Center of mass or center of gravity

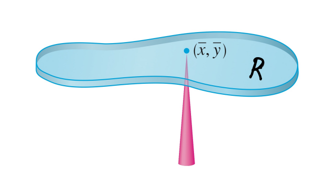
The total weight of the object concentrated in a single point is called object's center of gravity or it is a point on which object is in balance.

The coordinates $(\bar{x}, \bar{y})$ of center of mass of a lamina in xy plane occupying the region R and having density function $\rho(x,y)$ or $f(x,y)$ are

$$\boxed{\bar{x} = \frac{1}{M} \iint_R x \, \rho(x,y) \, dx \, dy \, , \quad \bar{y} = \frac{1}{M} \iint_R y \, f(x,y) \, dx \, dy}$$

where M is the total mass in the region R given by

$$\boxed{M = \iint_R \rho(x,y) \, dx \, dy}$$



For a solid if the density at the point $P(x,y,z)$ is $\rho(x,y,z)$ then the total mass of solid is

$$\boxed{M = \iiint_R \rho(x,y,z) \, dx \, dy \, dz}$$

The center of gravity of a mass in R has the coordinates $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{1}{M}\iiint_R x\, \rho(x,y,z)\,dx\,dy\,dz, \quad \bar{y} = \frac{1}{M}\iiint_R y\, \rho(x,y,z)\,dx\,dy\,dz,$$

$$\bar{z} = \frac{1}{M}\iiint_R z\, \rho(x,y,z)\,dx\,dy\,dz$$

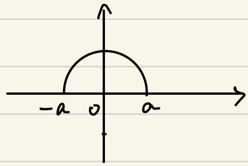
1) Find the center of gravity $(\bar{x}, \bar{y})$ of a mass of density $f(x,y) = 1$ in the region R $x^2 + y^2 \le a^2$, $y \ge 0$.

Sol: Given region is a circle with centre at origin and radius equal to a.

Let $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow x^2 + y^2 = r^2$ and $|J| = r$

Here r varies from 0 to a and θ varies from 0 to π



Mass is given by:

$$M = \iint_R f(x,y)\,dx\,dy = \int_0^{\pi}\int_0^{a} r\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^2\,d\theta = \frac{\pi a^2}{2}$$

The center of gravity $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{1}{M}\iint_R x\, f(x,y)\,dx\,dy = \frac{2}{\pi a^2}\int_0^{\pi}\int_0^{a} r\, r\cos\theta\,dr\,d\theta$$

$$= \frac{2}{\pi a^2}\int_0^{\pi}\cos\theta\,d\theta \times \int_0^{a} r^2\,dr$$

$$\bar{x} = \frac{2}{\pi a^2} \times \sin\theta\Big]_0^{\pi} \times \frac{r^3}{3}\Big]_0^{a} = 0$$

$$\bar{y} = \frac{1}{M} \iint_R y\, f(x,y)\, dx\, dy = \frac{2}{\pi a^2} \int_0^{\pi} \int_0^{a} r\, r \sin\theta\, dr\, d\theta$$

$$= \frac{2}{\pi a^2} \int_0^{\pi} \sin\theta\, d\theta \int_0^{a} r^2\, dr$$

$$= \frac{2}{\pi a^2} \times \left[-\cos\theta\right]_0^{\pi} \times \left\{\frac{r^3}{3}\right\}_0^{a} = \frac{-2}{\pi a^2} (-1-1) \times \frac{a^3}{3}$$

$$\therefore \bar{y} = \frac{4a}{3\pi}$$

$\therefore$ Center of gravity is $\left(0, \frac{4a}{3\pi}\right)$

2) Find the center of gravity in a volume of solid which is in the form of positive octant in the sphere $x^2+y^2+z^2=1$, the density ρ at any point (x,y) is given by $\rho = \mu xyz$ where μ is a constant.

