MEMORANDUM
RM-3408-PR
DECEMBER 1962

# AN ANALYTICAL STUDY OF THE PERT ASSUMPTIONS

K. R. MacCrimmon and C. A. Ryavec

PREPARED FOR:
UNITED STATES AIR FORCE PROJECT RAND

The RAND Corporation
SANTA MONICA • CALIFORNIA

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This research is sponsored by the United States Air Force under Project RAND — Contract No. AF 49(638)-700 — monitored by the Directorate of Development Planning, Deputy Chief of Staff, Research and Technology, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force. 

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# PREFACE

This Memorandum is part of a continuing research effort on network planning techniques for use in systems and project management. In this phase, the authors studied the PERT (Program Evaluation and Review Technique) system and analyzed its chief mathematical assumptions which have long been subject to question. This Memorandum presents the results of their attempt to evaluate the magnitude of the errors inherent in these assumptions.

Work is continuing on other applications of network planning, extensions into resource (cost) management, and procedures for integrating their product with current management information systems, for example, those described by Air Force Regulation 375 covering System Package Programming.

The Memorandum should be of use to various Air Force and other Governmental organizations concerned with the application of PERT and related systems. Offices which have requested an evaluation of this subject include the Office of the Assistant Secretary for Financial Management, the Directorate of Status Analysis at Systems Command, and the Directorate of Management Research and Evaluation at Aeronautical Systems Division.

K. R. MacCrimmon is a RAND consultant.

## SUMMARY

The purpose of this Memorandum is to present the results of a mathematical analysis of the standard assumptions used in PERT calculations. The principal objective of this analysis was to obtain an indication of the magnitude and direction of errors introduced by these assumptions. A secondary goal was to restate the mathematical aspects of the PERT model in order to provide a better understanding on which further study might be based.

This Memorandum is divided into two main sections. The first section deals with the analysis of those assumptions that are relevant to the individual activities. Three possible sources of errors are considered in this section: (1) the assumption of a beta distribution, (2) imprecise time estimates, and (3) the assumption of the standard deviation (one-sixth of the range) and the approximation formula for calculating the mean time. Since these errors can be either positive or negative, some degree of cancellation would be expected to occur in a network.

The second section deals with the PERT network as a whole and with an analysis of those calculations that relate to the project mean, variance, and probability statements. In general, a network with many independently parallel paths having approximately equal durations is found to give the largest errors in the PERT-calculated mean and variance. On the other hand, if many of the paths are cross-connected or if one path is significantly longer than any of the other paths, then the errors are reduced considerably.

Techniques for network reduction are also suggested in the second

section. Networks often contain many activities whose likelihood of being on a critical path is very low. Eliminating these activities from consideration may reduce the network considerably without affecting significantly the final results. In general, if the sum of the minimum times along one path is greater than the sum of the maximum times along another parallel path, then the latter path will not influence the calculation of the time distribution at the common end node.

As a result of this study, the authors have suggested that the concept of relative criticalness of an individual activity, regardless of its association with the critical path, may, perhaps, be more valid for the stochastic model (PERT) than is the critical path concept. This suggestion is based on the fact that the PERT-calculated critical path does not necessarily contain the most critical activities.

For computational ease, small network configurations containing representative properties of larger networks are used throughout this study. It is believed that the results obtained from this study, based on these typical networks, are applicable, to some extent, to larger and more complex networks.

## ACKNOWLEDGMENTS

The authors wish to express their appreciation to B. L. Fry and F. S. Pardee of The RAND Corporation for their helpful suggestions throughout the duration of the study and to J. R. Jackson of the University of California at Los Angeles for discussions which led to some of the concepts developed herein. In addition, the authors wish to thank T. E. Harris, E. M. Scheuer, and R. M. Van Slyke for their critical comments.

CONTENTS

# I. INTRODUCTION

## NETWORK MODELS

During the past several years, new techniques based on network models have been developed to aid management in planning and controlling large-scale projects. One such technique, which is discussed in this Memorandum, is PERT (Program Evaluation and Review Technique). The PERT technique has received widespread interest and is currently being used for many types of projects.

In general, the type of project for which PERT is often used comprises numerous activities -- sometimes thousands -- many of which may be interrelated in complex, and often subtle, ways. One of the significant features of PERT and other similar techniques is that the activities as well as the interrelationships are depicted in their entirety by a network of directed arcs (arcs with arrows, which denote the sequence of the activities they represent). The nodes, called events, represent instants in time when certain activities have been completed and others can then be started. All inwardly-directed activities at a node must be completed before any outwardly-directed activity of that node can be started. A path is defined as an unbroken chain of activities from the origin node (the beginning of the project) to some other node. An event is said to have occurred when all activities on all paths directed into the node representing that event have been completed.

## TIME ELEMENT

Each activity takes time to perform. Thus, it will have some duration associated with it. The time at which an event occurs is the maximum of the durations of the inwardly-directed paths* to that event, since all of the activities directed into the event must have been completed. The project duration is, then, the maximum of the elapsed times along all paths from the origin to the terminal node (the event marking the completion of the project). The path with the longest duration is called the "critical path," and the activities on it, "critical activities." Any delay in a critical activity will obviously cause a corresponding delay in the entire project.

The duration associated with an activity can be a single number (the deterministic case), or, as in PERT, it can be a random variable with some distribution (the stochastic case). The times used for each activity duration are based on time estimates made by the managers or engineers most directly concerned with the performance of the activity.

## PERT OUTLINE

The study reported herein will deal with those aspects of the PERT procedure which come after the establishment of the network itself. The following analysis assumes that a unique network representation has been established. However, whether or not a unique network representation can be established *a priori* is open to question. Nearly all of the assumptions made by PERT concerning its mathematical model are discussed in this Memorandum. These assumptions will be analyzed

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*Since the activities which form any path are connected serially, the duration of a path is the sum of the durations of those activities which compose it.

according to the following outline, distinguishing between those on the level of individual activities and those on the level of the whole network.

1. Activities

   a. Three time estimates are obtained for each activity.

   b. A beta distribution with its standard deviation equal to one-sixth of its range is fitted to the three time estimates to obtain a mean and variance, which are then used as the parameters of a normal distribution* of activity duration.

2. The Network

   a. The critical path is then determined using only the means of the activity durations.

   b. The Central-Limit Theorem is then invoked to obtain a normal project distribution, whose mean and variance are the sums of the means and variances, respectively, of those activities on the critical path. Probability statements concerning the project completion time are made from the normal distribution so determined.

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*See first footnote on page 6.

# II. ACTIVITIES

## UNCERTAINTY IN ACTIVITY DURATION

The activities in complex research and development programs are usually unique to a particular program and are seldom of a routine or repetitive nature. Those people most directly involved in the performance of these activities, however, usually have some experience in doing similar jobs. Thus, on the basis of their experience, it is felt that they can estimate how long some new activity will take to complete. On the other hand, the activities often require creative ability -- something which is hard to measure in individuals. By the nature of these activities, then, any estimate of their length must be an uncertain one.

In order to reflect this uncertainty, a stochastic model may be used; that is, one in which some measure of the possible variation in activity duration is given. This may take the form of a distribution showing the various probabilities that an activity will be completed in its various possible completion times. Alternatively, it may just be some number which represents the variance, or some other similar concept of variation. This latter method does not make any assumption about a distribution form.

## PERT ACTIVITY DURATION

PERT handles uncertainty by assuming that the probable duration of an activity is beta-distributed. The probability density function of the beta distribution is $f(t) = K \cdot (t-a)^{\alpha} \cdot (b-t)^{\beta}$. A few examples are plotted in Fig. 1.

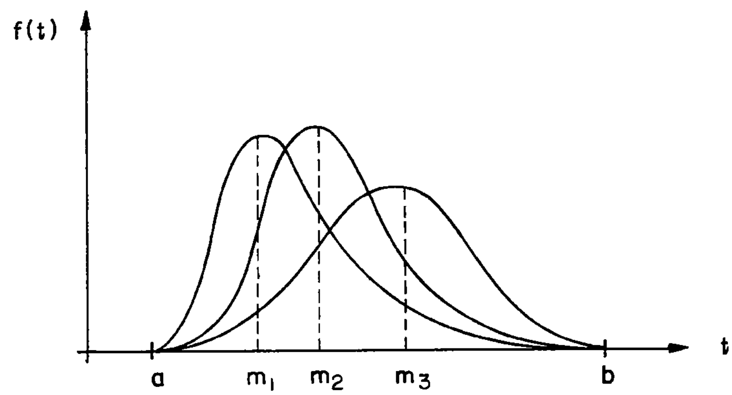


Fig. 1 -- Examples of Beta Distributions

In order to determine a unique beta distribution, the endpoints a and b, and the exponents $\alpha$ and $\beta$ must be specified. PERT uses two time estimates* (the optimistic time and the pessimistic time) to specify a and b. The optimistic time is that time earlier than which the activity could not be completed, and the pessimistic time is the longest time the activity could ever take to complete (barring "acts of God").** A third time estimate m, the most likely time, is also obtained. The value of m is the mode of the distribution, and this value, in conjunction with the PERT assumption that the standard deviation of the distribution is 1/6 of its range, serves to determine the two parameters, the exponents $\alpha$ and $\beta$.***

It is often convenient to consider only the mean and variance of a distribution, rather than the entire distribution. Sometimes these

*All time estimates are based on the use of a given, unchanging level of resources.

**There are slightly different interpretations of the meaning of a and b.

***The beta distribution so determined has parameters, $\alpha$ and $\beta$, around 2, 3, or 4. (See Appendix C.)

two values determine a unique distribution, as in the case of a normal distribution. With other distributions, such as the beta, the mean and variance alone do not determine a unique distribution. Even though PERT deals with a beta distribution, it is convenient to characterize the activity duration in terms of a mean and variance. The variance is $(b-a)^2/36$ (the square of the assumed standard deviation). In general, the determination of the mean involves the solution of a cubic equation. Values of the mean were calculated from the roots of cubic equations, and in order to simplify the future calculations of activity means, a linear approximation, $(a+4m+b)/6$, to these values was made. These expressions for the mean and variance are used to represent the activity duration in all future PERT calculations.*

## ACTUAL ACTIVITY DISTRIBUTIONS

Although the PERT model makes specific assumptions about the form of the activity distributions, the true distributions** are unknown. However, once an activity has been specified precisely, the distribution of that activity's duration has, thereby, been determined (although the distribution may be, and probably is, unknown).*** To the extent of the authors' knowledge, no empirical study has been made to determine the form of activity distributions. Indeed, there would be many problems

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*Although these two expressions were derived from a beta distribution, PERT is inconsistent about whether the activity durations are now normally or beta-distributed. This inconsistency appears in the original PERT report[4]. Appendix A of that report centers its discussion on a normal distribution (p. A2: "Each activity has a <u>time</u>. The time is stochastic and normally distributed ...."), whereas Appendix B discusses the beta distribution (p. B6: "As a model of the distribution of an activity time, we introduce the beta distribution ....").

**The term "distribution" used in this Memorandum will include the special case of a distribution with zero variance.

***It may be argued that there is no objective distribution *a priori* because of the nature of the PERT activities.

connected with such a study, not the least of which would be the non-repetitive nature of the activities. The choice of a particular distribution, such as the beta, while seeming rather arbitrary, has a heuristic justification, since it possesses certain features which an actual activity distribution could be expected to possess.

## EXPECTED PROPERTIES OF THE ACTUAL ACTIVITY DISTRIBUTION

Although a distribution may be polymodal, it is reasonable to expect that it is usually unimodal, since the probability that an activity will be completed in some small interval near the endpoints of the range of the activity duration is generally smaller than the probability that it will be completed in a similar interval at some intermediate point. Furthermore, as the interval moves to less extreme times and approaches some average, or most likely time, the probability that the activity will be completed in the interval should get larger and approach some maximum value.

Secondly, although the distribution need not be continuous, a distribution with this property is appropriate in many cases, and serves as a good approximation in others. Continuity reflects the property that if an activity has a particular probability of being completed in a certain small interval, the probability is only slightly increased when the size of the interval is slightly increased.

Lastly, the property that the distribution touches the abscissa at two non-negative points, although also not strictly necessary, has some merit. An activity can never be completed in a negative time. Therefore, the probability that an activity will be completed in the closed interval from minus infinity to zero is zero.* On the other

*Since the normal distribution assigns a positive -- although possibly very small -- probability to the completion of an activity in this range, it should technically be ruled out as a possible activity distribution.

hand, the probability that the activity will be completed in some small interval about the most likely time, m, is certainly positive. Thus, it is not unreasonable to postulate the existence of a point, $a \in [0,m)$, at which the distribution function, $f(t)$, satisfies: $f(t) = 0$, $t \in [-\infty,a)$; $f(t) \geq 0$, $t \in [a,m)$. A similar point, b, can also be postulated, where $b \in (m,\infty]$.* These two points, along with continuity, guarantee the third property.

POSSIBLE ACTIVITY-BASED ERRORS

Three possible sources of error (due to the PERT assumptions) in the PERT calculations of activity means and variances will be considered:

1. The true distribution of an activity (and its mean and standard deviation) is probably not known. Given that the distribution is continuous, unimodal, and that it touches the abscissa at two non-negative points, how much of an error would be introduced into the over-all PERT calculations of an activity mean and standard deviation by the assumption that the activity duration is beta-distributed?

2. If it is assumed that an activity is beta-distributed, with mean and standard deviation given by $(a+4m+b)/6$ and $(b-a)/6$, respectively, what errors can be introduced into the PERT calculations if the estimates of a, m, and b are inexact?

