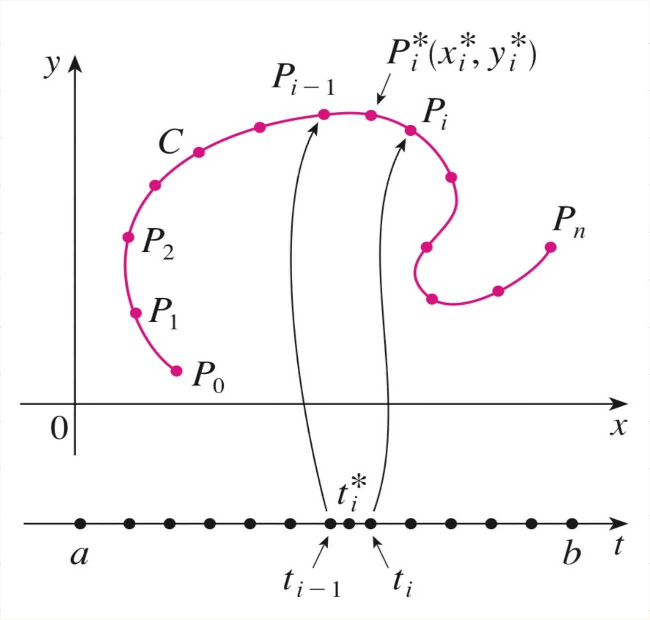
Vector Integration

Line integral :

An integral similar to a single integral except that instead of integrating over an interval $[a, b]$ we integrate over a curve C. Such integrals are called line integrals or curve integrals or path integrals.



We start with a plane curve C given by the parametric equations

$$\boxed{1} \qquad x = x(t) \qquad y = y(t) \qquad a \leqslant t \leqslant b$$

or, equivalently, by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, and we assume that C is a smooth curve. [This means that $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$. See Section 13.3.] If we divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width and we let $x_i = x(t_i)$ and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$. (See Figure 1.) We choose any point $P_i^*(x_i^*, y_i^*)$ in the ith subarc. (This corresponds to a point t_i^* in $[t_{i-1}, t_i]$.) Now if f is any function of two variables whose domain includes the curve C, we evaluate f at the point (x_i^*, y_i^*), multiply by the length Δs_i of the subarc, and form the sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

which is similar to a Riemann sum. Then we take the limit of these sums and make the following definition by analogy with a single integral.

DEFINITION If f is defined on a smooth curve C given by Equations 1, then the **line integral of f along C** is

$$\int_C f(x, y)\,ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

if this limit exists.

Geometrically, $\int_C f(x,y)\,ds$ represents the area of one side of the 'fence' in Fig 2, whose base is C and whose height above the point (x,y) is $f(x,y)$

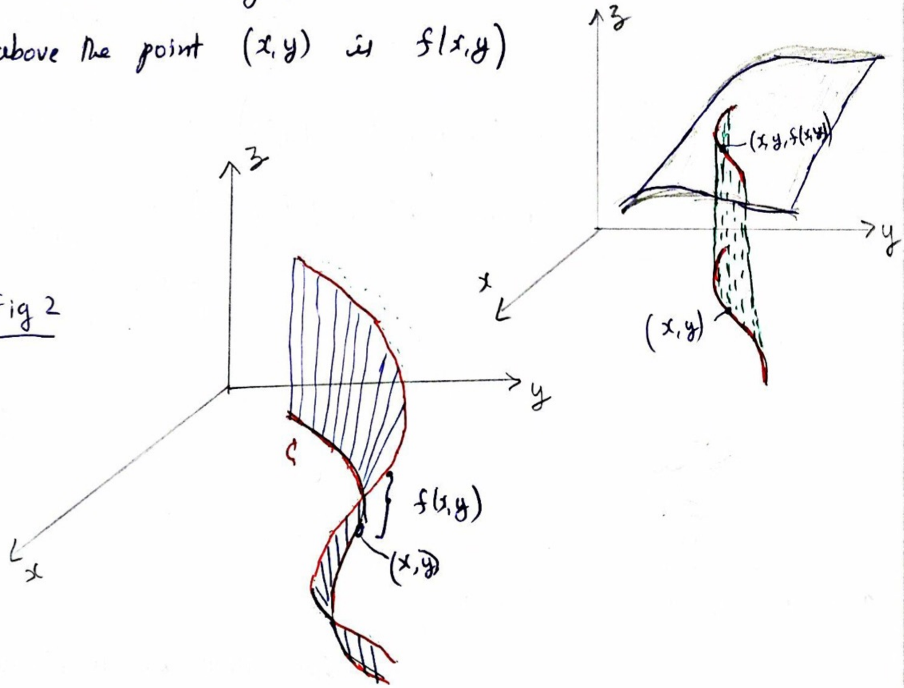


Fig 2

Line integral of vector function :

Let F be a vector field defined on curve C given by vector function $\vec{r}(t)$, $a \leq t \leq b$, then line integral of F along C is

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \left(\vec{F} \cdot \frac{d\vec{r}}{dt} \right) dt$$

1) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the circle $x^2+y^2=a^2$

where $\vec{F} = 3xy\,\hat{i} - y\hat{j} + 2z\hat{k}$.

Sol. The parametric eqns of given circle is

$x = a\cos t$, $y = a\sin t$, $z=0$, $0 \le t \le 2\pi$.

$dx = -a\sin t\, dt$, $dy = a\cos t\, dt$, $dz = 0$

$\vec{F} = 3(a\cos t)(a\sin t)\hat{i} - a\sin t\,\hat{j} + 0\,\hat{k}$

$\vec{F} = 3a^2 \sin t\cos t\,\hat{i} - a\sin t\,\hat{j}$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (-3a^3\sin^2 t\cos t - a^2\sin t\cos t)\,dt$$

$$= \int_0^{2\pi} \left\{ -a^2(3a\sin^2 t + \sin t)\cos t \right\} dt$$

$$= -a^2\left[\cancel{3}a\frac{\sin^3 t}{\cancel{3}} + \frac{1}{2}\sin^2 t\right]_0^{2\pi}$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = 0$$

$\int 3a\sin^2 t\cos t\,dt$

Let $\sin t = u$

$\cos t\,dt = du$

$\int 3a u^2\,du$

$= \cancel{3}a\frac{u^3}{\cancel{3}}$

2) If $\vec{F} = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$, evaluate the integral