sol: $$M = \iiint_R f(x,y,z)\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \mu xyz\, dz\, dy\, dx$$

$$= \mu \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left\{\frac{z^2}{2}\right\}_0^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \frac{\mu}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} xy\,(1-x^2-y^2)\, dy\, dx$$

$$= \frac{\mu}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} (xy - x^3y - xy^3)\, dy\, dx$$

$$= \frac{\mu}{2} \int_0^1 \left\{\frac{xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right\}_0^{\sqrt{1-x^2}} dx$$

$$= \frac{\mu}{2} \int_0^1 \left\{\frac{x(1-x^2)}{2} - \frac{x^3(1-x^2)}{2} - \frac{x(1-x^2)^2}{4}\right\} dx$$

$$= \frac{\mu}{2}\int_0^1 \left\{ \frac{x}{2} - \frac{x^3}{2} - \frac{x^3}{2} + \frac{x^5}{2} - \frac{x}{4}(1+x^4-2x^2) \right\} dx$$

$$= \frac{\mu}{2}\int_0^1 \left\{ \frac{x}{2} - x^3 + \frac{x^5}{2} - \frac{x}{4} - \frac{x^5}{4} + \frac{\cancel{2}x^3}{\cancel{4}_2} \right\} dx$$

$$= \frac{\mu}{2}\int_0^1 \left\{ \frac{x}{4} - \frac{x^3}{2} + \frac{x^5}{4} \right\} dx$$

$$= \frac{\mu}{2}\left\{ \frac{x^2}{8} - \frac{x^4}{8} + \frac{x^6}{24} \right\}_{x=0}^{1} = \frac{\mu}{2}\left(\frac{1}{24}\right) = \frac{\mu}{48}$$

Center of gravity is:

$$\bar{x} = \frac{1}{M}\iiint_R x\, f(x,y,z)\,dx\,dy\,dz = \frac{1}{M}\iiint_R \mu x^2 y z \,dz\,dy\,dx$$

$$= 48\int_0^1\int_0^{\sqrt{1-x^2}} x^2 y \frac{z^2}{2}\Big|_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= 24\int_0^1\int_0^{\sqrt{1-x^2}} x^2 y\,(1-x^2-y^2)\,dy\,dx$$

$$= 24\int_0^1\int_0^{\sqrt{1-x^2}} (x^2y - x^4y - x^2y^3)\,dy\,dx$$

$$= 24\int_0^1 \left\{ \frac{x^2y^2}{2} - \frac{x^4y^2}{2} - \frac{x^2y^4}{4} \right\}_0^{\sqrt{1-x^2}} dx$$

$$= 24\int_0^1 \left\{ \frac{x^2(1-x^2)}{2} - \frac{x^4(1-x^2)}{2} - \frac{x^2(1-x^2)^2}{4} \right\} dx$$

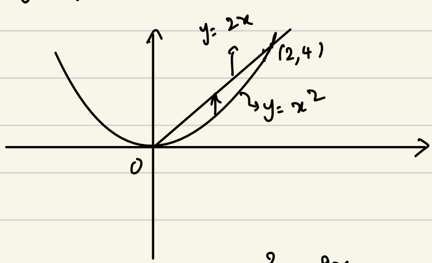
$$= 24\int_0^1 \left\{ \frac{x^3}{12} - \frac{x^5}{10} + \frac{x^7}{28} \right\} dx = \frac{16}{35}$$

$$\bar{y} = \frac{1}{M}\iiint_R y\, f(x,y,z)\,dz\,dy\,dx = \frac{1}{M}\iiint_R x y^2 z\,dz\,dy\,dx = \frac{16}{35}$$

$$\bar{z} = \frac{1}{M}\iiint_R z\, f(x,y,z)\,dz\,dy\,dx = \frac{1}{M}\iiint_R x y z^2\,dz\,dy\,dx = \frac{16}{35}$$

$\therefore$ Center of gravity is $\left(\frac{16}{35}, \frac{16}{35}, \frac{16}{35}\right)$

