Eigenvalues and eigenvectors

Let A be an $n \times n$ matrix. A scalar λ is said to be an eigenvalue or characteristic value or characteristic root of A if $\exists$ a non zero vector X such that $AX = \lambda X$. The vector X is said to be an eigenvector or a characteristic vector corresponding to λ.

Ex 1 : Consider permutation matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Eigenvector $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $Ax = x$, $\lambda = 1$

Other is $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $Ax = -x$, $\lambda = -1$

Properties of eigenvalues and eigenvectors:

1) Any square matrix A and its transpose A' have the same eigenvalues.

2) The sum of the eigenvalues of a matrix is equal to the trace of the matrix.

3) The product of the eigenvalues of a matrix A is equal to the determinant of A.

4) If $\lambda_1, \lambda_2, \dots \lambda_n$ are the eigenvalues of A, then the eigenvalues of

   a) KA are $K\lambda_1, K\lambda_2 \dots K\lambda_n$ b) A^m are $\lambda_1^m, \lambda_2^m, \dots, \lambda_n^m$

   c) A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots \frac{1}{\lambda_n}$

5) If all eigenvalues are positive then the determinant is positive. A matrix with negative determinant has atleast one negative eigenvalue.

6) If a matrix A is symmetric $(A = A^T)$ then its eigenvalues are real numbers.

7) If a matrix A is skew symmetric $(A = -A^T)$ then its eigenvalues are either 0 or purely imaginary

8) If A is a triangular matrix then its eigenvalues are diagonal entries of A.

9) Eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal.

10) If λ is eigenvalue of A, then $\lambda + k$ is eigenvalue of $A + kI$ and eigenvectors of A and $A+kI$ are same.

11) If sum of entries of all rows (columns) is a then a must be eigenvalue of the matrix and $X = (1, 1, \ldots 1)^T$ is corresponding eigenvector

12) Similar matrices have the same eigenvalues.
(Similar matrices: A and B are similar matrices if $\exists$ a non-singular matrix P such that $P^{-1}AP = B$)
However their eigenvectors are normally different.

ex:- $A = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -5 & -3 \\ 6 & 5 \end{bmatrix}$, $P = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

13) Eigenvalues of orthogonal matrix ($A^T = A^{-1}$) are -1 or 1

ex:- $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

14) Eigenvalues of idempotent matrix ($A^2 = A$) is either 0 or 1

ex:- $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$

15) Eigenvalues of involuntary matrices ($A^2 = I$) are either -1 or 1.

ex:- $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

16) 2 eigenvectors X_1 and X_2 are orthogonal vectors

if $X_1^T X_2 = 0$

Method to solve $AX = \lambda X$ for λ and X

Consider $AX = \lambda X$

Rewriting, we get

$$AX - \lambda X = 0$$

$$\Rightarrow [A - \lambda I] X = 0 \quad \rightarrow (1)$$

For non trivial solution of (1), $A - \lambda I$ must be singular i.e. $|A - \lambda I| = 0 \rightarrow (2)$

This is called characteristic equation. Solving we get λ, then find corresponding X from (1) by elimination method.

For instance

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Characteristic eq. $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$

$$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$$

$$\Rightarrow \lambda^2 - (a_{11}+a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$

$$\Rightarrow \lambda^2 - \text{trace}(A)\lambda + |A| = 0$$

For 3x3 matrix

The characteristic equation is

$$\lambda^3 - \text{trace}(A)\lambda^2 + (\text{sum of minor along diagonal})\lambda - |A| = 0$$

ex:- $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

Trace $(A) = 6$

Minors along the diagonal : $M_{11} = 6-0=6$, $M_{22} = 3-4 = -1$, $M_{33} = 2-0 = 2$

$|A| = 1(6-0) + 0 + 2(0-4) = 6-8 = -2$

The characteristic eq is $\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0$

Determine the eigenvalues and eigenvectors of the following matrices:

1) $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

The characteristic eq of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$$

$$\Rightarrow \lambda = 4, -1$$ are the eigenvalues of A

To find eigenvectors; For $\lambda_1 = 4$

Consider $(A - \lambda_1 I)X = 0$

$$\begin{bmatrix} 1-4 & 2 \\ 3 & 2-4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3x + 2y = 0$$

$$3x - 2y = 0$$

$$\Rightarrow 3x = 2y \quad \text{or} \quad x = \frac{2}{3}y$$

Thus $X_1 = \begin{bmatrix} 2/3\, K \\ K \end{bmatrix}$ is the eigenvector corresponding to $\lambda_1 = 4$.

