisms: $\mathrm{Hom}_{\mathrm{Ar}_\triangle\mathcal{D}}(A \xrightarrow{f} B, A' \xrightarrow{f'} B') = \left\{ \text{commutative squares} \quad \begin{tikzcd} A \arrow[r, "\cong"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r] & B' \end{tikzcd} \right\}$.

Definition 1.4 (CGW Category)**.** A *CGW category* $(\mathcal{C}, \varphi, c, k)$ consists of the following data

- A good double category $\mathcal{C} = (\mathcal{E}, \mathcal{M})$;
- An isomorphism of categories $\varphi\colon \mathrm{iso}\mathcal{M} \to \mathrm{iso}\mathcal{E}$ which is identity on objects;
- An equivalence of categories

$$k\colon \mathrm{Ar}_\Box\mathcal{E} \to \mathrm{Ar}_\triangle\mathcal{M} \qquad \text{and} \qquad c\colon \mathrm{Ar}_\Box\mathcal{M} \to \mathrm{Ar}_\triangle\mathcal{E};$$

satisfying the axioms:

(Z) *Basepoint object.* $\mathcal{C}$ contains an object O initial in both $\mathcal{E}$ and $\mathcal{M}$.

(I) *Stable Under Isomorphisms.* Let $\psi\colon A \to B$ be an isomorphism in ambient category $\mathcal{C}$. Then:
- ψ belongs to $\mathrm{iso}\mathcal{M}$, which we denote suggestively as $\psi\colon A \rightarrowtail B$.
- If $\mathcal{E}$ is a subcategory of $\mathcal{C}$, then $\varphi(\psi)\colon A \circ\!\!\to B$ corresponds to $\psi\colon A \to B$ in $\mathcal{C}$.
- If $\mathcal{E}^{\mathrm{op}}$ is a subcategory of $\mathcal{C}$, then $\varphi(\psi)\colon A \circ\!\!\to B$ corresponds to $\psi^{-1}\colon B \to A$ in $\mathcal{C}$.

(M) *Monicity.* Every morphism in $\mathcal{E}$ and $\mathcal{M}$ is monic.

(K) *Formal kernels and cokernels.* For any $f\colon A \rightarrowtail B$ in $\mathcal{M}$, there exists a *formal cokernel*, denoted $c(f)\colon \mathrm{coker}(f) \circ\!\!\to B$, and a distinguished square as below left.

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\Box"] & \mathrm{coker}(f) \arrow[d, "c(f)"] \\ A \arrow[r, tail, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\Box"] & A \arrow[d, "g"] \\ \ker(g) \arrow[r, tail, "k(g)"] & B \end{tikzcd}$$

Dually, for any $g\colon A \circ\!\!\to B$ in $\mathcal{E}$, there exists a *formal kernel*, denoted $k(g)\colon \ker(g) \rightarrowtail B$, and a distinguished square as above right.

These distinguished squares are unique up to isomorphism in the following sense. If there exists another $\mathcal{E}$-morphism $f'\colon C \circ\!\!\to B$ and a distinguished square

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\Box"] & C \arrow[d, "f'"] \\ A \arrow[r, tail, "f"] & B \end{tikzcd},$$

then there exists an isomorphism $\tau\colon \mathrm{coker}(f) \rightarrowtail C$ such that the rightmost square in

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\Box"] & \mathrm{coker}(f) \arrow[r, tail, "\tau"] \arrow[d, "c(f)"] \arrow[dr, phantom, "\Box"] & C \arrow[d, "f'"] \\ A \arrow[r, tail, "f"'] & B \arrow[r, tail, "1"'] & B \end{tikzcd}$$

commutes when regarded as a diagram in the ambient category $\mathcal{C}$. Notice that since distinguished squares interact well with isomorphisms, this implies the square is distingushed. Formal kernels are unique in the analogous sense.

There is a natural notion of structure-preserving functors and subcategories in the CGW context. A *CGW functor* of CGW categories is a double functor

$$F\colon (\mathcal{E}, \mathcal{M}) \to (\mathcal{E}', \mathcal{M}')$$

that commutes with the functors c and k in the following diagrams

$$\begin{tikzcd}
\mathrm{Ar}_{\square}\mathcal{E} \arrow[r, "k"] \arrow[d, "\mathrm{Ar}_{\square}F"'] & \mathrm{Ar}_{\triangle}\mathcal{M} \arrow[d, "\mathrm{Ar}_{\triangle}F"] \\
\mathrm{Ar}_{\square}\mathcal{E}' \arrow[r, "k'"] & \mathrm{Ar}_{\triangle}\mathcal{M}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Ar}_{\square}\mathcal{M} \arrow[r, "c"] \arrow[d, "\mathrm{Ar}_{\square}F"'] & \mathrm{Ar}_{\triangle}\mathcal{E} \arrow[d, "\mathrm{Ar}_{\triangle}F"] \\
\mathrm{Ar}_{\square}\mathcal{M}' \arrow[r, "c'"] & \mathrm{Ar}_{\triangle}\mathcal{E}'
\end{tikzcd}.$$

For a CGW category $(\mathcal{C}, \varphi, c, k)$ be a CGW category, a *CGW subcategory* is a sub-double category $\mathcal{D} \subseteq \mathcal{C}$ such that the obvious restrictions $(\mathcal{D}, \phi|_{\mathcal{D}}, c|_{\mathcal{D}}, k|_{\mathcal{D}})$ forms a CGW category.

Convention 1.5. When the context is clear, we will omit mentions of the CGW structure maps and refer to a CGW category $(\mathcal{C}, \varphi, c, k)$ by its underlying double category $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ or just $\mathcal{C}$.

We now discuss Axioms (K) and (I) in more detail below, followed by some illustrative examples.

Quotients in CGW Categories. A distinctive feature of CGW categories is that they are agnostic about whether formal cokernels arise from taking quotients in the *additive* setting (e.g. R-modules) or taking complements in the *non-additive setting* (e.g. finite sets, varieties etc.). Either way, the formal properties remain consistent. We adopt the following suggestive convention to reinforce this perspective.

Convention 1.6 ("Quotient")**.** We typically denote the formal cokernel of $f\colon A \rightarrowtail B$ as $\frac{B}{A}$, whenever the map f is clear from context. The object $\frac{B}{A}$ will typically be referred to as a *quotient*.

The following lemma is easy to check, and summarises a few key properties about quotients.

Lemma 1.7. *The following properties hold in any CGW category $\mathcal{C}$:*

(i) (Quotients respect Distinguished Squares). *Given any distinguished square*

$$\begin{tikzcd}
A \arrow[r, tail, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & B \arrow[d] \\
C \arrow[r, tail, "g"] & D
\end{tikzcd},$$

we have $\frac{B}{A} \cong \frac{D}{C}$.

(ii) (Quotients respect Filtrations). *Given $P_0 \overset{f_1}{\rightarrowtail} P_1 \overset{g_1}{\rightarrowtail} P_2$, one can construct the following diagram of distinguished squares*

$$\begin{tikzcd}
P_0 \arrow[r, tail, "f_1"] \arrow[dr, phantom, "\square"] & P_1 \arrow[r, tail, "g_1"] \arrow[dr, phantom, "\square"] & P_2 \\
O \arrow[r, tail] \arrow[u] & P_{1/0} \arrow[r, tail, "h_1"] \arrow[u, "f_2"] \arrow[dr, phantom, "\square"] & P_{2/0} \arrow[u, "g_2"'] \\
& O \arrow[r, tail] \arrow[u] & P_{2/1} \arrow[u, "h_2"']
\end{tikzcd}.$$

Isomorphisms in CGW Categories. In their original definition, CGW categories were only required to be double categories, not necessarily good. The hypothesis of goodness was used because it allows us to express what it means for distinguished squares to interact well with isomorphisms. Not only does this recover the original Axiom (I) in [CZ22, Definition 2.5], but it also plays a crucial role in our proofs (e.g. Lemmas B.1 and B.2). Ultimately, this is no real loss in generality since our definition still covers all the major examples in the original paper [CZ22]. The reason for this is the following observation.

Observation 1.8. Let $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ be any CGW category. Suppose

$$\begin{tikzcd}
A \arrow[r, tail, "f'"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\
C \arrow[r, tail, "f"] & D
\end{tikzcd} \tag{2}$$

defines a commutative diagram in the ambient category $\mathcal{C}$. If f and f' are isomorphisms, then Diagram (2) defines a pullback and a pushout in $\mathcal{C}$. The same holds if g and g' are isomorphisms.

We omit the proof since it is straightforward, and analogous to what was already worked out in [HMM$^+$22, Prop. 4.3]. Informally, Observation 1.8 says: pullback and pushout squares in the ambient category interact well with isomorphisms. As we shall see, this will be useful for verifying goodness.

Examples. We review in broad strokes some well-known examples of CGW categories, plus a couple new ones. Some further details can be found in [CZ22, §4].

Example 1.9 (Exact Categories)**.** For an exact category $\mathcal{C}$, define a CGW category $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ by setting

$$\mathcal{M} = \{\text{admissible monomorphisms}\} \qquad \mathcal{E} = \{\text{admissible epimorphisms}\}^{\mathrm{op}}.$$

The basepoint object is the zero object in $\mathcal{C}$.The distinguished squares are the biCartesian squares (= both pushouts and pullbacks in the ambient category $\mathcal{C}$). By Observation 1.8, $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ is a good double category. The equivalences k and c map admissible epis to kernels and admissible monics to cokernels, respectively. For more details on the other CGW axioms, see see [CZ22, Example 3.1].

Example 1.10 (Finite Sets)**.** Given FinSet, define a CGW category $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ by setting

$$\mathcal{E} = \mathcal{M} = \{\text{injections}\}.$$

The basepoint object O is the empty set, and the distinguished squares are the pushout squares. By Observation 1.8, $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ is a good double category. The equivalences c and k are given by taking any inclusion $A \hookrightarrow B$ to the inclusion $B \setminus A \hookrightarrow B$.

Example 1.11 (Varieties)**.** Given $\mathcal{V}\mathrm{ar}_k$, define $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ by setting

$$\mathcal{M} = \{\text{closed immersions}\} \qquad \mathcal{E} = \{\text{open immersions}\}.$$

The basepoint object O is the empty variety, and the distinguished squares are the pullback squares

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & B \arrow[d, circ, "g"] \\ C \arrow[r, tail, "f"'] & D \end{tikzcd}$$

in which $\mathrm{im} f \cup \mathrm{im} g = D$, which implies goodness of $\mathcal{C}$.[4] c and k takes a morphism to the inclusion of the complement. Axioms (I) and (M) follow from properties of closed and open immersions, while Axiom (K) holds as $D \setminus C \cong B \setminus A$ for any distinguished square

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & B \arrow[d, circ] \\ C \arrow[r, tail] & D \end{tikzcd}.$$

For those interested in model theory, the following example will be suggestive.

Example 1.12 (Definable Sets)**.** Fix Σ to be a first-order language and M a Σ-structure. The term *definable* will always mean with parameters from M. Following [KS00], denote $\mathrm{Def}(M)$ as the category with objects the definable sets of M and its powers M^n, and morphisms the definable functions. We upgrade this to a CGW category by setting

$$\mathcal{E} = \mathcal{M} = \{\text{Definable Injections}\}.$$

Axioms (I) and (M) are thus satisfied by definition. For Axiom (Z), set the basepoint object O as $\emptyset$. Define the distinguished squares as the pullback squares

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & B \arrow[d, circ, "g"] \\ C \arrow[r, tail, "f"'] & D \end{tikzcd}$$

[4]Notice that we get $\mathrm{im} f \cup \mathrm{im} g = D$ for free if either f or g are isomorphisms.

where $\mathrm{im}f \cup \mathrm{im}g = D$. Finally, any definable injection $f\colon C \rightarrowtail D$ can be mapped to its formal cokernel

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ" marking] & D \setminus f(C) \arrow[d, "c(f)"] \\ C \arrow[r, tail, "f"] \arrow[ur, phantom, "\square"] & D \end{tikzcd}$$

where $c(f)$ is the obvious inclusion of the complement. The analogous holds for the formal kernels.

Remark 1.13. For the curious non-logician: the category of definable sets $\mathrm{Def}(M)$ can be viewed as an abstraction of $\mathcal{V}\mathrm{ar}_k$. Consider $\mathbb{C}$ as an algebraically closed field with characteristic 0. In which case, the objects of $\mathrm{Def}(\mathbb{C})$ correspond to (Boolean combinations of) complex algebraic varieties.

Finally, let us mention another non-additive generalisation of exact categories: *proto-exact categories*, introduced by Dyckerhoff-Kapranov [DK19]. A particularly challenging example comes from [EJS20], which showed that the category of matroids form a proto-exact category, yielding a K-theory spectrum.

Informally, a *matroid* abstracts the notion of linear independence. It consists of a finite set E and a collection of subsets ("*flats*") that are closed under dependency – akin to how subspaces behave in vector spaces. Matroids bridge combinatorics and geometry, and have surprisingly deep links to algebraic geometry and Hodge Theory [Kat16, Bak18]. It is therefore very interesting that they also define a CGW category.

Example 1.14 (Matroids)**.** Let $M = (E, \mathcal{F}, \bullet_M)$ be a *pointed matroid*, where E is a finite set, and $\mathcal{F} \subseteq 2^E$ the set of flats of matroid M and $\bullet_M$ the distinguished base-point. A *strong map* of pointed matroids $f\colon M \to N$ is a function $f\colon E_M \to E_N$ such that $f(\bullet_M) = \bullet_N$ and $f^{-1}A \in \mathcal{F}(M)$ for all $A \in \mathcal{F}(N)$. By [EJS20, Lemma 2.12], pointed matroids and strong maps form a category $\mathrm{Mat}_\bullet$.

Next, denote $\widetilde{E} := E \setminus \{\bullet_M\}$. Given any $S \subseteq \widetilde{E}$, denote $M|S$ to be the *restriction of M to S* and M/S to be the *contraction of M to S* (for details, see e.g. [Oxl11] or [EJS20, §2].) We upgrade $\mathrm{Mat}_\bullet$ to a CGW category $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ by setting

$$\mathcal{M} = \{\text{strong maps that can be factored } N \xrightarrow{\sim} M|S \hookrightarrow M, \text{ for some } S \subseteq \widetilde{E}_M\}$$

$$\mathcal{E} = \{\text{strong maps that can be factored } M \twoheadrightarrow M/S \xrightarrow{\sim} N, \text{ for some } S \subseteq \widetilde{E}_M\}^{\mathrm{op}}.$$

Notice $\mathcal{E}$ is the *opposite* category of contractions, analogous to Example 1.9. In fact, one can apply [EJS20, Lemma 5.3] to check that $\mathcal{M}$ and $\mathcal{E}$ are closed under isomorphisms and composition, satisfying Axiom (I). Finally, let the distinguished squares be the biCartesian squares in $\mathrm{Mat}_\bullet$.

By [EJS20, Lemma 5.4]: a strong map f is monic in $\mathrm{Mat}_\bullet$ iff f is injective on the underlying set, and f is epi iff f is surjective. Thus, all morphisms in $\mathcal{M}$ and $\mathcal{E}$ are monic, and the pointed matroid $O := (\{*\}, *)$ is initial in both – satisfying Axioms (M) and (Z). In addition, translating [EJS20, Props. 5.7 and 5.8] to our setting: any $P \overset{i'}{\rightarrowtail} Q \overset{j'}{\circ\!\!\rightarrow} N$ or $P \overset{j}{\circ\!\!\rightarrow} M \overset{i}{\rightarrowtail} N$ can be completed into a distinguished square

$$\begin{tikzcd} P \arrow[r, tail, "i'"] \arrow[d, "j"'] & Q \arrow[d, "j'"] \\ M \arrow[r, tail, "i"] \arrow[ur, phantom, "\square"] & N \end{tikzcd}.$$

Setting $P = O$, this gives the formal kernels and cokernels required by Axiom (K), unique up to isomorphism by the biCartesian property.

1.2. **The K-Theory of pCGW categories.** The main result of [CZ22, §4] is that Quillen's Q-Construction [Qui73] can be applied to any CGW category to define a K-theory spectrum. However, this paper will focus on a particularly well-behaved class of CGW categories, which we call *pCGW categories.*

Informally, pCGW categories are CGW categories $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ whereby $\mathcal{M}$ is closed under a formal kind of pushout. This, of course, generalises the familiar fact that admissible monics are closed under pushouts in exact categories [Wei13, Exercise II.7.8], but it is instructive to understand why a generalisation is needed. Consider Example 1.10 where $\mathcal{M}$ is the category of finite sets and injections. In which case,

$$A \leftarrow \emptyset \rightarrow A \qquad \text{where } A \neq \emptyset$$

does not have a pushout in $\mathcal{M}$ since the map $A \coprod A \to A$ is not monic. Nonetheless, this issue can be circumvented by weakening the universal pushout property, as follows.

Definition 1.15 (Restricted Pushout, [CZ22, Def. 5.3])**.** Let $\mathcal{M}$ be a category whose morphsims are all monic. Suppose D is a span

$$C \leftarrow A \rightarrow B.$$

Define $\mathcal{M}_D$ to be the category of pullback squares in $\mathcal{M}$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & X \end{tikzcd},$$

where a morphism between squares is a natural transformation in which all components are equal to the identity except at X. The *restricted pushout* of D is the initial object in $\mathcal{M}_D$, which we denote by $B \star_A C$.

A useful fact is that restricted pushouts still behave functorially like a pushout in the following sense:

Fact 1.16. Consider the diagram

$$C \leftarrowtail A \rightarrowtail B \rightarrowtail B'$$

Then $B' \star_B (B \star_A C) \cong B' \star_A C$. More explicitly, the composite of restricted pushouts in Diagram (3) is the restricted pushout of the outer span.

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, tail] & B \arrow[r, tail] \arrow[d, tail] & B' \arrow[d, tail] \\ C \arrow[r, tail] & B \star_A C \arrow[r, tail] & B' \star_B (B \star_A C) \end{tikzcd} \tag{3}$$

We now introduce the definition of a pCGW category before reviewing a few key examples.

Definition 1.17 (pCGW Category)**.** Let $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ be a CGW category. We call $\mathcal{C}$ a *pCGW category* if $\mathcal{M}$ contains all restricted pushouts. In addition, restricted pushouts are required to satisfy:

(A) *Formal Direct Sums.* Denote the restricted pushout of $B \leftarrowtail O \rightarrowtail C$ as $B \oplus C := B \star_O C$, which we also call *formal direct sums*. Then, there exists a canonical pair of distinguished squares

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & B \arrow[d, "\circ"{marking, pos=0}, "q_B"] \\ C \arrow[r, tail, "p_C"] & B \oplus C \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & C \arrow[d, "\circ"{marking, pos=0}, "q_C"] \\ B \arrow[r, tail, "p_B"] & B \oplus C \end{tikzcd},$$

which we call *direct sum squares*.

(PQ) *Preserves quotients.* A restricted pushout

$$\begin{tikzcd} A \arrow[r, tail, "f"] \arrow[d, tail, "g"] & B \arrow[d, tail, "g'"] \\ C \arrow[r, tail, "f'"] & B \star_A C \end{tikzcd}$$

induces an isomorphism

$$\frac{B}{A} \cong \frac{B \star_A C}{C}.$$

(DS) *Compatibility with Distinguished Squares.* Given a diagram of distinguished squares

$$\begin{tikzcd} C \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & A \arrow[l, tail] \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & B \arrow[d, "\circ"{marking, pos=0}] \\ C' & A' \arrow[l, tail] \arrow[r, tail] & B' \end{tikzcd}$$

there is an induced map $B \star_A C \circ\!\!\longrightarrow B' \star_{A'} C'$ such that the two induced squares

$$\begin{tikzcd} B \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & B \star_A C \arrow[d, "\circ"{marking, pos=0}] \\ B' \arrow[r, tail] & B' \star_{A'} C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & B \star_A C \arrow[d, "\circ"{marking, pos=0}] \\ C' \arrow[r, tail] & B' \star_{A'} C \end{tikzcd}$$

are distinguished.

Remark 1.18. To avoid potential confusion, we emphasise that Definition 1.4 does not require $\mathcal{E}$ to contain all restricted pushouts – only $\mathcal{M}$.

Example 1.19 (Varieties)**.** Let $\star$ be the pushout of closed immersions in the category of schemes. Examining the work of [Sch05], it was noticed in [Cam19, §2] that a pushout of closed immersions of varieties produces a square of closed immersions of varieties. For clarity, we emphasise that these squares are not pushouts in the *category of closed immersions* but in the *entire category of schemes*. The fact that $\star$ satisfies Axioms (PQ) and (DS) follows from the universal property of pushouts. As for Axiom (A), let the direct sum squares be the coproduct squares with standard coprojection maps.

Example 1.20 (Exact Categories)**.** Let $\star$ be the usual pushout. Notice the pushout of any span of $\mathcal{M}$-morphisms is also a pullback, and thus defines the initial object in the relevant category of pullback squares. Axioms (PQ) and (DS) follow from the fact that pushouts preserve cokernels. For Axiom (A), recall that products and coproducts coincide in exact categories. Hence, let the distinguished squares correspond to the following biCartesian squares in the exact category

$$\begin{tikzcd} B \arrow[r, "p_B"] \arrow[d] & B \oplus C \arrow[d, "q_C"] \\ O \arrow[r] & C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "p_C"] \arrow[d] & B \oplus C \arrow[d, "q_B"] \\ O \arrow[r] & B \end{tikzcd},$$

where q_B, q_C are the natural projection maps, and p_B, p_C are the natural coprojection maps.

Example 1.21 (Definable Sets)**.** Assume M is a model of a theory $\mathbb{T}$ that eliminates imaginaries (and so, $\mathrm{Def}(M)$ admits coproducts and quotients of equivalence relations).[5] Given a span of definable injections $B \overset{f}{\leftarrowtail} A \overset{g}{\rightarrowtail} C$, define

$$B \star_A C := {B \coprod C}/_{\sim}$$

where $\sim$ is the finest equivalence relation generated by $f(a) \sim g(a), a \in A$. One easily checks that $\sim$ is expressible as a *definable* equivalence relation, with transitivity following from injectivity of f and g. In other words, $B \star_A C$ defines a pushout in $\mathrm{Def}(M)$, and so Axioms (PQ) and (DS) follow from the universal property. For Axiom (A), take the (definable) disjoint coproduct; notice the coproduct square is distinguished since $\mathcal{M} = \mathcal{E} = \{\text{Definable Injections}\}$.

Example 1.22 (Matroids)**.** There is a technical barrier. Suppose M_0 and M_1 are matroids with groundsets E_0 and E_1, and assume $M_0|T = M_1|T = N$ where $E_0 \cap E_1 = T$. An *amalgam* of M_0 and M_1 is a matroid M on $E_0 \cup E_1$ such that $M|E_0 = M_0$ and $M|E_1 = M_1$. Unfortunately, as noted in [Oxl11, §11.4], amalgams do not always exist. In our setting, this means an arbitrary span $M_0 \leftarrowtail N \rightarrowtail M_1$ may not be completable into a commutative square

$$\begin{tikzcd} N \arrow[r, tail] \arrow[d, tail] & M_0 \arrow[d, tail] \\ M_1 \arrow[r, tail] & M \end{tikzcd}$$

so long as we require that M has groundset $E_0 \cup E_1$. One solution is to relax this requirement, and allow M to have any groundset. The question then becomes: given a span $M_0 \leftarrowtail N \rightarrowtail M_1$, is it always possible to embed M_0 and M_1 into a larger matroid M? Can we regard this larger matroid as being initial in the sense of Definition 1.15? Further discussion of this problem is deferred to Section 5.4.

[5] Technically, to get coproducts we shall also need the hypothesis that $\mathbb{T}$ has a sort containing at least two distinct elements, but this is a fairly harmless condition. See e.g. [Har11].

We now setup the K-theory of pCGW categories via Waldhausen's $S_\bullet$-construction.

Construction 1.23 ($S_\bullet$-Construction)**.** Let $\mathcal{C}$ be a pCGW category. Define $S_\bullet\mathcal{C}$ to be the simplicial set with n-simplices $S_n\mathcal{C}$ given by flag diagrams

$$\begin{tikzcd}
M_{00} \arrow[r, tail] & M_{01} \arrow[r, tail] & M_{02} \arrow[r, tail] & \dots \arrow[r, tail] & M_{0n} \\
 & M_{11} \arrow[r, tail] \arrow[u, o->] & M_{12} \arrow[r, tail] \arrow[u, o->] & \dots \arrow[r, tail] & M_{1n} \arrow[u, o->] \\
 & & M_{22} \arrow[r, tail] \arrow[u, o->] & \dots \arrow[r, tail] & M_{2n} \arrow[u, o->] \\
 & & & \vdots \arrow[u, o->] & \vdots \arrow[u, o->] \\
 & & & & M_{nn} \arrow[u, o->]
\end{tikzcd}$$

subject to the conditions

(i) $M_{ii} = O$ for all i

(ii) Every subdiagram

$$\begin{tikzcd}
M_{ji} \arrow[r, tail] & M_{jl} \\
M_{ki} \arrow[r, tail] \arrow[u, o->] \arrow[ur, phantom, "\square"] & M_{kl} \arrow[u, o->]
\end{tikzcd}$$

for $j < k$ and $i < l$ is distinguished.

Since (formal) quotients respect filtrations (Lemma 1.7), we can represent an n-simplex as a sequence of $\mathcal{M}$-morphisms

$$O = M_0 \rightarrowtail M_1 \rightarrowtail M_2 \rightarrowtail \dots \rightarrowtail M_n$$

together with choices of quotients

$$M_{j/i} := \frac{M_j}{M_i} \qquad i < j.$$

Degeneracy maps are obtained by duplicating an M_i, face maps are obtained by forgetting an M_i, with the addendum that forgetting M_0 means factoring out by M_1.

Theorem 1.24 (Presentation Theorem)**.** *Let $\mathcal{C}$ be a pCGW category and define its K-theory spectrum*

$$K\mathcal{C} := \Omega|\mathcal{S}\mathcal{C}|,$$

with associated K-groups $K_n(\mathcal{C}) := \pi_n K\mathcal{C}$. Then $K_0(\mathcal{C})$ is the free abelian group generated by objects of $\mathcal{C}$ modulo the relation that for any distinguished square

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & B \arrow[d, o->] \\
D \arrow[r, tail] & C
\end{tikzcd},$$

we have $[D] + [B] = [A] + [C]$.

Proof. This translates Theorems 4.3 and 7.8 in [CZ22]. $\square$

Remark 1.25. In fact, Theorem 1.24 holds for any CGW category equipped with a formal direct sum satisfying Axiom (A), Definition 1.17. In the case of matroids, define $M_1 \oplus M_2$ to be the matroid on ground set $E_1 \coprod E_2$, with flats of the form $F_1 \coprod F_2$ where $F_1 \in \mathcal{F}(M_1)$ and $F_2 \in \mathcal{F}(M_2)$.

1.3. **Simplicial Loops & Fibers.** Re-examining Presentation Theorem 1.24, notice the loop space emerges naturally in our definition of $K\mathcal{C}$. Following [GG87, §2], we translate this construction to simplicial sets.

Convention 1.26. A simplicial set is a contravariant functor $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$. We sometimes write $X[n]$ as shorthand for $X([n])$.[6] If $A, B \in \Delta$, we write AB to mean the disjoint union of A followed by B, where elements of A are below those of B. A 0-simplex is sometimes called a *vertex*, a 1-simplex an *edge*.

