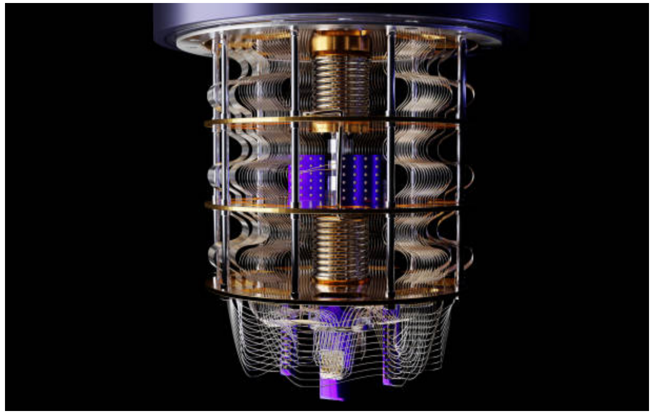
Unit II: Quantum computing



Quantum Computer

Quantum computing: A multidisciplinary field comprising aspects of Physics, computer science, and mathematics that utilizes quantum mechanics to solve complex problems faster than on classical computers.

Computing is a digital processing. Atoms, electrons or photons obey the laws of quantum mechanics. In quantum computing information is stored and processed using electrons, photons or isolated atoms hence the name quantum mechanics and the quantum computers perform the commutations or processing according to the quantum laws.

Why do we need quantum computers?

We need quantum computers if a supercomputer gets stumped, that's probably because the big classical machine was asked to solve a problem with a high degree of complexity

These computations in computers have potential applications in the field of cryptography. (Cryptography is the science of maintaining the secrecy and security in communication).

- An IBM Quantum processor is a wafer not much bigger than the one found in a laptop.
- A quantum hardware system is about the size of a car, made up mostly of cooling systems to keep the superconducting processor at its ultra-cold operational temperature.
- A classical processor uses classical bits to perform its operations.
- A quantum computer uses qubits (CUE-bits) to run multidimensional quantum algorithms.
- <u>Wave function in Dirac notation:</u>
- A wave function Ψ represents the physical state of a system.
- According to Paul Dirac the state of a system is described by a vector called state vector in Hilbert Space H. Depending on the degrees of freedom of system being considered H may be infinite in dimension.
- (Hilbert Space: It is a complex vector space and have the properties of vector space like vector addition, multiplication etc).
- *If ψis a wave function then the Dirac notation of the wave function is $|\psi\rangle$ and it is called ket vector of ψ* $|\psi\rangle$

If $\psi = A.e^{-ikx}$ then the in Dirac notation $|\psi\rangle = A.e^{-ikx}$ only the notation is changed

*The complex conjugate of ψ is ψ^*, in Dirac notation ψ^* is denoted as $\langle\psi|$ and it is called **bra vector** of ψ*

If $\psi = A.e^{-ikx}$ then $\psi^* = A^*.e^{+ikx}$ in Dirac notation $\langle\psi| = A^*.e^{+ikx}$. Only the notation is changed

Thus if $\psi = A.e^{-ikx}$ then $|\psi\rangle = A.e^{-ikx}$ and $\langle\psi| = A^*.e^{+ikx}$

Properties

To every ket vector $|\psi\rangle$ there corresponds a bra vector $\langle\psi|$ and vice versa $|\psi\rangle \leftrightarrow \langle\psi|$

- There is a one-to-one correspondence between bra and ket vectors.

$a|\psi\rangle + b|\psi\rangle \leftrightarrow a^*\langle\psi| + b^*\langle\psi|$ where a^* and b^* are complex numbers.

Basis: In quantum mechanics, the basis vector can be thought of a set of mutually perpendicular vectors one for each dimension of space in which the space vector is expressed. The magnitude of the basis vector is unity

- **MATRIX FORM OF WAVE FUNCTION**

A state vector ψ can be written as a linear combination of kets $|\varphi_1\rangle, |\varphi_2\rangle, |\varphi_{3...}\rangle$.... as

$\psi = a_1|\varphi_1\rangle + a_2|\varphi_2\rangle + a_3|\varphi_{3...}\rangle + \ldots + a_n\langle\psi_n|$ where a_n is a component of ψ along the vector $|\varphi_n\rangle$

Here the ket vector $|\psi\rangle$ is represented as a column matrix

$$|\psi\rangle = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ . \\ . \\ a_n \end{bmatrix} \qquad |\psi\rangle = \begin{bmatrix} \langle\varphi_1|\psi\rangle \\ \langle\varphi_2|\psi\rangle \\ \langle\varphi_3|\psi\rangle \\ . \\ \langle\varphi_n|\psi\rangle \end{bmatrix}$$

The bra vector $\langle\psi|$ is represented as a row matrix

$$\langle\psi| = [a_1^* \; a_2^* \; a_3^* \ldots\ldots a_n^*] \Rightarrow [\langle\psi_1||\varphi\rangle \; \langle\psi_2|\varphi\rangle \; \langle\psi_3|\varphi\rangle \ldots\ldots\ldots \langle\psi_n|\varphi\rangle]$$

- **MATRIX:** Matrix is a square or rectangular arrangement of elements (numbers or functions) that obey certain laws.
- Every element in a matrix belongs to a certain row and column.
- An element $\mathbf{a_{ij}}$ belongs to $\mathbf{i^{th}}$ **row** and $\mathbf{J^{th}}$ **column.**
- A matrix having m rows and n columns is mxn matrix.