$\int_C \vec{F}\cdot d\vec{r}$ where C is curve $x = 2t^2$, $y=t$, $z=t^3$

from the point $(0,0,0)$ to the point $(2,1,1)$

Sol: $x = 2t^2$, $y = t$, $z = t^3$

$\Rightarrow dx = 4t\,dt$, $dy = dt$, $dz = 3t^2\,dt$

$$\int_C \vec{F}\cdot\vec{dr} = \int_0^1 \left[(2t+3)\,4t + 2t^5 + (t^4-2t^2)\,3t^2\right] dt$$

$$= \int_0^1 \left\{ 8t^2 + 12t + 2t^5 + 3t^6 - 6t^4 \right\} dt$$

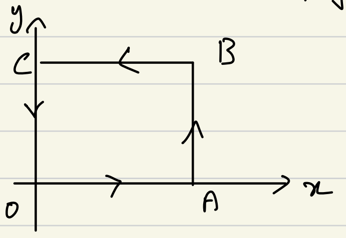
$$= \left\{ \frac{8t^3}{3} + \frac{12t^2}{2} + \frac{2t^6}{6} + \frac{3t^7}{7} - \frac{6t^5}{5} \right\}_0^1$$

$$= \frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5} = \frac{280+630+35+45-126}{105}$$

$$= \frac{864}{105} = \frac{288}{35}$$

3) Evaluate $\int_C \vec{F}\cdot\vec{dr}$ where $\vec{F} = x^2\hat{i} + xy\hat{j}$ and C is boundary of square in the plane $z=0$ and bounded by the lines $x=0$, $y=0$, $x=a$ and $y=a$.

Sol:



$$\int_C \vec{F}\cdot\vec{dr} = \int_{OA} \vec{F}\cdot\vec{dr} + \int_{AB} \vec{F}\cdot\vec{dr} + \int_{BC} \vec{F}\cdot\vec{dr} + \int_{CO} \vec{F}\cdot\vec{dr}$$

$$\vec{dr} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}, \quad \vec{F} = x^2\hat{i} + xy\,\hat{j}$$

$$\vec{F} \cdot d\vec{r} = x^2\,dx + xy\,dy \rightarrow (1)$$

On OA, $y = 0$ $\quad \therefore \vec{F} \cdot d\vec{r} = x^2\,dx$

$$\int_{OA} \vec{F} \cdot d\vec{r} = \int_0^a x^2\,dx = \frac{x^3}{3}\Big|_0^a = \frac{a^3}{3} \rightarrow (2)$$

On AB, $x = a$ $\quad \therefore dx = 0$

$(1) \Rightarrow \quad \vec{F} \cdot d\vec{r} = ay\,dy$

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_0^a ay\,dy = a\frac{y^2}{2}\Big|_0^a = \frac{a^3}{2} \rightarrow (3)$$

On BC, $y = a$ $\quad \therefore dy = 0$

$(1) \Rightarrow \quad \vec{F} \cdot d\vec{r} = x^2\,dx$

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_a^0 x^2\,dx = \frac{x^3}{3}\Big|_a^0 = -\frac{a^3}{3} \rightarrow (4)$$

On CO, $x = 0$, $\quad \therefore \vec{F} \cdot d\vec{r} = 0$

$$(1) \Rightarrow \quad \int_{CO} \vec{F} \cdot d\vec{r} = 0 \rightarrow (5)$$

Adding (2), (3), (4), (5), we get

$$\int_C \vec{F} \cdot d\vec{r} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0 = \frac{a^3}{2}.$$

Note: If $\int_A^B \vec{F} \cdot d\vec{r}$ is to be proved to be independent of path, then $\vec{F} = \nabla\phi$. Here $\vec{F}$ is called conservative (irrotational) vector field and ϕ is called scalar potential and $\nabla \times \vec{F} = \nabla \times \nabla \phi = 0$.

4) S.T. the integral $\int_{(1,2)}^{(3,4)} (xy^2 + y^3)dx + (x^2y + 3xy^2)dy$ is independent of the path joining the points $(1,2)$ and $(3,4)$. Hence, evaluate the integral.

Sol:

$$\int_{(1,2)}^{(3,4)} (xy^2 + y^3)dx + (x^2y + 3xy^2)dy =$$

$$\int_{(1,2)}^{(3,4)} \left[(xy^2 + y^3)\hat{i} + (x^2y + 3xy^2)\hat{j} \right] \cdot (dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_{(1,2)}^{(3,4)} \vec{F} \cdot d\vec{r}$$

This integral is independent of path if $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 + y^3 & x^2y + 3xy^2 & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(2xy + 3y^2 - 2xy - 3y^2)$$

$$= 0$$

$\exists$ $\nabla \times \vec{F} = 0$ then $\vec{F} = \nabla \phi$

$$\frac{\partial \phi}{\partial x} = xy^2 + y^3, \quad \frac{\partial \phi}{\partial y} = x^2y + 3xy^2$$

Integrating wrt x & y

$$\phi(x,y) = \frac{x^2}{2}y^2 + xy^3 + f_1(y)$$

$$\phi(x,y) = x^2\frac{y^2}{2} + xy^3 + f_2(x)$$

$$\therefore \phi(x,y) = \frac{x^2y^2}{2} + xy^3$$

$$\therefore [\phi]_{(1,2)}^{(3,4)} = \left.\frac{x^2y^2}{2} + xy^3\right]_{(1,2)}^{(3,4)} = \left[\frac{1}{2}(9)(16) + (3)(64)\right]$$
$$- \left[\frac{1}{2}(1)(4) + (1)(8)\right] = 254.$$

Practice problems:

1) If $\vec{F} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, evaluate $\int_C \vec{F}\cdot\vec{dr}$ from $(0,0,0)$ to $(1,1,1)$ along curve C given by $x=t$, $y=t^2$, $z=t^3$.

Ans: 5

2) Evaluate $\int_C \vec{F}\cdot\vec{dr}$ where $\vec{F} = 3xy\hat{i} - y^2\hat{j}$ and C is part of parabola $y=2x^2$ from the origin $(0,0)$ to the point $(1,2)$ Ans: $-7/6$

3) Find the work done by a force $\vec{F} = 2xy\hat{i} - 4z\hat{j} + 5x\hat{k}$ along the curve $x=t^2$, $y=2t+1$, $z=t^3$ from the point $t=1$ to $t=2$

Ans: $\frac{638}{5}$

4) Determine whether line integral

$$\int (2xyz^2)\,dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz$$ is independent of path of integration. If so, evaluate it from $(1, 0, 1)$ to $(0, \frac{\pi}{2}, 1)$. Ans: 1

5) Compute $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \dfrac{y\hat{i} - x\hat{j}}{x^2 + y^2}$ and C is circle $x^2 + y^2 = 1$ traversed counter clockwise.
Ans: -2π

6) Suppose $\vec{F}(x, y, z) = x^3\hat{i} + y\hat{j} + z\hat{k}$ is the force field. Find work done by $\vec{F}$ along the line from the points $(1, 2, 3)$ to $(3, 5, 7)$.
Ans: 50.5 units.

