Divergence of a vector

If $\vec{A} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ the divergence of $\vec{A}$ is defined as

$$\text{div}\,\vec{A} = \nabla\cdot\vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(a_x\hat{i} + a_y\hat{j} + a_z\hat{k}\right)$$

$$\text{div}\,\vec{A} = \nabla\cdot\vec{A} = \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right)$$

$\vec{A}$ is a vector function, but div $\vec{A}$ is a scalar function.
If div $\vec{A} = 0$, vector $\vec{A}$ is called solenoidal vector.

Significance of divergence:

1) The divergence of a vector field $F(x, y, z)$, namely div $F(x, y, z)$ measures tendancy of vector field quantity $F(x, y, z)$ to diverge from the point (x, y, z)

2) The vector field $F(x, y, z)$ can be a velocity vector field, gravitational field, electric field, magnetic field and so on. The quantity div $F(x, y, z)$ measures the flux of these fields passing through a given region of space.

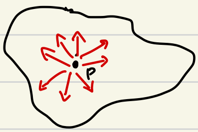
flux – process of flowing or passing

3) A solenoidal vector field is one where divergence is zero, meaning that flow is not diverging from any point but rather is closed or circulating around some region.

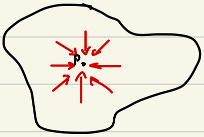
ex: magnetic field (as magnetic lines circulate around closed loops, rather than diverging outwards from any point)

4) The divergence of a vector field can also be interpreted as measure of rate of change of density of fluid at a point. In other words, div F is a measure of fluid's compressibility. If $\nabla \cdot F = 0$, the fluid is said to be incompressible (density of flowing fluid is constant during process of flow)

5)



a) div $F(P) > 0$; P is a source



b) div $F(P) < 0$; P is a sink

If div $F(P) > 0$, then P is said to be a source for F, since there is a net outward flow of fluid near P.
If div $F(P) < 0$, then P is said to be a sink for F, since there is a net inward flow of fluid near P.
If div $F(P) = 0$, there are no sources or sinks near P.

Curl of a vector field

If $\vec{A} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$, the curl of $\vec{A}$ is defined by

$$\text{curl } \vec{A} = \nabla \times \vec{A} = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \times \left(a_x \hat{i} + a_y \hat{j} + a_z \hat{k} \right)$$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix}$$

$\vec{A}$ is a vector function and curl $\vec{A}$ is also a vector function

Physical significance

1) Curl is a measure of how much a vector field circulates or rotates about a given point.
If curl $\vec{A} = \nabla \times \vec{A} = 0$, then $\vec{A}$ is said to be an irrotational vector.

2) Conservative vector field is a vector field that can be expressed as gradient of scalar potential function
In other words if vector field F is conservative then $\exists$ a scalar function ϕ such that $F = \nabla \phi$
Here $\nabla \rightarrow$ gradient operator, $\phi \rightarrow$ scalar potential function (scalar valued function that is used to describe the behavior of a conservative vector field).

* The curl of conservative vector fields is always zero. This is because curl of gradient is 0.

ex: Gravitational force b/w 2 masses is conservative vector field as masses move under influence of force. (work done in moving any particle / object from one position to other doesn't depend on path taken but only on initial and final positions of object).

$\cdot \star \cdot$ curl (grad ϕ) = 0 i.e. $\nabla \times (\nabla \phi) = 0$

$$\nabla \phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$\text{curl}(\nabla \phi) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial y \partial z}\right) - \hat{j}\left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial x \partial z}\right) + \hat{k}\left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y}\right) = 0$$

Component Test for conservative field

If $F(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$ is a conservative vector field in region R and P, Q are continuous and have first partial derivatives in R. Then

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

$P = \frac{\partial \phi}{\partial x}$, $Q = \frac{\partial \phi}{\partial y}$

$\frac{\partial P}{\partial y} = \frac{\partial^2 \phi}{\partial x \partial y}$, $\frac{\partial Q}{\partial x} = \frac{\partial^2 \phi}{\partial y \partial x}$

In 3D-space : If $F(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$ is conservative vector field then

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}}$$

Scalar potential of vector $\vec{A}$

If $\vec{A} = \nabla\phi$, then scalar function ϕ is said to be scalar potential function of $\vec{A}$

Laplacian of scalar field

Let $\phi = \phi(x, y, z)$ be a given scalar field. Then $\nabla\phi$ is a vector field given by,

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$\therefore$ Divergence of $\nabla\phi$ is

$$\text{div}(\nabla\phi) = \nabla\cdot(\nabla\phi) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\nabla\phi$$

$$\text{div}(\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

The RHS is called Laplacian of ϕ and denoted by $\nabla^2\phi$.

$$\therefore \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \rightarrow (1)$$

(1) can be written as

$$\nabla^2\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

∇^2 is the differential operator given by

$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$ and is called

Laplacian operator

Problems:

1) If $\vec{A} = 2xy^2\,\hat{i} + 3x^2z\,\hat{j} - 4xyz\,\hat{k}$ find

a) $\text{div}\,\vec{A}$ b) $\text{curl}\,\vec{A}$ at $(1,1,-2)$

Sol: a) $\text{div}\,\vec{A} = \nabla\cdot\vec{A} = \frac{\partial}{\partial x}(2xy^2) + \frac{\partial}{\partial y}(3x^2z) + \frac{\partial}{\partial z}(-4xyz)$

$$\therefore \nabla\cdot\vec{A} = 2y^2 + 0 - 4xy$$

At $(1,1,-2) \Rightarrow \text{div}\,\vec{A} = 2-4 = -2$

b) $\text{curl}\,\vec{A} = \nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2 & 3x^2z & -4xyz \end{vmatrix}$

$$= \hat{i}[-4xz - 3x^2] - \hat{j}[-4yz - 0] + \hat{k}[6xz - 4xy]$$

$$\text{curl}\,\vec{A} = (-4xz - 3x^2)\,\hat{i} + 4yz\,\hat{j} + (6xz - 4xy)\,\hat{k}$$

At $(1,1,-2) \Rightarrow \text{curl}\,\vec{A} = (8-3)\,\hat{i} - 8\,\hat{j} + (-12-4)\,\hat{k}$

$$\therefore \text{curl}\,\vec{A} = 5\,\hat{i} - 8\,\hat{j} - 16\,\hat{k}$$

2. A fluid motion is given by $\overrightarrow{A} = (y+z)\boldsymbol{i} + (z+x)\boldsymbol{j} + (x+y)\boldsymbol{k}$**.** Is this motion irrotational? If so, find its scalar potential.

Solution: we have $\vec{A} = (y+z)\boldsymbol{i} + (z+x)\boldsymbol{j} + (x+y)\boldsymbol{k}$

and $\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}$

Now

$$curl\,\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y+z & x+z & x+y \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(z+x)\right]\boldsymbol{i} - \left[\frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial z}(z+y)\right]\boldsymbol{j} + \boldsymbol{k}\left[\frac{\partial}{\partial x}(x+z) + \frac{\partial}{\partial y}(z+y)\right]$$

$$\Rightarrow [(1-1)\boldsymbol{i} + (1-1)\boldsymbol{j} + (1-1)\boldsymbol{k}] = 0$$

since $curl\,\vec{A} = 0$, $\vec{A}$ is irrotational.

Let ϕ be the scalar potential of $\vec{A}$

$\therefore \overrightarrow{\mathrm{A}} = \nabla\phi$

$$(y+z)\boldsymbol{i} + (z+x)\boldsymbol{j} + (x+y)\boldsymbol{k} = \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k}$$

Comparing the coefficients of i, j, k on both sides, we get

$\frac{\partial\phi}{\partial x} = y + z$ ----- (1),

$\frac{\partial\phi}{\partial y} = z + x$---- (2),

$\frac{\partial\phi}{\partial z} = x + y$ --- - (3)

Integrating with respect to x, y, z we get

$\phi(x,y,z) = yx + xz + f_1(y,z)$

$\phi(x,y,z) = xy + yz + f_2(x,z)$

$\phi(x,y,z) = xz + yz + f_3(x,y)$

By inspection,

$$\phi(x,y,z) = xy + yz + zx + c$$

3. Find a, b, c such that $\vec{A} = (x + y + az)\boldsymbol{i} + (bx + 2y - z)\boldsymbol{j} + (x + cy + 2z)$ **is** irrotational and hence find scalar potential ϕ.

Solution: The vector field is irrotational if $curl\,\vec{A} = 0$

$$curl\,\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y + az & bx + 2y - z & x + cy + 2z \end{vmatrix} = 0$$

$$= [c + 1]\boldsymbol{i} - [1 - a]\boldsymbol{j} + \boldsymbol{k}[b - 1] = 0$$

$$\therefore c + 1 = 0 \Rightarrow c = -1$$

$$1 - a = 0 \Rightarrow a = 1$$

$$b - 1 = 0 \Rightarrow c = 1$$

$$\therefore \vec{A} = (x + y + z)\boldsymbol{i} + (x + 2y - z)\boldsymbol{j} + (x - y + 2z)\boldsymbol{k}$$

Let ϕ be the scalar potential of $\vec{A}$

$\therefore \overrightarrow{A} = \nabla\phi$

$$(x + y + z)\boldsymbol{i} + (x + 2y - z)\boldsymbol{j} + (x - y + 2z)\boldsymbol{k} = \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k}$$

Comparing the coefficients of i, j, k on both sides, we get

$\frac{\partial\phi}{\partial x} = x + y + z$ ----- (1),

$\frac{\partial\phi}{\partial y} = x + 2y - z$---- (2),

$\frac{\partial\phi}{\partial x} = x - y + 2z$ --- - (3)

