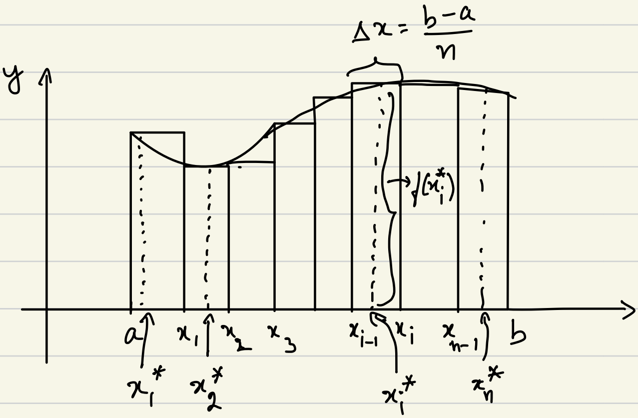
Multiple integrals

Review of definite integral



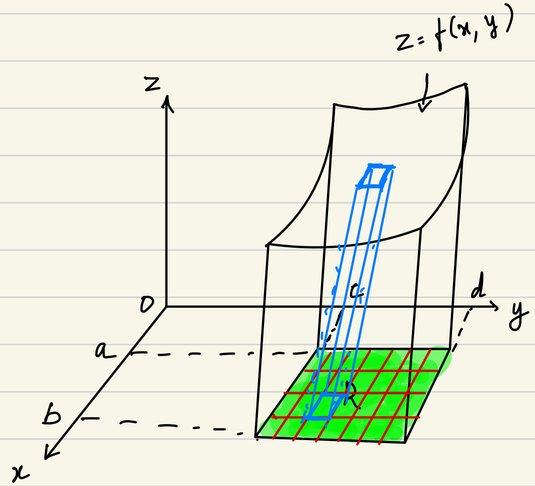
Let $f(x)$ be defined in $[a,b]$. We divide the interval $[a,b]$ into n subintervals of equal width, $\Delta x = \frac{b-a}{n}$. Pick a point out of each subinterval.

Measure the height of curve at that point and draw the rectangle whose width is Δx and whose height is function value at that point

$$\int_a^b f(x)\,dx = \underbrace{\lim_{n\to\infty}}_{\text{approximate area so we use limit}} \Big[\overbrace{\sum_{i=1}^{n} f(x_i^*)\,\Delta x}^{\text{Riemann sum}} \Big]$$

$\int_a^b f(x)\,dx \approx$ sum of areas of approximating rectangles as no of rectangles go to ∞

Double integrals over rectangles



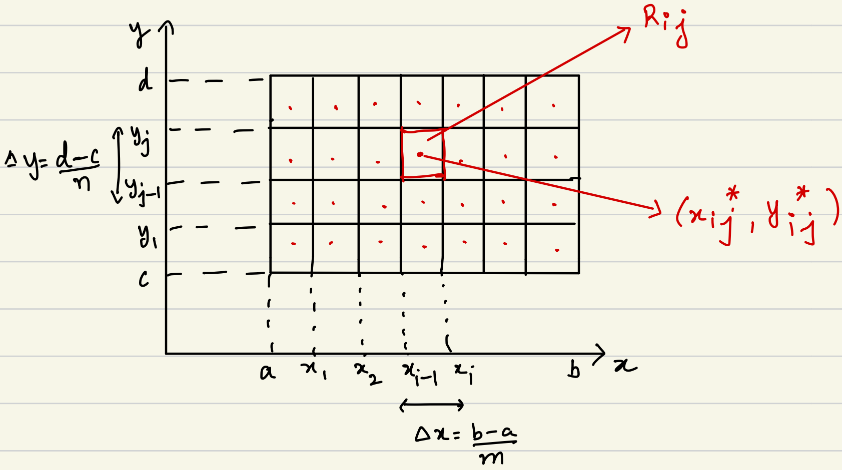
Consider a function of 2 variables defined on closed rectangle.

