$K_1(\mathcal{V}\mathrm{ar})$ IS GENERATED BY QUASI-AUTOMORPHISMS

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ABSTRACT. This paper provides a complete characterisation of $K_1(\mathcal{V}\mathrm{ar})$, the K_1 group of varieties, solving a problem left open in [Zak17c]. Our approach involves adapting Gillet-Grayson's G-Construction to define a new K-theory spectrum of varieties. There are two levels on which one can read the present paper. On one level, we streamline and extend a series of K-theory results on exact categories to a more general class of categories (including $\mathcal{V}\mathrm{ar}_k$). On another level, our investigations bring into focus an interesting generalisation of automorphisms ("double exact squares"), which generate K_1. Since our results apply to a wide range of non-additive contexts (e.g. varieties, matroids, definable sets etc.), this sets up a challenging question: what kind of information do these quasi-automorphisms calibrate?

Our understanding of K-theory is changing. Recent efforts to extend tools from classical algebraic K-theory to non-additive settings have led us to make decisions on what the essential features of the K-theory framework are. One perspective, influenced by Waldhausen's $S_\bullet$-construction [Wal87], is that K-theory is a framework for analysing the finite assembly and decompositions of objects; non-additive applications of this insight can be found in Campbell's $\widetilde{S}_\bullet$-construction [Cam19] as well as Zakharevich's use of finite disjoint covers in Assemblers [Zak17b]. A related perspective emphasises the view that K-theory breaks an object into two types of pieces. This underpins Campbell-Zakharevich's framework of *CGW categories* [CZ22], which formalises key similarities between exact categories and the category of varieties $\mathcal{V}\mathrm{ar}_k$.

In a different line of work: the study of $K_1(\mathcal{C})$ for arbitrary exact categories began with Gillet-Grayson's G-construction [GG87], which provided an elementary description of its generators. This description was refined by Sherman [She94, She98] and Nenashev [Nen96], culminating in Nenashev's characterisation of the complete set of relations for K_1 [Nen98b, Nen98a].

The present paper unites the two lines of investigation by extending the K_1 results to a subclass of CGW categories known as *pCGW categories*. These include not only exact categories and varieties, but also finite sets, matroids, and definable sets. As a result, we provide two alternative, complete characterisations of K_1 applicable to a broad range of non-additive contexts.

Overview. Let us develop the previous remark that K-theory is an abstract framework for breaking an object into two different types of pieces. Consider the following two definitions.

- Let R be a ring. We define $K_0(R)$ as

$$K_0(R) := \left\{ \begin{matrix} \text{free abelian group} \\ \text{fin. gen. proj. } R\text{-modules} \end{matrix} \right\} \Bigg/ \begin{matrix} [M] = [M'], \text{ if } M \cong M' \\ [M] = [M'] + [M''], \text{ if } M' \to M \to M'' \end{matrix}$$

  where $M' \to M \to M''$ is a short exact sequence.
- Let $\mathcal{V}\mathrm{ar}_k$ be the category of k-varieties, i.e. reduced separated schemes of finite type over field k. We define $K_0(\mathcal{V}\mathrm{ar}_k)$ as

$$K_0(\mathcal{V}\mathrm{ar}_k) := \left\{ \begin{matrix} \text{free abelian group} \\ k\text{-varieties} \end{matrix} \right\} \Bigg/ \begin{matrix} [X] = [X'], \text{ if } X \cong X' \\ [X] = [U] + [X \setminus U], \text{ if } U \hookrightarrow X \text{ is a closed immersion} \end{matrix}$$

The analogy is clear. A short exact sequence $M' \to M \to M''$ decomposes the R-module M into two distinct pieces, M' and M'', with $M' \to M$ an admissible monic and $M \to M''$ as an admissible epi. When viewed in $K_0(R)$, this translates to the equation $[M] = [M'] + [M'']$, reflecting that M is, in essence, constructed from these two components. Similarly, $K_0(\mathcal{V}\mathrm{ar}_k)$ decomposes a variety X into U and $X \setminus U$, with $U \hookrightarrow X$ a closed immersion and $X \setminus U \hookrightarrow X$ an open immersion.

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Both constructions follow the same principle: break an object into two different types of pieces and view it as an abstract sum of the two components in the group K_0. So what are the essential features of this mechanism?

Quillen [Qui73] introduced, and later characterised, an exact category as a pair $(\mathcal{C}, \mathcal{S})$, where $\mathcal{C}$ is additive, and $\mathcal{S}$ is a family of sequences $M' \to M \to M''$ satisfying specific properties. These properties, akin to those satisfed by short exact sequences in abelian categories, allowed Quillen to construct a K-theory spectrum $K\mathcal{C}$, recovering $K_0(\mathcal{C})$ as its π_0. In particular, this includes the natural condition that admissible monics $M' \to M$ are kernels of admissible epis $M \to M''$, and admissible epis are cokernels of admissible monics. A crucial move: [CZ22] relaxes this condition by no longer requiring the two classes of morphisms to compose. Their key insight is that to extend Quillen's framework, it suffices to encode the interaction between these morphisms formally. This flexibility allows for applications in non-additive settings like varieties, where sequences like $U \hookrightarrow X \hookleftarrow X \setminus U$ clearly do not compose.

Having established the existence of a K-theory spectrum $K\mathcal{V}\mathrm{ar}_k$ whose π_0 recovers the original $K_0(\mathcal{V}\mathrm{ar}_k)$, we can define the higher K-groups of varieties $K_n(\mathcal{V}\mathrm{ar}_k) := \pi_n(K\mathcal{V}\mathrm{ar}_k)$ and ask:

Question 1. What kind of information do the higher K-groups of varieties encode?

This is a challenging question. In classical algebraic K-theory, the coarseness of K_0 as an invariant may be measured by the fact that $K_0(F) = \mathbb{Z}$ for all fields F, whereas $K_1(F) \cong F^\times$. Is there an analogous story to be developed in the setting of varieties? More explicitly, how might we measure the loss of information in $K_0(\mathcal{V}\mathrm{ar}_k)$? To what extent can we recover this information in the higher K-groups? The following summary theorem gives a snapshot of the current landscape.

Summary Theorem 0.1. *Assume the base field k is of characeristic 0, and equip $K_0(\mathcal{V}\mathrm{ar}_k)$ with a ring structure by defining $[X] \cdot [Y] := [(X \times_k Y)_{\mathrm{red}}]$. Two k-varieties X, Y are said to be* piecewise isomorphic *if X and Y admit finite partitions*

$$X_1, \dots, X_n \qquad \textit{and} \qquad Y_1, \dots, Y_n$$

into locally closed subvarieties such that $X_i \cong Y_i$ for all n. The following is known:

- *(i) Define $SK_0(\mathcal{V}\mathrm{ar}_k)$ as the freely generated* semiring *on $[X]$ subject to $[X] = [Z] + [X \setminus Z]$. Then, two k-varieties X, Y are piecewise isomorphic iff $[X] = [Y]$ in $SK_0(\mathcal{V}\mathrm{ar}_k)$.*
- *(ii) Let X, Y k-varieties such that $\dim X \leq 1$. Then $[X] = [Y]$ in $K_0(\mathcal{V}\mathrm{ar}_k)$ iff they are piecewise isomorphic.*
- *(iii) There exists k-varieties X and Y such that $[X] = [Y]$ in $K_0(\mathcal{V}\mathrm{ar}_k)$ and yet fail to be piecewise isomorphic.*
- *(iv) Let X be a k-variety of any non-negative dimension containing only finitely many rational curves. Then for any k-variety Y, $[X] = [Y]$ in $K_0(\mathcal{V}\mathrm{ar}_k)$ iff they are piecewise isomorphic.*

Proof. (i) appears to be folklore, and is recorded in [Bek17] as well as [CLNS18, Cor. 1.4.9, Chapter 2]. (ii) is [LS10, Props. 5 and 6]. For (iii), various constructions are now known but the first example goes back to [Bor18]. (iv) is [LS10, Theorem 5]. □

Summary Theorem 0.1 sharpens our understanding of what is at stake. Given our high-level characterisation of K-theory as an abstract framework for analysing the finite assembly and decompositions of objects, the following question is natural:

Question 2 ([LL03, Question 1.2])**.** Is it true that two k-varieties are piecewise isomorphic iff they agree in $K_0(\mathcal{V}\mathrm{ar}_k)$?

In the setting of characteristic 0, item (iii) of the Summary Theorem answers no, signalling a loss of information on the level of K_0. Item (i) tells us the information is lost precisely because $K_0(\mathcal{V}\mathrm{ar}_k)$ involves group completion – akin to an Eilenberg Swindle. Item (ii) tells us that piecewise isomorphism and equivalence in $K_0(\mathcal{V}\mathrm{ar}_k)$ coincide so long as the varieties are of sufficiently low dimension. Put otherwise, the algebraic barriers to geometric information only occur at the higher dimensions. Item (iv) is subtler, and raises interesting questions about how taking piecewise isomorphisms of complex varieties relates to the ampleness of their canonical line bundles (cf. the algebraic hyperbolicity conjecture for surfaces).

In light of this discussion, let us return to Question 1. Some promising initial progress has been made thus far. Using the formalism of Assemblers, Zakharevich constructs a different (but equivalent) K-theory spectrum of varieties, before leveraging its connection with Waldhausen categories to obtain a partial characterisation of $K_1(\mathcal{V}\mathrm{ar}_k)$ [Zak17c, Theorem B]. Inspired by Borisov's work [Bor18], this was later developed in [Zak17a] to illuminate a subtle geometric insight: the failure to extend birational automorphisms of varieties to piecewise isomorphisms is tightly connected to the Lefschetz motive $[\mathbb{A}^1]$ being a zero divisor in $K_0(\mathcal{V}\mathrm{ar}_k)$. In a different vein: [CWZ19] identifies non-trivial elements in $K_n(\mathcal{V}\mathrm{ar}_k)$ by lifting various motivic measures $K_0(\mathcal{V}\mathrm{ar}_k) \to K_0(\mathcal{C})$ to the level of spectra $K\mathcal{V}\mathrm{ar}_k \to K\mathcal{C}$.

Discussion of Main Results. Until recently, a full characterisation of any higher K-group of varieties was not known. In her original paper, Zakharevich [Zak17c, Theorem B] identifies the generators of $K_1(\mathcal{V}\mathrm{ar}_k)$ and some key relations, but does not prove their completeness. Independently from us, an intriguing recent collaboration between algebraic topologists and experts in homological stability has uncovered a homological proof [KLM$^+$24, Prop. 4.1] that Zakharevich's presentation is in fact complete.

We take a different approach. Whereas [KLM$^+$24] utilises homological methods to analyse K_1, the present paper instead relies on techniques from simplicial homotopy theory. Further, whereas [Zak17c] relies on the connection between $\mathcal{V}\mathrm{ar}_k$ and Waldhausen categories, we instead focus on the (tighter) connection between $\mathcal{V}\mathrm{ar}_k$ and exact categories. This sets up the following theorem.

Theorem A (Theorem 2.12)**.** Let $\mathcal{C}$ be a pCGW category, and $\mathcal{SC}$ the simplicial set obtained by applying the $S_\bullet$-construction. Then, there exists a simplicial set $G\mathcal{C}$ such that there is a homotopy equivalence

$$|G\mathcal{C}| \simeq \Omega|\mathcal{SC}|.$$

In particular, $\pi_n|G\mathcal{C}| = K_n\mathcal{C}$ for all n.

In broad strokes: Theorem A extends Gillet-Grayson's G-construction on exact categories [GG87] to a wider class of categories including $\mathcal{V}\mathrm{ar}_k$. The beauty of the G-construction is that it translates a topological problem (i.e. characterising π_1 of a loop space) into a simplicial one, which is more combinatorial and thus easier to work with. To show that this gives us sufficient leverage to characterise K_1 will, of course, take the rest of the paper. Also, a technical footnote for the expert reader: while one can prove Theorem A by adapting the original proof in [GG87] to our setting, we provide a more streamlined argument (Theorem 2.7) inspired by Grayson's framework of dominant functors [Gra87].

The previous remarks underscore a more fundamental difference. Both [Zak17c] and [KLM$^+$24] are concerned with the K-theory of Assemblers, whereas our paper builds on [CZ22] to develop the K-theory of so-called *pCGW categories*. Precise definitions will be given in due course; for now, it suffices to think of Assemblers and pCGW categories as two distinct yet equivalent ways to define the K-theory spectrum of varieties.[1] This difference becomes apparent when comparing our respective presentations of $K_1(\mathcal{V}\mathrm{ar}_k)$. In our language:

Theorem B (Theorem 3.18 and Prop. 4.12)**.** Let $\mathcal{C}$ be a pCGW category. Then $K_1(\mathcal{C})$ is generated by *double exact squares*, i.e. by pairs of distinguished squares in $\mathcal{C}$ with identical nodes

$$l := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\square"] & C \arrow[d, hook, "g_1"] \\ A \arrow[r, tail, "f_1"] & B \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\square"] & C \arrow[d, hook, "g_2"] \\ A \arrow[r, tail, "f_2"] & B \end{tikzcd} \right), \tag{1}$$

modulo the following relations

(B1) $$\left\langle \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\square"] & A \arrow[d, hook, "1"] \\ O \arrow[r, tail] & A \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\square"] & A \arrow[d, hook, "1"] \\ O \arrow[r, tail] & A \end{tikzcd} \right) \right\rangle = 0;$$

[1]For the cautious reader: the weak equivalence of these spectra as spaces is [CZ22, Theorems 7.8 and 9.1].

(B2) $\left\langle \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & O \arrow[d] \\ A \arrow[r, tail, "1"'] & A \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & O \arrow[d] \\ A \arrow[r, tail, "1"'] & A \end{tikzcd} \right) \right\rangle = 0;$

(B3) Suppose $f_C \colon A \overset{f_A}{\rightarrowtail} B \overset{f_B}{\rightarrowtail} C$ and $f'_C \colon A \overset{f'_A}{\rightarrowtail} B \overset{f'_B}{\rightarrowtail} C$. Under technical conditions (imposed by the 2-simplices of $G\mathcal{C}$), the following splitting relation holds

$$\left\langle \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[d, "g_A"] \\ A \arrow[r, tail, "f_A"'] & B \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[d, "g'_A"] \\ A \arrow[r, tail, "f'_A"'] & B \end{tikzcd} \right) \right\rangle + \left\langle \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{C}{B} \arrow[d, "g_B"] \\ B \arrow[r, tail, "f_B"'] & C \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{C}{B} \arrow[d, "g'_B"] \\ B \arrow[r, tail, "f'_B"'] & C \end{tikzcd} \right) \right\rangle = \left\langle \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{C}{A} \arrow[d, "g_C"] \\ A \arrow[r, tail, "f_C"'] & C \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{C}{A} \arrow[d, "g'_C"] \\ A \arrow[r, tail, "f'_C"'] & C \end{tikzcd} \right) \right\rangle.$$

This presentation appears to be new, even in the context of exact categories. How does it compare with K_1 of an Assembler? The following informal discussion may be illuminating.

On Generators. Double exact squares describe how to break an object into two distinct parts – for instance, Equation (1) shows B being broken into A and C. Interestingly, these squares generalise the usual notion of an automorphism – see Example 3.17. By contrast, [Zak17c] shows that K_1 of an Assembler is generated by *piecewise automorphisms*, which break an object into n many pieces simultaneously. This difference reflects a trade-off between simplicity vs. flexibility. Our Theorem B presents a simpler set of generators for $K_1(\mathcal{V}\mathrm{ar}_k)$ than [Zak17c], which can be advantageous when e.g. constructing derived motivic measures, as done in [CWZ19].[2] On the other hand, the generality of piecewise automorphisms makes the Assemblers formalism better suited for investigating e.g. scissors congruence of convex polytopes, as done in [KLM$^+$24], where simultaneous decomposition is essential.[3]

On Relations. There is an interesting discrepancy regarding the relations of [Zak17c] and Theorem B. In Zakharevich's presentation, the composition of piecewise automorphisms always split in K_1. More precisely:

$$\left\langle A \underset{f_2}{\overset{f_1}{\rightrightarrows}} B \right\rangle + \left\langle B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C \right\rangle = \left\langle A \underset{g_2 f_2}{\overset{g_1 f_1}{\rightrightarrows}} C \right\rangle \qquad \text{in } K_1,$$

where f_i, g_i are piecewise automorphisms. Figure 1 gives an informal illustration.

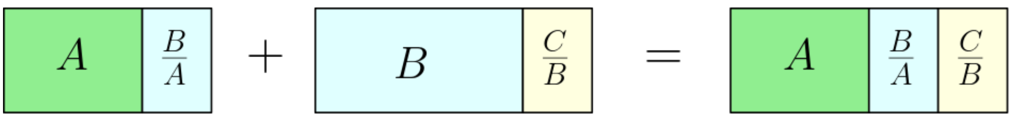

FIGURE 1. LHS: the piecewise automorphisms induced by closed immersions $A \overset{f}{\rightarrowtail} B$ and $B \overset{g}{\to} C$. RHS: the piecewise automorphism induced by their composition $A \overset{gf}{\rightarrowtail} C$.

By contrast, Relation (B3) of Theorem B asserts that composition splits in K_1 only when a technical condition is satisfied – see Warning 4.2 for details. Proposition 4.12 gives evidence that this condition is both non-trivial and necessary. This raises the question of whether an analogous condition should also appear in the Assemblers framework. Our analysis relies on Theorem C, an alternative presentation of K_1 inspired by Nenashev [Nen98a], and is the final main result of our paper.

Theorem C (Corollary 4.16)**.** Let $\mathcal{C}$ be a pCGW category. Then $K_1(\mathcal{C})$ is generated by double exact squares subject to the following relations:

[2]Technically, [CWZ19] views $\mathcal{V}\mathrm{ar}_k$ as a *subtractive category* before applying the $\widetilde{S}_\bullet$*-construction* as defined in [Cam19], but this is equivalent to viewing $\mathcal{V}\mathrm{ar}_k$ as a CGW category and applying the $S_\bullet$-construction; see [CZ22, Example 7.4].

[3]*Details.* Define two polytopes P and Q to be scissors congruent if: (i) $P = \bigcup_{i=1}^m P_i$ and $Q = \bigcup_{i=1}^m Q_i$ such that $P_i \cong Q_i$, and (ii) $P_i \cap P_j = Q_i \cap Q_j = \emptyset$ for $i \neq j$. The key hypothesis here is convexity. In particular, pairwise unions like $P_j \cup P_k$ may not form a convex polytope, so decomposition and reassembly must be done simultaneously.

(N1) $\langle l \rangle = 0$ if

$$l = \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \arrow[d, {Circle[open]}->, "g"] \\ A \arrow[r, tail, "f"] & B \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \arrow[d, {Circle[open]}->, "g"] \\ A \arrow[r, tail, "f"] & B \end{tikzcd} \right)$$

Notice that l is a pair of identical squares.

(N2) Given a good 3×3 diagram

$$\left(\begin{tikzcd} X_{00} \arrow[r, tail, "f_0"] \arrow[d, tail, "h_0"'] \arrow[dr, phantom, "\circlearrowleft"] & X_{01} \arrow[d, tail, "h_1"] & X_{02} \arrow[l, {Circle[open]}->, "g_0"'] \arrow[d, tail, "h_2"] \\ X_{10} \arrow[r, tail, "f_1"] & X_{11} \arrow[dr, phantom, "\circlearrowleft"] & X_{12} \arrow[l, {Circle[open]}->, "g_1"'] \\ X_{20} \arrow[r, tail, "f_2"] \arrow[u, {Circle[open]}->, "j_0"] & X_{21} \arrow[u, {Circle[open]}->, "j_1"] & X_{22} \arrow[l, {Circle[open]}->, "g_2"'] \arrow[u, {Circle[open]}->, "j_2"'] \end{tikzcd} , \begin{tikzcd} X_{00} \arrow[r, tail, "f'_0"] \arrow[d, tail, "h'_0"'] \arrow[dr, phantom, "\circlearrowleft"] & X_{01} \arrow[d, tail, "h'_1"] & X_{02} \arrow[l, {Circle[open]}->, "g'_0"'] \arrow[d, tail, "h'_2"] \\ X_{10} \arrow[r, tail, "f'_1"] & X_{11} \arrow[dr, phantom, "\circlearrowleft"] & X_{12} \arrow[l, {Circle[open]}->, "g'_1"'] \\ X_{20} \arrow[r, tail, "f'_2"] \arrow[u, {Circle[open]}->, "j'_0"] & X_{21} \arrow[u, {Circle[open]}->, "j'_1"] & X_{22} \arrow[l, {Circle[open]}->, "g'_2"'] \arrow[u, {Circle[open]}->, "j'_2"'] \end{tikzcd} \right)$$

defined by the following 6 double exact squares

$$l_i := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & X_{i2} \arrow[d, {Circle[open]}->, "g_i"] \\ X_{i0} \arrow[r, tail, "f_i"] & X_{i1} \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & X_{i2} \arrow[d, {Circle[open]}->, "g'_i"] \\ X_{i0} \arrow[r, tail, "f'_i"] & X_{i1} \end{tikzcd} \right) \qquad \text{for all } i \in \{0,1,2\}$$

$$l^i := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & X_{2i} \arrow[d, {Circle[open]}->, "j_i"] \\ X_{0i} \arrow[r, tail, "h_i"] & X_{1i} \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & X_{2i} \arrow[d, {Circle[open]}->, "j'_i"] \\ X_{0i} \arrow[r, tail, "h'_i"] & X_{1i} \end{tikzcd} \right) \qquad \text{for all } i \in \{0,1,2\},$$

the following 6-term relation holds

$$\langle l_0 \rangle + \langle l_2 \rangle - \langle l_1 \rangle = \langle l^0 \rangle + \langle l^2 \rangle - \langle l^1 \rangle$$

Does Theorem C imply a mistake in [Zak17c]? We do not assert this – further examination of the paper's argument will be needed. It also remains possible that the technical condition may be trivially satisfied in all cases. Nonetheless, this opens up an interesting discussion on whether this discrepancy arises from subtle differences between the scissors congruence of polytopes vs. varieties. In particular Theorem 2.1 of [Zak17c], inspired by Zakharevich's earlier work on polytopes [Zak12], states that one can model the K-theory of (closed) assemblers using the K-theory of Waldhausen categories whose cofibration sequences all split (up to weak equivalence). This is *a priori* surprising, particularly in the setting of varieties – for instance, the projective line features in the sequence $* \hookrightarrow \mathbb{P}^1 \hookleftarrow \mathbb{A}^1$, but it is clear that $\mathbb{P}^1 \not\cong \mathbb{A}^1 \coprod \{*\}$.

Implications for Characterising K_n. Many of the results of the present paper are inspired by Nenashev's work [Nen96] characterising $K_1(\mathcal{C})$ for an exact category $\mathcal{C}$. Grayson [Gra12] later extended this to characterise $K_n(\mathcal{C})$ for all n, and we expect our approach to generalise similarly to the higher K-groups of varieties (or, more generally, the higher K-groups of pCGW categories). It is currently unclear how one might analogously extend the methods from [Zak17c] or [KLM$^+$24].

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CONTENTS

1. Preliminaries

1.1. **CGW Categories.** The key definition in [CZ22] is the *CGW category*. It is essentially a category equipped with two subclasses of maps, $\mathcal{M}$ and $\mathcal{E}$ (analogous to admissible monics and epis in exact categories), along with a collection of square diagrams ("distinguished squares") which encode how $\mathcal{M}$ and $\mathcal{E}$-morphisms interact.

This is presented using the language of double categories. Recall that a *double category* $\mathcal{C}$ is an internal category in Cat. For the present paper, we will require the following refinement.

Definition 1.1. A *good double category* is a triple of categories $(\mathcal{C}, \mathcal{M}, \mathcal{E})$ presented by the data:

- *Objects.* All three categories have the same objects: $\mathrm{ob}(\mathcal{E}) = \mathrm{ob}(\mathcal{M}) = \mathrm{ob}(\mathcal{C})$.
- *Morphisms.*
  $\mathcal{M}$**-morphisms:** $\mathcal{M}$ is a subcategory of $\mathcal{C}$. Its morphisms are denoted $\rightarrowtail$.
  $\mathcal{E}$**-morphisms:** Either $\mathcal{E}$ or $\mathcal{E}^{\mathrm{op}}$ is a subcategory of $\mathcal{C}$. Its morphisms are denoted $\circ\!\!\rightarrow$.
- *Distinguished Squares.* A collection of square diagrams that encode how $\mathcal{M}$ and $\mathcal{E}$-morphisms interact. These are denoted

$$\begin{tikzcd} A \arrow[r, tail, "f'"] \arrow[d, o->, "g'"'] \arrow[dr, phantom, "\square"] & B \arrow[d, o->, "g"] \\ C \arrow[r, tail, "f"] & D \end{tikzcd}$$

  where $f, f' \in \mathcal{M}$ and $g, g' \in \mathcal{E}$. These squares closed under horizontal and vertical composition. They are also required to *interact well with isomorphisms* in the following sense: if

$$\begin{tikzcd} A \arrow[r, tail, "f'"] \arrow[d, o->, "g'"'] & B \arrow[d, o->, "g"] \\ C \arrow[r, tail, "f"] & D \end{tikzcd}$$

  defines a commutative diagram in $\mathcal{C}$, and either both $\mathcal{M}$-morphisms or both $\mathcal{E}$-morphisms are isomorphisms, then the square is distinguished.

Convention 1.2 (Ambient vs. Double Category)**.** We typically denote a good double category as $\mathcal{C} = (\mathcal{M}, \mathcal{E})$. When the context is clear, we simply write $\mathcal{C}$. When we wish to regard $\mathcal{C}$ as an ordinary 1-category (ignoring the double category structure), we refer to $\mathcal{C}$ as the *ambient category*, which by definition contains $\mathcal{M}$ and $\mathcal{E}$ (or $\mathcal{E}^{\mathrm{op}}$) as a subcategory.

We now introduce a couple of helper definitions, before defining what a CGW category is.

Definition 1.3. Let $C = (\mathcal{E}, \mathcal{M})$ be a good double category, and $\mathcal{D}$ be any (ordinary) category.

(1) Define $\mathrm{Ar}_\square\mathcal{E}$
  - Objects: Morphisms $A \circ\!\!\rightarrow B$ in $\mathcal{E}$.
  - Morphisms: $\mathrm{Hom}_{\mathrm{Ar}_\square\mathcal{E}}(A \overset{g}{\circ\!\!\rightarrow} B, A' \overset{g'}{\circ\!\!\rightarrow} B') = \left\{ \begin{array}{c} \text{distinguished} \\ \text{squares} \end{array} \begin{tikzcd} A \arrow[r, tail] \arrow[d, o->, "g"'] \arrow[dr, phantom, "\square"] & A' \arrow[d, o->] \\ B \arrow[r, tail, "g'"] & B' \end{tikzcd} \right\}.$

  $\mathrm{Ar}_\square\mathcal{M}$ is defined analogously.

(2) Define $\mathrm{Ar}_\triangle\mathcal{D}$
  - Objects: Morphisms $A \to B$ in $\mathcal{D}$.

- Morphisms: $\mathrm{Hom}_{\mathrm{Ar}_\triangle \mathcal{D}}(A \xrightarrow{f} B, A' \xrightarrow{f'} B') = \left\{ \begin{array}{c} \text{commutative} \\ \text{squares} \end{array} \quad \begin{tikzcd} A \arrow[r, "\cong"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r] & B' \end{tikzcd} \right\}.$

Definition 1.4 (CGW Category)**.** A *CGW category* $(\mathcal{C}, \varphi, c, k)$ consists of the following data:

- A good double category $\mathcal{C} = (\mathcal{E}, \mathcal{M})$;
- An isomorphism of categories $\varphi\colon \mathrm{iso}\mathcal{M} \to \mathrm{iso}\mathcal{E}$ which is identity on objects;
- An equivalence of categories

$$k\colon \mathrm{Ar}_\square \mathcal{E} \to \mathrm{Ar}_\triangle \mathcal{M} \qquad \text{and} \qquad c\colon \mathrm{Ar}_\square \mathcal{M} \to \mathrm{Ar}_\triangle \mathcal{E};$$

satisfying the axioms:

(Z) *Basepoint object.* $\mathcal{C}$ contains an object O initial in both $\mathcal{E}$ and $\mathcal{M}$.

(I) *Stable Under Isomorphisms.* Let $\psi\colon A \to B$ be an isomorphism in ambient category $\mathcal{C}$. Then:
  - ψ belongs to $\mathrm{iso}\mathcal{M}$, which we denote suggestively as $\psi\colon A \rightarrowtail B$.
  - If $\mathcal{E}$ is a subcategory of $\mathcal{C}$, then $\varphi(\psi)\colon A \circ\!\!\rightarrow B$ corresponds to $\psi\colon A \to B$ in $\mathcal{C}$.
  - If $\mathcal{E}^{\mathrm{op}}$ is a subcategory of $\mathcal{C}$, then $\varphi(\psi)\colon A \circ\!\!\rightarrow B$ corresponds to $\psi^{-1}\colon B \to A$ in $\mathcal{C}$.

(M) *Monicity.* Every morphism in $\mathcal{E}$ and $\mathcal{M}$ is monic.

(K) *Formal kernels and cokernels.* For any $f\colon A \rightarrowtail B$ in $\mathcal{M}$, there exists a *formal cokernel*, denoted $c(f)\colon \mathrm{coker}(f) \circ\!\!\rightarrow B$, and a distinguished square as below left.

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\square"] & \mathrm{coker}(f) \arrow[d, "c(f)"] \\ A \arrow[r, tail, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\square"] & A \arrow[d, "g"] \\ \ker(g) \arrow[r, tail, "k(g)"] & B \end{tikzcd}$$

Dually, for any $g\colon A \circ\!\!\rightarrow B$ in $\mathcal{E}$, there exists a *formal kernel*, denoted $k(g)\colon \ker(g) \rightarrowtail B$, and a distinguished square as above right.

These distinguished squares are unique up to isomorphism in the following sense: if there exists another $\mathcal{E}$-morphism $f'\colon C \circ\!\!\rightarrow B$ and a distinguished square

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\square"] & C \arrow[d, "f'"] \\ A \arrow[r, tail, "f"] & B \end{tikzcd}$$

then there exists an isomorphism $\tau\colon \mathrm{coker}(f) \rightarrowtail C$ such that the rightmost square in

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\square"] & \mathrm{coker}(f) \arrow[r, tail, "\tau"] \arrow[d, "c(f)"] \arrow[dr, phantom, "\square"] & C \arrow[d, "f'"] \\ A \arrow[r, tail, "f"'] & B \arrow[r, tail, "1"'] & B \end{tikzcd}$$

commutes when regarded as a diagram in the ambient category $\mathcal{C}$. Notice that since distinguished squares interact well with isomorphisms, this implies the square is distinguished. Formal kernels are unique in the analogous sense.[4]

There is a natural notion of structure-preserving functors and subcategories in the CGW context. A *CGW functor* of CGW categories is a double functor

$$F\colon (\mathcal{E}, \mathcal{M}) \to (\mathcal{E}', \mathcal{M}')$$

[4] **(M:) Be mindful of this; I want to say, cokernels are unique up to isomorphism. Can't be unique up to unique isomorphism since, e.g. one might permute $A \coprod A$ about if it features as a cokernel. Also want it to interact well with distinguished squares in some sense. This condition here essentially says, in a formal way, that $\tau \circ c(f) = f'$ or $f' \circ \tau = c(f)$. Both seem reasonable.**

that preserves the interaction between $\mathcal{M}$ and $\mathcal{E}$-morphisms. Explicitly, F commutes with the functors c and k in the following diagrams

$$\begin{tikzcd}
\mathrm{Ar}_\square \mathcal{E} \arrow[r, "k"] \arrow[d, "\mathrm{Ar}_\square F"'] & \mathrm{Ar}_\triangle \mathcal{M} \arrow[d, "\mathrm{Ar}_\triangle F"] \\
\mathrm{Ar}_\square \mathcal{E}' \arrow[r, "k'"] & \mathrm{Ar}_\triangle \mathcal{M}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Ar}_\square \mathcal{M} \arrow[r, "c"] \arrow[d, "\mathrm{Ar}_\square F"'] & \mathrm{Ar}_\triangle \mathcal{E} \arrow[d, "\mathrm{Ar}_\triangle F"] \\
\mathrm{Ar}_\square \mathcal{M}' \arrow[r, "c'"] & \mathrm{Ar}_\triangle \mathcal{E}'
\end{tikzcd} .$$

For a CGW category $(\mathcal{C}, \varphi, c, k)$ be a CGW category, a *CGW subcategory* is a sub-double category $\mathcal{D} \subseteq \mathcal{C}$ such that $(\mathcal{D}, \phi|_\mathcal{D}, c|_\mathcal{D}, k|_\mathcal{D})$ forms a CGW category. That is, the structure maps on $\mathcal{C}$ restrict to define a CGW category on $\mathcal{D}$.

Convention 1.5. When the context is clear, we will omit mentions of the CGW structure maps and refer to a CGW category $(\mathcal{C}, \varphi, c, k)$ by its underlying double category $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ or just $\mathcal{C}$.

We now discuss Axioms (K) and (I) in more detail below, followed by some illustrative examples.

Quotients in CGW Categories. A distinctive feature of CGW categories is that they are agnostic about whether formal cokernels arise from taking quotients in the *additive* setting (e.g .R-modules) or taking complements in the *non-additive setting* (e.g. finite sets, varieties etc.). Either way, the formal properties remain consistent. We adopt the following suggestive convention to reinforce this perspective.

Convention 1.6 ("Quotient")**.** We typically denote the formal cokernel of $f\colon A \rightarrowtail B$ as $\frac{B}{A}$, whenever the map f is clear from context. The object $\frac{B}{A}$ will typically be referred to as a *quotient*. This is, of course, an abuse of language, but this is justified by our framework which makes precise how e.g. open complements of closed immersion of varieties behave formally like quotients of abelian groups.

The following lemma summarises some key properties of quotients within CGW categories.

Lemma 1.7. *The following properties hold in any CGW category* $\mathcal{C}$*:*

(i) (Quotients respect Distinguished Squares). *Given any distinguished square*

$$\begin{tikzcd}
A \arrow[r, rightarrowtail, "f"] \arrow[d, o->] \arrow[dr, phantom, "\square"] & B \arrow[d, o->] \\
C \arrow[r, rightarrowtail, "g"] & D
\end{tikzcd} ,$$

we have $\frac{B}{A} \cong \frac{D}{C}$.

(ii) (Quotients respect Filtrations). *Given* $P_0 \overset{f_1}{\rightarrowtail} P_1 \overset{g_1}{\rightarrowtail} P_2$*, one can construct the following diagram of distinguished squares*

$$\begin{tikzcd}
P_0 \arrow[r, rightarrowtail, "f_1"] & P_1 \arrow[r, rightarrowtail, "g_1"] & P_2 \\
O \arrow[u, o->] \arrow[r, rightarrowtail] \arrow[ur, phantom, "\square"] & P_{1/0} \arrow[u, o->, "f_2"] \arrow[r, rightarrowtail, "h_1"] \arrow[ur, phantom, "\square"] & P_{2/0} \arrow[u, o->, "g_2"'] \\
& O \arrow[u, o->] \arrow[r, rightarrowtail] \arrow[ur, phantom, "\square"] & P_{2/1} \arrow[u, o->, "h_2"']
\end{tikzcd} .$$

Proof. (i): Since distinguished squares compose vertically, the following square

$$\begin{tikzcd}
O \arrow[r, rightarrowtail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[d, o->] \\
C \arrow[r, rightarrowtail, "g"] & D
\end{tikzcd}$$

is distinguished. Since formal cokernels are unique (up to isomorphism), conclude that $\frac{B}{A} \cong \frac{D}{C}$.