Integrating with respect to x, y, z we get

$$\phi(x, y, z) = \frac{x^2}{2} + xy + xz + f_1(y, z)$$

$$\phi(x, y, z) = xy + y^2 - yz + f_2(x, z)$$

$$\phi(x, y, z) = xz - yz + z^2 + f_3(x, y)$$

By inspection,

$$\phi(x, y, z) = \frac{x^2}{2} + xy + xz + y^2 - yz + z^2 + c$$

4. Show that $\vec{A} = y^2z^3\boldsymbol{i} + 2xyz^3\boldsymbol{j} + 3xy^2z^2\boldsymbol{k}$ is conservative.

Solution:

$$curl\,\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$\left[\frac{\partial}{\partial y}(3xy^2z^2) - \frac{\partial}{\partial z}(2xyz^3)\right]\boldsymbol{i} - \left[\frac{\partial}{\partial x}(3xy^2z^2) - \frac{\partial}{\partial z}(y^2z^3)\right]\boldsymbol{j} + \boldsymbol{k}\left[\frac{\partial}{\partial x}(2xyz^3) - \frac{\partial}{\partial y}(y^2z^3)\right]$$

$$\Rightarrow [(6xyz^2 - 6xyz^2)\boldsymbol{i} + (3y^2z^2 - 3y^2z^2)\boldsymbol{j} + (2yz^3 - 2yz^3)\boldsymbol{k}] = 0.$$

5. Show that the vector function $\vec{f} = 2xyzi + (xy - y^2z)j + (x^2 - zx)k$ is solenoidal.

Solution: Consider $div\,\vec{f} = \frac{\partial}{\partial x}(2xyz) + \frac{\partial}{\partial y}(xy - y^2z) + \frac{\partial}{\partial z}(x^2 - zx)$

$$div\,\vec{f} = 2yz + x - 2yz - x = 0$$

$\therefore \vec{f}$ is solenoidal vector function.

6) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$, S.T.

a) $div\,\vec{r} = 3$ b) $curl\,(\vec{r}) = 0$ c) $\nabla\left(\frac{1}{r}\right) = \frac{-r}{r^3}$

d) $\nabla(r^n) = n r^{n-2}\,\vec{r}$

Sol: a) $\nabla \cdot \vec{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$

$$\therefore \nabla \cdot \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

b) $$\nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0)$$

$$= 0$$

c)

$$r = |\vec{r}| = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{1/2}$$

$$\frac{1}{r} = (x^2+y^2+z^2)^{-1/2}$$

$$\nabla\left(\frac{1}{r}\right) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2+z^2)^{-1/2}$$

$$\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} = -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}\,\cancel{2}x = r^{-3}x$$

III^{ly} $\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2} = r^{-3}y$, $\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2} = r^{-3}z$

$$\therefore \nabla\left(\frac{1}{r}\right) = \hat{i}\,(r^{-3}x) + \hat{j}\,(r^{-3}y) + \hat{k}\,(r^{-3}z)$$

$$\therefore \nabla\left(\frac{1}{r}\right) = \frac{-1}{r^3}\left[x\hat{i} + y\hat{j} + z\hat{k}\right] = -\frac{\vec{r}}{r^3}$$

d)

$$\nabla r^n = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2+z^2)^{n/2}$$

$$\frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2} = \frac{n}{\cancel{2}}(x^2+y^2+z^2)^{n/2-1}(\cancel{2}x) = n(x^2+y^2+z^2)^{\frac{n-2}{2}}x$$

$$= n\left[(x^2+y^2+z^2)^{1/2}\right]^{n-2}x = n r^{n-2}x$$

III^{ly} $\frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2} = n r^{n-2}y$, $\frac{\partial}{\partial z}(x^2+y^2+z^2)^{n/2} = n r^{n-2}z$

$$\therefore \nabla r^n = n r^{n-2}x\,\hat{i} + n r^{n-2}y\,\hat{j} + n r^{n-2}z\,\hat{k}$$

$$\therefore \nabla r^n = n r^{n-2}(x\hat{i} + y\hat{j} + z\hat{k}) = n r^{n-2}\vec{r}$$

6) S.T. the vector field $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^3}$ is solenoidal and irrotational. Find its scalar potential ϕ.