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \,/\, a \leq x \leq b,\ c \leq y \leq d\}$$

Let S be the solid that lies above R and under the graph f. Our goal is to find volume of S.

We divide the rectangle R into subrectangles. We accomplish this by dividing the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{m}$ and dividing $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width

$$\Delta y = \frac{d-c}{n}$$



Pick a point as a sample point. Choose a sample point (x_{ij}^*, y_{ij}^*) in each subrectangle R_{ij}. Plug this value into the function i.e. height of the surface

If we choose a sample point then we can approximate S by a thin rectangular box whose volume is

$$V = f(x_{ij}^*, y_{ij}^*)\, \Delta A = f(x_{ij}^*, y_{ij}^*) \underbrace{\Delta x\, \Delta y}_{} \rightarrow \text{area of rectangle}$$

To find volume underneath whole surface, we add up all of these surfaces in x and y directions

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

Then we take limit

$$\iint_R f(x,y)\, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

Evaluating double integrals using limits of Riemann sums is tedious and rarely done.

We have more practical method that reduces double integral to 2 single (one-variable integral)

Hence

$$V = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy \rightarrow \text{Fubini's theorem}$$

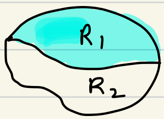
Properties of double integrals:

1) $\iint_R K f(x,y)\,dA = K \iint_R f(x,y)\,dA$, where K is a constant

2) $\iint_R [f(x,y) \pm g(x,y)]\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$

3) $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$

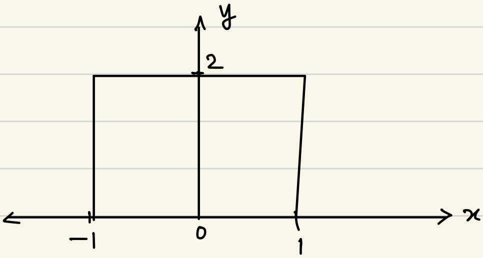
where R_1 and R_2 are subregions of R that do not overlap and $R = R_1 \cup R_2$.



$R = R_1 \cup R_2$

1) Find the volume of solid bounded by the surface $z = 4 + 9x^2y^2$ over the rectangular region

$R = \{ (x,y) : -1 \leq x \leq 1 , 0 \leq y \leq 2 \}$



$$V = \int_{x=-1}^{1}\int_{y=0}^{2} (4+9x^2y^2)\,dy\,dx = \int_{y=0}^{2}\int_{x=-1}^{1} (4+9x^2y^2)\,dx\,dy$$

$$V = \int_{-1}^{1} \left\{ \int_{0}^{2} (4+9x^2y^2)\,dy \right\} dx$$

$$= \int_{-1}^{1} \left\{ 4y + 9x^2 \frac{y^3}{3} \right\}_0^2 dx = \int_{-1}^{1} \left\{ 8 + 24x^2 \right\} dx$$

$$= \left\{ 8x + 24\frac{x^3}{3} \right\}_{-1}^{1} = 8(1+1) + 8(1+1)$$

$$= 16 + 16 = 32 \text{ cubic units.}$$

2) Evaluate the double integral $\iint_R (x-3y^2)\,dA$ where

$$R = \{ (x,y) \mid 0 \le x \le 1,\ 1 \le y \le 2 \}$$

Sol:

$$\iint_R (x-3y^2)\,dA = \int_1^2\int_0^1 (x-3y^2)\,dx\,dy$$

$$= \int_1^2 \left\{ \frac{x^2}{2} - 3y^2x \right\}_0^1 dy$$

$$= \int_1^2 \left\{ \frac{1}{2} - 3y^2 \right\} dy = \left\{ \frac{1}{2}y - y^3 \right\}_1^2$$

$$= \left\{ \frac{1}{2}(2-1) - (8-1) \right\} = \left\{ \frac{1}{2} - 7 \right\}$$

$$= -\frac{13}{2} \text{ cubic units}$$

or $\displaystyle\int_0^1\int_1^2 (x-3y^2)\,dy\,dx = \int_0^1 \left\{ xy - y^3 \right\}_1^2 dx = \int_0^1 \left\{ x(2-1) - (8-1) \right\} dx$

$$= \int_0^1 (x-7)\,dx = \left\{ \frac{x^2}{2} - 7x \right\}_0^1 = \frac{1}{2} - 7 = -\frac{13}{2}$$

