Chapter 1

Countervailing

These are the payoffs for Langlois $\begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix}$ and Axelrod $\begin{pmatrix} 3 & 0 \\ 5 & 1 \end{pmatrix}$ values of $\begin{pmatrix} R & S \\ T & P \end{pmatrix}$

$$
\begin{aligned}
&(1-x_i,\ \ x_i)\begin{pmatrix} R & S \\ T & P \end{pmatrix}\begin{pmatrix} 1-x_j \\ x_j \end{pmatrix} \\
=&(1-x_i)(1-x_j)R + (1-x_i)x_jS + x_i(1-x_j)T + x_ix_jP \\
=&(1-x_i-x_j+x_ix_j)R + (x_j - x_ix_j)S + (x_i - x_ix_j)T + x_ix_jP \\
=&R + (T-R)x_i + (S-R)x_j + (R-S-T+P)x_ix_j \\
=&\begin{cases} x_i - 2x_j & \text{Langlois} \\ 3 + 2x_i - 3x_j - x_ix_j & \text{Axelrod} \end{cases}
\end{aligned}
$$

Question about discounting

In the derivation of Zero Determinant strategies, Press and Dyson use the transition probability matrix

$$
\begin{bmatrix}
p_1q_1 & p_1(1-q_1) & (1-p_1)q_1 & (1-p_1)(1-q_1) \\
p_2q_3 & p_2(1-q_3) & (1-p_2)q_3 & (1-p_2)(1-q_3) \\
p_3q_2 & p_3(1-q_2) & (1-p_3)q_2 & (1-p_3)(1-q_2) \\
p_4q_4 & p_4(1-q_4) & (1-p_4)q_4 & (1-p_4)(1-q_4)
\end{bmatrix}
$$

for IPD with mixed strategies $\mathbf{p} = (p_1, p_2, p_3, p_4), \mathbf{q} = (q_1, q_2, q_3, q_4)$ for players X, Y respectively. If $\mathbf{v}$ is the stationary distribution of the above Markov chain, then they show that the dot product of $\mathbf{v}$ with an arbitrary vector $\mathbf{f}$ is given by the determinant

$$
\mathbf{v} \cdot \mathbf{f} \equiv D(\mathbf{p}, \mathbf{q}, \mathbf{f}) = \det \begin{bmatrix}
-1 + p_1q_1 & -1 + p_1 & -1 + q_1 & f_1 \\
p_2q_3 & -1 + p_2 & -q_3 & f_2 \\
p_3q_2 & p_3 & -1 + q_2 & f_3 \\
p_4q_4 & p_4 & q_4 & f_4
\end{bmatrix},
$$

so if $\mathbf{S}_X = (R, S, T, P)$, then on average X's payoff is $\mathbf{v} \cdot \mathbf{S}_X = D(\mathbf{p}, \mathbf{q}, \mathbf{S}_X)$. Press and Dyson use this average payoff for their definitions and proofs, so in their game there is no discounting.

Question: I'm not sure if this is a problem when comparing ZD strategies with countervailing strategies, which use a discount factor in their definition.

One way I could see doing this is making the game have five states: $(cc, cd, dc, dd, \text{sink})$, where sink is the end of game state associated with payoff of 0 for both players. I'm not sure what will happen to the ZD strategies if they are defined on this 5-state Markov chain.

The other way would be to say that if X's average payoff is $S_X = \mathbf{v} \cdot \mathbf{S}_X$, then X's discounted payoff should be $\frac{S_X}{1-w}$.

Is either of these approaches reasonable?

Are countervailing strategies ZD? Using your example.

The payoffs for S_X and S_Y are given by

$$S_X = \frac{\mathbf{v} \cdot \mathbf{S}_X}{\mathbf{v} \cdot \mathbf{1}} = \frac{D(\mathbf{p}, \mathbf{q}, \mathbf{S}_X)}{D(\mathbf{p}, \mathbf{q}, \mathbf{1})}$$

$$S_Y = \frac{\mathbf{v} \cdot \mathbf{S}_Y}{\mathbf{v} \cdot \mathbf{1}} = \frac{D(\mathbf{p}, \mathbf{q}, \mathbf{S}_Y)}{D(\mathbf{p}, \mathbf{q}, \mathbf{1})}$$

so a linear (affine?) combination of payoffs is given by

$$\alpha S_X + \beta S_Y + \gamma = \frac{D(\mathbf{p}, \mathbf{q}, \alpha\mathbf{S}_X + \beta\mathbf{S}_Y + \gamma\mathbf{1})}{D(\mathbf{p}, \mathbf{q}, \mathbf{1})}$$

from the countervailing example with $\psi_j(x_i, x_j) = \frac{x_i + x_j}{2.7}$ yielding $\mathbf{p} = (0, \frac{1}{2.7}, \frac{1}{2.7}, \frac{2}{2.7})^T$

$$\begin{pmatrix} 0 \\ \frac{1}{2.7} \\ \frac{1}{2.7} \\ \frac{2}{2.7} \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ -2 \\ 1 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ -2 \\ -1 \end{pmatrix} + \begin{pmatrix} \gamma \\ \gamma \\ \gamma \\ \gamma \end{pmatrix}$$

solving for α, β, γ, we get $\alpha = \beta = -\frac{1}{2.7}$, $\gamma = 0$, so the strategy $\mathbf{p}$ is the ZD strategy that enforces the relationship $-\frac{1}{2.7}S_X - \frac{1}{2.7}S_Y = 0$ or simply $S_X + S_Y = 0 = R$.