Sol: $$\vec{F} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$$

$$\nabla \cdot \vec{F} = \nabla \cdot \left(\frac{\vec{r}}{r^3}\right) = \nabla \cdot (\underbrace{r^{-3}}_{\phi} \underbrace{\vec{r}}_{\vec{A}})$$

$\nabla \cdot (\phi \vec{A}) = \phi(\nabla \cdot \vec{A}) + (\nabla\phi) \cdot \vec{A}$

$\nabla \cdot \vec{r} = 3$

$$= r^{-3}(\nabla \cdot \vec{r}) + (\nabla r^{-3}) \cdot \vec{r}$$

$$= 3r^{-3} + (-3r^{-5}\vec{r}) \cdot \vec{r}$$

$$= 3r^{-3} - 3r^{-5}(\vec{r} \cdot \vec{r})$$

$\nabla r^n = n r^{n-2} \vec{r}$

$\vec{r} \cdot \vec{r} = r^2$

$$\therefore \nabla \cdot \vec{F} = 3r^{-3} - 3r^{-3} = 0$$

$\Rightarrow \vec{F}$ is solenoidal.

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \nabla \times \left(\frac{\vec{r}}{r^3}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \dfrac{x}{(x^2+y^2+z^2)^{3/2}} & \dfrac{y}{(x^2+y^2+z^2)^{3/2}} & \dfrac{z}{(x^2+y^2+z^2)^{3/2}} \end{vmatrix}$$

$$= \hat{i}\left[-\frac{3}{2}\frac{1}{(x^2+y^2+z^2)^{5/2}}2yz + \frac{3}{2}\frac{1}{(x^2+y^2+z^2)^{5/2}}2yz\right]$$

$$- \hat{j}\left[-\frac{3}{2}\frac{1}{(x^2+y^2+z^2)^{5/2}}\cdot 2xz + \frac{3}{2}\frac{1}{(x^2+y^2+z^2)^{5/2}}2zx\right]$$

$$+ \hat{k}\left[-\frac{3}{2}\frac{1}{(x^2+y^2+z^2)^{5/2}}2xy + \frac{3}{2}\frac{1}{(x^2+y^2+z^2)^{5/2}}2yx\right] = 0$$

$\therefore$ curl $\vec{F} = 0$

$\Rightarrow \vec{F}$ is irrotational

7) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$ then S.T $\text{div}(r^n\vec{r}) = (n+3)r^n$

sol: $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$$\nabla\cdot(r^n\vec{r}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot r^n(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\nabla\cdot(r^n\vec{r}) = \frac{\partial}{\partial x}r^n x + \frac{\partial}{\partial y}r^n y + \frac{\partial}{\partial z}r^n z \rightarrow (1)$$

$$\frac{\partial}{\partial x}r^n x = n\cdot r^{n-1}\frac{\partial r}{\partial x}x + r^n(1), \quad \frac{\partial}{\partial y}r^n y = nr^{n-1}\frac{\partial r}{\partial y} + r^n(1)$$

$$\frac{\partial}{\partial z}r^n z = nr^{n-1}\frac{\partial r}{\partial z}\cdot z + r^n(1)$$

(1) becomes

$$\nabla\cdot(r^n\vec{r}) = nr^{n-1}\left(\frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}}\cdot\cancel{2}x\,x + \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}}\cancel{2}y\,y + \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}}\cdot\cancel{2}z\,z\right) + r^n(3)$$

$$\Rightarrow \nabla\cdot(r^n\vec{r}) = \frac{n r^{n-1}}{\sqrt{x^2+y^2+z^2}}(x^2+y^2+z^2) + 3r^n$$

$$= \frac{n r^{n-1}}{r}(r^2) + 3r^n = n\cdot r^{n-1} r + 3r^n$$

$$\therefore \nabla\cdot(r^n\vec{r}) = n\cdot r^n + 3r^n = (n+3) r^n$$

8) If $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$ and $r = |\vec{r}|$ then S.T. $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$

sol: $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$, $r^2 = x^2+y^2+z^2$

$$\Rightarrow \cancel{2} r \frac{\partial r}{\partial x} = \cancel{2} x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla^2 f(r) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(r) = \sum \frac{\partial^2}{\partial x^2}\left(f(r)\right) \rightarrow 1$$

$$\frac{\partial^2}{\partial x^2} f(r) = \frac{\partial}{\partial x}\left(f'(r)\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left(f'(r)\frac{x}{r}\right)$$

$$= \frac{r\left\{f'(r) + x f''(r)\frac{\partial r}{\partial x}\right\} - f'(r)\, x\frac{\partial r}{\partial x}}{r^2}$$

$$\therefore \frac{\partial^2}{\partial x^2} f(r) = \frac{r\left\{f'(r) + x f''(r)\frac{x}{r}\right\} - f'(r)\, x\cdot\frac{x}{r}}{r^2}$$

$$\frac{\partial^2}{\partial x^2} f(r) = \frac{f'(r)}{r} + \frac{f''(r)}{r^2} x^2 - \frac{f'(r)}{r^3} x^2$$

|||ly $$\frac{\partial^2}{\partial y^2} f(r) = \frac{f'(r)}{r} + \frac{f''(r)}{r^2} y^2 - \frac{f'(r)}{r^3} y^2$$

$$\frac{\partial^2}{\partial z^2} f(r) = \frac{f'(r)}{r} + \frac{f''(r)}{r^2} z^2 - \frac{f'(r)}{r^3} z^2$$