3) Find the mass and center of mass of the region that is bounded by the line $y = 2x$ and $y = x^2$ if the density function is $\rho(x, y) = x$



$$M = \iint_R \rho(x, y)\, dx\, dy = \int_{x=0}^{2} \int_{y=x^2}^{2x} x \, dy \, dx$$

$$= \int_{x=0}^{2} \left\{ xy \right\}_{x^2}^{2x} dx = \int_{x=0}^{2} (2x^2 - x^3)\, dx$$

$$= \frac{2x^3}{3} - \frac{x^4}{4} \Big|_0^2 = \frac{4}{3}$$

Center of mass:

$$\bar{x} = \frac{1}{M} \int_{x=0}^{2} \int_{y=x^2}^{2x} x \, \rho(x, y)\, dy\, dx$$

$$= \frac{3}{4} \int_{x=0}^{2} \int_{y=x^2}^{2x} x^2 \, dy\, dx = \frac{3}{4} \int_{x=0}^{2} x^2 y \Big|_{x^2}^{2x} dx$$

$$= \frac{3}{4} \int_{x=0}^{2} \left\{ 2x^3 - x^4 \right\} dx$$

$$= \frac{3}{4}\left\{\frac{2x^4}{4} - \frac{x^5}{5}\right\}_0^2 = \frac{3}{4}\left\{\frac{1}{2}(16) - \frac{1}{5}(32)\right\}$$

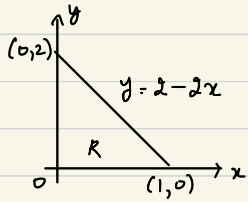
$$= \frac{3}{4}\left\{8 - \frac{32}{5}\right\} = \frac{3}{4} \times \frac{8}{5} = \frac{6}{5}$$

$$\bar{y} = \frac{1}{M}\int_{x=0}^{2}\int_{y=x^2}^{2x} y\,x\,dy\,dx = 2.$$

$$\therefore \text{Center of gravity} = \left(\frac{6}{5}, 2\right)$$

4) Find the mass and center of mass of a triangular lamina with vertices $(0,0)$, $(1,0)$ and $(0,2)$ if the density function $\rho(x,y) = 1+3x+y$.

Sol:



$$M = \iint_R \rho(x,y)\,dA = \int_0^1\int_0^{2-2x}(1+3x+y)\,dy\,dx = \frac{8}{3}$$

$$\bar{x} = \frac{1}{M}\iint_R x\,\rho(x,y)\,dA = \frac{3}{8}\int_0^1\int_0^{2-2x}(x+3x^2+xy)\,dy\,dx$$

$$\therefore \bar{x} = \frac{3}{8}$$

$$\bar{y} = \frac{1}{M}\iint_R y\,\rho(x,y)\,dA = \frac{3}{8}\int_0^1\int_0^{2-2x}(y+3xy+y^2)\,dy\,dx$$

$$\therefore \bar{y} = \frac{11}{16}$$

$$\therefore \text{center of mass is } \left(\frac{3}{8}, \frac{11}{16}\right)$$

Exercise:

1) Evaluate $\iint_R x \sin(xy)\, dA$ over the region $R = \{(x,y) \,/\, 0 \le x \le \pi,$
$1 \le y \le 2\}$ Ans: 0

2) Find the volume V of solid S that is bounded by elliptic paraboloid $2x^2 + y^2 + z = 27$ over the region
$R = \{(x,y) \,/\, 0 \le x \le 3,\ 0 \le y \le 3\}$ Ans: $\int_{y=0}^{3}\int_{x=0}^{3} (27 - 2x^2 - y^2)\, dx\, dy =$
162 cubic units.

3) Evaluate $\iint_R (2 - 3x^2 + y^2)\, dA$ where $R = \{(x,y) \,/\, 3 \le x \le 5,$
$-3 \le y \le 2\}$ Ans: $-\frac{1340}{3}$

4) Evaluate $\iint_D (x+y)\, dA$ where D is bounded by $y = \sqrt{x}$ and $y = x^2$ Ans: $3/10$

Determine the value of given integrals by change of order of integration:

1) $\int_0^\infty \int_0^\infty e^{-xy} \sin px\, dx\, dy$ and hence S.T. $\int_0^\infty \frac{\sin px}{x}\, dx = \frac{\pi}{2}$
Ans:- $\frac{\pi}{2}$