7) Find the work done in moving a particle in the force field $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along the curve $x^2 = 4y$ and $3x^3 = 8z$ from $x = 0$ to $x = 2$. Ans: 16 units

Surface integral → generalization of line integral

Any integral which is to be evaluated over a surface is called surface integral. (flux integral)

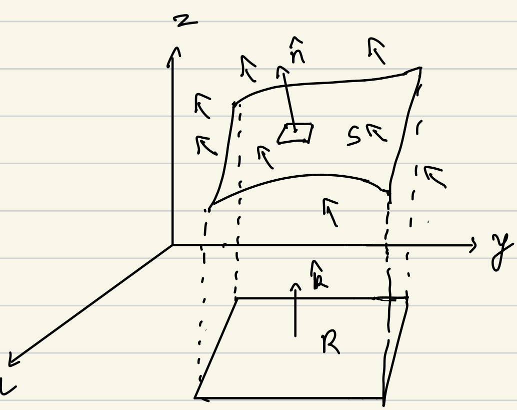
Physical interpretation: The surface integral of a vector function represents flux through a surface.

ex: If $\vec{F}$ represents velocity vector of a fluid, the surface integral represents the rate of flow of fluid through the surface.

Surface integral of vector function

If $\vec{F}$ is a continuous vector field defined on a surface S with unit normal vector $\hat{n}$, then the surface integral of $\vec{F}$ over S is

$$\boxed{\iint_S \vec{F}\cdot d\vec{s} = \iint_S \vec{F}\cdot\hat{n}\, ds}$$



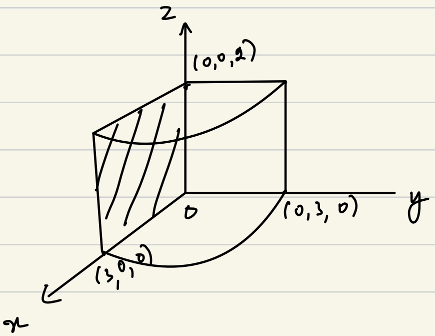
$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

if the projection is taken in xy plane

Note: If $\iint_S \vec{F}\cdot\hat{n}\, ds = 0$, then $\vec{F}$ is a solenoidal vector point function.

1) Evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$ where $\vec{F} = yz\,\hat{i} + 2y^2\,\hat{j} + xz^2\,\hat{k}$

and S is surface of cylinder $x^2+y^2=9$ contained in first octant between $z=0$ and $z=2$



Let $\phi = x^2+y^2-9$

$\nabla\phi = 2x\,\hat{i} + 2y\,\hat{j}$

$|\nabla\phi| = \sqrt{4x^2+4y^2} = 2\sqrt{9} = 6$

$\therefore\ \hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2(x\hat{i}+y\hat{j})}{6} = \dfrac{x\hat{i}+y\hat{j}}{3}$

$\vec{F}\cdot\hat{n} = \frac{1}{3}(xyz+2y^3)$

Projecting onto xz-plane

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_R \vec{F}\cdot\hat{n}\ \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$$

$$= \int_{z=0}^{2}\int_{x=0}^{3} \frac{1}{\cancel{3}}(xyz+2y^3)\,\frac{dx\,dz}{\left(\frac{y}{\cancel{3}}\right)}$$

$$= \int_{z=0}^{2} \int_{x=0}^{3} \left[xz + 2y^2 \right] dx\,dz$$

$$= \int_{z=0}^{2} \int_{x=0}^{3} \left[xz + 2(9-x^2) \right] dx\,dz$$

$$= \int_{z=0}^{2} z\,dz \int_{x=0}^{3} x\,dx + 2\int_{z=0}^{2} dz \int_{x=0}^{3} (9-x^2)\,dx$$

$$= \frac{z^2}{2}\Big|_0^2 \; \frac{x^2}{2}\Big|_0^3 + 2z\Big|_0^2 \cdot \left[9x - \frac{x^3}{3} \right]_0^3$$

$$= \left(2 \times \frac{9}{2}\right) + 4\left[27 - 9\right]$$

$$= 9 + 72 = 81$$

2) If S denotes the part of plane $2x+y+2z=6$ which lies in positive octant and $\vec{F} = 4x\hat{i} + y\hat{j} + z\hat{k}$, evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$

sol: The intercepts of given plane on positive x, y, z axes are 3, 6 and 3 resp.

$\therefore$ In first octant, $0 \le x \le 3$, $0 \le y \le 6$, $0 \le z \le 3$.

Let $\phi = 2x + y + 2z - 6$

$\nabla\phi = 2\hat{i} + \hat{j} + 2\hat{k}$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{4+1+4}} = \frac{1}{3}(2\hat{i} + \hat{j} + 2\hat{k})$$

$$\vec{F}\cdot\hat{n} = 4x \times \frac{2}{3} + \frac{y}{3} + \frac{2z}{3} = \frac{1}{3}(8x + y + 2z)$$

Projecting onto xy plane,

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_R \vec{F}\cdot\hat{n}\, \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$2x + y = 6$; x : 0 ~~to~~ 3 ; y : 0 to $6-2x$ and $|\hat{n}\cdot\hat{k}| = \frac{1}{\sqrt{3}}$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \int_0^3 \int_{y=0}^{6-2x} \frac{1}{\cancel{3}}\left(8x + y + (6 - 2x - y)\right) \frac{dy\,dx}{\frac{2}{\cancel{3}}}$$

$$= \int_0^3 \frac{1}{2}\left[8xy + \cancel{\frac{y^2}{2}} + 6y - 2xy - \cancel{\frac{y^2}{2}}\right]_0^{6-2x} dy\,dx$$

$$= \int_0^3 \frac{1}{2}\left[8 \times x(6-2x) + 6(6-2x) - 2x(6-2x)\right] dx$$

$$= \int_0^3 \frac{1}{2}\left[48x - 16x^2 + 36 - 12x - 12x + 4x^2\right] dx$$

$$= \frac{1}{2}\int_0^3 \left[24x - 12x^2 + 36\right] dx$$

$$= \frac{1}{2}\left[24\frac{x^2}{2} - 12\frac{x^3}{3} + 36x\right]_0^3 = \frac{1}{2}\left[108 - 108 + 108\right] = \frac{108}{2}$$

$$= 54$$

Practice problems :

1) Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where S is the part of surface of sphere $x^2+y^2+z^2=a^2$ in the first octant and $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ Ans: $\frac{3}{8}a^4$

2) If $\vec{F} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is part of the plane $2x+3y+6z=12$ in the first octant then find $\iint_S \vec{F}\cdot\hat{n}\,ds$ Ans: 24

3) Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where S is part of sphere $x^2+y^2+z^2=1$ above the xy- plane and $\vec{F} = y^2z^2\hat{i} + z^2x^2\hat{j} + x^2y^2\hat{k}$

Ans: $\frac{\pi}{24}$

Green's Theorem

It relates the line integral along a simple closed curve C to a double integral over the region bounded by C.

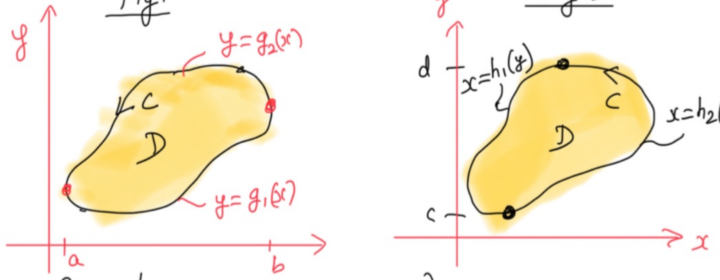
Green's theorem in the plane

If R is a closed region in xy plane, bounded by a simply closed curve C and if $P(x,y)$ and $Q(x,y)$, $\frac{\partial}{\partial x} Q(x,y)$, $\frac{\partial}{\partial y} P(x,y)$ be continuous functions at every point in R, then

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Proof:

Fig 1 Fig 2



$D=\{(x,y) \mid a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$ (Type-1)

$D=\{(x,y) \mid c\le y\le d,\ h_1(y)\le x\le h_2(y)\}$ (Type-2)

We show that

$$-\iint_D \frac{\partial P}{\partial y}\,dA = \oint_C P\,dx \quad \text{— (1)}$$

$$\iint_D \frac{\partial Q}{\partial x}\,dA = \oint_C Q\,dy \quad \text{— (2)}$$

Consider $-\iint_D \frac{\partial P}{\partial y} \, dA = -\int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y} \, dy \, dx$ (from fig 1)

$$= -\int_{x=a}^{b} P(x,y) \Big|_{y=g_1(x)}^{g_2(x)} dx$$

$$= -\int_{x=a}^{b} \left[P(x, g_2(x)) - P(x, g_1(x)) \right] dx$$

$$= \int_{x=a}^{b} P(x, g_1(x)) \, dx + \int_{x=b}^{a} P(x, g_2(x)) \, dx$$

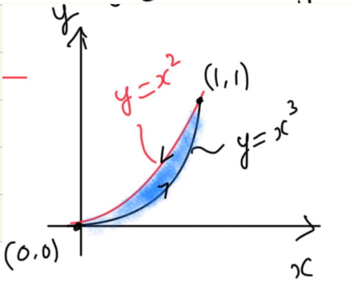
$$= \oint_C P(x,y) \, dx$$

IIIly from fig ②, we can show that

$$\iint_D \frac{\partial Q}{\partial x} \, dA = \oint_C Q(x,y) \, dx$$

This proves the theorem.

Ex 1: Evaluate $\oint_C (x^2-y^2)\,dx + (2y-x)\,dy$, where C consists of the boundary of the region in the 1st quadrant bounded by the graphs $y=x^2$ and $y=x^3$.



Using Green's thm

$$\oint_C \underbrace{(x^2-y^2)}_{P}\,dx + \underbrace{(2y-x)}_{Q}\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

where $\frac{\partial Q}{\partial x} = -1$, $\frac{\partial P}{\partial y} = -2y$

Thus

$$\oint_C (x^2-y^2)\,dx + (2y-x)\,dy = \iint_D (-1+2y)\,dA$$

$$\left(\text{where } D = \{(x,y) \mid 0 \le x \le 1,\ x^3 \le y \le x^2\}\right)$$

$$= \int_{x=0}^{1} \int_{y=x^3}^{x^2} (-1+2y)\,dy\,dx$$

$$= \int_{x=0}^{1} (-y+y^2)\Big|_{y=x^3}^{x^2}\,dx$$

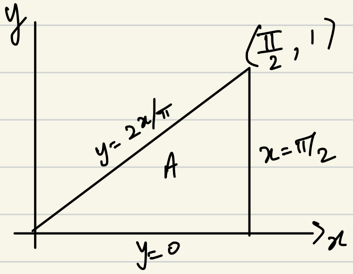
$$= \int_{x=0}^{1} \left(-x^2 + x^4 + x^3 - x^6\right) dx$$

$$= -\frac{x^3}{3} + \frac{x^5}{5} + \frac{x^4}{4} - \frac{x^7}{7} \Bigg|_0^1$$

$$= -1/3 + \frac{1}{5} + \frac{1}{4} - 1/7 = \frac{-11}{420}$$

2) By using Green's theorem evaluate $\int_C (y - \sin x)\,dx + \cos x\,dy$ where C is the triangle formed by the lines $y=0$, $x = \frac{\pi}{2}$ and $y = \frac{2x}{\pi}$

sol:



x increases from 0 to $\frac{\pi}{2}$ and y increases from 0 to $\frac{2x}{\pi}$.

