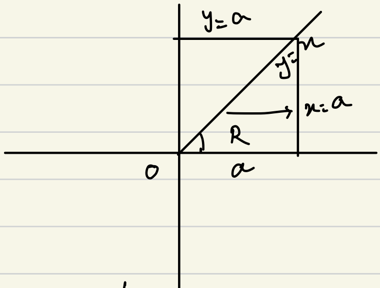
$$= \int_{v=0}^{5/6} 48\frac{u^2}{2}\Big|_{u=0}^{5/4} dv = 24\int_{v=0}^{5/6} \left(\frac{5}{4}\right)^2 dv$$

$$= 24\int_{v=0}^{5/6} \frac{25}{16}\, dv = \frac{3}{2} \times 25\, v\Big|_0^{5/6}$$

$$= \frac{\cancel{3}}{2} \times 25 \left(\frac{5}{\cancel{6}_2}\right) = \frac{125}{4}$$

Change of variables to polar coordinates: $\displaystyle\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\, dx\, dy$

1)



$$I = \int_{\theta=0}^{\pi/4} \int_{r=0}^{\frac{a}{\cos\theta}} \frac{r^2 \cos^2\theta}{\cancel{r}}\, \cancel{r}\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/4} \frac{r^3}{3}\Big|_0^{\frac{a}{\cos\theta}} \cos^2\theta\, d\theta$$

$$= \int_{\theta=0}^{\pi/4} \frac{1}{3} \frac{a^3}{\cos^{\cancel{3}}\theta}\, \cancel{\cos^2\theta}\, d\theta$$

$$= \int_{\theta=0}^{\pi/4} \frac{a^3}{3\cos\theta}\, d\theta = \int_0^{\pi/4} \frac{a^3}{3} \sec\theta\, d\theta$$

$$= \frac{a^3}{3} \log(\sec\theta + \tan\theta)\Big|_0^{\pi/4} = \frac{a^3}{3}\log(\sqrt{2}+1) - \frac{a^3}{3}\cancel{\log(1)}^{0}$$

$$= \frac{a^3}{3}\log(\sqrt{2}+1)$$

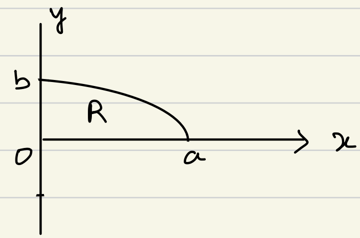
Applications of double integrals

1. In cartesian system, area of region $R = \iint_R dx\,dy$.

2. In polar coordinate system, area of region $R = \iint_R r\,dr\,d\theta$.

1) Find the area bounded by one quadrant of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

sol:-



$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y = b\left(1 - \frac{x^2}{a^2}\right)^{1/2}$$

$$R = \left\{ (x, y) \,\middle|\, 0 \le x \le a, \quad 0 \le y \le b\left(1 - \frac{x^2}{a^2}\right)^{1/2} \right\}$$

$$\therefore \quad A = \int_{x=0}^{a} \int_{y=0}^{b\left(1-\frac{x^2}{a^2}\right)^{1/2}} dy\,dx = \int_0^a \left[b\left(1 - \frac{x^2}{a^2}\right)^{1/2} \right] dx$$

$$= \frac{b}{a} \int_0^a \sqrt{a^2 - x^2}\, dx$$

or $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

Let $x = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$

When $x = 0 \Rightarrow \theta = 0$, $x = a \Rightarrow \theta = \pi/2$

$$= \frac{b}{a} \int_0^{\pi/2} a^2 \cos^2\theta \, d\theta = \frac{b}{a} \times a^2 \int_0^{\pi/2} \frac{(1 + \cos 2\theta)}{2} d\theta$$

$$= \frac{ab}{2} \left(\theta + \frac{\sin 2\theta}{2}\right)_0^{\pi/2} = \frac{ab}{2}\left\{ \frac{\pi}{2} + \frac{1}{2}\, 0 \right\} = \frac{\pi}{4} ab \text{ sq. units}$$

