

# $K_1(\mathcal{V}\text{ar})$ IS GENERATED BY QUASI-AUTOMORPHISMS

MING NG

**ABSTRACT.** This paper provides a complete characterisation of  $K_1(\mathcal{V}\text{ar})$ , the  $K_1$  group of varieties, solving a problem left open in [Zak17c]. Our approach involves adapting Gillet-Grayson’s  $G$ -Construction to define an un-delooed  $K$ -theory spectrum of varieties. There are two levels on which one can read the present paper. On one level, we streamline and extend a series of  $K$ -theory results on exact categories to a more general class of categories (including  $\mathcal{V}\text{ar}$ ). On another level, our investigations bring into focus an interesting generalisation of automorphisms (“double exact squares”), which generate  $K_1$ . Since our results apply to a wide range of non-additive contexts (e.g. varieties,  $o$ -minimal structures, definable sets etc.), this sets up a challenging question: what kind of information do these quasi-automorphisms calibrate?

Our understanding of  $K$ -theory is changing. Recent efforts to extend tools from classical algebraic  $K$ -theory to non-additive settings have led us to make decisions on what its essential features are. One perspective, influenced by Waldhausen’s  $S_\bullet$ -construction [Wal87], is that  $K$ -theory is a framework for analysing the finite assembly and disassembly of objects; non-additive applications of this insight can be found in Campbell’s  $\tilde{S}_\bullet$ -construction [Cam19] as well as Zakharevich’s Assemblers [Zak17b]. A related perspective emphasises the view that  $K$ -theory breaks an object into two types of pieces. This underpins Campbell-Zakharevich’s framework of *CGW categories* [CZ22], which formalises key similarities between exact categories and the category of varieties  $\mathcal{V}\text{ar}_k$ .

In a different line of work: the study of  $K_1(\mathcal{C})$  for arbitrary exact categories began with Gillet-Grayson’s  $G$ -construction [GG87], which provided an elementary description of its generators. This description was refined by Sherman [She94, She98] and Nenashev [Nen96], before culminating in Nenashev’s characterisation of the complete set of relations for  $K_1$  [Nen98b, Nen98a].

The present paper unites the two lines of investigation by extending the  $K_1$  results to a subclass of CGW categories known as *pCGW categories*. These include not only exact categories and varieties, but also finite sets,  $o$ -minimal structures, and definable sets. As a result, we provide two alternative, complete characterisations of  $K_1$  applicable to a broad range of non-additive contexts.

**Overview.** Let us develop the previous remark that  $K$ -theory is an abstract framework for breaking an object into two different types of pieces. Consider the following two definitions.

- Let  $R$  be a ring. We define  $K_0(R)$  as

$$K_0(R) := \left\{ \begin{array}{l} \text{free abelian group} \\ \text{fin. gen. proj. } R\text{-modules} \end{array} \right\} \Bigg/ \begin{array}{l} [M] = [M'], \text{ if } M \cong M' \\ [M] = [M'] + [M''], \text{ if } M' \rightarrow M \rightarrow M'' \end{array}$$

where  $M' \rightarrow M \rightarrow M''$  is a short exact sequence.

- Let  $\mathcal{V}\text{ar}_k$  be the category of  $k$ -varieties, i.e. reduced separated schemes of finite type over field  $k$ . We define  $K_0(\mathcal{V}\text{ar}_k)$  as

$$K_0(\mathcal{V}\text{ar}_k) := \left\{ \begin{array}{l} \text{free abelian group} \\ k\text{-varieties} \end{array} \right\} \Bigg/ \begin{array}{l} [X] = [X'], \text{ if } X \cong X' \\ [X] = [U] + [X \setminus U], \text{ if } U \hookrightarrow X \text{ is a closed immersion} \end{array}$$

The analogy is clear. A short exact sequence  $M' \rightarrow M \rightarrow M''$  decomposes the  $R$ -module  $M$  into two distinct pieces,  $M'$  and  $M''$ , with  $M' \rightarrow M$  an admissible monic and  $M \rightarrow M''$  as an admissible epi. Similarly,  $K_0(\mathcal{V}\text{ar}_k)$  decomposes a variety  $X$  into  $U$  and  $X \setminus U$ , with  $U \hookrightarrow X$  a closed immersion and

---

*Thanks:* Research partially supported by EPSRC Grant EP/V028812/1 and a FY2024 JSPS Postdoctoral Fellowship (Short-Term).

$X \setminus U \hookrightarrow X$  an open immersion. In both cases, an object that is broken into two pieces is viewed as an abstract sum of the two components. But what are the essential features of this decomposition?

Quillen [Qui73] introduced, and later characterised, an exact category as a pair  $(\mathcal{C}, \mathcal{S})$ , where  $\mathcal{C}$  is additive, and  $\mathcal{S}$  is a family of sequences  $M' \rightarrow M \rightarrow M''$  satisfying conditions analogous to short exact sequences in abelian categories. This includes the natural condition that admissible monics  $M' \rightarrow M$  are kernels of admissible epis  $M \rightarrow M''$ , and admissible epis are cokernels of admissible monics. These conditions enabled Quillen to construct a  $K$ -theory spectrum  $K\mathcal{C}$ , recovering  $K_0(R)$  on  $\pi_0$  when  $\mathcal{C}$  is the category of finitely-generated projective  $R$ -modules.

Campbell-Zakharevich [CZ22] re-examined Quillen's framework, and made a crucial observation: instead of requiring the two classes of morphisms to compose, it suffices to encode their interaction formally. This added flexibility allows us to extend Quillen's  $K$ -theory to non-additive settings like varieties, where sequences like  $U \hookrightarrow X \hookleftarrow X \setminus U$  clearly do not compose. This is made precise by [CZ22, Thm 4.3], which adapts Quillen's argument to construct a  $K$ -theory spectrum  $K\mathcal{V}\text{ar}_k$  recovering  $K_0(\mathcal{V}\text{ar}_k)$  on  $\pi_0$ .

In fact, we can define the  $K$ -groups  $K_n(\mathcal{V}\text{ar}_k) := \pi_n(K\mathcal{V}\text{ar}_k)$  for all  $n$ . It is natural to ask:

**Question 1.** What kind of information do the higher  $K$ -groups of varieties encode?

This is a challenging question. In classical algebraic  $K$ -theory, the coarseness of  $K_0$  as an invariant may be measured by the fact that  $K_0(F) = \mathbb{Z}$  for all fields  $F$ , whereas  $K_1(F) \cong F^\times$ . Is there an analogous story in the setting of varieties? More explicitly, how might we measure the loss of information in  $K_0(\mathcal{V}\text{ar}_k)$ ? To what extent can this information be recovered in the higher  $K$ -groups? The following summary theorem gives a snapshot of the current landscape.

**Summary Theorem 0.1.** *Assume the base field  $k$  is of characteristic 0, and equip  $K_0(\mathcal{V}\text{ar}_k)$  with a ring structure by defining  $[X] \cdot [Y] := [(X \times_k Y)_{\text{red}}]$ . Two  $k$ -varieties  $X, Y$  are said to be piecewise isomorphic if  $X$  and  $Y$  admit finite partitions*

$$X_1, \dots, X_n \quad \text{and} \quad Y_1, \dots, Y_n$$

*into locally closed subvarieties such that  $X_i \cong Y_i$  for all  $i$ . The following is known:*

- (i) *Define  $SK_0(\mathcal{V}\text{ar}_k)$  as the freely generated semiring on  $[X]$  subject to  $[X] = [Z] + [X \setminus Z]$ . Then, two  $k$ -varieties  $X, Y$  are piecewise isomorphic iff  $[X] = [Y]$  in  $SK_0(\mathcal{V}\text{ar}_k)$ .*
- (ii) *Let  $X, Y$   $k$ -varieties such that  $\dim X \leq 1$ . Then  $[X] = [Y]$  in  $K_0(\mathcal{V}\text{ar}_k)$  iff they are piecewise isomorphic.*
- (iii) *There exists  $k$ -varieties  $X$  and  $Y$  such that  $[X] = [Y]$  in  $K_0(\mathcal{V}\text{ar}_k)$  and yet fail to be piecewise isomorphic.*
- (iv) *Let  $X$  be a  $k$ -variety of any non-negative dimension containing only finitely many rational curves. Then for any  $k$ -variety  $Y$ ,  $[X] = [Y]$  in  $K_0(\mathcal{V}\text{ar}_k)$  iff they are piecewise isomorphic.*

*Proof.* (i) appears to be folklore, and is recorded in [Bek17] as well as [CLNS18, Cor. 1.4.9, Chapter 2]. (ii) is [LS10, Props. 5 and 6]. For (iii), various constructions are now known but the first example goes back to [Bor18]. (iv) is [LS10, Theorem 5].  $\square$

Summary Theorem 0.1 sharpens our understanding of what is at stake. Given our high-level characterisation of  $K$ -theory as an abstract framework for analysing the finite assembly and decompositions of objects, the following question was natural:

**Question 2** ([LL03, Question 1.2]). Is it true that two  $k$ -varieties are piecewise isomorphic iff they agree in  $K_0(\mathcal{V}\text{ar}_k)$ ?

In the setting of characteristic 0, item (iii) of the Summary Theorem answers no, signalling a loss of information on the level of  $K_0$ . Item (i) tells us the information is lost precisely because  $K_0(\mathcal{V}\text{ar}_k)$  involves group completion – akin to an Eilenberg Swindle. Item (ii) tells us that piecewise isomorphism and equivalence in  $K_0(\mathcal{V}\text{ar}_k)$  coincide so long as the varieties are of sufficiently low dimension. Put otherwise, the algebraic barriers to geometric information only occur at the higher dimensions. Item (iv) is subtler, and raises interesting questions about how taking piecewise isomorphisms of complex varieties relates to the ampleness of their canonical line bundles (cf. the algebraic hyperbolicity conjecture for surfaces).

In light of this discussion, let us return to Question 1. Some promising initial progress has been made. Using the formalism of Assemblers, Zakharevich constructs a different (but equivalent)  $K$ -theory spectrum of varieties, before leveraging its connection with Waldhausen categories to obtain a partial characterisation of  $K_1(\mathcal{V}\text{ar}_k)$  [Zak17c, Theorem B]. Inspired by Borisov’s work [Bor18], this was later developed in [Zak17a] to illuminate a subtle geometric insight: the failure to extend birational automorphisms of varieties to piecewise isomorphisms is tightly connected to the Lefschetz motive  $[\mathbb{A}^1]$  being a zero divisor in  $K_0(\mathcal{V}\text{ar}_k)$ . In a different vein: [CWZ19] identifies non-trivial elements in  $K_n(\mathcal{V}\text{ar}_k)$  by lifting various motivic measures  $K_0(\mathcal{V}\text{ar}_k) \rightarrow K_0(\mathcal{C})$  to the level of spectra  $K\mathcal{V}\text{ar}_k \rightarrow K\mathcal{C}$ .

**Discussion of Main Results.** Until recently, a full characterisation of any higher  $K$ -group of varieties was not known. In her original paper, Zakharevich [Zak17c, Theorem B] identifies the generators of  $K_1(\mathcal{V}\text{ar}_k)$  and some key relations, but does not prove their completeness. Independently from us, an intriguing recent collaboration between algebraic topologists and experts in homological stability has uncovered a homological proof [KLM<sup>+</sup>24, Prop. 4.1] that Zakharevich’s presentation is in fact complete.

We take a different approach. Whereas [KLM<sup>+</sup>24] utilises homological methods to analyse  $K_1$ , the present paper instead relies on techniques from simplicial homotopy theory. Further, whereas [Zak17c] relies on the connection between  $\mathcal{V}\text{ar}_k$  and Waldhausen categories, we instead focus on the (tighter) connection between  $\mathcal{V}\text{ar}_k$  and exact categories. This sets up the following theorem.

**Theorem A** (Theorem 2.11). Let  $\mathcal{C}$  be a pCGW category, and  $\mathcal{SC}$  the simplicial set obtained by applying the  $S_\bullet$ -construction. Then, there exists a simplicial set  $G\mathcal{C}$  such that there is a homotopy equivalence

$$|G\mathcal{C}| \simeq \Omega|\mathcal{SC}|.$$

In particular,  $\pi_n|G\mathcal{C}| = K_n\mathcal{C}$  for all  $n$ .

In broad strokes: Theorem A extends Gillet-Grayson’s  $G$ -construction on exact categories [GG87] to a wider class of categories including  $\mathcal{V}\text{ar}_k$ . The beauty of the  $G$ -construction is that it translates a topological problem (i.e. characterising  $\pi_1$  of a loop space) into a simplicial one, which is more combinatorial and thus easier to work with. To show that this gives us sufficient leverage to characterise  $K_1$  will, of course, take the rest of the paper. Let us also remark that while one can prove Theorem A by adapting the original proof [GG87] to our setting, we provide a more streamlined argument (Theorem 2.8) inspired by Grayson’s framework of dominant functors [Gra87].

The previous remarks underscore a more fundamental difference. Both [Zak17c] and [KLM<sup>+</sup>24] are concerned with the  $K$ -theory of Assemblers, whereas our paper builds on [CZ22] to develop the  $K$ -theory of so-called *pCGW categories*. Precise definitions will be given in due course; for now, it suffices to think of Assemblers and pCGW categories as two distinct yet equivalent ways of defining the  $K$ -theory spectrum of varieties.<sup>1</sup> This difference becomes apparent when comparing our respective presentations of  $K_1(\mathcal{V}\text{ar}_k)$ . In our language:

**Theorem B** (Theorem 3.16 and Prop. 4.12). Let  $\mathcal{C}$  be a pCGW category. Then  $K_1(\mathcal{C})$  is generated by *double exact squares*, i.e. by pairs of distinguished squares in  $\mathcal{C}$  with identical nodes

$$l := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & C \\ \downarrow & \square & \downarrow g_1 \\ A & \xrightarrow{f_1} & B \end{array} , \quad \begin{array}{ccc} O & \xrightarrow{\quad} & C \\ \downarrow & \square & \downarrow g_2 \\ A & \xrightarrow{f_2} & B \end{array} \right), \quad (1)$$

modulo the following relations

$$(B1) \left\langle \left( \begin{array}{ccc} O & \xrightarrow{\quad} & A \\ \downarrow & \square & \downarrow 1 \\ O & \xrightarrow{\quad} & A \end{array} , \quad \begin{array}{ccc} O & \xrightarrow{\quad} & A \\ \downarrow & \square & \downarrow 1 \\ O & \xrightarrow{\quad} & A \end{array} \right) \right\rangle = 0;$$

<sup>1</sup>For the cautious reader: the weak equivalence of these spectra as spaces is [CZ22, Theorems 7.8 and 9.1].

$$(B2) \left\langle \left( \begin{array}{ccc} O & \xrightarrow{\quad} & O \\ \downarrow & \square & \downarrow \\ A & \xrightarrow{1} & A \end{array}, \begin{array}{ccc} O & \xrightarrow{\quad} & O \\ \downarrow & \square & \downarrow \\ A & \xrightarrow{1} & A \end{array} \right) \right\rangle = 0;$$

(B3) Suppose  $f_C: A \xrightarrow{f_A} B \xrightarrow{f_B} C$  and  $f'_C: A \xrightarrow{f'_A} B \xrightarrow{f'_B} C$ . Under technical conditions (imposed by the 2-simplices of  $G\mathcal{C}$ ), the following splitting relation holds

$$\left\langle \left( \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{B}{A} \\ \downarrow & \square & \downarrow g_A \\ A & \xrightarrow{f_A} & B \end{array}, \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{B}{A} \\ \downarrow & \square & \downarrow g'_A \\ A & \xrightarrow{f'_A} & B \end{array} \right) \right\rangle + \left\langle \left( \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{C}{B} \\ \downarrow & \square & \downarrow g_B \\ B & \xrightarrow{f_B} & C \end{array}, \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{C}{B} \\ \downarrow & \square & \downarrow g'_B \\ B & \xrightarrow{f'_B} & C \end{array} \right) \right\rangle = \left\langle \left( \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{C}{A} \\ \downarrow & \square & \downarrow g_C \\ A & \xrightarrow{f_C} & C \end{array}, \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{C}{A} \\ \downarrow & \square & \downarrow g'_C \\ A & \xrightarrow{f'_C} & C \end{array} \right) \right\rangle.$$

This presentation appears to be new, even in the context of exact categories. How does it compare with  $K_1$  of an Assembler? The following informal discussion may be illuminating.

*On Generators.* Double exact squares describe how to break an object into two distinct parts – for instance, Equation (1) shows  $B$  being broken into  $A$  and  $C$ . Interestingly, these squares generalise the usual notion of an automorphism – see Example 3.15. By contrast, [Zak17c] shows that  $K_1$  of an Assembler is generated by *piecewise automorphisms*, which break an object into  $n$  many pieces simultaneously. This difference reflects a trade-off between simplicity vs. flexibility. Our Theorem B presents a simpler set of generators for  $K_1(\mathcal{V}\text{ar}_k)$  than [Zak17c], which can be advantageous when e.g. constructing derived motivic measures, as done in [CWZ19].<sup>2</sup> On the other hand, the generality of piecewise automorphisms makes the Assemblers formalism better suited for investigating e.g. scissors congruence of convex polytopes, as done in [KLM<sup>+</sup>24], where simultaneous decomposition is essential.<sup>3</sup>

*On Relations.* There is an interesting discrepancy regarding the relations of [Zak17c] and Theorem B. In Zakharevich’s presentation (Theorem 4.11), the composition of piecewise automorphisms always split in  $K_1$ . More precisely:

$$\left\langle A \xrightarrow[f_2]{f_1} B \right\rangle + \left\langle B \xrightarrow[g_2]{g_1} C \right\rangle = \left\langle A \xrightarrow[g_2 f_2]{g_1 f_1} C \right\rangle \quad \text{in } K_1,$$

where  $f_i, g_i$  are piecewise automorphisms. Figure 1 gives an informal illustration.

FIGURE 1. LHS: the piecewise automorphisms induced by closed immersions  $A \xrightarrow{f} B$  and  $B \xrightarrow{g} C$ . RHS: the piecewise automorphism induced by their composition  $A \xrightarrow{gf} C$ .

By contrast, Relation (B3) of Theorem B asserts that composition splits in  $K_1$  only when a technical condition is satisfied – see Warning 4.3 for details. Proposition 4.12 gives evidence that this condition is both non-trivial and necessary in general. Our analysis relies on Theorem C, an alternative presentation of  $K_1$  inspired by Nenashev [Nen98a], and is the final main result of our paper.

**Theorem C** (Corollary 4.16). Let  $\mathcal{C}$  be a pCGW category. Then  $K_1(\mathcal{C})$  is generated by double exact squares subject to the following relations:

<sup>2</sup>Technically, [CWZ19] views  $\mathcal{V}\text{ar}_k$  as a *subtractive category* before applying the  $\tilde{S}_\bullet$ -construction as defined in [Cam19], but this is equivalent to viewing  $\mathcal{V}\text{ar}_k$  as a CGW category and applying the  $S_\bullet$ -construction; see [CZ22, Example 7.4].

<sup>3</sup>*Details.* Define two polytopes  $P$  and  $Q$  to be scissors congruent if: (i)  $P = \bigcup_{i=1}^m P_i$  and  $Q = \bigcup_{i=1}^m Q_i$  such that  $P_i \cong Q_i$ , and (ii)  $P_i \cap P_j = Q_i \cap Q_j = \emptyset$  for  $i \neq j$ . The key hypothesis here is convexity. In particular, pairwise unions like  $P_j \cup P_k$  may not form a convex polytope, so decomposition and reassembly must be done simultaneously.

(N1)  $\langle l \rangle = 0$  if  $l$  is a pair of identical squares, say

$$l = \left( \begin{array}{ccc} O & \xrightarrow{\quad} & C \\ \downarrow & \square & \downarrow g \\ A & \xrightarrow{f} & B \end{array} , \quad \begin{array}{ccc} O & \xrightarrow{\quad} & C \\ \downarrow & \square & \downarrow g \\ A & \xrightarrow{f} & B \end{array} \right).$$

(N2) Given a good  $3 \times 3$  diagram

$$\left( \begin{array}{ccccc} X_{00} & \xrightarrow{f_0} & X_{01} & \xleftarrow{g_0} & X_{02} \\ h_0 \downarrow & \circlearrowleft & \downarrow h_1 & & \downarrow h_2 \\ X_{10} & \xrightarrow{f_1} & X_{11} & \xleftarrow{g_1} & X_{12} \\ j_0 \uparrow & & j_1 \uparrow & & j_2 \uparrow \\ X_{20} & \xrightarrow{f_2} & X_{21} & \xleftarrow{g_2} & X_{22} \end{array} , \quad \begin{array}{ccccc} X_{00} & \xrightarrow{f'_0} & X_{01} & \xleftarrow{g'_0} & X_{02} \\ h'_0 \downarrow & \circlearrowleft & \downarrow h'_1 & & \downarrow h'_2 \\ X_{10} & \xrightarrow{f'_1} & X_{11} & \xleftarrow{g'_1} & X_{12} \\ j'_0 \uparrow & & j'_1 \uparrow & & j'_2 \uparrow \\ X_{20} & \xrightarrow{f'_2} & X_{21} & \xleftarrow{g'_2} & X_{22} \end{array} \right)$$

defined by the following 6 double exact squares

$$l_i := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & X_{i2} \\ \downarrow & \square & \downarrow g_i \\ X_{i0} & \xrightarrow{f_i} & X_{i1} \end{array} , \quad \begin{array}{ccc} O & \xrightarrow{\quad} & X_{i2} \\ \downarrow & \square & \downarrow g'_i \\ X_{i0} & \xrightarrow{f'_i} & X_{i1} \end{array} \right) \quad l^i := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & X_{2i} \\ \downarrow & \square & \downarrow j_i \\ X_{0i} & \xrightarrow{h_i} & X_{1i} \end{array} , \quad \begin{array}{ccc} O & \xrightarrow{\quad} & X_{2i} \\ \downarrow & \square & \downarrow j'_i \\ X_{0i} & \xrightarrow{h'_i} & X_{1i} \end{array} \right)$$

for all  $i \in \{0, 1, 2\}$ , the following 6-term relation holds

$$\langle l_0 \rangle + \langle l_2 \rangle - \langle l_1 \rangle = \langle l^0 \rangle + \langle l^2 \rangle - \langle l^1 \rangle$$

Does Theorem C imply a mistake in [Zak17c]? Not necessarily. A key distinction lies in the different choices of weak equivalences. Zakharevich models Assemblers using a special class of Waldhausen categories whose cofibration sequences all split up to weak equivalence (see [Zak17c, Theorem 2.1]). This contrasts with the pCGW setting, where the weak equivalences are isomorphisms. To illustrate, consider the projective line in the sequence

$$* \hookrightarrow \mathbb{P}^1 \hookrightarrow \mathbb{A}^1.$$

In the pCGW setting, this sequence clearly fails to split since  $\mathbb{P}^1 \not\cong \mathbb{A}^1 \amalg \{*\}$ . By contrast, this sequence splits up to weak equivalence in Zakharevich's framework since

$$\mathbb{P}^1 \leftarrow \{ \mathbb{A}^1 \}, \{ * \} \xrightarrow{\cong} \{ \mathbb{A}^1 \}, \{ * \}$$

defines a weak equivalence in the sense of [Zak17c], Definition 1.7. We discuss the implications for the  $K$ -theory presentation more fully in Section 5.1.

Finally, we remark that our results extend to settings not modelled by the Assemblers framework. One obvious example is exact categories. More interestingly, there is suggestive evidence that they also apply to matroids (Example 1.15), although some details remain to be worked out (see Section 5.4).

*Implications for Characterising  $K_n$ .* Many of the results of the present paper are inspired by Nenashev's work [Nen96] characterising  $K_1(\mathcal{C})$  for an exact category  $\mathcal{C}$ . Grayson [Gra12] later extended this to characterise  $K_n(\mathcal{C})$  for all  $n$ , and we expect our approach to generalise similarly to the higher  $K$ -groups of varieties (or, more generally, the higher  $K$ -groups of pCGW categories). It is currently unclear how one might analogously extend the methods from [Zak17c] or [KLM<sup>+</sup>24].

**Acknowledgements.** This paper benefitted greatly from discussions with C. Eppolito, M. Kamsma, C. Malkiewich, A. Nenashev, J. Pajwani, S. Vasey, C. Weibel, C. Wings, T. Wittich and I. Zakharevich – it is a pleasure to thank them for their thoughtful insights. Special thanks are due to B. Noohi for patient guidance. The author also thanks M. Malliaris for encouragement, and for reminding him that a cup is not always a doughnut.

## CONTENTS

1.	Preliminaries	6
2.	A Technical Result on Right Fibers	16
3.	Generators of $K_1(\mathcal{C})$	21
4.	Relations of $K_1(\mathcal{C})$	32
5.	Some Test Problems	42
Appendix A. Properties of Restricted Pushouts		44
Appendix B. Technical Lemmas & Some 2-Simplices		49
References		63

### 1. PRELIMINARIES

**1.1. CGW Categories.** The key definition in [CZ22] is the *CGW category*. This is essentially a category equipped with two subclasses of maps,  $\mathcal{M}$  and  $\mathcal{E}$  (analogous to admissible monics and epis in exact categories), along with a collection of square diagrams (“distinguished squares”) that encode how  $\mathcal{M}$  and  $\mathcal{E}$ -morphisms interact.

This is presented using the language of double categories. Recall that a *double category*  $\mathcal{C}$  is an internal category in  $\mathbf{Cat}$ . For the present paper, we will require the following refinement.

**Definition 1.1.** A *good double category* is a triple of categories  $(\mathcal{C}, \mathcal{M}, \mathcal{E})$  presented by the data:

- *Objects.* All three categories have the same objects:  $\text{ob}(\mathcal{E}) = \text{ob}(\mathcal{M}) = \text{ob}(\mathcal{C})$ .
- *Morphisms.*
  - $\mathcal{M}$ -**morphisms:**  $\mathcal{M}$  is a subcategory of  $\mathcal{C}$ . Its morphisms are denoted  $\rightharpoonup$ .
  - $\mathcal{E}$ -**morphisms:** Either  $\mathcal{E}$  or  $\mathcal{E}^{\text{op}}$  is a subcategory of  $\mathcal{C}$ . Its morphisms are denoted  $\circ\!\!\rightarrow$ .
- *Distinguished Squares.* A collection of square diagrams, denoted

$$\begin{array}{ccc} A & \xrightarrow{f'} & B \\ g' \circ\!\!\downarrow & \square & \downarrow g \\ C & \xrightarrow{f} & D \end{array}$$

where  $f, f' \in \mathcal{M}$  and  $g, g' \in \mathcal{E}$ . These squares closed under horizontal and vertical composition. They are also required to *interact well with isomorphisms* in the following sense: if

$$\begin{array}{ccc} A & \xrightarrow{f'} & B \\ g' \circ\!\!\downarrow & & \downarrow g \\ C & \xrightarrow{f} & D \end{array}$$

defines a commutative diagram in  $\mathcal{C}$ , and either both  $\mathcal{M}$ -morphisms or both  $\mathcal{E}$ -morphisms are isomorphisms, then the square is distinguished.

**Convention 1.2.** We typically denote a good double category as  $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ . When the context is clear, we simply write  $\mathcal{C}$ . When we wish to regard  $\mathcal{C}$  as an ordinary 1-category, we refer to  $\mathcal{C}$  as the *ambient category* – since it contains  $\mathcal{M}$  and  $\mathcal{E}$  (or  $\mathcal{E}^{\text{op}}$ ) as subcategories.

We now introduce a couple of helper definitions, before defining a CGW category.

**Definition 1.3.** Let  $\mathcal{C} = (\mathcal{E}, \mathcal{M})$  be a good double category, and  $\mathcal{D}$  be any (ordinary) category.

(1) Define  $\text{Ar}_{\square}\mathcal{E}$

- *Objects:* Morphisms  $A \circ\!\!\rightarrow B$  in  $\mathcal{E}$ .
- *Morphisms:*  $\text{Hom}_{\text{Ar}_{\square}\mathcal{E}}(A \circ\!\!\xrightarrow{g} B, A' \circ\!\!\xrightarrow{g'} B') = \left\{ \begin{array}{c} \text{distinguished} \\ \text{squares} \end{array} \begin{array}{ccc} A & \xrightarrow{\quad} & A' \\ g \circ\!\!\downarrow & \square & \downarrow g' \\ B & \xrightarrow{\quad} & B' \end{array} \right\}.$

$\text{Ar}_{\square}\mathcal{M}$  is defined analogously.

(2) Define  $\text{Ar}_{\triangle}\mathcal{D}$

- Objects: Morphisms  $A \rightarrow B$  in  $\mathcal{D}$ .

$$\bullet \text{ Morphisms: } \text{Hom}_{\text{Ar}_{\triangle}\mathcal{D}}(A \xrightarrow{f} B, A' \xrightarrow{f'} B') = \left\{ \begin{array}{ccc} & A \xrightarrow{\cong} A' & \\ \text{commutative} & & \\ \text{squares} & \begin{array}{ccc} f \downarrow & & \downarrow f' \\ & B \longrightarrow B' & \end{array} & \end{array} \right\}.$$

**Definition 1.4** (CGW Category). A CGW category  $(\mathcal{C}, \varphi, c, k)$  consists of the following data

- A good double category  $\mathcal{C} = (\mathcal{E}, \mathcal{M})$ ;
- An isomorphism of categories  $\varphi: \text{iso}\mathcal{M} \rightarrow \text{iso}\mathcal{E}$  which is identity on objects;
- An equivalence of categories

$$k: \text{Ar}_{\square}\mathcal{E} \rightarrow \text{Ar}_{\triangle}\mathcal{M} \quad \text{and} \quad c: \text{Ar}_{\square}\mathcal{M} \rightarrow \text{Ar}_{\triangle}\mathcal{E};$$

satisfying the axioms:

(Z) *Basepoint object*.  $\mathcal{C}$  contains an object  $O$  initial in both  $\mathcal{E}$  and  $\mathcal{M}$ .

(I) *Stable Under Isomorphisms*. Let  $\psi: A \rightarrow B$  be an isomorphism in ambient category  $\mathcal{C}$ . Then:

- $\psi$  belongs to  $\text{iso}\mathcal{M}$ , which we denote suggestively as  $\psi: A \rightarrowtail B$ .
- If  $\mathcal{E}$  is a subcategory of  $\mathcal{C}$ , then  $\varphi(\psi): A \circrightarrow B$  corresponds to  $\psi: A \rightarrow B$  in  $\mathcal{C}$ .
- If  $\mathcal{E}^{\text{op}}$  is a subcategory of  $\mathcal{C}$ , then  $\varphi(\psi): A \circrightarrow B$  corresponds to  $\psi^{-1}: B \rightarrow A$  in  $\mathcal{C}$ .

(M) *Monicity*. Every  $\mathcal{M}$  and  $\mathcal{E}$ -morphism is monic. Furthermore:

- Every  $\mathcal{M}$ -morphism is monic in  $\mathcal{C}$ .
- If  $\mathcal{E} \subseteq \mathcal{C}$ , then every  $\mathcal{E}$ -morphism is monic in  $\mathcal{C}$ ; otherwise if  $\mathcal{E}^{\text{op}} \subseteq \mathcal{C}$  then every  $\mathcal{E}$ -morphism is epi in  $\mathcal{C}$ .

(K) *Formal kernels and cokernels*. For any  $f: A \rightarrowtail B$  in  $\mathcal{M}$ , there exists a *formal cokernel*, denoted  $c(f): \text{coker}(f) \circrightarrow B$ , and a distinguished square as below left.

$$\begin{array}{ccc} O \rightarrowtail \text{coker}(f) & & O \rightarrowtail A \\ \downarrow \circlearrowleft & \square & \downarrow \circlearrowleft \\ A \xrightarrow{f} B & & \text{ker}(g) \xrightarrow{k(g)} B \end{array}$$

Dually, for any  $g: A \circrightarrow B$  in  $\mathcal{E}$ , there exists a *formal kernel*, denoted  $k(g): \text{ker}(g) \rightarrowtail B$ , and a distinguished square as above right.

These distinguished squares are unique up to isomorphism in the following sense. If there exists another  $\mathcal{E}$ -morphism  $f': C \circrightarrow B$  and a distinguished square

$$\begin{array}{ccc} O \rightarrowtail C & & \\ \downarrow \circlearrowleft & \square & \downarrow \circlearrowleft \\ A \xrightarrow{f} B & & \end{array}$$

then there exists an isomorphism  $\tau: \text{coker}(f) \rightarrowtail C$  such that the rightmost square in

$$\begin{array}{ccccc} O \rightarrowtail \text{coker}(f) & \xrightarrow{\tau} & C & & \\ \downarrow \circlearrowleft & \square & \downarrow \circlearrowleft & \square & \downarrow \circlearrowleft \\ A \xrightarrow{f} B & \xrightarrow{1} & B & & \end{array}$$

commutes when regarded as a diagram in the ambient category  $\mathcal{C}$ . Notice that since distinguished squares interact well with isomorphisms, this implies the square is distinguished. Formal kernels are unique in the analogous sense.

**Remark 1.5.** By Axiom (K), one can deduce that functors  $c$  and  $k$  are inverse on objects – for details, see remarks after [CZ22, Definition 2.7].

There is a natural notion of structure-preserving functors and subcategories in the CGW context. A *CGW functor* of CGW categories is a double functor

$$F: (\mathcal{E}, \mathcal{M}) \rightarrow (\mathcal{E}', \mathcal{M}')$$

that commutes with the functors  $c$  and  $k$  in the following diagrams

$$\begin{array}{ccc} \text{Ar}_{\square} \mathcal{E} & \xrightarrow{k} & \text{Ar}_{\Delta} \mathcal{M} \\ \text{Ar}_{\square} F \downarrow & & \downarrow \text{Ar}_{\Delta} F \\ \text{Ar}_{\square} \mathcal{E}' & \xrightarrow{k'} & \text{Ar}_{\Delta} \mathcal{M}' \end{array} \quad \begin{array}{ccc} \text{Ar}_{\square} \mathcal{M} & \xrightarrow{c} & \text{Ar}_{\Delta} \mathcal{E} \\ \text{Ar}_{\square} F \downarrow & & \downarrow \text{Ar}_{\Delta} F \\ \text{Ar}_{\square} \mathcal{M}' & \xrightarrow{c'} & \text{Ar}_{\Delta} \mathcal{E}' \end{array} .$$

For a CGW category  $(\mathcal{C}, \varphi, c, k)$  be a CGW category, a *CGW subcategory* is a sub-double category  $\mathcal{D} \subseteq \mathcal{C}$  such that the obvious restrictions  $(\mathcal{D}, \phi|_{\mathcal{D}}, c|_{\mathcal{D}}, k|_{\mathcal{D}})$  forms a CGW category.

**Convention 1.6.** When the context is clear, we will omit mentions of the CGW structure maps and refer to a CGW category  $(\mathcal{C}, \varphi, c, k)$  by its underlying double category  $\mathcal{C} = (\mathcal{M}, \mathcal{E})$  or just  $\mathcal{C}$ .

We now discuss Axioms (K) and (I) in more detail below, followed by some illustrative examples.

*Quotients in CGW Categories.* A distinctive feature of CGW categories is that they are agnostic about whether formal cokernels arise from taking quotients in the *additive* setting (e.g.  $R$ -modules) or taking complements in the *non-additive* setting (e.g. finite sets, varieties etc.). Either way, the formal properties remain consistent. We adopt the following suggestive convention to reinforce this perspective.

**Convention 1.7** (“Quotient”). We typically denote the formal cokernel of  $f: A \rightarrowtail B$  as  $\frac{B}{A}$ , whenever the map  $f$  is clear from context. The object  $\frac{B}{A}$  will typically be referred to as a *quotient*.

One can leverage Helper Definition 1.3, to derive the following useful properties about quotients.

**Lemma 1.8** ([CZ22, Lemma 2.10]). *The following properties hold in any CGW category  $\mathcal{C}$ :*

(i) (Quotients respect Distinguished Squares). *Given any distinguished square*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \circlearrowleft & \square & \downarrow \circlearrowright \\ C & \xrightarrow{g} & D \end{array} ,$$

we have  $\frac{B}{A} \cong \frac{D}{C}$ .