Using Green's theorem, we get

$$\int_C (y-\sin x)\,dx + \cos x\,dy = \iint_R \left[\frac{\partial}{\partial x}\cos x - \frac{\partial}{\partial y}(y-\sin x)\right] dx\,dy$$

$$= -\iint_R (1+\sin x)\,dx\,dy$$

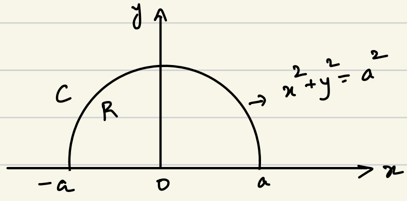
$$= -\int_{x=0}^{\pi/2}\int_{y=0}^{2x/\pi} (1+\sin x)\,dy\,dx = -\int_0^{\pi/2} \left((1+\sin x)y\right)\Big|_0^{\frac{2x}{\pi}} dx$$

$$= -\int_0^{\pi/2} \frac{2x}{\pi}(1+\sin x)\,dx = -\frac{2}{\pi}\left[\frac{x^2}{2} - x\cos x + \sin x\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left[\frac{\pi^2}{8}+1\right] = -\left(\frac{\pi}{4}+\frac{2}{\pi}\right)$$

3) By using Green's theorem, evaluate : $\int_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy$ where C is boundary of the area enclosed by x-axis and upper half of circle $x^2+y^2=a^2$

sol:



sol: $$\int_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy = \iint_R \left[\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(2x^2-y^2)\right] dx\,dy$$

$$= 2\iint_R (x+y)\,dx\,dy \rightarrow (1)$$

Here r increases from 0 to a and θ increases from

0 to π. (r, θ) are polar coordinates. Here $r = a$.

$\therefore dx\,dy = r\,dr\,d\theta. = a\,dr\,d\theta$

$$\iint_R (x+y)\,dx\,dy = \int_{\theta=0}^{\pi}\int_{r=0}^{a} (a\cos\theta + a\sin\theta)\,a\,dr\,d\theta$$

$$= \int_0^a a^2\,dr \int_0^{\pi} (\cos\theta + \sin\theta)\,d\theta$$

$$= \frac{a^3}{3}\Big|_0^a \times \Big(\sin\theta - \cos\theta\Big)_0^{\pi}$$

$$= \frac{a^3}{3} \times \Big[(0-(-1)) - (0-1)\Big]$$

$$= \frac{a^3}{3}\Big[1+1\Big] = \frac{2a^3}{3}$$

$\therefore$ (1) $\Rightarrow \int_C (2x^2 - y^2)\,dx + (x^2+y^2)\,dy = 2 \times \frac{2a^3}{3} = \frac{4a^3}{3}$

Practice problems:

1) Verify Green's theorem for $\int_C (xy + y^2)\,dx + x^2\,dy$

where C is the closed curve made up of the line $y = x$ and the parabola $y = x^2$.

Ans: $-\frac{1}{20}$

2) Using Green's theorem, evaluate $\int_C (x^2 y\,dx + x^2\,dy)$ where C is boundary described counter clockwise of triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$ Ans: $\frac{5}{12}$

3) verify Green's theorem for $\int_C (e^{-x} \sin y)\,dx + (e^{-x} \cos y)\,dy$ where C is the rectangle whose vertices are $(0,0)$, $(\pi, 0)$, $(\pi, \frac{\pi}{2})$, $(0, \frac{\pi}{2})$.

Ans: $2(e^{-\pi} - 1)$

4) If C is simple closed curve in xy-plane and containing the origin, calculate $\int_C \vec{F} \cdot d\vec{r}$ where

$$\vec{F} = \frac{-y\hat{i} + x\hat{j}}{x^2+y^2}$$

Ans: 0

5) verify Green's theorem for $\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$ where C is square with vertices $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$.

Ans: 8

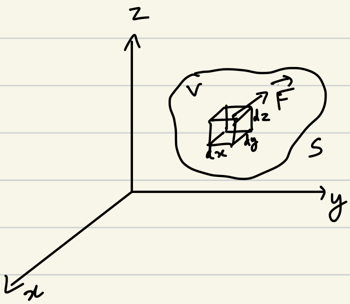
6) Using Green's theorem, find the area of the region enclosed between parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

Ans: $\frac{16a^3}{3}$

7) S.T. area bounded by a simple closed curve C is $\oint_C (x\,dy - y\,dx)$.

Volume integral

If $\vec{F}$ is a vector function and V is the volume enclosed by closed surface S then volume integral of $\vec{F}$ over V is defined and denoted by $\iiint_V \vec{F}\, dv$

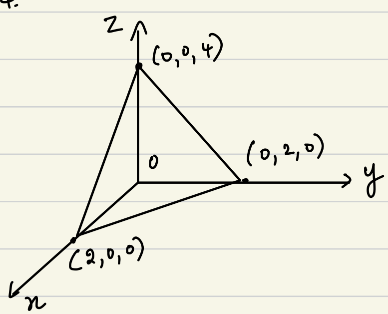


The surface S is divided into large no of volume elements such as $dv = dx\,dy\,dz$

Note: If ϕ is scalar function in volume V, then $\iiint_V \phi\, dv$ represents charge density (total charge enclosed in given volume)

1) If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\,\hat{j} - 4x\,\hat{k}$ evaluate $\iiint_V \nabla\cdot\vec{F}\, dv$ where V is closed region bounded by the planes $x=0,\ y=0,\ z=0$ and $2x+2y+z=4$.

Sol:



$$\nabla\cdot\vec{F} = 4x - 2x = 2x$$

$$\iiint_V \nabla\cdot\vec{F}\, dV = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x \, dz\, dy\, dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} \left[2xz\right]_0^{4-2x-2y} dy\, dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} 2x\,(4-2x-2y)\, dy\, dx$$

$$= \int_{x=0}^{2} \left[8xy - 4x^2y - 2xy^2\right]_0^{2-x} dx$$

$$= \int_{x=0}^{2} \left[8x(2-x) - 4x^2(2-x) - 2x(2-x)^2\right] dx$$

$$= \int_{x=0}^{2} \left[2x^3 - 8x^2 + 8x\right] dx$$

$$= \frac{x^4}{2} - \frac{8x^3}{3} + 4x^2 \Big|_0^2 = \frac{8}{3}$$

Practice problems:

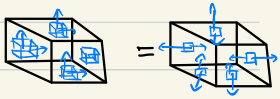
1) Evaluate $\iiint_V \phi \, dV$ where $\phi = 45x^2y$ and V is volume of closed region bounded by planes $4x+2y+z=8$, $x=0$, $y=0$, $z=0$. Ans: 128

2) Evaluate $\iiint_V \nabla\cdot\vec{F}\, dV$ where $\vec{F} = 2x^2y\,\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$, V is the region in first octant bounded by

$y^2 + z^2 = 9$ and plane $x = 2$. Ans: 180

(sum of sources and sinks of $\vec{F}$ within volume = total flux of $\vec{F}$ through surface of that volume)

Gauss Divergence theorem (gives relation b/w surface and volume integral)



The divergence theorem represents net rate of outward flux per unit volume. If there is net flow out of closed surface (i.e. div $\vec{F} > 0$) then integral is positive. If there is a net flow into the closed surface (i.e. div $\vec{F} < 0$) then the integral is negative.

