THE ARCHIMEDEAN PLACE IS A BLURRED INTERVAL AT INFINITY

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ABSTRACT.
- Topology & algebra are intrinsically linked, cannot be separated by (classical) set theory.
- Point-free perspective.
- Our investigation uses ideas from constructive analysis, point-free topology, descent techniques from topos theory and geometric logic.
- An organising theme of this paper is that while it is clear that the exponentiation of absolute values gives an algebraic action, characterising the point-free spaces quotiented by this action is a subtler issue.

1. INTRODUCTION

1.1. **Motivation from Arithmetic Geometry.** Much of the theory-building in number theory has been guided by a deep tension: while it is important to treat all the completions of the rationals[1] $\mathbb{Q}$ symmetrically (cf. the Hasse principle), it is also clear that there exist key disanalogies between the p-adics and the reals. The depth of these disanalogies can be measured by the fact that many powerful technologies work well in one setting but not the other.[2] Indeed, as Mazur muses [Maz93]:

> "A major theme in the development of Number Theory has been to try to bring $\mathbb{R}$ somewhat more into line with the p-adic fields; a major mystery is why $\mathbb{R}$ resists this attempt so strenuously."

This leads to a natural question, which will guide the investigations of this paper.

Question 1.1. What is the right perspective from which to understand this tension? That is, how can we treat the p-adics and the reals symmetrically whilst also accommodating their differences?

The number theorist is likely to have one of two reactions to Question 1.1 (and in fact, perhaps both). First, that our understanding of the reals and the p-adics should be guided by the function field analogy. Two, as already alluded to by Mazur, that we should strive to develop tools that work well for both settings. We discuss this in the context of Arakelov intersection theory [Ara74, PR21].

The Function Field Case. Consider a smooth affine curve C over an algebraically closed field k. Then, take the (unique) smooth compactification of C, which adds a finite number of points to yield a smooth projective curve $\overline{C}$. A *divisor* D on $\overline{C}$ is a finite formal linear combination of points on $\overline{C}$

$$D = \sum_{P \in \overline{C}} n_P \cdot P, \qquad n_P \in \mathbb{Z}. \tag{1}$$

In particular, for any non-zero *rational function* f on $\overline{C}$, one can define the divisor

$$(f) = \sum_{P \in \overline{C}} \mathrm{ord}_P(f) \cdot P, \tag{2}$$

where $\mathrm{ord}_P(f)$ denotes the multiplicity of f at P. One can then compute the *degree* of divisor (f) and deduce

$$\deg(f) = \sum_{P \in \overline{C}} \mathrm{ord}_P(f) = 0, \tag{3}$$

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[1] The same issue arises for a general number field, but this paper shall primarily focus on the basic case of $\mathbb{Q}$.

[2] One obvious example is how many choose to work with the finite adeles (i.e. just the p-adics, ignoring the reals) and not the full adele ring. See, for instance, Huber's work [Hub91] on the Beilinson-Parshin adeles, where she writes: "We want to stress that at this stage only a generalization of the finite adeles is found. It is not clear what one should take at infinity, or in fact even what the infinite 'places' should be."

a key result that allows us to develop a good intersection theory of divisors.

The Number Field Case. Consider $\mathbb{Q}$ and the spectrum of the ring of integers $\mathrm{Spec}(\mathbb{Z})$. Each non-zero prime $p \in \mathrm{Spec}(\mathbb{Z})$ corresponds to the p-adic numbers $\mathbb{Q}_p$. To account for $\mathbb{R}$, we formally add to $\mathrm{Spec}(\mathbb{Z})$ the set of complex embeddings $\sigma\colon \mathbb{Q} \hookrightarrow \mathbb{C}$; which gives a single embedding factoring through $\mathbb{R}$.[3] Denote this enlargement of $\mathrm{Spec}(\mathbb{Z})$ as $\Lambda_{\mathbb{Q}}$, which we shall call *the set of places of* $\mathbb{Q}$. Following standard conventions, we denote the "real prime" adjoined to $\mathrm{Spec}(\mathbb{Z})$ as ∞.

We then define the *Arakelov divisor* D on $\Lambda_{\mathbb{Q}}$ as the following finite formal linear combination

$$D = \sum_p n_p \cdot p + \alpha_\infty \cdot \infty, \qquad n_p \in \mathbb{Z},\ \alpha_\infty \in \mathbb{R}, \tag{4}$$

where the first sum runs over the set of non-zero primes in $\mathrm{Spec}(\mathbb{Z})$. As before, given any non-zero rational $f \in \mathbb{Q}$, one can define its Arakelov divisor (f), whose *Arakelov degree* can be computed to give

$$\widehat{\deg}(f) = \sum_{v \in \Lambda_{\mathbb{Q}}} \log |f|_v = 0. \tag{5}$$

Discussion: On Point-Set Reasoning. Considered side-by-side, the analogy between the two setups becomes clear, but notice the formal nature of the number field case. In the function field case, we added points to the smooth affine curve C via a geometric construction on C ("smooth compactification"). By contrast, the number field case starts with a formal abstraction: take the underlying **set** of $\mathrm{Spec}(\mathbb{Z})$. It is this formal move that allows us to combine the set of primes with the set of complex embeddings (even though they are *a priori* different objects), yielding a new set $\Lambda_{\mathbb{Q}}$ which we use to index the summands of the Arakelov divisor.

This style of point-set reasoning ("take the set of ...") is ubiquitous in classical mathematics, but here it presents a challenge to our understanding. For one, extending the function field analogy, one would like to regard $\Lambda_{\mathbb{Q}}$ as the compactification of $\mathrm{Spec}(\mathbb{Z})$. But on what grounds? Strictly speaking, $\Lambda_{\mathbb{Q}}$ is just a set of elements with no topology — it is only by analogy that one might regard it as morally being a kind of compactified affine curve. Further, notice that the construction of $\Lambda_{\mathbb{Q}}$ is still guided by an obvious case-split between the p-adics vs. the reals. In fact, as pointed out in [Bak08], Arakelov intersection theory uses very different-looking tools to deal with these two components, raising sharp questions about the extent to which Arakelov theory successfully resolves the lack of symmetry between the p-adics and the reals.

1.2. **The Point-free Perspective.** Having provided the number-theoretic context, we now shift gears and discuss the connection to the logical aspects of topos theory. Our main point of leverage is the following structure theorem:

Theorem 1.2 (Key Structure Theorem)**.** *Every (Grothendieck) topos $\mathcal{E}$ is a classifying topos of some geometric theory $\mathbb{T}_{\mathcal{E}}$. Conversely, every geometric theory $\mathbb{T}$ has a classifying topos $\mathcal{S}[\mathbb{T}]$.*

Precise definitions of terms will be deferred to Section 2. For now, we give an informal description:

- A *theory* $\mathbb{T}$ is a set of logical axioms that describes the structures of interest (e.g. groups, rings etc.);
- *Geometric logic* is a logic that is tailored to reflect topology, e.g. connectives $\wedge$ and $\bigvee$ to match intersection and union of opens. A *geometric theory* is a set of axioms expressed in geometric logic;
- A *model* $M_{\mathbb{T}}$ of a geometric theory $\mathbb{T}$ is a structure satisfying the description expressed by $\mathbb{T}$. For conciseness, we will often refer these structures as a $\mathbb{T}$-*model*;
- A *topos* $\mathcal{E}$ is some kind of category satisfying certain nice properties;[4]
- A *classifying topos of* $\mathbb{T}$, denoted $\mathcal{S}[\mathbb{T}]$, is a topos representing the universe of all $\mathbb{T}$-models. In particular, it contains a *generic model* $G_{\mathbb{T}}$, which is generic in the informal sense that it gives a blueprint from which all models $M_{\mathbb{T}}$ of $\mathbb{T}$ can be derived.[5]

Recalling Question 1.1, the authors were led to pose the following test problem.

[3] In the general case of a number field K, the construction involves adding $[K : \mathbb{Q}]$ many complex embeddings to $\mathrm{Spec}(\mathcal{O}_K)$.

[4] In this paper, the unqualified term "topos" will always mean a Grothendieck 1-topos, unless stated otherwise. The expert reader may take the phrase "nice properties" to mean Giraud's axiomatic characterisation of a topos.

[5] More precisely: given any $\mathbb{T}$-model $M_{\mathbb{T}}$ living in any topos $\mathcal{E}$, there exists a functor $f^* : \mathcal{S}[\mathbb{T}] \to \mathcal{E}$, unique up to isomorphism, such that $f^*(G_{\mathbb{T}}) \cong M_{\mathbb{T}}$ whilst also preserving colimits and finite limits.

Problem 1.3. Construct (if it exists) a geometric theory $\mathbb{T}_{\mathrm{comp}}$ whose models are the completions of $\mathbb{Q}$ (up to topological equivalence), and examine its properties.

Why might this be an interesting perspective? We give two natural reasons. First, if $\mathbb{T}_{\mathrm{comp}}$ exists then Structure Theorem 1.2 yields a generic model of $\mathbb{T}_{\mathrm{comp}}$ – call this *the generic completion of* $\mathbb{Q}$. It is well-known that the generic model $G_{\mathbb{T}}$ of any geometric theory $\mathbb{T}$ is conservative, i.e. given any property ϕ expressible in geometric logic, ϕ holds for $G_{\mathbb{T}}$ iff ϕ holds for all models of $\mathbb{T}$. There are a couple ways to read this in the present context. One interpretation: the generic completion of $\mathbb{Q}$ is a device that allows us to reason about properties that hold for all completions of $\mathbb{Q}$ in a symmetric manner — analogous to the adele ring $\mathbb{A}_{\mathbb{Q}}$ in classical number theory. Another interpretation: the generic completion of $\mathbb{Q}$ is a structure possessing no other properties besides being a completion of $\mathbb{Q}$. So if we wish to calibrate our understanding of the p-adics vs. the reals, it can be helpful to have a well-defined object that distills precisely what their shared similarities are.

The second, and more fundamental, reason is that the topos-theoretic perspective pulls Question 1.1 away from classical set theory, and opens it up to new tools from logic and category theory. This requires some explanation. To the uninitiated, the existence of serious interactions between number theory and logic may come as a surprise, but this itself is certainly not new. For instance, continuing our discussion of the function field analogy, model theorists may point to the Ax-Kochen-Eršov Theorem [AK65, Ers65]. Most notably, the theorem allows us to deduce that every homogeneous polynomial of degree $> d$ with more than d^2 variables has non-trivial zero in $\mathbb{Q}_p$ for all but finitely many p.

Herein lies a key difference between our approach and the model theorist's. The standard Ax-Kochen-Eršov Theorem relies on the existence of non-principal ultrafilters. It is well-known this necessarily invokes a weak form of choice [HL67, Bla77], but this is permissible for the model theorist who typically works in ZFC. By contrast, this paper will adhere to a strict regime of constructive mathematics known as *geometric mathematics* [Vic07, Vic14].[6] The payoff for working geometrically comes from Structure Theorem 1.2, which supports the view that a topos can be regarded as a *point-free space* in the following sense.

Definition 1.4.

(i) A *point-free space* is the universe of all models of some fixed geometric theory $\mathbb{T}$; the points of this space are the $\mathbb{T}$-models. For suggestiveness, we write "$x \in X$" to mean x is a model of the theory associated to X.

(ii) A map $f\colon X \to Y$ of (point-free) spaces is defined by taking the generic point $x \in X$ and *geometrically* constructing a point $f(x) \in Y$.

This unusual marriage of topology and logic, which we call "point-free topology", differs from the classical perspective in two important ways. One, unlike point-set topology, the points are no longer defined as elements of a set, but rather models of a logical theory. Two, point-free topology considers a wider range of models than those in classical model theory. For us, a model of a geometric theory $\mathbb{T}$ is no longer defined as a set decorated with the logical data of relations and/or functions (i.e. a $\mathbb{T}$-model in the topos Set), but a structure in any "universe sufficiently similar to Set" (i.e. a $\mathbb{T}$-model in an arbitrary topos $\mathcal{E}$). Further details will be explained in due course, but notice this discussion already gives some indication of how point-free topology systematically pulls our mathematics away from its underlying set theory.

Returning to our original context, what does the point-free perspective mean for Question 1.1? In practice, "working geometrically" means abandoning many classical tools in exchange for new ones. Unlike the model theorist, we do not have the axiom of choice, and so we shall prefer to work with the generic model of a theory rather than the ultraproducts of its models; and unlike the classical number theorist, we cannot take the underlying set of $\mathrm{Spec}(\mathbb{Z})$ (at least, not without losing geometricity), and so we must find other ways of dealing with the places of $\mathbb{Q}$. The classical mathematician may regard these as unwelcome restrictions, but we take a different view. A recent line of work, including the present paper, shows how geometric

[6] To work geometrically means to reason using constructions that are preserved by pullback along geometric morphisms between toposes. As will be explained in due course, this essentially means working with constructions/properties built out of finite limits and arbitrary colimits.

mathematics is sensitive to certain nuances previously elided by classical assumptions. Properly examined, these insights raise challenging new questions that go far beyond constructivist concerns.

1.3. **A Closer Look.** Section 1.1 gave the number-theoretic context, and discussed how point-set reasoning may distort some foundational aspects of arithmetic geometry. Section 1.2 discussed in broad strokes why the topos-theoretic perspective may have something useful to say about Question 1.1. This section develops this by taking a closer look at Problem 1.3 and reviewing the current state of our knowledge.

Convention 1.5. In this paper:

- The unqualified term "space" should be taken to mean a *point-free space* in the sense of Definition 1.4. If we wish to refer to the classical definition of a space [= a set of elements, equipped with a topology], we will use the term "topological space", or sometimes "point-set space" (for emphasis).
- The term "theory" will always mean a *geometric theory*, unless stated otherwise.
- The term "topos" will always mean a *Grothendieck 1-topos* in the sense of Giraud's axioms. For convenience, we work over the base topos Set, but any arbitrary elementary topos $\mathcal{S}$ with nno will work.[7] We denote the 2-category of toposes as $\mathfrak{Top}$.

As stated in Problem 1.3, we are interested in constructing a geometric theory $\mathbb{T}_{\text{comp}}$ whose models are completions of $\mathbb{Q}$ up to topological equivalence. By Structure Theorem 1.2, this is equivalent to constructing a topos $\mathcal{S}[\mathbb{T}_{\text{comp}}]$ that classifies $\mathbb{T}_{\text{comp}}$, and asking what it looks like as a (point-free) space. Concretely, this involves the following steps:

Step (1): Provide a geometric account of real exponentiation.
Step (2): Construct and characterise the classifying topos of absolute values of $\mathbb{Q}$.
Step (3): Construct and characterise the classifying topos of places of $\mathbb{Q}$.
Step (4): Construct and characterise the classifying topos of completions of $\mathbb{Q}$ (up to equivalence).

Let us elaborate. Classically, completions of $\mathbb{Q}$ are defined as point-set spaces whose points correspond to Cauchy sequences of $\mathbb{Q}$ with respect to some metric on $\mathbb{Q}$

$$\begin{aligned} |\cdot|\colon \mathbb{Q} &\longrightarrow [0,\infty) \\ x &\longmapsto |x| \end{aligned} \tag{6}$$

also known as an *absolute value*. As it turns out, there is an algebraic characterisation of when two completions of $\mathbb{Q}$ are topologically equivalent. Given two absolute values $|\cdot|_1, |\cdot|_2$, define an equivalence relation $\sim$ where

$$|\cdot|_1 \sim |\cdot|_2 \iff \exists \alpha \in (0,1]\,.\left(|x|_1^\alpha = |x|_2 \textit{ or } |x|_2^\alpha = |x|_1, \text{ for all non-zero } x \in \mathbb{Q}\right).$$

An equivalence class with respect to $\sim$ is called a *place*. A standard exercise shows that two absolute values belong to the same place iff their completions are homeomorphic. This suggests that before analysing the completions of $\mathbb{Q}$ [**Step (4)**], we should first construct and examine the topos of places [**Step (3)**].

In this regard, we start with an informal picture before filling the details. Recall our remark before Definition 1.4 that *any* topos can be regarded as a (point-free) space whose points are models of a theory. As such, consider the following diagram

$$(0,1] \times [av] \underset{ex}{\overset{\pi}{\rightrightarrows}} [av] \tag{7}$$

where

- $[av]$ denotes the space of absolute values (i.e. each point in $[av]$ corresponds to an absolute value).
- $(0,1]$ denotes the obvious subspace of Dedekind reals α such that $0 < \alpha \leq 1$.
- π is the projection map sending $(\alpha, |\cdot|) \mapsto |\cdot|$.
- ex is the exponentiation map sending $(\alpha, |\cdot|) \mapsto |\cdot|^\alpha$.

[7] **(M:) Rephrase?**

Suppose there exists a space X that acts as the coequaliser of Diagram (7). X can therefore be regarded as the space of places of $\mathbb{Q}$ since it is the universal space through which $|\cdot|$ and $|\cdot|^{\alpha}$ get mapped to the same point. **Step (3)** can thus be understood as asking for a complete description of X (if it exists).

A more rigorous account. It is easy to see that the subspace of (Dedekind) reals $(0,1]$ can be axiomatised by a geometric theory (see e.g. [NV22, Def. 1.21]). More recently, [NV23] gave an explicit construction of the theory of absolute values of $\mathbb{Q}$ [**Step (2)**]. Since all geometric theories possess a classifying topos (Structure Theorem 1.2), conclude that spaces $(0,1]$ and $[av]$ are well-defined. In fact, since the 2-category $\mathfrak{BTop}$ of toposes contains all finite limits [Joh02a, B4.1], and since geometric morphisms between toposes correspond to maps between point-free spaces[8], it follows that the projection map

$$\begin{aligned} \pi\colon (0,1]\times[av] &\longrightarrow [av] \\ (\alpha, |\cdot|) &\longmapsto |\cdot| \end{aligned} \tag{8}$$

exists and is well-defined. To show that the exponentiation map is well-defined requires more work, but is straightforward once we know that real exponentiation is geometric.[9] This was established in the paper [NV22] [**Step (1)**], before finding applications in [NV23]. In summary: Diagram (7) is a well-defined diagram of spaces. Moreover, it corresponds to a diagram of toposes in $\mathfrak{BTop}$. Since $\mathfrak{BTop}$ possesses all colimits [Joh02a, B3.4], deduce that there in fact exists a space X that acts as coequaliser of Diagram (7).

1.4. **Results.** This paper builds on work in [NV22, NV23] and applies descent techniques to study the places of $\mathbb{Q}$. This is where our earlier adherence to geometricity begins to pay off. Since point-free spaces can be dually regarded as toposes, committing to working geometrically now allows us to bring a deep collection of structure theorems from topos theory to bear on our analysis.

The picture we obtain is an unexpected one. One of the main contributions in [NV23] was to give a geometrically constructive proof[10] of Ostrowski's Theorem, stated below.

Theorem A ([NV23, Theorem 5.8]). *Let $|\cdot|$ be a non-trivial absolute value on $\mathbb{Q}$. Then, one of the following must hold:*

(i) $|\cdot| = |\cdot|_{\infty}^{\alpha}$ for some $\alpha\in(0,1]$; or
(ii) $|\cdot| = |\cdot|_{p}^{\alpha}$ for some $\alpha\in(0,\infty)$ and some (unique) prime $p\in\mathbb{N}_+$.

In fact, $[av_A]\cong(0,1]$ and $[av_{NA};p]\cong(0,\infty)$, where $[av_A]$ denotes the space of (non-trivial) Archimedean absolute values and $[av_{NA};p]$ denotes the space of non-Archimedean absolute values associated to prime p.

Theorem A classifies all places of $\mathbb{Q}$. This justifies a key move of this paper: first localise Diagram (7) to characterise the topos of a *single* place before moving to characterise the topos of *all* places. Our first main result shows that, given any non-Archimedean place of $\mathbb{Q}$, its corresponding topos $\mathcal{D}$ corresponds to a singleton, as one might expect.

Theorem B. $\mathcal{D}\simeq \mathrm{Set} = \mathcal{S}\{*\}$.

However, here comes the big surprise. When we investigate the topos of the Archimedean (i.e. the real) place, denoted $\mathcal{D}'$, we instead get:

Theorem C. $\mathcal{D}'\simeq \mathcal{S}\overleftarrow{[0,1]}$*, where* $\overleftarrow{[0,1]}$ *denotes the space of upper reals bounded between 0 and 1.*

This result overturns a longstanding classical assumption in number theory. Instead of corresponding to a singleton with no intrinsic features (as assumed in, e.g. Arakelov geometry), Theorem C indicates that the Archimedean place corresponds to a sort of blurred unit interval equipped with a non-Hausdorff topology.

What are the implications of Theorems B and C, e.g. for arithmetic geometry? At this critical juncture, our understanding is still incomplete and many urgent questions remain. Nonetheless, some partial answers are explored in the final two sections of the paper.

[8] Why? See Discussion 2.10 for details.

[9] Why is this? Recall that geometric transformations of models correspond to maps between spaces, Definition 1.4.

[10] For the expert reader: our proof is valid over any elementary topos $\mathcal{S}$ with nno.

- Section 6 gives a purely topos-theoretic account of the differences between the Archimedean and non-Archimedean places of $\mathbb{Q}$. In our language, the topos corresponding to the Archimedean place witnesses non-trivial forking in its sheaves whereas the topos corresponding to a non-Archimedean place eliminates all forking behaviour. This is summarised in Conclusion 6.9.
- Section 7 is expository, and brings into focus a key theme that has implicitly guided our investigations thus far: how do the connected and the disconnected interact? This basic question has a surprisingly far reach. As we shall discuss, thinking carefully about its placement in both topos theory and number theory forces a careful re-examination of definitions and our assumptions behind them, often from challenging angles.

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CONTENTS

2. PRELIMINARIES IN POINT-FREE TOPOLOGY

The discussion in the Introduction was premised on the claim that any topos corresponds to a point-free space (Definition 1.4). Section 2.1 provides the topos-theoretic justification for this perspective. We also explain what it means to "work geometrically". For the uneasy reader, let us remark: a guiding methodology of this paper is that one can reason with point-free spaces *as if* they were topological (point-set) spaces, so long as certain rules are obeyed.[11] Section 2.2 introduces two important examples of such spaces for this paper and proves a key lemma (Lifting Lemma 2.30).

2.1. **Why is a Topos a Point-free Space?**

There are (at least) two different levels on which to read the slogan: "A topos is a generalised space", both of which will be important to us. The mainstream view is that a topos $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, J)$ is a category of sheaves on a Grothendieck site $(\mathcal{C}, J)$; this is a generalised space insofar as the Grothendieck site categorifies a topological space [more correctly, the lattice of opens on a topological space]. The view that a topos can also be regarded as a point-free space is a more modern understanding [Vic99, Vic22] that exploits the topos' connection to *geometric theories*.

Definition 2.1 (Geometric Logic)**.** Let Σ be a first-order signature of sorts and symbols. Then over Σ, we define the following:

- A *geometric formula* is a logical formula built up from the symbols in Σ and a context of finitely many free variables, using truth $\top$, equality $=$, **finite** conjunctions $\wedge$, **arbitrary** (possibly infinite) disjunctions $\bigvee$, and $\exists$.
- A *geometric sequent* is an expression of the form $\forall xyz \ldots (\phi \rightarrow \psi)$, where ϕ and ψ are geometric formulae in the same context $\{x, y, z, \ldots\}$.
- A *geometric theory* over Σ is a set $\mathbb{T}$ of geometric sequents, the *axioms* of the theory.

[11] The first expressions of this methodology can be traced back to [Moe88].

- A *model* of a $\mathbb{T}$ is any mathematical structure (living in *any* topos) satisfying the axioms of $\mathbb{T}$.[12]

Notice the use of finite conjunctions and arbitrary disjunctions, which should remind the lay reader of opens of a topology being closed under finite intersections and arbitrary unions.

Example 2.2 (Commutative Rings)**.** The usual algebraic laws of commutative rings (with 1) can be formulated as geometric axioms, yielding a geometric theory $\mathbb{T}_{\text{com}}$. For instance, declaring R as our sort, and including $0, 1, +, \cdot$ as the obvious function symbols in Σ, we may express the distributivity law as

$$\forall xyz \in R. (\top \to x \cdot (y + z) = (x \cdot y) + (x \cdot z)).$$

The space of $\mathbb{T}_{\text{com}}$-models includes *all* discrete commutative rings living in *all* toposes.[13] Standard examples of $\mathbb{T}_{\text{com}}$-models include $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{F}_p$ etc.[14]

The connection between a Grothendieck topos and geometric logic is made precise via the beautiful key definition of a *classifying topos*. The next series of definitions and remarks develop this picture.

Definition 2.3. Let $\mathcal{E}, \mathcal{F}$ be toposes (= categories of sheaves over some Grothendieck site). A *geometric morphism* $f \colon \mathcal{E} \to \mathcal{F}$ is a pair of adjoint functors

$$\begin{tikzcd} \mathcal{E} \arrow[r, bend left, "f_*"] & \mathcal{F} \arrow[l, bend left, "f^*"] \end{tikzcd}$$

such that f^* preserves **finite** limits and **arbitrary** colimits. We call f^* the *inverse image functor* of f, and denote $\mathbf{Geom}(\mathcal{E}, \mathcal{F})$ to be the category of geometric morphisms $f \colon \mathcal{E} \to \mathcal{F}$.

Discussion 2.4 (Geometric Morphism = Generalised Continuous Map)**.** Two key points:

(i) The finite limits and arbitrary colimits in Definition 2.3 should be understood as corresponding to the finite conjunctions and arbitrary disjunctions we saw in Definition 2.1;

(ii) The fact that the inverse image functor f^* of a geometric morphism $f \colon \mathcal{E} \to \mathcal{F}$ preserves certain structures is analogous to the pre-image f^{-1} of an ordinary continuous map $f \colon X \to Y$ preserves the opens of Y. In particular, recalling our distinction between point-set vs. point-free topology, note that the pre-image f^{-1} is generally not well-defined on the set of *elements* of Y but it is well-defined on its lattice of *opens*.

Item (ii) justifies the perspective that a geometric morphism is a generalised continuous map. Combined with (i), this justifies the view that geometric logic possesses an intrinsic continuity.

Convention 2.5 ("Geometricity")**.** Discussion 2.4 suggests a less syntactic notion of what it means to work geometrically. Hereafter, we call a construction *geometric* if it is preserved by inverse image functors (or equivalently, if it is preserved by pullback along geometric morphisms) — essentially, if it is constructed from finite limits and arbitrary colimits.[15]

Definition 2.6. Denote $\mathbb{T}\text{-mod}(\mathcal{E})$ to be the category of $\mathbb{T}$-models living in topos $\mathcal{E}$. A *classifying topos* of a geometric theory $\mathbb{T}$ is a topos $\mathcal{S}[\mathbb{T}]$ that classifies the models of $\mathbb{T}$ in the following sense — for any topos $\mathcal{E}$, we have the equivalence of categories

$$\mathbf{Geom}(\mathcal{E}, \mathcal{S}[\mathbb{T}]) \simeq \mathbb{T}\text{-mod}(\mathcal{E}),$$

[12] An explicit definition using categorical semantics (e.g. following [Joh02b, D1.2]) is possible, but we shall not need this.

[13] The model theorist may wish to view our notion of "space" as a topos-theoretic generalisation of "elementary class".

[14] Warning: the Dedekind reals $(\mathbb{R}, +, \times)$ equipped with the usual addition and multiplication is *not* a $\mathbb{T}_{\text{com}}$-model. Classically, one is used to reasoning with the reals as if they were elements of a set (ignoring the topology); geometrically, however, one must always recognise the Dedekinds form a (non-discrete) space, and so $(\mathbb{R}, +, \times)$ cannot be regarded as a discrete commutative ring.

[15] A technical side-note: there are certain constructions, e.g. frames, which are technically not geometric but can be given a presentation which *are* geometric. Such constructions will continue to play a role in geometric mathematics – see. e.g. our use of Moerdijk's Stability Theorem 3.17.

natural in $\mathcal{E}$. That is, for any geometric morphism $f\colon \mathcal{E} \to \mathcal{F}$, we have a commutative square up to natural isomorphism

$$\begin{array}{ccc} \mathbf{Geom}(\mathcal{F}, \mathcal{S}[\mathbb{T}]) & \xrightarrow{\sim} & \mathbb{T}\text{-mod}(\mathcal{F}) \\ {\scriptstyle -\circ f}\downarrow & & \downarrow{\scriptstyle f^*(-)} \\ \mathbf{Geom}(\mathcal{E}, \mathcal{S}[\mathbb{T}]) & \xrightarrow{\sim} & \mathbb{T}\text{-mod}(\mathcal{E}) \end{array}$$

In particular, the *generic model* of $\mathbb{T}$ is the model correspond to the identity map $\mathrm{id}\colon \mathrm{Set}[\mathbb{T}] \to \mathrm{Set}[\mathbb{T}]$.

To refine our understanding of this definition, recall the analogy between maps between topological spaces and geometric morphisms between toposes in Discussion 2.4. Notice: any point of a topological space X corresponds to a map $x\colon * \to X$ from the singleton. Since the topos $\mathrm{Set} \simeq \mathcal{S}\{*\}$ (i.e. Set is equivalent to the category of sheaves on $*$), this justifies the following defintion.

Definition 2.7 (Points = Generalised Maps)**.** Let $\mathcal{E}, \mathcal{F}$ be toposes.

(i) A *generalised point* of $\mathcal{F}$ is any geometric morphism $f\colon \mathcal{E} \to \mathcal{F}$ whose codomain is $\mathcal{F}$.
(ii) In the special case where $f\colon \mathrm{Set} \to \mathcal{F}$, we call f a *global point* of $\mathcal{F}$.

In particular, the global points of the classifying topos $\mathcal{S}[\mathbb{T}]$ correspond to the standard models of $\mathbb{T}$ in Set.