① becomes

$$\nabla^2 f(r) = \frac{3 f'(r)}{r} + \frac{f''(r)}{r^2}(x^2+y^2+z^2) - \frac{f'(r)}{r^3}(x^2+y^2+z^2)$$

$$= \frac{3f'(r)}{r} + \frac{f''(r)}{\cancel{r^2}}\cancel{r^2} - \frac{f'(r)}{r^3} r^2$$

$$= \frac{3f'(r)}{r} + f''(r) - \frac{f'(r)}{r}$$

$$\therefore \nabla^2 f(r) = f''(r) + \frac{f'(r)}{r}(3-1) = f''(r) + \frac{2}{r} f'(r)$$

9) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$ the P.T. $\nabla^2(r^{n+1}) = (n+1)(n+2)r^{n-1}$

sol: $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$, $r^2 = x^2+y^2+z^2$, $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla^2 r^{n+1} = \sum \frac{\partial^2}{\partial x^2} r^{n+1} = (n+1)\sum \frac{\partial}{\partial x} r^n \cdot \frac{\partial r}{\partial x} = (n+1)\sum \frac{\partial}{\partial x} r^n \cdot \frac{x}{r}$$

$$\nabla^2 r^{n+1} = (n+1)\sum \frac{\partial}{\partial x}\left(x r^{n-1}\right) \rightarrow ①$$

$$\frac{\partial}{\partial x}(x r^{n-1}) = r^{n-1} + x(n-1) r^{n-2} \frac{\partial r}{\partial x} = r^{n-1} + (n-1) x\, r^{n-2} \frac{x}{r}$$

$$\frac{\partial}{\partial x}(x r^{n-1}) = r^{n-1} + (n-1) r^{n-3} x^2$$

$$\therefore \sum \frac{\partial}{\partial x}(x r^{n-1}) = 3r^{n-1} + (n-1) r^{n-3} r^2$$

$$= 3r^{n-1} + (n-1) r^{n-1} = r^{n-1}(3+n-1)$$

$$\sum \frac{\partial}{\partial x}(x r^{n-1}) = r^{n-1}(n+2)$$

$\therefore$ ① becomes

$$\nabla^2 r^{n+1} = (n+1)(n+2) r^{n-1}.$$

This result can be obtained by putting $f(r) = r^{n+1}$ in previous example.

$$\nabla^2 r^n = n(n+1) r^{n-2}$$

$$\nabla^2\left(\frac{1}{r}\right) = \nabla^2 r^{-1} = 0$$

10) If $\vec{f} = x^2 i - 2xzj + 2yzk$ then find $curl\left(curl\,\vec{f}\right)$.

Solution: $curl\,\vec{f} = \nabla \times \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -2xz & 2yz \end{vmatrix} = i(2z+2x) - j(0-0) + k(-2z)$

$$curl\left(curl\,\vec{f}\right) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x+2z) & 0 & -2z \end{vmatrix} = i(0-0) - j(0-2) + k(0-0) = 2j$$

11) For any differentiable vector function $\vec{f}$, P.T $div(curl\,\vec{f}) = 0$

sol:

$$curl\,\vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\nabla \times \vec{f} = \hat{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \hat{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$\nabla \cdot (\nabla \times \vec{f}) = div\,(curl\,\vec{f}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot curl\,\vec{f}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \cancel{\frac{\partial^2 f_3}{\partial x \partial y}} - \cancel{\frac{\partial^2 f_2}{\partial x \partial z}} - \cancel{\frac{\partial^2 f_3}{\partial y \partial x}} + \cancel{\frac{\partial^2 f_1}{\partial y \partial z}} + \cancel{\frac{\partial^2 f_2}{\partial z \partial x}} - \cancel{\frac{\partial^2 f_1}{\partial z \partial y}}$$

$\therefore$ $div\,(curl\,\vec{f}) = \nabla \cdot (\nabla \times \vec{f}) = 0.$

12. If $\vec{r} = xi + yj + zk$ and $r = |\vec{r}|$ then show that $r^n \vec{r}$ is irrotational for all values of n and solenoidal for $n = -3$.

Sol. $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$, $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

$r^n \vec{r} = r^n x\,\hat{i} + r^n y\,\hat{j} + r^n z\,\hat{k}$

$$\nabla \times (r^n \vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \sum \hat{i} \left[\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y) \right]$$

$$= \sum \hat{i} \left[n \cdot r^{n-1} \frac{\partial r}{\partial y} z - n \cdot r^{n-1} \frac{\partial r}{\partial z} y \right] = \sum \hat{i} \left[n r^{n-1} \frac{y}{r} z - n r^{n-1} \frac{z}{r} y \right]$$

$$\therefore \text{curl}(r^n \vec{r}) = \sum \hat{i}\, n r^{n-1} \left[z \cdot \frac{y}{r} - y \cdot \frac{z}{r} \right] = \sum \hat{i}\, n r^{n-1} (0) = 0$$

$\Rightarrow r^n \vec{r}$ is irrotational.