(ii) (Quotients respect Filtrations). *Given  $P_0 \xrightarrow{f_1} P_1 \xrightarrow{g_1} P_2$ , one can construct the following diagram of distinguished squares*

$$\begin{array}{ccccc} P_0 & \xrightarrow{f_1} & P_1 & \xrightarrow{g_1} & P_2 \\ \uparrow \circlearrowleft & \square & \uparrow \circlearrowleft & \square & \uparrow \circlearrowleft \\ O & \longrightarrow & P_{1/0} & \xrightarrow{h_1} & P_{2/0} \\ & & \uparrow \circlearrowleft & \square & \uparrow \circlearrowleft \\ & & O & \longrightarrow & P_{2/1} \end{array} .$$

*Isomorphisms in CGW Categories.* In their original definition, CGW categories were only required to be double categories, not necessarily good. The hypothesis of goodness was used because it allows us to express what it means for distinguished squares to interact well with isomorphisms. Not only does this recover the original Axiom (I) in [CZ22, Definition 2.5], but it also plays a crucial role in our proofs (e.g. Lemmas B.1 and B.2). Ultimately, this is no real loss in generality since our definition still covers all the major examples in the original paper [CZ22]. The underlying reason for this is the following observation.



**Observation 1.9.** Let  $\mathcal{C} = (\mathcal{M}, \mathcal{E})$  be any CGW category. Suppose

$$\begin{array}{ccc} A & \xrightarrow{f'} & B \\ g' \downarrow & & \downarrow g \\ C & \xrightarrow{f} & D \end{array} \quad (2)$$

defines a commutative diagram in the ambient category  $\mathcal{C}$ . If  $f$  and  $f'$  are isomorphisms, then Diagram (2) defines a pullback and a pushout in  $\mathcal{C}$ . The same holds if  $g$  and  $g'$  are isomorphisms.

We omit the proof since it is straightforward, and analogous to what was already worked out in [HMM<sup>+</sup>22, Prop. 4.3]. Informally, Observation 1.9 says: pullback and pushout squares in the ambient category interact well with isomorphisms. As we shall see, this will be useful for verifying goodness.

*Examples.* We review in broad strokes some well-known examples of CGW categories, plus a couple new ones. Some further details can be found in [CZ22, §4].

**Example 1.10** (Exact Categories). For an exact category  $\mathcal{C}$ , define a CGW category  $\mathcal{C} = (\mathcal{M}, \mathcal{E})$  by setting

$$\mathcal{M} = \{\text{admissible monomorphisms}\} \quad \mathcal{E} = \{\text{admissible epimorphisms}\}^{\text{op}}.$$

The basepoint object is the zero object in  $\mathcal{C}$ . The distinguished squares are the biCartesian squares (= both pushouts and pullbacks in the ambient category  $\mathcal{C}$ ). By Observation 1.9,  $\mathcal{C} = (\mathcal{M}, \mathcal{E})$  is a good double category. The equivalences  $k$  and  $c$  map admissible epis to kernels and admissible monics to cokernels, respectively. For more details on the other CGW axioms, see [CZ22, Example 3.1].

**Example 1.11** (Finite Sets). Given  $\text{FinSet}$ , define a CGW category  $\mathcal{C} = (\mathcal{M}, \mathcal{E})$  by setting

$$\mathcal{E} = \mathcal{M} = \{\text{injections}\}.$$

The basepoint object  $O$  is the empty set, and the distinguished squares are the pushout squares. By Observation 1.9,  $\mathcal{C} = (\mathcal{M}, \mathcal{E})$  is a good double category. The equivalences  $c$  and  $k$  are given by taking any inclusion  $A \hookrightarrow B$  to the inclusion  $B \setminus A \hookrightarrow B$ .

**Example 1.12** (Varieties). Given  $\text{Var}_k$ , define  $\mathcal{C} = (\mathcal{M}, \mathcal{E})$  by setting

$$\mathcal{M} = \{\text{closed immersions}\} \quad \mathcal{E} = \{\text{open immersions}\}.$$

The basepoint object  $O$  is the empty variety, and the distinguished squares are the pullback squares

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ \downarrow & \square & \downarrow g \\ C & \xrightarrow{f} & D \end{array}$$

in which  $\text{im } f \cup \text{im } g = D$ , which implies goodness of  $\mathcal{C}$ .<sup>4</sup>  $c$  and  $k$  takes a morphism to the inclusion of the complement. Axioms (I) and (M) follow from properties of closed and open immersions, while Axiom (K) holds as  $D \setminus C \cong B \setminus A$  for any distinguished square

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ \downarrow & \square & \downarrow \\ C & \xrightarrow{\quad} & D \end{array}.$$

For those interested in model theory, the following example will be suggestive.

**Example 1.13** (Definable Sets). Fix  $\Sigma$  to be a first-order language and  $M$  a  $\Sigma$ -structure. The term *definable* will always mean with parameters from  $M$ . Following [KS00], denote  $\text{Def}(M)$  as the category with objects the definable sets of  $M$  and its powers  $M^n$ , and morphisms the definable functions. We upgrade this to a CGW category by setting

$$\mathcal{E} = \mathcal{M} = \{\text{Definable Injections}\}.$$

<sup>4</sup>Notice that we get  $\text{im } f \cup \text{im } g = D$  for free if either  $f$  or  $g$  are isomorphisms.

Axioms (I) and (M) are thus satisfied by definition. For Axiom (Z), set the basepoint object  $O$  as  $\emptyset$ . Define the distinguished squares as the pullback squares

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ \downarrow & \square & \downarrow g \\ C & \xrightarrow{\quad f \quad} & D \end{array}$$

where  $\text{im} f \cup \text{img} = D$ . Finally, any definable injection  $f: C \hookrightarrow D$  can be mapped to its formal cokernel

$$\begin{array}{ccc} O & \xrightarrow{\quad} & D \setminus f(C) \\ \downarrow & \square & \downarrow c(f) \\ C & \xrightarrow{\quad f \quad} & D \end{array}$$

where  $c(f)$  is the obvious inclusion of the complement. The analogous holds for the formal kernels.

**Remark 1.14.** For the curious non-logician: the category of definable sets  $\text{Def}(M)$  can be viewed as an abstraction of  $\text{Var}_k$ . Consider  $\mathbb{C}$  as an algebraically closed field with characteristic 0. In which case, the objects of  $\text{Def}(\mathbb{C})$  correspond to (Boolean combinations of) complex algebraic varieties. See also [Bek17].

Finally, let us mention another non-additive generalisation of exact categories: *proto-exact categories*, introduced by Dyckerhoff-Kapranov [DK19]. A particularly challenging example comes from [EJS20], which showed that the category of matroids form a proto-exact category, yielding a  $K$ -theory spectrum.

Informally, a *matroid* abstracts the notion of linear independence. It consists of a finite set  $E$  and a collection of subsets (“*flats*”) that are closed under dependency – akin to how subspaces behave in vector spaces.<sup>5</sup> Matroids bridge combinatorics and geometry, and have surprisingly deep links to algebraic geometry and Hodge Theory [Kat16, Bak18]. It is therefore very interesting that they also define a CGW category.

**Example 1.15** (Matroids). Let  $M = (E, \mathcal{F}, \bullet_M)$  be a *pointed matroid*, where  $E$  is a finite set, and  $\mathcal{F} \subseteq 2^E$  the set of flats of matroid  $M$  and  $\bullet_M$  the distinguished base-point. A *strong map* of pointed matroids  $f: M \rightarrow N$  is a function  $f: E_M \rightarrow E_N$  such that  $f(\bullet_M) = \bullet_N$  and  $f^{-1}A \in \mathcal{F}(M)$  for all  $A \in \mathcal{F}(N)$ . By [EJS20, Lemma 2.12], pointed matroids and strong maps form a category  $\text{Mat}_\bullet$ .

Next, denote  $\tilde{E} := E \setminus \{\bullet_M\}$ . Given any  $S \subseteq \tilde{E}$ , denote  $M|S$  to be the *restriction of  $M$  to  $S$*  and  $M/S$  to be the *contraction of  $M$  to  $S$*  (for details, see e.g. [Oxl11] or [EJS20, §2].) We upgrade  $\text{Mat}_\bullet$  to a CGW category  $\mathcal{C} = (\mathcal{M}, \mathcal{E})$  by setting

$$\mathcal{M} = \{\text{strong maps that can be factored } N \xrightarrow{\sim} M|S \hookrightarrow M, \text{ for some } S \subseteq \tilde{E}_M\}$$

$$\mathcal{E} = \{\text{strong maps that can be factored } M \twoheadrightarrow M/S \xrightarrow{\sim} N, \text{ for some } S \subseteq \tilde{E}_M\}^{\text{op}}.$$

Notice  $\mathcal{E}$  is the *opposite* category of contractions, analogous to Example 1.10. In fact, one can apply [EJS20, Lemma 5.3] to check that  $\mathcal{M}$  and  $\mathcal{E}$  are closed under isomorphisms and composition, satisfying Axiom (I). Finally, let the distinguished squares be the biCartesian squares in  $\text{Mat}_\bullet$ .

Axioms (M) and (Z) follow from [EJS20, Lemma 5.4]: a strong map  $f$  is monic in  $\text{Mat}_\bullet$  iff  $f$  is injective on the underlying set, and  $f$  is epi iff  $f$  is surjective. In particular, this implies all morphisms in  $\mathcal{M}$  and  $\mathcal{E}$  are monic, and so the matroid  $O := (\{*\}, *)$  is initial in both. In addition, translating [EJS20, Props. 5.7 and 5.8] to our setting: any  $P \xrightarrow{i'} Q \xrightarrow{j'} N$  or  $P \xrightarrow{j} M \xrightarrow{i} N$  can be completed into a distinguished square

$$\begin{array}{ccc} P & \xrightarrow{i'} & Q \\ j \downarrow & \square & \downarrow j' \\ M & \xrightarrow{i} & N \end{array}$$

Setting  $P = O$ , this gives the formal kernels and cokernels required by Axiom (K), unique up to isomorphism by the biCartesian property.

<sup>5</sup>Those interested in homological stability may be reminded of the Tits Buildings.

**1.2. The  $K$ -Theory of pCGW categories.** The main result of [CZ22, §4] is that Quillen’s  $Q$ -Construction [Qui73] can be applied to any CGW category to define a  $K$ -theory spectrum. However, this paper will focus on a particularly well-behaved class of CGW categories, which we call *pCGW categories*.

Informally, pCGW categories are CGW categories  $\mathcal{C} = (\mathcal{M}, \mathcal{E})$  whereby  $\mathcal{M}$  is closed under a formal kind of pushout. This, of course, generalises the familiar fact that admissible monics are closed under pushouts in exact categories [Wei13, Exercise II.7.8], but it is instructive to understand why a generalisation is needed. Consider Example 1.11 where  $\mathcal{M}$  is the category of finite sets and injections. In which case,

$$A \leftarrow \emptyset \rightarrow A \quad \text{where } A \neq \emptyset$$

does not have a pushout in  $\mathcal{M}$  since the map  $A \amalg A \rightarrow A$  is not monic. Nonetheless, this issue can be circumvented by weakening the universal pushout property, as follows.

**Definition 1.16** (Restricted Pushout, [CZ22, Def. 5.3]). Let  $\mathcal{M}$  be a category whose morphisms are all monic. Suppose  $D$  is a span

$$C \leftarrow A \rightarrow B.$$

Define  $\mathcal{M}_D$  to be the category of pullback squares in  $\mathcal{M}$

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & X \end{array},$$

where a morphism between squares is a natural transformation in which all components are equal to the identity except at  $X$ . The *restricted pushout* of  $D$  is the initial object in  $\mathcal{M}_D$ , which we denote by  $B \star_A C$ .

A useful fact is that restricted pushouts still behave functorially like a pushout in the following sense:

**Fact 1.17.** Consider the diagram

$$C \leftarrow A \rightarrow B \rightarrow B'$$

Then  $B' \star_B (B \star_A C) \cong B' \star_A C$ . More explicitly, the composite of restricted pushouts in Diagram (3) is the restricted pushout of the outer span.

$$\begin{array}{ccccc} A & \rightarrow & B & \rightarrow & B' \\ \downarrow & & \downarrow & & \downarrow \\ C & \rightarrow & B \star_A C & \rightarrow & B' \star_B (B \star_A C) \end{array} \quad (3)$$

We now introduce the definition of a pCGW category before reviewing a few key examples.

**Definition 1.18** (pCGW Category). Let  $\mathcal{C} = (\mathcal{M}, \mathcal{E})$  be a CGW category. We call  $\mathcal{C}$  a *pCGW category* if  $\mathcal{M}$  contains all restricted pushouts. In addition, restricted pushouts are required to satisfy:

- (A) *Formal Direct Sums*. Denote the restricted pushout of  $B \leftarrow O \rightarrow C$  as  $B \oplus C := B \star_O C$ , which we also call *formal direct sums*. Then, there exists a canonical pair of distinguished squares

$$\begin{array}{ccc} O & \rightarrow & B \\ \downarrow & \square & \downarrow q_B \\ C & \xrightarrow{p_C} & B \oplus C \end{array} \quad \text{and} \quad \begin{array}{ccc} O & \rightarrow & C \\ \downarrow & \square & \downarrow q_C \\ B & \xrightarrow{p_B} & B \oplus C \end{array},$$

which we call *direct sum squares*.

- (PQ) *Preserves Quotients*. Given any  $B \star_A C$ , we have the quotient  $\frac{B \star_A C}{C} = \frac{B}{A}$ . This assembles into a commutative diagram in the ambient category  $\mathcal{C}$

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xleftarrow{\alpha} & \frac{B}{A} \\ \downarrow g & & \downarrow g' & \circlearrowleft & \downarrow = \\ C & \xrightarrow{f'} & B \star_A C & \xleftarrow{\beta} & \frac{B \star_A C}{C} \end{array}$$

We emphasise the RHS square need not be distinguished; we only require it commute in  $\mathcal{C}$ .

(DS) *Compatibility with Distinguished Squares*. Given a diagram of distinguished squares

$$\begin{array}{ccccc} C & \xleftarrow{\quad} & A & \xrightarrow{\quad} & B \\ \downarrow & \square & \downarrow & \square & \downarrow \\ C' & \xleftarrow{\quad} & A' & \xrightarrow{\quad} & B' \end{array}$$

there is an induced map  $B \star_A C \hookrightarrow B' \star_{A'} C'$  such that the two induced squares

$$\begin{array}{ccc} B & \xrightarrow{\quad} & B \star_A C \\ \downarrow & \square & \downarrow \\ B' & \xrightarrow{\quad} & B' \star_{A'} C \end{array} \quad \begin{array}{ccc} C & \xrightarrow{\quad} & B \star_A C \\ \downarrow & \square & \downarrow \\ C' & \xrightarrow{\quad} & B' \star_{A'} C \end{array}$$

are distinguished.

**Remark 1.19.** For transparency, we emphasise that Definition 1.4 does not require  $\mathcal{E}$  to contain all restricted pushouts – only  $\mathcal{M}$ .

**Example 1.20** (Varieties). Let  $\star$  be the pushout of closed immersions in the category of schemes. Examining the work of [Sch05], it was noticed in [Cam19, §2] that a pushout of closed immersions of varieties produces a square of closed immersions of varieties. For clarity, we emphasise that these squares are not pushouts in the *category of closed immersions* but in the *entire category of schemes*. The fact that  $\star$  satisfies Axioms (PQ) and (DS) follows from the universal property of pushouts. As for Axiom (A), let the direct sum squares be the coproduct squares with standard coprojection maps.

**Example 1.21** (Exact Categories). Let  $\star$  be the usual pushout. Notice the pushout of any span of  $\mathcal{M}$ -morphisms is also a pullback, and thus defines the initial object in the relevant category of pullback squares. Axioms (PQ) and (DS) follow from the fact that pushouts preserve cokernels. For Axiom (A), recall that products and coproducts coincide in exact categories. Hence, let the distinguished squares correspond to the following biCartesian squares in the exact category

$$\begin{array}{ccc} B & \xrightarrow{p_B} & B \oplus C \\ \downarrow & & \downarrow q_C \\ O & \longrightarrow & C \end{array} \quad \begin{array}{ccc} C & \xrightarrow{p_C} & B \oplus C \\ \downarrow & & \downarrow q_B \\ O & \longrightarrow & B \end{array},$$

where  $q_B, q_C$  are the natural projection maps, and  $p_B, p_C$  are the natural coprojection maps.

**Example 1.22** (Definable Sets). Let  $M \models \mathbb{T}$  eliminate imaginaries and contain at least two distinct elements. Then  $\text{Def}(M)$  admits coproducts<sup>6</sup> and quotients of equivalence relations. Given a span of definable injections  $B \xleftarrow{f} A \xrightarrow{g} C$ , define

$$B \star_A C := B \amalg C / \sim$$

where  $\sim$  is the finest equivalence relation generated by  $f(a) \sim g(a), a \in A$ . One easily checks that  $\sim$  is expressible as a *definable* equivalence relation, with transitivity following from injectivity of  $f$  and  $g$ . In other words,  $B \star_A C$  defines a pushout in  $\text{Def}(M)$ , and so Axioms (PQ) and (DS) follow from the universal property. For Axiom (A), take the (definable) disjoint coproduct; notice the coproduct square is distinguished since  $\mathcal{M} = \mathcal{E} = \{\text{Definable Injections}\}$ .

**Example 1.23** (Matroids). There is a technical barrier. Suppose  $M_0$  and  $M_1$  are matroids with groundsets  $E_0$  and  $E_1$ , and assume  $M_0|T = M_1|T = N$  where  $E_0 \cap E_1 = T$ . An *amalgam* of  $M_0$  and  $M_1$  is a matroid  $M$  on  $E_0 \cup E_1$  such that  $M|E_0 = M_0$  and  $M|E_1 = M_1$ . Unfortunately, as noted in [Oxl11, §11.4],

<sup>6</sup>This result is well-known for the weak syntactic category of  $\mathbb{T}$  [Har11], but the same argument works semantically. Fix definable sets  $D_0 \subseteq M^k$  and  $D_1 \subseteq M^l$ . Let  $P = M^k \times M^l \times M$ , and define an equivalence relation  $E \subseteq P^2$  by

$$(\bar{x}_0, \bar{y}_0, z_0) \sim (\bar{x}_1, \bar{y}_1, z_1) \iff (z_0 = z_1 \wedge \bar{x}_0 = \bar{x}_1) \vee (z_0 \neq z_1 \wedge \bar{y}_0 = \bar{y}_1).$$

By elimination of imaginaries,  $Q := P/E$  is definable. In fact, it can be definably identified with  $M^k \amalg M^l$ , and so there exists definable injections  $M^k, M^l \hookrightarrow Q$  whose disjoint images cover  $Q$ . Restricting to  $D_0$  and  $D_1$  yields the coproduct  $D_0 \amalg D_1 \subseteq Q$ .

amalgams do not always exist. In our setting, this means an arbitrary span  $M_0 \leftarrow N \rightarrow M_1$  may not be completable into a commutative square

$$\begin{array}{ccc} N & \xrightarrow{\quad} & M_0 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{\quad} & M \end{array}$$

so long as we require that  $M$  has groundset  $E_0 \cup E_1$ . One solution is to relax this requirement, and allow  $M$  to have any groundset. The question then becomes: given a span  $M_0 \leftarrow N \rightarrow M_1$ , is it always possible to embed  $M_0$  and  $M_1$  into a larger matroid  $M$ ? Can we regard this larger matroid as being initial in the sense of Definition 1.16? Further discussion of this problem is deferred to Section 5.4.

We now setup the  $K$ -theory of pCGW categories via Waldhausen's  $S_\bullet$ -construction.

**Construction 1.24** ( $S_\bullet$ -Construction). Let  $\mathcal{C}$  be a pCGW category. Define  $S_\bullet \mathcal{C}$  to be the simplicial set with  $n$ -simplices  $S_n \mathcal{C}$  given by flag diagrams

$$\begin{array}{ccccccc} M_{00} & \xrightarrow{\quad} & M_{01} & \xrightarrow{\quad} & M_{02} & \xrightarrow{\quad} & \dots & \xrightarrow{\quad} & M_{0n} \\ & & \uparrow \circ & & \uparrow \circ & & & & \uparrow \circ \\ & & M_{11} & \xrightarrow{\quad} & M_{12} & \xrightarrow{\quad} & \dots & \xrightarrow{\quad} & M_{1n} \\ & & & & \uparrow \circ & & & & \uparrow \circ \\ & & & & M_{22} & \xrightarrow{\quad} & \dots & \xrightarrow{\quad} & M_{2n} \\ & & & & & & \uparrow \circ & & \uparrow \circ \\ & & & & & & \vdots & & \vdots \\ & & & & & & & & \uparrow \circ \\ & & & & & & & & M_{nn} \end{array}$$

subject to the conditions

- (i)  $M_{ii} = O$  for all  $i$
- (ii) Every subdiagram

$$\begin{array}{ccc} M_{ji} & \xrightarrow{\quad} & M_{jl} \\ \uparrow \circ & \square & \uparrow \circ \\ M_{ki} & \xrightarrow{\quad} & M_{kl} \end{array}$$

for  $j < k$  and  $i < l$  is distinguished.

Since (formal) quotients respect filtrations (Lemma 1.8), we can represent an  $n$ -simplex as a sequence of  $\mathcal{M}$ -morphisms

$$O = M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \dots \rightarrow M_n$$

together with choices of quotients

$$M_{j/i} := \frac{M_j}{M_i} \quad i < j.$$

Degeneracy maps are obtained by duplicating an  $M_i$ , face maps are obtained by forgetting an  $M_i$ , with the addendum that forgetting  $M_0$  means factoring out by  $M_1$ .

This sets up the following translation of Theorems 4.3 and 7.8 in [CZ22].

**Theorem 1.25** (Presentation Theorem). *Let  $\mathcal{C}$  be a pCGW category and define its  $K$ -theory spectrum*

$$K\mathcal{C} := \Omega|\mathcal{SC}|,$$

with associated  $K$ -groups  $K_n(\mathcal{C}) := \pi_n K\mathcal{C}$ . Then  $K_0(\mathcal{C})$  is the free abelian group generated by objects of  $\mathcal{C}$  modulo the relation that for any distinguished square

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ \downarrow & \square & \downarrow \\ D & \xrightarrow{\quad} & C \end{array},$$

we have  $[D] + [B] = [A] + [C]$ .

**Remark 1.26.** In fact, Theorem 1.25 holds for any CGW category equipped with a formal direct sum satisfying Axiom (A), Definition 1.18. In the case of matroids, define  $M_1 \oplus M_2$  to be the matroid on ground set  $E_1 \coprod E_2$ , with flats of the form  $F_1 \coprod F_2$  where  $F_1 \in \mathcal{F}(M_1)$  and  $F_2 \in \mathcal{F}(M_2)$ .

**1.3. Simplicial Loops & Fibers.** Re-examining Presentation Theorem 1.25, notice the loop space features in our definition of  $K\mathcal{C}$ . Following [GG87, §2], we translate this construction to simplicial sets.

**Convention 1.27.** A simplicial set is a contravariant functor  $X: \Delta^{\text{op}} \rightarrow \text{Set}$ . We sometimes write  $X[n]$  as shorthand for  $X([n])$ . If  $A, B \in \Delta$ , we write  $AB$  to mean the disjoint union of  $A$  followed by  $B$ , where elements of  $A$  are below those of  $B$ . A 0-simplex is sometimes called a *vertex*, a 1-simplex an *edge*.

To motivate, recall that the loop space  $\Omega Z$  of a pointed topological space  $Z$  is the space of based loops  $\text{Map}(S^1, Z)$ . Now fix a simplicial set  $X$  with basepoint  $O \in X[0]$ . A simplicial loop may look like two 1-simplices glued together at the ends

$$O \xrightleftharpoons[g]{f} x,$$

while a homotopy of such loops may look like

where we glue 2-simplices in the triangles  $f_0 h f_1^{-1}$  and  $g_0 h g_1^{-1}$  along a shared 1-simplex  $h$ . One can then extend this picture in a natural way to the higher homotopies as follows:

**Construction 1.28** (Simplicial Loops). Given any simplicial set  $X$ , define

$$\Omega X(A) := \varprojlim \left( \begin{array}{ccccc} \{O\} & \hookrightarrow & X([0]) & \longleftarrow & X([0]A) \\ & & \uparrow & & \downarrow \\ & & X([0]A) & \longrightarrow & X(A) \end{array} \right).$$

Notice that Construction 1.28 works for any simplicial set  $X$ . This sets up the obvious definition:

**Definition 1.29** ( $G$ -Construction). Let  $\mathcal{C}$  be a pCGW category. The  $G$ -Construction on  $\mathcal{C}$  is defined by applying the Simplicial Loop Construction to  $\mathcal{S}\mathcal{C}$ ,

$$G\mathcal{C} := \Omega \mathcal{S}\mathcal{C}.$$

We define the associated  $K$ -groups as  $K_n^G(\mathcal{C}) = \pi_n |\Omega \mathcal{S}\mathcal{C}|$ .

Although the  $G$ -construction is well-defined for ordinary CGW categories, we will rely on the restricted pushouts of  $\mathcal{M}$ -morphisms to show that  $G\mathcal{C} \simeq K\mathcal{C}$  (and thus define isomorphic  $K$ -groups). Our approach mirrors what was done by Gillet-Grayson [GG87], who established  $G\mathcal{C} \simeq K\mathcal{C}$  for exact categories. A key notion in their argument is the so-called *right fiber*.

**Definition 1.30** (Right Fiber). Suppose  $F: X \rightarrow Y$  is a map of simplicial sets,  $A \in \Delta$  and  $\rho \in Y(A)$ . We define  $\rho|F$  (“the right fiber over  $\rho$ ”) by

$$(\rho|F)(B) := \lim_{\leftarrow} \left( \begin{array}{ccc} & & X(B) \\ & & \downarrow \\ & Y(AB) & \longrightarrow Y(B) \\ & \downarrow & \\ \{\rho\} & \hookrightarrow & Y(A) \end{array} \right).$$

We write  $\rho|Y$  for  $\rho|1_Y$ . We regard  $\rho|F$  as the simplicial analogue of the homotopy fiber  $|F|$  over  $\rho$ .

We can now restate our problem. To show that  $GC \simeq KC$ , we need to show that geometric realisation and the loop space constructions commute up to homotopy equivalence, i.e.

$$|\Omega SC| \simeq \Omega|SC|.$$

The following key observation tells us when this happens.

**Observation 1.31** (Key Observation). For any simplicial set  $X$ , consider the commutative square

$$\begin{array}{ccc} \Omega X & \xrightarrow{t} & O|X \\ \downarrow b & & \downarrow q \\ O|X & \xrightarrow{q} & X \end{array} . \quad (4)$$

where  $b$  and  $t$  are the projection maps and  $q$  the obvious face map. Then:

- (i)  $O|X$  is contractible.
- (ii)  $O|q \simeq \Omega X$
- (iii)  $|\Omega X| \simeq \Omega|X|$  iff this square is homotopy Cartesian.

*Proof.* (i) follows from [GG87, Lemma 1.4]. (ii) is clear from unpacking definitions. For (iii), first take the homotopy pullback  $P$  of

$$|O|X| \rightarrow |X| \leftarrow |O|X|$$

in the homotopy category of spaces. Taking the geometric realisation of Diagram (4), this yields a map  $|\Omega X| \rightarrow P$ , which is a homotopy equivalence iff Diagram (4) defines a homotopy pullback. Finally, since  $O|X$  is contractible, deduce that the homotopy pullback  $P$  is equivalent to the homotopy pullback of

$$* \rightarrow |X| \leftarrow *,$$

which is the loop space  $\Omega|X|$ . □

Key Observation 1.31 suggests the following proof strategy. By item (iii), in order to show  $GC \simeq KC$  it suffices to verify that the square

$$\begin{array}{ccc} \Omega SC & \xrightarrow{t} & O|SC \\ \downarrow b & & \downarrow q \\ O|SC & \xrightarrow{q} & SC \end{array} . \quad (5)$$

is homotopy Cartesian. By item (ii), this is equivalent to showing that

$$\begin{array}{ccc} O|q & \xrightarrow{t} & O|SC \\ \downarrow b & & \downarrow q \\ O|SC & \xrightarrow{q} & SC \end{array} . \quad (6)$$

is homotopy Cartesian. To show this, it suffices to analyse how  $q: O|SC \rightarrow SC$  behaves on the induced simplicial fibers in the following sense:

**Theorem 1.32** ([GG87], Theorem B'). Suppose  $F: X \rightarrow Y$  is a map of simplicial sets. Suppose for any  $A \in \Delta$ , any  $\rho \in Y(A)$ , and any  $f: A' \rightarrow A$  such that the induced map

$$\rho|_F \rightarrow f^* \rho|_F$$

is a homotopy equivalence. Then the square

$$\begin{array}{ccc} \rho|_F & \longrightarrow & X \\ \downarrow & & \downarrow \\ \rho|_Y & \longrightarrow & Y \end{array}$$

is homotopy Cartesian.

**Convention 1.33.** Following established usage in the literature [GG87, GG03], we shall refer to Theorem 1.32 as Theorem B', reflecting its role as the simplicial analogue of Quillen's Theorem B.

## 2. A TECHNICAL RESULT ON RIGHT FIBERS

**Convention 2.1.** Hereafter, any category denoted  $\mathcal{C}, \mathcal{D}$  should be assumed to be a pCGW category, unless stated otherwise.

The goal of this section is to prove Theorem 2.8, which essentially says: given a nice simplicial map  $F: Y \hookrightarrow \mathcal{S}\mathcal{C}$  where  $\mathcal{C}$  is a pCGW category, the right fiber  $O|_F$  admits a nice description. The results here are technical, and are inspired by Grayson's framework of *dominant functors* [Gra87].

There are two main applications of Theorem 2.8. First, the key result that  $G\mathcal{C} \simeq K\mathcal{C}$  is obtained as a straightforward corollary (Theorem 2.11). Second, it sets up the proof of Theorem 3.7, which gives an initial characterisation of the generators of  $K_1(\mathcal{C})$ ; the details will be deferred to Section 3.

**2.1. H-Space Structure.** We introduce the new notion of a *moral subset*, which is similar to a simplicial subset of  $\mathcal{S}\mathcal{C}$  except we allow for a shift in dimensions.

**Definition 2.2** (Moral Subset). Let  $\mathcal{C}$  be a pCGW category, and fix some  $k \in \mathbb{N}$ . A *k-moral subset* of  $\mathcal{S}\mathcal{C}$  is a simplicial map

$$F: Y \hookrightarrow \mathcal{S}\mathcal{C}$$

such that:

- (i) If  $k = 0$ ,  $Y[n] \subseteq \mathcal{S}\mathcal{C}[n] \forall n$ , and  $F$  is the inclusion map.
- (ii) If  $k \geq 1$ , fix a base-point  $O_Y := (O \rightharpoonup \dots \rightharpoonup O) \in \mathcal{S}\mathcal{C}[k-1]$ . Then all  $n$ -simplices in  $Y$  are of the form

$$(O_Y \rightharpoonup K_0 \rightharpoonup K_1 \rightharpoonup \dots \rightharpoonup K_n) \in \mathcal{S}\mathcal{C}([k-1][n])$$

The map  $F$  forgets  $O_Y$  and factors the rest by  $K_0$ ;

- (iii)  $Y$  is closed under direct sums.
- (iv)  $F$  commutes with face and degeneracy maps, i.e. for all  $f: [m] \rightarrow [n]$  in  $\Delta$ ,

$$\begin{array}{ccc} Y[m] & \xrightarrow{Ff} & Y[n] \\ F_m \downarrow & & \downarrow F_n \\ \mathcal{S}\mathcal{C}[m] & \xrightarrow{\mathcal{S}\mathcal{C}f} & \mathcal{S}\mathcal{C}[n] \end{array}$$

Notice a 0-moral subset is an honest simplicial subset of  $\mathcal{S}\mathcal{C}$ .

**Example 2.3.** The quotient map  $q: O|\mathcal{S}\mathcal{C} \rightarrow \mathcal{S}\mathcal{C}$  from Observation 1.31 is a 1-moral subset.

**Construction 2.4** (Addition Map). Fix  $A \in \Delta$ , where  $A = [a]$ . Given a  $k$ -moral subset  $F: Y \rightarrow \mathcal{S}\mathcal{C}$  and  $\overline{M} \in \mathcal{S}\mathcal{C}(A)$ , represent a  $q$ -simplex  $W$  in  $\overline{M}|_F$  as

$$W = \left( \frac{W_{\text{top}}}{W_{\text{bot}}} \right) = \left( O \rightharpoonup M_1 \rightharpoonup \dots \rightharpoonup M_a \rightharpoonup \frac{K_0 \rightharpoonup K_1 \rightharpoonup \dots \rightharpoonup K_q}{L_0 \rightharpoonup L_1 \rightharpoonup \dots \rightharpoonup L_q} \right) \quad (7)$$



where  $W_{\text{top}}$  is a simplex of  $Y$ ,  $W_{\text{bot}}$  a simplex of  $\mathcal{SC}$  and the double line represents the identity

$$\begin{array}{ccc} O = F\left(\frac{K_0}{K_0}\right) & \rightsquigarrow & \dots \rightsquigarrow F\left(\frac{K_q}{K_0}\right) \\ \parallel & & \parallel \\ O = \frac{L_0}{L_0} & \rightsquigarrow & \dots \rightsquigarrow \frac{L_q}{L_0} \end{array}.$$

Parentheses are placed around  $(O_Y \rightsquigarrow)$  in Equation (7) to indicate that when  $k = 0$ , we delete  $O_Y \rightsquigarrow$  from  $W_{\text{top}}$  and set  $K_0 = O$ .

Define addition  $(W, W') \mapsto W + W'$  by setting

$$W + W' := \left( \begin{array}{ccc} (O_Y \rightsquigarrow) & & K_0 \oplus K'_0 \rightsquigarrow \dots \rightsquigarrow K_q \oplus K'_q \\ O \rightsquigarrow \dots \rightsquigarrow M_a \rightsquigarrow & & \frac{L_0 \star_{M_a} L'_0 \rightsquigarrow \dots \rightsquigarrow L_q \star_{M_a} L'_q}{L_0 \star_{M_a} L'_0 \rightsquigarrow \dots \rightsquigarrow L_q \star_{M_a} L'_q} \end{array} \right),$$

with the following quotients:

- For  $K_i \oplus K'_i$ :

$$\frac{K_i \oplus K'_i}{K_j \oplus K'_j} := \frac{K_i}{K_j} \oplus \frac{K'_i}{K'_j}.$$

- For  $L_i \star_{M_a} L'_i$ :

$$\frac{L_i \star_{M_a} L'_i}{M_j} := \begin{cases} \frac{L_i \star_{M_a} L'_i}{M_j} & \text{by Axiom (K), if } 1 \leq j < a \\ \frac{L_i}{M_a} \oplus \frac{L'_i}{M_a} & j = a \end{cases},$$

and recursively:

$$\frac{L_i \star_{M_a} L'_i}{L_j \star_{M_a} L'_j} := F\left(\frac{K_i \oplus K'_i}{K_j \oplus K'_j}\right).$$

**Claim 2.5.** *The addition map above turns  $|\overline{M}|F|$  into a homotopy associative and commutative  $H$ -space, making  $\pi_0(\overline{M}|F|)$  a monoid.*

*Proof.* This follows from a series of basic checks:

- Well-definedness.* Choices of quotients are valid in  $\mathcal{SC}$  by Lemma A.1. Further, the top row of  $W + W'$  also defines a simplex in  $Y$  since moral subsets are closed under direct sums.
- Identity.* The 0-simplex

$$\left( \begin{array}{ccc} (O_Y \rightsquigarrow) & & \underline{O} \\ O \rightsquigarrow M_1 \rightsquigarrow \dots \rightsquigarrow M_a \rightsquigarrow & \xrightarrow{1} & M_a \end{array} \right) \quad (8)$$

serves as additive identity. To see why, use the fact that restricted pushouts are initial to deduce  $K \oplus O \cong K$  for any  $K \in \mathcal{C}$ , and  $L_i \star_{M_a} M_a \cong L_i$ .