To motivate, recall that the loop space ΩZ of a pointed topological space Z is the space of based loops $\mathrm{Map}(S^1, Z)$. Now fix a simplicial set X with basepoint $O \in X[0]$. A simplicial loop may look like two 1 simplices glued together at the end

$$O \overset{f}{\underset{g}{\rightrightarrows}} x,$$

while a homotopy of such loops may look like

$$\begin{tikzcd} O \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[rr, bend left=70, "f'"] \arrow[rr, bend right=70, "g'"'] & x \arrow[r, "h"] & y \end{tikzcd}$$

where we glue 2-simplices in the triangle $h \circ f = f'$ and $h \circ g = g'$ along a shared 1-simplex h. One can then extend this picture in a natural way to the higher homotopies as follows:

Construction 1.27 (Simplicial Loops)**.** Given any simplicial set X, define

$$\Omega X(A) := \lim_{\leftarrow} \left(\begin{tikzcd} \{O\} \arrow[r, hook] & X([0]) & X([0]A) \arrow[l] \arrow[d] \\ & X([0]A) \arrow[u] \arrow[r] & X(A) \end{tikzcd} \right).$$

Notice that Construction 1.27 works for any simplicial set X. This sets up the obvious definition:

Definition 1.28 (G-Construction)**.** Let $\mathcal{C}$ be a pCGW category. The *G-Construction* on $\mathcal{C}$ is defined by applying the Simplicial Loop Construction to $\mathcal{S}\mathcal{C}$,

$$G\mathcal{C} := \Omega S\mathcal{C}.$$

We define the associated K-groups as $K_n^G(\mathcal{C}) = \pi_n|\Omega\mathcal{S}\mathcal{C}|$.

The G-construction is well-defined for ordinary CGW categories. Nonetheless, we rely on the restricted pushouts of $\mathcal{M}$-morphisms to show that $G\mathcal{C} \simeq K\mathcal{C}$ (and thus define isomorphic K-groups). Our approach mirrors what was done by Gillet-Grayson [GG87], who established $G\mathcal{C} \simeq K\mathcal{C}$ for exact categories. A central notion in their argument is the so-called *right fiber*.

Definition 1.29 (Right Fiber)**.** Suppose $F\colon X \to Y$ is a map of simplicial sets, $A \in \Delta$ and $\rho \in Y(A)$. We define $\rho|F$ ("the right fiber over ρ") by

$$(\rho|F)(B) := \lim_{\leftarrow} \left(\begin{tikzcd} & & X(B) \arrow[d] \\ & Y(AB) \arrow[r] \arrow[d] & Y(B) \\ \{\rho\} \arrow[r, hook] & Y(A) & \end{tikzcd} \right).$$

We write $\rho|Y$ for $\rho|1_Y$. We regard $\rho|F$ as the simplicial analogue of the homotopy fiber $|F|$ over ρ.

[6] **(M:) Do we use this shorthand X_n? I feel like I eliminated this cos it was potentially confusing.**

We can now restate our problem. To show that $G\mathcal{C} \simeq K\mathcal{C}$, we need to show that geometric realisation and the loop space constructions commute up to homotopy equivalence, i.e.

$$|\Omega\mathcal{S}\mathcal{C}| \simeq \Omega|\mathcal{S}\mathcal{C}|.$$

The following key observation tells us when this happens.

Observation 1.30 (Key Observation)**.** For any simplicial set X, consider the commutative square

$$\begin{array}{ccc} \Omega X & \xrightarrow{t} & O|X \\ \Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle q} \\ O|X & \xrightarrow{q} & X \end{array} \quad . \tag{4}$$

where b and t are the projection maps and q the obvious face map. Then:

(i) $O|X$ is contractible.
(ii) $O|q \simeq \Omega X$
(iii) $|\Omega X| \simeq \Omega|X|$ iff this square is homotopy Cartesian.

Proof. (i) follows from [GG87, Lemma 1.4]. (ii) is clear from unpacking definitions. For (iii), first take the homotopy pullback P of

$$|O|X| \to |X| \leftarrow |O|X|$$

in the homotopy category of spaces. Taking the geometric realisation of Diagram (4), this yields a map $|\Omega X| \to P$, which is a homotopy equivalence iff Diagram (4) defines a homotopy pullback. Finally, since $O|X$ is contractible, deduce that the homotopy pullback P is equivalent to the homotopy pullback of

$$* \to |X| \leftarrow *,$$

which is the loop space $\Omega|X|$. □

Key Observation 1.30 suggests the following proof strategy. By item (iii), in order to show $G\mathcal{C} \simeq K\mathcal{C}$ it suffices to verify that the square

$$\begin{array}{ccc} \Omega\mathcal{S}\mathcal{C} & \xrightarrow{t} & O|\mathcal{S}\mathcal{C} \\ \Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle q} \\ O|\mathcal{S}\mathcal{C} & \xrightarrow{q} & \mathcal{S}\mathcal{C} \end{array} \quad . \tag{5}$$

is homotopy Cartesian. By item (ii), this is equivalent to showing that

$$\begin{array}{ccc} O|q & \xrightarrow{t} & O|\mathcal{S}\mathcal{C} \\ \Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle q} \\ O|\mathcal{S}\mathcal{C} & \xrightarrow{q} & \mathcal{S}\mathcal{C} \end{array} \quad . \tag{6}$$

is homotopy Cartesian. To show this, it suffices to analyse how $q\colon O|\mathcal{S}\mathcal{C} \to \mathcal{S}\mathcal{C}$ behaves on the induced simplicial fibers in the following sense:

Theorem 1.31 ([GG87], Theorem B')**.** *Suppose $F\colon X \to Y$ is a map of simplicial sets. Suppose for any $A \in \Delta$, any $\rho \in Y(A)$, and any $f\colon A' \to A$ such that the induced map*

$$\rho|F \to f^*\rho|F$$

is a homotopy equivalence. Then the square

$$\begin{array}{ccc} \rho|F & \longrightarrow & X \\ \Big\downarrow & & \Big\downarrow \\ \rho|Y & \longrightarrow & Y \end{array}$$

is homotopy Cartesian.

Convention 1.32. Following established usage in the literature [GG87, GG03], we shall refer to Theorem 1.31 as Theorem B', reflecting its role as the simplicial analogue of Quillen's Theorem B.

2. A Technical Result on Right Fibers

Convention 2.1. Hereafter, any category denoted $\mathcal{C}, \mathcal{D}$ should be assumed to be a pCGW category, unless stated otherwise.

The goal of this section is to prove Theorem 2.8, which essentially says: given a nice simplicial map $F\colon Y \hookrightarrow \mathcal{SC}$ where $\mathcal{C}$ is a pCGW category, the right fiber $O|F$ admits a nice description. The results here are technical, and are inspired by Grayson's framework of *dominant functors* [Gra87].

There are two main applications of Theorem 2.8. First, the key result that $G\mathcal{C} \simeq K\mathcal{C}$ is obtained as a straightforward corollary (Theorem 2.11). Second, it sets up the proof of Theorem 3.7, which gives an initial characterisation of the generators of $K_1(\mathcal{C})$; the details will be deferred to Section 3.

2.1. **H-Space Structure.** We introduce the new notion of a *moral subset*, which is similar to a simplicial subset of $\mathcal{SC}$ except we allow for a shift in dimensions.

Definition 2.2 (Moral Subset)**.** Let $\mathcal{C}$ be a pCGW category, and fix some $k \in \mathbb{N}$. A *k-moral subset* of $\mathcal{SC}$ is a simplicial map

$$F\colon Y \hookrightarrow \mathcal{SC}$$

such that:

(i) If $k = 0$, $Y[n] \subseteq \mathcal{SC}[n]\ \forall n$, and F is the inclusion map.

(ii) If $k \geq 1$, fix a base-point $O_Y := (O \rightarrowtail \ldots \rightarrowtail O) \in \mathcal{SC}[k-1]$. Then all n-simplices in Y are of the form

$$(O_Y \rightarrowtail K_0 \rightarrowtail K_1 \rightarrowtail \ldots \rightarrowtail K_n) \in \mathcal{SC}([k-1][n])$$

The map F forgets O_Y and factors the rest by K_0;

(iii) Y is closed under direct sums.

(iv) F commutes with face and degeneracy maps, i.e. for all $f\colon [m] \to [n]$ in Δ,

$$\begin{tikzcd} Y[m] \arrow[r, "Ff"] \arrow[d, "F_m"'] & Y[n] \arrow[d, "F_n"] \\ \mathcal{SC}[m] \arrow[r, "\mathcal{SC}f"] & \mathcal{SC}[n] \end{tikzcd} .$$

Notice a 0-moral subset is an honest simplicial subset of $\mathcal{SC}$.

Example 2.3. The quotient map $q\colon O|\mathcal{SC} \to \mathcal{SC}$ from Observation 1.30 is a 1-moral subset.

Construction 2.4 (Addition Map)**.** Fix $A \in \Delta$, where $A = [a]$. Given a k-moral subset $F\colon Y \to \mathcal{SC}$ and $\overline{M} \in \mathcal{SC}(A)$, represent a q-simplex W in $\overline{M}|F$ as

$$W = \begin{pmatrix} W_{\text{top}} \\ \hline\hline W_{\text{bot}} \end{pmatrix} = \begin{pmatrix} & & & & (O_Y & \rightarrowtail) & K_0 \rightarrowtail K_1 \rightarrowtail \ldots \rightarrowtail K_q \\ O \rightarrowtail M_1 & \rightarrowtail & \ldots & \rightarrowtail & M_a & \rightarrowtail & \overline{\overline{L_0 \rightarrowtail L_1 \rightarrowtail \ldots \rightarrowtail L_q}} \end{pmatrix} \tag{7}$$

where W_{top} is a simplex of Y, W_{bot} a simplex of $\mathcal{SC}$ and the double line represents the identity

$$\begin{tikzcd} O = F(K_0) \arrow[r, tail] \arrow[d, equal] & \ldots \arrow[r, tail] & F(K_q) \arrow[d, equal] \\ O = \frac{L_0}{L_0} \arrow[r, tail] & \ldots \arrow[r, tail] & \frac{L_q}{L_0} \end{tikzcd} .$$

Parentheses are placed around $(O_Y \rightarrowtail)$ in Equation (7) to indicate that when $k = 0$, we delete $O_Y \rightarrowtail$ from W_{top} and set $K_0 = O$.

Define addition $(W, W') \mapsto W + W'$ by setting

$$W + W' := \begin{pmatrix} & (O_Y & \rightarrowtail) & K_0 \oplus K_0' \rightarrowtail \ldots \rightarrowtail K_q \oplus K_q' \\ O \rightarrowtail \ldots \rightarrowtail & M_a & \rightarrowtail & \overline{\overline{L_0 \star_{M_a} L_0' \rightarrowtail \ldots \rightarrowtail L_q \star_{M_a} L_q'}} \end{pmatrix},$$

with the following quotients:

- For $K_i \oplus K'_i$:

$$\frac{K_i \oplus K'_i}{K_j \oplus K'_j} := \frac{K_i}{K_j} \oplus \frac{K'_i}{K'_j}.$$

- For $L_i \star_{M_a} L'_i$:

$$\frac{L_i \star_{M_a} L'_i}{M_j} := \begin{cases} \frac{L_i \star_{M_a} L'_i}{M_j} & \text{by Axiom (K), if } 1 \leq j < a \\ \frac{L_i}{M_a} \oplus \frac{L'_i}{M_a} & j = a \end{cases},$$

  and recursively:

$$\frac{L_i \star_{M_a} L'_i}{L_j \star_{M_a} L'_j} := F\left(\frac{K_i \oplus K'_i}{K_j \oplus K'_j}\right).$$

Claim 2.5. *The addition map above turns $|\overline{M}|F|$ into a homotopy associative and commutative H-space, making $\pi_0(\overline{M}|F)$ a monoid.*

Proof. This follows from a series of basic checks:

(a) *Well-definedness.* Choices of quotients are valid in $\mathcal{SC}$ by Lemma A.1. Further, the top row of $W + W'$ also defines a simplex in Y since moral subsets are closed under direct sums.

(b) *Identity.* The 0-simplex

$$\begin{pmatrix} & & & (O_Y \rightarrowtail) & \underline{\underline{O}} \\ O \rightarrowtail M_1 \rightarrowtail \ldots \rightarrowtail & & & M_a \overset{1}{\rightarrowtail} & M_a \end{pmatrix}$$

serves as additive identity. To see why, use the fact that restricted pushouts are initial to deduce $K \oplus O \cong K$ for any $K \in \mathcal{C}$, and $L_i \star_{M_a} M_a \cong L_i$.

(c) *Associativity and Commutativity.* Since restricted pushouts (and direct sums) are initial, they are associative and commutative up to natural isomorphism. These induce simplicial homotopies that show $|\overline{M}|F|$ is a homotopy associative and commutative H-space.

□

2.2. **The Main Result.** We now leverage the H-space structure of $O|F$ to describe the right fiber, assuming F satisfies a key technical condition, which we call *cofinality.*

Definition 2.6 (Cofinality)**.** Suppose $F\colon Y \hookrightarrow \mathcal{SC}$ is a k-moral subset. We say F has *cofinal image* if:

(i) **Case 1:** $k = 0$**.** For any $O \rightarrowtail M \in \mathcal{SC}$, there exists $T \in \mathrm{ob}\mathcal{C}$ such that

$$O \rightarrowtail M \oplus T \quad \in Y[1].$$

(ii) **Case 2:** $k \geq 1$**.** For any $C \in \mathrm{ob}\mathcal{C}$,

$$O_Y \rightarrowtail C \ \in Y[0] \qquad \text{and} \qquad O_Y \rightarrowtail O \rightarrowtail C \in Y[1].$$

Example 2.7. It is clear that the simplicial map $q\colon O|\mathcal{SC} \to \mathcal{SC}$ has cofinal image.

This sets up the main theorem of this section.

Theorem 2.8. *Let $\mathcal{C}$ be a pCGW category, and $F\colon Y \hookrightarrow \mathcal{SC}$ a moral subset with cofinal image. Then*

$$\begin{array}{ccc} O|F & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ O|\mathcal{SC} & \longrightarrow & \mathcal{SC} \end{array} \tag{8}$$

is a homotopy Cartesian square.

Proof. By Claim 2.5, $\pi_0(\overline{M}|F)$ is a monoid with respect to $+$. Say that F is *dominant* if $\pi_0(\overline{M}|F)$ is also a group given *any* $\overline{M} \in \mathcal{SC}([a])$ for *any* $[a] \in \Delta$. The proof then follows from two main implications.

- Step 1: If F has cofinal image, then F is dominant.
- Step 2: If F is dominant, then Diagram (8) is a homotopy Cartesian square.

Step 1: Cofinality implies dominance. Fixing some $\overline{M} \in \mathcal{SC}([a])$, $\pi_0(\overline{M}|F)$ can be presented as

- Generators: Vertices of $\overline{M}|F$, represented as

$$W = \left(\begin{array}{ccccccc} & & & & (O_Y & \rightarrowtail) & \underline{\underline{K_0}} \\ O \rightarrowtail M_1 & \rightarrowtail & \dots & \rightarrowtail & M_a & \rightarrowtail & N \end{array}\right)$$

- Relations: 1-simplices of $\overline{M}|F$.

As before, we omit the choice of quotients, but any two vertices with the same presentation will have isomorphic quotients, and are thus connected by a 1-simplex.[7] The generators can therefore be simplified to

$$W = \left(\begin{array}{ccc} (O & \rightarrowtail) & \underline{\underline{K_0}} \\ M_a & \rightarrowtail & N \end{array}\right). \tag{9}$$

To construct its inverse in $\pi_0(\overline{M}|F)$, it will be helpful to consider the case of 0-moral subsets separately.

Case I: $k = 0$. Since F has cofinal image, there exists some $T \in \mathrm{ob}\mathcal{C}$ such that

$$O \rightarrowtail \frac{N}{M_a} \oplus T \quad \in Y[1].$$

Setting $N' := M_a \oplus T$, it follows from Fact 1.16 that

$$N \star_{M_a} N' \cong N \oplus T.$$

Thus, define a new vertex W' whereby

$$W + \underbrace{\left(\begin{array}{ccc} & & \underline{\underline{O}} \\ M_a & \rightarrowtail & N' \end{array}\right)}_{W'} = \left(\begin{array}{ccc} & & \underline{\underline{O}} \\ M_a & \rightarrowtail & N \oplus T \end{array}\right).$$

To show W' is the inverse of W, we claim that $W + W'$ lies in the connected component of the additive identity. This is because

$$\left(\begin{array}{ccc} & & \underline{\underline{O \rightarrowtail \frac{N}{M_a} \oplus T}} \\ M_a & \rightarrowtail & M_a \rightarrowtail N \oplus T \end{array}\right)$$

defines a 1-simplex in $\pi_0(\overline{M}|F)$, since $\frac{N}{M_a} \oplus T \cong \frac{N \oplus T}{M_a}$ by Lemma A.2. Since the 0^{th} face map acts by forgetting the second M_a on the bottom row and factoring the top by $\frac{N}{M_a} \oplus T$, conclude that the 1-simplex connects the additive identity (see proof of Claim 2.5) to $W + W'$.

Case II: $k \geq 1$. The argument is similar. Given W as above, use cofinality to define a vertex W' whereby

$$W + \underbrace{\left(\begin{array}{ccc} O & \rightarrowtail & \underline{\underline{\frac{N}{M_a}}} \\ M_a & \rightarrowtail & M_a \oplus K_0 \end{array}\right)}_{W'} = \left(\begin{array}{ccc} O & \rightarrowtail & \underline{\underline{\frac{N}{M_a} \oplus K_0}} \\ M_a & \rightarrowtail & N \oplus K_0 \end{array}\right).$$

The sum $W + W'$ is then connected to the additive inverse via the 1-simplex

$$\left(\begin{array}{ccc} O & \rightarrowtail & \underline{\underline{O \rightarrowtail \frac{N}{M_a} \oplus K_0}} \\ M_a & \rightarrowtail & M_a \rightarrowtail N \oplus K_0 \end{array}\right).$$

In sum: given any $W \in \pi_0(\overline{M}|F)$, we can use cofinality to construct its inverse with respect to $+$. Since $\overline{M} \in \mathcal{SC}(A)$ and $A \in \Delta$ were chosen arbitrarily, this shows that F is dominant.

Step 2: $O|F$ as a homotopy pullback. Fix $\overline{M} \in \mathcal{SC}(A)$ for some $A \in \Delta$, and fix $f\colon A' \to A$. By Theorem B' (Theorem 1.31), it suffices to show that the base-change map $f^*\colon \overline{M}|F \to f^*\overline{M}|F$ is a homotopy equivalence. By Step 1, we can use the fact that $\pi_0(\overline{M}|F)$ is a group since F is dominant.

[7]The choice of 1-simplex is obvious, but justification takes some work. Informal proof sketch: construct the obvious simplex for the bottom row using the fact that formal quotients respect filtrations (Lemma 1.7) and are unique up to isomorphism (Axiom (K)). To show it is indeed a simplex, we must show the rightmost column of isomorphism squares are all distinguished; by goodness, it suffices to show they commute in the ambient category. We get the top square for free by Axiom (K). To see the square below it also commutes, use the fact that all $\mathcal{E}$-morphisms are monic (if $\mathcal{E} \subseteq \mathcal{C}$) or epi (if $\mathcal{E}^{\mathrm{op}} \subseteq \mathcal{C}$); this uses Axiom (M). Keep going.

Step 2a: A reduction. Let $g\colon [0] \to A'$ be any morphism in Δ. To show that f^* is a homotopy equivalence, it suffices to show that $(fg)^* = g^* f^*$ and g^* are. In fact, since both fg and g have $[0]$ as source, it suffices to show that

$$f_i\colon [0] \to A, \qquad f_i(0) = i \text{ for } i \in A,$$

induces a homotopy equivalence for any $A \in \Delta$. Notice f_i^* defines a map

$$f_i^*\colon \overline{M}|F \to O|F$$

since O is the only vertex of $\mathcal{SC}$.

Step 2b: The base case. Define a map

$$h_0\colon O|F \longrightarrow \overline{M}|F$$

$$\begin{pmatrix} (O_Y \rightarrowtail) & \underline{\underline{K_0 \rightarrowtail \ldots \rightarrowtail K_q}} \\ O \rightarrowtail & L_0 \rightarrowtail \ldots \rightarrowtail L_q \end{pmatrix} \longmapsto \begin{pmatrix} & (O_Y \rightarrowtail) & \underline{\underline{K_0 \rightarrowtail \ldots \rightarrowtail K_q}} \\ O \rightarrowtail \ldots \rightarrowtail & M_a \rightarrowtail & M_a \oplus L_0 \rightarrowtail \ldots \rightarrowtail M_a \oplus L_q \end{pmatrix}$$

with quotients defined as

- $\frac{M_a \oplus L_j}{M_a \oplus L_k} := \frac{L_j}{L_k} \left(= F\left(\frac{K_j}{K_k}\right)\right)$,
- $\frac{M_a \oplus L_j}{M_a} := L_j, \qquad \frac{M_a \oplus L_j}{M_i} := \frac{M_a}{M_i} \oplus L_j$

To show that f_i^* is a homotopy equivalence (for arbitrary i), it suffices to establish the following claim.

Claim 2.9. *The maps $f_i^* \circ h_0$ and h_0 are homotopy equivalences.*

Proof of Claim. Two main checks.

(i) *On $f_i^* \circ h_0$.* The map $f_i^* \circ h_0\colon O|F \to O|F$ sends

$$\begin{pmatrix} (O_Y \rightarrowtail) & \underline{\underline{K_0 \rightarrowtail \ldots \rightarrowtail K_q}} \\ O \rightarrowtail & L_0 \rightarrowtail \ldots \rightarrowtail L_q \end{pmatrix} \longmapsto \begin{pmatrix} (O_Y \rightarrowtail) & \underline{\underline{K_0 \rightarrowtail \ldots \rightarrowtail K_q}} \\ O \rightarrowtail & \frac{M_a}{M_i} \oplus L_0 \rightarrowtail \ldots \rightarrowtail \frac{M_a}{M_i} \oplus L_q \end{pmatrix}$$

for $0 \leq i \leq a$. Since $O|F$ is an H-space, we can reformulate $f_i^* \circ h_0$ more suggestively as

$$(f_i^* \circ h_0)(W) = W + \begin{pmatrix} (O_Y \rightarrowtail) & \underline{\underline{O}} \\ O \rightarrowtail & \frac{M_a}{M_i} \end{pmatrix}.$$

Since $\pi_0(O|F)$ is a group on the vertices of $O|F$, there exists a vertex V such that

$$\begin{pmatrix} (O_Y \rightarrowtail) & \underline{\underline{O}} \\ O \rightarrowtail & \frac{M_a}{M_i} \end{pmatrix} + V \sim \begin{pmatrix} (O_Y \rightarrowtail) & \underline{\underline{O}} \\ O \rightarrowtail & O \end{pmatrix}.$$

Define $h_1\colon O|F \to O|F$ as mapping

$$h_1(W) = W + V$$

for any simplex W. Since $+$ is homotopy associative and homotopy commutative, deduce that

$$f_i^* \circ h_0 \circ h_1 \sim 1, \qquad h_1 \circ f_i^* \circ h_0 \sim 1.$$

(ii) *On h_0.* Notice: $f_a^* \circ h_0$ is isomorphic to the identity map on $O|F$. It therefore suffices to show $h_0 \circ f_a^*$ is homotopic to the identity map 1 on $\overline{M}|F$. But this follows from the natural isomorphism

$$h_0 \circ f_a^* \cong 1,$$

induced by the isomorphism

$$M_a \oplus \frac{L_j}{M_a} \cong \frac{M_a \oplus L_j}{M_a} = L_j, \qquad \text{for all } j,$$

a consequence of Lemma A.2 and the choice of quotients by h_0. [Notice: the specific choice of quotients by h_0 is crucial; otherwise, the isomorphism may fail to hold since e.g. not all short exact sequences split.]

This completes proof of Claim 2.9. □

Step 3: Finish. Fix a k-moral subset $F\colon Y \to \mathcal{SC}$ that has cofinal image. Step 1 showed that F is dominant, i.e. $\pi_0(\overline{M}|F)$ is a group for any $M \in \mathcal{SC}([a])$. Step 2 combines this with Theorem B' to show that Diagram 8 is homotopy Cartesian. This proves the theorem. □

The following corollary justifies viewing the right fiber (Definition 1.29) as the simplicial analogue of a homotopy fiber, and will be useful later.

Corollary 2.10. *Suppose $F\colon Y \hookrightarrow \mathcal{SC}$ is k-moral subset with cofinal image. Then $|O|F|$ is homotopy equivalent to the homotopy fiber of $|F|$.*

Proof. By Observation 1.30, $O|\mathcal{SC}$ is contractible. Since the homotopy fiber of $|F|$ is the homotopy pullback of the cospan $* \to |\mathcal{SC}| \xleftarrow{|F|} Y$, the statement follows. □

In addition, we now obtain a key result of this paper regarding the G-construction.

Theorem 2.11. *Let $\mathcal{C}$ be a pCGW category. Then, there is a homotopy equivalence*

$$|G\mathcal{C}| \xrightarrow{\simeq} \Omega|S\mathcal{C}|.$$

Further, direct sum induces an H-space structure on $G\mathcal{C}$.

Proof. Let us review Key Observation 3.12. By item (iii), $|G\mathcal{C}| = |\Omega\mathcal{SC}| \simeq \Omega|\mathcal{SC}|$ if

$$\begin{tikzcd} \Omega\mathcal{SC} \arrow[r, "t"] \arrow[d, "b"] & O|\mathcal{SC} \arrow[d, "q"] \\ O|\mathcal{SC} \arrow[r, "q"] & \mathcal{SC} \end{tikzcd}$$

is homotopy Cartesian. By item (ii), we have $O|q \simeq \Omega\mathcal{SC}$. Since q is a 1-moral subset with cofinal image, the rest follows from Theorem 2.8; the H-space structure on $G\mathcal{C}$ comes from Construction 2.4. □

Remark 2.12. The equivalence in Theorem 2.11 established between $G\mathcal{C}$ and $K\mathcal{C}$ is one of topological spaces, not of infinite loop spaces or spectra.

Discussion 2.13 (Comparison with other proofs)**.** We are aware of two existing proofs of Theorem 2.11 for exact categories in the literature. While we all utilise Theorem B' in some way, key differences emerge in the proof strategies (even after accounting for the fact that our result extends to pCGW categories).

In broad strokes, Theorem B' says: if the induced map on fibers $\rho|F \to f^*\rho|F$ is a homotopy equivalence for *any* $f\colon A' \to A$, then $\rho|F$ is a homotopy pullback. In the original paper [GG87], Gillet-Grayson simplifies this condition by restricting to the maps $f_0, f_1\colon [0] \to [1]$; compare this with Step 2 of our proof of Theorem 2.8. Their argument is technical, but reflects the informal intuition: to understand how objects break into finitely many pieces, it suffices to understand how to break a single object into two.

A different proof appears in [Gra87, Thm 8.2], where Grayson uses Theorem B' to study *dominant* exact functors. By contrast, notice that we define dominance for *simplicial maps*. This adjustment accounts for the fact that $O|\mathcal{SC}$ *a priori* does not correspond to a pCGW category. Further, our definitions of moral subsets and cofinality were designed for our arguments to go through smoothly; no real attempts at generality were made. Whereas Grayson gives a full characterisation of dominant functors [Gra87, Thm 2.1], his proof does not translate well to our setting; for our purposes, it suffices to identify a sufficient criterion for dominance, i.e. cofinality.

3. GENERATORS OF $K_1(\mathcal{C})$

Having established Theorem 2.11, we now begin to deliver on our promise that the G-construction leads to an explicit description of $K_1(\mathcal{C})$. Section 3.1 unpacks the definition of the $G\mathcal{C}$-construction. Sections 3.2 and 3.3 work to obtain an increasingly sharp description of the generators of K_1; their analysis extends various results from [She94, She98, Nen96].