Discussion 2.8 (Topos = Point-free Space)**.** A classifying topos $\mathcal{S}[\mathbb{T}]$ classifies $\mathbb{T}$ in the sense that $\mathbb{T}$-models living in a topos $\mathcal{E}$ correspond to geometric morphisms $f\colon \mathcal{E} \to \mathcal{S}[\mathbb{T}]$. Since these geometric morphisms can be regarded as generalised points of $\mathcal{S}[\mathbb{T}]$, we regard the collection of all $\mathbb{T}$-models (ranging over all toposes $\mathcal{E}$) as a generalised space. Hereafter, we denote the space corresponding to a topos as $[\mathbb{T}]$, i.e. the "universe of all $\mathbb{T}$-models".[16] Since any topos is a classifying topos of some geometric theory (Structure Theorem 1.2), this justifies viewing a topos as a point-free space.

Convention 2.9 (Notation: $[\mathbb{T}]$ vs. $\mathcal{S}[\mathbb{T}]$)**.** Having established two different ways of viewing a topos, we fix a convention to indicate which perspective we are considering. Consider some geometric theory $\mathbb{T}$.

- If we wish to view its classifying topos as a category whose objects are sheaves, then we denote the topos as $\mathcal{S}[\mathbb{T}]$.
- If we wish to consider the universe of $\mathbb{T}$-models associated to the topos – say as a pseudo-functor $[\mathbb{T}]\colon \mathfrak{Top}^{\mathrm{op}} \to \mathfrak{CAT}$ in the sense of Footnote 16 – then we shall denote this universe as $[\mathbb{T}]$.

We write $F \in \mathcal{S}[\mathbb{T}]$ to mean that F is a sheaf in $\mathcal{S}[\mathbb{T}]$, and $x \in [\mathbb{T}]$ to mean that x is a point in $[\mathbb{T}]$. For clarity, we emphasise: F is an *object* in the category of sheaves $\mathcal{S}[\mathbb{T}]$ whereas x corresponds to a *geometric morphism* $\mathcal{E} \to \mathcal{S}[\mathbb{T}]$.

Discussion 2.10 (Maps as Point-Transformers)**.** Consider a *map* $f\colon [\mathbb{T}] \to [\mathbb{T}']$, as described in Definition 1.4. To define it, we declare "let x be a point of $[\mathbb{T}]$", and then work geometrically to construct a point $f(x)$ of $[\mathbb{T}']$. We claim that in order to define f, it suffices to define the geometric construction when x corresponds to the generic model, which we denote $G_{\mathbb{T}}$.

Why? Recall that $G_{\mathbb{T}}$ corresponds to geometric morphism $\mathrm{id}\colon \mathcal{S}[\mathbb{T}] \to \mathcal{S}[\mathbb{T}]$. A standard check shows that all $\mathbb{T}$-models $M_{\mathbb{T}}$ can be represented as $M_{\mathbb{T}} \cong g^*(G_{\mathbb{T}})$ for some appropriate geometric morphism $g\colon \mathcal{E} \to \mathcal{S}[\mathbb{T}]$.[17] Since geometric constructions are precisely those which are preserved by inverse image functors, it follows that the generic construction suffices to describe all the instances of more specific points of $[\mathbb{T}]$. The generic point $G_{\mathbb{T}}$ thus plays the role of *formal parameter* x in the definition of $f(x)$, and actual parameters are substituted by transporting constructions along the functors.

To summarise: a map $f\colon [\mathbb{T}] \to [\mathbb{T}']$ defines a point $f(G_{\mathbb{T}})$ of $[\mathbb{T}']$, constructed geometrically in $\mathcal{S}[\mathbb{T}]$. But that is in turn equivalent to a functor $f^*\colon \mathcal{S}[\mathbb{T}'] \to \mathcal{S}[\mathbb{T}]$ – note the reversal of direction – that preserves colimits and finite limits, and takes $G_{\mathbb{T}'}$ to $f(G_{\mathbb{T}})$. From preservation of colimits we can get a right adjoint $f_*\colon \mathcal{S}[\mathbb{T}] \to \mathcal{S}[\mathbb{T}']$, and we have arrived at the usual definition of geometric morphism. This establishes

[16] Again, we will not need the technical definition of $[\mathbb{T}]$ — it suffices for us to view it as some meta-universe containing all the $\mathbb{T}$-models. But for the curious reader, see [Joh02a, B4.2]. In particular, one can view $[\mathbb{T}]$ as a pseudo-functor $[\mathbb{T}]\colon \mathfrak{Top}^{\mathrm{op}} \to \mathfrak{CAT}$ that assigns to each topos $\mathcal{E}$ a category $\mathbb{T}(\mathcal{E})$, whose objects are the $\mathbb{T}$-models in $\mathcal{E}$, with the topos $\mathcal{S}[\mathbb{T}]$ as the representing object for the pseudo-functor. Put otherwise, $[\mathbb{T}]$ is a $\mathfrak{Top}$-indexed category cataloguing all the points of $\mathcal{S}[\mathbb{T}]$.

[17] Why? Notice: the equivalence of categories $\mathbf{Geom}(\mathcal{E}, \mathcal{S}[\mathbb{T}]) \simeq \mathbb{T}\text{-mod}(\mathcal{E})$ induced by $\mathcal{S}[\mathbb{T}]$ is required to be natural in $\mathcal{E}$.

the correspondence between point-free maps and geometric morphisms, as the reader may have already anticipated by Discussion 2.4.

Finally, we conclude with a key methodological fact about this dual perspective of toposes.[18]

Fact 2.11 (Translation of [Ng23, Prop. 2.1.28])**.** Given any two geometric theories $\mathbb{T}, \mathbb{T}'$, $\mathcal{S}[\mathbb{T}] \simeq \mathcal{S}[\mathbb{T}]$ iff $[\mathbb{T}] \cong [\mathbb{T}']$. That is, given two theories $\mathbb{T}$ and $\mathbb{T}'$, their classifying toposes are equivalent as categories iff their spaces have equivalent points.

Remark 2.12. **(M:) Perhaps a remark on constructive aspects.**

2.2. **Localic Spaces & Essentially Propositional Theories.** A (geometric) theory $\mathbb{T}$ is called a *propositional theory* if its signature Σ has no sorts.[19] In particular, its axioms are constructed only from constant symbols in Σ, $\top$ (true), finite $\wedge$ and arbitrary $\bigvee$. An *essentially propositional theory* $\mathbb{T}$ is one such that $\mathcal{S}[\mathbb{T}] \simeq \mathcal{S}[\mathbb{T}']$ for some propositional theory $\mathbb{T}$.[20] Informally, this says that the space of $\mathbb{T}$-models is equivalent to the space of $\mathbb{T}'$-models, even if $\mathbb{T}$ possesses sorts in its signature.[21]. In fact, there exist various kinds of sorts that can be added to a theory's signature without essentially changing its models. Although, to our knowledge, a complete description of such sorts is still unknown, one important class of such sorts are the free algebra constructions:

Fact 2.13 ([Vic07, Vic17], but see also [Joh02a, pp. 108])**.**

- (i) The free algebra constructions are geometric constructions (cf. Convention 2.5), and are uniquely determined up to isomorphism. These include the natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$ and the rationals $\mathbb{Q}$, along with their usual arithmetic structure (e.g. addition, multiplication, strict order etc.).
- (ii) Let $\mathbb{T}$ be a geometric theory. If we perform free algebra constructions on the sorts already present in its signature to construct new "derived sorts", we obtain a new theory whose models are equivalent to those of $\mathbb{T}$. In particular, if $\mathbb{T}$ has only free algebra constructions as is its sorts (e.g. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, etc.), then it is essentially propositional.

Definition 2.14. Let $\mathbb{T}$ be a geometric theory.

- (i) If $\mathbb{T}$ is essentially propositional, we call $[\mathbb{T}]$ a *localic space*. They define a category Loc, which has (point-free) maps as morphisms.
- (ii) If we wish to emphasise that $\mathbb{T}$ is an arbitrary theory (i.e. not necessarily essentially propositional), we will call $[\mathbb{T}]$ a *generalised space*.

Localic spaces are the primary objects of interest in this paper. They occupy a conceptual sweet spot in point-free topology, where the interactions between topology and logic become especially clear. To illustrate, consider the key definition of a frame.

Definition 2.15 (Frame)**.**

- (i) A *frame* is a complete lattice A possessing all small joins $\bigvee$ and all finite meets $\wedge$, such that the following distributivity law holds
$$a \wedge \bigvee S = \bigvee \{a \wedge b \,|\, b \in S\}$$
where $a \in A, S \subseteq A$.
- (ii) A *frame homomorphism* is a function between frames that preserves arbitrary joins and finite meets.

Discussion 2.16. It is clear the opens of any topological space X define a frame, which we denote Ω_X. On the side of logic, recall that a propositional theory $\mathbb{T}$ is defined as having no sorts. As such, its formulae are constructed from symbols in the signature using only $\top$, $\wedge$ and $\bigvee$. Extending this: define the Lindenbaum Algebra of $\mathbb{T}$ as the lattice of all formulae of $\mathbb{T}$ modulo provable equivalence, which also clearly defines a frame, say $\Omega_{\mathbb{T}}$. We therefore refer to the Lindenbaum Algebra of $\mathbb{T}$ as its *frame of opens*, where we regard propositional formulae as the point-free analogue for opens. In fact, given any map $f\colon X \to X'$

[18] **(M:) Rephrase.**

[19] And so there can be no variables or terms, nor existential quantification.

[20] Recall: by Structure Theorem 1.2 every geometric theory $\mathbb{T}$ has a classifying topos $\mathcal{S}[\mathbb{T}]$.

[21] Why? Two classifying toposes are equivalent as categories iff their space of models are equivalent – see [Ng23, Prop. 2.1.28].

between localic spaces, a well-known result shows that the associated inverse image functors $f^*\colon \mathcal{S}X' \to \mathcal{S}X$ correspond to frame homomorphisms $f^{-1}\colon \Omega_{X'} \to \Omega_X$ [Joh02b, C1.3-4], [Vic07, §2.2].

In Section 3, we shall see that one can leverage this connection to translate many standard notions in point-set topology to point-free topology via the language of frames. Further details can also be found in [Vic07, §2.2]. The remainder of this section will be devoted to discussing two key examples of localic spaces in this paper: (a) localic reals, and (b) localic spectra.

2.3. **The Localic Reals.** This paper uses two different types of reals: Dedekind reals and the one-sided reals. Both reals are built up from the rationals but in different ways; this results in different topologies and therefore different subtleties in their analysis.

Definition 2.17 (Dedekind Reals)**.** The localic space $\mathbb{R}$ is a space whose points are the Dedekind sections of the rationals. That is, a point of $\mathbb{R}$ is a pair (L, R) of subsets of $\mathbb{Q}$ such that

(1) R is rounded upper (i.e. $q \in R$ iff there exists $q' \in R$ with $q' < q$) and inhabited (i.e. there exists $q \in R$).
(2) L is rounded lower and inhabited.
(3) L and R are *separated*: if $q \in L$ and $r \in R$, then $q < r$.
(4) L and R are *located*: if $q < r$ are rationals, then either $q \in L$ or $r \in R$.

The two relations L, R correspond to the left and right Dedekind sections of a real number.

Convention 2.18. We shall often denote a point of $\mathbb{R}$ as x, instead of explicitly writing out the pair of relations representing it: (L_x, R_x). We will also use $q < x$ to mean $q \in L_x$ and use $x < r$ to mean $r \in R_x$.

Definition 2.19 (One-Sided Reals)**.** There are two kinds of one-sided reals.

(i) The space of *upper reals*, denoted $\overleftarrow{[-\infty, \infty]}$, has as points the rounded upper subsets of $\mathbb{Q}$.
(ii) The space of *lower reals*, denoted $\overrightarrow{[-\infty, \infty]}$, has as points the rounded lower subsets of $\mathbb{Q}$.

Notice this definition allows the upper (resp. lower) reals to be empty, which correspond to ∞ (resp. $-\infty$).

Informally, an inhabited upper real (resp. lower real) approximates a number from above (resp. below), whereas a Dedekind real approximates the number from both directions. That is,

Fact 2.20. There exist natural maps

$$L\colon \mathbb{R} \longrightarrow \overrightarrow{(-\infty, \infty]}$$
$$R\colon \mathbb{R} \longrightarrow \overleftarrow{[-\infty, \infty)}$$

where given a Dedekind real $x = (L_x, R_x)$, L sends $x \mapsto L_x$ and R sends $x \mapsto R_x$.

Remark 2.21. The motivated reader can check that the axioms in Definition 2.17 can be written in the syntax of geometric logic. Notice that the theory of $\mathbb{R}$ would feature $\mathbb{Q}$ as its only sort with L, R as unary predicates. By Fact 2.13, this means $\mathbb{R}$ and the one-sided reals are in fact localic spaces.

Convention 2.22. Extending Convention 2.18, given an upper real x, we often write $x < q$ to mean that q belongs to the subset of rationals constituting x. We write $x \leq y$ for two upper reals x, y just in case $y < q \to x < q$. The analogous convention holds for lower reals. When we wish to collectively refer to both the lower and upper reals, we use the term 'one-sided reals' or just the 'one-sideds'.

We will, of course, usually be looking at subspaces of the localic reals rather than the entire spaces themselves. Defining subspaces in our context requires some technical care, the details of which we shall suppress (this can be found in [Ng23, §2.1 - 2.2]). For our purposes, the following slogan will suffice.

Convention 2.23 (Subspaces of Reals)**.** Subspaces of $\mathbb{R}$ can be defined in the obvious way whereas subspaces of one-sided reals require a bit more care due to their non-standard topology. In more detail:

(i) $(0, \infty)$ denotes the open subspace of positive Dedekinds. $(0, 1]$ denotes the closed subspace of $(0, \infty)$ bounded above by 1.
(ii) Since one-sided reals are honest subsets of $\mathbb{Q}$, subset inclusion induces a specialisation order on the points of these spaces:

- For lower reals, $x \sqsubseteq y$ iff $x \leq y$.
- For upper reals, $x \sqsubseteq y$ iff $x \geq y$.

Observe that $\sqsubseteq$ for the lower reals agrees with the numerical order, whereas for the upper reals it is the opposite. To reflect this, we shall use arrows on top of the spaces to show the direction of the refinement under subset inclusion. For instance, consider the space $(0, \infty)$ — we then denote the corresponding space of lower reals as $\overrightarrow{(0, \infty]}$ and the corresponding space of upper reals as $\overleftarrow{[0, \infty)}$.

(iii) Notice that the previous one-sided intervals were closed at the arrowhead — e.g. ∞ was included in $\overrightarrow{(0, \infty]}$ and 0 in $\overleftarrow{[0, \infty)}$. In fact, this is canonical — *all* one-sided intervals must be closed at the arrowhead. [Why? $\sqsubseteq$ induces an order topology on the one-sideds, also known as the *Scott topology*. In particular, all subspaces of the one-sideds must be closed under arbitrary directed joins with respect to $\sqsubseteq$.]

2.4. **The Lifting Lemma.** Although perhaps unusual to the classical mathematician, the one-sided reals arise as natural examples of a well-known construction in domain theory known as *rounded ideal completions* [Smy77, Vic93].

Definition 2.24 (Rounded Ideal Completions)**.** Consider $(Y, \prec)$ where Y is a set equipped with a dense[22] transitive order $\prec$. We emphasise that we do not require $\prec$ to be linear or strict here.

(i) An *ideal* in Y is a subset $I \subseteq Y$ that is downward-closed and contains an upper bound for each of its finite subsets (with respect to $\prec$). In particular, if the set $I_q := \{q' \in Y \mid q' \prec q\}$ is an ideal for all $q \in Y$, then we call $(Y, \prec)$ an *R-structure*.

(ii) A subset $S \subseteq Y$ is called *rounded* if for any $q \in S$, there exists $q' \in S$ such that $q \prec q'$. It is clear that all ideals of Y are rounded. We therefore typically refer to the ideals of Y as *rounded ideals*.

(iii) The *rounded ideal completion* of an R-structure $(Y, \prec)$ is the space $\mathsf{RIdl}(Y, \prec)$ of all ideals of Y. The specialisation order $\sqsubseteq$ is then the partial order by inclusion, inducing a Scott topology.

A subset $D \subset Y$ is called *directed* if it is inhabited, and any two elements of D have an upper bound in D. The *directed join* of ideals over a directed subset D is defined as $\bigvee^{\uparrow}_{q \in D} I_q := \bigcup_{q \in D} I_q$. In particular, directed joins interact with the topology of $\mathsf{RIdl}(Y, \prec)$ in a natural way.

Fact 2.25. Let $(Y, \prec)$ be an R-structure. Given a rounded ideal $I \in \mathsf{RIdl}(Y, \prec)$, a space Z equipped with a specialisation order, then the following is true:

(i) $I = \bigvee^{\uparrow}_{q \in I} I_q$.

(ii) The space $\mathsf{RIdl}(Y, \prec)$ is closed under directed joins, i.e. if $I = \bigvee^{\uparrow}_{q \in D} I_q$ for any directed subset $D \subset Y$, then $I \in \mathsf{RIdl}(Y, \prec)$.

(iii) Any continuous map

$$f \colon \mathsf{RIdl}(Y, \prec) \longrightarrow Z$$

preserves directed joins, i.e. $f(\bigvee^{\uparrow}_{q \in D} I_q) = \bigvee^{\uparrow}_{q \in D} f(I_q)$.

(iv) Suppose we have two continuous maps

$$f \colon \mathsf{RIdl}(Y, \prec) \to Z$$

$$g \colon \mathsf{RIdl}(Y, \prec) \to Z$$

such that $f(I_q) \sqsubseteq g(I_q)$ for all $q \in I$. Then $f(I) \sqsubseteq g(I)$.

Proof. (i) and (ii) are obvious. (iii) is [Vic89, Theorem 7.3.1]. For (iv), note that (i) and (iii) give

$$f(I) = \bigvee\nolimits^{\uparrow}_{q \in I} f(I_q) \quad \text{and} \quad g(I) = \bigvee\nolimits^{\uparrow}_{q \in I} g(I_q).$$

Now suppose $t \in f(I)$. Then there exists $q' \in I$ such that $t \in f(I_{q'})$. But since $f(I_{q'}) \sqsubseteq g(I_{q'})$ by hypothesis, this implies $t \in g(I)$, and so $f(I) \sqsubseteq g(I)$. $\square$

[22] Recall: $(Y, \prec)$ is said to have a *dense* order if for any $q, q' \in Y$ such that $q \prec q'$, there exists $q'' \in Y$ such that $q \prec q'' \prec q'$. This property goes by a variety of names — e.g. in [Vic93, Definition 2.1], the same property is referred to as being 'interpolative'.

The fact that Definition 2.24 does not require $\prec$ to be a strict order gives us considerable flexibility. In fact, a subspace of one-sided reals is often representable as a rounded ideal completion $\mathsf{RIdl}(Y, \prec)$, where Y is some subset of $\mathbb{Q}$ and $\prec$ is the standard order $<$ on $\mathbb{Q}$ except possibly reversed or modified to permit edge cases. This fact was used extensively in [NV22]. The most relevant example for us is the following.

Example 2.26. Denote $\mathbb{Q}_{(0,1]} := \{q \in \mathbb{Q} \,|\, 0 < q \leq 1\}$ and define $x \prec y$ iff $x > y$ or $x = y = 1$. Then

$$\overleftarrow{[0,1]} \cong \mathsf{RIdl}(\mathbb{Q}_{(0,1]}, \prec).$$

Convention 2.27. Example 2.26 justifies the view that a point $\gamma \in \overleftarrow{[0,1]}$ is simultaneously an upper real in the usual sense, as well as a rounded ideal $I_\gamma \in \mathsf{RIdl}(\mathbb{Q}_{(0,1]}, \prec)$ in the sense of Definition 2.24. Hereafter, we shall use both representations interchangeably, depending on convenience.

The language of rounded ideal completions allows us to reduce many questions about the one-sided reals to questions about the rationals, which are comparatively easier to work with. The following results develop this remark.

Lemma 2.28. *Let $f\colon X \to Y$ be a map of spaces (where X and Y need not be localic). Then f preserves filtered colimits of points.*

Proof. Standard, but we elaborate. Denote $\mathrm{colim}_{i\in J}\, x_i$ to be a set-indexed filtered colimit of W-points of X, i.e. each point x_i can be represented as:

$$W \xrightarrow{x_i} X \xrightarrow{f} Y$$

Since the filtered colimit is computed pointwise, it is clear that $f(\mathrm{colim}_{i\in J}\, x_i) \cong \mathrm{colim}_{i\in J}\, f(x_i)$.[23] $\square$

Lemma 2.29. *Let $\mathsf{RIdl}(Y, \prec)$ be the rounded ideal completion of R-structure $(Y, \prec)$. Then, there exists a canonical map*

$$\begin{aligned} \psi\colon Y &\longrightarrow \mathsf{RIdl}(Y, \prec) \\ q &\longmapsto I_q := \{q' \in Y \,|\, q' \prec q\}, \end{aligned} \tag{9}$$

which is an epimorphism of spaces.

Proof. It is clear that the canonical map ψ is well-defined. [Why? Note that Y is an R-structure, and so I_q is a rounded ideal of Y by definition.] To show that ψ is an epimorphism of spaces, suppose we have two maps $g_1, g_2\colon \mathsf{RIdl}(Y, \prec) \to Z$ such that $g_1 \circ \psi \cong g_2 \circ \psi$. Note: every ideal $I \in \mathsf{RIdl}(Y, \prec)$ can be represented as a directed join $I = {\bigvee\!\!\!\!\nearrow}_{q\in I}\, I_q$, which is a filtered colimit. Hence, apply Lemma 2.28 to compute

$$g_1(I) = g_1({\textstyle\bigvee\!\!\!\!\nearrow}_{q\in I}\, I_q) \cong \operatorname*{colim}_{q\in I} g_1(I_q) \cong \operatorname*{colim}_{q\in I} g_1 \circ \psi(q)$$

and

$$g_2(I) = \operatorname*{colim}_{q\in I} g_2 \circ \psi(q).$$

Since $g_1 \circ \psi \cong g_2 \circ \psi$ by hypothesis, it follows that $g_1 \cong g_2$, i.e. ψ is indeed an epimorphism. $\square$

This sets up the following key lemma.

Lemma 2.30 (Lifting Lemma)**.** *As our setup,*

- *Let X be a space;*
- *Let $(Y, \prec)$ be an R-structure.*

Then, the epimorphism from Lemma 2.29 induces an equivalence between:

(i) A map $f\colon Y \longrightarrow X$ satisfying the following continuity conditions:
  - (Cocycle condition) *For all $q, q' \in Y$, we have f maps $q' \prec q$ to a map $\theta_{q'q}\colon f(q') \to f(q)$ such that if $q'' \prec q' \prec q$ then $\theta_{q''q} = \theta_{q'q} \circ \theta_{q''q'}$;*
  - (Colimit condition) *The map $\theta_q\colon \operatorname*{colim}_{q'\prec q} f(q') \to f(q)$ is an isomorphism.*

[23] Note that this generalises Fact 2.25 (iii), except we now use isomorphisms rather than writing $f(\mathrm{colim}_{i\in J}\, x_i) = \mathrm{colim}_{i\in J}\, f(x_i)$, since spaces here may not be localic.

(ii) A map $\overline{f}\colon \mathsf{RIdl}(Y,\prec) \longrightarrow X$.

Proof. The proof splits into two main stages.

Step 1: Transforming the given map. Suppose $f\colon Y \longrightarrow X$ is a map satisfying the continuity conditions of the lemma. Then for any $I \in \mathsf{RIdl}(Y,\prec)$, one easily checks that $\operatorname*{colim}_{q\in I} f(q)$ is a filtered colimit due to the cocycle condition. As such, since toposes possess all set-indexed filtered colimits of their points [Joh77b, Corollary 7.14], the following map is well-defined:

$$\begin{aligned} \overline{f}\colon \mathsf{RIdl}(Y,\prec) &\longrightarrow X \\ I &\longmapsto \operatorname*{colim}_{q\in I} f(q). \end{aligned} \tag{10}$$

Conversely, suppose we have a map $\overline{f}\colon \mathsf{RIdl}(Y,\prec) \to X$. We can then define a map

$$\begin{aligned} f\colon Y &\longrightarrow X \\ q &\longmapsto \overline{f}(I_q) \end{aligned} \tag{11}$$

where I_q as in Equation (9). That f as defined in Equation (11) satisfies the cocycle condition is immediate from functoriality of $\overline{f}$. That f also satisifes the colimit condition follows from applying Lemma 2.28, which gives

$$\operatorname*{colim}_{q'\prec q} f(q') = \operatorname*{colim}_{q'\prec q} \overline{f}(I_{q'}) \cong \overline{f}(I_q) = f(q).$$

Step 2: Proving Equivalence. Suppose we are given $f\colon Y \to X$. Following Step 1, define

$$\begin{aligned} g\colon Y &\longrightarrow \mathcal{E} \\ q &\longmapsto \operatorname*{colim}_{q'\in I_q} f(I_q). \end{aligned}$$

We claim that $g \cong f$. Why? By Lemma 2.29, there exists a canonical epimorphism $\psi\colon Y \to \mathsf{RIdl}(Y,\prec)$ such that $\psi(q) = I_q$. Note that $g(q) = \overline{f}\circ\psi(q)$ where $\overline{f}$ is defined as in Equation (10). As such,

$$g(q) = \overline{f}\circ\psi(q) = \overline{f}(I_q) = \operatorname*{colim}_{q'\prec q} f(q') \cong f(q),$$

where the final isomorphism follows from the colimit condition.

Conversely, suppose we are given $\overline{f}\colon \mathsf{RIdl}(Y,\prec) \to X$. By Step 1, define

$$\begin{aligned} \overline{g}\colon \mathsf{RIdl}(Y,\prec) &\longrightarrow X \\ I &\longmapsto \operatorname*{colim}_{q\in I} \overline{f}(I_q) \end{aligned}$$

By Lemma 2.28, it is clear that $\overline{g}\circ\psi \cong \overline{f}\circ\psi$. Since ψ is epi by Lemma 2.29, this gives $\overline{f} \cong \overline{g}$, finishing the proof. ☐

Remark 2.31. The Lifting Lemma 2.30 generalises [NV22, Lemma 1.29]: here we no longer require X to be localic. Once the appropriate adjustments have been made, the original argument goes through almost immediately.

An important application of the Lifting Lemma 2.30 is the following characterisation of sheaves on $\overleftarrow{[0,1]}$. This will play a key role in our analysis of the Archimedean place.

Observation 2.32. As our setup,

- Let F be an object in the category of sheaves $\mathcal{S}\overleftarrow{[0,1]}$ (cf. Convention 2.9)
- Denote $\mathbb{O}$ to be the theory of objects, i.e. it has one sort, and no functions, predicates or axioms.
- Denote $[\mathbb{O}]$ to be the *object classifier*, i.e. the space of models of $\mathbb{O}$.[24]

Then, F can be equivalently characterised as:

(i) F is a sheaf over $\overleftarrow{[0,1]}$;

[24]Warning: not to be confused with the *subobject classifier*, which is an *object* living in each topos $\mathcal{E}$. By contrast, the *object classifier* corresponds to an actual *topos*, i.e. the classifying topos of $\mathbb{O}$.

(ii) $F\colon \overleftarrow{[0,1]} \to [\mathbb{O}]$;

(iii) $F\colon \mathbb{Q}_{(0,1]} \to [\mathbb{O}]$ is a map satisfying the continuity conditions of Lemma 2.30.

Proof.

(i) $\iff$ (ii): Immediate from the universal property of the object classifier $[\mathbb{O}]$ – see Fact 3.13.

(ii) $\iff$ (iii): Immediate from the fact that $\overleftarrow{[0,1]} \cong \mathsf{RIdl}(\mathbb{Q}_{(0,1]}, \prec)$ (Example 2.26) and the Lifting Lemma 2.30. ☐

2.5. **Localic Spectra.** Let us revisit the discussion on Arakelov Divisors from Section 1.1. Classically, the spectrum of a commutative ring R, which we denote $\mathrm{spec}(R)$ (with small s), is defined as the set of all prime ideals of R. This can be transformed into a topological space by equipping it with a suitable topology (e.g. Zariski, coZariski, constructible). By contrast, to define a point-free space requires specifying a theory $\mathbb{T}$ – but this already defines the points (= the $\mathbb{T}$-models) and the topology (= the Lindenbaum Algebra of $\mathbb{T}$). In other words, the topology and points in point-free topology must be defined simultaneously and cannot be separated.

Reviewing our context, how should we define the spectrum of $\mathbb{Z}$? Work of Cole [Col16] and Johnstone [Joh77a] on topos-theoretic spectra suggests two natural candidates. Note: in both cases, R denotes a commutative ring with 1.

Example 2.33 (The Zariski Spectrum)**.** The *Zariski Spectrum* $\mathrm{LSpec}(R)$ for R denotes the space of prime filters of R. Explicitly, they are the subobjects[25] $S \rightarrowtail R$ satisfying the axioms:

- $\top \to 1 \in S$, and $0 \in S \to \bot$;
- $(\forall a, a' \in R).\ aa' \in S \leftrightarrow a \in S \wedge a' \in S$;
- $(\forall a, a' \in R).\ a + a' \in S \leftrightarrow a \in S \vee a' \in S$.

Notice: in Set, these axioms say that S is the complement of a [non-trivial] prime ideal of R. In fact, working classically[26], $\mathrm{LSpec}(R)$ defines a topological space whose underlying set of points are $\mathrm{spec}(R)$, equipped with the the *Zariski topology*, generated by the basic Zariski open sets $D(a) = \{\mathfrak{p} \in \mathrm{spec}(R) \mid a \notin \mathfrak{p}\}$.