2) Find the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

Sol: The points of intersection of $y^2 = 4ax$ and $x^2 = 4ay$ are

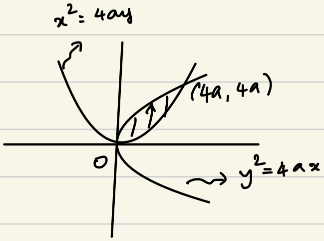
$$\left(\frac{x^2}{4a}\right)^2 = 4ax \Rightarrow \frac{x^4}{16a^2} = 4ax$$

$$\Rightarrow x^4 = 64a^3x \Rightarrow x^4 - 64a^3x = 0$$

$$\Rightarrow x(x^3 - 64a^3) = 0$$

$$\Rightarrow x = 0, \quad x = 4a$$

$$\therefore y = 0, \quad y = 4a$$



$$\therefore \text{Required area} = \int_{x=0}^{4a}\int_{y=\frac{x^2}{4a}}^{\sqrt{4ax}} dy\,dx$$

$$= \int_{x=0}^{4a} y\Big]_{y=x^2/4a}^{\sqrt{4ax}} dx = \int_0^{4a} \left\{\sqrt{4ax} - \frac{x^2}{4a}\right\} dx$$

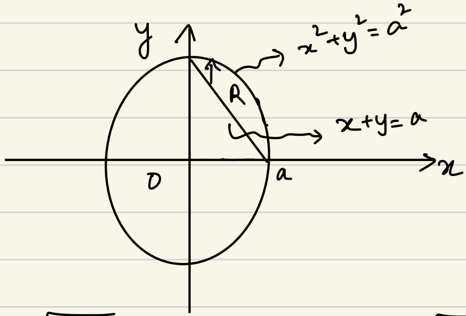
$$= 2a^{1/2}\, x^{3/2} \times \frac{2}{3} - \frac{1}{4a}\,\frac{x^3}{3}\Big]_{x=0}^{4a}$$

$$= \frac{4\sqrt{a}}{3}\, x^{3/2} - \frac{1}{12a}\, x^3\Big]_0^{4a} = \frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{1}{12a}(4a)^3$$

$$= \frac{32a^2}{3} - \frac{64a^3}{12a} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3} \text{ sq. units}$$

3) Find the area bounded by circle $x^2+y^2=a^2$ and line $x+y=a$ in the first quadrant

Sol:



$$A = \int_{x=0}^{a} \int_{y=a-x}^{\sqrt{a^2-x^2}} dy\, dx = \int_0^a y \Big]_{a-x}^{\sqrt{a^2-x^2}}$$

$$= \int_0^a \left\{ \sqrt{a^2-x^2} - (a-x) \right\} dx$$

$$= \left\{ \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) - ax + \frac{x^2}{2} \right\}_{x=0}^{a}$$

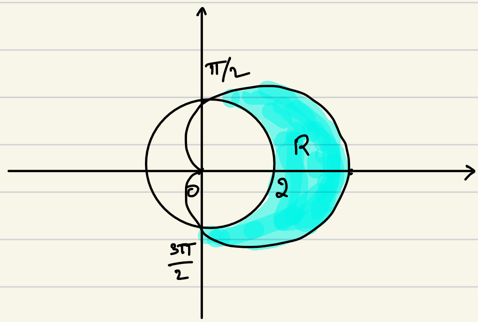
$$= \frac{x}{2}\sqrt{a^2-a^2} + \frac{a^2}{2}\sin^{-1}(1) - a^2 + \frac{a^2}{2} - \left(0 + \frac{a^2}{2}\cdot\sin^{-1}(0) - 0 + 0 \right)$$

$$= \frac{a^2}{2}\cdot\frac{\pi}{2} - a^2 + \frac{a^2}{2} - \frac{a^2}{2}\times(0) = \frac{\pi a^2}{4} - \frac{a^2}{2}$$

$$= \frac{a^2}{4}(\pi - 2) \text{ sq. units}$$

4) Find the area which is inside the cardioid $r = 2(1+\cos\theta)$ and outside the circle $r = 2$.

Sol:



To find point of intersection:

$$2(1+\cos\theta) = 2$$

$$\cancel{2} + 2\cos\theta = \cancel{2}$$

$$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$A = 2\int_{\theta=0}^{\pi/2}\int_{2}^{2(1+\cos\theta)} r\,dr\,d\theta = \cancel{2}\int_{\theta=0}^{\pi/2} \left\{\frac{r^2}{\cancel{2}}\right\}_{2}^{2(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi/2} \left\{4(1+\cos\theta)^2 - 4\right\} d\theta$$

$$= \int_0^{\pi/2} \left\{4\cos^2\theta + 8\cos\theta\right\} d\theta$$

$$= \left\{4\frac{\theta}{2} + 4\frac{\sin 2\theta}{4} + 8\sin\theta\right\}_0^{\pi/2}$$

$$= \left\{2\theta + \sin 2\theta + 8\sin\theta\right\}_0^{\pi/2}$$

$$= \left\{ 2 \times \frac{\pi}{2} + 8 \right\} = \pi + 8$$

Evaluation of triple integrals

$$\iiint_R f(x,y,z)\, dV = \lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V$$

