COMS W4701: Artificial Intelligence, Spring 2025

Homework 4

Instructions: Compile all written solutions for this assignment in a single, **typed** PDF file. If you use Python or any other computational tool, append the code to your PDF file. Turn in your submission on Gradescope, and **tag all pages**. Please be mindful of the deadline and late policy, as well as our policies on citations and academic honesty.

For Problems 4-5, you may use the Bayes Net applet to verify your answers, but you should carry out all computations yourself, either by hand or using Python (please attach all notebooks/code). When computing distributions, show or print out each intermediate factor resulting from each variable summation step.

Problem 1: Measles Diagnosis (18 points)

We have the following joint probability distribution describing three Boolean random variables. M describes whether one has caught measles, S describes whether one is presenting measles symptoms, and T describes whether a measles test comes up positive or negative.

M	S	T	$\Pr(M, S, T)$
$+m$	$+s$	$+t$	0.108
$+m$	$+s$	$-t$	0.012
$+m$	$-s$	$+t$	0.072
$+m$	$-s$	$-t$	0.008
$-m$	$+s$	$+t$	0.016
$-m$	$+s$	$-t$	0.064
$-m$	$-s$	$+t$	0.144
$-m$	$-s$	$-t$	0.576

(a) (3 pts) Find the marginal distributions of each of the three random variables.

(b) (6 pts) Find the joint distributions of each of the three different *pairs* of the random variables.

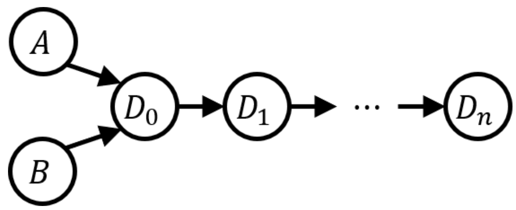
(c) (3 pts) Using the distributions you found above, show whether each pair of random variables is independent.

(d) (4 pts) Find the following conditional distributions: $\Pr(S|+m)$, $\Pr(T|+m)$, $\Pr(S, T|+m)$.

(e) (2 pts) Explain whether we can conclude from the three conditional distributions that you found above that S and T are conditionally independent given M. If we cannot conclude this, what additional information do we need?

Problem 2: Descendants of Colliders (18 points)

Here we will investigate the absence of conditional independence guarantees between two random variables in a Bayes net when an arbitrary common descendant is observed. We will consider the case of a simple chain of descendants, as shown below.



Suppose that all random variables are binary. The marginal distributions of A and B are both uniform $(0.5, 0.5)$, and the CPTs of the collider D_0 and its descendants are as follows:

A	B	$\Pr(+d_0 \mid A, B)$
$+a$	$+b$	0.0
$+a$	$-b$	0.5
$-a$	$+b$	0.5
$-a$	$-b$	1.0

D_{i-1}	$\Pr(+d_i \mid D_{i-1})$
$+d_{i-1}$	0.0
$-d_{i-1}$	1.0

(a) (6 pts) Compute the distributions $\Pr(A, B| + d_0)$ and $\Pr(A, B| - d_0)$. Show whether A is conditionally independent of B given D_0.

(b) (3 pts) Find an analytical expression for the joint probabilities $\Pr(A, B, d_n)$. The expression should only contain terms in the Bayes net CPTs and involve product and sum operations.

(c) (6 pts) Suppose that n is odd. Starting from your expression in (b), use the Bayes net CPTs to numerically compute the joint conditional distributions $\Pr(A, B| + d_n)$ and $\Pr(A, B| - d_n)$. Show whether A is conditionally independent of B given D_n.

(d) (3 pts) What happens when we attempt to compute the distributions $\Pr(A, B| + d_0, +d_n)$ and $\Pr(A, B| - d_0, -d_n)$, starting from your expression in (b)? Explain the results. Are these proper distributions?

Problem 3: Car Start Bayes Net (24 points)

This problem investigates the “Car Starting” Bayes net from the Sample Problems of the Belief and Decision Networks tool on AIspace.

(a) (4 pts) Suppose we observe no variables. (i) What variables, if any, are guaranteed to be independent of “Starter System OK”? (ii) What variables, if any, are guaranteed to be independent of “Spark Adequate”?

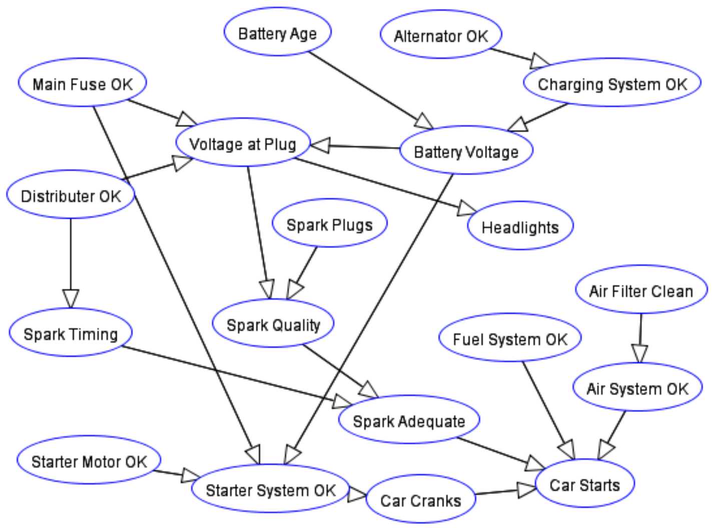
(b) (4 pts) Suppose we observe “Voltage at Plug”. (i) Relative to (a), which variables, if any, gain guarantees of independence to ”Starter System OK”? (ii) Which variables, if any, lose guarantees of independence to ”Starter System OK”?