3. Finally, if it is assumed that an activity distribution is a beta, and that the expression for this function is known exactly, what errors are introduced into the PERT calculations by the assumption $\sigma_e = (b-a)/6$ and the estimate $t_e = (a+4m+b)/6$, if a, m, and b are known exactly?

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*The beta is an example of a distribution where b is finite. The gamma is an example where b is infinite.

## POSSIBLE ERROR INTRODUCED BY THE ASSUMPTION OF A BETA DISTRIBUTION

If the actual activity distribution possesses the aforementioned three properties (i.e., unimodality, continuity, and two non-negative abscissa intercepts), then the beta approximation to this distribution is at least correct with regard to its general shape. Different distributions, which merely possess these properties, however, could well have very different means and variances; and hence -- at least theoretically -- an imprecise knowledge of the actual activity distribution could contribute significantly to any over-all error between the PERT-calculated mean and variance of an activity and its actual mean and variance.

Consider, for example, three distributions shown in Fig. 2. Each of these distributions possesses the three properties discussed previously; and since the use of a beta distribution, $D_1$, proceeded from intuitive grounds -- from the belief that the actual activity distribution should satisfy the aforementioned three properties -- then it can be assumed that $D_2$ and $D_3$ can also be possible activity distributions. With this assumption, the extent and direction of any possible errors due to the

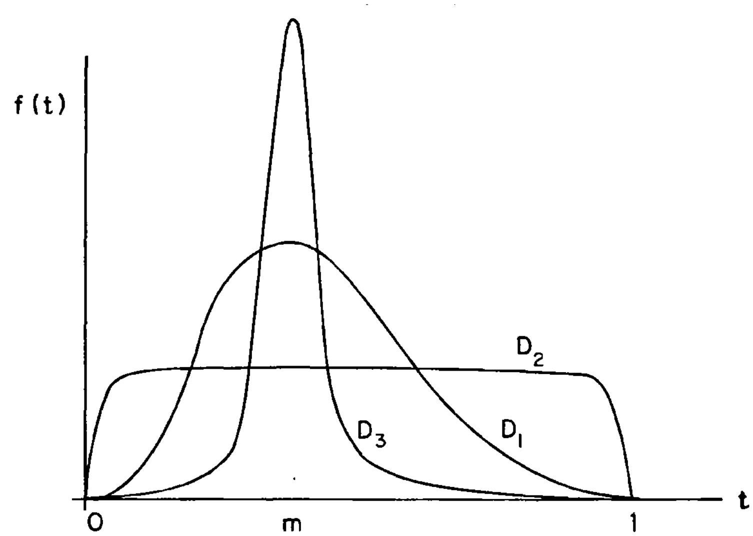


Fig. 2 -- Examples of Possible Activity Distributions

use of a beta distribution can be determined. The three distributions have the range $[0,1]^*$ and have their modes at m.

$D_1$ represents a beta distribution with a standard deviation equal to 1/6 of the range (the standard PERT assumption). $D_2$ represents a quasi-uniform distribution. Therefore, its mean and standard deviation will be very close to 1/2 and $\sqrt{1/12}$, respectively. $D_3$ is a quasi-delta function with its mean very close to its mode and its standard deviation very close to zero. Although $D_2$ and $D_3$ are extreme examples of possible activity distributions -- and hence rather unlikely -- they serve to put bounds on possible errors in the calculation of an activity mean and standard deviation due to the use of an incorrect activity distribution.

It was found that this error depended upon the mode. If the mode was allowed to vary between zero and one, it was found (see Appendix A) that the PERT-calculated mean and standard deviation could be in error by as much as 33 per cent and 16 per cent of the range, respectively. It was also found that the errors could be both positive and negative; and, thus, it could be expected that some degree of cancellation would occur when the individual activities were combined in series in a network. The extent, and the net result of such cancellation, are dependent on three factors: (1) the number of activities in series; (2) the ranges of the activity durations; and (3) the skewness of the activity distributions. If, in a network, there are a large number of activities in series, and if their ranges are about equal, then a relatively high

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*Zero and one were chosen as endpoints of the range for computational ease. The results, expressed as a per cent of the range, can be extended to an arbitrary range $[a,b]$.

degree of cancellation could be expected (assuming that the extent and direction of the skewness of activity distributions are arbitrary). A net error, equal to the sum of the worst positive and the worst negative possible errors, has been calculated for this case. The values of the net error in the mean and standard deviation are 17 per cent and 4.4 per cent of the range, respectively.*

It can be noted that in practice, however, the skewness of the activities tends to be biased to the right** and that the range of activity durations can often differ by an order of magnitude. Moreover, many networks have a large number of activities in parallel, thus offering no chance for error cancellation. For these reasons, there may be little cancellation of the above errors.

These errors, although possible, were felt to be a misleading representation of the errors that could be expected in general, since the extreme errors were found when the mode was close to the endpoints of the range, whereas the mode is rather centralized in practice. Thus, the possible errors in the mean were also calculated for modal values, m, such that $\left|\frac{1}{2}-m\right| \leq 1/6$. In this case, the PERT-calculated mean could be off from the actual mean by about 10 per cent of the range; and the sum of the worst positive and the worst negative errors -- or the net error -- was found to be about 5 per cent of the range.

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*Note that the net error might be biased since 1/6 is not in the middle of the total range of possible values the standard deviation might assume $(0, \sqrt{1/12})$, and thus the PERT-assumed standard deviation might be too low in general.

**The mean is to the right of the mode.

POSSIBLE ERROR IN THE THREE TIME ESTIMATES

Even if the random variable representing the duration of an activity is assumed to be beta-distributed, it is highly unlikely that any procedure could be devised to determine the exact parameters of the distribution, since, ultimately, any such procedure must rely on human estimates.* Thus it is desirable to determine the contribution to the error in the PERT-calculated mean and the PERT-assumed standard deviation resulting from the PERT-estimating procedure itself.

In order to determine the magnitude and direction of possible errors in the estimates, it is assumed that the values a, m, and b are the actual values of the lower bound, mode, and upper bound, respectively, of a beta distribution. The estimates of these values are $t_a$, $t_m$, and $t_b$ and it will be assumed that they could be incorrect to the following extent: $0.8a \leq t_a \leq 1.1a$; $0.9m \leq t_m \leq 1.1m$; $0.9b \leq t_b \leq 1.2b$.** This is depicted in Fig. 3.

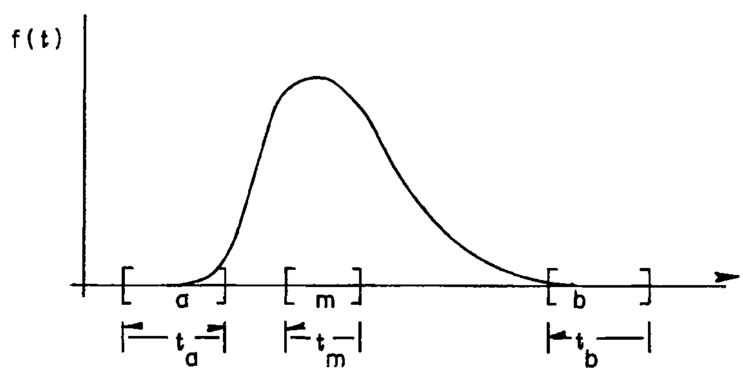


Fig. 3 -- Beta Distribution with Assumed Errors in a, m, and b

*In the case of a repetitive activity, it might be possible to dispense with human estimates altogether; but as was previously pointed out, very few of the activities handled by PERT will be of this nature.

**These intervals are based on discussions with managers having experience with PERT techniques. If the estimates are further in error, the absolute error in the mean and standard deviation will be increased.

The sensitivity of the PERT expressions $t_e = (a+4m+b)/6$ and $\sigma_e = (b-a)/6$ to incorrect estimates of a, m, and b can be seen from the data in Appendix B. Table 1 gives these results for values of a, b where b = 2a.

Table 1

RESULTS OF IMPRECISE TIME ESTIMATES

| | Worst Possible Error (% of Range) | | Net Error (% of Range) | |
|---|---|---|---|---|
| Mode | Mean | Standard Deviation | Mean | Standard Deviation |
| a | 15.0 | 10 | 1.6 | 5 |
| a+(b-a)/4 | 16.7 | 10 | 1.6 | 5 |
| a+(b-a)/3 | 16.9 | 10 | 1.6 | 5 |
| a+(b-a)/2 | 18.3 | 10 | 1.6 | 5 |
| a+2(b-a)/3 | 19.5 | 10 | 1.6 | 5 |
| a+3(b-a)/4 | 20.0 | 10 | 1.6 | 5 |
| b | 21.7 | 10 | 1.6 | 5 |

Although the worst possible absolute error in the mean runs between 15 per cent and 22 per cent of the range for b = 2a, the net error (sum of the worst positive and negative errors) is only 1.6 per cent of the range. Here, then, the possible effects of cancellation are significant.

## POSSIBLE ERROR DUE TO THE STANDARD DEVIATION ASSUMPTIONS AND THE APPROXIMATION OF THE MEAN

Appendix C shows that the assumption $\sigma_e = (b-a)/6$ and the approximation $t_e = (a+4m+b)/6$ can introduce absolute errors in the mean and standard deviation as great as 27.8 per cent and 12.1 per cent of the range,* respectively. These high values -- especially that of the

*The range of the activity duration was again assumed to be $[0,1]$.

mean -- are due to very unlikely values of the mode: $\left|\frac{1}{2}-m\right| \geq 1/6$. For values of the mode satisfying $\left|\frac{1}{2}-m\right| \leq 1/6$, the above errors reduce to 11.1 per cent and 12.1 per cent of the range. If partial cancellation can be assumed, these errors will be further reduced (to about 7 per cent and 4 per cent of the range, respectively).

SUMMARY OF THE ACTIVITY SECTION

As has been shown, the three factors discussed previously can each cause absolute errors in the PERT-calculated mean and the PERT-assumed standard deviation on the order of 30 per cent and 15 per cent of the range, respectively. The possible error due to one of these factors -- the estimates of a, m, and b -- was based on the assumption that these estimates would be incorrect to only a certain extent, i.e., $\pm$ 10 or 20 per cent of the range. Although these figures are thought to be conservative,* the degree to which the estimates of a, m, and b are imprecise will vary with each individual activity. Thus, the errors in the mean and standard deviation due to imprecise estimates will likely be larger than the results obtained from the calculations in this Memorandum (20 per cent, 10 per cent).

On the other hand, the errors in the mean and standard deviation can either be positive or negative, so that it can be assumed that some degree of cancellation of each of these errors will occur when all of the activities are combined in a network.** Furthermore, since many of the cases considered -- although theoretically possible -- are rather

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*Unless, of course, the individuals are working to schedule, i.e., they try to meet their estimate of m or the PERT-computed mean, $t_e$.

**An interesting experiment would be a Monte Carlo analysis of the effects of the three types of activity errors on an actual network. This might give some clue to the extent of possible cancellation of activity errors, since the individual activity errors themselves would be known.

unlikely to occur in practice, the possible errors in the mean and standard deviation were calculated from cases which are more likely to occur in practice (and assuming some cancellation in the network). Under these conditions, the errors may be reduced from the 30 and 15 per cent stated above.

ADDENDUM

It is interesting to note that -- except for the last type of error discussed -- the previous error analysis would have yielded approximately the same results if PERT had employed a triangular distribution instead of a beta distribution in its stochastic model. There would be no contribution to the error in the mean and standard deviation (or variance) from the third factor, since for a triangular distribution, the mean and variance are given exactly by $t_e = (a+m+b)/3$ and $\sigma_e^2 = \left[(b-a)^2 + (m-a)\cdot(m-b)\right]/18$. (See Appendix D.) These are the values that would be used in the calculation of the mean and variance, rather than the approximations used now.

Furthermore, the possible range of the variance of a triangular distribution is more centralized than is the PERT-assumed variance of $(b-a)^2/36$ in the total range of possible variances 0 to $(b-a)^2/12$ (again assuming the three properties discussed previously). However, if it is known or observed in practice that activity variances cluster more about $(b-a)^2/36$ than about $(b-a)^2/18$ to $(b-a)^2/24$ (the range of possible variances of a triangular distribution), then the use of a beta over a triangular distribution would be supported on this point.

However, when the mode and the range of a triangular distribution are specified (for example, by three time estimates) the entire distribution is then determined -- and so, of course, the variance is also determined. This is not the case for the class of beta distributions: there are an infinite number of unimodal beta distributions with the same range and mode (their variances lie between 0 and $(b-a)^2/12$). Thus, the class of beta distributions is more flexible than the class of triangular distributions since the former can handle more activity data. PERT, however, does not take advantage of this added flexibility, since it assumes that $\sigma_e^2 = (b-a)^2/36$ in all cases. With this assumption, a beta distribution is also completely determined when the range and mode are specified. Thus, since there is no _a priori_ justification for either function as an activity distribution, and since the actual variances of activities are unknown, the fact that the mean and variance can be given exactly for a triangular distribution make it an equally meaningful and more manageable distribution.

# III. THE NETWORK

## NETWORK CONSIDERATIONS

Up to this point, this study has been on the level of the individual activities within a PERT network. Now attention will be directed to the network as a whole. After the mean and variance of each activity have been computed, they can be used to determine some measure of the criticalness of all activities taken together and to aid in the estimation of the completion time distribution of the whole project.*

As it has been shown, the possible errors in the individual activities could, by themselves, cause errors in the calculation of a project mean and variance, although the extent and direction of these errors might be difficult to determine. However, even if the data (i.e., the mean, variance, and distribution) that PERT obtains for each activity are correct, significant errors can still be introduced into the calculation of a network mean and variance. As a result, probability statements concerning the various completion times of a project can also be incorrect.