3.1. **Review of G-Construction.**

Construction 3.1 (G-Construction)**.** An n-simplex of $G\mathcal{C}$ is a pair of flag diagrams of the form

$$
\begin{tikzcd}
P_0 \arrow[r, tail] & P_1 \arrow[r, tail] & P_2 \arrow[r, tail] & \dots \arrow[r, tail] & P_n \\
& P_{1/0} \arrow[u, o->] \arrow[r, tail] & P_{2/0} \arrow[u, o->] \arrow[r, tail] & \dots \arrow[r, tail] & P_{n/0} \arrow[u, o->] \\
& & P_{2/1} \arrow[u, o->] \arrow[r, tail] & \dots \arrow[r, tail] & P_{n/1} \arrow[u, o->] \\
& & & \vdots \arrow[u, o->] & \vdots \arrow[u, o->] \\
& & & & P_{n/n-1} \arrow[u, o->]
\end{tikzcd}
\qquad
\begin{tikzcd}
P'_0 \arrow[r, tail] & P'_1 \arrow[r, tail] & P'_2 \arrow[r, tail] & \dots \arrow[r, tail] & P'_n \\
& P_{1/0} \arrow[u, o->] \arrow[r, tail] & P_{2/0} \arrow[u, o->] \arrow[r, tail] & \dots \arrow[r, tail] & P_{n/0} \arrow[u, o->] \\
& & P_{2/1} \arrow[u, o->] \arrow[r, tail] & \dots \arrow[r, tail] & P_{n/1} \arrow[u, o->] \\
& & & \vdots \arrow[u, o->] & \vdots \arrow[u, o->] \\
& & & & P_{n/n-1} \arrow[u, o->]
\end{tikzcd}
$$

subject to the conditions:

(i) Every quotient index square

$$
\begin{tikzcd}
P_{j/i} \arrow[r, tail] & P_{k/i} \\
P_{j/l} \arrow[u, o->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & P_{k/l} \arrow[u, o->]
\end{tikzcd}
\qquad \text{where } i < l < j < k
$$

is distinguished, and coincide in both flag diagrams.

(ii) Every quotient index triangle defines a distinguished square

$$
\begin{tikzcd}
P_{j/i} \arrow[r, tail] & P_{k/i} \\
O \arrow[u, o->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & P_{k/j} \arrow[u, o->]
\end{tikzcd}
\qquad \text{for any } i < j < k,
$$

and coincide in both flag diagrams.

(iii) Any $P_j \rightarrowtail P_k$ and $P'_j \rightarrowtail P'_k$ in the filtration can be completed into distinguished squares

$$
\begin{tikzcd}
P_j \arrow[r, tail] & P_k \\
O \arrow[u, o->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & P_{k/j} \arrow[u, o->]
\end{tikzcd}
,\quad
\begin{tikzcd}
P'_j \arrow[r, tail] & P'_k \\
O \arrow[u, o->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & P_{k/j} \arrow[u, o->]
\end{tikzcd}
\qquad \text{for any } i < j < k.
$$

Convention 3.2. Technically, an n-simplex of $G\mathcal{C}$ is a pair of $(n+1)$-simplices in $\mathcal{S}\mathcal{C}$

$$O \rightarrowtail P_0 \rightarrowtail \dots \rightarrowtail P_n \qquad\qquad O \rightarrowtail P'_0 \rightarrowtail \dots \rightarrowtail P'_n$$

such that the 0^{th} faces (= forgetting O and quotienting by P_0) agree. Here we omit the basepoint O for simplicity. In particular, a *vertex* of $G\mathcal{C}$ is a pair $(M, N) \in \mathcal{C} \times \mathcal{C}$, and an *edge* $(M, N) \to (M', N')$ is a pair of distinguished squares with identical quotient

$$
\left(
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & C \arrow[d, o->] \\
M \arrow[r, tail] & M'
\end{tikzcd}
,\;
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & C \arrow[d, o->] \\
N \arrow[r, tail] & N'
\end{tikzcd}
\right).
$$

We sometimes also represent a vertex as $\begin{pmatrix} M \\ N \end{pmatrix}$, without any double lines.

3.2. **Sherman Loops & Splitting.** We now extend Sherman's analysis of K_1 for exact categories [She94, She98] to pCGW categories. A guiding principle in his approach is that restricting to split exact sequences can clarify the general case. This section adapts this insight to pCGW categories, first by introducing some key definitions, before giving a first characterisation of the generators of $K_1(\mathcal{C})$ (Theorem 3.7).

Construction 3.3 (Sherman Loop)**.** A *Sherman triple* (α, β, θ) consists of the following data:

- Two $\mathcal{M}$-morphisms $A \overset{\alpha}{\rightarrowtail} B$, $A' \overset{\beta}{\rightarrowtail} B'$;
- An isomorphism $\theta\colon A \oplus C \oplus B' \rightarrowtail A' \oplus C' \oplus B$, where C and C' are specific choices of quotients

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d, "\delta"] \\ A \arrow[r, tail, "\alpha"] & B \end{tikzcd} \qquad \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C' \arrow[d, "\gamma"] \\ A' \arrow[r, tail, "\beta"] & B' \end{tikzcd} . \tag{10}$$

Its associated *Sherman loop* is the homotopy class $G(\alpha, \beta, \theta)$ in $\pi_1(|G\mathcal{C}|)$ represented by the loop

$$\begin{pmatrix} O \\ O \end{pmatrix} \to \begin{pmatrix} A \\ A \end{pmatrix} \to \begin{pmatrix} A \oplus C \oplus B' \\ B \oplus B' \end{pmatrix} \to \begin{pmatrix} A' \oplus C' \oplus B \\ B' \oplus B \end{pmatrix} \leftarrow \begin{pmatrix} A' \\ A' \end{pmatrix} \leftarrow \begin{pmatrix} O \\ O \end{pmatrix}, \tag{11}$$

where the arrows denote the obvious 1-simplices.[8]

Remark 3.4. Fix a pair of $\mathcal{M}$-morphisms α, β. One easily checks that any two Sherman triples (α, β, θ) and (α, β, θ') define the same Sherman loop in $K_1(\mathcal{C})$ up to homotopy – see e.g. [She94, §1].

Definition 3.5 (Split)**.** Call a distinguished square of the form

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d, "g"] \\ A \arrow[r, tail, "f"] & B \end{tikzcd}$$

an *exact square*.

(i) Call an exact square *split* if there exists an isomorphism

$$\Psi\colon B \rightarrowtail A \oplus C$$

such that the following squares commute

$$\begin{tikzcd} A \arrow[r, tail, "f"] \arrow[d, tail, "1_A"'] & B \arrow[d, tail, "\Psi"] \\ A \arrow[r, tail, "p_A"] & A \oplus C \end{tikzcd} \qquad \begin{tikzcd} B \arrow[d, "\varphi(\Psi)"'] & C \arrow[l, "g"'] \arrow[d, "1_C"] \\ A \oplus C & C \arrow[l, "q_C"] \end{tikzcd}$$

in $\mathcal{M}$ and $\mathcal{E}$ respectively, where p_A and q_C are the morphisms from the obvious direct sum square.

(ii) Call an $\mathcal{M}$-morphism $A \rightarrowtail B$ is called *split* if its corresponding exact square [obtained by taking formal quotients] is split.

Remark 3.6. When $\mathcal{C}$ is an exact category, and so $\varphi(\Psi)$ corresponds to

$$\Psi^{-1}\colon A \oplus C \to B$$

by Axiom (I). Hence, a split exact square means there exists an isomorphism Ψ such that $\Psi \circ f = p_A$ and $g \circ \Psi^{-1} = q_C$ in the ambient category. Since $g \circ \Psi^{-1} = q_C \iff g = q_C \circ \Psi$, this recovers the usual definition of a split exact sequence.

Theorem 3.7. *$K_1(\mathcal{C})$ is generated by Sherman loops $G(\alpha, \beta, \theta)$, where $\mathcal{C}$ is a pCGW category.*

Proof. The proof relies on two helper constructions: (i) $\mathcal{C}^{\oplus}$, a pCGW subcategory of $\mathcal{C}$; and (ii) $\widehat{G}\mathcal{C}$, a simplicial subset of $G\mathcal{C}$. Informally, they record the splitting data of $\mathcal{C}$ and $G\mathcal{C}$ respectively, and so admit particularly nice presentations on the level of π_1. One can then relate this to $\pi_1|G\mathcal{C}|$ to establish the theorem. Proceed in stages.

[8] *Details.* The middle arrow in Equation (11) applies θ on the top row and the canonical isomorphism $B \oplus B' \to B' \oplus B$ on the bottom; the rest of the 1-simplices are defined by applying Axiom (DS), Definition 1.17.

Step 0: Setup. This step introduces the two helper constructions, and records some preliminary observations.

Construction 3.8. Define $\mathcal{C}^{\oplus}$ to be the CGW subcategory of $\mathcal{C}$ whose exact squares are precisely the split exact squares of $\mathcal{C}$.

An easy check shows that $\mathcal{C}^{\oplus}$ is in fact a pCGW subcategory, and that the simplicial map

$$\mathcal{S}F\colon \mathcal{S}\mathcal{C}^{\oplus} \hookrightarrow \mathcal{S}\mathcal{C}$$

induced by the inclusion CGW functor $F\colon \mathcal{C}^{\oplus} \hookrightarrow \mathcal{C}$ is a 0-moral subset. In fact, since $\mathcal{S}\mathcal{C}^{\oplus}[1] = \mathcal{S}\mathcal{C}[1]$, $\mathcal{S}F$ has cofinal image. One can therefore apply Theorem 2.8 to construct a homotopy Cartesian diagram

$$\begin{array}{ccc} O|\mathcal{S}F & \xrightarrow{p} & \mathcal{S}\mathcal{C}^{\oplus} \\ \big\downarrow{\scriptstyle r} & & \big\downarrow{\scriptstyle \mathcal{S}F} \\ O|\mathcal{S}\mathcal{C} & \xrightarrow{q} & \mathcal{S}\mathcal{C} \end{array}. \tag{12}$$

Construction 3.9. Following the convention of Construction 2.4, represent an n-simplex in $G\mathcal{C}$ as

$$\begin{pmatrix} O \rightarrowtail & \underline{\underline{L_0 \rightarrowtail \ldots \rightarrowtail L_n}} \\ O \rightarrowtail & M_0 \rightarrowtail \ldots \rightarrowtail M_n \end{pmatrix}$$

Define $\widehat{G}\mathcal{C}$ to be the simplicial subset of $G\mathcal{C}$ where the $\mathcal{M}$-morphisms in the top row are all split.

Notice this mirrors $O|\mathcal{S}F$, except the top row of an n-simplex in $\widehat{G}\mathcal{C}$ is an $(n+1)$-simplex in $\mathcal{S}\mathcal{C}$ whereas the top row in $O|\mathcal{S}F$ is an n-simplex (whose $\mathcal{M}$-morphisms are also all split). Notice also that the simplices used to define $G(\alpha, \beta, \theta)$ in fact belong to $\widehat{G}\mathcal{C}$, and so $G(\alpha, \beta, \theta) \in \pi_1(|\widehat{G}\mathcal{C}|)$.

Step 1: Relating $\widehat{G}\mathcal{C}$ and $\mathcal{S}\mathcal{C}^{\oplus}$. We establish a series of homotopy equivalences.

Claim 3.10. *The map r induces a homotopy equivalence $O|p \to O|q$.*

Proof of Claim. As our setup, notice:

- $O|q \simeq \Omega\mathcal{S}\mathcal{C}$, by Observation 1.30.
- An n-simplex of $O|\mathcal{S}F$ is of the form

  $$W = \begin{pmatrix} & \underline{\underline{O = K_0 \rightarrowtail \ldots \rightarrowtail K_n}} \\ O \rightarrowtail & L_0 \rightarrowtail \ldots \rightarrowtail L_n \end{pmatrix};$$

  the map p acts by projecting the top row, the map r projects the bottom row. An n-simplex of $O|p$ is therefore a triple

  $$V = \begin{pmatrix} & \underline{\underline{O = K_0 \rightarrowtail \ldots \rightarrowtail K_n}} \\ O \rightarrowtail & \underline{\underline{L_0 \rightarrowtail \ldots \rightarrowtail L_n}} \\ O \rightarrowtail & M_0 \rightarrowtail \ldots \rightarrowtail M_n \end{pmatrix}$$

  where the double lines indicate equality of the corresponding quotients.

The induced map $r^*\colon O|p \to \Omega\mathcal{S}\mathcal{C}$ acts by forgetting the top row of V. We define its homotopy inverse as follows: given any n-simplex

$$\alpha = \begin{pmatrix} O \rightarrowtail \underline{\underline{L_0 \rightarrowtail \ldots \rightarrowtail L_n}} \\ O \rightarrowtail M_0 \rightarrowtail \ldots \rightarrowtail M_n \end{pmatrix} \in \Omega\mathcal{S}\mathcal{C},$$

define the map

$$
\begin{aligned}
s\colon \Omega\mathcal{SC} &\longrightarrow O|p \\
\alpha &\longmapsto \left(\begin{array}{ccc}
 & O = \dfrac{L_0}{L_0} \rightarrowtail \cdots\cdots\cdots\cdots \rightarrowtail \displaystyle\bigoplus_{m=0}^{i} \frac{L_m}{L_0} \rightarrowtail \cdots\cdots\cdots \rightarrowtail \bigoplus_{m=0}^{n} \frac{L_m}{L_0} \\
\hline\hline
O \rightarrowtail & L_0 \rightarrowtail \ldots \rightarrowtail L_i \oplus \left(\displaystyle\bigoplus_{m=0}^{i-1} \frac{L_m}{L_0}\right) \rightarrowtail \ldots \rightarrowtail L_n \oplus \left(\displaystyle\bigoplus_{m=0}^{n-1} \frac{L_m}{L_0}\right) \\
\hline\hline
O \rightarrowtail & M_0 \rightarrowtail \ldots \rightarrowtail M_i \oplus \left(\displaystyle\bigoplus_{m=0}^{i-1} \frac{M_m}{M_0}\right) \rightarrowtail \ldots \rightarrowtail M_n \oplus \left(\displaystyle\bigoplus_{m=0}^{n-1} \frac{M_m}{M_0}\right)
\end{array}\right),
\end{aligned}
$$

equipped with the obvious quotients to make $s(\alpha)$ a simplex of $O|p$.

To construct a simplicial homotopy from $1 \to r^* \circ s$, define

$$
\begin{aligned}
h\colon \Omega\mathcal{SC} \times [1]([n]) &\longrightarrow \Omega\mathcal{SC}([n]) \\
(\alpha, \beta) &\longmapsto \left(\begin{array}{ll}
O \rightarrowtail & L_0 \rightarrowtail \ldots \rightarrowtail L_i \rightarrowtail L_{i+1} \oplus \left(\displaystyle\bigoplus_{m=0}^{i} \frac{L_m}{L_0}\right) \rightarrowtail \ldots \rightarrowtail L_n \oplus \left(\displaystyle\bigoplus_{m=0}^{n-1} \frac{L_m}{L_0}\right) \\
\hline\hline
O \rightarrowtail & M_0 \rightarrowtail \ldots \rightarrowtail M_i \rightarrowtail M_{i+1} \oplus \left(\displaystyle\bigoplus_{m=0}^{i} \frac{M_m}{M_0}\right) \rightarrowtail \ldots \rightarrowtail M_n \oplus \left(\displaystyle\bigoplus_{m=0}^{n-1} \frac{M_m}{M_0}\right)
\end{array}\right)
\end{aligned}
$$

where i is determined by β via the rule:

- $i = n$, if $\beta = 0$ (interpreted as taking no direct sums);
- $i = 0$, if $\beta = 1$;
- Otherwise, i is the largest index where $\beta(m) = 0$.

In other words, $h(\alpha, \beta)$ acts on $\alpha \in \Omega\mathcal{SC}$ by adding direct sums of preceding quotients starting from the $(i+1)^{\mathrm{th}}$ place onwards, where i is determined by β. In particular, notice if $\beta = 0$ then $h(\alpha, \beta) = \alpha$, whereas if $\beta = 1$ then $h(\alpha, \beta) = r^* \circ s(\alpha)$. One can check that h is a simplicial map, and thus defines a simplicial homotopy $1 \to r^* \circ s$. A similar argument defines a simplicial homotopy $1 \to s \circ r^*$. □

Claim 3.10 is a simplicial analogue of the well-known fact that a homotopy Cartesian square of spaces induces weak equivalences on its homotopy fibers. As an application, the next two results now follow almost immediately, streamlining Sherman's original argument in [She98].

Corollary 3.11. *$\widehat{G}\mathcal{C}$ is homotopy equivalent to $O|p$ and $G\mathcal{C}$.*

Proof of Corollary. Observation 1.30 (ii) notes that $G\mathcal{C} \simeq O|q$, essentially by unpacking definitions. One can similarly verify that $\widehat{G}\mathcal{C} \simeq O|p$. By Claim 3.10, conclude that $G\mathcal{C} \simeq O|q \simeq O|p \simeq \widehat{G}\mathcal{C}$. □

In particular, consider the homotopy fiber sequence associated to $|O|p| \to |O|\mathcal{S}F| \xrightarrow{p} |\mathcal{SC}^{\oplus}|$. Applying Corollary 3.11 gives a basic but key observation.

Observation 3.12. The exact sequence

$$\pi_1(\Omega|\mathcal{SC}^{\oplus}|) \longrightarrow \pi_1(|O|p|) \longrightarrow \pi_1(|O|\mathcal{S}F|) \xrightarrow{p_*} \pi_1(|\mathcal{SC}^{\oplus}|) \tag{13}$$

may be reformulated as

$$\pi_1(\Omega|\mathcal{SC}^{\oplus}|) \longrightarrow \pi_1(|\widehat{G}\mathcal{C}|) \xrightarrow{v} \pi_1(|O|\mathcal{S}F|) \xrightarrow{p_*} \pi_1(|\mathcal{SC}^{\oplus}|), \tag{14}$$

for some map v induced by $O|p \simeq \widehat{G}\mathcal{C}$. For details on the exact sequence, see [GG87, Cor. 1.8].

Step 2: Generators of $\pi_1(|O|\mathcal{S}F|)$. The argument is standard – no surprises. The 1-simplices of the form

$$\begin{pmatrix} & O \rightarrowtail N \\ O \rightarrowtail & \overline{\overline{O \rightarrowtail N}} \end{pmatrix} \tag{15}$$

yield a maximal tree for the 1-skeleton of $|O|\mathcal{S}F|$, connecting the base-point of $O|\mathcal{S}F$ to any of its vertices. Thus by [Wei13, Lemma IV.3.4], the total set of 1-simplices of $O|\mathcal{S}F$ generate $\pi_1(|O|\mathcal{S}F|)$. We therefore represent the generators of $\pi_1(O|\mathcal{S}F)$ as

$$\begin{pmatrix} & O \rightarrowtail A \\ O \rightarrowtail & \overline{\overline{O \rightarrowtail A}} \end{pmatrix} \begin{pmatrix} & O \rightarrowtail C \\ O \rightarrowtail & \overline{\overline{A \rightarrowtail B}} \end{pmatrix} \begin{pmatrix} & O \rightarrowtail B \\ O \rightarrowtail & \overline{\overline{O \rightarrowtail B}} \end{pmatrix}^{-1}. \tag{16}$$

For simplicity, generators are sometimes represented just by the middle term above, since the other two edges can be recovered from the maximal tree.

Step 3: Constructing a Sherman Loop. Recall Homotopy Exact Sequence (14). Given $v(x) \in \pi_1(|O|\mathcal{S}F|)$ for any $x \in \pi(|\widehat{G}\mathcal{C}|)$, we can use its presentation from Step 2 to construct a Sherman Loop $G(\alpha, \beta, \theta)$.

- *The two* $\mathcal{M}$*-morphisms.* Since $O|\mathcal{S}F$ is an H-space, $v(x)$ can be expressed as a difference of two 1-simplices, let us say

$$\begin{pmatrix} & O \rightarrowtail C \\ O \rightarrowtail & \overline{\overline{A \rightarrowtail B}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} & O \rightarrowtail C' \\ O \rightarrowtail & \overline{\overline{A' \rightarrowtail B'}} \end{pmatrix}. \tag{17}$$

  This yields the $\mathcal{M}$-morphisms $\alpha\colon A \rightarrowtail B$ and $\beta\colon A' \rightarrowtail B'$.
- The isomorphism θ. Recall the map $p\colon O|\mathcal{S}F \to \mathcal{S}\mathcal{C}^{\oplus}$ acts by projection on the top row. Thus $p_* v(x) \in \pi_1(|\mathcal{S}\mathcal{C}^{\oplus}|)$ corresponds to the difference between

$$(O \rightarrowtail A)(O \rightarrowtail C)(O \rightarrowtail B')^{-1} \qquad \text{and} \qquad (O \rightarrowtail A')(O \rightarrowtail C')(O \rightarrowtail B')^{-1}. \tag{18}$$

  By Presentation Theorem 1.24, we know that $\pi_1(|\mathcal{S}\mathcal{C}^{\oplus}|) = K_0(\mathcal{C}^{\oplus})$, so let us rewrite the above as

$$[A] + [C] - [B] \qquad \text{and} \qquad [A'] + [C'] - [B']. \tag{19}$$

  We don't know if, e.g. $[A] + [C] = [B]$ in $K_0(\mathcal{C}^{\oplus})$ since $A \rightarrowtail B$ may not be split. Nonetheless, by exactness of Homotopy Sequence (14), we can deduce

$$[A] + [C] - [B] - \big([A'] + [C'] - [B']\big) = 0, \tag{20}$$

  and so

$$[A] + [C] + [B'] = [A'] + [C'] + [B]. \tag{21}$$

  Since $[M] = [N]$ in $K_0(\mathcal{C}^{\oplus})$ iff M and N are stably isomorphic, there exists some $Z \in \mathcal{C}$ such that

$$A \oplus C \oplus B' \oplus Z \cong A' \oplus C' \oplus B \oplus Z. \tag{22}$$

  Now notice that

$$\begin{pmatrix} & O \rightarrowtail Z \\ O \rightarrowtail & \overline{\overline{O \rightarrowtail Z}} \end{pmatrix} \begin{pmatrix} & O \rightarrowtail O \\ O \rightarrowtail & \overline{\overline{Z \rightarrowtail Z}} \end{pmatrix} \begin{pmatrix} & O \rightarrowtail Z \\ O \rightarrowtail & \overline{\overline{O \rightarrowtail Z}} \end{pmatrix}^{-1} \tag{23}$$

  is null-homotopic, Hence, we can always modify the representation of $v(x)$ by adding Loop (23) to Equation (18) without changing the homotopy class. As such, without loss of generality, assume $Z = O$, giving the isomorphism

$$\theta\colon A \oplus C \oplus B' \xrightarrow{\cong} A' \oplus C' \oplus B. \tag{24}$$

Step 4: A Reduction. The following claim tells us that any $x \in \pi_1(|\widehat{G}\mathcal{C}|)$ looks like a Sherman loop when viewed in $\pi_1(O|\mathcal{S}F)$.

Claim 3.13. *Given any $x \in \pi_1(|\widehat{G}\mathcal{C}|)$, there exists a Sherman loop $G(\alpha, \beta, \theta)$ such that $v(x)$ and $v(G(\alpha, \beta, \theta))$ are homotopic.*

Proof. Given $x \in \pi_1(|\widehat{G}\mathcal{C}|)$, construct a Sherman Loop $G(\alpha, \beta, \theta)$ as in Step 3. In particular, notice or recall that $G(\alpha, \beta, \theta) \in \pi_1(|\widehat{G}\mathcal{C}|)$, and so $v(G(\alpha, \beta, \theta))$ is well-defined.

Consider the following diagram in $\pi_1(|O|\mathcal{S}F|)$

$$\begin{tikzcd}
\left(O \rightarrowtail \frac{O}{\overline{A}}\right) \arrow[r, blue] \arrow[d, red] \arrow[dr, phantom, "(1)"] & \left(O \rightarrowtail \frac{O}{\overline{B \oplus B'}}\right) \arrow[r, blue] \arrow[dr, phantom, "(3)\quad(4)"] & \left(O \rightarrowtail \frac{O}{\overline{B' \oplus B}}\right) \arrow[dr, phantom, "(5)"] & \left(O \rightarrowtail \frac{O}{\overline{A'}}\right) \arrow[l, blue] \arrow[dl, red] \\
\left(O \rightarrowtail \frac{O}{\overline{B}}\right) \arrow[ur] \arrow[r, phantom, "(2)"] & \left(O \quad \frac{O}{\overline{O}}\right) \arrow[l, red] \arrow[r, red] \arrow[u] \arrow[ur] & \left(O \rightarrowtail \frac{O}{\overline{B'}}\right) \arrow[u] &
\end{tikzcd} \tag{25}$$

The edges of the diagram are obvious, and record various paths between vertices

$$\left(O \rightarrowtail \frac{O}{\overline{A}}\right) \dashrightarrow \left(O \rightarrowtail \frac{O}{\overline{A'}}\right), \tag{26}$$

e.g. by composing along the blue edges, by composing along the red edges, etc.

A couple of key observations. First, notice that all triangles in Diagram (25) define boundaries of 2-simplices, listed below.

$$(1) \left(O \rightarrowtail \frac{O \rightarrowtail C \rightarrowtail C \oplus B'}{\overline{\overline{A \rightarrowtail B \rightarrowtail B \oplus B'}}}\right),\ (2) \left(O \rightarrowtail \frac{O \rightarrowtail B \rightarrowtail B \oplus B'}{\overline{\overline{O \rightarrowtail B \rightarrowtail B \oplus B'}}}\right),\ (3) \left(O \rightarrowtail \frac{O \rightarrowtail B \oplus B' \rightarrowtail B' \oplus B}{\overline{\overline{O \rightarrowtail B \oplus B' \rightarrowtail B' \oplus B}}}\right)$$

$$(4) \left(O \rightarrowtail \frac{O \rightarrowtail B' \rightarrowtail B' \oplus B}{\overline{\overline{O \rightarrowtail B' \rightarrowtail B' \oplus B}}}\right),\ (5) \left(O \rightarrowtail \frac{O \rightarrowtail C' \rightarrowtail C' \oplus B}{\overline{\overline{A' \rightarrowtail B' \rightarrowtail B' \oplus B}}}\right).$$

Hence, the blue and red paths in Diagram (25) between the two vertices (26) are homotopic.

Second, $v(G(\alpha, \beta, \theta))$ corresponds to the loop

$$\left(O \rightarrowtail \frac{O \rightarrowtail A}{\overline{O \rightarrowtail A}}\right) \left(O \rightarrowtail \frac{O \rightarrowtail C \oplus B'}{\overline{A \rightarrowtail B \oplus B'}}\right) \left(O \rightarrowtail \frac{O \rightarrowtail O}{\overline{B \oplus B' \rightarrowtail B' \oplus B}}\right) \left(O \rightarrowtail \frac{O \rightarrowtail C \oplus B'}{\overline{A' \rightarrowtail B' \oplus B}}\right)^{-1} \left(O \rightarrowtail \frac{O \rightarrowtail A'}{\overline{O \rightarrowtail A'}}\right)^{-1}$$

whereas $v(x)$, the difference between two generators, corresponds to the loop

$$\left(O \rightarrowtail \frac{O \rightarrowtail A}{\overline{O \rightarrowtail A}}\right) \left(O \rightarrowtail \frac{O \rightarrowtail C}{\overline{A \rightarrowtail B}}\right) \left(O \rightarrowtail \frac{O \rightarrowtail B}{\overline{O \rightarrowtail B}}\right)^{-1} \left(O \rightarrowtail \frac{O \rightarrowtail B'}{\overline{O \rightarrowtail B'}}\right) \left(O \rightarrowtail \frac{O \rightarrowtail C'}{\overline{A' \rightarrowtail B'}}\right)^{-1} \left(O \rightarrowtail \frac{O \rightarrowtail A'}{\overline{O \rightarrowtail A'}}\right)^{-1}.$$

In particular, the loop $v(G(\alpha, \beta))$ corresponds to composing along the blue edges in Diagram (25) whereas $v(x)$ corresponds to composing along the red edges – which are homotopy equivalent by our previous observation. Conclude that $v(G(\alpha, \beta, \theta))$ and $v(x)$ have the same homotopy class. □

Step 5: Finish. Let x be an element of $K_1(\mathcal{C})$. By Theorem 2.11 and Corollary 3.11, we know

$$\Omega|\mathcal{S}\mathcal{C}| \simeq |G\mathcal{C}| \simeq |\widehat{G}\mathcal{C}|,$$

and so regard $x \in \pi_1(|\widehat{G}\mathcal{C}|)$. In particular, x is an element in Homotopy Sequence (14). By Claim 3.13, there exists a Sherman loop $G(\alpha, \beta, \theta)$ such that $v(G(\alpha, \beta, \theta)) = v(x)$ in $\pi_1(|O|\mathcal{S}F|)$. In other words, the difference $x - G(\alpha, \beta, \theta)$ vanishes in $\pi_1(|O|\mathcal{S}F|)$, and thus lies in the image of

$$K_1(\mathcal{C}^{\oplus}) = \pi_1(\Omega|\mathcal{S}\mathcal{C}^{\oplus}|) \longrightarrow \pi_1(|\widehat{G}\mathcal{C}|) = K_1(\mathcal{C}).$$

To finish, we quote a couple of technical facts about Sherman Loops whose proof we defer to Appendix B.1. By Lemmas B.2 and B.3, $K_1(\mathcal{C}^{\oplus})$ is generated by Sherman Loops, and thus so is its image in $K_1(\mathcal{C})$. By Lemma B.1, the sum of two Sherman Loops is still a Sherman Loop. As such, since

$$x = x - G(\alpha, \beta, \theta) + G(\alpha, \beta, \theta),$$

conclude that x is homotopic to a Sherman Loop in $K_1(\mathcal{C})$. □

Discussion 3.14. Our proof broadly follows Sherman's argument in [She98], except that Sherman often works on the level of geometric realisations whereas we work simplicially wherever possible.