- Associativity and Commutativity.* Since restricted pushouts (and direct sums) are initial, they are associative and commutative up to natural isomorphism. These induce simplicial homotopies that show  $|\overline{M}|F|$  is a homotopy associative and commutative  $H$ -space.

□

**2.2. The Main Result.** We now leverage the  $H$ -space structure of  $O|F|$  to describe the right fiber, assuming  $F$  satisfies a key technical condition, which we call *cofinality*.

**Definition 2.6** (Cofinality). Suppose  $F: Y \hookrightarrow \mathcal{SC}$  is a  $k$ -moral subset. We say  $F$  has *cofinal image* if:

- Case 1:**  $k = 0$ . For any  $O \rightsquigarrow M \in \mathcal{SC}$ , there exists  $T \in \text{ob}\mathcal{C}$  such that

$$O \rightsquigarrow M \oplus T \in Y[1].$$

- Case 2:**  $k \geq 1$ . For any  $C \in \text{ob}\mathcal{C}$ ,

$$O_Y \rightsquigarrow C \in Y[0] \quad \text{and} \quad O_Y \rightsquigarrow O \rightsquigarrow C \in Y[1].$$

**Example 2.7.** It is clear that the simplicial map  $q: O|\mathcal{SC} \rightarrow \mathcal{SC}$  has cofinal image.

This sets up the main theorem of this section.

**Theorem 2.8.** *Let  $\mathcal{C}$  be a pCGW category, and  $F: Y \hookrightarrow \mathcal{SC}$  a moral subset with cofinal image. Then*

$$\begin{array}{ccc} O|F & \longrightarrow & Y \\ \downarrow & & \downarrow \\ O|\mathcal{SC} & \longrightarrow & \mathcal{SC} \end{array} \quad (9)$$

is a homotopy Cartesian square.

*Proof.* By Claim 2.5,  $\pi_0(\overline{M}|F)$  is a monoid with respect to  $+$ . Say that  $F$  is *dominant* if  $\pi_0(\overline{M}|F)$  is also a group given any  $\overline{M} \in \mathcal{SC}([a])$  for any  $[a] \in \Delta$ . The proof then follows from two main implications.

- Step 1: If  $F$  has cofinal image, then  $F$  is dominant.
- Step 2: If  $F$  is dominant, then Diagram (9) is a homotopy Cartesian square.

*Step 1: Cofinality implies dominance.* Fixing some  $\overline{M} \in \mathcal{SC}([a])$ ,  $\pi_0(\overline{M}|F)$  can be presented as

- Generators: Vertices of  $\overline{M}|F$ , represented as

$$W = \left( \begin{array}{ccccccc} & & & & (O_Y \twoheadrightarrow) & \frac{K_0}{N} \\ O \twoheadrightarrow M_1 & \twoheadrightarrow & \dots & \twoheadrightarrow & M_a & \twoheadrightarrow & \end{array} \right)$$

- Relations: 1-simplices of  $\overline{M}|F$ .

As before, we omit the choice of quotients, but any two vertices with the same presentation will have isomorphic quotients, and are thus connected by a 1-simplex.<sup>7</sup> The generators can therefore be simplified to

$$W = \left( \begin{array}{ccc} (O \twoheadrightarrow) & \frac{K_0}{N} \\ M_a & \twoheadrightarrow & \end{array} \right). \quad (10)$$

To construct its inverse in  $\pi_0(\overline{M}|F)$ , it will be helpful to consider the case of 0-moral subsets separately.

**Case I:**  $k = 0$ . Since  $F$  has cofinal image, there exists some  $T \in \text{ob } \mathcal{C}$  such that

$$O \twoheadrightarrow \frac{N}{M_a} \oplus T \in Y[1].$$

Setting  $N' := M_a \oplus T$ , it follows from Fact 1.17 that

$$N \star_{M_a} N' \cong N \oplus T.$$

Thus, define a new vertex  $W'$  whereby

$$W + \underbrace{\left( \begin{array}{ccc} & \frac{O}{N'} \\ M_a & \twoheadrightarrow & \end{array} \right)}_{W'} = \left( \begin{array}{ccc} & \frac{O}{N \oplus T} \\ M_a & \twoheadrightarrow & \end{array} \right).$$

To show  $W'$  is the inverse of  $W$ , we claim that  $W + W'$  lies in the connected component of the additive identity. This is because

$$\left( \begin{array}{ccc} & \frac{O \twoheadrightarrow \frac{N}{M_a} \oplus T}{M_a \twoheadrightarrow N \oplus T} \\ M_a & \twoheadrightarrow & \end{array} \right)$$

defines a 1-simplex in  $\pi_0(\overline{M}|F)$ , since  $\frac{N}{M_a} \oplus T \cong \frac{N \oplus T}{M_a}$  by Lemma A.3 (ii). Since the 0<sup>th</sup> face map acts by forgetting the second  $M_a$  on the bottom row and factoring the top by  $\frac{N}{M_a} \oplus T$ , conclude that the 1-simplex connects the additive identity (Equation (8)) to  $W + W'$ .

<sup>7</sup>The choice of 1-simplex is obvious, but justification takes some work. Informal proof sketch: construct the obvious simplex for the bottom row using the fact that formal quotients respect filtrations (Lemma 1.8) and are unique up to isomorphism (Axiom (K)). To show it is indeed a simplex, we must show the rightmost column of isomorphism squares are all distinguished; by goodness, it suffices to show they commute in the ambient category. We get the top square for free by Axiom (K). To see the square below it also commutes, use the fact that all  $\mathcal{E}$ -morphisms are monic (if  $\mathcal{E} \subseteq \mathcal{C}$ ) or epi (if  $\mathcal{E}^{\text{op}} \subseteq \mathcal{C}$ ); this uses Axiom (M). Keep going.

**Case II:**  $k \geq 1$ . The argument is similar. Given  $W$  as above, use cofinality to define a vertex  $W'$  whereby

$$W + \underbrace{\begin{pmatrix} O & \rightharpoonup & \frac{N}{M_a} \\ M_a & \rightharpoonup & \overline{M_a \oplus K_0} \end{pmatrix}}_{W'} = \begin{pmatrix} O & \rightharpoonup & \frac{N}{M_a \oplus K_0} \\ M_a & \rightharpoonup & \overline{N \oplus K_0} \end{pmatrix}.$$

The sum  $W + W'$  is then connected to the additive inverse via the 1-simplex

$$\begin{pmatrix} O & \rightharpoonup & O \rightharpoonup \frac{N}{M_a \oplus K_0} \\ M_a & \rightharpoonup & \overline{M_a \rightharpoonup N \oplus K_0} \end{pmatrix}.$$

In sum: given any  $W \in \pi_0(\overline{M}|F)$ , we can use cofinality to construct its inverse with respect to  $+$ . Since  $\overline{M} \in \mathcal{SC}(A)$  and  $A \in \Delta$  were chosen arbitrarily, this shows that  $F$  is dominant.

*Step 2:  $O|F$  as a homotopy pullback.* Fix  $\overline{M} \in \mathcal{SC}(A)$  for some  $A \in \Delta$ , and fix  $f: A' \rightarrow A$ . By Theorem B' (Theorem 1.32), it suffices to show that the base-change map  $f^*: \overline{M}|F \rightarrow f^*\overline{M}|F$  is a homotopy equivalence. By Step 1, we can use the fact that  $\pi_0(\overline{M}|F)$  is a group since  $F$  is dominant.

*Step 2a: A reduction.* Let  $g: [0] \rightarrow A'$  be any morphism in  $\Delta$ . To show that  $f^*$  is a homotopy equivalence, it suffices to show that  $(fg)^* = g^*f^*$  and  $g^*$  are. In fact, since both  $fg$  and  $g$  have  $[0]$  as source, it suffices to show that

$$f_i: [0] \rightarrow A, \quad f_i(0) = i \text{ for } i \in A,$$

induces a homotopy equivalence for any  $A \in \Delta$ . Notice  $f_i^*$  defines a map

$$f_i^*: \overline{M}|F \rightarrow O|F$$

since  $O$  is the only vertex of  $\mathcal{SC}$ .

*Step 2b: The base case.* Define a map

$$h_0: O|F \longrightarrow \overline{M}|F$$

$$\left( \begin{pmatrix} (O_Y \rightharpoonup) & K_0 \rightharpoonup \dots \rightharpoonup K_q \\ O \rightharpoonup & \overline{L_0 \rightharpoonup \dots \rightharpoonup L_q} \end{pmatrix} \right) \mapsto \left( \begin{pmatrix} (O_Y \rightharpoonup) & K_0 \rightharpoonup \dots \rightharpoonup K_q \\ O \rightharpoonup \dots \rightharpoonup M_a \rightharpoonup & \overline{M_a \oplus L_0 \rightharpoonup \dots \rightharpoonup M_a \oplus L_q} \end{pmatrix} \right)$$

with quotients defined as

- $\frac{M_a \oplus L_j}{M_a \oplus L_k} := \frac{L_j}{L_k} \left( = F \left( \frac{K_j}{K_k} \right) \right),$
- $\frac{M_a \oplus L_j}{M_a} := L_j, \quad \frac{M_a \oplus L_j}{M_i} := \frac{M_a}{M_i} \oplus L_j$

To show that  $f_i^*$  is a homotopy equivalence (for arbitrary  $i$ ), it suffices to establish the following claim.

**Claim 2.9.** *The maps  $f_i^* \circ h_0$  and  $h_0$  are homotopy equivalences.*

*Proof of Claim.* Two main checks.

(i) On  $f_i^* \circ h_0$ . The map  $f_i^* \circ h_0: O|F \rightarrow O|F$  sends

$$\left( \begin{pmatrix} (O_Y \rightharpoonup) & K_0 \rightharpoonup \dots \rightharpoonup K_q \\ O \rightharpoonup & \overline{L_0 \rightharpoonup \dots \rightharpoonup L_q} \end{pmatrix} \right) \mapsto \left( \begin{pmatrix} (O_Y \rightharpoonup) & K_0 \rightharpoonup \dots \rightharpoonup K_q \\ O \rightharpoonup & \overline{\frac{M_a}{M_i} \oplus L_0 \rightharpoonup \dots \rightharpoonup \frac{M_a}{M_i} \oplus L_q} \end{pmatrix} \right)$$

for  $0 \leq i \leq a$ . Since  $O|F$  is an  $H$ -space, we can reformulate  $f_i^* \circ h_0$  more suggestively as

$$(f_i^* \circ h_0)(W) = W + \begin{pmatrix} (O_Y \rightharpoonup) & O \\ O \rightharpoonup & \overline{\frac{M_a}{M_i}} \end{pmatrix}.$$

Since  $\pi_0(O|F)$  is a group on the vertices of  $O|F$ , there exists a vertex  $V$  such that

$$\begin{pmatrix} (O_Y \rightharpoonup) & O \\ O \rightharpoonup & \overline{\frac{M_a}{M_i}} \end{pmatrix} + V \sim \begin{pmatrix} (O_Y \rightharpoonup) & O \\ O \rightharpoonup & \overline{O} \end{pmatrix}.$$

Define  $h_1: O|F \rightarrow O|F$  as mapping

$$h_1(W) = W + V$$

for any simplex  $W$ . Since  $+$  is homotopy associative and homotopy commutative, deduce that

$$f_i^* \circ h_0 \circ h_1 \sim 1, \quad h_1 \circ f_i^* \circ h_0 \sim 1.$$

- (ii) On  $h_0$ . Notice:  $f_a^* \circ h_0$  is isomorphic to the identity map on  $O|F$ . It therefore suffices to show  $h_0 \circ f_a^*$  is homotopic to the identity map 1 on  $\overline{M}|F$ . But this follows from the natural isomorphism

$$h_0 \circ f_a^* \cong 1,$$

induced by the isomorphism

$$M_a \oplus \frac{L_j}{M_a} \cong \frac{M_a \oplus L_j}{M_a} = L_j, \quad \text{for all } j,$$

a consequence of Lemma A.3 (ii) and the choice of quotients by  $h_0$ . [Notice: the specific choice of quotients by  $h_0$  is crucial; otherwise, the isomorphism may fail to hold since e.g. not all short exact sequences split.]

This completes proof of Claim 2.9.  $\square$

*Step 3: Finish.* Fix a  $k$ -moral subset  $F: Y \rightarrow \mathcal{SC}$  that has cofinal image. Step 1 showed that  $F$  is dominant, i.e.  $\pi_0(\overline{M}|F)$  is a group for any  $M \in \mathcal{SC}([a])$ . Step 2 combines this with Theorem B' to show that Diagram 9 is homotopy Cartesian. This proves the theorem.  $\square$

The following corollary justifies viewing the right fiber (Definition 1.30) as the simplicial analogue of a homotopy fiber, and will be useful later.

**Corollary 2.10.** *Suppose  $F: Y \hookrightarrow \mathcal{SC}$  is  $k$ -moral subset with cofinal image. Then  $|O|F|$  is homotopy equivalent to the homotopy fiber of  $|F|$ .*

*Proof.* By Observation 1.31,  $O|\mathcal{SC}|$  is contractible. Since the homotopy fiber of  $|F|$  is the homotopy pullback of the cospan  $* \rightarrow |\mathcal{SC}| \xleftarrow{|F|} Y$ , the statement follows.  $\square$

In addition, we now obtain a key result of this paper regarding the  $G$ -construction.

**Theorem 2.11.** *Let  $\mathcal{C}$  be a  $pCGW$  category. Then, there is a homotopy equivalence*

$$|G\mathcal{C}| \xrightarrow{\sim} \Omega|\mathcal{SC}|.$$

*Further, direct sum induces an  $H$ -space structure on  $G\mathcal{C}$ .*

*Proof.* Let us review Key Observation 3.11. By item (iii),  $|G\mathcal{C}| = |\Omega\mathcal{SC}| \simeq \Omega|\mathcal{SC}|$  if

$$\begin{array}{ccc} \Omega\mathcal{SC} & \xrightarrow{t} & O|\mathcal{SC}| \\ \downarrow b & & \downarrow q \\ O|\mathcal{SC}| & \xrightarrow{q} & \mathcal{SC} \end{array}$$

is homotopy Cartesian. By item (ii), we have  $O|q| \simeq \Omega\mathcal{SC}$ . Since  $q$  is a 1-moral subset with cofinal image, the rest follows from Theorem 2.8; the  $H$ -space structure on  $G\mathcal{C}$  comes from Construction 2.4.  $\square$

**Remark 2.12.** The equivalence in Theorem 2.11 established between  $G\mathcal{C}$  and  $K\mathcal{C}$  is one of topological spaces, not of infinite loop spaces or spectra.

**Discussion 2.13** (Comparison with other proofs). Our approach combines ideas from two existing proofs of Theorem 2.11 when  $\mathcal{C}$  is an exact category. While all three proofs rely on Theorem B', there are some key differences worth mentioning.

In broad strokes, Theorem B' says: if the induced map on fibers  $\rho|F \rightarrow f^*\rho|F$  is a homotopy equivalence for *any*  $f: A' \rightarrow A$ , then  $\rho|F$  is a homotopy pullback. In the original paper [GG87], Gillet-Grayson simplifies this condition by restricting to the maps  $f_0, f_1: [0] \rightarrow [1]$ ; compare this with Step 2 of our proof

of Theorem 2.8. Their argument is technical, but reflects the informal intuition: to understand how objects break into finitely many pieces, it suffices to understand how to break a single object into two.

A different proof appears in [Gra87, Thm 8.2], where Grayson uses Theorem B' to study *dominant* exact functors. By contrast, notice we define dominance for *simplicial maps*. This adjustment accounts for the fact that  $O|\mathcal{SC}$  *a priori* does not correspond to a pCGW category. Further, our definitions of moral subsets and cofinality were designed for our arguments to go through smoothly; no real attempts at generality were made. Whereas Grayson gives a full characterisation of dominant functors [Gra87, Thm 2.1], his proof does not translate well to our setting; for our purposes, it suffices to identify a sufficient criterion for dominance, i.e. cofinality.

### 3. GENERATORS OF $K_1(\mathcal{C})$

Having established Theorem 2.11, we now begin to deliver on our promise that the  $G$ -construction leads to an explicit description of  $K_1(\mathcal{C})$ . Section 3.1 unpacks the definition of the  $G\mathcal{C}$ -construction. Sections 3.2 and 3.3 work to obtain an increasingly sharp description of the generators of  $K_1$ ; our analysis extends various results from [She94, She98, Nen96].

#### 3.1. Review of $G$ -Construction.

**Construction 3.1** ( $G$ -Construction). An  $n$ -simplex of  $G\mathcal{C}$  is a pair of flag diagrams of the form

$$\begin{array}{ccccccc}
 P_0 & \twoheadrightarrow & P_1 & \twoheadrightarrow & P_2 & \twoheadrightarrow & \dots \twoheadrightarrow P_n \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & P_{1/0} & \twoheadrightarrow & P_{2/0} & \twoheadrightarrow & \dots \twoheadrightarrow P_{n/0} \\
 & & & & \uparrow & & \uparrow \\
 & & & & P_{2/1} & \twoheadrightarrow & \dots \twoheadrightarrow P_{n/1} \\
 & & & & & & \uparrow \\
 & & & & & & \vdots \\
 & & & & & & \uparrow \\
 & & & & & & P_{n/n-1}
 \end{array}
 \qquad
 \begin{array}{ccccccc}
 P'_0 & \twoheadrightarrow & P'_1 & \twoheadrightarrow & P'_2 & \twoheadrightarrow & \dots \twoheadrightarrow P'_n \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & P'_{1/0} & \twoheadrightarrow & P'_{2/0} & \twoheadrightarrow & \dots \twoheadrightarrow P'_{n/0} \\
 & & & & \uparrow & & \uparrow \\
 & & & & P'_{2/1} & \twoheadrightarrow & \dots \twoheadrightarrow P'_{n/1} \\
 & & & & & & \uparrow \\
 & & & & & & \vdots \\
 & & & & & & \uparrow \\
 & & & & & & P'_{n/n-1}
 \end{array}$$

subject to the conditions:

- (i) Every quotient index square

$$\begin{array}{ccc}
 P_{j/i} & \twoheadrightarrow & P_{k/i} \\
 \uparrow & \square & \uparrow \\
 P_{j/l} & \twoheadrightarrow & P_{k/l}
 \end{array}
 \quad \text{where } i < l < j < k$$

is distinguished, and coincide in both flag diagrams.

- (ii) Every quotient index triangle defines a distinguished square

$$\begin{array}{ccc}
 P_{j/i} & \twoheadrightarrow & P_{k/i} \\
 \uparrow & \square & \uparrow \\
 O & \twoheadrightarrow & P_{k/j}
 \end{array}
 \quad \text{for any } i < j < k,$$

and coincide in both flag diagrams.

(iii) Any  $P_j \twoheadrightarrow P_k$  and  $P'_j \twoheadrightarrow P'_k$  in the filtration can be completed into distinguished squares

$$\begin{array}{ccc} P_j & \twoheadrightarrow & P_k \\ \uparrow & \square & \uparrow \\ O & \twoheadrightarrow & P_{k/j} \end{array}, \quad \begin{array}{ccc} P'_j & \twoheadrightarrow & P'_k \\ \uparrow & \square & \uparrow \\ O & \twoheadrightarrow & P_{k/j} \end{array} \quad \text{for any } i < j < k.$$

**Convention 3.2.** Technically, an  $n$ -simplex of  $G\mathcal{C}$  is a pair of  $(n+1)$ -simplices in  $\mathcal{SC}$

$$O \twoheadrightarrow P_0 \twoheadrightarrow \dots \twoheadrightarrow P_n \quad O \twoheadrightarrow P'_0 \twoheadrightarrow \dots \twoheadrightarrow P'_n$$

such that the  $0^{\text{th}}$  faces (= forgetting  $O$  and quotienting by  $P_0$ ) agree. Here we omit the basepoint  $O$  for simplicity. Therefore, a *vertex* of  $G\mathcal{C}$  is a pair  $(M, N) \in \mathcal{C} \times \mathcal{C}$ , and an *edge*  $(M, N) \rightarrow (M', N')$  is a pair of distinguished squares with identical quotient

$$\left( \begin{array}{ccc} O & \twoheadrightarrow & C \\ \downarrow & \square & \downarrow \\ M & \twoheadrightarrow & M' \end{array}, \begin{array}{ccc} O & \twoheadrightarrow & C \\ \downarrow & \square & \downarrow \\ N & \twoheadrightarrow & N' \end{array} \right).$$

We sometimes also represent a vertex as  $\begin{pmatrix} M \\ N \end{pmatrix}$ , without any double lines.

**3.2. Sherman Loops & Splitting.** Here we extend Sherman's analysis of  $K_1$  for exact categories [She94, She98] to pCGW categories. A guiding principle in his approach is that restricting to split exact sequences can clarify the general case. We adapt this insight to pCGW categories, first by introducing some key definitions, before giving a first characterisation of the generators of  $K_1(\mathcal{C})$  (Theorem 3.7).

**Construction 3.3** (Sherman Loop). A *Sherman triple*  $(\alpha, \beta, \theta)$  consists of the following data:

- Two  $\mathcal{M}$ -morphisms  $A \xrightarrow{\alpha} B$ ,  $A' \xrightarrow{\beta} B'$ ;
- An isomorphism  $\theta: A \oplus C \oplus B' \xrightarrow{\sim} A' \oplus C' \oplus B$ , where  $C$  and  $C'$  are specific choices of quotients

$$\begin{array}{ccc} O & \twoheadrightarrow & C \\ \downarrow & \square & \downarrow \delta \\ A & \xrightarrow{\alpha} & B \end{array} \quad \begin{array}{ccc} O & \twoheadrightarrow & C' \\ \downarrow & \square & \downarrow \gamma \\ A' & \xrightarrow{\beta} & B' \end{array}. \quad (11)$$

Its associated *Sherman loop* is the homotopy class  $G(\alpha, \beta, \theta)$  in  $\pi_1(|G\mathcal{C}|)$  represented by the loop

$$\begin{pmatrix} O \\ O \end{pmatrix} \rightarrow \begin{pmatrix} A \\ A \end{pmatrix} \rightarrow \begin{pmatrix} A \oplus C \oplus B' \\ B \oplus B' \end{pmatrix} \rightarrow \begin{pmatrix} A' \oplus C' \oplus B \\ B' \oplus B \end{pmatrix} \leftarrow \begin{pmatrix} A' \\ A' \end{pmatrix} \leftarrow \begin{pmatrix} O \\ O \end{pmatrix}, \quad (12)$$

where the arrows denote the obvious 1-simplices.<sup>8</sup>

**Remark 3.4.** Fix a pair of  $\mathcal{M}$ -morphisms  $\alpha, \beta$ . One easily checks that any two Sherman triples  $(\alpha, \beta, \theta)$  and  $(\alpha, \beta, \theta')$  define the same Sherman loop in  $K_1(\mathcal{C})$  up to homotopy – see e.g. [She94, §1].

**Definition 3.5** (Split). Call a distinguished square of the form

$$\begin{array}{ccc} O & \twoheadrightarrow & C \\ \downarrow & \square & \downarrow g \\ A & \xrightarrow{f} & B \end{array}$$

an *exact square*.

<sup>8</sup>*Details.* The middle arrow in Equation (12) applies  $\theta$  on the top row and the canonical isomorphism  $B \oplus B' \rightarrow B' \oplus B$  on the bottom; the rest of the 1-simplices are defined by applying Axiom (DS), Definition 1.18.

- (i) Call an exact square *split* if there exists an isomorphism

$$\Psi: B \rightarrow A \oplus C$$

such that the following squares commute

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow 1_A & & \downarrow \Psi \\ A & \xrightarrow{p_A} & A \oplus C \end{array} \quad \begin{array}{ccc} B & \xleftarrow{g} & C \\ \downarrow \varphi(\Psi) & & \downarrow 1_C \\ A \oplus C & \xleftarrow{q_C} & C \end{array}$$

in  $\mathcal{M}$  and  $\mathcal{E}$  respectively, where  $p_A$  and  $q_C$  are the morphisms from the obvious direct sum square.

- (ii) Call an  $\mathcal{M}$ -morphism  $A \rightarrow B$  is called *split* if its corresponding exact square [obtained by taking formal quotients] is split.

**Remark 3.6.** A sanity check: suppose  $\mathcal{C}$  is an exact category, and so  $\varphi(\Psi)$  corresponds to

$$\Psi^{-1}: A \oplus C \rightarrow B$$

by Axiom (I). A split exact square means there exists an isomorphism  $\Psi$  such that  $\Psi \circ f = p_A$  and  $g \circ \Psi^{-1} = q_C$  in the ambient category. Since  $g \circ \Psi^{-1} = q_C \iff g = q_C \circ \Psi$ , this recovers the usual definition of a split exact sequence.

**Theorem 3.7.**  $K_1(\mathcal{C})$  is generated by Sherman loops  $G(\alpha, \beta, \theta)$ , where  $\mathcal{C}$  is a pCGW category.

*Proof.* The proof relies on two helper constructions: (i)  $\mathcal{C}^\oplus$ , a pCGW subcategory of  $\mathcal{C}$ ; and (ii)  $\widehat{G}\mathcal{C}$ , a simplicial subset of  $G\mathcal{C}$ . Informally, they record the splitting data of  $\mathcal{C}$  and  $G\mathcal{C}$  respectively, and so admit particularly nice presentations on the level of  $\pi_1$ . One can then relate this to  $\pi_1|G\mathcal{C}|$  to establish the theorem. Proceed in stages.

*Step 0: Setup.* This step introduces the two helper constructions, and records some preliminary observations.

**Construction 3.8.** Define  $\mathcal{C}^\oplus$  to be the CGW subcategory of  $\mathcal{C}$  whose exact squares are precisely the split exact squares of  $\mathcal{C}$ .

An easy check shows that  $\mathcal{C}^\oplus$  is in fact a pCGW subcategory, and that the simplicial map

$$\mathcal{S}F: \mathcal{S}\mathcal{C}^\oplus \hookrightarrow \mathcal{S}\mathcal{C}$$

induced by the inclusion CGW functor  $F: \mathcal{C}^\oplus \hookrightarrow \mathcal{C}$  is a 0-moral subset. In fact, since  $\mathcal{S}\mathcal{C}^\oplus[1] = \mathcal{S}\mathcal{C}[1]$ ,  $\mathcal{S}F$  has cofinal image. One can therefore apply Theorem 2.8 to construct a homotopy Cartesian diagram

$$\begin{array}{ccc} O|\mathcal{S}F & \xrightarrow{p} & \mathcal{S}\mathcal{C}^\oplus \\ \downarrow r & & \downarrow \mathcal{S}F \\ O|\mathcal{S}\mathcal{C} & \xrightarrow{q} & \mathcal{S}\mathcal{C} \end{array} \quad (13)$$

**Construction 3.9.** Following the convention of Construction 2.4, represent an  $n$ -simplex in  $G\mathcal{C}$  as

$$\left( \begin{array}{c} O \rightarrow L_0 \rightarrow \dots \rightarrow L_n \\ O \rightarrow M_0 \rightarrow \dots \rightarrow M_n \end{array} \right)$$

Define  $\widehat{G}\mathcal{C}$  to be the simplicial subset of  $G\mathcal{C}$  where the  $\mathcal{M}$ -morphisms in the top row are all split.

Notice this mirrors  $O|\mathcal{S}F$ , except the top row of an  $n$ -simplex in  $\widehat{G}\mathcal{C}$  is an  $(n+1)$ -simplex in  $\mathcal{S}\mathcal{C}$  whereas the top row in  $O|\mathcal{S}F$  is an  $n$ -simplex (whose  $\mathcal{M}$ -morphisms are also all split). Notice also that the simplices used to define  $G(\alpha, \beta, \theta)$  in fact belong to  $\widehat{G}\mathcal{C}$ , and so  $G(\alpha, \beta, \theta) \in \pi_1(|\widehat{G}\mathcal{C}|)$ .

*Step 1: Relating  $\widehat{G}\mathcal{C}$  and  $\mathcal{S}\mathcal{C}^\oplus$ .* We work towards extracting a homotopy fiber sequence from Diagram (13).

**Claim 3.10.**  $|\widehat{G}\mathcal{C}|$  is homotopy equivalent to  $|O|p|$  and  $|G\mathcal{C}|$ .

*Proof.* Take the geometric realisation of Diagram (13) to obtain a homotopy Cartesian square of spaces. As is well-known, the map  $r$  induces a homotopy equivalence on the homotopy fibers  $|O|p| \rightarrow |O|q|$ . Next, Observation 1.31 (ii) notes that  $G\mathcal{C} \simeq O|q|$ , essentially by unpacking definitions. One can similarly verify that  $\widehat{G}\mathcal{C} \simeq O|p|$ . Assembling all the equivalences, conclude that  $|G\mathcal{C}| \simeq |O|q| \simeq |O|p| \simeq |\widehat{G}\mathcal{C}|$ .  $\square$

In particular, consider the homotopy fiber sequence associated to  $|O|p| \rightarrow |O|\mathcal{S}F| \xrightarrow{p} |\mathcal{S}\mathcal{C}^\oplus|$ . Applying Claim 3.10 gives a basic but key observation.

**Observation 3.11.** The exact sequence

$$\pi_1(\Omega|\mathcal{S}\mathcal{C}^\oplus|) \longrightarrow \pi_1(|O|p|) \longrightarrow \pi_1(|O|\mathcal{S}F|) \xrightarrow{p_*} \pi_1(|\mathcal{S}\mathcal{C}^\oplus|) \quad (14)$$

may be reformulated as

$$\pi_1(\Omega|\mathcal{S}\mathcal{C}^\oplus|) \longrightarrow \pi_1(|\widehat{G}\mathcal{C}|) \xrightarrow{v} \pi_1(|O|\mathcal{S}F|) \xrightarrow{p_*} \pi_1(|\mathcal{S}\mathcal{C}^\oplus|), \quad (15)$$

for some map  $v$  induced by  $O|p| \simeq \widehat{G}\mathcal{C}$ . For details on the exact sequence, see [GG87, Cor. 1.8].

*Step 2: Generators of  $\pi_1(|O|\mathcal{S}F|)$ .* The argument is standard – no surprises. The 1-simplices of the form

$$\left( O \rightharpoonup \frac{O \rightharpoonup N}{O \rightharpoonup N} \right) \quad (16)$$

yield a maximal tree for the 1-skeleton of  $|O|\mathcal{S}F|$ , connecting the base-point of  $O|\mathcal{S}F|$  to any of its vertices. Thus by [Wei13, Lemma IV.3.4], the total set of 1-simplices of  $O|\mathcal{S}F|$  generate  $\pi_1(|O|\mathcal{S}F|)$ . We therefore represent the generators of  $\pi_1(O|\mathcal{S}F|)$  as

$$\left( O \rightharpoonup \frac{O \rightharpoonup A}{O \rightharpoonup A} \right) \left( O \rightharpoonup \frac{O \rightharpoonup C}{A \rightharpoonup B} \right) \left( O \rightharpoonup \frac{O \rightharpoonup B}{O \rightharpoonup B} \right)^{-1}. \quad (17)$$

For simplicity, generators are sometimes represented just by the middle term above, since the other two edges can be recovered from the maximal tree.

*Step 3: Constructing a Sherman Loop.* Recall Homotopy Exact Sequence (15). Given  $v(x) \in \pi_1(|O|\mathcal{S}F|)$  for any  $x \in \pi(|\widehat{G}\mathcal{C}|)$ , we can use its presentation from Step 2 to construct a Sherman Loop  $G(\alpha, \beta, \theta)$ .

- *The two  $\mathcal{M}$ -morphisms.* Since  $O|\mathcal{S}F|$  is an  $H$ -space,  $v(x)$  can be expressed as a difference of two 1-simplices, let us say

$$\left( O \rightharpoonup \frac{O \rightharpoonup C}{A \rightharpoonup B} \right) \quad \text{and} \quad \left( O \rightharpoonup \frac{O \rightharpoonup C'}{A' \rightharpoonup B'} \right). \quad (18)$$

This yields the  $\mathcal{M}$ -morphisms  $\alpha: A \rightharpoonup B$  and  $\beta: A' \rightharpoonup B'$ .

- *The isomorphism  $\theta$ .* Recall the map  $p: O|\mathcal{S}F| \rightarrow \mathcal{S}\mathcal{C}^\oplus$  acts by projection on the top row. Thus  $p_*v(x) \in \pi_1(|\mathcal{S}\mathcal{C}^\oplus|)$  corresponds to the difference between

$$(O \rightharpoonup A)(O \rightharpoonup C)(O \rightharpoonup B')^{-1} \quad \text{and} \quad (O \rightharpoonup A')(O \rightharpoonup C')(O \rightharpoonup B')^{-1}. \quad (19)$$

By Presentation Theorem 1.25, we know that  $\pi_1(|\mathcal{S}\mathcal{C}^\oplus|) = K_0(\mathcal{C}^\oplus)$ , so let us rewrite the above as

$$[A] + [C] - [B] \quad \text{and} \quad [A'] + [C'] - [B']. \quad (20)$$

We don't know if, e.g.  $[A] + [C] = [B]$  in  $K_0(\mathcal{C}^\oplus)$  since  $A \rightharpoonup B$  may not be split. Nonetheless, by exactness of Homotopy Sequence (15), we can deduce

$$[A] + [C] - [B] - ([A'] + [C'] - [B']) = 0, \quad (21)$$

and so

$$[A] + [C] + [B'] = [A'] + [C'] + [B]. \quad (22)$$



Since  $[M] = [N]$  in  $K_0(\mathcal{C}^\oplus)$  iff  $M$  and  $N$  are stably isomorphic, there exists some  $Z \in \mathcal{C}$  such that

$$A \oplus C \oplus B' \oplus Z \cong A' \oplus C' \oplus B \oplus Z. \quad (23)$$

Now notice that

$$\left( O \rightharpoonup \frac{O \rightharpoonup Z}{O \rightharpoonup Z} \right) \left( O \rightharpoonup \frac{O \rightharpoonup O}{Z \rightharpoonup Z} \right) \left( O \rightharpoonup \frac{O \rightharpoonup Z}{O \rightharpoonup Z} \right)^{-1} \quad (24)$$

is null-homotopic, Hence, we can always modify the representation of  $v(x)$  by adding Loop (24) to Equation (19) without changing the homotopy class. As such, without loss of generality, assume  $Z = O$ , giving the isomorphism

$$\theta: A \oplus C \oplus B' \xrightarrow{\cong} A' \oplus C' \oplus B. \quad (25)$$

**Step 4: A Reduction.** The following claim tells us that any  $x \in \pi_1(|\widehat{G\mathcal{C}}|)$  looks like a Sherman loop when viewed in  $\pi_1(O|\mathcal{S}F)$ .

**Claim 3.12.** *Given any  $x \in \pi_1(|\widehat{G\mathcal{C}}|)$ , there exists a Sherman loop  $G(\alpha, \beta, \theta)$  such that  $v(x)$  and  $v(G(\alpha, \beta, \theta))$  are homotopic.*

*Proof.* Given  $x \in \pi_1(|\widehat{G\mathcal{C}}|)$ , construct a Sherman Loop  $G(\alpha, \beta, \theta)$  as in Step 3. In particular, notice or recall that  $G(\alpha, \beta, \theta) \in \pi_1(|\widehat{G\mathcal{C}}|)$ , and so  $v(G(\alpha, \beta, \theta))$  is well-defined.