(ii): First, apply Axiom (K) to obtain distinguished squares

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]->}] \arrow[dr, phantom, "\square"] & P_{1/0} \arrow[d, {Circle[open]->}, "f_2"] \\ P_0 \arrow[r, tail, "f_1"] & P_1 \end{tikzcd} \qquad \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]->}] \arrow[dr, phantom, "\square"] & P_{2/0} \arrow[d, {Circle[open]->}, "g_2"] \\ P_0 \arrow[r, tail, "g_1 f_1"] & P_2 \end{tikzcd} \qquad \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]->}] \arrow[dr, phantom, "\square"] & P_{2/1} \arrow[d, {Circle[open]->}] \\ P_1 \arrow[r, tail, "g_1"] & P_2 \end{tikzcd} .$$

Notice that $P_0 \rightarrowtail P_1 \rightarrowtail P_2$ yields a morphism

$$(P_0 \overset{f_1}{\rightarrowtail} P_1) \overset{g_1}{\rightarrowtail} (P_0 \overset{g_1 f_1}{\rightarrowtail} P_2)$$

in $\mathrm{Ar}_\triangle \mathcal{M}$. Applying k^{-1} and Axiom (K), this yields the distinguished square

$$\begin{tikzcd} P_1 \arrow[r, tail, "g_1"] \arrow[d, {Circle[open]->}, "f_2"'] \arrow[dr, phantom, "\square"] & P_2 \arrow[d, {Circle[open]->}, "g_2"] \\ P_{1/0} \arrow[r, tail, "h_1"] & P_{2/0} \end{tikzcd} .$$

This in turn can be interpreted as a morphism between g_1 and h_1 in $\mathrm{Ar}_\square \mathcal{M}$. Applying c yields an isomorphism between $c(P_1 \overset{g_1}{\rightarrowtail} P_2) = P_{2/1}$ and $c(P_{1/0} \overset{h_1}{\rightarrowtail} P_{2/0})$, and so this gives the bottom square.[5] $\square$

Isomorphisms in CGW Categories. In their original definition, CGW categories were only required to be double categories, not necessarily good double categories. The hypothesis of goodness was used because it allows us to express what it means for distinguished squares to interact well with isomorphisms[6] – this will play a crucial role in our proofs (see e.g. Lemmas B.2 and B.3). In addition, goodness allows us to streamline and generalise the original Axiom (I) in [CZ22, Definition 2.5] as follows.

Lemma 1.8. *If $A' \overset{f}{\rightarrowtail} A$ and $B' \overset{f'}{\rightarrowtail} B$ are both isomorphisms, and $A \overset{g}{\circ\!\!\rightarrow} B$ is a morphism in $\mathcal{E}$, then*

$$\begin{tikzcd} A' \arrow[r, tail, "f"] \arrow[d, {Circle[open]->}, "\varphi(f'^{-1})\circ g \circ \varphi(f)"'] \arrow[dr, phantom, "\square"] & A \arrow[d, {Circle[open]->}, "g"] \\ B' \arrow[r, tail, "f'"] & B \end{tikzcd}$$

is distinguished. Dually, if $A' \overset{g'}{\circ\!\!\rightarrow} B'$ and $A \overset{g}{\circ\!\!\rightarrow} B$ are isomorphisms, and $A \rightarrowtail B$ is a morphism in $\mathcal{M}$, then

$$\begin{tikzcd} A' \arrow[r, tail, "\varphi^{-1}(g^{-1})\circ f \circ \varphi^{-1}(g')"] \arrow[d, {Circle[open]->}, "g'"'] \arrow[dr, phantom, "\square"] & A \arrow[d, {Circle[open]->}, "g"] \\ B' \arrow[r, tail, "f"] & B \end{tikzcd}$$

is also distinguished.

The proof proceeds by a straightforward diagram-chase, which we leave to the reader. Let us also remark that imposing goodness is no real loss in generality since it still covers all the major examples mentioned in the original paper [CZ22]. The reason for this lies in the following observation.

Observation 1.9. Let $\mathcal{C}$ be any CGW category. Then pullback and/or pushout squares in the ambient category interact well with isomorphisms (in the sense of Definition 1.1).

Proof. Suppose

$$\begin{tikzcd} A \arrow[r, tail, "f'"] \arrow[d, {Circle[open]->}, "g'"'] & B \arrow[d, {Circle[open]->}, "g"] \\ C \arrow[r, tail, "f"] & D \end{tikzcd}$$

[5] **(M:) Double-check proof.**

[6] This appears in a different language in [HMM+22, Def. 3.1], which develops the (non-additive) K-theory of manifolds.

defines a commutative diagram in the ambient category $\mathcal{C}$. We first want to show that if either f' and f are isomorphisms, or g and g' are isomorphisms, then this diagram defines a pullback square in the ambient category. There are two main cases to check.

- **Case 1:** $\mathcal{E}^{\mathrm{op}}$ *is a subcategory of* $\mathcal{C}$. In which case, consider the following diagram in $\mathcal{C}$

$$\begin{tikzcd} R \arrow[drr, bend left, "j_1"] \arrow[dr, dashed, "\exists"] \arrow[ddr, bend right, "j_0"'] & & \\ & C \arrow[r, "f"] \arrow[d, "g'"'] & D \arrow[d, "g"] \\ & A \arrow[r, "f'"] & B \end{tikzcd}$$

  where the solid arrows define a commutative diagram. Notice the reversal of the vertical arrows. Suppose f and f' are isomorphisms. A straightforward exercise shows that $\exists := f^{-1} \circ j_1$ is the unique map making the whole diagram commute, proving that this defines a pullback square. The case when g and g' are isomorphisms follows by symmetry.
- **Case 2:** $\mathcal{E}$ *is a subcategory of* $\mathcal{C}$. Analogous to Case 1.

In summary: we have shown pulback squares interact well with isomorphisms. The argument for pushout squares is entirely analogous, and in fact was already worked out in [HMM$^+$22, Lemma 4.3]. $\square$

Examples. We review in broad strokes several motivating examples of CGW categories, as well as including a few new ones. For further details, see [CZ22, §4].

Example 1.10 (Exact Categories)**.** For an exact category $\mathcal{C}$, define a CGW category $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ by setting

$$\mathcal{M} = \{\text{admissible monomorphisms}\} \qquad \mathcal{E} = \{\text{admissible epimorphisms}\}^{\mathrm{op}}.$$

The basepoint object is the zero object in $\mathcal{C}$,[7] The distinguished squares are the biCartesian squares (= both pushouts and pullbacks in the ambient category $\mathcal{C}$). By Observation 1.9, $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ is a good double category. The equivalences k and c map admissible epis to kernels and admissible monos to cokernels, respectively. For more details on the other CGW axioms, see see [CZ22, Example 3.1].

Example 1.11 (Finite Sets)**.** Given FinSet, define a CGW category $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ by setting

$$\mathcal{E} = \mathcal{M} = \{\text{injections}\}.$$

The basepoint object O is the empty set, and the distinguished squares are the pushout squares. By Observation 1.9, $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ is a good double category. The equivalences c and k are given by taking any inclusion $A \hookrightarrow B$ to the inclusion $B \setminus A \hookrightarrow B$.

Example 1.12 (Varieties)**.** Given $\mathcal{V}\mathrm{ar}_k$, define $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ by setting

$$\mathcal{M} = \{\text{closed immersions}\} \qquad \mathcal{E} = \{\text{open immersions}\}.$$

The basepoint object O is the empty variety, and the distinguished squares are the pullback squares

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & B \arrow[d, circ, "g"] \\ C \arrow[r, tail, "f"] & D \end{tikzcd}$$

in which $\mathrm{im} f \cup \mathrm{im} g = D$, which implies goodness of $\mathcal{C}$.[8] c and k takes a morphism to the inclusion of the complement. Axioms (I) and (M) follow from properties of closed and open immersions, while Axiom (K)

[7] Notice that we take the opposite category for $\mathcal{E}$, and so the zero object is initial in $\mathcal{E}$ as required by Axiom (Z).

[8] Notice that we get $\mathrm{im} f \cup \mathrm{im} g = D$ for free if either f or g are isomorphisms.

holds as $D \setminus C \cong B \setminus A$ for any distinguished square

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\square"] & B \arrow[d, "\circ" marking] \\ C \arrow[r, tail] & D \end{tikzcd} .$$

For those interested in model theory, the following example will be suggestive.

Example 1.13 (Definable Sets)**.** Fix Σ to be a first-order language, M a Σ-structure, and $A \subseteq M$ the subset of parameters. Define a CGW category $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ where the objects are A-definable subsets of M, and

$$\mathcal{E} = \mathcal{M} = \{A\text{-definable injective functions}\}.$$

Axioms (I) and (M) are thus satisfied by definition. For Axiom (Z), set the basepoint object O as $\emptyset$. Next, since A-definable sets are closed under finite unions, define the distinguished squares to be the pushout squares, corresponding to disjunctions of the corresponding formulae. Finally, consider a definable injection $C \rightarrowtail D$ where C is defined by $\phi(x; \overline{a})$ and D is defined by $\psi(x; \overline{b})$. The equivalences c and k are given by sending $C \rightarrowtail D$ to $C' \rightarrowtail D$ where C' is defined by $\psi(x; \overline{b}) \wedge \neg\phi(x; \overline{a})$.[9]

Remark 1.14. A note for the curious non-logician: the example of definable sets is an abstraction of $\mathcal{V}\mathrm{ar}_k$, at least when k is an algebraically-closed field. For example, let us view $\mathbb{C}$ as a model of the usual theory of algebraically-closed fields, with $\mathbb{C}$ as our parameter set. The formulae then correspond to polynomials with coefficients in $\mathbb{C}$, and so the objects in $\mathrm{Def}(\mathbb{C})$ correspond to (Boolean combinations of) affine $\mathbb{C}$-varieties.[10]

Finally, let us mention another non-additive generalisation of exact categories known as *proto-exact categories*, introduced by Dyckerhoff-Kapranov [DK19]. A particularly challenging example comes from a recent result in [EJS20], which shows that the category of matroids form a proto-exact category, thereby admitting a K-theory spectrum.

An informal overview: a *matroid* abstracts the notion of linear independence, consisting of a finite set E and a collection of subsets called *flats*, which are maximal dependent sets whose proper subsets are independent. These combinatorial gadgets have surprisingly deep links with algebraic and tropical geometry. For instance, the characteristic polynomial of matroids admits a motivic interpretation in $K_0(\mathcal{V}\mathrm{ar}_k)$, and substantive breakthroughs have been made by the newly-developed Hodge theory for matroids.[11] Matroids are also suggestive from a homological perspective because of their family resemblance to Tits buildings – in particular, any finite vector space V gives rise to a matroid $M(V)$, whose flats are its vector subspaces. These observations motivate many interesting questions, particularly in light of the unique way matroids bridges combinatorics and geometry. Here we refine our understanding by translating the results of [EJS20] to show that their category of matroids is also a CGW category.

Example 1.15 (Matroids)**.** Let $M = (E, \mathcal{F}, \bullet_M)$ be a *pointed matroid*, where E is a finite set, and $\mathcal{F} \subseteq 2^E$ the set of *flats* of matroid M and $\bullet_M$ the distinguished base-point. To ease notation, denote $\widetilde{E} := E \setminus \{\bullet_M\}$. Given any $S \subseteq \widetilde{E}$, denote

$\mathcal{F}(M|S) := \{((A \cap S, \bullet_M) \mid A \in \mathcal{F}(M)\}$ to be the *restriction of M to S*;
$\mathcal{F}(M/S) = \{(A \setminus S, \bullet_M) \mid S \subseteq A \in \mathcal{F}(M)\}$ to be the *contraction of M to S*.

A *strong map* of pointed matroids $f\colon M \to N$ is a function $f\colon E_M \to E_N$ such that $f(\bullet_M) = \bullet_N$ and $f^{-1}A \in \mathcal{F}(M)$ for all $A \in \mathcal{F}(N)$. By [EJS20, Lemma 2.12], pointed matroids and strong maps form a category $\mathrm{Mat}_\bullet$. Now define a CGW category $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ by setting

$$\mathcal{M} = \{\text{strong maps that can be factored } N \xrightarrow{\sim} M|S \hookrightarrow M, \text{ for some } S \subseteq \widetilde{E}_M\}$$

$$\mathcal{E} = \{\text{strong maps that can be factored } M \twoheadrightarrow M/S \xrightarrow{\sim} N, \text{ for some } S \subseteq \widetilde{E}_M\}^{\mathrm{op}}.$$

Applying [EJS20, Lemma 5.2],$\mathcal{M}$ and $\mathcal{E}$ are closed under isomorphisms and composition, satisfying Axiom (I). We define the distinguished squares to be the biCartesian squares in $\mathrm{Mat}_\bullet$, which interact well with

[9] **(M:) Be careful; is any inclusion of a definable set into another one necessarily definable? Or does it not matter?**
[10] **(M:) Double-check the bit about definability vs. 0-definability.**
[11] **(M:) Cite Matt Baker, Chris Eur and Eric Katz Surveys.**

isomorphisms. Next, [EJS20, Lemma 5.4] says: a strong map f is monic in $\mathrm{Mat}_\bullet$ iff f is injective on the underlying set, and f is epi iff f is surjective. Thus, all morphisms in $\mathcal{M}$ and $\mathcal{E}$ are monic, and the pointed matroid $O := (\{*\}, *)$ initial in both – satisfying Axioms (M) and (Z). Finally, translating [EJS20, Props. 5.7 and 5.8] to our setting: any $P \overset{i'}{\rightarrowtail} Q \overset{j'}{\circ\!\!\rightarrow} N$ or $P \overset{j}{\circ\!\!\rightarrow} M \overset{i}{\rightarrowtail} N$ can be completed into a distinguished square

$$\begin{tikzcd} P \arrow[r, tail, "i'"] \arrow[d, "j"'] \arrow[dr, phantom, "\square"] & Q \arrow[d, "j'"] \\ M \arrow[r, tail, "i"] & N \end{tikzcd} .$$

Setting $P = O$, this gives the formal kernels and cokernels required by Axiom (K), unique up to isomorphism due to the biCartesian property.

1.2. **The K-Theory of pCGW categories.** The main result of [CZ22, §4] is that Quillen's Q-Construction [Qui73] can be applied to any CGW category to define its corresponding K-theory spectrum. However, the present paper will focus on a particularly well-behaved class of CGW categories, which we call *pCGW categories.*

Informally, pCGW categories are CGW categories $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ whereby $\mathcal{M}$ is closed under a formal kind of pushout. This, of course, generalises the familiar fact that admissible monics are closed under pushouts in exact categories [Wei13, Exercise II.7.8], but it is instructive to understand why this generalisation is needed. Consider Example 1.11 where $\mathcal{M}$ is the category of finite sets and injections. In which case,

$$A \leftarrow \emptyset \rightarrow A \qquad \text{where } A \neq \emptyset$$

does not have a pushout in $\mathcal{M}$ since the map $A \coprod A \to A$ is not monic. Nonetheless, this issue can be circumvented by placing suitable restrictions on the universal pushout property. The following key definition makes this precise.

Definition 1.16 (Restricted Pushout, [CZ22, Def. 5.3])**.** Let $\mathcal{M}$ be a category whose morphsims are all monic, and let

$$C \leftarrow A \rightarrow B$$

be a span. The *restricted pushout* is the initial object (if it exists) in the category of pullback squares in $\mathcal{M}$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & X \end{tikzcd} .$$

A morphism between squares is a natural transformation in which all components are equal to the identity except at X. Restricted pushouts are denoted by $B \star_A C$.

A useful fact is that restricted pushouts still behave functorially like a pushout in the following sense:

Fact 1.17. Consider the diagram

$$C \leftarrowtail A \rightarrowtail B \rightarrowtail B'$$

Then $B' \star_B (B \star_A C) \cong B' \star_A C$. More explicitly, the composite of restricted pushouts in Diagram (2) is the restricted pushout of the outer span.

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, tail] & B \arrow[r, tail] \arrow[d, tail] & B' \arrow[d, tail] \\ C \arrow[r, tail] & B \star_A C \arrow[r, tail] & B' \star_B (B \star_A C) \end{tikzcd} \tag{2}$$

Proof. This follows from [SS21, Corollary A.2] – their framework uses a generalisation[12] of restricted pushouts but the proof is analogous. □

We now introduce the definition of a pCGW category before reviewing a few key examples.

[12] **(M:) Double-check this!**

Definition 1.18. Let $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ be a CGW category. We call $\mathcal{C}$ a *pCGW category* if $\mathcal{M}$ contains all restricted pushouts. In addition, restricted pushouts are required to satisfy:

(A) *Formal Direct Sums.* Denote the restricted pushout of $B \leftarrowtail O \rightarrowtail C$ as $B \oplus C := B \star_O C$, which we also call *formal direct sums*. Then, there exists a canonical pair of distinguished squares

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & B \arrow[d, {Circle[open]}->, "q_B"] \\ C \arrow[r, tail, "p_C"] & B \oplus C \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \arrow[d, {Circle[open]}->, "q_C"] \\ B \arrow[r, tail, "p_B"] & B \oplus C \end{tikzcd},$$

which we call *direct sum squares*.

(PQ) *Preserves quotients.* A restricted pushout

$$\begin{tikzcd} A \arrow[r, tail, "f"] \arrow[d, tail, "g"] & B \arrow[d, tail, "g'"] \\ C \arrow[r, tail, "f'"] & B \star_A C \end{tikzcd}$$

induces an isomorphism

$$\frac{B}{A} \cong \frac{B \star_A C}{C}.$$

(DS) *Compatibility with Distinguished Squares.* Given a diagram of distinguished squares

$$\begin{tikzcd} C \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & A \arrow[l, tail] \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & B \arrow[d, {Circle[open]}->] \\ C' & A' \arrow[l, tail] \arrow[r, tail] & B' \end{tikzcd}$$

there is an induced map $B \star_A C \circ\!\!\rightarrow B' \star_{A'} C'$ such that the two induced squares

$$\begin{tikzcd} B \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & B \star_A C \arrow[d, {Circle[open]}->] \\ B' \arrow[r, tail] & B' \star_{A'} C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & B \star_A C \arrow[d, {Circle[open]}->] \\ C' \arrow[r, tail] & B' \star_{A'} C \end{tikzcd}$$

are distinguished.

Remark 1.19. To remove potential confusion, we point out that Definition 1.4 does not require $\mathcal{E}$ to contain all restricted pushouts – only $\mathcal{M}$. This is contrast to the definition of ACGW categories in [CZ22, Def 5.6], which imposes more conditions than we do.

Example 1.20 (Exact Categories)**.** Admissible monics are closed under pushouts in exact categories so let this be our notion of restricted pushouts. Axioms (PQ) and (DS) follow from the fact that pushouts preserve cokernels. For Axiom (A), leverage the fact that $B \oplus C$ is a biproduct in an exact category, and define the following squares

$$\begin{tikzcd} B \arrow[r, "p_B"] \arrow[d] & B \oplus C \arrow[d, "q_C"] \\ O \arrow[r] & C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "p_C"] \arrow[d] & B \oplus C \arrow[d, "q_B"] \\ O \arrow[r] & B \end{tikzcd},$$

where q_B, q_C are the natural projection maps, and p_B, p_C are the natural coprojection maps. Applying the universal properties of $B \oplus C$ as a biproduct, a standard exercise shows that the above squares are both pushouts and pullbacks, and thus define distinguished squares.

Example 1.21 (Varieties)**.** Let $\star$ be the pushout of closed immersions in the category of schemes. Examining the work of [Sch05], it was noticed in [Cam19, §2] that a pushout of closed immersions of varieties produces a square of closed immersions of varieties. For transparency, we emphasise that these squares are not pushouts in the *category of closed immersions* but rather in the *entire category of schemes*. The fact that $\star$ satisfies Axioms (PQ) and (DS) follows from the universal property of pushouts. As for Axiom (A), let the direct sum squares be the coproduct squares with standard coprojection maps.

Example 1.22 (Finite & Definable Sets)**.** Let $\star$ be the pushout along the $\mathcal{M}$-morphisms. Notice that since $\mathcal{M} = \mathcal{E}$ in these two examples, a restricted pushout corresponds to a distinguished square.

We now setup the K-theory of pCGW categories by way of Waldhausen's $S_\bullet$-construction.

Construction 1.23 ($S_\bullet$-Construction)**.** Let $\mathcal{C}$ be a pCGW category. Define $S_\bullet\mathcal{C}$ to be the simplicial set with n-simplices $S_n\mathcal{C}$ given by flag diagrams

$$\begin{tikzcd}
M_{00} \arrow[r, tail] & M_{01} \arrow[r, tail] & M_{02} \arrow[r, tail] & \dots \arrow[r, tail] & M_{0n} \\
 & M_{11} \arrow[r, tail] \arrow[u, {Circle[open]}->] & M_{12} \arrow[r, tail] \arrow[u, {Circle[open]}->] & \dots \arrow[r, tail] & M_{1n} \arrow[u, {Circle[open]}->] \\
 & & M_{22} \arrow[r, tail] \arrow[u, {Circle[open]}->] & \dots \arrow[r, tail] & M_{2n} \arrow[u, {Circle[open]}->] \\
 & & & \vdots \arrow[u, {Circle[open]}->] & \vdots \arrow[u, {Circle[open]}->] \\
 & & & & M_{nn} \arrow[u, {Circle[open]}->]
\end{tikzcd}$$

subject to the conditions

(i) $M_{ii} = O$ for all i

(ii) Every subdiagram

$$\begin{tikzcd}
M_{ki} \arrow[r, tail] & M_{kl} \\
M_{ji} \arrow[r, tail] \arrow[u, {Circle[open]}->] \arrow[ur, phantom, "\square"] & M_{jl} \arrow[u, {Circle[open]}->]
\end{tikzcd}$$

for $k < j$ and $i < l$ is distinguished.

We shall often represent an n-simplex as a sequence of $\mathcal{M}$-morphisms

$$O = M_0 \rightarrowtail M_1 \rightarrowtail M_2 \rightarrowtail \dots \rightarrowtail M_n$$

together with choice of (formal) quotients

$$M_{j/i} := \frac{M_j}{M_i} \qquad i < j.$$

Face maps are obtained by forgetting an M_i, degeneracy maps by duplicating an M_i, with the exception that forgetting M_0 means factoring out by M_1.

Theorem 1.24 (Presentation Theorem)**.** *Let $\mathcal{C}$ be a pCGW category and define its K-theory spectrum*

$$K\mathcal{C} := \Omega|\mathcal{S}\mathcal{C}|,$$

with associated K-groups $K_n(\mathcal{C}) := \pi_n K\mathcal{C}$. Then $K_0(\mathcal{C})$ is the free abelian group generated by objects of $\mathcal{C}$ modulo the relation that for any distinguished square

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & B \arrow[d, {Circle[open]}->] \\
D \arrow[r, tail] & C
\end{tikzcd},$$

we have $[D] + [B] = [A] + [C]$.

Proof. There are various ways to see this; here is one such proof. Start by applying the Q-construction to $\mathcal{C}$ to obtain the spectra $K^Q(\mathcal{C})$. By [CZ22, Thm 4.3], $\pi_0(K^Q(\mathcal{C}))$ is precisely the free abelian group on objects of $\mathcal{C}$ modulo the distinguished square relation above. Apply the standard edgewise subdivision argument to show that $K\mathcal{C}$ and $K^Q(\mathcal{C})$ are weakly equivalent as spaces [CZ22, Thm. 7.8]. $\square$

Remark 1.25. In fact, Theorem 1.24 holds for any CGW category equipped with some notion of a formal direct sum, as in Axiom (A) Definition 1.18.

Summary Example 1.26. *Tutte-Grothendieck, all the different π_0's, brief discussion.*

1.3. **Simplicial Loops & Fibers.** Let us take a closer look at the statement of Presentation Theorem 1.24. Notice that the loop space emerges naturally in our definition of $K\mathcal{C}$. How might we translate this construction to simplicial sets? We follow the approach in [GG87, §2].

Convention 1.27. A simplicial set is a contravariant functor $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$. To ease exposition, we sometimes write X_n to mean $X([n])$. If $A, B \in \Delta$, we write AB to mean the disjoint union of A followed by B, where elements of A are below those of B.

To motivate, recall that the loop space ΩZ of a pointed topological space Z is the space of based loops $\mathrm{Map}(S^1, Z)$. Now fix a simplicial set X with basepoint $O \in X([0])$. A simplicial loop may look like two 1 simplices glued together at the end

$$0 \underset{g}{\overset{f}{\rightrightarrows}} x,$$

while a homotopy of such loops may look like

$$\begin{tikzcd} 0 \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[rr, bend left=60, "f'"] \arrow[rr, bend right=60, "g'"'] & x \arrow[r, "h"] & y \end{tikzcd}$$

where we glue 2-simplices in the triangle $h \circ f = f'$ and $h \circ g = g'$ along a shared 1-simplex h. One can then extend this picture in a natural way to the higher homotopies as follows:

Construction 1.28 (Simplicial Loops)**.** Given any simplicial set X, define

$$\Omega X(A) := \varprojlim \left(\begin{tikzcd} \{0\} \arrow[r, hook] & X([0]) & X([0]A) \arrow[l] \arrow[d] \\ & X([0]A) \arrow[u] \arrow[r] & X(A) \end{tikzcd} \right).$$

Notice that Construction 1.28 works for any simplicial set X. This sets up the obvious definition:

Definition 1.29 (G-Construction)**.** Let $\mathcal{C}$ be a pCGW category. The *G-Construction* on $\mathcal{C}$ is defined by applying the Simplicial Loop Construction to $\mathcal{S}\mathcal{C}$,

$$G\mathcal{C} := \Omega S\mathcal{C}.$$

We define the associated K-groups as $K_n^G(\mathcal{C}) = \pi_n|\Omega\mathcal{S}\mathcal{C}|$.

The G-construction is well-defined for ordinary CGW categories. However, we will need the restricted pushouts of $\mathcal{M}$-morphisms in order to show that $G\mathcal{C}$ and $K\mathcal{C}$ are homotopy equivalent (and thus define isomorphic K-groups). Our approach mirrors what was done by Gillet-Grayson [GG87], who established $G\mathcal{C} \simeq K\mathcal{C}$ for exact categories. Their argument proceeded by translating standard homotopical notions to the simplicial setting, before constructing the desired equivalences through well-chosen pushouts of admissible monics.

A central notion in their paper [GG87] is the *right fiber*.

Definition 1.30 (Right Fiber)**.** Suppose $F\colon X \to Y$ is a map of simplicial sets, $A \in \Delta$ and $\rho \in Y(A)$. We define $\rho|F$ ("the right fiber over ρ") by

$$(\rho|F)(B) := \lim_{\leftarrow} \left(\begin{tikzcd} & & X(B) \arrow[d] \\ & Y(AB) \arrow[r] \arrow[d] & Y(B) \\ \{\rho\} \arrow[r, hook] & Y(A) & \end{tikzcd} \right).$$

We write $\rho|Y$ for $\rho|1_Y$. We regard $\rho|F$ as the simplicial analogue of the homotopy fiber $|F|$ over ρ.

We can now restate our problem. To show that $G\mathcal{C} \simeq K\mathcal{C}$, we need to show that the geometric realisation and the loop space constructions commute up to homotopy equivalence, i.e.

$$|\Omega\mathcal{S}\mathcal{C}| \simeq \Omega|\mathcal{S}\mathcal{C}|.$$

The following key observation tells us when this happens.

Observation 1.31 (Key Observation)**.** For any simplicial set X, consider the commutative square

$$\begin{tikzcd} \Omega X \arrow[r, "t"] \arrow[d, "b"] & O|X \arrow[d, "q"] \\ O|X \arrow[r, "q"] & X \end{tikzcd} . \tag{3}$$

where b and t are the projection maps (forgetting one of the components $X([0]A)$) and q is the obvious face map. Then:

(i) $O|X$ is contractible.
(ii) $O|q \simeq \Omega X$
(iii) $|\Omega X| \simeq \Omega|X|$ iff this square is homotopy Cartesian.

Proof. (i) follows from [GG87, Lemma 1.4]. (ii) is clear from unpacking definitions. Explicitly, notice

$$(O|q)(B) := \lim_{\leftarrow} \left(\begin{tikzcd} & & O|X(B) \arrow[d, "q"] \\ & X([0]B) \arrow[r] \arrow[d] & X(B) \\ \{O\} \arrow[r, hook] & X([0]) & \end{tikzcd} \right)$$

where

$$(O|X)(B) := \lim_{\leftarrow} \left(\begin{tikzcd} & X([0]B) \arrow[r, "q"] \arrow[d] & X(B) \\ \{O\} \arrow[r, hook] & X([0]) & \end{tikzcd} \right).$$

In other words, the B-simplices of $(O|q)$ correspond to a pair of $B+1$ simplices in X that agree at the base-point O, and on the Bth face. For (iii), first take the homotopy pullback P of

$$|O|X| \to |X| \leftarrow |O|X|$$

in the homotopy category of spaces. Taking the geometric realisation of Diagram (3), notice that this gives rise to a map $|\Omega X| \to P$, which is a homotopy equivalence iff Diagram (3) defines a homotopy pullback. Finally, since $O|X$ is contractible, deduce that the homotopy pullback P is equivalent to the homotopy pullback of

$$* \to |X| \leftarrow *$$

which is the loop space $\Omega|X|$. □

Key Observation 1.31 suggests the following proof strategy. By item (iii), in order to show $G\mathcal{C} \simeq K\mathcal{C}$ it suffices to verify that the square

$$\begin{tikzcd} \Omega\mathcal{S}\mathcal{C} \arrow[r, "t"] \arrow[d, "b"] & O|\mathcal{S}\mathcal{C} \arrow[d, "q"] \\ O|\mathcal{S}\mathcal{C} \arrow[r, "q"] & \mathcal{S}\mathcal{C} \end{tikzcd} . \tag{4}$$

is homotopy Cartesian. By item (ii), this is equivalent to showing that

$$\begin{tikzcd} O|q \arrow[r, "t"] \arrow[d, "b"] & O|\mathcal{S}\mathcal{C} \arrow[d, "q"] \\ O|\mathcal{S}\mathcal{C} \arrow[r, "q"] & \mathcal{S}\mathcal{C} \end{tikzcd} . \tag{5}$$

is homotopy Cartesian.[13] To show this, it suffices to analyse how $q\colon O|\mathcal{S}\mathcal{C} \to \mathcal{S}\mathcal{C}$ behaves on the induced simplicial fibers in the following sense:

Theorem 1.32 ([GG87], Theorem B'). *Suppose $F\colon X \to Y$ is a map of simplicial sets. Suppose for any $A \in \Delta$, any $\rho \in Y(A)$, and any $f\colon A' \to A$ such that the induced map*

$$\rho|F \to f^*\rho|F$$

is a homotopy equivalence. Then the square

$$\begin{tikzcd} \rho|F \arrow[r] \arrow[d] & X \arrow[d] \\ \rho|Y \arrow[r] & Y \end{tikzcd}$$

is homotopy Cartesian.

Remark 1.33. Theorem 1.32 is a simplicial analogue of Quillen's Theorem B, and is proved by imitating Quillen's original argument. Although our proof that $K\mathcal{C} \simeq G\mathcal{C}$ is different from the one presented in [GG87], this result still plays a crucial role in our argument. [A sidenote: the original proof of Theorem B' has a small error, but this has since been corrected in [GG03].]

2. A Technical Result on Right Fibers

Convention 2.1. Hereafter, any category denoted $\mathcal{C}, \mathcal{D}$ should be assumed to be a pCGW category, unless stated otherwise.

The goal of this section is to prove Theorem 2.7, which informally states: given a nice simplicial map $F\colon Y \hookrightarrow \mathcal{S}\mathcal{C}$ where $\mathcal{C}$ is a pCGW category, the right fiber $O|F$ admits a nice description as a homotopy pullback (Theorem 2.7). In addition, we show that direct sum induces an H-space structure on $O|F$. The results here are technical, and our approach relies on a simplicial translation of Grayson's framework of *dominant functors* [Gra87].

There are two main applications of Theorem 2.7. First, the key result that $G\mathcal{C} \simeq K\mathcal{C}$ is obtained as a straightforward corollary (Theorem 2.12). Second, it also sets up the proof of Theorem 3.7, which gives an initial characterisation of the generators of $K_1(\mathcal{C})$; details of this will be deferred to Section 3.

2.1. H-Space Structure.

Recall: an *H-space* is a triple $(X, e, \bullet)$ whereby X is a space, $e \in X$ is a point, and $\bullet\colon X \times X \to X$ is a continuous map such that $e \bullet e = e$ and the maps $x \mapsto x \bullet e$ and $x \mapsto e \bullet x$ are homotopic to the identity map.

[13] To streamline notation, we have chosen to leave the labels of the maps in the new diagram unchanged; we hope this will not cause too much confusion.

Construction 2.2 (Addition Map on the Right Fiber)**.** Let $\mathcal{C}$ be a pCGW category, and let

$$F\colon Y \hookrightarrow \mathcal{SC}$$

be the inclusion[14] of a subsimplicial set $Y \subseteq \mathcal{SC}$. In particular, notice that $F(O) = O$.

(i) *An Explicit Description.* Let $A = [a] = \{0 < 1 < \cdots < a\}$, and $\overline{M} \in \mathcal{SC}(A)$. We represent a q-simplex W of $\overline{M}|F$ (cf. Definition 1.30) as

$$W = \left(O \rightarrowtail M_1 \rightarrowtail \ldots \rightarrowtail \quad M_a \rightarrowtail \quad \frac{\underline{\underline{O = K_0 \rightarrowtail \ldots \rightarrowtail K_q}}}{L_0 \quad \rightarrowtail \ldots \rightarrowtail L_q} \right)$$

where
- The top row represents a q-simplex of Y;
- The bottom row represents a $q + a + 1$-simplex of $\mathcal{SC}$;
- The double line represents the identity

$$\begin{array}{ccc} O = F(O) & \rightarrowtail \ldots \rightarrowtail & FK_q \\ \| & & \| \\ O = L_0/L_0 & \rightarrowtail \ldots \rightarrowtail & L_q/L_0 \end{array}$$

In the case where $F = \mathrm{id}$ and $\overline{M} = O$, it will be convenient to represent the n-simplices of $O|\mathcal{SC}$ as filtrations of the form $O \rightarrowtail K_0 \to \ldots K_n$ [where K_0 need not be O].[15]

(ii) *Defining the addition map.* We use restricted pushouts to define an operation

$$+\colon \overline{M}|F \times \overline{M}|F \to \overline{M}|F$$

by setting

$$W + W' := \left(O \rightarrowtail \ldots \rightarrowtail M_a \rightarrowtail \quad \frac{\underline{\underline{O \rightarrowtail \ldots \rightarrowtail K_q \oplus K_q'}}}{L_0 \star_{M_a} L_0' \rightarrowtail \ldots \rightarrowtail L_q \star_{M_a} L_q'} \right),$$

with the quotients specified by
- $\frac{K_i \oplus K_i'}{K_j \oplus K_j'} := \frac{K_i}{K_j} \oplus \frac{K_i'}{K_j'}$, $\qquad \frac{L_i \star_{M_a} L_i'}{L_j \star_{M_a} L_j'} := F\left(\frac{K_i \oplus K_i'}{K_j \oplus K_j'}\right)$
- $\frac{L_i \star_{M_a} L_i'}{M_a} := \frac{L_i}{M_a} \oplus \frac{L_i'}{M_a}$
- $\frac{L_i \star_{M_a} L_i'}{M_j}$ defined by applying Axiom (K), Definition 1.4, for $1 \leq j < a$.

Convention 2.3. Say something about having double-underline, and without underline being just pairs of things.

Claim 2.4. *The addition map defined in Construction 2.2 turns $|\overline{M}|F|$ into a homotopy associative and homotopy commutative H-space. In particular, $\pi_0(\overline{M}|\mathcal{F})$ equipped with $+$ is a monoid.*

Proof. Some basic observations.

(a) The addition map is well-defined.
[Why? Apply Lemma A.1 to verify the new filtrations exist and the quotients make sense.]