Statement: If V is volume enclosed by a closed surface S and $\vec{F}$ is a vector point function having continuous derivatives then

$$\iint_S \underbrace{\vec{F}\cdot\hat{n}}_{\text{flux across C}}\, ds = \iiint_V \underbrace{\nabla\cdot\vec{F}}_{\text{flux per unit area (flux density)}}\, dV$$

where $\hat{n}$ is unit normal drawn to S. (i.e. outward unit normal vector away from the surface)

1) Verify divergence theorem for $\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ taken over the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$

Sol: By G.D.T.

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dV$$

$$\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$$

$$\nabla\cdot\vec{F} = 4z - 2y + y = 4z - y$$

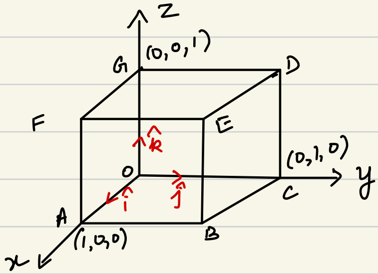
R.H.S. : $\iiint_V \nabla\cdot\vec{F}\, dv = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (4z-y)\, dz\, dy\, dx$

$$= \int_0^1\int_0^1 4\frac{z^2}{2} - yz\Big|_{z=0}^{1} dy\, dx$$

$$= \int_0^1\int_0^1 (2-y)\, dy\, dx = \int_{x=0}^{1} 2y - \frac{y^2}{2}\Big|_0^1 dx$$

$$= \int_0^1 \left(2-\frac{1}{2}\right) dx = \int_0^1 \frac{3}{2}\, dx = \frac{3}{2}x\Big|_0^1 = \frac{3}{2}$$

L.H.S



$$\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$$

Here S is surface of cube bounded by 6 faces

S_1 : For ABEF which is parallel to yz plane

The eq. is $x=1 \Rightarrow dx = 0$, $\hat{n} = \hat{i}$, $ds = dy\, dz$

$$\therefore \iint_{S_1} \vec{F}\cdot\hat{n}\, ds = \int_{z=0}^{1}\int_{y=0}^{1} 4xz\, dy\, dz$$

$$= \int_{z=0}^{1}\int_{y=0}^{1} 4z\, dy\, dz \qquad (\because x = 1)$$

$$= \int_{z=0}^{1} 4yz\Big|_0^1 dz = \int_{z=0}^{1} 4z\, dz = 4\frac{z^2}{2}\Big|_0^1$$

$$= 2$$

S_2 : For OCDG which is yz plane

The eq is $x=0 \Rightarrow dx=0$, $\hat{n}=-\hat{i}$, $ds=dy\,dz$

$$\therefore \iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \int_{z=0}^{1}\int_{y=0}^{1} (-4xz)\,dy\,dz = 0 \quad (\because x=0)$$

S_3 : For BCDE which is parallel to xz plane

The eq is $y=1 \Rightarrow dy=0$, $\hat{n}=\hat{j}$, $ds=dx\,dz$

$$\therefore \iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 -y^2\,dx\,dz = \int_0^1\int_0^1 (-1)\,dx\,dz$$

$$= -1$$

S_4 : For AOGF which is xz plane

The eq is $y=0 \Rightarrow dy=0$, $\hat{n}=-\hat{j}$, $ds=dx\,dz$

$$\therefore \iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 -y^2\,dx\,dz = 0 \quad (\because y=0)$$

S_5 : For DEFG which is parallel to xy plane

The eq is $z=1 \Rightarrow dz=0$, $\hat{n}=\hat{k}$, $ds=dx\,dy$

$$\therefore \iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 yz\,dx\,dy = \int_{y=0}^{1}\int_{x=0}^{1} y\,dx\,dy$$

$$= \frac{1}{2}$$

S_6 : For OABC which is xy plane

The eq is $z=0 \Rightarrow dz=0$, $\hat{n}=-\hat{k}$, $ds=dx\,dy$

$$\therefore \iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 -yz\,dx\,dy = 0 \quad (\because z=0)$$

Hence $\iint_S \vec{F}\cdot\hat{n}\,ds = 2+0-1+0+\frac{1}{2}+0 = \frac{3}{2}$

$\therefore$ G.D.T. is verified.

2) Use Gauss divergence theorem to evaluate $\iint_S (x^3\hat{i} + y^3\hat{j} + z^3\hat{k})\cdot\hat{n}\,ds$ where S is surface of sphere $x^2+y^2+z^2=9$.

Sol:
$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dV$$

$$\vec{F} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$$

$$\nabla\cdot\vec{F} = 3x^2 + 3y^2 + 3z^2 = 3(x^2+y^2+z^2)$$

Transforming the variables to spherical polar coordinates

$$x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta$$

$$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \iiint 3r^2\,dx\,dy\,dz$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{3} 3r^2\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 3\,\phi\Big|_0^{2\pi}\Big[-\cos\theta\Big]_0^{\pi}\left[\frac{r^5}{5}\right]_0^3$$

$$= 3\times 2\pi\times 2\times\frac{3^5}{5} = \frac{2^2\,3^6\,\pi}{5} = \frac{2916\pi}{5}$$

3) If $\vec{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$, a, b, c are constants S.T. $\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{4}{3}\pi(a+b+c)$, where S is surface of unit sphere.

Sol:
$$\nabla\cdot\vec{F} = a+b+c$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (a+b+c)\,dv$$

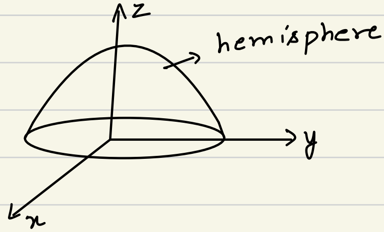
$$= a+b+c\iiint_V dv = a+b+c \times \text{volume of unit sphere}$$

$$= (a+b+c)\,\frac{4}{3}\pi r^3 = \frac{4}{3}\pi(a+b+c)$$ ($\because r=1$ for unit sphere)

4) Evaluate $\iint_S (y^2z^2\,\hat{i} + z^2x^2\,\hat{j} + z^2y^2\,\hat{k}) \cdot \hat{n}\, ds$ where

S is part of sphere $x^2+y^2+z^2=1$ above xy-plane and bounded by xy plane.