Consider the following diagram in  $\pi_1(O|\mathcal{S}F)$

$$\begin{array}{ccccccc} \left( O \rightharpoonup \frac{O}{A} \right) & \xrightarrow{\text{blue}} & \left( O \rightharpoonup \frac{O}{B \oplus B'} \right) & \xrightarrow{\text{blue}} & \left( O \rightharpoonup \frac{O}{B' \oplus B} \right) & \xleftarrow{\text{blue}} & \left( O \rightharpoonup \frac{O}{A'} \right) \\ \downarrow \text{red} & (1) \nearrow & \uparrow (2) & (3) \nearrow & \uparrow (4) & (5) \nwarrow & \downarrow \text{red} \\ \left( O \rightharpoonup \frac{O}{B} \right) & \xleftarrow{\text{red}} & \left( O \rightharpoonup \frac{O}{O} \right) & \xrightarrow{\text{red}} & \left( O \rightharpoonup \frac{O}{B'} \right) & & \end{array} \quad (26)$$

The edges of the diagram are obvious, and record various paths between vertices

$$\left( O \rightharpoonup \frac{O}{A} \right) \dashrightarrow \left( O \rightharpoonup \frac{O}{A'} \right), \quad (27)$$

e.g. by composing along the blue edges, by composing along the red edges, etc.

A couple of key observations. First, notice that all triangles in Diagram (26) define boundaries of 2-simplices, listed below.

$$\begin{aligned} (1) \left( O \rightharpoonup \frac{O \rightharpoonup C \rightharpoonup C \oplus B'}{A \rightharpoonup B \rightharpoonup B \oplus B'} \right), \quad (2) \left( O \rightharpoonup \frac{O \rightharpoonup B \rightharpoonup B \oplus B'}{O \rightharpoonup B \rightharpoonup B \oplus B'} \right), \quad (3) \left( O \rightharpoonup \frac{O \rightharpoonup B \oplus B' \rightharpoonup B' \oplus B}{O \rightharpoonup B \oplus B' \rightharpoonup B' \oplus B} \right) \\ (4) \left( O \rightharpoonup \frac{O \rightharpoonup B' \rightharpoonup B' \oplus B}{O \rightharpoonup B' \rightharpoonup B' \oplus B} \right), \quad (5) \left( O \rightharpoonup \frac{O \rightharpoonup C' \rightharpoonup C' \oplus B}{A' \rightharpoonup B' \rightharpoonup B' \oplus B} \right). \end{aligned}$$

Hence, the blue and red paths in Diagram (26) between the two vertices (27) are homotopic.

Second,  $v(G(\alpha, \beta, \theta))$  corresponds to the loop

$$\left( O \rightharpoonup \frac{O \rightharpoonup A}{O \rightharpoonup A} \right) \left( O \rightharpoonup \frac{O \rightharpoonup C \oplus B'}{A \rightharpoonup B \oplus B'} \right) \left( O \rightharpoonup \frac{O \rightharpoonup O}{B \oplus B' \rightharpoonup B' \oplus B} \right) \left( O \rightharpoonup \frac{O \rightharpoonup C \oplus B'}{A' \rightharpoonup B' \oplus B} \right)^{-1} \left( O \rightharpoonup \frac{O \rightharpoonup A'}{O \rightharpoonup A'} \right)^{-1}$$

whereas  $v(x)$ , the difference between two generators, corresponds to the loop

$$\left( O \rightharpoonup \frac{O \rightharpoonup A}{O \rightharpoonup A} \right) \left( O \rightharpoonup \frac{O \rightharpoonup C}{A \rightharpoonup B} \right) \left( O \rightharpoonup \frac{O \rightharpoonup B}{O \rightharpoonup B} \right)^{-1} \left( O \rightharpoonup \frac{O \rightharpoonup B'}{O \rightharpoonup B'} \right) \left( O \rightharpoonup \frac{O \rightharpoonup C'}{A' \rightharpoonup B'} \right)^{-1} \left( O \rightharpoonup \frac{O \rightharpoonup A'}{O \rightharpoonup A'} \right)^{-1}.$$

In particular, the loop  $v(G(\alpha, \beta, \theta))$  corresponds to composing along the blue edges in Diagram (26) whereas  $v(x)$  corresponds to composing along the red edges – which are homotopy equivalent by our previous observation. Conclude that  $v(G(\alpha, \beta, \theta))$  and  $v(x)$  have the same homotopy class.  $\square$

*Step 5: Finish.* Let  $x$  be an element of  $K_1(\mathcal{C})$ . By Theorem 2.11 and Claim 3.10, we know

$$\Omega|\mathcal{SC}| \simeq |G\mathcal{C}| \simeq |\widehat{G}\mathcal{C}|,$$

and so regard  $x \in \pi_1(|\widehat{G}\mathcal{C}|)$ . In particular,  $x$  is an element in Homotopy Sequence (15). By Claim 3.12, there exists a Sherman loop  $G(\alpha, \beta, \theta)$  such that  $v(G(\alpha, \beta, \theta)) = v(x)$  in  $\pi_1(|O|\mathcal{SF}|)$ . In other words, the difference  $x - G(\alpha, \beta, \theta)$  vanishes in  $\pi_1(|O|\mathcal{SF}|)$ , and thus lies in the image of

$$K_1(\mathcal{C}^\oplus) = \pi_1(\Omega|\mathcal{SC}^\oplus|) \longrightarrow \pi_1(|\widehat{G}\mathcal{C}|) = K_1(\mathcal{C}).$$

To finish, we quote a couple of technical facts about Sherman Loops whose proof we defer to Appendix B.1. By Lemmas B.2 and B.3,  $K_1(\mathcal{C}^\oplus)$  is generated by Sherman Loops, and thus so is its image in  $K_1(\mathcal{C})$ . By Lemma B.1, the sum of two Sherman Loops is still a Sherman Loop. As such, since

$$x = x - G(\alpha, \beta, \theta) + G(\alpha, \beta, \theta),$$

conclude that  $x$  is homotopic to a Sherman Loop in  $K_1(\mathcal{C})$ .  $\square$

**Discussion 3.13.** Our proof broadly follows Sherman's approach in [She98], except we give a more direct argument for Claim 3.10. Sherman's original argument proceeds by defining a pair of exact functors

$$\Delta: \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$$

$$\Delta': \mathcal{C}^\oplus \rightarrow \mathcal{C}^\oplus \times \mathcal{C},$$

where  $\Delta$  is the diagonal, and  $\Delta'$  is the diagonal composed with the obvious inclusion map. Sherman then asserts as obvious that the cofiber of  $\mathcal{S}\Delta$  is homotopy equivalent to  $|\mathcal{SC}|$ , presumably by analogy with short exact sequences of abelian groups. However, there are subtleties. It is not generally true that  $\text{cofib}(\Delta) \simeq X$  for a diagonal map of spaces – e.g. consider  $\Delta: S^1 \rightarrow S^1 \times S^1$ . One potential remedy would be to first prove  $\text{cofib}(\mathcal{S}\Delta)$  and  $\mathcal{SC}$  are equivalent as  $\mathbb{E}_\infty$ -spaces before recovering the desired result, but this is rather involved. A cleaner argument is possible by unpacking definitions, as in Claim 3.10.

**3.3. Double Exact Squares.** Given an exact category, Nenashev [Nen98b, Nen96] shows that its  $K_1$  is generated by so-called *double short exact sequences* – sharpening Sherman's original result. We adapt his analysis to the pCGW setting.

**Definition 3.14** (Double Exact Squares). A *double exact square* is a pair of distinguished squares with identical nodes

$$l := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & C \\ \downarrow & \square & \downarrow g_1 \\ A & \xrightarrow{f_1} & B \end{array} , \begin{array}{ccc} O & \xrightarrow{\quad} & C \\ \downarrow & \square & \downarrow g_2 \\ A & \xrightarrow{f_2} & B \end{array} \right).$$

Since this defines an edge  $(A, A) \rightarrow (B, B)$ , any double exact square defines a loop

$$\begin{array}{ccc} (A, A) & \xrightarrow{l} & (B, B) \\ & \nwarrow e(A) \quad \nearrow e(B) & \\ & (O, O) & \end{array} \quad (28)$$

where  $e(A)$  and  $e(B)$  are the obvious edges from the base-point, e.g.

$$e(A) := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & A \\ \downarrow & \square & \downarrow 1_A \\ O & \xrightarrow{\quad} & A \end{array} , \begin{array}{ccc} O & \xrightarrow{\quad} & A \\ \downarrow & \square & \downarrow 1_A \\ O & \xrightarrow{\quad} & A \end{array} \right).$$

We call this the *canonical loop of  $l$* , and denote it as  $\mu(l)$ . We denote  $\langle l \rangle$  to be its homotopy class in  $K_1(\mathcal{C})$ .

As the following example illustrates, double exact squares generalise automorphisms in  $\mathcal{C}$ .

**Example 3.15** (Automorphisms). If  $(A, \alpha) \in \text{Aut}(\mathcal{C})$  is an automorphism, we write

$$l(\alpha) = \left( \begin{array}{ccc} O & \xrightarrow{\quad} & O \\ \downarrow & \square & \downarrow \\ A & \xrightarrow{\alpha} & A \end{array} , \begin{array}{ccc} O & \xrightarrow{\quad} & O \\ \downarrow & \square & \downarrow \\ A & \xrightarrow{1_A} & A \end{array} \right).$$

To prove that  $K_1(\mathcal{C})$  is generated by double exact squares, it suffices to show that any Sherman Loop is homotopic to the canonical loop of a double exact square; the rest follows from Theorem 3.7.

**Theorem 3.16.** *Let  $\mathcal{C}$  be a pCGW category. Given any  $x \in K_1(\mathcal{C})$ , there exists a double exact square  $l$  such that  $x = \mu(l)$ .*

*Proof.* By Theorem 3.7, we may assume  $x$  is a Sherman Loop  $G(\alpha, \beta, \theta)$  arising from a pair of exact squares

$$\begin{array}{ccc} O & \xrightarrow{\quad} & C \\ \downarrow & \square & \downarrow \delta \\ A & \xrightarrow{\alpha} & B \end{array} , \begin{array}{ccc} O & \xrightarrow{\quad} & C' \\ \downarrow & \square & \downarrow \delta' \\ A' & \xrightarrow{\alpha'} & B' \end{array} \quad (29)$$

and an isomorphism  $\theta: A \oplus C \oplus B' \xrightarrow{\cong} A' \oplus C' \oplus B$ . To turn this into a double exact square, first construct the following distinguished squares

$$s_0 := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & C \oplus C' \\ \downarrow & \square & \downarrow g_0 \\ A \oplus A' & \xrightarrow{f_0} & A \oplus C \oplus B' \end{array} \right) , \quad s_1 := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & C \oplus C' \\ \downarrow & \square & \downarrow g_1 \\ A \oplus A' & \xrightarrow{f_1} & A' \oplus C' \oplus B \end{array} \right) \quad (30)$$

$$f_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \alpha' \end{pmatrix} , \quad f_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ \alpha & 0 \end{pmatrix} , \quad g_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \delta' \end{pmatrix} , \quad g_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ \delta & 0 \end{pmatrix} . \quad (31)$$

Then apply the isomorphism  $\theta$  to define

$$l(x) := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & C \oplus C' \\ \downarrow & \square & \downarrow \varphi(\theta) \circ g_0 \\ A \oplus A' & \xrightarrow{\theta \circ f_0} & A' \oplus C' \oplus B \end{array} , \begin{array}{ccc} O & \xrightarrow{\quad} & C \oplus C' \\ \downarrow & \square & \downarrow g_1 \\ A \oplus A' & \xrightarrow{f_1} & A' \oplus C' \oplus B \end{array} \right) . \quad (32)$$

We leave the reader to check that  $s_0$  and  $s_1$  are in fact distinguished squares; this follows from Lemmas A.4 (i) and A.3.

**Convention 3.17** (Coordinate-wise Definition of Maps). Consider two exact squares

$$h_0 := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & M'' \\ \downarrow & \square & \downarrow m_1 \\ M' & \xrightarrow{m_0} & M \end{array} \right) \quad h_1 := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & N'' \\ \downarrow & \square & \downarrow n_1 \\ N' & \xrightarrow{n_0} & N \end{array} \right).$$

- If  $M'' = N''$ , we usually denote the corresponding 1-simplex as  $(h_0, h_1): (M', N') \rightarrow (M, N)$ .
- If  $M'' = O = N''$ , we denote the corresponding 1-simplex as  $(m_0, n_0): (M', N') \rightarrow (M, N)$ .
- If we wish to take their direct sum, this will be denoted

$$h_0 \oplus h_1 := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & M'' \oplus N'' \\ \downarrow & \square & \downarrow m_1 \oplus n_1 \\ M' \oplus N' & \xrightarrow{m_0 \oplus n_0} & M \oplus N \end{array} \right).$$

- We can also “add” an object  $C$  to  $h_0$  in the obvious way (see Lemma A.3) as follows:

$$h_0 \oplus C := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & M'' \oplus C \\ \downarrow & \square & \downarrow m_1 \oplus 1 \\ M' & \xrightarrow{m_0 \oplus C} & M \oplus C \end{array} \right).$$

For more involved definitions, we use the matrix notation of Equation (31) to indicate, e.g. the morphism  $f_1: A \oplus A' \rightarrow A' \oplus C' \oplus B$  is determined by  $\alpha: A \rightarrow B$  and  $1: A' \rightarrow A'$ .

Let us resume our proof. Starting with a Sherman Loop  $G(\alpha, \beta, \theta)$ , we defined a double exact square  $l(x)$  with corresponding loop  $\mu(l(x))$ . It thus remains to show that  $\mu(l(x))$  is homotopic to  $G(\alpha, \beta, \theta)$  in  $K_1(\mathcal{C})$ . In fact, since  $K_1(\mathcal{C})$  is abelian, it suffices to show that they are freely homotopic. This is accomplished by the following series of lemmas.

**Convention 3.18.** To ease notation, denote  $P := A \oplus C \oplus B'$  and  $Q := A' \oplus C' \oplus B$ .

**Lemma 3.19.**  $\mu(l(x))$  is freely homotopic to the loop

$$\begin{array}{ccc} (P, Q) & \xrightarrow{(\theta, 1)} & (Q, Q) \\ & \swarrow (s_0, s_1) \quad \searrow (s_1, s_1) & \\ & (A \oplus A', A \oplus A') & \end{array} \quad (33)$$

*Proof of Lemma.* Consider the diagram

$$\begin{array}{ccccc} & & (P, Q) & & \\ & \nearrow (s_0, s_1) & & \searrow (\theta, 1) & \\ & (A \oplus A', A \oplus A') & & (Q, Q) & \\ & \nwarrow e(A \oplus A') & \xrightarrow{l(x)} & \nearrow e(Q) & \\ & (O, O) & & & \end{array} \quad (34)$$

The statement follows from observing that Triangles (1) and (2) form 2-simplices. Triangle (2) is obvious. Triangle (1) is given by

$$\begin{array}{ccccc} A \oplus A' & \xrightarrow{f_0} & P & \xrightarrow{\theta} & Q \\ \uparrow g_0 & & \square & & \uparrow \varphi(\theta) \circ g_0 \\ C \oplus C' & \xrightarrow{1} & C \oplus C' & & \\ & & \uparrow & & \\ & & O & & \end{array} \quad \begin{array}{ccccc} A \oplus A' & \xrightarrow{f_1} & Q & \xrightarrow{1} & Q \\ \uparrow g_1 & & \square & & \uparrow g_1 \\ C \oplus C' & \xrightarrow{1} & C \oplus C' & & \\ & & \uparrow & & \\ & & O & & \end{array}$$

□

**Lemma 3.20.** The loop  $G(\alpha, \beta, \theta)$  is freely homotopic to the loop

$$\begin{array}{ccc} (P, B \oplus B') & \xrightarrow{(\theta, 1)} & (Q, B \oplus B') \\ & \swarrow (s_0, s) \quad \searrow (s_1, s) & \\ & (A \oplus A', A \oplus A') & \end{array} \quad (35)$$

where  $s$  is the distinguished square

$$s := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & C \oplus C' \\ \downarrow & \square & \downarrow \delta \oplus \delta' \\ A \oplus A' & \xrightarrow{\alpha \oplus \alpha'} & B \oplus B' \end{array} \right) . \quad (36)$$

*Proof of Lemma.* Consider the diagram

$$\begin{array}{ccccc} (P, B \oplus B') & \xrightarrow{(\theta, 1)} & (Q, B \oplus B') & & \\ \uparrow (a_0, b_0) & \nwarrow (s_0, s) & \nearrow (s_1, s) & \uparrow (a_1, b_1) & \\ (A, A) & \xrightarrow{\quad} & (A \oplus A', A \oplus A') & \xleftarrow{\quad} & (A', A') \\ \nwarrow e(A) & \uparrow e(A \oplus A') & \nearrow e(A') & & \\ (O, O) & & & & \end{array} \quad (\star) \quad (37)$$

where

$$(a_0, b_0) := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & C \oplus B' \\ \downarrow & \square & \downarrow \\ A & \xrightarrow{1 \oplus C \oplus B'} & A \oplus C \oplus B' \end{array} , \begin{array}{ccc} O & \xrightarrow{\quad} & C \oplus B' \\ \downarrow & \square & \downarrow \\ A & \xrightarrow{\alpha \oplus B'} & B \oplus B' \end{array} \right)$$

$$(a_1, b_1) := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & C' \oplus B \\ \downarrow & \square & \downarrow \\ A' & \xrightarrow{1 \oplus C' \oplus B} & A' \oplus C' \oplus B \end{array} , \begin{array}{ccc} O & \xrightarrow{\quad} & C' \oplus B \\ \downarrow & \square & \downarrow \\ A' & \xrightarrow{B \oplus \alpha'} & B \oplus B' \end{array} \right)$$

Notice: the outer loop of Diagram (37) is equivalent<sup>9</sup> to the Sherman Loop  $G(\alpha, \beta, \theta)$  while the triangle  $(\star)$  is Loop (35). It remains to check that all other triangles in Diagram (37) bound 2-simplices. The bottom two triangles are immediate. The top left triangle bounds

$$\begin{array}{ccc} A \xrightarrow{\quad} A \oplus A' \xrightarrow{f_0} P & & A \xrightarrow{\quad} A \oplus A' \xrightarrow{\alpha \oplus \alpha'} B \oplus B' \\ \uparrow \quad \square \quad \uparrow A \oplus 1 & & \uparrow \quad \square \quad \uparrow \delta \oplus 1 \\ A' \xrightarrow{C \oplus \alpha'} C \oplus B' & & A' \xrightarrow{C \oplus \alpha'} C \oplus B' \\ \uparrow \quad \square \quad \uparrow 1 \oplus \delta' & & \uparrow \quad \square \quad \uparrow 1 \oplus \delta' \\ O \xrightarrow{\quad} C \oplus C' & & O \xrightarrow{\quad} C \oplus C' \end{array}$$

That this defines a 2-simplex essentially follows from Lemma 1.8 (“Quotients Respect Filtrations”); details are left to the reader. The top right triangle can be handled similarly.  $\square$

<sup>9</sup>A small difference: the top edge here is  $(\theta, 1): (P, B \oplus B') \rightarrow (Q, B \oplus B')$ , as opposed to  $(\theta, \tau): (P, B \oplus B') \rightarrow (Q, B' \oplus B)$ , but the twist is already encoded in  $(a_1, b_1): (A, A') \rightarrow (Q, B \oplus B')$ .

**Lemma 3.21.** Denote  $V := (B \oplus B') \star_{(A \oplus A')} Q$  and  $W := C \oplus C' \oplus C \oplus C'$ . Then, there exists distinguished squares of the form

$$t := \left( \begin{array}{ccc} C \oplus C' & \xrightarrow{\quad} & W \\ g_1 \downarrow & \square & \downarrow j_t \\ Q & \xrightarrow{h_t} & V \end{array} \right) \quad t' := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & C \oplus C' \\ \downarrow & \square & \downarrow k_t \\ Q & \xrightarrow{h_t} & V \end{array} \right) \quad (38)$$

$$u := \left( \begin{array}{ccc} C' \oplus C' & \xrightarrow{\quad} & W \\ \delta \oplus \delta' \downarrow & \square & \downarrow j_u \\ B \oplus B' & \xrightarrow{h_u} & V \end{array} \right) \quad u' := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & C \oplus C' \\ \downarrow & \square & \downarrow k_u \\ B \oplus B' & \xrightarrow{h_u} & V \end{array} \right). \quad (39)$$

*Proof.* Applying Axiom (DS), Definition 1.18 to the diagram

$$\begin{array}{ccccc} C \oplus C' & \xleftarrow{\quad} & O & \xrightarrow{\quad} & C \oplus C' \\ \downarrow \delta \oplus \delta' & \square & \downarrow & \square & \downarrow g_1 \\ B \oplus B' & \xleftarrow{\alpha \oplus \alpha'} & A \oplus A' & \xrightarrow{f_1} & Q \end{array},$$

conclude that there exists a distinguished square of the form  $t$ . Since  $t$  is distinguished, deduce that  $\frac{V}{Q} \cong C \oplus C'$  (Lemma 1.8 (i)), and thus there exists a distinguished square of the form  $t'$ . The same argument shows there exists distinguished squares of the form  $u$  and  $u'$ .  $\square$

**Lemma 3.22.** Loops (33) and (35) are homotopic.

*Proof of Lemma.* Recall  $V := (B \oplus B') \star_{(A \oplus A')} Q$  and the definition of  $t'$  from Lemma 3.21. In addition, define  $\tilde{s}$  as the horizontal composition of  $s_1$  and  $t$

$$\tilde{s} := \left( \begin{array}{ccccc} O & \xrightarrow{\quad} & C \oplus C' & \xrightarrow{\quad} & W \\ \downarrow & \square & \downarrow g_1 & \square & \downarrow j_t \\ A \oplus A' & \xrightarrow{f_1} & Q & \xrightarrow{h_t} & V \end{array} \right).$$

Notice: the  $\mathcal{M}$ -morphism  $h_t \circ f_1$  is equal to

$$h_u \circ (\alpha \oplus \alpha'): A \oplus A' \rightarrow B \oplus B \rightarrow V$$

by construction of  $V$ , and so  $\tilde{s}$  is also equivalent to the horizontal composition of  $s$  and  $u$ . We can therefore construct a loop  $L$  that Loops (33) and (35) are both homotopic to. Consider the diagram

$$\begin{array}{ccc} (P \oplus C \oplus C', V) & \xrightarrow{(\theta \oplus 1, 1)} & (Q \oplus C \oplus C', V) \\ \uparrow (1 \oplus C \oplus C', t') & \nearrow (\theta \oplus C \oplus C', t') & \uparrow (1 \oplus C \oplus C', t') \\ (P, Q) & \xrightarrow{(3)} & (Q, Q) \\ \xrightarrow{(4)} & \xrightarrow{(\theta, 1)} & \\ \nwarrow (s_0, s_1) & & \nearrow (s_1, s_1) \\ (A \oplus A', A \oplus A') & & \\ \nwarrow (s_0 \oplus C \oplus C', \tilde{s}) & & \nearrow (s_1 \oplus C \oplus C', \tilde{s}) \end{array} \quad (40)$$

The purple edges are Loop (33), the blue edges form an outer loop, which we denote  $L$ . To show that the two loops are homotopic, it suffices to check that Triangles (1) - (4) are boundaries of 2-simplices – this is worked out explicitly in Claim B.4. Analogously, one can construct the diagram

$$\begin{array}{ccc}
 (P \oplus C \oplus C', V) & \xrightarrow{(\theta \oplus 1, 1)} & (Q \oplus C \oplus C', V) \\
 \uparrow (1 \oplus C \oplus C', u') & \nearrow (\theta \oplus C \oplus C', u') & \uparrow (1 \oplus C \oplus C', u') \\
 (P, B \oplus B') & \xrightarrow{(\theta, 1)} & (Q, B \oplus B') \\
 \nwarrow (s_0, s) & & \nearrow (s_1, s) \\
 (A \oplus A', A \oplus A') & & 
 \end{array}
 \quad (41)$$

where the red edges are Loop (35) and the blue edges are loop  $L$ . A similar check shows Triangles (1') - (4') are also boundaries of 2-simplices (details in Claim B.5). Conclude that Loop (33) and Loop (35) are both homotopic to loop  $L$ , and thus homotopic to each other as well.  $\square$

*Finish.* Given any Sherman Loop  $G(\alpha, \beta, \theta) \in K_1(\mathcal{C})$ , we can construct another loop  $\mu(l(x))$  where  $l(x)$  is a double exact square. Recall that:

- Lemma 3.19 shows  $\mu(l(x))$  is freely homotopic to Loop (33).
- Lemma 3.20 shows  $G(\alpha, \beta, \theta)$  is freely homotopic to Loop (35).
- Lemma 3.22 shows Loops (33) and (35) are homotopic.

Since  $K_1(\mathcal{C})$  is an abelian group, deduce that  $G(\alpha, \beta, \theta)$  is homotopic to  $\mu(l(x))$ . Since  $K_1(\mathcal{C})$  is generated by Sherman Loops (Theorem 3.7), conclude that  $K_1(\mathcal{C})$  is generated by double exact squares.  $\square$

**Discussion 3.23.** Although we were guided by Nenashev's proof in [Nen96], a direct translation to our setting does not work. The key issue is the isomorphism

$$B \oplus \frac{B}{A} \cong B \star_A B,$$

which holds in exact categories and is central to [Nen96, Lemma 2.6], but (as Inna Zakharevich pointed out to us) fails in  $\mathcal{V}\text{ar}_k$  – see Figure 2. To get around this, we construct the homotopies explicitly, verifying by hand that the chosen diagrams define valid 2-simplices. The methodological upshot: while pushouts in exact categories are arguably better behaved, restricted pushouts still retain enough good properties to make the analysis go through (cf. Lemmas 3.21 and A.3).



FIGURE 2.  $\mathbb{A}^1 \amalg (\mathbb{A}^1 \setminus \{*\})$  is not isomorphic to  $\mathbb{A}^1 \star_{\{*\}} \mathbb{A}^1$ .

#### 4. RELATIONS OF $K_1(\mathcal{C})$

Having characterised the generators of  $K_1(\mathcal{C})$  for pCGW categories, we now work to determine its relations. We first give a baseline characterisation in Proposition 4.1. We then sharpen our understanding by comparing this to other descriptions of  $K_1$  by Nenashev [Nen98b, Nen98a] (for exact categories) and Zakharevich [Zak17c] (for Assemblers). A guiding observation is Warning 4.3, which highlights a technical subtlety regarding the composition of 1-simplices in  $K_1$ . Interestingly, this brings into focus an apparent discrepancy between our account and Zakharevich's regarding the correct relations of  $K_1$ .

**4.1. A Baseline Argument.** Denote the base-point component of  $G\mathcal{C}$  as  $G\mathcal{C}^o$ . Since  $K_1(\mathcal{C}) = \pi_1(G\mathcal{C}^o)$ , one can therefore apply the standard description of the fundamental group of a connected simplicial space to get the following presentation.

**Proposition 4.1.**  $K_1(\mathcal{C})$  is generated by isomorphism classes of double exact squares  $\langle f \rangle$  modulo the following relations:

(B1) Given any  $A \in \mathcal{C}$ , the standard edge  $e(A): (O, O) \rightarrow (A, A)$  of  $G\mathcal{C}$  vanishes. That is,

$$\left\langle \left( \begin{array}{ccc} O & \xrightarrow{\quad} & A \\ \downarrow & \square & \downarrow 1 \\ O & \xrightarrow{\quad} & A \end{array} \right) \right\rangle = 0.$$

(B2) Given any  $A \in \mathcal{C}$ , the degenerate 1-simplex  $\text{id}_A: (A, A) \rightarrow (A, A)$  of  $G\mathcal{C}$  vanishes. That is,

$$\left\langle \left( \begin{array}{ccc} O & \xrightarrow{\quad} & O \\ \downarrow & \square & \downarrow \\ A & \xrightarrow{1} & A \end{array} \right) \right\rangle = 0.$$

(B3) Given double exact squares of the form

$$l_A := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{B}{A} \\ \downarrow & \square & \downarrow g_A \\ A & \xrightarrow{f_A} & B \end{array} \right), \quad l_B := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{C}{B} \\ \downarrow & \square & \downarrow g_B \\ B & \xrightarrow{f_B} & C \end{array} \right) \quad (42)$$

$$l_C := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{C}{A} \\ \downarrow & \square & \downarrow g_C \\ A & \xrightarrow{f_C} & C \end{array} \right) \quad (43)$$

that assemble into a 2-simplex in  $G\mathcal{C}$

$$\begin{array}{ccc} A & \xrightarrow{f_A} & B & \xrightarrow{f_B} & C \\ & \uparrow g_A & \square & \uparrow g_C \\ & \frac{B}{A} & \xrightarrow{h_1} & \frac{C}{A} \\ & \uparrow & \square & \uparrow h_2 \\ O & \xrightarrow{\quad} & \frac{C}{B} \end{array} \quad \begin{array}{ccc} A & \xrightarrow{f'_A} & B & \xrightarrow{f'_B} & C \\ & \uparrow g'_A & \square & \uparrow g'_C \\ & \frac{B}{A} & \xrightarrow{h_1} & \frac{C}{A} \\ & \uparrow & \square & \uparrow h_2 \\ O & \xrightarrow{\quad} & \frac{C}{B} \end{array} \quad (44)$$

we have that

$$\langle l_A \rangle + \langle l_B \rangle = \langle l_C \rangle.$$

*Proof.* Let  $X$  be a connected simplicial set with  $\Gamma$  as the maximal tree for its 1-skeleton. It is well-known that  $\pi_1|X|$  has the following presentation

$$\pi_1|X| := \pi_0(X[1]) \left/ \begin{array}{l} \langle t \rangle = 0 \text{ if } t \in \Gamma, \text{ and } \langle \text{id}_A \rangle = 0 \text{ for any degenerate 1-simplex} \\ d_1(x) = d_2(x) + d_0(x), \quad \forall x \in \pi_0(X[2]) \end{array} \right. .$$



Let  $X$  be the (connected) simplicial subset of  $G\mathcal{C}^o$  featuring only double exact squares. By Theorem 3.16, any loop in  $K_1(\mathcal{C})$  is homotopic to the canonical loop of a double exact square, and so  $K_1(\mathcal{C}) = \pi_1|G\mathcal{C}^o| \cong \pi_1|X|$ . Notice: the set  $\pi_0(X[1])$  denotes the isomorphism classes of double exact squares, and any  $x \in \pi_0(X[2])$  can be represented as Diagram 44, where  $d_0(x) = l_B$ ,  $d_2(x) = l_A$  and  $d_1 = l_C$ . Next, define  $\Gamma$  as the set of 1-simplices  $\{(O, O) \rightarrow (A, A)\}$ ; this is clearly a maximal tree spanning the 1-skeleton of  $X$ . Thus, apply the above presentation of  $\pi_1$  to obtain the claimed relations in the proposition.  $\square$

**Remark 4.2.** Was appealing to Theorem 3.16 necessary? In principle, one could apply the above argument to  $G\mathcal{C}^o$  directly, but there are complications. Notice the obvious choice set of 1-simplices

$$\{(O, O) \rightarrow (A, A')\}$$

fails to form a maximal tree spanning  $G\mathcal{C}^o$  in general (cf. Convention 3.2). Of course, this can always be upgraded to a maximal tree  $T$  by Zorn's Lemma, but we would no longer have an explicit description of  $T$  – and thus no explicit description of the relations for  $\pi_1|G\mathcal{C}^o|$  either.

**Warning 4.3** (Composition of 1-simplices). Proposition 4.1 does *not* assert that any three double exact squares  $l_A, l_B, l_C$  satisfy

$$\langle l_A \rangle + \langle l_B \rangle = \langle l_C \rangle, \quad \text{whenever } f_A \circ f_B = f_C \text{ and } g_A \circ g_B = g_C.$$

Why? While composition in  $\mathcal{SC}$  yields a pair of flag diagrams, these define a 2-simplex in  $G\mathcal{C}$  only if the quotient index triangles in Equation (44)

$$\left( \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{C}{B} \\ \downarrow & \square & \downarrow h_2 \\ \frac{B}{A} & \xrightarrow{h_1} & \frac{C}{A} \end{array} \quad , \quad \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{C}{B} \\ \downarrow & \square & \downarrow h_2 \\ \frac{B}{A} & \xrightarrow{h_1} & \frac{C}{A} \end{array} \right)$$

are identical in both diagrams.

Keeping Warning 4.3 in mind will help us appreciate the work done in the subsequent sections. On one level, we extend Nenashev's work on exact categories [Nen98b, Nen98a] to our setting, providing yet another characterisation of  $K_1(\mathcal{C})$ . On another level, Nenashev's presentation clarifies *how* composition of 1-simplices split in  $K_1$ , illuminating the difference between our approach and Zakharevich's.

**4.2. Admissible Triples.** A triangle contour  $\mathcal{T}$  in  $G\mathcal{C}$

$$\begin{array}{ccc} & (P_1, P'_1) & \\ e_0 \nearrow & & \searrow e_1 \\ (P_0, P'_0) & \xrightarrow{e_2} & (P_2, P'_2) \end{array} \quad (45)$$

is given by three pairs of distinguished squares of the form

$$\begin{aligned} e_0 &:= \left( \begin{array}{ccc} O & \xrightarrow{\quad} & P_{1/0} \\ \downarrow & \square & \downarrow \alpha_{1/0,1} \\ P_0 & \xrightarrow{\alpha_{0,1}} & P_1 \end{array} \quad , \quad \begin{array}{ccc} O & \xrightarrow{\quad} & P_{1/0} \\ \downarrow & \square & \downarrow \alpha'_{1/0,1} \\ P'_0 & \xrightarrow{\alpha'_{0,1}} & P'_1 \end{array} \right) \quad e_1 := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & P_{2/1} \\ \downarrow & \square & \downarrow \alpha_{2/1,2} \\ P_1 & \xrightarrow{\alpha_{1,2}} & P_2 \end{array} \quad , \quad \begin{array}{ccc} O & \xrightarrow{\quad} & P_{2/1} \\ \downarrow & \square & \downarrow \alpha'_{2/1,2} \\ P'_1 & \xrightarrow{\alpha'_{1,2}} & P'_2 \end{array} \right) \\ e_2 &:= \left( \begin{array}{ccc} O & \xrightarrow{\quad} & P_{2/0} \\ \downarrow & \square & \downarrow \alpha_{2/0,2} \\ P_0 & \xrightarrow{\alpha_{0,2}} & P_2 \end{array} \quad , \quad \begin{array}{ccc} O & \xrightarrow{\quad} & P_{2/0} \\ \downarrow & \square & \downarrow \alpha'_{2/0,2} \\ P'_0 & \xrightarrow{\alpha'_{0,2}} & P'_2 \end{array} \right). \end{aligned}$$

**Definition 4.4** (Admissible Triple). We call a triple  $\tau = (e_0, e_1, e_2)$  of the above form *admissible* if

$$\alpha_{1,2} \circ \alpha_{0,1} = \alpha_{0,2} \quad \text{and} \quad \alpha'_{1,2} \circ \alpha'_{0,1} = \alpha'_{0,2}.$$

In which case, one can apply Lemma 1.8 and complete  $\tau$  to the following pair of diagrams:

$$\begin{array}{ccc}
P_0 & \xrightarrow{\alpha_{0,1}} & P_1 & \xrightarrow{\alpha_{1,2}} & P_2 \\
& \alpha_{1/0,1} \uparrow & \square & \uparrow \alpha_{2/0,2} & \\
& P_{1/0} & \xrightarrow{\alpha_{1/0,2/0}} & P_{2/0} & \\
& \uparrow & \square & \uparrow \alpha_{2/1,2/0} & \\
O & \xrightarrow{\quad} & P_{2/1} & & 
\end{array}
\quad
\begin{array}{ccc}
P'_0 & \xrightarrow{\alpha'_{0,1}} & P'_1 & \xrightarrow{\alpha'_{1,2}} & P'_2 \\
& \alpha'_{1/0,1} \uparrow & \square & \uparrow \alpha'_{2/0,2} & \\
& P'_{1/0} & \xrightarrow{\alpha'_{1/0,2/0}} & P'_{2/0} & \\
& \uparrow & \square & \uparrow \alpha'_{2/1,2/0} & \\
O & \xrightarrow{\quad} & P'_{2/1} & & 
\end{array}
\tag{46}$$

In particular, define

$$l(\tau) := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & P_{2/1} \\ \downarrow & \square & \downarrow \alpha_{2/1,2/0} \\ P_{1/0} & \xrightarrow{\alpha_{1/0,2/0}} & P_{2/0} \end{array} \quad \begin{array}{ccc} O & \xrightarrow{\quad} & P_{2/1} \\ \downarrow & \square & \downarrow \alpha'_{2/1,2/0} \\ P_{1/0} & \xrightarrow{\alpha'_{1/0,2/0}} & P_{2/0} \end{array} \right) \tag{47}$$

to be the *double exact square associated to admissible triple*  $\tau$ . Following Definition 3.14, denote  $\mu(l(\tau))$  to be the canonical loop of  $l(\tau)$ .