For $\lambda_2 = -1$

Consider $(A - \lambda_2 I)X = 0$

$$\begin{bmatrix} 1+1 & 2 \\ 3 & 2+1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 2x + 2y = 0 \quad \Rightarrow \quad x = -y$$
$$3x + 3y = 0 \quad \Rightarrow \quad x = -y$$

Let $y = K \Rightarrow x = -K$ (K is a scalar, $K \neq 0$)

Thus $X_2 = \begin{bmatrix} -K \\ K \end{bmatrix}$ is the eigenvector corresponding to $\lambda_2 = -1$

2) $B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$

Sol:- The characteristic eq. is $|B - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 8\lambda + 15 = 0$$

$\Rightarrow \lambda = 5, 3$ are the eigenvalues of B

$15\lambda^2$

$-5\lambda \quad -3\lambda$

To find eigenvectors :

For $\lambda_1 = 5$

Consider $(B - \lambda_1 I)X = 0$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x + y = 0$$

Let $y = K \Rightarrow x = K$ (K is a scalar, $K \neq 0$)

Thus $X_1 = \begin{bmatrix} K \\ K \end{bmatrix}$ is the eigenvector corresponding to $\lambda_1 = 5$

For $\lambda_2 = 3$

Consider $(B - \lambda_2 I)X = 0$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x + y = 0$

Let $y = K \Rightarrow x = -K$

Thus $X_2 = \begin{bmatrix} -K \\ K \end{bmatrix}$ is the eigenvector corresponding to $\lambda_2 = 3$

3) $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$

Sol: The characteristic eq is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix} = 0$$

$|A| = 2(-4+3) + 3(2-1) + 1(-3+2) = -2+3-1 = 0$

$M_{11} = -1$

$M_{22} = 3$

$M_{33} = -1$

$\lambda^3 - 2\lambda^2 + \lambda + 0 = 0$

$\Rightarrow \lambda(\lambda^2 - 2\lambda + 1) = 0$

$\Rightarrow \lambda(\lambda - 1)^2 = 0$

$\therefore \lambda = 0, 1, 1$ are the eigenvalues.

To find eigenvectors:

For $\lambda_1 = 0$

Consider $(A - \lambda_1 I) X = 0$

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider $\begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 1 \\ 1 & -3 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x - 2y + z = 0$$

$$y - z = 0$$

Let $z = K$ be the free variable

$$\Rightarrow y = K$$

$$\therefore \; x = K$$

$\therefore$ Eigenvector corresponding to $\lambda_1 = 0$ is $X_1 = \begin{bmatrix} K \\ K \\ K \end{bmatrix}$

For $\lambda_2 = 1$

Consider $(A - \lambda_2 I) X = 0$

$$\begin{bmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x - 3y + z = 0$$

Let $y = K_1$ and $z = K_2$

$$\Rightarrow x = 3K_1 - K_2$$

$\therefore$ Eigenvector corresponding to $\lambda = 1$ is

$$X_2 = \begin{bmatrix} 3K_1 - K_2 \\ K_1 \\ K_2 \end{bmatrix} = K_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + K_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

4) $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

sol:- The characteristic eq is $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

The eigenvalues of A are $2, 3, 5$

To find eigenvectors:

For $\lambda_1 = 2$

$$(A - \lambda_1 I) X = 0$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider $\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix}$

$$R_3 \rightarrow R_3 - \frac{1}{2} R_2$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x+y+4z=0, \quad 6z=0 \Rightarrow z=0$$

$$x+y=0$$

Let $y=K \Rightarrow x=-K$

Thus $X_1 = \begin{bmatrix} -K \\ K \\ 0 \end{bmatrix}$ is the eigenvector corresponding to $\lambda=2$

For $\lambda_2 = 3$

$$(A-\lambda_2 I)X = 0$$

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y+4z=0, \quad -y+6z=0, \quad 2z=0$$

i.e. $y=0, \ z=0$

Let $x=K$

Thus $X_2 = \begin{bmatrix} K \\ 0 \\ 0 \end{bmatrix}$ is the eigenvector corresponding to $\lambda=3$

For $\lambda=5$

$$(A-\lambda_3 I)X = 0$$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x+y+4z=0$, $-3y+6z=0 \Rightarrow y=2z$

$\Rightarrow -2x+6z=0$

Let $z=k$, $y=2k$, $x=3k$

Thus $X_3 = \begin{bmatrix} 3k \\ 2k \\ k \end{bmatrix}$ is the eigenvector corresponding to $\lambda=5$

5) If 3 and 15 are the 2 eigenvalues of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

find $|A|$, without expanding the determinant.