## CRITICALNESS

To obtain a measure of the criticalness of each activity, PERT uses the critical path concept discussed earlier. Criticalness of an activity is a measure of the relative importance of the activity to the on-time completion of the over-all project. Some activities can obviously be delayed without delaying the project, while others cannot.

*Or the distribution of a selected sequence of milestones.

In PERT, only the means of the activity durations are used in determining the critical path. The stochastic element -- the variance of the activity duration -- is not incorporated. Thus, the model is reduced to a deterministic form. In a deterministic model (where no uncertainty in the activity durations is recognized) the longest path can be calculated by simple addition.

In a stochastic model, each path has a specific probability (in general, nonzero) of being the longest path at any particular time. However, if the network is large, the probability that any given path is the critical one may be very small. (An analogous situation would be one where a coin was tossed 1000 times. The most probable number of heads is 500, but the probability of getting exactly 500 heads is very small.) Thus, the most probable critical path may occur only rarely, and an activity that has a high probability of being on a longest path may not be on this most probable critical path. The following example may clarify these points.

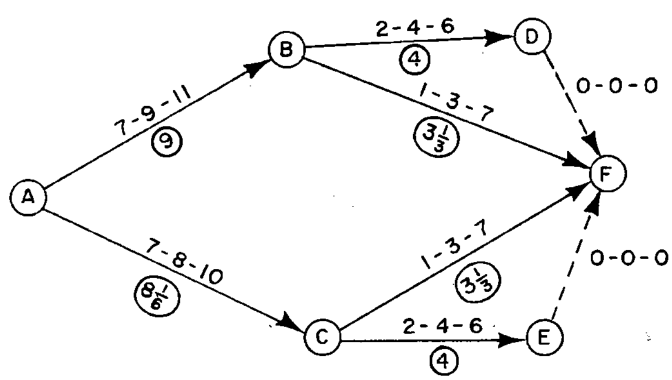


Fig. 4 -- PERT Network Showing Activities with Associated Times

Consider the network depicted in Fig. 4, with the customary time estimates of a, m, and b shown beside the corresponding activity. Using the PERT-calculated mean times (given in the circles below the activities), the PERT procedure would choose ABDF as the critical path because it has the maximum sum of means (13). If the activity time estimates are assigned equal weight, for computational simplicity, calculations (see Appendix E) will show that path ABF has the probability 0.33 of being the longest path, and this is a larger probability than any of the other three paths. The probability of each activity being on the longest path is: AB, 0.63; AC, 0.48; BF, 0.33; BD(F), 0.30; CF, 0.26; CE(F), 0.22. Note that although path ABF is the most probable longest path, it does not contain activity AC which is more critical than activity BF, which is on this most probable longest path.*

This example suggests that a critical activity concept may be more valid in a stochastic model than a critical path concept, especially since the PERT-calculated critical path is not even necessarily the most probable longest path. The computation of some sort of index of criticalness, such as that indicated above, would supplement (or possible replace) the slack determination, as done in the present PERT procedure.

## PROJECT DISTRIBUTION -- PERT AND ACTUAL

The PERT procedure for obtaining the project completion time distribution may be stated as follows. Assume that in a network

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*Similar results could be obtained using distributions other than a uniform, such as a beta distribution.

there are n different paths $P_1, P_2, \ldots, P_n$, which connect the origin node and the terminal node.* Let $p_1, p_2, \ldots, p_n$ denote the n random variables which represent, respectively, the durations of the n paths $P_1, P_2, \ldots, P_n$. One of these random variables, say $p_1$, will have an expected value which is not less than the expected value of the other n-1 variables, $p_2, p_3, \ldots, p_n$. Let the expected value and the variance of $p_i$ be $E(p_i)$ and $\sigma_i^2$, respectively. Thus, $E(p_1)$ is the expected or mean duration of path $P_1$, and $\sigma_1^2$ is its variance.

PERT now uses $E(p_1)$ and $\sigma_1^2$ as the project mean and variance, and assumes that the project duration is normally distributed and given by $F_p(t) = K \exp(-[t-E(p_1)]^2/2\sigma_1^2)$. Path $P_1$ is called the critical path.** In actuality, the project distribution is given by $F(t) = \Pr(\max_i p_i \leq t)$. Clearly, the expected value of the random variable $u = \max_i p_i$ is not less than the expected value of any one of the $p_i$. Hence, the PERT-calculated mean is generally less than, and never greater than, the true project mean. In general, the PERT-calculated variance is greater than the actual variance. If the distributions are symmetric, the variance of the random variable u will be less than any of the $\sigma_i^2$. This result is shown for two identical distributions in Appendix F, and can be extended to the general case. However, if the distributions are considerably skewed to the right (such as $e^{-t}$) the reverse may be true.

In order to determine the error in the mean and variance made by PERT in a particular network, it is necessary to calculate the

---

*Some of these paths may have a portion of their activities in common.

**If there is more than one path with the largest expected value, PERT labels them all as critical paths, and uses the one with the largest variance as $P_1$.

actual project means and variances from the data of each of the activities. The procedure used in this study to obtain the project mean and variance relies exclusively on the calculation of an exact project distribution from the individual activity distributions by analytical methods.* Such a calculation is extremely difficult in all but a few simple networks, regardless of the distributions on the activities themselves. These difficulties are discussed in Appendix G.

To get a feeling for the errors PERT makes by assuming that the project mean and variance are given by $E(p_1)$ and $\sigma_1^2$, respectively, various simple networks are analyzed both analytically and according to the PERT procedure.** Since calculations with beta distributions and other continuous distributions are rather lengthy, the distributions used in the network analyses which follow are, in general, discrete. Some results, however, have been obtained for the beta, uniform, and normal distributions.

## EXAMPLES OF NETWORK-BASED ERROR

The possible errors in PERT networks depend on the particular network configurations; so generalizations can be made only to a very limited extent. The examples studied in this paper are of a very simple form because of the computational problems discussed

*An alternative method, which could provide a close approximation to the project mean and variance, would be to use Monte Carlo techniques.

**It is assumed that the individual activity distributions are known exactly. This allows the determination of the errors made on the network level alone, without confounding them with possible errors made in the activities.

earlier, and because larger networks are really no more "typical" than small networks. They were chosen because they possessed some of the properties that may cause significant errors in regular PERT networks. The emphasis is on the factors causing the errors, the direction of the errors, and the magnitude of them. The subsection on network decomposition discusses the possible application of the results to much larger networks, such as those found in practice.

EXAMPLES -- SIMPLE SERIES AND PARALLEL

The first network configuration to be considered -- a simple path -- is one in which PERT makes no errors in calculating the project mean and variance. This case, depicted in Fig. 5,* will actually occur when there is one path through a network that is so much longer

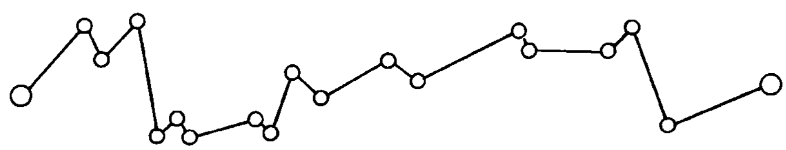


Fig. 5 -- Series Path

than any of the other paths that all other paths have no effect whatsoever on the determination of the project completion time distribution. The Central-Limit Theorem is applicable here, and the correct way to obtain the project mean and variance is by adding the activity means and variances along this (critical) path, the same procedure that PERT uses. Thus, whenever the network reduces to one very much

*The path may be viewed as containing a large number of intermediate nodes.

longer* path, the only PERT errors that can occur will be on the level of the individual activities.

Consider next the case where there are two paths of approximately the same length through the network. This results in the parallel configuration shown in Fig. 6. One may wish to view a large number of intermediate nodes on each path (however, there cannot be any connection between any of the nodes of the two paths). The PERT

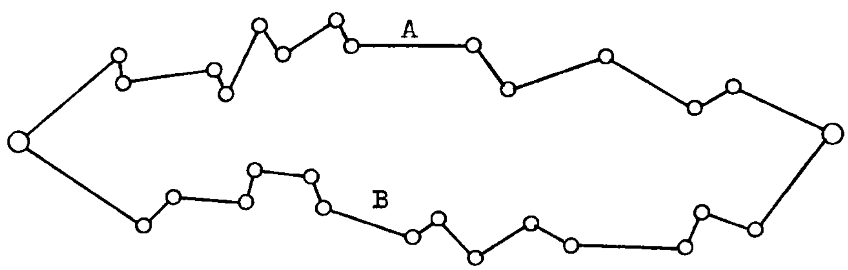



Fig. 6 -- Two Paths in Parallel

procedure will take as the mean and variance of the project, the sum of the means and the sum of the variances along that path with the largest mean. However, if the other path has a mean very close to the first (for a limiting case assume they are equal), the activities on this second path (the one PERT ignores) will also be a major determinant of the project completion time distribution.

As an example, if paths** A and B are beta-distributed on the interval $[0,1]$ with parameters $\alpha = \beta = 1$, then each distribution has a mean at 1/2. However, the mean of the maximum time distribution of the two distributions*** is not at 0.50, but rather 0.63. (See

---

*The meaning of "very much longer" will be discussed in the subsections on the effect of slack and on network decomposition.

**A path may be a single activity.

***The distributions are also independent, since it was assumed that there is no cross connection between paths.

Appendix H.) Similarly, if the distributions are assumed to be normally and identically distributed,* with mean $\mu$ and standard deviation $\sigma$, the mean of the maximum time distribution is $\mu + \sigma/\sqrt{\pi}$.

If a third path is present that is approximately as long as the other two, and has no cross connections with them, then there are three independent paths in parallel, and the error in the PERT-calculation mean increases. For example, in the beta distribution example above, the presence of a third path, with the same distribution as the other two, would raise the mean of the maximum time distribution to 0.69.

## EXAMPLES -- PARALLELS AND CROSS CONNECTIONS

As indicated above, the more parallelism in a network, the larger will be the error in the PERT-calculated mean, other things being equal. However, there is a counter-balancing factor -- correlation -- which tends to offset the error resulting from parallelism. When activities are common to two or more paths, the paths are correlated. Thus, when one path has a very long duration, other paths which have activities in common with this first path are likely to have a long duration also.

The extent to which these two factors tend to compensate depends on the network configuration. Since parallelism tends to cause the actual mean to be larger than the PERT-calculated mean, the more

*The distributions are also independent, since it was assumed that there is no cross connection between paths.

parallelism will result in a larger discrepency. On the other hand, the more common are the activities in the network, the greater will be the tendency for the PERT-calculated mean and the actual mean to be closer together. A comparison between a parallel configuration and a common activity configuration is given in the following example.

Consider the four-event example in Fig. 7. There are four activities, and the particular discrete distribution used on each activity can be identified by the corresponding mean on the network diagram.

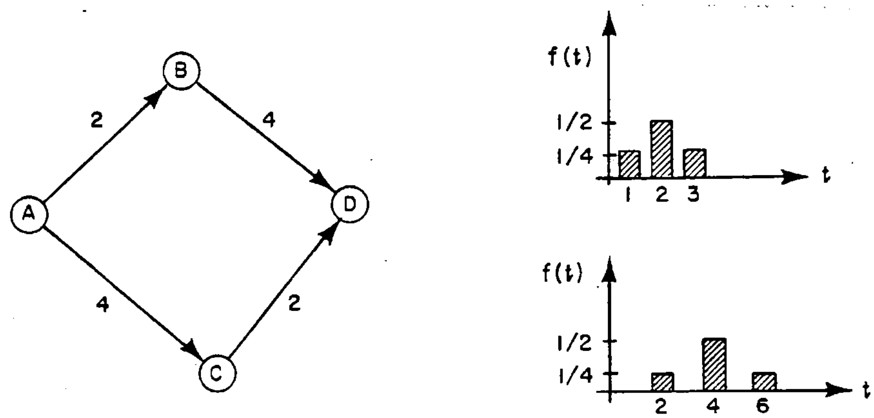


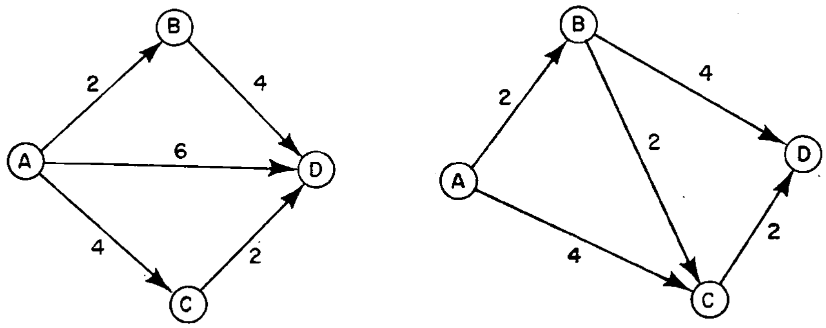

Fig. 7 -- Four-event Parallel Network

There are two paths, ABD and ACD, both having a mean length of 6.* The mean of the maximum time distribution is 6.89 (see Appendix I). Thus the error in the PERT-calculated mean is 12.9 per cent of the actual mean.

There are two possible ways a third path, with a mean length of 6,* may be created by adding one more activity. In one case the path may be completely independent of the other two paths, thus resulting in a third parallel element, AD, as depicted in Fig. 8(a). Alternatively,

*These are extreme cases since all paths have the same expected length.

an activity BC can be added with a mean time of 2, thus creating path ABCD, shown in Fig. 8(b). In both cases there are three paths, all




(a) Three Parallel Paths (b) Cross-Connected Paths

Fig. 8 -- Three-Path Networks

of mean length 6, and the network has four events and five activities.

The addition of the third path in parallel (Fig. 8(a)) leads to an increase in the deviation of the PERT-calculated mean (still 6) from the actual mean. The actual mean of the network in Fig. 8(a) is 7.336; thus the error has increased to 18.2 per cent.