(b) The 0-simplex

$$\left(O = M_0 \rightarrowtail \ldots \rightarrowtail \quad M_a \overset{1}{\rightarrowtail} \quad \frac{\underline{\underline{O}}}{M_a} \right)$$

serves as additive identity. (Notation: $M_a \overset{1}{\rightarrowtail} M_a$ denotes the identity map.)
[Why? Since formal cokernels are unique (up to isomorphism), and O is initial with respect to $\mathcal{M}$-morphisms, it is clear $O \oplus K_i \cong K_i$. Further, since restricted pushouts are initial[16], deduce $L_i \star_{M_a} M_a \cong L_i$.]

[14](M:) **I don't really require the thing to be an inclusion. Double-check, and if so, get rid of it.**

[15](M:) **Double-check this later.**

[16](M:) **Double-check. What exactly is meant by being initial in this category?**

(c) $|\overline{M}|F|$ is a homotopy associative and homotopy commutative H-space.
[Why? That $|\overline{M}|F|$ is an H-space follows from Observations (a) and (b). As for the rest, since restricted products (and direct sums) are initial, they are associative and commutative up to natural isomorphism. These define natural transformations that turn $|\overline{M}|F|$ into a homotopy associative and commutative H-space.[17]]

$\square$

2.2. **The Main Result.** One can leverage the H-space structure of $O|F$ to obtain an elegant description of the right fiber, so long as F satisfies a certain technical condition.

Definition 2.5 (Cofinality)**.** Let $\mathcal{C}$ be a pCGW category, and $F\colon Y \hookrightarrow \mathcal{S}\mathcal{C}$ be the inclusion of a subsimplicial set $Y \subseteq \mathcal{S}\mathcal{C}$. Define the *image of F* as

$$\mathrm{im}F := \{M \in \mathcal{S}\mathcal{C}[1] \mid M \cong F(K) \text{ for some } \ K \in Y[1]\}.$$

We call $\mathrm{im}F$ *cofinal* in $\mathcal{S}\mathcal{C}$ if for any $T \in \mathcal{S}\mathcal{C}[1]$, there exists $T' \in \mathcal{S}\mathcal{C}[1]$ such that $T \oplus T' \in \mathrm{im}F$.

Remark 2.6. The notation $\mathrm{im}F$ is suggestive. Recall that $\mathcal{S}\mathcal{C}[1]$ is isomorphic to $\mathcal{C}$. Hence, if $Y = \mathcal{S}\mathcal{D}$ for some CGW subcategory $\mathcal{D} \subseteq \mathcal{C}$, then $\mathrm{im}F$ corresponds to the set of objects in $\mathcal{D}$ under the inclusion functor. This gives a simplicial translation of the original definition of cofinality [Gra87], which was done on the level of functors.

We can now state the main technical result of this section.

Theorem 2.7. *Let $\mathcal{C}$ be a pCGW category, and $F\colon Y \hookrightarrow \mathcal{S}\mathcal{C}$ be the inclusion of a subsimplicial set $Y \subseteq \mathcal{S}\mathcal{C}$. Suppose $\mathrm{im}F$ is cofinal in $\mathcal{S}\mathcal{C}$. <u>Then</u> the square*

$$\begin{tikzcd} O|F \arrow[r] \arrow[d] & Y \arrow[d] \\ O|\mathcal{S}\mathcal{C} \arrow[r] & \mathcal{S}\mathcal{C} \end{tikzcd} \tag{6}$$

is homotopy cartesian.

Proof. By Claim 2.4, $\pi_0(\overline{M}|F)$ is a monoid with respect to $+$. Say that F is *dominant* if $\pi_0(\overline{M}|F)$ is a group [not just a monoid] given *any* $\overline{M} \in \mathcal{S}\mathcal{C}(A)$ for *any* $A \in \Delta$.[18] The proof of the theorem then follows from establishing two main implications.

- Step 1: If $\mathrm{im}F$ is cofinal in $\mathcal{S}\mathcal{C}$, then F is also dominant.
- Step 2: If F is dominant, then Diagram (6) is a homotopy cartesian square.

Step 1: F is dominant. Fix some $\overline{M} \in \mathcal{S}\mathcal{C}(A)$ for some $A \in \Delta$. We want to show $\pi_0(\overline{M}|F)$ is a group. We start by introducing a helper definition.

Definition 2.8 (F-mono)**.** Let $A \rightarrowtail B$ be an $\mathcal{M}$-morphism in $\mathcal{C}$. If $\frac{B}{A} \cong F(K)$ for some $K \in Y[1]$, then call F an *F-mono*.

In [Gra87, Theorem 2.1], Grayson gives a characterisation of *dominant functors* via F-monos in the setting of exact categories. The following claim adapts his argument to our setting.

Claim 2.9. *$\pi_0(\overline{M}|F)$ is a group iff for each $\mathcal{M}$-morphism $M \rightarrowtail N$ of $\mathcal{C}$, there exists another $\mathcal{M}$-morphism $M \rightarrowtail N'$ and a commutative diagram*

$$\begin{tikzcd} M \arrow[r, equal] \arrow[d, tail] & L_0 \arrow[r, tail] & \dots \arrow[r, tail] & L_s \arrow[d, equal] \\ N \star_M N' \arrow[r, equal] & N_0 \arrow[r, tail] & \dots \arrow[r, tail] & N_q \end{tikzcd}$$

such that the horizontal arrows are F-monos.

[17](M:) **Double-check later, maybe with Behrang.**

[18](M:) **Changed definitino, used dominant, because M has to vary.**

With Claim 2.9 in hand, one easily proves the desired implication. For suppose $M \rightarrowtail N$ in $\mathcal{C}$. Since $\mathrm{im}F$ is cofinal in $\mathcal{SC}$, find T' so that

$$\frac{N}{M} \oplus T' \in \mathrm{im}F.$$

Setting $N' := M \oplus T'$, deduce the following:

- $N \star_M N' \cong N \oplus T'$, by applying Fact 1.17 to $T' \twoheadleftarrow O \rightarrowtail M \rightarrowtail N$.
- $M \rightarrowtail N \star_M N'$ is an F-mono, since $\frac{N}{M} \oplus T' \cong \frac{N \oplus T'}{M}$ by Lemma A.2.
- $\mathrm{id}\colon N \star_M N' \rightarrowtail N \star_M N'$ is an F-mono, since $\frac{N \star_M N'}{N \star_M N'} = O = F(O)$.

In other words, given any $M \rightarrowtail N$ in $\mathcal{C}$, one can define another $\mathcal{M}$-morphism $M \rightarrowtail N'$ and a diagram

$$\begin{tikzcd} M \arrow[r, tail] \arrow[d, tail] & N \star_M N' \arrow[d, equal] \\ N \star_M N' \arrow[r, tail] & N \star_M N' \end{tikzcd}$$

such that the horizontal arrows are F-monos. By Claim 2.9, conclude that $\pi_0(\overline{M}|F)$ is a group.

Thus to finish Step 1, it remains to prove the stated claim.

Proof of Claim 2.9. [19] Proceed by examining the generators and relations of $\pi_0(\overline{M}|F)$. Consider the standard presentation

- Generators: Vertices of $\overline{M}|F$, e.g.

$$W = \left(\begin{tikzcd}[row sep=small] & & & & O \arrow[d, equal] \\ O \arrow[r, tail] & M_1 \arrow[r, tail] & \dots \arrow[r, tail] & M_a \arrow[r, tail] & N \end{tikzcd} \right).$$

- Relations: 1-simplices of $\overline{M}|F$.

Let us refine the generators. No real information is lost by fogetting the top row, which is identically O for all vertices. Now consider two vertices of $\overline{M}|F$, which we represent as

$$W := O \rightarrowtail M_1 \rightarrowtail \dots M_a \rightarrowtail N_0$$

$$W' := O \rightarrowtail M_1 \rightarrowtail \dots M_a \rightarrowtail N_1$$

whereby $M_a \rightarrowtail N_0 = M_a \rightarrowtail N_1$. In other words, W and W' are identical sequences of $\mathcal{M}$-morphisms that only (potentially) differ in their choices of quotients. By Axiom (K), these quotients are all isomorphic. One can therefore leverage these isomorphisms to define a 1-simplex $W \to W'$, and thus W and W' are equivalent in $\pi_0(\overline{M}|F)$. (An explicit construction of this 1-simplex is given in Section A.2.)

By the above analysis, there is no real loss of information if we forget quotients and simply represent the generators of $\overline{M}|F$ as $M_a \rightarrowtail N$. This suggests an alternative presentation of $\pi_0(\overline{M}|F)$:

- Generators: All $\mathcal{M}$-morphisms $M_a \rightarrowtail N$ in $\mathcal{C}$; M_a is fixed and part of $\overline{M}$, and N is variable.
- Elementary Relations: Say

$$(M_a \rightarrowtail N_0) \sim_E (M_a \rightarrowtail N_1)$$

  if there is an F-mono $i\colon N_0 \rightarrowtail N_1$ in $\mathcal{C}$ such that

$$\begin{tikzcd} M_a \arrow[r, tail] \arrow[d, equal] & N_0 \arrow[d, tail] \\ M_a \arrow[r, tail] & N_1 \end{tikzcd}$$

[19] **(M:) Seems good, the final bit about why we get a commutative diagram may need a bit more thought. CGPT: analogous to contracting a loop back to its base-point?**

commutes in $\mathcal{M}$. It is easy to check the elementary relations give the precise condition required to construct the top row of the usual 1-simplex of $\overline{M}|F$. Define the equivalence relation

$$(M_a \rightarrowtail N_0) \sim (M_a \rightarrowtail N_1)$$

if $(M_a \rightarrowtail N_0)$ and $(M_a \rightarrowtail N_1)$ are related by a (finite) chain of elementary relations.

The operation $+$ acts on the generators by[20]

$$(M_a \rightarrowtail N_0) + (M_a \rightarrowtail N_1) := (M_a \rightarrowtail N_0 \star_{M_a} N_1).$$

By definition, $\pi_0(\overline{M}|F)$ is a group iff for any $M_a \rightarrowtail N_0$, there exists $M_a \rightarrowtail N_1$ such that

$$(M_a \rightarrowtail N_0) + (M_a \rightarrowtail N_1) = (M_a \rightarrowtail N_0 \star_{M_a} N_1) \sim (M_a \overset{1}{\rightarrowtail} M_a)$$

iff there exists a diagram

$$\begin{tikzcd}
& M_a \arrow[r, equal] \arrow[d, tail] & M_a \arrow[r, equal] \arrow[d, tail] & \dots \arrow[r, equal] & M_a \arrow[d, equal] \\
N_0 \star_{M_a} N_1 = & L_0 & L_1 \arrow[l, tail] \arrow[r, tail] & \dots \arrow[r, tail] & L_n = M_a
\end{tikzcd} \tag{7}$$

where the bottom row is a zig-zag of F-monos (some of which may be identity maps).

Observe: if $L_i \leftarrowtail L_j \rightarrowtail L_k$ are F-monos, then all arrows in the square

$$\begin{tikzcd}
L_j \arrow[r, tail] \arrow[d, tail] & L_k \arrow[d, tail] \\
L_i \arrow[r, tail] & L_i \star_{L_j} L_k
\end{tikzcd}$$

are also F-monos by Axiom (PQ) of Definition 1.18. Repeated applications of this fact allows us to convert Diagram (7) to chains of the form

$$\begin{tikzcd}
M_a \arrow[r, equal] \arrow[d, tail] & \dots \arrow[r, equal] & M_a \arrow[r, equal] \arrow[d, tail] & M_a \arrow[r, equal] \arrow[d, tail] & \dots \arrow[r, equal] & M_a \arrow[d, equal] \\
N_0 \star_{M_a} N_1 = V_0 \arrow[r, tail] & \dots \arrow[r, tail] & V_i \arrow[r, equal] & V_j & \dots \arrow[l, tail] & V_0' = M_a \arrow[l, tail]
\end{tikzcd} \tag{8}$$

where the bottom row arrows are all F-monos. Since the right end of the diagram is $1\colon M_a \rightarrowtail M_a$, we can represent it more suggestively as

$$\begin{tikzcd}
M_a = V_0' \arrow[r, tail] \arrow[d, tail] & \dots \arrow[r, tail] & V_i \arrow[d, equal] \\
N_0 \star_{M_a} N_1 = V_0 \arrow[r, tail] & \dots \arrow[r, tail] & V_j
\end{tikzcd} \tag{9}$$

such that all horizontal arrows are F-monos. □

In sum: having established Claim 2.9, we know that F is cofinal implies that $\pi_0(\overline{M}|F)$ is a group. Since $\overline{M} \in \mathcal{SC}(A)$ and $A \in \Delta$ were chosen arbitrarily, this shows that F is dominant

Step 2: $O|F$ as a homotopy pullback. Fix $\overline{M} \in \mathcal{SC}(A)$ for some $A \in \Delta$. Applying Theorem 1.32, it suffices to show that for any $f\colon A' \to A$ in Δ, the base-change map $f^*\colon \overline{M}|F \to f^*\overline{M}|F$ is a homotopy equivalence. By Step 1, we also know that F is dominant and so $\pi_0(\overline{M}|F)$ is a group. Hereafter, we fix some $f\colon A' \to A$.

[20] **(M:) Deleted stuff about respecting relations. I think we just need the check regarding restricted pushouts downstairs.**

Step 2a: A reduction. Let $g\colon [0] \to A'$ be any morphism in Δ. To show that f^* is a homotopy equivalence, it suffices to show that $(fg)^* = g^* f^*$ and g^* are. In fact, it suffices to show that

$$f_i\colon [0] \to A, \qquad f_i(0) = i \text{ for } i \in A$$

induces a homotopy equivalence for any $A \in \Delta$ [since fg and g both have $[0]$ as source]. Notice f_i^* defines a map

$$f_i^*\colon \overline{M}|F \to O|F$$

since O is the only vertex of $\mathcal{SC}$.

Step 2b: The base case. Define a map

$$H\colon O|F \longrightarrow \overline{M}|F \tag{10}$$

$$\left(O \rightarrowtail \frac{O \rightarrowtail \dots \rightarrowtail K_q}{L_0 \rightarrowtail \dots \rightarrowtail L_q} \right) \longmapsto \left(O \rightarrowtail \dots \rightarrowtail M_a \rightarrowtail \frac{O \rightarrowtail \dots \rightarrowtail K_q}{M_a \oplus L_0 \rightarrowtail \dots \rightarrowtail M_a \oplus L_q} \right)$$

with quotients defined as

- $\frac{M_a \oplus L_j}{M_a \oplus L_k} := \frac{L_j}{L_k} \left(= F\left(\frac{K_j}{K_k}\right)\right)$,
- $\frac{M_a \oplus L_j}{M_a} := L_j, \qquad \frac{M_a \oplus L_j}{M_i} := \frac{M_a}{M_i} \oplus L_j$

To show that f_i^* is a homotopy equivalence [for any i], it suffices to establish the following claim.

Claim 2.10. *The maps $f_i^* \circ H$ and H are homotopy equivalences.*

Proof of Claim. Two main checks.

(i) *On $f_i^* \circ H$.* The map $f_i^* \circ H\colon O|F \to O|F$ sends

$$\left(O \rightarrowtail \frac{O \rightarrowtail \dots \rightarrowtail K_q}{L_0 \rightarrowtail \dots \rightarrowtail L_q} \right) \longmapsto \left(O \rightarrowtail \frac{O \rightarrowtail \dots \rightarrowtail K_q}{\frac{M_a}{M_i} \oplus L_0 \rightarrowtail \dots \rightarrowtail \frac{M_a}{M_i} \oplus L_q} \right)$$

for $0 \leq i \leq a$. Recall $O|F$ is an H-space. We can formulate $f_i^* \circ H$ more suggestively as adding a vertex

$$(f_i^* \circ H)(W) = W + \left(O \rightarrowtail \frac{O}{\frac{M_a}{M_i}} \right).$$

to any simplex W. Since $\pi_0(O|F)$ is a group on the vertices of $O|F$, there exists a vertex V such that

$$\left(O \rightarrowtail \frac{O}{\frac{M_a}{M_i}} \right) + V \sim \left(O \rightarrowtail \frac{O}{O} \right).$$

Define $h\colon O|F \to O|F$ as mapping

$$h(W) = W + V$$

for any simplex W. Since $+$ is homotopy associative and homotopy commutative, deduce that[21]

$$f_i^* \circ H \circ h \sim 1, \qquad h \circ f_i^* \circ H \sim 1.$$

(ii) *On H.* Notice: $f_a^* \circ H$ is isomorphic to the identity map on $O|F$. It therefore suffices to show $H \circ f_a^*$ is homotopic to the identity map 1 on $\overline{M}|F$. But this follows from the natural isomorphism

$$H \circ f_a^* \cong 1,$$

or more explicitly, the isomorphism

$$M_a \oplus \frac{L_j}{M_a} \cong \frac{M_a \oplus L_j}{M_a} = L_j, \qquad \text{for all } j,$$

[21] **(M:) Seems resaonable, but double check reasoning.**

which is a consequence of Lemma A.2 and the choice of quotients by H. [Notice: the specific choice of quotients by H is crucial; otherwise, the isomorphism may fail to hold since e.g. not all short exact sequences split.]

This completes proof of Claim 2.10. $\square$

Step 3: Finish. Fix a simplicial map $F\colon Y \to \mathcal{SC}$ satisfying hypotheses of the theorem. Step 1 showed if $\mathrm{im}F$ is cofinal in $\mathcal{SC}$, then F is dominant. Step 2 showed that if F is dominant, then one can leverage the group structure of $\pi_0(O|F)$ to show that Diagram 6 is homotopy Cartesian. Putting the two together yields the Theorem. $\square$

The following corollary justifies viewing the right fiber (Definition 1.30) as the simplicial analogue of a homotopy fiber, and will be useful later.

Corollary 2.11. *Suppose $F\colon Y \hookrightarrow \mathcal{SC}$ is a simplicial map satisfying the same conditions as in Theorem 2.7. Then $|O|F|$ is homotopy equivalent to the homotopy fiber of $|F|$.*

Proof. By Observation 1.31, $O|\mathcal{SC}$ is contractible. Since the homotopy fiber of $|F|$ is the homotopy pullback of the cospan $* \to |\mathcal{SC}| \xleftarrow{|F|} Y$, the statement follows. $\square$

In addition, we now obtain a key result of the paper regarding the G-construction.

Theorem 2.12. *Let $\mathcal{C}$ be a pCGW category. Then, there is a homotopy equivalence*

$$|G\mathcal{C}| \xrightarrow{\sim} \Omega|S\mathcal{C}|.$$

Further, direct sum induces an H-space structure on $G\mathcal{C}$.

Proof. Let us review Key Observation 3.13. By item (iii), $|G\mathcal{C}| = |\Omega\mathcal{SC}| \simeq \Omega|\mathcal{SC}|$ if

$$\begin{array}{ccc} \Omega\mathcal{SC} & \xrightarrow{t} & O|\mathcal{SC} \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle q} \\ O|\mathcal{SC} & \xrightarrow{q} & \mathcal{SC} \end{array}$$

is homotopy Cartesian. By item (ii), we have $O|q \simeq \Omega\mathcal{SC}$. Finally, it is clear that $O|\mathcal{SC}$ is a subsimplicial set of $\mathcal{SC}$ since any n-simplex of $O|\mathcal{SC}$ is an $n+1$-simplex of $\mathcal{SC}$ by construction, and inherits all the structure maps in the obvious way. In particular, q is cofinal since given any $O \rightarrowtail M \in \mathcal{SC}[1]$, we may pick $(O \rightarrowtail O \rightarrowtail M) \in O|\mathcal{SC}[1]$ so that

$$q(O \rightarrowtail O \rightarrowtail M) = O \rightarrowtail M.$$

The rest follows from Theorem 2.7. Finally, the H-space structure on $G\mathcal{C}$ comes from Construction 2.2. $\square$

Remark 2.13. The equivalence in Theorem 2.12 established between $G\mathcal{C}$ and $K\mathcal{C}$ is one of topological spaces, not of inifinite loop spaces or spectra.

Discussion 2.14 (Comparison with other proofs)**.** We are aware of two existing proofs of Theorem 2.12 for exact categories in the literature. While all three approaches (including ours) make use of Theorem B' (Theorem 1.32) in some form, there are key differences in the proof strategy; this is even after we account for the fact that our result applies to pCGW categories, not just exact categories.

In broad strokes, Theorem B' says: once we know that the induced map on fibers $\rho|F \to f^*\rho|F$ is a homotopy equivalence for *any* $f\colon A' \to A$, then $\rho|F$ can be described as a homotopy pullback. In the original paper [GG87], Gillet-Grayson first simplifies this condition to just checking homotopy equivalence for the two maps $f_0, f_1\colon [0] \to [1]$, before giving a technical analysis of how f_0 and f_1 behave on the fibers. Very informally, this argument makes precise the intuition: if we want to understand how objects break into finitely many pieces, it suffices to understand how to break a single object into two.

A second (implicit) proof appears in [Gra87, Thm 8.2], where Grayson uses Theorem B' to develop the theory of *dominant functors* between exact categories $F\colon \mathcal{D} \to \mathcal{C}$. While this overlaps with our approach, Grayson does not use Key Observation 1.31. Instead, he relies on a new construction $\mathcal{C}_n$, whose objects are the exact sequences of length n in the original exact category $\mathcal{C}$. Another difference is that Grayson defines dominance for functors, whereas we define it for simplicial maps. This adjusts for the fact that $O|\mathcal{SC}$, while a subsimplicial set of $\mathcal{SC}$, does not directly correspond to a CGW category.[22]

3. Generators of $K_1(\mathcal{C})$

Having established Theorem 2.12, we now begin to deliver on our promise that the G-construction allows for a more explicit description of elements in $K_1(\mathcal{C})$. Section 3.1 unpacks the definition of the $G\mathcal{C}$-construction. Sections 3.2 and 3.3 work to obtain an increasingly sharp description of the generators of K_1; their analysis extends various results from [She94, She98, Nen96].

3.1. **Review of G-Construction.** Recall that the G-Construction for $\mathcal{C}$ is defined as $G\mathcal{C} := \Omega\mathcal{SC}$. We unpack this definition explicitly below.

Construction 3.1 (G-Construction)**.** An n-simplex of $G\mathcal{C}$ is a pair of flag diagrams of the form

$$\begin{tikzcd}
P_0 \arrow[r, tail] & P_1 \arrow[r, tail] & P_2 \arrow[r, tail] & \dots \arrow[r, tail] & P_n \\
& P_{1/0} \arrow[r, tail] \arrow[u] & P_{2/0} \arrow[r, tail] \arrow[u] & \dots \arrow[r, tail] & P_{n/0} \arrow[u] \\
& & P_{2/1} \arrow[r, tail] \arrow[u] & \dots \arrow[r, tail] & P_{n/1} \arrow[u] \\
& & & \vdots \arrow[u] & \vdots \arrow[u] \\
& & & & P_{n/n-1} \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
P'_0 \arrow[r, tail] & P'_1 \arrow[r, tail] & P'_2 \arrow[r, tail] & \dots \arrow[r, tail] & P'_n \\
& P_{1/0} \arrow[r, tail] \arrow[u] & P_{2/0} \arrow[r, tail] \arrow[u] & \dots \arrow[r, tail] & P_{n/0} \arrow[u] \\
& & P_{2/1} \arrow[r, tail] \arrow[u] & \dots \arrow[r, tail] & P_{n/1} \arrow[u] \\
& & & \vdots \arrow[u] & \vdots \arrow[u] \\
& & & & P_{n/n-1} \arrow[u]
\end{tikzcd}$$

subject to the conditions:

(i) Every quotient index square

$$\begin{tikzcd}
P_{j/i} \arrow[r, tail] \arrow[dr, phantom, "\square"] & P_{k/i} \\
P_{j/l} \arrow[r, tail] \arrow[u] & P_{k/l} \arrow[u]
\end{tikzcd} \qquad \text{where } i < l \text{ and } j < k$$

is distinguished, and coincide in both flag diagrams.

(ii) Every quotient index triangle defines a distinguished square

$$\begin{tikzcd}
P_{j/i} \arrow[r, tail] \arrow[dr, phantom, "\square"] & P_{k/i} \\
O \arrow[r, tail] \arrow[u] & P_{k/j} \arrow[u]
\end{tikzcd} \qquad \text{for any } i < j < k,$$

and coincide in both flag diagrams.

(iii) Any $P_j \rightarrowtail P_k$ and $P'_j \rightarrowtail P'_k$ in the filtration can be completed into distinguished squares

$$\begin{tikzcd}
P_j \arrow[r, tail] \arrow[dr, phantom, "\square"] & P_k \\
O \arrow[r, tail] \arrow[u] & P_{k/j} \arrow[u]
\end{tikzcd}, \quad
\begin{tikzcd}
P'_j \arrow[r, tail] \arrow[dr, phantom, "\square"] & P'_k \\
O \arrow[r, tail] \arrow[u] & P_{k/j} \arrow[u]
\end{tikzcd} \qquad \text{for any } i < j < k.$$

[22] (M:) **Double-check what Grayson did, particularly the definition of $\mathcal{C}_n$.**

Convention 3.2. Technically, an n-simplex of $G\mathcal{C}$ is a pair of $(n+1)$-simplices in $\mathcal{SC}$

$$O \rightarrowtail P_0 \rightarrowtail \ldots \rightarrowtail P_n \qquad\qquad O \rightarrowtail P'_0 \rightarrowtail \ldots \rightarrowtail P'_n$$

such that the obvious n-faces agree (obtained by forgetting O and quotienting by P_0). Here we omit the basepoint O for simplicity. In particular, a *vertex* of $G\mathcal{C}$ for us is a pair $(M, N) \in \mathcal{C} \times \mathcal{C}$, and an *edge* or 1-*simplex* connecting $(M, N) \to (M', N')$ is given by a pair of distinguished squares

$$\left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\square"] & C \arrow[d, "\circ" marking] \\ M \arrow[r, tail] & M' \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\square"] & C \arrow[d, "\circ" marking] \\ N \arrow[r, tail] & N' \end{tikzcd} \right).$$

Remark 3.3. The requirement that the quotients must coincide imposes a coherence condition between the two flag diagrams, introducing subtleties. For instance, given an edge $(M, N) \to (M', N')$, the associated $\mathcal{M}$-morphisms $M \rightarrowtail M'$ and $N \rightarrowtail N'$ are required to have the same quotient C. This means not every vertex $(M, N) \in G\mathcal{C}$ is connected to the base point (O, O) – unlike in $\mathcal{SC}$.

3.2. **Sherman Loops & Splitting.** Suppose $\mathcal{J}$ is an exact category. A guiding principle of Sherman's work in [She94, She98] is that if we wish to describe the generators of $K_1(\mathcal{J})$, it is helpful to restrict to the class of split exact sequences. This section extends this insight to the setting of pCGW categories. We start by introducing some key definitions, before giving a first characterisation of the generators of $K_1(\mathcal{C})$ (Theorem 3.7).

Construction 3.4 (Sherman Loop)**.** A *Sherman triple* (α, β, θ) consists of the following data:

- Two $\mathcal{M}$-morphism $A \overset{\alpha}{\rightarrowtail} B$, $A' \overset{\beta}{\rightarrowtail} B'$;
- An isomorphism $\theta \colon A \oplus C \oplus B' \rightarrowtail A' \oplus C' \oplus B$, where C and C' are specific choices of quotients

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\square"] & C \arrow[d, "\circ" marking, "\delta"] \\ A \arrow[r, tail, "\alpha"] & B \end{tikzcd} \qquad \begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\square"] & C' \arrow[d, "\circ" marking, "\gamma"] \\ A' \arrow[r, tail, "\beta"] & B' \end{tikzcd} . \tag{11}$$

Its associated *Sherman loop* is the homotopy class $G(\alpha, \beta, \theta)$ in $\pi_1(|G\mathcal{C}|)$ represented by the loop

$$\begin{pmatrix} O \\ O \end{pmatrix} \to \begin{pmatrix} A \\ A \end{pmatrix} \to \begin{pmatrix} A \oplus C \oplus B' \\ B \oplus B' \end{pmatrix} \to \begin{pmatrix} A' \oplus C' \oplus B \\ B' \oplus B \end{pmatrix} \leftarrow \begin{pmatrix} A' \\ A' \end{pmatrix} \leftarrow \begin{pmatrix} O \\ O \end{pmatrix}, \tag{12}$$

where the arrows denote the obvious 1-simplices.[23]

Remark 3.5. Fix a pair of $\mathcal{M}$-morphisms α, β. A straightforward exercise shows that any two Sherman triples (α, β, θ) and (α, β, θ') define the same Sherman loop in $K_1(\mathcal{C})$ – see e.g. [She94, §1].

Definition 3.6 (Split)**.** Call a distinguished square of the form

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\square"] & C \arrow[d, "\circ" marking, "g"] \\ A \arrow[r, tail, "f"] & B \end{tikzcd} \tag{13}$$

an *exact square*.

(i) Call an exact square *split* if there exists an isomorphism

$$\Psi \colon B \rightarrowtail A \oplus C$$

[23]*Details.* The middle arrow in Equation (12) applies θ on the top row and the canonical isomorphism $B \oplus B' \to B' \oplus B$ on the bottom; the rest of the 1-simplices are defined by applying Axiom (DS), Definition 1.18.

such that the following squares commute

$$\begin{tikzcd} A \arrow[r, rightarrowtail, "f"] \arrow[d, rightarrowtail, "1"'] & B \arrow[d, rightarrowtail, "\Psi"] \\ A \arrow[r, rightarrowtail, "p_A"] & A \oplus C \end{tikzcd} \qquad \begin{tikzcd} B & C \arrow[l, "g"'] \arrow[d, "1_C"] \\ A \oplus C & C \arrow[l, "q_C"'] \end{tikzcd}$$

in $\mathcal{M}$ and $\mathcal{E}$ respectively, where p_A and q_C are the morphisms from the obvious direct sum square.

(ii) Call an $\mathcal{M}$-morphism $A \rightarrowtail B$ is called *split* if its corresponding exact square [obtained by taking formal quotients] is split.

Theorem 3.7. *$K_1(\mathcal{C})$ is generated by Sherman loops $G(\alpha, \beta, \theta)$, where $\mathcal{C}$ is a pCGW category.*

Proof. The proof relies on two helper constructions: (i) $\mathcal{C}^\oplus$, a pCGW subcategory of $\mathcal{C}$; and (ii) $\widehat{G}\mathcal{C}$, a simplicial subset of $G\mathcal{C}$. Informally, they record the splitting data of $\mathcal{C}$ and $G\mathcal{C}$ respectively, and so admit particularly nice presentations on the level of π_1. One can then relate this to $\pi_1|G\mathcal{C}|$ to establish the theorem. Proceed in stages.

Step 0: Setup. This step introduces the two helper constructions, and records some preliminary observations.

Construction 3.8. Define $\mathcal{C}^\oplus$ to be the CGW subcategory of $\mathcal{C}$ where all exact squares are split.

An easy exercise verifies that $\mathcal{C}^\oplus$ is a CGW subcategory, and so the obvious inclusion CGW functor $F\colon \mathcal{C}^\oplus \to \mathcal{C}$ induces a simplicial inclusion

$$\mathcal{S}F\colon \mathcal{S}\mathcal{C}^\oplus \hookrightarrow \mathcal{S}\mathcal{C}.$$

In particular, notice $\mathrm{im}\mathcal{S}F = \mathcal{S}\mathcal{C}[1]$ since $\mathcal{C}^\oplus$ only throws out the exact squares that are not split. In other words, $\mathcal{S}\mathcal{F}$ is cofinal in $\mathcal{S}\mathcal{C}$. One can therefore apply Theorem 2.7 to construct a homotopy Cartesian diagram

$$\begin{tikzcd} O|\mathcal{S}F \arrow[r, "p"] \arrow[d, "r"] & \mathcal{S}\mathcal{C}^\oplus \arrow[d, "\mathcal{S}F"] \\ O|\mathcal{S}\mathcal{C} \arrow[r, "q"] & \mathcal{S}\mathcal{C} \end{tikzcd} . \tag{14}$$

Construction 3.9. Define $\widehat{G}\mathcal{C}$ to be the simplicial subset of $G\mathcal{C}$ where the $\mathcal{M}$-morphisms in the top row must be split.

Notice this mirrors $O|\mathcal{S}F$, whose top row is a filtration from $C^\oplus$ and thus must also be split.[24] Notice also that the simplices used to define $G(\alpha, \beta, \theta)$ in fact belong to $\widehat{G}\mathcal{C}$, and so $G(\alpha, \beta, \theta) \in \pi_1(|\widehat{G}\mathcal{C}|)$.

Step 1: Relating the constructions $\widehat{G}\mathcal{C}$ and $\mathcal{S}\mathcal{C}^\oplus$. We establish a series of homotopy equivalences.

Claim 3.10. *The map r induces a homotopy equivalence $O|p \to O|q$.*

Proof of Claim. As our setup, notice:

- $O|q \simeq \Omega\mathcal{S}\mathcal{C}$, by Observation 1.31.
- An n-simplex of $O|\mathcal{S}F$ is of the form

$$W = \left(\begin{array}{cc} & \underline{\underline{O = K_0 \rightarrowtail \dots \rightarrowtail K_n}} \\ O \rightarrowtail & L_0 \quad \rightarrowtail \dots \rightarrowtail L_n \end{array} \right); \tag{15}$$

the map p acts by projecting the top row, the map r projects the bottom row. An n-simplex of $O|p$ is therefore a triple

$$\left(\begin{array}{cc} & \underline{\underline{O = K_0 \rightarrowtail \dots \rightarrowtail K_n}} \\ O \rightarrowtail & \underline{\underline{L_0 \quad \rightarrowtail \dots \rightarrowtail L_n}} \\ O \rightarrowtail & M_0 \quad \rightarrowtail \dots \rightarrowtail M_n \end{array} \right)$$

where the double lines indicate equality of the corresponding n-faces.

[24] **(M:) Not just any filtration, but a filtration representing a flag of quotients, which must be split.**

The induced map $r^*\colon O|p \to \Omega\mathcal{SC}$ acts by forgetting the top row. To construct the homotopy inverse, we must account for the top row filtration being split, which imposes restrictions on the bottom two filtrations. On this front, given any n-simplex

$$\alpha = \begin{pmatrix} O \rightarrowtail \dfrac{L_0 \rightarrowtail \ldots \rightarrowtail K_n}{M_0 \rightarrowtail \ldots \rightarrowtail M_n} \\ O \rightarrowtail \end{pmatrix} \in \mathcal{SC}$$

define the map

$$s\colon \Omega\mathcal{SC} \longrightarrow O|p$$

$$\alpha \longmapsto \begin{pmatrix} & O = \frac{L_0}{L_0} \rightarrowtail \ldots \rightarrowtail \bigoplus_{m=0}^{i} \dfrac{L_m}{L_0} \rightarrowtail \ldots \rightarrowtail \bigoplus_{m=0}^{n} \dfrac{L_m}{L_0} \\ \hline\hline O \rightarrowtail & L_0 \rightarrowtail \ldots\ldots \rightarrowtail \bigoplus_{m=0}^{i} L_m \rightarrowtail \ldots\ldots \rightarrowtail \bigoplus_{m=0}^{n} L_m \\ \hline\hline O \rightarrowtail & M_0 \rightarrowtail \ldots\ldots \rightarrowtail \bigoplus_{m=0}^{i} M_m \rightarrowtail \ldots\ldots \rightarrowtail \bigoplus_{m=0}^{n} M_m \end{pmatrix} \tag{16}$$

where we turn α into a pair of split filtrations by taking direct sums.