3) Evaluate $\iint_R y\sin(xy)\,dA$ where $R=[1,2]\times[0,\pi]$

Sol:
$$\iint_R y\sin(xy)\,dA = \int_{y=0}^{\pi}\int_{x=1}^{2} y\sin(xy)\,dx\,dy$$
$$= \int_0^{\pi}\left\{-\cancel{y}\frac{\cos(xy)}{\cancel{y}}\right\}_1^2 dy$$
$$= \int_0^{\pi}\left[-\cos(2y)+\cos y\right]dy = \left[-\frac{\sin(2y)}{2}+\sin y\right]_0^{\pi}$$
$$= 0$$

4) Evaluate $\iint_R ye^{xy}\,dA$ where $R=\{(x,y): 0\le x\le 1,$
$0\le y\le \ln 2\}$

Sol:
$$\iint_R ye^{xy}\,dA = \int_0^{\ln 2}\int_0^1 ye^{xy}\,dx\,dy$$
$$= \int_0^{\ln 2}\left\{\frac{\cancel{y}e^{xy}}{\cancel{y}}\right\}_0^1 dy$$
$$= \int_0^{\ln 2}\left\{e^y-1\right\}dy$$
$$= \left(e^y-y\right)_0^{\ln 2} = \left(e^{\ln 2}-e^0\right)-(\ln 2-0)$$
$$= 2-1-\ln 2 = 1-\ln 2.$$

$$= 1 - \ln 2.$$

5) Evaluate $\iint_R e^y \cos x \, dA$ over the region

$$R = \left\{ (x,y) \,\middle|\, 0 \le x \le \frac{\pi}{2} \,,\ 0 \le y \le 1 \right\}$$

Sol:- $$\iint_R e^y \cos x \, dA = \int_0^1 \int_0^{\pi/2} e^y \cos x \, dx \, dy$$

$$= \int_0^1 \left\{ e^y \sin x \right\}_0^{\pi/2} dy = \int_0^1 \left\{ e^y \sin \frac{\pi}{2} \right\} dy$$

$$= \int_0^1 e^y \, dy = e^y \Big|_0^1 = e - 1$$

or

$$\int_0^{\pi/2} \int_0^1 e^y \cos x \, dy \, dx = \int_0^{\pi/2} \left\{ e^y \cos x \right\}_0^1 dx$$

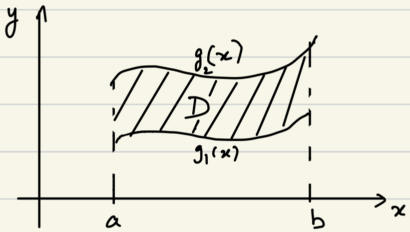
$$= \int_0^{\pi/2} (e \cos x - \cos x) \, dx$$

$$= \int_0^{\pi/2} \cos x \, (e-1) \, dx = (e-1) \left\{ \sin x \right\}_0^{\pi/2}$$

$$= e - 1$$

Double integrals over general region (regions which are not rectangular)

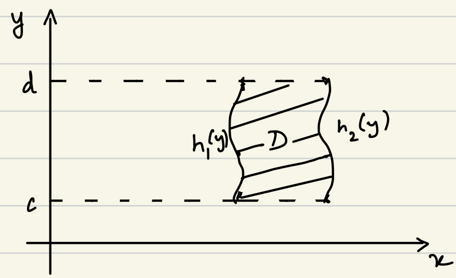
Type-1 region: If the region is bounded above and below by a function of x, we integrate wrt y first with limits of integration wrt y (i.e. $g_1(x)$ and $g_2(x)$)



$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

$$\iint_D f(x,y)\, dA = \int_a^b \int_{y=g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