Figure 8(b), on the other hand, is a network configuration where there is a cross connection between two parallel paths. Since there are three paths, one would expect a larger error than in a similar network with only two paths (such as Fig. 7), although not as large an error as in Fig. 8(a), where the three paths are in parallel. The correlation (resulting from the common activities) in the network of Fig. 8(b) does indeed have the effect discussed, and the mean of the maximum time distribution lies between these two bounds, being 7.074. The error as a per cent of the actual mean is 15.2 per cent.

EFFECT OF SLACK IN NETWORKS

The examples given in the two previous sections are extreme cases since all the paths have the same expected duration -- hence, they are all critical paths. If the durations of some paths are shorter than the duration of the longest path, their effect on the project mean and variance would not be as great. However, if they have a mean duration very close to the mean duration of the critical path, they would not be critical but they would have an effect almost as significant as the examples of the previous sections. The following examples shown in Fig. 9 indicate the effect of slack in a path length.

The simple network has only two paths, ABC and AC. All activities are assumed to be normally distributed with variance equal to 1, and the appropriate mean given on the diagram. It may be noted from

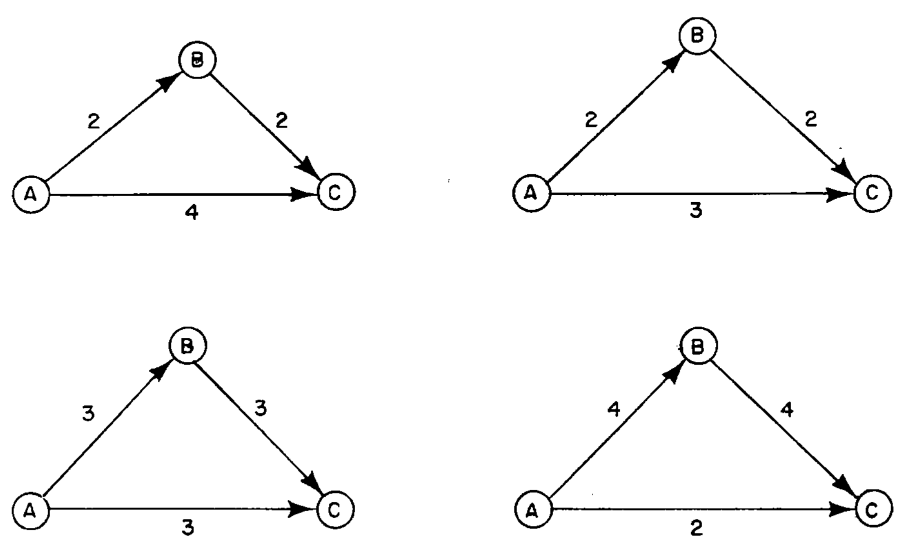



Fig. 9 -- Networks with Slack Paths

the diagrams that various lengths were assumed for paths ABC and AC, ranging from both of them being of equal length, to path AC being only 1/4 the length of path ABC. Table 2 summarizes the results. Further computational detail is furnished in Appendix J.

Table 2

SUMMARY OF RESULTS FROM FIGURE 9

| Ratio of lengths: $\frac{\text{path AC}}{\text{path ABC}}$ | 1/1 | 3/4 | 1/2 | 1/4 |
|---|---|---|---|---|
| PERT-calculated mean | 4 | 4 | 6 | 8 |
| Analytically-calculated mean | 4.69 | 4.30 | 6.03 | 8.00 |
| Per cent error (PERT from actual mean) | -17% | -8% | -0.5% | -0.00% |
| PERT-calculated standard deviation | 1 or 1.414 | 1.414 | 1.414 | 1.414 |
| Analytically-calculated standard deviation | 1.015 | 1.149 | 1.364 | 1.414 |
| Per cent error (PERT from actual std. dev.) | -1% or +39% | + 23% | + 4% | + 0.00% |

This example indicates that the deviation of the PERT-calculated mean and variance from the actual mean and variance may be quite large when the paths are about equal in length, but the difference decreases substantially as the path lengths become farther apart.

## FOUR-EVENT EXAMPLE -- CONTINUOUS DISTRIBUTIONS

If a fourth node and two connecting activities are added to the network configuration just studied for the effect of slack, the network of Fig. 8(b) may be obtained. This network, reproduced in Fig. 10,

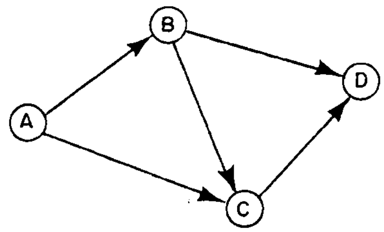



Fig. 10 -- Four-Event, Cross-Connected Network

has some desirable properties. It has three paths, maximums at two nodes, and is the smallest network that has two activities in common between two paths (activities AB and CD are common to two paths).

Various continuous distributions have been placed on the activities of this network. Complete results have only been obtained for the uniform and normal distributions. In addition, results, which are in agreement with the uniform and normal, for other networks have been obtained for the gamma, beta, sine, and a few polynomials.

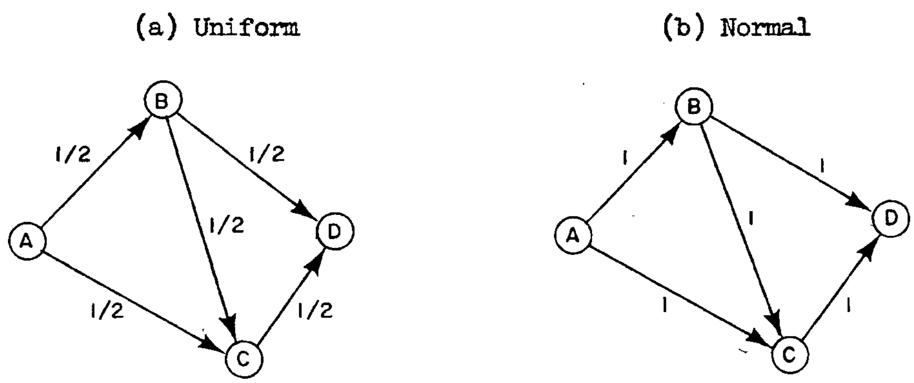



Fig. 11 -- Four-Event, Cross-Connected Networks with Continuous Activity Distributions

The means of the activity distributions are all equal and are shown in Fig. 11. It should be noted that the critical path, ABCD, has a duration half again as long as the duration of either of the other two paths; so the calculations will not give an upper bound,* but rather will represent an intermediate situation in terms of possible error. The results for the examples of Fig. 11 are given in Table 3. The details of the calculations are found in Appendix K.

Table 3

SUMMARY OF RESULTS FOR FIGURE 11

| Type of Distribution | Uniform | Normal |
|---|---|---|
| Activity mean duration | 0.50 | 1.00 |
| PERT-calculated project mean | 1.50 | 3.00 |
| Analytically-calculated project mean | 1.59 | 3.48 |
| Per cent error (PERT from actual mean) | 5.7 | 13.8 |

FOUR EVENT NETWORK AND ITS EXPANSION -- DISCRETE DISTRIBUTIONS

The examples of the previous section will now be expanded. In order to simplify calculations, discrete distributions will be used. The mean times and the corresponding discrete distributions are given in Fig. 12.

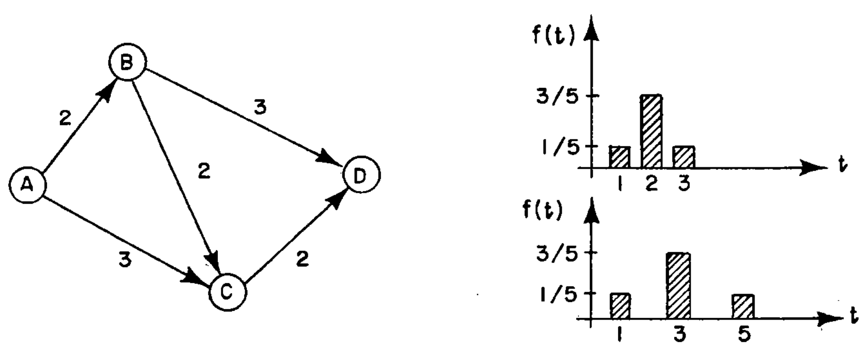



Fig. 12 -- Four-Event, Cross-Connected Network with Mean Times and Corresponding Discrete Distributions

*As they would if all paths had the same duration.

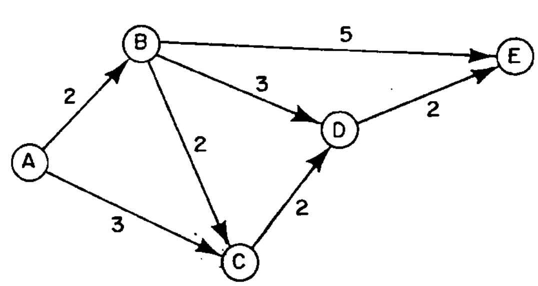
Note that in this example, the noncritical paths have durations 5/6 as long as the critical path, in contrast to 2/3 as long for the continuous cases above. The results for this network, given in Table 4, can be seen to be similar to those of the continuous cases. Computational details are given in Appendix L.

Table 4

SUMMARY OF RESULTS FOR FIGURE 12

| Four Event-Five Activity Network | Event B | Event C | Event D |
|---|---|---|---|
| PERT-calculated mean | 2 | 4 | 6 |
| Analytically-calculated mean | 2 | 4.23 | 6.42 |
| Per cent error (PERT from actual mean) | 0 | -5.4 | -7.4 |
| PERT-calculated standard deviation | .63 | .89 | 1.10 |
| Analytically-calculated standard deviation | .63 | .87 | 1.03 |
| Per cent error (PERT from actual standard deviation) | 0 | +2.3 | +6.8 |

If another event, E, is added to the network, along with two connecting activities, the configuration shown in Fig. 13 is obtained. The distribution of the activity, BE, with the longer duration is given beside the network.



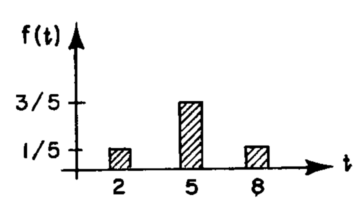




Fig. 13 -- Expansion of Network in Fig. 12 to Five Events

Table 5 summarizes the results.

Table 5

SUMMARY OF RESULTS FOR FIGURE 13

| Five Event-Seven Activity Network | Event E |
|---|---|
| PERT-calculated mean | 8.0 |
| Analytically-calculated mean | 8.77 |
| Per cent error (PERT from actual mean) | -9.5 |
| PERT-calculated standard deviation | 1.26 |
| Analytically-calculated standard deviation | 1.25 |
| Per cent error (PERT from actual standard deviation) | +0.8 |

Finally, if the origin and terminal nodes are connected, i.e., activity AE is added (with the distribution given below), the error in the PERT-calculated mean goes up even more. The network is shown in Fig. 14 and Table 6 summarizes the results.

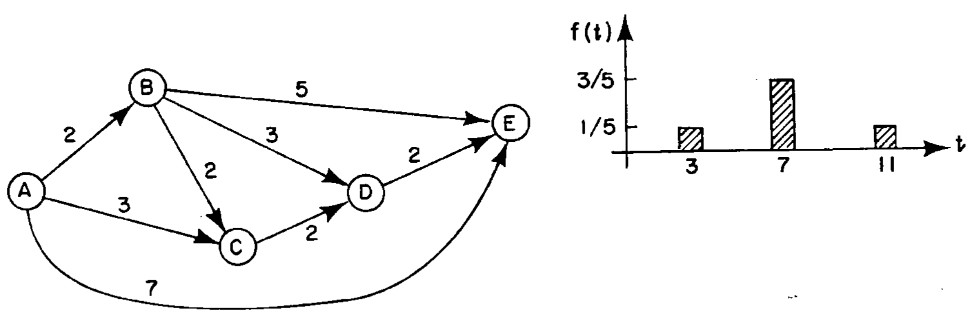



Fig. 14 -- Expansion of Network in Fig. 13 to Include Activity AE

Table 6

SUMMARY OF RESULTS FOR FIGURE 14

| Five Node-Eight Activity Network | Event E |
|---|---|
| PERT-calculated mean | 8.0 |
| Analytically-calculated mean | 9.23 |
| Per cent error (PERT from actual mean) | -13.3 |
| PERT-calculated standard deviation | 1.26 |
| Analytically-calculated standard deviation | 1.39 |
| Per cent error (PERT from actual standard deviation) | -9.4 |

Note that throughout this example, the error in the PERT-calculated mean has increased as the network has expanded. In addition, one may observe that the error in the PERT-calculated standard deviation changed sign (i.e., it went from being larger than the actual standard deviation to being smaller).

COMBINATIONS OF SIMPLE SERIES AND PARALLEL

The most elementary series and parallel elements discussed earlier will now be studied again. Combinations of these elements will be taken.

Consider the network in Fig. 15, which is a simple connection of

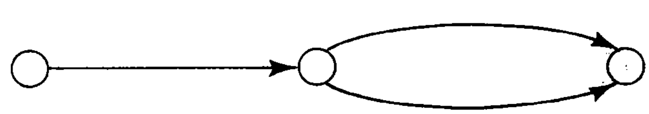


Fig. 15 -- Simple Series-Parallel Network

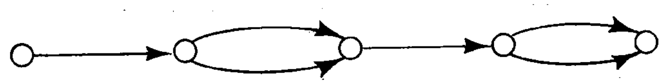
a series and parallel element.* As noted previously, no error will be made in combining activities along the series element. However, the two paths comprising the parallel element lead to error in the PERT calculation. The error in the whole network is at some intermediate value between these two extremes. The location of this intermediate value in the possible interval of error depends on which element is the dominant one. If the series element is dominant, the parallel error would not have much effect, and the net error in the PERT-calculated mean for the whole network will be small. However, if the parallel combination is the dominant one, the error in the whole network may be nearly as large as it is in the parallel configuration alone.