Any admissible triple  $\tau = (e_0, e_1, e_2)$  defines a loop  $e_0 e_1 e_2^{-1}$ , which we also denote using  $\tau$ . Notice: if

$$\alpha_{1/0,2/0} = \alpha'_{1/0,2/0} \quad \text{and} \quad \alpha_{2/1,2/0} = \alpha'_{2/1,2/0}$$

in Diagram (46), then the loop  $\tau$  bounds a 2-simplex in  $G\mathcal{C}$  since the quotient index triangles of both diagrams now coincide. However, even when this condition does not hold, we can still say something meaningful about the (free) homotopy class of  $\tau$ , as below.

**Lemma 4.5.** *Let  $\tau$  be an admissible triple, with  $l(\tau)$  its associated double exact square. If the 1-simplices of  $\tau$  are all double exact squares, then the loop  $\tau$  is freely homotopic to  $\mu(l(\tau))$ .*

This lemma extends [Nen98b, Cor. 4.3]. The proof is more involved than Nenashev's original argument (for the same reasons as in Discussion 3.23), and will be deferred to Section B.3.

**4.3. Nenashev Relations.** To define the relations on  $K_1(\mathcal{C})$ , we shall need the following generalisation of double exact squares. A  $3 \times 3$  *diagram* in a pCGW category  $\mathcal{C}$  is a pair of diagrams

$$\left( \begin{array}{ccccc} X_{00} & \xrightarrow{f_0} & X_{01} & \xleftarrow{g_0} & X_{02} \\ h_0 \downarrow & \circlearrowleft & \downarrow h_1 & & \downarrow h_2 \\ X_{10} & \xrightarrow{f_1} & X_{11} & \xleftarrow{g_1} & X_{12} \\ j_0 \uparrow & & j_1 \uparrow & & j_2 \uparrow \\ X_{20} & \xrightarrow{f_2} & X_{21} & \xleftarrow{g_2} & X_{22} \end{array} \quad , \quad \begin{array}{ccccc} X_{00} & \xrightarrow{f'_0} & X_{01} & \xleftarrow{g'_0} & X_{02} \\ h'_0 \downarrow & \circlearrowleft & \downarrow h'_1 & & \downarrow h'_2 \\ X_{10} & \xrightarrow{f'_1} & X_{11} & \xleftarrow{g'_1} & X_{12} \\ j'_0 \uparrow & & j'_1 \uparrow & & j'_2 \uparrow \\ X_{20} & \xrightarrow{f'_2} & X_{21} & \xleftarrow{g'_2} & X_{22} \end{array} \right)$$

on the same objects subject to the following conditions:

- The horizontal and vertical rows of each diagram define exact squares. Explicitly, a  $3 \times 3$  diagram is defined by 6 double exact squares:

$$l_i := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & X_{i2} \\ \downarrow & \square & \downarrow g_i \\ X_{i0} & \xrightarrow{f_i} & X_{i1} \end{array} \quad , \quad \begin{array}{ccc} O & \xrightarrow{\quad} & X_{i2} \\ \downarrow & \square & \downarrow g'_i \\ X_{i0} & \xrightarrow{f'_i} & X_{i1} \end{array} \right) \quad \text{for all } i \in \{0, 1, 2\}$$

$$l^i := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & X_{2i} \\ \downarrow & \square & \downarrow j_i \\ X_{0i} & \xrightarrow{h_i} & X_{1i} \end{array} \quad , \quad \begin{array}{ccc} O & \xrightarrow{\quad} & X_{2i} \\ \downarrow & \square & \downarrow j'_i \\ X_{0i} & \xrightarrow{h'_i} & X_{1i} \end{array} \right) \quad \text{for all } i \in \{0, 1, 2\}$$

- The top left square in  $\mathcal{M}$  must commute. By contrast, we impose no conditions on the other squares.

**Definition 4.6** (Good  $3 \times 3$  Diagrams). A  $3 \times 3$  diagram in  $\mathcal{C}$  is called *good* if there exists objects  $Z$  and  $Z'$ , maps  $v: Z \rightarrow X_{11}$  and  $v': Z' \rightarrow X_{11}$  inducing the following diagrams

(P1)

$$\begin{array}{ccccc} X_{01} & \xrightarrow{\quad} & Z & \xrightarrow{v} & X_{11} \\ \uparrow & \square & \uparrow & \square & j_1 \uparrow \\ O & \xrightarrow{\quad} & X_{20} & \xrightarrow{f_2} & X_{21} \\ & & \uparrow & \square & g_2 \uparrow \\ & & O & \xrightarrow{\quad} & X_{22} \end{array} \quad \begin{array}{ccccc} X_{01} & \xrightarrow{\quad} & Z' & \xrightarrow{v'} & X_{11} \\ \uparrow & \square & \uparrow & \square & j'_1 \uparrow \\ O & \xrightarrow{\quad} & X_{20} & \xrightarrow{f'_2} & X_{21} \\ & & \uparrow & \square & g'_2 \uparrow \\ & & O & \xrightarrow{\quad} & X_{22} \end{array}$$

(P2)

$$\begin{array}{ccccc} X_{10} & \xrightarrow{\quad} & Z & \xrightarrow{v} & X_{11} \\ \uparrow & \square & \uparrow & \square & g_1 \uparrow \\ O & \xrightarrow{\quad} & X_{02} & \xrightarrow{h_2} & X_{12} \\ & & \uparrow & \square & j_2 \uparrow \\ & & O & \xrightarrow{\quad} & X_{22} \end{array} \quad \begin{array}{ccccc} X_{10} & \xrightarrow{\quad} & Z' & \xrightarrow{v'} & X_{11} \\ \uparrow & \square & \uparrow & \square & g'_1 \uparrow \\ O & \xrightarrow{\quad} & X_{02} & \xrightarrow{h'_2} & X_{12} \\ & & \uparrow & \square & j'_2 \uparrow \\ & & O & \xrightarrow{\quad} & X_{22} \end{array}$$

(P3)

$$\begin{array}{ccccc} X_{00} & \xrightarrow{f_0} & X_{01} & \xrightarrow{\quad} & Z \\ g_0 \uparrow & \square & \uparrow & & \\ X_{02} & \xrightarrow{\quad} & X_{02} \oplus X_{20} & & \\ & & \uparrow & & \\ & & X_{20} & & \end{array} \quad \begin{array}{ccccc} X_{00} & \xrightarrow{f'_0} & X_{01} & \xrightarrow{\quad} & Z' \\ g'_0 \uparrow & \square & \uparrow & & \\ X_{02} & \xrightarrow{\quad} & X_{02} \oplus X_{20} & & \\ & & \uparrow & & \\ & & X_{20} & & \end{array}$$

(P4)

$$\begin{array}{ccccc} X_{00} & \xrightarrow{h_0} & X_{10} & \xrightarrow{\quad} & Z \\ j_0 \uparrow & \square & \uparrow & & \\ X_{20} & \xrightarrow{\quad} & X_{02} \oplus X_{20} & & \\ & & \uparrow & & \\ & & X_{02} & & \end{array} \quad \begin{array}{ccccc} X_{00} & \xrightarrow{h'_0} & X_{10} & \xrightarrow{\quad} & Z' \\ j'_0 \uparrow & \square & \uparrow & & \\ X_{20} & \xrightarrow{\quad} & X_{02} \oplus X_{20} & & \\ & & \uparrow & & \\ & & X_{02} & & \end{array}$$

**Remark 4.7.** These so-called  $3 \times 3$  diagrams were originally defined as commutative diagrams in exact categories, where goodness automatically follows from the universal property of pushouts [Nen98b, Prop. 5.1]. However, this setup breaks down for general pCGW categories: if we define  $Z := X_{01} \star_{X_{00}} X_{10}$ , we may lack an  $\mathcal{M}$ -morphism  $v: Z \rightarrow X_{11}$  unless the original  $\mathcal{M}$ -square is a pullback. We circumvent this problem by hardcoding the desired properties into our definition of good  $3 \times 3$  diagrams.

**Definition 4.8.** Let  $\mathcal{C}$  be a pCGW category. Define  $\mathcal{D}(\mathcal{C})$  to be the abelian group with generators  $\langle l \rangle$  for all double exact squares  $l$  in  $\mathcal{C}$  subject to the following relations.

(N1)  $\langle l \rangle = 0$  if

$$l = \left( \begin{array}{ccccc} O & \xrightarrow{\quad} & C \\ \downarrow & \square & \downarrow g \\ A & \xrightarrow{f} & B \end{array} , \quad \begin{array}{ccccc} O & \xrightarrow{\quad} & C \\ \downarrow & \square & \downarrow g \\ A & \xrightarrow{f} & B \end{array} \right) \quad (48)$$

Any identical pair of exact squares will be called a *diagonal* double exact square.

(N2) Given a good  $3 \times 3$  diagram

$$\left( \begin{array}{ccc} X_{00} \xrightarrow{f_0} X_{01} \xleftarrow{g_0} X_{02} & & X_{00} \xrightarrow{f'_0} X_{01} \xleftarrow{g'_0} X_{02} \\ h_0 \downarrow \quad \circlearrowleft \quad \downarrow h_1 & & h'_0 \downarrow \quad \circlearrowleft \quad \downarrow h'_1 \\ X_{10} \xrightarrow{f_1} X_{11} \xleftarrow{g_1} X_{12} & , & X_{10} \xrightarrow{f'_1} X_{11} \xleftarrow{g'_1} X_{12} \\ j_0 \uparrow \quad \quad \quad \uparrow j_1 & & j'_0 \uparrow \quad \quad \quad \uparrow j'_1 \\ X_{20} \xrightarrow{f_2} X_{21} \xleftarrow{g_2} X_{22} & & X_{20} \xrightarrow{f'_2} X_{21} \xleftarrow{g'_2} X_{22} \end{array} \right) \quad (49)$$

defined by the following 6 double exact squares

$$l_i := \left( \begin{array}{ccc} O \xrightarrow{\quad} X_{i2} & O \xrightarrow{\quad} X_{i2} \\ \downarrow \quad \square \quad \downarrow g_i & \downarrow \quad \square \quad \downarrow g'_i \\ X_{i0} \xrightarrow{f_i} X_{i1} & X_{i0} \xrightarrow{f'_i} X_{i1} \end{array} \right) \quad l^i := \left( \begin{array}{ccc} O \xrightarrow{\quad} X_{2i} & O \xrightarrow{\quad} X_{2i} \\ \downarrow \quad \square \quad \downarrow j_i & \downarrow \quad \square \quad \downarrow j'_i \\ X_{0i} \xrightarrow{h_i} X_{1i} & X_{0i} \xrightarrow{h'_i} X_{1i} \end{array} \right)$$

for all  $i \in \{0, 1, 2\}$ , the following 6-term relation holds

$$\langle l_0 \rangle + \langle l_2 \rangle - \langle l_1 \rangle = \langle l^0 \rangle + \langle l^2 \rangle - \langle l^1 \rangle \quad (50)$$

**Theorem 4.9.** *Given a pCGW category  $\mathcal{C}$  there exists a well-defined homomorphism*

$$m: \mathcal{D}(\mathcal{C}) \longrightarrow K_1(\mathcal{C}) \quad (51)$$

that is surjective. In other words, the two relations of  $\mathcal{D}(\mathcal{C})$  also hold in  $K_1(\mathcal{C})$ .

*Proof.* By Theorem 3.16, we know that  $K_1(\mathcal{C})$  is generated by double exact squares so both groups have the same generators. It remains to check the relations.

(N1): Let  $l$  be as in Equation (48). The corresponding loop  $\mu(l)$  bounds the 2-simplex

$$\begin{array}{ccc} O \xrightarrow{\quad} A \xrightarrow{f} B & & O \xrightarrow{\quad} A \xrightarrow{f} B \\ \uparrow \quad \square \quad \uparrow & & \uparrow \quad \square \quad \uparrow \\ A \xrightarrow{f} B & & A \xrightarrow{f} B \\ \uparrow g & & \uparrow g \\ C & & C \end{array}$$

in  $G\mathcal{C}$ , and so  $\langle l \rangle = 0$ .

(N2): Leveraging the fact that the  $3 \times 3$  diagram is good, construct the diagram

$$\begin{array}{ccc} (X_{00}, X_{00}) & \xrightarrow{\quad l_0 \quad} & (X_{01}, X_{01}) \\ & \searrow \alpha_0 & \swarrow \alpha_3 \\ & (Z, Z') & \\ & \swarrow \alpha_1 & \searrow \alpha_2 \\ (X_{10}, X_{10}) & \xrightarrow{\quad l_1 \quad} & (X_{11}, X_{11}) \end{array} \quad (52)$$

where outer blue edges  $\alpha := l_0 l^1 (l_1)^{-1} (l^0)^{-1}$  form a loop, and the inner edges are given by

$$\alpha_0 := \left( \begin{array}{ccc} O \xrightarrow{\quad} X_{02} \oplus X_{20} & O \xrightarrow{\quad} X_{02} \oplus X_{20} \\ \downarrow \quad \square \quad \downarrow & \downarrow \quad \square \quad \downarrow \\ X_{00} \xrightarrow{\quad} Z & X_{00} \xrightarrow{\quad} Z' \end{array} \right) \quad \alpha_1 := \left( \begin{array}{ccc} O \xrightarrow{\quad} X_{02} & O \xrightarrow{\quad} X_{02} \\ \downarrow \quad \square \quad \downarrow & \downarrow \quad \square \quad \downarrow \\ X_{10} \xrightarrow{\quad} Z & X_{10} \xrightarrow{\quad} Z' \end{array} \right)$$

$$\alpha_2 := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & X_{22} \\ \downarrow & \square & \downarrow \\ Z & \xrightarrow{v} & X_{11} \end{array} , \begin{array}{ccc} O & \xrightarrow{\quad} & X_{22} \\ \downarrow & \square & \downarrow \\ Z' & \xrightarrow{v'} & X_{11} \end{array} \right) \quad \alpha_3 := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & X_{20} \\ \downarrow & \square & \downarrow \\ X_{01} & \xrightarrow{\quad} & Z \end{array} , \begin{array}{ccc} O & \xrightarrow{\quad} & X_{20} \\ \downarrow & \square & \downarrow \\ X_{01} & \xrightarrow{\quad} & Z' \end{array} \right)$$

For orientation, we start with a basic observation.

**Lemma 4.10.** *Given any closed loop  $l = e_0 \dots e_n$  whose edges are all double exact squares,*

$$\langle l \rangle = \sum_{i=0}^n (-1)^{\epsilon_i} \langle e_i \rangle$$

in  $K_1(\mathbb{C})$ , where the coefficient  $(-1)^{\epsilon_i}$  reflects the orientation of edge  $e_i$  in  $l$ .

*Proof of Lemma.* The proof is obvious, but let us work this out explicitly for the case of  $\alpha$ . Consider the diagram

$$\begin{array}{ccccc} (X_{00}, X_{00}) & \xrightarrow{l_0} & (X_{01}, X_{01}) & & \\ & \nwarrow & \nearrow & & \\ & (O, O) & & & \\ & \swarrow & \searrow & & \\ (X_{10}, X_{10}) & \xrightarrow{l_1} & (X_{11}, X_{11}) & & \\ & \nwarrow & \nearrow & & \\ & (O, O) & & & \\ & \swarrow & \searrow & & \\ & (X_{00}, X_{00}) & & & \end{array} \quad \begin{array}{c} \downarrow l^0 \\ \downarrow l^1 \end{array} \quad (53)$$

featuring  $\alpha$  as the outer loop but now with the base-point  $(O, O)$  at the center. Since the triangles all bound 2-simplices, the loop  $\mu(l_0)\mu(l^1)\mu(l_1)^{-1}\mu(l^0)^{-1}$  is freely homotopic to  $\alpha$ , and so

$$\langle \alpha \rangle = \langle l_0 \rangle + \langle l^1 \rangle - \langle l_1 \rangle - \langle l^0 \rangle.$$

□

Next, appealing once more to the goodness of the  $3 \times 3$  diagram, notice:

- Diagrams (P3) and (P4) are 2-simplices, which are bounded by the loops  $l^0\alpha_1\alpha_0^{-1}$  and  $l_0\alpha_3\alpha_0^{-1}$ . Therefore, deduce that

$$\begin{aligned} \langle l^0 \rangle + \langle \alpha_1 \rangle - \langle \alpha_0 \rangle &= 0 \\ \langle l_0 \rangle + \langle \alpha_3 \rangle - \langle \alpha_0 \rangle &= 0 \end{aligned} \quad (54)$$

- Diagrams (P1) and (P2) are admissible triples whose associated exact squares are  $l_2$  and  $l^2$  respectively. It is also clear there exists an edge  $(O, O) \rightarrow (X_{11}, X_{11})$ . Thus, applying Lemma 4.5, deduce that

$$\begin{aligned} \langle \alpha_3 \rangle + \langle \alpha_2 \rangle - \langle l^1 \rangle &= \langle l_2 \rangle \\ \langle \alpha_1 \rangle + \langle \alpha_2 \rangle - \langle l_1 \rangle &= \langle l^2 \rangle. \end{aligned} \quad (55)$$

Combining Equations (54) and (55),

$$\begin{aligned} \langle l_2 \rangle - \langle l^2 \rangle &= \langle \alpha_3 \rangle + \langle \alpha_2 \rangle - \langle l^1 \rangle - \langle \alpha_1 \rangle - \langle \alpha_2 \rangle + \langle l_1 \rangle \\ &= \langle \alpha_3 \rangle - \langle \alpha_0 \rangle - \langle l^1 \rangle - \langle \alpha_1 \rangle + \langle \alpha_0 \rangle + \langle l_1 \rangle \\ &= -\langle l_0 \rangle - \langle l^1 \rangle + \langle l^0 \rangle + \langle l_1 \rangle, \end{aligned}$$

and so by rearranging terms, conclude

$$\langle l_0 \rangle + \langle l_2 \rangle - \langle l_1 \rangle = \langle l^0 \rangle + \langle l^2 \rangle - \langle l^1 \rangle. \quad (56)$$

□

**4.4. Assembler Relations.** We now apply Theorem 4.9 to compare our relations with Zakharevich's  $K_1$  of an Assembler (Proposition 4.12). As a corollary, we prove that  $\mathcal{D}(\mathcal{C}) \cong K_1(\mathcal{C})$ , and so  $\mathcal{D}(\mathcal{C})$  gives an alternative presentation of  $K_1(\mathcal{C})$  for pCGW categories (Corollary 4.16).

We start with an informal overview. An *Assembler* is a Grothendieck site  $\mathcal{A}$  whose topology encodes how an object  $A$  may be covered by a finite set of disjoint subobjects  $\{A_i\}_{i \in I}$ . In particular, given any Assembler  $\mathcal{A}$ , one can associate to it the category  $\mathcal{W}(\mathcal{A})$  whereby

**Objects:** Finite sets of objects  $\{A_i\}_{i \in I}$  in  $\mathcal{A}$ ;

**Morphisms:** Piecewise automorphisms in  $\mathcal{A}$ . Explicitly, a morphism  $f: \{A_i\}_{i \in I} \rightarrow \{B_j\}_{j \in J}$  in  $\mathcal{W}(\mathcal{A})$  is a tuple of morphisms  $f_i: A_i \rightarrow B_{f(i)}$  such that  $\{f_i: A_i \rightarrow B_{f^{-1}(j)}\}_{i \in f^{-1}(j)}$  is a finite disjoint covering family.

Details can be found in [Zak17b, §2], which introduces and develops the  $K$ -theory of Assemblers. Its relevance to our paper is that one can also define  $K\mathcal{V}\text{ar}_k$  via Assemblers, which was shown to be equivalent to the CGW model of  $K\mathcal{V}\text{ar}_k$  by [CZ22, Theorems 7.8 and 9.1]. Extending Muro-Tonks' model of  $K_1$  of a Waldhausen Category [MT08], Zakharevich proved the following.

**Theorem 4.11** ([Zak17c, Theorem B]). *Let  $\mathcal{A}$  be an Assembler whose morphisms are closed under pulback. Then  $K_1(\mathcal{A})$  is generated by a pair of morphisms*

$$A \begin{smallmatrix} f \\ \rightrightarrows \\ g \end{smallmatrix} B$$

in  $\mathcal{W}(\mathcal{A})$ . These satisfy the relations

$$\begin{aligned} (Z1) \quad \langle A \begin{smallmatrix} f \\ \rightrightarrows \\ f \end{smallmatrix} B \rangle &= 0; \\ (Z2) \quad \langle A \begin{smallmatrix} f_1 \\ \rightrightarrows \\ f_2 \end{smallmatrix} B \rangle + \langle C \begin{smallmatrix} g_1 \\ \rightrightarrows \\ g_2 \end{smallmatrix} D \rangle &= \langle A \amalg C \begin{smallmatrix} f_1 \amalg g_1 \\ \rightrightarrows \\ f_2 \amalg g_2 \end{smallmatrix} B \amalg D \rangle; \\ (Z3) \quad \langle B \begin{smallmatrix} g_1 \\ \rightrightarrows \\ g_2 \end{smallmatrix} C \rangle + \langle A \begin{smallmatrix} f_1 \\ \rightrightarrows \\ f_2 \end{smallmatrix} B \rangle &= \langle A \begin{smallmatrix} g_1 f_1 \\ \rightrightarrows \\ g_2 f_2 \end{smallmatrix} B \rangle. \end{aligned}$$

A couple remarks are in order. First, [Zak17c, Theorem B] leaves open the possibility that there may be more relations on  $K_1$  to be identified. This incompleteness is inherited from Muro-Tonks' original model of  $K_1$ : although [MT08, Prop 6.3] shows that their model coincides with Nenashev's model for exact categories (and is thus complete), they were unable to show the same for all Waldhausen categories. Second, Relation (Z1) clearly corresponds to the diagonal relation (N1) of  $\mathcal{D}(\mathcal{C})$ . It remains to investigate Relations (Z2) and (Z3) in our context, which we work out below.

**Proposition 4.12** (Assembler Relations). *Let  $f: (A, A) \rightarrow (B, B)$  and  $g: (C, C) \rightarrow (D, D)$  be two double exact squares given by*

$$f := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{B}{A} \\ \downarrow & \square & \downarrow f_2 \\ A & \xrightarrow{f_1} & B \end{array}, \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{B}{A} \\ \downarrow & \square & \downarrow f'_2 \\ A & \xrightarrow{f'_1} & B \end{array} \right), \quad g := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{D}{C} \\ \downarrow & \square & \downarrow g_2 \\ C & \xrightarrow{g_1} & D \end{array}, \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{D}{C} \\ \downarrow & \square & \downarrow g'_2 \\ C & \xrightarrow{g'_1} & D \end{array} \right).$$

Then the following relations hold in  $K_1(\mathcal{C})$ :

- (A1) (Formal Direct Sums).  $\langle f \rangle + \langle g \rangle = \langle f \oplus g \rangle$ .
- (A2) (Restricted Composition). Suppose  $(B, B) = (C, C)$ . Then

$$\langle f \rangle + \langle g \rangle = \langle g \circ f \rangle + \langle l_2 \rangle,$$

where  $l_2$  is the associated double exact square (cf. Definition 4.4)

$$l_2 := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{D}{B} \\ \downarrow & \square & \downarrow j_1 \\ \frac{B}{A} & \xrightarrow{h_1} & \frac{D}{A} \end{array}, \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{D}{B} \\ \downarrow & \square & \downarrow j'_1 \\ \frac{B}{A} & \xrightarrow{h'_1} & \frac{D}{A} \end{array} \right) \quad \text{and} \quad \langle g \circ f \rangle := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{D}{A} \\ \downarrow & \square & \downarrow \\ A & \xrightarrow{g_1 f_1} & D \end{array}, \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{D}{A} \\ \downarrow & \square & \downarrow \\ A & \xrightarrow{g'_1 f'_1} & D \end{array} \right).$$

*Proof.* The argument proceeds by constructing the obvious  $3 \times 3$  diagram, and applying Theorem 4.9 to perform our calculations.

(i): We claim that the following is a good  $3 \times 3$  diagram:

$$\left( \begin{array}{ccc} A \xrightarrow{f_1} B \xleftarrow{f_2} \circ \frac{B}{A} \\ \downarrow \quad \circ \quad \downarrow \\ A \oplus C \xrightarrow{f_1 \oplus g_1} B \oplus D \xleftarrow{f_2 \oplus g_2} \frac{B}{A} \oplus \frac{D}{C} \\ \uparrow \quad \quad \uparrow \\ C \xrightarrow{g_1} D \xleftarrow{g_2} \circ \frac{D}{C} \end{array} \quad , \quad \begin{array}{ccc} A \xrightarrow{f'_1} B \xleftarrow{f'_2} \circ \frac{B}{A} \\ \downarrow \quad \circ \quad \downarrow \\ A \oplus C \xrightarrow{f'_1 \oplus g'_1} B \oplus D \xleftarrow{f'_2 \oplus g'_2} \frac{B}{A} \oplus \frac{D}{C} \\ \uparrow \quad \quad \uparrow \\ C \xrightarrow{g'_1} D \xleftarrow{g'_2} \circ \frac{D}{C} \end{array} \right) \quad (57)$$

The vertical columns define exact squares arising from formal direct sums. The top and bottom rows correspond to the exact squares from  $f$  and  $g$ . The middle rows correspond to  $f \oplus g$ ; the fact this indeed defines a double exact square follows from Lemma A.1. One easily checks that the  $\mathcal{M}$  square commutes, and thus this defines a  $3 \times 3$  diagram.

It remains to check goodness. Taking repeated restricted pushouts, Fact 1.17 yields

$$\begin{array}{ccc} O \xrightarrow{\quad} A \xrightarrow{f_1} B \\ \downarrow \quad \downarrow \quad \downarrow \\ C \xrightarrow{\quad} A \oplus C \xrightarrow{\quad} B \oplus C \\ \downarrow g_1 \quad \downarrow \quad \downarrow v \\ D \xrightarrow{\quad} A \oplus D \xrightarrow{\quad} B \oplus D \end{array} \quad \begin{array}{ccc} O \xrightarrow{\quad} A \xrightarrow{f'_1} B \\ \downarrow \quad \downarrow \quad \downarrow \\ C \xrightarrow{\quad} A \oplus C \xrightarrow{\quad} B \oplus C \\ \downarrow g'_1 \quad \downarrow \quad \downarrow v' \\ D \xrightarrow{\quad} A \oplus D \xrightarrow{\quad} B \oplus D \end{array} . \quad (58)$$

Since we now have  $\mathcal{M}$ -morphisms  $v, v': B \oplus C \rightarrow B \oplus D$ , we can construct the required diagrams:  
(P1):

$$\begin{array}{ccc} B \xrightarrow{\quad} B \oplus C \xrightarrow{v} B \oplus D \\ \uparrow \quad \square \quad \uparrow \quad \square \quad \uparrow \\ O \xrightarrow{\quad} C \xrightarrow{g_1} D \\ \quad \uparrow \quad \square \quad \uparrow \\ \quad O \xrightarrow{\quad} \frac{D}{C} \end{array} \quad \begin{array}{ccc} B \xrightarrow{\quad} B \oplus C \xrightarrow{v'} B \oplus D \\ \uparrow \quad \square \quad \uparrow \quad \square \quad \uparrow \\ O \xrightarrow{\quad} C \xrightarrow{g'_1} D \\ \quad \uparrow \quad \square \quad \uparrow \\ \quad O \xrightarrow{\quad} \frac{D}{C} \end{array}$$

(P2):

$$\begin{array}{ccc} A \oplus C \xrightarrow{\quad} B \oplus C \xrightarrow{v} B \oplus D \\ \uparrow \quad \square \quad \uparrow \quad \square \quad \uparrow \\ O \xrightarrow{\quad} \frac{B}{A} \xrightarrow{\quad} \frac{B}{A} \oplus \frac{D}{C} \\ \quad \uparrow \quad \square \quad \uparrow \\ \quad O \xrightarrow{\quad} \frac{D}{C} \end{array} \quad \begin{array}{ccc} A \oplus C \xrightarrow{\quad} B \oplus C \xrightarrow{v'} B \oplus D \\ \uparrow \quad \square \quad \uparrow \quad \square \quad \uparrow \\ O \xrightarrow{\quad} \frac{B}{A} \xrightarrow{\quad} \frac{B}{A} \oplus \frac{D}{C} \\ \quad \uparrow \quad \square \quad \uparrow \\ \quad O \xrightarrow{\quad} \frac{D}{C} \end{array}$$

(P3):

$$\begin{array}{ccc}
 A \xrightarrow{f_1} B \longrightarrow B \oplus C & A \xrightarrow{f'_1} B \longrightarrow B \oplus C \\
 \uparrow f_2 \quad \square \quad \uparrow & \uparrow f'_2 \quad \square \quad \uparrow \\
 \frac{B}{A} \longrightarrow \frac{B}{A} \oplus C & \frac{B}{A} \longrightarrow \frac{B}{A} \oplus C \\
 \uparrow & \uparrow \\
 C & C
 \end{array}$$

(P4):

$$\begin{array}{ccc}
 A \longrightarrow A \oplus C \longrightarrow B \oplus C & A \longrightarrow A \oplus C \longrightarrow B \oplus C \\
 \uparrow \quad \square \quad \uparrow & \uparrow \quad \square \quad \uparrow \\
 C \longrightarrow \frac{B}{A} \oplus C & C \longrightarrow \frac{B}{A} \oplus C \\
 \uparrow & \uparrow \\
 \frac{B}{A} & \frac{B}{A}
 \end{array}$$

One can verify that the indicated squares are distinguished by repeated applications of Lemma A.3.

Finally, to prove the relation, review Diagram (57). Denote the double exact squares corresponding to the vertical columns as  $l^0, l^1$  and  $l^2$ , from left to right. Since Diagram (57) is a good  $3 \times 3$  diagram, apply Relation (N2) to get

$$\langle f \rangle + \langle g \rangle - \langle f \oplus g \rangle = \langle l^0 \rangle + \langle l^2 \rangle - \langle l^1 \rangle.$$

Since  $l^0, l^1$  and  $l^2$  are all diagonal, deduce from (N1) that  $\langle l^0 \rangle = \langle l^1 \rangle = \langle l^2 \rangle = 0$ , and so conclude

$$\langle f \rangle + \langle g \rangle = \langle f \oplus g \rangle.$$

(ii): Given  $(B, B) = (C, C)$ , construct the following diagram

$$\left( \begin{array}{ccc}
 A \xrightarrow{=} A \xleftarrow{\circ} O & A \xrightarrow{=} A \xleftarrow{\circ} O \\
 f_1 \downarrow \quad \circ \quad \downarrow g_1 f_1 & f'_1 \downarrow \quad \circ \quad \downarrow g'_1 f'_1 \\
 B \xrightarrow{g_1} D \xleftarrow{\circ} \frac{D}{B} & B \xrightarrow{g'_1} D \xleftarrow{\circ} \frac{D}{B} \\
 f_2 \uparrow & f'_2 \uparrow \\
 \frac{B}{A} \xrightarrow{h_1} \frac{D}{A} \xleftarrow{\circ} \frac{D}{B} & \frac{B}{A} \xrightarrow{h'_1} \frac{D}{A} \xleftarrow{\circ} \frac{D}{B}
 \end{array} \right), \quad (59)$$

Following Definition 4.8, label the horizontal rows as  $l_0, l_1$  and  $l_2$  and the vertical columns as  $l^0, l^1$  and  $l^2$ , where  $l_1 = g$ ,  $l^0 = f$  and  $l^1 = g \circ f$ . The fact that  $l_2$  defines a double exact square comes from Lemma 1.8 (“quotients respect filtrations”); the remaining rows are obvious. It is also obvious that the  $\mathcal{M}$ -square commutes, and so the above defines a  $3 \times 3$  diagram. Finally, taking restricted pushouts of  $A \xleftarrow{=} A \xrightarrow{f_1} B$  and  $A \xleftarrow{=} A \xrightarrow{f'_1} B$ , define

$$v := g_1: A \star_A B = B \rightarrow D$$

$$v' := g'_1: A \star_A B = B \rightarrow D.$$

It is then straightforward to verify that Diagram (59) is good.

Now apply Relation (N2) to get

$$\langle l_0 \rangle + \langle l_2 \rangle - \langle g \rangle = \langle f \rangle + \langle l^2 \rangle - \langle g \circ f \rangle.$$

Since  $l_0$  and  $l^2$  are diagonal, deduce from (N1) that  $\langle l_0 \rangle = \langle l^2 \rangle = 0$ , and thus conclude that

$$\langle g \circ f \rangle + \langle l_2 \rangle = \langle f \rangle + \langle g \rangle.$$

□



Relation (A2) improves upon Proposition 4.1 by determining what happens to *any* admissible triple in  $K_1$  (Definition 4.4), clarifying Warning 4.3. Namely, given an admissible triple  $\tau = (f, g, g \circ f)$ , we now know that

$$\langle g \circ f \rangle + \langle l_2 \rangle = \langle f \rangle + \langle g \rangle \quad \text{in } K_1.$$

Unlike Theorem 4.11, where composition always splits in  $K_1$ , here an obstruction term  $\langle l_2 \rangle$  appears. Of course, if  $\langle l_2 \rangle = 0$  then Relations (A2) and (Z3) coincide, but this is not true in general, as below.

**Example 4.13.** Given any automorphism  $\alpha: B \rightarrow B$ , construct the pair diagram

$$\begin{array}{ccccc} A & \xrightarrow{\quad} & A \oplus B & \xrightarrow{1_A \oplus \alpha} & A \oplus B \\ \uparrow \circlearrowleft & \square & \uparrow \circlearrowleft & \square & \uparrow \circlearrowleft \\ O & \xrightarrow{\quad} & B & \xrightarrow{\alpha} & B \\ & & \uparrow \circlearrowleft & \square & \uparrow \circlearrowleft \\ & & O & \xrightarrow{\quad} & O \end{array} \quad \begin{array}{ccccc} A & \xrightarrow{\quad} & A \oplus B & \xrightarrow{1_A \oplus 1_B} & A \oplus B \\ \uparrow \circlearrowleft & \square & \uparrow \circlearrowleft & \square & \uparrow \circlearrowleft \\ O & \xrightarrow{\quad} & B & \xrightarrow{1_B} & B \\ & & \uparrow \circlearrowleft & \square & \uparrow \circlearrowleft \\ & & O & \xrightarrow{\quad} & O \end{array} .$$

This defines an admissible triple where the obstruction term is  $\langle l_2 \rangle = l(\alpha)$ , the double exact square associated to  $\alpha$  (Example 3.15). It is known that  $l(\alpha) \neq 0$  in general – see e.g. [Zak22].

**Discussion 4.14.** We suspect this discrepancy with Zakharevich’s account stems from [Zak17c, Theorem 2.1], a key ingredient in Zakharevich’s proof of Theorem 4.11, which models the  $K$ -theory of (nice) Assemblers using Waldhausen categories whose cofibration sequences all split (up to weak equivalence). We emphasise that this relies on a non-standard notion of weak equivalence [Zak17c, Def. 1.7]. In the CGW setting, it is clearly false that all exact squares split in  $\mathcal{V}\text{ar}_k$  – consider e.g.

$$\begin{array}{ccc} O & \xrightarrow{\quad} & \mathbb{A}^1 \\ \downarrow \circlearrowleft & \square & \downarrow \circlearrowleft \\ \{*\} & \xrightarrow{\quad} & \mathbb{P}^1 \end{array} .$$

**Remark 4.15.** Proposition 4.12 illustrates the kind of leverage Theorem 4.9 gives us, but one can also prove it directly. Relation (A1) can be deduced from the  $H$ -space structure of  $G\mathcal{C}$  and applying the standard Eckmann-Hilton argument; Relation (A2) translates Lemma 4.5, though the proof is comparatively involved (see Section B.3).