2) Type-2 region: If the region is bounded to the right and left by a function of y, we integrate wrt x first and limits of integration wrt x (ie. $h_1(y)$ and $h_2(y)$)



$$D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

$$\iint_R f(x,y)\, dA = \int_c^d \int_{x=h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

Note:-

1) For type 1 region, arrow is drawn vertically from lower boundary to upper boundary (lower limit to upper limit)

2) For type 2 region, arrow is drawn horizontally from left boundary to right boundary

Examples:

1) Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$

Sol:- This is a Type-1 region

$$\int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx = \int_{x=0}^1 \left\{ x \frac{y^2}{2} \right\}_{y=x}^{\sqrt{x}} dx = \int_{x=0}^1 \frac{x}{2} \left\{ x - x^2 \right\} dx$$

$$= \frac{1}{2} \int_0^1 (x^2 - x^3)\, dx = \frac{1}{2} \left\{ \frac{x^3}{3} - \frac{x^4}{4} \right\}_0^1 = \frac{1}{2} \left\{ \frac{1}{3} - \frac{1}{4} \right\}$$

$$= \frac{1}{2} \left\{ \frac{4-3}{12} \right\} = \frac{1}{24}$$

2) $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y\, dx\, dy$

Sol: This is a type-2 region

$$\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y\, dx\, dy = \int_0^1 \left\{ y \frac{x^4}{4} \right\}_{x=0}^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 \frac{y}{4}\left\{ (1-y^2)^2 - 0 \right\} dy = \frac{y}{4}\int_0^1 \left\{ 1 + y^4 - 2y^2 \right\} dy$$

$$\frac{1}{4}\int_0^1 \left\{ y + y^5 - 2y^3 \right\} dy = \frac{1}{4}\left\{ \frac{y^2}{2} + \frac{y^6}{6} - 2\frac{y^4}{4} \right\}_0^1 = \frac{1}{4}\left\{ \frac{1}{2} + \frac{1}{6} - \frac{1}{2} \right\}$$

$$\therefore \int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y \, dx\, dy = \frac{1}{24}$$

3) $\displaystyle\int_0^{\pi} \int_0^{\cos\theta} r \sin\theta \, dr\, d\theta$

Sol:- $$\int_0^{\pi} \int_0^{\cos\theta} r\sin\theta \, dr\, d\theta = \int_0^{\pi} \sin\theta \left\{ \frac{r^2}{2} \right\}_0^{\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \sin\theta \cos^2\theta \, d\theta$$

Let $\cos\theta = t \Rightarrow -\sin\theta \, d\theta = dt$

If $\theta = \pi \Rightarrow t = -1$, $\theta = 0 \Rightarrow t = 1$

$$\therefore \int_0^{\pi} \int_0^{\cos\theta} r\sin\theta \, dr\, d\theta = -\frac{1}{2}\int_1^{-1} t^2 \, dt = -\frac{1}{2}\frac{t^3}{3}\Big]_1^{-1}$$

$$= -\frac{1}{6}\left\{ -1 - 1 \right\} = \frac{2}{6} = \frac{1}{3}$$

4) S.T. $\displaystyle\int_0^{\infty} \int_y^{\infty} x e^{-x^2/y} \, dx\, dy = \frac{1}{2}$

$$\int_0^{\infty} \int_y^{\infty} x e^{-x^2/y} \, dx\, dy = \int_0^{\infty} \left[\int_y^{\infty} \left(-\frac{y}{2} e^{-x^2/y} \left(-\frac{2x}{y} \right) \right) dx \right] dy$$