Example 2.34 (The coZariski Spectrum)**.** The *coZariski Spectrum* $\mathrm{ISpec}(R)$ for R denotes the space whose points are the *prime ideals* of R. The axioms can be obtained using contrapositives of those for $\mathrm{LSpec}(R)$. Regarded as a point-set space, $\mathrm{ISpec}(R)$ is $\mathrm{spec}(R)$ equipped with the *coZariski topology*, which is generated by the sub-basic open set $V(a) = \{\mathfrak{p} \in \mathrm{spec}(R) \mid a \in \mathfrak{p}\}$.

We now give the big picture. When viewed as point-set spaces, the points of $\mathrm{LSpec}(R)$ and $\mathrm{ISpec}(R)$ become classically equivalent; one only sees the difference between these spaces in their topologies. By contrast, when viewed as point-free spaces, the points of $\mathrm{LSpec}(R)$ and $\mathrm{ISpec}(R)$ are constructively different. These constructive differences turn out to have surprising implications for our analysis.

Observation 2.35 (Extending Observation 5.12, [NV23])**.** Define $[av_U]$ to be the space of absolute values on $\mathbb{Q}$ satisfying the ultrametric inequality. In light of Theorem A, it would be natural to expect there to exist a quotient map

$$\mathrm{quot}\colon [av_U] \longrightarrow [\mathrm{places}_U] \tag{12}$$

sending an ultrametric absolute value to its corresponding place. However, if $[\mathrm{places}_U] = \mathrm{LSpec}(\mathbb{Z})$, then [NV23, Observation 5.12] tells us that no such map exists. By contrast, the natural map

$$\begin{aligned} \mathfrak{I}\colon [av_U] &\longrightarrow \mathrm{ISpec}(\mathbb{Z}) \\ |\cdot| &\longmapsto \{n \mid |n| < 1\} \end{aligned} \tag{13}$$

is well-defined, and can be easily checked to send both trivial and non-trivial non-Archimedean absolute values to their corresponding prime ideals.

Observation 2.35 gives strong evidence that it is more natural [in the point-free setting] to work with the coZariski spectrum $\mathrm{ISpec}(\mathbb{Z})$ than the standard Zariski spectrum $\mathrm{LSpec}(\mathbb{Z})$ in classical algebraic geometry. We therefore fix the following hypothesis/convention.

[25] The reader may wish to substitute mentions of "subobject" with "subsets" without too much harm — see [Tie76].

[26] i.e. by restricting to the global points of $\mathrm{LSpec}(R)$

Convention 2.36. In the paper, the space of primes in $\mathbb{Z}$ will always be $\mathrm{ISpec}(\mathbb{Z})$. Notice: while $\mathrm{ISpec}(R)$ is not a localic space for a general commutative ring R, the space $\mathrm{ISpec}(\mathbb{Z})$ *is* in fact localic since $\mathbb{Z}$ is a free algebra construction (cf. Fact 2.13).

3. Descent in Topos Theory

This section collects some basic results and folklore on descent in topos theory. Much of the material is standard (see, e.g. [JT84, Moe88]), except that we shall rework the key constructions in the language of point-free topology. We shall also be interested in both standard descent and lax descent, the latter of which appears to be less studied (but see [Pit86, Joh02b, Bun15]).

3.1. **Standard Descent.**

By way of motivation: consider a discrete set X and a discrete group G acting upon it. One then typically defines the *quotient of X by the G-action* as the set of G-orbits on X, i.e.

$$X/G := \{\mathrm{Orb}(x)\,|\, x \in X\}\,,$$

where $\mathrm{Orb}(x) := \{y\,|\,\exists g \in G \text{ s.t. } gx = y\}$. Of course, a direct translation of this construction to point-free spaces is obstructed by its point-set formulation — not only does X/G start with a *set of elements*, but it gives no information on how topology should interact with the quotienting since X and G are both discrete. Additional care is therefore needed to work out the correct analogue of a quotient construction in the point-free setting, which we outline via the framework of descent.

Convention 3.1. In this section, caligraphic letters $\mathcal{E}, \mathcal{E}' \ldots$ denote generalised spaces, and $\mathcal{S}\mathcal{E}, \mathcal{S}\mathcal{E}' \ldots$ denote their corresponding categories of sheaves (cf. Convention 2.9). If we wish to work in the localic setting, we use standard capital letters $X, Y \ldots$ to denote the localic spaces and $\mathcal{S}X, \mathcal{S}Y$ to denote their corresponding categories of sheaves. We remind the reader that localic spaces form a category, which we denote Loc.

Definition 3.2. A *2-truncated simplicial space* $\mathcal{E}_\bullet$ is a diagram of (generalised) spaces of the form

$$\begin{tikzcd} \mathcal{E}_2 \arrow[r, bend left, "\widehat{d}_0"] \arrow[r, "\widehat{d}_1"] \arrow[r, bend right, "\widehat{d}_2"'] & \mathcal{E}_1 \arrow[r, bend left, "d_0"] \arrow[r, bend right, "d_1"'] & \mathcal{E}_0 \arrow[l, "s_0"'] \end{tikzcd} \tag{14}$$

We require that the maps in Equation (14) commute up to isomorphism. More explicitly, this means that the following simplicial identities hold:

$$d_0 \circ \widehat{d}_1 \cong d_0 \circ \widehat{d}_0, \qquad d_0 \circ \widehat{d}_2 \cong d_1 \circ \widehat{d}_0, \qquad d_1 \circ \widehat{d}_2 \cong d_1 \circ \widehat{d}_1. \tag{15}$$

$$d_0 \circ s_0 \cong \mathrm{id}, \qquad\qquad d_1 \circ s_0 \cong \mathrm{id}. \tag{16}$$

In the case where $\mathcal{E}_\bullet$ is a diagram in Loc, we shall ask that Equations (15) and (16) hold up to *equality* (since points of localic spaces do not possess automorphisms).

Construction 3.3 (Standard Descent)**.** Given any 2-truncated simplicial space $\mathcal{E}_\bullet$, we can construct its *universal descent cocone* in two main steps.

- **First**, taking the sheaves on the spaces of Diagram (14), we obtain the following diagram:

$$\begin{tikzcd} \mathcal{S}\mathcal{E}_2 & \mathcal{S}\mathcal{E}_1 \arrow[l, bend right, "\widehat{d}_0^*"'] \arrow[l, "\widehat{d}_1^*"] \arrow[l, bend left, "\widehat{d}_2^*"] \arrow[r, "s_0^*"] & \mathcal{S}\mathcal{E}_0 \arrow[l, bend right, "d_0^*"'] \arrow[l, bend left, "d_1^*"] \end{tikzcd} \tag{17}$$

  Note the reversal of arrows: since the objects of Diagram (17) now denote categories of sheaves (as opposed to spaces), the correct arrows between them are the inverse image functors [corresponding to the maps of Diagram (14)].
- **Next**, we define the following two pieces of data:
  (1) The *descent category* Des, which is a category possessing
     Objects: (F, θ), where F is an object of $\mathcal{S}\mathcal{E}_0$ and $\theta : d_0^*(F) \xrightarrow{\sim} d_1^*(F)$, also known as the *descent data*, is an isomorphism satisfying the identities

(i) *(Unit Condition)* $s_0^*(\theta) \cong \mathrm{id}$;
(ii) *(Cocycle Condition)* $\widehat{d_0^*}(\theta) \circ \widehat{d_2^*}(\theta) \cong \widehat{d_1^*}(\theta)$.

Morphisms: $\alpha\colon (F,\theta) \to (F',\xi)$, where $u\colon F \to F'$ is a morphism in $\mathcal{SE}_0$ that is compatible with the descent data, i.e. $d_1^*(u) \circ \theta = \xi \circ d_0^*(u)$.

(2) p^* is the forgetful functor

$$\begin{aligned} p^*\colon \mathrm{Des} &\longrightarrow \mathcal{SE}_0 \\ (F,\theta) &\longmapsto F \end{aligned}$$

Notice: Des as defined is just a category. [Moe88, §3] proves that Des is in fact a topos, that p^* defines the inverse image of a universal geometric morphism $p\colon \mathcal{SE}_0 \to \mathrm{Des}$, and that (Des, p) defines the pseudo-colimit of Diagram (17) in $\mathfrak{Top}$ (the 2-category of toposes). Translated to the point-free setting, this yields the following diagram of spaces

$$p\colon \mathcal{E}_\bullet \longrightarrow [\mathbb{T}_{\mathrm{Des}}], \tag{18}$$

where $[\mathbb{T}_{\mathrm{Des}}]$ is the point-free space corresponding to Des. Informally, we may view $[\mathbb{T}_{\mathrm{Des}}]$ as acting as a coequaliser between d_0 and d_1 satisfying some additional descent conditions.

Two main examples of Construction 3.3 will be important for this paper.

Example 3.4 (Descent Topos of a Groupoid)**.** Translating [Bor94, Theorem 1.4.3] to our setting, we know that pullbacks exist in Loc.[27] One can therefore define a groupoid $G := (G_0, G_1)$ in Loc

$$\begin{tikzcd} G_1 \times_{G_0} G_1 \arrow[r, bend left, "\pi_1"] \arrow[r, "\mathfrak{m}"'] \arrow[r, bend right, "\pi_0"'] & G_1 \arrow[r, bend left, "d_0"] \arrow[r, bend right, "d_1"'] & G_0 \arrow[l, "s"] \end{tikzcd} \tag{19}$$

whereby

- $d_0, d_1\colon G_1 \to G_0$ are the domain and codomain maps,
- $s\colon G_0 \to G_1$ is the unit map,
- $G_1 \times_{G_0} G_1$ is the pullback of $G_1 \xrightarrow{d_0} G_0 \xleftarrow{d_1} G_1$, and we call $\mathfrak{m}\colon G_1 \times_{G_0} G_1 \to G_1$ the multiplication map. Notation: we write $\mathfrak{m}(h,g) = h \cdot g$ for $g, h \in G_1$.
- We also require G_1 to have inverses with respect to $\mathfrak{m}$ in the appropriate sense.[28]

Two key observations on the setup.

(a) It is clear Diagram (19) defines a 2-truncated simplicial space. Thus, applying Construction 3.3 yields the following diagram of spaces

$$p\colon G \longrightarrow\!\!\!\!\rightarrow [\mathbb{T}_{\mathrm{Des}}]. \tag{20}$$

where $[\mathbb{T}_{\mathrm{Des}}]$ is the space corresponding to the descent topos Des. [29]

(b) Informally, G_0 corresponds to the space of objects and G_1 the space of arrows acting on G_0. Examining how their points interact in Diagram (19) illuminates the groupoid structure.
[*Details.* A point of G_1 is a map $g\colon \mathcal{E} \to G_1$. Define $x := d_0 \circ g$ and $gx := d_1 \circ g$. It is clear x and gx are both points of G_0, for any $g \in G_1$. Combined with the simplicial identities, this justifies viewing the points of G_1 as arrows acting on the points of G_0. As for when when these

[27] For [Bor94], the category of locales $\mathbf{Loc} := \mathrm{Frm}^{\mathrm{op}}$ is the opposite of the usual category of frames. But this is equivalent to our category of localic spaces Loc by Discussion 2.16. **(M:) Be careful about this reference; proof might be classical. Ask Vickers if there's a better reference.**

[28] More formally, we require the maps

$$G_1 \times_{G_0} G_1 \xrightarrow{(\mathfrak{m},\pi_1)} G_1 \times_{G_0} G_1$$

and $(\pi_0, \mathfrak{m})$ to be isomorphisms over G_1.

[29] A small warning: while G is a diagram of localic spaces, it is not necessary that $[\mathbb{T}_{\mathrm{Des}}]$ be localic as well. In fact, as was discovered by [JT84], one can characterise all (Grothendieck) toposes as descent toposes with respect to a suitable localic groupoid.

arrows are composable, this is recorded by the multiplication map: $h \cdot gx$ is well-defined just in case $(h, g) \in G_1 \times_{G_0} G_1$.]

Notice: $[\mathbb{T}_{\mathrm{Des}}]$ coequalises the maps d_0 and d_1 in G. In the language of Observation (b), this means $p(x) \cong p(gx)$, for any $g \in G_1$. In fact, examining the unit condition and cocycle conditions yields the identities $p(\mathrm{id}_x x) \cong p(x)$ and $p(g \cdot hx) \cong p(g) \cdot p(hx)$ respectively. Since $[\mathbb{T}_{\mathrm{Des}}]$ is the universal space for which these identities hold, one can therefore view $[\mathbb{T}_{\mathrm{Des}}]$ as the quotient space of G_0 by the G_1-action.

Example 3.5. Let $\Phi\colon \mathcal{E}' \to \mathcal{E}$ be a map of spaces. Since pullbacks of (generalised) spaces exist (see e.g. [Vic07, Theorem 1.43]), construct the diagram

$$\begin{tikzcd} \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' \arrow[r, "\pi_{02}"] \arrow[r, bend left, "\pi_{01}"] \arrow[r, bend right, "\pi_{12}"'] & \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' & \mathcal{E}' \arrow[l, "\Delta"'] \arrow[l, bend right, "\pi_0"'] \arrow[l, bend left, "\pi_1"] \arrow[r, "\Phi"] & \mathcal{E} \end{tikzcd} \tag{21}$$

where

- $\Delta\colon \mathcal{E}' \to \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}'$ is the diagonal map sending $x \mapsto (x, x)$;
- π_{01}, π_{02} and π_{12} are the projection maps such that, e.g. π_{01} maps $(x_0, x_1, x_2) \mapsto (x_0, x_1)$.

It is clear Diagram (21) is a 2-truncated simplicial space (in fact, a groupoid – the inverse is given by swapping projections). One can therefore construct the associated descent category, which we denote as $\mathrm{Des}(\Phi)$, whose objects are pairs (F, θ) such that

- $F \in \mathcal{S}\mathcal{E}'$; and
- $\theta\colon \pi_0^*(F) \to \pi_1^*(F)$ is an isomorphism in $\mathcal{S}(\mathcal{E}' \times_{\mathcal{E}} \mathcal{E}')$ satisfying:
  $\Delta^*(\theta) \cong \mathrm{id}$;
  $\pi_{02}^*(\theta) \cong \pi_{12}^*(\theta) \circ \pi_{01}^*(\theta)$.

The analogy between Example 3.4 and the earlier example of X/G is clear, and illustrates how descent can be regarded as a quotient construction. Example 3.5 emphasises something different. Since $\mathrm{Des}(\Phi)$ can be constructed from any map $\Phi\colon \mathcal{E}' \to \mathcal{E}$ of spaces, it allows us to link topological properties of the map Φ to various structural relationships between the toposes $\mathrm{Des}(\Phi)$, $\mathcal{S}\mathcal{E}$ and $\mathcal{S}\mathcal{E}'$. The next series of definitions and results develop this remark.

Definition 3.6. Continuing with the setup of Example 3.5, the associated inverse image functor of Φ, i.e.

$$\Phi^*\colon \mathcal{S}\mathcal{E} \to \mathcal{S}\mathcal{E}', \tag{22}$$

induces a functor $\chi\colon \mathcal{S}\mathcal{E} \to \mathrm{Des}(\Phi)$ such that

$$\begin{tikzcd} \mathcal{S}\mathcal{E} \arrow[rr, "\chi"] \arrow[dr, "\Phi^*"'] & & \mathrm{Des}(\Phi) \arrow[dl, "U"] \\ & \mathcal{S}\mathcal{E}' & \end{tikzcd} \tag{23}$$

commutes, where U is the forgetful functor $U(F, \theta) = F$. We say that Φ satisfies *effective descent* when the induced functor χ is an equivalence.[30]

Remark 3.7. For more details on the construction of the functor χ, see Remark A.1.

This definition sets up a key result in Joyal-Tierney's groundbreaking monograph [JT84].

Theorem 3.8 ([JT84, Theorem VIII.2.1])**.** *Open surjections of toposes are effective descent morphisms.*

[30]This definition is traditionally applied to geometric morphisms, but we already saw in Discussion 2.10 how point-free maps correspond to geometric morphisms.

We will primarily be interested in open surjections between *localic* spaces. By way of motivation: consider a continuous map $f\colon X \to Y$ of topological spaces, and denote Ω_X and Ω_Y to be the frame of opens belonging to X and Y respectively. The pre-image f^{-1} maps opens of Y to opens of X. If f is an open map, then it maps open sets to open sets, i.e. there exists a map $\exists_f\colon \Omega_X \to \Omega_Y$. In fact, one easily checks the following:

(a) $\exists_f(U) \subseteq V$ iff $U \subseteq f^{-1}(V)$;
(b) $\exists_f(U \cap f^{-1}(V)) = \exists_f(U) \cap V$,

for any $U \in \Omega_X$ and $V \in \Omega_Y$. Notice Condition (a) says: $\exists_f$ is left adjoint to f^{-1}. Recalling Discussion 2.16 that we may regard the Lindenbaum Algebra of a propositional theory as its frame of opens, one is led to the following definition.

Definition 3.9. Let $f\colon X \to X'$ be a map of localic spaces, and consider its corresponding frame homomorphism on its frame of opens $f^{-1}\colon \Omega_{X'} \to \Omega_X$. We call f ...

(i) ... a *surjection* if f^{-1} is 1-to-1.
(ii) ... an *open map* if f^{-1} has a left adjoint satisfying the Frobenius reciprocity condition
$$f_!(U \wedge f^{-1}(V)) = f_!(U) \wedge V,$$
for all $U \in \Omega_X, V \in \Omega_{X'}$.

Remark 3.10. Strictly speaking, Definition 3.9 only defines open surjections for frame homomorphisms, not point-free maps in the sense of Definition 1.4 — but this is justified by Discussion 2.16. In fact, applying [Joh02b, Prop. C1.5.1 and Theorem C1.5.4], one can show that f^{-1} is an open surjection in the sense of Definition 3.9 iff the associated geometric morphism is an open surjection in the topos-theoretic sense, justifying our choice of definition. For the technical definition of an open surjection between arbitrary toposes (not needed for this paper), see [Joh02b, C3.1].

The definition of an open map brings to light new ways of understanding toposes and their sheaves. Following Joyal-Tierney [JT84, Chapter V], Definition 3.9 can be used to give the point-free analogue of a local homeomorphism.[31]

Definition 3.11 (Étale Bundles)**.** A map $f\colon Y \to X$ in Loc is *étale* if the maps f and its diagonal $\Delta\colon Y \to Y \times_X Y$ are both open. We call any étale map with codomain X an *étale bundle on* X.

Remark 3.12 (Étale = Fibrewise Discrete)**.** Two helpful intuitions regarding étale bundles.

(i) Informally, a bundle $p\colon Y \to X$ is just a map but thought of in an inverse way: given each $x \in X$, we define a space $p^{-1}(x)$ over x (i.e. the fiber over x), and we require the fibres to vary continuously with the base-point. As is usual in point-free topology, it suffices for us to just define the fiber $Y(x)$ over the generic point $x \in X$: so long as $Y(x)$ is a geometric construction, it will be preserved by pullback and thus extend continuously to all points in X. For more details, see [Vic22, §9], especially on the differences with point-set topology.
(ii) When $X = \{*\}$ the canonical projection $f\colon Y \to \{*\}$ is étale iff Y is a discrete space [JT84, Theorem V.5.1]. This gives rise to a useful slogan: "étale bundles = fibrewise discrete bundles".

This sets up the following fact, which gives three different ways of characterising sheaves of a localic topos $\mathcal{S}X$. (We remind the reader that the theory of objects $\mathbb{O}$ has only one sort, with no function symbols, predicates or axioms.)

Fact 3.13. Let X be a localic space. A sheaf $F \in \mathcal{S}X$ can be characterised in following equivalent ways.

(i) A functor $F\colon \Omega_X^{\mathrm{op}} \to \mathrm{Set}$ subject to the standard gluing conditions, where the frame of opens Ω_X is viewed as a (small) category.
(ii) A map $F\colon X \to [\mathbb{O}]$ from X to the object classifier.
(iii) An étale bundle $f\colon Y \to X$.

Proof.

[31] **(M:) Reference on why this is the point-free analogue of local homeomorphism? I recall that there's something in the Elephant.**

(i) This is the standard definition – see e.g. [Joh02a, Example A.2.1.8].

(i) $\iff$ (ii) Denote FinSet as (a small skeleton of) the category of finite sets. By [Joh02a, Example 3.2.9], the functor category $[\mathrm{FinSet}, \mathrm{Set}]$ is an object classifier in $\mathfrak{Top}$ in the sense that

$$\mathfrak{Top}(\mathcal{E}, [\mathrm{FinSet}, \mathrm{Set}]) \simeq \mathcal{E}$$

for any topos $\mathcal{E}$. The fact that $[\mathrm{FinSet}, \mathrm{Set}]$ classifies the theory $\mathbb{O}$ is [Vic07, Example 1.47]. Translated to the point-free setting: any model of $\mathbb{O}$ in any topos $\mathcal{E}$ corresponds to an object of $\mathcal{E}$.[32]

(i) $\iff$ (iii) This is [JT84, Prop. VI.3.3].

$\square$

Convention 3.14 ("Substituting x"). Consider a geometric morphism between two localic toposes[33]

$$g\colon \mathcal{S}X' \to \mathcal{S}X.$$

Under the inverse image functor g^*, an object $F \in \mathcal{S}X$ gets mapped to another object $g^*F \in \mathcal{S}X'$. But how does this translate to the other characterisations of $F \in \mathcal{S}X$ in Fact 3.13?

- Consider $F\colon X \to [\mathbb{O}]$ as a map to the object classifier. Then, g^* acts by precomposition

$$g \circ F\colon X' \to X \to [\mathbb{O}],$$

  where $g\colon X' \to X$ is the map corresponding to the geometric morphism g. Informally, g^* acts by a change of variables: whereas F maps $x \mapsto F(x)$ for the generic point $x \in X$, $g \circ F$ maps $x' \mapsto F(g(x'))$ for $x' \in X$.
- Consider $F \in \mathcal{S}X$ as an étale bundle $f\colon Y \to X$. g^* then acts by pullback along $g\colon X' \to X$

$$\begin{tikzcd} g^*Y \arrow[r, "i_0"] \arrow[d, "i_1"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

  where $g^*Y = \{(x', y) \in X' \times Y \mid g(x') = f(y)\}$. Extending this, a morphism $\theta\colon F \to F'$ gets sent to a bundle map $g^*\theta$ over X' satisfying the condition[34]

$$\begin{tikzcd} g^*Y \arrow[r, "g^*\theta"] \arrow[d, "i_0"] & g^*Y' \arrow[d, "i'_0"] \\ Y \arrow[r, "\theta"] & Y' \end{tikzcd}$$

In fact, leveraging descent techniques, a remarkable structure theorem extends this description of sheaves as étale bundles to characterise sheaves of arbitrary toposes (not necessarily localic).

Theorem 3.15. *As our setup,*

- *Let $G := (G_0, G_1)$ be an open localic groupoid (= d_0, d_1 are open maps);*
- *Define an* étale G-space *to be an étale bundle $E \xrightarrow{p} G_0$ equipped with an action $E \times_{G_0} G_1 \xrightarrow{\bullet} E$ satisfying the usual axioms (the pullback here is along $G_1 \xrightarrow{d_1} G_0$). Denote BG to be the* category of étale G-spaces*;*
- *Denote* Des *to be the descent category associated to G (as in Example 3.4).*

Then, Des $\simeq BG$. *Furthermore, any topos is equivalent to BG for some open localic groupoid G.*

Proof. The fact that $\mathrm{Des} \simeq BG$ is [Moe88, §4.2 and §5.2]. The fact that any topos is equivalent to BG for some open localic groupoid G is [JT84, Theorem VIII.3.2] by Joyal-Tierney. $\square$

[32] For those interested in the topos-validity of our analysis: any 2-category $\mathfrak{BTop}/\mathcal{S}$ of bounded toposes over $\mathcal{S}$ has an object classifier so long as $\mathcal{S}$ is an elementary topos with natural number object [Joh02a, Theorem B4.2.11].

[33] **(M:) Polish this later to streamline later discussion regarding pullbacks.**

[34] **(M:) This is natural, but double-check.**

Discussion 3.16 (Descent and Automorphisms)**.** Let's pause to appreciate Theorem 3.15. It tells us *all* toposes are of the form BG, or equivalently, all toposes are a colimit of some open localic groupoid G. As remarked by Johnstone [Joh02b, C5.1], this resonates with an informal picture, dating back to Grothendieck's work on étale cohomology of schemes, that a topos is "a space whose points have enough internal structure to allow them to possess non-trivial automorphisms".

Comparing with Fact 3.13 (iii), we see that localic toposes correspond to BG with trivial G_1-action, i.e. their points have no automorphisms.[35] But if we consider instead, e.g. a connected atomic topos $\mathcal{S}\mathcal{E}$ with a global point $p\colon \mathrm{Set} \to \mathcal{S}\mathcal{E}$, then one can show that $\mathcal{S}\mathcal{E} \simeq BG$, where G is the localic group of automorphisms of the point p [Joh02b, Remark C5.2.14(c)].

3.2. **Working Internally and Base-Changes.** Since pullbacks exist in $\mathfrak{Top}$, one may ask: given a construction A over a topos $\mathcal{E}$, can we pullback A along some map $\Phi\colon \mathcal{E}' \to \mathcal{E}$ to get a construction $\phi^{\#}(A)$ over $\mathcal{E}'$? A key result in [Moe88] guarantees that, under mild hypotheses, one can indeed pullback the descent construction BG (i.e. it is "stable under base-change").

Denote $\mathbf{Loc}(X)$ to be the slice category of localic spaces over X.[36] If $\Phi\colon X' \to X$ is a map of localic spaces, then Φ induces an adjunction

$$\begin{tikzcd} \mathbf{Loc}(X) \arrow[r, bend left, "\Phi^{\#}"] & \mathbf{Loc}(X') \arrow[l, bend left, "\Phi_!"] \end{tikzcd}, \qquad \Phi_! \dashv \Phi^{\#}. \tag{24}$$

The functor $\Phi^{\#}$ should be understood as pulling back a localic map $f\colon Y \to X$ along Φ. In fact, given any G in $\mathrm{Loc}(X)$, a straightforward check shows that pulling back along Φ yields another groupoid $\Phi^{\#}(G)$ in $\mathrm{Loc}(X')$. The same descent construction as in Example 3.4 can be applied to groupoids internal to $\mathbf{Loc}(X)$ for any localic X, but let us denote the resulting topos as $B(\mathcal{S}X, G)$ for clarity.

Theorem 3.17 (Moerdijk's Stability Theorem [Moe88, Theorem 6.7])**.** *As our setup,*

- *Let $\Phi\colon X' \to X$ be a map of localic spaces;*
- *Let G be an open localic groupoid over X.*

<u>Then</u>, the canonical geometric morphism

$$B(\mathcal{S}X', \Phi^{\#}(G)) \xrightarrow{\sim} \mathcal{S}X' \times_{\mathcal{S}X} B(\mathcal{S}X, G)$$

is an equivalence of toposes.

Remark 3.18. Various simplifications have been made when translating Moerdijk's result to our context. In [Moe88], $\mathbf{Loc}(X)$ is taken to be the category of internal locales in the topos $\mathcal{S}X$, but this is equivalent to the slice category of localic spaces over X by [Joh02b, C1.6] and Discussion 2.16. In fact, one can define the category of internal locales $\mathbf{Loc}(\mathcal{E})$ in an arbitrary topos $\mathcal{S}\mathcal{E}$ [not necessarily localic], and the obvious restatement of Moerdijk's Stability Theorem still holds.

We remark that the point-free perspective is particularly helpful in clarifying the underlying mathematics of working internally to some topos. The following convention outlines an important methodological principle, already used in [NV22].

Convention 3.19 ("Fixing x")**.** Suppose we wish to construct a map with multiple arguments, such as

$$f\colon [\mathbb{T}] \times [\mathbb{U}] \to [\mathbb{U}'].$$

To do this, we shall often say "fix $x \in [\mathbb{T}]$" and then, by the usual process, construct a map

$$f_x\colon [\mathbb{U}] \to [\mathbb{U}'].$$

[35] **(M:) Double-check - seems reasonable, but does it follow?**

[36] **(M:) I'm restricting to localic spaces X, but that's only because I haven't found a reference for generalised spaces yet.**

To spell out the underlying topos theory, first take the obvious products

$$\begin{tikzcd} {[\mathbb{T}] \times [\mathbb{U}]} \arrow[r] \arrow[d, "p"] & {[\mathbb{U}]} \arrow[d] \\ {[\mathbb{T}]} \arrow[r] & * \end{tikzcd} \qquad \begin{tikzcd} {[\mathbb{T}] \times [\mathbb{U}']} \arrow[r] \arrow[d, "p"] & {[\mathbb{U}']} \arrow[d] \\ {[\mathbb{T}]} \arrow[r] & * \end{tikzcd}, \tag{25}$$

which are well-defined since pullback of spaces exist. The declaration "fix $x \in [\mathbb{T}]$" means that we work over $[\mathbb{T}]$ (that is, we work internally to the topos of sheaves $\mathcal{S}[\mathbb{T}]$). Hence, to define f_x as above is actually to define $\langle p, f\rangle$ such that the following triangle commutes –

$$\begin{tikzcd} {[\mathbb{T}] \times [\mathbb{U}]} \arrow[rr, "\langle p,f\rangle"] \arrow[dr, "p"'] & & {[\mathbb{T}] \times [\mathbb{U}']} \arrow[dl, "p"] \\ & {[\mathbb{T}]} & \end{tikzcd}. \tag{26}$$

Notice this defines a morphism in the slice category of spaces over $[\mathbb{T}]$. Notice also that this is clearly equivalent to the f we wanted.

3.3. **Lax Descent.** There is also an important weakening of Construction 3.3 known as lax descent.

Construction 3.20 (Lax Descent)**.** Given a 2-truncated simplicial space $\mathcal{E}_\bullet$, the *lax descent category* of $\mathcal{E}_\bullet$, which we denote LDes, is the same as the standard descent category Des of Construction 3.3 except we omit the requirement that descent data θ be an isomorphism.