We end with one final application. In order to show that  $\mathcal{D}(\mathcal{C}) \cong K_1(\mathcal{C})$  for exact categories, Nenashev [Nen98a] constructs a homomorphism

$$b: K_1(\mathcal{C}) \rightarrow \mathcal{D}(\mathcal{C})$$

and shows that it is inverse to the map  $m: \mathcal{D}(\mathcal{C}) \rightarrow K_1(\mathcal{C})$  from Equation (51). Notice the naive map sending  $\langle f \rangle \rightarrow \langle f \rangle$  is *a priori* not well-defined since  $K_1(\mathcal{C})$  may have more relations than  $\mathcal{D}(\mathcal{C})$ . Indeed, the original construction of  $b$  in [Nen98a] requires a fair bit of technical legwork to show that it yields a well-defined homomorphism. However, Propositions 4.1 and 4.12 combine to give a shorter direct proof.

**Corollary 4.16.** *Given any pCGW category  $\mathcal{C}$ , there is an isomorphism*

$$\mathcal{D}(\mathcal{C}) \cong K_1(\mathcal{C}).$$

*Proof.* It suffices to show the naive map  $b: K_1(\mathcal{C}) \rightarrow \mathcal{D}(\mathcal{C})$  sending  $\langle f \rangle \rightarrow \langle f \rangle$  is in fact well-defined. Any equivalence  $\langle f \rangle = \langle g \rangle$  in  $K_1(\mathcal{C})$  must be generated by the relations of Proposition 4.1. It therefore suffices to check that  $\mathcal{D}(\mathcal{C})$  satisfies those relations as well. Relations (B1) and (B2) are diagonal relations, and follow immediately from Relation (N1). Relation (B3) is a special case of Restricted Composition (A2) in Proposition 4.12, where  $\langle l_2 \rangle = 0$  since  $l_2$  is diagonal by assumption. And we are done.  $\square$

**Remark 4.17.** In fact, the proof of Corollary 4.16 shows  $K_1(\mathcal{C})$  can be presented as the free abelian group generated by double exact squares, modulo the Diagonal Relation (N1) (Definition 4.8) and Restricted Composition Relation (A2) (Proposition 4.12)

## 5. SOME TEST PROBLEMS

This paper began with Question 1: what information do the higher  $K$ -groups of varieties encode? Progress on this question requires advances on two fronts: developing the framework of non-additive  $K$ -theory on the one hand, and concrete applications of the newly-developed ( $K$ -theory) tools on the other. This paper is of the first kind, with a view towards laying the groundwork for future theorems. We conclude with some test problems and discussion.

**5.1. Non-Additive  $K$ -theory.** Having characterised  $K_1$ , the natural next step is the following problem.

**Problem 5.1.** Characterise  $K_n(\mathcal{C})$  for  $n > 1$  for pCGW categories.

**Discussion 5.2.** In a substantial generalisation of Nenashev's work, Grayson gave a complete characterisation of  $K_n$  for all  $n$  in the setting of exact categories [Gra12]. Encouraged by the results in this paper, the natural proof strategy would be to extend Grayson's argument to the pCGW setting.

However, there is an obvious barrier. Grayson characterises the  $K$ -groups of exact categories via binary chain complexes, and so invokes the Gillet-Waldhausen Theorem. Although an analogue of this result has been shown for extensive categories (e.g.  $\mathbf{FinSet}$ ) [SS21], a Gillet-Waldhausen Theorem for  $\mathbf{Var}_k$  has not been worked out yet. Nonetheless, even the case of  $\mathbf{FinSet}$  is interesting. By Barratt-Priddy-Quillen, the  $K$ -groups of  $\mathbf{FinSet}$  correspond to the stable homotopy groups of spheres, whose complete description remains a longstanding open problem in homotopy theory. What might this presentation of  $K_n(\mathbf{FinSet})$  tell us about the stable homotopy groups of spheres? About the  $J$ -homomorphism and Adams  $e$ -invariant?

In light of the comparisons with Zakharevich's  $K_1$ , perhaps a more urgent problem is the following.

**Problem 5.3.** Resolve the discrepancy between our presentation of  $K_1(\mathbf{Var}_k)$  and Zakharevich's.

**Discussion 5.4.** Here's one line of investigation. Our proof of Theorem 3.7 makes crucial use of Lemma B.2, which states: if  $\mathcal{C}$  is a pCGW category whose exact squares are all split, then  $K_1(\mathcal{C})$  is generated by automorphisms. In which case, it is easy to check that composition splits in  $K_1$  in the manner of (Z3), Theorem 4.11. In light of [Zak17c, Theorem 2.1], can we extend Lemma B.2 to Waldhausen categories whose cofibration sequences all split?

**Discussion 5.5.** It is also worth carefully re-examining Zakharevich's argument in [Zak17c], and exploring the extent to which this extends to exact categories.

Another natural question, posed to us by Emanuele Dotto, is the following.

**Problem 5.6.** Does the  $K$ -theory of pCGW categories commute with infinite products?

**Discussion 5.7.** Relevantly: Zakharevich conjectures in [Zak22, Remark 2.4] that the  $K$ -theory of Assemblers commutes with infinite products. However, she notes that such this result seems presently out of reach since previous results of this form were worked out for Waldhausen categories with cylinder functors (which Assemblers do not have) and exact categories.

**5.2. The Motivic Euler Characteristic.** There is a well-known enrichment of the Euler Characteristic, known as the *motivic Euler Characteristic* or *compactly supported  $\mathbb{A}^1$ -Euler Characteristic*, which can be defined as a ring homomorphism

$$\chi^{\text{mot}}: K_0(\mathbf{Var}_k) \rightarrow \text{GW}(k)$$

valued in  $\text{GW}(k)$ , the Grothendieck-Witt ring of quadratic forms over field  $k$ . An exposition of its construction can be found in [AMBO<sup>+</sup>22]. It is natural to ask if one can lift this to the level of  $K$ -theory spectra, which was proved in the affirmative by Nanavaty [Nan24, Theorem 1.1]. Explicitly, he constructs a map of spectra

$$K\mathbf{Var}_k \rightarrow \text{End}(\mathbb{1}_k)$$

where  $\text{End}(\mathbb{1}_k)$  is the *Endomorphism Spectrum of the unit object in the motivic stable homotopy category*, recovering  $\chi^{\text{mot}}$  on  $\pi_0$ . This sets up the problem:

**Problem 5.8.** Define a natural map  $K_1(\mathbf{Var}_k) \rightarrow \pi_{1,0}(\mathbb{1}_k)$ . What geometric information does it encode?

**Discussion 5.9.** The homotopy groups of  $\text{End}(\mathbb{1}_k)$  are defined as  $\pi_{*,0}$ . A foundational result, due to Morel [Mor06], shows that  $\pi_{0,0}(\mathbb{1}_k) \cong \text{GW}(k)$ , so one may regard the higher homotopy groups as defining higher Grothendieck-Witt Groups. What geometric information is detected at the higher levels? Recent work by [RSØ19] tells us

$$0 \rightarrow K_2^M(k)/24 \rightarrow \pi_{1,0}(\mathbb{1}_k) \rightarrow k^\times/2 \oplus \mathbb{Z}/2 \rightarrow 0, \quad (60)$$

where  $K_*^M(k)$  denotes the Milnor  $K$ -theory of  $k$ . Combined with Theorems B and/or C, we now have an explicit description of both groups in Problem 5.8. It remains to map the double exact squares in  $\text{Var}_k$  to  $\pi_{1,0}(\mathbb{1}_k)$  in a natural way, but it is presently unclear, e.g. how these generalised automorphisms ought to interact with the Milnor  $K$ -theory term, and what this means geometrically.

Note: the canonical unit map  $\text{End}(\mathbb{1}_k) \rightarrow KQ$  (i.e. the Hermitian  $K$ -theory spectrum) induces an isomorphism on the level of  $\pi_0$ . We may therefore also think of  $\chi^{\text{mot}}$  as a map on  $\pi_0$  of  $K\text{Var}_k \rightarrow KQ$ , which may be more tractable than working in the setting of motivic homotopy theory. For those interested in the power structures on  $K_0(\text{Var}_k)$ , a natural question may be:

**Problem 5.10.** Lift the symmetric power structure on  $K_0(\text{Var}_k)$  to the level of spectra on  $K\text{Var}_k$ . In particular, if we regard  $\chi^{\text{mot}}$  as a map

$$\chi^{\text{mot}}: \pi_0(K\text{Var}_k) \rightarrow \pi_0(KQ),$$

can we deduce the compatibility of  $\chi^{\text{mot}}$  with the symmetric power structures on the level of  $\pi_0$  from formal properties on their underlying simplicial sets?

**Discussion 5.11.** To our knowledge, compatibility of  $\chi^{\text{mot}}$  with power structures on  $K_0$  and  $\text{GW}(k)$  has only been shown in special cases, and the proofs rely on deep arithmetic (e.g. [PP23, PRV24]). Problem 5.10 calls for a shift in perspective: what if we approach the problem homotopically instead?

The jury is still out on how much mileage this gives us, but we have a first clue: Grayson [Gra92] relies on the  $G$ -construction to provide an explicit combinatorial description of the Adams Operations on higher  $K$ -groups induced by symmetric powers on the exact category. Since we know the  $G$ -construction behaves as expected on  $\text{Var}_k$  by Theorem A, this tells us where to start. We plan to investigate this in future work.

**Remark 5.12.** Lifting the symmetric power structure on  $K_0(\text{Var}_k)$  also ought to have useful implications for lifting Kapranov’s motivic zeta function to a map of  $K$ -theory spectra – see [CWZ19, Question 7.6].

**5.3. Combinatorics of Definable Sets.** Theorem 3.16 showed that  $K_1(\mathbb{C})$  of a pCGW category is generated by double exact squares, which generalise automorphisms (see Example 3.15). For suggestiveness, call the generators of  $K_1$  *quasi-automorphisms*. Automorphisms play a key role in many different areas of mathematics – can this picture be extended to quasi-automorphisms in a productive way?

For the model theorist, the automorphism group  $\text{Aut}(M)$  of a countable first-order structure  $M$  encodes important information about  $M$  – e.g.  $\text{Aut}(M)$  measures the homogeneity of  $M$ , but there are other insights [MK94, Eva97]. Some interesting questions:

**Problem 5.13.** How might  $K_1(M)$  be useful for studying non-homogeneous structures? Structures with small  $\text{Aut}(M)$ ?

**Problem 5.14.** A countable structure  $M$  is said to be *homogeneous* just in case every finite partial isomorphism extends to a global automorphism of  $M$ . Can we adapt the obstruction theory developed in [Zak17a] to analyse the barriers to homogeneity? What about uncountable structures?

**Problem 5.15.** In the converse direction, by viewing  $\text{Aut}(M)$  as a topological group, model theorists were able to analyse its structure productively using a wide range of tools. Can we apply the same analysis to  $K_1(M)$ ? What kind of (new) insights does this give? Work in [KLM<sup>+</sup>24] may be relevant.

It is also worth revisiting the original papers [Kra00, KS00] where the Grothendieck ring of Definable Sets was first investigated. In particular, [KS00] introduces the so-called strong and weak Euler Characteristics on first-order structures before asking which fields admit a non-trivial strong Euler Characteristic. In light of our present work, a natural problem may be:

**Problem 5.16.** Lift the weak/strong Euler Characteristic on first-order structures to the level of spectra. Analyse what happens on  $K_1$  – what information does it detect? In addition, are there examples of fields with strong Euler characteristics that are trivial on  $K_0$  but non-trivial on  $K_1$ ?

**5.4. Matroids.** Similar questions about quasi-automorphisms may be posed regarding matroids. However, in light of Example 1.23, a more urgent problem is the following:

**Problem 5.17.** What is the right notion of restricted pushouts for matroids?

**Discussion 5.18.** As pointed out to us by Chris Eppolito, the restricted pushout of  $M_0 \leftarrow N \rightarrow M_1$  cannot be the pushout in the ambient category  $\text{Mat}_\bullet$  since this may not exist. Consider, for example, when  $N = \{a, b, c, \bullet\}$ ,  $M_0 = \{a, b, c, d, \bullet\}$  and  $M_1 = \{a, b, c, e, \bullet\}$ , where the ground sets are endowed with the uniform matroid structure of rank 2, with point  $\bullet$ .

**Discussion 5.19** (Independent Squares). A related problem was considered by the model theorists. Let  $\mathcal{D}$  be the category whose morphisms are all monic, and suppose  $\mathcal{D}$  admits a *stable independence notion*, i.e. a class of so-called *independent squares*

$$\begin{array}{ccc} N & \xrightarrow{\quad} & M_1 \\ \downarrow & \lrcorner & \downarrow \\ M_0 & \xrightarrow{\quad} & M \end{array}$$

satisfying the axioms listed in [Vas19, Definition 5.5]. This includes the condition that any span in  $\mathcal{D}$

$$M_0 \leftarrow N \rightarrow M_1$$

can be completed into an independent square. Examples include  $\text{FinSet}$ , the category of vector spaces over a fixed fields (with injective linear transformations), and the category of graphs (with subgraph embeddings).

Here is the key insight from [Vas19, §5.1]. Under certain technical conditions, the category of independent squares associated to a given span has a weakly initial object, which is unique up to (not necessarily unique) isomorphism. Model theorists call this the *prime object* over the span. Some natural questions:

- (1) Do prime objects satisfy the required properties of restricted pushouts for pCGW categories?
- (2) Can specific classes of matroids (e.g. vectorial, or graphic matroids) admit such a structure? All classes?
- (3) How does the model-theoretic notion of independent squares align with flats and ranks in matroids?

This will be investigated in future work.

## APPENDIX A. PROPERTIES OF RESTRICTED PUSHOUTS

As explained in Section 1.2, it is unreasonable to ask for  $\mathcal{M}$ -morphisms of CGW categories to be closed under pushouts, so we instead ask for them to be closed under *restricted pushouts*, a weaker notion. This section collects various technical facts about them, which hold in any pCGW category  $\mathcal{C}$ . Whenever relevant, we follow Convention 3.17 when labelling morphisms.

**Lemma A.1.**

(i) Given a span  $B \leftarrow A \rightarrow C$ ,

$$\frac{B}{A} \oplus \frac{C}{A} \cong \frac{B \star_A C}{A}.$$

(ii) Given a span  $B \leftarrow A \rightarrow C$ , along with  $\mathcal{M}$ -morphisms  $B \rightarrow B'$  and  $C \rightarrow C'$ ,

$$\frac{B' \star_A C'}{B \star_A C} \cong \frac{B'}{B} \oplus \frac{C'}{C}.$$

*Proof.*

(i): Consider the diagram

$$\begin{array}{ccccc} \frac{C}{A} & \xleftarrow{\quad} & O & \xrightarrow{\quad} & \frac{B}{A} \\ \downarrow \circlearrowleft & \square & \downarrow \circlearrowleft & \square & \downarrow \circlearrowleft \\ C & \xleftarrow{\quad} & A & \xrightarrow{\quad} & B \end{array}$$

Apply Axiom (DS) to obtain the left diagram below

$$\begin{array}{ccc}
 O & \xrightarrow{\quad} & \frac{B}{A} \xrightarrow{\quad} \frac{B}{A} \oplus \frac{C}{A} \\
 \downarrow \circlearrowleft & \square & \downarrow \circlearrowleft \\
 A & \xrightarrow{f} & B \xrightarrow{f'} B \star_A C
 \end{array}
 \qquad
 \begin{array}{ccc}
 O & \xrightarrow{\quad} & \frac{B \star_A C}{A} \\
 \downarrow \circlearrowleft & \square & \downarrow \circlearrowleft \\
 A & \xrightarrow{f'f} & B \star_A C
 \end{array}
 \tag{61}$$

Since distinguished squares compose, the outermost rectangle of the left diagram also defines a distinguished square. On the other hand, by Axiom (K), the right distinguished square above exists. Since formal cokernels are unique (up to unique isomorphism), conclude that  $\frac{B}{A} \oplus \frac{C}{A} \cong \frac{B \star_A C}{A}$ .

(ii): Repeated applications of Fact 1.17 yields

$$\begin{array}{ccccc}
 A & \xrightarrow{\quad} & B & \xrightarrow{\quad} & B' \\
 \downarrow & & \downarrow & & \downarrow \\
 C & \xrightarrow{\quad} & B \star_A C & \xrightarrow{\quad} & B' \star_A C \\
 \downarrow & & \downarrow & & \downarrow \\
 C' & \xrightarrow{\quad} & B \star_A C' & \xrightarrow{\quad} & B' \star_A C'
 \end{array}
 \tag{62}$$

In particular, there exists an  $\mathcal{M}$ -morphism  $B \star_A C \rightarrow B' \star_A C'$ , so  $\frac{B' \star_A C'}{B \star_A C}$  is well-defined and exists by Axiom (K). By Axiom (PQ), restricted pushouts preserve quotients, and so

$$\frac{B'}{B} \cong \frac{B' \star_A C}{B \star_A C} \quad \text{and} \quad \frac{C'}{C} \cong \frac{B \star_A C'}{B \star_A C}.
 \tag{63}$$

Now consider the diagram

$$\begin{array}{ccccc}
 \frac{B \star_A C'}{B \star_A C} & \xleftarrow{\quad} & O & \xrightarrow{\quad} & \frac{B' \star_A C}{B \star_A C} \\
 \downarrow \circlearrowleft & & \downarrow \circlearrowleft & & \downarrow \circlearrowleft \\
 B \star_A C' & \xleftarrow{\quad} & B \star_A C & \xrightarrow{\quad} & B' \star_A C
 \end{array}
 \tag{64}$$

Applying the same argument from (i) to Equations (63) and (64), deduce the desired isomorphism.  $\square$

**Corollary A.2.** *Given any span of 1-simplices in  $G\mathcal{C}$*

$$\begin{pmatrix} M' \\ L' \end{pmatrix} \leftarrow \begin{pmatrix} M \\ L \end{pmatrix} \rightarrow \begin{pmatrix} M'' \\ L'' \end{pmatrix},
 \tag{65}$$

*taking restricted pushouts yields the following diagram*

$$\begin{array}{ccc}
 \begin{pmatrix} M \\ L \end{pmatrix} & \xrightarrow{\quad} & \begin{pmatrix} M'' \\ L'' \end{pmatrix} \\
 \downarrow & \searrow & \downarrow \\
 \begin{pmatrix} M' \\ L' \end{pmatrix} & \xrightarrow{\quad} & \begin{pmatrix} M' \star_M M'' \\ L' \star_L L'' \end{pmatrix}
 \end{array},
 \tag{66}$$

*where both triangles bound 2-simplices. In particular, this defines a homotopy between Diagram (65) and*

$$\begin{pmatrix} M' \\ L' \end{pmatrix} \rightarrow \begin{pmatrix} M' \star_M M'' \\ L' \star_L L'' \end{pmatrix} \leftarrow \begin{pmatrix} M'' \\ L'' \end{pmatrix}.
 \tag{67}$$

*Proof.* Given Diagram (66) in  $G\mathcal{C}$ , notice that  $\frac{M'}{M} = \frac{L'}{L}$  and  $\frac{M''}{M} = \frac{L''}{L}$  (cf. Convention 3.2). Now form restricted pushouts  $P := M' \star_M M''$  and  $Q := L' \star_L L''$ . Apply Lemma A.1 to deduce

$$\frac{P}{M} \cong \frac{M'}{M} \oplus \frac{M''}{M} \quad \text{and} \quad \frac{Q}{L} \cong \frac{L'}{L} \oplus \frac{L''}{L} = \frac{M'}{M} \oplus \frac{M''}{M}.$$

In fact, Equation (61) of the proof tells us the data assembles into diagram pairs, such as

$$\begin{array}{ccc}
 M \rightharpoonup M' \rightharpoonup P & L \rightharpoonup L' \rightharpoonup Q \\
 \uparrow \square \uparrow \square \uparrow & \uparrow \square \uparrow \square \uparrow \\
 O \rightharpoonup \frac{M'}{M} \rightharpoonup \frac{M'}{M} \oplus \frac{M''}{M} & O \rightharpoonup \frac{M'}{M} \rightharpoonup \frac{M'}{M} \oplus \frac{M''}{M} \\
 \uparrow \square \uparrow & \uparrow \square \uparrow \\
 O \rightharpoonup \frac{M''}{M} & O \rightharpoonup \frac{M''}{M}
 \end{array} .$$

This defines 2-simplices in  $G\mathcal{C}$

$$\left( \begin{array}{c} O \rightharpoonup \frac{M \rightharpoonup M' \rightharpoonup P}{L \rightharpoonup L' \rightharpoonup Q} \\ O \rightharpoonup \frac{M \rightharpoonup M'' \rightharpoonup P}{L \rightharpoonup L'' \rightharpoonup Q} \end{array} \right) \quad \text{and} \quad \left( \begin{array}{c} O \rightharpoonup \frac{M \rightharpoonup M'' \rightharpoonup P}{L \rightharpoonup L'' \rightharpoonup Q} \\ O \rightharpoonup \frac{M \rightharpoonup M' \rightharpoonup P}{L \rightharpoonup L' \rightharpoonup Q} \end{array} \right),$$

which assembles into Diagram (66). □

**Lemma A.3.** *Given any distinguished square*

$$\phi := \left( \begin{array}{ccc} O & \rightharpoonup & C \\ \downarrow \square \downarrow g & & \\ A & \rightharpoonup & B \end{array} \right) \quad (68)$$

*the following squares are also distinguished*

(i) *Given any  $D \in \mathcal{C}$ :*

$$\begin{array}{ccc}
 O \rightharpoonup C \oplus D & & O \rightharpoonup C \\
 \downarrow \square \downarrow g \oplus 1 & \text{and} & \downarrow \square \downarrow g \oplus D \\
 A \rightharpoonup B \oplus D & & A \oplus D \rightharpoonup B \oplus D
 \end{array} \quad (69)$$

$$\begin{array}{ccc}
 C \xrightarrow{1 \oplus D} C \oplus D & & D \xrightarrow{C \oplus 1} C \oplus D \\
 g \downarrow \square \downarrow g \oplus D & \text{and} & A \oplus 1 \downarrow \square \downarrow g \oplus 1 \\
 B \xrightarrow{1 \oplus D} B \oplus D & & A \oplus D \xrightarrow{f \oplus 1} B \oplus D
 \end{array} \quad (70)$$

(ii) *Given  $A \xrightarrow{f \oplus D} B \oplus D$ ,*

$$\frac{B \oplus D}{A} \cong \frac{B}{A} \oplus D.$$

(iii) *Given any  $f': A' \rightharpoonup B'$ :*

$$\begin{array}{ccc}
 C \oplus A' \xrightarrow{1 \oplus f'} C \oplus B' & & \\
 g \oplus 1 \downarrow \square \downarrow g \oplus 1 & & \\
 B \oplus A' \xrightarrow{1 \oplus f'} B \oplus B' & &
 \end{array} \quad (71)$$

*Proof.* To ease notation, we omit the labels of the morphisms when they are clear from context.

(i): First, apply Fact 1.17 to

$$D \leftarrow O \rightharpoonup A \rightharpoonup B,$$

and obtain the isomorphism  $B \oplus D \cong (A \oplus D) \star_A B$ . Then apply Axiom (DS) of Definition 1.18 to the diagram

$$\begin{array}{ccccc}
 D & \longleftarrow & O & \longrightarrow & C \\
 \downarrow & & \downarrow & & \downarrow \\
 A \oplus D & \longleftarrow & A & \longrightarrow & B
 \end{array}$$

to obtain the following distinguished squares

$$\psi := \left( \begin{array}{ccc} C & \xrightarrow{\quad} & C \oplus D \\ \downarrow & \square & \downarrow \\ B & \xrightarrow{\quad} & B \oplus D \end{array} \right) \quad \psi' := \left( \begin{array}{ccc} D & \xrightarrow{\quad} & C \oplus D \\ \downarrow & \square & \downarrow \\ A \oplus D & \xrightarrow{\quad} & B \oplus D \end{array} \right).$$

Then, obtain Equation (69) by horizontal composition of  $\psi$  with  $\phi$ , and vertical composition of  $\psi'$  with the formal sum square

$$\begin{array}{ccc} O & \xrightarrow{\quad} & C \\ \downarrow & \square & \downarrow \\ D & \xrightarrow{\quad} & C \oplus D \end{array}.$$

(ii): Apply Axiom (K) to the left exact square of Diagram (69).

(iii): We will need Helper Definition 1.3, namely the following equivalence of categories:

$$c: \text{Ar}_{\square} \mathcal{M} \rightarrow \text{Ar}_{\Delta} \mathcal{E}$$

$$k: \text{Ar}_{\square} \mathcal{E} \rightarrow \text{Ar}_{\Delta} \mathcal{M}.$$

By item (i), the following squares are distinguished

$$\begin{array}{ccc} O & \xrightarrow{\quad} & C & \xrightarrow{\quad} & C \oplus A' \\ \downarrow & \square & \downarrow & \square & \downarrow \\ A & \xrightarrow{\quad} & B & \xrightarrow{\quad} & B \oplus A' \end{array} \quad \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{B'}{A'} \\ \downarrow & \square & \downarrow \\ B \oplus A' & \xrightarrow{\quad} & B \oplus B' \end{array},$$

and so we may construct the following diagram

$$\begin{array}{ccc} C \oplus A' & & \\ g \oplus 1 \downarrow & & \\ B \oplus A' & \xrightarrow{1 \oplus f'} & B \oplus B' \end{array} \quad (72)$$

and  $\mathcal{M}$ -morphism  $v := \left( A \xrightarrow{\quad} B \oplus A' \xrightarrow{\quad} B \oplus B' \right)$ . Since  $v$  factors through  $A \rightarrow B$  by construction, apply Item (i) to obtain

$$\begin{array}{ccc} O & \xrightarrow{\quad} & C \oplus B' \\ \downarrow & \square & \downarrow g \oplus 1 \\ A & \xrightarrow{v} & B \oplus B' \end{array}, \quad (73)$$

which is distinguished. Notice  $v$  also represents a morphism

$$(A \rightarrow B \oplus A') \rightarrow (A \rightarrow B \oplus B') \in \text{Ar}_{\Delta} \mathcal{M}.$$

Applying functor  $k^{-1}$  to this morphism produces the distinguished square

$$\begin{array}{ccc} C \oplus A' & \xrightarrow{u} & C \oplus B' \\ g \oplus 1 \downarrow & \square & \downarrow g \oplus 1 \\ B \oplus A' & \xrightarrow{1 \oplus f'} & B \oplus B' \end{array}, \quad (74)$$

where we use the fact that  $c$  and  $k$  are inverse on objects (Remark 1.5).

It remains to show  $u = 1 \oplus f'$ . Towards this end, consider the diagram

$$\begin{array}{ccc} C \oplus A' & \xrightarrow{1 \oplus f'} & C \oplus B' \\ & & \downarrow g \oplus 1 \\ & & B \oplus B' \end{array}, \quad (75)$$

and apply the dual argument (as above) to obtain a distinguished square

$$\begin{array}{ccc} C \oplus A' & \xrightarrow{1 \oplus f'} & C \oplus B' \\ v \downarrow & \square & \downarrow g \oplus 1 \\ B \oplus A' & \xrightarrow{1 \oplus f'} & B \oplus B' \end{array} . \quad (76)$$

Next, apply functor  $c$  to Diagrams (74) and (76) to obtain

$$\begin{array}{ccc} \frac{B'}{A'} \circ \xrightarrow{=} \frac{B'}{A'} & & \frac{B'}{A'} \circ \xrightarrow{=} \frac{B'}{A'} \\ p_u \downarrow & \downarrow q & p_v \downarrow \quad \downarrow q \\ C \oplus B' \circ \xrightarrow{g \oplus 1} B \oplus B' & & C \oplus B' \circ \xrightarrow{g \oplus 1} B \oplus B' \end{array} , \quad (77)$$

where the LHS is induced by Diagram (74) and the RHS by Diagram (76). Notice we have  $q: \frac{B'}{A'} \circ \rightarrow B \oplus B'$  in both squares, since this  $\mathcal{E}$ -morphism is induced by  $1 \oplus f': B \oplus A' \rightarrow B \oplus B'$ . Further, since  $\mathcal{E}$ -morphisms are monic by Axiom (M), this implies  $p_u = p_v$ . In other words, Diagrams (74) and (76) map to the same diagram in  $\text{Ar}_\Delta \mathcal{E}$  under  $c$ . Applying  $c^{-1}$  to Diagram (77), deduce that  $u = 1 \oplus f'$ ; once again, this follows from  $c$  and  $k$  being inverse on objects.  $\square$

**Lemma A.4** (Direct Sums).

(i) Given any pair of exact squares

$$\begin{array}{ccc} O & \xrightarrow{\quad} & \frac{B}{A} \\ \downarrow & \square & \downarrow f_2 \\ A & \xrightarrow{f_1} & B \end{array} , \quad \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{D}{C} \\ \downarrow & \square & \downarrow g_2 \\ C & \xrightarrow{g_1} & D \end{array}$$

we can construct the obvious exact square via direct sums

$$\begin{array}{ccc} O & \xrightarrow{\quad} & \frac{B}{A} \oplus \frac{D}{C} \\ \downarrow & \square & \downarrow f_2 \oplus g_2 \\ A \oplus C & \xrightarrow{f_1 \oplus g_1} & B \oplus D \end{array}$$

Notice this implies  $\frac{B \oplus D}{A \oplus C} \cong \frac{B}{A} \oplus \frac{D}{C}$ .

(ii) Given a pair of flag diagrams

$$\begin{array}{ccccc} A & \xrightarrow{f_0} & B & \xrightarrow{g_0} & C \\ \uparrow & \square & \uparrow f_1 & \square & \uparrow g_1 \\ O & \xrightarrow{\quad} & \frac{B}{A} & \xrightarrow{h_0} & \frac{C}{A} \\ & & \uparrow & \square & \uparrow h_1 \\ & & O & \xrightarrow{\quad} & \frac{C}{B} \end{array} \quad \begin{array}{ccccc} A' & \xrightarrow{f'_0} & B' & \xrightarrow{g'_0} & C' \\ \uparrow & \square & \uparrow f'_1 & \square & \uparrow g'_1 \\ O & \xrightarrow{\quad} & \frac{B'}{A'} & \xrightarrow{h'_0} & \frac{C'}{A'} \\ & & \uparrow & \square & \uparrow h'_1 \\ & & O & \xrightarrow{\quad} & \frac{C'}{B'} \end{array} .$$

we can construct the following distinguished square

$$\begin{array}{ccc} \frac{B}{A} \oplus \frac{B'}{A'} & \xrightarrow{h_0 \oplus h'_0} & \frac{C}{A} \oplus \frac{C'}{A'} \\ f_1 \oplus f'_1 \downarrow & \square & \downarrow g_1 \oplus g'_1 \\ B \oplus B' & \xrightarrow{g_0 \oplus g'_0} & C \oplus C' \end{array}$$



*Proof.* (i): Take repeated restricted pushouts to get

$$\begin{array}{ccccc}
O & \xrightarrow{\quad} & C & \xrightarrow{g_1} & D \\
\downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{\quad} & A \oplus C & \xrightarrow{1 \oplus g_1} & A \oplus D \\
\downarrow f_1 & & \downarrow f_1 \oplus 1 & & \downarrow f_1 \oplus 1 \\
B & \xrightarrow{\quad} & B \oplus C & \xrightarrow{1 \oplus g_1} & B \oplus D
\end{array}$$

Since restricted pushouts preserve quotients, construct the diagram

$$\begin{array}{ccccc}
\frac{D}{C} & \xleftarrow{\quad} & O & \xrightarrow{\quad} & \frac{B}{A} \\
g_2 \downarrow & \square & \downarrow & \square & \downarrow f_2 \\
A \oplus D & \xleftarrow{1 \oplus g_1} & A \oplus C & \xrightarrow{f_1 \oplus 1} & B \oplus C
\end{array}$$

The rest follows from applying Axiom (DS) and the fact that distinguished squares compose horizontally.

(ii): Part (i) and Lemma 1.8 gives

$$\begin{array}{ccccc}
\frac{B}{A} \oplus \frac{B'}{A'} & \xrightarrow{v} & \frac{C}{A} \oplus \frac{C'}{A'} \\
f_1 \oplus f_1' \downarrow & \square & \downarrow g_1 \oplus g_1' \\
B \oplus B' & \xrightarrow{g_0 \oplus g_0'} & C \oplus C'
\end{array}$$

It remains to check  $v = h_0 \oplus h_0'$ , but this follows by a similar argument to Lemma A.3 (iii).  $\square$

## APPENDIX B. TECHNICAL LEMMAS & SOME 2-SIMPLICES

**B.1. Technical Facts about Sherman Loops.** To finish the proof of Theorem 3.7, we will require the following three technical lemmas. A high-level summary: the results here extend the arguments from [She94, §2] to the setting of pCGW categories.

**Lemma B.1.** *The sum of two Sherman loops is equivalent to a Sherman loop. Explicitly, consider two pairs of  $\mathcal{M}$ -morphisms*

$$\alpha_i: A_i \rightarrow X_i \quad , \quad \beta_i: B_i \rightarrow Y_i, \quad \text{for } i = 1, 2;$$

and isomorphisms

$$\theta_i: A_i \oplus \frac{X_i}{A_i} \oplus Y_i \longrightarrow B_i \oplus \frac{Y_i}{B_i} \oplus X_i, \quad \text{for } i = 1, 2.$$

*Then*

$$G(\alpha_1, \beta_1, \theta_1) + G(\alpha_2, \beta_2, \theta_2) = G(\alpha_1 \oplus \alpha_2, \beta_1 \oplus \beta_2, T_2(\theta_1 \oplus \theta_2)T_1^{-1})$$

in  $K_1(\mathcal{C})$ , whereby  $T_1$  and  $T_2$  are the canonical permutation isomorphisms

$$T_1: A_1 \oplus \frac{X_1}{A_1} \oplus Y_1 \oplus A_2 \oplus \frac{X_2}{A_2} \oplus Y_2 \rightarrow A_1 \oplus A_2 \oplus \frac{X_1}{A_1} \oplus \frac{X_2}{A_2} \oplus Y_1 \oplus Y_2$$

$$T_2: B_1 \oplus \frac{Y_1}{B_1} \oplus X_1 \oplus B_2 \oplus \frac{Y_2}{B_2} \oplus X_2 \rightarrow B_1 \oplus B_2 \oplus \frac{Y_1}{B_1} \oplus \frac{Y_2}{B_2} \oplus X_1 \oplus X_2.$$

*Proof.* There are no surprises – the argument is the same as in [She94, Prop. 1]. Apply the  $H$ -space structure of  $G\mathcal{C}$  to construct the loop corresponding to  $G(\alpha_1, \beta_1, \theta_1) + G(\alpha_2, \beta_2, \theta_2)$ . Notice it is almost identical to  $G(\alpha_1 \oplus \alpha_2, \beta_1 \oplus \beta_2, T_2(\theta_1 \oplus \theta_2)T_1^{-1})$  – the only difference being that the direct summands of certain vertices are arranged in a different order.

To establish homotopy equivalence, use natural isomorphisms to permute these summands and construct a sequence of 2-simplices connecting the two loops. Finally, check the choices of 2-simplices are valid by applying the fact that distinguished squares interact well with isomorphisms (Definition 1.1).  $\square$

**Lemma B.2.** *Let  $\mathcal{C}$  be a  $p\text{CGW}$  category whose exact squares all split. Then, every element of  $K_1(\mathcal{C})$  corresponds to the loop*

$$G(A, \alpha) := \left( \begin{array}{ccc} (A, A) & \xrightarrow{l(\alpha)} & (A, A) \\ & \nwarrow \quad \nearrow & \\ & (O, O) & \end{array} \right) \quad (78)$$

where  $l(\alpha)$  is the 1-simplex

$$l(\alpha) := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & O \\ \downarrow & \square & \downarrow \\ A & \xrightarrow{1} & A \end{array} , \begin{array}{ccc} O & \xrightarrow{\quad} & O \\ \downarrow & \square & \downarrow \\ A & \xrightarrow{\alpha} & A \end{array} \right)$$

for some automorphism  $(A, \alpha) \in \text{Aut}(\mathcal{C})$ .