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \rightarrow \text{This ia a 3x2 matrix}$$

- **Properties of matrices**

• 1.Equality of matrices.

If $A = \begin{bmatrix} i & j & k \\ x & y & z \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 & 8 \\ 2 & 4 & 7 \end{bmatrix}$ then A = B only if each element of A is equal to the corresponding element in B

i.e $i = 1, j = 5, k = 8, x = 2, y = 4$ and $z = 7$.

• Addition of matrices

$$\text{If } A = \begin{bmatrix} i & j \\ k & l \end{bmatrix} \text{ and } B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } A + B = \begin{bmatrix} (i+a) & (j+b) \\ (k+c) & (l+d) \end{bmatrix}$$

- **H.W verify**
- **i. Commutative law : A+B = B+A ii. Associative law (A+B)+C= A+(B+C)**
- **iii. Distributive Law: $\lambda(a_{ii}+b_{ij}) = \lambda a_{ii} + \lambda b_{ij}$).**

- # 3. Multiplication of Matrices.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \text{ then } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} . \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{12} + a_{12}b_{22}) \\ (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) \\ (a_{31}b_{11} + a_{32}b_{21}) & (a_{31}b_{12} + a_{32}b_{22}) \end{bmatrix}$$

$$\text{If } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \text{ S.T } AB = \begin{bmatrix} 08 & 16 \\ 18 & 38 \end{bmatrix}$$

S.T AB≠BA and AB=BA only if A=B

- **TYPES OF MATRICES**
- **1.Square matrix**: a matrix with equal number of rows and columns is a square matrix. Ex. mxm matrix.
- **2.Diagonal matrix**: A square matrix in which principle or leading diagonal elements are only non-zero elements.

$$eg.\ a_{ij} \begin{bmatrix} =0 \text{ for } i \neq j \\ \neq 0 \text{ for } i = j \end{bmatrix} \Rightarrow \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$$

- **3. Scalar matrix:** A scalar matrix in which all the principal diagonal elements are the same.

$$eg.\ a_{ij} \begin{bmatrix} =0 \text{ for } i \neq j \\ =\lambda \text{ for } i = j \end{bmatrix} \Rightarrow \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

- **4. Identity Matrix:** A scalar matrix in which each principal diagonal element is unity. The identity matrix is denoted by I.

$$I = \begin{bmatrix} 100 \\ 010 \\ 001 \end{bmatrix}$$

Note: If a matrix is multiplied by an identity matrix, then the product is identity matrix itself. IA=A

- **Transpose of a matrix:** A matrix obtained by interchanging the rows and columns of a given matrix is called the transpose of the given matrix. If P(mxn) is a given matrix then Q(nxm) is the transpose of P and it is denoted by P^T.

$$If\ P = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \text{ then } P^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}$$

Verify: 1). $[(A)^T]^T = A$ 2). $[A + B]^T = A^T + B^T$ 3. $(A.B)^T = B^T.A^T$.

- **Conjugate of a matrix:** If A is a matrix with complex elements then the conjugate of A (denoted as A^*) is obtained by replacing the complex elements in A by their complex conjugates.

$$\text{If } A = \begin{bmatrix} x+3i & 8 \\ 1+2i & 3-i \end{bmatrix} \text{ then } A^* = \begin{bmatrix} x-3i & 8 \\ 1-2i & 3+i \end{bmatrix}$$

$$\text{If } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \text{ then } B^* = \begin{bmatrix} b_{11}^* & b_{12}^* \\ b_{21}^* & b_{22}^* \end{bmatrix}$$

Verify: 1. $[A^*]^* = A$ 2. $[A + B]^* = A^* + B^*$ 3. $[AB]^* = A^* x\, B^*$ 4. $[\lambda A]^* = \lambda A^*$

- **Unitary Matrix:** It is square matrix of complex elements. The product of conjugate transpose of a unitary matrix with the unitary matrix is Identity matrix. If $[U^*]^T U=I$ then U is a unitory matrix.

$$\text{Let } U=\frac{1}{3}\begin{bmatrix}2 & -2+i\\ 2+i & 2\end{bmatrix} \qquad \text{Then } U^*=\frac{1}{3}\begin{bmatrix}2 & -2-i\\ 2-i & 2\end{bmatrix} \qquad [U^*]^T=\frac{1}{3}\begin{bmatrix}2 & 2-i\\ -2-i & 2\end{bmatrix}$$

$$[U^*]^T U=\frac{1}{3}\begin{bmatrix}2 & 2-i\\ -2-i & 2\end{bmatrix}\cdot\frac{1}{3}\begin{bmatrix}2 & -2+i\\ 2+i & 2\end{bmatrix}=\frac{1}{9}\begin{bmatrix}9 & 0\\ 0 & 9\end{bmatrix}=\frac{1}{9}9\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$$ **As $[U^*]^T U=I$, U is a unitary matrix.**

- **Hermitian Matrix:** A Hermitian matrix is **a square matrix that is equal to the transpose of its conjugate matrix**
  - The diagonal elements of a Hermitian matrix are all real numbers, and the element of the (i, j) position is equal to the conjugate of the element in the (j, i) position.

