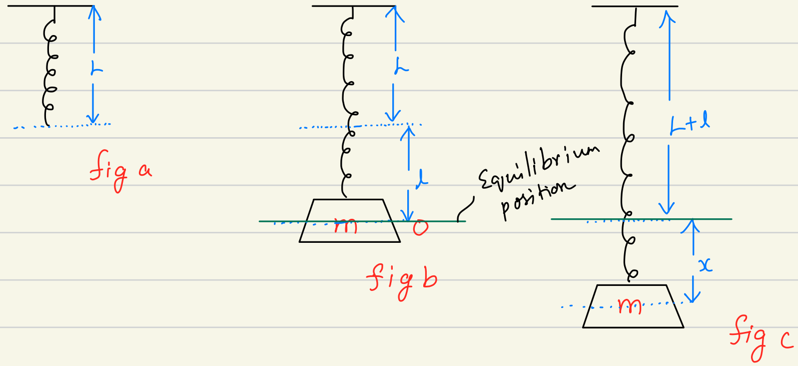
Applications of second order ODE

Mass and spring system

The differential equation of the vibrations of a mass on a spring.

The Basic problem:



- fig a shows natural length L of a spring, i.e, unstretch length of a spring.
- Now we attach a mass to the lower end of the spring and allow it to come to rest in an equilibrium position as given in fig b.
- The system is then set in motion. Let $x(t)$ denote the displacement of the mass from 0 along the line as shown in fig c.
- x is considered to be +ve, zero or −ve according to whether the mass is below, at or above its equilibrium position.

Problem is to determine the resulting motion of the mass on the spring.

We set up differential eqn to the problem, in order to do so we consider two laws of physics

1) Hooke's law
2) Newton's second law of motion

Hooke's law states that the magnitude of the force needed to produce certain elongation is directly proportional to the amount of its elongation

i.e $$\boxed{|F| \propto l \Rightarrow |F| = kl,}$$

where F is the force, l is the amount of elongation and k is a constant of proportionality called spring constant.

Newton's 2nd law states that net forces acting on an object is equal to the mass times acceleration of the object.

i.e $$\boxed{\sum F = m\frac{d^2x}{dt^2}.}$$

Here, we assume that forces tending to pull the mass downward are +ve, while those tending to pull it upward are negative.

In fig b The mass is in equibrium, Thus net forces on the mass is equal to zero.

Forces acting are i) Force of gravity (mg, g is acceleration due to gravity)

ii) Restoring force of the spring, equal to kl from Hooke's law

i.e $mg - kl = 0$ (mg acts downward (+ve) and kl acts upward (-ve))

$$\Rightarrow \boxed{mg = kl} \quad \text{—— } (1)$$

In fig c, the mass is in motion, x is the displacement of the mass at time $t>0$

1) F_1, the force of gravity

$$\boxed{F_1 = mg} \quad (2) \quad (\text{it acts downward } (+ve))$$

2) F_2, the restoring force of the spring

$$F_2 = -k(x+l)$$

$$\Rightarrow \boxed{F_2 = -kx - mg} \quad (3) \quad (\because kl = mg \text{ from } (1))$$

3) F_3, the resisting force of the medium, called the damping force

$$\boxed{F_3 = -a\frac{dx}{dt}} \quad (4) \quad (-ve \text{ because it is against the motion of mass})$$

Here $a>0$, is called damping constant.

4) F_4, any external forces that act upon the mass.

Let $f(t)$ be the resultant of all external forces.

$$\boxed{F_4 = f(t)} \quad (5)$$

From Newton's law

$$F_1 + F_2 + F_3 + F_4 = m\frac{d^2x}{dt^2}$$

$$\Rightarrow \cancel{mg} - kx - \cancel{mg} - a\frac{dx}{dt} + f(t) = m\frac{d^2x}{dt^2}$$

$$\Rightarrow m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = f(t) \quad (6)$$

This is a second order non-homogenous LDE with constant coefficients.

This we take as the DE for the motion of the mass on the spring.

Note: If $a=0$ the motion is called undamped. Otherwise it is called damped.

If $f(t)=0\ \forall t$ the motion is called free, otherwise it is called forced.

Ex1: An 8-lb weight is placed upon the lower end of a coil spring suspended from the ceiling. The weight comes to rest in its equilibrium position, thereby stretching the spring 6 in. The weight is then pulled down 3in below its equilibrium position and released at $t=0$ with an initial velocity of 1ft/sec, directed downward. Neglecting the resistance of the medium and assuming that no external forces are present, determine the displacement of the weight and hence determine amplitude, period and frequency of the resulting motion.