Let $-\frac{x^2}{y} = t \Rightarrow -\frac{2x}{y}\,dx = dt$

If $x = y \Rightarrow t = -y$, $x = \infty \Rightarrow t = -\infty$

$$= \int_0^\infty \int_{-y}^{-\infty} -\frac{y}{2} e^{t}\,dt\,dy$$

$$= \int_0^\infty -\frac{y}{2}\left[e^{t}\right]_{-y}^{-\infty} dy$$

$$= \int_0^\infty -\frac{y}{2}\left[0 - e^{-y}\right] dy$$

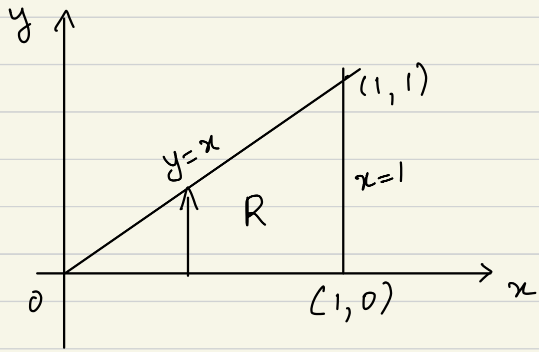
$$= \int_0^\infty \frac{y}{2} e^{-y}\,dy = \frac{1}{2}\left\{-y e^{-y} - \int 1 \cdot \frac{e^{-y}}{-1}\right\}_0^\infty$$

$$= \frac{1}{2}\left\{-ye^{-y} - e^{-y}\right\}_0^\infty = \frac{1}{2}\left\{0 - (0-1)\right\}$$

$$\therefore \int_0^\infty \int_y^\infty x e^{-x^2/y}\,dx\,dy = \frac{1}{2}$$

5) Evaluate $\iint_R (x^2+y^2)\,dx\,dy$ where R is the triangle bounded by the lines $y=0$, $y=x$ and $x=1$.

sol:



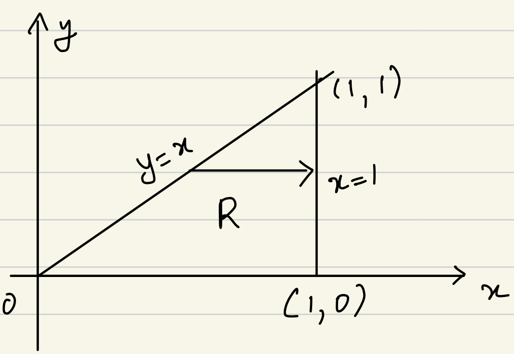
$$\iint_R (x^2+y^2)\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{x} (x^2+y^2)\;dy\,dx$$

$$= \int_{x=0}^{1} \left\{ x^2 y + \frac{y^3}{3} \right\}_0^x dx$$

$$= \int_{x=0}^{1} \left\{ x^3 + \frac{x^3}{3} \right\} dx = \left[\frac{x^4}{4} + \frac{x^4}{12} \right]_0^1$$

$$= \frac{1}{4} + \frac{1}{12} = \frac{3+1}{12} = \frac{4}{12} = \frac{1}{3}$$

or



$$\iint_R (x^2+y^2)\,dx\,dy = \int_{y=0}^{1}\int_{x=y}^{1} (x^2+y^2)\,dx\,dy$$

$$= \int_{y=0}^{1} \left(\frac{x^3}{3} + y^2 x\right)_{x=y}^{1} dy$$

$$= \int_{y=0}^{1} \left\{ \frac{1}{3} + y^2 - \frac{y^3}{3} - y^3 \right\} dy$$

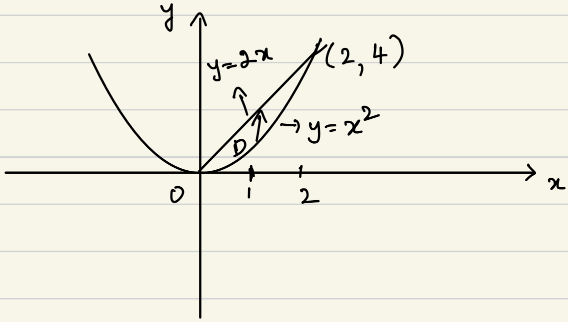
$$= \int_{y=0}^{1} \left\{ \frac{1}{3} + y^2 - \frac{4y^3}{3} \right\} dy$$

$$= \left\{ \frac{1}{3}y + \frac{y^3}{3} - \frac{4}{3}\frac{y^4}{4} \right\}_0^1 = \frac{1}{3} + \frac{1}{3} - \frac{1}{3}$$

$$\therefore \iint_R (x^2+y^2)\,dx\,dy = \frac{1}{3}$$

6) Find the volume of solid that lies under the paraboloid $z = x^2+y^2$ and above the region D bounded by the line $y = 2x$ and $y = x^2$.

