Exercise:

1) Find the total derivative of u wrt 't' when $u = e^x \sin y$, where $x = \log t$, $y = t^2$ Ans: $\frac{du}{dt} = \sin t^2 + 2t^2 \cdot \cos t^2$

2) If $u = e^x \sin(yz)$, where $x = t^2$, $y = t-1$, $z = \frac{1}{t}$, find $\frac{du}{dt}$ at $t = 1$. Ans: $\frac{du}{dt} = e$

3) Find $\frac{du}{dt}$ if $u = x^2 + y^2 + z^2$, $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$ Ans: $\frac{du}{dt} = 4e^{2t}$

4) If $z = f(u, v)$, where $u = x^2 - y^2$, $v = 2xy$, P.T.

a) $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2(x^2+y^2)\frac{\partial z}{\partial u}$

b) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4(x^2+y^2)\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$

5) If $u = e^x \sin y$, $v = e^x \cos y$ and $w = f(u, v)$, P.T. $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = (u^2+v^2)\left(\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2}\right)$

6) If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, P.T. $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

7) If $u = f(r, s, t)$ and $r = \frac{x}{y}$, $s = \frac{y}{z}$ and $t = \frac{z}{x}$, S.T. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

8) If $u = x + \frac{y^2}{x}$, $v = \frac{y^2}{x}$, find $\frac{\partial(u, v)}{\partial(x, y)}$ Ans: $\frac{2y}{x}$

9) If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, find $\frac{\partial(x, y, z)}{\partial(r, \theta, z)}$

((r, θ, z) defined above are cylindrical polar coordinates of the point (x, y, z) in space) Ans: r

10) If $u = xyz$, $v = xy + yz + zx$, $w = x + y + z$, S.T.

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (x-y)(y-z)(z-x)$$

11) If $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$, S.T. $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 4$

12) If $u = z - x$, $v = y - z$, $w = x + y + z$, find Jacobians

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} \quad \text{and} \quad J' = \frac{\partial(x, y, z)}{\partial(u, v, w)}.$$ Also verify $JJ' = 1$.

13) Determine the extreme values of following functions $f(x, y)$:

a) $2xy - 5x^2 - 2y^2 + 4x + 4y - 6$ Ans: $\left(\frac{2}{3}, \frac{4}{3}\right)$ is point of maximum and $f\left(\frac{2}{3}, \frac{4}{3}\right) = -2$

b) $x^3 y^2 (1 - x - y)$ Ans: $(0, 0)$ and $\left(\frac{1}{2}, \frac{1}{3}\right)$ are critical points, $\left(\frac{1}{2}, \frac{1}{3}\right)$ is point of maximum and $f\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{432}$

c) $x^3 + y^3 - 3xy$ Ans: $(0, 0)$ and $(1, 1)$ are critical points, $(0, 0)$ is a saddle point, $(1, 1)$ is a point of minimum and $f(1, 1) = -1$

d) $x^4 + 2x^2 y - x^2 + 3y^2$ Ans: $(0, 0)$, $\left(\frac{\sqrt{3}}{2}, \frac{-1}{4}\right)$, $\left(-\frac{\sqrt{3}}{2}, \frac{-1}{4}\right)$ are critical points, $(0, 0) \rightarrow$ saddle point, $\left(\frac{\sqrt{3}}{2}, -\frac{1}{4}\right) \rightarrow$ minimum point,

$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{4}\right) \rightarrow$ minimum point, $f\left(\frac{\sqrt{3}}{2}, -\frac{1}{4}\right) = f\left(-\frac{\sqrt{3}}{2}, -\frac{1}{4}\right) = -\frac{3}{8}$

e) $2x^3 + x^2y + 5x^2 + y^2$ Ans: min. value $= 0$ at $(0,0)$, max. value $= \frac{125}{7}$ at $\left(-\frac{5}{3}, 0\right)$ and $(-1, 2), (-1, -2) \rightarrow$ saddle points

f) $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ Ans: max. value $= 112$ at $(4, 0)$

14) A rectangular box without a lid is to be made from $12\,m^2$ cardbox. Find the maximum value of such a box using Lagrange's method of multipliers

Ans: $V(2, 2, 1) = 4\,m^3$

15) Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$ Ans: Max. value is $f(0, \pm 1) = 2$, Min. value $f(\pm 1, 0) = 1$.

16) Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$

Ans: Closest point is $\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right)$, Farthest point is $\left(\frac{-6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$

17) A rectangular box open at the top is to have volume of 32 cubic feet. Find dimensions of box requiring least material for construction.

Ans: Minimum surface area, represents least material required for construction and dimensions are $x = y = 4$ ft, $z = 2$ ft

Maximum and minimum values (Extreme values) of function.