Soln: Weight $W = 8$ lb

8 lb weight streches the spring by 6 in $= \frac{1}{2}$ ft $(= l)$

If $x(t)$ is the displacement of the mass at time t,

$x(0) = 3$ in $= \frac{1}{4}$ ft,

$\frac{dx}{dt}(0) = 1$ ft/sec, (+ve because directed downward)

accelaration due to gravity, $g = 32$ ft/sec^2

$$W = mg \Rightarrow m = \frac{W}{g} = \frac{8}{32} = \frac{1}{4}$$

Given: $a=0$, $f(t)=0$ (external forces)

Spring constant $k = \frac{mg}{l} = \frac{8}{\frac{1}{2}} = 16$ lb/ft $\quad (mg = kl)$

This is free and undamped motion, thus the DE is

$$m\frac{d^2x}{dt^2} + kx = 0$$

$$\text{or} \quad \frac{1}{4}\frac{d^2x}{dt^2} + 16x = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + 64x = 0 \qquad (\lambda^2 = 64)$$

AE : $m^2 + 64 = 0$

Roots are $m = \pm 8i$

Independent solns are $\cos 8t$, $\sin 8t$

G.S is $\boxed{x = c_1 \cos 8t + c_2 \sin 8t}$ —— (1)

IC's are $x(0) = \frac{1}{4}$ ft, $\frac{dx}{dt}(0) = 1$ ft/sec

Diff (1) wrt t,

$$\frac{dx}{dt} = -c_1 8 \sin 8t + 8 c_2 \cos 8t \quad \text{—— (2)}$$

At $t=0$,

(1) $\Rightarrow$ $\boxed{\frac{1}{4} = c_1}$

(2) $\Rightarrow$ $1 = 8c_2$ $\Rightarrow$ $\boxed{c_2 = \frac{1}{8}}$

Sub for c_1 and c_2 in (1),

$$\boxed{x(t) = \frac{1}{4} \cos 8t + \frac{1}{8} \sin 8t}$$

Amplitude is $\sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2} = \frac{\sqrt{5}}{8}$ ft.

frequency is $\frac{8}{2\pi} = \frac{4}{\pi}$ osc/sec.

Time period is $\frac{\pi}{4}$ sec.

Free, damped motion

Let $a>0$ be the damping constanst and $f(t)=0\ \forall t$. The basic DE reduces to

$$m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + Kx = 0$$

$$r = \frac{-a \pm \sqrt{a^2-4mK}}{2m}$$

Depending on nature of these roots, 3 distinct cases occur.

Case 1: $a^2 - 4mK > 0$ (over damping)

Case 2: $a^2 - 4mK = 0$ (critical damping)

Case 3: $a^2 - 4mK < 0$ (under damping)

Ex2: A 32 lb - weight is attached to the lower end of a coil spring suspended from the ceiling. The weight comes to rest in its equilibrium position, thereby stretching the spring 2 ft. The weight is then pulled down 6 in below its equilibrium position and released at $t=0$. No external forces are present; but the resistance of the medium in pounds is equal to $4\frac{dx}{dt}$, where $\frac{dx}{dt}$ is the instantaneous velocity in feet per sec.

Detemine the resulting motion of the weight on the spring. Interpret the motion.

Soln: Given Weight $W = 32\,lb$

It stretches the spring by 2 ft. By Hooke's law

$W = kl$ or $32 = k\,2$ $\Rightarrow$ $\boxed{k = 16\ lb/ft}$

mass $= \frac{W}{g} = \frac{32}{32} = 1$ $\quad | \ g = 32\ ft/sec^2$

$\Rightarrow$ $\boxed{m = 1}$

Damping constant $\boxed{a = 4}$

External forces $\boxed{F(t) = 0}$

The DE of the problem:

$$\boxed{\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 16x = 0}$$

where $x(t)$ is a displacement of the mass at time t.

IC's $\quad x(0) = \frac{1}{2}$ $\quad$ Since it is pulled 6 in $= \frac{1}{2}$ ft below its equilibrium position at

$x'(0) = 0$ $\quad$ $t=0$ and released it with no initial velocity.