Sol: Let $\lambda_1 = 3$, $\lambda_2 = 15$

WKT sum of eigenvalues = Trace of matrix

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 8+7+3$

$18 + \lambda_3 = 18$

$\therefore \lambda_3 = 0$

Product of eigenvalues = $|A|$

$\Rightarrow (3)(15)(0) = |A|$

$\therefore |A| = 0$

6) 2 of the eigenvalues of $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are 3 and 6.

Find the eigenvalues of A^{-1}.

Sol:- $\lambda_1 + \lambda_2 + \lambda_3 = 3+5+3 = 11$

$9 + \lambda_3 = 11$ $\therefore \lambda_3 = 2$

The eigenvalues of A^{-1} are $\frac{1}{3}, \frac{1}{6}, \frac{1}{2}$.

Obtain the eigenvalues and eigenvectors of the following matrices:

1) $A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$

Ans:- $\lambda = 3, 3$

$X = \begin{bmatrix} -K \\ K \end{bmatrix}$

2) $A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$

Ans:- $\lambda_1 = 8, \lambda_2 = -2$

$X_1 = \begin{bmatrix} 3K \\ K \end{bmatrix}, \quad X_2 = \begin{bmatrix} K \\ -3K \end{bmatrix}$

3) $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Ans: $\lambda_1 = -2, \lambda_2 = 3, \lambda_3 = 6$

$X_1 = \begin{bmatrix} -K \\ 0 \\ K \end{bmatrix}, \quad X_2 = \begin{bmatrix} K \\ -K \\ K \end{bmatrix}, \quad X_3 = \begin{bmatrix} K \\ 2K \\ K \end{bmatrix}$

4) $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

Ans: $\lambda_1 = 1, \lambda_2 = \sqrt{5}, \lambda_3 = -\sqrt{5}$

$X_1 = \begin{bmatrix} K \\ 0 \\ -K \end{bmatrix}, \quad X_2 = \begin{bmatrix} (1-\sqrt{5})K \\ -K \\ K \end{bmatrix}, \quad X_3 = \begin{bmatrix} (1+\sqrt{5})K \\ -K \\ K \end{bmatrix}$

5) Two eigenvalues of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal to 1 each. Find eigenvalues of A^{-1}.

Rayleigh's Power method to find the largest eigenvalue:

Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be the eigenvalues of a $n \times n$ matrix A. λ_1 is called the dominant eigenvalue of A if it is larger in absolute value than all other eigenvalues.

i.e. $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \ldots \geq |\lambda_n|$

The eigenvector corresponding to λ_1 is called dominant eigenvector of A.

Let x_0 be the initial approximation such that $Ax_0 = x \rightarrow$ (1). In order to get a convergent sequence of eigenvectors simultaneously scaling method is adopted.

At each stage, each components of the resultant approximate vector is to be divided by its absolutely largest component.

Accordingly x in (1) can be scaled by dividing each of its components by absolutely largest component of it. Thus

$$Ax_0 = x = \lambda_1 x_1 \; ; \; x_1 \text{ is the scaled vector of } x.$$

Now the scaled vector x_1 is used in the next iteration to obtain $Ax_1 = x = \lambda_2 x_2$. Proceeding this way we get $Ax_n = \lambda_{n+1} x_{n+1}$; $n = 0, 1, 2 \ldots$ where λ_{n+1} is the numerically largest eigenvalue upto desired accuracy and x_{n+1} is the corresponding eigenvector.

Apply power method to $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ with initial $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and estimate the dominant eigenvalue and corresponding eigenvector of A.

Sol:

$$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Ax_0 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = c_1 x_1$$

$$Ax_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 2 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ 0.8 \end{bmatrix} = c_2 x_2$$

$$Ax_2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 2.6 \\ 2.8 \end{bmatrix} = 2.8 \begin{bmatrix} 0.93 \\ 1 \end{bmatrix} = c_3 x_3$$

$$Ax_3 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0.93 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.93 \\ 2.86 \end{bmatrix} = 2.93 \begin{bmatrix} 1 \\ 0.98 \end{bmatrix} = c_4 x_4$$

$$Ax_4 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.98 \end{bmatrix} = \begin{bmatrix} 2.96 \\ 2.98 \end{bmatrix} = 2.98 \begin{bmatrix} 0.99 \\ 1 \end{bmatrix} = c_5 x_5$$

$$Ax_5 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0.99 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.99 \\ 2.98 \end{bmatrix} = 2.99 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_6 x_6$$

$$Ax_6 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_7 x_7$$

Since $x_6 = x_7$, $\lambda = 3$ is the dominant eigenvalue and

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the dominant eigenvector.