(c) (4 pts) (i) Suppose we observe “Battery Voltage” and “Spark Adequate”. (i) Relative to (a), which variables, if any, gain guarantees of independence to ”Starter System OK”? (ii) Which variables, if any, lose guarantees of independence to ”Starter System OK”?

(d) (4 pts) Write an analytical expression for the marginal distribution Pr(Voltage at Plug). Your expression should only include Bayes net parameters (CPTs) and sum and product operations. You may abbreviate the variable names using just the first letters of the words, i.e. $\Pr(VAP)$ in place of Pr(Voltage at Plug).

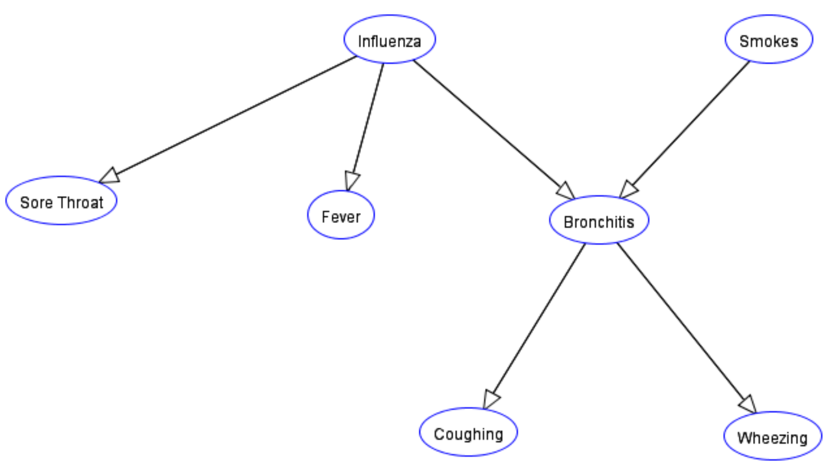
(e) (4 pts) Repeat the above for Pr(Spark Adequate|voltage at plug). "voltage at plug" represents an *observed* variable.

(f) (4 pts) Repeat the above for Pr(Air Filter Clean|spark adequate, car cranks, car starts). Your expression may also include a normalization (division) operation, if necessary.



Problem 4: Diagnostic Bayes Net (20 points)

The following Bayes net is the "Simple Diagnostic Example" from the Sample Problems of the Belief and Decision Networks tool on AIspace. All variables are binary.



(a) (6 pts) We are interested in computing the distribution Pr(Fever | Wheezing = True). Find a minimal analytical expression for this distribution (or its unnormalized version) in terms of the Bayes net CPTs, and then rewrite each term as a factor explicitly showing the variable dependencies. What is the maximum size of the intermediate factor if all marginalization is done at the end?

(b) (6 pts) We employ variable elimination to solve the query above. Consider the following two variable orderings (you may only need to sum over a subset of each):

(i) influenza, smokes, sore throat, fever, bronchitis, coughing, wheezing

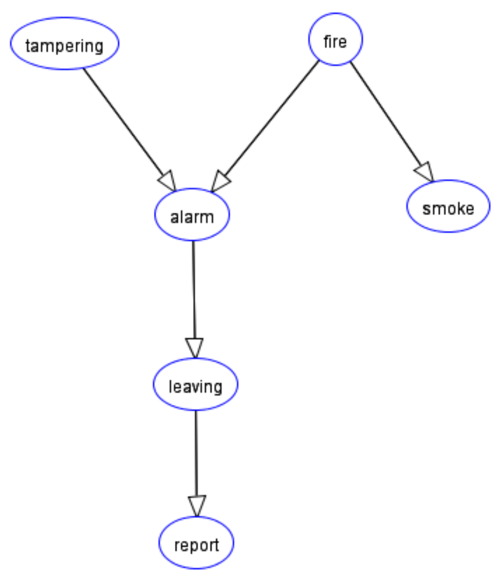
(ii) wheezing, coughing, bronchitis, fever, sore throat, smokes, influenza

For each of these orderings, rewrite your analytical expression by splitting up the sum over each variable, and factor out the maximal number of terms from each summation. Remember that sums are evaluated from right to left, or from inside to out. What is the size of the largest intermediate factor in each?

(c) (8 pts) Follow the more efficient ordering from above to numerically compute Pr(Fever | Wheezing = True) using the applet parameters.

Problem 5: Fire Alarm Bayes Net (20 points)

The following Bayes net is the "Fire Alarm Belief Network" from the Sample Problems of the Belief and Decision Networks tool on AIspace. All variables are binary.



(a) (6 pts) We are interested in computing the distribution Pr(Smoke | Report = True). Find a minimal analytical expression for this distribution (or its unnormalized version) in terms of the Bayes net CPTs, and then rewrite each term as a factor explicitly showing the variable dependencies. What is the maximum size of the intermediate factor if all marginalization is done at the end?

(b) (6 pts) We employ variable elimination to solve the query above. Consider the following two variable orderings (you may only need to sum over a subset of each):

(i) tampering, fire, alarm, smoke, leaving, report

(ii) report, leaving, smoke, alarm, fire, tampering

For each of these orderings, rewrite your analytical expression by splitting up the sum over each variable, and factor out the maximal number of terms from each summation. Remember that sums are evaluated from right to left, or from inside to out. What is the size of the largest intermediate factor in each?

(c) (8 pts) Follow the more efficient (in terms of space) ordering from above to numerically compute $\Pr(\text{Smoke} \mid \text{Report} = \text{True})$ using the applet parameters.