1) Show that $z = xy(a-x-y)$, $a>0$ is maximum at the point $\left(\frac{a}{3}, \frac{a}{3}\right)$

sol:- Let $z = f(x,y) = y(ax - x^2 - xy) = axy - x^2y - xy^2$

$p = f_x = y(a-2x-y)$, $q = f_y = x(a-x-2y)$

$r = f_{xx} = -2y$, $s = f_{xy} = a-2x-2y$, $t = f_{yy} = -2x$

We shall find points (x,y) such that $f_x = 0$ and $f_y = 0$

i.e. $y(a-2x-y) = 0$ and $x(a-x-2y) = 0$

1) $y=0$, $x=0 \Rightarrow (0,0)$
2) $y=0$, $a-x-2y=0 \Rightarrow (a,0)$
3) $a-2x-y=0$, $x=0 \Rightarrow (0,a)$
4) $a-2x-y=0$, $a-x-2y=0 \Rightarrow \left(\frac{a}{3}, \frac{a}{3}\right)$

$2x + y = a \rightarrow ①$
$x + 2y = a \quad \times 2$
$2x + 4y = 2a \rightarrow ②$
① − ②
$-3y = -a$
$\therefore y = \frac{a}{3}$

So $(0,0)$, $(a,0)$, $(0,a)$, $\left(\frac{a}{3}, \frac{a}{3}\right)$ are critical points. We shall examine these points for maxima and minima

	$(0,0)$	$(a,0)$	$(0,a)$	$\left(\frac{a}{3}, \frac{a}{3}\right)$
$r=-2y$	0	0	$-2a<0$	$-2a/3 < 0$
$t=-2x$	0	$-2a$	0	$-2a/3$
$s=a-2x-2y$	a	$-a$	$-a$	$-a/3$
$rt-s^2$	$-a^2<0$	$-a^2<0$	$-a^2<0$	$a^2/3>0$
Conclusion	Saddle point	Saddle pt.	Saddle pt.	Max. point

$\left(\frac{a}{3}, \frac{a}{3}\right)$ is a point of maximum since $r<0$ and $rt-s^2>0$.

Thus $f(x,y)$ is maximum at the point $\left(\frac{a}{3}, \frac{a}{3}\right)$

The maximum value of $f(x,y)$ is $\frac{1}{27}a^3$.

2) Determine extreme values of function

$$f(x,y) = x^3 + y^3 - 63(x+y) + 12xy$$

Sol:- $p = f_x = 3x^2 - 63 + 12y$, $q = f_y = 3y^2 - 63 + 12x$

$r = f_{xx} = 6x$, $s = f_{xy} = 12$, $t = f_{yy} = 6y$

consider $f_x = 0$, $f_y = 0$

$$x^2 - 21 + 4y = 0, \quad y^2 - 21 + 4x = 0$$

on subtracting,

$$(x^2 - y^2) + 4(y - x) = 0 \Rightarrow (x+y)(x-y) - 4(x-y) = 0$$

$$\Rightarrow (x-y)(x+y-4) = 0$$

$$\Rightarrow x = y, \quad x + y = 4$$

Putting $x = y$ in $x^2 - 21 + 4y = 0 \Rightarrow y^2 - 21 + 4y = 0$

$$\Rightarrow (y+7)(y-3) = 0 \Rightarrow y = -7, 3$$

Since $x = y$, $(-7,-7)$ and $(3,3)$ are stationary points

Put $x = 4 - y$ in $x^2 - 21 + 4y = 0 \Rightarrow (4-y)^2 - 21 + 4y = 0$

$$\Rightarrow y^2 - 4y - 5 = 0 \Rightarrow (y-5)(y+1) = 0 \Rightarrow y = 5, -1$$

Since $x = 4 - y \Rightarrow (-1, 5)$ and $(5,-1)$ are stationary points

Let us examine these 4 points for maxima & minima

	$(-7,-7)$	$(3,3)$	$(-1,5)$	$(5,-1)$
$r = 6x$	$-42 < 0$	$18 > 0$	$-6 < 0$	$30 > 0$
$t = 6y$	-42	18	30	-6
$s = 12$	12	12	12	12
$rt - s^2$	$1620 > 0$	$180 > 0$	$-324 < 0$	$-324 < 0$
Conclusion	Max. point	Min. point	Saddle point	Saddle point

$(-7, -7)$ is a point of maximum
$(3, 3)$ is a point of minimum
The maximum value of $f(x, y) = f(-7, -7) = 784$
The minimum value of $f(x, y) = f(3, 3) = -216$

3) Examine the function $f(x, y) = x^4 + y^4 - 2(x - y)^2$ for extreme values

Sol: $p = f_x = 4x^3 - 4(x - y)$, $q = f_y = 4y^3 + 4(x - y)$

$r = f_{xx} = 12x^2 - 4$, $s = f_{xy} = 4$, $t = f_{yy} = 12y^2 - 4$

Consider $f_x = 0$, $f_y = 0$ $\Rightarrow$ $\underbrace{x^3 - (x - y) = 0}_{①}$, $\underbrace{y^3 + (x - y) = 0}_{②}$

Adding we get $x^3 + y^3 = 0 \Rightarrow (x + y)(x^2 + y^2 - xy) = 0$
$\Rightarrow y = -x$; $x^2 + y^2 - xy = 0$
Put $y = -x$ in ① $x^3 - 2x = 0 \Rightarrow x(x^2 - 2) = 0$
$\Rightarrow x = 0, \pm\sqrt{2}$
Since $y = -x \Rightarrow y = 0, \mp\sqrt{2}$
Hence $(0, 0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$ are set of stationary points.