The simplicial approach has its advantages. For instance, a rigorous proof that $|\widehat{G}\mathcal{C}| \simeq |G\mathcal{C}|$ becomes more involved via Sherman's approach. The original argument proceeds by defining a pair of exact functors

$$\Delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$$

$$\Delta'\colon \mathcal{C}^{\oplus} \to \mathcal{C}^{\oplus} \times \mathcal{C},$$

where Δ is the diagonal, and Δ' is the diagonal composed with the obvious inclusion map. Sherman then asserts that the cofiber of $\mathcal{S}\Delta$ is homotopy equivalent to $|\mathcal{S}\mathcal{C}|$, but there are subtleties. It is not generally true that $\mathrm{cofib}(\Delta) \simeq X$ for a diagonal map of spaces – e.g. consider $\Delta\colon S^1 \to S^1 \times S^1$, which embeds a circle S^1 into a torus. One potential remedy is to first prove $\mathrm{cofib}(\mathcal{S}\Delta)$ and $\mathcal{S}\mathcal{C}$ are equivalent as $\mathbb{E}_\infty$-spaces before recovering the desired result, but this invokes additional machinery. Alternatively, one can show $|\widehat{G}\mathcal{C}| \simeq |G\mathcal{C}|$ by explicit analysis of the relevant right fibers, as was done in Claim 3.10.

3.3. **Double Exact Squares.** Given an exact category, Nenashev [Nen98b, Nen96] shows that its K_1 is generated by so-called *double short exact sequences* – sharpening Sherman's original result. We adapt his analysis to the pCGW setting.

Definition 3.15 (Double Exact Squares)**.** A *double exact square* is a pair of distinguished squares with identical nodes

$$l := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\square"] & C \arrow[d, hook, "g_1"] \\ A \arrow[r, tail, "f_1"] & B \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\square"] & C \arrow[d, hook, "g_2"] \\ A \arrow[r, tail, "f_2"] & B \end{tikzcd} \right).$$

Since this defines an edge $(A, A) \to (B, B)$, any double exact square defines a loop

$$\begin{tikzcd} (A,A) \arrow[rr, "l"] & & (B,B) \\ & (O,O) \arrow[ul, "e(A)"] \arrow[ur, "e(B)"'] & \end{tikzcd} \tag{27}$$

where $e(A)$ and $e(B)$ are the obvious edges from the base-point, e.g.

$$e(A) := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\square"] & A \arrow[d, hook, "1_A"] \\ O \arrow[r, tail] & A \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\square"] & A \arrow[d, hook, "1_A"] \\ O \arrow[r, tail] & A \end{tikzcd} \right).$$

We call this the *canonical loop of* l, and denote it as $\mu(l)$. We denote $\langle l \rangle$ to be its homotopy class in $K_1(\mathcal{C})$.

As the following example illustrates, double exact squares generalise automorphisms in $\mathcal{C}$.

Example 3.16 (Automorphisms)**.** If $(A, \alpha) \in \mathrm{Aut}(\mathcal{C})$ is an automorphism, we write

$$l(\alpha) = \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\square"] & A \arrow[d, hook, "1_A"] \\ O \arrow[r, tail] & A \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\square"] & A \arrow[d, hook, "\alpha"] \\ O \arrow[r, tail] & A \end{tikzcd} \right).$$

To prove that $K_1(\mathcal{C})$ is generated by double exact squares, it suffices to show that any Sherman Loop is homotopic to the canonical loop of a double exact square; the rest follows from Theorem 3.7.

Theorem 3.17. *Let $\mathcal{C}$ be a pCGW category. Given any $x \in K_1(\mathcal{C})$, there exists a double exact square l such that $x = \mu(l)$.*

Proof. By Theorem 3.7, we may assume x is a Sherman Loop $G(\alpha, \beta, \theta)$ arising from a pair of exact squares

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & C \arrow[d, o->, "\delta"] \\ A \arrow[r, tail, "\alpha"] & B \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & C' \arrow[d, o->, "\delta'"] \\ A' \arrow[r, tail, "\alpha'"] & B' \end{tikzcd} \tag{28}$$

and an isomorphism $\theta\colon A \oplus C \oplus B' \overset{\cong}{\rightarrowtail} A' \oplus C' \oplus B$. To turn this into a double exact square, first construct the following distinguished squares

$$s_0 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & C \oplus C' \arrow[d, o->, "g_0"] \\ A \oplus A' \arrow[r, tail, "f_0"] & A \oplus C \oplus B' \end{tikzcd} \right) \quad , \quad s_1 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & C \oplus C' \arrow[d, o->, "g_1"] \\ A \oplus A' \arrow[r, tail, "f_1"] & A' \oplus C' \oplus B \end{tikzcd} \right) \tag{29}$$

$$f_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \alpha' \end{pmatrix}, \; f_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ \alpha & 0 \end{pmatrix}, \; g_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \delta' \end{pmatrix}, \; g_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ \delta & 0 \end{pmatrix}. \tag{30}$$

Then apply the isomorphism θ to define

$$l(x) := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & C \oplus C' \arrow[d, o->, "\theta \circ g_0"] \\ A \oplus A' \arrow[r, tail, "\theta \circ f_0"] & A' \oplus C' \oplus B \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & C \oplus C' \arrow[d, o->, "g_1"] \\ A \oplus A' \arrow[r, tail, "f_1"] & A' \oplus C' \oplus B \end{tikzcd} \right). \tag{31}$$

Remark 3.18. Some work is required to check that s_0 and s_1 are in fact distinguished, which we leave to the reader. This essentially follows by constructing the obvious exact squares via restricted pushouts, before taking their direct sums (see Lemma A.3).

Convention 3.19 (Coordinate-wise Definition of Maps)**.** Consider two exact squares

$$h_0 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & M'' \arrow[d, o->, "m_2"] \\ M' \arrow[r, tail, "m_1"] & M \end{tikzcd} \right) \qquad h_1 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & N'' \arrow[d, o->, "n_2"] \\ N' \arrow[r, tail, "n_1"] & N \end{tikzcd} \right).$$

If $M'' = N''$, we denote the corresponding 1-simplex as $(h_0, h_1)\colon (M', N') \to (M, N)$. On the other hand, if we wish to take their direct sum, then this will be denoted

$$h_0 \oplus h_1 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & M'' \oplus N'' \arrow[d, o->, "m_2 \oplus n_2"] \\ M' \oplus N' \arrow[r, tail, "m_1 \oplus n_1"] & M \oplus N \end{tikzcd} \right).$$

We can also "add" an object C to h_0 in the obvious way (see Lemma A.4) as follows:

$$h_0 \oplus C := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & M'' \oplus C \arrow[d, o->, "m_2 \oplus C"] \\ M' \arrow[r, tail, "m_1 \oplus C"] & M \oplus C \end{tikzcd} \right).$$

For more involved definitions, we use the matrix notation of Equation (30) to indicate, e.g. the morphism $f_1\colon A \oplus A' \rightarrowtail A' \oplus C' \oplus B$ is determined by $\alpha\colon A \rightarrowtail B$ and $1\colon A' \rightarrowtail A'$.

Let us resume our proof. Starting with a Sherman Loop $G(\alpha, \beta, \theta)$, we defined a double exact square $l(x)$ with corresponding loop $\mu(l(x))$. It thus remains to show that $\mu(l(x))$ is homotopic to $G(\alpha, \beta, \theta)$ in $K_1(\mathcal{C})$. In fact, since $K_1(\mathcal{C})$ is abelian, it suffices to show that they are freely homotopic. This is accomplished by the following series of lemmas.

Convention 3.20. To ease notation, denote $P := A \oplus C \oplus B'$ and $Q := A' \oplus C' \oplus B$.

Lemma 3.21. *$\mu(l(x))$ is freely homotopic to the loop*

$$\begin{tikzcd}
(P,Q) \arrow[rr, "{(\theta,1)}"] & & (Q,Q) \\
& (A \oplus A', A \oplus A') \arrow[ul, "{(s_0,s_1)}"] \arrow[ur, "{(s_1,s_1)}"'] &
\end{tikzcd} \tag{32}$$

Proof of Lemma. Consider the diagram

$$\begin{tikzcd}
& (P,Q) \arrow[dr, "{(\theta,1)}"] & \\
(A \oplus A', A \oplus A') \arrow[ur, "{(s_0,s_1)}"] \arrow[rr, bend left, "l(x)"] \arrow[rr, bend right, "{(s_1,s_1)}"'] & {} \arrow[u, phantom, "(1)"] \arrow[d, phantom, "(2)"] & (Q,Q) \\
& (O,O) \arrow[ul, "e(A\oplus A')"] \arrow[ur, "e(Q)"'] &
\end{tikzcd} \tag{33}$$

The statement follows from observing that Triangles (1) and (2) form 2-simplices. Triangle (2) is obvious. Triangle (1) is given by

$$\begin{tikzcd}
A \oplus A' \arrow[r, tail, "f_0"] & P \arrow[r, tail, "\theta"] & Q \\
& C \oplus C' \arrow[u, "g_0"] \arrow[r, tail, "1"] \arrow[ur, phantom, "\square"] & C \oplus C' \arrow[u, "\varphi(\theta)\circ g_0"'] \\
& & O \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \oplus A' \arrow[r, tail, "f_1"] & Q \arrow[r, tail, "1"] & Q \\
& C \oplus C' \arrow[u, "g_1"] \arrow[r, tail, "1"] \arrow[ur, phantom, "\square"] & C \oplus C' \arrow[u, "g_1"] \\
& & O \arrow[u]
\end{tikzcd}$$

$\square$

Lemma 3.22. *The loop $G(\alpha, \beta, \theta)$ is freely homotopic to the loop*

$$\begin{tikzcd}
(P, B \oplus B') \arrow[rr, "{(\theta,1)}"] & & (Q, B \oplus B') \\
& (A \oplus A', A \oplus A') \arrow[ul, "{(s_0,s)}"] \arrow[ur, "{(s_1,s)}"'] &
\end{tikzcd} \tag{34}$$

where s is the distinguished square

$$s := \left(\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C \oplus C' \arrow[d, "\delta\oplus\delta'"] \\
A \oplus A' \arrow[r, tail, "\alpha\oplus\alpha'"] & B \oplus B'
\end{tikzcd}\right) . \tag{35}$$

Proof of Lemma. Consider the diagram

$$
\begin{tikzcd}
(P, B \oplus B') \arrow[rrrr, "(\theta,1)"] & & & & (Q, B \oplus B') \\
(A, A) \arrow[u, "(a_0,b_0)"] \arrow[rr] & & (A \oplus A', A \oplus A') \arrow[ull, "(s_0,s)"'] \arrow[urr, "(s_1,s)"] \arrow[u, phantom, "(\star)"] & & (A', A') \arrow[ll] \arrow[u, "(a_1,b_1)"'] \\
& & (O, O) \arrow[ull, "e(A)"] \arrow[u, "e(A\oplus A')"] \arrow[urr, "e(A')"'] & &
\end{tikzcd}
\tag{36}
$$

where

$$
(a_0, b_0) := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \oplus B' \arrow[d, {Circle[open]}->] \\ A \arrow[r, tail, "1\oplus C\oplus B'"'] & A \oplus C \oplus B' \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \oplus B' \arrow[d, {Circle[open]}->] \\ A \arrow[r, tail, "\alpha\oplus B'"'] & B \oplus B' \end{tikzcd} \right)
$$

$$
(a_1, b_1) := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C' \oplus B \arrow[d, {Circle[open]}->] \\ A' \arrow[r, tail, "1\oplus C'\oplus B"'] & A' \oplus C' \oplus B \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C' \oplus B \arrow[d, {Circle[open]}->] \\ A' \arrow[r, tail, "B\oplus \alpha'"'] & B \oplus B' \end{tikzcd} \right)
$$

Notice: the outer loop of Diagram (36) is equivalent[9] to the Sherman Loop $G(\alpha, \beta, \theta)$ while the triangle $(\star)$ is Loop (34). It remains to check that all other triangles in Diagram (36) bound 2-simplices. The bottom two triangles are immediate. The top left triangle bounds

$$
\begin{tikzcd}
A \arrow[r, tail] & A \oplus A' \arrow[r, tail, "f_0"] & P \\
& A' \arrow[u, {Circle[open]}->] \arrow[r, tail, "C\oplus\alpha'"] \arrow[ur, phantom, "\square"] & C \oplus B' \arrow[u, {Circle[open]}->, "A\oplus 1"'] \\
& O \arrow[u, {Circle[open]}->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & C \oplus C' \arrow[u, {Circle[open]}->, "1\oplus\delta'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, tail] & A \oplus A' \arrow[r, tail, "\alpha\oplus\alpha'"] & B \oplus B' \\
& A' \arrow[u, {Circle[open]}->] \arrow[r, tail, "C\oplus\alpha'"] \arrow[ur, phantom, "\square"] & C \oplus B' \arrow[u, {Circle[open]}->, "\delta\oplus 1"'] \\
& O \arrow[u, {Circle[open]}->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & C \oplus C' \arrow[u, {Circle[open]}->, "1\oplus\delta'"']
\end{tikzcd}
$$

That this defines a 2-simplex essentially follows from Lemma 1.7 ("Quotients Respect Filtrations"); details are left to the reader. The top right triangle can be handled similarly. ☐

Lemma 3.23. *Denote $V := (B \oplus B')\star_{(A\oplus A')} Q$ and $W := C\oplus C'\oplus C\oplus C'$. Then, there exists distinguished squares of the form*

$$
t := \left(\begin{tikzcd} C \oplus C' \arrow[r, tail] \arrow[d, {Circle[open]}->, "g_1"'] \arrow[dr, phantom, "\square"] & W \arrow[d, {Circle[open]}->, "j_t"] \\ Q \arrow[r, tail, "h_t"] & V \end{tikzcd} \right) \qquad t' := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \oplus C' \arrow[d, {Circle[open]}->, "k_t"] \\ Q \arrow[r, tail, "h_t"] & V \end{tikzcd} \right)
\tag{37}
$$

$$
u := \left(\begin{tikzcd} C' \oplus C' \arrow[r, tail] \arrow[d, {Circle[open]}->, "\delta\oplus\delta'"'] \arrow[dr, phantom, "\square"] & W \arrow[d, {Circle[open]}->, "j_u"] \\ B \oplus B' \arrow[r, tail, "h_u"] & V \end{tikzcd} \right) \qquad u' := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \oplus C' \arrow[d, {Circle[open]}->, "k_u"] \\ B \oplus B' \arrow[r, tail, "h_u"] & V \end{tikzcd} \right).
\tag{38}
$$

[9] A small difference: the top edge here is $(\theta, 1)\colon (P, B\oplus B') \to (Q, B\oplus B')$, as opposed to $(\theta, \tau)\colon (P, B\oplus B') \to (Q, B'\oplus B)$, but the twist is already encoded in $(a_1, b_1)\colon (A', A') \to (Q, B \oplus B')$.

Proof. Applying Axiom (DS), Definition 1.17 to the diagram

$$
\begin{tikzcd}
C \oplus C' & O \arrow[l, tail] \arrow[r, tail] \arrow[d] & C \oplus C' \arrow[d, "g_1"] \\
B \oplus B' \arrow[from=u, "\delta \oplus \delta'"] \arrow[ur, phantom, "\square"] & A \oplus A' \arrow[l, tail, "\alpha \oplus \alpha'"'] \arrow[r, tail, "f_1"] \arrow[ur, phantom, "\square"] & Q
\end{tikzcd},
$$

conclude that there exists a distinguished square of the form t. Since t is distinguished, deduce that $\frac{V}{Q} \cong C \oplus C'$ (Lemma 1.7 (i)), and thus there exists a distinguished square of the form t'. The same argument shows there exists distinguished squares of the form u and u'. $\square$

Lemma 3.24. *Loops* (32) *and* (34) *are homotopic.*

Proof of Lemma. Recall $V := (B \oplus B') \star_{(A \oplus A')} Q$ and the definition of t' from Lemma 3.23. In addition, define $\tilde{s}$ as the horizontal composition of s_1 and t

$$
\tilde{s} := \left(\begin{tikzcd}
O \arrow[r, tail] \arrow[d] & C \oplus C' \arrow[r, tail] \arrow[d, "g_1"] & W \arrow[d, "j_t"] \\
A \oplus A' \arrow[r, tail, "f_1"] \arrow[ur, phantom, "\square"] & Q \arrow[r, tail, "h_t"] \arrow[ur, phantom, "\square"] & V
\end{tikzcd} \right).
$$

Notice: the $\mathcal{M}$-morphism $h_t \circ f_1$ is equal to

$$h_u \circ (\alpha \oplus \alpha') \colon A \oplus A' \rightarrowtail B \oplus B \rightarrowtail V$$

by construction of V, and so $\tilde{s}$ is equivalent to the horizontal composition of s and u. We can therefore construct a loop L that Loops (32) and (34) are both homotopic to. Consider the diagram

$$
\begin{tikzcd}
(P \oplus C \oplus C', V) \arrow[rr, "(\theta \oplus 1{,}1)"] & & (Q \oplus C \oplus C', V) \\
(P, Q) \arrow[u, "(1 \oplus C \oplus C'{,}t')"] \arrow[rr, "(\theta{,}1)"'] \arrow[urr, "(\theta \oplus C \oplus C'{,}t')"] \arrow[ur, phantom, "(3)"] & (4) & (Q, Q) \arrow[u, "(1 \oplus C \oplus C'{,}t')"'] \\
(1) & (A \oplus A', A \oplus A') \arrow[ul, "(s_0{,}s_1)"] \arrow[ur, "(s_1{,}s_1)"'] \arrow[uul, bend left=60, "(s_0 \oplus C \oplus C'{,}\ \tilde{s})"] \arrow[uur, bend right=60, "(s_1 \oplus C \oplus C'{,}\tilde{s})"'] & (2)
\end{tikzcd}
\tag{39}
$$

The purple edges are Loop (32), the blue edges form an outer loop, which we denote L. To show that the two loops are homotopic, it suffices to check that Triangles (1) - (4) are boundaries of 2-simplices – this is worked out explicitly in Claim B.4. Analogously, one can construct the diagram

$$
\begin{tikzcd}
(P \oplus C \oplus C', V) \arrow[rr, "(\theta\oplus 1{,}1)"] & & (Q \oplus C \oplus C', V) \\
(P, B \oplus B') \arrow[u, "(1\oplus C\oplus C'{,}u')"] \arrow[rr, "(\theta{,}1)"'] \arrow[urr, "(\theta\oplus C\oplus C'{,}\, u')"] \arrow[ur, phantom, "(3')"] & (4') & (Q, B \oplus B') \arrow[u, "(1\oplus C\oplus C'{,}u')"'] \\
(1') & (A \oplus A', A \oplus A') \arrow[ul, "(s_0{,}s)"'] \arrow[ur, "(s_1{,}s)"] \arrow[uul, bend left=60, "(s_0\oplus C\oplus C'{,}\ \tilde{s})"] \arrow[uur, bend right=60, "(s_1\oplus C\oplus C'{,}\tilde{s})"'] & (2')
\end{tikzcd}
\tag{40}
$$

where the red edges are Loop (34) and the blue edges are loop L. A similar check shows Triangles (1') - (4') are also boundaries of 2-simplices (details in Claim B.5). Conclude that Loop (32) and Loop (34) are both homotopic to loop L, and thus homotopic to each other as well. $\square$

Finish. Given any Sherman Loop $G(\alpha, \beta, \theta) \in K_1(\mathcal{C})$, we can construct another loop $\mu(l(x))$ where $l(x)$ is a double exact square. Recall that:

- Lemma 3.21 shows $\mu(l(x))$ is freely homotopic to Loop (32).
- Lemma 3.22 shows $G(\alpha, \beta, \theta)$ is freely homotopic to Loop (34).
- Lemma 3.24 shows Loops (32) and (34) are homotopic.

Since $K_1(\mathcal{C})$ is an abelian group, deduce that $G(\alpha, \beta, \theta)$ is homotopic to $\mu(l(x))$. Since $K_1(\mathcal{C})$ is generated by Sherman Loops (Theorem 3.7), conclude that $K_1(\mathcal{C})$ is generated by double exact squares. $\square$

Discussion 3.25. Although we were guided by Nenashev's proof in [Nen96], a direct translation to our setting does not work. The key issue is the isomorphism

$$B \oplus \frac{B}{A} \cong B \star_A B,$$

which holds in exact categories and is central to [Nen96, Lemma 2.6], but (as Inna Zakharevich pointed out to us) fails in $\mathcal{V}\mathrm{ar}_k$ – see Figure 2. To get around this, we construct the homotopies explicitly, verifying by hand that the chosen diagrams define valid 2-simplices. The methodological upshot: while pushouts in exact categories are arguably better behaved, restricted pushouts still retain enough good properties to make the analysis go through (cf. Lemmas 3.23 and A.4).

(A) (B)

FIGURE 2. $\mathbb{A}^1 \coprod (\mathbb{A}^1 \setminus \{*\})$ is not isomorphic to $\mathbb{A}^1 \star_{\{*\}} \mathbb{A}^1$.

4. RELATIONS OF $K_1(\mathcal{C})$

Having characterised the generators of $K_1(\mathcal{C})$ for pCGW categories, we now work to determine its relations. We first give a baseline characterisation in Proposition 4.1. We then sharpen our understanding by comparing this to other descriptions of K_1 by Nenashev [Nen98b, Nen98a] (for exact categories) and Zakharevich [Zak17c] (for Assemblers). A guiding observation is Warning 4.2, which highlights a technical subtlety regarding the composition of 1-simplices in K_1. Interestingly, this brings into focus an apparent discrepancy between our account and Zakharevich's regarding the correct relations of K_1.

4.1. **A Baseline Argument.** Observe that any double exact square lies in the base-point component of $G\mathcal{C}$, which we denote $G\mathcal{C}^o$. Since double exact squares generate $K_1(\mathcal{C})$ (Theorem 3.17), it follows that $K_1(\mathcal{C}) = \pi_1(G\mathcal{C}^o)$. One can therefore apply the standard description of the fundamental group of a connected simplicial spsace to get the following presentation.

Proposition 4.1. *$K_1(\mathcal{C})$ is generated by isomorphism classes of double exact squares $\langle f\rangle$ modulo the following relations:*

(B1) Given any $A \in \mathcal{C}$, the standard edge $\mathrm{e}(A)\colon (O,O) \to (A,A)$ of $G\mathcal{C}$ vanishes. That is,

$$\left\langle \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & A \arrow[d, circ, "1"] \\ O \arrow[r, tail] & A \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & A \arrow[d, circ, "1"] \\ O \arrow[r, tail] & A \end{tikzcd} \right) \right\rangle = 0.$$

(B2) Given any $A \in \mathcal{C}$, the degenerate 1-simplex $\mathrm{id}_A\colon (A,A) \to (A,A)$ of $G\mathcal{C}$ vanishes. That is,

$$\left\langle \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & O \arrow[d, circ] \\ A \arrow[r, tail, "1"] & A \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & O \arrow[d, circ] \\ A \arrow[r, tail, "1"] & A \end{tikzcd} \right) \right\rangle = 0.$$

(B3) Given double exact squares of the form

$$l_A := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[d, circ, "g_A"] \\ A \arrow[r, tail, "f_A"] & B \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[d, circ, "g'_A"] \\ A \arrow[r, tail, "f'_A"] & B \end{tikzcd} \right) \quad l_B := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & \frac{C}{B} \arrow[d, circ, "g_B"] \\ B \arrow[r, tail, "f_B"] & C \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & \frac{C}{B} \arrow[d, circ, "g'_B"] \\ B \arrow[r, tail, "f'_B"] & C \end{tikzcd} \right) \tag{41}$$

$$l_C := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & \frac{C}{A} \arrow[d, circ, "g_C"] \\ A \arrow[r, tail, "f_C"] & C \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & \frac{C}{A} \arrow[d, circ, "g'_C"] \\ A \arrow[r, tail, "f'_C"] & C \end{tikzcd} \right) \tag{42}$$

that assemble into a 2-simplex in $G\mathcal{C}$

$$\begin{tikzcd} A \arrow[r, tail, "f_A"] & B \arrow[r, tail, "f_B"] \arrow[dr, phantom, "\square"] & C \\ & \frac{B}{A} \arrow[u, circ, "g_A"] \arrow[r, tail, "h_1"'] \arrow[dr, phantom, "\square"] & \frac{C}{A} \arrow[u, circ, "g_C"'] \\ & O \arrow[u, circ] \arrow[r, tail] & \frac{C}{B} \arrow[u, circ, "h_2"'] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, tail, "f'_A"] & B \arrow[r, tail, "f'_B"] \arrow[dr, phantom, "\square"] & C \\ & \frac{B}{A} \arrow[u, circ, "g'_A"] \arrow[r, tail, "h_1"'] \arrow[dr, phantom, "\square"] & \frac{C}{A} \arrow[u, circ, "g'_C"'] \\ & O \arrow[u, circ] \arrow[r, tail] & \frac{C}{B} \arrow[u, circ, "h_2"'] \end{tikzcd} \tag{43}$$

we have that

$$\langle l_A\rangle + \langle l_B\rangle = \langle l_C\rangle.$$

Proof. Let X be a connected simplicial set with Γ as the maximal tree for its 1-skeleton. It is well-known that $\pi_1|X|$ has the following presentation

$$\pi_1|X| := \pi_0(X[1]) \Big/ \begin{array}{c} \langle t\rangle = 0 \text{ if } t\in\Gamma, \text{ and } \langle \mathrm{id}_A\rangle = 0 \text{ for any degenerate 1-simplex} \\ d_1(x) = d_2(x) + d_0(x), \quad \forall x \in \pi_0(X[2]) \end{array} .$$

In our case, $X = G\mathcal{C}^o$. In particular:

- $\pi_0(G\mathcal{C}^o[1])$ is the set of isomorphism classes of 1-simplices of $G\mathcal{C}^o$. By Theorem 3.17, we may restrict this to the isomorphism classes of double exact squares.
- The obvious set of 1-simplices $(O,O) \to (A,A')$ defines a maximal subtree of the 1-skeleton of $G\mathcal{C}^o$. Since we restrict to just the double exact squares for our generators, we may assume $A = A'$.
- Any $x \in \pi_0(G\mathcal{C}^o[2])$ can be represented as Diagram 43, where $d_0(x) = l_B$, $d_2(x) = l_A$ and $d_1 = l_C$.

The proposition then follows from our earlier observation that $K_1(\mathcal{C}) = \pi_1(G\mathcal{C}^o)$. ☐

Warning 4.2 (Composition of 1-simplices)**.** Proposition 4.1 does *not* assert that any three double exact squares l_A, l_B, l_C satisfy

$$\langle l_A\rangle + \langle l_B\rangle = \langle l_C\rangle, \qquad \text{whenever } \; f_A \circ f_B = f_C \; \text{ and } \; g_A \circ g_B = g_C.$$

Why? While composition in $\mathcal{SC}$ yields a pair of flag diagrams, these define a 2-simplex in $G\mathcal{C}$ only if the quotient index triangles in Equation (43)

$$\left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ" description] \arrow[dr, phantom, "\square"] & \frac{C}{B} \arrow[d, "h_2"] \\ \frac{B}{A} \arrow[r, tail, "h_1"] & \frac{C}{A} \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ" description] \arrow[dr, phantom, "\square"] & \frac{C}{B} \arrow[d, "h_2"] \\ \frac{B}{A} \arrow[r, tail, "h_1"] & \frac{C}{A} \end{tikzcd} \right)$$

are identical in both diagrams.