Consider, $\nabla \cdot (r^n \vec{r}) = \text{div}(r^n \vec{r}) = \sum \frac{\partial}{\partial x}(r^n x)$

$$= \sum \left[n r^{n-1} \frac{\partial r}{\partial x} x + r^n \right] = \sum \left[n r^{n-1} \frac{x}{r} x + r^n \right]$$

$$= \sum \left[n \cdot r^{n-2} x^2 + r^n \right] = n r^{n-2} \sum x^2 + \sum r^n$$

$$\nabla \cdot (r^n \vec{r}) = n r^{n-2} r^2 + 3 r^n = n r^n + 3 r^n = (n+3) r^n \rightarrow (1)$$

Now $r^n \vec{r}$ is solenoidal if $\text{div}(r^n \vec{r}) = 0$

From (1) $\Rightarrow$ $(n+3)\, r^{n} = 0 \Rightarrow n = -3$

12. Show that $\vec{f} = (\sin y + z)i + (x\cos y - z)j + (x - y)k$ is irrotational. Find the function ϕ such that $\vec{f} = grad\, \emptyset$.

Solution: Given that $\vec{f} = (\sin y + z)i + (x\cos y - z)j + (x - y)k$

Consider $curl\, \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y + z & xcoy - z & x - y \end{vmatrix}$

$$curl\, \vec{f} = i[-1+1] - j[1-1] + k[\cos y - \cos y] = i(0) + j(0) + k(0) = 0$$

Therefore $\vec{f}$ is irrotational.

To find : ϕ such that $\vec{f} = \nabla \phi$

$\frac{\partial \phi}{\partial x} = \sin y + z \rightarrow (1)$, $\frac{\partial \phi}{\partial y} = x\cos y - z \rightarrow (2)$, $\frac{\partial \phi}{\partial z} = x - y \rightarrow (3)$

Integrating (1), (2), (3) wrt x, y, z

$$\phi = x\sin y + xz + f_1(y, z)$$
$$\phi = x\sin y - yz + f_2(x, z)$$
$$\phi = xz - yz + f_3(x, y)$$

By inspection, $\phi = x\sin y + xz - yz + c$

Practice problems:

1) If $\phi = 2x^3y^2z^4$, find $\operatorname{div}(\operatorname{grad}\phi)$

Ans: $4xz^2(3y^2z^2 + x^2z^2 + 6x^2y^2)$

2) If $\phi = x^2 - y^2 + 4z$, S.T. $\nabla^2\phi = 0$

3) Find the constant a such that $\vec{f} = y(ax^2+z)\hat{i} + x(y^2-z^2)\hat{j} + 2xy(z-xy)\hat{k}$ is solenoidal Ans: $a = -2$

4) If $\vec{r}$ is position vector of a point and $r = |\vec{r}|$, S.T. $\nabla\cdot(r^3\vec{r}) = 6r^3$

5) S.T. $\nabla\cdot\left[r\nabla\left(\frac{1}{r^3}\right)\right] = \frac{3}{r^4}$, where $r^2 = x^2+y^2+z^2$

6) If $\phi = xyz$, $\vec{f} = x^2yz\hat{i} + y^2zx\hat{j} + z^2xy\hat{k}$, find $\nabla\cdot(\phi\vec{f})$

Ans: $9x^2y^2z^2$

7) S.T. the vector $\vec{f} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$ is irrotational

8) Prove that $\vec{F} = (y^2 + 2xz^2)\boldsymbol{i} + (2xy - z)\boldsymbol{j} + (2x^2z - y + 2z)\boldsymbol{k}$ is irrotational and hence find the scalar potential ϕ such that $\vec{F} = \nabla\phi$.

Ans: $curl\vec{F} = (1-1)i - j(4xz - 4xz) + k(2y - 2y) = 0$ and scalar potential $\phi = y^2x + x^2z^2 - yz + z^2 + c$

9. Show that $\vec{A} = y^2z^3\boldsymbol{i} + 2xyz^3\boldsymbol{j} + 3xy^2z^2\boldsymbol{k}$ is conservative.

10. S.T. $F = (e^x \cos y + yz)\hat{i} + (xz - e^x \sin y)\hat{j} + (xy+z)\hat{k}$ is conservative over its natural domain and find a potential function for it.