The following remarks clarify the distinction between standard vs. lax descent by examining the significance of requiring the descent data to be an isomorphism.

Remark 3.21 (Coinserters vs. Coequalisers)**.** Let $\mathcal{C}$ be a category, and consider the following diagram in $\mathcal{C}$:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd} \tag{27}$$

As is well-known, the *coequaliser* of (f, g) (if it exists) gives the universal solution to the problem of finding a morphism $h\colon B \to C$ such that $hf = hg$. Less familiar is a weaker construction known as the *coinserter* of (f, g), which gives the universal solution to the problem of finding a morphism $h\colon B \to C$ together with a 2-cell $hf \to hg$. Notice, in particular, the 2-cell provided by the universal property of the coinserter is not required to be invertible, unlike the coequaliser.

In our setting, the standard descent category Des can be regarded as a (pseudo-)coequaliser in $\mathfrak{Top}$ subject to specific descent conditions and the lax descent category LDes as a (pseudo-)coinserter subject to the same conditions. The two constructions define colimits that occasionally coincide but generally do not.

Example 3.22 (Standard Descent & Group Completion)**.** Consider an internal category $\mathbb{M}$ in the topos Set which we represent as

$$\begin{tikzcd} \mathbb{M}_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbb{M}_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathbb{M}_0 \arrow[l] \end{tikzcd} \tag{28}$$

where the arrows commute in the obvious way. We may therefore regard $\mathbb{M}_1$ as a discrete monoid acting on the set $\mathbb{M}_0$. Unpacking Remark 3.21, the lax descent topos essentially quotients $\mathbb{M}_0$ by the monoidal action $\mathbb{M}_1$ whereas the standard descent topos quotients $\mathbb{M}_0$ by the group completion of $\mathbb{M}_1$.

This association of standard descent with group completion is suggestive, particularly because group completion signals a potential loss of information[37], which alerts us to the same possibility when using standard descent. In fact, this phenomena that standard descent adjoins inverses for all morphisms of an internal category occurs more generally – see [Joh02a, Example B3.4.14].

[37]To illustrate, consider the example of the monoid $(\mathbb{N} \cup \{\infty\}, +)$ equipped with standard addition. This monoid is clearly non-trivial, yet its group completion is trivial: $n + \infty = \infty$ and thus $n = 0$ for all $n \in \mathbb{N} \cup \{\infty\}$. This general style of argument is called an Eilenberg swindle, whose basic moral is: group completions typically trivialise whenever we introduce infinities. There are also subtler ways in which group completion may result in a loss of information – see e.g. [CLNS18, Corollary 1.4.9] and [Bor18] in regards to $K_0(\mathrm{Var})$, the Grothendieck ring of varieties.

Remark 3.23. A natural question: is LDes a topos, just as in Construction 3.3? The answer is yes, but the result seems to only exist as folklore. For a sketch of the proof, see [Ng23, Theorem 6.1.16], which develops an earlier observation by Johnstone [Joh02a, Remark B3.4.10]. We note that we will not need this result for this paper – Theorem C proves directly that our lax descent category of interest is indeed a topos.

4. Non-Archimedean Places

Leveraging Ostrowski's Theorem A, we now begin our analysis of the non-Archimedean [hereafter: NA] places. The key takeaway is the following basic observation.

Observation 4.1. Let $|\cdot|$ be an NA absolute value on $\mathbb{Q}$.

(i) $|\cdot|^\alpha$ is also an NA absolute value, for any Dedekind $\alpha \in (0,\infty)$.
(ii) $|\cdot| = |\cdot_p^{|}\,\alpha$ for some $\alpha \in (0,\infty)$ and some unique prime $p \in \mathbb{N}_+$.

As written, Observation 4.1 is standard elementary number theory; the significance is that we now know it also holds geometrically (in the sense of Convention 2.5). Recall our objective of characterising the space of places of $\mathbb{Q}$. As a first approximation, we defined the space as the coequaliser of the Diagram (7), reproduced below.

$$[av] \times (0,1] \underset{ex}{\overset{\pi}{\rightrightarrows}} [av]$$

Since each non-Archimedean $|\cdot|$ is uniquely associated to some prime, this suggests a natural first reduction: instead of considering all places of $\mathbb{Q}$ at once, start by first "localising" and working prime by prime, before recovering the "global" picture. Further, since $|\cdot|^\alpha$ is still an NA absolute value for any $\alpha \in (0,\infty)$, we should view the space of NA absolute values as being acted on by $(0,\infty)$ (and not just $(0,1]$, as indicated above).

4.1. Local: At single $\mathfrak{p}$.

Denote $\mathrm{ISpec}(\mathbb{Z})_{\neq(0)}$ as *the space of non-trivial prime ideals of* $\mathbb{Z}$, i.e. prime ideals that possess a non-zero integer. Throughout this subsection, we fix a single $\mathfrak{p} \in \mathrm{ISpec}(\mathbb{Z})_{\neq(0)}$. By [NV23, Lemma 1.14], there exists a prime $p \in \mathbb{N}_+$ such that $\mathfrak{p} = (p)$.[38] As such, let us construct the following diagram of (point-free) spaces

$$(0,\infty) \times [av_{NA}; p] \underset{\mathrm{ex}}{\overset{\pi}{\rightrightarrows}} [av_{NA}; p] \tag{29}$$

whereby:

- $[av_{NA}; p]$ is the space of non-Archimedean absolute values such that $|p| < 1$;
- π is the projection map sending $(\alpha, |\cdot|) \mapsto |\cdot|$;
- ex sends $(\alpha, |\cdot|) \mapsto |\cdot|^\alpha$.

Note that ex is a well-defined map by Observation 4.1 and strict monotonicity of positive Dedekind exponentiation (which gives $|p| < 1 \implies |p|^\alpha < 1^\alpha = 1$). In fact, since $[av_{NA}; p] \cong (0,\infty)$ by [NV23, Prop. 5.5], we may reformulate Diagram (29) as

$$(0,\infty) \times (0,\infty) \underset{\mathcal{M}}{\overset{\pi}{\rightrightarrows}} (0,\infty) \tag{30}$$

where $\mathcal{M}$ is the multiplication map sending $(\alpha,\beta) \mapsto \alpha\cdot\beta$. The β should be understood as representing $|\cdot| \in [av_{NA}; p]$ via the relation $|\cdot| = |\cdot|_p^\beta$, whereas the multiplication action should be understood as corresponding to $|\cdot|^\alpha = \left(|\cdot|_p^\beta\right)^\alpha = |\cdot|_p^{\alpha\cdot\beta}$.

A final reformulation. Classically, a place of $\mathbb{Q}$ is defined as *just* an equivalence class of absolute values. However, upon closer examination, it becomes clear that ex should be understood as an algebraic action: indeed, note that $|\cdot|^1 = |\cdot|$ and $(|\cdot|^\alpha)^\lambda = |\cdot|^{\alpha\cdot\lambda}$, for any $|\cdot| \in [av_{NA}; p]$ and any $\alpha, \lambda \in (0,\infty)$. As such, let us reformulate Diagram (30) as follows:

[38] We quote this lemma explicitly to indicate there exists a geometric proof of this elementary fact. We emphasise we do not get this for free since the standard proof (see e.g. [vdW91, §3.7]) is not geometric – see [NV23, Remark 1.15] for details.

Construction 4.2. Define the localic groupoid $\mathbf{G} := (G_0, G_1)$ as

$$\begin{tikzcd} (G_1 \times_{G_0} G_1) \arrow[r, "\mathfrak{m}"'] \arrow[r, bend left, "\pi_0"] \arrow[r, bend right, "\pi_1"'] & (0,\infty)\times(0,\infty) & (0,\infty) \arrow[l, "s"] \arrow[l, bend right, "\pi"'] \arrow[l, bend left, "\mathcal{M}"] \end{tikzcd} \tag{31}$$

where

- $G_0 := (0,\infty)$ and $G_1 := (0,\infty)\times(0,\infty)$;
- π and $\mathcal{M}$ correspond to the projection and multiplication maps from Diagram (30);
- $\mathfrak{m}$ corresponds to the obvious multiplication map, s corresponds to the unit map sending $\beta \mapsto (1,\beta)$, and π_0, π_1 are the obvious projection maps.

Construction 4.2 leads to the following key definition, and sets up the main test problem of this section.

Definition 4.3 (A Non-Archimedean Place)**.** Define $\mathcal{D}$ to be the descent category corresponding to the universal descent cocone of Diagram (31) (as in Example 3.4). Call $\mathcal{D}$ the *topos of the non-Archimedean place associated to prime* p.

Problem 4.4. Give a useful characterisation of $\mathcal{D}$.

Discussion 4.5. Some orienting remarks. On the topos-theoretic side, Construction 3.3 tells us $\mathcal{D}$ is well-defined since the universal descent cocone of Diagram (31) exists, and that $\mathcal{D}$ is in fact a topos. On the number-theoretic side, the point-free perspective is illuminating. Since $\mathcal{D}$ is a topos, denote $[\mathbb{T}_{\mathcal{D}}]$ to be its corresponding point-free space. Recall from Example 3.4 that $[\mathbb{T}_{\mathcal{D}}]$ can be regarded as "the quotient space of G_0 by the G_1-action". As such, since:

(a) $G_0 := (0,\infty)$ represents the space of all NA absolute values associated to prime p; and
(b) Two (non-trivial) NA absolute values are equivalent iff they are both associated to the same prime iff they are related by a G_1-action,

this justifies our definition of $\mathcal{D}$ as the non-Archimedean place associated to p.

In what follows, we work to improve our understanding of $[\mathbb{T}_{\mathcal{D}}]$, the (quotient) space associated to the topos $\mathcal{D}$. To start, one may first observe G_1 represents a free transitive action of $(0,\infty)$ on G_0, and deduce that there exists a single G_1-orbit on G_0. One may also recall Discussion 3.16, which gave this informal picture of a topos as a generalised space whose points potentially carry non-trivial automorphisms. Put together, the following guess is reasonable.

Guess 4.6. $[\mathbb{T}_{\mathcal{D}}]$ is the singleton space $\{*\}$, with $(0,\infty)$ as the group of automorphisms acting on $\{*\}$.

Very interestingly, Guess 4.6 turns out to be wrong. The fundamental reason behind this has to do with the misplaced expectation that $[\mathbb{T}_{\mathcal{D}}]$ possesses non-trivial automorphisms. A baseline observation: non-triviality of the G_1-action does not imply non-triviality of the quotient space – e.g. $BG \simeq \mathrm{Set}$ for any connected topological group G. A similar issue arises in our setting:

Theorem B. $\mathcal{D} \simeq \mathrm{Set}$. *Or, equivalently,* $[\mathbb{T}_{\mathcal{D}}] \cong \{*\}$.

In other words, the quotient space $[\mathbb{T}_{\mathcal{D}}]$ is trivial: it is the singleton $\{*\}$ with *no* non-trivial automorphisms. Comparing our groupoid $\mathbf{G}$ with the previous example, one might then suspect that the connectedness of $(0,\infty)$ is the main culprit behind the trivialisation, but this is again a red herring. In fact, Theorem B follows from a more general result:

Theorem 4.7. *Consider the following groupoid* $\mathbf{H}$ *in* Loc

$$\begin{tikzcd} (G\times M)\times_M (G\times M) \arrow[r, "\mathfrak{m}"'] \arrow[r, bend left, "\pi_0"] \arrow[r, bend right, "\pi_1"'] & G\times M & M \arrow[l, "s"] \arrow[l, bend right, "\pi"'] \arrow[l, bend left, "\mathcal{M}"] \end{tikzcd} \tag{32}$$

where

- *The unique map $\rho\colon M \to \{*\}$ is an open surjection.*
- *π is the projection map sending $(g,m) \mapsto m$;*
- *G induces a free transitive action on M. More explicitly, denoting $\mathcal{M}(g,m) = g \cdot m$,*
  (a) Given any $m \in M$ and $h, h' \in G$, we have $h \cdot m = h' \cdot m$ implies $h = h'$;
  (b) Given any $m_0, m_1 \in M$, there exists $g \in G$ such that $g \cdot m_0 = m_1$.
- *s is the unit map sending $m \mapsto (\mathrm{id}_G, m)$, where $\mathrm{id}_G \in G$ represents the unit of the G-action.*

Then:

(i) $B\boldsymbol{H}$, the category of étale $\boldsymbol{H}$-spaces (Theorem 3.15), is equivalent to the descent category of the simplicial topos associated to Diagram (32)*;*
(ii) $B\boldsymbol{H} \simeq \mathrm{Set}$.

Proof. The proof proceeds in stages.

Step 0: Setup. Construction 3.3 gives us an explicit description of the universal descent cocone of the simplicial topos associated to Diagram (32). Denote this descent category to be Des. Our analysis of Des rests on identifying which sheaves over M (i.e. objects of topos $\mathcal{S}M$) are able to support the descent data conditions required by Construction 3.3.

Step 1: Warm-up. Following Example 3.5, construct the following diagram in Loc by taking iterated pullbacks of $\rho\colon M \to \{*\}$:

$$\begin{tikzcd} M \times M \times M \arrow[r, bend left, "\pi_{01}"] \arrow[r, "\pi_{02}"] \arrow[r, bend right, "\pi_{12}"'] & M \times M & M \arrow[l, "\Delta"'] \arrow[l, bend right, "\pi_1"'] \arrow[l, bend left, "\pi_0"] \arrow[r, "\rho"] & \{*\} \end{tikzcd} \tag{33}$$

where

- $\Delta\colon M \to M \times M$ is the diagonal map sending $m \mapsto (m,m)$;
- $\pi_{0,1}, \pi_{02}$ and π_{12} are the projection maps such that, e.g. π_{01} maps $(m_0, m_1, m_2) \mapsto (m_0, m_1)$.

In particular, Diagram (33) defines a groupoid, which we shall denote as $\mathbf{M}$.

Next, taking the category of sheaves on Diagram (33), construct the corresponding descent category $\mathrm{Des}(\rho)$. By hypothesis, $M \xrightarrow{\rho} \{*\}$ is an open surjection in Loc, and thus so is the corresponding inverse image functor $\rho^*\colon \mathrm{Set} \to \mathcal{S}M$ (Remark 3.10). Applying Theorem 3.8, this means that ρ is an effective descent morphism, i.e. $\mathrm{Des}(\rho) \simeq \mathrm{Set}$.

Step 2: $\boldsymbol{H}$ and $\boldsymbol{M}$ are isomorphic. Consider the following diagram in Loc

$$\begin{tikzcd} G \times M \arrow[rr, dashed] \arrow[dr, shift left, "\pi"] \arrow[dr, shift right, "\mathcal{M}"'] & & M \times M \arrow[dl, shift right, "\pi_0"'] \arrow[dl, shift left, "\pi_1"] \\ & M & \end{tikzcd} \tag{34}$$

Next, define the map

$$\begin{aligned} \langle \pi, \mathcal{M} \rangle\colon G \times M &\longrightarrow M \times M \\ (g,m) &\longmapsto (m, g \cdot m). \end{aligned} \tag{35}$$

This sets up the following key claim.

Claim 4.8. *$\langle \pi, \mathcal{M} \rangle$ induces an isomorphism of groupoids $\boldsymbol{H} \cong \boldsymbol{M}$.*

Proof. To define an isomorphism between groupoids $\mathbf{H} = (H_0, H_1)$ and $\mathbf{M} = (M_0, M_1)$, it suffices to define isomorphisms $M_0 \cong H_0$ and $M_1 \cong H_1$ that commute with the groupoid structure maps.[39] This follows from a series of observations.

[39]Consider two groupoids $\mathcal{G}, \mathcal{G}'$, viewed as categories. To show they are isomorphic, it suffices to construct an invertible functor $F\colon \mathcal{G} \to \mathcal{G}'$. Showing that our assignment commutes with the groupoid structure maps amounts to checking that our assigment is (in some sense) functorial. **(M:) Possibly rephrase.**

(a) $\langle \pi, \mathcal{M} \rangle$ makes the inner and outer triangles of Diagram (34) commute.
[Why? A quick diagram chase shows that

$$\pi(g, m) = m = \pi_0 \circ \langle \pi, \mathcal{M} \rangle (g, m)$$
$$\mathcal{M}(g, m) = g \cdot m = \pi_1 \circ \langle \pi, \mathcal{M} \rangle (g, m).]$$

(b) $\langle \pi, \mathcal{M} \rangle$ defines an isomorphism between $G \times M \xrightarrow{\sim} M \times M$.
[Why? The standard argument works, so long as we are careful when reasoning point-wise. More explicitly, define

$$\begin{aligned} \langle \pi, \mathcal{M} \rangle^{-1} \colon M \times M &\longrightarrow G \times M \\ (m_0, m_1) &\longmapsto (g_{m_0,m_1}, m_0) \end{aligned} \tag{36}$$

where we denote $g_{m_0,m_1} \in G$ to be the point of G such that $g_{m_0,m_1} \cdot m_0 = m_1$, given to us by transitivity of the G-action. Then, the the two maps are clearly inverse to each other, since

$$\langle \pi, \mathcal{M} \rangle \circ \langle \pi, \mathcal{M} \rangle^{-1} (m_0, m_1) = (m_0, g_{m_0,m_1} \cdot m_0) = (m_0, m_1)$$
$$\langle \pi, \mathcal{M} \rangle^{-1} \circ \langle \pi, \mathcal{M} \rangle (g, m_0) = (g_{m_0, g \cdot m_0}, m_0) = (g, m_0)$$

where the fact that $g_{m_0, g\cdot m_0} = g$ follows from the hypothesis that the G-action is free.]

(c) $\langle \pi, \mathcal{M} \rangle$ commutes with the multiplication maps. In fact, it induces an isomorphism

$$(G \times M) \times_M (G \times M) \xrightarrow{\cong} M \times M \times M.$$

[Why? Recall that $(G \times M) \times_M (G \times M)$ and $M \times M \times M$ are constructed via pullbacks of the domain and codomain maps of **H** and **M** respectively:

$$\begin{tikzcd} (G \times M) \times_M (G \times M) \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & G \times M \arrow[d, "\mathcal{M}"] \\ G \times M \arrow[r, "\pi"] & M \end{tikzcd} \qquad \begin{tikzcd} M \times M \times M \arrow[r, "\pi_{01}"] \arrow[d, "\pi_{12}"'] & M \times M \arrow[d, "\pi_1"] \\ M \times M \arrow[r, "\pi_0"] & M \end{tikzcd} \tag{37}$$

Consider the following diagram

$$\begin{tikzcd} (G \times M) \times_M (G \times M) \arrow[r, "\langle \pi, \mathcal{M} \rangle \circ \pi_1"] \arrow[d, "\langle \pi, \mathcal{M} \rangle \circ \pi_0"'] & M \times M \arrow[d, "\pi_1"] \\ M \times M \arrow[r, "\pi_0"] & M \end{tikzcd} \tag{38}$$

One easily checks that it commutes, since:

$$\begin{aligned} \pi_0 \circ \langle \pi, \mathcal{M} \rangle \circ \pi_0 (g_1, g_0 \cdot m, g_0, m) &= \pi \circ \pi_0 (g_1, g_0 \cdot m, g_0, m) && \text{[By item (a)]} \\ &= \mathcal{M} \circ \pi_1 (g_1, g_0 \cdot m, g_0, m) && \text{[By Diagram (37)]} \\ &= \pi_1 \circ \langle \pi, \mathcal{M} \rangle \circ \pi_1 (g_1, g_0 \cdot m, g_0, m) && \text{[By item (a)]} \end{aligned}$$

and so by the universal pullback property of $M \times M \times M$, we obtain a map

$$i \colon (G \times M) \times_M (G \times M) \longrightarrow M \times M \times M \tag{39}$$

such that $\pi_{12} \circ i = \langle \pi, \mathcal{M} \rangle \circ \pi_0$ and $\pi_{01} \circ i = \langle \pi, \mathcal{M} \rangle \circ \pi_1$. A similar argument yields a map

$$i^{-1} \colon M \times M \times M \longrightarrow (G \times M) \times_M (G \times M) \tag{40}$$

such that $\pi_0 \circ i^{-1} = \langle \pi, \mathcal{M} \rangle^{-1} \circ \pi_{12}$ and $\pi_1 \circ i^{-1} = \langle \pi, \mathcal{M} \rangle^{-1} \circ \pi_{01}$.

By item (b), $\langle \pi, \mathcal{M} \rangle$ and $\langle \pi, \mathcal{M} \rangle^{-1}$ are inverses. Hence, one computes that

$$\pi_0 \circ i^{-1} \circ i = \langle \pi, \mathcal{M} \rangle^{-1} \circ \pi_{12} \circ i = \langle \pi, \mathcal{M} \rangle^{-1} \circ \langle \pi, \mathcal{M} \rangle \circ \pi_0 = \pi_0, \tag{41}$$

and that $\pi_1 \circ i^{-1} \circ i = \pi_1$. Since π_0, π_1 are jointly monic, this implies $i^{-1} \circ i = \mathrm{id}$. An analogous argument gives $i \circ i^{-1} = \mathrm{id}$. This shows that $\langle \pi, \mathcal{M} \rangle$ induces an isomorphism of spaces between $(G \times M) \times_M (G \times M)$ and $M \times M \times M$ via the pullback property.]

(d) $\langle \pi, \mathcal{M}\rangle$ and $\langle \pi, \mathcal{M}\rangle^{-1}$ commutes with the structure maps of **H** and **M**.
[Why? Item (a) showed that $\langle \pi, \mathcal{M}\rangle$ commutes with the domain and codomain maps of **H** and **M**. The case for $\langle \pi, \mathcal{M}\rangle^{-1}$ can be similarly verified. Item (c) showed that $\langle \pi, \mathcal{M}\rangle$ commutes with the multiplication map. As for commuting with the unit maps, this follows from computing

$$\langle \pi, \mathcal{M}\rangle \circ s(m) = (m, m) = \Delta(m)$$
$$\langle \pi, \mathcal{M}\rangle^{-1} \circ \Delta(m) = (g_{m,m}, m) = (\mathrm{id}_G, m) = s(m),$$

where the second equation once again follows from the freeness of the G-action.]

□

*Step 3: **H** and **M** are open groupoids.* Theorem 3.15 tells us that $\mathrm{Des} \simeq \mathrm{B}\mathbf{H}$ if **H** is an open groupoid, i.e. if its domain and codomain maps

$$\pi, \mathcal{M} \colon G \times M \rightrightarrows M \tag{42}$$

are open maps. By Claim 4.8, we know that Diagram (42) is equivalent to the diagram

$$\pi_0, \pi_1 \colon M \times M \rightrightarrows M, \tag{43}$$

and so it suffices to prove that the projection maps are open. But this is straightforward. First note that these projection maps can be obtained via the kernel pair of $\rho \colon M \to \{*\}$, as depicted:

$$\begin{tikzcd} M \times M \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & M \arrow[d, "\rho"] \\ M \arrow[r, "\rho"] & \{*\} \end{tikzcd} \tag{44}$$

Since ρ is an open map by hypothesis, and since open maps are stable under pullback [JT84, §V.4], deduce that π_0, π_1 must also be open.

Step 4: Reduce to Step 1. By construction, both Des and $\mathrm{Des}(\rho)$ select certain sheaves of $\mathcal{S}M$ compatible with their respective definitions for descent data. This suggests the following heuristic: if the *descent data* of Des and $\mathrm{Des}(\rho)$ are equivalent (in some appropriate sense), then this should imply $\mathrm{Des} \simeq \mathrm{Des}(\rho)$.

To prove this, one can apply Claim 4.8 and explicitly check that the isomorphism $\mathbf{H} \cong \mathbf{M}$ induces the desired equivalence of descent data. Let us however give a more conceptual argument. By Step 3, both **H** and **M** are open groupoids. Applying Theorem 3.15, we obtain the equivalences $\mathrm{Des} \simeq \mathrm{B}\mathbf{H}$ and $\mathrm{Des}(\rho) \simeq \mathrm{B}\mathbf{M}$. Finally, since $\langle \pi, \mathcal{M}\rangle \colon \mathbf{H} \xrightarrow{\sim} \mathbf{M}$ induces an isomorphism of groupoids, we can therefore apply [Moe88, Summary Theorem 5.15] to deduce

$$\mathrm{Des} \simeq \mathrm{B}\mathbf{H} \simeq \mathrm{B}\mathbf{M} \simeq \mathrm{Des}(\rho).$$

Step 5: Finish. By Step 1, we obtain the equivalence

$$\mathrm{Des}(\rho) \simeq \mathrm{Set}.$$

By Steps 2-4, we obtain the characterisation

$$\mathrm{Des} \simeq \mathrm{B}\mathbf{H} \simeq \mathrm{Des}(\rho),$$

and so putting everything together gives

$$\mathrm{B}\mathbf{H} \simeq \mathrm{Des} \simeq \mathrm{Set}.$$

□

Discussion 4.9 (On the hypothesis "open")**.** The locale theorist may ask: why did we require $M \xrightarrow{\rho} \{*\}$ to be an *open* surjection in Theorem 4.7? After all, as Borceux proves in [Bor94, Example 1.6.5c], the unique map $L \to \{*\}$ can be shown to be open for any given locale L. This result appears to indicate that the additional hypothesis of openness is unnecessary.

The answer can be found in Borceux's argument. Recall that Ω, i.e. the frame of opens on $\{*\}$, corresponds to the frame of truth values. In his argument, Borceux represents this as the classical frame of Boolean truth values $\Omega = \{\bot, \top\}$, which is constructively inequivalent to the geometric frame of truth values. In other words, Borceux's argument only holds true classically; it is constructively *false* that $L \to \{*\}$

is open for a general locale L (or indeed that all projection maps are open). Since we want our analysis to be topos-valid, this justifies our original hypothesis.

As an application of Theorem 4.7, we obtain the more quotable result:

Theorem B. $\mathcal{D} \simeq \mathrm{Set}$. *Or, equivalently,* $[\mathbb{T}_{\mathcal{D}}] \cong \{*\}$.

Proof. Recall that $\mathcal{D}$ is the descent category associated to the groupoid $\mathbf{G}$ defined in Construction 4.2. Examining the hypotheses of Theorem 4.7, it thus suffices to show that:

(a) The $\mathcal{M}$-action of $\mathbf{G}$ is both free and transitive;
(b) The unique map $!\colon (0,\infty) \to \{*\}$ is an open surjection.

(a) is easy. As for (b), the fact that $!\colon (0,\infty) \to \{*\}$ is an open map[40] follows from [Vic09, Corollary 6.2] and the fact that $(0,\infty)$ is a generalised metric space. To show that ! is a surjection, it suffices to show it has a right-sided inverse (since this shows that ! is an epi in Loc). Let us define one such possible map:

$$\begin{aligned} !^{-1}\colon \{*\} &\longrightarrow (0,\infty) \\ \{*\} &\longmapsto 1. \end{aligned} \tag{45}$$

□

4.2. **Global: Over all $\mathfrak{p}$.** *Some recollections.* A *non-Archimedean* absolute value $|\cdot|$ is one which satisfies the ultrametric inequality $|x+y| \leq \max\{|x|,|y|\}$. A *trivial* absolute value $|\cdot|_0$ is defined by mapping $|x| = 1$ for all rationals $x \neq 0$, and all other absolute values are called *non-trivial*.

By Theorem B, we know that each non-Archimedean place (including the trivial place) corresponds to a singleton – as is classically assumed in number theory. Combined with Observation 2.35, this gives compelling evidence that the space of NA places ought to correspond to $\mathrm{ISpec}(\mathbb{Z})$. Perhaps surprisingly, we still do not know how to justify this geometrically. This has to do with topological subtleties in reconciling the trivial and non-trivial non-Archimedean absolute values, already discussed in [NV23, §6]. Nonetheless, if we restrict to the *non-trivial* NA places, then we can characterise the entire space of non-trivial NA places by way of a base-change argument.

Let us review our work in Section 4.1. Given some fixed $\mathfrak{p} \in \mathrm{ISpec}(\mathbb{Z})_{\neq(0)}$, we defined a groupoid $\mathbf{G}$ that expressed how the exponentiation acts on the space $[av_{NA}; p]$. We then defined the non-Archimedean place as corresponding to the quotient space $[\mathbb{T}_{\mathcal{D}}]$ of this action, before deducing that $\mathcal{S}[\mathbb{T}_{\mathcal{D}}] \simeq \mathrm{Set}$, or equivalently $[\mathbb{T}_{\mathcal{D}}] \cong \{*\}$ (Theorem B). However, notice the groupoid $\mathbf{G}$ was defined for an arbitrary fixed non-trivial $\mathfrak{p} = (p)$. In the language of Convention 3.19, we were working internally within the topos $\mathcal{S}(\mathrm{ISpec}(\mathbb{Z})_{\neq(0)})$. As such, in order to characterise the entire space of non-trivial NA places, we shall need to externalise the descent construction.

The first step is to assemble the following pullback diagram of spaces

$$\begin{tikzcd} v^{\#}(\mathbf{G}) \arrow[r] \arrow[d] & \mathbf{G} \arrow[d] \\ \mathrm{ISpec}(\mathbb{Z})_{\neq(0)} \arrow[r, "v"] & \{*\} \end{tikzcd} \tag{46}$$

where we explicitly denote $v\colon \mathrm{ISpec}(\mathbb{Z})_{\neq(0)} \to \{*\}$ as the unique terminal map in Loc. Then, define the topos of non-trivial NA places as $\mathcal{S}[\mathrm{places}_{NA\neq 0}] := \mathrm{B}(\mathcal{S}(\mathrm{ISpec}(\mathbb{Z})_{\neq(0)}), v^{\#}(\mathbf{G}))$, which we also regard as the category of sheaves on some space $[\mathrm{places}_{NA\neq 0}]$. The following theorem verifies that we obtain the expected characterisation.