Keeping Warning 4.2 in mind will help us appreciate the work done in the subsequent sections. On one level, we extend Nenashev's work on exact categories [Nen98b, Nen98a] to our setting, providing yet another characterisation of $K_1(\mathcal{C})$. On another level, Nenashev's presentation clarifies *how* composition of 1-simplices split in K_1, illuminating the difference between our approach and Zakharevich's.

4.2. **Admissible Triples.** A *triangle contour* $\mathcal{T}$ in $G\mathcal{C}$

$$\begin{tikzcd} & (P_1, P_1') \arrow[dr, "e_1"] & \\ (P_0, P_0') \arrow[ur, "e_0"] \arrow[rr, "e_2"] & & (P_2, P_2') \end{tikzcd} \tag{44}$$

is given by three pairs of distinguished squares of the form

$$e_0 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & P_{1/0} \arrow[d, "\alpha_{1/0,1}"] \\ P_0 \arrow[r, tail, "\alpha_{0,1}"] & P_1 \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & P_{1/0} \arrow[d, "\alpha'_{1/0,1}"] \\ P_0' \arrow[r, tail, "\alpha'_{0,1}"] & P_1' \end{tikzcd} \right) \quad e_1 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & P_{2/1} \arrow[d, "\alpha_{2/1,2}"] \\ P_1 \arrow[r, tail, "\alpha_{1,2}"] & P_2 \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & P_{2/1} \arrow[d, "\alpha'_{2/1,2}"] \\ P_1' \arrow[r, tail, "\alpha'_{1,2}"] & P_2' \end{tikzcd} \right)$$

$$e_2 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & P_{2/0} \arrow[d, "\alpha_{2/0,2}"] \\ P_0 \arrow[r, tail, "\alpha_{0,2}"] & P_2 \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & P_{2/0} \arrow[d, "\alpha'_{2/0,2}"] \\ P_0' \arrow[r, tail, "\alpha'_{0,2}"] & P_2' \end{tikzcd} \right).$$

In particular, given any vertex $(A, A') \in G\mathcal{C}$, one can apply Lemma A.4 to construct a new triangle contour $(A, A') \oplus \mathcal{T}$ by formal direct sum

$$\begin{tikzcd}
 & (P_1 \oplus A, P_1' \oplus A') \arrow[dr, "e_1 \oplus (A,A')"] & \\
(P_0 \oplus A, P_0' \oplus A') \arrow[ur, "e_0 \oplus (A,A')"] \arrow[rr, "e_2 \oplus (A,A')"] & & (P_2 \oplus A, P_2' \oplus A')
\end{tikzcd} \tag{45}$$

Definition 4.3 (Admissible Triple)**.** We call a triple $\tau = (e_0, e_1, e_2)$ of the above form *admissible* if

$$\alpha_{1,2} \circ \alpha_{0,1} = \alpha_{0,2} \qquad \text{and} \qquad \alpha'_{1,2} \circ \alpha'_{0,1} = \alpha'_{0,2} \ .$$

By Lemma 1.7, τ can be completed to the following pair of diagrams:

$$\begin{tikzcd}
P_0 \arrow[r, tail, "\alpha_{0,1}"] & P_1 \arrow[r, tail, "\alpha_{1,2}"] & P_2 \\
 & P_{1/0} \arrow[u, "\alpha_{1/0,1}"] \arrow[r, tail, "\alpha_{1/0,2/0}"'] \arrow[ur, phantom, "\square"] & P_{2/0} \arrow[u, "\alpha_{2/0,2}"'] \\
 & O \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & P_{2/1} \arrow[u, "\alpha_{2/1,2/0}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
P_0' \arrow[r, tail, "\alpha'_{0,1}"] & P_1' \arrow[r, tail, "\alpha'_{1,2}"] & P_2' \\
 & P_{1/0} \arrow[u, "\alpha'_{1/0,1}"] \arrow[r, tail, "\alpha'_{1/0,2/0}"'] \arrow[ur, phantom, "\square"] & P_{2/0} \arrow[u, "\alpha'_{2/0,2}"'] \\
 & O \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & P_{2/1} \arrow[u, "\alpha'_{2/1,2/0}"']
\end{tikzcd} \tag{46}$$

In particular, define

$$l(\tau) := \left(\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & P_{2/1} \arrow[d, "\alpha_{2/1,2/0}"] \\
P_{1/0} \arrow[r, tail, "\alpha_{1/0,2/0}"] & P_{2/0}
\end{tikzcd}
\qquad
\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & P_{2/1} \arrow[d, "\alpha'_{2/1,2/0}"] \\
P_{1/0} \arrow[r, tail, "\alpha'_{1/0,2/0}"] & P_{2/0}
\end{tikzcd} \right) \tag{47}$$

to be the *double exact square associated to admissible triple* τ.

Any admissible triple $\tau = (e_0, e_1, e_2)$ defines a loop $e_0 e_1 e_2^{-1}$, which we also denote using τ. Notice: if

$$\alpha_{1/0,2/0} = \alpha'_{1/0,2/0} \qquad \text{and} \qquad \alpha_{2/1,2/0} = \alpha'_{2/1,2/0}$$

in Diagram (46), then the loop τ bounds a 2-simplex in $G\mathcal{C}$ since the quotient index triangles of both diagrams now coincide. However, even when this condition does not hold, we can still say something meaningful about the (free) homotopy class of τ.

Lemma 4.4. *Setup:*

- *Let $\tau = (e_0, e_1, e_2)$ be an admissible triple.*
- *Let $l(\tau)$ be the double exact square associated to τ, and $\mu(l(\tau))$ be its canonical loop.*

Then the loop $\tau = e_0 e_1 e_2^{-1}$ is freely homotopic to

$$(P_2, P_2') \oplus \mu(l(\tau)).$$

Proof. Construct the obvious diagram

$$\begin{tikzcd}
(P_2 \oplus P_{1/0}, P_2' \oplus P_{1/0}) \arrow[rrrr, red, "(P_2, P_2') \oplus l(\tau)"] & & & & (P_2 \oplus P_{2/0}, P_2' \oplus P_{2/0}) \\
 & (P_1, P_1') \arrow[ul] \arrow[urrr, bend left=10] \arrow[rr, blue, "e_1"] \arrow[u, phantom, "(1)"] \arrow[l, phantom, "(3)"] & & (P_2, P_2') \arrow[ur] \arrow[u, phantom, "(2)"] \arrow[r, phantom, "(4)"] & \\
 & & (P_0, P_0') \arrow[ul, blue, "e_0"] \arrow[ur, blue, "e_2"'] \arrow[uull, bend left] \arrow[uurr, bend right] \arrow[d, "e_2"] \arrow[ll, phantom, "(5)"] \arrow[rr, phantom, "(6)"] & & \\
 & & (P_2, P_2') \arrow[uuull, red, bend left, "(P_2, P_2') \oplus e(P_{1/0})"] \arrow[uuurr, red, bend right, "(P_2, P_2') \oplus e(P_{2/0})"'] & &
\end{tikzcd} \tag{48}$$

The red edges form $(P_2, P_2') \oplus \mu(l(\tau))$, the blue edges form τ. To show that they are freely homotopic, it suffices to show that all the triangles bound 2-simplices in $G\mathcal{C}$ – details are given in Section B.3. $\square$

Corollary 4.5. *If an admissible triple τ lies in $G\mathcal{C}^o$, then its loop is freely homotopic to $\mu(l(\tau))$.*

Proof. Consider the operation

$$(A, A') \oplus (\text{—})\colon G\mathcal{C} \to G\mathcal{C}, \tag{49}$$

which adds a vertex (A, A') to all the nodes of an n-simplex of $G\mathcal{C}$. Given any edge $(A, A') \to (B, B')$, this induces a simplicial homotopy between the maps

$$(A, A') \oplus (\text{—}) \longrightarrow (B, B') \oplus (\text{—}). \tag{50}$$

In particular, if there exists an edge $(O, O) \to (P_2, P_2')$, then $(P_2, P_2') \oplus \mu(l(\tau))$ is homotopic to $\mu(l(\tau))$. $\square$

4.3. **Nenashev Relations.** To define the relations on $K_1(\mathcal{C})$, we shall need the following generalisation of double exact squares. A 3×3 *diagram* in a pCGW category $\mathcal{C}$ is a pair of diagrams

$$\left(\begin{tikzcd}
X_{00} \arrow[r, tail, "f_0"] \arrow[d, tail, "h_0"'] \arrow[dr, phantom, "\circlearrowleft"] & X_{01} \arrow[d, tail, "h_1"] & X_{02} \arrow[l, {Circle[open]}->, "g_0"'] \arrow[d, tail, "h_2"] \\
X_{10} \arrow[r, tail, "f_1"] & X_{11} \arrow[dr, phantom, "\circlearrowleft"] & X_{12} \arrow[l, {Circle[open]}->, "g_1"'] \\
X_{20} \arrow[r, tail, "f_2"] \arrow[u, {Circle[open]}->, "j_0"] & X_{21} \arrow[u, {Circle[open]}->, "j_1"] & X_{22} \arrow[l, {Circle[open]}->, "g_2"'] \arrow[u, {Circle[open]}->, "j_2"']
\end{tikzcd}, \quad
\begin{tikzcd}
X_{00} \arrow[r, tail, "f_0'"] \arrow[d, tail, "h_0'"'] \arrow[dr, phantom, "\circlearrowleft"] & X_{01} \arrow[d, tail, "h_1'"] & X_{02} \arrow[l, {Circle[open]}->, "g_0'"'] \arrow[d, tail, "h_2'"] \\
X_{10} \arrow[r, tail, "f_1'"] & X_{11} \arrow[dr, phantom, "\circlearrowleft"] & X_{12} \arrow[l, {Circle[open]}->, "g_1'"'] \\
X_{20} \arrow[r, tail, "f_2'"] \arrow[u, {Circle[open]}->, "j_0'"] & X_{21} \arrow[u, {Circle[open]}->, "j_1'"] & X_{22} \arrow[l, {Circle[open]}->, "g_2'"'] \arrow[u, {Circle[open]}->, "j_2'"']
\end{tikzcd}\right)$$

on the same objects subject to the following conditions:

- The horizontal and vertical rows of each diagram define exact squares. Explicitly, a 3×3 diagram is defined by 6 double exact squares:

$$l_i := \left(\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & X_{i2} \arrow[d, {Circle[open]}->, "g_i"] \\
X_{i0} \arrow[r, tail, "f_i"] & X_{i1}
\end{tikzcd}, \quad
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & X_{i2} \arrow[d, {Circle[open]}->, "g_i'"] \\
X_{i0} \arrow[r, tail, "f_i'"] & X_{i1}
\end{tikzcd}\right) \qquad \text{for all } i \in \{0, 1, 2\}$$

$$l^i := \left(\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & X_{2i} \arrow[d, {Circle[open]}->, "j_i"] \\
X_{0i} \arrow[r, tail, "h_i"] & X_{1i}
\end{tikzcd}, \quad
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & X_{2i} \arrow[d, {Circle[open]}->, "j_i'"] \\
X_{0i} \arrow[r, tail, "h_i'"] & X_{1i}
\end{tikzcd}\right) \qquad \text{for all } i \in \{0, 1, 2\}$$

- The top left square and bottom left squares of each diagram must commute in $\mathcal{M}$ and $\mathcal{E}$ respectively. By contrast, we impose no conditions on the "mixed" squares.

Definition 4.6 (Good 3×3 Diagrams)**.** A 3×3 diagram in $\mathcal{C}$ is called *good* if there exists objects Z and Z', maps $v\colon Z \rightarrowtail X_{11}$ and $v'\colon Z' \rightarrowtail X_{11}$ inducing the following diagrams

(P1)

$$\begin{tikzcd}
X_{01} \arrow[r, tail] \arrow[dr, phantom, "\square"] & Z \arrow[r, tail, "v"] \arrow[dr, phantom, "\square"] & X_{11} \\
O \arrow[r, tail] \arrow[u, {Circle[open]}->] & X_{20} \arrow[r, tail, "f_2"] \arrow[u, {Circle[open]}->] \arrow[dr, phantom, "\square"] & X_{21} \arrow[u, {Circle[open]}->, "j_1"'] \\
& O \arrow[r, tail] \arrow[u, {Circle[open]}->] & X_{22} \arrow[u, {Circle[open]}->, "g_2"']
\end{tikzcd} \qquad
\begin{tikzcd}
X_{01} \arrow[r, tail] \arrow[dr, phantom, "\square"] & Z' \arrow[r, tail, "v'"] \arrow[dr, phantom, "\square"] & X_{11} \\
O \arrow[r, tail] \arrow[u, {Circle[open]}->] & X_{20} \arrow[r, tail, "f_2'"] \arrow[u, {Circle[open]}->] \arrow[dr, phantom, "\square"] & X_{21} \arrow[u, {Circle[open]}->, "j_1'"'] \\
& O \arrow[r, tail] \arrow[u, {Circle[open]}->] & X_{22} \arrow[u, {Circle[open]}->, "g_2'"']
\end{tikzcd}$$

(P2)

$$
\begin{tikzcd}
X_{10} \arrow[r, tail] & Z \arrow[r, tail, "v"] & X_{11} \\
O \arrow[u, {Circle[open]}->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & X_{02} \arrow[u, {Circle[open]}->] \arrow[r, tail, "h_2"] \arrow[ur, phantom, "\square"] & X_{12} \arrow[u, {Circle[open]}->, "g_1"'] \\
 & O \arrow[u, {Circle[open]}->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & X_{22} \arrow[u, {Circle[open]}->, "j_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{10} \arrow[r, tail] & Z' \arrow[r, tail, "v'"] & X_{11} \\
O \arrow[u, {Circle[open]}->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & X_{02} \arrow[u, {Circle[open]}->] \arrow[r, tail, "h'_2"] \arrow[ur, phantom, "\square"] & X_{12} \arrow[u, {Circle[open]}->, "g'_1"'] \\
 & O \arrow[u, {Circle[open]}->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & X_{22} \arrow[u, {Circle[open]}->, "j'_2"']
\end{tikzcd}
$$

(P3)

$$
\begin{tikzcd}
X_{00} \arrow[r, tail, "f_0"] & X_{01} \arrow[r, tail] & Z \\
 & X_{02} \arrow[u, {Circle[open]}->, "g_0"] \arrow[r, tail] \arrow[ur, phantom, "\square"] & X_{02} \oplus X_{20} \arrow[u, {Circle[open]}->] \\
 & & X_{20} \arrow[u, {Circle[open]}->]
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{00} \arrow[r, tail, "f'_0"] & X_{01} \arrow[r, tail] & Z' \\
 & X_{02} \arrow[u, {Circle[open]}->, "g'_0"] \arrow[r, tail] \arrow[ur, phantom, "\square"] & X_{02} \oplus X_{20} \arrow[u, {Circle[open]}->] \\
 & & X_{20} \arrow[u, {Circle[open]}->]
\end{tikzcd}
$$

(P4)

$$
\begin{tikzcd}
X_{00} \arrow[r, tail, "h_0"] & X_{10} \arrow[r, tail] & Z \\
 & X_{20} \arrow[u, {Circle[open]}->, "j_0"] \arrow[r, tail] \arrow[ur, phantom, "\square"] & X_{02} \oplus X_{20} \arrow[u, {Circle[open]}->] \\
 & & X_{02} \arrow[u, {Circle[open]}->]
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{00} \arrow[r, tail, "h'_0"] & X_{10} \arrow[r, tail] & Z' \\
 & X_{20} \arrow[u, {Circle[open]}->, "j'_0"] \arrow[r, tail] \arrow[ur, phantom, "\square"] & X_{02} \oplus X_{20} \arrow[u, {Circle[open]}->] \\
 & & X_{02} \arrow[u, {Circle[open]}->]
\end{tikzcd}
$$

Remark 4.7. These so-called 3×3 diagrams were originally defined as commutative diagrams in exact categories, where goodness automatically follows from the universal property of pushouts [Nen98b, Prop. 5.1]. However, this setup breaks down for general pCGW categories: if we define $Z := X_{01} \star_{X_{00}} X_{10}$, we may lack an $\mathcal{M}$-morphism $v \colon Z \rightarrowtail X_{11}$ unless the original $\mathcal{M}$-square is a pullback. To address this, we hardcode the desired properties into our definition of good 3×3 diagrams.

Definition 4.8. Let $\mathcal{C}$ be a pCGW category. Define $\mathcal{D}(\mathcal{C})$ to be the abelian group with generators $\langle l \rangle$ for all double exact squares l in $\mathcal{C}$ subject to the following relations.

(N1) $\langle l \rangle = 0$ if

$$
l = \left(\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \arrow[d, {Circle[open]}->, "g"] \\
A \arrow[r, tail, "f"] & B
\end{tikzcd} , \begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \arrow[d, {Circle[open]}->, "g"] \\
A \arrow[r, tail, "f"] & B
\end{tikzcd} \right) \tag{51}
$$

Any identical pair of exact squares will be called a *diagonal* double exact square.

(N2) Given a good 3×3 diagram

$$
\left(\begin{tikzcd}
X_{00} \arrow[r, tail, "f_0"] \arrow[d, tail, "h_0"'] \arrow[dr, phantom, "\circlearrowleft"] & X_{01} \arrow[d, tail, "h_1"] & X_{02} \arrow[l, {Circle[open]}->, "g_0"'] \arrow[d, tail, "h_2"] \\
X_{10} \arrow[r, tail, "f_1"] & X_{11} & X_{12} \arrow[l, {Circle[open]}->, "g_1"'] \\
X_{20} \arrow[u, {Circle[open]}->, "j_0"] \arrow[r, tail, "f_2"] & X_{21} \arrow[u, {Circle[open]}->, "j_1"'] \arrow[ur, phantom, "\circlearrowleft"] & X_{22} \arrow[l, {Circle[open]}->, "g_2"'] \arrow[u, {Circle[open]}->, "j_2"']
\end{tikzcd} , \begin{tikzcd}
X_{00} \arrow[r, tail, "f'_0"] \arrow[d, tail, "h'_0"'] \arrow[dr, phantom, "\circlearrowleft"] & X_{01} \arrow[d, tail, "h'_1"] & X_{02} \arrow[l, {Circle[open]}->, "g'_0"'] \arrow[d, tail, "h'_2"] \\
X_{10} \arrow[r, tail, "f'_1"] & X_{11} & X_{12} \arrow[l, {Circle[open]}->, "g'_1"'] \\
X_{20} \arrow[u, {Circle[open]}->, "j'_0"] \arrow[r, tail, "f'_2"] & X_{21} \arrow[u, {Circle[open]}->, "j'_1"'] \arrow[ur, phantom, "\circlearrowleft"] & X_{22} \arrow[l, {Circle[open]}->, "g'_2"'] \arrow[u, {Circle[open]}->, "j'_2"']
\end{tikzcd} \right) \tag{52}
$$

defined by the following 6 double exact squares

$$l_i := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & X_{i2} \arrow[d, "g_i"] \\ X_{i0} \arrow[r, tail, "f_i"] & X_{i1} \end{tikzcd}, \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & X_{i2} \arrow[d, "g'_i"] \\ X_{i0} \arrow[r, tail, "f'_i"] & X_{i1} \end{tikzcd}\right) \quad \text{for all } i \in \{0,1,2\} \tag{53}$$

$$l^i := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & X_{2i} \arrow[d, "j_i"] \\ X_{0i} \arrow[r, tail, "h_i"] & X_{1i} \end{tikzcd}, \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & X_{2i} \arrow[d, "j'_i"] \\ X_{0i} \arrow[r, tail, "h'_i"] & X_{1i} \end{tikzcd}\right) \quad \text{for all } i \in \{0,1,2\}, \tag{54}$$

the following 6-term relation holds

$$\langle l_0\rangle + \langle l_2\rangle - \langle l_1\rangle = \langle l^0\rangle + \langle l^2\rangle - \langle l^1\rangle \tag{55}$$

Theorem 4.9. *Given a pCGW category $\mathcal{C}$ there exists a well-defined homomorphism*

$$m \colon \mathcal{D}(\mathcal{C}) \longrightarrow K_1(\mathcal{C}) \tag{56}$$

that is surjective. In other words, the two relations of $\mathcal{D}(\mathcal{C})$ also hold in $K_1(\mathcal{C})$.

Proof. By Theorem3.17, we know that $K_1(\mathcal{C})$ is generated by double exact squares so both groups have the same generators. It remains to check the relations.

(N1): Let l be as in Equation (51). The corresponding loop $\mu(l)$ bounds the 2-simplex

$$\begin{tikzcd} O \arrow[r, tail] & A \arrow[r, tail, "f"] \arrow[dr, phantom, "\square"] & B \\ & A \arrow[u] \arrow[r, tail, "f"] & B \arrow[u] \\ & & C \arrow[u, "g"] \end{tikzcd} \qquad \begin{tikzcd} O \arrow[r, tail] & A \arrow[r, tail, "f"] \arrow[dr, phantom, "\square"] & B \\ & A \arrow[u] \arrow[r, tail, "f"] & B \arrow[u] \\ & & C \arrow[u, "g"] \end{tikzcd}$$

in $G\mathcal{C}$, and so $\langle l\rangle = 0$.

(N2): Leveraging the fact that the 3×3 diagram is good, construct the diagram

$$\begin{tikzcd} (X_{00}, X_{00}) \arrow[rr, "l_0"] \arrow[dd, "l^0"] \arrow[dr, "\alpha_0"] & & (X_{01}, X_{01}) \arrow[dd, "l^1"] \arrow[dl, "\alpha_3"'] \\ & (Z, Z') \arrow[dr, "\alpha_2"] & \\ (X_{10}, X_{10}) \arrow[rr, "l_1"] \arrow[ur, "\alpha_1"] & & (X_{11}, X_{11}) \end{tikzcd} \tag{57}$$

where outer blue edges $\alpha := l_0 l^1 (l_1)^{-1} (l^0)^{-1}$ form a loop, and the inner edges are given by

$$\alpha_0 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & X_{02} \oplus X_{20} \arrow[d] \\ X_{00} \arrow[r, tail] & Z \end{tikzcd}, \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & X_{02} \oplus X_{20} \arrow[d] \\ X_{00} \arrow[r, tail] & Z' \end{tikzcd}\right) \quad \alpha_1 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & X_{02} \arrow[d] \\ X_{10} \arrow[r, tail] & Z \end{tikzcd}, \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & X_{02} \arrow[d] \\ X_{10} \arrow[r, tail] & Z' \end{tikzcd}\right)$$

$$\alpha_2 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & X_{22} \arrow[d] \\ Z \arrow[r, tail, "v"] & X_{11} \end{tikzcd}, \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & X_{22} \arrow[d] \\ Z' \arrow[r, tail, "v'"] & X_{11} \end{tikzcd}\right) \quad \alpha_3 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & X_{20} \arrow[d] \\ X_{01} \arrow[r, tail] & Z \end{tikzcd}, \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & X_{20} \arrow[d] \\ X_{01} \arrow[r, tail] & Z' \end{tikzcd}\right)$$

For orientation, we start with a basic observation.

Lemma 4.10. *Given any closed loop $l = e_0 \dots e_n$ whose edges are all double exact squares,*

$$\langle l \rangle = \sum_{i=0}^{n} (-1)^{\epsilon_i} \langle e_i \rangle$$

in $K_1(\mathcal{C})$, where the coefficient $(-1)^{\epsilon_i}$ reflects the orientation of edge e_i in l.

Proof of Lemma. The proof is obvious, but let us work this out explicitly for the case of α. Consider the diagram

$$\begin{tikzcd}
(X_{00}, X_{00}) \arrow[rr, "l_0"] \arrow[dd, "l^0"] & & (X_{01}, X_{01}) \arrow[dd, "l^1"] \\
& (O, O) \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \\
(X_{10}, X_{10}) \arrow[rr, "l_1"] & & (X_{11}, X_{11})
\end{tikzcd} \tag{58}$$

featuring α as the outer loop but now with the base-point (O, O) at the center. Since the triangles all bound 2-simplices, the loop $\mu(l_0)\mu(l^1)\mu(l_1)^{-1}\mu(l^0)^{-1}$ is freely homotopic to α, and so

$$\langle \alpha \rangle = \langle l_0 \rangle + \langle l^1 \rangle - \langle l_1 \rangle - \langle l^0 \rangle.$$

$\square$

Next, leveraging the fact that the 3×3 diagram is good, notice:

- Diagrams (P3) and (P4) are 2-simplices, which are bounded by the loops $l^0\alpha_1\alpha_0^{-1}$ and $l_0\alpha_3\alpha_0^{-1}$. Therefore, deduce that

$$\begin{aligned} \langle l^0 \rangle + \langle \alpha_1 \rangle - \langle \alpha_0 \rangle &= 0 \\ \langle l_0 \rangle + \langle \alpha_3 \rangle - \langle \alpha_0 \rangle &= 0 \end{aligned} \tag{59}$$

- Diagrams (P1) and (P2) are admissible triples whose associated exact squares are l_2 and l^2 respectively. It is also clear there exists an edge $(O, O) \to (X_{11}, X_{11})$. Thus, applying Corollary 4.5, deduce that

$$\begin{aligned} \alpha_3\alpha_2(l^1)^{-1} &\sim l_2 \\ \alpha_1\alpha_2(l_1)^{-1} &\sim l^2 \end{aligned} \tag{60}$$

  and so

$$\begin{aligned} \langle \alpha_3 \rangle + \langle \alpha_2 \rangle - \langle l^1 \rangle &= \langle l_2 \rangle \\ \langle \alpha_1 \rangle + \langle \alpha_2 \rangle - \langle l_1 \rangle &= \langle l^2 \rangle. \end{aligned} \tag{61}$$

Combining Equations (59) and (61),

$$\begin{aligned} \langle l_2 \rangle - \langle l^2 \rangle &= \langle \alpha_3 \rangle + \langle \alpha_2 \rangle - \langle l^1 \rangle - \langle \alpha_1 \rangle - \langle \alpha_2 \rangle + \langle l_1 \rangle \\ &= \langle \alpha_3 \rangle - \langle \alpha_0 \rangle - \langle l^1 \rangle - \langle \alpha_1 \rangle + \langle \alpha_0 \rangle + \langle l_1 \rangle \\ &= -\langle l_0 \rangle - \langle l^1 \rangle + \langle l^0 \rangle + \langle l_1 \rangle, \end{aligned}$$

and so by rearranging terms, conclude

$$\langle l_0 \rangle + \langle l_2 \rangle - \langle l_1 \rangle = \langle l^0 \rangle + \langle l^2 \rangle - \langle l^1 \rangle. \tag{62}$$

$\square$

4.4. **Assembler Relations.** We now apply Theorem 4.9 to compare our relations with Zakharevich's K_1 of an Assembler (Proposition 4.12). As a corollary, we prove that $\mathcal{D}(\mathcal{C}) \cong K_1(\mathcal{C})$, and so $\mathcal{D}(\mathcal{C})$ gives an alternative presentation of $K_1(\mathcal{C})$ for pCGW categories (Corollary 4.14).

We start with an informal overview. An *Assembler* is a Grothendieck site $\mathcal{A}$ whose topology encodes how an object A may be covered by a finite set of disjoint subobjects $\{A_i\}_{i\in I}$. In particular, given any Assembler $\mathcal{A}$, one can associate to it the category $\mathcal{W}(\mathcal{A})$ whereby

Objects: Finite sets of objects $\{A_i\}_{i\in I}$ in $\mathcal{A}$;
Morphisms: Piecewise automorphisms in $\mathcal{A}$. Explicitly, a morphism $f\colon \{A_i\}_{i\in I} \to \{B_j\}_{j\in J}$ in $\mathcal{W}(\mathcal{A})$ is a tuple of morphisms $f_i\colon A_i \to B_{f(i)}$ such that $\{f_i\colon A_i \to B_j\}_{i\in f^{-1}(j)}$ is a finite disjoint covering family.