2) $\int_0^1 \int_{e^x}^{e} \frac{1}{\log y}\, dy\, dx$ Ans: $e - 1$

3) $\int_0^1 \int_x^1 \sin y^2\, dy\, dx$ Ans: $\frac{1}{2}\{1 - \cos 1\}$

4) $\int_0^a \int_{\sqrt{ax}}^{a} \frac{y^2\, dx\, dy}{\sqrt{(y^4 - a^2x^2)}}$ Ans: $\frac{\pi a^2}{6}$

5) $\int_0^1 \int_x^1 e^{x/y}\, dy\, dx$ Ans: $\frac{e-1}{2}$

6) $\int_0^2 \int_1^{e^x} dx\, dy$ Ans: e^2-3

(Hint: $\int_1^{e^2} \int_{\log y}^{2} dx\, dy$)

7) $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$ Ans: $\pi a/4$

8) $\int_0^1 \int_{\sqrt{y}}^{2-y} xy\, dx\, dy$ Ans: $7/24$

Evaluate the following integrals by change of variables:

1) $\iint_D xy\, dx\, dy$ where D is the portion of circle with centre 0, radius 1 that lies in first quadrant

Ans: $\frac{1}{8}$

2) Let D be the region in the first quadrant bounded by $xy=1$, $xy=9$ and the lines $y=x$ and $y=4x$

Evaluate $\iint_D \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\, dy$ Ans: $8 + \frac{52}{3} \ln 2$

3) $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$ by changing to polar coordinates

Ans: $\frac{\pi}{4}$

4) Evaluate $\iint_R (3x+4y^2)\, dA$, where R is the region in the upper half plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$. Ans: $\frac{15\pi}{2}$

5) $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\, dy\, dx}{\sqrt{x^2+y^2}}$ by changing to polar coordinates

Ans: $\frac{4}{3}$

6) $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\, dx\, dy$ by changing to polar coordinates

(Ans: $\frac{\pi a^4}{8}$)

7) $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\, dx\, dy$ by changing to polar coordinates

(Ans: $\frac{a^3}{3} \log(1+\sqrt{2})$)

8) $\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} x^2\, dy\, dx$ by changing to polar coordinates

(Ans: $\frac{5}{8} \pi a^4$)

9) $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2 \sqrt{x^2+y^2}\, dy\, dx$

by changing to polar coordinates

(Ans: $\frac{\pi}{20} a^5$)

10) By transforming to polar coordinates, evaluate

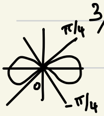
$\iint \frac{x^2y^2}{x^2+y^2}\, dx\, dy$ over region bounded b/w circles $x^2+y^2=a^2$ & $x^2+y^2=b^2$ with $b>a$

(Ans: $\frac{\pi}{16}(b^4-a^4)$)

Problems on Area enclosed by the curves

1) Find area enclosed by one loop of the four leaved rose $r = \cos 2\theta$ (Hint: $D = \{(r,\theta) \mid -\frac{\pi}{4} \le \theta \le \frac{\pi}{4},\ 0 \le r \le \cos 2\theta\}$) Ans: $\frac{\pi}{8}$ sq. units

2) Find the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1+\cos\theta)$ Ans: $a^2\left(1-\frac{\pi}{4}\right)$ sq. units



3) Find the area bounded by lemniscate $r^2 = a^2\cos 2\theta$

(Hint: $\int_{\theta=-\pi/4}^{\pi/4} \int_{r=0}^{a\sqrt{\cos 2\theta}} r\, dr\, d\theta$) Ans: a^2 sq. units

4) Find the area common to the circles $r = a\cos\theta$, $r = a\sin\theta$ Ans: $\frac{a^2}{4}$

5) Find area enclosed by the cardioid $r = a(1+\cos\theta)$ b/w $\theta = 0$ and $\theta = \pi$. Ans: $\frac{3\pi a^2}{4}$

Evaluate the foll. triple integrals:

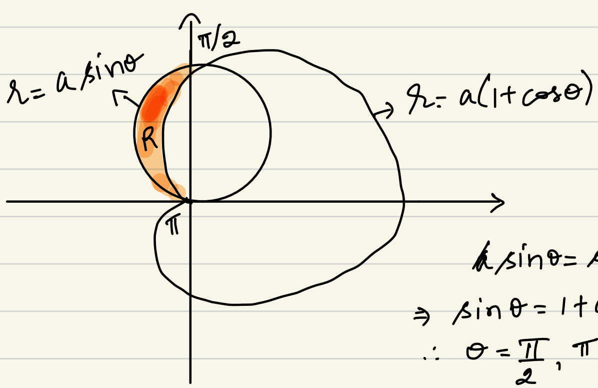
1) $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz\, dy\, dx$ (Ans: $\frac{1}{48}$)

2) $\int_0^{\log 2} \int_0^{x} \int_0^{x+\log y} e^{x+y+z} \, dz\, dy\, dx$ (Ans: $\frac{8}{3}\log 2 - \frac{19}{9}$)

3) $\int_0^{\log 2} \int_0^{x} \int_0^{x+y} e^{x+y+z} \, dz\, dy\, dx$ (Ans: $\frac{5}{8}$)

4) $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z) \, dz\, dy\, dx$ (Ans: $1/8$)

5) $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyz \, dz\, dy\, dx$ (Ans: $\frac{1}{720}$)



$a\sin\theta = a(1+\cos\theta)$

$\Rightarrow \sin\theta = 1 + \cos\theta$

$\therefore \theta = \frac{\pi}{2}, \pi$

6) Find the volume in the first octant bounded by the cylinder $x^2+y^2=2$ and the planes $z=x+y$, $y=x$, $z=0$ and $x=0$. Ans: $\frac{2\sqrt{2}}{3}$

(Hint: $\int_{\theta=\frac{\pi}{4}}^{\pi/2}\int_{r=0}^{\sqrt{2}}\int_{z=0}^{r(\cos\theta+\sin\theta)} r\,dz\,dr\,d\theta$)

7) Find the volume of the solid bounded by the parabolic $y^2+z^2=4x$ and the plane $x=5$.

(Hint: $\int_0^5\int_{-2\sqrt{x}}^{2\sqrt{x}}\int_{-\sqrt{4x-y^2}}^{\sqrt{4x-y^2}} dz\,dy\,dx = 4\int_0^5\int_0^{2\sqrt{x}}\int_0^{\sqrt{4x-y^2}} dz\,dy\,dx = 50\pi$)

Center of gravity or center of mass

1) Find the center of gravity of the area in the first quadrant lying b/w the curves $y^2=x^3$ and $y=x$. Consider $\rho(x,y)=K$. (Ans: $M=\frac{K}{10}$, $\bar{x}=\frac{10}{21}$, $\bar{y}=\frac{5}{12}$)

2) Find the center of gravity of the area bounded by the parabola $y^2=x$ and the line $x+y=2$. Let $\rho(x,y)=K$.

(Ans: $M=\frac{9}{2}K$, $\bar{x}=\frac{8}{5}$, $\bar{y}=-\frac{1}{2}$)

3) Find the center of gravity of a triangular lamina with vertices $(0,0)$, $(3,0)$ and $(0,3)$ if the density function $\rho(x,y)=xy$. (Ans: $M=\frac{27}{8}$, $\bar{x}=\frac{6}{5}$, $\bar{y}=\frac{6}{5}$)

Evaluation of double integrals using Beta and Gamma functions

Gamma function : It is a definite integral defined as

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx \quad \text{where } n \text{ is +ve real no} \quad (1)$$

The advantage of this function is we evaluate the integral without actually integrating.

ex: Evaluate $\int_0^\infty \underset{v}{e^{-x}}\ \underset{u}{x}\ dx$

Normally, we use integration by parts to evaluate this

$$I = -x e^{-x}\Big|_0^\infty - \int_0^\infty 1\cdot(-e^{-x})\,dx$$

$$= 0 + \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty$$

$$= -(e^{-\infty} - e^{0}) = -(0-1) = 1$$

$\int (uv)\,dx = uv_1 - u'v_2 + u''v_3 - \ldots$ where dashes represent differentiation & suffixes represent integration

Instead of actually integrating the above function, gamma function helps to evaluate the above integral without integrating.