*Proof.* The proof combines an argument from [GG87, §5] and [She94, Prop. 2]. Proceed in stages.

*Step 1: Combinatorial Loops in  $K_1(\mathcal{C})$ .* Suppose  $z \in K_1(\mathcal{C}) = \pi_1|G\mathcal{C}|$ . By the simplicial approximation theorem,  $z$  can be represented by a loop formed combinatorially from 1-simplices of  $G\mathcal{C}$

$$\begin{pmatrix} O \\ O \end{pmatrix} \rightarrow \bullet \leftarrow \bullet \rightarrow \dots \leftarrow \bullet \rightarrow \bullet \leftarrow \begin{pmatrix} O \\ O \end{pmatrix} \quad (79)$$

where we draw the 1-simplices as arrows. Consider one of the configurations in Diagram (79), e.g.

$$\begin{pmatrix} M' \\ L' \end{pmatrix} \leftarrow \begin{pmatrix} M \\ L \end{pmatrix} \rightarrow \begin{pmatrix} M'' \\ L'' \end{pmatrix}.$$

By Corollary A.2, this is homotopic to

$$\begin{pmatrix} M' \\ L' \end{pmatrix} \rightarrow \begin{pmatrix} M' \star_M M'' \\ L' \star_L L'' \end{pmatrix} \leftarrow \begin{pmatrix} M'' \\ L'' \end{pmatrix}.$$

Applying this trick multiple times, we can deform Loop (79) into one of the form

$$\begin{array}{ccccccc} \begin{pmatrix} O \\ O \end{pmatrix} & \longrightarrow & \begin{pmatrix} M_0 \\ L_0 \end{pmatrix} & \longrightarrow & \dots & \longrightarrow & \begin{pmatrix} M_{q-1} \\ L_{q-1} \end{pmatrix} \\ \downarrow & & & & & & \downarrow \\ \begin{pmatrix} M'_0 \\ L'_0 \end{pmatrix} & \longrightarrow & \dots & \longrightarrow & \begin{pmatrix} M'_{q-1} \\ L'_{q-1} \end{pmatrix} & \longrightarrow & \begin{pmatrix} M \\ L \end{pmatrix} \end{array} \quad (80)$$

*Step 2: The Base Case.* Start by analysing the component

$$\begin{pmatrix} O \\ O \end{pmatrix} \xrightarrow{l_0} \begin{pmatrix} M_0 \\ L_0 \end{pmatrix} \xrightarrow{l_1} \begin{pmatrix} M_1 \\ L_1 \end{pmatrix} \quad (81)$$

of Loop (80) in  $K_1(\mathcal{C})$ . Suppose  $l_0$  is defined by the following pair of exact squares

$$l_0 := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & \widehat{M}_0 \\ \downarrow & \square & \downarrow \eta_0 \\ O & \xrightarrow{\quad} & M_0 \end{array} , \begin{array}{ccc} O & \xrightarrow{\quad} & \widehat{M}_0 \\ \downarrow & \square & \downarrow \mu_0 \\ O & \xrightarrow{\quad} & L_0 \end{array} \right), \quad (82)$$

with isomorphisms  $\eta_0$  and  $\mu_0$ . Recall that CGW categories require an isomorphism of categories

$$\varphi: \text{iso}\mathcal{M} \rightarrow \text{iso}\mathcal{E}.$$

We can therefore define two 1-simplices

$$l'_0 := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & \widehat{M}_0 \\ \downarrow & \square & \downarrow 1 \\ O & \xrightarrow{\quad} & \widehat{M}_0 \end{array} , \begin{array}{ccc} O & \xrightarrow{\quad} & \widehat{M}_0 \\ \downarrow & \square & \downarrow 1 \\ O & \xrightarrow{\quad} & \widehat{M}_0 \end{array} \right) \quad \text{and} \quad l''_0 := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & O \\ \downarrow & \square & \downarrow \\ \widehat{M}_0 & \xrightarrow{\varphi^{-1}(\eta_0)} & M_0 \end{array} , \begin{array}{ccc} O & \xrightarrow{\quad} & O \\ \downarrow & \square & \downarrow \\ \widehat{M}_0 & \xrightarrow{\varphi^{-1}(\mu_0)} & L_0 \end{array} \right),$$

which assemble into the following 2-simplex

$$\begin{array}{ccccc} O & \xrightarrow{\quad} & \widehat{M}_0 & \xrightarrow{\varphi^{-1}(\eta_0)} & M_0 \\ \uparrow & \square & \uparrow 1 & \square & \uparrow \eta_0 \\ O & \xrightarrow{\quad} & \widehat{M}_0 & \xrightarrow{1} & \widehat{M}_0 \\ & & \uparrow & \square & \uparrow \\ & & O & \xrightarrow{\quad} & O \end{array} \quad \begin{array}{ccccc} O & \xrightarrow{\quad} & \widehat{M}_0 & \xrightarrow{\varphi^{-1}(\mu_0)} & L_0 \\ \uparrow & \square & \uparrow 1 & \square & \uparrow \mu_0 \\ O & \xrightarrow{\quad} & \widehat{M}_0 & \xrightarrow{1} & \widehat{M}_0 \\ & & \uparrow & \square & \uparrow \\ & & O & \xrightarrow{\quad} & O \end{array}.$$

We remark that the top right squares are distinguished by Axiom (I) of Definition 1.4. We then assemble the following diagram

$$\begin{array}{ccccc} & & \begin{pmatrix} \widehat{M}_0 \\ \widehat{M}_0 \end{pmatrix} & & \\ & \nearrow l'_0 & \downarrow l''_0 & \nwarrow l_1 \circ l''_0 & \\ \begin{pmatrix} O \\ O \end{pmatrix} & \xrightarrow{l_0} & \begin{pmatrix} M_0 \\ L_0 \end{pmatrix} & \xrightarrow{l_1} & \begin{pmatrix} M_1 \\ L_1 \end{pmatrix} \end{array} \quad (83)$$

One easily checks the added triangle also bounds a 2-simplex, which implies the red path of 1-simplices is homotopic to the blue path. Hence, assume without loss of generality that both  $\mu_0$  and  $\eta_0$  of Equation (82) are the identity  $1: M_0 \rightarrow M_0$ .

*Step 3: The Inductive Step.* Proceeding along Loop (81), consider  $l_1$ , to be represented as

$$l_1 := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{M_1}{M_0} \\ \downarrow & \square & \downarrow \eta'_1 \\ M_0 & \xrightarrow{\eta_1} & M_1 \end{array} , \begin{array}{ccc} O & \xrightarrow{\quad} & \frac{M_1}{M_0} \\ \downarrow & \square & \downarrow \mu'_1 \\ M_0 & \xrightarrow{\mu_1} & L_1 \end{array} \right).$$

Since all exact squares split by hypothesis, this yields isomorphisms

$$\psi_M: M_1 \rightarrow M_0 \oplus \frac{M_1}{M_0} \quad \text{and} \quad \psi_L: L_1 \rightarrow M_0 \oplus \frac{M_1}{M_0},$$

which we shall consider as  $\mathcal{M}$ -morphisms. This defines the following distinguished squares

$$\begin{array}{ccccc} O & \xrightarrow{\quad} & \frac{M_1}{M_0} & \xrightarrow{1} & \frac{M_1}{M_0} \\ \downarrow & \square & \downarrow \eta'_1 & \square & \downarrow \varphi(\psi_M) \circ \eta'_1 \\ M_0 & \xrightarrow{\eta_1} & M_1 & \xrightarrow{\psi_M} & M_0 \oplus \frac{M_1}{M_0} \end{array} \quad \begin{array}{ccccc} O & \xrightarrow{\quad} & \frac{M_1}{M_0} & \xrightarrow{1} & \frac{M_1}{M_0} \\ \downarrow & \square & \downarrow \eta'_1 & \square & \downarrow \varphi(\psi_L) \circ \mu'_1 \\ M_0 & \xrightarrow{\mu_1} & L_1 & \xrightarrow{\psi_L} & M_0 \oplus \frac{M_1}{M_0} \end{array}$$

The right squares are distinguished by Axiom (I), and thus the horizontal compositions define two exact squares. To ease notation, denote

$$\begin{aligned} v &:= \psi_M \circ \eta_1 & \text{and} & & v' &:= \varphi(\psi_M) \circ \eta'_1, \\ w &:= \psi_L \circ \mu_1 & \text{and} & & w' &:= \varphi(\psi_L) \circ \mu'_1. \end{aligned}$$

In fact, we can say more. Denote

$$\begin{array}{ccc} O & \xrightarrow{\quad} & \frac{M_1}{M_0} \\ \downarrow & \square & \downarrow q \\ M_0 & \xrightarrow{p} & M_0 \oplus \frac{M_1}{M_0} \end{array}$$

to be the canonical direct sum square of  $M_0 \oplus \frac{M_1}{M_0}$ . Since both exact squares in  $l_1$  are split, we know that

$$v = \psi_M \circ \eta_1 = p = \psi_L \circ \mu_1 = w \quad (84)$$

$$v' = \varphi(\psi_M) \circ \eta'_1 = q = \varphi(\psi_L) \circ \mu'_1 = w'.$$

Leverage these identities to construct the following 2-simplex

$$\begin{array}{ccc} O \xrightarrow{\quad} M_0 \xrightarrow{\eta_1} M_1 & O \xrightarrow{\quad} M_0 \xrightarrow{\mu_1} L_1 & \\ \uparrow \square \uparrow 1 \uparrow \square \uparrow \varphi(\psi_M^{-1}) & \uparrow \square \uparrow 1 \uparrow \square \uparrow \varphi(\psi_L^{-1}) & \\ O \xrightarrow{\quad} M_0 \xrightarrow{v} M_0 \oplus \frac{M_1}{M_0} & O \xrightarrow{\quad} M_0 \xrightarrow{v} M_0 \oplus \frac{M_1}{M_0} & \\ \uparrow \square \uparrow v' & \uparrow \square \uparrow v' & \\ O \xrightarrow{\quad} \frac{M_1}{M_0} & O \xrightarrow{\quad} \frac{M_1}{M_0} & \end{array} \quad (85)$$

To see why the square indicated in red is distinguished, notice:

- **Case 1.** Suppose  $\mathcal{E}$  is a subcategory of  $\mathcal{C}$ . Then

$$\varphi(\psi_L^{-1}) \circ v = \psi_L^{-1} \circ v = \mu_1 \quad \text{in } \mathcal{C}$$

if and only if

$$v = \psi_L \circ \mu_1,$$

which holds by Identity (84).

- **Case 2.** Suppose  $\mathcal{E}^{\text{op}}$  is a subcategory of  $\mathcal{C}$ . Applying Identity (84) once more, deduce

$$v = \varphi(\psi_L^{-1}) \circ \mu_1 = \psi_L \circ \mu_1.$$

The claim then follows from the fact that distinguished squares interact well with isomorphisms. The case for the blue-indicated square is analogous.

Having checked that Diagram (85) defines a 2-simplex, the gears line up and the inductive argument falls into place. First note that Diagram (85) defines a homotopy between

$$\begin{pmatrix} O \\ O \end{pmatrix} \xrightarrow{l_0} \begin{pmatrix} M_0 \\ L_0 \end{pmatrix} \xrightarrow{l_1} \begin{pmatrix} M_1 \\ L_1 \end{pmatrix}$$

and a 1-simplex of the form

$$l'_1: \begin{pmatrix} O \\ O \end{pmatrix} \rightarrow \begin{pmatrix} M_1 \\ L_1 \end{pmatrix}$$

whereby

$$l'_1 := \left( \begin{array}{ccc} O \xrightarrow{\quad} \widehat{M}_1 & & O \xrightarrow{\quad} \widehat{M}_1 \\ \downarrow \square \downarrow \alpha_1 & & \downarrow \square \downarrow \beta_1 \\ O \xrightarrow{\quad} M_1 & & O \xrightarrow{\quad} L_1 \end{array} \right).$$

Hence, the initial segment of Loop (80) is homotopic to

$$\begin{pmatrix} O \\ O \end{pmatrix} \xrightarrow{l'_1} \begin{pmatrix} M_1 \\ L_1 \end{pmatrix} \xrightarrow{l_2} \begin{pmatrix} M_2 \\ L_2 \end{pmatrix},$$

deleting a term. Then, apply the Base Case argument (Step 2) to justify presenting  $l'_1$  as

$$l'_1 := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & M_1 \\ \downarrow & \square & \downarrow_1 \\ O & \xrightarrow{\quad} & M_1 \end{array} , \quad \begin{array}{ccc} O & \xrightarrow{\quad} & M_1 \\ \downarrow & \square & \downarrow_1 \\ O & \xrightarrow{\quad} & M_1 \end{array} \right),$$

which sets up our inductive step again. Keep going for the rest of Loop (80) on both sides, until we finally obtain a loop of the form

$$\begin{pmatrix} O \\ O \end{pmatrix} \xrightarrow{\kappa} (M, L) \xleftarrow{\gamma} \begin{pmatrix} O \\ O \end{pmatrix}, \quad (86)$$

where

$$\kappa := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & \widehat{M} \\ \downarrow & \square & \downarrow_{\kappa_0} \\ O & \xrightarrow{\quad} & M \end{array} , \quad \begin{array}{ccc} O & \xrightarrow{\quad} & \widehat{M} \\ \downarrow & \square & \downarrow_{\kappa_1} \\ O & \xrightarrow{\quad} & L \end{array} \right) \quad \text{and} \quad \gamma := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & \widehat{N} \\ \downarrow & \square & \downarrow_{\gamma_0} \\ O & \xrightarrow{\quad} & M \end{array} , \quad \begin{array}{ccc} O & \xrightarrow{\quad} & \widehat{N} \\ \downarrow & \square & \downarrow_{\gamma_1} \\ O & \xrightarrow{\quad} & L \end{array} \right)$$

Notice we can no longer apply the Base Case argument to simplify  $\kappa$  or  $\gamma$  since the arrows of Diagram (86) are in the wrong direction.

*Step 4: Finish.* A technical observation: both  $\kappa$  and  $\gamma$  define isomorphisms  $L \xrightarrow{\cong} M$  in  $\mathcal{C}$  but the presentation will differ depending on whether  $\mathcal{E}^{\text{op}}$  or  $\mathcal{E}$  is a subcategory of  $\mathcal{C}$ .

**Case 1:**  $\mathcal{E} \subseteq \mathcal{C}$ . In which case, define  $\omega := \kappa_0 \circ \kappa_1^{-1}$  and  $\lambda := \gamma_0 \circ \gamma_1^{-1}$  in  $\mathcal{C}$

**Case 2:**  $\mathcal{E}^{\text{op}} \subseteq \mathcal{C}$ . In which case, define  $\omega := \kappa_0^{-1} \circ \kappa_1$  and  $\lambda := \gamma_0^{-1} \circ \gamma_1$  in  $\mathcal{C}$ .

By Axiom (I), we may regard  $\omega$  and  $\lambda$  as  $\mathcal{M}$ -morphisms as well. We now construct the obvious diagram

$$\begin{array}{ccccc} & & \begin{pmatrix} M \\ M \end{pmatrix} & \xrightarrow{g} & \begin{pmatrix} M \\ M \end{pmatrix} \\ & \nearrow f_0 & \uparrow f_1 & \nearrow f_2 & \uparrow f_3 \\ \begin{pmatrix} O \\ O \end{pmatrix} & \xrightarrow{\kappa} & \begin{pmatrix} M \\ L \end{pmatrix} & \xleftarrow{\gamma} & \begin{pmatrix} O \\ O \end{pmatrix} \end{array} \quad (87)$$

whereby

$$g := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & O \\ \downarrow & \square & \downarrow \\ M & \xrightarrow{1} & M \end{array} , \quad \begin{array}{ccc} O & \xrightarrow{\quad} & O \\ \downarrow & \square & \downarrow \\ M & \xrightarrow{\lambda \circ \omega^{-1}} & M \end{array} \right)$$

$$f_0 := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & \widehat{M} \\ \downarrow & \square & \downarrow_{\kappa_0} \\ O & \xrightarrow{\quad} & M \end{array} , \quad \begin{array}{ccc} O & \xrightarrow{\quad} & \widehat{M} \\ \downarrow & \square & \downarrow_{\kappa_0} \\ O & \xrightarrow{\quad} & M \end{array} \right) \quad f_3 := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & \widehat{N} \\ \downarrow & \square & \downarrow_{\gamma_0} \\ O & \xrightarrow{\quad} & M \end{array} , \quad \begin{array}{ccc} O & \xrightarrow{\quad} & \widehat{N} \\ \downarrow & \square & \downarrow_{\gamma_0} \\ O & \xrightarrow{\quad} & M \end{array} \right).$$

$$f_1 := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & O \\ \downarrow & \square & \downarrow \\ M & \xrightarrow{1} & M \end{array} , \quad \begin{array}{ccc} O & \xrightarrow{\quad} & O \\ \downarrow & \square & \downarrow \\ L & \xrightarrow{\omega} & M \end{array} \right) \quad f_2 := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & O \\ \downarrow & \square & \downarrow \\ M & \xrightarrow{1} & M \end{array} , \quad \begin{array}{ccc} O & \xrightarrow{\quad} & O \\ \downarrow & \square & \downarrow \\ L & \xrightarrow{\lambda} & M \end{array} \right).$$

In both cases ( $\mathcal{E} \subseteq \mathcal{C}$  or  $\mathcal{E}^{\text{op}} \subseteq \mathcal{C}$ ), it is easy to check that the triangles of Diagram (87) bound the following 2-simplices:

$$\begin{aligned}
f_1 \kappa &= f_0 & \left( \begin{array}{ccc} O \rightharpoonup M \xrightarrow{1} M & O \rightharpoonup L \xrightarrow{\omega} M \\ \uparrow \kappa_0 \square \uparrow \kappa_0 & \uparrow \kappa_1 \square \uparrow \kappa_0 \\ \widehat{M} \xrightarrow{1} \widehat{M} & \widehat{M} \xrightarrow{1} \widehat{M} \\ \uparrow & \uparrow \\ O & O \end{array} \right) \\
f_2 \gamma &= f_3 & \left( \begin{array}{ccc} O \rightharpoonup M \xrightarrow{1} M & O \rightharpoonup L \xrightarrow{\lambda} M \\ \uparrow \gamma_0 \square \uparrow \gamma_0 & \uparrow \gamma_1 \square \uparrow \gamma_0 \\ \widehat{N} \xrightarrow{1} \widehat{N} & \widehat{N} \xrightarrow{1} \widehat{N} \\ \uparrow & \uparrow \\ O & O \end{array} \right) \\
gf_1 &= f_2 & \left( \begin{array}{ccc} M \xrightarrow{1} M \xrightarrow{1} M & L \xrightarrow{\omega} M \xrightarrow{\lambda \circ \omega^{-1}} M \\ \uparrow \square \uparrow & \uparrow \square \uparrow \\ O \rightharpoonup O & O \rightharpoonup O \\ \uparrow & \uparrow \\ O & O \end{array} \right)
\end{aligned}$$

Conclude that the red loop in Diagram (87) is homotopic to the blue loop. Notice the blue loop is precisely of the form  $G(A, \alpha)$  as claimed in lemma statement, with  $A := M$  and  $\alpha := \lambda \circ \omega^{-1}$ .  $\square$

**Lemma B.3.** *The automorphism loop  $G(A, \alpha)$  in Lemma B.2 is equivalent to a Sherman Loop.*

*Proof.* Let  $p_A: A \rightharpoonup A \oplus A$  be an  $\mathcal{M}$ -morphism arising from direct sum squares and denote  $\tau_A: A \oplus A \rightharpoonup A \oplus A$  to be the isomorphism swapping components. Define the following loop

$$\begin{pmatrix} O \\ O \end{pmatrix} \rightarrow \begin{pmatrix} A \\ A \end{pmatrix} \xrightarrow{\iota_\alpha} \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \xrightarrow{l_\tau} \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \leftarrow \begin{pmatrix} O \\ O \end{pmatrix} \quad (88)$$

where

$$\iota_\alpha := \left( \begin{array}{ccc} O \rightharpoonup A & O \rightharpoonup A \\ \downarrow \square \downarrow q_A & \downarrow \square \downarrow q_A \\ A \xrightarrow{p_A} A \oplus A & A \xrightarrow{p_A \circ \alpha} A \oplus A \end{array} \right), \quad l_\tau := \left( \begin{array}{ccc} O \rightharpoonup O & O \rightharpoonup O \\ \downarrow \square \downarrow & \downarrow \square \downarrow \\ A \oplus A \xrightarrow{\tau_A} A \oplus A & A \oplus A \xrightarrow{\tau_A} A \oplus A \end{array} \right)$$

This is a Sherman Loop  $G(\alpha, 0, \tau_A)$ , where 0 denotes the  $\mathcal{M}$ -morphism  $O \rightharpoonup A$ .

To show  $G(A, \alpha) \sim G(\alpha, 0, \tau_A)$  in  $\pi_1(|G\mathcal{C}|)$  involves modifying  $G(A, \alpha)$  in sensible ways that respects its homotopy class. An initial observation: the following Loop

$$\begin{pmatrix} O \\ O \end{pmatrix} \rightarrow \begin{pmatrix} A \\ A \end{pmatrix} \xrightarrow{\iota_\alpha} \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \leftarrow \begin{pmatrix} O \\ O \end{pmatrix}. \quad (89)$$

is homotopic to  $G(A, \alpha)$ , since the following diagram

$$\begin{array}{ccc} \begin{pmatrix} A \\ A \end{pmatrix} & \xrightarrow{\iota_\alpha} & \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \\ & \searrow \alpha & \nearrow \iota_A \\ & \begin{pmatrix} A \\ A \end{pmatrix} & \end{array}$$

bounds a 2-simplex, where  $\iota_A$  corresponds to the canonical direct sum square  $A \oplus A$ . Further, the Loop (89) is homotopic to

$$\begin{pmatrix} O \\ O \end{pmatrix} \rightarrow \begin{pmatrix} A \\ A \end{pmatrix} \xrightarrow{\iota_\alpha} \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \xrightarrow{1_{A \oplus A}} \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \leftarrow \begin{pmatrix} O \\ O \end{pmatrix}, \quad (90)$$

since all we did was insert a degenerate 1-simplex  $1_{A \oplus A}$ . It remains to show that  $G(\alpha, 0, \tau_A)$  is homotopic to Loop (90). But this follows from noting that the triangles in the diagram below bound 2-simplices

$$\begin{array}{ccc} \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} & \xrightarrow{1_{A \oplus A}} & \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} & \leftarrow & \begin{pmatrix} O \\ O \end{pmatrix} \\ & \searrow \iota_\tau & \uparrow \iota_\tau & \swarrow & \\ & \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} & & & \end{array}.$$

□

**B.2. Explicit Descriptions of 2-Simplices.** This section explicitly constructs the two key homotopies claimed by Lemma 3.22, necessary to finish the proof of Theorem 3.16

**Claim B.4.** *Loop (33) is homotopic to loop  $L$ .*

*Proof.* Recall: in order to establish that the two loops are homotopic, it suffices to show that the indicated triangles of the Diagram (40) are boundaries of 2-simplices. We describe the 2-simplices explicitly below.

- Triangle (1). Consider

$$\begin{array}{ccc} A \oplus A' \xrightarrow{f_0} P \xrightarrow{1 \oplus C \oplus C'} P \oplus C \oplus C' & A \oplus A' \xrightarrow{f_1} Q \xrightarrow{h_t} V \\ \uparrow g_0 \quad \square \quad \uparrow g_0 \oplus 1 & \uparrow g_1 \quad \square \quad \uparrow j_t \\ C \oplus C' \longrightarrow C \oplus C' \oplus C \oplus C' & C \oplus C' \longrightarrow C \oplus C' \oplus C \oplus C' \\ \uparrow & \uparrow \\ C \oplus C' & C \oplus C' \end{array}$$

[To show this is a 2-simplex, we need to check that the relevant subdiagrams define distinguished squares. The indicated square on the right diagram is the distinguished square  $t$  of Lemma 3.21. The left indicated square is distinguished by Lemma A.3 (i). The rest are obvious.]

- Triangle (2).

$$\begin{array}{ccc} A \oplus A' \xrightarrow{f_1} Q \xrightarrow{1 \oplus C \oplus C'} Q \oplus C \oplus C' & A \oplus A' \xrightarrow{f_1} Q \xrightarrow{h_t} V \\ \uparrow g_1 \quad \square \quad \uparrow g_1 \oplus 1 & \uparrow g_1 \quad \square \quad \uparrow j_t \\ C \oplus C' \longrightarrow C \oplus C' \oplus C \oplus C' & C \oplus C' \longrightarrow C \oplus C' \oplus C \oplus C' \\ \uparrow & \uparrow \\ C \oplus C' & C \oplus C' \end{array}$$

[Why is this a 2-simplex? Analogous to Triangle (1).]

- Triangle (3).

$$\begin{array}{ccc} P \xrightarrow{1 \oplus C \oplus C'} P \oplus C \oplus C' \xrightarrow{\theta \oplus 1} Q \oplus C \oplus C' & Q \xrightarrow{h_t} V \xrightarrow{1} V \\ \uparrow P \oplus 1 \quad \square \quad \uparrow Q \oplus 1 & \uparrow k_t \quad \square \quad \uparrow k_t \\ C \oplus C' \xrightarrow{1} C \oplus C' & C \oplus C' \xrightarrow{1} C \oplus C' \\ \uparrow & \uparrow \\ O & O \end{array}$$

[To show that this is a 2-simplex, notice Lemma 3.21 already verified that

$$t' := \left( \begin{array}{ccc} O & \xrightarrow{\quad} & C \oplus C' \\ \downarrow & \square & \downarrow k_t \\ Q & \xrightarrow{h_t} & V \end{array} \right)$$

is a distinguished square. The rest either follow from Lemma A.3 or are immediate.]

- Triangle (4).

$$\begin{array}{ccc} P \xrightarrow{\theta} Q \xrightarrow{1 \oplus C \oplus C'} Q \oplus C \oplus C' & Q \xrightarrow{1} Q \xrightarrow{h_t} V \\ \uparrow \circlearrowleft & \square & \uparrow \circlearrowleft \\ O \xrightarrow{\quad} C \oplus C' & O \xrightarrow{\quad} C \oplus C' \\ \uparrow \circlearrowleft & & \uparrow \circlearrowleft \\ C \oplus C' & & C \oplus C' \end{array}$$

[Why is this a 2-simplex? Obvious.]

□

**Claim B.5.** *Loop (35) is homotopic to loop L.*

*Proof.* By analogy with Claim B.4, all indicated triangles Diagram (41) can be shown to bound 2-simplices. The only subtlety is that we need to verify that Triangles (1') and (2') do in fact bound the 2-simplices below, but this follows from  $V$  being a restricted pushout (see remarks above Diagram (40)).

- Triangle (1').

$$\begin{array}{ccc} A \oplus A' \xrightarrow{f_0} P \xrightarrow{1 \oplus C \oplus C'} P \oplus C \oplus C' & A \oplus A' \xrightarrow{\alpha \oplus \alpha'} B \oplus B' \xrightarrow{h_u} V \\ \uparrow g_0 \circlearrowleft & \square & \uparrow j_u \circlearrowleft \\ C \oplus C' \xrightarrow{\quad} C \oplus C' \oplus C \oplus C' & C \oplus C' \xrightarrow{\quad} C \oplus C' \oplus C \oplus C' \\ \uparrow \circlearrowleft & & \uparrow \circlearrowleft \\ C \oplus C' & & C \oplus C' \end{array}$$

- Triangle (2').

$$\begin{array}{ccc} A \oplus A' \xrightarrow{f_1} Q \xrightarrow{1 \oplus C \oplus C'} Q \oplus C \oplus C' & A \oplus A' \xrightarrow{\alpha \oplus \alpha'} B \oplus B' \xrightarrow{h_u} V \\ \uparrow g_1 \circlearrowleft & \square & \uparrow j_u \circlearrowleft \\ C \oplus C' \xrightarrow{\quad} C \oplus C' \oplus C \oplus C' & C \oplus C' \xrightarrow{\quad} C \oplus C' \oplus C \oplus C' \\ \uparrow \circlearrowleft & & \uparrow \circlearrowleft \\ C' & & C' \end{array}$$

- Triangle (3').

$$\begin{array}{ccc} P \xrightarrow{1 \oplus C \oplus C'} P \oplus C \oplus C' \xrightarrow{\theta \oplus 1} Q \oplus C \oplus C' & B \oplus B' \xrightarrow{h_u} V \xrightarrow{1} V \\ \uparrow P \oplus 1 \circlearrowleft & \square & \uparrow k_u \circlearrowleft \\ C \oplus C' \xrightarrow{1} C \oplus C' & C \oplus C' \xrightarrow{1} C \oplus C' \\ \uparrow \circlearrowleft & & \uparrow \circlearrowleft \\ O & & O \end{array}$$



- Triangle (4').

$$\begin{array}{ccc}
P \xrightarrow{\theta} Q \xrightarrow{1 \oplus C \oplus C'} Q \oplus C \oplus C' & B \oplus B' \xrightarrow{1} B \oplus B' \xrightarrow{h_u} V \\
\uparrow \circlearrowleft \quad \square \quad \uparrow \circlearrowleft & \uparrow \circlearrowleft \quad \square \quad \uparrow \circlearrowleft \\
O \longrightarrow C \oplus C' & O \longrightarrow C \oplus C' \\
\uparrow \circlearrowleft & \uparrow \circlearrowleft \\
C \oplus C' & C \oplus C'
\end{array}$$

□

**B.3. Admissible Triples.** This section proves Lemma 4.5, which characterises the free homotopy class of admissible triples (satisfying some mild conditions). The result is obtained as Corollary B.8 of a key technical lemma, which we prove below. As before, recall that  $G\mathcal{C}^o$  denotes the base-point component of the simplicial set  $G\mathcal{C}$ , and  $K_1(\mathcal{C}) = \pi_1(|G\mathcal{C}^o|)$ .

**Construction B.6.** Let  $\mathcal{T}$  be a triangle contour, presented as in Diagram (45). Given any vertex  $(A, A') \in G\mathcal{C}$ , one can construct a new triangle contour  $(A, A') \oplus \mathcal{T}$  by formal direct sum

$$\begin{array}{ccc}
& (P_1 \oplus A, P'_1 \oplus A') & \\
e_0 \oplus (A, A') \nearrow & & \searrow e_1 \oplus (A, A') \\
(P_0 \oplus A, P'_0 \oplus A') & \xrightarrow{e_2 \oplus (A, A')} & (P_2 \oplus A, P'_2 \oplus A')
\end{array} \tag{91}$$

The fact that  $(A, A') \oplus \mathcal{T}$  is well-defined follows from Lemma A.3.

**Lemma B.7.** *Setup:*

- Let  $\tau = (e_0, e_1, e_2)$  be an admissible triple, where each  $e_i$  is a double exact square. This assembles into a pair of flag diagrams, presented below.

$$\begin{array}{ccc}
P_0 \xrightarrow{\alpha_{0,1}} P_1 \xrightarrow{\alpha_{1/2}} P_2 & P_0 \xrightarrow{\alpha'_{0,1}} P_1 \xrightarrow{\alpha'_{1/2}} P_2 \\
\alpha_{1/0,1} \uparrow \circlearrowleft \quad \square \quad \uparrow \alpha_{2/0,2} & \alpha'_{1/0,1} \uparrow \circlearrowleft \quad \square \quad \uparrow \alpha'_{2/0,2} \\
P_{1/0} \xrightarrow{\alpha_{1/0,2/0}} P_{2/0} & P_{1/0} \xrightarrow{\alpha'_{1/0,2/0}} P_{2/0} \\
\uparrow \alpha_{2/1,2/0} & \uparrow \alpha'_{2/1,2/0} \\
P_{2/1} & P_{2/1}
\end{array} \tag{92}$$

- Let  $l(\tau)$  be the double exact square associated to  $\tau$ , and  $\mu(l(\tau))$  be its canonical loop.

Then the loop  $\tau = e_0 e_1 e_2^{-1}$  is freely homotopic to

$$(P_2, P_2) \oplus \mu(l(\tau)).$$

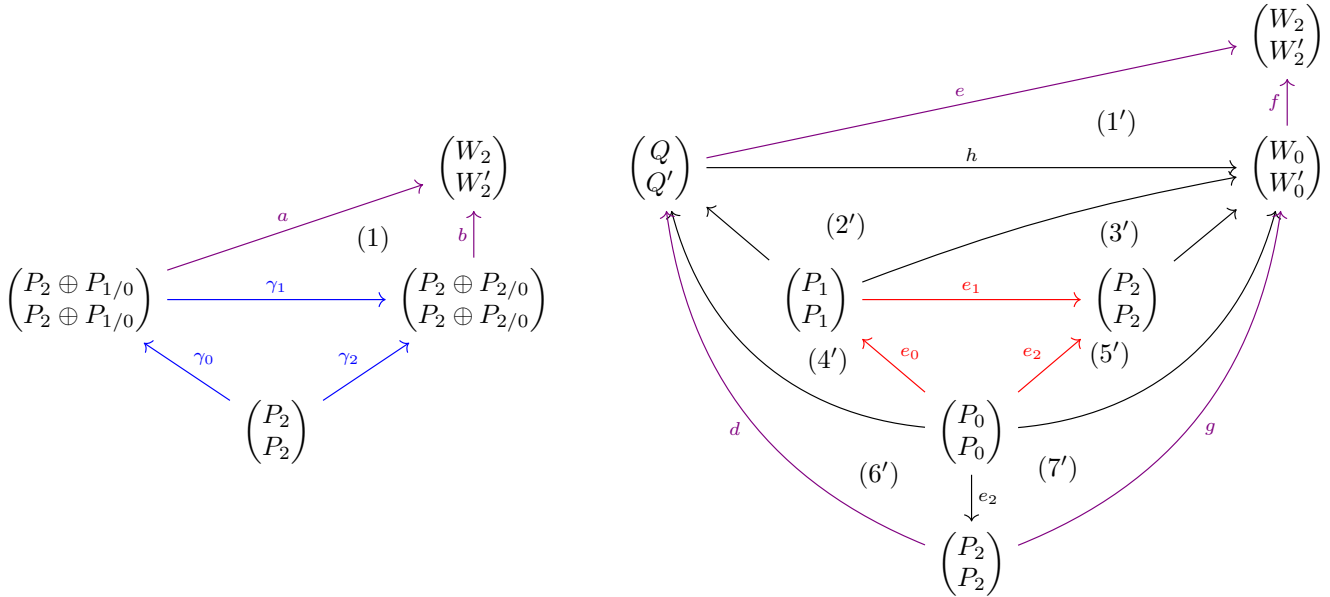
*Proof.*

*Step 0: Setup.* Start by taking repeated restricted pushouts

$$\begin{array}{ccc}
O & \xrightarrow{\quad} & P_{1/0} \xrightarrow{\alpha_{1/0,2/0}} P_{2/0} \\
\downarrow & & \downarrow \\
P_0 & \xrightarrow{\alpha_{0,2}} P_2 \xrightarrow{1 \oplus P_{1/0}} P_2 \oplus P_{1/0} \xrightarrow{1 \oplus \alpha_{1/0,2/0}} P_2 \oplus P_{2/0} \\
\downarrow \alpha_{0,1} & & \downarrow \\
P_1 & \xrightarrow{\quad} Q \xrightarrow{\quad} Z_0 \xrightarrow{\quad} Z_1 \\
\downarrow \alpha_{1,2} & & \downarrow \\
P_2 & \xrightarrow{\quad} W_0 \xrightarrow{\quad} W_1 \xrightarrow{\quad} W_2
\end{array}
\quad
\begin{array}{ccc}
O & \xrightarrow{\quad} & P_{1/0} \xrightarrow{\alpha'_{1/0,2/0}} P_{2/0} \\
\downarrow & & \downarrow \\
P_0 & \xrightarrow{\alpha'_{0,2}} P_2 \xrightarrow{1 \oplus P_{1/0}} P_2 \oplus P_{1/0} \xrightarrow{1 \oplus \alpha'_{1/0,2/0}} P_2 \oplus P_{2/0} \\
\downarrow \alpha'_{0,1} & & \downarrow \\
P_1 & \xrightarrow{\quad} Q' \xrightarrow{\quad} Z'_0 \xrightarrow{\quad} Z'_1 \\
\downarrow \alpha'_{1,2} & & \downarrow \\
P_2 & \xrightarrow{\quad} W'_0 \xrightarrow{\quad} W'_1 \xrightarrow{\quad} W'_2
\end{array}, \quad (93)$$

where commutativity of the diagrams is given by Fact 1.17. For readability, we have renamed the restricted pushouts so that, e.g.  $Q := P_2 \star_{P_0} P_1$  is the restricted pushout induced by  $P_1 \xleftarrow{\alpha_{0,1}} P_0 \xrightarrow{\alpha_{0,2}} P_2$ . Notice: all pairs of  $\mathcal{M}$ -morphisms in Diagram (93) form 1-simplices. This fact, combined with Corollary A.2, gives a roadmap for how to connect the vertices of the two loops.