Now suppose that two identical series-parallel combinations are joined together in series. The per cent error in the total will be the same as the per cent error in either individual one. This configuration is shown in Fig. 16(a). However, if in another arrangement, three identical combinations are joined in a series-parallel arrangement, as shown in Fig. 16(b), the error in the total will be greater than the error in any of the three individual series-parallel elements.

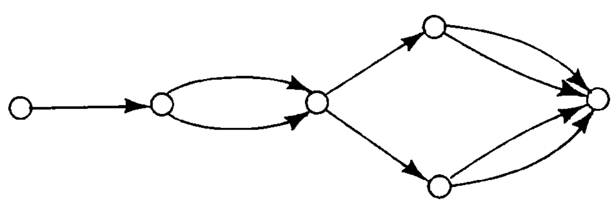
Some numerical results have been obtained when simple discrete distributions were placed on the activities (see Appendix M). The results are not surprising. When there is no error in the series element, and a 19 per cent error in the parallel element, the error

*It is realized that this representation is not in accordance with PERT networking techniques since a predecessor event and successor event uniquely determine only one activity. However, throughout this Memorandum, an arc may be considered as the resultant of numerous activities or subnetworks. The purpose of this type of representation is to highlight the symmetry of the particular network configuration.

in the mean of the whole network is around one-half of 19 per cent -- actually it is 7 per cent (slightly less than half since the series element has a slightly larger mean). Taking two of these combinations together, as in Fig. 16(a), still gives a per cent error in the mean of 7 per cent. Three of them, in a Fig. 16(b) configuration, produce an error of 14 per cent. By a similar procedure one might be able to determine the approximate error in simple series and parallel combinations, if the error in the individual elements were known along with the various relative values of the corresponding means.



(a) Arrangement of Two Identical Series-Parallel Combinations



(b) Arrangement of Three Identical Series-Parallel Combinations

Fig. 16 -- Series-Parallel Arrangements

PROBABILITY STATEMENTS

At this stage, it should be obvious that the PERT probability statements concerning the various possible project completion times may be considerably in error. Since these statements are based on a normal distribution having the PERT-calculated mean and variance as parameters, and since it has been shown that these PERT calculations can be seriously in error, considerable doubt is cast on the validity of these statements. In addition, the normal approximation to the project distribution may be a poor one, since the parallelism in a network will tend to skew the distribution to the left.

## DECOMPOSITION OF NETWORKS

Although the networks analyzed in this Memorandum are very small, the results obtained are applicable, to some extent, to much larger networks. Most of the examples here have dealt with networks whose activities were all critical (i.e., they all made some contribution to the project distribution). However, in large networks many activities are not of a critical nature. Dropping all of the noncritical activities from consideration may reduce the network substantially. Another procedure would be to try to identify simple series and parallel elements in a network and to collapse parts of the network on the basis of the network configuration alone.

Examining this latter procedure first, one may find that some networks, or at least parts of them, are composed of simple series and parallel elements with few cross connections. Two activities in series can be treated as one larger activity by adding the two durations. Two activities in parallel can be treated as one activity by taking the maximum of the two durations. By such reduction, a large network can possibly be broken down into a number of small networks for which approximate results are known. Then, since the effects of combining errors in simple series and parallel arrangements are roughly known, an estimate of the error in the whole network may be obtained.

Unfortunately, this technique does not generally reduce the PERT network very much because of the numerous cross connections. However, if use is made of the time estimates given for each activity, and not just the network configuration alone, many noncritical activities can be eliminated from consideration. One method would be to compare the sum of the minimum times (i.e., optimistic estimates) of all activities along every path to a given node with the sum of the maximum times (i.e., pessimistic estimates) of all activities along every path to the same node. If the sum of the minimum times along one path is greater than the sum of the maximum times along another path, then the latter path cannot (by the definition of these times) be a determinant of the time distribution at that node. The latter path can thus be disregarded in the computation of the distribution at that node. The activities that are unique to this latter path can be removed from the analysis of distributions at the given node and all nodes further along.

## SUMMARY OF NETWORK EXAMPLES

The examples of this section demonstrate the possible sources of error in the PERT calculation of the project mean and variance.* They should also provide an indication of the magnitude and direction of the possible error in some very basic network configurations. The errors in the PERT-calculated mean and standard deviation for the examples studied were around 10 to 30 per cent.

The PERT-calculated mean will always be biased optimistically, but the PERT-calculated variance may be biased in either direction. Precise statements about the magnitude of the errors, however, cannot be made since errors in the project mean and variance vary with different network configurations. If there is one path through a network that is significantly longer than any other path, then the PERT procedure for calculating the project mean and variance will give approximately

*Assuming that individual activity distributions are known exactly.

correct results. However, if there are a large number of paths having approximately the same length, and having few activities in common, errors will be introduced in the PERT-calculated project mean and variance. The more parallel paths there are through the network, the larger will be the errors. If, however, the paths share a large number of common activities, the errors will tend to be lower. The extent to which these two factors compensate depends on the particular network configuration.

The errors in the PERT-calculated project mean and variance will tend to be large if many noncritical paths each have a duration approximately equal to the duration of the critical path. However, the more slack there is in each of the noncritical paths, the smaller will be the error.

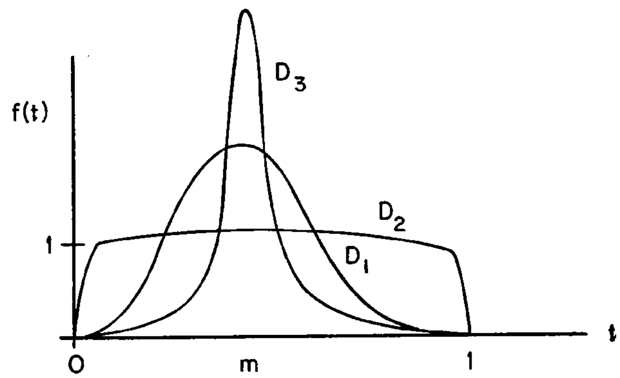
Because of the possible errors in the PERT-calculated project mean and variance, there may be correspondingly large errors in the probability statements that are based on these parameters.

It is suggested that for a stochastic model (such as PERT) a critical activity concept is more valid than, and probably as useful as, a critical path concept. This is based on the fact that the PERT-calculated critical path does not necessarily contain the most critical activities.

Networks very often contain many activities that are not of a critical nature. Eliminating these activities from consideration may reduce the network considerably without affecting to any large extent the final results. In general, if the sum of the minimum times along one path is greater than the sum of the maximum times along a parallel path, then the latter path will not influence the calculation of the time distribution at the common end node.

Appendix A

## POSSIBLE ERROR INTRODUCED BY THE ASSUMPTION OF A BETA DISTRIBUTION




Range: $[0,1]$

Mode: $0 \leq m \leq \frac{1}{2}$

$D_1$ (PERT) beta distribution with mode at m and variance 1/36.

$D_2$ Quasi-uniform distribution which can be made to have a mean and variance as close to 1/2 and 1/12, respectively, as desired.

$D_3$ Quasi-delta function which can be made to have a mean and variance as close to m and 0, respectively, as desired.

Mean:

Worst absolute error, as a proportion* of the range.

$$= \max\left[\left|(4m+1)/6 - 1/2\right|, \left|(4m+1)/6 - m\right|\right],$$
$$= \max\left[\left|(4m-2)/6\right|, \left|(1-2m)/6\right|\right],$$
$$= \max\left[\left|(2m-1)/3\right|, \left|(1-2m)/6\right|\right],$$
$$= (1-2m)/3.$$

Net error (sum of worst positive error and worst negative error) as a proportion of the range.

$$= \left[2(2m-1) + (1-2m)\right]/6 \quad = (2m-1)/6,$$
$$= -(1-2m)/6.$$

*To convert from proportion of the range to per cent of the range multiply the former by 100.

Variance:

Worst absolute error as a proportion of the range.

$$= \max\left[\left|1/12 - 1/36\right|, \ \left|0 - 1/36\right|\right],$$

$$= 1/18.$$

Net error as a proportion of range.

$$= 1/12 - 1/36 + 0 - 1/36,$$

$$= 1/36.$$

Standard Deviation:

Worst absolute error as a proportion of the range.

$$= \max\left|\sqrt{1/12} - 1/6, \ 0 - 1/6\right|,$$

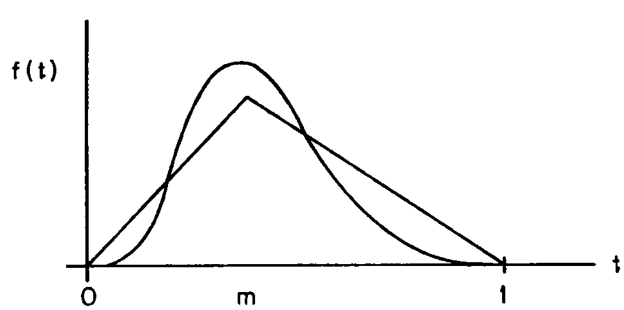
$$= 1/6 = 0.167.$$

Net error as a proportion of the range.

$$= \sqrt{1/12} - 1/6 + 0 - 1/6,$$

$$= -0.044.$$

Since $D_2$ and $D_3$ are unlikely distributions a comparison is made between a triangular and a beta distribution.



| Mode | Beta (PERT) | | Triangular | |
|---|---|---|---|---|
| | Mean | Variance | Mean | Variance |
| 1/2 | 1/2 = .500 | 1/36 = .028 | 1/2 = .500 | 1/24 = .042 |
| 1/3 | 7/18 = .389 | 1/36 = .028 | 4/9 = .444 | 7/162 = .043 |
| 1/4 | 1/3 = .333 | 1/36 = .028 | 5/12 = .417 | 13/16·18 = .045 |
| 1/6 | 5/18 = .278 | 1/36 = .028 | 7/18 = .389 | 31/18·36 = .048 |
| 1/12 | 2/9 = .222 | 1/36 = .028 | 13/36 = .361 | 133/18·144 = .051 |

The means of the triangular distribution are all greater than those of the beta distribution for $0 < m \leqslant \frac{1}{2}$. They are less by a corresponding amount for modal values = 1-m.

## Appendix B

### POSSIBLE ERROR DUE TO IMPRECISE TIME ESTIMATES

Let a,m, and b be the true values for the least, most likely, and greatest times at which the activity could be completed. Let $a \leqslant m \leqslant \frac{a+b}{2}$.

Now consider:

$$\left.\begin{array}{l} 0.8a \leqslant t_a \leqslant 1.1a \\ 0.9a \leqslant t_m \leqslant 1.1m \\ 0.9b \leqslant t_b \leqslant 1.2b \end{array}\right\}$$

where a,m, and b are such that $t_a < t_m < t_b$.

Mean: Estimate -- $t_e = (a+4m+b)/6$

Worst absolute error as a proportion of the range.

$$= \frac{1}{b-a} \max\left[\left|\frac{(.8a+3.6m+.9b) - (a+4m+b)}{6}\right|, \left|\frac{(1.1a+4.4m+1.2b)-(a+4m+b)}{6}\right|\right],$$

$$= \frac{1}{60}\left(\frac{a+4m+2b}{b-a}\right).$$

Worst net error (sum of worst positive and worst negative errors) as a proportion of the range.

$$= \frac{1}{b-a}\left[\frac{-.2a-.4m-.1b+.1a+.4m+.2b}{6}\right],$$

$$= 1/60.$$

Standard Deviation: Assumption -- $\sigma_e = (b-a)/6$

Worst absolute error as a proporation of the range.

$$= \frac{1}{b-a} \max\left[\left|\frac{(.9b-1.1a)-(b-a)}{6}\right|, \left|\frac{(1.2b-.8a)-(b-a)}{6}\right|\right],$$

$$= \frac{1}{30}\left(\frac{b+a}{b-a}\right).$$

Worst net error as a proportion of the range.

$$= \frac{1}{b-a}\left[\frac{(.9b-1.1a)-(b-a)+(1.2b-.8a)-(b-a)}{6}\right],$$

$$= \frac{1}{60}\left(\frac{b+a}{b-a}\right).$$

Effect of mode:

| Mode | Worst Absolute Error* Mean | Worst Absolute Error* Standard Deviation | Worst Net Error* Mean | Worst Net Error* Standard Deviation |
|---|---|---|---|---|
| a | $\frac{5a+2b}{60(b-a)}$ | $\frac{1}{30}\left(\frac{b+a}{b-a}\right)$ | 1.6 | $\frac{1}{60}\left(\frac{b+a}{b-a}\right)$ |
| $\frac{3a+b}{4}$ | $\frac{4a+3b}{60(b-a)}$ | " | " | " |
| $\frac{2a+b}{3}$ | $\frac{11a+10b}{180(b-a)}$ | " | " | " |
| $\frac{a+b}{2}$ | $\frac{3a+4b}{60(b-a)}$ | " | " | " |
| $\frac{a+2b}{3}$ | $\frac{7a+14b}{180(b-a)}$ | " | " | " |
| $\frac{a+3b}{4}$ | $\frac{2a+5b}{60(b-a)}$ | " | " | " |
| b | $\frac{a+6b}{60(b-a)}$ | " | " | " |

This table, which was derived from the results on the previous page, shows that the value of the mode only affects the absolute error of the mean.

*As a proportion of the range.