When you obtain integrals of above type use the following properties of gamma function. (Proof not needed)

$$\Gamma\left(1+\tfrac{1}{2}\right)=\tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right)$$
$$=\frac{\sqrt{\pi}}{2}$$

properties of gamma functions

* For all non-negative real number n, we have

$$\boxed{\Gamma(n+1)=n\,\Gamma(n)}$$

Proof: WKT. $\Gamma(n)=\int_0^\infty e^{-x}\,x^{n-1}\,dx \rightarrow (1)$

Let $x=t^2$, $\Gamma(n)=2\int_0^\infty e^{-t^2}\,t^{2n-2}\,dt$

Changing the variables of integration from t to x

$$\Gamma(n)=2\int_0^\infty x^{2n-1}\,e^{-x^2}\,dx$$

From (1), $\Gamma(n+1)=\int_0^\infty e^{-x}\,x^n\,dx$

$$=-x^n e^{-x}\Big|_0^\infty+n\int_0^\infty x^{n-1}\,e^{-x}\,dx$$

(using by parts)

$\therefore \Gamma(n+1)=0+n\,\Gamma(n)$

$$\Rightarrow \Gamma(n+1)=n\,\Gamma(n) \rightarrow (2)$$

Note: (1) $\Gamma(1)=\int_0^\infty e^{-x}\,dx=-e^{-x}\Big|_0^\infty=-(e^{-\infty}-1)=1$

$\Gamma(2)=\Gamma(1+1)=1\,\Gamma(1)=1\cdot 1=1$ (using $\Gamma(n+1)=n\Gamma(1)$)

$\Gamma(3)=\Gamma(2+1)=2\,\Gamma(2)=2\cdot 1=2=2!$

$\Gamma(4)=\Gamma(3+1)=3\,\Gamma(3)=3\cdot 2\cdot 1=6=3!$

Hence

$$\boxed{\Gamma(n+1)=n!} \quad \text{or } \Gamma(n)=(n-1)!$$

(2) $\Gamma(1/2)=\sqrt{\pi}$ (Proof not needed)

(Remember this result)

Beta function : It is defined as

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$ where m and n are +ve real numbers

Note: $B(m,n) = B(n,m)$

i.e. $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^1 (1-x)^{m-1} x^{n-1}\,dx$

other forms of $B(m,n)$

1) $$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta.$$

Relation between Beta and gamma function

$$\boxed{B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}}$$

Some sample problems on beta and gamma functions

1) Evaluate $\Gamma(3/2)$

$$\Gamma(3/2) = \Gamma(1/2+1) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$ (using $\Gamma(n+1) = n\Gamma(n)$)

2) Evaluate $\Gamma(5/2)$

$$\Gamma(5/2) = \Gamma(3/2+1) = 3/2\;\Gamma(3/2) = \frac{3}{2} \times \frac{\sqrt{\pi}}{2}$$

$$\therefore \Gamma(5/2) = \frac{3\sqrt{\pi}}{4}$$

∴ Hence $\Gamma(n+1/2) = \dfrac{1\cdot 3\cdot 5 \ldots\ldots (2n-1)\sqrt{\pi}}{2^n}$

$|||^{ly}$ $\Gamma(7/2) = \Gamma(3+1/2)$

Comparing $\Gamma(3+1/2)$ with $\Gamma(n+1/2)$

$\Rightarrow n = 3$

$\therefore \Gamma(3+1/2) = \Gamma(7) = \dfrac{1\cdot 3\cdot 5\sqrt{\pi}}{2^3} = \dfrac{15\sqrt{\pi}}{8}$

Also $\Gamma(5/2)$ can be obtained in similar manner

$\Gamma(5/2) = \Gamma(2+1/2)$

Here $n = 2$

$\therefore \Gamma(2+1/2) = \dfrac{1\cdot 3}{2^2}\sqrt{\pi} = \dfrac{3}{4}\sqrt{\pi}$

3) Evaluate $\Gamma(7)$

$\Gamma(7) = (7-1)! = 6! = 720$ (using $\Gamma(n) = (n-1)!$)

This means

WKT $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

$\Gamma(7) = \int_0^\infty e^{-x} x^{7-1}\, dx = \int_0^\infty e^{-x} x^6\, dx = 720$

i.e. We have evaluated the above integral without integrating.