Theorem 4.10. $[\mathrm{places}_{NA\neq 0}] \cong \mathrm{ISpec}(\mathbb{Z})_{\neq(0)}$.

Proof. Regard G as a localic groupoid internal to the topos Set. By Step 3 of the proof of Theorem 4.7, G is an open groupoid. Hence, apply Moerdijk's Stability Theorem 3.17 to obtain the following equivalence of toposes:

$$\mathrm{B}(\mathcal{S}(\mathrm{ISpec}(\mathbb{Z})_{\neq(0)}), v^{\#}(\mathbf{G})) \simeq \mathcal{S}(\mathrm{ISpec}(\mathbb{Z})_{\neq(0)}) \times_{\mathrm{Set}} \mathrm{B}(\mathrm{Set}, \mathbf{G}) \tag{47}$$

[40]Note: in the language of [Vic09], a localic space Y such that the unique map $!\colon Y \to \{*\}$ is an open map is called *overt*.

By Theorem B , we know that $\mathrm{B}(\mathrm{Set}, \mathbf{G}) \simeq \mathrm{Set}$. Translating Equation (47) into the language of point-free spaces therefore gives

$$[\mathrm{places}_{NA\neq 0}] \cong \mathrm{ISpec}(\mathbb{Z})_{\neq(0)} \times \{*\} \cong \mathrm{ISpec}(\mathbb{Z})_{\neq(0)}, \tag{48}$$

as claimed. $\square$

5. The Archimedean Place

Convention 5.1. Unless stated otherwise, this section will work with non-trivial Archimedean absolute values, i.e. $|\cdot|$ such that exists some non-zero integer n whereby $|n| > 1$. ([NV23, §2] discusses other definitions that allow for the trivial absolute value, but let us postpone this discussion for later.)

Recall that $[av_A]$ denotes the space of (non-trivial) Archimedean absolute values. By Theorem A, we know that $[av_A] \cong (0,1]$. Playing the same game as we did for the non-Archimedean place, this suggests the following reformulation of the algebraic action of exponentiation on $[av_A]$.

Construction 5.2. Define the following diagram in Loc:

$$\begin{tikzcd} M_1 \times_{(0,1]} M_1 \arrow[r, bend left, "\pi_1"] \arrow[r, "\mathfrak{m}"'] \arrow[r, bend right, "\pi_2"'] & (0,1] \times (0,1] \arrow[r, bend left, "\pi"] \arrow[r, bend right, "\mathcal{M}"'] & (0,1] \arrow[l, "s"'] \end{tikzcd} \tag{49}$$

where

- $M_1 := (0,1] \times (0,1]$;
- π corresponds to the projection map sending $(\alpha, \beta) \mapsto \beta$ and $\mathcal{M}$ corresponds to the multiplication map sending $(\alpha, \beta) \mapsto \alpha \cdot \beta$.
- $\mathfrak{m}$ corresponds to the obvious multiplication map, s corresponds to the unit map sending $\beta \mapsto (1, \beta)$, and π_1, π_2 are the obvious projection maps.

Remark 5.3. For clarity, here's how we should understand the parameters in Diagram (49):

- $\beta \in (0,1]$ in Diagram (49) represents $|\cdot| \in [av_A]$ via the relation $|\cdot| = |\cdot|_\infty^\beta$
- The $\mathcal{M}$-action mapping $(\alpha, \beta) \mapsto \alpha \cdot \beta$ represents $|\cdot|^\alpha = \left(|\cdot|_\infty^\beta\right)^\alpha = |\cdot|_\infty^{\alpha\cdot\beta}$.

The set-up is entirely analogous to Construction 4.2 except for one key difference: Diagram (49) is not a groupoid since the $\mathcal{M}$-action is not invertible. This is a crucial detail. Whereas Observation 4.1 tells us we can extend the $(0,1]$-action to an $(0,\infty)$-action on the NA absolute values, the same does not hold for Archimedean absolute values. [Why? Notice if $|\cdot|$ and $\alpha \in (0,\infty)$ is Archimedean then the triangle inequality clearly fails for $|\cdot|^\alpha$ since $(1+1)^\alpha > 1+1$.]

This is a basic algebraic observation, but brings to light certain subtleties in the topos theory. We want to define the Archimedean place as some kind of colimit of Diagram (49) — but which one? Discussion 3.22 alerts us to the possibility that the standard descent construction freely inverts the $\mathcal{M}$-action in the colimit, signalling a potential loss of information. Since we want to quotient $(0,1]$ by a non-invertible monoid action [as opposed to its group completion], we are naturally led to use the lax descent construction instead.

Definition 5.4 (Archimedean Place)**.** Following Construction 3.20, define $\mathcal{D}'$ to be the lax descent category of Diagram (49). Call $\mathcal{D}'$ the *topos of the Archimedean place*. Denote $[\mathbb{T}_{\mathcal{D}'}]$ to be the corresponding space of points of $\mathcal{D}'$.

The justification for Definition 5.4 is entirely analogous to Discussion 4.5 in the previous section. Remark 3.23 gives that $\mathcal{D}'$ is a topos, so we may ask about its corresponding space of points. Diagram (49) indicates that $[\mathbb{T}_{\mathcal{D}'}]$ can be regarded as the quotient space of $(0,1]$ by the $(0,1]$-action, just as in the case of the (single) non-Archimedean place. It is then natural to wonder if we get the same result as before, i.e. if $[\mathbb{T}_{\mathcal{D}'}]$ corresponds to the singleton space $\{*\}$. It does not. In fact, we get the following surprising result.

Theorem C. $\mathcal{D}' \simeq \mathcal{S}\overleftarrow{[0,1]}$*, or equivalently,* $[\mathbb{T}_{\mathcal{D}'}] \cong \overleftarrow{[0,1]}$.

Aside from the obvious difference with the non-Archimedean case, why else might Theorem C be surprising? One answer would be its number-theoretic implications, which we postpone till Section 7 for proper discussion. Here we give two other observations to round out our perspective:

(a) The appearance of the upper reals is unexpected.[41] Informally, Theorem C says that if we quotient the real interval $(0,1]$ by the multiplicative action of the monoid $(0,1]$, then we kill off all the left Dedekind sections of the reals in $(0,1]$ — something which is *a priori* not obvious.

(b) Although we were careful to exclude the trivial place — note that we considered $(0,1]$ instead of the closed interval $[0,1]$ — the fact that $[\mathbb{T}_{\mathcal{D}'}] \cong \overleftarrow{[0,1]}$ suggests that the (non-trivial) Archimedean place and trivial place cannot be definably separated.[42] This raises interesting questions on how we should understand the generic Archimedean completion, especially since $\mathbb{Q}$ and $\mathbb{R}$ are clearly not homeomorphic.

Before proceeding, a few words about strategy. The basic plan of attack for proving Theorem C is simple: construct two functors

$$\begin{tikzcd} \mathfrak{J}\colon \mathcal{D}' \arrow[rr, bend left] & & \mathcal{S}\overleftarrow{[0,1]} \colon \mathfrak{K} \arrow[ll, bend left] \end{tikzcd}$$

and verify that $\mathfrak{J}$ and $\mathfrak{K}$ are inverse to each other. The mathematical devil, unsurprisingly, lies in the details. Nonetheless, though the constructions are involved, they are (implicitly) guided by a key topological insight regarding the Archimedean vs. non-Archimedean case: *$\mathcal{D}'$ witnesses non-trivial forking in the connected components of its sheaves whereas $\mathcal{D}$ does not*. Many of the arguments in this section can be understood as adjusting for this difference. Further discussion of this so-called forking phenomena will be deferred to Section 6; for now, we focus on establishing the key moves of the proof.

5.1. **First Direction.** We start by working to construct the functor:

$$\begin{aligned} \mathfrak{J}\colon \mathcal{D}' &\longrightarrow \mathcal{S}\overleftarrow{[0,1]} \\ (F,\theta) &\mapsto \quad ? \end{aligned} \tag{50}$$

We start by applying Fact 3.13 to obtain useful characterisations of the sheaves in $\mathcal{S}(0,1]$ selected by the lax descent topos $\mathcal{D}'$. Throughout this section, we fix the following presentation.

Setup 5.5. *Let $(F,\theta) \in \mathcal{D}'$.*

(i) F can be equivalently characterised as:
- *F is a sheaf over $(0,1]$;*
- *$F\colon (0,1] \to [\mathbb{O}]$ is a map to the object classifier;*
- *F corresponds to an étale bundle $f\colon Y \to (0,1]$, where F can be viewed as a fibrewise definition of the bundle space of f, i.e. $F(\gamma) = f^{-1}(\gamma)$ for any $\gamma \in (0,1]$. In particular, notice: each fibre $F(\gamma)$ defines a set since étale bundles are fibrewise discrete (Remark 3.12).*

Throughout this section, we shall move freely between these different characterisations of F, depending on convenience.

(ii) The pullback of f along π and $\mathcal{M}$ gives

$$\begin{tikzcd} \pi^*(Y) \arrow[r, "\phi"] \arrow[d, "\delta"] & Y \arrow[d, "f"] \\ (0,1]\times(0,1] \arrow[r, "\pi"] & (0,1] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M}^*(Y) \arrow[r, "\phi'"] \arrow[d, "\delta'"] & Y \arrow[d, "f"] \\ (0,1]\times(0,1] \arrow[r, "\mathcal{M}"] & (0,1] \end{tikzcd}$$

which can be represented as

$$\pi^*(Y) = Y \times (0,1]$$
$$\mathcal{M}^*(Y) = \{(y,\alpha,\beta) \in Y \times (0,1] \times (0,1] \,\big|\, f(y) = \alpha\cdot\beta)\}$$

[41] Although, in hindsight, perhaps less surprising once we step away from classical number theory and examine the lax descent construction by itself: the quotient converts actions by the monoid into 2-cells, which introduces the one-sidedness.

[42] Why? Recall that any subspace of the upper reals must be closed under arbitrary joins (cf. Convention 2.23).

and where δ sends $(y,\beta) \mapsto (\beta, f(y))$ and ϕ sending $(y,\beta) \mapsto y$, whereas δ' maps $(y,\alpha,\beta) \mapsto (\alpha,\beta)$ and ϕ' maps $(y,\alpha,\beta) \mapsto y$.

(iii) Correspondingly, since the data $\theta\colon \pi^*(Y) \to \mathcal{M}^*(Y)$ defines a bundle map over $(0,1] \times (0,1]$, it is required to make the following diagram commute:

$$\begin{tikzcd} \pi^*(Y) \arrow[rr, "\theta"] \arrow[dr, "\delta"'] & & \mathcal{M}^*(Y) \arrow[dl, "\delta'"] \\ & (0,1] \times (0,1] & \end{tikzcd}$$

As such, we can express θ coordinate-wise as the following:

$$\begin{aligned} \theta\colon Y \times (0,1] &\longrightarrow \mathcal{M}^*(Y) \\ (y,\beta) &\longmapsto (\theta_0(y,\beta), \beta, f(y)) \end{aligned}$$

In particular, notice: $f(\theta_0(y,\beta)) = f(y)\cdot\beta$.[43]

(iv) For later quotation, let us also reformulate the descent conditions in the language of (iii).

- *Since $s^*\theta = \mathrm{id}$ by the unit condition, this implies*

$$\theta_0(y,1) = y.$$

[Details. Under the unit map $s\colon (0,1] \to (0,1] \times (0,1]$, one obtains the bundle map

$$\begin{tikzcd} s^*\pi^*(Y) \arrow[rr, "s^*\theta"] \arrow[dr, "s^*\delta"'] & & s^*\mathcal{M}^*(Y) \arrow[dl, "s^*\delta'"] \\ & (0,1] & \end{tikzcd}$$

with the bundle subspaces

$$s^*\pi^*(Y) = \{(y,\beta) \in Y \times (0,1] \mid \beta = 1\}$$

$$s^*\mathcal{M}^*(Y) = \{(y,\alpha,\beta) \in \mathcal{M}^*(Y) \mid \alpha = 1\}$$

and the maps $s^*\delta$ mapping $(y,1) \to f(y)$ and $s^*\delta$ mapping $(y,1,\beta) = 1\cdot\beta = f(y)$. Since $s^*\theta = \mathrm{id}$, the claim follows.]

- *Since $\pi_1^*\theta \circ \pi_0^*\theta = \mathfrak{m}^*\theta$ by the cocycle condition,*[44] *this implies*

$$\theta_0(\theta_0(y,\alpha''),\alpha') = \theta_0(y,\alpha),$$

where $\alpha'\cdot\alpha'' = \alpha$.

[Details. Recall that a pair $(\alpha,\beta) \in (0,1] \times (0,1]$ represents a point β being acted upon α via multiplication (Remark 5.3). The multiplication map

$$\mathfrak{m}\colon (0,1] \times (0,1] \times_{(0,1]} (0,1] \times (0,1] \longrightarrow (0,1] \times (0,1]$$

sends $(\alpha,\beta,\alpha',\beta') \longmapsto (\alpha\cdot\alpha',\beta')$.]

Remark 5.6. Notice the lax descent data was defined as $\theta\colon \pi^*(Y) \to \mathcal{M}^*(Y)$ as opposed to going the opposite direction $\theta\colon \mathcal{M}^*(Y) \to \pi^*(Y)$. In principle, one could have defined the lax descent data going in the latter direction, which would yield in a different lax descent topos[45]. However, we have chosen the former since we feel it is more natural to regard the projection map π as corresponding to the domain map d_0 of Construction 3.20 as opposed to the codomain map d_1.

[43] Why? By construction of $\mathcal{M}^*(Y)$.

[44] **(M:) This needs polishing/tightening.**

[45] Note: in the case of standard descent, the choice of direction does not affect the resulting descent topos since the descent data is required to be an isomorphism.

Constructing $\mathfrak{J}(F,\theta)$. We now work to show how to construct a new sheaf $\overline{F} \in \mathcal{S}\overleftarrow{[0,1]}$ from the original $(F,\theta) \in \mathcal{D}'$. First, consider the canonical map

$$\Psi : \mathbb{Q}_{(0,1]} \longrightarrow (0,1] \tag{51}$$

which sends a (discrete) point of $\mathbb{Q}_{(0,1]} := \{q \in \mathbb{Q} | 0 < q \leq 1\}$ to its canonical representative in $(0,1]$. Regarding F as an étale bundle $f\colon Y \to (0,1]$, we can pullback f along Ψ to obtain:

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "f_{\text{res}}"'] & Y \arrow[d, "f"] \\ \mathbb{Q}_{(0,1]} \arrow[r, "\Psi"] & (0,1] \end{tikzcd}$$

In other words, $f_{\text{res}}\colon Z \to \mathbb{Q}_{(0,1]}$ can be viewed as restricting the domain of $F\colon (0,1] \to [\mathbb{O}]$ to the map $F_{\text{res}}\colon \mathbb{Q}_{(0,1]} \to [\mathbb{O}]$.[46] In particular, if F_{res} satisfies the continuity conditions stipulated by the Lifting Lemma 2.30, then Observation 2.32 tells us that it canonically defines a sheaf on $\overleftarrow{[0,1]}$. We verify this is indeed the case by proving a more general claim.

Claim 5.7 (Key Claim)**.** *Given $(F,\theta) \in \mathcal{D}'$, the descent data θ induces a map*[47]

$$\theta_{\gamma'\gamma}\colon F(\gamma') \to F(\gamma),$$

for any $\gamma, \gamma' \in (0,1]$ such that $\gamma' \geq \gamma$, satisfying the following conditions:

(i) For all $\gamma \in (0,1]$, $\theta_{\gamma\gamma} = \mathrm{id}$ for all $\gamma \in (0,1]$;

(ii) If $\gamma, \gamma', \gamma'' \in (0,1]$ such that $\gamma'' \geq \gamma' \geq \gamma$, then $\theta_{\gamma''\gamma} = \theta_{\gamma'\gamma} \circ \theta_{\gamma''\gamma'}$;

(iii) For any $\gamma \in (0,1]$, denote

$$I_\gamma := \{q | \gamma < q < 1\} \cup \{1\}$$

to be its associated rounded ideal in $\mathsf{RIdl}(\mathbb{Q}_{(0,1]}, \prec)$, as defined in Example 2.26.
Then, the induced map

$$\theta_\gamma\colon \operatorname*{colim}_{q \in I_\gamma} F(q) \to F(\gamma)$$

is an isomorphism.

Proof. The proof involves performing various technical checks, but they all follow the same basic strategy: examine how the descent data θ imposes specific conditions on F, before leveraging them to deduce Conditions (i) - (iii).

Step 0: Setup. We start by reformulating the action of the descent data. Given $\gamma, \gamma' \in (0,1]$ such that $\gamma' \geq \gamma$ and $z \in F(\gamma')$, define the following function of sets:

$$\begin{aligned} \theta_{\gamma'\gamma}\colon F(\gamma') &\longrightarrow F(\gamma) \\ z &\longmapsto \theta_0\left(z, \frac{\gamma}{\gamma'}\right) \end{aligned} \tag{52}$$

where $\theta_0\colon Y \times (0,1] \to Y$ is the first coordinate map of θ as in Setup 5.5. We record two quick observations:

- Item (iii) of Setup 5.5 tells us that the following identity holds:

$$f(\theta_0(z, \frac{\gamma}{\gamma'})) = f(z) \cdot \frac{\gamma}{\gamma'}. \tag{53}$$

  This equation makes precise how the multiplicative action on the base space $(0,1]$ (i.e. mapping $\gamma' \mapsto \gamma$) lifts to an action on the bundle space Y (i.e. mapping $F(\gamma') \to F(\gamma)$).
- $\theta_{\gamma'\gamma}$ is well-defined.
  [Why? Obvious that $z \in F(\gamma') \subset Y$ and $\frac{\gamma}{\gamma'} \in (0,1]$. Further, since $f(\theta_0(z, \frac{\gamma}{\gamma'})) = \gamma' \cdot \frac{\gamma}{\gamma'}$, deduce $\theta_0(z, \frac{\gamma}{\gamma'}) \in F(\gamma)$.]

[46]Warning: $\mathbb{Q}_{(0,1]}$ cannot be thought of as a naive subspace of $(0,1]$ since $\mathbb{Q}_{(0,1]}$ is discrete space and thus its topology is not the subspace topology inherited from $(0,1]$.

[47]In fact, a function on sets — recall that étale bundles are fibrewise discrete (Remark 3.12.)

Step 1: Verifying Conditions (i) and (ii). Conditions (i) and (ii) essentially follows from the unit and cocycle condition on the descent data. [Details: For (i), the unit condition gives $\theta_0(y, \frac{\gamma}{\gamma}) = \theta_0(y, 1) = y$. Similarly for (ii), suppose $\gamma'' \geq \gamma' \geq \gamma$ in $(0, 1]$, and denote $y'' \in Y$ such that $f(y'') = \gamma''$. The cocycle condition yields $\theta_0(y'', \frac{\gamma}{\gamma''}) = \theta_0(\theta_0(y'', \frac{\gamma'}{\gamma''}), \frac{\gamma}{\gamma'})$, which by Equation (52) gives $\theta_{\gamma''\gamma} = \theta_{\gamma'\gamma} \circ \theta_{\gamma''\gamma'}$.]

Step 2: Reformulating Condition (iii). Notice: $\operatorname*{colim}_{q \in I_\gamma} F(q)$ is a filtered colimit in Set, and so admits the canonical description

$$\operatorname*{colim}_{q \in I_\gamma} F(q) = \coprod_{q \in I_\gamma} F(q)/\sim \tag{54}$$

as a coproduct quotiented by the equivalence relation

$$(x, F(q)) \sim (y, F(q')) \leftrightarrow \exists r \in I_\gamma. \left(q \prec r \wedge q' \prec r \wedge \theta_{qr}(x) = \theta_{q'r}(y)\right),$$

where "$(x, F(q))$" denotes $x \in F(q)$ and "$(y, F(q'))$" denotes $y \in F(q')$. Recall from Example 2.26 that $q \prec r$ iff $q > r$ or $q = r = 1$.

To verify Condition (iii) involves verifying that the induced map $\theta_\gamma \colon \operatorname*{colim}_{q \in I_\gamma} F(q) \longrightarrow F(\gamma)$ is an isomorphism (in fact, a bijection of sets) for any $\gamma \in (0, 1]$. Applying our above description of $\operatorname*{colim}_{q \in I_\gamma} F(q)$, this means verifying the following two sequents:

(a) $x \in F(\gamma) \longrightarrow \exists q \in I_\gamma. \left(\exists y \in F(q) \wedge x = \theta_{q\gamma}(y)\right)$
(b) $y, z \in F(q), \theta_{q\gamma}(y) = \theta_{q\gamma}(z) \longrightarrow \exists r \in I_\gamma. (q \prec r \wedge \theta_{qr}(y) = \theta_{qr}(z))$

which correspond to verifying surjectivity and injectivity of θ_γ respectively.

Step 3: Verifying surjectivity. We structure the proof of Sequent (a) into two stages.

Step 3a: Setup. Suppose $x \in F(\gamma)$. Since $f \colon Y \to (0, 1]$ is a local homeomorphism, there exists some open $U \subset Y$ such that $x \in U$ as well as a partial section $u \colon f(U) \to U$ where u induces an isomorphism $U \cong f(U)$. In particular, note $u(\gamma) = x$. In addition, since $(0, 1]$ has a base of rational-ended open intervals, we may assume without loss of generality that $f(U)$ is of that form. Explicitly: assume $f(U)$ is of the form (α, β) or $(\alpha, 1]$ where $0 \leq \alpha < \beta \leq 1$ for $\alpha, \beta \in \mathbb{Q}$.

Step 3b: Exploiting topology and the unit condition. Denote

$$X := f(U) \cap [\gamma, 1]$$

where $[\gamma, 1] := \{\gamma' \in (0, 1] \mid \gamma \leq \gamma'\}$ denotes the obvious closed interval.[48] We then use the partial section $u \colon f(U) \to U$ to define the following map:

$$\begin{aligned} \Theta \colon X &\longrightarrow F(\gamma) \\ a &\longmapsto \theta_{a\gamma}(u(a)) \end{aligned}$$

Notice that X is inhabited, since $\gamma \in f(U) \cap [\gamma, 1]$ by construction. In fact, since $f(U) = (\alpha, \beta)$ or $f(U) = (\alpha, 1]$ by Step 3a, it follows X can be characterised as one of the following (connected) subspaces of $(0, 1]$:

- Case #1: $X = [\gamma, 1]$;
- Case #2: $X = [\gamma, \beta)$.

This description of X has two important implications. First, since X is a connected space and $F(\gamma)$ is a discrete space, the image of $\Theta(X)$ is constant. In particular, since

$$\Theta(\gamma) = \theta_{\gamma\gamma}(u(\gamma)) = \theta_0(x, 1) = x,$$

where the final equality is by the unit condition, this implies $\Theta(a) = x$ for all $a \in X$. Second, note that in both cases, there exists some rational $q \in X$ such that $q \in I_\gamma$. [Why? Case #2 is obvious. For Case #1, let $q = 1$. Notice this works even when $X = \{1\}$ since we allow $1 \prec 1$.] Thus for $u(q) \in F(q)$, we obtain the identity $\Theta(q) = \theta_{q\gamma}(u(q)) = x$, proving Sequent (a).

Step 4: Verifying injectivity. The proof of Sequent (b) also proceeds in stages.

[48] Notice this definition allows for the degenerate case $\gamma = 1$, in which case $[\gamma, 1] = \{1\}$.

Step 4a: Setup. As our hypothesis, suppose $y, z \in F(q)$ such that $\theta_{q\gamma}(y) = \theta_{q\gamma}(z)$. For explicitness, denote $x := \theta_{q\gamma}(y) = \theta_{q\gamma}(z)$, which we observe to be an element of $F(\gamma)$. Next, define two maps $v, v' \colon (0, q] \to Y$ whereby $v(a) := \theta_{qa}(y)$ and $v'(a) := \theta_{qa}(z)$ respectively. Notice:

- The images of v and v' coincide on γ, since

$$v(\gamma) = \theta_{q\gamma}(y) = x = \theta_{q\gamma}(z) = v'(\gamma).$$

- v and v' are (partial) section maps of f since, e.g.:

$$f \circ v(a) = f \circ (\theta_{qa}(y)) = a,$$

  for any $a \in (0, q]$.

Finally, just as in Step 3a, let $U \subset Y$ be an open subspace such that $x \in U$ equipped with a section $u \colon f(U) \xrightarrow{\sim} U$.[49]

Step 4b: Refinement of open subspaces. We work to identify an open subspace of $(0, 1]$ on which all the section maps u, v, v' all agree. Notice for our chosen $q \in I_\gamma$, either $\gamma < q$ or $q = 1$ (or both). As such, define the following subspace

$$V := \begin{cases} f(U) \cap (0, q), & \text{if } \gamma < q \\ f(U), & \text{if } q = 1 \end{cases} \tag{55}$$

In either case, one easily checks:

- V is an open subspace of $(0, 1]$, and $\gamma \in V$;
- Both u and v are well-defined sections of f on the whole of V. In particular, the following diagram commutes:[50]

$$\begin{tikzcd} V \arrow[r, hook, "u"] \arrow[d, "v"] & U \arrow[d, "f"] \\ Y \arrow[r, "f"] & (0,1] \end{tikzcd} \tag{56}$$

Extending this, construct the obvious pullback:

$$\begin{tikzcd} V_u \arrow[r] \arrow[d, "v_u"'] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, hook, "u"] \\ V \arrow[r, "v"] & Y \end{tikzcd} \tag{57}$$

In particular, observe the following:

- $\gamma \in V_u$.
  [Why? Recall $u(\gamma) = x = v(\gamma)$ by Step 4a.]
- $V_u \subset V$ defines an open subspace on which both v and u agree on.
  [Why? The fact that u, v agree on V_u follows from construction. The fact that V_u is an open subspace follows from noting that V itself is an open subspace, that $u \colon V \to Y$ is homeomorphic onto its image, and the general fact that pullbacks preserve open inclusions (see e.g. [Joh02b, pp. 504]).]

Repeat the process to obtain an open inclusion $V'_u \hookrightarrow V$ of an open subspace V'_u on which u and v' agree, and also $\gamma \in V'_u$. Repeat one last time to obtain an open subspace $P \subset (0, 1]$ on which u, v, v' all agree, and also $\gamma \in P$. By the same argument as in Step 3a, we assume without loss of generality that P is an open interval of the form (α, β) or $(\alpha, 1]$ where $\alpha, \beta \in \mathbb{Q}$ such that $0 \leq \alpha < \beta \leq q$.

[49] Unlike Step 3a, we do not require $f(U)$ to be connected.

[50] To ease notation, we will not use "$u|_V$" (resp. "$f|_U$") to express the restriction of u to V (resp. the restriction of f to U).

Step 4c: Finish. By Step 4b, since $\gamma \in P$, we know that $\gamma \in (\alpha, \beta)$ or $(\alpha, 1]$ for appropriate rationals α, β. It is therefore clear there exists $r \in I_\gamma$ such that $q \prec r$ and $\theta_{qr}(y) = v(r) = v'(r) = \theta_{qr}(z)$. For instance, if $\gamma \in (\alpha, 1]$ then let $r = 1$, and if $\gamma \in (\alpha, \beta)$, then pick some rational r where $\alpha < \gamma < r < \beta$. $\square$

As an immediate corollary of Claim 5.7, we get:

Corollary 5.8. *Any $(F, \theta) \in \mathcal{D}'$ defines a sheaf $\overline{F} \in \mathcal{S}\overleftarrow{[0,1]}$.*

Proof. Represent $F \in \mathcal{S}(0,1]$ as a map $F\colon (0,1] \to [\mathbb{O}]$ to the object classifier, and consider its restriction $F_{\text{res}}\colon \mathbb{Q}_{(0,1]} \to [\mathbb{O}]$. By Claim 5.7, F_{res} satisfies the continuity conditions required by item (iii) of Observation 2.32, and so (F, θ) defines a sheaf $\overline{F}$ on $\overleftarrow{[0,1]}$. $\square$

Discussion 5.9 (Choice vs. Existential Quantifiers)**.** Let us flag a possible source of confusion. Recall the following argument employed in Steps 3a and 4a: "given a point $x \in F(\gamma)$, since $f\colon Y \to (0,1]$ is a local homeomorphism, we can pick an open $U \subset Y$ such that $x \in U \cong f(U)$". The cautious reader may ask: why is this argument geometrically justified? After all, given $x \in F(\gamma)$, there may in principle exist numerous opens $U \subset Y$ such that $x \in U \cong f(U)$. Hence, by picking a single open U, are we not implicitly invoking choice?

This touches upon a common misconception regarding "choice" vs. "existence" in constructive mathematics. To paraphrase Bauer [Bau17, §1.3], if we know (constructively) that "there exists an x satisfying property $\phi(x)$", then picking such an x is *not* an application of choice but rather an elimination of an existential quantifier. In our setting, recall from Remark 3.12 that local homeomorphisms can be characterised as maps $f\colon Y \to X$ such that both f and the diagonal $\Delta\colon Y \to Y \times_X Y$ are open. From this, one can deduce (constructively) that Y has a base of opens of the form U equipped with a unique section u. Hence, for any point $y \in Y$, one concludes that there exists an open $U \ni y$ with a section such that $u\colon f(U) \xrightarrow{\sim} U$.[51]

The Main Construction. We can now define our functor $\mathfrak{J}$. On the level of objects, we map:

$$\begin{aligned} \mathfrak{J}\colon \mathcal{D}' &\longrightarrow \mathcal{S}\overleftarrow{[0,1]} \\ (F, \theta) &\mapsto \quad \overline{F} \end{aligned}$$

where $\overline{F}$ is the sheaf over $\overleftarrow{[0,1]}$ associated to (F, θ) by Corollary 5.8, a consequence of Lifting Lemma 2.30.

