

Chapter 1

Countervailing

These are the payoffs for Langlois $\begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix}$ and Axelrod $\begin{pmatrix} 3 & 1 \\ 5 & 0 \end{pmatrix}$ values of $\begin{pmatrix} R & S \\ T & P \end{pmatrix}$

$$\begin{aligned}
 & (1 - x_i, x_i) \begin{pmatrix} R & S \\ T & P \end{pmatrix} \begin{pmatrix} 1 - x_j \\ x_j \end{pmatrix} \\
 &= (1 - x_i)(1 - x_j)R + (1 - x_i)x_jS + x_i(1 - x_j)T + x_ix_jP \\
 &= (1 - x_i - x_j + x_ix_j)R + (x_j - x_ix_j)S + (x_i - x_ix_j)T + x_ix_jP \\
 &= \begin{cases} -2x_j + 2x_ix_j + x_i - x_ix_j - x_ix_j & \text{Langlois} \\ 3 + 2x_i - 3x_j - x_ix_j & \text{Axelrod} \end{cases} \\
 &= \begin{cases} x_i - 2x_ix_j & \text{Langlois} \\ 3 + 2x_i - 3x_j - x_ix_j & \text{Axelrod} \end{cases}
 \end{aligned}$$

Question about discounting

In the derivation of Zero Determinant strategies, Press and Dyson use the transition probability matrix

$$\begin{bmatrix} p_1q_1 & p_1(1 - q_1) & (1 - p_1)q_1 & (1 - p_1)(1 - q_1) \\ p_2q_3 & p_2(1 - q_3) & (1 - p_2)q_3 & (1 - p_2)(1 - q_3) \\ p_3q_2 & p_3(1 - q_2) & (1 - p_3)q_2 & (1 - p_3)(1 - q_2) \\ p_4q_4 & p_4(1 - q_4) & (1 - p_4)q_4 & (1 - p_4)(1 - q_4) \end{bmatrix}$$

for IPD with mixed strategies $\mathbf{p} = (p_1, p_2, p_3, p_4)$, $\mathbf{q} = (q_1, q_2, q_3, q_4)$ for players X, Y respectively. If \mathbf{v} is the stationary distribution of the above Markov chain, then they show that the dot product of \mathbf{v} with an arbitrary vector \mathbf{f} is given by the determinant

$$\mathbf{v} \cdot \mathbf{f} \equiv D(\mathbf{p}, \mathbf{q}, \mathbf{f}) = \det \begin{bmatrix} -1 + p_1q_1 & -1 + p_1 & -1 + q_1 & f_1 \\ p_2q_3 & -1 + p_2 & -q_3 & f_2 \\ p_3q_2 & p_3 & -1 + q_2 & f_3 \\ p_4q_4 & p_4 & q_4 & f_4 \end{bmatrix},$$

so if $\mathbf{S}_X = (R, S, T, P)$, then on average X 's payoff is $\mathbf{v} \cdot \mathbf{S}_X = D(\mathbf{p}, \mathbf{q}, \mathbf{S}_X)$. Press and Dyson use this average payoff for their definitions and proofs, so in their game there is no discounting.

Question: I'm not sure if this is a problem when comparing ZD strategies with countervailing strategies, which use a discount factor in their definition.

One way I could see doing this is making the game have five states: $(cc, cd, dc, dd, \text{sink})$, where sink is the end of game state associated with payoff of 0 for both players. I'm not sure what will happen to the ZD strategies if they are defined on this 5-state Markov chain.

The other way would be to say that if X 's average payoff is $S_X = \mathbf{v} \cdot \mathbf{S}_X$, then X 's discounted payoff should be $\frac{S_X}{1-w}$.

Is either of these approaches reasonable?

Are countervailing strategies ZD? Using your example.

The payoffs for S_X and S_Y are given by

$$S_X = \frac{\mathbf{v} \cdot \mathbf{S}_X}{\mathbf{v} \cdot \mathbf{1}} = \frac{D(\mathbf{p}, \mathbf{q}, \mathbf{S}_X)}{D(\mathbf{p}, \mathbf{q}, \mathbf{1})}$$

$$S_Y = \frac{\mathbf{v} \cdot \mathbf{S}_Y}{\mathbf{v} \cdot \mathbf{1}} = \frac{D(\mathbf{p}, \mathbf{q}, \mathbf{S}_Y)}{D(\mathbf{p}, \mathbf{q}, \mathbf{1})}$$

so a linear (affine?) combination of payoffs is given by

$$\alpha S_X + \beta S_Y + \gamma = \frac{D(\mathbf{p}, \mathbf{q}, \alpha \mathbf{S}_X + \beta \mathbf{S}_Y + \gamma \mathbf{1})}{D(\mathbf{p}, \mathbf{q}, \mathbf{1})}$$

from the countervailing example with $\psi_j(x_i, x_j) = \frac{x_i + x_j}{2.7}$ yielding $\mathbf{p} = (0, \frac{1}{2.7}, \frac{1}{2.7}, \frac{2}{2.7})^T$

$$\begin{pmatrix} 0 \\ \frac{1}{2.7} \\ \frac{1}{2.7} \\ \frac{2}{2.7} \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ -2 \\ 1 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ -2 \\ -1 \end{pmatrix} + \begin{pmatrix} \gamma \\ \gamma \\ \gamma \\ \gamma \end{pmatrix}$$

solving for α, β, γ , we get $\alpha = \beta = -\frac{1}{2.7}$, $\gamma = 0$, so the strategy \mathbf{p} is the ZD strategy that enforces the relationship $-\frac{1}{2.7}S_X - \frac{1}{2.7}S_Y = 0$ or simply $S_X + S_Y = 0 = R$.