If $A=[A^*]^T$ Then A is Hermitian matrix. It is denoted by $\bar{A}$

$$\text{Let } M=\begin{bmatrix}1 & i\\ -i & 1\end{bmatrix} \quad M^*=\begin{bmatrix}1 & -i\\ i & 1\end{bmatrix} \text{ and } [A^*]^T=\begin{bmatrix}1 & i\\ -i & 1\end{bmatrix}=M \text{ Thus M is Hermitian.}$$

- If A and B are two additional conformable Hermitian matrices, then A + B is also Hermitian

- Conjugate of a Hermitian matrix is also Hermitian.

- If A is Hermitian, then A*A and AA* is also Hermitian.

- If A is Hermitian matrix, then A^n is also Hermitian for all positive integers n.

S.T Following matrices are hermitian matrices

$$1.A=\begin{bmatrix} 2 & 4-i \\ 4+i & 8 \end{bmatrix} \quad 2.N=\begin{bmatrix} 1 & 2+i & 6+i \\ 2-i & 5 & 4-i \\ 6-i & 4+i & 9 \end{bmatrix}$$

$$3.N=\begin{bmatrix} 1 & i & 3i \\ -i & 0 & 2-i \\ -3i & 2+i & 9 \end{bmatrix} \quad 4.X=\begin{bmatrix} 2 & -2i \\ +2i & 8 \end{bmatrix}$$

Note: If a matrix of real numbers is equal to its transpose, then it is called a symmetric matrix

PAULI'S MATRICES

"Pauli's matrices are a set of 2x2 matrices and they are Unitary and Hermitian matrices". These matrices are used to represent spin angular momentum. Usually indicated by the Greek letter sigma (σ).

- Following are the Pauli's matrices

$$x = \sigma_1 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad y = \sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad z = \sigma_3 = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- *<u>TST Pauli's Matrices are Hermitian and Unit matrices.</u>*

$$1)\ \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_x^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad [\sigma_x^*]^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma_x. \quad \textit{Thus } \sigma_x \text{ is Hermitian}$$

$$[\sigma_x^*]^T.\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \qquad \textit{Thus } \sigma_x \text{ is Unitary}$$

$$2).\ \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_y^* = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad [\sigma_y^*]^T = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \sigma_y \qquad \textit{Thus } \sigma_y \text{ is Hermitian}$$

$$[\sigma_y^*]^T.\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \qquad \textit{Thus } \sigma_y \text{ is Unitary}$$

$$3).\ \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_z^* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad [\sigma_z^*]^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_z \qquad \textit{Thus } \sigma_z \text{ is Hermitian}$$

$$[\sigma_z^*]^T.\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \sigma_z \qquad \textit{Thus } \sigma_z \text{ is Unitary}$$

Matrix form of Dirac vectors

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad and \; |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Operation of identity matrix on Dirac matrices

$$|0\rangle I = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \; Thus \; |0\rangle I = |0\rangle$$

$$|1\rangle I = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \; Thus \; |1\rangle I = |1\rangle$$

Pauli's Matrices on Dirac vectors $|0\rangle$ ***and*** $|1\rangle$

$$1. \sigma_x|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \therefore \; \sigma_x|0\rangle = |1\rangle$$

$$\sigma_x|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \therefore \; \sigma_x|1\rangle = |0\rangle$$

$$2. \sigma_y|0\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ i \end{bmatrix} = i\begin{bmatrix} 1 \\ 0 \end{bmatrix} = i|1\rangle. \; \therefore \; \sigma_y|0\rangle = \mathrm{i}|1\rangle$$

$$\sigma_y|1\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -i \\ 0 \end{bmatrix} = -i\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -i|0\rangle. \; \therefore \; \sigma_y|1\rangle = -\mathrm{i}|0\rangle$$

$$3. \sigma_z|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle. \; \therefore \; \sigma_z|0\rangle = |0\rangle$$

$$\sigma_z|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -|1\rangle. \; \therefore \; \sigma_z|1\rangle = -1|1\rangle$$

- **Inner product of wavefunctions**

If $\psi(x)$ and $\varphi(x)$ are two wave functions then $\int \psi^*(x).\varphi(x)dx$ is called the inner product of the two vectors.

The inner product is denoted as (ψ,ϕ).

$$\therefore (\psi,\varphi) = \int \psi^*(x).\varphi(x)dx$$

In Dirac notation the inner product is represented as $\langle\psi|\varphi\rangle$

$$\therefore \langle\psi|\varphi\rangle = \int \psi^*(x).\varphi(x)dx$$

The inner product is a complex number

$\therefore$ *We can show that* $\langle\psi|\varphi\rangle = \langle\varphi|\psi\rangle^*$.

We have $(\psi,\varphi) = \int \psi^*(x).\varphi(x)dx$

$$\therefore \langle\varphi|\psi\rangle^* = \left(\int \varphi^*(x)\psi(x)dx\right)^* = \int \psi^*(x).\varphi(x)dx = \langle\psi|\varphi\rangle$$

$$\therefore \langle\psi|\varphi\rangle = \langle\varphi|\psi\rangle^*$$

- **Quantum superposition and probability**

Consider the state $|\psi\rangle$ *which is a* superposition *of* $|0\rangle$ *and* $|1\rangle$

$$\therefore |\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$= \alpha\begin{bmatrix}1\\0\end{bmatrix} + \beta\begin{bmatrix}0\\1\end{bmatrix}$$

$$= \begin{bmatrix}\alpha\\0\end{bmatrix} + \begin{bmatrix}0\\\beta\end{bmatrix}$$

$$\therefore |\psi\rangle = \begin{bmatrix}\alpha\\\beta\end{bmatrix}$$

$$\langle\psi| = [\alpha^* \ \beta^*]$$

$\therefore$ *The inner product* $= \langle\psi|\psi\rangle = [\alpha^* \ \beta^*].\begin{bmatrix}\alpha\\\beta\end{bmatrix} = [\alpha^*\alpha + \beta^*\beta] = [\alpha^2 + \beta^2]$