4) Evaluate $\int_0^1 x^{1/2}(1-x)^{-1/2}\,dx$

This cannot be integrated by usual integration so we use beta and gamma functions.

WKT. $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$.

$$\int_0^1 x^{1/2}(1-x)^{-1/2}\,dx = \int_0^1 x^{3/2-1}(1-x)^{1/2-1}\,dx$$

$$\int_0^1 x^{1/2}(1-x)^{-1/2}\,dx = \beta\left(\tfrac{3}{2},\tfrac{1}{2}\right)$$

$m-1 = 1/2$

$\therefore m = 3/2$

$n-1 = -1/2$

$\therefore n = 1-1/2 = 1/2$

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\therefore \beta\left(\tfrac{3}{2},\tfrac{1}{2}\right) = \frac{\Gamma(3/2)\,\Gamma(1/2)}{\Gamma(3/2+1/2)} = \frac{\frac{\sqrt{\pi}}{2}\cdot\sqrt{\pi}}{\Gamma(2)} = \frac{\pi/2}{1} = \frac{\pi}{2}$$

5) Evaluate $\int_0^{\pi/2} \sin^5\theta \cos^4\theta \, d\theta$

It's hectic to solve by usual integration, so we use beta and gamma function.

WKT $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

$$\Rightarrow \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2}\beta(m,n)$$

Given $\int_0^{\pi/2} \sin^5\theta \cos^4\theta \, d\theta = \frac{1}{2}\beta\left(3,\frac{5}{2}\right)$

$2m-1 = 5$

$\therefore m = 3$

$2n-1 = 4$

$\therefore n = 5/2$

$$= \frac{1}{2}\,\frac{\Gamma(3)\,\Gamma(5/2)}{\Gamma(3+5/2)}$$

using $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$$= c\int_0^1 x^{1/3}\,dx \int_0^1 t^{1/2-1}(1-t)^{3/2-1}\,dt$$

$$= c\int_0^1 x^{1/3}\,dx \times B\left(1/2, 3/2\right) = \int_0^1 c\,x^{1/3}\,dx \times \frac{\Gamma(1/2)\Gamma(3/2)}{\Gamma(1/2+3/2)}$$

$$= \int_0^1 (2-x)\,x^{1/3}\,dx \times \frac{\sqrt{\pi}\times\frac{\sqrt{\pi}}{4}}{\Gamma(2)} \quad (\because\ 2-x=c)$$

$$= \int_0^1 (2-x)\,x^{1/3}\,dx \times \frac{\pi}{2} = \frac{\pi}{2}\int_0^1 \left(2x^{1/3} - x^{4/3}\right)dx$$

$$= \frac{\pi}{2}\left\{ \frac{2x^{1/3+1}}{1/3+1} - \frac{x^{\frac{4}{3}+1}}{\frac{4}{3}+1} \right\}_0^1 = \frac{\pi}{2}\left\{ \frac{2x^{4/3}}{4/3} - \frac{x^{7/3}}{7/3} \right\}_0^1$$

$$= \frac{\pi}{2}\left\{ 2\times\frac{3}{4}x^{4/3} - \frac{3}{7}x^{7/3} \right\}_0^1 = \frac{\pi}{2}\left\{ \frac{6}{4} - \frac{3}{7} \right\}$$

$$= \frac{\pi}{2}\left\{ \frac{42-12}{28} \right\} = \frac{\pi}{2}\left\{ \frac{\cancel{30}^{\,15}}{\cancel{28}_{\,14}} \right\} = \frac{15\pi}{28}$$