Sol:



$$\nabla \cdot \vec{F} = 2y^2 z$$

By G.D.T,

$$\iiint_V \nabla\cdot\vec{F}\, dv = \iiint_V 2y^2 z \; dz\, dy\, dx \rightarrow (1)$$

Changing to spherical polar coordinates

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$, $0 \le r \le 1$,

$0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le 2\pi$

$$\therefore (1) \Rightarrow \iiint_V \nabla\cdot\vec{F}\, dv = 2\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} (r\cos\theta)(r^2\sin^2\theta\sin^2\phi)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= 2\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} \sin^3\theta\cos\theta\sin^2\phi \int_{r=0}^{1} r^5\, dr\, d\theta\, d\phi$$

$$= \frac{2}{6}\int_0^{2\pi}\int_0^{\pi/2} \sin^3\theta\cos\theta\, \sin^2\phi \cdot d\theta\, d\phi$$

$$= \frac{1}{3}\int_{\phi=0}^{2\pi} \sin^2\phi \left[\int_{\theta=0}^{\pi/2} \sin^3\theta\cos\theta\, d\theta\right] d\phi$$

$$= \frac{1}{3} \int_{\phi=0}^{2\pi} \sin^2\phi \left[\frac{\sin^4\theta}{4} \right]_{\theta=0}^{\pi/2} d\phi$$

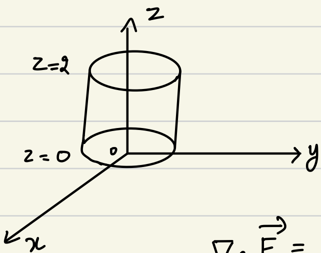
$$= \frac{1}{3} \times \frac{1}{4} \int_{\phi=0}^{2\pi} \sin^2\phi \, d\phi$$

$$= \frac{1}{12} \int_{\phi=0}^{2\pi} \left(\frac{1-\cos 2\phi}{2} \right) d\phi$$

$$= \frac{1}{24} \left[\phi - \frac{\sin 2\phi}{2} \right]_0^{2\pi} = \frac{1}{24} \left[2\pi - \frac{\sin 4\pi}{2} \right] = \frac{\pi}{12}$$

5) Let S be surface of a cylindrical solid D whose boundary is $x^2+y^2 = 4$, $z=0$ and $z=1$. Let $\vec{F} = (x^3, y^3, z^2)$ Evaluate the surface integral.

Sol:



$$\nabla \cdot \vec{F} = 3x^2 + 3y^2 + 2z$$

By G.D.T., $\iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dV$.

Using cylindrical polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$dx\,dy\,dz = r\,dr\,d\theta\,dz$

$$\iiint_V \nabla \cdot \vec{F} \, dV = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=0}^{1} (3r^2 + 2z) \, r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 3r^3 z + r z^2 \Big|_0^1 \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 (3r^3 + r) \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{3r^4}{4} + \frac{r^2}{2} \Big|_0^2 \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{48}{4} + \frac{4}{2} \right) d\theta = \int_0^{2\pi} 14 \, d\theta$$

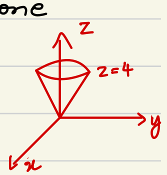
$$= 14 \cdot \theta \Big|_0^{2\pi} = 28\pi$$

Practice problems:

1) Evaluate $\iint_S \vec{F} \cdot \hat{n} \, ds$ for $\vec{F} = x\hat{i} - y\hat{j} + (z^2 - 1)\hat{k}$ where S is boundary surface of solid bounded by planes $z=0$, $z=1$ and cylinder $x^2+y^2=1$ Ans: 24π

2) Verify Gauss divergence theorem for the function $\vec{F} = (x^3 - yz)\hat{i} - 2x^2y\,\hat{j} + 2\hat{k}$ over a cube bounded by $x=0$, $x=a$, $y=0$, $y=a$, $z=0$ and $z=a$ Ans: $\frac{a^5}{3}$

3) Evaluate $\iint_S \vec{F} \cdot \hat{n} \, ds$ over the entire surface of the region above xy-plane bounded by the cone $z^2 = x^2+y^2$ and the plane $z = 4$ where $\vec{F} = 4xz\,\hat{i} + xyz^2\,\hat{j} + 3z\,\hat{k}$ Ans: 320π



4) Verify G.D.T. for $\vec{F} = 2xy\,\hat{i} + yz^2\,\hat{j} + xz\,\hat{k}$ and S is total surface of the rectangular parallelopiped bounded by the planes $x=0$, $y=0$, $z=0$, $x=1$, $y=2$, $z=3$
Ans: 33

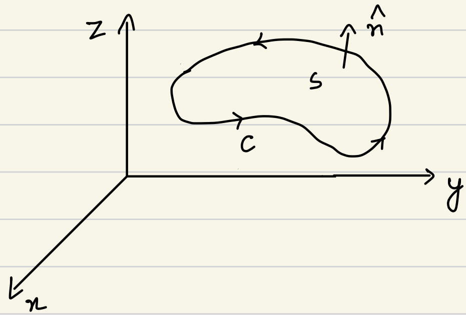
5) Using G.D.T., evaluate $\iint_S \vec{F} \cdot \hat{n} \, ds$, where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and S is surface bounding the region $x^2+y^2=4$, $z=0$, $z=3$
Ans: 84π

Stoke's Theorem (generalization of Green's Theorem and relates line integrals in 3 dimensions with surface integrals)

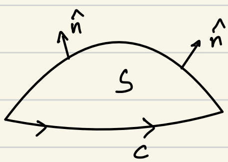
Statement: If S is an open surface bounded by a simple closed curve C and $\vec{F}$ is any vector point function having continuous first order partial derivatives then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\, \vec{F}\cdot \hat{n}\, ds = \iint_S \text{curl}\, \vec{F}\cdot d\vec{s}$$

where $\hat{n}$ is outward drawn unit normal at any point to S.

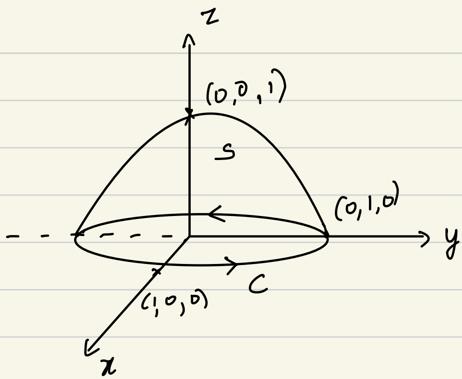


ex: S as the ballon and C is string of balloon.