Examining the proof of the Lifting Lemma, $\overline{F}$ is defined by restricting F to $\mathbb{Q}_{(0,1]}$ before taking the appropriate colimit in the upper reals.[52] The same procedure can be carried out on the level of morphisms. Explicitly, suppose we are given a morphism $u\colon (F, \theta) \to (G, \xi)$ in the lax descent category $\mathcal{D}'$. Representing $F, G\colon (0,1] \to [\mathbb{O}]$ as maps to the object classifier, the morphism u can be formulated as a (geometric) natural transformation, calculated point-wise[53]

$$\begin{tikzcd} F(\gamma) \arrow[r, "u_\gamma"] & G(\gamma) \\ F(\gamma') \arrow[u, "\theta_{\gamma'\gamma}"] \arrow[r, "u_{\gamma'}"] & G(\gamma') \arrow[u, "\xi_{\gamma'\gamma}"'] \end{tikzcd}, \tag{58}$$

where $\theta_{\gamma'\gamma}$ and $\xi_{\gamma'\gamma}$ are the maps induced by the respective descent data θ and ξ in the sense of Claim 5.7.

[51] **(M:) Some discussion of why topos-valid mathematics is constructive is needed in the preliminaries.**

[52] Details: By Example 2.26 (ii), any upper real $\gamma \in \overleftarrow{[0,1]}$ corresponds to a rounded ideal $I_\gamma \in \mathsf{RIdl}(\mathbb{Q}_{(0,1]}, \prec)$. Applying Equation (10) from the proof of the Lifting Lemma 2.30, this gives the explicit characterisation $\overline{F}(I_\gamma) := \operatorname*{colim}_{q \in I_\gamma} F(q)$ and $\overline{G}(I_\gamma) := \operatorname*{colim}_{q \in I_\gamma} G(q)$.

[53] **(M:) Double-check this. A geometric transformation between geometric morphisms is is a natural transformation $f^* \to g^*$ between the inverse image functors. It should suffice to check the points for the same reason that two toposes are equivalent as categories iff they have equivalent points, but check the details. Also, inverse image functors act by precomposition.**

We now give a point-wise definition of $\mathfrak{J}(u)\colon \overline{F} \to \overline{G}$. Given an upper real $I_\gamma \in \overleftarrow{[0,1]}$, construct the following vertical composition of maps

$$\begin{tikzcd}
& \operatorname*{colim}_{q\in I_\gamma} G(q) & \\
G(q') \arrow[ur] \arrow[rr, "\xi_{q'q}"] & & G(q) \arrow[ul] \\
F(q') \arrow[u, "u_{q'}"] \arrow[rr, "\theta_{q'q}"] & & F(q) \arrow[u, "u_q"']
\end{tikzcd} \tag{59}$$

where
- $q', q' \in I_\gamma$;
- The commutative square is the natural transformation square associated to $u\colon F \to G$;
- The upper triangle is the cocone associated to $\overline{G}(I_\gamma) = \operatorname*{colim}_{q\in I_\gamma} G(q)$.

It is clear Diagram (59) defines a cocone over the diagram

$$\left\{ F(q') \xrightarrow{\theta_{q'q}} F(q) \right\}_{q,q'\in I_\gamma}$$

and thus by the universal property of colimits, this induces a map

$$\mathfrak{J}(u)(I_\gamma)\colon \overline{F}(I_\gamma) \to \overline{G}(I_\gamma)$$

which we define to be the image of morphism u under $\mathfrak{J}$. An easy check shows that $\mathfrak{J}$ as defined is indeed functorial.

5.2. **Second Direction.** We now construct a functor inverse to $\mathfrak{J}$, i.e.

$$\begin{aligned} \mathfrak{K}\colon \mathcal{S}\overleftarrow{[0,1]} &\longrightarrow \mathcal{D}' \\ F &\mapsto \ ? \end{aligned} \tag{60}$$

Start by considering the following diagram of spaces:

$$(0,1] \times (0,1] \underset{\mathcal{M}}{\overset{\pi}{\rightrightarrows}} (0,1] \xrightarrow{\ r\ } \overleftarrow{[0,1]}$$

where $r\colon [0,1] \longrightarrow \overleftarrow{[0,1]}$ sends a Dedekind $\gamma \in (0,1]$ to its right Dedekind section. On the level of sheaves, the arrows reverse and the associated inverse image functors yield the following diagram

$$\mathcal{S}(0,1] \times \mathcal{S}(0,1] \underset{\mathcal{M}^*}{\overset{\pi^*}{\leftleftarrows}} \mathcal{S}(0,1] \xleftarrow{\ r^*\ } \mathcal{S}\overleftarrow{[0,1]}. \tag{61}$$

Recall from e.g. Convention 3.14 that sheaves can be regarded as maps to the object classifier $[\mathbb{O}]$. The following observation unpacks Diagram (61) in light of this fact.

Observation 5.10. Let $F \in \mathcal{S}\overleftarrow{[0,1]}$ be a sheaf.

(i) *On r^*.* The functor r^* sends F to a sheaf $\widehat{F}$ over $(0,1]$.

(ii) *Pullback via π^* and $\mathcal{M}^*$.* Regarding $\widehat{F} \in \mathcal{S}(0,1]$ as a map $\widehat{F}\colon (0,1] \to [\mathbb{O}]$, we obtain:
- $\pi^*(\widehat{F})\colon (0,1] \times (0,1] \to [\mathbb{O}]$ is a map that sends $(\alpha,\beta) \mapsto \widehat{F}(\alpha)$;
- $\mathcal{M}^*(\widehat{F})\colon (0,1] \times (0,1] \to [\mathbb{O}]$ is a map that sends $(\alpha,\beta) \mapsto \widehat{F}(\alpha\cdot\beta)$.

(iii) *Map between Fibres.* Recall for any $\gamma \in (0,1]$, we denote

$$I_\gamma := \{q \in \mathbb{Q}_{(0,1]} \mid \gamma < q \text{ or } q = 1\} \in \mathsf{RIdl}(\mathbb{Q}_{(0,1]}, \prec)$$

to be its associated rounded ideal. In particular, one can regard r as sending $\gamma \in (0,1]$ to I_γ — the difference between its right Dedekind section and I_γ lays only in the presentation.

Now for any $\gamma, \gamma', \in (0,1]$ such that $I_{\gamma'} \sqsubseteq I_\gamma$, functoriality of F yields a morphism

$$F(I_{\gamma'}) \xrightarrow{s_{\gamma'\gamma}} F(I_\gamma).$$

Notice $s_{\gamma'\gamma}$ can be viewed as a map of cocones. Details: first apply F to the cocone diagram of $I_{\gamma'} = \operatorname*{colim}_{q \in I_{\gamma'}} I_q$ to get:

$$\begin{tikzcd}
 & \operatorname*{colim}_{q\in I_{\gamma'}} I_q & \\
I_{q'} \arrow[ur, "\sqsubseteq"] \arrow[rr, "\sqsubseteq"] & & I_q \arrow[ul, "\sqsubseteq"']
\end{tikzcd}
\overset{F}{\rightsquigarrow}
\begin{tikzcd}
 & \operatorname*{colim}_{q\in I_{\gamma'}} F(I_q) & \\
F(I_{q'}) \arrow[ur, "s_{q'\gamma'}"] \arrow[rr, "s_{q'q}"] & & F(I_q) \arrow[ul, "s_{q\gamma'}"']
\end{tikzcd}$$

where the presentation of the RHS diagram is justified by the fact that

$$F(I_{\gamma'}) = F(\operatorname*{colim}_{q\in I_{\gamma'}} I_q) = \operatorname*{colim}_{q\in I_{\gamma'}} F(I_q),$$

since maps preserve filtered colimits (Lemma 2.28). It is clear $F(I_\gamma)$ and $F(I_{\gamma'})$ can both be viewed as cocones over the diagram

$$\left\{ F(I_{q'}) \xrightarrow{s_{q'q}} F(I_q) \right\}_{q',q\in I_{\gamma'}}, \tag{62}$$

and that functoriality of F makes $s_{\gamma'\gamma} \colon F(I_{\gamma'}) \to F(I_\gamma)$ into a cocone map. In fact, by the universal colimit property, $s_{\gamma'\gamma}$ is the unique map making the obvious cocone diagrams commute.

In fact, one can leverage Observation 5.10 to prove the following.

Claim 5.11. *Given $F \in \mathcal{S}\overleftarrow{[0,1]}$, there exists an $\mathcal{S}(0,1] \times \mathcal{S}(0,1]$-morphism*

$$\widehat{\theta} \colon \pi^*(\widehat{F}) \longrightarrow \mathcal{M}^*(\widehat{F})$$

satisfying the unit and cocycle conditions.

Proof. For any $\alpha, \beta \in (0,1]$, we have $\alpha \cdot \beta \leq \beta$, or equivalently $I_\beta \sqsubseteq I_{\alpha\cdot\beta}$. By Observation 5.10 (iii), this gives the morphism

$$s_{\beta(\alpha\cdot\beta)} \colon F(I_\beta) \longrightarrow F(I_{\alpha\cdot\beta}). \tag{63}$$

Applying the functor $r^* \colon \mathcal{S}\overleftarrow{[0,1]} \to \mathcal{S}(0,1]$ to Equation (63), we get the morphism

$$\widehat{\theta}_{\beta(\alpha\cdot\beta)} \colon \widehat{F}(\beta) \longrightarrow \widehat{F}(\alpha\cdot\beta)$$

where $\widehat{\theta}_{\beta(\alpha\cdot\beta)} := r^*\big(s_{\beta(\alpha\cdot\beta)}\big)$. In the language of Observation 5.10 (ii), this defines an $\mathcal{S}(0,1] \times \mathcal{S}(0,1]$-morphism

$$\widehat{\theta} \colon \pi^*(\widehat{F}) \longrightarrow \mathcal{M}^*(\widehat{F}).$$

It remains to check the descent conditions.

- $\widehat{\theta}$ *satisfies the unit condition.* This follows from the fact that $\beta \cdot 1 = \beta$.
  [Details: Since $\beta \cdot 1 = \beta$, it is clear that

  $$F(I_\beta) \xrightarrow{s_{\beta(1\cdot\beta)}} F(I_{1\cdot\beta})$$

  is the identity morphism, by functoriality of $F \colon \overleftarrow{[0,1]} \to [\mathbb{O}]$. Applying r^* gives that

  $$\widehat{\theta}_{\beta(1\cdot\beta)} \colon \widehat{F}(\beta) \to \widehat{F}(1\cdot\beta)$$

  is also the identity morphism, again by functoriality.]

- $\widehat{\theta}$ *satisfies the cocycle condition.* This follows from the fact that multiplication is associative. [Details: Suppose we are given $\alpha, \alpha', \beta \in (0,1]$. It is obvious
$$I_{\alpha\cdot(\alpha'\beta)} = I_{(\alpha\alpha')\cdot\beta}$$
since $\alpha\cdot(\alpha'\beta) = (\alpha\alpha')\cdot\beta$. It is also clear that the morphism
$$I_\beta \sqsubseteq I_{\alpha\alpha'\cdot\beta}$$
is equivalent to the morphism
$$I_\beta \sqsubseteq I_{\alpha'\cdot\beta} \sqsubseteq I_{\alpha\cdot\alpha'\beta},$$
since the specialisation order $\sqsubseteq$ on $\overleftarrow{[0,1]}$ defines a unique morphism between its points. Appealing to functoriality once more, conclude that the induced morphisms
$$\widehat{\theta}_{\beta(\alpha\alpha'\cdot\beta)}\colon \widehat{F}(\beta) \longrightarrow \widehat{F}(\alpha\alpha'\cdot\beta)$$
$$\widehat{\theta}_{\alpha\cdot(\alpha'\beta)} \circ \widehat{\theta}_{\beta(\alpha'\cdot\beta)}\colon \widehat{F}(\alpha) \longrightarrow \widehat{F}(\alpha'\cdot\beta) \longrightarrow \widehat{F}(\alpha\cdot\alpha'\beta)$$
are isomorphic, proving the cocycle condition.]

□

We now define our functor $\mathfrak{K}$. On the level of objects, we map:
$$\begin{aligned}\mathfrak{K}\colon \mathcal{S}\overleftarrow{[0,1]} &\longrightarrow \mathcal{D}'\\ F &\longmapsto (\widehat{F}, \widehat{\theta})\end{aligned}$$
where $\widehat{F} = r^*(F)$ as in Observation 5.10 (i), and $\widehat{\theta}$ as in Claim 5.11. $\mathfrak{K}$ is clearly well-defined on objects: $\widehat{F} = r^*(F)$ is clearly a sheaf over $(0,1]$ and we've already checked that $\widehat{\theta}$ satisfies the descent conditions.

On the level of morphsims: for any $\mathcal{S}\overleftarrow{[0,1]}$-morphism $u\colon F \to G$ where $F, G\colon \overleftarrow{[0,1]} \to [\mathbb{O}]$, we define
$$\mathfrak{K}(u) := r^*(u)\colon \widehat{F} \to \widehat{G}.$$
with r^* as in Diagram (61). This defines an $\mathcal{S}\overleftarrow{[0,1]}$-morphism functorially, so it remains to check that $r^*(u)$ is also a $\mathcal{D}'$-morphism[54]. The following claim verifies this.

Claim 5.12. *As our set-up, let*

- *$u\colon F \to G$ be a morphism in $F, G \in \mathcal{S}\overleftarrow{[0,1]}$;*
- *$(\widehat{F}, \widehat{\theta}_F), (\widehat{G}, \widehat{\theta}_G) \in \mathcal{D}'$ be obtained by applying $\mathfrak{K}$ to sheaves $F, G \in \mathcal{S}\overleftarrow{[0,1]}$.*

Then the following identity holds:
$$\mathcal{M}^*(\mathfrak{K}(u)) \circ \widehat{\theta}_F = \widehat{\theta}_G \circ \pi^*(\mathfrak{K}(u)). \tag{64}$$

Proof. In the language of Observation 5.10 (ii) and Claim 5.11, Equation (64) says that the following square commutes:
$$\begin{tikzcd} \widehat{F}(\alpha) \arrow[r, "\widehat{\theta}_F"] \arrow[d, "\mathfrak{K}(u)"'] & \widehat{F}(\alpha\cdot\beta) \arrow[d, "\mathfrak{K}(u)"] \\ \widehat{G}(\alpha) \arrow[r, "\widehat{\theta}_G"] & \widehat{G}(\alpha\cdot\beta) \end{tikzcd}$$
for generic point $(\alpha, \beta) \in (0,1] \times (0,1]$. But this follows from the more general fact that
$$\begin{tikzcd} \widehat{F}(\gamma') \arrow[r, "\widehat{\theta}_F"] \arrow[d, "\mathfrak{K}(u)"'] & \widehat{F}(\gamma) \arrow[d, "\mathfrak{K}(u)"] \\ \widehat{G}(\gamma') \arrow[r, "\widehat{\theta}_G"] & \widehat{G}(\gamma) \end{tikzcd}$$
for any $\gamma, \gamma' \in (0,1]$ such that $\gamma \le \gamma'$, since $\mathfrak{K}(u)$ is a geometric natural transformation. □

[54] **(M:) Include definition of what morphisms in the descent category look like.**

5.3. **Assemble and Finish.** With the key constructions completed, all the gears line up and we now prove the main result of the section. Here's a high-level picture of what's going on. It is clear any Dedekind real $\gamma \in (0,1]$ is uniquely determined by its right Dedekind section, which we represent as an upper real $I_\gamma \in \overleftarrow{[0,1]}$. Our theorem extends this correspondence to the level of sheaves: any sheaf on $(0,1]$ satisfying the descent conditions corresponds to a sheaf on $\overleftarrow{[0,1]}$.

Theorem C. $\mathcal{D}' \simeq \mathcal{S}\overleftarrow{[0,1]}$.

Proof. Let $\mathfrak{J}\colon \mathcal{D}' \to \overleftarrow{[0,1]}$ and $\mathfrak{K}\colon \mathcal{S}\overleftarrow{[0,1]} \to \mathcal{D}'$ be the two functors as defined in Sections 5.1 and 5.2. We now check that $\mathfrak{J}, \mathfrak{K}$ are inverse to each other.

Step 1: Verifying $\mathfrak{K}\circ\mathfrak{J} \cong \mathrm{id}_{\mathcal{D}'}$. Let $(F,\theta) \in \mathcal{D}'$. Recall from Key Claim 5.7 (iii) that there exists a map θ_γ inducing an isomorphism

$$\theta_\gamma\colon \operatorname*{colim}_{q\in I_\gamma} F(q) \xrightarrow{\sim} F(\gamma).$$

In fact, θ_γ induces much more.

Claim 5.13. *Let $(\widehat{\overline{F}}, \widehat{\theta}) := \mathfrak{K}\circ\mathfrak{J}(F,\theta)$. Then, θ_γ induces an isomorphism $(F,\theta) \cong (\widehat{\overline{F}}, \widehat{\theta})$.*

Proof of Claim. This amounts to checking the following:

(a) $\widehat{\overline{F}} \cong F$ *as sheaves.* In the language of Observation 5.10 (iii), the map $r\colon (0,1] \to \overleftarrow{[0,1]}$ sends $\gamma \in (0,1]$ to its associated rounded ideal $I_\gamma \in \overleftarrow{[0,1]}$. Unwinding definitions, deduce that

$$\widehat{\overline{F}}(\gamma) = \overline{F}\big(r(\gamma)\big) = \overline{F}(I_\gamma) = \operatorname*{colim}_{q\in I_\gamma} F(q).$$

Since $\operatorname*{colim}_{q\in I_\gamma} F(q) \cong F(\gamma)$, conclude that $\widehat{\overline{F}}(\gamma) \cong F(\gamma)$ for any $\gamma \in (0,1]$.

(b) $\widehat{\theta} \cong \theta$ *as morphisms.* The analysis proceeds by successive refinements of the original claim.

The first refinement. In the language of Observation 5.10 (ii), the claim that θ_γ induces an isomorphism between $\widehat{\theta}$ and θ as morphisms of sheaves is equivalent to saying that the diagram [55]:

$$\begin{tikzcd} \widehat{\overline{F}}(\alpha) \arrow[r, "\theta_\alpha"] \arrow[d, "\widehat{\theta}"'] & F(\alpha) \arrow[d, "\theta"] \\ \widehat{\overline{F}}(\alpha\cdot\beta) \arrow[r, "\theta_{\alpha\cdot\beta}"] & F(\alpha\cdot\beta) \end{tikzcd} \tag{65}$$

commutes for generic $(\alpha,\beta) \in (0,1]\times(0,1]$.

The second refinement. Since maps preserve filtered colimits, we know that

$$\widehat{\overline{F}}(\gamma) = \operatorname*{colim}_{q\in I_\gamma} F(q) = \operatorname*{colim}_{q\in I_\gamma} F(\operatorname*{colim}_{q'\in I_q} q') = \operatorname*{colim}_{q\in I_\gamma} \overline{F}(I_q)$$

for any $\gamma \in (0,1]$. Hence, we may reformulate Diagram (65) as

$$\begin{tikzcd} \operatorname*{colim}_{q\in I_\alpha} \overline{F}(I_q) = \operatorname*{colim}_{q\in I_\alpha} F(q) \arrow[r, "\theta_\alpha"] \arrow[d, "\widehat{\theta}"'] & F(\alpha) \arrow[d, "\theta"] \\ \operatorname*{colim}_{q\in I_{\alpha\cdot\beta}} \overline{F}(I_q) = \operatorname*{colim}_{q\in I_{\alpha\cdot\beta}} F(q) \arrow[r, "\theta_{\alpha\cdot\beta}"] & F(\alpha\cdot\beta) \end{tikzcd}\ . \tag{66}$$

[55] A remark on notation: We have been careful to denote the multiplication $\mathcal{M}(\alpha,\beta) = \alpha\cdot\beta$ as opposed to just $\mathcal{M}(\alpha,\beta) = \alpha\beta$. One reason for this is to emphasise that an algebraic action has taken place. Another reason is to reduce potential confusion between the morphism $\theta_{(\alpha\cdot\beta)}$ in Diagram (65) and the morphism $\theta_{\alpha\beta}\colon F(\alpha) \to F(\beta)$ as defined in Equation (52).

The re-appearance of colimits is suggestive. In particular, we make the following key observation. Recalling the morphisms defined in Observation 5.10 (iii), suppose all four corners of Diagram (66) can be regarded as cocones over the diagram

$$\left\{ s_{q'q} \colon \overline{F}(I_{q'}) \to \overline{F}(I_q) \right\}_{q',q \in I_\alpha} \tag{67}$$

By the universal colimit property, we know there exists a unique cocone map

$$\operatorname*{colim}_{q \in I_\alpha} F(q) \to F(\alpha \cdot \beta).$$

In particular, if $\theta \circ \theta_\alpha$ and $\theta_{\alpha\cdot\beta} \circ \widehat{\theta}$ both define cocone maps $\operatorname*{colim}_{q \in I_\alpha} F(q) \to F(\alpha \cdot \beta)$ over Diagram (67), then they must both be equivalent, i.e. Diagram (65) commutes.

The third and final refinement. We now work to clarify: how might $F(\alpha \cdot \beta)$ be regarded as a cocone over Diagram (67)? Start with $\operatorname{colim}_{q \in I_\alpha} \overline{F}(I_q)$. By Observation 5.10 (iii), we get the cocone

$$\begin{tikzcd} & \operatorname*{colim}_{q \in I_\alpha} \overline{F}(I_q) & \\ \overline{F}(I_{q'}) \arrow[ur, "s_{q'\alpha}"] \arrow[rr, "s_{q'q}"] & & \overline{F}(I_q) \arrow[ul, "s_{q\alpha}"'] \end{tikzcd}, \tag{68}$$

along with the fact that

$$\widehat{\theta} \colon \operatorname*{colim}_{q \in I_\alpha} \overline{F}(I_q) \longrightarrow \operatorname*{colim}_{q \in I_{\alpha\cdot\beta}} \overline{F}(I_q)$$

corresponds to the (unique) cocone map

$$s_{\alpha(\alpha\cdot\beta)} \colon \operatorname*{colim}_{q \in I_\alpha} \overline{F}(I_q) \longrightarrow \operatorname*{colim}_{q \in I_{\alpha\cdot\beta}} \overline{F}(I_q). \tag{69}$$

Composing Diagram (68) with $\theta_{\alpha\cdot\beta} \circ \widehat{\theta}$ in the obvious way, we get the following representation of $F(\alpha \cdot \beta)$ as a cocone over Diagram (67):

$$\begin{tikzcd} & F(\alpha \cdot \beta) & \\ \overline{F}(I_{q'}) \arrow[ur, "\theta_{\alpha\cdot\beta} \circ s_{q'(\alpha\cdot\beta)}"] \arrow[rr, "s_{q'q}"] & & \overline{F}(I_q) \arrow[ul, "\theta_{\alpha\cdot\beta} \circ s_{q(\alpha\cdot\beta)}"'] \end{tikzcd} \tag{70}$$

By a similar argument, the map $\theta \circ \theta_\alpha$ induces the cocone:

$$\begin{tikzcd} & F(\alpha \cdot \beta) & \\ \overline{F}(I_{q'}) \arrow[ur, "\theta \circ \theta_\alpha \circ s_{q'\alpha}"] \arrow[rr, "s_{q'q}"] & & \overline{F}(I_q) \arrow[ul, "\theta \circ \theta_\alpha \circ s_{q\alpha}"'] \end{tikzcd} \tag{71}$$

Our present task now reduces to understanding how Diagrams (70) and (71) define the same cocone diagram. More explicitly, we wish to prove the identity

$$\theta_{\alpha\cdot\beta} \circ s_{q(\alpha\cdot\beta)} = \theta \circ \theta_\alpha \circ s_{q\alpha} \tag{72}$$

for any $q \in I_\alpha$. To do this, let us review our proof of Key Claim 5.7. Read in our present context, it defines a cocone isomorphism $\theta_\gamma \colon \operatorname*{colim}_{q \in I_\gamma} F(q) \to F(\gamma)$ for any $\gamma \in (0, 1]$ such that the diagram

$$\begin{tikzcd} \operatorname*{colim}_{q \in I_\gamma} \overline{F}(I_q) = \operatorname*{colim}_{q \in I_\gamma} F(q) \arrow[r, "\theta_\gamma"] & F(\gamma) \\ \overline{F}(I_q) = F(q) \arrow[u, "s_{q\gamma}"] \arrow[ur, "\theta_{q\gamma}"'] & \end{tikzcd} \tag{73}$$

commutes. In particular, this means

$$\theta_{\alpha\cdot\beta} \circ s_{q(\alpha\cdot\beta)} = \theta_{q(\alpha\cdot\beta)}. \tag{74}$$

Further, by examining Equation (52) in Key Claim 5.7, it is also clear that the original descent morphism

$$\theta \colon F(\alpha) \to F(\alpha \cdot \beta)$$

coincides with the induced map

$$\theta_{\alpha(\alpha\cdot\beta)} \colon F(\alpha) \to F(\alpha \cdot \beta).$$

In particular, compute that:

$$\begin{aligned}
\theta \circ \theta_\alpha \circ s_{q\alpha} &= \theta_{\alpha(\alpha\cdot\beta)} \circ \theta_\alpha \circ s_{q\alpha} && [\text{since } \theta = \theta_{\alpha(\alpha\cdot\beta)}] \\
&= \theta_{\alpha(\alpha\cdot\beta)} \circ \theta_{q\alpha} && [\text{by Diagram (73)}] \\
&= \theta_{q(\alpha\cdot\beta)} && [\text{by Key Claim 5.7 (ii)}]
\end{aligned} \tag{75}$$

By Equations (74) and (75), we deduce Equation (72), proving our claim that $\widehat{\theta} \cong \theta$.

□

To complete Step 1, we need to check one final claim:

Claim 5.14. *Let $u \colon (F, \theta) \to (G, \xi)$ be a $\mathcal{D}'$-morphism. Then, $\mathfrak{K} \circ \mathfrak{J}(u) \cong u$.*

Proof of Claim. It suffices to show that the diagram

$$\begin{tikzcd}
F(\gamma) \arrow[r, "u_\gamma"] & G(\gamma) \\
\operatorname*{colim}_{q\in I_\gamma} F(q) \arrow[u, "\theta_\gamma"] \arrow[r, "\mathfrak{K}\circ\mathfrak{J}(u)"] & \operatorname*{colim}_{q\in I_\gamma} G(q) \arrow[u, "\xi_\gamma"']
\end{tikzcd} \tag{76}$$

commutes for any $\gamma \in (0, 1]$. It is clear θ_γ and $\mathfrak{K} \circ \mathfrak{J}(u)$ define cocone maps over the diagram

$$\left\{ F(q') \xrightarrow{\theta_{q'q}} F(q) \right\}_{q,q'\in I_\gamma}. \tag{77}$$

The claim then follows from checking that the following diagram commutes

$$\begin{tikzcd}
& F(q) \arrow[dl, "s_{q\gamma}"'] \arrow[d, "\theta_{q\gamma}"] \arrow[r, "u_q"] & G(q) \arrow[d, "\xi_{q\gamma}"] \arrow[dr, "s_{q\gamma}"] & \\
\operatorname*{colim}_{q\in I_\gamma} F(q) \arrow[r, "\theta_\gamma"'] & F(\gamma) \arrow[r, "u_\gamma"'] & G(\gamma) & \operatorname*{colim}_{q\in I_\gamma} G(q) \arrow[l, "\xi_\gamma"]
\end{tikzcd}. \tag{78}$$

□

In summary, since

- $\mathfrak{K} \circ \mathfrak{J}(F, \theta) \cong (F, \theta)$ for any $(F, \theta) \in \mathcal{D}'$, by Claim 5.13
- $\mathfrak{K} \circ \mathfrak{J}(u) \cong u$ for any $\mathcal{D}'$-morphism $u : (F, \theta) \to (G, \xi)$, by Claim 5.14

conclude that $\mathfrak{K} \circ \mathfrak{J} \cong \mathrm{id}_{\mathcal{D}'}$, finishing Step 1.

Step 2: Verifying $\mathfrak{J} \circ \mathfrak{K} \cong \mathrm{id}_{\mathcal{S}\overleftarrow{[0,1]}}$. As our setup:

- Let F be a sheaf over $\overleftarrow{[0,1]}$;
- Denote $(\widehat{F}, \widehat{\theta}) := \mathfrak{K}(F)$ and $\overline{\widehat{F}} := \mathfrak{J} \circ \mathfrak{K}(F)$;
- Let $u \colon F \to G$ be a morphism of sheaves over $\overleftarrow{[0,1]}$.

To prove that $\mathfrak{J} \circ \mathfrak{K} \cong \mathrm{id}_{\mathcal{S}\overleftarrow{[0,1]}}$, we need to check two things:

(a) $\overline{\widehat{F}} \cong F$ *as sheaves.* For any upper real $I_\gamma \in \overleftarrow{[0,1]}$, one easily verifies that

$$\overline{\widehat{F}}(I_\gamma) = \operatorname*{colim}_{q\in I_\gamma} \widehat{F}(q) = \operatorname*{colim}_{q\in I_\gamma} F(r(q)) = \operatorname*{colim}_{q\in I_\gamma} F(I_q) = F(I_\gamma).$$

(b) $\mathfrak{J}\circ\mathfrak{K}(u) \cong u$ *as morphisms.* For any upper real $I_\gamma \in \overleftarrow{[0,1]}$, it is clear $u\colon F \to G$ can be defined level-wise as a map of cocones

$$\begin{tikzcd} \operatorname*{colim}_{q\in I_\gamma} F(I_q) \arrow[r, dashed, "u"] & \operatorname*{colim}_{q\in I_\gamma} G(I_q) \\ F(I_q) \arrow[u, "s_{q\gamma}"] \arrow[ur, swap, "u\circ s_{q\gamma}"] & \end{tikzcd} \tag{79}$$

over the diagram

$$\left\{ F(I_{q'}) \xrightarrow{s_{q'q}} F(I_q) \right\}_{q,q'\in I_\gamma}, \tag{80}$$

where $s_{q'q}$ and $s_{q\gamma}$ are the morphisms from Observation 5.10 (iii). Unpacking definitions, one easily checks that $\mathfrak{J}\circ\mathfrak{K}(u)$ defines the same cocone map.