Here $a^2 - 4mk = 16 - (4 \times 1 \times 16) = 16 - 64 = -48 < 0$

Since $a^2 - 4mk < 0$

So the motion is damped oscillatory (under damping)

AE of the DE is

$$r^2 + 4r + 16 = 0$$

Roots are $\frac{-4 \pm \sqrt{16-64}}{2} = -2 \pm 2\sqrt{3}$

indep. solns are $e^{-2t}\cos 2\sqrt{3}\,t$, $e^{-2t}\sin 2\sqrt{3}\,t$

G.S is

$$\boxed{x = e^{-2t}\left(c_1 \cos(2\sqrt{3}\,t) + c_2 \sin(2\sqrt{3}\,t)\right)}$$

Given $x(0) = \frac{1}{2}$

$$\therefore \quad \frac{1}{2} = c_1$$

$$x'(t) = e^{-2t}\left(-2\sqrt{3}\,c_1 \sin(2\sqrt{3}\,t) + 2\sqrt{3}\,c_2 \cos(2\sqrt{3}\,t)\right) + (-2)\,e^{-2t}\left(c_1 \cos(2\sqrt{3}\,t) + c_2 \sin(2\sqrt{3}\,t)\right)$$

But $x'(0) = 0$

$$\therefore \quad 0 = 2\sqrt{3}\,c_2 - 2c_1$$

$$\Rightarrow \quad c_2 = \frac{1}{2\sqrt{3}}$$

Sub for c_1 and c_2 in the G.S,

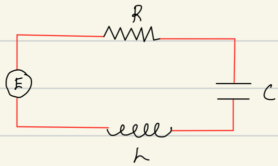
$$x(t) = e^{-2t}\left(\frac{1}{2}\cos(2\sqrt{3}\,t) + \frac{1}{2\sqrt{3}}\sin(2\sqrt{3}\,t)\right)$$

Exercise:

1) A 16 lb weight is attached to the lower end of a coil spring suspended from the ceiling the spring constant of the spring being 10 lb/ft. The weight comes to rest in its equilibrium position. Beginning at $t=0$ an external force given by $F(t) = 5\cos 2t$ is applied to the system. Determine the resulting motion if the damping force in pounds is equal to $2\frac{dx}{dt}$, where $\frac{dx}{dt}$ is the instantaneous velocity in ft. per second.

Ans: $x = e^{-2t}\left(-\frac{1}{2}\cos 4t - \frac{3}{8}\sin 4t\right) + \frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t$

Electric circuit problems



Let E be emf, let i be the current in the circuit

Voltage drop across a resistor is iR

Voltage drop across an inductor is $L\frac{di}{dt}$

Voltage drop across a capacitor is $\frac{1}{C}q = \frac{1}{C}\int i\,dt$

Kirchhoff's voltage law (KVL) :

The sum of the voltage drops across resistors, inductors and capacitors is equal to the total emf in a closed circuit.

That is $$L\frac{di}{dt} + iR + \frac{1}{C}q = E(t) \quad \text{—— (1)}$$

WKT $\frac{dq}{dt} = i$

Thus, (1) $\Rightarrow$ $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

Also
$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E'(t)$$

Ex: 1) Find the charge on the capacitor in a LCR series circuit when $L = \frac{5}{3}$ H, $R = 10\,\Omega$, $C = \left(\frac{1}{30}\right)$ F, $E(t) = 300$ V, $q(0) = 0$, $\frac{dq}{dt}(0) = 0$

Sol: $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E(t)$$

$$\frac{5}{3}\frac{d^2q}{dt^2} + 10\frac{dq}{dt} + 30q = 300$$

$$\Rightarrow 5\frac{d^2q}{dt^2} + 30\frac{dq}{dt} + 90q = 900$$

A.E: $5m^2 + 30m + 90 = 0$

$\therefore\ m = -3 \pm 3i$

$$q_c = e^{-3t}(c_1 \cos 3t + c_2 \sin 3t)$$

$$q_p = \frac{1}{5D^2+30D+90}\, 900 \cdot e^{0t} = \frac{1}{0+0+90} \times 900 = 10$$

G.S. is

$$q(t) = e^{-3t}(c_1 \cos 3t + c_2 \sin 3t) + 10 \rightarrow (1)$$

Using $q(0) = 0$, $(1) \Rightarrow 0 = c_1 + 10 \Rightarrow c_1 = -10$

$\frac{dq}{dt} = 0$, $(1) \Rightarrow c_2 = -10$

$\therefore$ Charge on capacitor is:

$$q(t) = 10 - 10e^{-3t}[\cos 3t + \sin 3t]$$