Sol:-



$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0, 2$

This is a type-1 region.

$$D = \{(x,y) \mid 0 \le x \le 2, \quad x^2 \le y \le 2x\}$$

The point of intersection is

$$x^2 = 2x \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0$$

$$x = 0, 2$$

When $x = 0 \Rightarrow y = 0$, $\quad x = 2 \Rightarrow y = 4$

$$V = \int_{x=0}^{2} \int_{y=x^2}^{2x} (x^2+y^2)\, dy\, dx = \int_0^2 \left\{ x^2 y + \frac{y^3}{3} \right\}_{y=x^2}^{2x} dx$$

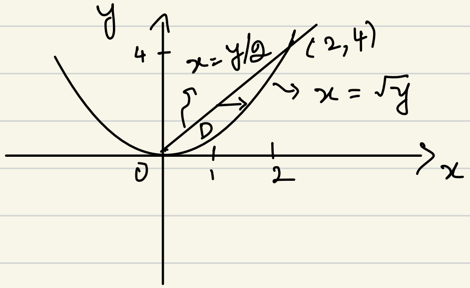
$$= \int_0^2 \left\{ x^2(2x) + \frac{(2x)^3}{3} - x^2 \cdot x^2 - \frac{(x^2)^3}{3} \right\} dx$$

$$= \int_0^2 \left(\frac{14x^3}{3} - x^4 - \frac{x^6}{3} \right) dx$$

$$= \left\{ \frac{14}{3} \cdot \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^7}{21} \right\}_0^2$$

$$= \left\{ \frac{7x^4}{6} - \frac{x^5}{5} - \frac{x^7}{21} \right\}_0^2 = \frac{216}{35}$$

or



$$D = \left\{ (x, y) \;\middle|\; 0 \le y \le 4,\ \frac{y}{2} \le x \le \sqrt{y} \right\}$$

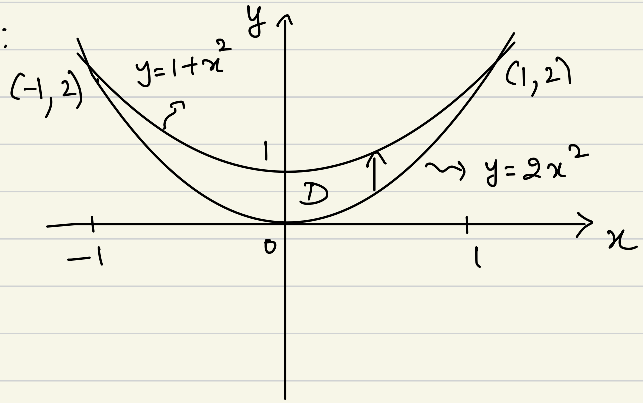
$$V = \iint_D (x^2+y^2)\, dA$$

$$= \int_{y=0}^{4} \int_{x=y/2}^{\sqrt{y}} (x^2+y^2)\, dx\, dy$$

$$= \frac{216}{35}$$

7) Evaluate $\iint_D (x+2y)\, dA$ where D is the region bounded by parabolas $y = 2x^2$ and $y = 1+x^2$.