We now define the homotopy between $r^* \circ s \to 1$, as below.

$$h\colon \Omega\mathcal{SC} \times [1]([n]) \longrightarrow \Omega\mathcal{SC}([n])$$

$$(\alpha,\beta) \longmapsto \begin{pmatrix} O \rightarrowtail & L_0 \rightarrowtail \ldots \rightarrowtail L_i \rightarrowtail \bigoplus_{m=0}^{i+1} L_m \rightarrowtail \ldots\ldots \rightarrowtail \bigoplus_{m=0}^{n} L_m \\ \hline\hline O \rightarrowtail & M_0 \rightarrowtail \ldots \rightarrowtail M_i \rightarrowtail \bigoplus_{m=0}^{i+1} M_m \rightarrowtail \ldots\ldots \rightarrowtail \bigoplus_{m=0}^{n} M_m \end{pmatrix}$$

where i is an integer $-1 \leq i \leq n$ chosen such that $\beta(0) = \ldots \beta(i) = 0$ and $\beta(i+1) = \cdots = \beta(n) = 1$. Applying Lemma A.2, we make the obvious choices for the quotients to make $h(\alpha,\beta)$ a simplex of $\Omega\mathcal{SC}$.

Unpacking this construction, h takes the direct sum of all preceding terms in the filtration from the $(i+1)$-place onwards, where i is determined by β. In particular,

- If $\beta(m) = 0$ for all $m \in [q]$ then $h(\alpha,\beta) = r^* \circ s(\alpha)$; whereas
- If $\beta(m) = 1$ for all $m \in [q]$ then $h(\alpha,\beta) = \alpha$.

A straightforward check shows that h is a simplicial map, and thus defines a simplicial homotopy $r^* \circ s \to 1$. The converse direction $s \circ r^* \to 1$ can be proved analogously. $\square$

Discussion 3.11. Claim 3.10 is the simplicial analogue of the well-known fact that a diagram of spaces is homotopy Cartesian iff it induces a weak equivalence on all relevant homotopy fibers. The explicit description of the homotopy equivalence allows us to streamline the original argument in [She98]. In particular, the next two results now follow almost immediately.

Corollary 3.12. *$\widehat{G}\mathcal{C}$ is homotopy equivalent to $O|p$ and $G\mathcal{C}$.*

Proof of Corollary. Observation 1.31 (ii) notes that $G\mathcal{C} := \Omega\mathcal{SC} \simeq O|q$, essentially by unpacking definitions. One can similarly verify that $\widehat{G}\mathcal{C} \simeq O|p$. By Claim 3.10, conclude that $G\mathcal{C} \simeq O|q \simeq O|p \simeq \widehat{G}\mathcal{C}$. $\square$

Applying Corollary 3.12, a basic but key observation:

Observation 3.13. π_1 of the homotopy sequence associated[25] to p

$$\pi_1(\Omega|\mathcal{SC}^{\oplus}|) \longrightarrow \pi_1(|O|p|) \longrightarrow \pi_1(|O|\mathcal{S}F|) \xrightarrow{p_*} \pi_1(|\mathcal{SC}^{\oplus}|) \tag{17}$$

can be reformulated as

$$\pi_1(\Omega|\mathcal{SC}^{\oplus}|) \longrightarrow \pi_1(|\widehat{G}\mathcal{C}|) \xrightarrow{v} \pi_1(|O|\mathcal{S}F|) \xrightarrow{p_*} \pi_1(|\mathcal{SC}^{\oplus}|) \tag{18}$$

[25] **(M:) Double-check where this comes from.**

for some map v induced by $O|p \simeq \widehat{G}\mathcal{C}$.

Step 2: Generators of $\pi_1(|O|\mathcal{S}F|)$. The argument is standard – no surprises. The 1-simplices of the form

$$\left(O \rightarrowtail \begin{array}{c} O \rightarrowtail N \\ \hline\hline O \rightarrowtail N \end{array} \right) \tag{19}$$

form a maximal tree for the 1-skeleton of $|O|\mathcal{S}F|$, connecting the base-point of $O|\mathcal{S}F$ to any of its vertices. Thus by [Wei13, Lemma IV.3.4], the total set of 1-simplices of $O|\mathcal{S}F$ generate $\pi_1(|O|\mathcal{S}F|)$.

Details. The 1-simplices of $O|\mathcal{S}F$ are of the form

$$\left(O \rightarrowtail \begin{array}{c} O \rightarrowtail C \\ \hline\hline A \rightarrowtail B \end{array} \right) \tag{20}$$

where $C = \frac{B}{A}$. [The reader may wish to view the 1-simplex as corresponding to a short exact sequence, or better yet, an exact square (not necessarily split).] Its corresponding generator in $\pi_1(|O|\mathcal{S}F|)$ is

$$\left(O \rightarrowtail \begin{array}{c} O \rightarrowtail A \\ \hline\hline O \rightarrowtail A \end{array} \right) \left(O \rightarrowtail \begin{array}{c} O \rightarrowtail C \\ \hline\hline A \rightarrowtail B \end{array} \right) \left(O \rightarrowtail \begin{array}{c} O \rightarrowtail B \\ \hline\hline O \rightarrowtail B \end{array} \right)^{-1} \tag{21}$$

which codes the loop

$$\begin{tikzcd}[column sep=small]
\left(O \rightarrowtail \begin{array}{c} O \\ \hline\hline A \end{array} \right) \arrow[rr, "\left(O \rightarrowtail \begin{array}{c} O \rightarrowtail C \\ \hline\hline A \rightarrowtail B \end{array} \right)"] & & \left(O \rightarrowtail \begin{array}{c} O \\ \hline\hline B \end{array} \right) \\
& \left(O \rightarrowtail \begin{array}{c} O \\ \hline\hline O \end{array} \right) \arrow[ul, "\left(O \rightarrowtail \begin{array}{c} O \rightarrowtail A \\ \hline\hline O \rightarrowtail A \end{array} \right)"] \arrow[ur, "\left(O \rightarrowtail \begin{array}{c} O \rightarrowtail B \\ \hline\hline O \rightarrowtail B \end{array} \right)"'] &
\end{tikzcd} . \tag{22}$$

Step 3: Examining $\pi_1(|\widehat{G}\mathcal{C}|)$ *via* $\pi_1(|\mathcal{S}\mathcal{C}^{\oplus}|)$. Let us review the extended homotopy sequence (18) from Observation 3.13. Recall that $O|\mathcal{S}F$ has an H-space structure (Construction 2.2). Hence, given any $x \in \pi_1(|\widehat{G}\mathcal{C}|)$, deduce $v(x) \in \pi_1(|O|\mathcal{S}F|)$ can be expressed as a difference of two 1-simplices, let us say

$$\left(O \rightarrowtail \begin{array}{c} O \rightarrowtail C \\ \hline\hline A \rightarrowtail B \end{array} \right) \qquad \left(O \rightarrowtail \begin{array}{c} O \rightarrowtail C' \\ \hline\hline A' \rightarrowtail B' \end{array} \right) \tag{23}$$

Since the map $p\colon O|\mathcal{S}F \to \mathcal{S}\mathcal{C}^{\oplus}$ acts by projection on the top row, this means $p_*v(x) \in \pi_1(|\mathcal{S}\mathcal{C}^{\oplus}|)$ corresponds to the difference of

$$(O \rightarrowtail A)(O \rightarrowtail C)(O \rightarrowtail B')^{-1} \qquad \text{and} \qquad (O \rightarrowtail A')(O \rightarrowtail C')(O \rightarrowtail B')^{-1}. \tag{24}$$

Leveraging the fact that $\pi_1(|\mathcal{S}\mathcal{C}^{\oplus}|) \cong \pi_1(|Q\mathcal{C}^{\oplus}|) = K_0(\mathcal{C}^{\oplus})$ [CZ22, Thm 7.8], we rewrite this equation more suggestively as

$$[A] + [C] - [B] \qquad \text{and} \qquad [A'] + [C'] - [B']. \tag{25}$$

Since Equation (18) is exact[26], deduce that[27]

$$[A] + [C] - [B] - \big([A'] + [C'] - [B']\big) = 0, \tag{26}$$

and so

$$[A] + [C] + [B'] = [A'] + [C'] + [B]. \tag{27}$$

[26] **(M:) Double-check that this is true.**

[27] Exactness of Equation (18) plays a key role here. Given any 1-simplex $x' \in \pi(|O|\mathcal{S}F|)$ (or difference of 1-simplices), which corresponds to an exact square, its image $p_*(x') \in \pi_1(|\S\mathcal{C}^{\oplus}|)$ need not be 0 in $\pi_1(|\mathcal{S}\mathcal{C}^{\oplus}|)$ since its corresponding exact square need not be split.

Since this equation holds in $K_0(\mathcal{C}^{\oplus})$, a standard exercise[28] shows there exists some $Z \in \mathcal{C}$ such that

$$A \oplus C \oplus B' \oplus Z \cong A' \oplus C' \oplus B \oplus Z. \tag{28}$$

Further, since

$$\left(O \rightarrowtail \; \frac{O \rightarrowtail Z}{O \rightarrowtail Z} \right) \left(O \rightarrowtail \; \frac{O \rightarrowtail O}{Z \rightarrowtail Z} \right) \left(O \rightarrowtail \; \frac{O \rightarrowtail Z}{O \rightarrowtail Z} \right)^{-1} \tag{29}$$

is null-homotopic, we can add it to the generator $v(x)$ without changing the homotopy class.[29] As such, assume without loss of generality that $Z = O$ and so there is an isomorphism

$$\theta \colon A \oplus C \oplus B' \xrightarrow{\cong} A' \oplus C' \oplus B. \tag{30}$$

Step 4: Relation to Sherman Loops. So far we have worked with a generic $x \in \pi_1(|\widehat{G}\mathcal{C}|)$. The following claim tells us that x nonetheless looks like a Sherman loop when viewed in $\pi_1(O|\mathcal{S}F)$.

Claim 3.14. *Given any $x \in \pi_1(|\widehat{G}\mathcal{C}|)$, there exists a Sherman loop $G(\alpha, \beta, \theta)$ such that $v(x)$ and $v(G(\alpha, \beta, \theta))$ are homotopic.*

Proof. Reviewing Step 3: Equation (23) yields a pair of $\mathcal{M}$-morphisms $\alpha \colon A \rightarrowtail B$ and $\beta \colon A' \rightarrowtail B'$ and Equation (30) yields an isomorphism θ. This forms a Sherman triple, and thus we can define the corresponding Sherman Loop $G(\alpha, \beta, \theta)$. [Recall that the 1-simplices defining Sherman loop are all split, and so we may view $G(\alpha, \beta, \theta) \in \pi_1(|\widehat{G}\mathcal{C}|)$.]

Now consider the following diagram in $\pi_1(|O|\mathcal{S}F|)$

$$\begin{tikzcd}
\left(O \rightarrowtail \frac{O}{A} \right) \arrow[r, blue] \arrow[d, red] \arrow[dr, phantom, "(1)"] & \left(O \rightarrowtail \frac{O}{B \oplus B'} \right) \arrow[r, blue] \arrow[dr, phantom, "(3)"] & \left(O \rightarrowtail \frac{O}{B' \oplus B} \right) \arrow[dr, phantom, "(5)"] & \left(O \rightarrowtail \frac{O}{A'} \right) \arrow[l, blue] \arrow[dl, red] \\
\left(O \rightarrowtail \frac{O}{B} \right) \arrow[ur] \arrow[r, phantom, "(2)"] & \left(O \; \frac{O}{O} \right) \arrow[l, red] \arrow[u] \arrow[ur] \arrow[r, red] \arrow[r, phantom, "(4)"'] & \left(O \rightarrowtail \frac{O}{B'} \right) \arrow[u] &
\end{tikzcd} \tag{31}$$

The edges of the diagram are obvious (see e.g. Footnote 23). The diagram shows various different paths between vertices

$$\left(O \rightarrowtail \frac{O}{A} \right) \dashrightarrow \left(O \rightarrowtail \frac{O}{A'} \right), \tag{32}$$

e.g. by composing along the blue edges, by composing along the red edges, etc.

A couple of key observations. First, notice that all triangles in Diagram (31) define boundaries of 2-simplices, listed below.

$$(1) \left(O \rightarrowtail \; \frac{O \rightarrowtail C \rightarrowtail C \oplus B'}{A \rightarrowtail B \rightarrowtail B \oplus B'} \right), \; (2) \left(O \rightarrowtail \; \frac{O \rightarrowtail B \rightarrowtail B \oplus B'}{O \rightarrowtail B \rightarrowtail B' \oplus B} \right), \; (3) \left(O \rightarrowtail \; \frac{O \rightarrowtail O \rightarrowtail B' \oplus B'}{B \rightarrowtail B \oplus B' \rightarrowtail B' \oplus B'} \right)$$

$$(4) \left(O \rightarrowtail \; \frac{O \rightarrowtail B' \rightarrowtail B' \oplus B}{O \rightarrowtail B' \rightarrowtail B' \oplus B} \right), \; (5) \left(O \rightarrowtail \; \frac{O \rightarrowtail C' \rightarrowtail C' \oplus B}{A' \rightarrowtail B' \rightarrowtail B' \oplus B} \right).$$

Hence, the blue and red paths in Diagram (31) between the two vertices (32) are homotopic.

Second, $v(G(\alpha, \beta, \theta))$ corresponds to the loop[30]

$$\left(O \rightarrowtail \; \frac{O \rightarrowtail A}{O \rightarrowtail A} \right) \left(O \rightarrowtail \; \frac{O \rightarrowtail C \oplus B'}{A \rightarrowtail B \oplus B'} \right) \left(O \rightarrowtail \; \frac{O \rightarrowtail O}{B \oplus B' \rightarrowtail B' \oplus B} \right) \left(O \rightarrowtail \; \frac{O \rightarrowtail C \oplus B'}{A' \rightarrowtail B' \oplus B} \right)^{-1} \left(O \rightarrowtail \; \frac{O \rightarrowtail A'}{O \rightarrowtail A'} \right)^{-1}$$

---

[28] *Details.* By [CZ22, Thm 4.3], $K_0(\mathcal{C}^{\oplus}) = F/R$ is the free abelian group F generated by objects in $\mathcal{C}^{\oplus}$ modulo the relation R that $[P] + [Q] = [Z]$ iff $P \oplus Q \cong Z$. Suppose $[M] = [N]$ in $K_0(\mathcal{C}^{\oplus})$. On the level of the free group F, this implies $\overline{M} - \overline{N} = \sum_i^n \left(\overline{P_i} + \overline{Q_i} - \overline{P_i \oplus Q_i} \right)$, for some finite set of $P_i, Q_i \in \mathcal{C}$. Rearranging terms and quotienting by relation R gives $[M] + \sum_i^n [P_i \oplus Q_i] = [N] + \sum_i^n [P_i] + \sum_i^n [Q_i]$, and so $M \oplus Z \cong N \oplus Z$ where $Z \cong \sum_i^n P_i \oplus Q_i \cong \sum_i^n P_i \oplus \sum_i^n Q_i$. The rest follows from noting $[A \oplus C \oplus B'] = [A] + [C] + [B']$ and $[A' \oplus C' \oplus B] = [A'] + [C'] + [B]$. **(M:) Double-check, but should be OK.**

[29] **(M:) Double-check this later. I believe it just means adding an extra null-homotopic loop round Diagram (22)**

[30] **(M:) Double-check this later. Seems plausible.**

whereas $v(x)$ corresponds to the loop

$$\left(O\rightarrowtail \;\; \frac{O\rightarrowtail A}{O\rightarrowtail A}\right)\left(O\rightarrowtail \;\; \frac{O\rightarrowtail C}{A\rightarrowtail B}\right)\left(O\rightarrowtail \;\; \frac{O\rightarrowtail B}{O\rightarrowtail B}\right)^{-1}\left(O\rightarrowtail \;\; \frac{O\rightarrowtail B'}{O\rightarrowtail B'}\right)\left(O\rightarrowtail \;\; \frac{O\rightarrowtail C'}{A'\rightarrowtail B'}\right)^{-1}\left(O\rightarrowtail \;\; \frac{O\rightarrowtail A'}{O\rightarrowtail A'}\right)^{-1}.$$

In particular, the loop $v(G(\alpha, \beta))$ corresponds to composing along the blue edges in Diagram (31) whereas $v(x)$ corresponds to composing along the red edges – which we already know to be homotopy equivalent by our previous observation. Conclude that $v(G(\alpha, \beta, \theta))$ and $v(x)$ have the same homotopy class. $\square$

Step 5: Finish. Let x be an element of $K_1(\mathcal{C})$. By Theorem 2.12 and Corollary 3.12, we know

$$\Omega|\mathcal{S}\mathcal{C}| \simeq |G\mathcal{C}| \simeq |\widehat{G}\mathcal{C}|,$$

and so regard $x \in \pi_1(|\widehat{G}\mathcal{C}|)$. In particular, x is an element in Homotopy Sequence (18). By Claim 3.14, there exists a Sherman loop $G(\alpha, \beta, \theta)$ such that $v(G(\alpha, \beta, \theta)) = v(x)$ in $\pi_1(|O|\mathcal{S}F|)$. In other words, the difference $x - G(\alpha, \beta, \theta)$ vanishes in $\pi_1(|O|\mathcal{S}F|)$, and thus lies in the image of

$$K_1(\mathcal{C}^{\oplus}) = \pi_1(\Omega|\mathcal{S}\mathcal{C}^{\oplus}|) \to \pi_1(|\widehat{G}\mathcal{C}|) = K_1(\mathcal{C}).$$

To finish, we quote a couple of technical facts about Sherman Loops whose proof we defer to Appendix B.1. By Lemmas B.3 and B.4, $K_1(\mathcal{C}^{\oplus})$ is generated by Sherman Loops, and thus so is its image in $K_1(\mathcal{C})$.[31] By Lemma B.2, the sum of two Sherman Loops is still a Sherman Loop. Put together, conclude that

$$x = x - G(\alpha, \beta, \theta) + G(\alpha, \beta, \theta)$$

is indeed a Sherman Loop. $\square$

Discussion 3.15. Our proof strategy follows Sherman's argument in [She98], except that Sherman primarily works on the level of geometric realisations whereas we work simplicially wherever possible.

The simplicial approach has its advantages: a fully rigorous proof that $|\widehat{G}\mathcal{C}| \simeq |G\mathcal{C}|$ becomes more intricate via Sherman's approach. His original argument proceeds by defining a pair of exact functors

$$\Delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$$
$$\Delta'\colon \mathcal{C}^{\oplus} \to \mathcal{C}^{\oplus} \times \mathcal{C},$$

where Δ is the diagonal, and Δ' is the diagonal composed with the obvious inclusion map. It was then claimed as obvious that the cofiber of $\mathcal{S}\Delta$ is homotopy equivalent to $|\mathcal{S}\mathcal{C}|$, but there are subtleties here. It is not generally true that $\mathrm{cofib}(\Delta) \simeq X$ for a diagonal map of spaces – e.g. consider $\Delta\colon S^1 \to S^1 \times S^1$, which embeds a circle S^1 into a diagonal line on the torus. One potential remedy is to prove that $\mathrm{cofib}(\mathcal{S}\Delta)$ and $\mathcal{S}\mathcal{C}$ are equivalent as $\mathbb{E}_\infty$-spaces, but this involves invoking additional theory. By contrast, we avoid these complications by working out an explicit description of the simplicial fibers (as in Claim 3.10), which gives a more direct path to establishing $|\widehat{G}\mathcal{C}| \simeq |G\mathcal{C}|$.

3.3. **Double Exact Squares.** Given an exact category $\mathcal{J}$, Nenashev [Nen98b, Nen96] shows that $K_1(\mathcal{J})$ is in fact generated by so-called double short exact sequences – sharpening Sherman's original result. We adapt his argument to the pCGW setting.

Definition 3.16 (Double Exact Squares)**.** A *double exact square* is a pair of distinguished squares with identical nodes

$$l := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\square"] & C \arrow[d, hook, "g_1"] \\ A \arrow[r, tail, "f_1"] & B \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\square"] & C \arrow[d, hook, "g_2"] \\ A \arrow[r, tail, "f_2"] & B \end{tikzcd}\right).$$

Notice this defines an edge from $(A, A) \to (B, B)$ in $G\mathcal{C}$. In particular, given any object $A \in \mathcal{C}$, denote the standard edge from (O, O) to (A, A) as

$$e(A) := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\square"] & A \arrow[d, hook, "1_A"] \\ O \arrow[r, tail] & A \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\square"] & A \arrow[d, hook, "1_A"] \\ O \arrow[r, tail] & A \end{tikzcd}\right).$$

[31] **(M:) Why? Double-check.**

Any double exact square therefore defines a loop

$$
\begin{tikzcd}
(A,A) \arrow[rr, "l"] & & (B,B) \\
& (O,O) \arrow[ul, "e(A)"] \arrow[ur, "e(B)"'] &
\end{tikzcd}
\tag{33}
$$

We call this the *canonical loop of* l, and denote it as $\mu(l)$. We denote $\langle l \rangle$ to be its homotopy class in $K_1(\mathcal{C})$.

As the following example illustrates, double exact squares can be regarded as a generalisation of automorphisms in $\mathcal{C}$.

Example 3.17 (Automorphisms)**.** If $(A,\alpha) \in \mathrm{Aut}(\mathcal{C})$ is an automorphism, we write

$$
l(\alpha) = \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & A \arrow[d, {Circle[open]}->, "1_A"] \\ O \arrow[r, tail] & A \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & A \arrow[d, {Circle[open]}->, "\alpha"] \\ O \arrow[r, tail] & A \end{tikzcd} \right).
$$

To prove that $K_1(\mathcal{C})$ is generated by double exact squares, it suffices to show that any Sherman Loop can be associated to a pair of double exact squares; the rest follows from Theorem 3.7.

Theorem 3.18. *Let $\mathcal{C}$ be a pCGW category. Given any $x \in K_1(\mathcal{C})$, there exists a double exact square l such that $x = \mu(l)$.*

Proof. By Theorem 3.7, assume without loss of generality that x is a Sherman Loop $G(\alpha,\beta,\theta)$ arising from a pair of exact squares

$$
\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \arrow[d, {Circle[open]}->, "\delta"] \\ A \arrow[r, tail, "\alpha"] & B \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C' \arrow[d, {Circle[open]}->, "\delta'"] \\ A' \arrow[r, tail, "\alpha'"] & B' \end{tikzcd}
\tag{34}
$$

and an isomorphism $\theta\colon A \oplus C \oplus B' \overset{\cong}{\rightarrowtail} A' \oplus C' \oplus B$. From this, construct another pair of exact squares, denoted

$$
s_0 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \oplus C' \arrow[d, {Circle[open]}->, "g_0"] \\ A \oplus A' \arrow[r, tail, "f_0"] & A \oplus C \oplus B' \end{tikzcd} \right) \quad , \quad s_1 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \oplus C' \arrow[d, {Circle[open]}->, "g_1"] \\ A \oplus A' \arrow[r, tail, "f_1"] & A' \oplus C' \oplus B \end{tikzcd} \right)
\tag{35}
$$

where

$$
f_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \alpha' \end{pmatrix}, \; f_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ \alpha & 0 \end{pmatrix}, \; g_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \delta' \end{pmatrix}, \; g_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ \delta & 0 \end{pmatrix}.
$$

[*Technical Note.* Some work is required to check that s_0 and s_1 are indeed distinguished, which we leave to the reader. This essentially follows from repeated applications of Lemma A.4, and permuting summands.] Applying the isomorphism θ, we obtain the obvious double exact square, which we denote

$$
l(x) := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \oplus C' \arrow[d, {Circle[open]}->, "\theta \circ g_0"] \\ A \oplus A' \arrow[r, tail, "\theta \circ f_0"] & A' \oplus C' \oplus B \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \oplus C' \arrow[d, {Circle[open]}->, "g_1"] \\ A \oplus A' \arrow[r, tail, "f_1"] & A' \oplus C' \oplus B \end{tikzcd} \right).
\tag{36}
$$

Convention 3.19. To ease notation, denote $P := A \oplus C \oplus B'$ and $Q := A' \oplus C' \oplus B$.

Convention 3.20. We fix the following convention when defining maps coordinate-wise. Consider two exact squares

$$h_0 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, ""] & M'' \arrow[d, "m_2"] \\ M' \arrow[r, tail, "m_1"] \arrow[ur, phantom, "\square"] & M \end{tikzcd} \right) \qquad h_1 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] & N'' \arrow[d, "n_2"] \\ N' \arrow[r, tail, "n_1"] \arrow[ur, phantom, "\square"] & N \end{tikzcd} \right).$$

If $M'' = N''$, we denote the corresponding 1-simplex as $(h_0, h_1)\colon (M', N') \to (M, N)$. On the other hand, if we wish to take their direct sum, then this will be denoted

$$h_0 \oplus h_1 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] & M'' \oplus N'' \arrow[d, "m_2 \oplus n_2"] \\ M' \oplus N' \arrow[r, tail, "m_1 \oplus n_1"] \arrow[ur, phantom, "\square"] & M \oplus N \end{tikzcd} \right).$$

If we simply wish to add a component C to h_0 via the $\mathcal{M}$-morphism, then this will be denoted

$$h_0 \oplus C := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] & M'' \arrow[d, "m_2"] \\ M' \oplus C \arrow[r, tail, "m_1 \oplus C"] \arrow[ur, phantom, "\square"] & M \oplus C \end{tikzcd} \right).$$

Let us resume our proof. Starting with a Sherman Loop $G(\alpha, \beta, \theta)$, we defined a double exact square $l(x)$ with corresponding loop $\mu(l(x))$. It thus remains to show that $\mu(l(x))$ is homotopic to $G(\alpha, \beta, \theta)$ in $K_1(\mathcal{C})$. In fact, since $K_1(\mathcal{C})$ is abelian, it suffices to show that they are freely homotopic. This is accomplished by the following series of lemmas.

Lemma 3.21. *$\mu(l(x))$ is freely homotopic to the loop*

$$\begin{tikzcd} (P, Q) \arrow[rr, "(\theta,1)"] & & (Q, Q) \\ & (A \oplus A', A \oplus A') \arrow[ul, "(s_0,s_1)"] \arrow[ur, "(s_1,s_1)"'] & \end{tikzcd} \tag{37}$$

Proof of Lemma. Consider the diagram

$$\begin{tikzcd} & (P, Q) \arrow[dr, "(\theta,1)"] \arrow[d, phantom, "(1)"] & \\ (A \oplus A', A \oplus A') \arrow[ur, "(s_0,s_1)"] \arrow[rr, bend left=10, "l(x)"] \arrow[rr, bend right=10, "(s_0,s_1)"'] & {} \arrow[d, phantom, "(2)"] & (Q, Q) \\ & (O, O) \arrow[ul, "e(A \oplus A')"] \arrow[ur, "e(Q)"'] & \end{tikzcd} \tag{38}$$

The statement follows from observing that Triangles (1) and (2) form 2-simplices. Triangle (2) is obvious. Triangle (1) is given by

$$
\begin{tikzcd}
A \oplus A' \arrow[r, tail, "f_0"] & P \arrow[r, tail, "\theta"] & Q \\
& C \oplus C' \arrow[u, "g_0"] \arrow[r, tail, "1"] \arrow[ur, phantom, "\square"] & C \oplus C' \arrow[u, "\theta \circ g_0"'] \\
& & O \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \oplus A' \arrow[r, tail, "f_1"] & Q \arrow[r, tail, "1"] & Q \\
& C \oplus C' \arrow[u, "g_1"] \arrow[r, tail, "1"] \arrow[ur, phantom, "\square"] & C \oplus C' \arrow[u, "g_1"'] \\
& & O \arrow[u]
\end{tikzcd}
$$

□

Lemma 3.22. *The loop $G(\alpha, \beta, \theta)$ is freely homotopic to the loop*

$$
\begin{tikzcd}
(P, B \oplus B') \arrow[rr, "{(\theta,1)}"] & & (Q, B \oplus B') \\
& (A \oplus A', A \oplus A') \arrow[ul, "{(s_0,s)}"] \arrow[ur, "{(s_1,s)}"'] &
\end{tikzcd}
\tag{39}
$$

where s is the distinguished square

$$
s := \left(
\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C \oplus C' \arrow[d, "{(\delta,\delta')}"] \\
A \oplus A' \arrow[r, tail, "{(\alpha,\alpha')}"] & B \oplus B'
\end{tikzcd}
\right) \quad . \tag{40}
$$

Proof of Lemma. Consider the diagram

$$
\begin{tikzcd}
(P, B \oplus B') \arrow[rrrr, "{(\theta,1)}"] & & & & (Q, B \oplus B') \\
& & (\star) & & \\
(A, A) \arrow[uu, "{(a_0,b_0)}"] \arrow[rr] & & (A \oplus A', A \oplus A') \arrow[uull, "{(s_0,s)}"] \arrow[uurr, "{(s_1,s)}"'] & & (A', A') \arrow[uu, "{(a_1,b_1)}"'] \arrow[ll] \\
& & (O, O) \arrow[ull, "e(A)"] \arrow[u, "e(A \oplus A')"] \arrow[urr, "e(A')"'] & &
\end{tikzcd}
\tag{41}
$$

where

$$
(a_0, b_0) := \left(
\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C \oplus B' \arrow[d] \\
A \arrow[r, tail, "{(1,0,0)}"] & A \oplus C \oplus B'
\end{tikzcd}
,
\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C \oplus B' \arrow[d] \\
A \arrow[r, tail, "{(\alpha,0)}"] & B \oplus B'
\end{tikzcd}
\right)
$$

$$
(a_1, b_1) := \left(
\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C' \oplus B' \arrow[d] \\
A' \arrow[r, tail, "{(1,0,0)}"] & A' \oplus C' \oplus B
\end{tikzcd}
,
\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & B \oplus C' \arrow[d] \\
A' \arrow[r, tail, "{(0,\alpha')}"] & B \oplus B'
\end{tikzcd}
\right) .
$$

Notice the outer loop of Diagram (41) is the Sherman Loop $G(\alpha, \beta, \theta)$ while the triangle $(\star)$ is Loop (39). An easy check shows that all triangles in Diagram (41) except $(\star)$ are 2-simplices.[32] Conclude $G(\alpha, \beta, \theta)$ and Loop (39) are indeed freely homotopic. □

[32] **(M:) Check later.**

Lemma 3.23. *Denote $V := (B \oplus B') \star_{(A\oplus A')} Q$. Then, there exists distinguished squares of the form*

$$t := \left(\begin{tikzcd} C \oplus C' \arrow[r, tail, "1\oplus C'"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\square"] & C \oplus C' \oplus C' \arrow[d, "j_t"] \\ Q \arrow[r, tail, "h_t"] & V \end{tikzcd}\right) \qquad t' := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C' \arrow[d, "k_t"] \\ Q \arrow[r, tail, "h_t"] & V \end{tikzcd}\right) \tag{42}$$

$$u := \left(\begin{tikzcd} C' \oplus C' \arrow[r, tail, "1\oplus C'"] \arrow[d, "(\delta,\delta')"'] \arrow[dr, phantom, "\square"] & C \oplus C' \oplus C' \arrow[d, "j_u"] \\ B \oplus B' \arrow[r, tail, "h_u"] & V \end{tikzcd}\right) \qquad u' := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C' \arrow[d, "k_u"] \\ B \oplus B' \arrow[r, tail, "h_u"] & V \end{tikzcd}\right). \tag{43}$$

Proof. Applying Axiom (DS), Definition 1.18 to the diagram

$$\begin{tikzcd} C' \arrow[d] \arrow[dr, phantom, "\square"] & O \arrow[l, tail] \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C \oplus C' \arrow[d, "g_1"] \\ B \oplus B' & A \oplus A' \arrow[l, tail, "(\alpha,\alpha')"'] \arrow[r, tail, "f_1"] & Q \end{tikzcd} ,$$

conclude that there exists a distinguished square of the form t. By Axiom (PQ), we know that

$$\frac{C \oplus C' \oplus C'}{C \oplus C'} \cong C'.$$

Since t is a distinguished square, deduce that $\frac{V}{Q} \cong C'$ (Lemma 1.7), and thus there must exist a distinguished square of the form t'. To verify there exist distinguished squares of the form u and u', apply the same argument to the diagram

$$\begin{tikzcd} C \oplus C' \arrow[d, "(\delta,\delta')"'] \arrow[dr, phantom, "\square"] & O \arrow[l, tail] \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C' \arrow[d] \\ B \oplus B' & A \oplus A' \arrow[l, tail, "(\alpha,\alpha')"'] \arrow[r, tail, "f_1"] & Q \end{tikzcd} .$$

$\square$

Lemma 3.24. *Loops* (37) *and* (39) *are homotopic.*

Proof of Lemma. Recall $V := (B \oplus B') \star_{(A\oplus A')} Q$ and the definition of t' from Lemma 3.23. In addition, define $\tilde{s}$ as the horizontal composition of s_1 and t

$$\tilde{s} := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C \oplus C' \arrow[r, tail, "1\oplus C'"] \arrow[d, "g_1"] \arrow[dr, phantom, "\square"] & C \oplus C' \oplus C' \arrow[d, "j"] \\ A \oplus A' \arrow[r, tail, "f_1"] & Q \arrow[r, tail, "h_t"] & V \end{tikzcd}\right).$$

We now construct a loop L that Loops (37) and (39) are both homotopic to. Consider the diagram

$$\begin{tikzcd}
(P \oplus C', V) \arrow[rr, "(\theta\oplus 1{,}1)"] & \arrow[d, phantom, "(3)"] & (Q \oplus C', V) \\
(P, Q) \arrow[u, "(1\oplus C'{,}t')"] \arrow[urr, "(\theta\oplus C'{,}\, t')"] \arrow[rr, "(\theta{,}1)"'] & \arrow[phantom, "(4)"] & (Q, Q) \arrow[u, "(1\oplus C'{,}t')"'] \\
\arrow[r, phantom, "(1)"] & (A \oplus A', A \oplus A') \arrow[ul, "(s_0{,}s_1)"'] \arrow[ur, "(s_1{,}s_1)"] \arrow[uul, bend left=60, "(s_0\oplus C'{,}\, \tilde{s})"] \arrow[uur, bend right=60, "(s_1\oplus C'{,}\tilde{s})"'] \arrow[r, phantom, "(2)"] &
\end{tikzcd} \tag{44}$$

The red edges are Loop (37), the blue edges form an outer loop, which we denote L. To show that the two loops are homotopic, it suffices to check that Triangles (1) - (4) are boundaries of 2-simplices – this is worked out explicitly in Claim B.5. Analogously, one can construct the diagram

$$\begin{tikzcd}
(P \oplus C', V) \arrow[rr, "(\theta\oplus 1{,}1)"] & \arrow[d, phantom, "(3')"] & (Q \oplus C', V) \\
(P, B \oplus B') \arrow[u, "(1\oplus C'{,}u')"] \arrow[urr, "(\theta\oplus C'{,}\, u')"] \arrow[rr, "(\theta{,}1)"'] & \arrow[phantom, "(4')"] & (Q, B \oplus B') \arrow[u, "(1\oplus C'{,}u')"'] \\
\arrow[r, phantom, "(1')"] & (A \oplus A', A \oplus A') \arrow[ul, "(s_0{,}s)"'] \arrow[ur, "(s_1{,}s)"] \arrow[uul, bend left=60, "(s_0\oplus C'{,}\, \tilde{s})"] \arrow[uur, bend right=60, "(s_1\oplus C'{,}\tilde{s})"'] \arrow[r, phantom, "(2')"] &
\end{tikzcd} \tag{45}$$

where the teal edges are Loop (39) and the blue edges are loop L. A similar check shows Triangles (1') - (4') are also boundaries of 2-simplices (details in Claim B.6). Conclude that Loop (37) and Loop (39) are both homotopic to loop L, and thus homotopic to each other as well. $\square$

Finish. Given any Sherman Loop $G(\alpha, \beta, \theta) \in K_1(\mathcal{C})$, we can construct another loop $\mu(l(x))$ where $l(x))$ is a double exact square. Recall that:

- Lemma 3.21 shows $\mu(l(x))$ is freely homotopic to Loop (37).
- Lemma 3.22 shows $G(\alpha, \beta, \theta)$ is freely homotopic to Loop (39).
- Lemma 3.24 shows Loops (37) and (39) are homotopic.