$\therefore \langle\psi|\psi\rangle = [\alpha^2 + \beta^2]$ *If* α^2 *and* β^2 *are the probabilities then* $[\alpha^2 + \beta^2]$ *is the total probality*

Normalisation

If $|\psi\rangle$ *is a wave function then the condition for normalisation is* $\langle\psi|\psi\rangle = 1$.

- **Orthogonality**
- Two states are said to be orthogonal if their inner product is zero

The states $|\psi\rangle$ *and* $|\varphi\rangle$ *are said to be orthognoal if* $\langle\psi|\varphi\rangle = 0$

T.S.T $|0\rangle$ *and* $|1\rangle$ *are orthogonal*

We have $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \langle 0| = [10]$ *and* $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$

$$\therefore \langle 0|1\rangle = [10]\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = 0$$

As $\langle 0|1\rangle = 0$ $|0\rangle$ *and* $|1\rangle$ *are orthogonal*

Orthonormal

Two states are said to be orthonormal if

i. Each of them are normalised.
ii. They are orthogonal

The states $|\psi\rangle$ *and* $|\varphi\rangle$ *are said to be orthonormal if*

i). $\langle\psi|\psi\rangle = 1, \langle\varphi|\varphi\rangle = 1$ *ii*). $\langle\psi|\varphi\rangle = o$

Ex. Consider $|0\rangle$ *and* $|1\rangle$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \langle 0| = [1 0]$$

$$\therefore \langle 0|0\rangle = [1 0]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \langle 1| = [0 1]$$

$$\therefore \langle 1|1\rangle = [0 1]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$$

$$\langle 0|1\rangle = [1 0]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

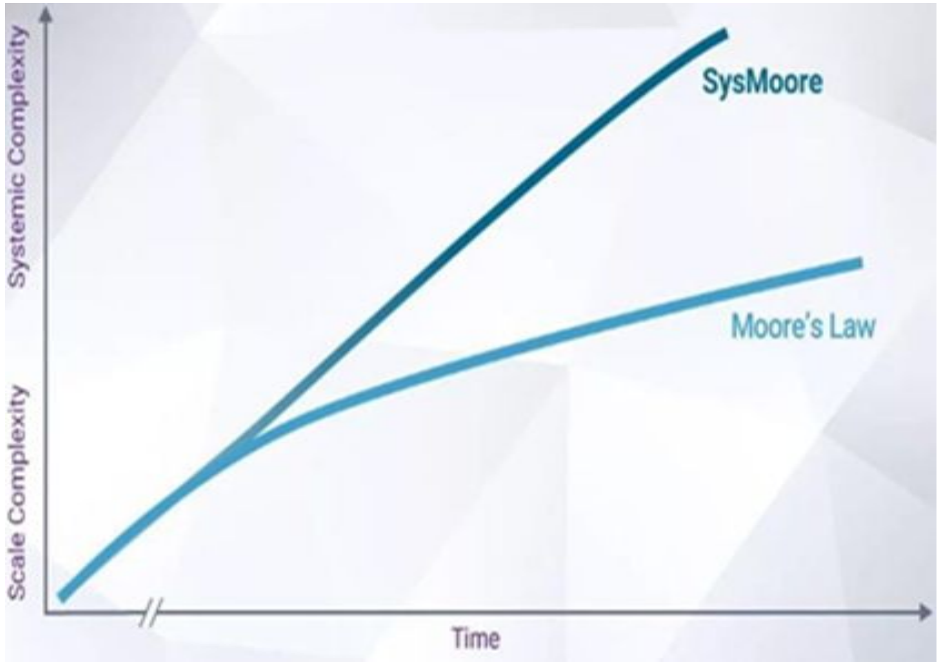
Thus the states $|0\rangle$ and $|1\rangle$ *are ortho normal*

- ***Classica computing and Moore's Law***

- Moor's law is an observation by Gordon G Moore the founder of intel.

- **Statement: "The number of transistors on a computer chip and its computation power doubles every two years"**

> - For many years Moore's law drove the semiconductor industries exponential growth. Computers became faster, cheaper and smaller.
> - The Moore's law now seems to be slowed down since the transistor count started doubling after three years not two years.
> - The industry is finding other methods to maintain the growth. Some aspects of this new approach to design have been dubbed as " More than Moore" (eg: The 3 D integration technique)