Sol:



$(x-h)^2 = 4a(y-K)$

$x^2 = \frac{y}{2}$

$\therefore (h, K) = (0, 0)$

$x^2 = y-1$

$\therefore (h, K) = (0, 1)$

The parabolas intersect when

$$1+x^2 = 2x^2$$

$$\Rightarrow x^2 = 1 \quad \therefore x = \pm 1$$

When $x = 1 \Rightarrow y = 2$

$x = -1 \Rightarrow y = 2$

This is a type-1 region

$$D = \{(x, y) \mid -1 \le x \le 1,\ 2x^2 \le y \le 1+x^2\}$$

$$\iint_D (x+2y)\, dA = \int_{x=-1}^{1} \int_{y=2x^2}^{1+x^2} (x+2y)\, dy\, dx$$

$$= \int_{-1}^{1} \{xy + y^2\}_{2x^2}^{1+x^2} dx$$

$$= \int_{-1}^{1} \{x(1+x^2) + (1+x^2)^2 - 2x^3 - 4x^4\} dx$$

$$= \int_{-1}^{1} \{x + x^3 + 1 + x^4 + 2x^2 - 2x^3 - 4x^4\} dx$$

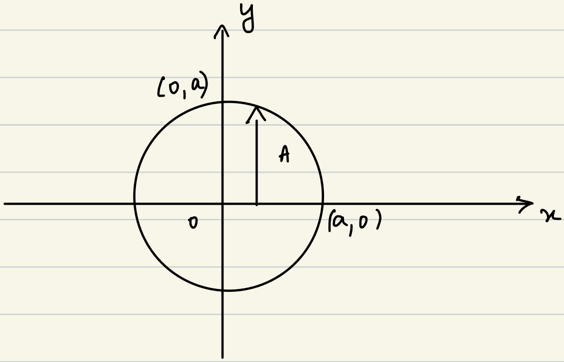
$$= \int_{-1}^{1} \{-3x^4 - x^3 + 2x^2 + x + 1\} dx$$

$$= \left\{-3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\right\}_{-1}^{1}$$

$$= \frac{32}{15}.$$

8) Evaluate $\iint_A xy\,dx\,dy$ where A is area bounded by the circle $x^2+y^2=a^2$ in the first quadrant.

Sol:



$$\iint_A xy\,dx\,dy = \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy\,dy\,dx$$

This is a type-1 region

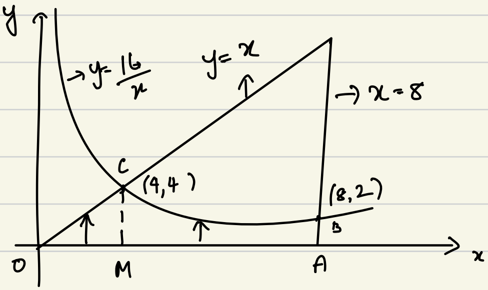
$$= \int_{x=0}^{a} \left\{x\,\frac{y^2}{2}\right\}_0^{\sqrt{a^2-x^2}} dx = \int_0^a \left\{\frac{x}{2}(a^2-x^2)\right\}dx$$

$$= \frac{1}{2}\int_0^a \left\{a^2x - x^3\right\}dx = \frac{1}{2}\left\{a^2\frac{x^2}{2} - \frac{x^4}{4}\right\}_0^a$$

$$\therefore \iint_A xy\,dx\,dy = \frac{1}{2}\left\{\frac{a^4}{2} - \frac{a^4}{4}\right\} = \frac{1}{2}\left\{\frac{2a^4-a^4}{4}\right\} = \frac{a^4}{8}$$

9) Evaluate $\iint_R x^2\,dx\,dy$ where R is the region in the first quadrant bounded by the lines $x=y$, $y=0$, $x=8$ and the curve $xy=16$

Sol:



$$\iint_R x^2\,dx\,dy = \int_{x=0}^{4}\int_{y=0}^{x} x^2\,dy\,dx + \int_{x=4}^{8}\int_{y=0}^{16/x} x^2\,dy\,dx$$

$$= \int_{x=0}^{4}\left\{x^2 y\right\}_{y=0}^{x} dx + \int_{x=4}^{8}\left\{x^2 y\right\}_{y=0}^{16/x} dx$$

$$= \int_{x=0}^{4} x^3\,dx + \int_{x=4}^{8} 16x\,dx = \left.\frac{x^4}{4}\right|_0^4 + \left.16\frac{x^2}{2}\right|_4^8$$