Since $K_1(\mathcal{C})$ is an abelian group, deduce that $G(\alpha, \beta, \theta)$ is homotopic to $\mu(l(x))$. Since $K_1(\mathcal{C})$ is generated by Sherman Loops (Theorem 3.7), conclude that $K_1(\mathcal{C})$ is generated by double exact squares. $\square$

Discussion 3.25. Although the proof strategy behind Theorem 3.18 is similar to Nenashev's original proof [Nen96] for exact categories, a naive translation of his argument to our setting does not work. Consider, for instance, the isomorphism

$$B \oplus \frac{B}{A} \cong B \star_A B$$

induced by some $\mathcal{M}$-morphism $f\colon A \rightarrowtail B$ in $\mathcal{C}$. If $\mathcal{C}$ is an exact category and the restricted pushout is the usual pushout, then the isomorphism holds; indeed, this isomorphism plays a key role in [Nen96, Lemma 2.6] of the original proof. However, as pointed out to us by I. Zakharevich, this isomorphism fails in general if $\mathcal{C} = \mathcal{V}\mathrm{ar}_k$. To see why, consider the closed immersion of a point into the affine line $f\colon \{*\} \hookrightarrow \mathbb{A}^1$. Our proof therefore establishes the desired homotopies by constructing the required 2-simplices by hand.

(A) (B)

FIGURE 2. Figure (A) is $\mathbb{A}^1 \oplus \left(\mathbb{A}^1 \setminus \{*\}\right)$, while Figure (B) is $\mathbb{A}^1 \star_{\{*\}} \mathbb{A}^1$.

4. Relations of $K_1(\mathcal{C})$

Having characterised the generators of $K_1(\mathcal{C})$ for pCGW categories, we now work to determine the full list of its relations. We first give a baseline characterisation in Proposition 4.1. We then sharpen our understanding by comparing this to other descriptions of K_1 by Nenashev [Nen98b, Nen98a] (in the setting of exact categories) and Zakharevich [Zak17c] (in the setting of Assemblers). A guiding observation is Warning 4.2, which highlights a technical subtlety regarding the composition of 1-simplices in K_1. Interestingly, this brings into focus an apparent discrepancy between our account and Zakharevich's regarding the correct relations of K_1.

4.1. **A Baseline Argument.** Observe that any double exact square lies in the base-point component of $G\mathcal{C}$, which we denote $G\mathcal{C}^o$. Since double exact squares generate $K_1(\mathcal{C})$ (Theorem 3.18), it follows that $K_1(\mathcal{C}) = \pi_1(G\mathcal{C}^o)$. One can therefore apply the standard description of the fundamental group of a connected simplicial spsace to get the following presentation.

Proposition 4.1. *$K_1(\mathcal{C})$ is generated by isomorphism classes of double exact squares $\langle f \rangle$ modulo the following relations:*

(B1) Given any $A \in \mathcal{C}$, the standard edge $\mathrm{e}(A)\colon (O,O) \to (A,A)$ of $G\mathcal{C}$ vanishes. That is,

$$\left\langle \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & A \arrow[d, "1"] \\ O \arrow[r, tail] & A \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & A \arrow[d, "1"] \\ O \arrow[r, tail] & A \end{tikzcd} \right) \right\rangle = 0.$$

(B2) Given any $A \in \mathcal{C}$, the degenerate 1-simplex $\mathrm{id}_A\colon (A,A) \to (A,A)$ of $G\mathcal{C}$ vanishes. That is,

$$\left\langle \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & O \arrow[d] \\ A \arrow[r, tail, "1"] & A \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & O \arrow[d] \\ A \arrow[r, tail, "1"] & A \end{tikzcd} \right) \right\rangle = 0.$$

(B3) Given double exact squares of the form

$$l_A := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[d, "g_A"] \\ A \arrow[r, tail, "f_A"] & B \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[d, "g'_A"] \\ A \arrow[r, tail, "f'_A"] & B \end{tikzcd} \right) \quad l_B := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{C}{B} \arrow[d, "g_B"] \\ B \arrow[r, tail, "f_B"] & C \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{C}{B} \arrow[d, "g'_B"] \\ B \arrow[r, tail, "f'_B"] & C \end{tikzcd} \right) \tag{46}$$

$$l_C := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{C}{A} \arrow[d, "g_C"] \\ A \arrow[r, tail, "f_C"] & C \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{C}{A} \arrow[d, "g'_C"] \\ A \arrow[r, tail, "f'_C"] & C \end{tikzcd} \right) \tag{47}$$

that assemble into a 2-simplex in $G\mathcal{C}$

$$\begin{tikzcd} A \arrow[r, tail, "f_A"] & B \arrow[r, tail, "f_B"] \arrow[dr, phantom, "\square"] & C \\ & \frac{B}{A} \arrow[u, "g_A"] \arrow[r, tail, "h_1"'] \arrow[dr, phantom, "\square"] & \frac{C}{A} \arrow[u, "g_C"'] \\ & O \arrow[u] \arrow[r, tail] & \frac{C}{B} \arrow[u, "h_2"'] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, tail, "f'_A"] & B \arrow[r, tail, "f'_B"] \arrow[dr, phantom, "\square"] & C \\ & \frac{B}{A} \arrow[u, "g'_A"] \arrow[r, tail, "h_1"'] \arrow[dr, phantom, "\square"] & \frac{C}{A} \arrow[u, "g'_C"'] \\ & O \arrow[u] \arrow[r, tail] & \frac{C}{B} \arrow[u, "h_2"'] \end{tikzcd} \tag{48}$$

we have that

$$\langle l_A \rangle + \langle l_B \rangle = \langle l_C \rangle.$$

Proof. Let us divide the proof into two main steps.

Step 1: A General Description. Let X be a connected simplicial set with Γ as the maximal tree for its 1-skeleton. It is well-known folklore that $\pi_1|X|$ has the following presentation

$$\pi_1|X| := \pi_0(X[1]) \Big/ \begin{array}{c} \langle t \rangle = 0 \text{ if } t \in \Gamma, \text{ and } \langle \mathrm{id}_A \rangle = 0 \text{ for any degenerate 1-simplex} \\ d_1(x) = d_2(x)d_0(x), \quad \forall x \in \pi_0(X[2]) \end{array} .$$

Here is a sketch of the argument. Suppose X be a simplicial set as above. The fundamental group of its geometric realisation is determined by its 2-skeleta. Thus, as a first reduction, take the 2-truncation of X. Next, collapse the maximal spanning tree Γ in X to a single vertex O. Notice that since Γ is contractible, this implies $|X| \simeq |X/\Gamma|$, and so $\pi_1(|X|) \cong \pi_1(|X/\Gamma|)$. Geometrically, collapsing Γ has the effect of turning all 1-simplices $t \notin \Gamma$ into loops based at vertex O.

It is well-established (e.g. [Wei13, Prop. IV.8.4]) that if $X_\bullet$ is any simplicial space with $X[0] = \{*\}$, then $\pi_1|X_\bullet|$ is the free group on the 1-simplices modulo the relations $d_1(x) = d_2(x)d_0(x)$ for every $x \in \pi_0(X[2])$.[33] Notice this gives us the right generators of $\pi_1|X/\Gamma|$ and one of its relations. To get the remaining relations, notice that obviously $\langle t \rangle = 0$ if $t \in \Gamma$ since we contracted X by Γ. Finally, the geometric realisation of any simplicial set associates an n-cell to any *non-degenerate* n-simplex. Put otherwise, the degenerate 1-simplices of X do not contribute to $\pi_1|X/\Gamma|$ and so $\langle id_A \rangle = 0$. And we are done.

Step 2: Application. In our case, our connected simplicial set $X = G\mathcal{C}^o$. In particular:

- $\pi_0(G\mathcal{C}^o[1])$ is the set of isomorphism classes of 1-simplices of $G\mathcal{C}^o$. By Theorem 3.18, we may restrict this to the isomorphism classes of double exact squares.
- The obvious set of 1-simplices $(O, O) \to (A, A')$ defines a maximal subtree of the 1-skeleton of $G\mathcal{C}^o$. Since we restrict to just the double exact squares for our generators, we may assume $A = A'$.
- $\pi_0(G\mathcal{C}^o[2])$ is the set of equivalence classes of 2-simplices of $G\mathcal{C}^o$. In particular, any $x \in \pi_0(G\mathcal{C}^o[2])$ can be represented in the form of Diagram 48, where $d_0(x) = l_B$, $d_2(x) = l_A$ and $d_1 = l_C$.

The proposition then follows from our earlier observation that $K_1(\mathcal{C}) = \pi_1(G\mathcal{C}^o)$. $\square$

[33] Alternatively, one may wish to prove this directly by applying Van Kampen's Theorem to the skeletal filtration of X.

Warning 4.2 (Composition of 1-simplices)**.** There is a technical fine-print in Proposition 4.1. Notice that the quotient index triangles in Equation (48)

$$\left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\square"] & \frac{C}{B} \arrow[d, "h_2"] \\ \frac{B}{A} \arrow[r, tail, "h_1"] & \frac{C}{A} \end{tikzcd}, \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{C}{B} \arrow[d, "h_2"] \\ \frac{B}{A} \arrow[r, tail, "h_1"] & \frac{C}{A} \end{tikzcd}\right)$$

are identical in both diagrams; otherwise this no longer defines a 2-simplex in $G\mathcal{C}$. In particular, suppose we have three double exact squares

$$l_A, l_B, l_C$$

of the form in Equations (46)-(47) with no further information. We emphasise that Proposition 4.1 does *not* say: if $f_A \circ f_B = f_C$ and $g_A \circ g_B = g_C$ in $\mathcal{C}$, then

$$\langle l_A \rangle + \langle l_B \rangle = \langle l_C \rangle.$$

Why not? The informal answer: even if two pairs of 1-simplices compose in $S\mathcal{C}$, their composition still may not define a 2-simplex in $G\mathcal{C}$. The puzzled reader should review the G-Construction 3.1 and remind themselves that an n-simplex in $G\mathcal{C}$ is more than just a pair of n-simplices in $S\mathcal{C}$ sharing the same vertices.

Keeping Warning 4.2 in mind will help us appreciate the work done in the subsequent sections. On a basic level, these results carefully translate Nenashev's work on K_1 of exact categories [Nen98b, Nen98a] to our more general setting, providing yet another characterisation of $K_1(\mathcal{C})$. On another level, Nenashev's presentation clarifies exactly *how* composition of 1-simplices are split in $K_1(\mathcal{C})$, illuminating the abovementioned discrepancy between our account of K_1 and Zakharevich's.

4.2. **Admissible Triples.** A triangle contour $\mathcal{T}$ in $G\mathcal{C}$

$$\begin{tikzcd} & (P_1, P_1') \arrow[dr, "e_1"] & \\ (P_0, P_0') \arrow[ur, "e_0"] \arrow[rr, "e_2"] & & (P_2, P_2') \end{tikzcd} \tag{49}$$

is given by three pairs of distinguished squares of the form

$$e_0 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & P_{1/0} \arrow[d, "\alpha_{1/0,1}"] \\ P_0 \arrow[r, tail, "\alpha_{0,1}"] & P_1 \end{tikzcd}, \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & P_{1/0} \arrow[d, "\alpha'_{1/0,1}"] \\ P_0' \arrow[r, tail, "\alpha'_{0,1}"] & P_1' \end{tikzcd}\right) \quad e_1 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & P_{2/1} \arrow[d, "\alpha_{2/1,2}"] \\ P_1 \arrow[r, tail, "\alpha_{1,2}"] & P_2 \end{tikzcd}, \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & P_{2/1} \arrow[d, "\alpha'_{2/1,2}"] \\ P_1' \arrow[r, tail, "\alpha'_{1,2}"] & P_2' \end{tikzcd}\right)$$

$$e_2 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & P_{2/0} \arrow[d, "\alpha_{2/0,2}"] \\ P_0 \arrow[r, tail, "\alpha_{0,2}"] & P_2 \end{tikzcd}, \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & P_{2/0} \arrow[d, "\alpha'_{2/0,2}"] \\ P_0' \arrow[r, tail, "\alpha'_{0,2}"] & P_2' \end{tikzcd}\right).$$

A simple but key observation: given any vertex $(A, A') \in G\mathcal{C}$, one can construct a new triangle contour $(A, A') \oplus \mathcal{T}$ by formal direct sum:

$$\begin{tikzcd} & (P_1 \oplus A, P_1' \oplus A') \arrow[dr, "e_1 \oplus (A,A')"] & \\ (P_0 \oplus A, P_0' \oplus A') \arrow[ur, "e_0 \oplus (A,A')"] \arrow[rr, "e_2 \oplus (A,A')"] & & (P_2 \oplus A, P_2' \oplus A') \end{tikzcd} \tag{50}$$

Definition 4.3 (Admissible Triple)**.** We call a triple $\tau = (e_0, e_1, e_2)$ of the above form *admissible* if it can be completed to the following pair of diagrams:

$$
\begin{tikzcd}
P_0 \arrow[r, tail, "\alpha_{0,1}"] & P_1 \arrow[r, tail, "\alpha_{1,2}"] & P_2 \\
& P_{1/0} \arrow[u, "\alpha_{1/0,1}"] \arrow[r, tail, "\alpha_{1/0,2/0}"'] \arrow[ur, phantom, "\square"] & P_{2/0} \arrow[u, "\alpha_{2/0,2}"'] \\
& O \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & P_{2/1} \arrow[u, "\alpha_{2/1,2/0}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
P'_0 \arrow[r, tail, "\alpha'_{0,1}"] & P'_1 \arrow[r, tail, "\alpha'_{1,2}"] & P'_2 \\
& P_{1/0} \arrow[u, "\alpha'_{1/0,1}"] \arrow[r, tail, "\alpha'_{1/0,2/0}"'] \arrow[ur, phantom, "\square"] & P_{2/0} \arrow[u, "\alpha'_{2/0,2}"'] \\
& O \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & P_{2/1} \arrow[u, "\alpha'_{2/1,2/0}"']
\end{tikzcd}
\tag{51}
$$

In particular, define

$$
l(\tau) := \left(
\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & P_{2/1} \arrow[d, "\alpha_{2/1,2/0}"] \\
P_{1/0} \arrow[r, tail, "\alpha_{1/0,2/0}"] & P_{2/0}
\end{tikzcd}
\qquad
\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & P_{2/1} \arrow[d, "\alpha'_{2/1,2/0}"] \\
P_{1/0} \arrow[r, tail, "\alpha'_{1/0,2/0}"] & P_{2/0}
\end{tikzcd}
\right)
\tag{52}
$$

to be the *double exact square associated to admissible triple* τ.

Any admissible triple $\tau = (e_0, e_1, e_2)$ defines a loop $e_0 e_1 e_2^{-1}$, which we also denote using τ. Notice: if

$$\alpha_{1/0,2/0} = \alpha'_{1/0,2/0}$$

in Diagram (51), then the loop τ bounds a 2-simplex in $G\mathcal{C}$ since the quotient index triangles of both diagrams now coincide. However, even if this condition does not hold (cf. Warning 4.2), we can still say something meaningful about the (free) homotopy class of τ in general.

Lemma 4.4. *Setup:*

- *Let* $\tau = (e_0, e_1, e_2)$ *be an admissible triple.*
- *Let* $l(\tau)$ *be the double exact square associated to* τ*, and* $\mu(l(\tau))$ *be its canonical loop.*

Then the loop $\tau = e_0 e_1 e_2^{-1}$ *is freely homotopic to*

$$(P_2, P'_2) \oplus \mu(l(\tau)).$$

Proof. Construct the obvious diagram

$$
\begin{tikzcd}[column sep=small]
(P_2 \oplus P_{1/0}\,, P'_2 \oplus P_{1/0}) \arrow[rrrr, red, "{(P_2\,,P'_2)\oplus l(\tau)}"] & & & & (P_2 \oplus P_{2/0}\,, P'_2 \oplus P_{2/0}) \\
& (P_1\,,P'_1) \arrow[ul] \arrow[urrr] \arrow[rr, blue, "e_1"] & & (P_2\,,P'_2) \arrow[ur] & \\
& & (P_0\,,P'_0) \arrow[ul, blue, "e_0"] \arrow[ur, blue, "e_2"'] \arrow[uull, bend left] \arrow[uurr, bend right] \arrow[d, "e_2"] & & \\
& & (P_2\,,P'_2) \arrow[uuull, red, bend left, "{(P_2\,,P'_2)\oplus e(P_{1/0})}"] \arrow[uuurr, red, bend right, "{(P_2\,,P'_2)\oplus e(P_{2/0})}"'] & &
\end{tikzcd}
\tag{53}
$$

The red edges form the loop $(P_2, P'_2) \oplus \mu(l(\tau))$, the blue edges form the loop τ. To show that the two loops are freely homotopic, it suffices to show that all the triangles are in fact 2-simplices of $G\mathcal{C}$.

Once again, a naive attempt to translate Nenashev's original argument [Nen96, Lemma 4.1] does not work because pushouts in exact categories are better behaved than restricted pushouts in CGW categories (cf. Discussion 3.25). Nonetheless, this can be circumvented by working out the 2-simplices explicitly – details are given in Section B.3. Some non-trivial work is still needed to verify that the chosen diagrams in fact define 2-simplices of $G\mathcal{C}$, but thankfully restricted pushouts have enough good properties to make the calculations go through (see Lemma A.3). $\square$

Corollary 4.5. *If an admissible triple τ lies in the base component of $G\mathcal{C}$, then the loop τ is freely homotopic to $\mu(l(\tau))$.*

Proof. Recall the construction $(A, A') \oplus \mathcal{T}$ obtained by adding a vertex (A, A') to a triangular contour $\mathcal{T}$. This can be extended in the obvious way to define an action

$$(A, A') \oplus (—)\colon G\mathcal{C} \to G\mathcal{C}, \tag{54}$$

which takes an n-simplex in $G\mathcal{C}$ and adds (A, A') to all the relevant nodes.

Now suppose there exists an edge between vertices $(A, A') \to (B, B')$. By the the same argument[34] as in Claim 3.10, this induces a simplicial homotopy between the maps

$$(A, A') \oplus (—) \longrightarrow (B, B') \oplus (—). \tag{55}$$

In particular, if there exists an edge $(O, O) \to (P_2, P_2')$, then $(P_2, P_2') \oplus \mu(l(\tau))$ is homotopic to $\mu(l(\tau))$. $\square$

4.3. **Nenashev Relations.** To define the relations on $K_1(\mathcal{C})$, we shall need the following generalisation of double exact squares. A 3×3 *diagram* in a pCGW category $\mathcal{C}$ is a pair of diagrams

$$\left(\begin{tikzcd}
X_{00} \arrow[r, tail, "f_0"] \arrow[d, tail, "h_0"'] \arrow[dr, phantom, "\circlearrowright"] & X_{01} \arrow[d, tail, "h_1"] & X_{02} \arrow[l, "g_0"'] \arrow[d, tail, "h_2"] \\
X_{10} \arrow[r, tail, "f_1"] & X_{11} \arrow[dr, phantom, "\circlearrowright"] & X_{12} \arrow[l, "g_1"'] \\
X_{20} \arrow[u, "j_0"] \arrow[r, tail, "f_2"] & X_{21} \arrow[u, "j_1"] & X_{22} \arrow[l, "g_2"'] \arrow[u, "j_2"']
\end{tikzcd}, \quad
\begin{tikzcd}
X_{00} \arrow[r, tail, "f_0'"] \arrow[d, tail, "h_0'"'] \arrow[dr, phantom, "\circlearrowright"] & X_{01} \arrow[d, tail, "h_1'"] & X_{02} \arrow[l, "g_0'"'] \arrow[d, tail, "h_2'"] \\
X_{10} \arrow[r, tail, "f_1'"] & X_{11} \arrow[dr, phantom, "\circlearrowright"] & X_{12} \arrow[l, "g_1'"'] \\
X_{20} \arrow[u, "j_0'"] \arrow[r, tail, "f_2'"] & X_{21} \arrow[u, "j_1'"] & X_{22} \arrow[l, "g_2'"'] \arrow[u, "j_2'"']
\end{tikzcd}\right)$$

on the same objects subject to the following conditions:

- The horizontal and vertical rows of each diagram define exact squares. Explicitly, a 3×3 diagram is defined by 6 double exact squares:

$$l_i := \left(\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & X_{i2} \arrow[d, "g_i"] \\
X_{i0} \arrow[r, tail, "f_i"] & X_{i1}
\end{tikzcd}, \quad
\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & X_{i2} \arrow[d, "g_i'"] \\
X_{i0} \arrow[r, tail, "f_i'"] & X_{i1}
\end{tikzcd}\right) \qquad \text{for all } i \in \{0, 1, 2\}$$

$$l^i := \left(\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & X_{2i} \arrow[d, "j_i"] \\
X_{0i} \arrow[r, tail, "h_i"] & X_{1i}
\end{tikzcd}, \quad
\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & X_{2i} \arrow[d, "j_i'"] \\
X_{0i} \arrow[r, tail, "h_i'"] & X_{1i}
\end{tikzcd}\right) \qquad \text{for all } i \in \{0, 1, 2\}$$

- The top left square and bottom left squares of each diagram are required to commute in $\mathcal{M}$ and $\mathcal{E}$ respectively. By contrast, we impose no conditions on the "mixed" squares – in particular, they need not be distinguished.

Definition 4.6 (Good 3×3 Diagrams)**.** A 3×3 diagram in $\mathcal{C}$ is called *good* if there exists objects Z and Z', maps $v\colon Z \rightarrowtail X_{11}$ and $v'\colon Z' \rightarrowtail X_{11}$ inducing the following diagrams[35]:

(P1)

$$\begin{tikzcd}
X_{01} \arrow[r, tail] \arrow[dr, phantom, "\square"] & Z \arrow[r, tail] \arrow[dr, phantom, "\square"] & X_{11} \\
O \arrow[u] \arrow[r, tail] & X_{20} \arrow[u] \arrow[r, tail, "f_2"] \arrow[dr, phantom, "\square"] & X_{21} \arrow[u, "j_1"'] \\
& O \arrow[u] \arrow[r, tail] & X_{22} \arrow[u, "g_2"']
\end{tikzcd} \qquad
\begin{tikzcd}
X_{01} \arrow[r, tail] \arrow[dr, phantom, "\square"] & Z' \arrow[r, tail] \arrow[dr, phantom, "\square"] & X_{11} \\
O \arrow[u] \arrow[r, tail] & X_{20} \arrow[u] \arrow[r, tail, "f_2'"] \arrow[dr, phantom, "\square"] & X_{21} \arrow[u, "j_1'"'] \\
& O \arrow[u] \arrow[r, tail] & X_{22} \arrow[u, "g_2'"']
\end{tikzcd}$$

[34] **(M:) Double-check this – basically, just change the summand one direct sum at a time.**

[35] **(M:) Might need to be more explicit about the morphisms featuring Z; or I could just write it in the diagram rather doing this upfront**

(P2)

$$
\begin{tikzcd}
X_{10} \arrow[r, tail] & Z \arrow[r, tail] & X_{11} \\
O \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & X_{02} \arrow[u] \arrow[r, tail, "h_2"] \arrow[ur, phantom, "\square"] & X_{12} \arrow[u, "g_1"'] \\
 & O \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & X_{22} \arrow[u, "j_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{10} \arrow[r, tail] & Z' \arrow[r, tail] & X_{11} \\
O \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & X_{02} \arrow[u] \arrow[r, tail, "h_2'"] \arrow[ur, phantom, "\square"] & X_{12} \arrow[u, "g_1'"'] \\
 & O \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & X_{22} \arrow[u, "j_2'"']
\end{tikzcd}
$$

(P3)

$$
\begin{tikzcd}
X_{00} \arrow[r, tail, "f_0"] & X_{01} \arrow[r, tail] & Z \\
 & X_{02} \arrow[u, "g_0"] \arrow[r, tail] \arrow[ur, phantom, "\square"] & X_{02} \oplus X_{20} \arrow[u] \\
 & & X_{20} \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{00} \arrow[r, tail, "f_0'"] & X_{01} \arrow[r, tail] & Z' \\
 & X_{02} \arrow[u, "g_0'"] \arrow[r, tail] \arrow[ur, phantom, "\square"] & X_{02} \oplus X_{20} \arrow[u] \\
 & & X_{20} \arrow[u]
\end{tikzcd}
$$

(P4)

$$
\begin{tikzcd}
X_{00} \arrow[r, tail, "h_0"] & X_{10} \arrow[r, tail] & Z \\
 & X_{20} \arrow[u, "j_0"] \arrow[r, tail] \arrow[ur, phantom, "\square"] & X_{02} \oplus X_{20} \arrow[u] \\
 & & X_{02} \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{00} \arrow[r, tail, "h_0'"] & X_{10} \arrow[r, tail] & Z' \\
 & X_{20} \arrow[u, "j_0'"] \arrow[r, tail] \arrow[ur, phantom, "\square"] & X_{02} \oplus X_{20} \arrow[u] \\
 & & X_{02} \arrow[u]
\end{tikzcd}
$$

Discussion 4.7 (Limitations of Restricted Pushouts)**.** In the setting of exact categories, the admissible monics and epis are all morphisms of a larger category, so one can define a 3×3 diagram as one which all the squares commute. One can then prove, as in [Nen98b, Prop. 5.1], that such a 3×3 diagram is automatically good since one can obtain the desired Z and Z' by taking pushouts. Unfortunately, this argument does not work in our setting. We can certainly take e.g. the restricted pushout of $X_{01} \leftarrow X_{00} \rightarrow X_{10}$ and apply Axiom (PQ) to construct the following exact squares

$$
\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & X_{20} \arrow[d] \\
X_{01} \arrow[r, tail] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & X_{02} \arrow[d] \\
X_{10} \arrow[r, tail] & Z
\end{tikzcd}
.
$$

However, we still cannot construct Diagram (P1) because we do not know if there exists $\mathcal{M}$-morphism $Z \to X_{11}$ since restricted pushouts need not satisfy the universal property of pushouts.[36] There are various ways to get around this problem, but the most straightforward option is to formally hardcode the desired properties into our definition of good 3×3 diagrams.

Definition 4.8. Let $\mathcal{C}$ be a pCGW category. Define $\mathcal{D}(\mathcal{C})$ to be the abelian group with generators $\langle l \rangle$ for all double exact squares l in $\mathcal{C}$ subject to the following relations.

(N1) $\langle l \rangle = 0$ if

$$
l = \left(\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d, "g"] \\
A \arrow[r, tail, "f"] & B
\end{tikzcd}
,
\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d, "g"] \\
A \arrow[r, tail, "f"] & B
\end{tikzcd} \right) \tag{56}
$$

Any identical pair of exact squares will be called a *diagonal* double exact square.

[36] **(M:) Such a morphism will exist if the $\mathcal{M}$-morphism square is a pullback. Might make life more difficult if we want to prove that we have the whole list of results. We would have to introduce extra axioms to discuss how these pullbacks interact with distinguished squares, which would be complicated. Easier to just formally hardcode what we need into the definition.**

(N2) Given a good 3×3 diagram

$$\left(\begin{tikzcd} X_{00} \arrow[r, tail, "f_0"] \arrow[d, tail, "h_0"'] \arrow[dr, phantom, "\circlearrowleft"] & X_{01} \arrow[d, tail, "h_1"] & X_{02} \arrow[l, o->, "g_0"'] \arrow[d, tail, "h_2"] \\ X_{10} \arrow[r, tail, "f_1"] & X_{11} & X_{12} \arrow[l, o->, "g_1"'] \\ X_{20} \arrow[u, o->, "j_0"] \arrow[r, tail, "f_2"] & X_{21} \arrow[u, o->, "j_1"] \arrow[ur, phantom, "\circlearrowleft"] & X_{22} \arrow[l, o->, "g_2"'] \arrow[u, o->, "j_2"'] \end{tikzcd} \quad , \quad \begin{tikzcd} X_{00} \arrow[r, tail, "f'_0"] \arrow[d, tail, "h'_0"'] \arrow[dr, phantom, "\circlearrowleft"] & X_{01} \arrow[d, tail, "h'_1"] & X_{02} \arrow[l, o->, "g'_0"'] \arrow[d, tail, "h'_2"] \\ X_{10} \arrow[r, tail, "f'_1"] & X_{11} & X_{12} \arrow[l, o->, "g'_1"'] \\ X_{20} \arrow[u, o->, "j'_0"] \arrow[r, tail, "f'_2"] & X_{21} \arrow[u, o->, "j'_1"] \arrow[ur, phantom, "\circlearrowleft"] & X_{22} \arrow[l, o->, "g'_2"'] \arrow[u, o->, "j'_2"'] \end{tikzcd}\right) \tag{57}$$

defined by the following 6 double exact squares

$$l_i := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & X_{i2} \arrow[d, o->, "g_i"] \\ X_{i0} \arrow[r, tail, "f_i"] & X_{i1} \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & X_{i2} \arrow[d, o->, "g'_i"] \\ X_{i0} \arrow[r, tail, "f'_i"] & X_{i1} \end{tikzcd}\right) \qquad \text{for all } i \in \{0,1,2\} \tag{58}$$

$$l^i := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & X_{2i} \arrow[d, o->, "j_i"] \\ X_{0i} \arrow[r, tail, "h_i"] & X_{1i} \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, o->] \arrow[dr, phantom, "\square"] & X_{2i} \arrow[d, o->, "j'_i"] \\ X_{0i} \arrow[r, tail, "h'_i"] & X_{1i} \end{tikzcd}\right) \qquad \text{for all } i \in \{0,1,2\}, \tag{59}$$

the following 6-term relation holds

$$\langle l_0 \rangle + \langle l_2 \rangle - \langle l_1 \rangle = \langle l^0 \rangle + \langle l^2 \rangle - \langle l^1 \rangle \tag{60}$$

Theorem 4.9. *Given a pCGW category $\mathcal{C}$ there exists a well-defined homomorphism*

$$m \colon \mathcal{D}(\mathcal{C}) \longrightarrow K_1(\mathcal{C}) \tag{61}$$

that is surjective. In other words, the two relations of $\mathcal{D}(\mathcal{C})$ also hold in $K_1(\mathcal{C})$.

Proof. By Theorem3.18, we know that $K_1(\mathcal{C})$ is generated by double exact squares so both groups have the same generators. It remains to check the relations.

(N1): Let l be as in Equation (56). The corresponding loop $\mu(l)$ bounds the 2-simplex

$$\begin{tikzcd} O \arrow[r, tail] & A \arrow[r, tail, "f"] \arrow[dr, phantom, "\square"] & B \\ & A \arrow[u, o->] \arrow[r, tail, "f"] & B \arrow[u, o->] \\ & & C \arrow[u, o->, "g"] \end{tikzcd} \qquad \begin{tikzcd} O \arrow[r, tail] & A \arrow[r, tail, "f"] \arrow[dr, phantom, "\square"] & B \\ & A \arrow[u, o->] \arrow[r, tail, "f"] & B \arrow[u, o->] \\ & & C \arrow[u, o->, "g"] \end{tikzcd}$$

in $G\mathcal{C}$, and so $\langle l \rangle = 0$.

(N2): Leveraging the fact that the 3×3 diagram is good, construct the diagram

$$\begin{tikzcd} (X_{00}, X_{00}) \arrow[rr, "l_0"] \arrow[dd, "l_1"] \arrow[dr, "\alpha_0"] & & (X_{01}, X_{01}) \arrow[dd, "l^1"] \arrow[dl, "\alpha_3"] \\ & (Z, Z') \arrow[dr, "\alpha_2"] & \\ (X_{10}, X_{10}) \arrow[rr, "l^0"] \arrow[ur, "\alpha_1"] & & (X_{11}, X_{11}) \end{tikzcd} \tag{62}$$

where outer blue edges $\alpha := l_1 l^0 (l^1)^{-1}(l_0)^{-1}$ form a loop, and the inner edges are given by

$$\alpha_0 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & X_{02}\oplus X_{20} \arrow[d, "\circ"{marking, pos=0}] \\ X_{00} \arrow[r, tail] & Z \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & X_{02}\oplus X_{20} \arrow[d, "\circ"{marking, pos=0}] \\ X_{00} \arrow[r, tail] & Z' \end{tikzcd} \right) \quad \alpha_1 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & X_{02} \arrow[d, "\circ"{marking, pos=0}] \\ X_{10} \arrow[r, tail] & Z \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & X_{02} \arrow[d, "\circ"{marking, pos=0}] \\ X_{10} \arrow[r, tail] & Z' \end{tikzcd} \right)$$

$$\alpha_2 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & X_{22} \arrow[d, "\circ"{marking, pos=0}] \\ Z \arrow[r, tail] & X_{11} \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & X_{22} \arrow[d, "\circ"{marking, pos=0}] \\ Z' \arrow[r, tail] & X_{11} \end{tikzcd} \right) \quad \alpha_3 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & X_{20} \arrow[d, "\circ"{marking, pos=0}] \\ X_{01} \arrow[r, tail] & Z \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, pos=0}] \arrow[dr, phantom, "\square"] & X_{20} \arrow[d, "\circ"{marking, pos=0}] \\ X_{01} \arrow[r, tail] & Z' \end{tikzcd} \right)$$

For orientation, we start with a basic observation.

Lemma 4.10. *Given any closed loop $l = e_0 \dots e_n$ whose edges are all double exact squares,*

$$\langle l \rangle = \sum_{i=0}^{n} (-1)^{\epsilon_i} \langle e_i \rangle$$

in $K_1(\mathcal{C})$, where the coefficient $(-1)^{\epsilon_i}$ reflects the orientation of edge e_i in l.

Proof of Lemma. The proof is obvious, but let us work this out explicitly for the case of α. Consider the diagram

$$\begin{tikzcd} (X_{00}\,, X_{00}) \arrow[rr, "l_0"] \arrow[dd, "l_1"] & & (X_{01}\,, X_{01}) \arrow[dd, "l^1"] \\ & (O\,, O) \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \\ (X_{10}\,, X_{10}) \arrow[rr, "l^0"] & & (X_{11}\,, X_{11}) \end{tikzcd} \tag{63}$$

featuring α as the outer loop but now with the base-point (O, O) at the center. Since the triangles all bound 2-simplices, the loop $\mu(l_1)\mu(l^0)\mu(l^1)^{-1}\mu(l_0)^{-1}$ is freely homotopic to α, and so

$$\langle \alpha \rangle = \langle l_1 \rangle + \langle l^0 \rangle - \langle l^1 \rangle - \langle l_0 \rangle.$$

□

Next, leveraging the fact that the 3×3 diagram is good, notice:

- Diagrams (P3) and (P4) are 2-simplices, which are bounded by the loops $l_1\alpha_1\alpha_0^{-1}$ and $l_0\alpha_3\alpha_0^{-1}$. Therefore, deduce that

$$\begin{aligned} \langle l_1 \rangle + \langle \alpha_1 \rangle - \langle \alpha_0 \rangle &= 0 \\ \langle l_0 \rangle + \langle \alpha_3 \rangle - \langle \alpha_0 \rangle &= 0 \end{aligned} \tag{64}$$

- Diagrams (P1) and (P2) are admissible triples whose associated exact squares are l_2 and l^2 respectively. It is also clear there exists an edge $(O, O) \to (X_{11}, X_{11})$. Thus, applying Corollary 4.5, deduce that

$$\begin{aligned} \alpha_3\alpha_2(l^1)^{-1} &\sim l_2 \\ \alpha_1\alpha_2(l^0)^{-1} &\sim l^2 \end{aligned} \tag{65}$$

  and so

$$\begin{aligned} \langle \alpha_3 \rangle + \langle \alpha_2 \rangle - \langle l^1 \rangle &= \langle l_2 \rangle \\ \langle \alpha_1 \rangle + \langle \alpha_2 \rangle - \langle l^0 \rangle &= \langle l^2 \rangle. \end{aligned} \tag{66}$$

Combining Equations (64) and (66),

$$\begin{aligned}\langle l_2\rangle - \langle l^2\rangle &= \langle \alpha_3\rangle + \langle \alpha_2\rangle - \langle l^1\rangle - \langle \alpha_1\rangle - \langle \alpha_2\rangle + \langle l^0\rangle\\ &= \langle \alpha_3\rangle - \langle \alpha_0\rangle - \langle l^1\rangle - \langle \alpha_1\rangle + \langle \alpha_0\rangle + \langle l^0\rangle\\ &= -\langle l_0\rangle + \langle l^0\rangle + \langle l_1\rangle - \langle l^1\rangle,\end{aligned}$$

and so by rearranging terms, conclude

$$\langle l_0\rangle + \langle l_2\rangle - \langle l_1\rangle = \langle l^0\rangle + \langle l^2\rangle - \langle l^1\rangle. \tag{67}$$

□

4.4. **Assembler Relations.** We conclude by applying Theorem 4.9 to compare our relations with Zakharevich's K_1 of an Assembler (Proposition 4.12). As a corollary, we prove that $\mathcal{D}(\mathcal{C}) \cong K_1(\mathcal{C})$, and so $\mathcal{D}(\mathcal{C})$ gives an alternative presentation of $K_1(\mathcal{C})$ for pCGW categories (Corollary 4.16).