• Beam Splitter

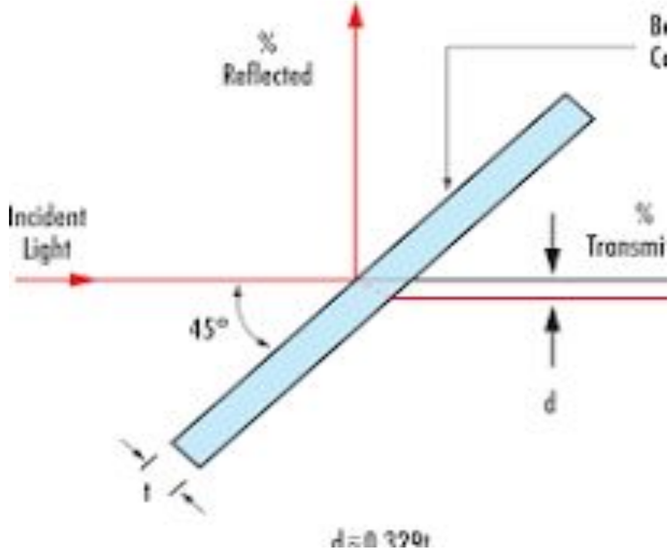
- A beam splitter is an optical device which is a half-silvered mirror. This is composed of an optical substrate, which is often a sheet of glass or transparent plastic, with a partially transparent thin coating of metal.
- The thickness of the coating is so that part (typically half) of the light, which is incident at a 45-degree angle and not absorbed by the coating or substrate material, is transmitted and the remainder is reflected in mutually perpendicular directions.

1.When the thickness of the mirror is very small the lateral shift in the transmitted light can be minimised and it will be along the incident direction

2.If the light is reflected on the denser medium there will be an additional phase difference of π or a path difference of λ/2 in the reflected light.

3. If the reflection is on the rarer medium, there will be no additional path difference introduced in the reflected light

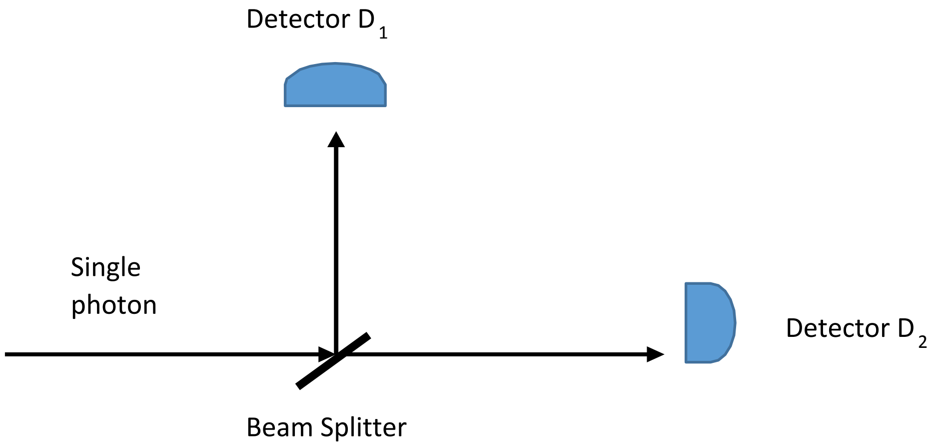
4. If the beam is refracted through the mirror, there will be no additional path difference introduced in the refracted beam



Single photon interference experiment

- A laser that emits one photon at a time is used in the experimental setup (a).
- This photon is then split by a beam splitter.
- The beam splitter reflects half of the light that strikes it and allows the other half to pass through.
- Photon detectors D_1 and D_2 detect the photon with equal probabilities.
- Hence it can be concluded that during any one run the photon has traveled

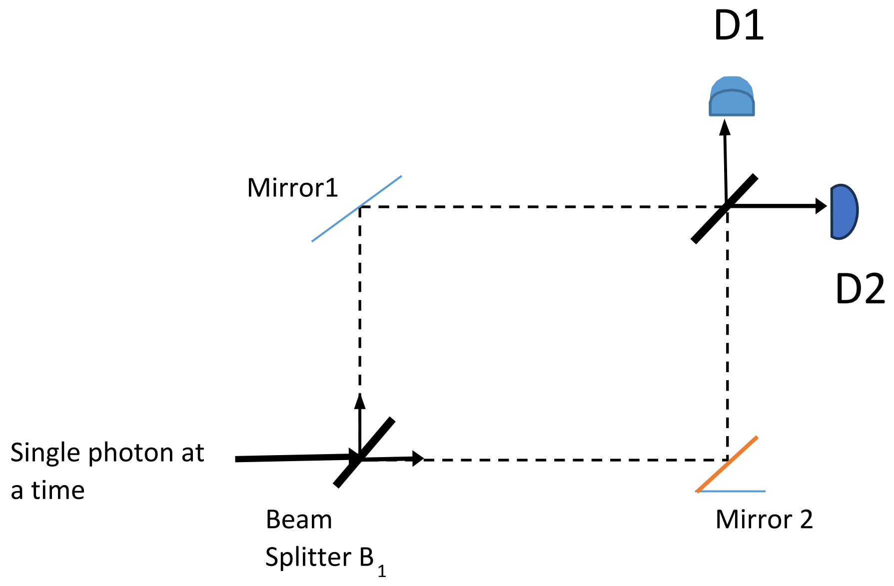
  one of the paths since it cannot be split into two.
- *However, this assumption is not true.*