## Appendix C

### POSSIBLE ERROR DUE TO THE STANDARD DEVIATION ASSUMPTION AND THE APPROXIMATION OF THE MEAN

Beta distribution: $f(t) = \frac{(\alpha + \beta + 1)!}{\alpha!\,\beta!}\, t^{\alpha}\ (1-t)^{\beta}.$

Mode: $m = \frac{\alpha}{\alpha + \beta}$ .

Mean: $t_e = \frac{\alpha + 1}{\alpha + \beta + 2} = \frac{m(\alpha + 1)}{\alpha + 2m}.$

Variance: $\sigma_e^2 = E(x^2) - \left[E(x)\right]^2 = \frac{(\alpha+1)(\alpha+2)}{(\alpha+\beta+2)(\alpha+\beta+3)} - \frac{(\alpha+1)^2}{(\alpha+\beta+2)^2},$

$$= \frac{(\alpha+1)(\beta+1)}{(\alpha+\beta+2)^2\,(\alpha+\beta+3)}.$$

The table below gives values of the variance for various values of $\alpha$ and m.

| | $\alpha$ | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Mode | 0 | 1/10 | 1/6 | .29 | 1/2 | .70 | 3/4 | 1 | 1.21 | 1.82 | 2 | 3 | 6 | 10 |
| 1/2 | .083 | .078 | .075 | | .063 | | .056 | .050 | | | .036 | .028 | .017 | .011 |
| 1/3 | .083 | .076 | .071 | | .054 | | .046 | .040 | | .028 | .026 | .019 | .011 | .007 |
| 1/4 | .083 | .073 | .067 | | .047 | | .038 | .032 | .028 | | .019 | .013 | .007 | .005 |
| 1/6 | .083 | .068 | .059 | | .035 | .028 | .026 | .021 | | | .011 | .008 | .004 | .002 |
| 1/12 | .083 | .054 | .041 | .028 | .017 | | .011 | .008 | | | .004 | .002 | .001 | .001 |

Note that .028 is the decimal equivalent of 1/36 (i.e., the PERT variance).

Tables showing the PERT mean and variance and the actual mean and variance* for greatest and least variances* at each mode (as obtained from the table on the previous page) for (1) an extreme case, and (2) a reasonable case.**

| Mode | | (1) Extreme Case | | | PERT | | (2) Reasonable Case | | |
|---|---|---|---|---|---|---|---|---|---|
| | | Variance | Standard Deviation | Actual Mean | Mean | Standard Deviation | Variance | Standard Deviation | Actual Mean |
| 1/2 | Greatest | .083 | .288 | .500 | .500 | .167 | .036 | .190 | .500 |
| | Least | .017 | .105 | .500 | | | .017 | .130 | .500 |
| 1/3 | Greatest | .083 | .288 | .500 | .389 | .167 | .040 | .200 | .400 |
| | Least | .007 | .084 | .343 | | | .019 | .138 | .363 |
| 1/4 | Greatest | .083 | .288 | .500 | .333 | .167 | .032 | .179 | .333 |
| | Least | .005 | .071 | .262 | | | .019 | .138 | .300 |
| 1/6 | Greatest | .083 | .288 | .500 | .278 | .167 | .035 | .187 | .300 |
| | Least | .002 | .045 | .177 | | | .021 | .145 | .250 |
| 1/12 | Greatest | .083 | .288 | .500 | .222 | .167 | .041 | .202 | .292 |
| | Least | .001 | .032 | .090 | | | .017 | .130 | .188 |

*Standard deviation is also shown.

**"Reasonable" here refers to beta distributions whose parameters $\alpha, \beta$ are near 1 or 2.

Table showing the per cent errors between the PERT mean, variance, and standard deviation and the actual mean, variance, and standard deviation for values shown in the table on the preceding page.

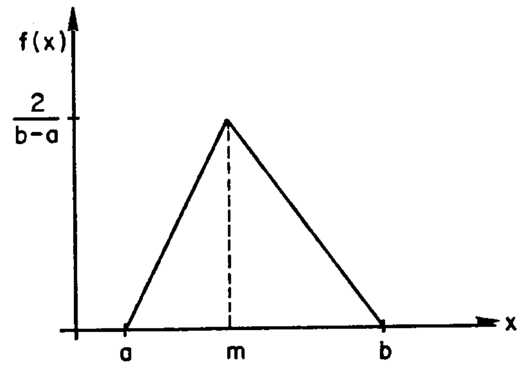
| Mode | | Worst Absolute Error (% of Range) | Reasonable Absolute Error (% of Range) | Worst Net Error* (% of Range) | Reasonable Net Error* (% of Range) |
|---|---|---|---|---|---|
| 1/2 | Mean | 0.0 | 0.0 | 0.0 | 0.0 |
| | Variance | 5.5 | 1.2 | 3.8 | -0.3 |
| | Standard Deviation | 12.1 | 3.7 | 5.9 | -1.4 |
| 1/3 | Mean | 11.1 | 2.6 | 6.5 | -1.5 |
| | Variance | 5.5 | 1.2 | 3.4 | 0.3 |
| | Standard Deviation | 12.1 | 3.3 | 3.8 | 0.4 |
| 1/4 | Mean | 16.3 | 3.3 | 9.6 | -3.3 |
| | Variance | 5.5 | 0.9 | 3.2 | -0.5 |
| | Standard Deviation | 12.1 | 2.9 | 2.5 | -1.7 |
| 1/6 | Mean | 22.2 | 2.8 | 12.1 | -0.6 |
| | Variance | 5.5 | 0.7 | 2.9 | 0.0 |
| | Standard Deviation | 12.1 | 2.2 | -0.1 | -0.2 |
| 1/12 | Mean | 27.8 | 7.0 | 15.4 | 3.6 |
| | Variance | 5.5 | 1.3 | 2.8 | 0.2 |
| | Standard Deviation | 12.1 | 3.7 | -1.4 | -0.2 |

*The net error is the sum of the positive and negative errors.

# Appendix D

## TRIANGULAR DISTRIBUTION

$$f(x) = \begin{cases} = \dfrac{2(x-a)}{(m-a)(b-a)}, & a \leq x \leq m \\ = \dfrac{2(b-x)}{(b-m)(b-a)}. & m \leq x \leq b \end{cases}$$




$$\text{Mean} = t_e = \int_a^m \frac{2x(x-a)}{(m-a)(b-a)}\,dx + \int_m^b \frac{2x(b-x)}{(b-m)(b-a)}\,dx,$$

$$= \frac{a+m+b}{3}.$$

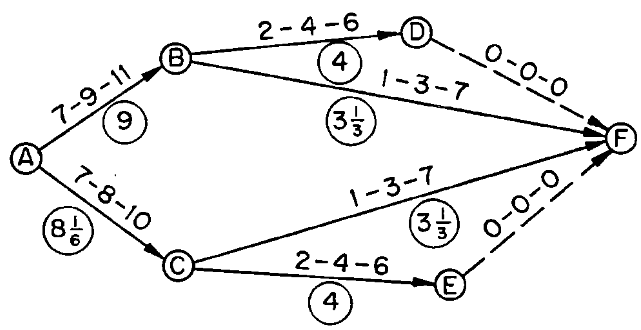
$$\text{Variance} = \sigma_e^2 = E(x^2) - \left[E(x)\right]^2 = \int_a^m \frac{2x^2(x-a)}{(m-a)(b-a)}\,dx + \int_m^b \frac{2x^2(b-x)}{(b-m)(b-a)}\,dx - E(x)^2,$$

$$= \frac{(b-a)^2 + (m-a)(m-b)}{18}.$$

## Appendix E

## CRITICALNESS IN A STOCHASTIC MODEL

Consider the following network with the indicated times for each activity.




Problem: Obtain a measure of criticalness of each activity.

PERT (deterministic) solution:

From the expression $t_e = (a+4m+b)/6$, the activity means are calculated. These are shown in the circles on the network. The longest path through the network using these means is ABDF with a mean length of 13. This is the PERT (deterministic) critical path.

A possible stochastic solution:

The probability that an activity will be on the longest path would serve as a measure of criticalness. If the activities are assumed to be uniformly distributed (for computational ease), this probability may be determined by taking each possible activity time on one path with all possible times on all other paths. This may be done by setting up the tables shown below. The entries in the tables are the maximum of the corresponding parellel activities plus the corresponding series activity (e.g., in the top table, entry = max $\left[\text{BF, BD(F)}\right]$ + AB). The arrow indicates which parallel activity was the maximum.

Top Paths (ABDF and ABF)

| | | Activity BF | | |
|---|---|---|---|---|
| Activity AB | Activity BD(F) | 1 | 3 | 7 |
| 7 | 2 | 9 ← (0-1) | 10 ↑ (1-2) | 14 ↑ (12-7) |
| | 4 | 11 ← (3-3) | 11 ← (3-3) | 14 ↑ (12-7) |
| | 6 | 13 ← (9-3) | 13 ← (9-3) | 14 ↑ (12-7) |
| 9 | 2 | 11 ← (3-3) | 12 ↑ (6-3) | 16 ↑ (22-2) |
| | 4 | 13 ← (9-3) | 13 ← (9-3) | 16 ↑ (22-2) |
| | 6 | 15 ← (19-3) | 15 ← (19-3) | 16 ↑ (22-2) |
| 11 | 2 | 13 ← (9-3) | 14 ↑ (12-7) | 18 ↑ (27) |
| | 4 | 15 ← (19-3) | 15 ← (19-3) | 18 ↑ (27) |
| | 6 | 17 ← (24-3) | 17 ← (24-3) | 18 ↑ (27) |

Bottom Paths (ACEF and ACF)

| | | Activity CF | | |
|---|---|---|---|---|
| Activity AC | Activity CE(F) | 1 | 3 | 7 |
| 7 | 2 | 9 ← (0-1) | 10 ↑ (1-1) | 14 ↑ (11-4) |
| | 4 | 11 ← (2-3) | 11 ← (2-3) | 14 ↑ (11-4) |
| | 6 | 13 ← (6-5) | 13 ← (6-5) | 14 ↑ (11-4) |
| 8 | 2 | 10 ← (1-1) | 11 ↑ (2-3) | 15 ↑ (15-4) |
| | 4 | 12 ← (5-1) | 12 ← (5-1) | 15 ↑ (15-4) |
| | 6 | 14 ← (11-4) | 14 ← (11-4) | 15 ↑ (15-4) |
| 10 | 2 | 12 ← (5-1) | 13 ↑ (6-5) | 17 ↑ (22-2) |
| | 4 | 14 ← (11-4) | 14 ← (11-4) | 17 ↑ (22-2) |
| | 6 | 16 ← (19-3) | 16 ← (19-3) | 17 ↑ (22-2) |

Each top path combination is taken with every bottom path combination, and vice versa. The numbers in the parentheses indicate how many times the particular combination was larger (L) or tied (T) for the largest (L-T) with all the combinations in the other table. The table below summarizes this for each activity.

| Activity | Larger | Tied | Total Larger or Tied | Total Larger or Tied Divided by the Total Possible Combinations ($3^6 = 729$) |
|---|---|---|---|---|
| AB | 380 | 82 | 462 | 0.63 |
| AC | 267 | 82 | 349 | 0.48 |
| BD(F) | 178 | 43 | 221 | 0.30 |
| BF | 202 | 39 | 241 | 0.33 |
| CE(F) | 114 | 43 | 157 | 0.22 |
| CF | 153 | 39 | 192 | 0.26 |

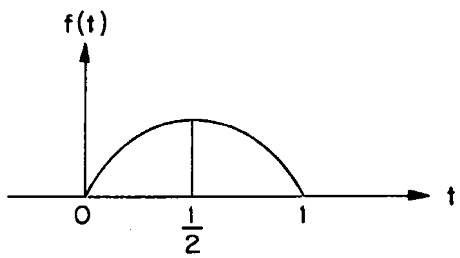
## Appendix F

## VARIANCE OF THE MAXIMUM OF TWO SYMMETRICALLY-DISTRIBUTED RANDOM VARIABLES

If $t_1$, $t_2$ have identical distributions, which are symmetric, continuous, and of compact support,* then the variance of the distribution of the max $(t_1, t_2)$ is less than the variance of the distribution of either $t_1$ or $t_2$.

Let $t_1$ have the following distribution: $f(t_i)$

and cumulative distribution: $F(t_i)$




$f(t_i) = f(1 - t_i)$

$F(t_i) = 1 - F(1 - t_i)$

Mean of $t_i = \mu = \frac{1}{2}$

Mode of $t_i = m = \frac{1}{2}$

Variance of $t_i = \sigma^2$

Let $t^* = \max(t_1, t_2)$

Then the cumulative distribution of $t^*$, $G(u) = Pr(t^* \leq u)$

$$= F^2(u);$$

and $t^*$ has the following distribution: $g(u) = 2f(u)F(u)$.