Details can be found in [Zak17b, §2], which introduces and develops the K-theory of Assemblers. Its relevance to our paper is that one can also define $K\mathcal{V}\mathrm{ar}_k$ via Assemblers, which was shown to be equivalent to the CGW model of $K\mathcal{V}\mathrm{ar}_k$ by [CZ22, Theorems 7.8 and 9.1]. Extending Muro-Tonks' model of K_1 of a Waldhausen Category [MT08], Zakharevich proved the following.

Theorem 4.11 ([Zak17c, Theorem B])**.** *Let $\mathcal{A}$ be an Assembler whose morphisms are closed under pullback. Then $K_1(\mathcal{A})$ is generated by a pair of morphisms*

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

in $\mathcal{W}(\mathcal{A})$. These satisfy the relations

(Z1) $\langle A \underset{f}{\overset{f}{\rightrightarrows}} B\rangle = 0$;
(Z2) $\langle A \underset{f_2}{\overset{f_1}{\rightrightarrows}} B\rangle + \langle C \underset{g_2}{\overset{g_1}{\rightrightarrows}} D\rangle = \langle A \coprod C \underset{f_2\coprod g_2}{\overset{f_1\coprod g_1}{\rightrightarrows}} B \coprod D\rangle$;
(Z3) $\langle B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C\rangle + \langle A \underset{f_2}{\overset{f_1}{\rightrightarrows}} B\rangle = \langle A \underset{g_2 f_2}{\overset{g_1 f_1}{\rightrightarrows}} B\rangle$.

A couple remarks are in order. First, [Zak17c, Theorem B] leaves open the possibility that there may be more relations on K_1 to be identified. This incompleteness is inherited from Muro-Tonks' original model of K_1: although [MT08, Prop 6.3] shows that their model coincides with Nenashev's model for exact categories (and is thus complete), they were unable to show the same for all Waldhausen categories. Second, Relation (Z1) clearly corresponds to the diagonal relation (N1) of $\mathcal{D}(\mathcal{C})$. It remains to investigate Relations (Z2) and (Z3) in our context, which we work out below.

Proposition 4.12 (Assembler Relations)**.** *Let $f\colon (A, A) \to (B, B)$ and $g\colon (C, C) \to (D, D)$ be two double exact squares given by*

$$f := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[d, "f_2"] \\ A \arrow[r, tail, "f_1"] & B \end{tikzcd}\ ,\quad \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[d, "f_2'"] \\ A \arrow[r, tail, "f_1'"] & B \end{tikzcd}\right) \qquad g := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{D}{C} \arrow[d, "g_2"] \\ C \arrow[r, tail, "g_1"] & D \end{tikzcd}\ ,\quad \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{D}{C} \arrow[d, "g_2'"] \\ C \arrow[r, tail, "g_1'"] & D \end{tikzcd}\right).$$

Then the following relations hold in $K_1(\mathcal{C})$:
(A1) (Formal Direct Sums). $\langle f\rangle + \langle g\rangle = \langle f \oplus g\rangle$.
(A2) (Restricted Composition). *Suppose $(B, B) = (C, C)$. Then*

$$\langle f\rangle + \langle g\rangle = \langle g \circ f\rangle + \langle l_2\rangle,$$

where l_2 is the induced double exact square

$$l_2 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{B}{D} \arrow[d, "j_1"] \\ \frac{B}{A} \arrow[r, tail, "h_1"] & \frac{D}{A} \end{tikzcd}\ ,\quad \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{B}{D} \arrow[d, "j_1'"] \\ \frac{B}{A} \arrow[r, tail, "h_1'"] & \frac{D}{A} \end{tikzcd}\right) \quad \textit{and} \quad \langle g\circ f\rangle := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{D}{A} \arrow[d] \\ A \arrow[r, tail, "g_1 f_1"] & D \end{tikzcd}\ ,\quad \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{D}{A} \arrow[d] \\ A \arrow[r, tail, "g_1' f_1'"] & D \end{tikzcd}\right).$$

Proof. The argument proceeds by constructing the obvious 3×3 diagram, and applying Theorem 4.9 to perform our calculations.

(i): We claim that the following is a good 3×3 diagram:

$$\left(\begin{tikzcd} A \arrow[r, tail, "f_1"] \arrow[d, tail] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, tail] & \frac{B}{A} \arrow[l, o->, "f_2"'] \arrow[d, tail] \\ A \oplus C \arrow[r, tail, "f_1 \oplus g_1"] & B \oplus D & \frac{B}{A} \oplus \frac{D}{C} \arrow[l, o->, "f_2 \oplus g_2"'] \\ C \arrow[u, o->] \arrow[r, tail, "g_1"] & D \arrow[u, o->] \arrow[ur, phantom, "\circlearrowleft"] & \frac{D}{C} \arrow[l, o->, "g_2"'] \arrow[u, o->] \end{tikzcd} \quad , \quad \begin{tikzcd} A \arrow[r, tail, "f_1'"] \arrow[d, tail] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, tail] & \frac{B}{A} \arrow[l, o->, "f_2'"'] \arrow[d, tail] \\ A \oplus C \arrow[r, tail, "f_1' \oplus g_1'"] & B \oplus D & \frac{B}{A} \oplus \frac{D}{C} \arrow[l, o->, "f_2' \oplus g_2'"'] \\ C \arrow[u, o->] \arrow[r, tail, "g_1'"] & D \arrow[u, o->] \arrow[ur, phantom, "\circlearrowleft"] & \frac{D}{C} \arrow[l, o->, "g_2'"'] \arrow[u, o->] \end{tikzcd}\right) \tag{63}$$

The vertical columns define exact squares arising from formal direct sums. The top and bottom rows correspond to the exact squares from f and g. The middle rows correspond to $f \oplus g$; the fact this indeed defines a double exact square follows from Lemma A.1. One easily checks that the top left and bottom right squares are pullback squares in $\mathcal{M}$ and $\mathcal{E}$ respectively, and thus they commute. Therefore, we have a 3×3 diagram.

It remains to check goodness. Taking repeated restricted pushouts, Fact 1.16 yields

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, tail] & A \arrow[r, tail, "f_1"] \arrow[d, tail] & B \arrow[d, tail] \\ C \arrow[r, tail] \arrow[d, tail, "g_1"] & A \oplus C \arrow[r, tail] \arrow[d, tail] & B \oplus C \arrow[d, tail, "v"] \\ D \arrow[r, tail] & A \oplus D \arrow[r, tail] & B \oplus D \end{tikzcd} \qquad \begin{tikzcd} O \arrow[r, tail] \arrow[d, tail] & A \arrow[r, tail, "f_1'"] \arrow[d, tail] & B \arrow[d, tail] \\ C \arrow[r, tail] \arrow[d, tail, "g_1'"] & A \oplus C \arrow[r, tail] \arrow[d, tail] & B \oplus C \arrow[d, tail, "v'"] \\ D \arrow[r, tail] & A \oplus D \arrow[r, tail] & B \oplus D \end{tikzcd} . \tag{64}$$

Since we now have $\mathcal{M}$-morphisms $v, v' \colon B \oplus C \rightarrowtail B \oplus D$, we can construct the required diagrams:
(P1):

$$\begin{tikzcd} B \arrow[r, tail] & B \oplus C \arrow[r, tail, "v"] & B \oplus D \\ O \arrow[u, o->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & C \arrow[u, o->] \arrow[r, tail, "g_1"] \arrow[ur, phantom, "\square"] & D \arrow[u, o->] \\ & O \arrow[u, o->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{D}{C} \arrow[u, o->, "g_2"'] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, tail] & B \oplus C \arrow[r, tail, "v'"] & B \oplus D \\ O \arrow[u, o->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & C \arrow[u, o->] \arrow[r, tail, "g_1'"] \arrow[ur, phantom, "\square"] & D \arrow[u, o->] \\ & O \arrow[u, o->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{D}{C} \arrow[u, o->, "g_2'"'] \end{tikzcd}$$

(P2):

$$\begin{tikzcd} A \oplus C \arrow[r, tail] & B \oplus C \arrow[r, tail, "v"] & B \oplus D \\ O \arrow[u, o->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{B}{A} \arrow[u, o->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{B}{A} \oplus \frac{D}{C} \arrow[u, o->] \\ & O \arrow[u, o->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{D}{C} \arrow[u, o->] \end{tikzcd} \qquad \begin{tikzcd} A \oplus C \arrow[r, tail] & B \oplus C \arrow[r, tail, "v'"] & B \oplus D \\ O \arrow[u, o->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{B}{A} \arrow[u, o->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{B}{A} \oplus \frac{D}{C} \arrow[u, o->] \\ & O \arrow[u, o->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{D}{C} \arrow[u, o->] \end{tikzcd}$$

(P3):

$$
\begin{tikzcd}
A \arrow[r, tail, "f_1"] & B \arrow[r, tail] & B \oplus C \\
& \frac{B}{A} \arrow[u, {Circle[open]}->, "f_2"] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{B}{A} \oplus C \arrow[u, {Circle[open]}->] \\
& & C \arrow[u, {Circle[open]}->]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, tail, "f_1'"] & B \arrow[r, tail] & B \oplus C \\
& \frac{B}{A} \arrow[u, {Circle[open]}->, "f_2'"] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{B}{A} \oplus C \arrow[u, {Circle[open]}->] \\
& & C \arrow[u, {Circle[open]}->]
\end{tikzcd}
$$

(P4):

$$
\begin{tikzcd}
A \arrow[r, tail] & A \oplus C \arrow[r, tail] & B \oplus C \\
& C \arrow[u, {Circle[open]}->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{B}{A} \oplus C \arrow[u, {Circle[open]}->] \\
& & \frac{B}{A} \arrow[u, {Circle[open]}->]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, tail] & A \oplus C \arrow[r, tail] & B \oplus C \\
& C \arrow[u, {Circle[open]}->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{B}{A} \oplus C \arrow[u, {Circle[open]}->] \\
& & \frac{B}{A} \arrow[u, {Circle[open]}->]
\end{tikzcd}
$$

One can verify that the indicated squares are distinguished by repeated applications of Lemma A.4.

Finally, to prove the relation, let us review Diagram (63). Denote the double exact squares corresponding to the vertical columns as l^0, l^1 and l^2, from left to right. Since Diagram (63) is a good 3×3 diagram, apply Relation (N2) to get

$$\langle f \rangle + \langle g \rangle - \langle f \oplus g \rangle = \langle l^0 \rangle + \langle l^2 \rangle - \langle l^1 \rangle.$$

Further, since l^0, l^1 and l^2 are all identical pairs of formal direct sum squares, Relation (N1) gives

$$\langle l^0 \rangle = \langle l^1 \rangle = \langle l^2 \rangle = 0,$$

and so conclude

$$\langle f \rangle + \langle g \rangle = \langle f \oplus g \rangle.$$

(ii): Given $(B, B) = (C, C)$, construct the following diagram

$$
\left(
\begin{tikzcd}
A \arrow[r, tail, "="] \arrow[d, tail, "f_1"'] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d, tail, "g_1 f_1"] & O \arrow[l, {Circle[open]}->] \arrow[d, tail] \\
B \arrow[r, tail, "g_1"] & D & \frac{D}{B} \arrow[l, {Circle[open]}->, "g_2"'] \\
\frac{B}{A} \arrow[u, {Circle[open]}->, "f_2"] \arrow[r, tail, "h_1"] & \frac{D}{A} \arrow[u, {Circle[open]}->] \arrow[ur, phantom, "\circlearrowleft"] & \frac{D}{B} \arrow[u, {Circle[open]}->, "="'] \arrow[l, {Circle[open]}->, "j_1"]
\end{tikzcd}
,
\begin{tikzcd}
A \arrow[r, tail, "="] \arrow[d, tail, "f_1'"'] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d, tail, "g_1' f_1'"] & O \arrow[l, {Circle[open]}->] \arrow[d, tail] \\
B \arrow[r, tail, "g_1'"] & D & \frac{D}{B} \arrow[l, {Circle[open]}->, "g_2'"'] \\
\frac{B}{A} \arrow[u, {Circle[open]}->, "f_2'"] \arrow[r, tail, "h_1'"] & \frac{D}{A} \arrow[u, {Circle[open]}->] \arrow[ur, phantom, "\circlearrowleft"] & \frac{D}{B} \arrow[u, {Circle[open]}->, "="'] \arrow[l, {Circle[open]}->, "j_1'"]
\end{tikzcd}
\right).
\tag{65}
$$

Following the convention from Definition 4.8, label the horizontal rows as l_0, l_1 and l_2 and the vertical columns as l^0, l^1 and l^2. In particular, we have $l_1 = g$, $l^0 = f$ and $l^1 = g \circ f$. The fact that l_2 defines a double exact square comes from Lemma 1.7 ("quotients respect filtrations"); the remaining rows are obvious. It is also clear the top left and bottom right squares are pullback squares in $\mathcal{M}$ and $\mathcal{E}$ respectively, and therefore commute. Finally, take the restricted pushout of $A \overset{=}{\hookleftarrow} A \overset{f_1}{\rightarrowtail} B$. Since we have an $\mathcal{M}$-morphism $g_1 \colon A \star_A B = B \to D$, the same argument as part (i) shows that Diagram (65) is in fact a good 3×3 diagram.

Now apply Relation (N2) to get

$$\langle l_0 \rangle + \langle l_2 \rangle - \langle g \rangle = \langle f \rangle + \langle l^2 \rangle - \langle g \circ f \rangle.$$

Since l_0 and l^2 are diagonal, deduce from (N1) that $\langle l_0 \rangle = \langle l^2 \rangle = 0$, and thus conclude that

$$\langle g \circ f \rangle + \langle l_2 \rangle = \langle f \rangle + \langle g \rangle.$$

□

Discussion 4.13 (On Restricted Composition)**.** Relation (A2) improves on Proposition 4.1 by determining what happens to *any* admissible triple in K_1 (Definition 4.3). This gives a clearer answer to the issue raised in Warning 4.2. Namely, given an admissible triple $\tau = (f, g, g \circ f)$, we now know that

$$\langle g \circ f \rangle + \langle l_2 \rangle = \langle f \rangle + \langle g \rangle \qquad \text{in } K_1.$$

Interestingly, this obstruction $\langle l_2 \rangle$ does not appear in Theorem 4.11, where composition of piecewise automorphisms always split in K_1. Of course, if $\langle l_2 \rangle = 0$ then Relations (A2) and (Z3) coincide, but it is unclear if this holds in general.

We suspect the discrepancy with Zakharevich's account of K_1 is due to [Zak17c, Theorem 2.1], a key ingredient in the proof of Theorem 4.11. The result states that the K-theory of (nice) Assemblers can be modelled by Waldhausen categories whose cofibration sequences all split (up to weak equivalence). We emphasise that this relies on a non-standard notion of weak equivalence [Zak17c, Def. 1.7]. In the CGW setting, it is clearly false that all exact squares split in $\mathcal{V}\mathrm{ar}_k$ – consider e.g.

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\square"] & \mathbb{A}^1 \arrow[d, "\circ" marking] \\ \{*\} \arrow[r, tail] & \mathbb{P}^1 \end{tikzcd} .$$

We end with one final surprise. In order to show that $\mathcal{D}(\mathcal{C}) \cong K_1(\mathcal{C})$ for exact categories, Nenashev [Nen98a] constructs a homomorphism

$$b \colon K_1(\mathcal{C}) \to \mathcal{D}(\mathcal{C})$$

and shows that it is inverse to the map $m \colon \mathcal{D}(\mathcal{C}) \to K_1(\mathcal{C})$ from Equation (56). Notice the naive map sending $\langle f \rangle \to \langle f \rangle$ is *a priori* not well-defined since $K_1(\mathcal{C})$ may have more relations than $\mathcal{D}(\mathcal{C})$. Indeed, the original construction of b in [Nen98a] requires a fair bit of technical legwork to show that it yields a well-defined homomorphism. However, Propositions 4.1 and 4.12 combine to give a shorter direct proof.

Corollary 4.14. *Given any pCGW category $\mathcal{C}$, there is an isomorphism*

$$\mathcal{D}(\mathcal{C}) \cong K_1(\mathcal{C}).$$

Proof. It suffices to show the naive map $b \colon K_1(\mathcal{C}) \to \mathcal{D}(\mathcal{C})$ sending $\langle f \rangle \to \langle f \rangle$ is in fact well-defined. Any equivalence $\langle f \rangle = \langle g \rangle$ in $K_1(\mathcal{C})$ must be generated by the relations of Proposition 4.1. It therefore suffices to check that $\mathcal{D}(\mathcal{C})$ satisfies those relations as well. Relations (B1) and (B2) are diagonal relations, and follow immediately from Relation (N1). Relation (B3) is a special case of Restricted Composition (A2) in Proposition 4.12, where $\langle l_2 \rangle = 0$ since l_2 is diagonal by assumption. And we are done. □

5. Some Test Problems

This paper began with Question 1: what information do the higher K-groups of varieties encode? Progress on this question requires advances on two fronts: developing the framework of non-additive K-theory on the one hand, and concrete applications of the newly-developed (K-theory) tools on the other. This paper is of the first kind, with a view towards laying the groundwork for future theorems. We conclude with some test problems and discussion.

5.1. **Non-Additive K-theory.**

Having characterised K_1, the natural next step is the following problem.

Problem 5.1. Characterise $K_n(\mathcal{C})$ for $n > 1$ for pCGW categories.

Discussion 5.2. In a substantial generalisation of Nenashev's work, Grayson gave a complete characterisation of K_n for all n in the setting of exact categories [Gra12]. Encouraged by the results in this paper, the natural proof strategy would be to extend Grayson's argument to the pCGW setting.

However, there is an obvious barrier. Grayson characterises the K-groups of exact categories via binary chain complexes, and so invokes the Gillet-Waldhausen Theorem. Although an analogue of this result has been shown for extensive categories (e.g. FinSet) [SS21], a Gillet-Waldhausen Theorem for $\mathcal{V}\mathrm{ar}_k$ has not been worked out yet. Nonetheless, even the case of FinSet is interesting. By Barratt-Priddy-Quillen, the K-groups of FinSet correspond to the stable homotopy groups of spheres, whose complete description remains a longstanding open problem in homotopy theory. What might this presentation of $K_n(\mathrm{Finset})$ tell us about the stable homotopy groups of spheres? About the J-homomorphism and Adams e-invariant?

In light of the comparisons with Zakharevich's K_1, perhaps a more urgent question is the following.

Problem 5.3. Is it generally true that $\langle l_2 \rangle = 0$ for Relation (A2) in Proposition 4.12?

Discussion 5.4. We suspect no, although we have yet to construct a counter-example in $\mathcal{V}\mathrm{ar}_k$. The difficulty is that we do not have a complete characterisation of double exact squares that trivialise in K_1. In particular, there are non-diagonal double exact squares that trivialise, e.g.

$$l_\tau := \left(\begin{array}{ccc} O & \rightarrowtail & A \oplus A \\ \downarrow & \square & \downarrow \tau \\ O & \rightarrowtail & A \oplus A \end{array} \quad , \quad \begin{array}{ccc} O & \rightarrowtail & A \oplus A \\ \downarrow & \square & \downarrow 1 \\ O & \rightarrowtail & A \oplus A \end{array} \right)$$

where $\tau \colon A \oplus A \xrightarrow{\sim} A \oplus A$ is the *twist automorphism* that swaps components.

Discussion 5.5. Many details in Zakharevich's paper [Zak17c] were omitted, since the analysis is based on her previous work on the scissors congruence of polytopes [Zak12]. Our present discussion prompts a careful re-examination of her arguments. Some guiding observations and questions:

(1) The key hypothesis in Theorem 4.11 is that the Assembler is closed under pullbacks – this is used for defining composition of weak equivalences in the associated Waldhausen category [Zak17c, Theorem 2.1]. Can Zakharevich's result be extended to exact categories whose admissible monics are closed under pullbacks? If so, there may be interesting applications – see e.g. Previdi's work on partially abelian exact categories [Pre12].
(2) Pullbacks in the Assembler are also used to define pushouts in the associated Waldhausen category. How do these relate to the restricted pushouts of pCGW categories?
(3) Our proof of Theorem 3.7 makes crucial use of Lemma B.2, which states: if $\mathcal{C}$ is a pCGW category whose exact squares are all split, then $K_1(\mathcal{C})$ is generated by automorphisms. In which case, it is straightforward to check that composition splits in K_1 in the manner of $(Z3)$, Theorem 4.11. Does an analogue of Lemma B.2 hold for Assemblers?

Answers in any direction could be very interesting. Positive answers may point to stronger presentation theorems for the K-theory. Negative answers may reveal subtle but important differences between the scissors congruence of varieties vs. polytopes, previously overlooked.

Another natural question, posed to us by Emanuele Dotto, is the following.

Problem 5.6. Does the K-theory of pCGW categories commute with infinite products?

Discussion 5.7. Relevantly: Zakharevich conjectures in [Zak22, Remark 2.4] that the K-theory of Assemblers commutes with infinite products. However, she notes that such a result seems presently out of reach since previous results of this form were worked out for Waldhausen categories with cylinder functors (which Assemblers do not have) and exact categories.

5.2. **The Motivic Euler Characteristic.** There is a well-known enrichment of the Euler Characteristic, known as the *motivic Euler Characteristic* or *compactly supported* $\mathbb{A}^1$*-Euler Characteristic*, which can be defined as a ring homomorphism

$$\chi^{\mathrm{mot}} \colon K_0(\mathcal{V}\mathrm{ar}_k) \to \mathrm{GW}(k)$$

valued in $GW(k)$, the Grothendieck-Witt ring of quadratic forms over field k. An exposition of its construction can be found in [AMBO+22]. It is natural to ask if one can lift this to the level of K-theory spectra, which was proved in the affirmative by Nanavaty [Nan24, Theorem 1.1]. Explicitly, he constructs a map of spectra

$$K\mathcal{V}\mathrm{ar}_k \to \mathrm{End}(\mathbb{1}_k)$$

where $\mathrm{End}(\mathbb{1}_k)$ is the *Endomorphism Spectrum of the unit object in the motivic stable homotopy category*, recovering χ^{mot} on π_0. This sets up the problem:

Problem 5.8. Define a natural map $K_1(\mathcal{V}\mathrm{ar}_k) \to \pi_{1,0}(\mathbb{1}_k)$. What geometric information does it encode?

Discussion 5.9. The homotopy groups of $\mathrm{End}(\mathbb{1}_k)$ are defined as $\pi_{*,0}$. A foundational result, due to Morel [Mor06], shows that $\pi_{0,0}(\mathbb{1}_k) \cong \mathrm{GW}(k)$, so one may regard the higher homotopy groups as defining higher Grothendieck-Witt Groups. What geometric information is detected at the higher levels? Recent work by [RSØ19] tells us

$$0 \to K_2^M(k)/24 \to \pi_{1,0}(\mathbb{1}_k) \to k^\times/2 \oplus \mathbb{Z}/2 \to 0, \tag{66}$$

where $K_*^M(k)$ denotes the Milnor K-theory of k. Combined with Theorems B and/or C, we now have an explicit description of both groups in Problem 5.8. It remains to map the double exact squares in $\mathcal{V}\mathrm{ar}_k$ to $\pi_{1,0}(\mathbb{1}_k)$ in a natural way, but it is presently unclear, e.g. how these generalised automorphisms ought to interact with the Milnor K-theory term, and what this means geometrically.

Note: the canonical unit map $\mathrm{End}(\mathbb{1}_k) \to KQ$ (i.e. the Hermitian K-theory spectrum) induces an isomorphism on the level of π_0. We may therefore also think of χ^{mot} as a map on π_0 of $K\mathcal{V}\mathrm{ar}_k \to KQ$, which may be more tractable than working in the setting of motivic homotopy theory. For those interested in the exterior powers of $K_0(\mathcal{V}\mathrm{ar}_k)$, a natural question may be:

Problem 5.10. Lift the symmetric power structure on $K_0(\mathcal{V}\mathrm{ar}_k)$ to the level of spectra on $K\mathcal{V}\mathrm{ar}_k$. In particular, if we regard χ^{mot} as a map

$$\chi^{\mathrm{mot}} \colon \pi_0(K\mathcal{V}\mathrm{ar}_k) \to \pi_0(KQ),$$

can we deduce the compatibility of χ^{mot} with the symmetric power structures on the level of π_0 from formal properties on the level of K-theory spectra?

Discussion 5.11. To our knowledge, compatibility of χ^{mot} with power structures on K_0 and $GW(k)$ has only been shown in special cases, and the proofs rely on deep arithmetic (e.g. [PP23, PRV24]). Problem 5.10 calls for a shift in perspective: what if we approach the problem homotopically instead?

The jury is still out on how much mileage this gives us, but we have a first clue: Grayson [Gra92] relies on the G-construction on exact categories to provide an explicit combinatorial description of the Adams Operations on higher K-groups induced by symmetric powers on the exact category. Since we know the G-construction behaves as expected on $\mathcal{V}\mathrm{ar}_k$ by Theorem A, this tells us where to start.

Remark 5.12. Lifting the symmetric power structure on $K_0(\mathcal{V}\mathrm{ar}_k)$ also ought to have useful implications for lifting Kapranov's motivic zeta function to a map of K-theory spectra – see e.g. [CZ22, Question 7.3].

5.3. **Combinatorics of Definable Sets.** Theorem 3.17 showed that $K_1(\mathcal{C})$ of a pCGW category is generated by double exact squares, which generalise automorphisms (see Example 3.16). For suggestiveness, call the generators of K_1 *quasi-automorphisms*. Automorphisms play a key role in many different areas of mathematics – can this picture be extended to quasi-automorphisms in a productive way?

For the model theorist, the automorphism group $\mathrm{Aut}(M)$ of a countable first-order structure M encodes important information about M. For instance, $\mathrm{Aut}(M)$ measures the homogeneity of M, but there are other examples [MK94, Eva97]. Some interesting questions:

Problem 5.13. How might $K_1(M)$ be useful for studying non-homogeneous structures?

Problem 5.14. If all exact squares of $\mathrm{Def}(M)$ splits, then $K_1(M)$ is generated by automorphisms of M by Lemma B.2. In which case, can we adapt the obstruction theory developed in [Zak17a] to analyse e.g. the barriers to extending partial isomorphisms of M to global automorphisms? What about the non-split case?

Problem 5.15. In the converse direction, by viewing $\mathrm{Aut}(M)$ as a topological group, model theorists were able to analyse its structure productively using a wide range of tools. Can we apply the same analysis to $K_1(M)$? What kind of (new) insights does this give? Work in [KLM$^+$24, §7] may be relevant.

It is also worth revisiting the original papers [Kra00, KS00] where the Grothendieck ring of Definable Sets $K_0(\mathrm{Def})$ was first investigated. In particular, [KS00] introduces the so-called strong and weak Euler Characteristics on first-order structures before asking which fields admit a non-trivial strong Euler Characteristic. In light of our present work, a natural problem may be:

Problem 5.16. Lift the weak/strong Euler Characteristic on first-order structures to the level of spectra. Analyse what happens on K_1 – what information does it detect? In addition, are there examples of fields with strong Euler characteristics that are trivial on K_0 but non-trivial on K_1?

5.4. **Matroids.** Similar questions about quasi-automorphisms may be posed regarding matroids. However, in light of Example 1.22, a more urgent problem is the following:

Problem 5.17. What is the right notion of restricted pushouts for matroids?

Discussion 5.18. As pointed out to us by Chris Eppolito, the restricted pushout of $M_0 \leftarrowtail N \rightarrowtail M_1$ cannot be the pushout in the ambient category $\mathrm{Mat}_\bullet$ since this may not exist. Consider, for example, when $N = \{a, b, c, \bullet\}$, $M_0 = \{a, b, c, d, \bullet\}$ and $M_1 = \{a, b, c, e, \bullet\}$, where the ground sets are endowed with the uniform matroid structure of rank 2, with point $\bullet$.