Fubini's theorem for triple integrals:

$$\iiint_R f(x,y,z)\, dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\, dx\, dy\, dz$$

$$= \int_a^b \int_r^s \int_c^d f(x,y,z)\, dy\, dz\, dx$$

Note: 1) If $f(x,y,z) = 1$ then the triple integral gives the volume of region.

1. Evaluate $\iiint_B xyz^2\, dV$, where B is the rectangular box given by $B = \{(x,y,z) \mid 0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3\}$

Sol: $$\iiint_B xyz^2\, dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\, dx\, dy\, dz = \int_{z=0}^3 \int_{y=-1}^2 \left\{ \frac{x^2}{2} y z^2 \right\}_{x=0}^1 dy\, dz$$

$$= \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\, dy\, dz = \int_0^3 \left\{ \frac{y^2 z^2}{4} \right\}_{y=-1}^2 dz$$

$$= \frac{1}{4} \int_0^3 (4z^2 - z^2)\, dz = \frac{1}{4} \int_0^3 3z^2\, dz$$

$$= \frac{1}{4}\left\{z^3\right\}_0^3 = \frac{1}{4}\left\{27-0\right\} = \frac{27}{4}$$

2) $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3}\, dz\, dy\, dx$

Sol: $$I = \int_{x=0}^{1} \int_{y=0}^{1-x} \left\{ \int_{z=0}^{1-x-y} \frac{1}{(x+y+z+1)^3}\, dz \right\} dy\, dx$$

$$= \int_0^1 \int_0^1 \left[\frac{1}{-2(x+y+z+1)^2} \right]_{z=0}^{1-x-y} dy\, dx$$

$$= -\frac{1}{2} \int_0^1 \int_0^{1-x} \left[\frac{1}{4} - \frac{1}{(x+y+1)^2} \right] dy\, dx$$

$$= -\frac{1}{2} \int_0^1 \left\{ \frac{y}{4} + \frac{1}{x+y+1} \right\}_{y=0}^{1-x} dx$$

$$= \frac{-1}{2} \int_0^1 \left\{ \frac{1-x}{4} + \frac{1}{2} - \frac{1}{1+x} \right\} dx$$

$$= \frac{-1}{2} \left[-\frac{(1-x)^2}{8} + \frac{x}{2} - \log(1+x) \right]_0^1$$

$$= \frac{-1}{2}\left[\frac{1}{2} - \log 2 - \left(-\frac{1}{8}\right) \right] = \frac{-1}{2}\left[\frac{5}{8} - \log 2 \right]$$

$$= \frac{\log 2}{2} - \frac{5}{16}.$$

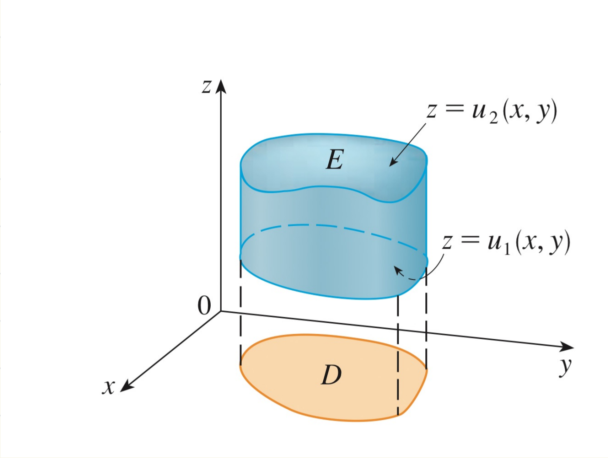
3) $\int_{r=0}^{a} \int_{0}^{\pi/2} \int_{0}^{\pi/2} r^2 \sin\theta \, d\phi \, d\theta \, dr$

$$I = \int_{r=0}^{a} r^2 \, dr \int_{\theta=0}^{\pi/2} \sin\theta \, d\theta \int_{\phi=0}^{\pi/2} d\phi$$

$$= \frac{r^3}{3}\Big|_0^a \times -\cos\theta\Big|_0^{\pi/2} \times \phi\Big|_0^{\pi/2}$$

$$= \frac{a^3}{3} \times -(0-1) \times \frac{\pi}{2} = \frac{\pi a^3}{6}.$$

Triple integral over a general bounded region E in 3 dimensional space:



Type 1 solid region

Section 15.3).