We start with an informal overview. An *Assembler* is a Grothendieck site $\mathcal{A}$ whose topology encodes how an object A may be covered by a finite set of disjoint subobjects $\{A_i\}_{i\in I}$. In particular, given any Assembler $\mathcal{A}$, one can associate to it the category $\mathcal{W}(\mathcal{A})$ whereby

Objects: Finite sets of objects $\{A_i\}_{i\in I}$ in $\mathcal{A}$;
Morphisms: Piecewise automorphisms in $\mathcal{A}$. Explicitly, a morphism $f\colon \{A_i\}_{i\in I} \to \{B_j\}_{j\in J}$ is in $\mathcal{W}(\mathcal{A})$ is a tuple of morphisms $f_i\colon A_i \to B_{f(i)}$ such that $\{f_i\colon A_i \to B_j\}_{i\in f^{-1}(j)}$ is a finite disjoint covering family.

For more details see [Zak17b, §2], which introduces and develops the K-theory of Assemblers. Its relevance to our paper is that one can also define $K\mathcal{V}\mathrm{ar}_k$ via Assemblers, which was shown to be equivalent to the $K\mathcal{V}\mathrm{ar}_k$ defined via CGW categories by [CZ22, Theorems 7.8 and 9.1]. Extending Muro-Tonks' model of K_1 of a Waldhausen Category [MT08], Zakharevich proved the following.

Theorem 4.11 ([Zak17c, Theorem B]). *For any Assembler $\mathcal{A}$, $K_1(\mathcal{A})$ is generated by a pair of morphisms*

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

in $\mathcal{W}(\mathcal{A})$. These satisfy the relations

(Z1) $\langle A \underset{f}{\overset{f}{\rightrightarrows}} B\rangle = 0$*;*
(Z2) $\langle A \underset{f_2}{\overset{f_1}{\rightrightarrows}} B\rangle + \langle C \underset{g_2}{\overset{g_1}{\rightrightarrows}} D\rangle = \langle A \coprod C \underset{f_2\coprod g_2}{\overset{f_1\coprod g_1}{\rightrightarrows}} B \coprod D\rangle$*;*
(Z3) $\langle B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C\rangle + \langle A \underset{f_2}{\overset{f_1}{\rightrightarrows}} B\rangle = \langle A \underset{g_2f_2}{\overset{g_1f_1}{\rightrightarrows}} B\rangle$*.*

A couple remarks are in order. First, [Zak17c, Theorem B] leaves open the possibility that there may be more relations on K_1 to be identified. This incompleteness is inherited from Muro-Tonks' original model of K_1. Although it was shown in [MT08, Prop 6.3] that their model coincides with Nenashev's model for exact categories (and is thus complete), they were unable to show the same for general Waldhausen categories. Second, Relation (Z1) clearly corresponds to the diagonal relation (N1) of $\mathcal{D}(\mathcal{C})$. It remains to investigate Relations (Z2) and (Z3) in our context, as worked out below.

Proposition 4.12 (Assembler Relations). *Let $f\colon (A,A) \to (B,B)$ and $g\colon (C,C) \to (D,D)$ be two double exact squares given by*

$$f := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[d, circ, "f_2"] \\ A \arrow[r, tail, "f_1"] & B \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[d, circ, "f_2'"] \\ A \arrow[r, tail, "f_1'"] & B \end{tikzcd}\right) \qquad g := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & \frac{D}{C} \arrow[d, circ, "g_2"] \\ C \arrow[r, tail, "g_1"] & D \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, circ] \arrow[dr, phantom, "\square"] & \frac{D}{C} \arrow[d, circ, "g_2'"] \\ C \arrow[r, tail, "g_1'"] & D \end{tikzcd}\right).$$

Then the following relations hold in $K_1(\mathcal{C})$:

(A1) Formal Direct Sums. *$\langle f\rangle + \langle g\rangle = \langle f \oplus g\rangle$, where*

$$f \oplus g := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{B}{A} \oplus \frac{D}{C} \arrow[d, {Circle[open]}->, "f_2 \oplus g_2"] \\ A \oplus C \arrow[r, tail, "f_1 \oplus g_1"] & B \oplus D \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{B}{A} \oplus \frac{D}{C} \arrow[d, {Circle[open]}->, "f'_2 \oplus g'_2"] \\ A \oplus C \arrow[r, tail, "f'_1 \oplus g'_1"] & B \oplus D \end{tikzcd} \right) \tag{68}$$

(A2) Restricted Composition. *Suppose $(B, B) = (C, C)$. Then $\langle f\rangle + \langle g\rangle = \langle g \circ f\rangle + \langle l_2\rangle$, where l_2 is the induced double exact square*

$$l_2 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{B}{D} \arrow[d, {Circle[open]}->, "j_1"] \\ \frac{B}{A} \arrow[r, tail, "h_1"] & \frac{D}{A} \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{B}{D} \arrow[d, {Circle[open]}->, "j'_1"] \\ \frac{B}{A} \arrow[r, tail, "h'_1"] & \frac{D}{A} \end{tikzcd} \right).$$

and

$$\langle g \circ f\rangle := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{D}{A} \arrow[d, {Circle[open]}->] \\ A \arrow[r, tail, "g_1 f_1"] & D \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{D}{A} \arrow[d, {Circle[open]}->] \\ A \arrow[r, tail, "g'_1 f'_1"] & D \end{tikzcd} \right). \tag{69}$$

Remark 4.13. To avoid confusion, a quick remark on a mild abuse of notation. Notice e.g. in Equation (4.12) that we use $\frac{B}{A}$ to denote the quotient of B with respect to $f_1 \colon A \rightarrowtail B$ as well as $f_2 \colon A \rightarrowtail B$ even though f_1 and f_2 may be different. However, this is justified since a double exact square by definition is a pair of distinguished squares with identical nodes.

Proof. The argument proceeds by constructing the obvious 3×3 diagram, and applying Theorem 4.9 to perform our calculations.

(i): We claim that the following is a good 3×3 diagram:

$$\left(\begin{tikzcd} A \arrow[r, tail, "f_1"] \arrow[d, tail] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, tail] & \frac{B}{A} \arrow[l, {Circle[open]}->, "f_2"'] \arrow[d, tail] \\ A \oplus C \arrow[r, tail, "f_1 \oplus g_1"] & B \oplus D \arrow[dr, phantom, "\circlearrowleft"] & \frac{B}{A} \oplus \frac{D}{C} \arrow[l, {Circle[open]}->, "f_2 \oplus g_2"'] \\ C \arrow[r, tail, "g_1"] \arrow[u, {Circle[open]}->] & D \arrow[u, {Circle[open]}->] & \frac{D}{C} \arrow[l, {Circle[open]}->, "g_2"'] \arrow[u, {Circle[open]}->] \end{tikzcd} \quad , \quad \begin{tikzcd} A \arrow[r, tail, "f'_1"] \arrow[d, tail] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, tail] & \frac{B}{A} \arrow[l, {Circle[open]}->, "f'_2"'] \arrow[d, tail] \\ A \oplus C \arrow[r, tail, "f'_1 \oplus g'_1"] & B \oplus D \arrow[dr, phantom, "\circlearrowleft"] & \frac{B}{A} \oplus \frac{D}{C} \arrow[l, {Circle[open]}->, "f'_2 \oplus g'_2"'] \\ C \arrow[r, tail, "g'_1"] \arrow[u, {Circle[open]}->] & D \arrow[u, {Circle[open]}->] & \frac{D}{C} \arrow[l, {Circle[open]}->, "g'_2"'] \arrow[u, {Circle[open]}->] \end{tikzcd} \right) \tag{70}$$

The vertical columns define exact squares arising from formal direct sums. The top and bottom rows correspond to the exact squares from f and g. The middle rows correspond to $f \oplus g$; the fact this indeed defines a double exact square follows from Lemma A.1. One easily checks that the top left and bottom right squares are pullback squares in $\mathcal{M}$ and $\mathcal{E}$ respectively, and thus they commute. Therefore, we have a 3×3 diagram.

It remains to check goodness. Taking repeated restricted pushouts, Fact 1.17 yields

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, tail] & A \arrow[r, tail, "f_1"] \arrow[d, tail] & B \arrow[d, tail] \\ C \arrow[r, tail] \arrow[d, tail, "g_1"] & A \oplus C \arrow[r, tail] \arrow[d, tail] & B \oplus C \arrow[d, tail, "v"] \\ D \arrow[r, tail] & A \oplus D \arrow[r, tail] & B \oplus D \end{tikzcd} \qquad \begin{tikzcd} O \arrow[r, tail] \arrow[d, tail] & A \arrow[r, tail, "f'_1"] \arrow[d, tail] & B \arrow[d, tail] \\ C \arrow[r, tail] \arrow[d, tail, "g'_1"] & A \oplus C \arrow[r, tail] \arrow[d, tail] & B \oplus C \arrow[d, tail, "v'"] \\ D \arrow[r, tail] & A \oplus D \arrow[r, tail] & B \oplus D \end{tikzcd} . \tag{71}$$

Since we now have $\mathcal{M}$-morphisms $v, v' \colon B \oplus C \rightarrowtail B \oplus D$ (unlike in Discussion 4.7), we can construct the required diagrams:

(P1):

$$\begin{tikzcd}
B \arrow[r, tail] & B \oplus C \arrow[r, tail, "v"] & B \oplus D \\
O \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & C \arrow[u] \arrow[r, tail, "g_1"] \arrow[ur, phantom, "\square"] & D \arrow[u] \\
& O \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{D}{C} \arrow[u, "g_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, tail] & B \oplus C \arrow[r, tail, "v'"] & B \oplus D \\
O \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & C \arrow[u] \arrow[r, tail, "g_1'"] \arrow[ur, phantom, "\square"] & D \arrow[u] \\
& O \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{D}{C} \arrow[u, "g_2'"']
\end{tikzcd}$$

[The top left and bottom right squares are obviously distinguished. The top right square is distinguished by Lemma A.3.]

(P2):

$$\begin{tikzcd}
A \oplus C \arrow[r, tail] & B \oplus C \arrow[r, tail, "v"] & B \oplus D \\
O \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{B}{A} \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{B}{A} \oplus \frac{D}{C} \arrow[u] \\
& O \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{D}{C} \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \oplus C \arrow[r, tail] & B \oplus C \arrow[r, tail, "v'"] & B \oplus D \\
O \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{B}{A} \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{B}{A} \oplus \frac{D}{C} \arrow[u] \\
& O \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{D}{C} \arrow[u]
\end{tikzcd}$$

[The bottom right square is a formal direct sum square, and thus distinguished. The top left square is distinguished by applying Axiom (PQ) to the restricted pushout of $A \oplus C \leftarrowtail A \rightarrowtail B$. The top right square is the vertical composition of two distinguished squares

$$\begin{tikzcd}
B \oplus C \arrow[r, tail] & B \oplus D \\
B \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & B \oplus \frac{D}{C} \arrow[u] \\
\frac{B}{A} \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{B}{A} \oplus \frac{D}{C} \arrow[u]
\end{tikzcd} \quad ; \tag{72}$$

the fact that these two squares are indeed distinguished follows from Lemma A.3.]

(P3):

$$\begin{tikzcd}
A \arrow[r, tail, "f_1"] & B \arrow[r, tail] & B \oplus C \\
& \frac{B}{A} \arrow[u, "f_2"] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{B}{A} \oplus C \arrow[u] \\
& & C \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, tail, "f_1'"] & B \arrow[r, tail] & B \oplus C \\
& \frac{B}{A} \arrow[u, "f_2'"] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{B}{A} \oplus C \arrow[u] \\
& & C \arrow[u]
\end{tikzcd}$$

[The indicated square is distinguished by Lemma A.3.]

(P4):

$$\begin{tikzcd}
A \arrow[r, tail] & A \oplus C \arrow[r, tail] & B \oplus C \\
& C \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{B}{A} \oplus C \arrow[u] \\
& & \frac{B}{A} \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, tail] & A \oplus C \arrow[r, tail] & B \oplus C \\
& C \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{B}{A} \oplus C \arrow[u] \\
& & \frac{B}{A} \arrow[u]
\end{tikzcd}$$

[The indicated square is distinguished by Lemma A.3.]

To prove the relation, let us review Diagram (70). Denote the double exact squares corresponding to the vertical columns as l^0, l^1 and l^2, from left to right. Since Diagram (70) is a good 3×3 diagram, apply Relation (N2) to get

$$\langle f\rangle + \langle g\rangle - \langle f \oplus g\rangle = \langle l^0\rangle + \langle l^2\rangle - \langle l^1\rangle.$$

Further, since l^0, l^1 and l^2 are all identical pairs of formal direct sum squares, Relation (N1) gives

$$\langle l^0\rangle = \langle l^1\rangle = \langle l^2\rangle = 0,$$

and so conclude

$$\langle f\rangle + \langle g\rangle = \langle f \oplus g\rangle.$$

(ii): Given $(B, B) = (C, C)$, construct the following diagram

$$\left(\begin{tikzcd} A \arrow[r, tail, "="] \arrow[d, tail, "f_1"'] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d, tail, "g_1 f_1"] & O \arrow[l] \arrow[d, tail] \\ B \arrow[r, tail, "g_1"] & D & \frac{D}{B} \arrow[l, "g_2"'] \\ \frac{B}{A} \arrow[u, "f_2"] \arrow[r, tail, "h_1"] & \frac{D}{A} \arrow[u] \arrow[ur, phantom, "\circlearrowleft"] & \frac{D}{B} \arrow[l, "j_1"] \arrow[u, "="'] \end{tikzcd} , \quad \begin{tikzcd} A \arrow[r, tail, "="] \arrow[d, tail, "f_1'"'] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d, tail, "g_1' f_1'"] & O \arrow[l] \arrow[d, tail] \\ B \arrow[r, tail, "g_1'"] & D & \frac{D}{B} \arrow[l, "g_2'"'] \\ \frac{B}{A} \arrow[u, "f_2'"] \arrow[r, tail, "h_1'"] & \frac{D}{A} \arrow[u] \arrow[ur, phantom, "\circlearrowleft"] & \frac{D}{B} \arrow[l, "j_1'"] \arrow[u, "="'] \end{tikzcd} \right). \tag{73}$$

Following the convention from Definition 4.8, label the horizontal rows as l_0, l_1 and l_2 and the vertical columns as l^0, l^1 and l^2. In particular, we have $l_1 = g$, $l^0 = f$ and $l^1 = g \circ f$. The fact that l_2 defines a double exact square comes from Lemma 1.7 ("quotients respect filtrations"); the remaining rows are obvious. It is also clear the top left and bottom right squares are pullback squares in $\mathcal{M}$ and $\mathcal{E}$ respectively, and therefore commute. Finally, take the restricted pushout of $A \overset{=}{\hookleftarrow} A \overset{f_1}{\rightarrowtail} B$. Since we have an $\mathcal{M}$-morphism $g_1 \colon A \star_A B = B \to D$, the same argument as part (i) shows that Diagram (73) is in fact a good 3×3 diagram.

Now apply Relation (N2) to get

$$\langle l_0\rangle + \langle l_2\rangle - \langle g\rangle = \langle f\rangle + \langle l^2\rangle - \langle g \circ f\rangle.$$

Since l_0 and l^2 are diagonal, deduce from (N1) that $\langle l_0\rangle = \langle l^2\rangle = 0$, and thus conclude that

$$\langle g \circ f\rangle + \langle l_2\rangle = \langle f\rangle + \langle g\rangle.$$

□

Discussion 4.14 (Restricted Composition)**.** Relation (A2) improves on Proposition 4.1 by determining what happens to *any* admissible triple in K_1 (Definition 4.3). This gives a clearer answer to the issue raised in Warning 4.2. Namely, given an admissible triple $\tau = (f, g, g \circ f)$, we now know that

$$\langle g \circ f\rangle + \langle l_2\rangle = \langle f\rangle + \langle g\rangle \qquad \text{in } K_1.$$

Interestingly, this obstruction $\langle l_2\rangle$ does not appear in Theorem 4.11, where composition of piecewise automorphisms always split in K_1. Of course, if $\langle l_2\rangle = 0$ then Relations (A2) and (Z3) coincide, but it is unclear if this holds in general.

We suspect this discrepancy is due to [Zak17c, Theorem 2.1], a key ingredient in proving Theorem 4.11. Roughly speaking, this result states that one can model the K-theory of Assemblers via Waldhausen categories whose cofibration sequences all split (up to weak equivalence). Here's an interesing clue: Lemma B.3 tells us that if $\mathcal{C}$ is a pCGW category whose exact squares are all split, then $K_1(\mathcal{C})$ is generated by automorphisms. In which case, it is straightforward to check that composition splits in K_1 in the manner of $(Z3)$.

The problem is that not all exact squares split in $\mathcal{V}\mathrm{ar}_k$ – consider, for instance

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \mathbb{A}^1 \arrow[d, hook] \\ \{*\} \arrow[r, tail] & \mathbb{P}^1 \end{tikzcd}.$$

Remark 4.15. If it turns out [Zak17c, Theorem 2.1] does not apply to $\mathcal{V}\mathrm{ar}_k$ (as discussed above), then this impacts the proof of [CZ22, Theorem 9.1], which shows that the K-theory spectrum of varieties via Assemblers and CGW categories are equivalent.

We end with one final surprise. In order to show that $\mathcal{D}(\mathcal{C}) \cong K_1(\mathcal{C})$ for exact categories, Nenashev [Nen98a] constructs a homomorphism

$$b\colon K_1(\mathcal{C}) \to \mathcal{D}(\mathcal{C})$$

and shows that it is inverse to the map $m\colon \mathcal{D}(\mathcal{C}) \to K_1(\mathcal{C})$ from Equation (61). Notice the naive map sending $\langle f\rangle \to \langle f\rangle$ is *a priori* not well-defined since $K_1(\mathcal{C})$ may have more relations than $\mathcal{D}(\mathcal{C})$. Indeed, the original construction of b in [Nen98a] is intricate, and a fair bit of technical legwork is required to show it defines a well-defined homomorphism. In our case, however, Propositions 4.1 and 4.12 combine to give a shorter direct proof.

Corollary 4.16. *Given any pCGW category $\mathcal{C}$, there is an ismorphism*

$$\mathcal{D}(\mathcal{C}) \cong K_1(\mathcal{C}).$$

Proof. We show the naive map $b\colon K_1(\mathcal{C}) \to \mathcal{D}(\mathcal{C})$ sending $\langle f\rangle \to \langle f\rangle$ is in fact well-defined. Any equivalence $\langle f\rangle = \langle g\rangle$ in $K_1(\mathcal{C})$ must be generated by the relations of Proposition 4.1. It therefore suffices to check that $\mathcal{D}(\mathcal{C})$ satisfies those relations as well. Relations (B1) and (B2) are diagonal relations, and follow immediately from Relation (N1). Relation (B3) is a special case of Restricted Composition (A2) in Proposition 4.12, where $\langle l_2\rangle = 0$ since l_2 is diagonal by assumption. And we are done. $\square$

5. Some Test Problems

This paper began with Question 1: what information do the higher K-groups of varieties encode? Progress on this question requires advances on two fronts: developing the framework of non-additive K-theory on the one hand, and concrete applications of the newly-developed (K-theory) tools on the other. This paper is of the first kind, with a view towards laying the groundwork for future theorems. Let us therefore conclude with some test problems and discussion.

5.1. Non-Additive K-theory.

Having characterised K_1, the natural next step is the following problem.

Problem 5.1. Characterise $K_n(\mathcal{C})$ for $n > 1$ for pCGW categories.

Discussion 5.2. In a substantial generalisation of Nenashev's work, Grayson gave a complete characterisation of K_n for all n in the setting of exact categories [Gra12]. Encouraged by the results in this paper, the natural proof strategy would be to extend Grayson's argument to the pCGW setting.

However, there is an obvious barrier. Grayson characterises the K-groups of exact categories via binary chain complexes, and so invokes the Gillet-Waldhausen Theorem. Although an analogue of this result[37] has been shown for FinSet [SS21], a Gillet-Waldhausen Theorem for $\mathcal{V}\mathrm{ar}_k$ has not been worked out yet. Nonetheless, even working out the case of FinSet is interesting. By Barratt-Priddy-Quillen, the K-groups of FinSet correspond to the stable homotopy groups of spheres, whose complete description remains a longstanding open problem in homotopy theory. What might this presentation of $K_n(\mathrm{Finset})$ tell us about the stable homotopy groups of spheres? About the J-homomorphism and Adams e-invariant?

In light of the comparisons with Zakharevich's K_1, perhaps a more urgent question is the following.

Problem 5.3. Is it always true that $\langle l_2\rangle = 0$ for Relation (A2) in Proposition 4.12?

[37] In fact, the authors in [SS21] establish a Gillet-Waldhausen Theorem for extensive categories, not just FinSet.

Discussion 5.4. We suspect no, although we have yet to construct a counter-example. The difficulty is that we do not have a complete characterisation of double exact squares that trivialise in K_1. In particular, there are non-diagonal double exact squares that trivialise, e.g.

$$l_\tau := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\square"] & A \oplus A \arrow[d, hook, "\tau"] \\ O \arrow[r, tail] & A \oplus A \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\square"] & A \oplus A \arrow[d, hook, "1"] \\ O \arrow[r, tail] & A \oplus A \end{tikzcd} \right)$$

where $\tau\colon A \oplus A \xrightarrow{\sim} A \oplus A$ is the *twist automorphism* that swaps components.

Given Discussion 4.14, it is also worth revisiting the proof of [Zak17c, Theorem 2.1] and [CZ22, Theorem 9.1], which defines comparison maps from the Assembler K-theory of $\mathcal{V}\mathrm{ar}_k$ to the K-theory of Waldhausen Categories (whose cofibration sequences all split) and the K-theory of CGW categories respectively. What happens to the sequence $\{*\} \hookrightarrow \mathbb{P}^1 \hookleftarrow \mathbb{A}^1$ under these comparison maps?

Another natural question, posed to us by E. Dotto, is the following.

Problem 5.5. Does the K-theory of pCGW categories commute with infinite products?

Discussion 5.6. Relevantly: Zakharevich conjectures in [Zak22, Remark 2.4] that the K-theory of Assemblers commutes with infinite products. However, she notes that such a result seems presently out of reach since previous results of this form were worked out for Waldhausen categories with cylinder functors (which Assemblers do not have) and exact categories.

5.2. **The Motivic Euler Characteristic.** There is a well-known enrichment of the Euler Characteristic, known as the *motivic Euler Characteristic* or *compactly supported* $\mathbb{A}^1$*-Euler Characteristic*, which is a ring homomorphism

$$\chi^{\mathrm{mot}}\colon K_0(\mathcal{V}\mathrm{ar}_k) \to \mathrm{GW}(k)$$

where $GW(k)$ denotes the Grothendieck-Witt ring of quadratic forms over field k. It is natural to ask if one can lift this to the level of K-theory spectra, which was proved in the affirmative by Nanavaty [Nan24, Theorem 1.1]. Explicitly, he constructs a map of spectra

$$K\mathcal{V}\mathrm{ar}_k \to \mathrm{End}(\mathbb{1}_k)$$

where $\mathrm{End}(\mathbb{1}_k)$ is the *Endomorphism Spectrum of the unit object in the motivic stable homotopy category*, recovering χ^{mot} on π_0. This sets up the problem:

Problem 5.7. Define a natural map $K_1(\mathcal{V}\mathrm{ar}_k) \to \pi_{1,0}(\mathbb{1}_k)$. What geometric information does it encode?

Discussion 5.8. The homotopy groups of $\mathrm{End}(\mathbb{1}_k)$ are defined as $\pi_{*,0}$. A foundational result, due to Morel[38], shows that $\pi_{0,0}(\mathbb{1}_k) \cong \mathrm{GW}(k)$, so one may regard the higher homotopy groups as defining higher Grothendieck-Witt Groups. What geometric information is detected at the higher levels? Recent work by [RSØ19] tells us

$$0 \to K_2^M(k)/24 \to \pi_{1,0}(\mathbb{1}_k) \to k^\times/2 \oplus \mathbb{Z}/2 \to 0. \tag{74}$$

where $K_*^M(k)$ denotes the Milnor K-theory of k. Combined with Theorems B and/or C, we now have an explicit description of both groups in Problem 5.7. It remains to determine a natural way of mapping the double exact squares in $\mathcal{V}\mathrm{ar}_k$ to $\pi_{1,0}(\mathbb{1}_k)$, but it is not clear, e.g. how these generalised automorphisms ought to interact with the Milnor K-theory term, and what this means geometrically.

Note: the canonical unit map $\mathrm{End}(\mathbb{1}_k) \to KQ$ (i.e. the Hermitian K-theory spectrum) induces an isomorphism on the level of π_0. We may therefore also think of χ^{mot} as a map on π_0 of $K\mathcal{V}\mathrm{ar}_k \to KQ$, which may be more tractable than working in the setting of motivic homotopy theory. For those interested in the exterior powers of $K_0(\mathcal{V}\mathrm{ar}_k)$,[39], a natural question may be:

[38] **(M:) Cite.**

[39] **(M:) Cite Pajwani, Pal.**

Problem 5.9. Lift the symmetric power structure on $K_0(\mathcal{V}\mathrm{ar}_k)$ to the level of spectra on $K\mathcal{V}\mathrm{ar}_k$. If we regard χ^{mot} as a map

$$\chi^{\mathrm{mot}}\colon \pi_0(K\mathcal{V}\mathrm{ar}_k) \to \pi_0(KQ),$$

can we deduce the compatibility of χ^{mot} with the symmetric power structures on the level of π_0 from formal properties on the level of K-theory spectra?

Discussion 5.10. A helpful first clue: Grayson [Gra92] relies on the G-construction on exact categories to provide an explicit combinatorial description of the Adams Operations on higher K-groups induced by symmetric powers on the exact category. Since we know the G-construction behaves as expected on $\mathcal{V}\mathrm{ar}_k$ by Theorem A, this tells us where to start. In addition, lifting the symmetric power structure on $K_0(\mathcal{V}\mathrm{ar}_k)$ ought to advance our understanding towards lifting Kapranov's motivic zeta function to a map of K-theory spectra – see e.g. [CZ22, Question 7.3].

5.3. **Other Non-Additive Settings.** Theorem 3.18 showed that $K_1(\mathcal{C})$ of a pCGW category is generated by double exact squares, which are generalisations of automorphisms (see Example 3.17). For suggetiveness, call the generators of K_1 *quasi-automorphisms*. Automorphisms play a key role in many different areas of mathematics – can this picture be extended to quasi-automorphisms in a productive way?

For the model theorist, the automorphism group $\mathrm{Aut}(M)$ of a countable first-order structure M encodes important information about M. One obvious example is that $\mathrm{Aut}(M)$ measures the homogeneity of M, but there are many others [MK94, Eva97]. This suggests the following general problem:

Problem 5.11. Generalise previous model-theoretic analyses on $\mathrm{Aut}(M)$ to $K_1(M)$ – e.g. by extending *homogeneity* in the obvious way, or perhaps by defining the quasi-automorphism group of field extensions before building a new kind of Galois Theory, or perhaps simply examining $K_1(M)$ for some interesting choice of M etc. What new information do these new definitions or frameworks calibrate about M that was previously inaccessible?

Remark 5.12. Thinking about o-minimal structures may be a good warmup problem. It would also be interesting to see if the cell decomposition theorem has a productive translation to K-theory.

It is also worth revisiting the original papers [Kra00, KS00] where the Grothendieck ring of Definable Sets $K_0(\mathrm{Def})$ was first investigated. In particular, [KS00] introduces the so-called strong and weak Euler Characteristics on first-order structures before asking which fields admit a non-trivial strong Euler Characteristic. In light of our present work, a natural problem may be:

Problem 5.13. Lift the weak/strong Euler Characteristic on first-order structures to the level of spectra. Analyse what happens in K_1 – what information does it detect? Are there examples of fields with strong Euler characteristics that are trivial on K_0 but non-trivial on K_1? Might K_1 highlight relevant combinatorial features of definable sets that force a strong Euler characteristic to be trivial?

One can pose similar questions to the matroid theorists. To our knowledge, the K-theory spectrum of matroids was first defined in [EJS20], where it was also showed that the higher K-groups are non-trivial (Theorem 6.4 of the paper). The following problems are natural.

Problem 5.14. Compute $K_1(\mathrm{Mat}_\bullet)$. What aspect of matroids does K_1 measure? Extending the perspective from algebraic K-theory, one can view K_1 as a generalised determinant – what does this perspective tell us about the abstract independence relations modelled by matroids?

Problem 5.15. Lift the Tutte Polynomial

$$T\colon K_0(\mathrm{Mat}_\bullet) \to \mathbb{Z}[x, y]$$

to the level of K-theory spectra. What information is encoded on the level of K_1?

Appendix A. Technical Lemmas for understanding the G-Construction

A.1. **Properties of Restricted Pushouts.** Lemma A.1 justifies Claim 2.4 that the addition map $+\colon Y \times Y \to Y$ is well-defined.

Lemma A.1.

(i) Given a span $B \leftarrowtail A \rightarrowtail C$,

$$\frac{B}{A} \oplus \frac{C}{A} \cong \frac{B \star_A C}{A}.$$

(ii) Given a span $B \leftarrowtail A \rightarrowtail C$, along with $\mathcal{M}$-morphisms $B \rightarrowtail B'$ and $C \rightarrowtail C'$,

$$\frac{B' \star_A C'}{B \star_A C} \cong \frac{B'}{B} \oplus \frac{C'}{C}.$$

Proof.

(i): Consider the diagram

$$\begin{tikzcd}
\frac{C}{A} \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & O \arrow[l, tail] \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[d, {Circle[open]}->] \\
C & A \arrow[l, tail] \arrow[r, tail] & B
\end{tikzcd}$$

Apply Axiom (DS) to obtain the left diagram below

$$\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{B}{A} \oplus \frac{C}{A} \arrow[d, {Circle[open]}->] \\
A \arrow[r, tail, "f"] & B \arrow[r, tail, "f'"] & B \star_A C
\end{tikzcd}
\qquad
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{B \star_A C}{A} \arrow[d, {Circle[open]}->] \\
A \arrow[r, tail, "f'f"] & B \star_A C
\end{tikzcd} \tag{75}$$

Since distinguished squares compose, the outermost rectangle of the left diagram also defines a distinguished square. On the other hand, by Axiom (K), the right distinguished square above exists. Since formal cokernels are unique (up to unique isomorphism), conclude that $\frac{B}{A} \oplus \frac{C}{A} \cong \frac{B \star_A C}{A}$.

(ii): Repeated applications of Fact 1.17 yields

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, tail] & B \arrow[r, tail] \arrow[d, tail] & B' \arrow[d, tail] \\
C \arrow[r, tail] \arrow[d, tail] & B \star_A C \arrow[r, tail] \arrow[d, tail] & B' \star_A C \arrow[d, tail] \\
C' \arrow[r, tail] & B \star_A C' \arrow[r, tail] & B' \star_A C'
\end{tikzcd} \tag{76}$$

In particular, there exists an $\mathcal{M}$-morphism $B \star_A C \rightarrowtail B' \star_A C'$, so $\frac{B' \star_A C'}{B \star_A C}$ is well-defined and exists by Axiom (K). By Axiom (PQ), restricted pushouts preserve quotients, and so

$$\frac{B'}{B} \cong \frac{B' \star_A C}{B \star_A C} \quad \text{and} \quad \frac{C'}{C} \cong \frac{B \star_A C'}{B \star_A C}. \tag{77}$$

Now consider the diagram

$$\begin{tikzcd}
\frac{B \star_A C'}{B \star_A C} \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & O \arrow[l, tail] \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{B' \star_A C}{B \star_A C} \arrow[d, {Circle[open]}->] \\
B \star_A C' & B \star_A C \arrow[l, tail] \arrow[r, tail] & B' \star_A C
\end{tikzcd} \tag{78}$$

Applying the same argument from (i) to Equations (77) and (78), deduce the desired isomorphism. □

Lemma A.2. *Suppose $A \rightarrowtail B$. Then $\frac{B \oplus C}{A} \cong \frac{B}{A} \oplus C$, for any $C \in \mathcal{C}$.*

Proof. Apply Fact 1.17 to the diagram $C \leftarrowtail O \rightarrowtail A \rightarrowtail B$, deduce that $B \star_A (A \oplus C) \cong B \oplus C$. Next, apply Axiom (DS) to the diagram

$$\begin{tikzcd}
C \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & O \arrow[l, tail] \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[d, {Circle[open]}->] \\
A \oplus C & A \arrow[l, tail] \arrow[r, tail] & B
\end{tikzcd} \tag{79}$$

and obtain

$$
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{B}{A} \oplus C \arrow[d, {Circle[open]}->] \\
A \arrow[r, tail, "f"] & B \oplus C
\end{tikzcd}
\tag{80}
$$

as a distinguished square. Applying Axiom (K) to $f\colon A \rightarrowtail B \oplus C$, conclude that $\frac{B \oplus C}{A} \cong \frac{B}{A} \oplus C$.[40] $\square$

Lemma A.3. *Given any distinguished square in pCGW category $\mathcal{C}$*

$$
\phi := \left(
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \arrow[d, {Circle[open]}->] \\
A \arrow[r, tail] & B
\end{tikzcd}
\right)
\tag{81}
$$

the following squares are also distinguished

(i) For any $D \in \mathcal{C}$:

$$
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \oplus D \arrow[d, {Circle[open]}->] \\
A \arrow[r, tail] & B \oplus D
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \arrow[d, {Circle[open]}->] \\
A \oplus D \arrow[r, tail] & B \oplus D
\end{tikzcd}
\; ;
\tag{82}
$$

$$
\begin{tikzcd}
C \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \oplus D \arrow[d, {Circle[open]}->] \\
B \arrow[r, tail] & B \oplus D
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
D \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \oplus D \arrow[d, {Circle[open]}->] \\
A \oplus D \arrow[r, tail] & B \oplus D
\end{tikzcd}
\; .
\tag{83}
$$

(ii) Given any $A' \rightarrowtail B'$:

$$
\begin{tikzcd}
C \oplus A' \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \oplus B' \arrow[d, {Circle[open]}->] \\
B \oplus A' \arrow[r, tail] & B \oplus B'
\end{tikzcd}
\tag{84}
$$

Proof. (i): First, apply Fact 1.17 to

$$D \leftarrowtail O \rightarrowtail A \rightarrowtail B,$$

and obtain the isomorphism $B \oplus D \cong (A \oplus D) \star_A B$. Then apply Axiom (DS) of Definition 1.18 to the diagram

$$
\begin{tikzcd}
D \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & O \arrow[l, tail] \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \arrow[d, {Circle[open]}->] \\
A \oplus D & A \arrow[l, tail] \arrow[r, tail] & B
\end{tikzcd}
$$

to obtain the following distinguished squares

$$
\psi := \left(
\begin{tikzcd}
C \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \oplus D \arrow[d, {Circle[open]}->] \\
B \arrow[r, tail] & B \oplus D
\end{tikzcd}
\right)
\qquad
\psi' := \left(
\begin{tikzcd}
D \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & C \oplus D \arrow[d, {Circle[open]}->] \\
A \oplus D \arrow[r, tail] & B \oplus D
\end{tikzcd}
\right).
$$

Since distinguished squares are closed under composition, obtain Equation (82) by horizontal composition of ψ with the original distinguished square ϕ, and vertical composition of ψ' with the formal

[40] **(M:) There may be multiple different ways of including into $A \rightarrowtail B \oplus C$, the same way that there are different ways of including $\mathbb{Z} \rightarrowtail \mathbb{Z}$. So just because you have two short exact sequences with $A \to B \to C$ and $A \to B \to C'$, you cannot conclude that $C \cong C'$. But here we fix the same $A \to B$.**

sum square

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, very near start}] \arrow[dr, phantom, "\square"] & C \arrow[d, "\circ"{marking, very near start}] \\ D \arrow[r, tail] & C \oplus D \end{tikzcd} .$$

(ii): The argument is analogous to the proof of [CZ22, Lemma 2.9]. By item (i), the following squares are distinguished

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, very near start}] \arrow[dr, phantom, "\square"] & C \arrow[r, tail] \arrow[d, "\circ"{marking, very near start}] \arrow[dr, phantom, "\square"] & C \oplus A' \arrow[d, "\circ"{marking, very near start}] \\ A \arrow[r, tail] & B \arrow[r, tail] & B \oplus A' \end{tikzcd} \qquad \begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, very near start}] \arrow[dr, phantom, "\square"] & \frac{B'}{A'} \arrow[d, "\circ"{marking, very near start}] \\ B \oplus A' \arrow[r, tail] & B \oplus B' \end{tikzcd} ,$$

and so we may construct the following diagram

$$\begin{tikzcd} C \oplus A' \arrow[d, "\circ"{marking, very near start}] & \\ B \oplus A' \arrow[r, tail] & B \oplus B' \end{tikzcd} \tag{85}$$

and $\mathcal{M}$-morphism

$$v := \Big(A \rightarrowtail B \oplus A' \rightarrowtail B \oplus B' \Big). \tag{86}$$

Since v factors through $A \rightarrowtail B$ by construction, we can apply Lemma A.2 to obtain the distinguished square

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, very near start}] \arrow[dr, phantom, "\square"] & C \oplus B' \arrow[d, "\circ"{marking, very near start}] \\ A \arrow[r, tail, "v"] & B \oplus B' \end{tikzcd} . \tag{87}$$

Notice v represents a morphism

$$(A \rightarrowtail B \oplus A') \rightarrowtail (A \rightarrowtail B \oplus B') \in \mathrm{Ar}_{\Delta}\mathcal{M}.$$

Applying k^{-1} to this morphism produces the desired distinguished square

$$\begin{tikzcd} C \oplus A' \arrow[r, tail] \arrow[d, "\circ"{marking, very near start}] \arrow[dr, phantom, "\square"] & C \oplus B' \arrow[d, "\circ"{marking, very near start}] \\ B \oplus A' \arrow[r, tail] & B \oplus B' \end{tikzcd} ,$$

where we use the fact that c and k are inverse on objects.