Setup a

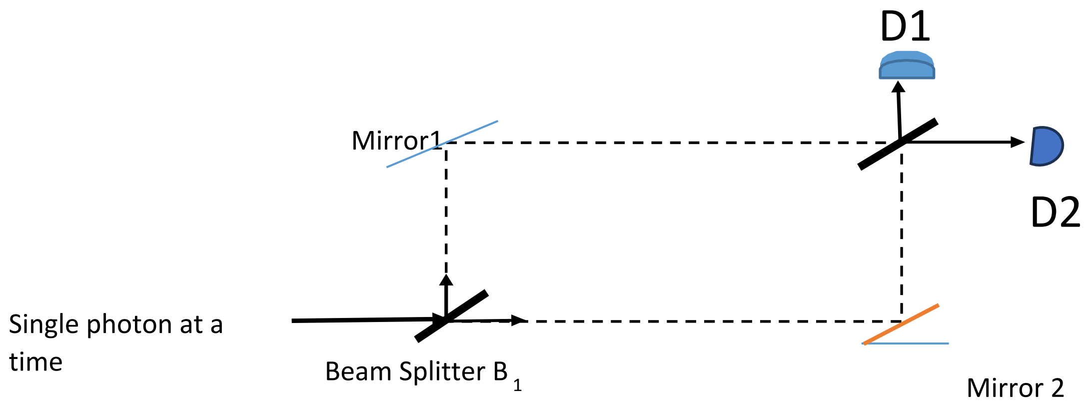
Consider the experimental set up b

- In the experimental set up (b), a single photon may travel horizontally, gets deflected by a mirror 2 and reaches the detectors
- Another possibility is that it passes vertically and gets deflected by a mirror 1 and reaches the detectors.
- Hence if the photon really takes a single path through the apparatus, both detectors would detect it with equal probabilities
- However, this does not happen. The photon is always detected by detector D_1 and never detector D_2.
- If we change the path length by half the wave length by introducing a half-wave glass plate in one of the paths (say vertical one), the photon is detected by detector D_2 and never by detector D_1.
- It means that the photon was in a superposition state, and it travelled through both paths simultaneously.
- At the second beam splitter the 2 components interfered constructively or destructively, and get detected by one of the detectors.



Set up b

- The phenomenon can be understood as follows.
- Light from B_1 going up and reaching D_1 suffers a phase change of π each at B_1, mirror1 and at B_2 thus a total phase change of 3π.
- The light from B_1 going horizontally and reaching D_1 suffers a phase difference of π at the mirror2. Thus the phase difference between the lights reaching D_1 is 2π (3π-π) or a path difference of λ and causes constructive interference.



Set up 2

- # Light from B_1 going vertically and reaching D_2 suffers a phase change of π each at B_1 and at mirror1 thus a total phase change of 2π. The light from B_1 going horizontally and reaching D_2 suffers a phase difference of π at the mirror2 (This is because there is no phase change due to refraction through B_1, due to the reflection at the rarer medium at B_2). Thus, the phase difference between the lights reaching D_2 is π or a path difference of λ/2 and causes destructive interference.
- To confirm this, if we introduce a half-wave plate which introduces a path difference of λ/2 in any one of the paths the condition at D_1 is destructive interference and the condition at D_2 constructive and the photon is detected only at D_2.
- Thus the single photon is undergoing interference.

Quantum Superposition

- Quantum superposition is a phenomenon associated with quantum systems such as nuclei, electrons and photons, for which wave-particle duality and other non-classical effects are observed.
- A quantum system can exist in more than one state at the same time.
- The result of the measurement is the observation of some definite state with a given probability.
- Eg: When a coin is tossed the probability its landing is 50:50 but when it lands it has a definite state.
- The coin can be in either heads, tails or a combination of heads and tails, all these are called states of the coin.
- The state of the coin when it is in air is a superposition of heads and tails.
- The measurement destroys the superposition.
- **Qubit:**
- **" Qubit is a basic unit of information in quantum computing"**
- A qubit, like a bit, also makes use of two states $|0\rangle$ and $|1\rangle$ to hold information.
- Mathematically the qubits |0> and |1> can be represented as column matrices"

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Unlike classical bits a qubit, |Ψ> can also be in a superposition state of |0> and |1> states. It can be written as |Ψ> =α |0> + β |1>
- where α and β are generally complex numbers which represent the probability amplitudes of the states.
- When a qubit is measured, it only results in either |0> or |1>.
- **Summation of probabilities**
- The probability of measuring the qubit in state |0> is $|\alpha|^2$, and the probability of measuring the qubit in state |1> is $|\beta|^2$. Since the total probability of observing all the states of the quantum system must add up to 100%, the modulus squared of the amplitudes must add up to 1. Thus, we have the following constraint: $|\alpha|2 + |\beta|2 = 1$

Physical Realization of Qubits:

- In a classical computer, the 0 and 1 bit mathematically represent the two allowed voltages across a wire in a classical circuit. Semiconductor devices called transistors are used to control what happens to these voltages.

“What is a qubit made out of?”