We restrict our attention to continuous functions f and to certain simple types of regions. A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

where D is the projection of E onto the xy-plane as shown in Figure 2. Notice that the upper boundary of the solid E is the surface with equation $z = u_2(x, y)$, while the lower boundary is the surface $z = u_1(x, y)$.

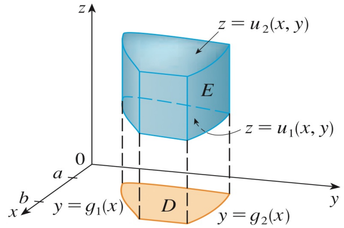


FIGURE 3
A type 1 solid region where the projection D is a type I plane region

$$\boxed{\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dy\, dx}$$

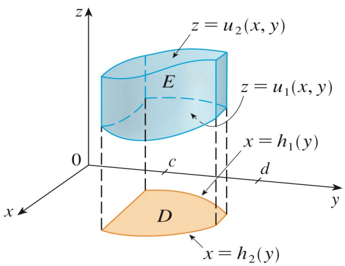


FIGURE 4
A type 1 solid region with a type II projection

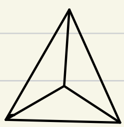
$$\boxed{\iiint_E f(x, y, z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dx\, dy}$$

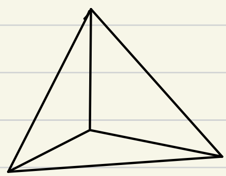
Volume of solids:

The volume V of the region R is given by

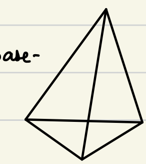
$$V = \iiint_R dx\,dy\,dz$$

1) Find volume of the tetrahedron $x \geq 0$, $y \geq 0$, $z \geq 0$, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1$





Pyramid on triangular base - tetrahedron



sol:

$$V = \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dz\,dy\,dx$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} \{z\}_0^{c(1-\frac{x}{a}-\frac{y}{b})} dy\,dx$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dy\,dx$$

$$= c\int_0^a \left\{\left(1-\frac{x}{a}\right)y - \frac{y^2}{2b}\right\}_{y=0}^{b(1-\frac{x}{a})} dx$$

$$= c\int_0^a \left\{ b\left(1-\frac{x}{a}\right)^2 - \frac{b\left(1-\frac{x}{a}\right)^2}{2}\right\} dx$$

$$= \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\int_0^a \left(1-\frac{2x}{a}+\frac{x^2}{a^2}\right) dx$$

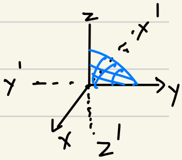
$$= \frac{bc}{2}\left[x - \frac{x^2}{a} + \frac{x^3}{3a^2}\right]_0^a = \frac{abc}{6}$$

2) Find the volume of sphere $x^2+y^2+z^2=a^2$

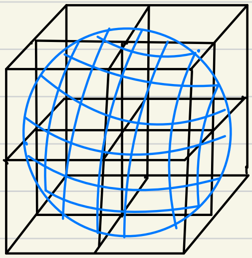
Sol. Using polar spherical coordinates

$$x = r\sin\theta\cos\phi \;,\; y = r\sin\theta\sin\phi,\quad z = r\cos\theta$$

$$\Rightarrow dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$$



Volume of sphere is 8 times the volume of its portion in positive octant for which r varies from 0 to a, θ varies from 0 to $\frac{\pi}{2}$ and ϕ varies from 0 to $\frac{\pi}{2}$



$$\therefore V = 8\int_0^a\int_0^{\pi/2}\int_0^{\pi/2} r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 8\int_0^a r^2\,dr \cdot \int_0^{\pi/2}\sin\theta\,d\theta \cdot \int_0^{\pi/2} d\phi = 8\cdot \frac{r^3}{3}\Big|_0^a \cdot (-\cos\theta)_0^{\pi/2} \cdot \frac{\pi}{2}$$

$$= \frac{4\pi a^3}{3}$$