$\square$

Lemma A.4 (Direct Sums)**.** *Given any pair of exact squares*

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, very near start}] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[d, "\circ"{marking, very near start}, "f_2"] \\ A \arrow[r, tail, "f_1"] & B \end{tikzcd} , \qquad \begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, very near start}] \arrow[dr, phantom, "\square"] & \frac{D}{C} \arrow[d, "\circ"{marking, very near start}, "g_2"] \\ C \arrow[r, tail, "g_1"] & D \end{tikzcd}$$

we can construct the obvious exact square via direct sums

$$\left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, very near start}] \arrow[dr, phantom, "\square"] & \frac{B}{A} \oplus \frac{D}{C} \arrow[d, "\circ"{marking, very near start}, "f_2 \oplus g_2"] \\ A \oplus C \arrow[r, tail, "f_1 \oplus g_1"] & B \oplus D \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, "\circ"{marking, very near start}] \arrow[dr, phantom, "\square"] & \frac{B}{A} \oplus \frac{D}{C} \arrow[d, "\circ"{marking, very near start}, "f_2' \oplus g_2'"] \\ A \oplus C \arrow[r, tail, "f_1' \oplus g_1'"] & B \oplus D \end{tikzcd} \right).$$

Proof. Take repeated restricted pushouts to get

$$
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, tail] & C \arrow[r, tail, "g_1"] \arrow[d, tail] & D \arrow[d, tail] \\
A \arrow[r, tail] \arrow[d, tail, "f_1"] & A \oplus C \arrow[r, tail, "1\oplus g_1"] \arrow[d, tail, "f_1\oplus 1"] & A \oplus D \arrow[d, tail, "f_1\oplus 1"] \\
B \arrow[r, tail] & B \oplus C \arrow[r, tail, "1\oplus g_1"'] & B \oplus D
\end{tikzcd}
\tag{88}
$$

Since restricted pushouts preserve quotients, construct the diagram

$$
\begin{tikzcd}
\frac{D}{C} \arrow[d, {Circle[open]}->, "g_2"'] \arrow[dr, phantom, "\square"] & O \arrow[l, tail] \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{B}{A} \arrow[d, {Circle[open]}->, "f_2"] \\
A \oplus D & A \oplus C \arrow[l, tail, "1\oplus g_1"] \arrow[r, tail, "1\oplus f_1"'] & B \oplus C
\end{tikzcd}
.
$$

Apply Axiom (DS) and the fact that distinguished squares compose horizontally to get

$$
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{B}{A} \oplus \frac{D}{C} \arrow[d, {Circle[open]}->, "f_2\oplus g_2"] \\
A \oplus C \arrow[r, tail, "f_1\oplus g_1"] & B \oplus D
\end{tikzcd}
.
$$

□

A.2. **Constructing a 1-Simplex for Claim 2.9.** (M:) **Quick restatement of problem; we have two generators $\pi_0(\overline{M}|F)$, they may differ in terms of their quotients. Show that choice of quotient does matter. Justify notation.**

[*Details.* Since $M_a \rightarrowtail N_0 = M_a \rightarrowtail N_1$, this in particular implies $N_0 = N_1$ and so there exists an identity map $1\colon N_0 \rightarrowtail N_1$. This notation was chosen to help distinguish their respective choice of quotients. Now define a 1-simplex $W \to W'$ by constructing the obvious $a+2$-simplex in $\mathcal{SC}$:

$$
\begin{tikzcd}
O = M_0 \arrow[r, tail] & M_1 \arrow[r, tail] & M_2 \arrow[r, tail] & \dots \arrow[r, tail] & M_a \arrow[r, tail] & N_0 \arrow[r, tail, "1"] \arrow[dr, phantom, "\square"] & N_1 \\
& M_{1/0} \arrow[u, {Circle[open]}->] \arrow[r, tail] & M_{2/0} \arrow[u, {Circle[open]}->] \arrow[r, tail] & \dots \arrow[r, tail] & M_{a/0} \arrow[u, {Circle[open]}->] \arrow[r, tail, "f_0"] & \frac{N_0}{M_0} \arrow[u, {Circle[open]}->, "g_0"] \arrow[r, tail, "\psi_0"] \arrow[dr, phantom, "\square"] & \frac{N_1}{M_0} \arrow[u, {Circle[open]}->, "g_0'"'] \\
& & M_{2/1} \arrow[u, {Circle[open]}->] \arrow[r, tail] & \dots \arrow[r, tail] & M_{a/1} \arrow[u, {Circle[open]}->] \arrow[r, tail, "f_1"] & \frac{N_0}{M_1} \arrow[u, {Circle[open]}->, "g_1"] \arrow[r, tail, "\psi_1"] & \frac{N_1}{M_1} \arrow[u, {Circle[open]}->, "g_1'"'] \\
& & & \vdots \arrow[u, {Circle[open]}->] & \vdots \arrow[u, {Circle[open]}->] & \vdots \arrow[u, {Circle[open]}->] & \vdots \arrow[u, {Circle[open]}->] \\
& & & & & \frac{N_0}{M_a} \arrow[u, {Circle[open]}->] \arrow[r, tail, "\psi_n"] & \frac{N_1}{M_a} \arrow[u, {Circle[open]}->] \\
& & & & & O \arrow[u, {Circle[open]}->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & O \arrow[u, {Circle[open]}->]
\end{tikzcd}
\tag{89}
$$

where the ψ_i are the relevant isomorphisms from Axiom (K). To verify that Diagram (89) is an $a+2$-simplex, it suffices to check the indicated squares in the rightmost column are distinguished. Let us examine this in detail.

- **Top and Bottom Squares.** The top square is distinguished by definition of ψ_0. The bottom square is also obviously distinguished.

- **Intermediate Squares.** Consider the square

$$\begin{tikzcd} \frac{N_0}{M_1} \arrow[r, tail, "\psi_1"] \arrow[d, "g_1"] & \frac{N_1}{M_i} \arrow[d, "g_1'"] \\ \frac{N_0}{M_0} \arrow[r, tail, "\psi_0"] & \frac{N_1}{M_0}. \end{tikzcd} \tag{90}$$

  - *Case 1:* $\mathcal{E} \subseteq \mathcal{C}$. By Axiom (K) and Lemma 1.7 ("Quotients respect filtrations"), we have that $g_0' g_1' \psi_1 = g_0 g_1$ and $g_0' \psi_0 = g_0$ in $\mathcal{C}$. This yields
$$g_0' g_1' \psi_1 = g_0' \psi_0 g_1,$$
which in turn implies
$$g_1' \psi_1 = \psi_0 g_1$$
since all morphsims in $\mathcal{E}$ are monic and $\mathcal{E}$ contains all isomorphisms. Conclude that the given square is distinguished since distinguished squares interact well with isomorphisms.
  - *Case 2:* $\mathcal{E}^{\mathrm{op}} \subseteq \mathcal{C}$. Analogously, we get $g_0' = \psi_0 g_0$ and $\psi_1 g_1 g_0 = g_1' g_0'$ in $\mathcal{C}$, and so
$$\psi_1 g_1 g_0 = g_1' \psi_0 g_0$$
in $\mathcal{C}$. Since all morphisms in $\mathcal{E}$ are monic, all morphisms in $\mathcal{E}^{\mathrm{op}}$ are epi, and so conclude
$$\psi_1 g_1 = g_1' \psi_0.$$

  By Cases 1 and 2, we've shown Diagram (90) is a distinguished square for any pCGW category. An inductive argument applies the same reasoning to the remaining squares, verifying that each is distinguished.

Summarising: we've shown that Diagram (89) is an $a + 2$-simplex of $S\mathcal{C}$. It is clear that forgetting the final or second last column corresponds to W and W' in $\overline{M}|F$ respectively, and so this defines a 1-simplex $W \to W'$.]

APPENDIX B. MORE ON GENERATORS OF $K_1(\mathcal{C})$

B.1. **Technical Facts about Sherman Loops.** Let $\mathcal{C}$ be a pCGW category.

Claim B.1. *The homotopy class of $G(\alpha, \beta, \Theta)$ in $K_1(\mathcal{C})$ depends only on the choice of α, β and Θ.*

Proof. Keeping the presentation from Construction 3.4, consider another pair of distinguished squares

$$\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C_1 \arrow[d, "\delta_1"] \\ A \arrow[r, tail, "\alpha"] & X \end{tikzcd} \qquad \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & D_1 \arrow[d, "\gamma_1"] \\ B \arrow[r, tail, "\beta"] & Y \end{tikzcd} \tag{91}$$

with isomorphism $\theta_1 \colon A \oplus C_1 \oplus Y \to B \oplus D_1 \oplus X$ representing Θ. There exist unique isomorphisms $\tau \colon C \circ\!\!\to C_1$, $\delta \colon D \circ\!\!\to D_1$ such that

$$\begin{tikzcd} C \arrow[rr, "\tau"] \arrow[dr, "\delta"'] & & C_1 \arrow[dl, "\delta_1"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} D \arrow[rr, "\sigma"] \arrow[dr, "\gamma"'] & & D_1 \arrow[dl, "\gamma_1"] \\ & Y & \end{tikzcd} . \tag{92}$$

Obviously, this yields a commutative diagram of isomorphisms

$$\begin{tikzcd} A \oplus C \oplus Y \arrow[r, "\theta"] \arrow[d, "1\oplus\tau\oplus1"'] & B \oplus D \oplus X \arrow[d, "1\oplus\sigma\oplus1"] \\ A \oplus C_1 \oplus Y \arrow[r, "\theta_1"] & B \oplus D_1 \oplus X \end{tikzcd} \tag{93}$$

which we use to construct the diagram below.

$$
\begin{tikzcd}
& \begin{pmatrix} A\oplus C_1\oplus Y \\ X\oplus Y \end{pmatrix} \arrow[rr] & & \begin{pmatrix} B\oplus D_1\oplus X \\ Y\oplus X \end{pmatrix} & \\
\begin{pmatrix} A \\ A \end{pmatrix} \arrow[r] \arrow[ur] & \begin{pmatrix} A\oplus C\oplus Y \\ X\oplus Y \end{pmatrix} \arrow[r] \arrow[u] \arrow[urr] & \begin{pmatrix} B\oplus D\oplus X \\ Y\oplus X \end{pmatrix} \arrow[ur] & & \begin{pmatrix} B \\ B \end{pmatrix} \arrow[ul] \arrow[ll]
\end{tikzcd}
\tag{94}
$$

As a warm-up, notice the isomorphisms allow us to construct the following 2-simplex in $G\mathcal{C}$

$$
\left(
\begin{tikzcd}
A \arrow[r, tail] & A\oplus C\oplus Y \arrow[r, tail] & A\oplus C_1\oplus Y \\
& C\oplus Y \arrow[u, circ] \arrow[r, tail] & C_1\oplus Y \arrow[u, circ] \\
& & O \arrow[u, circ]
\end{tikzcd}
,\quad
\begin{tikzcd}
A \arrow[r, tail] & X\oplus Y \arrow[r, tail] & X\oplus Y \\
& C\oplus Y \arrow[u, circ] \arrow[r, tail] & C_1\oplus Y \arrow[u, circ] \\
& & O \arrow[u, circ]
\end{tikzcd}
\right)
$$

and so the leftmost triangle in Diagram (94) bounds a 2-simplex. A similar argument shows the other triangles in the diagram also bound a 2-simplex. The claim thus follows.

□

To finish the proof of Theorem 3.7, we will require the following three technical lemmas. We follow the argument from [She94, §2].

Lemma B.2. *The sum of two Sherman loops is equivalent to a Sherman loop. Explicitly, consider two pairs of $\mathcal{M}$-morphisms*

$$\alpha_i\colon A_i \to X_i \qquad , \qquad \beta\colon B_i \to Y_i, \qquad \textit{for } i = 1,2$$

and isomorphisms

$$\theta_i\colon A_i \oplus \frac{X_i}{A_i} \oplus Y_i \longrightarrow B_i \oplus \frac{Y_i}{B_i} \oplus X_i, \qquad \textit{for } i = 1,2.$$

Then

$$G(\alpha_1,\beta_1,\theta_1) + G(\alpha_2,\beta_2,\theta_2) = G\left(\alpha_1\oplus\alpha_2, \beta_1\oplus\beta_2, T_2(\theta_1\oplus\theta_2)T_1^{-1}\right)$$

whereby T_1 and T_2 are the canonical permutation isomorphisms

$$T_1\colon A_1 \oplus \frac{X_1}{A_1} \oplus Y_1 \oplus A_2 \oplus \frac{X_2}{A_2} \oplus Y_2 \to A_1 \oplus A_2 \oplus \frac{X_1}{A_1} \oplus \frac{X_2}{A_2} \oplus Y_1 \oplus Y_2$$

$$T_2\colon B_1 \oplus \frac{Y_1}{B_1} \oplus X_1 \oplus B_2 \oplus \frac{Y_2}{B_2} \oplus X_2 \to B_1 \oplus B_2 \oplus \frac{Y_1}{B_1} \oplus \frac{Y_2}{B_2} \oplus X_1 \oplus X_2.$$

Proof. There are no surprises. Applying the H-space structure of $|G\mathcal{C}|$, observe that $G(\alpha_1, \beta_1, \theta_1) + G(\alpha_2, \beta_2, \theta_2)$ is represented by the loop

$$\begin{tikzcd}
\begin{pmatrix} A_1 \oplus \frac{X_1}{A_1} \oplus Y_1 \oplus A_2 \oplus \frac{X_2}{A_2} \oplus Y_2 \\ X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 \end{pmatrix} \arrow[rr, "\begin{pmatrix} \theta_1 \oplus \theta_2 \\ \tau_1 \end{pmatrix}"] & & \begin{pmatrix} B_1 \oplus \frac{Y_1}{B_1} \oplus X_1 \oplus B_2 \oplus \frac{Y_2}{B_2} \oplus X_2 \\ Y_1 \oplus X_1 \oplus Y_2 \oplus X_2 \end{pmatrix} \\
\begin{pmatrix} A_1 \oplus A_2 \\ A_1 \oplus A_2 \end{pmatrix} \arrow[u] & & \begin{pmatrix} B_1 \oplus B_2 \\ B_1 \oplus B_2 \end{pmatrix} \arrow[u] \\
& \begin{pmatrix} O \\ O \end{pmatrix} \arrow[ul] \arrow[ur] &
\end{tikzcd} \tag{95}$$

The 1-simplices are the obvious ones, with the middle 1-simplex applying $\theta_1 \oplus \theta_2$ on the top row, and the permutation isomorphism τ_1 on the bottom. The loop corresponding to $G\left(\alpha_1 \oplus \alpha_2, \beta_1 \oplus \beta_2, T_2(\theta_1 \oplus \theta_2)T_1^{-1}\right)$ is defined similarly, except with the summands permuted.

$$\begin{tikzcd}
\begin{pmatrix} A_1 \oplus A_2 \oplus \frac{X_1}{A_1} \oplus \frac{X_2}{A_2} \oplus Y_1 \oplus Y_2 \\ X_1 \oplus X_2 \oplus Y_1 \oplus Y_2 \end{pmatrix} \arrow[rr, "\begin{pmatrix} T_2(\theta_1 \oplus \theta_2)T_1^{-1} \\ \tau_2 \end{pmatrix}"] & & \begin{pmatrix} B_1 \oplus B_2 \oplus \frac{Y_1}{B_1} \oplus \frac{Y_2}{B_2} \oplus X_1 \oplus X_2 \\ Y_1 \oplus Y_2 \oplus X_1 \oplus X_2 \end{pmatrix} \\
\begin{pmatrix} A_1 \oplus A_2 \\ A_1 \oplus A_2 \end{pmatrix} \arrow[u] & & \begin{pmatrix} B_1 \oplus B_2 \\ B_1 \oplus B_2 \end{pmatrix} \arrow[u] \\
& \begin{pmatrix} O \\ O \end{pmatrix} \arrow[ul] \arrow[ur] &
\end{tikzcd} \tag{96}$$

where τ_2 is the obvious permutation isomorphism on the bottom row. To show the two loops are equivalent, consider the diagram below.

$$
\begin{tikzcd}[column sep=small]
& \begin{pmatrix} A_1 \oplus A_2 \\ A_1 \oplus A_2 \end{pmatrix} \arrow[dl] \arrow[dr] & \\
\begin{pmatrix} A_1 \oplus \frac{X_1}{A_1} \oplus Y_1 \oplus A_2 \oplus \frac{X_2}{A_2} \oplus Y_2 \\ X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 \end{pmatrix} \arrow[rr, "\begin{pmatrix} T_1 \\ \tau_3 \end{pmatrix}"] \arrow[d, blue, "\begin{pmatrix} \theta_1 \oplus \theta_2 \\ \tau_1 \end{pmatrix}"'] \arrow[drr, "\begin{pmatrix} T_2(\theta_1 \oplus \theta_2) \\ \tau_4 \circ \tau_1 \end{pmatrix}"'] & & \begin{pmatrix} A_1 \oplus A_2 \oplus \frac{X_1}{A_1} \oplus \frac{X_2}{A_2} \oplus Y_1 \oplus Y_2 \\ X_1 \oplus X_2 \oplus Y_1 \oplus Y_2 \end{pmatrix} \arrow[d, red, "\begin{pmatrix} T_2(\theta_1 \oplus \theta_2) T_1^{-1} \\ \tau_2 \end{pmatrix}"] \\
\begin{pmatrix} B_1 \oplus \frac{Y_1}{B_1} \oplus X_1 \oplus B_2 \oplus \frac{Y_2}{B_2} \oplus X_2 \\ Y_1 \oplus X_1 \oplus Y_2 \oplus X_2 \end{pmatrix} \arrow[rr, "\begin{pmatrix} T_2 \\ \tau_4 \end{pmatrix}"'] & & \begin{pmatrix} B_1 \oplus B_2 \oplus \frac{Y_1}{B_1} \oplus \frac{Y_2}{B_2} \oplus X_1 \oplus X_2 \\ Y_1 \oplus Y_2 \oplus X_1 \oplus X_2 \end{pmatrix} \\
& \begin{pmatrix} B_1 \oplus B_2 \\ B_1 \oplus B_2 \end{pmatrix} \arrow[ul] \arrow[ur] &
\end{tikzcd}
\tag{97}
$$

where τ_3, τ_4 are the obvious permutation isomorphisms. An easy check shows all the triangles in Diagram (97) are boundaries of 2-simplices. For instance, the top triangle bounds the 2-simplex

$$
\begin{tikzcd}
A_1 \oplus A_2 \arrow[r, tail] & A_1 \oplus \frac{X_1}{A_1} \oplus Y_1 \oplus A_2 \oplus \frac{X_2}{A_2} \oplus Y_2 \arrow[r, tail] & A_1 \oplus A_2 \oplus \frac{X_1}{A_1} \oplus \frac{X_2}{A_2} \oplus Y_1 \oplus Y_2 \\
& \frac{X_1}{A_1} \oplus Y_1 \oplus \frac{X_2}{A_2} \oplus Y_2 \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{X_1}{A_1} \oplus \frac{X_2}{A_2} \oplus Y_1 \oplus Y_2 \arrow[u] \\
& & O \arrow[u]
\end{tikzcd}
$$

$$
\begin{tikzcd}
A_1 \oplus A_2 \arrow[r, tail] & X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 \arrow[r, tail] & X_1 \oplus X_2 \oplus Y_1 \oplus Y_2 \\
& \frac{X_1}{A_1} \oplus Y_1 \oplus \frac{X_2}{A_2} \oplus Y_2 \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\square"] & \frac{X_1}{A_1} \oplus \frac{X_2}{A_2} \oplus Y_1 \oplus Y_2 \arrow[u] \\
& & O \arrow[u]
\end{tikzcd}
$$

Notice the indicated squares are distinguished because distinguished squares interact well with isomorphisms (see Definition 1.1). And thus proves the lemma. $\square$

Lemma B.3. *Let $\mathcal{C}$ be a pCGW category whose exact squares all split.* <u>*Then,*</u> *every element of $K_1(\mathcal{C})$ corresponds to the loop*

$$
G(A, \alpha) := \left(\begin{tikzcd} (A, A) \arrow[rr, "l(\alpha)"] & & (A, A) \\ & (O, O) \arrow[ul] \arrow[ur] & \end{tikzcd} \right)
\tag{98}
$$

where $l(\alpha)$ is the 1-simplex

$$
l(\alpha) := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & O \arrow[d] \\ A \arrow[r, tail, "1"] & A \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & O \arrow[d] \\ A \arrow[r, tail, "\alpha"] & A \end{tikzcd} \right)
$$

for some automorphism $(A, \alpha) \in \mathrm{Aut}(\mathcal{C})$.

Proof. The proof combines an argument from [GG87, §5] and [She94, Prop. 2]. Proceed in stages.

Step 1: Combinatorial Loops in $K_1(\mathcal{C})$. Suppose $z \in K_1(\mathcal{C}) = \pi_1|G\mathcal{C}|$. By the simplicial approximation theorem, z can be represented by a loop formed combinatorially from 1-simplices of $G\mathcal{C}$

$$\begin{pmatrix} O \\ O \end{pmatrix} \to \bullet \leftarrow \bullet \to \cdots \leftarrow \bullet \to \bullet \leftarrow \begin{pmatrix} O \\ O \end{pmatrix} \tag{99}$$

where we draw the 1-simplices as arrows. We claim this loop is homotopic to one of the form

$$\begin{pmatrix} O \\ O \end{pmatrix} \to \bullet \to \cdots \to \bullet \leftarrow \cdots \leftarrow \bullet \leftarrow \begin{pmatrix} O \\ O \end{pmatrix}.$$

Consider one of the configurations in Diagram (99), e.g.

$$\begin{pmatrix} M' \\ L' \end{pmatrix} \leftarrow \begin{pmatrix} M \\ L \end{pmatrix} \to \begin{pmatrix} M'' \\ L'' \end{pmatrix}.$$

Since the arrows are 1-simplices in $G\mathcal{C}$, the relevant quotients agree, i.e. $\frac{M'}{M} = \frac{L'}{L}$ and $\frac{M''}{M} = \frac{L''}{L}$. Now form restricted pushouts $P := K' \star_K K''$ and $Q := L' \star_L L''$. Apply Lemma A.1 to deduce

$$\frac{P}{M} \cong \frac{M'}{M} \oplus \frac{M''}{M} \qquad \text{and} \qquad \frac{Q}{L} \cong \frac{L'}{L} \oplus \frac{L''}{L} = \frac{M'}{M} \oplus \frac{M''}{M}.$$

In particular, Equation (75) of the proof tells us the data assembles into diagram pairs, such as

$$\begin{tikzcd}
M \arrow[r, tail] \arrow[dr, phantom, "\square"] & M' \arrow[r, tail] \arrow[dr, phantom, "\square"] & P \\
O \arrow[u, {Circle[open]}->] \arrow[r, tail] & \frac{M'}{M} \arrow[u, {Circle[open]}->] \arrow[r, tail] \arrow[dr, phantom, "\square"] & \frac{M'}{M} \oplus \frac{M''}{M} \arrow[u, {Circle[open]}->] \\
& O \arrow[u, {Circle[open]}->] \arrow[r, tail] & \frac{M''}{M} \arrow[u, {Circle[open]}->]
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r, tail] \arrow[dr, phantom, "\square"] & L' \arrow[r, tail] \arrow[dr, phantom, "\square"] & Q \\
O \arrow[u, {Circle[open]}->] \arrow[r, tail] & \frac{M'}{M} \arrow[u, {Circle[open]}->] \arrow[r, tail] \arrow[dr, phantom, "\square"] & \frac{M'}{M} \oplus \frac{M''}{M} \arrow[u, {Circle[open]}->] \\
& O \arrow[u, {Circle[open]}->] \arrow[r, tail] & \frac{M''}{M} \arrow[u, {Circle[open]}->]
\end{tikzcd}.$$

Here is the upshot. By the above argument, use the proof of Lemma A.1 to define two 2-simplices in $G\mathcal{C}$

$$\left(\begin{array}{cc} O \rightarrowtail & \underline{\underline{M \rightarrowtail M' \rightarrowtail P}} \\ O \rightarrowtail & L \rightarrowtail L' \rightarrowtail Q \end{array} \right) \quad \text{and} \quad \left(\begin{array}{cc} O \rightarrowtail & \underline{\underline{M \rightarrowtail M'' \rightarrowtail P}} \\ O \rightarrowtail & L \rightarrowtail L'' \rightarrowtail Q \end{array} \right),$$

which fill in the two triangles of the diagram

$$\begin{tikzcd}
\begin{pmatrix} M \\ L \end{pmatrix} \arrow[r] \arrow[d] \arrow[dr] & \begin{pmatrix} M'' \\ L'' \end{pmatrix} \arrow[d] \\
\begin{pmatrix} M' \\ L' \end{pmatrix} \arrow[r] & \begin{pmatrix} P \\ Q \end{pmatrix}
\end{tikzcd}.$$

Abstractly, this turns a configuration

$$\bullet \leftarrow \bullet \to \bullet \qquad \text{into} \qquad \bullet \to \bullet \qquad .$$

Applying this trick multiple times, we can deform Loop (99) into one of the form

$$\begin{tikzcd}
\begin{pmatrix} O \\ O \end{pmatrix} \arrow[r] \arrow[d] & \begin{pmatrix} M_0 \\ L_0 \end{pmatrix} \arrow[r] & \cdots \arrow[r] & \begin{pmatrix} M_{q-1} \\ L_{q-1} \end{pmatrix} \arrow[d] \\
\begin{pmatrix} M'_0 \\ L'_0 \end{pmatrix} \arrow[r] & \cdots \arrow[r] & \begin{pmatrix} M'_{q-1} \\ L'_{q-1} \end{pmatrix} \arrow[r] & \begin{pmatrix} M \\ L \end{pmatrix}
\end{tikzcd} \tag{100}$$

Step 2: The Base Case. Start by analysing the component

$$\begin{pmatrix} O \\ O \end{pmatrix} \xrightarrow{l_0} \begin{pmatrix} M_0 \\ L_0 \end{pmatrix} \xrightarrow{l_1} \begin{pmatrix} M_1 \\ L_1 \end{pmatrix} \tag{101}$$

of Loop (100) in $K_1(\mathcal{C})$. Suppose l_0 is defined by the following pair of exact squares

$$l_0 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \widehat{M_0} \arrow[d, "\eta_0"] \\ O \arrow[r, tail] & M_0 \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \widehat{M_0} \arrow[d, "\mu_0"] \\ O \arrow[r, tail] & L_0 \end{tikzcd} \right), \tag{102}$$

with isomorphisms η_0 and μ_0. Now recall that the CGW category structure includes an isomorphism of categories

$$\varphi\colon \mathrm{iso}\mathcal{M} \to \mathrm{iso}\mathcal{E}.$$

We can therefore define two 1-simplices

$$l_0' := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \widehat{M_0} \arrow[d, "1"] \\ O \arrow[r, tail] & \widehat{M_0} \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \widehat{M_0} \arrow[d, "1"] \\ O \arrow[r, tail] & \widehat{M_0} \end{tikzcd} \right) \quad \text{and} \quad l_0'' := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & O \arrow[d] \\ \widehat{M_0} \arrow[r, tail, "\varphi^{-1}(\eta_0)"] & M_0 \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & O \arrow[d] \\ \widehat{M_0} \arrow[r, tail, "\varphi^{-1}(\mu_0)"] & L_0 \end{tikzcd} \right),$$

which assemble into the following 2-simplex

$$\begin{tikzcd} O \arrow[r, tail] \arrow[dr, phantom, "\square"] & \widehat{M_0} \arrow[r, tail, "\varphi^{-1}(\eta_0)"] \arrow[dr, phantom, "\square"] & M_0 \\ O \arrow[r, tail] \arrow[u] & \widehat{M_0} \arrow[r, tail, "1"] \arrow[u, "1"] \arrow[dr, phantom, "\square"] & \widehat{M_0} \arrow[u, "\eta_0"'] \\ & O \arrow[r, tail] \arrow[u] & O \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} O \arrow[r, tail] \arrow[dr, phantom, "\square"] & \widehat{M_0} \arrow[r, tail, "\varphi^{-1}(\mu_0)"] \arrow[dr, phantom, "\square"] & L_0 \\ O \arrow[r, tail] \arrow[u] & \widehat{M_0} \arrow[r, tail, "1"] \arrow[u, "1"] \arrow[dr, phantom, "\square"] & \widehat{M_0} \arrow[u, "\mu_0"'] \\ & O \arrow[r, tail] \arrow[u] & O \arrow[u] \end{tikzcd}.$$

We remark that the top right squares are distinguished by Axiom (I) of Definition 1.4. We then assemble the following diagram

$$\begin{tikzcd} & \begin{pmatrix} \widehat{M_0} \\ \widehat{M_0} \end{pmatrix} \arrow[d, "l_0''"] \arrow[dr, blue, "l_1 \circ l_0''"] & \\ \begin{pmatrix} O \\ O \end{pmatrix} \arrow[ur, blue, "l_0'"] \arrow[r, red, "l_0"] & \begin{pmatrix} M_0 \\ L_0 \end{pmatrix} \arrow[r, red, "l_1"] & \begin{pmatrix} M_1 \\ L_1 \end{pmatrix} \end{tikzcd}, \tag{103}$$

where $l_1 \circ l_0''$ is defined by horizontally composing l_1 with l_0'' in the obvious way. One easily checks the added triangle also bounds a 2-simplex, which implies the red path of 1-simplices is homotopic to the blue path.[41] Hence, without loss of generality, let us assume that both μ_0 and η_0 of Equation (102) are the identity $1\colon M_0 \to M_0$.

Step 3: The Inductive Step. Proceeding along Loop (101), consider $l_1\colon (M_0, M_0) \to (M_1, L_1)$ whereby

$$l_1 := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{M_1}{M_0} \arrow[d, "\eta_1'"] \\ M_0 \arrow[r, tail, "\eta_1"] & M_1 \end{tikzcd} \quad , \quad \begin{tikzcd} O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{M_1}{M_0} \arrow[d, "\mu_1'"] \\ M_0 \arrow[r, tail, "\mu_1"] & L_1 \end{tikzcd} \right).$$

[41] **(M:) Double-check.**

Since all exact squares split by hypothesis, this yields isomorphisms

$$\psi_M \colon M_1 \rightarrowtail M_0 \oplus \frac{M_1}{M_0} \qquad \text{and} \qquad \psi_L \colon L_1 \rightarrowtail M_0 \oplus \frac{M_1}{M_0},$$

which we shall consider as $\mathcal{M}$-morphisms. Notice this defines the following distinguished squares

$$\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{M_1}{M_0} \arrow[r, tail, "1"] \arrow[d, {Circle[open]}->, "\eta_1'"] \arrow[dr, phantom, "\square"] & \frac{M_1}{M_0} \arrow[d, {Circle[open]}->, "\varphi(\psi_M)\circ\eta_1'"] \\
M_0 \arrow[r, tail, "\eta_1"] & M_1 \arrow[r, tail, "\psi_M"] & M_0 \oplus \frac{M_1}{M_0}
\end{tikzcd}
\qquad
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{M_1}{M_0} \arrow[r, tail, "1"] \arrow[d, {Circle[open]}->, "\eta_1'"] \arrow[dr, phantom, "\square"] & \frac{M_1}{M_0} \arrow[d, {Circle[open]}->, "\varphi(\psi_L)\circ\mu_1'"] \\
M_0 \arrow[r, tail, "\mu_1"] & L_1 \arrow[r, tail, "\psi_L"] & M_0 \oplus \frac{M_1}{M_0}
\end{tikzcd}$$

The right squares are distinguished by Axiom (I) of Definition 1.4, and thus the horizontal compositions define two exact squares. To ease notation, denote

$$v := \psi_M \circ \eta_1 \qquad \text{and} \qquad v' := \varphi(\psi_M) \circ \eta_1',$$
$$w := \psi_L \circ \mu_1 \qquad \text{and} \qquad w' := \varphi(\psi_L) \circ \mu_1'.$$

In fact, we can say more. Denote

$$\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] \arrow[dr, phantom, "\square"] & \frac{M_1}{M_0} \arrow[d, {Circle[open]}->, "q"] \\
M_0 \arrow[r, tail, "p"] & M_0 \oplus \frac{M_1}{M_0}
\end{tikzcd}$$

to be the canonical direct sum square of $M_0 \oplus \frac{M_1}{M_0}$. Since both exact squares in l_1 are split, it follows that

$$v = \psi_M \circ \eta_1 = p = \psi_L \circ \mu_1 = w \tag{104}$$
$$v' = \varphi(\psi_M) \circ \eta_1' = q = \varphi(\psi_L) \circ \mu_1' = w'.$$

Leverage these identities to construct the following 2-simplex

$$\begin{tikzcd}
O \arrow[r, tail] \arrow[dr, phantom, "\square"] & M_0 \arrow[r, tail, "\eta_1"] \arrow[dr, phantom, "\square"] & M_1 \\
O \arrow[r, tail] \arrow[u, {Circle[open]}->] & M_0 \arrow[r, tail, "v"] \arrow[u, {Circle[open]}->, "1"] \arrow[dr, phantom, "\square"] & M_0 \oplus \frac{M_1}{M_0} \arrow[u, {Circle[open]}->, "\varphi(\psi_M^{-1})"'] \\
& O \arrow[r, tail] \arrow[u, {Circle[open]}->] & \frac{M_1}{M_0} \arrow[u, {Circle[open]}->, "v'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
O \arrow[r, tail] \arrow[dr, phantom, "\square"] & M_0 \arrow[r, tail, "\mu_1"] \arrow[dr, phantom, "\square"] & L_1 \\
O \arrow[r, tail] \arrow[u, {Circle[open]}->] & M_0 \arrow[r, tail, "v"] \arrow[u, {Circle[open]}->, "1"] \arrow[dr, phantom, "\square"] & M_0 \oplus \frac{M_1}{M_0} \arrow[u, {Circle[open]}->, "\varphi(\psi_L^{-1})"'] \\
& O \arrow[r, tail] \arrow[u, {Circle[open]}->] & \frac{M_1}{M_0} \arrow[u, {Circle[open]}->, "v'"']
\end{tikzcd}
\quad . \tag{105}$$

To see why the square indicated in red is distinguished, notice:

- **Case 1.** Suppose $\mathcal{E}$ is a subcategory of $\mathcal{C}$. Then

  $$\varphi(\psi_L^{-1}) \circ v = \psi_L^{-1} \circ v = \mu_1 \qquad \text{in } \mathcal{C}$$

  if and only if

  $$v = \psi_L \circ \mu_1 = w,$$

  which holds by Identity (104).
- **Case 2.** Suppose $\mathcal{E}^{\mathrm{op}}$ is a subcategory of $\mathcal{C}$. Applying Identity (104) once more, deduce

  $$v = \varphi(\psi_L^{-1}) \circ \mu_1 = \psi_L \circ \mu_1 = w.$$

The claim then follows from the fact that distinguished squares interact well with isomorphisms. The case for the blue-indicated square is analogous.