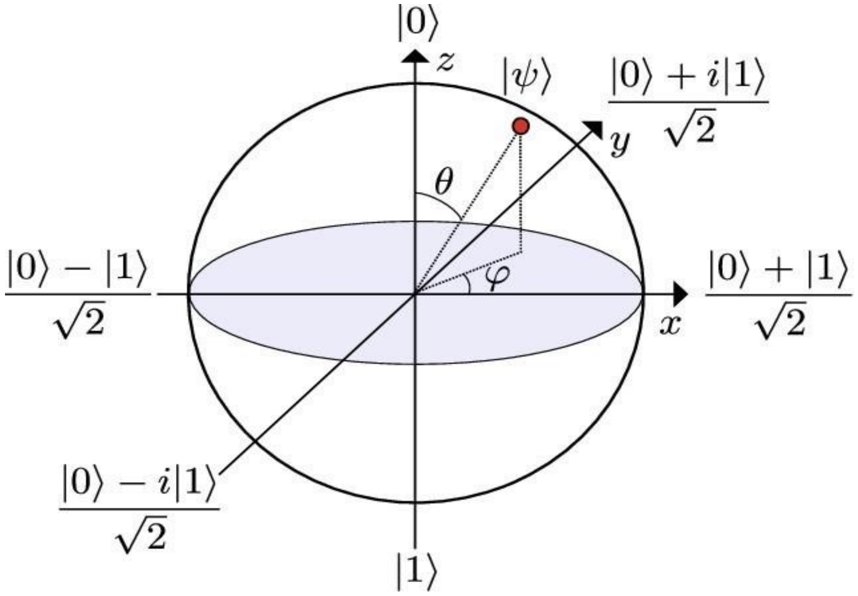
- **Energy levels of an atom:** Consider the electron in a hydrogen atom. It can be in its ground state (i.e. an s orbital) or in an excited state. So we can also store a qubit of information in the quantum state of the electron, i.e., in the superposition of the Ground state |0> and the Excited state |1>

- **Spin:** Elementary particles like electrons and protons carry an intrinsic angular momentum called spin.

- Their spins can be used as qubits with |0 >= | ↑>, |1 >= | ↓

• Representation of qubit by Bloch Sphere

- The distinct states |0> and |1> can be represented as north and south poles
- respectively of a sphere of unit radius called **Bloch Sphere.**
- The state of a qubit is represented by a vector $|\psi >$ as shown in the figure.



- Using the spherical coordinate system, an arbitrary position of the state vector of a qubit can be written in terms of the angles θ (elevation, the state vector makes from z-axis) and φ (azimuth, the angle of projection of the state vector in the x-y plane from the x-axis)
- The $|\psi >$ is given by $|\psi\rangle = cos(\theta/2)|0\rangle + e^{i\varphi} sin(\theta/2)|1\rangle$

- For ϕ = 0 and θ = 0, the state $|\psi >$ corresponds to |0> and is along z-axis.
- For ϕ = 0 and θ = 180o the state $|\psi >$corresponds to |1> and is along -z-axis.
- **<u>When θ = 90o, $|\psi >$ is in the x-y plane.</u>**

$$When\ \theta = 90^0\ \varphi = 90^0,\ |\psi\rangle = \frac{(|0\rangle + i|1\rangle)}{\sqrt{2}},\ is\ a\ superposition\ state\ along\ the\ \ +ve\ y\ axis.$$

$$When\ \theta = 90^0\ \varphi = -90^0,\ |\psi\rangle = \frac{(|0\rangle - i|1\rangle)}{\sqrt{2}},\ is\ a\ superposition\ state\ along\ the\ \ -ve\ y\ axis.$$

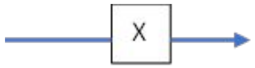
$$When\ \theta = 90^0\ \varphi = 0^0,\ |\psi\rangle = \frac{(|0\rangle + |1\rangle)}{\sqrt{2}},\ is\ a\ superposition\ state\ along\ the\ +ve\ x-axis.$$

$$When\ \theta = 90^0\ \varphi = 180^0,\ |\psi\rangle = \frac{(|0\rangle - |1\rangle)}{\sqrt{2}},\ is\ a\ superposition\ state\ along\ the\ -ve\ x-axis.$$

Quantum Gates

- Classical computers manipulate bits using logic gates such as OR, AND, NOT, NAND etc., Similarly quantum computers manipulate qubits using quantum gates.
- A quantum gate which acts on n qubits is represented by 2nx2n unitary matrix. The number of qubits in the input and output of the gate are to be equal.
- Each quantum gate is represented by a matrix. The action of the gate is found by multiplying the matrix representing the gate with the vector which represents the quantum state.
- Quantum gates are rotating gates, which corresponds to the rotation of qubit about X, Y and Zaxes of Bloch's sphere.

1.Quantum NOT gate or Pauli's X gate:



•The application of Quantum NOT gate or Pauli's X gate rotates the qubit by 180^0 along the x-axis.

It transforms $|0\rangle$ *to* $|1\rangle$ *and* $|1\rangle$ *to* $|0\rangle$ *i.e* $X|1\rangle = |0\rangle$ *and* $X|0\rangle = |1\rangle$

When a qubit is in a superposition state then the superposition also flips

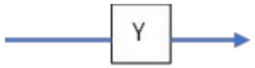
$$\alpha|0\rangle + \beta|1\rangle \xrightarrow{X} \alpha|1\rangle + \beta|0\rangle$$

$\alpha|0\rangle + \beta|1\rangle$

The matrix of quantum X − gate is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Truth table for NOT or X gate	
Input	output
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$
$\alpha\|1\rangle + \beta\|0\rangle$	$\alpha\|0\rangle + \beta\|1\rangle$

2 Quantum Y gate or Pauli's Y gate:



•The application of Quantum Pauli's Y gate rotates the qubit by 1800 along the y-axis.