Ans: $\phi = e^x \cos y + xyz + \frac{z^2}{2} + c$

11) If $r = (x^2+y^2+z^2)^{1/2}$ then S.T. $\nabla^2 (\log r) = \frac{1}{r^2}$

12) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, S.T. ① $\text{grad } r = \frac{\vec{r}}{r}$

② $\text{grad}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

Vector Identities

If $\vec{f}$ and $\vec{g}$ are any 2 vector point functions & ϕ and ψ are 2 scalar point functions then the foll are true

1) $\text{div}(\phi \vec{f}) = \phi \, \text{div} \vec{f} + \vec{f} \cdot \text{grad} \, \phi$

i.e. $\nabla \cdot (\phi \vec{f}) = \phi(\nabla \cdot \vec{f}) + \vec{f} \cdot (\nabla \phi)$

Proof:

$$\phi \vec{f} = \phi f_1 \hat{i} + \phi f_2 \hat{j} + \phi f_3 \hat{k}$$

$$\text{div}(\phi \vec{f}) = \frac{\partial}{\partial x}(\phi f_1) + \frac{\partial}{\partial y}(\phi f_2) + \frac{\partial}{\partial z}(\phi f_3)$$

$$\text{div}(\phi \vec{f}) = \phi \frac{\partial f_1}{\partial x} + \frac{\partial \phi}{\partial x} f_1 + \phi \frac{\partial f_2}{\partial y} + \frac{\partial \phi}{\partial y} f_2 + \phi \frac{\partial f_3}{\partial z} + \frac{\partial \phi}{\partial z} f_3$$

$$\text{div}(\phi \vec{f}) = \phi \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) + f_1 \frac{\partial \phi}{\partial x} + f_2 \frac{\partial \phi}{\partial y} + f_3 \frac{\partial \phi}{\partial z}$$

$$\text{div}(\phi \vec{f}) = \phi \, \text{div} \vec{f} + (f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k}) \cdot \left(\frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \right)$$

$$\therefore \text{div}(\phi \vec{f}) = \phi \, \text{div} \vec{f} + f \cdot \text{grad} \, \phi$$

2) $\text{curl}(\phi \vec{f}) = \phi \, \text{curl} \vec{f} + (\text{grad} \, \phi) \times \vec{f}$

i.e. $\nabla \times (\phi \vec{f}) = \phi(\nabla \times \vec{f}) + \nabla \phi \times \vec{f}$

Proof: $\quad \phi\vec{f} = \phi f_1\hat{i} + \phi f_2\hat{j} + \phi f_3\hat{k}$

$$\nabla\times\phi\vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \phi f_1 & \phi f_2 & \phi f_3 \end{vmatrix}$$

$$= \sum \hat{i}\left[\frac{\partial}{\partial y}(\phi f_3) - \frac{\partial}{\partial z}(\phi f_2)\right]$$

$$= \sum \hat{i}\left[\phi\frac{\partial f_3}{\partial y} + \frac{\partial\phi}{\partial y}f_3 - \phi\frac{\partial f_2}{\partial z} - \frac{\partial\phi}{\partial z}f_2\right]$$

$$= \phi\sum\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right]\hat{i} + \sum\left[\frac{\partial\phi}{\partial y}f_3 - \frac{\partial\phi}{\partial z}f_2\right]\hat{i}$$

$$= \phi\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\text{curl}(\phi\vec{f}) = \phi(\nabla\times\vec{f}) + \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\times(f_1\hat{i} + f_2\hat{j} + f_3\hat{k})$$

$$\therefore \text{curl}(\phi\vec{f}) = \phi(\nabla\times\vec{f}) + (\nabla\phi)\times\vec{f}$$

3) $\text{div}(\vec{f}\times\vec{g}) = \vec{g}\,\text{curl}\,\vec{f} - \vec{f}\,\text{curl}\,\vec{g}$

i.e. $\nabla\cdot(\vec{f}\times\vec{g}) = \vec{g}\cdot(\nabla\times\vec{f}) - \vec{f}\cdot(\nabla\times\vec{g})$

Proof: Consider

$$\nabla \times \vec{f} = \text{curl}\,\vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\text{curl}\,\vec{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k}$$

$$\vec{g}\cdot\text{curl}\,\vec{f} = g_1\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + g_2\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + g_3\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \rightarrow (1)$$

$$\text{III}^{ly}\quad \vec{f}\cdot\text{curl}\,\vec{g} = f_1\left(\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z}\right) + f_2\left(\frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}\right) + f_3\left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right) \rightarrow (2)$$