Having checked that Diagram (105) defines a 2-simplex, the gears line up and the inductive argument falls into place. First note that Diagram (105) defines a homotopy between

$$\begin{pmatrix} O \\ O \end{pmatrix} \xrightarrow{l_0} \begin{pmatrix} M_0 \\ L_0 \end{pmatrix} \xrightarrow{l_1} \begin{pmatrix} M_1 \\ L_1 \end{pmatrix}$$

and a 1-simplex of the form

$$l'_1 \colon \begin{pmatrix} O \\ O \end{pmatrix} \to \begin{pmatrix} M_1 \\ L_1 \end{pmatrix}$$

whereby

$$l'_1 := \left(\begin{array}{ccc} O & \rightarrowtail & \widehat{M_1} \\ \downarrow & \square & \downarrow{\scriptstyle \alpha_1} \\ O & \rightarrowtail & M_1 \end{array} \quad , \quad \begin{array}{ccc} O & \rightarrowtail & \widehat{M_1} \\ \downarrow & \square & \downarrow{\scriptstyle \beta_1} \\ O & \rightarrowtail & L_1 \end{array} \right).$$

Hence, the initial segment of Loop (106) is homotopic to

$$\begin{pmatrix} O \\ O \end{pmatrix} \xrightarrow{l'_1} \begin{pmatrix} M_1 \\ L_1 \end{pmatrix} \xrightarrow{l_2} \begin{pmatrix} M_2 \\ L_2 \end{pmatrix}.$$

We can therefore apply the Base Case argument (Step 2) to justify presenting l'_1 as

$$l'_1 := \left(\begin{array}{ccc} O & \rightarrowtail & M_1 \\ \downarrow & \square & \downarrow{\scriptstyle 1} \\ O & \rightarrowtail & M_1 \end{array} \quad , \quad \begin{array}{ccc} O & \rightarrowtail & \widehat{M_1} \\ \downarrow & \square & \downarrow{\scriptstyle 1} \\ O & \rightarrowtail & M_1 \end{array} \right),$$

which sets up our inductive step again. Keep going for the rest of Loop (100) on both sides, until we finally obtain a loop of the form

$$\begin{pmatrix} O \\ O \end{pmatrix} \to \begin{pmatrix} M_{q-1} \\ M_{q-1} \end{pmatrix} \to (M, L) \leftarrow \begin{pmatrix} M'_{q-1} \\ M'_{q-1} \end{pmatrix} \leftarrow \begin{pmatrix} O \\ O \end{pmatrix} \tag{106}$$

At which point, we can apply the inductive argument on both sides once more to obtain the loop

$$\begin{pmatrix} O \\ O \end{pmatrix} \xrightarrow{\kappa} (M, L) \xleftarrow{\gamma} \begin{pmatrix} O \\ O \end{pmatrix}, \tag{107}$$

where

$$\kappa := \left(\begin{array}{ccc} O & \rightarrowtail & \widehat{M} \\ \downarrow & \square & \downarrow{\scriptstyle \kappa_0} \\ O & \rightarrowtail & M \end{array} \quad , \quad \begin{array}{ccc} O & \rightarrowtail & \widehat{M} \\ \downarrow & \square & \downarrow{\scriptstyle \kappa_1} \\ O & \rightarrowtail & L \end{array} \right) \quad \text{and} \quad \gamma := \left(\begin{array}{ccc} O & \rightarrowtail & \widehat{N} \\ \downarrow & \square & \downarrow{\scriptstyle \gamma_0} \\ O & \rightarrowtail & M \end{array} \quad , \quad \begin{array}{ccc} O & \rightarrowtail & \widehat{N} \\ \downarrow & \square & \downarrow{\scriptstyle \gamma_1} \\ O & \rightarrowtail & L \end{array} \right)$$

Notice, however, we can no longer apply the Base Case argument to simplify κ or γ since the arrows of Diagram (107) are in the wrong direction.

Step 4: Finish. A technical observation: both κ and γ define isomorphisms $L \xrightarrow{\cong} M$ in $\mathcal{C}$ but the presentation will differ depending on whether $\mathcal{E}^{\mathrm{op}}$ or $\mathcal{E}$ is a subcategory of $\mathcal{C}$.

Case 1: $\mathcal{E} \subseteq \mathcal{C}$. In which case, define $\omega := \kappa_0 \circ \kappa_1^{-1}$ and $\lambda := \gamma_0 \circ \gamma_1^{-1}$ in $\mathcal{C}$

Case 2: $\mathcal{E}^{\mathrm{op}} \subseteq \mathcal{C}$. In which case, define $\omega := \kappa_0^{-1} \circ \kappa_1$ and $\lambda := \gamma_0^{-1} \circ \gamma_1$ in $\mathcal{C}$.

Since $\mathcal{M}$ is always a subcategory of $\mathcal{C}$, the isomorphisms ω and λ define $\mathcal{M}$-morphisms as well. We now construct the obvious diagram

$$\begin{tikzcd} & \begin{pmatrix} M \\ M \end{pmatrix} \arrow[r, "g"] & \begin{pmatrix} M \\ M \end{pmatrix} \\ \begin{pmatrix} O \\ O \end{pmatrix} \arrow[ur, "f_0"] \arrow[r, "\kappa"] & \begin{pmatrix} M \\ L \end{pmatrix} \arrow[u, "f_1"] \arrow[ur, "f_2"] & \begin{pmatrix} O \\ O \end{pmatrix} \arrow[l, "\gamma"'] \arrow[u, "f_3"'] \end{tikzcd} \tag{108}$$

whereby

$$
g := \left(
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] & O \arrow[d, {Circle[open]}->] \\
M \arrow[r, tail, "1"] \arrow[ur, phantom, "\square"] & M
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] & O \arrow[d, {Circle[open]}->] \\
M \arrow[r, tail, "\lambda\circ\omega^{-1}"] \arrow[ur, phantom, "\square"] & M
\end{tikzcd}
\right)
$$

$$
f_0 := \left(
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] & \widehat{N} \arrow[d, {Circle[open]}->, "\kappa_0"] \\
O \arrow[r, tail] \arrow[ur, phantom, "\square"] & M
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] & \widehat{N} \arrow[d, {Circle[open]}->, "\kappa_0"] \\
O \arrow[r, tail] \arrow[ur, phantom, "\square"] & M
\end{tikzcd}
\right)
\qquad
f_1 := \left(
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] & O \arrow[d, {Circle[open]}->] \\
M \arrow[r, tail, "1"] \arrow[ur, phantom, "\square"] & M
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] & O \arrow[d, {Circle[open]}->] \\
L \arrow[r, tail, "\omega"] \arrow[ur, phantom, "\square"] & M
\end{tikzcd}
\right).
$$

$$
f_2 := \left(
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] & O \arrow[d, {Circle[open]}->] \\
M \arrow[r, tail, "1"] \arrow[ur, phantom, "\square"] & M
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] & O \arrow[d, {Circle[open]}->] \\
L \arrow[r, tail, "\lambda"] \arrow[ur, phantom, "\square"] & M
\end{tikzcd}
\right)
\qquad
f_3 := \left(
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] & \widehat{N} \arrow[d, {Circle[open]}->, "\gamma_0"] \\
O \arrow[r, tail] \arrow[ur, phantom, "\square"] & M
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
O \arrow[r, tail] \arrow[d, {Circle[open]}->] & \widehat{N} \arrow[d, {Circle[open]}->, "\gamma_0"] \\
O \arrow[r, tail] \arrow[ur, phantom, "\square"] & M
\end{tikzcd}
\right).
$$

In both cases ($\mathcal{E} \subseteq \mathcal{C}$ or $\mathcal{E}^{\mathrm{op}} \subseteq \mathcal{C}$), it is easy to check that the triangles of Diagram (108) bound the following 2-simplices:

$$
f_1\kappa = f_0 \qquad \left(
\begin{tikzcd}
O \arrow[r, tail] & M \arrow[r, tail, "1"] & M \\
 & \widehat{M} \arrow[u, {Circle[open]}->, "\kappa_0"] \arrow[r, tail, "1"] \arrow[ur, phantom, "\square"] & \widehat{M} \arrow[u, {Circle[open]}->, "\kappa_0"'] \\
 & & O \arrow[u, {Circle[open]}->]
\end{tikzcd}
\qquad
\begin{tikzcd}
O \arrow[r, tail] & L \arrow[r, tail, "\omega"] & M \\
 & \widehat{N} \arrow[u, {Circle[open]}->, "\kappa_1"] \arrow[r, tail, "1"] \arrow[ur, phantom, "\square"] & \widehat{N} \arrow[u, {Circle[open]}->, "\kappa_0"'] \\
 & & O \arrow[u, {Circle[open]}->]
\end{tikzcd}
\right).
$$

$$
f_2\gamma = f_3 \qquad \left(
\begin{tikzcd}
O \arrow[r, tail] & M \arrow[r, tail, "1"] & M \\
 & \widehat{N} \arrow[u, {Circle[open]}->, "\gamma_0"] \arrow[r, tail, "1"] \arrow[ur, phantom, "\square"] & \widehat{N} \arrow[u, {Circle[open]}->, "\gamma_0"'] \\
 & & O \arrow[u, {Circle[open]}->]
\end{tikzcd}
\qquad
\begin{tikzcd}
O \arrow[r, tail] & L \arrow[r, tail, "\lambda"] & M \\
 & \widehat{N} \arrow[u, {Circle[open]}->, "\gamma_1"] \arrow[r, tail, "1"] \arrow[ur, phantom, "\square"] & \widehat{N} \arrow[u, {Circle[open]}->, "\gamma_0"'] \\
 & & O \arrow[u, {Circle[open]}->]
\end{tikzcd}
\right).
$$

$$
gf_1 = f_2 \qquad \left(
\begin{tikzcd}
M \arrow[r, tail, "1"] & M \arrow[r, tail, "1"] & M \\
 & O \arrow[u, {Circle[open]}->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & O \arrow[u, {Circle[open]}->] \\
 & & O \arrow[u, {Circle[open]}->]
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r, tail, "\omega"] & M \arrow[r, tail, "\lambda\circ\omega^{-1}"] & M \\
 & O \arrow[u, {Circle[open]}->] \arrow[r, tail] \arrow[ur, phantom, "\square"] & O \arrow[u, {Circle[open]}->] \\
 & & O \arrow[u, {Circle[open]}->]
\end{tikzcd}
\right).
$$

Conclude that the red loop in Diagram (108) is homotopic to the blue loop. Notice the blue loop is precisely of the form $G(A, \alpha)$ as claimed in lemma statement, with $A := M$ and $\alpha := \lambda \circ \omega^{-1}$. $\square$

Lemma B.4. *The automorphism loop $G(A, \alpha)$ in Lemma B.3 is equivalent to a Sherman Loop.*

Proof. Let $p_A \colon A \to A \oplus A$ be the usual coproduct morphism arising from direct sum squares and let $\tau_A \colon A \oplus A \to A \oplus A$ as the isomorphism swapping components. Define the following loop

$$
\begin{pmatrix} O \\ O \end{pmatrix} \to \begin{pmatrix} A \\ A \end{pmatrix} \xrightarrow{\iota_\alpha} \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \xrightarrow{l_\tau} \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \leftarrow \begin{pmatrix} O \\ O \end{pmatrix} \tag{109}
$$

where

$$\iota_\alpha := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\square"] & A \arrow[d, hook, "p_A"] \\ A \arrow[r, tail, "p_A"] & A \oplus A \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\square"] & A \arrow[d, hook, "p_A"] \\ A \arrow[r, tail, "p_A \circ \alpha"] & A \oplus A \end{tikzcd} \right) \quad l_\tau := \left(\begin{tikzcd} O \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\square"] & O \arrow[d, hook] \\ A \oplus A \arrow[r, tail, "\tau_A"] & A \oplus A \end{tikzcd} , \begin{tikzcd} O \arrow[r, tail] \arrow[d, hook] \arrow[dr, phantom, "\square"] & O \arrow[d, hook] \\ A \oplus A \arrow[r, tail, "\tau_A"] & A \oplus A \end{tikzcd} \right)$$

This is a Sherman Loop $G(\alpha, 0, \tau_A)$, where 0 denotes the $\mathcal{M}$-morphism $O \rightarrowtail A$.[42]

To show $G(A,\alpha) \sim G(\alpha, 0, \tau_A)$ in $\pi_1(|G\mathcal{C}|)$ involves modifying $G(A,\alpha)$ in sensible ways that respects its homotopy class. Consider the loop

$$\begin{pmatrix} O \\ O \end{pmatrix} \to \begin{pmatrix} A \\ A \end{pmatrix} \xrightarrow{\iota_\alpha} \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \xrightarrow{1_{A\oplus A}} \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \leftarrow \begin{pmatrix} O \\ O \end{pmatrix} \tag{110}$$

Since we inserted a degenerate 1-simplex $1_{A\oplus A}$, this is homotopic to

$$\begin{pmatrix} O \\ O \end{pmatrix} \to \begin{pmatrix} A \\ A \end{pmatrix} \xrightarrow{\iota_\alpha} \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \leftarrow \begin{pmatrix} O \\ O \end{pmatrix}. \tag{111}$$

The fact that Loop (111) is homotopic to $G(A,\alpha)$ follows observing that

$$\begin{tikzcd} \begin{pmatrix} A \\ A \end{pmatrix} \arrow[rr, "\iota_\alpha"] \arrow[dr, "\alpha"'] & & \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \\ & \begin{pmatrix} A \\ A \end{pmatrix} \arrow[ur, "\iota_A"'] & \end{tikzcd}$$

bounds a 2-simplex, where ι_A corresponds to the canonical direct sum square $A \oplus A$.

It thus remains to show that $G(\alpha, 0, \tau_A)$ is homotopic to Loop 110. But this follows from noting the following triangles bound 2-simplices

$$\begin{tikzcd} \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \arrow[r, "1_{A\oplus A}"] \arrow[dr, "l_\tau"'] & \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} & \begin{pmatrix} O \\ O \end{pmatrix} \arrow[l] \arrow[dl] \\ & \begin{pmatrix} A \oplus A \\ A \oplus A \end{pmatrix} \arrow[u, "l_\tau"] & \end{tikzcd}.$$

$\square$

B.2. **Explicit Descriptions of 2-Simplices.** This section gives the technical details of the proof of Lemma 3.24. Recall [proof overview].

Claim B.5. *Loop* (37) *is homotopic to loop* L.

Proof. Recall: in order to establish that the two loops are homotopic, it suffices to show that the indicated triangles of the Diagram (44) are boundaries of 2-simplices. We describe the 2-simplices explicitly below.

- Triangle (1). Consider

$$\begin{tikzcd} A \oplus A' \arrow[r, tail, "f_0"] & P \arrow[r, tail, "1\oplus C'"] \arrow[dr, phantom, "\square"] & P \oplus C' \\ & C \oplus C' \arrow[u, hook, "g_0"] \arrow[r, tail, "1 \oplus C'"] & C \oplus C' \oplus C' \arrow[u, hook, "g_0 \oplus 1"'] \\ & & C' \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} A \oplus A' \arrow[r, tail, "f_1"] & Q \arrow[r, tail, "h_t"] \arrow[dr, phantom, "\square"] & V \\ & C \oplus C' \arrow[u, hook, "g_1"] \arrow[r, tail, "1 \oplus C'"] & C \oplus C' \oplus C' \arrow[u, hook, "j"'] \\ & & C' \arrow[u, hook] \end{tikzcd} \tag{112}$$

42(M:) **Double-check this.**

To show this is a 2-simplex in $G\mathcal{C}$, we need to check that the triangles and the given squares define distinguished squares. The given square on the right diagram is the distinguished square t of Lemma 3.23. It is also obvious the following squares are distinguished:

$$
\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C \oplus C' \arrow[d, "g_0"] \\
A \oplus A' \arrow[r, tail, "f_0"] & P
\end{tikzcd}
\quad
\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C \oplus C' \arrow[d, "g_0"] \\
A \oplus A' \arrow[r, tail, "f_0"] & Q
\end{tikzcd}
\quad
\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C' \arrow[d] \\
C \oplus C' \arrow[r, tail, "1 \oplus C'"] & C \oplus C' \oplus C'
\end{tikzcd}
.
$$

[Why? The first two squares are the distinguished squares in Equation (35). The third square is distinguished because it arises from a formal direct sum – see Axiom (A), Definition 1.18.] Finally, apply Axiom (DS) of Definition 1.18 to the diagram

$$
\begin{tikzcd}
C' \arrow[d, "\cong"'] \arrow[dr, phantom, "\square"] & O \arrow[l, tail] \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C \oplus C' \arrow[d, "g_0"] \\
C' & O \arrow[l, tail] \arrow[r, tail] & P
\end{tikzcd}
$$

to deduce that

$$
\begin{tikzcd}
C \oplus C' \arrow[r, tail, "1 \oplus C'"] \arrow[d, "g_0"'] \arrow[dr, phantom, "\square"] & C \oplus C' \oplus C' \arrow[d, "g_0 \oplus 1"] \\
P \arrow[r, tail, "1 \oplus C'"] & P \oplus C'
\end{tikzcd}
$$

is indeed distinguished.

- Triangle (2).

$$
\begin{tikzcd}
A \oplus A' \arrow[r, tail, "f_1"] & Q \arrow[r, tail, "1 \oplus C'"] & Q \oplus C' \\
& C \oplus C' \arrow[u, "g_1"] \arrow[r, tail, "1 \oplus C'"] \arrow[ur, phantom, "\square"] & C \oplus C' \oplus C' \arrow[u, "g_1 \oplus C'"'] \\
& & C' \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \oplus A' \arrow[r, tail, "f_1"] & Q \arrow[r, tail, "h_t"] & V \\
& C \oplus C' \arrow[u, "g_1"] \arrow[r, tail, "1 \oplus C'"] \arrow[ur, phantom, "\square"] & C \oplus C' \oplus C' \arrow[u, "j"'] \\
& & C' \arrow[u]
\end{tikzcd}
\tag{113}
$$

The argument for why this defines a 2-simplex is analogous to the case of Triangle (1).

- Triangle (3).

$$
\begin{tikzcd}
P \arrow[r, tail, "1 \oplus C'"] & P \oplus C' \arrow[r, tail, "\theta \oplus 1"] & Q \oplus C' \\
& C' \arrow[u, "P \oplus 1"] \arrow[r, tail, "1"] \arrow[ur, phantom, "\square"] & C' \arrow[u, "Q \oplus 1"'] \\
& & O \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \arrow[r, tail, "h_t"] & V \arrow[r, tail, "1"] & V \\
& C' \arrow[u, "k_t"] \arrow[r, tail, "1"] \arrow[ur, phantom, "\square"] & C' \arrow[u, "k_t"'] \\
& & O \arrow[u]
\end{tikzcd}
\tag{114}
$$

To show that this is a 2-simplex, notice Lemma 3.23 already verified that

$$
t' := \left(
\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C' \arrow[d, "k_t"] \\
Q \arrow[r, tail, "h_t"] & V
\end{tikzcd}
\right)
$$

is a distinguished square. The other subdiagrams are obvious.

- Triangle (4).

$$
\begin{tikzcd}
P \arrow[r, tail, "\theta"] & Q \arrow[r, tail, "1\oplus C'"] \arrow[dr, phantom, "\square"] & Q \oplus C' & & Q \arrow[r, tail, "1"] & Q \arrow[r, tail, "h_t"] \arrow[dr, phantom, "\square"] & V \\
& O \arrow[u] \arrow[r, tail] & C' \arrow[u, "Q\oplus 1"'] & & & O \arrow[u] \arrow[r, tail] & C' \arrow[u, "k_t"'] \\
& & C' \arrow[u] & & & & C' \arrow[u]
\end{tikzcd}
\tag{115}
$$

Notice the given square in the left diagram is a formal direct sum square, and is thus distinguished; the remaining subdiagrams are obviously distinguished squares.

$\square$

Claim B.6. *Loop* (39) *is homotopic to loop L.*

Proof. We show that all the indicated triangles of Diagram (45) are boundaries of 2-simplices. The proof that these diagrams do in fact define 2-simplices in $G\mathcal{C}$ is completely analogous to the proof in Claim B.5.

- Triangle (1').

$$
\begin{tikzcd}
A \oplus A' \arrow[r, tail, "f_0"] & P \arrow[r, tail, "1\oplus C'"] \arrow[dr, phantom, "\square"] & P \oplus C' & & A \oplus A' \arrow[r, tail, "(\alpha{,}\alpha')"] & B \oplus B' \arrow[r, tail, "h_u"] \arrow[dr, phantom, "\square"] & V \\
& C \oplus C' \arrow[u, "g_0"] \arrow[r, tail, "1\oplus C'"] & C \oplus C' \oplus C' \arrow[u, "g_0\oplus 1"] & & & C \oplus C' \arrow[u, "(\delta{,}\delta')"] \arrow[r, tail, "1\oplus C'"] & C \oplus C' \oplus C' \arrow[u, "j_u"] \\
& & C' \arrow[u] & & & & C' \arrow[u]
\end{tikzcd}
\tag{116}
$$

To verify that this defines a 2-simplex of $G\mathcal{C}$, notice the given square on the right diagram is the distinguished square u in Lemma 3.23. The remaining subdiagrams can be shown to be distinguished squares by the same argument for Triangle (1) in Claim B.5.

- Triangle (2').

$$
\begin{tikzcd}
A \oplus A' \arrow[r, tail, "f_1"] & Q \arrow[r, tail, "1\oplus C'"] \arrow[dr, phantom, "\square"] & Q \oplus C' & & A \oplus A' \arrow[r, tail, "(\alpha{,}\alpha')"] & B \oplus B' \arrow[r, tail, "h_u"] \arrow[dr, phantom, "\square"] & V \\
& C \oplus C' \arrow[u, "g_1"] \arrow[r, tail, "1\oplus C'"] & C \oplus C' \oplus C' \arrow[u, "g_1\oplus C'"] & & & C \oplus C' \arrow[u, "(\delta{,}\delta')"] \arrow[r, tail, "1\oplus C'"] & C \oplus C' \oplus C' \arrow[u, "j_u"] \\
& & C' \arrow[u] & & & & C' \arrow[u]
\end{tikzcd}
\tag{117}
$$

- Triangle (3').

$$
\begin{tikzcd}
P \arrow[r, tail, "1\oplus C'"] & P \oplus C' \arrow[r, tail, "\theta\oplus 1"] \arrow[dr, phantom, "\square"] & Q \oplus C' & & B \oplus B' \arrow[r, tail, "h_u"] & V \arrow[r, tail, "1"] \arrow[dr, phantom, "\square"] & V \\
& C' \arrow[u, "P\oplus 1"] \arrow[r, tail, "1"] & C' \arrow[u, "Q\oplus 1"] & & & C' \arrow[u, "k_u"] \arrow[r, tail, "1"] & C' \arrow[u, "k_u"] \\
& & O \arrow[u] & & & & O \arrow[u]
\end{tikzcd}
\tag{118}
$$

To show that this is a 2-simplex, notice Lemma 3.23 already verified that

$$
u' := \left(\begin{tikzcd}
O \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & C' \arrow[d, "k_u"] \\
B \oplus B' \arrow[r, tail, "h_u"] & V
\end{tikzcd} \right)
$$

is a distinguished square. The remaining subdiagrams are obvious.

- Triangle (4').

$$
\begin{tikzcd}
P \arrow[r, tail, "\theta"] & Q \arrow[r, tail, "1\oplus C'"] \arrow[dr, phantom, "\square"] & Q\oplus C' \\
& O \arrow[u] \arrow[r, tail] & C' \arrow[u, "Q\oplus 1"'] \\
& & C' \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
B\oplus B' \arrow[r, "1"] & B\oplus B' \arrow[r, tail, "h_t"] \arrow[dr, phantom, "\square"] & V \\
& O \arrow[u] \arrow[r, tail] & C' \arrow[u, "k_t"'] \\
& & C' \arrow[u]
\end{tikzcd}
\tag{119}
$$

$\square$

B.3. **Admissible Triples.** Recall: Lemma 4.4 essentially asks to prove that τ (= the blue loop) is freely homotopic to $(P_2, P_2') \oplus \mu(l(\tau))$ (= the red loop).

$$
\begin{tikzcd}[column sep=small]
(P_2\oplus P_{1/0}\,,P_2'\oplus P_{1/0}) \arrow[rrrr, red, "{(P_2\,,P_2')\oplus l(\tau)}"] \arrow[drr, phantom, "(1)"] & & & & (P_2\oplus P_{2/0}\,,P_2'\oplus P_{2/0}) \\
& (P_1\,,P_1') \arrow[ul] \arrow[urrr, bend left=10] \arrow[rr, blue, "e_1"] & \arrow[ur, phantom, "(2)"] & (P_2\,,P_2') \arrow[ur] & \\
\arrow[r, phantom, "(3)"] & & (P_0\,,P_0') \arrow[ul, blue, "e_0"] \arrow[ur, blue, "e_2"'] \arrow[uull, bend left=30] \arrow[uurr, bend right=30] \arrow[d, "e_2"] & & \arrow[l, phantom, "(4)"] \\
\arrow[r, phantom, "(5)"] & & (P_2\,,P_2') \arrow[uuull, red, bend left=40, "{(P_2\,,P_2')\oplus e(P_{1/0})}"] \arrow[uuurr, red, bend right=40, "{(P_2\,,P_2')\oplus e(P_{2/0})}"'] & & \arrow[l, phantom, "(6)"]
\end{tikzcd}
$$

To prove this, it suffices to show that all six triangles in the diagram are 2-simplices in $G\mathcal{C}$. Here we provide the explicit details.

- Triangle (1)

$$
\begin{tikzcd}
P_1 \arrow[r, tail, "\alpha_{1,2}\oplus P_{1/0}"] & P_2\oplus P_{1/0} \arrow[r, tail, "{(1\,,\alpha_{1/0,2/0})}"] \arrow[dr, phantom, "\square"] & P_2\oplus P_{2/0} \\
& P_{2/1}\oplus P_{1/0} \arrow[u, "{(\alpha_{2/1,2}\,,1)}"] \arrow[r, tail, "{(1\,,\alpha_{1/0,2/0})}"'] & P_{2/1}\oplus P_{2/0} \arrow[u, "{(\alpha_{2/1,2}\,,1)}"'] \\
& & P_{2/1} \arrow[u, "P_{2/1}\oplus\alpha_{2/1,2/0}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
P_1' \arrow[r, tail, "\alpha'_{1,2}\oplus P_{1/0}"] & P_2'\oplus P_{1/0} \arrow[r, tail, "{(1\,,\alpha_{1/0,2/0})}"] \arrow[dr, phantom, "\square"] & P_2'\oplus P_{2/0} \\
& P_{2/1}\oplus P_{1/0} \arrow[u, "{(\alpha'_{2/1,2}\,,1)}"] \arrow[r, tail, "{(1\,,\alpha_{1/0,2/0})}"'] & P_{2/1}\oplus P_{2/0} \arrow[u, "{(\alpha_{2/1,2}\,,1)}"'] \\
& & P_{2/1} \arrow[u, "P_{2/1}\oplus\alpha_{2/1,2/0}"']
\end{tikzcd}
$$

- Triangle (2)

$$
\begin{tikzcd}
P_1 \arrow[r, tail, "\alpha_{1,2}"] & P_2 \arrow[r, tail, "1\oplus P_{2/0}"] \arrow[dr, phantom, "\square"] & P_2\oplus P_{2/0} \\
& P_{2/1} \arrow[u, "\alpha_{2/1,2}"] \arrow[r, tail, "1\oplus P_{2/0}"'] & P_{2/1}\oplus P_{2/0} \arrow[u, "{(\alpha_{2/1,2}\,,1)}"'] \\
& & P_{2/0} \arrow[u, "P_{2/1}\oplus 1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
P_1' \arrow[r, tail, "\alpha'_{1,2}"] & P_2' \arrow[r, tail, "1\oplus P_{2/0}"] \arrow[dr, phantom, "\square"] & P_2'\oplus P_{2/0} \\
& P_{2/1} \arrow[u, "\alpha'_{2/1,2}"] \arrow[r, tail, "1\oplus P_{2/0}"'] & P_{2/1}\oplus P_{2/0} \arrow[u, "{(\alpha_{2/1,2}\,,1)}"'] \\
& & P_{2/0} \arrow[u, "P_{2/1}\oplus 1"']
\end{tikzcd}
$$

- Triangle (3)

$$
\begin{tikzcd}
P_0 \arrow[r, tail, "\alpha_{0,1}"] & P_1 \arrow[r, tail, "\alpha_{1,2}\oplus P_{1,0}"] \arrow[dr, phantom, "\square"] & P_2 \oplus P_{1/0} \\
& P_{1/0} \arrow[u, "{(\alpha_{2/1,2}\,,1)}"] \arrow[r, tail, "\alpha_{1/0,2/0}\oplus P_{1/0}"'] & P_{2/0} \oplus P_{1/0} \arrow[u, "{(\alpha_{2/0,2}\,,1)}"'] \\
& & P_{2/1} \oplus P_{1/0} \arrow[u, "{(\alpha_{2/1,2/0}\,,1)}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
P'_0 \arrow[r, tail, "\alpha'_{0,1}"] & P'_1 \arrow[r, tail, "\alpha'_{1,2}\oplus P_{1,0}"] \arrow[dr, phantom, "\square"] & P'_2 \oplus P_{1/0} \\
& P_{1/0} \arrow[u, "{(\alpha'_{2/1,2}\,,1)}"] \arrow[r, tail, "\alpha_{1/0,2/0}\oplus P_{1/0}"'] & P_{2/0} \oplus P_{1/0} \arrow[u, "{(\alpha_{2/0,2}\,,1)}"'] \\
& & P_{2/1} \oplus P_{1/0} \arrow[u, "{(\alpha_{2/1,2/0}\,,1)}"']
\end{tikzcd}
$$

- Triangle (4)

$$
\begin{tikzcd}
P_0 \arrow[r, tail, "\alpha_{0,2}"] & P_2 \arrow[r, tail, "1\oplus P_{2,0}"] \arrow[dr, phantom, "\square"] & P_2 \oplus P_{2/0} \\
& P_{2/0} \arrow[u, "\alpha_{2/0,2}"] \arrow[r, tail, "1\oplus P_{2/0}"'] & P_{2/0} \oplus P_{2/0} \arrow[u, "{(\alpha_{2/0,2}\,,1)}"'] \\
& & P_{2/0} \arrow[u, "P_{2/0}\oplus 1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
P'_0 \arrow[r, tail, "\alpha'_{0,2}"] & P'_2 \arrow[r, tail, "1\oplus P_{2,0}"] \arrow[dr, phantom, "\square"] & P'_2 \oplus P_{2/0} \\
& P_{2/0} \arrow[u, "\alpha'_{2/0,2}"] \arrow[r, tail, "1\oplus P_{2/0}"'] & P_{2/0} \oplus P_{2/0} \arrow[u, "{(\alpha'_{2/0,2}\,,1)}"'] \\
& & P_{2/0} \arrow[u, "P_{2/0}\oplus 1"']
\end{tikzcd}
$$

- Triangle (5)

$$
\begin{tikzcd}
P_0 \arrow[r, tail, "\alpha_{0,2}"] & P_2 \arrow[r, tail, "1\oplus P_{1,0}"] \arrow[dr, phantom, "\square"] & P_2 \oplus P_{1/0} \\
& P_{2/0} \arrow[u, "\alpha_{2/0,2}"] \arrow[r, tail, "1\oplus P_{1/0}"'] & P_{2/0} \oplus P_{1/0} \arrow[u, "{(\alpha_{2/0,2}\,,1)}"'] \\
& & P_{1/0} \arrow[u, "P_{2/0}\oplus 1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
P'_0 \arrow[r, tail, "\alpha'_{0,2}"] & P'_2 \arrow[r, tail, "1\oplus P_{1,0}"] \arrow[dr, phantom, "\square"] & P'_2 \oplus P_{1/0} \\
& P_{2/0} \arrow[u, "\alpha'_{2/0,2}"] \arrow[r, tail, "1\oplus P_{1/0}"'] & P_{2/0} \oplus P_{1/0} \arrow[u, "{(\alpha'_{2/0,2}\,,1)}"'] \\
& & P_{1/0} \arrow[u, "P_{2/0}\oplus 1"']
\end{tikzcd}
$$

- Triangle (6)

$$
\begin{tikzcd}
P_0 \arrow[r, tail, "\alpha_{0,2}"] & P_2 \arrow[r, tail, "1\oplus P_{2,0}"] \arrow[dr, phantom, "\square"] & P_2 \oplus P_{2/0} \\
& P_{2/0} \arrow[u, "\alpha_{2/0,2}"] \arrow[r, tail, "1\oplus P_{2/0}"'] & P_{2/0} \oplus P_{2/0} \arrow[u, "{(\alpha_{2/0,2}\,,1)}"'] \\
& & P_{2/0} \arrow[u, "P_{2/0}\oplus 1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
P'_0 \arrow[r, tail, "\alpha'_{0,2}"] & P'_2 \arrow[r, tail, "1\oplus P_{2,0}"] \arrow[dr, phantom, "\square"] & P'_2 \oplus P_{2/0} \\
& P_{2/0} \arrow[u, "\alpha'_{2/0,2}"] \arrow[r, tail, "1\oplus P_{2/0}"'] & P_{2/0} \oplus P_{2/0} \arrow[u, "{(\alpha'_{2/0,2}\,,1)}"'] \\
& & P_{2/0} \arrow[u, "P_{2/0}\oplus 1"']
\end{tikzcd}
$$

These diagrams are the obvious choices – no surprises here. It remains to justify that they indeed define 2-simplices in $G\mathcal{C}$ by checking that all the relevant subdiagrams define distinguished squares. This is a straightforward exercise once we know Lemma A.3.

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