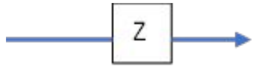
It transforms $|0\rangle$ *to* $i|1\rangle$ *and* $|1\rangle$ *to* $-i|0\rangle$

$$\alpha|0\rangle + \beta|1\rangle \xrightarrow{Y} \alpha i|1\rangle - \beta i|0\rangle$$

$i.e\ Y|0\rangle = i|1\rangle \quad and \quad Y|1\rangle = -i|0\rangle$

The matrix of quantum Y − gate is $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

Truth table for Y gate	
Input	output
$\|0\rangle$	$i\|1\rangle$
$\|1\rangle$	$-i\|0\rangle$
$\alpha\|1\rangle + \beta\|0\rangle$	$\alpha i\|1\rangle - \beta i\|0\rangle$

3 Quantum Z gate or Pauli's Z gate: 

The application of Quantum Pauli's Z gate rotates the qubit by 180^0 along the z-axis.

It leaves $|0\rangle$ un changed and flips sign $|1\rangle$ to $-\ |1\rangle$

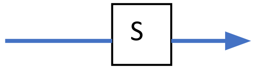
In Dirac form $Z|0\rangle = |0\rangle$ and $Z|1\rangle = -|1\rangle$

$$\alpha|0\rangle + \beta|1\rangle \xrightarrow{Z} \alpha|0\rangle - \beta|1\rangle$$

The matrix of quantum Z – gate is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Truth table for Z gate	
Input	output
$\lvert 0\rangle$	$\lvert 0\rangle$
$\lvert 1\rangle$	$-\lvert 1\rangle$
$\alpha\lvert 1\rangle + \beta\lvert 0\rangle$	$\alpha\lvert 0\rangle - \beta\lvert 1\rangle$

4 Phase Gate or S-Gate:



It rotates the qubit by $\pi/2$ along the Z-axis

It modifies the phase of the state

It leaves $|0\rangle$ *un changed and changes* $|1\rangle$ *to* $e^{\frac{i\pi}{2}}|1\rangle$ *or* $i|1\rangle$ $\qquad [e^{i\theta} = Cos\theta + iSin\theta]$

$$i.e\ S|0\rangle = |0\rangle \qquad and\ \ S|1\rangle = e^{\frac{i\pi}{2}}|1\rangle\ or\ i|1\rangle$$

$$\alpha|0\rangle + \beta|1\rangle \xrightarrow{S} \alpha|0\rangle + \beta i|1\rangle$$

The matrix of quantum S – gate is $\begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{2}} \end{bmatrix}$ *or* $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

Truth table for S gate	
Input	output
$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$i\|1\rangle$
$\alpha\|1\rangle + \beta\|0\rangle$	$\alpha\|0\rangle + \beta i\|1\rangle$

5 T Gate: $\longrightarrow\boxed{T}\longrightarrow$

It rotates the qubit by π/4 along the Z-axis

It modifies the phase of the state

It leaves $|0\rangle$ *un changed and changes* $|1\rangle$ *to* $e^{\frac{i\pi}{4}}|1\rangle$ *or* $\frac{1+i}{\sqrt{2}}|1\rangle$

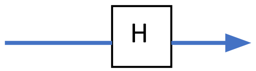
$$\alpha|0\rangle + \beta|1\rangle \xrightarrow{T} \alpha|0\rangle + \beta\frac{1+i}{\sqrt{2}}|1\rangle$$

The matrix of quantum T – gate is $\begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix}$ *or* $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} \end{bmatrix}$ *or* $e^{\frac{i\pi}{8}}\begin{bmatrix} e^{-\frac{i\pi}{8}} & 0 \\ 0 & e^{\frac{i\pi}{8}} \end{bmatrix}$

The T – Gate is also known as $e^{\frac{i\pi}{8}}$ *gate*

Truth table for T gate	
Input	output
$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$\frac{1+i}{\sqrt{2}}\|1\rangle$
$\alpha\|1\rangle + \beta\|0\rangle$	$\alpha\|0\rangle + \beta\frac{1+i}{\sqrt{2}}\|1\rangle$

6. Hadamard Gate or H-Gate:



H-Gate is an important gate in quantum computing.

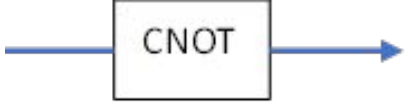
In Dirac Notation

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$The\ matrix\ of\ quantum\ H - gate\ is \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Truth table for T gate	
Input	output
$\|0\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$
$\|1\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$
$\alpha\|1\rangle + \beta\|0\rangle$	$\alpha\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle) + \beta\frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$

7CNOT - Gate



- It is a multiple qubit gate which acts on two qubits.
- It is a controlled NOT gate.
- It performs NOT operation on the second qubit only when the first qubit is

•It performs NOT operation on the second qubit only when the first qubit is $|1\rangle$
otherwise leaves it un changed

Dirac form of CNOT

$$CNOT|00\rangle = |00\rangle$$
$$CNOT|01\rangle = |01\rangle$$
$$CNOT|10\rangle = |11\rangle$$
$$CNOT|11\rangle = |10\rangle$$

$$\text{The matrix form of } C-NOT \text{ gate is } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Truth table for T gate			
Control qubit	Target qubit	Control qubit	Target qubit
$|0\rangle$	$|0\rangle$	$|0\rangle$	$|0\rangle$
$|0\rangle$	$|1\rangle$	$|0\rangle$	$|1\rangle$
$|1\rangle$	$|0\rangle$	$|1\rangle$	$|1\rangle$
$|1\rangle$	$|1\rangle$	$|1\rangle$	$|0\rangle$