2) Perform six iterations of the power method with scaling to approximate a dominant eigenvector of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$$

Use $x_0 = (1, 1, 1)$ as the initial approximation.

Sol:-

1st iteration: $Ax_0 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 0.60 \\ 0.20 \\ 1 \end{bmatrix} = c_1 x_1$

2nd iteration: $Ax_1 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.60 \\ 0.20 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2.20 \end{bmatrix} = 2.20 \begin{bmatrix} 0.45 \\ 0.45 \\ 1 \end{bmatrix} = c_2 x_2$

3rd iteration: $Ax_2 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.45 \\ 0.45 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.35 \\ 1.55 \\ 2.8 \end{bmatrix} = 2.8 \begin{bmatrix} 0.48 \\ 0.55 \\ 1 \end{bmatrix} = c_3 x_3$

4th iteration: $Ax_3 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.48 \\ 0.55 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.58 \\ 1.59 \\ 3.13 \end{bmatrix} = 3.13 \begin{bmatrix} 0.50 \\ 0.51 \\ 1 \end{bmatrix} = c_4 x_4$

5th iteration: $Ax_4 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.50 \\ 0.51 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.52 \\ 1.51 \\ 3.03 \end{bmatrix} = 3.03 \begin{bmatrix} 0.50 \\ 0.50 \\ 1 \end{bmatrix} = c_5 x_5$

6^{th} iteration: $Ax_5 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.50 \\ 0.50 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.5 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 0.5 \\ 0.5 \\ 1 \end{bmatrix} = c_6 x_6$

Since $x_5 = x_6$, $\lambda = 3$ is the dominant eigenvalue and

$\begin{bmatrix} 0.5 \\ 0.5 \\ 1 \end{bmatrix}$ is the dominant eigenvector.

3) $\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}$

sol:- det $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$Ax_0 = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = c_1 x_1$$

$$Ax_1 = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 6 \end{bmatrix} = 8 \begin{bmatrix} -1 \\ 0.75 \end{bmatrix} = c_2 x_2$$

$$Ax_2 = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 0.75 \end{bmatrix} = \begin{bmatrix} -6.75 \\ 5 \end{bmatrix} = 6.75 \begin{bmatrix} -1 \\ 0.741 \end{bmatrix} = c_3 x_3$$

$$Ax_3 = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 0.741 \end{bmatrix} = \begin{bmatrix} -6.705 \\ 4.964 \end{bmatrix} = 6.705 \begin{bmatrix} -1 \\ 0.7403 \end{bmatrix} = c_4 x_4$$

$$Ax_4 = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 0.7403 \end{bmatrix} = \begin{bmatrix} -6.7015 \\ 4.9612 \end{bmatrix} = 6.7015 \begin{bmatrix} -1 \\ 0.7403 \end{bmatrix} = c_5 x_5$$

Since $x_4 = x_5$, $\lambda = 6.701$ is the dominant eigenvalue and

$\begin{bmatrix} -1 \\ 0.7403 \end{bmatrix}$ is the dominant eigenvector.

Exercise:

1) Apply power method to foll matrices with initial x_0 and estimate the dominant eigenvalue and corresponding eigenvector of A.

a) $A = \begin{bmatrix} 5 & -2 & -2 \\ -3 & 5 & 0 \\ 23 & -19 & -6 \end{bmatrix}$, $x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Ans: $\lambda = 2$, $X = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

b) $A = \begin{bmatrix} -7 & 2 \\ 8 & -1 \end{bmatrix}$, $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Ans: $\lambda = -9$, $X = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

2) Perform six iterations of the power method with scaling to approximate a dominant eigenvector of the matrix. Use $x_0 = (1, 1, 1)$ as the initial approximation.

a) $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Ans: $\lambda = 14.9992 \simeq 15$,

$X = \begin{bmatrix} 1 \\ -0.9999 \\ 0.4999 \end{bmatrix} \approx \begin{bmatrix} 1 \\ -1 \\ 0.5 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$

Ans: $\lambda = 3.4593$

$X = \begin{bmatrix} 1 \\ 0 \\ 0.8157 \end{bmatrix}$