(1) - (2) $\Rightarrow$

$$\vec{g}.curl\,\vec{f} - \vec{f}.curl\,\vec{g} = [g_1\frac{\partial f_3}{\partial y} + f_3\frac{\partial g_1}{\partial y} + g_2\frac{\partial f_1}{\partial z} + f_1\frac{\partial g_1}{\partial z} + g_3\frac{\partial f_2}{\partial x} + f_2\frac{\partial g_3}{\partial x}]$$

$$-\left[g_1\frac{\partial f_2}{\partial z} + f_2\frac{\partial g_1}{\partial z} + g_2\frac{\partial f_3}{\partial x} + f_3\frac{\partial g_2}{\partial x} + g_3\frac{\partial f_1}{\partial y} + f_1\frac{\partial g_3}{\partial y}\right]$$

$$\vec{g}.curl\,\vec{f} - \vec{f}\,curl\,\vec{g} = \left[\frac{\partial}{\partial y}(f_3g_1) + \frac{\partial}{\partial z}(f_1g_2) + \frac{\partial}{\partial x}(f_2g_3) - \frac{\partial}{\partial z}(f_2g_1) - \frac{\partial}{\partial x}(f_3g_2) - \frac{\partial}{\partial y}(f_1g_3)\right]$$

$$\vec{g}.curl\,\vec{f} - \vec{f}.curl\,\vec{g} = \frac{\partial}{\partial x}(f_2g_3 - f_3g_2) + \frac{\partial}{\partial y}(f_3g_1 - f_1g_3) + \frac{\partial}{\partial z}(f_1g_2 - f_2g_1)\ldots\ldots\ldots\ldots \quad (3)$$

Consider $\vec{f}\times\vec{g} = \begin{vmatrix} i & j & k \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{vmatrix} = (f_2g_3 - f_3g_2)i + (f_3g_1 - f_1g_3)j + (f_1g_2 - f_2g_1)k$

Now $div\left(\vec{f}\times\vec{g}\right) = \frac{\partial}{\partial x}(f_2g_3 - f_3g_2) + \frac{\partial}{\partial y}(f_3g_1 - f_1g_3) + \frac{\partial}{\partial z}(f_1g_2 - f_2g_1)\ldots\ldots\ldots\ldots\ldots \quad (4)$

From (3) and (4) we have

$div\left(\vec{f}\times\vec{g}\right) = \vec{g}.curl\,\vec{f} - \vec{f}.curl\,\vec{g}$ i.e. $\nabla.\left(\vec{f}\times\vec{g}\right) = \vec{g}.\left(\nabla\times\vec{f}\right) - \vec{f}.\left(\nabla\times\vec{g}\right)$

4) curl $(\vec{f}\times\vec{g}) = (\operatorname{div}\vec{g})\,\vec{f} + (\vec{g}\cdot\nabla)\vec{f} - (\operatorname{div}\vec{f})\,\vec{g} - (\vec{f}\cdot\nabla)\vec{g}$

i.e. $\nabla\times(\vec{f}\times\vec{g}) = (\nabla\cdot\vec{g})\,\vec{f} + (\vec{g}\cdot\nabla)\vec{f} - (\nabla\cdot\vec{f})\,\vec{g} - (\vec{f}\cdot\nabla)\vec{g}$

Proof : (Exercise : Go through reference books)

5)

$$curl\left(curl\,\vec{f}\right) = grad\left(div\,\vec{f}\right) - \nabla^2\vec{f} \text{ i.e. } \nabla\times\left(\nabla\times\vec{f}\right) = \nabla\left(\nabla.\vec{f}\right) - \nabla^2\vec{f}$$

Proof: Now $curl\,\vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right]i + \left[\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right]j + \left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]k$

$$curl\left(curl\,\vec{f}\right) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right] & \left[\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right] & \left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right] \end{vmatrix}$$

$$curl\left(curl\,\vec{f}\right) = \sum\left[\frac{\partial}{\partial y}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\right].i$$

$$curl\left(curl\,\vec{f}\right) = \sum\left[\frac{\partial^2 f_2}{\partial y.\partial x} - \frac{\partial^2 f_1}{\partial y^2} - \frac{\partial^2 f_1}{\partial z^2} + \frac{\partial^2 f_3}{\partial z.\partial x}\right].i$$

Add and subtract $\frac{\partial^2 f_1}{\partial x^2}$

$$curl\left(curl\,\vec{f}\right) = \sum\left[\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_2}{\partial y,\partial x} + \frac{\partial^2 f_3}{\partial z.\partial x} - \left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}\right)\right].i$$

$$curl\left(curl\,\vec{f}\right) = \sum\frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right).i - \sum\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z^2}\right)f_1 i$$

$$curl\left(curl\,\vec{f}\right) = \sum\frac{\partial}{\partial x}\left(div\,\vec{f}\right).i - \sum\nabla^2 f_1 i$$

$$curl\left(curl\,\vec{f}\right) = grad\left(div\,\vec{f}\right) - \nabla^2\vec{f} \text{ i.e. } \nabla\times\left(\nabla\times\vec{f}\right) = \nabla\left(\nabla.\vec{f}\right) - \nabla^2\vec{f}$$

6) $\text{div}(\text{curl}\,\vec{f}) = 0$ (Proved earlier)

7) $\text{curl}(\text{grad}\,\phi) = 0$ (Proved earlier)

Practice problem

1) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, S.T. $\text{curl}(r^n \vec{r}) = 0$, where $r = |r|$. That is $r^n \vec{r}$ is irrotational for all n.