Discussion 5.19 (Independent Squares)**.** A related problem was considered by the model theorists. Let $\mathcal{D}$ be the category whose morphisms are all monic, and suppose $\mathcal{D}$ admits a *stable independence notion*, i.e. a class of so-called *independent squares*

$$\begin{tikzcd} N \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\perp"] & M_1 \arrow[d, tail] \\ M_0 \arrow[r, tail] & M \end{tikzcd}$$

satisfying the axioms listed in [Vas19, Definition 5.5]. This includes the condition that any span in $\mathcal{D}$

$$M_0 \leftarrowtail N \rightarrowtail M_1$$

can be completed into an independent square. Examples include FinSet, the category of vector spaces over a fixed fields (with injective linear transformations), and the category of graphs (with subgraph embeddings).

Here is the key insight from [Vas19, §5.1]. Under certain technical conditions, the category of independent squares associated to a given span has a weakly initial object, which is unique up to (not necessarily unique) isomorphism. Model theorists call this the *prime object* over the span. Some natural questions:

(1) Do prime objects satisfy the required properties of restricted pushouts for pCGW categories?
(2) Can specific classes of matroids (e.g. vectorial, or graphic matroids) admit such a structure? All classes?
(3) How does the model-theoretic notion of independent squares align with flats and ranks in matroids?

This will be investigated in future work.

Appendix A. Properties of Restricted Pushouts

As explained in Section 1.2, it is unreasonable to ask for $\mathcal{M}$-morphisms of CGW categories to be closed under pushouts, so we instead ask for them to be closed under *restricted pushouts*, a weaker notion. This section collects various technical facts about them.

Lemma A.1.

(i) Given a span $B \leftarrowtail A \rightarrowtail C$,

$$\frac{B}{A} \oplus \frac{C}{A} \cong \frac{B \star_A C}{A}.$$

(ii) Given a span $B \leftarrowtail A \rightarrowtail C$, along with $\mathcal{M}$-morphisms $B \rightarrowtail B'$ and $C \rightarrowtail C'$,

$$\frac{B' \star_A C'}{B \star_A C} \cong \frac{B'}{B} \oplus \frac{C'}{C}.$$

Proof.

(i): Consider the diagram

$$\begin{tikzcd} \frac{C}{A} \arrow[d, circ] \arrow[dr, phantom, "\square"] & O \arrow[l, tail] \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[d, circ] \\ C & A \arrow[l, tail] \arrow[r, tail] & B \end{tikzcd}$$

Apply Axiom (DS) to obtain the left diagram below

$$
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{B}{A} \oplus \frac{C}{A} \arrow[d, {Circle[open]}->] \\
A \arrow[r, tail, "f"] & B \arrow[r, tail, "f'"] & B \star_A C
\end{tikzcd}
\qquad
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{B \star_A C}{A} \arrow[d, {Circle[open]}->] \\
A \arrow[r, tail, "f'f"] & B \star_A C
\end{tikzcd}
\tag{67}
$$

Since distinguished squares compose, the outermost rectangle of the left diagram also defines a distinguished square. On the other hand, by Axiom (K), the right distinguished square above exists. Since formal cokernels are unique (up to unique isomorphism), conclude that $\frac{B}{A} \oplus \frac{C}{A} \cong \frac{B \star_A C}{A}$.

(ii): Repeated applications of Fact 1.16 yields

$$
\begin{tikzcd}
A \arrow[r, tail] \arrow[d, tail] & B \arrow[r, tail] \arrow[d, tail] & B' \arrow[d, tail] \\
C \arrow[r, tail] \arrow[d, tail] & B \star_A C \arrow[r, tail] \arrow[d, tail] & B' \star_A C \arrow[d, tail] \\
C' \arrow[r, tail] & B \star_A C' \arrow[r, tail] & B' \star_A C'
\end{tikzcd}
\tag{68}
$$

In particular, there exists an $\mathcal{M}$-morphism $B \star_A C \rightarrowtail B' \star_A C'$, so $\frac{B' \star_A C'}{B \star_A C}$ is well-defined and exists by Axiom (K). By Axiom (PQ), restricted pushouts preserve quotients, and so

$$
\frac{B'}{B} \cong \frac{B' \star_A C}{B \star_A C} \quad \text{and} \quad \frac{C'}{C} \cong \frac{B \star_A C'}{B \star_A C}. \tag{69}
$$

Now consider the diagram

$$
\begin{tikzcd}
\frac{B \star_A C'}{B \star_A C} \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & O \arrow[l, tail] \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{B' \star_A C}{B \star_A C} \arrow[d, {Circle[open]}->] \\
B \star_A C' & B \star_A C \arrow[l, tail] \arrow[r, tail] & B' \star_A C
\end{tikzcd}
\tag{70}
$$

Applying the same argument from (i) to Equations (69) and (70), deduce the desired isomorphism. $\square$

Lemma A.2. *Suppose $A \rightarrowtail B$. Then $\frac{B \oplus C}{A} \cong \frac{B}{A} \oplus C$, for any $C \in \mathcal{C}$.*

Proof. Apply Fact 1.16 to the diagram $C \leftarrowtail O \rightarrowtail A \rightarrowtail B$, deduce that $B \star_A (A \oplus C) \cong B \oplus C$. Next, apply Axiom (DS) to the diagram

$$
\begin{tikzcd}
C \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & O \arrow[l, tail] \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[d, {Circle[open]}->] \\
A \oplus C & A \arrow[l, tail] \arrow[r, tail] & B
\end{tikzcd}
\tag{71}
$$

and obtain

$$
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{B}{A} \oplus C \arrow[d, {Circle[open]}->] \\
A \arrow[r, tail, "f"] & B \oplus C
\end{tikzcd}
\tag{72}
$$

as a distinguished square. Applying Axiom (K) to $f \colon A \rightarrowtail B \oplus C$, conclude that $\frac{B \oplus C}{A} \cong \frac{B}{A} \oplus C$. $\square$

Lemma A.3 (Direct Sums)**.** *Given any pair of exact squares*

$$
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[d, {Circle[open]}->, "f_2"] \\
A \arrow[r, tail, "f_1"] & B
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{D}{C} \arrow[d, {Circle[open]}->, "g_2"] \\
C \arrow[r, tail, "g_1"] & D
\end{tikzcd}
$$

we can construct the obvious exact square via direct sums

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\square"] & \frac{B}{A} \oplus \frac{D}{C} \arrow[d, "f_2 \oplus g_2"] \\ A \oplus C \arrow[r, tail, "f_1 \oplus g_1"] & B \oplus D \end{tikzcd}$$

Proof. Take repeated restricted pushouts to get

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, tail] & C \arrow[r, tail, "g_1"] \arrow[d, tail] & D \arrow[d, tail] \\ A \arrow[r, tail] \arrow[d, tail, "f_1"] & A \oplus C \arrow[r, tail, "A \oplus g_1"] \arrow[d, tail, "f_1 \oplus C"] & A \oplus D \arrow[d, tail, "f_1 \oplus D"] \\ B \arrow[r, tail] & B \oplus C \arrow[r, tail, "B \oplus g_1"'] & B \oplus D \end{tikzcd}$$

Since restricted pushouts preserve quotients, construct the diagram

$$\begin{tikzcd} \frac{D}{C} \arrow[d, "g_2"'] & O \arrow[l, tail] \arrow[r, tail] \arrow[d] \arrow[dl, phantom, "\square"] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[d, "f_2"] \\ A \oplus D & A \oplus C \arrow[l, tail, "A \oplus g_1"] \arrow[r, tail, "f_1 \oplus C"'] & B \oplus C \end{tikzcd} .$$

The rest follows from applying Axiom (DS) and the fact that distinguished squares compose horizontally. □

Lemma A.4. *Given any distinguished square in pCGW category $\mathcal{C}$*

$$\phi := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d] \\ A \arrow[r, tail] & B \end{tikzcd} \right) \tag{73}$$

the following squares are also distinguished

(i) For any $D \in \mathcal{C}$:

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C \oplus D \arrow[d] \\ A \arrow[r, tail] & B \oplus D \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d] \\ A \oplus D \arrow[r, tail] & B \oplus D \end{tikzcd} ; \tag{74}$$

$$\begin{tikzcd} C \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C \oplus D \arrow[d] \\ B \arrow[r, tail] & B \oplus D \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} D \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C \oplus D \arrow[d] \\ A \oplus D \arrow[r, tail] & B \oplus D \end{tikzcd} . \tag{75}$$

(ii) Given any $A' \rightarrowtail B'$:

$$\begin{tikzcd} C \oplus A' \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C \oplus B' \arrow[d] \\ B \oplus A' \arrow[r, tail] & B \oplus B' \end{tikzcd} \tag{76}$$

Proof. (i): First, apply Fact 1.16 to

$$D \leftarrowtail O \rightarrowtail A \rightarrowtail B,$$

and obtain the isomorphism $B \oplus D \cong (A \oplus D) \star_A B$. Then apply Axiom (DS) of Definition 1.17 to the diagram

$$\begin{tikzcd} D \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & O \arrow[l, tail] \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & C \arrow[d, "\circ"{marking, pos=0}] \\ A \oplus D & A \arrow[l, tail] \arrow[r, tail] & B \end{tikzcd}$$

to obtain the following distinguished squares

$$\psi := \left(\begin{tikzcd} C \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & C \oplus D \arrow[d, "\circ"{marking, pos=0}] \\ B \arrow[r, tail] & B \oplus D \end{tikzcd} \right) \qquad \psi' := \left(\begin{tikzcd} D \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & C \oplus D \arrow[d, "\circ"{marking, pos=0}] \\ A \oplus D \arrow[r, tail] & B \oplus D \end{tikzcd} \right).$$

Since distinguished squares are closed under composition, obtain Equation (74) by horizontal composition of ψ with the original distinguished square ϕ, and vertical composition of ψ' with the formal sum square

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & C \arrow[d, "\circ"{marking, pos=0}] \\ D \arrow[r, tail] & C \oplus D \end{tikzcd} .$$

(ii): By item (i), the following squares are distinguished

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & C \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & C \oplus A' \arrow[d, "\circ"{marking, pos=0}] \\ A \arrow[r, tail] & B \arrow[r, tail] & B \oplus A' \end{tikzcd} \qquad \begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & \frac{B'}{A'} \arrow[d, "\circ"{marking, pos=0}] \\ B \oplus A' \arrow[r, tail] & B \oplus B' \end{tikzcd} ,$$

and so we may construct the following diagram

$$\begin{tikzcd} C \oplus A' \arrow[d, "\circ"{marking, pos=0}] & \\ B \oplus A' \arrow[r, tail] & B \oplus B' \end{tikzcd} \tag{77}$$

and $\mathcal{M}$-morphism $v := \Big(A \rightarrowtail B \oplus A' \rightarrowtail B \oplus B' \Big)$. Since v factors through $A \rightarrowtail B$ by construction, apply Lemma A.2 to obtain

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & C \oplus B' \arrow[d, "\circ"{marking, pos=0}] \\ A \arrow[r, tail, "v"] & B \oplus B' \end{tikzcd} , \tag{78}$$

which is distinguished. Notice v also represents a morphism

$$(A \rightarrowtail B \oplus A') \rightarrowtail (A \rightarrowtail B \oplus B') \in \mathrm{Ar}_\Delta \mathcal{M}.$$

Applying k^{-1} to this morphism produces the desired distinguished square

$$\begin{tikzcd} C \oplus A' \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & C \oplus B' \arrow[d, "\circ"{marking, pos=0}] \\ B \oplus A' \arrow[r, tail] & B \oplus B' \end{tikzcd} ,$$

where we use the fact that c and k are inverse on objects.

$\square$

APPENDIX B. TECHNICAL LEMMAS & SOME 2-SIMPLICES

B.1. **Technical Facts about Sherman Loops.** To finish the proof of Theorem 3.7, we will require the following three technical lemmas. A high-level summary: the results here extend the arguments from [She94, §2] to the setting of pCGW categories.

Lemma B.1. *The sum of two Sherman loops is equivalent to a Sherman loop. Explicitly, consider two pairs of $\mathcal{M}$-morphisms*

$$\alpha_i\colon A_i \to X_i \qquad , \qquad \beta_i\colon B_i \to Y_i, \qquad \textit{for } i = 1,2;$$

and isomorphisms

$$\theta_i\colon A_i \oplus \frac{X_i}{A_i} \oplus Y_i \longrightarrow B_i \oplus \frac{Y_i}{B_i} \oplus X_i, \qquad \textit{for } i = 1,2.$$

Then

$$G(\alpha_1,\beta_1,\theta_1) + G(\alpha_2,\beta_2,\theta_2) = G\left(\alpha_1 \oplus \alpha_2, \beta_1 \oplus \beta_2, T_2(\theta_1 \oplus \theta_2)T_1^{-1}\right)$$

in $K_1(\mathcal{C})$, whereby T_1 and T_2 are the canonical permutation isomorphisms

$$T_1\colon A_1 \oplus \frac{X_1}{A_1} \oplus Y_1 \oplus A_2 \oplus \frac{X_2}{A_2} \oplus Y_2 \to A_1 \oplus A_2 \oplus \frac{X_1}{A_1} \oplus \frac{X_2}{A_2} \oplus Y_1 \oplus Y_2$$

$$T_2\colon B_1 \oplus \frac{Y_1}{B_1} \oplus X_1 \oplus B_2 \oplus \frac{Y_2}{B_2} \oplus X_2 \to B_1 \oplus B_2 \oplus \frac{Y_1}{B_1} \oplus \frac{Y_2}{B_2} \oplus X_1 \oplus X_2.$$

Proof Sketch. There are no surprises – the argument is the same as in [She94, Prop. 1]. Apply the H-space structure of $G\mathcal{C}$ to construct the loop corresponding to $G(\alpha_1,\beta_1,\theta_1) + G(\alpha_2,\beta_2,\theta_2)$. Notice it is almost identical to $G\left(\alpha_1 \oplus \alpha_2, \beta_1 \oplus \beta_2, T_2(\theta_1 \oplus \theta_2)T_1^{-1}\right)$ – the only difference being that the direct summands of certain vertices are arranged in a different order.

To establish homotopy equivalence, use natural isomorphisms to permute these summands and construct a natural sequence of 2-simplices connecting the two loops. It remains to check that the obvious choices of 2-simplices are in fact 2-simplices in $G\mathcal{C}$ – but this follows almost immediately from the fact that distinguished squares interact well with isomorphisms (Definition 1.1).

$\square$

Lemma B.2. *Let $\mathcal{C}$ be a pCGW category whose exact squares all split. Then, every element of $K_1(\mathcal{C})$ corresponds to the loop*

$$G(A,\alpha) := \left(\begin{tikzcd} (A,A) \arrow[rr, "l(\alpha)"] & & (A,A) \\ & (O,O) \arrow[ul] \arrow[ur] & \end{tikzcd}\right) \tag{79}$$

where $l(\alpha)$ is the 1-simplex

$$l(\alpha) := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & O \arrow[d, tail] \\ A \arrow[r, tail, "1"] & A \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & O \arrow[d, tail] \\ A \arrow[r, tail, "\alpha"] & A \end{tikzcd}\right)$$

for some automorphism $(A,\alpha) \in \mathrm{Aut}(\mathcal{C})$.

Proof. The proof combines an argument from [GG87, §5] and [She94, Prop. 2]. Proceed in stages.

Step 1: Combinatorial Loops in $K_1(\mathcal{C})$. Suppose $z \in K_1(\mathcal{C}) = \pi_1|G\mathcal{C}|$. By the simplicial approximation theorem, z can be represented by a loop formed combinatorially from 1-simplices of $G\mathcal{C}$

$$\begin{pmatrix} O \\ O \end{pmatrix} \to \bullet \leftarrow \bullet \to \cdots \leftarrow \bullet \to \bullet \leftarrow \begin{pmatrix} O \\ O \end{pmatrix} \tag{80}$$

where we draw the 1-simplices as arrows. Consider one of the configurations in Diagram (80), e.g.

$$\begin{pmatrix} M' \\ L' \end{pmatrix} \leftarrow \begin{pmatrix} M \\ L \end{pmatrix} \to \begin{pmatrix} M'' \\ L'' \end{pmatrix}.$$

Since the arrows are 1-simplices in $G\mathcal{C}$, the relevant quotients agree, i.e. $\frac{M'}{M} = \frac{L'}{L}$ and $\frac{M''}{M} = \frac{L''}{L}$. Now form restricted pushouts $P := M' \star_M M''$ and $Q := L' \star_L L''$. Apply Lemma A.1 to deduce

$$\frac{P}{M} \cong \frac{M'}{M} \oplus \frac{M''}{M} \qquad \text{and} \qquad \frac{Q}{L} \cong \frac{L'}{L} \oplus \frac{L''}{L} = \frac{M'}{M} \oplus \frac{M''}{M}.$$

In fact, Equation (67) of the proof tells us the data assembles into diagram pairs, such as

$$\begin{tikzcd}
M \arrow[r, tail] \arrow[d, phantom, "\square" near start, shift left=4ex] & M' \arrow[r, tail] & P \\
O \arrow[u, "\circ" description] \arrow[r, tail] & \frac{M'}{M} \arrow[u, "\circ" description] \arrow[r, tail] & \frac{M'}{M} \oplus \frac{M''}{M} \arrow[u, "\circ" description] \\
 & O \arrow[u, "\circ" description] \arrow[r, tail] & \frac{M''}{M} \arrow[u, "\circ" description]
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r, tail] & L' \arrow[r, tail] & Q \\
O \arrow[u, "\circ" description] \arrow[r, tail] & \frac{M'}{M} \arrow[u, "\circ" description] \arrow[r, tail] & \frac{M'}{M} \oplus \frac{M''}{M} \arrow[u, "\circ" description] \\
 & O \arrow[u, "\circ" description] \arrow[r, tail] & \frac{M''}{M} \arrow[u, "\circ" description]
\end{tikzcd}.$$

Here is the upshot. By the above argument, use the proof of Lemma A.1 to define two 2-simplices in $G\mathcal{C}$

$$\left(\begin{matrix} O \rightarrowtail \\ O \rightarrowtail \end{matrix} \quad \begin{matrix} M \rightarrowtail M' \rightarrowtail P \\ \hline\hline L \rightarrowtail L' \rightarrowtail Q \end{matrix} \right) \quad \text{and} \quad \left(\begin{matrix} O \rightarrowtail \\ O \rightarrowtail \end{matrix} \quad \begin{matrix} M \rightarrowtail M'' \rightarrowtail P \\ \hline\hline L \rightarrowtail L'' \rightarrowtail Q \end{matrix} \right),$$

which fill in the two triangles of the diagram

$$\begin{tikzcd}
\begin{pmatrix} M \\ L \end{pmatrix} \arrow[r] \arrow[d] \arrow[dr] & \begin{pmatrix} M'' \\ L'' \end{pmatrix} \arrow[d] \\
\begin{pmatrix} M' \\ L' \end{pmatrix} \arrow[r] & \begin{pmatrix} P \\ Q \end{pmatrix}
\end{tikzcd}.$$

Abstractly, this turns a configuration

$$\bullet \leftarrow \bullet \to \bullet \qquad \text{into} \qquad \bullet \to \bullet \leftarrow \bullet \qquad .$$

Applying this trick multiple times, we can deform Loop (80) into one of the form

$$\begin{tikzcd}
\begin{pmatrix} O \\ O \end{pmatrix} \arrow[r] \arrow[d] & \begin{pmatrix} M_0 \\ L_0 \end{pmatrix} \arrow[r] & \ldots \arrow[r] & \begin{pmatrix} M_{q-1} \\ L_{q-1} \end{pmatrix} \arrow[d] \\
\begin{pmatrix} M'_0 \\ L'_0 \end{pmatrix} \arrow[r] & \ldots \arrow[r] & \begin{pmatrix} M'_{q-1} \\ L'_{q-1} \end{pmatrix} \arrow[r] & \begin{pmatrix} M \\ L \end{pmatrix}
\end{tikzcd} \tag{81}$$

Step 2: The Base Case. Start by analysing the component

$$\begin{pmatrix} O \\ O \end{pmatrix} \xrightarrow{l_0} \begin{pmatrix} M_0 \\ L_0 \end{pmatrix} \xrightarrow{l_1} \begin{pmatrix} M_1 \\ L_1 \end{pmatrix} \tag{82}$$

of Loop (81) in $K_1(\mathcal{C})$. Suppose l_0 is defined by the following pair of exact squares

$$l_0 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & \widehat{M}_0 \arrow[d, o->, "\eta_0"] \\ O \arrow[r, tail] & M_0 \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & \widehat{M}_0 \arrow[d, o->, "\mu_0"] \\ O \arrow[r, tail] & L_0 \end{tikzcd} \right), \tag{83}$$

with isomorphisms η_0 and μ_0. Recall that CGW categories require an isomorphism of categories

$$\varphi\colon \mathrm{iso}\mathcal{M} \to \mathrm{iso}\mathcal{E}.$$

We can therefore define two 1-simplices

$$l_0' := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & \widehat{M}_0 \arrow[d, o->, "1"] \\ O \arrow[r, tail] & \widehat{M}_0 \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & \widehat{M}_0 \arrow[d, o->, "1"] \\ O \arrow[r, tail] & \widehat{M}_0 \end{tikzcd} \right) \quad \text{and} \quad l_0'' := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & O \arrow[d, o->] \\ \widehat{M}_0 \arrow[r, tail, "\varphi^{-1}(\eta_0)"] & M_0 \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & O \arrow[d, o->] \\ \widehat{M}_0 \arrow[r, tail, "\varphi^{-1}(\mu_0)"] & L_0 \end{tikzcd} \right),$$

which assemble into the following 2-simplex

$$\begin{tikzcd} O \arrow[r, tail] & \widehat{M}_0 \arrow[r, tail, "\varphi^{-1}(\eta_0)"] & M_0 \\ O \arrow[r, tail] \arrow[u, o->] \arrow[ur, phantom, "\square"] & \widehat{M}_0 \arrow[r, tail, "1"] \arrow[u, o->, "1"] \arrow[ur, phantom, "\square"] & \widehat{M}_0 \arrow[u, o->, "\eta_0"'] \\ & O \arrow[r, tail] \arrow[u, o->] \arrow[ur, phantom, "\square"] & O \arrow[u, o->] \end{tikzcd} \quad \begin{tikzcd} O \arrow[r, tail] & \widehat{M}_0 \arrow[r, tail, "\varphi^{-1}(\mu_0)"] & L_0 \\ O \arrow[r, tail] \arrow[u, o->] \arrow[ur, phantom, "\square"] & \widehat{M}_0 \arrow[r, tail, "1"] \arrow[u, o->, "1"] \arrow[ur, phantom, "\square"] & \widehat{M}_0 \arrow[u, o->, "\mu_0"'] \\ & O \arrow[r, tail] \arrow[u, o->] \arrow[ur, phantom, "\square"] & O \arrow[u, o->] \end{tikzcd}.$$

We remark that the top right squares are distinguished by Axiom (I) of Definition 1.4. We then assemble the following diagram

$$\begin{tikzcd} & \begin{pmatrix} \widehat{M}_0 \\ \widehat{M}_0 \end{pmatrix} \arrow[d, "l_0''"] \arrow[dr, "l_1 \circ l_0''"] & \\ \begin{pmatrix} O \\ O \end{pmatrix} \arrow[ur, "l_0'"] \arrow[r, "l_0"] & \begin{pmatrix} M_0 \\ L_0 \end{pmatrix} \arrow[r, "l_1"] & \begin{pmatrix} M_1 \\ L_1 \end{pmatrix} \end{tikzcd}. \tag{84}$$

One easily checks the added triangle also bounds a 2-simplex, which implies the red path of 1-simplices is homotopic to the blue path. Hence, assume without loss of generality that both μ_0 and η_0 of Equation (83) are the identity $1\colon M_0 \to M_0$.

Step 3: The Inductive Step. Proceeding along Loop (82), consider l_1, to be represented as

$$l_1 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & \frac{M_1}{M_0} \arrow[d, o->, "\eta_1'"] \\ M_0 \arrow[r, tail, "\eta_1"] & M_1 \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & \frac{M_1}{M_0} \arrow[d, o->, "\mu_1'"] \\ M_0 \arrow[r, tail, "\mu_1"] & L_1 \end{tikzcd} \right).$$

Since all exact squares split by hypothesis, this yields isomorphisms

$$\psi_M\colon M_1 \rightarrowtail M_0 \oplus \frac{M_1}{M_0} \qquad \text{and} \qquad \psi_L\colon L_1 \rightarrowtail M_0 \oplus \frac{M_1}{M_0},$$

which we shall consider as $\mathcal{M}$-morphisms. This defines the following distinguished squares

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{M_1}{M_0} \arrow[r, tail, "1"] \arrow[d, "\eta_1'"] \arrow[dr, phantom, "\square"] & \frac{M_1}{M_0} \arrow[d, "\varphi(\psi_M)\circ\eta_1'"] \\ M_0 \arrow[r, tail, "\eta_1"] & M_1 \arrow[r, tail, "\psi_M"] & M_0 \oplus \frac{M_1}{M_0} \end{tikzcd} \qquad \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{M_1}{M_0} \arrow[r, tail, "1"] \arrow[d, "\eta_1'"] \arrow[dr, phantom, "\square"] & \frac{M_1}{M_0} \arrow[d, "\varphi(\psi_L)\circ\mu_1'"] \\ M_0 \arrow[r, tail, "\mu_1"] & L_1 \arrow[r, tail, "\psi_L"] & M_0 \oplus \frac{M_1}{M_0} \end{tikzcd}$$

The right squares are distinguished by Axiom (I), and thus the horizontal compositions define two exact squares. To ease notation, denote

$$\begin{aligned} v &:= \psi_M \circ \eta_1 \qquad \text{and} \qquad v' := \varphi(\psi_M) \circ \eta_1', \\ w &:= \psi_L \circ \mu_1 \qquad \text{and} \qquad w' := \varphi(\psi_L) \circ \mu_1'. \end{aligned}$$

In fact, we can say more. Denote

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{M_1}{M_0} \arrow[d, "q"] \\ M_0 \arrow[r, tail, "p"] & M_0 \oplus \frac{M_1}{M_0} \end{tikzcd}$$

to be the canonical direct sum square of $M_0 \oplus \frac{M_1}{M_0}$. Since both exact squares in l_1 are split, we know that

$$\begin{aligned} v = \psi_M \circ \eta_1 = p = \psi_L \circ \mu_1 = w \\ v' = \varphi(\psi_M) \circ \eta_1' = q = \varphi(\psi_L) \circ \mu_1' = w'. \end{aligned} \tag{85}$$

Leverage these identities to construct the following 2-simplex

$$\begin{tikzcd} O \arrow[r, tail] \arrow[dr, phantom, "\square"] & M_0 \arrow[r, tail, "\eta_1"] \arrow[dr, phantom, "\square"] & M_1 \\ O \arrow[r, tail] \arrow[u] & M_0 \arrow[r, tail, "v"] \arrow[u, "1"] \arrow[dr, phantom, "\square"] & M_0 \oplus \frac{M_1}{M_0} \arrow[u, "\varphi(\psi_M^{-1})"'] \\ & O \arrow[r, tail] \arrow[u] & \frac{M_1}{M_0} \arrow[u, "v'"'] \end{tikzcd} \qquad \begin{tikzcd} O \arrow[r, tail] \arrow[dr, phantom, "\square"] & M_0 \arrow[r, tail, "\mu_1"] \arrow[dr, phantom, "\square"] & L_1 \\ O \arrow[r, tail] \arrow[u] & M_0 \arrow[r, tail, "v"] \arrow[u, "1"] \arrow[dr, phantom, "\square"] & M_0 \oplus \frac{M_1}{M_0} \arrow[u, "\varphi(\psi_L^{-1})"'] \\ & O \arrow[r, tail] \arrow[u] & \frac{M_1}{M_0} \arrow[u, "v'"'] \end{tikzcd} . \tag{86}$$

To see why the square indicated in red is distinguished, notice:

- **Case 1.** Suppose $\mathcal{E}$ is a subcategory of $\mathcal{C}$. Then
  $$\varphi(\psi_L^{-1}) \circ v = \psi_L^{-1} \circ v = \mu_1 \qquad \text{in } \mathcal{C}$$
  if and only if
  $$v = \psi_L \circ \mu_1,$$
  which holds by Identity (85).
- **Case 2.** Suppose $\mathcal{E}^{\mathrm{op}}$ is a subcategory of $\mathcal{C}$. Applying Identity (85) once more, deduce
  $$v = \varphi(\psi_L^{-1}) \circ \mu_1 = \psi_L \circ \mu_1.$$

The claim then follows from the fact that distinguished squares interact well with isomorphisms. The case for the blue-indicated square is analogous.