*Step 1: Free Homotopy via Restricted Pushouts.* Present  $(P_2, P_2) \oplus \mu(l(\tau))$  in terms of its 1-simplices as  $\gamma_0 \gamma_1 \gamma_2^{-1}$ , which correspond to the blue edges in the left diagram below. Similarly, the loop  $\tau = e_0 e_1 e_2^{-1}$  corresponds to the red edges in the right diagram.



Some explanatory remarks:

(a) On the RHS Diagram: the loop  $dhg^{-1}$  is obtained by taking restricted pushouts of

$$\begin{pmatrix} P_2 \\ P_2 \end{pmatrix} \leftarrow \begin{pmatrix} P_0 \\ P_0 \end{pmatrix} \rightarrow \begin{pmatrix} P_1 \\ P_1 \end{pmatrix} \quad \begin{pmatrix} P_2 \\ P_2 \end{pmatrix} \leftarrow \begin{pmatrix} P_0 \\ P_0 \end{pmatrix} \rightarrow \begin{pmatrix} P_2 \\ P_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Q \\ Q' \end{pmatrix} \leftarrow \begin{pmatrix} P_1 \\ P_1 \end{pmatrix} \rightarrow \begin{pmatrix} P_2 \\ P_2 \end{pmatrix}.$$

By Corollary A.2, Triangles (2') - (7') are all 2-simplices. It is also easy to see that they assemble into the diagram above. The only non-trivial check is verifying Triangles (3') and (5') share the same face  $(P_2, P'_2) \rightarrow (W_0, W'_0)$  – but this follows from the loop  $\tau$  being an *admissible* triple. Conclude that the loop  $\tau$  is freely homotopic to the loop  $dhg^{-1}$ .

(b) Triangles (1) and (1') are obtained by taking restricted pushouts of

$$\begin{pmatrix} W_1 \\ W'_1 \end{pmatrix} \leftarrow \begin{pmatrix} P_2 \oplus P_{1/0} \\ P_2 \oplus P_{1/0} \end{pmatrix} \rightarrow \begin{pmatrix} P_2 \oplus P_{2/0} \\ P_2 \oplus P_{2/0} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Z_1 \\ Z'_1 \end{pmatrix} \leftarrow \begin{pmatrix} Q \\ Q' \end{pmatrix} \rightarrow \begin{pmatrix} W_0 \\ W'_0 \end{pmatrix}$$

respectively, before applying Corollary A.2. Conclude that  $\tau$  is freely homotopic to the outer loop  $def^{-1}g^{-1}$ , and  $(P_2, P'_2) \oplus \mu(l(\tau))$  is freely homotopic to  $\gamma_0 ab^{-1}\gamma_2^{-1}$ .

- (c) The composition of 1-simplices  $\gamma_2 b$  is homotopic to  $gf$  by applying Corollary A.2 to the restricted pushout

$$\begin{array}{ccc} \begin{pmatrix} P_2 \\ P_2 \end{pmatrix} & \xrightarrow{g} & \begin{pmatrix} W_0 \\ W'_0 \end{pmatrix} \\ \gamma_2 \downarrow & \searrow & \downarrow f \\ \begin{pmatrix} P_2 \oplus P_{2/0} \\ P_2 \oplus P_{2/0} \end{pmatrix} & \xrightarrow{b} & \begin{pmatrix} W_2 \\ W'_2 \end{pmatrix} \end{array} .$$

- (d) Let us clarify the composition behaviour in Item (c). Given two 1-simplices

$$z_0: (A, A') \rightarrow (B, B') \quad \text{and} \quad z_1: (B, B') \rightarrow (C, C')$$

whose underlying  $\mathcal{M}$ -morphisms fit into composable chains

$$A \rightharpoonup B \rightharpoonup C \quad \text{and} \quad A' \rightharpoonup B' \rightharpoonup C',$$

their composite

$$\overline{z_0 z_1}: (A, A') \rightarrow (C, C')$$

does not in general satisfy

$$\langle z_0 \rangle + \langle z_1 \rangle = \langle \overline{z_0 z_1} \rangle$$

in  $K_1(\mathcal{C})$ , unless the three are related by a 2-simplex (cf. Warning 4.3, Remark 4.2). In the case of Item (c), however, such a 2-simplex exists, and so we have

$$\langle g \rangle + \langle f \rangle = \langle \overline{gf} \rangle = \langle \overline{\gamma_2 b} \rangle = \langle \gamma_2 \rangle + \langle b \rangle.$$

In sum: by items (a) and (b),  $\tau$  is freely homotopic to the loop  $def^{-1}g^{-1}$  and  $(P_2, P'_2) \oplus \mu(l(\tau))$  is freely homotopic to  $\gamma_0 ab^{-1}\gamma_2^{-1}$ . Hence, to establish the lemma, it suffices for us to show that  $def^{-1}g^{-1}$  and  $\gamma_0 ab^{-1}\gamma_2^{-1}$  are freely homotopic to each other. By item (c), we know that  $f^{-1}g^{-1}$  is homotopic to  $b^{-1}\gamma_2^{-1}$ . It therefore remains to show that  $\gamma_0 a$  is (freely) homotopic to  $de$ .

*Step 2:  $H$ -space action on  $\pi_1$ .* By a standard Eckmann-Hilton argument, the group action of  $\pi_1(|G\mathcal{C}^o|)$  defined by concatenation of loops is homotopic to the  $H$ -space action induced by the  $G$ -construction – which, by Theorem 2.11, corresponds to taking direct sums.

Let us clarify how this works in our setting. Given any 1-simplex in  $G\mathcal{C}^o$ , say

$$X := \left( \begin{array}{ccc} O \rightharpoonup & \underline{A \xrightarrow{\alpha} B} \\ O \rightharpoonup & A' \xrightarrow{\alpha'} B' \end{array} \right),$$

define  $Y$  by swapping the top and bottom rows

$$Y := \left( \begin{array}{ccc} O \rightharpoonup & \underline{A' \xrightarrow{\alpha'} B'} \\ O \rightharpoonup & A \xrightarrow{\alpha} B \end{array} \right).$$

Notice that we may view  $X$  and  $Y$  as generators of  $\pi_1(G\mathcal{C}^o)$  – this is justified by Remark 4.2.

Taking their sum, we get

$$X + Y = \left( \begin{array}{ccc} O \rightharpoonup & \underline{A \oplus A' \xrightarrow{\alpha \oplus \alpha'} B \oplus B'} \\ O \rightharpoonup & A' \oplus A \xrightarrow{\alpha' \oplus \alpha} B' \oplus B \end{array} \right) \sim \left( \begin{array}{ccc} O \rightharpoonup & \underline{A \oplus A' \xrightarrow{\alpha \oplus \alpha'} B \oplus B'} \\ O \rightharpoonup & A \oplus A' \xrightarrow{\alpha \oplus \alpha'} B \oplus B' \end{array} \right).$$

with the obvious quotients. We leave the reader to check that this is homotopic to the (diagonal) double exact square on the RHS, obtained by permuting the summands in the bottom row.<sup>10</sup> The key point: the canonical loop of the RHS bounds the 2-simplex

$$\left( \begin{array}{c} O \rightarrow O \rightarrow A \oplus A' \xrightarrow{\alpha \oplus \alpha'} B \oplus B' \\ O \rightarrow O \rightarrow A \oplus A' \xrightarrow{\alpha \oplus \alpha'} B \oplus B' \end{array} \right),$$

and so  $X + Y$  trivialises in  $\pi_1(|G\mathcal{C}^o|)$ . In other words,  $Y$  is the inverse of  $X$  under the induced  $H$ -space action.

*Step 3: A New 2-Simplex.* We extend the insight from Step 2 to compare  $\tau_0 a$  and  $de$  from Step 1. To show that they are homotopic, it suffices to show in  $\pi_1(|G\mathcal{C}|)$  that

$$\langle \tau_0 \rangle + \langle a \rangle - \langle d \rangle - \langle e \rangle = 0.$$

Towards this end, we would like a diagram

$$\left( \begin{array}{c} O \rightarrow P_2 \oplus P_2 \rightarrow P_2 \oplus P_{1/0} \oplus Q' \rightarrow W_2 \oplus W_2' \\ O \rightarrow P_2 \oplus P_2 \rightarrow P_2 \oplus P_{1/0} \oplus Q \rightarrow W_2' \oplus W_2 \end{array} \right),$$

corresponding to a 2-simplex of the form

$$\begin{array}{ccccc} P_2 \oplus P_2 & \rightarrow & P_2 \oplus P_{1/0} \oplus Q' & \rightarrow & W_2 \oplus W_2' \\ \uparrow & & \square & & \uparrow \\ P_{1/0} \oplus P_{1/0} & \rightarrow & P_{2/0} \oplus P_{2/0} \oplus P_{2/0} \oplus P_{2/0} & & \\ & & \uparrow & & \\ & & P_{2/0} \oplus P_{2/1} \oplus P_{2/0} \oplus P_{2/1} & & \end{array} \quad \begin{array}{ccccc} P_2 \oplus P_2 & \rightarrow & P_2 \oplus P_{1/0} \oplus Q & \rightarrow & W_2' \oplus W_2 \\ \uparrow & & \square & & \uparrow \\ P_{1/0} \oplus P_{1/0} & \rightarrow & P_{2/0} \oplus P_{2/0} \oplus P_{2/0} \oplus P_{2/0} & & \\ & & \uparrow & & \\ & & P_{2/0} \oplus P_{2/1} \oplus P_{2/0} \oplus P_{2/1} & & \end{array}$$

We build this 2-simplex in stages.

*Step 3a.* By Axiom (PQ), we have the following exact squares

$$\begin{array}{ccc} O \rightarrow P_{1/0} & & O \rightarrow P_{1/0} \\ \downarrow \circlearrowleft & \square & \downarrow \circlearrowleft \\ P_2 \xrightarrow{q_1} Q & & P_2 \xrightarrow{q'_1} Q' \end{array} \quad \text{and} \quad \begin{array}{ccc} O \rightarrow P_{1/0} & & O \rightarrow P_{1/0} \\ \downarrow \circlearrowleft & \square & \downarrow \circlearrowleft \\ P_2 \xrightarrow{q'_1} Q' & & P_2 \xrightarrow{q'_1} Q' \end{array} \quad (94)$$

Applying Lemma A.4 (i), construct the following pair of distinguished squares<sup>11</sup>

$$\begin{array}{ccc} O \rightarrow P_{1/0} \oplus P_{1/0} & & O \rightarrow P_{1/0} \oplus P_{1/0} \\ \downarrow \circlearrowleft & \square & \downarrow \circlearrowleft \\ P_2 \oplus P_2 \xrightarrow{s_0} P_2 \oplus P_{1/0} \oplus Q' & & P_2 \oplus P_2 \xrightarrow{s_1} P_2 \oplus P_{1/0} \oplus Q \end{array} \quad (95)$$

$$s_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & q'_1 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & q_1 \end{pmatrix}, \quad t_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \beta'_{1/0} \end{pmatrix}, \quad t_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ \beta_{1/0} & 0 \end{pmatrix}.$$

<sup>10</sup> *Proof Sketch.* The goal is to show that the canonical loop associated to  $X + Y$  is homotopic to the canonical loop of a diagonal double exact square. Start by extending  $X + Y$  to a 2-simplex by permuting  $B' \oplus B$  on the bottom row [to get  $B \oplus B'$ ]. Next, by the same argument as in Lemma B.2 Step 2 (Base Case), show that  $(O, O) \rightarrow (A \oplus A', A' \oplus A) \rightarrow (B \oplus B', B \oplus B')$  is homotopic to  $(O, O) \rightarrow (A \oplus A', A \oplus A') \rightarrow (B \oplus B', B \oplus B')$ ; notice that we have permuted  $A' \oplus A$  on the bottom row to get  $A \oplus A'$ . Assembling everything together yields the desired homotopy.

<sup>11</sup> Sidenote: Lemma A.4 will not get us all the way. We will also need to permute the summands – but this can be done by constructing a distinguished square with the obvious permutation isomorphism. We can then modify the original distinguished square by composition.

Step 3b. As a warm-up, compare the following two flag diagrams.

$$\begin{array}{ccc}
P_0 \xrightarrow{\alpha_{0,1}} P_1 \xrightarrow{\alpha_{1/2}} P_2 & & P_2 \xrightarrow{q_1} Q \xrightarrow{w_0} W_0 \\
\alpha_{1/0,1} \uparrow \square \uparrow \alpha_{2/0,2} & & \beta_{1/0} \uparrow \square \uparrow \beta_{2/0} \\
P_{1/0} \xrightarrow{\alpha_{1/0,2/0}} P_{2/0} & & P_{1/0} \xrightarrow{\beta_{1/0,2/0}} P_{2/0} \\
\alpha_{2/1,2/0} \uparrow & & \beta_{2/1,2/0} \uparrow \\
P_{2/1} & & P_{2/1}
\end{array} \quad (96)$$

The LHS is from Diagram (92). The RHS applies Lemma 1.8 and Axiom (PQ) to the filtration

$$P_2 \twoheadrightarrow Q \twoheadrightarrow W_0$$

from Step 0. We claim that  $\alpha_{1/0,2/0} = \beta_{1/0,2/0}$  and  $\alpha_{2/1,2/0} = \beta_{2/1,2/0}$ .

Why? First notice that Axiom (PQ) also defines the following diagrams, which commute in the ambient pCGW category  $\mathcal{C}$ .

$$\begin{array}{ccc}
P_1 \xrightarrow{\alpha_{1,2}} P_2 \xleftarrow{\alpha_{2/1,2}} P_{2/1} & & P_0 \xrightarrow{\alpha_{0,2}} P_2 \xleftarrow{\alpha_{2/0,2}} P_{2/0} \\
q_1 \downarrow \quad q_2 \downarrow \quad \circlearrowright \quad \downarrow = & \text{and} & \alpha_{0,2} \downarrow \quad q_2 \downarrow \quad \circlearrowright \quad \downarrow = \\
Q \xrightarrow{w_0} W_0 \xleftarrow{\beta_{2/1}} P_{2/1} & & P_2 \xrightarrow{q_2} W_0 \xleftarrow{\beta_{2/0}} P_{2/0}
\end{array} \quad (97)$$

Next, apply the functor  $c: \text{Ar}_{\square}\mathcal{M} \rightarrow \text{Ar}_{\Delta}\mathcal{E}$  to the indicated distinguished squares in Diagram (96). This yields the following commutative diagrams in  $\mathcal{E}$ .

$$\begin{array}{ccc}
P_{2/1} \xrightarrow{\circ} P_{2/1} & & P_{2/1} \xrightarrow{\circ} P_{2/1} \\
\alpha_{2/1,2/0} \downarrow \quad \alpha_{2/1,2} \downarrow & & \beta_{2/1,2/0} \downarrow \quad \beta_{2/1} \downarrow \\
P_{2/0} \xrightarrow{\alpha_{2/0,2}} P_2 & & P_{2/0} \xrightarrow{\beta_{2/0}} W_0
\end{array} \quad (98)$$

Our argument then proceeds by translating Diagrams (97) and (98) into commutative diagrams in  $\mathcal{C}$ , and performing a diagram-chase. There are two cases to check.

- **Case 1:**  $\mathcal{E} \subseteq \mathcal{C}$ . In which case, we get the identity

$$\beta_{2/0} \circ \beta_{2/1,2/0} = \beta_{2/1} = q_2 \circ \alpha_{2/1,2}.$$

Since

$$\alpha_{2/1,2} = \alpha_{2/0,2} \circ \alpha_{2/1,2/0},$$

this yields

$$\beta_{2/0} \circ \beta_{2/1,2/0} = q_2 \circ \alpha_{2/0,2} \circ \alpha_{2/1,2/0}.$$

Notice that  $\beta_{2/0} = q_2 \circ \alpha_{2/0,2}$  by Diagram (97). Hence, since  $\mathcal{E}$ -morphisms are monic in  $\mathcal{C}$  by Axiom (M), deduce that

$$\beta_{2/1,2/0} = \alpha_{2/1,2/0}.$$

In particular, since  $c^{-1} = k$  on objects, this implies

$$\alpha_{1/0,2/0} = \beta_{1/0,2/0}.$$

- **Case 2:**  $\mathcal{E}^{\text{op}} \subseteq \mathcal{C}$ . The argument is dual to Case 1.

*Step 3c.* Step 3b clarifies how restricted pushouts respect the structure of quotients. Combine the argument with Axiom (DS) to obtain the following distinguished squares

$$\begin{array}{ccc}
P_{1/0} \xrightarrow{u_0} P_{2/0} \oplus P_{2/0} & & P_{1/0} \xrightarrow{u_1} P_{2/0} \oplus P_{2/0} \\
\beta_{1/0} \downarrow \quad \square \quad \downarrow v_0 & & \beta'_{1/0} \downarrow \quad \square \quad \downarrow v_1 \\
Q \xrightarrow{\quad} W_2 & & Q' \xrightarrow{\quad} W'_2
\end{array} \quad (99)$$

$$u_0 = \begin{pmatrix} \alpha_{1/0,2/0} \\ 0 \end{pmatrix} \quad u_1 = \begin{pmatrix} \alpha'_{1/0,2/0} \\ 0 \end{pmatrix}$$

An explicit description of the morphisms  $v_0, v_1$  is not needed for our argument, so we have suppressed it. Likewise, we also have distinguished squares

$$\begin{array}{ccc}
P_{1/0} \xrightarrow{u_0} P_{2/0} \oplus P_{2/0} & & P_{1/0} \xrightarrow{u_1} P_{2/0} \oplus P_{2/0} \\
P_2 \oplus 1 \downarrow \quad \square \quad \downarrow w_0 & & P_2 \oplus 1 \downarrow \quad \square \quad \downarrow w_1 \\
P_2 \oplus P_{1/0} \xrightarrow{\quad} W_2 & & P_2 \oplus P_{1/0} \xrightarrow{\quad} W'_2
\end{array} \quad (100)$$

We now play the same game as in Step 3a. Applying Lemma A.4 (ii), combine Diagrams (99) and (100) to construct

$$\begin{array}{ccc}
P_{1/0} \oplus P_{1/0} \xrightarrow{u_3} P_{2/0} \oplus P_{2/0} \oplus P_{2/0} \oplus P_{2/0} & & P_{1/0} \oplus P_{1/0} \xrightarrow{u_3} P_{2/0} \oplus P_{2/0} \oplus P_{2/0} \oplus P_{2/0} \\
t_0 \downarrow \quad \square \quad \downarrow z_0 & & t_1 \downarrow \quad \square \quad \downarrow z_1 \\
P_2 \oplus P_{1/0} \oplus Q' \xrightarrow{\quad} W_2 \oplus W'_2 & & P_2 \oplus P_{1/0} \oplus Q \xrightarrow{\quad} W'_2 \oplus W_2
\end{array} \quad (101)$$

whereby

$$u_3 = \begin{pmatrix} \alpha_{1/0,2/0} & 0 \\ 0 & 0 \\ 0 & \alpha'_{1/0,2/0} \\ 0 & 0 \end{pmatrix} \quad z_0 = \begin{pmatrix} v_0 & 0 \\ 0 & w_1 \end{pmatrix} \quad z_1 = \begin{pmatrix} 0 & v_1 \\ w_0 & 0 \end{pmatrix}$$

and  $t_0, t_1$  are defined as in Step 3a.

*Step 3d.* We can now construct our 2-simplex. Start by horizontally composing Diagrams (95) and (101). Then, vertically compose Diagram (101) with the following (identical) pair of exact squares, obtained by applying Axiom (K):

$$\begin{array}{ccc}
O \xrightarrow{\quad} P_{2/0} \oplus P_{2/1} \oplus P_{2/0} \oplus P_{2/1} & & O \xrightarrow{\quad} P_{2/0} \oplus P_{2/1} \oplus P_{2/0} \oplus P_{2/1} \\
\downarrow \quad \square \quad \downarrow & & \downarrow \quad \square \quad \downarrow \\
P_{1/0} \oplus P_{1/0} \xrightarrow{u_3} P_{2/0} \oplus P_{2/0} \oplus P_{2/0} \oplus P_{2/0} & & P_{1/0} \oplus P_{1/0} \xrightarrow{u_3} P_{2/0} \oplus P_{2/0} \oplus P_{2/0} \oplus P_{2/0}
\end{array} \quad (102)$$

And we are done.

*Step 4: Finish.* Our goal is to prove that  $(P_2, P_2) \oplus \mu(l(\tau))$  is freely homotopic to the loop  $\tau$ . Let us review our work so far.

- **Step 1:**  $\tau$  is freely homotopic to  $def^{-1}g^{-1}$ .
- **Step 1:**  $(P_2, P_2) \oplus \mu(l(\tau))$  is homotopic to  $\gamma_0 ab^{-1}\gamma_2^{-1}$ .
- **Step 1:** The concatenation of 1-simplices  $gf$  is homotopic to  $\gamma_2 b$ . In particular, their composites agree, i.e.

$$\langle \overline{gf} \rangle = \langle \overline{\gamma_2 b} \rangle$$

- **Step 3:** We have the following 2-simplex

$$\left( \begin{array}{c} O \rightrightarrows \frac{P_2 \oplus P_2 \rightrightarrows P_2 \oplus P_{1/0} \oplus Q' \rightrightarrows W_2 \oplus W'_2}{P_2 \oplus P_2 \rightrightarrows P_2 \oplus P_{1/0} \oplus Q \rightrightarrows W'_2 \oplus W_2} \end{array} \right).$$

By Step 1, our problem reduces to showing that the concatenation  $de$  is homotopic to  $\gamma_0 a$ . Expressed using the generators of  $\pi_1(|G\mathcal{C}^o|)$ , this amounts to showing

$$\langle \gamma_0 \rangle + \langle a \rangle - \langle d \rangle - \langle e \rangle = 0.$$

By Step 3, the 2-simplex yields the equation

$$\left\langle \left( \begin{array}{c} O \rightharpoonup \frac{P_2 \oplus P_2 \rightharpoonup P_2 \oplus P_{1/0} \oplus Q'}{P_2 \oplus P_2 \rightharpoonup P_2 \oplus P_{1/0} \oplus Q} \\ O \rightharpoonup \frac{P_2 \oplus P_2 \rightharpoonup P_2 \oplus P_{1/0} \oplus Q'}{P_2 \oplus P_2 \rightharpoonup P_2 \oplus P_{1/0} \oplus Q} \end{array} \right) \right\rangle + \left\langle \left( \begin{array}{c} O \rightharpoonup \frac{P_2 \oplus P_{1/0} \oplus Q' \rightharpoonup W_2 \oplus W'_2}{P_2 \oplus P_{1/0} \oplus Q \rightharpoonup W'_2 \oplus W_2} \\ O \rightharpoonup \frac{P_2 \oplus P_{1/0} \oplus Q' \rightharpoonup W_2 \oplus W'_2}{P_2 \oplus P_{1/0} \oplus Q \rightharpoonup W'_2 \oplus W_2} \end{array} \right) \right\rangle \\ = \left\langle \left( \begin{array}{c} O \rightharpoonup \frac{P_2 \oplus P_2 \rightharpoonup W_2 \oplus W'_2}{P_2 \oplus P_2 \rightharpoonup W'_2 \oplus W_2} \\ O \rightharpoonup \frac{P_2 \oplus P_2 \rightharpoonup W_2 \oplus W'_2}{P_2 \oplus P_2 \rightharpoonup W'_2 \oplus W_2} \end{array} \right) \right\rangle. \quad (103)$$

Applying Step 2, one can check that the 1-simplex resulting from

$$\langle \gamma_0 \rangle - \langle d \rangle - \left\langle \left( \begin{array}{c} O \rightharpoonup \frac{P_2 \oplus P_2 \rightharpoonup P_2 \oplus P_{1/0} \oplus Q'}{P_2 \oplus P_2 \rightharpoonup P_2 \oplus P_{1/0} \oplus Q} \\ O \rightharpoonup \frac{P_2 \oplus P_2 \rightharpoonup P_2 \oplus P_{1/0} \oplus Q'}{P_2 \oplus P_2 \rightharpoonup P_2 \oplus P_{1/0} \oplus Q} \end{array} \right) \right\rangle$$

is homotopic to a *diagonal* double exact square. This follows by permuting the summands on, say, the top row to match the bottom row, and constructing the relevant 2-simplices (as in Footnote 10). As such, deduce that

$$\langle \gamma_0 \rangle - \langle d \rangle = \left\langle \left( \begin{array}{c} O \rightharpoonup \frac{P_2 \oplus P_2 \rightharpoonup P_2 \oplus P_{1/0} \oplus Q'}{P_2 \oplus P_2 \rightharpoonup P_2 \oplus P_{1/0} \oplus Q} \\ O \rightharpoonup \frac{P_2 \oplus P_2 \rightharpoonup P_2 \oplus P_{1/0} \oplus Q'}{P_2 \oplus P_2 \rightharpoonup P_2 \oplus P_{1/0} \oplus Q} \end{array} \right) \right\rangle.$$

Applying the same argument to the other terms of Equation (103), conclude

$$\langle \gamma_0 \rangle - \langle d \rangle + \langle a \rangle - \langle e \rangle = \langle \overline{\gamma_2 b} \rangle - \langle \overline{g f} \rangle = 0,$$

where the final equality is justified by Step 1. This completes the proof.  $\square$

**Corollary B.8.** *If an admissible triple  $\tau$  features only double exact squares, then its loop is freely homotopic to  $\mu(l(\tau))$ .*

*Proof.* Consider the operation

$$(A, A') \oplus (—): G\mathcal{C} \rightarrow G\mathcal{C}, \quad (104)$$

which adds a vertex  $(A, A')$  to all the nodes of an  $n$ -simplex of  $G\mathcal{C}$ . Given any edge  $(A, A') \rightarrow (B, B')$ , this induces a simplicial homotopy between the maps

$$(A, A') \oplus (—) \rightarrow (B, B') \oplus (—). \quad (105)$$

In particular, if the loop  $\tau$  features only double exact squares, then there exists an edge  $(O, O) \rightarrow (P_2, P_2)$ . By the above, this implies  $(P_2, P_2) \oplus \mu(l(\tau))$  is homotopic to  $\mu(l(\tau))$ .  $\square$

## REFERENCES

- [AMBO<sup>+</sup>22] Niny Arcila-Maya, Candace Bethea, Morgan Opie, Kirsten Wickelgren, and Inna Zakharevich. Compactly supported A1-Euler characteristic and the Hochschild complex. *Topology and its Applications*, 316:108108, 2022. Women in Topology III.
- [Bak18] Matt Baker. Hodge theory in combinatorics. *Bulletin of the American Mathematical Society*, 55(1):57–80, 2018.
- [Bek17] Tibor Beke. The Grothendieck ring of varieties, and the theory of algebraically closed fields. *Journal of Pure and Applied Algebra*, 221(2):393–400, 2017.
- [Bor18] Lev A. Borisov. The class of the affine line is a zero divisor in the Grothendieck ring. *Journal of Algebraic Geometry*, 27:203–209, 2018.
- [Cam19] Jonathan Campbell. The  $K$ -theory spectrum of varieties. *Transactions of the AMS*, 371(11):7845–7884, 2019.
- [CLNS18] Antoine Chambert-Loir, Johannes Nicaise, and Julien Sebag. *Motivic Integration*, volume 325 of *Progress in Mathematics*. Birkhäuser, 2018.
- [CWZ19] Jonathan Campbell, Jesse Wolfson, and Inna Zakharevich. Derived  $\ell$ -adic zeta functions. *Advances in Mathematics*, 354:106760, 2019.
- [CZ22] Jonathan Campbell and Inna Zakharevich. Dévissage and localization for the Grothendieck spectrum of varieties. *Advances in Mathematics*, 411, December 2022.
- [DK19] Tobias Dyckerhoff and Mikhail Kapranov. *Higher Segal Spaces*, volume 2244 of *Lecture Notes in Mathematics*. Springer, 2019.
- [EJS20] Chris Eppolito, Jaiung Jun, and Matt Szczesny. Proto-exact categories of matroids, Hall algebras, and  $K$ -theory. *Mathematische Zeitschrift*, 296:147–167, 2020.

- [Eva97] David Evans, editor. *Model Theory of Groups and Automorphism Groups*, volume 244 of *LMS Lecture Note Series*. Cambridge University Press, 1997.
- [GG87] Henri Gillet and Daniel Grayson. The loop space of the  $Q$ -construction. *Illinois Journal of Math*, 31(4):574–597, 1987.
- [GG03] Henri Gillet and Daniel Grayson. Erratum to “The loop space of the  $Q$ -Construction”. *Illinois Journal of Math*, 47(3):745–748, 2003.
- [Gra87] Daniel Grayson. Exact sequences in Algebraic  $K$ -Theory. *Illinois Journal of Math*, 31(4), 1987.
- [Gra92] Daniel Grayson. Adams operations on higher  $K$ -theory. *K-Theory*, 6(2):97–111, 1992.
- [Gra12] Daniel Grayson. Algebraic  $K$ -theory via binary complexes. *Journal of the American Mathematical Society*, 25(4):1149–1167, 2012.
- [Har11] Victor Harnik. Model theory vs. categorical logic: two approaches to pretopos completion (a.k.a.  $t^{\text{eq}}$ ). In *Models, logics, and higher-dimensional categories*, volume 53 of *CRM Proc. Lecture Notes*, pages 83–134. American Mathematical Society, 2011.
- [HMM<sup>+</sup>22] Renee S. Hoekzema, Mona Merling, Laura Murray, Carmen Rovi, and Julia Semikina. Cut and paste invariants of manifolds via algebraic  $K$ -theory. *Topology and its Applications*, 316:108105, 2022. Women in Topology III.
- [Kat16] Eric Katz. *Nonarchimedean and Tropical Geometry*, *Simons Symposium*, chapter Matroid theory for algebric geometers. Springer, Cham, 2016.
- [KLM<sup>+</sup>24] Alexander Kupers, Ezekiel Lemann, Cary Malkiewich, Jeremy Miller, and Robin Sroka. Scissors automorphism groups and their homology. arXiv:2408.08081, 2024.
- [Kra00] Jan Krajíček. Uniform families of polynomial equations over a finite field and structures admitting an Euler characteristic of definable sets. *Proc. London Mathematical Society*, 3(81):257–284, 2000.
- [KS00] Jan Krajíček and Tom Scanlon. Combinatorics with definable sets: Euler characteristics and Grothendieck rings. *Bulletin of Symbolic Logic*, 3(3):311–330, 2000.
- [LL03] Michael Larsen and Valery Lunts. Motivic measures and stable birational geometry. *Moscow Mathematical Journal*, 3(1):85–95, 2003.
- [LS10] Qing Liu and Julien Sebag. The Grothendieck ring of varieties and piecewise isomorphisms. *Mathematische Zeitschrift*, 265:321–342, 2010.
- [MK94] Dugald MacPherson and Richard Kaye, editors. *Automorphisms of first-order structures*. Oxford University Press, 1994.
- [Mor06] *International Congress of Mathematicians*, volume II, chapter  $\mathbb{A}^1$ -Algebraic Topology, pages 1035–1059. Eur. Math. Soc., 2006.
- [MT08] Fernando Muro and Andrew Tonks. On  $K_1$  of a Waldhausen category. *K-theory and Noncommutative geometry*, EMS Ser. Congr. Rep., Eur. Math. Soc.:91–115, 2008.
- [Nan24] Anubhav Nanavaty. Weight Filtrations and Derived Motivic Measures. arXiv:2401.06879, 2024.
- [Nen96] A. Nenashev. Double short exact sequences produce all elements of Quillen’s  $K_1$ . *Algebraic K-Theory*, 199:151–160, 1996.
- [Nen98a] A. Nenashev.  $K_1$  by generators and relations. *Journal of Pure and Applied Algebra*, 131(2):195–212, 1998.
- [Nen98b] Alexander Nenashev. Double short exact sequences and  $K_1$  of an exact category. *K-Theory*, 14(1):23–41, January 1998.
- [Oxl11] James Oxley. *Matroid Theory*. Oxford University Press, 2011.
- [PP23] Jesse Pajwani and Ambrus Pál. Power structures on the Grothendieck-Witt ring and the Motivic Euler characteristic. 2023.
- [PRV24] Jesse Pajwani, Herman Rohrbach, and Anna M. Viergever. Compactly supported  $\mathbb{A}^1$ -Euler characteristics of symmetric powers of cellular varieties. arXiv:2404.08486, 2024.
- [Qui73] Daniel Quillen. Algebraic  $K$ -theory, I: Higher  $K$ -theories. In *Higher Algebraic K-Theory: I*, pages 85–147. Springer, 1973.
- [RSØ19] Oliver Röndigs, Markus Spitzweck, and Paul Østvær. The first stable homotopy groups of motivic spheres. *Annals of Mathematics*, 189(1):1 – 74, 2019.
- [Sch05] Karl Schwede. Gluing schemes and a scheme without closed points. *Recent progress in arithmetic and algebraic geometry: Contemporary Mathematics*, 386:157–172, 2005.
- [She94] Clayton Sherman. On  $K_1$  of an abelian category. *Journal of Algebra*, 163(2), 1994.
- [She98] Clayton Sherman. On  $K_1$  of an exact category. *K-Theory*, 14:1–22, 1998.
- [SS21] Maru Sarazola and Brandon Shapiro. A Gillet-waldhausen theorem for chain complexes of sets. arXiv:2107.07701, 2021.
- [Vas19] Sebastian Vasey. Accessible categories, set theory, and model theory: an invitation. arXiv:1904.11307, 2019.
- [Wal87] Friedhelm Waldhausen. Algebraic  $K$ -theory of spaces, concordance, and stable homotopy theory. In *Algebraic topology and algebraic K-theory (Princeton, N.J., 1983)*, volume 113 of *Ann. of Math. Stud.*, pages 392–417. Princeton Univ. Press, Princeton, NJ, 1987.
- [Wei13] Charles A. Weibel. *The K-book: An Introduction to Algebraic K-theory*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, 2013.
- [Zak17a] Inna Zakharevich. Annihilator of the Lefschetz motive. *Duke Math Journal*, 166(11), 2017.
- [Zak17b] Inna Zakharevich. The  $K$ -theory of Assemblers. *Advances in Mathematics*, 304:1176–1218, 2017.



- [Zak17c] Inna Zakharevich. On  $K_1$  of an Assembler. *Journal of Pure and Applied Algebra*, 221(7):1867–1898, 2017.
- [Zak22] Inna Zakharevich.  $E_\infty$ -ring structures on the  $K$ -theory of Assemblers and point counting. arXiv:2209.08640, 2022.

GRADUATE SCHOOL OF INFORMATICS, NAGOYA UNIVERSITY  
*Email address:* ng.ming.k0@a.mail.nagoya-u.ac.jp