This completes Step 2, and we are done. □

Let's step back for a moment. What would happened had we used standard descent to define the Archimedean place [instead of lax descent]? Given the association between group completion and standard descent (cf. Discussion 3.22), it is natural to expect this results in a loss of information. The following observation confirms this.

Observation 5.15. Denote Z to be the space corresponding to the standard descent topos of Construction 5.2. Then $Z = \{*\}$.

Proof. By construction, Z is the coequaliser of $\pi, \mathcal{M}\colon (0,1]\times(0,1] \to (0,1]$ regarded as a diagram of spaces (subject, of course, to the descent conditions). In particular, denote:

- $p\colon (0,1] \to Z$ to be the (universal) quotient map;
- Z' to be the image of $(0,1]\times(0,1]$ under the map $p\circ\pi$ (or equivalently, $p\circ\mathcal{M}$), along with the obvious inclusion map $i\colon Z' \hookrightarrow Z$.

Now notice that:

- $Z' = \{*\}$.
  [Why? Let $(\alpha,\beta), (\alpha',\beta') \in (0,1]\times(0,1]$ be any two (pairs of) points in $(0,1]\times(0,1]$. Notice that there exists[56] $\gamma\in(0,1]$ such that $\gamma\cdot\beta' = \beta$ or $\gamma\cdot\beta = \beta'$. If $\gamma\cdot\beta' = \beta$, one computes
  $$p\circ\pi(\alpha,\beta) = p(\beta) = p(\gamma\cdot\beta') = p\circ\mathcal{M}(\gamma,\beta') = p\circ\pi(\alpha',\beta').$$
  The argument when $\gamma\cdot\beta = \beta'$ is entirely symmetric, and so deduce that $Z' = \{*\}$.]
- There exists a unique map $p'\colon (0,1] \to \{*\}$ into the singleton space.

We assemble the data into the following diagram

$$\begin{tikzcd} (0,1]\times(0,1] \arrow[r, shift left, "\pi"] \arrow[r, shift right, swap, "\mathcal{M}"] & (0,1] \arrow[r, "p"] \arrow[dr, swap, "p'"] & Z \arrow[d, dashed, swap, "j"] \\ & & Z' = \{*\} \arrow[u, bend right, swap, "i"] \end{tikzcd} \tag{81}$$

where j is obtained via the universal property of coequalisers. It remains to show that $Z \cong \{*\}$. The fact that $j\circ i = \mathrm{id}_{\{*\}}$ is obvious. For the converse direction, compute for any $\beta\in(0,1]$:

$$i\circ j\circ p(\beta) = i\circ p'(\beta) = i\circ p'\circ\pi(\beta,\beta) = p\circ\pi(\beta,\beta) = p(\beta),$$

[56] **(M:) Do we know this geometrically? Order is not decidable. But I think this can be circumvented by the locatedness axiom.**

where $i \circ p' = p$ by definition of Z'. Since p is an epi, this implies $i \circ j = \mathrm{id}_Z$. □

Remark 5.16. The proof strategy of Observation 5.15 can be adapted to give an alternative proof that a single non-trivial NA place corresponds to a singleton. Notice, however, we still need to verify that Diagram (32) is an open groupoid if order to apply Moerdijk's Stability Theorem 3.17 [as done in Theorem 4.10 to show $[\text{places}_{NA\neq 0}] \cong \mathrm{ISpec}(\mathbb{Z})_{\neq(0)}$.] The presented proof of Theorem B accomplishes both at once.

6. Discussion: Non-Trivial Forking of Sheaves

This section works out the topos-theoretic differences between the Archimedean vs. non-Archimedean place. By Theorems B and C, we already know that $\mathcal{D} \simeq \mathrm{Set}$ while $\mathcal{D}' \simeq \mathcal{S}\overleftarrow{[0,1]}$. Motivated by this, we ask:

Question 6.1. What kinds of sheaves are eliminated by standard vs. lax descent data? Alternatively, how wild or complicated are the sheaves of $\mathcal{D}'$ compared to those of $\mathcal{D}$?

The following basic observation tells us where to start looking.

Observation 6.2. As our setup, let:
- X be a locally connected localic space;
- F be a sheaf on X, which we represent as an étale bundle $f\colon Y \to X$.

Then, Y is locally connected. In particular, there exists a pairwise disjoint decomposition of Y into (a set of) connected open subspaces:

$$Y = \coprod_{i\in I} U_i. \tag{82}$$

Proof. By definition, since X is locally connected, every open subspace of X is expressible as a union of connected open subspaces. Since f is a local homeomorphism, this gives Y an open cover of locally connected subspaces, and so Y is locally connected as well. In particular, applying [Joh02b, Lemma C.1.5.8], one obtains a pairwise disjoint decomposition of Y into open connected components. □

In particular, recall that:
- Any $(F,\theta) \in \mathcal{D}$ defines a sheaf F on $(0,\infty)$;
- Any $(F',\theta') \in \mathcal{D}'$ defines a sheaf F' on $(0,1]$.

Since $(0,\infty)$ and $(0,1]$ are both locally connected localic spaces, Observation 6.2 suggests that analysis of sheaves in $\mathcal{D}$ or $\mathcal{D}'$ ought to be reducible to analysis of their connected components. Leveraging this insight, we establish the next series of observations.

Observation 6.3. As our setup, let:
- $\mathcal{D}$ be the topos of a single non-trivial non-Archimedean place, as in Theorem B;
- $(F,\theta) \in \mathcal{D}$, where F corresponds to an étale bundle $f\colon Y \to (0,\infty)$;
- $\mathrm{id}_{(0,\infty)}\colon (0,\infty) \to (0,\infty)$

Then, the étale bundle f can be represented as the following disjoint (set-indexed) coproduct

$$f \cong \coprod_I \mathrm{id}_{(0,\infty)}.$$

Proof. By Theorem B, the (inverse image functor of the) unique geometric morphism

$$\gamma^*\colon \mathrm{Set} \to \mathcal{D}$$

induces an equivalence of categories.[57] The following observations clarify our setup:

(a) Since γ^* induces an equivalence of categories, it must be essentially surjective, and so there exists $I \in \mathrm{Set}$ [viewed as an étale bundle over $\{*\}$] such that $\gamma^*(I) \cong f$.
(b) Represent a singleton $\{*\} \in \mathrm{Set}$ as the bundle $\mathrm{id}_{\{*\}}\colon \{*\} \to \{*\}$. It is obvious any set S can be represented as the disjoint coproduct $S \cong \coprod_S \mathrm{id}_{\{*\}}$;

[57] **(M:) This makes sense when we think about these toposes as spaces and we examine why equivalent toposes have equivalent points, but is there a more direct way of seeing this?**

(c) It is clear that $\gamma^*(\mathrm{id}_{\{*\}}) \cong \mathrm{id}_{(0,\infty)}$, since pulling back along $(0,\infty) \to \{*\}$ gives

$$\begin{tikzcd} (0,\infty) \arrow[r] \arrow[d, "\mathrm{id}_{(0,\infty)}"'] & \{*\} \arrow[d, "\mathrm{id}_{\{*\}}"] \\ (0,\infty) \arrow[r] & \{*\} \end{tikzcd}$$

Since γ^* preserves arbitrary colimits, Observations (a) - (c) give

$$f \cong \gamma^*(I) = \gamma^*(\coprod_I \mathrm{id}_{\{*\}}) = \coprod_I \gamma^*(\mathrm{id}_{\{*\}}) \cong \coprod_I \mathrm{id}_{(0,\infty)}.$$

□

Here is the upshot. Suppose $(F,\theta) \in \mathcal{D}$ is a connected sheaf, i.e. F corresponds to an étale bundle $f\colon Y \to (0,\infty)$ where Y is connected. Observation 6.3 then forces $Y \cong (0,\infty)$, as illustrated in Figure 1.

FIGURE 1. A connected sheaf F of $\mathcal{D}$

Since $\mathcal{D}' \not\simeq \mathrm{Set}$ by Theorem C, one naturally suspects that the connected sheaves of $\mathcal{D}'$ are no longer quite as simple. The following example gives the first indications of this.

Example 6.4. Define $(F,\theta) \in \mathcal{D}'$ where
- F corresponds to the inclusion map $f\colon (0,\alpha) \to (0,1]$, regarded as an étale bundle over $(0,1]$;
- θ is the descent data whose first coordinate map is defined as

$$\begin{aligned} \theta_0\colon (0,\alpha) \times (0,1] &\longrightarrow (0,\alpha) \\ (y,\beta) &\longmapsto (y \cdot \beta) \end{aligned} \tag{83}$$

[Why is this sufficient? Recall from Setup 5.5 that descent data θ is determined by the first coordinate map θ_0. One then easily verifies that our θ_0 satisfies the unit and cocycle conditions.]

Notice $(0,\alpha)$ is connected and $f((0,\alpha)) \cong (0,\alpha)$ by construction.

Example 6.4 signals an interesting difference with $\mathcal{D}$: the connected sheaves of $\mathcal{D}'$ need not be homeomorphic to the base space $(0,1]$. In fact, they can turn out to be much more complicated:

Example 6.5 (Tuning Fork Sheaf)**.** Following Example 6.4, define $(F,\theta),(F',\theta') \in \mathcal{D}'$ whereby:
- F corresponds to the inclusion map $f\colon (0,\frac{1}{2}) \hookrightarrow (0,1]$; and
- F' corresponds to the identity map $f'\colon (0,1] \to (0,1]$,
- The corresponding descent data θ and θ' both act by multiplication, analogous to Equation (83).

Now observe: the inclusion map $(0,\frac{1}{2}) \hookrightarrow (0,1]$ also induces a bundle map between f and f'. Since $\mathcal{D}'$ is a topos and toposes possess all pushouts, the cokernel pair of this bundle map exists, which we illustrate in Figure 2. For obvious reasons, we shall call this pushout sheaf the *Tuning Fork Sheaf*. In particular, since the pushout construction glues two connected spaces along a common subspace, one easily checks that the Tuning Fork sheaf is itself connected.[58]

[58]Details: denote $g\colon Z \to (0,1]$ to be the étale bundle corresponding to the Tuning Fork Sheaf. To show that Z is connected, it suffices to show that any map $h\colon Z \to S$ to a discrete space S is constant. As such, define two global sections $p_1, p_2\colon (0,1] \to Z$, one which maps the subspace $[\frac{1}{2},1]$ to the lower branch of Z, while the other maps it to the upper branch. Since $(0,1]$ is connected, we know that $h \circ p_1$ and $h \circ p_2$ are both constant. Now let $\gamma \in [\frac{1}{2},1]$ and $\gamma' \in (0,\frac{1}{2})$. Since $p_1(\gamma') = p_2(\gamma')$, conclude that h is constant by observing: $h \circ p_1(\gamma) = h \circ p_1(\gamma') = h \circ p_2(\gamma') = h \circ p_2(\gamma)$.

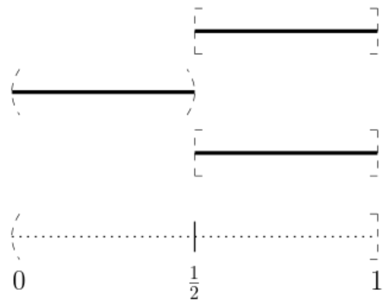
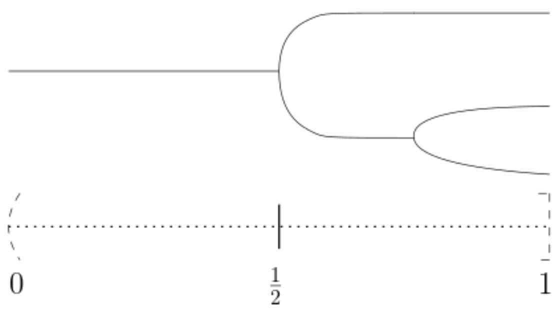


FIGURE 2. The Tuning Fork Sheaf of $\mathcal{D}'$

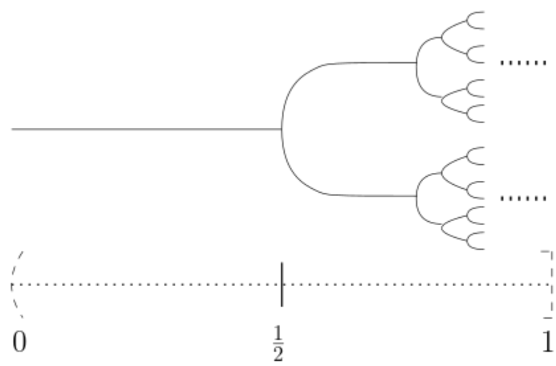
Discussion 6.6. The construction in Example 6.5 is fairly flexible, and can be used to construct a wide variety of forking structures in the [connected components of the] sheaves of $\mathcal{D}'$. This gives a new way to read the difference between standard descent vs. lax descent. In the non-Archimedean case, the rich sheafy structure over the original base space is completely flattened by the descent data: as shown in Observation 6.3 and illustrated by Figure 1, the connected sheaves are forced to be homeomorphic to $(0, \infty)$. In the Archimedean case, where the lax descent is comparatively weaker, this is no longer true. Example 6.5 gives an example of non-trivial forking persisting in the connected sheaves of $\mathcal{D}'$.

Discussion 6.6 gives an insight into the difference between the non-Archimedean vs. Archimedean case by identifying the kinds of sheaves present in $\mathcal{D}'$ (but absent in $\mathcal{D}$). For the rest of this section, we round out our understanding by identifying the kinds of sheaves which do *not* exist in $\mathcal{D}'$.

Example 6.7. Developing Discussion 6.6, note that there was nothing special about our choice of inclusion map $(0, \frac{1}{2}) \hookrightarrow (0, 1]$ in Example 6.5. In fact, one can iterate the argument to obtain the sheaf as illustrated in Figure 3a. However, a warning: there are limits to how far we can push this. For instance, we cannot iterate the forking construction for each branch of the 'fork' indefinitely, as illustrated in Figure 3b. Why? Note that the bundle space over 1 in Figure 3b gives the Cantor set, which is profinite. Since étale bundles must be fibrewise discrete, this means that Figure 3b no longer defines a sheaf over $(0, 1]$.



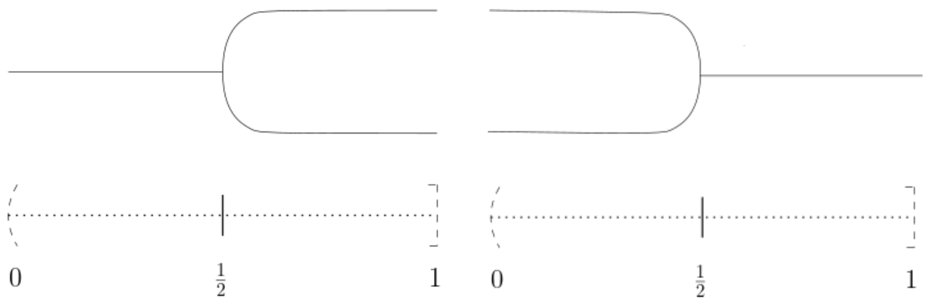
(A) Iterated Forking



(B) A Forbidden Configuration

FIGURE 3

Let us sharpen our language regarding this forking phenomena. Given an arbitrary sheaf $F \in \mathcal{S}(0, 1]$, say that F witnesses *upper bound forking* if there exists a connected component (cf. Observation 6.2) with two branches on the right of the branching point and one on its left (as illustrated in Figure 4a). Analogously, say that F witnesses *lower bound forking* if there exists two branches on the left of the branching point and one on its right (as illustrated in Figure 4b). In principle, there may be multiple instances of forking (see, e.g. Figure 3a), but we shall always assume that the branches of the fork do not 'join' back up.



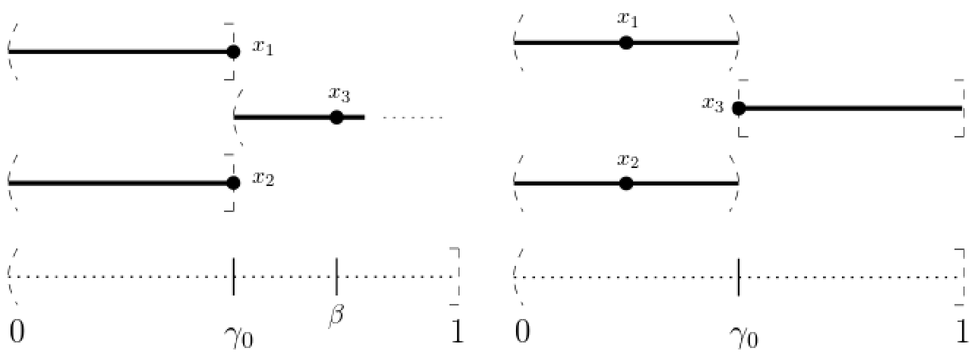
(A) Upper Bound Forking (B) Lower Bound Forking

FIGURE 4. Two Types of Forking with Branching Point at $\frac{1}{2}$

Claim 6.8. *For any $(F, \theta) \in \mathcal{D}'$, $F \in \mathcal{S}(0,1]$ does not witness lower bound forking.*

Proof. The argument proceeds in stages.

Step 0: Setup. Let $(F, \theta) \in \mathcal{D}'$. Suppose, for contradiction, that F witnesses an instance of lower bound forking. In principle, there may be multiple instances of forking in Y_i, but let us first assume for simplicity there only exists a single instance of lower bound forking — say in component Y_i at some $\gamma_0 \in (0,1)$. We can give the following explicit representation of Y_i: regard the obvious inclusions $(\gamma_0, 1] \hookrightarrow (0,1]$ and $(0,1] \hookrightarrow (0,1]$ as étale bundles over $(0,1]$, before obtaining Y_i as the cokernel pair of $(\gamma_0, 1] \hookrightarrow (0,1]$. See Figure 5a below for an illustration.



(A) The Connected Component Y_i (B) Another Forbidden Configuration

FIGURE 5

Step 1: On the fibre at γ_0. Moving from right to left, note that the forking begins *at* γ_0 and not *after*. More explicitly, our construction gives $Y_i \cap F(\alpha) = 2$ for all $\alpha \in (0, \gamma_0]$, where 2 denotes the discrete space of two points.

In fact, this is canonical. Why? Suppose Y_i witnesses an instance of lower bound forking at γ_0 such that $Y_i \cap F(\gamma_0) = \{*\}$, as illustrated in Figure 5b. However, this is equivalent to taking the cokernel pair of $[\gamma_0, 1] \hookrightarrow (0,1]$, which no longer defines an étale bundle — in particular, notice any open containing x_3 in the bundle space of Figure 5b will always contain parts of the two branches, obstructing local homeomorphism.[59] In plainer English: despite the family resemblance with Figure 5a, the bundle space of Figure 5b does not depict a sheaf in $\mathcal{S}(0,1]$.

[59] **(M:) Can I make this more precise? This would be in the footnotes.**

Step 2: Analysis of Descent Data. For orientation, we ask: how does the descent data θ of (F, θ) interact with the forking structure of Y_i? Recall from Key Claim 5.7 that θ induces maps on the fibres of F

$$\theta_{\gamma'\gamma}\colon F(\gamma') \to F(\gamma),$$

for any pair of Dedekinds $\gamma', \gamma \in (0,1]$ such that $\gamma \leq \gamma'$. In particular, since θ satisfies the unit condition, we get the identity $\theta_{\gamma\gamma}(x) = x$, given any $\gamma \in (0,1]$ and any $x \in F(\gamma)$. Read in the present context, this allows us to make a series of useful deductions.

Step 2a: θ restricts nicely to Y_i. Let $\beta \in (\gamma_0, 1]$. By Step 1, there exists distinct points $x_1, x_2, x_3 \in Y_i$ such that $x_1, x_2 \in F(\gamma_0)$ and $x_3 \in F(\beta)$, as depicted in Figure 5a. Given that $\theta_{\beta\beta}(x_3) = x_3 \in Y_i$, and that Y_i is a connected component disjoint from other components of F, deduce that $\theta_{\beta\gamma}(x_3) \in Y_i$ for any $\gamma \in (0, \beta]$. In particular, this implies that either $\theta_{\beta\gamma_0}(x_3) = x_1$ or $\theta_{\beta\gamma_0}(x_3) = x_2$.

Step 2b: No jumps. By Step 2a, assume without loss of generality that $\theta_{\beta\gamma_0}(x_3) = x_1$. Then, for any $y \in F(\gamma) \cap Y_i$ whereby $\gamma \in (\gamma_0, \beta]$, we claim that $\theta_{\gamma\gamma_0}(y) = x_1$. Why? First note that $F(\gamma) \cap Y_i = \{*\}$, essentially by construction. Since Step 2a gives $\theta_{\beta\gamma}(x_3) \in Y_i$, deduce that $\theta_{\beta\gamma}(x_3) = y$. Then, apply Key Claim 5.7 (ii) to get

$$x_1 = \theta_{\beta\gamma_0}(x_3) = \theta_{\gamma\gamma_0} \circ \theta_{\beta\gamma}(x_3) = \theta_{\gamma\gamma_0}(y).$$

That is, the descent data will always map $y \mapsto x_1$ in the upper branch and not "jump" to the lower branch.

Step 2c: A contradiction. Recall that the proof of Key Claim 5.7 (iii) involved verifying two sequents, which we reproduce below for the reader's convenience:

(a) $x \in F(\gamma) \longrightarrow \exists q \in I_\gamma.\big(\exists y \in F(q).(x = \theta_{q\gamma}(y)\big)$
(b) $y, z \in F(q), \theta_{q\gamma}(y) = \theta_{q\gamma}(z) \longrightarrow \exists r \in I_\gamma.(q \prec r \wedge \theta_{qr}(y) = \theta_{qr}(z))$

Applied to our setting, Sequent (a) says: given $x_2 \in F(\gamma_0)$, which lives on the lower branch of Figure 5a, there exists[60] some $q > \gamma_0$, and some $y \in F(q)$ such that $x_2 = \theta_{q\gamma_0}(y)$. But Step 2b forces the identity $\theta_{q\gamma_0}(y) = x_1 \neq x_2$, giving a contradiction.

Step 3: Extend and Finish. The same argument extends naturally to sheaves witnessing more than just a single instance of lower bound forking. We give an informal sketch. Suppose $(F, \theta) \in \mathcal{D}'$ and F witnesses an instance of lower bound forking (of possibly many instances) in some connected component Y_i. Then, find a sufficiently small neighbourhood of Y_i such that only a single instance of lower bound forking is witnessed. Adapt Step 2 accordingly to obtain the same contradiction, and conclude that such an instance of lower bound forking cannot occur. This completes the proof of the Claim. $\square$

Reviewing our work in this section, we present the following summary answer to Question 6.1.

Conclusion 6.9.

(i) Standard descent eliminates all forms of forking in the sheaves of $\mathcal{D}$.
(ii) Although upper bound forking persists in the sheaves of $\mathcal{D}'$, lax descent eliminates lower bound forking.
(iii) Lax descent also 'stretches' the sheaves of $\mathcal{S}(0,1]$ downwards. More precisely, if $(F, \theta) \in \mathcal{D}'$ and F corresponds to an étale bundle $f\colon Y \to (0,1]$, then $f(Y)$ must be a downward-closed interval in $(0,1]$.

Proof. (i) is by Observation 6.3, (ii) is Example 6.5 and Claim 6.8. (iii) is straightforward, but we elaborate for clarity. Suppose $\gamma \in f(Y)$, i.e. there exists some $y \in Y$ such that $f(y) = \gamma$. Recall from Setup 5.5 that the (lax) descent data gives $\theta_0(y, \beta) \in Y$ such that

$$f(\theta_0(y, \beta)) = f(y) \cdot \beta,$$

for any $\beta \in (0,1]$. Now suppose $\gamma' \in (0, \gamma]$. Since $\gamma^{-1} \cdot \gamma' \in (0,1]$, the lax descent data thus gives us a $y' \in Y$ such that

$$f(y') = f(y) \cdot \gamma^{-1} \cdot \gamma' = \gamma \cdot \gamma^{-1} \cdot \gamma' = \gamma',$$

i.e. that $\gamma' \in f(Y)$, proving downward closure. Notice: in contrast to the standard descent case, we do not get upward closure of $f(Y)$ since we only have a (non-invertible) monoidal action induced by $(0,1]$. $\square$

[60]Notice $\gamma_0 \in (0,1)$ by hypothesis, so we avoid the situation where the only $q \in I_\gamma$ such that $q \prec \gamma_0 = 1$ is $q = 1$.

7. A STRANGE WOODS

> *That is what one achieves, and in a very satisfactory manner, too, in the theory of "valuations" [...] To define a prime ideal in a field (a field given abstractly) is to represent the field "isomorphically" in a p-adic field: to represent it in the same way in the field of real or complex numbers, is (in this theory) to define a "prime ideal at infinity".*
>
> **— André Weil, letter to his sister** [Wei05]

In mathematics, difficult problems are often approached by breaking them into smaller, more manageable pieces. This method leads to two fundamental questions:

(1) How do we account for all the different pieces of the problem?
(2) How and when can we glue the pieces to form a global solution, and what are the obstructions to this reassembly?

In classical number theory, the Hasse Local-Global Principle addresses how solutions to polynomial equations over $\mathbb{Q}$ can be reassembled from its solutions over completions of $\mathbb{Q}$, i.e. the reals $\mathbb{R}$ and the p-adics $\mathbb{Q}_p$. The classical position, apparently going back to Hasse and/or Artin [Wei05], is to regard $\mathbb{Q}_p$ as corresponding to prime ideals of $\mathbb{Z}$, while $\mathbb{R}$ corresponds to a single formal prime at infinity. This understanding of how the local pieces ought to fit together forms the basis of powerful technologies, such as Arakelov Geometry already mentioned in Section 1. It is therefore surprising that the point-free perspective overturns this longstanding assumption. In particular, Theorem C shows that the real place, far from being an individual point with no intrinsic features, actually resembles a blurred unit interval $\overleftarrow{[0,1]}$.

The general reader may be forgiven for thinking that point-free mathematics simply reproves well-known results in a more constrained framework. Still, our results challenge this assumption, and demonstrate the potential of the point-free perspective to uncover insights previously missed. The underlying reason for this can be found in its methodology. By working geometrically, we pull classical mathematics away from the set theory, which in turn reveals a deep nerve connecting topology and algebra, with unexpected nuances. This primarily expository section explores some implications of this new perspective. At a high-level, we reframe the classical algebraic question, "How do we justify viewing the real place as a prime?" into a broader topological question: "How should the connected and the disconnected interact?"

7.1. What can Classifying Toposes Classify?

An important clue in our investigation of the non-Archimedean places was the following example by Bunge [Bun90], which we now discuss more fully:

Example 7.1. Let $G := (G_0, G_1)$ be a connected localic group. Then $BG \simeq \mathrm{Set}$.

Proof. An object of BG is an étale G-space, i.e. an étale bundle $E \xrightarrow{p} G_0$ equipped with a G_1-action $G_1 \times_{G_0} E \xrightarrow{\bullet} E$ satisfying the usual axioms. Since G is a group, this means that $G_0 = \{*\}$, and so deduce from Remark 3.12 that the bundle space of any étale G-space $E \xrightarrow{p} G_0$ is also discrete, equipped with G_1-action $G_1 \times E \xrightarrow{\bullet} E$. Given $e \in E$, this defines a natural map

$$\begin{aligned}(—)\bullet e \colon G_1 &\longrightarrow E\\ g &\longmapsto g\bullet e.\end{aligned}$$

Since E is discrete and G_1 is connected, this map must be constant; in fact, $g \bullet e = e$ for all $g \in G_1$ since the G_1-action forces the identity $s(*) \bullet e = e$ where $s \colon G_0 \to G_1$ is the unit map. Since the objects of BG are just sets (equipped with trivial G_1-action), conclude that $BG \simeq \mathrm{Set}$. □

The following comments give some context as to why Example 7.1 is interesting.

Discussion 7.2. Example 7.1 gave us our first indication that the topos-theoretic characterisation of a single non-Archimedean place may in fact be a trivialisation result — contrary to the expectations of Guess 4.6. Although the eventual proof of Theorem B did not require the hypothesis that the $(0,\infty)$-action of the groupoid in Construction 4.2 is connected, it is in fact possible to extend the argument of Example 7.1 to give an alternate (though much more involved) proof of the theorem.[61]

Discussion 7.3 (Connected vs. Fibrewise Discrete)**.** The core mechanism of Example 7.1's argument rests on two general facts:

(a) All sheaves over localic spaces can be characterised as fibrewise discrete bundles (Remark 3.12).
(b) All maps from connected spaces into discrete sets must be constant.

Put together, this suggests that the present issues with connectedness is not a bug but rather a feature of toposes (since toposes are, after all, defined to be categories of sheaves).

Discussion 7.4. The trivialisation result is also striking because it contravenes a basic expectation from the discrete setting. By Diaconescu's Theorem, we know that the presheaf topos $BG \simeq [G, \mathrm{Set}]$ classifies all G-torsors for any discrete group G. Yet when G is a connected group, the fact that $BG \simeq \mathrm{Set}$ implies that for each topological space X there exists essentially only one geometric moprhism $\mathcal{S}X \to BG$, even though for suitable X and G there may exist many non-isomorphic G-torsors. [62]

To clarify Discussion 7.4, one may ask: is the failure to classify G-torsors due to the nature of the BG construction, or due to the very nature of toposes themselves (as suggested by Discussion 7.3)? That is, even in cases where BG does not classify G-torsors, can we find some other topos $\mathcal{S}\mathcal{E}$ that does? The following observation by Lurie gives a general instance where this cannot happen.