Having checked that Diagram (86) defines a 2-simplex, the gears line up and the inductive argument falls into place. First note that Diagram (86) defines a homotopy between

$$\begin{pmatrix} O \\ O \end{pmatrix} \xrightarrow{l_0} \begin{pmatrix} M_0 \\ L_0 \end{pmatrix} \xrightarrow{l_1} \begin{pmatrix} M_1 \\ L_1 \end{pmatrix}$$

and a 1-simplex of the form

$$l_1' \colon \begin{pmatrix} O \\ O \end{pmatrix} \to \begin{pmatrix} M_1 \\ L_1 \end{pmatrix}$$

whereby

$$
l_1' := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \widehat{M_1} \arrow[d, {Circle[open]}->, "\alpha_1"] \\ O \arrow[r, tail] & M_1 \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \widehat{M_1} \arrow[d, {Circle[open]}->, "\beta_1"] \\ O \arrow[r, tail] & L_1 \end{tikzcd} \right).
$$

Hence, the initial segment of Loop (81) is homotopic to

$$
\begin{pmatrix} O \\ O \end{pmatrix} \xrightarrow{l_1'} \begin{pmatrix} M_1 \\ L_1 \end{pmatrix} \xrightarrow{l_2} \begin{pmatrix} M_2 \\ L_2 \end{pmatrix},
$$

deleting a term. Then, apply the Base Case argument (Step 2) to justify presenting l_1' as

$$
l_1' := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & M_1 \arrow[d, {Circle[open]}->, "1"] \\ O \arrow[r, tail] & M_1 \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & M_1 \arrow[d, {Circle[open]}->, "1"] \\ O \arrow[r, tail] & M_1 \end{tikzcd} \right),
$$

which sets up our inductive step again. Keep going for the rest of Loop (81) on both sides, until we finally obtain a loop of the form

$$
\begin{pmatrix} O \\ O \end{pmatrix} \xrightarrow{\kappa} (M, L) \xleftarrow{\gamma} \begin{pmatrix} O \\ O \end{pmatrix}, \tag{87}
$$

where

$$
\kappa := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \widehat{M} \arrow[d, {Circle[open]}->, "\kappa_0"] \\ O \arrow[r, tail] & M \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \widehat{M} \arrow[d, {Circle[open]}->, "\kappa_1"] \\ O \arrow[r, tail] & L \end{tikzcd} \right) \quad \text{and} \quad \gamma := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \widehat{N} \arrow[d, {Circle[open]}->, "\gamma_0"] \\ O \arrow[r, tail] & M \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \widehat{N} \arrow[d, {Circle[open]}->, "\gamma_1"] \\ O \arrow[r, tail] & L \end{tikzcd} \right)
$$

Notice, however, we can no longer apply the Base Case argument to simplify κ or γ since the arrows of Diagram (87) are in the wrong direction.

Step 4: Finish. A technical observation: both κ and γ define isomorphisms $L \xrightarrow{\cong} M$ in $\mathcal{C}$ but the presentation will differ depending on whether $\mathcal{E}^{\mathrm{op}}$ or $\mathcal{E}$ is a subcategory of $\mathcal{C}$.

Case 1: $\mathcal{E} \subseteq \mathcal{C}$. In which case, define $\omega := \kappa_0 \circ \kappa_1^{-1}$ and $\lambda := \gamma_0 \circ \gamma_1^{-1}$ in $\mathcal{C}$

Case 2: $\mathcal{E}^{\mathrm{op}} \subseteq \mathcal{C}$. In which case, define $\omega := \kappa_0^{-1} \circ \kappa_1$ and $\lambda := \gamma_0^{-1} \circ \gamma_1$ in $\mathcal{C}$.

By Axiom (I), we may regard ω and λ as $\mathcal{M}$-morphisms as well. We now construct the obvious diagram

$$
\begin{tikzcd} & \begin{pmatrix} M \\ M \end{pmatrix} \arrow[r, blue, "g"] & \begin{pmatrix} M \\ M \end{pmatrix} \\ \begin{pmatrix} O \\ O \end{pmatrix} \arrow[ur, blue, "f_0"] \arrow[r, red, "\kappa"] & \begin{pmatrix} M \\ L \end{pmatrix} \arrow[u, "f_1"] \arrow[ur, "f_2"] & \begin{pmatrix} O \\ O \end{pmatrix} \arrow[l, red, "\gamma"'] \arrow[u, blue, "f_3"'] \end{tikzcd} \tag{88}
$$

whereby

$$
g := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & O \arrow[d, {Circle[open]}->] \\ M \arrow[r, tail, "1"] & M \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & O \arrow[d, {Circle[open]}->] \\ M \arrow[r, tail, "\lambda\circ\omega^{-1}"] & M \end{tikzcd} \right)
$$

$$
f_0 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \widehat{M} \arrow[d, {Circle[open]}->, "\kappa_0"] \\ O \arrow[r, tail] & M \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \widehat{M} \arrow[d, {Circle[open]}->, "\kappa_0"] \\ O \arrow[r, tail] & M \end{tikzcd} \right) \qquad f_3 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \widehat{N} \arrow[d, {Circle[open]}->, "\gamma_0"] \\ O \arrow[r, tail] & M \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \widehat{N} \arrow[d, {Circle[open]}->, "\gamma_0"] \\ O \arrow[r, tail] & M \end{tikzcd} \right).
$$

$$f_1 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & O \arrow[d, {Circle[open]}->] \\ M \arrow[r, tail, "1"] & M \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & O \arrow[d, {Circle[open]}->] \\ L \arrow[r, tail, "\omega"] & M \end{tikzcd} \right) \qquad f_2 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & O \arrow[d, {Circle[open]}->] \\ M \arrow[r, tail, "1"] & M \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & O \arrow[d, {Circle[open]}->] \\ L \arrow[r, tail, "\lambda"] & M \end{tikzcd} \right).$$

In both cases ($\mathcal{E} \subseteq \mathcal{C}$ or $\mathcal{E}^{\mathrm{op}} \subseteq \mathcal{C}$), it is easy to check that the triangles of Diagram (88) bound the following 2-simplices:

$$f_1\kappa = f_0 \qquad \left(\begin{tikzcd} O \arrow[r, tail] & M \arrow[r, tail, "1"] & M \\ & \widehat{M} \arrow[u, {Circle[open]}->, "\kappa_0"] \arrow[r, tail, "1"] \arrow[ur, phantom, "\square"] & \widehat{M} \arrow[u, {Circle[open]}->, "\kappa_0"'] \\ & & O \arrow[u, {Circle[open]}->] \end{tikzcd} \qquad \begin{tikzcd} O \arrow[r, tail] & L \arrow[r, tail, "\omega"] & M \\ & \widehat{M} \arrow[u, {Circle[open]}->, "\kappa_1"] \arrow[r, tail, "1"] \arrow[ur, phantom, "\square"] & \widehat{M} \arrow[u, {Circle[open]}->, "\kappa_0"'] \\ & & O \arrow[u, {Circle[open]}->] \end{tikzcd} \right).$$

$$f_2\gamma = f_3 \qquad \left(\begin{tikzcd} O \arrow[r, tail] & M \arrow[r, tail, "1"] & M \\ & \widehat{N} \arrow[u, {Circle[open]}->, "\gamma_0"] \arrow[r, tail, "1"] \arrow[ur, phantom, "\square"] & \widehat{N} \arrow[u, {Circle[open]}->, "\gamma_0"'] \\ & & O \arrow[u, {Circle[open]}->] \end{tikzcd} \qquad \begin{tikzcd} O \arrow[r, tail] & L \arrow[r, tail, "\lambda"] & M \\ & \widehat{N} \arrow[u, {Circle[open]}->, "\gamma_1"] \arrow[r, tail, "1"] \arrow[ur, phantom, "\square"] & \widehat{N} \arrow[u, {Circle[open]}->, "\gamma_0"'] \\ & & O \arrow[u, {Circle[open]}->] \end{tikzcd} \right).$$

$$gf_1 = f_2 \qquad \left(\begin{tikzcd} M \arrow[r, tail, "1"] & M \arrow[r, tail, "1"] & M \\ & O \arrow[u, {Circle[open]}->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & O \arrow[u, {Circle[open]}->] \\ & & O \arrow[u, {Circle[open]}->] \end{tikzcd} \qquad \begin{tikzcd} L \arrow[r, tail, "\omega"] & M \arrow[r, tail, "\lambda\circ\omega^{-1}"] & M \\ & O \arrow[u, {Circle[open]}->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & O \arrow[u, {Circle[open]}->] \\ & & O \arrow[u, {Circle[open]}->] \end{tikzcd} \right).$$

Conclude that the red loop in Diagram (88) is homotopic to the blue loop. Notice the blue loop is precisely of the form $G(A, \alpha)$ as claimed in lemma statement, with $A := M$ and $\alpha := \lambda \circ \omega^{-1}$. $\square$

Lemma B.3. *The automorphism loop $G(A, \alpha)$ in Lemma B.2 is equivalent to a Sherman Loop.*

Proof. Let $p_A\colon A \rightarrowtail A \oplus A$ be an $\mathcal{M}$-morphism arising from direct sum squares and let $\tau_A\colon A \oplus A \rightarrowtail A \oplus A$ as the isomorphism swapping components. Define the following loop

$$\begin{pmatrix} O \\ O \end{pmatrix} \to \begin{pmatrix} A \\ A \end{pmatrix} \xrightarrow{\iota_\alpha} \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \xrightarrow{l_\tau} \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \leftarrow \begin{pmatrix} O \\ O \end{pmatrix} \tag{89}$$

where

$$\iota_\alpha := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & A \arrow[d, {Circle[open]}->, "q_A"] \\ A \arrow[r, tail, "p_A"] & A \oplus A \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & A \arrow[d, {Circle[open]}->, "q_A"] \\ A \arrow[r, tail, "p_A\circ\alpha"] & A \oplus A \end{tikzcd} \right) \quad l_\tau := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & O \arrow[d, {Circle[open]}->] \\ A \oplus A \arrow[r, tail, "\tau_A"] & A \oplus A \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & O \arrow[d, {Circle[open]}->] \\ A \oplus A \arrow[r, tail, "\tau_A"] & A \oplus A \end{tikzcd} \right)$$

This is a Sherman Loop $G(\alpha, 0, \tau_A)$, where 0 denotes the $\mathcal{M}$-morphism $O \rightarrowtail A$.

To show $G(A, \alpha) \sim G(\alpha, 0, \tau_A)$ in $\pi_1(|G\mathcal{C}|)$ involves modifying $G(A, \alpha)$ in sensible ways that respects its homotopy class. An initial observation: the following Loop

$$\begin{pmatrix} O \\ O \end{pmatrix} \to \begin{pmatrix} A \\ A \end{pmatrix} \xrightarrow{\iota_\alpha} \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \leftarrow \begin{pmatrix} O \\ O \end{pmatrix}. \tag{90}$$

is homotopic to $G(A, \alpha)$, since the following diagram

$$\begin{tikzcd}
\begin{pmatrix} A \\ A \end{pmatrix} \arrow[rr, "\iota_\alpha"] \arrow[dr, "\alpha"'] & & \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \\
& \begin{pmatrix} A \\ A \end{pmatrix} \arrow[ur, "\iota_A"'] &
\end{tikzcd}$$

bounds a 2-simplex, where ι_A corresponds to the canonical direct sum square $A \oplus A$. Further, the Loop (90) is homotopic to

$$\begin{pmatrix} O \\ O \end{pmatrix} \to \begin{pmatrix} A \\ A \end{pmatrix} \xrightarrow{\iota_\alpha} \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \xrightarrow{1_{A\oplus A}} \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \leftarrow \begin{pmatrix} O \\ O \end{pmatrix}, \tag{91}$$

since all we did was insert a degenerate 1-simplex $1_{A\oplus A}$. It remains to show that $G(\alpha, 0, \tau_A)$ is homotopic to Loop (91). But this follows from noting that the triangles in the diagram below bound 2-simplices

$$\begin{tikzcd}
\begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \arrow[r, "1_{A\oplus A}"] \arrow[dr, "l_\tau"'] & \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} & \begin{pmatrix} O \\ O \end{pmatrix} \arrow[l] \arrow[dl] \\
& \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \arrow[u, "l_\tau"] &
\end{tikzcd}.$$

$\square$

B.2. **Explicit Descriptions of 2-Simplices.** This section explicitly constructs the two key homotopies claimed by Lemma 3.24, necessary to finish the proof of Theorem 3.17

Claim B.4. *Loop* (32) *is homotopic to loop L.*

Proof. Recall: in order to establish that the two loops are homotopic, it suffices to show that the indicated triangles of the Diagram (39) are boundaries of 2-simplices. We describe the 2-simplices explicitly below.

- Triangle (1). Consider

$$\begin{tikzcd}
A \oplus A' \arrow[r, tail, "f_0"] & P \arrow[r, tail, "1\oplus C\oplus C'"] \arrow[dr, phantom, "\square"] & P \oplus C \oplus C' \\
& C \oplus C' \arrow[u, o->, "g_0"] \arrow[r, tail] & C \oplus C' \oplus C \oplus C' \arrow[u, o->, "g_0\oplus 1"'] \\
& & C \oplus C' \arrow[u, o->]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \oplus A' \arrow[r, tail, "f_1"] & Q \arrow[r, tail, "h_t"] \arrow[dr, phantom, "\square"] & V \\
& C \oplus C' \arrow[u, o->, "g_1"] \arrow[r, tail] & C \oplus C' \oplus C \oplus C' \arrow[u, o->, "j_t"'] \\
& & C \oplus C' \arrow[u, o->]
\end{tikzcd}$$

  [To show this is a 2-simplex, we need to check that the relevant subdiagrams define distinguished squares. The indicated square on the right diagram is the distinguished square t of Lemma 3.23. The left indicated square is distinguished by Lemma A.4 (i). The rest are obvious.]

- Triangle (2).

$$\begin{tikzcd}
A \oplus A' \arrow[r, tail, "f_1"] & Q \arrow[r, tail, "1\oplus C\oplus C'"] \arrow[dr, phantom, "\square"] & Q \oplus C \oplus C' \\
& C \oplus C' \arrow[u, o->, "g_1"] \arrow[r, tail] & C \oplus C' \oplus C \oplus C' \arrow[u, o->, "g_1\oplus 1"'] \\
& & C \oplus C' \arrow[u, o->]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \oplus A' \arrow[r, tail, "f_1"] & Q \arrow[r, tail, "h_t"] \arrow[dr, phantom, "\square"] & V \\
& C \oplus C' \arrow[u, o->, "g_1"] \arrow[r, tail] & C \oplus C' \oplus C \oplus C' \arrow[u, o->, "j_t"'] \\
& & C \oplus C' \arrow[u, o->]
\end{tikzcd}$$

  [Why is this a 2-simplex? Analogous to Triangle (1).]

- Triangle (3).

$$
\begin{tikzcd}
P \arrow[r, tail, "1\oplus C\oplus C'"] & P\oplus C\oplus C' \arrow[r, tail, "\theta\oplus 1"] \arrow[dr, phantom, "\square"] & Q\oplus C\oplus C' \\
& C\oplus C' \arrow[u, "P\oplus 1"] \arrow[r, tail, "1"] & C\oplus C' \arrow[u, "Q\oplus 1"'] \\
& & O \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \arrow[r, tail, "h_t"] & V \arrow[r, tail, "1"] \arrow[dr, phantom, "\square"] & V \\
& C\oplus C' \arrow[u, "k_t"] \arrow[r, tail, "1"] & C\oplus C' \arrow[u, "k_t"'] \\
& & O \arrow[u]
\end{tikzcd}
$$

[To show that this is a 2-simplex, notice Lemma 3.23 already verified that

$$
t' := \left(
\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C\oplus C' \arrow[d, "k_t"] \\
Q \arrow[r, tail, "h_t"] & V
\end{tikzcd}
\right)
$$

is a distinguished square. The rest either follow from Lemma A.4 or are immediate.]

- Triangle (4).

$$
\begin{tikzcd}
P \arrow[r, tail, "\theta"] & Q \arrow[r, tail, "1\oplus C\oplus C'"] \arrow[dr, phantom, "\square"] & Q\oplus C\oplus C' \\
& O \arrow[u] \arrow[r, tail] & C\oplus C' \arrow[u, "Q\oplus 1"'] \\
& & C\oplus C' \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \arrow[r, tail, "1"] & Q \arrow[r, tail, "h_t"] \arrow[dr, phantom, "\square"] & V \\
& O \arrow[u] \arrow[r, tail] & C\oplus C' \arrow[u, "k_t"'] \\
& & C\oplus C' \arrow[u]
\end{tikzcd}
$$

[Why is this a 2-simplex? Obvious.]

□

Claim B.5. *Loop* (34) *is homotopic to loop L.*

Proof. By analogy with Claim B.4, all indicated triangles of Diagram (40) are boundaries of 2-simplices. The only subtlety is that we need to verify that Triangles (1') and (2') do in fact bound the 2-simplices below, but this follows from V being a restricted pushout (see remarks above Diagram (39)).

- Triangle (1').

$$
\begin{tikzcd}
A\oplus A' \arrow[r, tail, "f_0"] & P \arrow[r, tail, "1\oplus C\oplus C'"] \arrow[dr, phantom, "\square"] & P\oplus C\oplus C' \\
& C\oplus C' \arrow[u, "g_0"] \arrow[r, tail] & C\oplus C'\oplus C\oplus C' \arrow[u, "g_0\oplus 1"'] \\
& & C\oplus C' \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
A\oplus A' \arrow[r, tail, "\alpha\oplus\alpha'"] & B\oplus B' \arrow[r, tail, "h_u"] \arrow[dr, phantom, "\square"] & V \\
& C\oplus C' \arrow[u, "\delta\oplus\delta'"] \arrow[r, tail] & C\oplus C'\oplus C\oplus C' \arrow[u, "j_u"'] \\
& & C\oplus C' \arrow[u]
\end{tikzcd}
$$

- Triangle (2').

$$
\begin{tikzcd}
A\oplus A' \arrow[r, tail, "f_1"] & Q \arrow[r, tail, "1\oplus C\oplus C'"] \arrow[dr, phantom, "\square"] & Q\oplus C\oplus C' \\
& C\oplus C' \arrow[u, "g_1"] \arrow[r, tail] & C\oplus C'\oplus C\oplus C' \arrow[u, "g_1\oplus 1"'] \\
& & C' \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
A\oplus A' \arrow[r, tail, "\alpha\oplus\alpha'"] & B\oplus B' \arrow[r, tail, "h_u"] \arrow[dr, phantom, "\square"] & V \\
& C\oplus C' \arrow[u, "\delta\oplus\delta'"] \arrow[r, tail] & C\oplus C'\oplus C\oplus C' \arrow[u, "j_u"'] \\
& & C' \arrow[u]
\end{tikzcd}
$$

- Triangle (3').

$$\begin{tikzcd}
P \arrow[r, tail, "1\oplus C\oplus C'"] & P\oplus C\oplus C' \arrow[r, tail, "\theta\oplus 1"] \arrow[dr, phantom, "\square"] & Q\oplus C\oplus C' \\
& C\oplus C' \arrow[u, "P\oplus 1"] \arrow[r, tail, "1"] & C\oplus C' \arrow[u, "Q\oplus 1"'] \\
& & O \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
B\oplus B' \arrow[r, tail, "h_u"] & V \arrow[r, tail, "1"] \arrow[dr, phantom, "\square"] & V \\
& C\oplus C' \arrow[u, "k_u"] \arrow[r, tail, "1"] & C\oplus C' \arrow[u, "k_u"'] \\
& & O \arrow[u]
\end{tikzcd}$$

- Triangle (4').

$$\begin{tikzcd}
P \arrow[r, tail, "\theta"] & Q \arrow[r, tail, "1\oplus C\oplus C'"] \arrow[dr, phantom, "\square"] & Q\oplus C\oplus C' \\
& O \arrow[u] \arrow[r, tail] & C\oplus C' \arrow[u, "Q\oplus 1"'] \\
& & C\oplus C' \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
B\oplus B' \arrow[r, tail, "1"] & B\oplus B' \arrow[r, tail, "h_u"] \arrow[dr, phantom, "\square"] & V \\
& O \arrow[u] \arrow[r, tail] & C\oplus C' \arrow[u, "k_u"'] \\
& & C\oplus C' \arrow[u]
\end{tikzcd}$$

∎

B.3. **Admissible Triples.** Recall: Lemma 4.4 claims τ is freely homotopic to $(P_2, P_2')\oplus\mu(l(\tau))$. We prove this by showing that all six triangles in Diagram (48) bound 2-simplices in $G\mathcal{C}$, as below.

- Triangle (1)

$$\begin{tikzcd}
P_1 \arrow[r, tail, "\alpha_{1,2}\oplus P_{1/0}"] & P_2\oplus P_{1/0} \arrow[r, tail, "1\oplus\alpha_{1/0,2/0}"] \arrow[dr, phantom, "\square"] & P_2\oplus P_{2/0} \\
& P_{2/1}\oplus P_{1/0} \arrow[u, "\alpha_{2/1,2}\oplus 1"] \arrow[r, tail, "1\oplus\alpha_{1/0,2/0}"'] & P_{2/1}\oplus P_{2/0} \arrow[u, "\alpha_{2/1,2}\oplus 1"'] \\
& & P_{2/1} \arrow[u, "P_{2/1}\oplus\alpha_{2/1,2/0}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
P_1' \arrow[r, tail, "\alpha'_{1,2}\oplus P_{1/0}"] & P_2'\oplus P_{1/0} \arrow[r, tail, "1\oplus\alpha_{1/0,2/0}"] \arrow[dr, phantom, "\square"] & P_2'\oplus P_{2/0} \\
& P_{2/1}\oplus P_{1/0} \arrow[u, "\alpha'_{2/1,2}\oplus 1"] \arrow[r, tail, "1\oplus\alpha_{1/0,2/0}"'] & P_{2/1}\oplus P_{2/0} \arrow[u, "\alpha'_{2/1,2}\oplus 1"'] \\
& & P_{2/1} \arrow[u, "P_{2/1}\oplus\alpha_{2/1,2/0}"']
\end{tikzcd}$$

- Triangle (2).

$$\begin{tikzcd}
P_1 \arrow[r, tail, "\alpha_{1,2}"] & P_2 \arrow[r, tail, "1\oplus P_{2/0}"] \arrow[dr, phantom, "\square"] & P_2\oplus P_{2/0} \\
& P_{2/1} \arrow[u, "\alpha_{2/1,2}"] \arrow[r, tail, "1\oplus P_{2/0}"'] & P_{2/1}\oplus P_{2/0} \arrow[u, "\alpha_{2/1,2}\oplus 1"'] \\
& & P_{2/0} \arrow[u, "1\oplus P_{2/0}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
P_1' \arrow[r, tail, "\alpha'_{1,2}"] & P_2' \arrow[r, tail, "1\oplus P_{2/0}"] \arrow[dr, phantom, "\square"] & P_2'\oplus P_{2/0} \\
& P_{2/1} \arrow[u, "\alpha'_{2/1,2}"] \arrow[r, tail, "1\oplus P_{2/0}"'] & P_{2/1}\oplus P_{2/0} \arrow[u, "\alpha'_{2/1,2}\oplus 1"'] \\
& & P_{2/0} \arrow[u, "1\oplus P_{2/0}"']
\end{tikzcd}$$

- Triangle (3)

$$\begin{tikzcd}
P_0 \arrow[r, tail, "\alpha_{0,1}"] & P_1 \arrow[r, tail, "\alpha_{1,2}\oplus P_{1,0}"] \arrow[dr, phantom, "\square"] & P_2\oplus P_{1/0} \\
& P_{1/0} \arrow[u, "\alpha_{1/0,1}"] \arrow[r, tail, "\alpha_{1/0,2/0}\oplus P_{1/0}"'] & P_{2/0}\oplus P_{1/0} \arrow[u, "\alpha_{2/0,2}\oplus 1"'] \\
& & P_{2/1}\oplus P_{1/0} \arrow[u, "\alpha_{2/1,2/0}\oplus 1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
P_0' \arrow[r, tail, "\alpha'_{0,1}"] & P_1' \arrow[r, tail, "\alpha'_{1,2}\oplus P_{1,0}"] \arrow[dr, phantom, "\square"] & P_2'\oplus P_{1/0} \\
& P_{1/0}' \arrow[u, "\alpha'_{1/0,1}"] \arrow[r, tail, "\alpha_{1/0,2/0}\oplus P_{1/0}"'] & P_{2/0}'\oplus P_{1/0} \arrow[u, "\alpha_{2/0,2}\oplus 1"'] \\
& & P_{2/1}'\oplus P_{1/0} \arrow[u, "\alpha_{2/1,2/0}\oplus 1"']
\end{tikzcd}$$

- Triangles (4) & (6)

$$\begin{tikzcd}
P_0 \arrow[r, tail, "\alpha_{0,2}"] & P_2 \arrow[r, tail, "1\oplus P_{2/0}"] \arrow[dr, phantom, "\square"] & P_2 \oplus P_{2/0} \\
& P_{2/0} \arrow[u, "\alpha_{2/0,2}"] \arrow[r, tail] & P_{2/0} \oplus P_{2/0} \arrow[u, "\alpha_{2/0,2}\oplus 1"'] \\
& & P_{2/0} \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
P_0' \arrow[r, tail, "\alpha_{0,2}'"] & P_2' \arrow[r, tail, "1\oplus P_{2/0}"] \arrow[dr, phantom, "\square"] & P_2' \oplus P_{2/0} \\
& P_{2/0} \arrow[u, "\alpha_{2/0,2}'"] \arrow[r, tail] & P_{2/0} \oplus P_{2/0} \arrow[u, "\alpha_{2/0,2}'\oplus 1"'] \\
& & P_{2/0} \arrow[u]
\end{tikzcd}$$

- Triangle (5)

$$\begin{tikzcd}
P_0 \arrow[r, tail, "\alpha_{0,2}"] & P_2 \arrow[r, tail, "1\oplus P_{1/0}"] \arrow[dr, phantom, "\square"] & P_2 \oplus P_{1/0} \\
& P_{2/0} \arrow[u, "\alpha_{2/0,2}"] \arrow[r, tail] & P_{2/0} \oplus P_{1/0} \arrow[u, "\alpha_{2/0,2}\oplus 1"'] \\
& & P_{1/0} \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
P_0' \arrow[r, tail, "\alpha_{0,2}'"] & P_2' \arrow[r, tail, "1\oplus P_{1/0}"] \arrow[dr, phantom, "\square"] & P_2' \oplus P_{1/0} \\
& P_{2/0} \arrow[u, "\alpha_{2/0,2}'"] \arrow[r, tail] & P_{2/0} \oplus P_{1/0} \arrow[u, "\alpha_{2/0,2}'\oplus 1"'] \\
& & P_{1/0} \arrow[u]
\end{tikzcd}$$

These diagrams are the obvious choices – no surprises here. To verify that the relevant subdiagrams define distinguished squares, apply Lemma A.4. That these 2-simplices assemble into Diagram (48) (= share the relevant common faces) is either clear by inspection or taking repeated restricted pushouts, e.g.

$$\begin{tikzcd}
P_1 \arrow[r, tail, "1\oplus P_{1/0}"] \arrow[d, tail, "\alpha_{1,2}"'] & P_1 \oplus P_{1/0} \arrow[r, tail, "1\oplus\alpha_{1/0,2/0}"] \arrow[d, tail, "\alpha_{1,2}\oplus 1"] & P_1 \oplus P_{2/0} \arrow[d, tail] \\
P_2 \arrow[r, tail, "1\oplus P_{1/0}"'] & P_2 \oplus P_{1/0} \arrow[r, tail, "1\oplus\alpha_{1/0,2/0}"'] & P_2 \oplus P_{2/0}
\end{tikzcd}$$

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