Mean of $t^* = \mu^* > \frac{1}{2}$ Variance of $t^* = \sigma^{*2}$

$(\mu^* - \mu)$ and $(\mu^* + \mu - 1)$ are both positive, hence

$$(\mu^* - \mu)(\mu^* + \mu - 1) > 0,$$

$$\Rightarrow \quad -\tfrac{1}{2} + \mu + \tfrac{1}{2} - \mu^* + (\mu^* - \mu)(\mu^* + \mu) > 0,$$

$$\Rightarrow \quad \tfrac{1}{2}\int_0^1 f(t)\left[-1 + 2t\right]\left[1 - 2F(t)\right]dt + (\mu^* - \mu)(\mu^* + \mu) > 0,$$

$$= \tfrac{1}{2}\int_0^m f(t)\left[-1 + 2t\right]\left[1 - 2F(t)\right]dt + \tfrac{1}{2}\int_m^1 f(t)\left[-1 + 2t\right]\left[1 - 2F(t)\right]dt + \mu^{*2} - \mu^2 > 0,$$

---

*Here the interval of support is [0,1]. The result is, however, applicable to any compact interval[a,b].

$$= \int_0^m f(t)\left[-1 + 2t\right]\left[1 - 2F(t)\right]dt + \mu^{*2} - \mu^2 > 0,$$

$$= \int_0^m f(t)\Big[t^2 - 2t^2F(t) + 1 - 2t + t^2 - 2 + 4t - 2t^2 + 2F(t) - 4tF(t)$$
$$+ 2t^2F(t)\Big]dt + \mu^{*2} - \mu^2 > 0,$$

$$= \int_0^m t^2f(t)\left[1 - 2F(t)\right]dt + \int_0^m (1-t)^2f(1-t)\left[1 - 2F(1-t)\right]dt$$
$$+ \mu^{*2} - \mu^2 > 0,$$

$$= \int_0^m t^2f(t)\left[1 - 2F(t)\right]dt + \int_m^1 t^2f(t)\left[1 - 2F(t)\right]dt + \mu^{*2} - \mu^2 > 0,$$

$$= \int_0^1 t^2f(t)\left[1 - 2F(t)\right]dt + \left[\int_0^1 2tf(t)F(t)dt\right]^2 - \left[\int_0^1 tf(t)dt\right]^2 > 0,$$

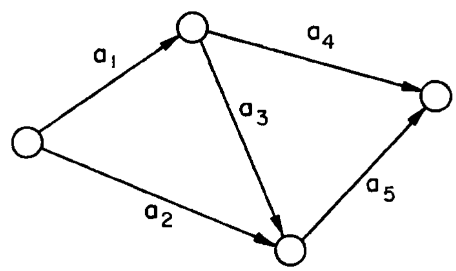
$$\Rightarrow \int_0^1 t^2f(t)dt - \left[\int_0^1 tf(t)dt\right]^2 > \int_0^1 t^2g(t)dt - \left[\int_0^1 tg(t)dt\right]^2,$$

$$\Rightarrow \qquad \sigma^2 \quad > \quad \sigma^{*2}.$$

Appendix G

## DIFFICULTIES IN THE ANALYTICAL CALCULATION OF THE PROJECT DISTRIBUTION

As was stated before, the main difficulty in determining the errors made by PERT in particular networks is the calculation of the actual network distribution from the individual activity distributions. Consider, for example, the following network.




Let $t_i$ be the random variable which represents the duration of activity $a_i$, $i = 1, 2, 3, 4, 5$; and let the density function of $t_i$ be given by $f(t_i)$, then the cumulative distribution of the network is

$F(u) = \Pr\left(\max\left[t_1 + t_4,\ t_1 + t_3 + t_5,\ t_2 + t_5\right] \leq u\right)$,

$$\int_{-\infty}^{u}\left[\int_{-\infty}^{u-t_1}\left(\int_{-\infty}^{u-t_1} f(t_4)\,dt_4 \int_{-\infty}^{u-t_2} f(t_2)\,dt_2 \int_{-\infty}^{u-t_1-t_5} f(t_3)\,dt_3\right) f(t_5)\,dt_5\right] f(t_1)\,dt_1 .$$

Two types of unimodal, continuous distributions (i.e., density functions) are considered:

(1) Those which are nonzero over an infinite domain, such as:

$$f(t_i) = 0 \qquad -\infty \leq t_i < 0 \qquad\qquad f(t_i) = 0 \qquad -\infty \leq t_i < 0\,,$$
$$= t_i e^{-t_i} \qquad 0 \leq t_i \leq \infty \qquad \text{and} \qquad = \frac{Kt_i}{1+t_i^4} \quad 0 \leq t_i \leq \infty\,,$$

(2) Those which are nonzero only over some finite domain, such as:

$$\begin{aligned} f(t) &= 0 && -\infty \leq t < a & \qquad f(t) &= 0 && -\infty \leq t < 0 \\ &= K(t-a)\cdot(b-t) && a \leq t \leq b & \text{and} \qquad &= 1 && 0 \leq t \leq 1 \\ &= 0 && b < t \leq \infty & &= 0 && 1 < t \leq \infty \end{aligned}$$

The calculation of the network distribution from functions of class (1) is almost impossible, since if the functions themselves are integrable in a closed and simple expression -- which is not always the case -- the iterated integrals become very complicated. For the two functions given, the project distribution "reduces" to the evaluation of:

$$F(u) = \int_0^u \left[ \int_0^{u-t_1} \left[1-(u+1-t_1-t_5)e^{-(u-t_1-t_5)}\right]\left[1-(u+1-t_1)e^{-(u-t_1)}\right]\cdot \right.$$
$$\left. \cdot\left[1-(a+1-t_5)e^{-(u-t_5)}\right] f(t_5)dt_5 \right] f(t_1)dt_1,$$

and

$$F(u) = K^5 \int_0^u \left[ \int_0^{u-t_1} \frac{\tan^{-1}(u-t_1-t_5)^2 \tan^{-1}(u-t_1)^2 \tan^{-1}(u-t_5)^2\, t_1 t_5}{(1+t_5^4)\ (1+t_1^4)} dt_5 \right] dt_1\,.$$

Whereas F(u) would be difficult to evaluate in the first case, it is doubtful that F(u) can even be expressed as an elementary function in the latter case.

Although the functions in class (2) are easier to integrate, in general, they have to be defined in two or more parts. This causes much difficulty in the evaluation of F(u), since the single integral expression must be broken up into a number of parts.

$$
\begin{aligned}
\text{If } f(t) &= 0 \qquad t \in \left[-\infty, 0\right) \cup \left(1, \infty\right] \qquad \text{(See Appendix K)} \\
&= 1 \qquad t \in \left[0, 1\right]
\end{aligned}
$$

then,

$$
\begin{aligned}
F(u) &= \frac{11}{120} u^5, && 0 \leq u \leq 1 \\
&= \frac{1}{120} (-u^5 - 20u^4 + 100u^3 - 120u^2 + 80u - 28), && 1 \leq u \leq 2 \\
&= \frac{1}{6} (u^3 - 9u^2 + 27u - 21), && 2 \leq u \leq 3 \\
&= 0. && \text{Otherwise}
\end{aligned}
$$

If the activity density functions have a more complicated form, F(u) becomes more involved and more difficult to calculate.

## Appendix H

## MAXIMUMS OF SIMPLE (INDEPENDENT) BETA DISTRIBUTIONS

Take the beta distribution $f_1(t)$ on the interval $\left[0,1\right]$ with $\alpha=\beta=1$.

$$f_1(t) = \frac{(\alpha + \beta + 1)!}{\alpha!\ \beta!}\ t^{\alpha}(1-t)^{\beta},$$

$$= 6t(1-t).$$

$$F_1(t) = \int_0^t f_1(u)du = \int_0^t 6u\ (1-u)du = 3t^2 - 2t^3.$$

$$F_2(t) = F_1(t)\ F_1(t),$$

$$= (3t^2 - 2t^3)^2,$$

$$= 9t^4 - 12t^5 + 4t^6.$$

$$f_2(t) = \frac{dF_2(t)}{dt} = 36t^3 - 60t^4 + 24t^5.$$

$$F_3(t) = F_1(t) \cdot F_1(t) \cdot F_1(t),$$

$$= (3t^2 - 2t^3)^3,$$

$$= 27t^6 - 54t^7 + 36t^8 - 8t^9.$$

$$f_3(t) = \frac{dF_3(t)}{dt} = 162t^5 - 378t^6 + 288t^7 - 72t^8.$$

$$\mu_i = \int_0^1 t\ f_i(t)dt.$$

$$\sigma_i^2 = \int_0^1 t^2 f_i(t)dt - \mu_i^2.$$

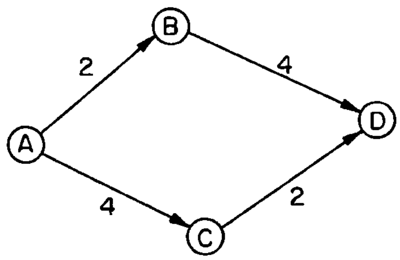
$$\mu_1 = 0.50,$$

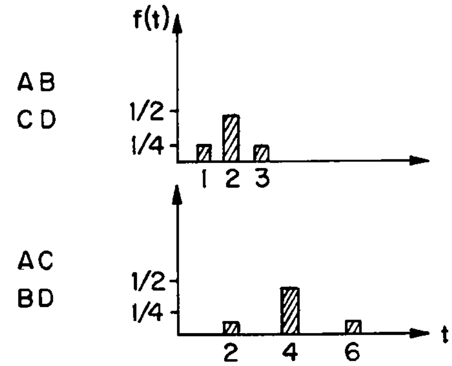
$$\mu_2 = 0.63,$$

$$\mu_3 = 0.69.$$

Appendix I

## EXAMPLES -- PARALLELISM AND CROSS CONNECTIONS




Distribution of Paths ABD and ACD

ABD = AB + BD

ACD = AC + CD

Sum

| f(t) | | 1/4 | 1/2 | 1/4 |
|---|---|---|---|---|
| | t | 2 | 4 | 6 |
| 1/4 | 1 | 3 | 5 | 7 |
| 1/2 | 2 | 4 | 6 | 8 |
| 1/4 | 3 | 5 | 7 | 9 |

| t | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|
| f(t) | 1/16 | 2/16 | 3/16 | 4/16 | 3/16 | 2/16 | 1/16 |

Distribution of Event D

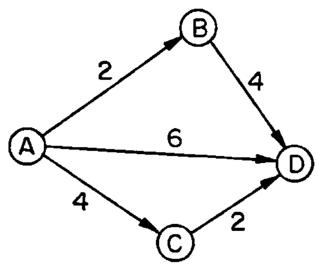
D = Max [ABD, ACD]

Max

| f(t) | | 1/16 | 2/16 | 3/16 | 4/16 | 3/16 | 2/16 | 1/16 |
|---|---|---|---|---|---|---|---|---|
| | t | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1/16 | 3 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2/16 | 4 | 4 | 4 | 5 | 6 | 7 | 8 | 9 |
| 3/16 | 5 | 5 | 5 | 5 | 6 | 7 | 8 | 9 |
| 4/16 | 6 | 6 | 6 | 6 | 6 | 7 | 8 | 9 |
| 3/16 | 7 | 7 | 7 | 7 | 7 | 7 | 8 | 9 |
| 2/16 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 9 |
| 1/16 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |

| t | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|
| f(t) | 1/256 | 8/256 | 27/256 | 64/256 | 69/256 | 56/256 | 31/256 |

Mean = 6.89




AD

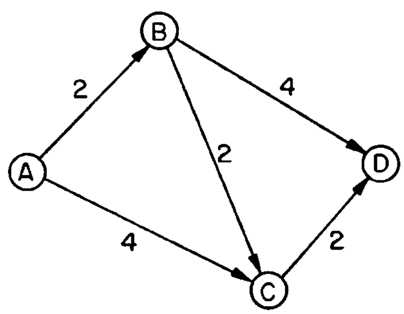
| t | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|
| f(t) | 1/16 | 2/16 | 3/16 | 4/16 | 3/16 | 2/16 | 1/16 |

Note that this is the sum of the distributions with means 2 and 4.

$$D = \text{Max}\left[ABD,\ ACD,\ AD\right]$$

| t | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|
| f(t) | 1/4096 | 26/4096 | 189/4096 | 784/4096 | 1197/4096 | 1178/4096 | 721/4096 |

Mean = 7.336




$$D = p(B=1)\cdot\left[D/(B=1)\right] + p(B=2)\cdot\left[D/(B=2)\right] + p(B=3)\cdot\left[D/(B=3)\right]$$

$$D/(B=1) = \text{Max}\left[1 + BD,\ C/(B=1)\right]$$

$$C/(B=1) = \text{Max}\left[1 + BC,\ AC\right] \quad \text{etc.}$$

D

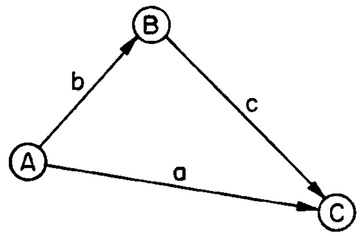
| t | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|
| f(t) | 1/1024 | 6/1024 | 75/1024 | 248/1024 | 315/1024 | 262/1024 | 117/1024 |

Mean = 7.074

## Appendix J

## EFFECT OF SLACK

Consider the network below.




Activities a, b, and c are all normally distributed with variance = 1. Means are given below. Formulas are from Clark [1].