1) Verify Stokes' theorem for $\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ where S is upper half of sphere $x^2+y^2+z^2=1$ and C is its boundary.

Sol:



By Stokes theorem $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot \hat{n}\, ds$

Since C is a circle of unit radius in xy plane

$x = \cos\phi,\ y = \sin\phi$ so that $\vec{r} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$

$d\vec{r} = -\sin\phi\, d\phi\,\hat{i} + \cos\phi\, d\phi\,\hat{j}$

On the boundary C, $z=0$ so that

$\vec{F} = (2x-y)\hat{i} = (2\cos\phi - \sin\phi)\hat{i}$

Thus

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (2\cos\phi - \sin\phi)\,\hat{i}\cdot(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})\,d\phi$$

$$= \int_0^{2\pi} (-2\sin\phi\cos\phi + \sin^2\phi)\,d\phi$$

$$= \int_0^{2\pi} \left[-\sin 2\phi + \frac{1}{2}(1-\cos 2\phi)\right] d\phi$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \left.\frac{\cos 2\phi}{2} + \frac{1}{2}\phi - \frac{1}{2}\frac{\sin 2\phi}{2}\right|_0^{2\pi}$$

$$= \frac{1}{2}(1-1) + \frac{1}{2}(2\pi - 0) + \frac{1}{4}(\sin 4\pi - \sin 0)$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \pi$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= \hat{i}\left[-2yz + 2yz\right] - \hat{j}\left[0-0\right] + \hat{k}\left[0+1\right] = \hat{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{2\sqrt{x^2+y^2+z^2}} = x\hat{i}+y\hat{j}+z\hat{k}$$

$$\iint_S \text{curl } \vec{F}\cdot\hat{n}\, ds = \iint_S \hat{k}\cdot(x\hat{i}+y\hat{j}+z\hat{k})\, ds$$

$$= \iint_S z\, ds = \int_0^{2\pi}\int_0^{\pi/2} \cos\theta \sin\theta\, d\theta\, d\phi$$

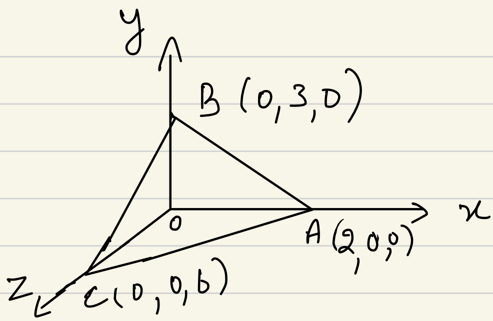
$$= \int_0^{2\pi} d\phi \int_0^{\pi/2} \frac{\sin 2\theta}{2}\, d\theta = -\cancel{2}\pi \times \frac{1}{\cancel{2}} \left.\frac{\cos 2\theta}{2}\right|_0^{\pi/2}$$

$$= \frac{-\pi}{2}\left[\cos\pi - 1\right] = \frac{-\pi}{2}(-2) = \pi$$

Hence $\oint_C \vec{F}\cdot d\vec{x} = \iint_S \text{curl}\vec{F}\cdot\hat{n}\, ds$

2) Use Stokes' theorem to evaluate $\int_C (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz$, where C is the boundary of triangle with vertices $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$.

Sol:



$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F})\cdot \hat{n}\, ds$$

Eq. of triangular surface is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$\Rightarrow \frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1 \Rightarrow 3x + 2y + z = 6$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y & 2x-z & y+z \end{vmatrix} = 2\hat{i} + \hat{k}$$

Since the given surface neither lies nor parallel to any plane we can project onto any plane. Let us project onto xy plane.

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}, \qquad \hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

Let $\phi = 3x + 2y + z - 6$

$$\therefore \hat{n} = \frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}}$$

$$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\, ds = \int_{x=0}^{2}\int_{y=0}^{\frac{6-3x}{2}} (2\hat{i}+\hat{k})\cdot\left(\frac{3}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{1}{\sqrt{14}}\hat{k}\right)\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$= \int_{x=0}^{2} \int_{y=0}^{\frac{6-3x}{2}} \left(\frac{6}{\sqrt{14}} + 0 + \frac{1}{\sqrt{14}} \right) \frac{dy\,dx}{\frac{1}{\sqrt{14}}}$$

$$= \int_{0}^{2} \int_{y=0}^{\frac{6-3x}{2}} \frac{7}{\sqrt{14}} \times \sqrt{14} \; dy\,dx$$

$$= \int_{0}^{2} 7y \Big|_{0}^{\frac{6-3x}{2}} dx = \frac{7}{2} \int_{0}^{2} (6-3x)\,dx$$

$$= \frac{7}{2} \left(6x - \frac{3x^2}{2} \right)_0^2 = \frac{7}{2} \left[(12-6) - (0-0) \right]$$

$$= \frac{7}{2} \times 6 = 21$$

Practice problems:

1) If C is the triangle with vertices at $P(1,0,0)$, $Q(0,2,0)$ and $R(0,0,3)$, evaluate $\int_C (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz$ by using Stokes' theorem

Ans: 7

2) Verify Stokes' theorem for $\vec{F} = -y^3\,\hat{i} + x^3\,\hat{j}$ where S is the circular disc $x^2+y^2 \leq 1$, $z=0$.

Ans: $\frac{3}{2}\pi$

3) Verify Stokes' theorem for the vector field $\vec{F} = (x^2-y^2)\,\hat{i} + 2xy\,\hat{j}$ over the rectangular box bounded by the planes $x=0$, $x=a$; $y=0$, $y=b$; $z=0$, $z=c$ with the face $z=0$ removed

Ans: $2ab^2$

4) Using Stokes' theorem evaluate $\int_c \vec{F} \cdot d\vec{r}$ where $\vec{F} = 2y\,\hat{i} + 3x\,\hat{j} - z^2\,\hat{k}$ over the upper half of sphere $x^2 + y^2 + z^2 = 9$ bounded by its projection on xy plane Ans: 9π

5) Verify Stokes' theorem for the function $\vec{F} = x^2\,\hat{i} + xy\,\hat{j}$ integrated round the square in xy-plane whose sides are along the lines $x = 0$, $y = 0$, $x = a$, $y = a$

Ans: $\frac{a^3}{2}$