Observation 7.5 (Lurie's Observation [Lur14])**.** Let $G := (\{*\}, G_1)$ be a localic group such that there exists a non-constant continuous map $\nu\colon \mathbb{R} \to G_1$. Then, there cannot exist a topos $\mathcal{S}\mathcal{E}$ that classifies G-torsors over localic spaces, i.e. there does not exist a topos $\mathcal{S}\mathcal{E}$ such that

$$\mathbf{Geom}(\mathcal{S}X, \mathcal{S}\mathcal{E}) \simeq \mathrm{Tor}_G(X),$$

where X is a localic space and $\mathrm{Tor}_G(X)$ denotes the category of G-torsors over X.

Proof. Here's a sketch of the proof (for details, see [Ng23, Obs 6.5.5]). Since $\mathbb{R}$ is connected, the unique projection map $p\colon \mathbb{R} \to \{*\}$ induces a fully faithful embedding on the level of their sheaf toposes

$$p^*\colon \mathrm{Set} \hookrightarrow \mathcal{S}\mathbb{R}.$$

Lurie's Observation then follows from the following two claims:

(a) For any topos $\mathcal{S}\mathcal{E}$, p^* induces a fully faithful embedding

$$\mathbf{Geom}(\mathrm{Set}, \mathcal{S}\mathcal{E}) \hookrightarrow \mathbf{Geom}(\mathcal{S}\mathbb{R}, \mathcal{S}\mathcal{E}).$$

(b) There does not exist a fully faithful embedding

$$\mathrm{Tor}_G(\{*\}) \hookrightarrow \mathrm{Tor}_G(\mathbb{R}).$$

Now suppose there exists a topos $\mathcal{S}\mathcal{E}$ such that

$$\mathbf{Geom}(\mathcal{S}X, \mathcal{S}\mathcal{E}) \simeq \mathrm{Tor}_G(X),$$

for any localic space X. By Claim (a), this equivalence yields a fully faithful embedding

$$\mathrm{Tor}_G(\{*\}) \simeq \mathbf{Geom}(\mathrm{Set}, \mathcal{S}\mathcal{E}) \hookrightarrow \mathbf{Geom}(\mathcal{S}\mathbb{R}, \mathcal{S}\mathcal{E}) \simeq \mathrm{Tor}_G(\mathbb{R}),$$

contradicting Claim (b). □

[61] In fact, this was how our original proof of Theorem B went before being shortened to its present form. The main idea was to define a functor $G\colon \mathrm{Set} \to \mathcal{D}$ and prove that G was both fully faithful and essentially surjective. The argument that G was a fully faithful functor is similar to the argument of Example 7.1; the key difficulty was showing that G was essentially surjective. The crux move was to note that for any pair $(F, \theta) \in \mathcal{D}$, F is a sheaf over a locally connected space $(0, \infty)$; hence, applying Observation 6.2, analysis of F reduces to analysis of its connected components. After which, one then shows that the descent restrictions force the connected components of F to all be homeomorphic to $(0, \infty)$ (cf. Observation 6.3). After which, essential surjectivity follows by a straightfoward (if involved) book-keeping argument.

[62] **(M:) Do we have an example? We've thought about this previously.**

Lurie's Observation 7.5 vindicates Bunge's original suspicion in [Bun90] that "toposes are not the right kind of structures to consider when dealing with G-bundles for a general G". This gives us a sharper understanding of the topos' limitations, and raises challenging questions about its intended role in modern applications.

Discussion 7.6 (Caramello's Bridge Technique)**.** Going back to [Car10], an attractive proposal was developed by Caramello on how toposes might play a unifying role in mathematics. This programme is motivated a basic observation: even if two geometric theories $\mathbb{T}, \mathbb{T}'$ look very different (e.g. they may have different signatures, they may not be bi-interpretable etc.), their classifying toposes can still be equivalent, as depicted:

$$\begin{tikzcd} & \mathcal{E}_{\mathbb{T}} \simeq \mathcal{E}_{\mathbb{T}'} \arrow[dr, dashed, bend left] & \\ \mathbb{T} \arrow[ur, dashed, bend left] & & \mathbb{T}' \end{tikzcd}$$

FIGURE 6. Classifying Toposes as "Bridges" [Car17]

This suggests that the topos plays a natural role in understanding the structural relationships between theories $\mathbb{T}$ and $\mathbb{T}'$. Extending this idea significantly, Caramello argues that the topos provides an abstract framework for the analysis of much more general structural connections in mathematics — in her language, the topos serves as a "bridge" to transfer information between different mathematical contexts.

This interesting programme has strong ambitions. Throughout her work, Caramello has proposed potential applications of the Bridge technique to the Langlands correspondence, to mirror symmetry, to the study of motives, and to the AdS/CFT correspondence [CL16, Car21]. However, Example 7.1 and Lurie's Observation 7.5 raise questions about whether this overstates the unifying power of toposes. We now know that toposes sometimes lose important (cohomological) information when attempting to classify G-torsors. Given the potential lossy-ness of toposes, one is naturally led to ask: why are toposes the right framework for analysing a given structural connection? If we wish to use toposes as "bridges" to transfer information, how do we know that relevant information isn't being lost in the process?

Discussion 7.7 (∞-toposes)**.** Very interestingly, Lurie [Lur14] points out that the argument of Observation 7.5 can be extended to show that ∞-toposes also do not classify G-torsors for all topological groups G either. This is *a priori* surprising: one may have expected that the generality of ∞-toposes would resolve the previous issues faced by the standard topos.[63] In any case, Lurie's remark suggests that new ideas are needed if we wish to deal with G-torsors for general topological/localic G.

These discussions set up the following test problem:

Problem 7.8. What generalised space classifies G-torsors for all topological/localic groupoids G? What is its relationship to geometric logic?

7.2. **Local-Global Questions.** An interesting asymmetry has emerged in the present state of arithmetic geometry. Whenever one wants to import analytical methods from the Archimedean setting (e.g. complex analytification of varieties) to the non-Archimedean setting, the barriers to this translation are clear: non-Archimedean fields are totally disconnected, and so e.g. the naive analytification of non-Archimedean varieties is of limited usefulness, unlike the complex case. There are now various well-established solutions to remedy this problem of disconnectedness — e.g. we might modify the classical notion of topology (as in Tate's rigid analytic geometry, see [Pay15]) or perhaps we might "fill in the gaps" induced by the disconnected base field (as in Berkovich geometry, see [Ber90]).

Moving in the opposite direction, however, seems more difficult: how should we incorporate the analytic world into an algebraic framework? We already know that the classical solution regards the real place as a

[63]This becomes less surprising when we re-examine the motivations behind higher topos theory. Higher topos theory, as detailed in [Lur09], aims to develop a categorical framework for interpreting higher cohomology classes, analogous to how G-torsors describe first cohomology classes. Thus, the objective is distinct from classifying all G-torsors when G is topological/localic.

formal prime at infinity, but Section 1 already discussed why this formal construction is dissatisfying. What is interesting is how the literature has repeatedly framed this as an algebraic problem, to be resolved once we find the correct generalisation of commutative rings (see e.g. [Dur07, Har07]).

Our results suggest a different picture. If the real place corresponds to $\overleftarrow{[0,1]}$ (Theorem C), then this indicates that completing respect to this place does not just give a single completion of $\mathbb{Q}$ but rather a parametrised family of completions. This is in contrast to completing with respect to the non-Archimedean places, where we expect to obtain the usual p-adics $\mathbb{Q}_p$ [more precisely: the equivalence class of $\mathbb{Q}_p$] since the non-Archimedean places are just singletons (Theorem B). More work, of course, will be needed to make this precise, but notice this picture already tells us something interesting about the connected and the disconnected.

Observation 7.9. As our setup,

- Let $|\cdot|_\infty$ be the standard Euclidean norm on $\mathbb{Q}$ and $|\cdot|_0$ be the trivial norm.
- Let $\alpha \in \mathbb{Q}_{(0,1]}$.[64]

It is clear $|\cdot|_\infty^\alpha = |\cdot|_\infty$ when $\alpha = 1$ and that $|\cdot|_\infty^\alpha = |\cdot|_0$ when $\alpha = 0$. In particular, note that the completion of $\mathbb{Q}$ with respect to $|\cdot|_\infty^\alpha$ is $\mathbb{R}$ when $\alpha = 1$ and $\mathbb{Q}$ when $\alpha = 0$.

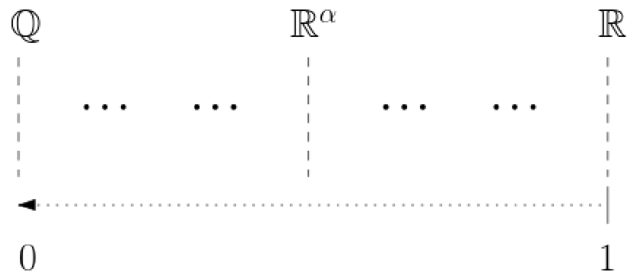


FIGURE 7. A parametrised family of completions over the Archimedean place

Discussion 7.10. If the usual finite primes $\mathfrak{p} \in \mathrm{Spec}(\mathbb{Z})$ measure the divisibility of an integer, what exactly does the infinite prime (or the real place) measure? Observation 7.9 gives some interesting clues. It is obvious that $\mathbb{Q}$ is disconnected whereas $\mathbb{R}$ is connected. Further, notice: although completions of $\mathbb{Q}$ with respect to $|\cdot|_\infty^\alpha$ are all connected (in fact, homeomorphic to $\mathbb{R}$) so long as $\alpha \in (0,1]$, the nearby points of these completions become increasingly "spaced out" as $\alpha \to 0$, where the completion finally becomes disconnected in the limit. Combined with Theorem C, this suggests the real place may be understood as measuring the "disconnectedness"[65] of the Archimedean completions.[66]

Discussion 7.11. A tentative picture is starting to emerge. The question of reconciling the p-adics with the reals (cf. Question 1.1) appears to be more closely bound up with the question of reconciling the connected with the disconnected than previously thought. Following Berkovich [Ber90], we have a robust strategy for transforming certain disconnected spaces over non-Archimedean fields into locally connected ones by "filling in the gaps". Conversely, in order to relate analytic structures over $\mathbb{R}$ to the (disconnected) arithmetic setting, Theorem C and Observation 7.9 suggest that we should look to somehow parametrise certain families of analytic structures $\{\mathcal{M}_q\}$, perhaps over $\mathbb{Q}_{(0,1]}$, before examining the behaviour of $\{\mathcal{M}_q\}$ as we scale $q \to 0$ in the limit. The challenge will be to develop a framework that makes this rigorous.

We are now at a strange mathematical juncture. Certain long-held assumptions have been subverted by new findings, calling for a significant reorientation in our approach to understanding Local-Global questions. Helpfully, certain key themes have emerged in the analysis — in particular, the subtle interplay between the connected and disconnected — which sets some basic expectations going forward. Nonetheless, our

[64] Why not let α be an upper real from $\overleftarrow{[0,1]}$ instead of a rational in $\mathbb{Q}_{(0,1]}$? In the former case, notice $|\cdot|^\alpha$ would be an upper-valued absolute value on $\mathbb{Q}$, creating the same issues regarding multiplicative inverses raised by [NV23, Obs. 0.1].

[65] Quotes are placed because these completions are still technically connected (since they are homeomorphic to $\mathbb{R}$), so we shall need a subtler notion. This will be discussed in Section 7.2.3 on q-liquidity.

[66] Although, one should still bear in mind the issues raised in Footnote 64.

current picture is still a tentative one. New ideas will be needed to develop the suggestions mentioned in Discussion 7.11 and we also don't know the current extent of our blindness. As such, to guide the development of our understanding, we include below a varied list of test problems.

7.2.1. *Compactifying and Living Below* $\mathrm{Spec}(\mathbb{Z})$. Although we have characterised the trivial place and non-Archimedean places of $\mathbb{Q}$ as singletons, and the Archimedean place as $\overleftarrow{[0,1]}$, this only gives a piecewise account of the space of places of $\mathbb{Q}$. Taking seriously our objective of treating the places of $\mathbb{Q}$ as an actual space (as opposed to an indexing set), let us restate the following problem from the Introduction.

Problem 7.12. Characterise the (entire) point-free space of places of $\mathbb{Q}$.

Discussion 7.13. The problem is more interesting than first appears. Even if we prove that space of non-Archimedean places is equivalent to $\mathrm{ISpec}(\mathbb{Z})$, it remains unclear how the Archimedean and the non-Archimedean places fit together.

On this front, recall Artin-Whaples's classical result [AW45] that $\mathbb{Q}$ (in fact, all global fields) satisfies a product formula

$$\prod_{v\in\Lambda_{\mathbb{Q}}} |x|_v = 1, \qquad \text{for all } x \neq 0 \qquad , \tag{84}$$

where v ranges over all the places of $\mathbb{Q}$, including the Archimedean. This formula holds for any normalisation of the absolute values. In particular: if we exponentiate $|\cdot|_\infty$ by $\alpha \in [0,1]$, the p-adic norms $|\cdot|_p$ must also be exponentiated by α for the product formula to hold. This gives another way of understanding how the Archimedean place may function as a parameter space (cf. Discussion 7.10). Rephrased in the language of Convention 3.19, since the normalisation at the Archimedean place determines the normalisation across all places, this suggests that the Archimedean Place lives *below* $\mathrm{Spec}(\mathbb{Z})$:

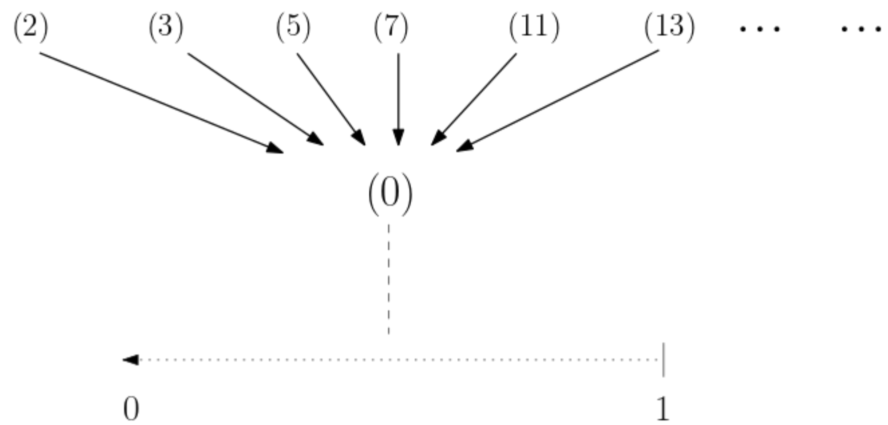


FIGURE 8. A candidate picture for the space of places

At this preliminary stage, we make no assertions about the correct characterisation of the space of places and its implications. Nonetheless, there are various suggestive aspects to this candidate picture that are worth recording for further meditation:

(a) The (point-free) space of places sharpens our understanding of the Arakelov compactification of $\mathrm{Spec}(\mathbb{Z})$ (cf. Section 1); [67]
(b) The perspective that the Archimedean place lives below $\mathrm{Spec}(\mathbb{Z})$ will be suggestive to those familiar with $\mathbb{F}_1$-geometry (see e.g. [nL11]);
(c) In our setting, the Archimedean place "adds" numerous extra points to $\mathrm{Spec}(\mathbb{Z})$, which form a non-Hausdorff space $\overleftarrow{[0,1]}$ equipped with a specialisation order. As a first step towards developing our results into a robust algebro-geometric framework, one may ask: are these extra points analogous to

[67] A. Connes encouraged us to think about how the structure sheaf of $\mathbb{S}$-algebras on the (classical) Arakelov compactification, as introduced in [CC16], can be extended via a one parameter deformation by replacing the usual norm $||\cdot||$ with a real parameter $\alpha \in (0,1]$, corresponding to $||\cdot||^\alpha$ – for details, see [CC20, Prop. 4.1].

the Type V points of adic spaces (see e.g. [Wei19, §1.7])? It would also be interesting to see if the language of vertical/horizontal specialisation (see [Mor19, §I.3.1]) finds a useful translation to our setting.

Let us also recall our original motivating test problem.

Problem 7.14. Characterise and analyse the space of completions of $\mathbb{Q}$.

Discussion 7.15. Of particular interest to us are the following: (a) the generic Archimedean completion, especially since $\mathbb{Q}$ and $\mathbb{R}$ are not homeomorphic and since working with upper reals introduces technical subtleties; (b) how the trivial and non-trivial completions of $\mathbb{Q}$ interact; (c) the role of local compactness, since the non-trivial completions are locally compact (as is desired by the number theorist) but not the trivial completion $\mathbb{Q}$.

It is also worth revisiting the function field analogy, and ask:

Problem 7.16. In what sense is the space of places a (smooth) compactification of $\mathrm{Spec}(\mathbb{Z})$?

Discussion 7.17 ("Smooth")**.** We expect the Arakelov compactification of $\mathrm{Spec}(\mathbb{Z})$ to be the canonical compactification of $\mathrm{Spec}(\mathbb{Z})$. However, since $\mathrm{Spec}(\mathbb{Z})$ is meant to be analogous to an affine curve, we should be mindful there may be different compactifications for the same affine curve. This issue is resolved if we require smoothness, as smooth compactifications over perfect fields (e.g. $\mathbb{Q}$) are always unique.[68] In addition, smoothness is also a standard hypothesis in intersection theory. Hence, taking the function field analogy seriously, one is therefore naturally led to ask: how can we regard the usual compactification of $\mathrm{Spec}(\mathbb{Z})$ as a *smooth* compactification?[69]

This is a challenging question. It is unclear how the standard Arakelov compactification of $\mathrm{Spec}(\mathbb{Z})$ can be regarded as being "smooth" at $p = \infty$; in fact, it is arguably highly singular since the element ∞ is formally adjoined to $\mathrm{Spec}(\mathbb{Z})$. On the other hand, this question is more reasonable in the point-free setting. Since the Archimedean place corresponds to $\overleftarrow{[0,1]}$, perhaps the [point-free] space of places of $\mathbb{Q}$ can be regarded as smoothing out the formal adjoining of ∞ to $\mathrm{Spec}(\mathbb{Z})$.

7.2.2. *Revisiting Homotopy Theory.* One of our original motivations behind this research programme was to explore if geometric reasoning could be applied to investigate Local-Global Principles in Arithmetic Geometry. In particular, it was asked in [Ng23] if we could we reformulate number-theoretic statements of the form

"ϕ holds over $\mathbb{Q}$ iff ϕ holds over *all* (non-trivial) completions of $\mathbb{Q}$"

as

"ϕ holds over $\mathbb{Q}$ iff ϕ holds for the *generic* (non-trivial) completion of $\mathbb{Q}$" ?

Unfortunately, since the Archimedean place is equivalent to $\overleftarrow{[0,1]}$, this suggests that the non-trivial Archimedean completion(s) cannot be definably separated from the trivial completion.[70] In other words, barring unexpected surprises, the evidence suggests that we are unable to reason geometrically about the generic *non-trivial* completion of $\mathbb{Q}$, only the generic completion of $\mathbb{Q}$.

Nonetheless, this only appears to be an immediate problem for Local-Global Principles so long as

"Global = $\mathbb{Q}$", and "Local = Non-Trivial Completions of $\mathbb{Q}$".

[68] Otherwise, notice that simply taking the usual projective closure of an affine curve may yield singularities, yielding a different choice of compactification.

[69] Another perspective, suggested to the authors by J.I.B. Gil, is to consider normality as opposed to smoothness. In particular, notice: (a) if C is a normal curve, then this allows us to define the degree of regular functions on C at $P \in C$, which we require when computing the degree of the associated divisor; and (b) normal algebraic curves are already smooth. **(M:) I need to double-check this; otherwise I'll delete it.**

[70] Why? Recall Convention 2.23 that we cannot have $\overleftarrow{(0,1]}$.

However, we already know that there are other interesting (and deep) variants of the same basic idea. For instance, recall the Arithmetic Square, which is a pullback square of rings

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_p \widehat{\mathbb{Z}}_p \arrow[d] \\ \mathbb{Q} \arrow[r] & \mathbb{Q} \otimes_{\mathbb{Z}} \prod_p \widehat{\mathbb{Z}}_p \end{tikzcd} , \tag{85}$$

where $\widehat{\mathbb{Z}}_p$ denotes the p-adic integers. An influential interpretation: since the Arithmetic Square is a pullback, it tells us how $\mathbb{Z}$ can be reconstructed from a rational piece [localisation at zero] and infinitely many p-adic pieces [profinite completion]. In other words, we have:

"Global = $\mathbb{Z}$", and "Local = $\mathbb{Q}$ and $\widehat{\mathbb{Z}}_p$ for all primes p".

Notice that $\mathbb{Q}$ now features as a local piece rather than a global piece. The power of this perspective can be seen in Sullivan's groundbreaking lecture notes [Sul05], where this algebraic picture was extended to the setting of homotopy theory to show that any sufficiently nice space X can be understood as being built from infinitely many p-adic pieces and one rational piece.

Now the fact $\mathbb{Q}$ may not be definably separable from $\mathbb{R}$ now becomes quite interesting, because it dovetails with the discussion in [Sul05, p. 87-88] on augmenting the finite adele type to include $\mathbb{R}$.

Problem 7.18. Notice that the Arithmetic Square in Diagram (85) features only the finite adele ring $\mathbb{A}_{\mathbb{Q}}^{\mathrm{fin}}$ and not the complete adele ring $\mathbb{A}_{\mathbb{Q}}$. In light of Theorem C, formulate a plausible conjecture on how to augment the Arithmetic Square to include $\mathbb{R}$.

Discussion 7.19. Let X be a space. Sullivian originally only asked how one might might augment its (finite) adele type X_A to the complete adele type $X_A \times X_{\mathbb{R}}$, where X_A is the finite adele type and $X_{\mathbb{R}}$ is the real homotopy type. Returning to the setting of (topological) rings, Problem 7.18 asks how one might incorporate $\mathbb{R}$ into the Arithmetic Square itself so that we naturally obtain the complete adele type (as opposed to just taking the direct product of the finite adeles and $\mathbb{R}$). Footnote 64 may be relevant — especially since working with the upper reals inclines us to look at the absolute values on $\mathbb{Z}$ rather than $\mathbb{Q}$.

7.2.3. *Interactions with q-liquidity.* Independently, recent work of Clausen-Scholze on condensed mathematics [Sch19a, Sch19b] have also explored the interactions between topology and algebra, albeit from a very different angle. Many very interesting things can be said about how the condensed perspective and point-free perspective may interact, but here we shall focus specifically on how the condensed formalism engages with the p-adics vs. the reals.

Denote $\mathrm{Cond(Ab)}$ to be the category of condensed abelian groups. Compare the following two structure theorems:

(a) [Sch19b, Theorems 5.8 and 6.2]: The inclusion of the category of solid abelian groups into $\mathrm{Cond(Ab)}$ admits a left adjoint $M \mapsto M^{\blacksquare}$, known as *solidification*. In particular, there exists a unique tensor product on solid abelian groups making the solidification functor $M \mapsto M^{\blacksquare}$ symmetric monoidal (i.e. compatible with the tensor product).

(b) [CS22, Theorem 3.11]: Fix any $0 < q \leq 1$. Then, the inclusion of the category of q-liquid $\mathbb{R}$-vector spaces[71] into $\mathrm{Cond(Ab)}$ admits a left adjoint, known as *q-liquidification*, which is the unique colimit-preserving extension of

$$\mathbb{Z}[S] \mapsto \mathcal{M}_{<q}(S) = \varinjlim_{q'<q} \mathcal{M}_{q'}(S).$$

In particular, there exists a unique tensor product of q-liquid $\mathbb{R}$-vector spaces making q-liquidification symmetric monoidal.

For details, see the cited references. Notice that both structure theorems work to construct a canonical tensor product for two different classes of condensed abelian groups. In fact, this is by necessity: whereas

[71] This is originally refered to as the category of *p-liquid $\mathbb{R}$-vector spaces*, but we have opted for the term q-liquid since the primes are already denoted as p.

solid abelian groups contain all the usual algebraic suspects (e.g. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}_p$ etc.), it does *not* contain the (condensed) reals and so a different approach to tensoring condensed $\mathbb{R}$-vector spaces is required.

Examined closely, these results reveal an interesting disanalogy: defining a canonical tensor product for $\mathbb{R}$-vector spaces requires a form of scaling based on choice of $0 < q \leq 1$ while no such scaling is necessary to define a tensor product for the solid abelian groups.[72] This disanalogy mirrors our geometric investigation of the places of $\mathbb{Q}$, where the Archiemdean place corresponds to $\overleftarrow{[0,1]}$ and the non-Archimedean places are singletons. It is natural to ask if this convergence in perspectives hints at a deeper difference between the Archimedean vs. non-Archimedean settings. Again, the challenge will be to make this precise.

Discussion 7.20 (Convexity)**.** The mechanics underlying the proof of [Sch19a, Theorem 6.5] remain somewhat mysterious but have recently been clarified through a key step formalized in LEAN, as discussed in Scholze's recent blog posts [Sch20, Sch21]. This formalisation highlights a core mechanism of the argument: reducing a non-convex problem over the reals to a convex problem over the integers. It is interesting to ask how these insights might provide useful clues for Problems 7.12 and 7.14 – with a view towards reconciling the non-Archimedean and the Archimedean.

Discussion 7.21. Returning to Problem 7.18 on augmenting the Arithmetic Square: what is the condensed perspective on the adele ring? [73]

Appendix A. On Effective Descent

Recount Example 3.5, including the diagram obtained via iterated pullbacks.

Remark A.1. Continuing with the setup of Example 3.5, the associated inverse image functor of Φ, i.e.

$$\Phi^*\colon \mathcal{S}\mathcal{E} \to \mathcal{S}\mathcal{E}', \tag{86}$$

induces a functor $\chi\colon \mathcal{S}\mathcal{E} \to \mathrm{Des}(\Phi)$ such that

$$\begin{tikzcd} \mathcal{S}\mathcal{E} \arrow[rr, "\chi"] \arrow[dr, "\Phi^*"'] & & \mathrm{Des}(\Phi) \arrow[dl, "U"] \\ & \mathcal{S}\mathcal{E}' & \end{tikzcd} \tag{87}$$

commutes, where U is the forgetful functor $U(F,\theta) = F$.

Proof. Any sheaf $F \in \mathcal{S}\mathcal{E}$ corresponds to a map $F\colon \mathcal{E} \to [\mathbb{O}]$, where $[\mathbb{O}]$ is the object classifier. It is clear that $F \circ \Phi \circ \pi_0 \cong F \circ \Phi \circ \pi_1$. Restated in a different language: there exists an isomorphism $\theta\colon \pi_0^*(F) \to \pi_1^*(F)$. It is also clear that $\Delta^*(\theta) \cong \mathrm{id}$ since the following pullback diagram commutes up to isomorphism:

$$\begin{tikzcd} \mathcal{E}' \arrow[dr, dashed, "\Delta"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & & \\ & \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' \arrow[r, "\pi_1"] \arrow[d, "\pi_0"] \arrow[dr, phantom, "\Nearrow \theta"] & \mathcal{E}' \arrow[d, "\Phi"] & \\ & \mathcal{E}' \arrow[r, "\Phi"'] & \mathcal{E} \arrow[r, "F"] & {[\mathbb{O}]} \end{tikzcd} \tag{88}$$

[72] It is worth noting that one can still define a notion of q-liquid $\mathbb{Q}_p$-vector spaces, except now we allow q to range over $0 < q < \infty$ [CS22, Rem 3.13].

[73] **(M:) I had some thoughts about this, but I want to be careful not say anything foolish. Potentially delete. Will email Johann Commelin first.**

Analogously, the simplicial identities (15) guarantee the following diagram commutes up to isomorphism

$$\begin{tikzcd}
 & & \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' \arrow[r, blue, "\pi_{02}"] \arrow[d, red, "\pi_{01}"] \arrow[dl, equal] \arrow[dr, phantom, "\cong"] & \mathcal{E}' \times_{\mathcal{E}} \mathcal{E} \arrow[d, blue, "\pi_0"] \\
 & \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' \arrow[r, teal, "\pi_{01}"] \arrow[dl, blue, "\pi_{02}"'] \arrow[d, teal, "\pi_{12}"] \arrow[dr, phantom, "\cong"] & \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' \arrow[r, red, "\pi_0"] \arrow[d, red, "\pi_1"] \arrow[dr, phantom, "\Swarrow\theta"] & \mathcal{E}' \arrow[d, blue, "\Phi"] \\
\mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' \arrow[r, phantom, "\cong"] \arrow[dr, blue, "\pi_1"'] & \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' \arrow[r, teal, "\pi_0"] \arrow[d, teal, "\pi_1"] \arrow[dr, phantom, "\Swarrow\theta"] & \mathcal{E}' \arrow[r, red, "\Phi"'] \arrow[d, red, "\Phi"] & \mathcal{E} \arrow[dl, equal] \\
 & \mathcal{E}' \arrow[r, blue, "\Phi"'] & \mathcal{E} &
\end{tikzcd} \tag{89}$$

The simplicial identities are marked by " $\cong$ ". The blue arrows pick out the 2-cell

$$\pi_{02}^*(\theta)\colon \Phi \circ \pi_0 \circ \pi_{02} \Rightarrow \Phi \circ \pi_1 \circ \pi_{02},$$

the red arrows pick out the 2-cell $\pi_{01}^*(\theta)$ while the teal arrows pick out the 2-cell $\pi_{12}^*(\theta)$. Conclude that θ also satisfies the cocycle condition,[74]. In particular, Φ^* induces a functor sending $F \mapsto (\Phi^* F, \theta)$, making Diagram (87) commute. □

[74] **(M:) Double-check this gives the right notion of composition. May need to rephrase this to make things clearer, but at least the diagram is correct. Seems clunky.**

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