Also $x^2 + y^2 - xy = 0 \Rightarrow x(x - y) + y^2 = 0 \Rightarrow x^4 + y^2 = 0$
which clearly shows that we do not have any real value satisfying the above eq.

	$(0,0)$	$(\sqrt{2},-\sqrt{2})$	$(-\sqrt{2},\sqrt{2})$
$r=12x^2-4$	$-4<0$	$20>0$	$20>0$
$t=12y^2-4$	-4	20	20
$s=4$	4	4	4
$rt-s^2$	0	$384>0$	$384>0$
Conclusion	—	Min. point	Min. point

$(\sqrt{2},-\sqrt{2})$ and $(-\sqrt{2},\sqrt{2})$ are points of minima

The minimum value of $f(x,y) = f(\sqrt{2},-\sqrt{2}) = f(-\sqrt{2},\sqrt{2}) = -8$

4) Determine the maxima/minima of the functions

$\sin x + \sin y + \sin(x+y)$

Sol: Let $f(x,y) = \sin x + \sin y + \sin(x+y)$

$p = f_x = \cos x + \cos(x+y)$, $r = f_{xx} = -\sin x - \sin(x+y)$, $s = f_{xy} = -\sin(x+y)$

$q = f_y = \cos y + \cos(x+y)$, $t = f_{yy} = -\sin y - \sin(x+y)$

Consider $f_x = 0$ and $f_y = 0$

$\cos(x+y) = -\cos x$ and $\cos(x+y) = -\cos y$

$-\cos y = -\cos x \Rightarrow \cos x = \cos y$

$\Rightarrow x = y$ or $x = -y$

If $x = y \Rightarrow \cos 2x = -\cos x \Rightarrow \cos 2x = \cos(\pi + x)$ or $\cos 2x = \cos(\pi - x)$

$\Rightarrow x = \pi, \pi/3$

$\therefore (\pi,\pi), \left(\frac{\pi}{3},\frac{\pi}{3}\right)$ are critical points

If $x = -y \Rightarrow \cos 0 = -\cos x \Rightarrow \cos x = -1 \Rightarrow x = \pi$

$\therefore (\pi,-\pi)$ is critical point.

$\sin(120^\circ) = \sin(180-60)$
$= \sin 60 = \frac{\sqrt{3}}{2}$

At $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$, $r = f_{xx} = -\sin\frac{\pi}{3} - \sin\frac{2\pi}{3} = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$

$t = f_{yy} = -\sin\frac{\pi}{3} - \sin\frac{2\pi}{3} = -\sqrt{3}$, $s = -\sin\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(\frac{\sqrt{3}}{2}\right)^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$

At (π, π) and $(\pi, -\pi)$ $\Rightarrow$ $rt - s^2 = 0 \Rightarrow$ Inconclusive.

Also $r = -\sqrt{3} < 0$

$\therefore \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is a point of maximum.

The maximum value of $f(x,y) = f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

5) A rectangle box, open at the top to have a volume of 32 cubic units, what must be the dimension so that total surface area of box is a minimum.

Sol: Let x, y and z be the length, breadth and height of box. Given: $V = xyz$

Surface area$(S) = xy + 2yz + 2zx$

$V = 32 \Rightarrow xyz = 32 \Rightarrow z = \frac{32}{xy}$

$$S = xy + 2y\frac{32}{xy} + 2x\frac{32}{xy} = xy + \frac{64}{x} + \frac{64}{y} = f(x,y)$$

$f_x = y - \frac{64}{x^2}$, $f_y = x - \frac{64}{y^2}$, $r = f_{xx} = \frac{128}{x^3}$, $s = f_{xy} = 1$, $t = f_{yy} = \frac{128}{y^3}$

Consider $f_x = 0$, $f_y = 0 \Rightarrow y - \frac{64}{x^2} = 0$, $x - \frac{64}{y^2} = 0$
(1) (2)

(1) $\Rightarrow y = \frac{64}{x^2}$ in (2)

$$x - \frac{\cancel{64}}{\frac{\cancel{64}}{x^2}\cdot\frac{64}{x^2}} = 0 \Rightarrow x - \frac{x^4}{64} = 0$$

$$\Rightarrow 64x - x^4 = 0 \Rightarrow x(64 - x^3) = 0$$

$$\therefore x^3 = 64 \Rightarrow x = 4$$

(1) $\Rightarrow y = \frac{64}{16} = 4$

$\therefore (4, 4)$ is critical point