---

**I.** Means: a = 4, b = 2, c = 2

Event

A $E(A) = 0$

$V(A) = 0$

B $E(B) = 2$

$V(B) = 1$

C $C = \text{Max }(a, B+C)$ $E(a)=4$ $E(B+c)=4$

$V(a)=1$ $V(B+c)=2$

$$g = \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho = 1+2-0 = 3$$

$$\alpha = (\mu_1 - \mu_2)/g = 0$$

$$\nu_1 = \mu_1 \Phi(\alpha) + \mu_2 \Phi(-\alpha) + g\varphi(\alpha)$$

$$= 4\ \Phi(0) + 4\ \Phi(0) + \sqrt{3}\ \varphi(0)$$

$$= 4.69$$

$$\nu_2 = (\mu_1^2 + \sigma_1^2)\Phi(\alpha) + (\mu_2^2 + \sigma_2^2)\Phi(-\alpha) + (\mu_1 + \mu_2) g \varphi(\alpha)$$

$$= (17)\Phi(0) + (18)\Phi(0) + 8(3)^{\frac{1}{2}}\varphi(0)$$

$$= 23.03$$

$$E(c) = \nu_1 = 4.69$$

$$V(c) = \nu_2 - \nu_1^2 = 23.03 - 22.00 = 1.03$$

---

**II.** Means: a = 3, b = 2, c = 2

Event

A $E(A) = 0$

$V(A) = 0$

B $E(B) = 2$

$V(B) = 1$

C $C = \text{Max }(a, B+c)$ $E(a)=3$ $E(B+c)=4$

$V(a)=1$ $V(B+c)=2$

$$g^2 = 3$$

$$\alpha = -0.577$$

$$\nu_1 = 4.303$$

$$\nu_2 = 19.839$$

$$E(c) = 4.303$$

$$V(c) = 1.323$$

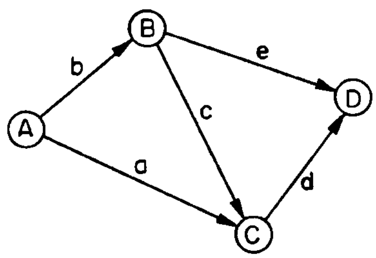
| III. Means: $a = 3$, $b = 3$, $c = 3$ | IV. Means: $a = 2$, $b = 4$, $c = 4$ |
|---|---|
| **Event** | **Event** |
| A $\quad E(A) = 0$ | A $\quad E(A) = 0$ |
| $V(A) = 0$ | $V(A) = 0$ |
| B $\quad E(B) = 3$ | B $\quad E(B) = 4$ |
| $V(B) = 1$ | $V(B) = 1$ |
| C $\quad C = \text{Max}\ (a, B+c)$ $\quad E(a)=3$ $E(B+c)=6$ | C $\quad C = \text{Max}\ (a, B+c)$ $\quad E(a)=2$ $E(B+c)=8$ |
| $V(a)=1$ $V(B+c)=2$ | $V(a)=1$ $V(B+c)=2$ |
| $g^2 = 3$ | $g^2 = 3$ |
| $\alpha = -1.732$ | $\alpha = -3.464$ |
| $\nu_1 = 6.03$ | $\nu_1 = 8.000$ |
| $\nu_2 = 38.22$ | $\nu_2 = 66.000$ |
| $E(c) = 6.03$ | $E(c) = 8.00$ |
| $V(c) = 1.86$ | $V(c) = 2.00$ |

## Appendix K

### FOUR EVENT NETWORK -- CONTINUOUS DISTRIBUTIONS

Normal:

Analytical calculation of project mean and variance for the network below, all distributions being normally distributed. The formulas used are from Clark, [1].




Means:

$E(a) = E(b) = E(c) = E(d) = E(e) = 1.$

Variances:

$V(a) = V(b) = V(c) = V(d) = V(e) = 1.$

Event

A $\quad$ Mean $= E(A) = 0,$ $\quad$ by definition

Variance $= V(A) = 0.$

B $\quad E(B) = E(b) + E(A) = 1,$

$V(B) = V(b) + V(A) = 1.$

C $\quad C = \text{Max}\ (a, B + c).$ $\quad E(a) = 1 \quad E(B+c) = 2$

$V(a) = 1 \quad V(B+c) = 2$

$$g^2 = \sigma_1^2 + \sigma_2^2 - 2\,\sigma_1\sigma_2\rho,$$

$$= V(a) + V(B+c) - 2V^{\frac{1}{2}}(a)\ V^{\frac{1}{2}}(B+c)\cdot r(a,B+c) = 1 + 2 - 0 = 3.$$

$$\alpha^2 = (\mu_1 - \mu_2)/g,$$

$$= \left[E(a) - E(B+c)\right]/\sqrt{3} = -0.577.$$

$$\nu_1 = \mu_1\Phi(\alpha) + \mu_2\Phi(-\alpha) + g\,\phi(\alpha),$$

$$= E(a)\ (-0.577) + E(B+c)\ (0.577) + 1.732\ (-0.577),$$

$$= 1\ (0.2820) + 2\ (0.7180) + 1.732\ (0.3377) = 2.203.$$

C (continued) $\mathcal{V}_2 = (\mu_1^2 + \sigma_1^2)\Phi(\alpha) + (\mu_2^2 + \sigma_2^2) \quad (-\alpha) + (\mu_1 + \mu_2)\, g\,\phi(\alpha),$

$$= (1 + 1)\,(-0.577) + (4 + 2)\,(0.577) + (1 + 2)\sqrt{3}\,(-0.577),$$

$$= 0.564 + 4.308 + 1.755 = 6.627.$$

$$E(C) = \mathcal{V}_1 = 2.203.$$

$$V(C) = \mathcal{V}_2 - \mathcal{V}_1^2 = 6.627 - 4.853 = 1.774.$$

D $\quad D = \text{Max}\,(B + e,\ C + d) \qquad E(B + e) = 2 \quad E(C + d) = 3.203$

$$V(B + e) = 2 \quad V(C + d) = 2.774$$

$$r(B + e,\ C + d) = \frac{V^{\frac{1}{2}}(B)\ V^{\frac{1}{2}}(C)\ r(B,C)}{V^{\frac{1}{2}}(B + e)\ V^{\frac{1}{2}}(C + d)}.$$

$$r(B,C) = r\left[b,\ \max\,(a,\ b + c)\right],$$

$$= \frac{\sigma_1 \rho_1 \Phi(\alpha) + \sigma_2 \rho_2 \Phi(-\alpha)}{(\mathcal{V}_2 - \mathcal{V}_1^2)^{\frac{1}{2}}}.$$

$$\rho_1 = r(a,\ b) = 0.$$

$$\rho_2 = r(b + c,\ b) = \frac{V^{\frac{1}{2}}(b)}{V^{\frac{1}{2}}(b + c)} = \frac{1}{\sqrt{2}} = 0.702.$$

$$r(B,C) = \frac{(1)(0)\Phi(-0.577) + (\sqrt{2})\ (0.702)\Phi(0.577)}{(1.774)^{\frac{1}{2}}},$$

$$= \frac{(.718)}{(1.774)^{\frac{1}{2}}}.$$

$$r(B{+}e,C{+}d) = \frac{(1)^{\frac{1}{2}}\ (1.774)^{\frac{1}{2}}}{(2)^{\frac{1}{2}}\ (2.774)^{\frac{1}{2}}} \cdot \frac{(.718)}{(1.774)^{\frac{1}{2}}} = 0.3055.$$

$$g^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho = 2 + 2.774 - (2)(2)^{\frac{1}{2}}(2.774)^{\frac{1}{2}}(.3055) = 3.338.$$

$$\alpha = (\mu_1 - \mu_2)/g = (-1.203)/1.827 = -0.658.$$

D (continued) $\mathcal{V}_1 = \mu_1\Phi(\alpha) + \mu_2\Phi(-\alpha) + g\phi(\alpha) = 2(.2552) + 3.203\ (.7448)$

$+ 1.827\ (.3214),$

$= 3.4832.$

$\mathcal{V}_2 = (\mu_1^2 + \sigma_1^2)\Phi(\alpha) + (\mu_2^2 + \sigma_2^2)\Phi(-\alpha) + (\mu_1 + \mu_2)\ g\phi(\alpha),$

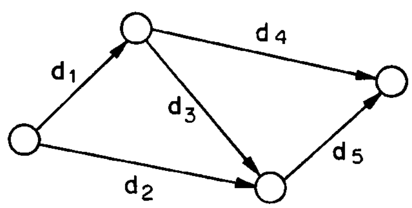
$= (6)\ (.2552) + (13.033)\ (.7448) + (5.203)\ (1.827)\ (.3214),$

$= 14.293.$

$E(D) = \mathcal{V}_1 = 3.483.$

$V(D) = \mathcal{V}_2 - \mathcal{V}_1^2 = 14.2934 - 12.1313 = 2.162.$

Uniform




$t_i$ : duration of activity $d_i$

density of $t_i$ : $\begin{cases} 1 & 0 \le t_i \le 1 \\ 0 & \text{otherwise} \end{cases}$

Project Cumulative Distribution $F(u)$

For $1 \le u \le 2$,

$$F(u) = \int_0^{u-1} \left[ \int_0^{u-1-t_1} dt_5 \right] dt_1$$

$$+ \int_0^{u-1} \left[ \int_{u-1-t_1}^{u-1} (u - t_1 - t_5) dt_5 \right] dt_1$$

$$+ \int_{u-1}^{1} \left[ \int_0^{u-1} (u - t_1)\ (u - t_1 - t_5) dt_5 \right] dt_1$$

$$+\int_{0}^{u-1}\left[\int_{u-1}^{1} (u - t_5)\ (u - t_1 - t_5)dt_5\right]dt_1$$

$$+\int_{u-1}^{1}\left[\int_{u-1}^{u-t_1} (u - t_1)\ (u - t_5)\ (u - t_1 - t_5)dt_5\right]dt_1$$

$$= \frac{1}{120}\left[-u^5 - 20u^4 + 100u^3 - 120u^2 + 80u - 28\right].$$

For $0 \leqslant u \leqslant 1$,

$$F(u) = \int_{0}^{u}\left[\int_{0}^{u-t_1} (u - t_1)\ (u - t_5)\ (u - t_1 - t_5)dt_5\right]dt_1,$$

$$= \frac{11}{120}\, u^5.$$

For $2 \leqslant u \leqslant 3$,

$$F(u) = 1 - \frac{1}{6}\,(3 - u)^3,$$

$$= \frac{1}{6}\,(u^3 - 9u^2 + 27u - 21).$$

# Appendix L

## FOUR-EVENT EXAMPLE -- DISCRETE DISTRIBUTIONS

The same calculation procedure is used as was used in Appendix I.

The distributions are:

Event

D

| t | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|
| f(t) | $\frac{1}{3125}$ | $\frac{87}{3125}$ | $\frac{528}{3125}$ | $\frac{941}{3125}$ | $\frac{1139}{3125}$ | $\frac{404}{3125}$ | $\frac{25}{3125}$ |

Mean = 6.42

Variance = 1.059

E (without activity AE)

| t | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|
| f(t) | $\frac{1}{78125}$ | $\frac{90}{78125}$ | $\frac{2305}{78125}$ | $\frac{11156}{78125}$ | $\frac{18380}{78125}$ | $\frac{21978}{78125}$ | $\frac{18914}{78125}$ | $\frac{5176}{78125}$ | $\frac{125}{78125}$ |

Mean = 8.77

Variance = 1.557

E (with activity AE)

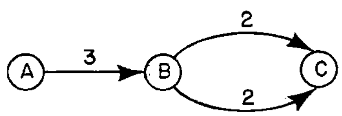
| t | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|
| f(t) | $\frac{1}{390625}$ | $\frac{90}{390625}$ | $\frac{2305}{390625}$ | $\frac{51812}{390625}$ | $\frac{73520}{390625}$ | $\frac{87912}{390625}$ | $\frac{75656}{390625}$ | $\frac{98704}{390625}$ | $\frac{625}{390625}$ |

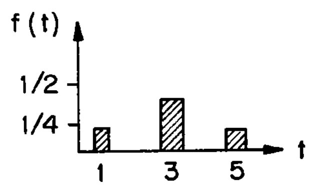
Mean = 9.23

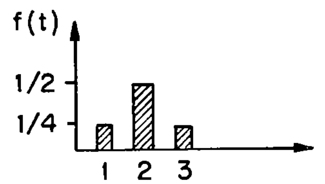
Variance = 1.941

Appendix M

EXAMPLES -- COMBINING SIMPLE SERIES AND PARALLEL




The calculation procedure is similar to that given in Appendix I.

Taking parallel element BC

Distribution of C:

| t | 1 | 2 | 3 |
|---|---|---|---|
| f(t) | 1/16 | 8/16 | 7/16 |

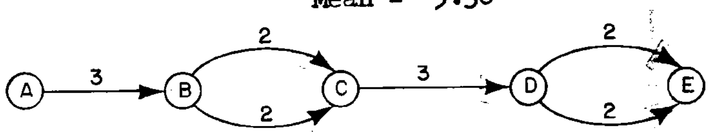
Mean = 2.38

Adding series element AB

Distribution of C:

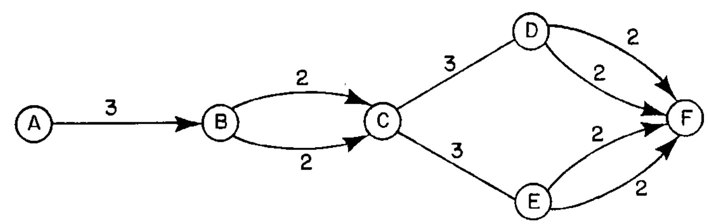
| t | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| f(t) | 1/64 | 8/64 | 9/64 | 16/64 | 15/64 | 8/64 | 7/64 |

Mean = 5.38



Distribution of E

| t | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| f(t) | $\frac{1}{4096}$ | $\frac{16}{4096}$ | $\frac{82}{4096}$ | $\frac{176}{4096}$ | $\frac{367}{4096}$ | $\frac{544}{4096}$ | $\frac{668}{4096}$ | $\frac{736}{4096}$ | $\frac{607}{4096}$ | $\frac{464}{4096}$ | $\frac{274}{4096}$ | $\frac{112}{4096}$ | $\frac{49}{4096}$ |




| t | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| f(t) | $\frac{4}{3044608}$ | $\frac{440}{3044608}$ | $\frac{5352}{3044608}$ | $\frac{24584}{3044608}$ | $\frac{91064}{3044608}$ | $\frac{210528}{3044608}$ | $\frac{364400}{3044608}$ | $\frac{542432}{3044608}$ | $\frac{578676}{3044608}$ | $\frac{546728}{3044608}$ | $\frac{394856}{3044608}$ | $\frac{190680}{3044608}$ | $\frac{94864}{3044608}$ |

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