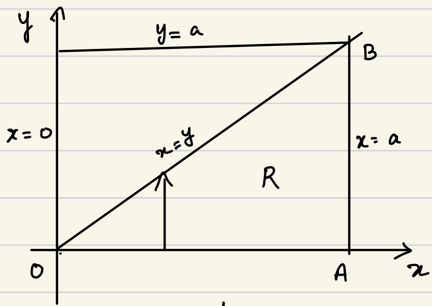
$$= \frac{1}{4}\left(4^4-0\right) + 8\left(8^2-4^2\right) = \frac{1}{4}(256) + 8(64-16)$$

$$= 64 + 384 = 448$$

Evaluation of double integrals by change of order of integration

1) Evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$ by changing order of integration

Sol:



OAB is the region of integration. To change the order of integration we divide the region of integration into vertical strips.

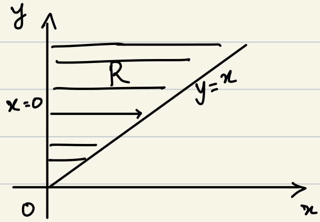
$$\int_{y=0}^{a} \int_{x=y}^{a} \frac{x}{x^2+y^2}\, dx\, dy = \int_{x=0}^{a} \int_{y=0}^{x} \frac{x}{x^2+y^2}\, dy\, dx$$

$$= \int_{x=0}^{a} x \left\{ \frac{1}{x} \cdot \tan^{-1}\left(\frac{y}{x}\right) \right\}_{y=0}^{x} dx$$

$$= \int_{x=0}^{a} \left\{ \tan^{-1}(1) - \tan^{-1} 0 \right\} dx = \int_{x=0}^{a} \left\{ \frac{\pi}{4} - 0 \right\} dx$$

$$= \frac{\pi}{4} \left\{ x \right\}_0^a = \frac{\pi a}{4}$$

2)
$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$$

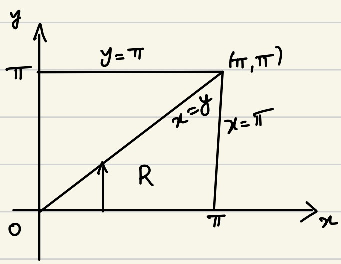


$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_{y=0}^{\infty} \int_{x=0}^{y} \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_{y=0}^{\infty} \left\{ x \cdot \frac{e^{-y}}{y} \right\}_{x=0}^{y} dy = \int_{y=0}^{\infty} \cancel{y} \cdot \frac{e^{-y}}{\cancel{y}}\, dy$$

$$= -e^{-y} \Big|_{y=0}^{\infty} = -(0-1) = 1$$

3)
$$\int_0^\pi \int_y^\pi \frac{\sin x}{x}\, dx\, dy$$



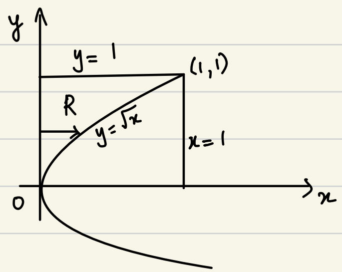
$$\int_0^\pi \int_y^\pi \frac{\sin x}{x}\, dx\, dy = \int_{x=0}^{\pi} \int_{y=0}^{x} \frac{\sin x}{x}\, dy\, dx$$

$$= \int_{x=0}^{\pi} \left\{ y \cdot \frac{\sin x}{x} \right\}_{y=0}^{x} dx = \int_{x=0}^{\pi} \left\{ \cancel{x} \frac{\sin x}{\cancel{x}} - 0 \right\} dx$$

$$= \int_{x=0}^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi} = -(\cos\pi - \cos 0)$$

$$= -(-1-1) = 2$$

4) $\displaystyle\int_0^1 \int_{\sqrt{x}}^1 \sqrt{1+y^3} \, dy \, dx$



$$\int_0^1 \int_{\sqrt{x}}^1 \sqrt{1+y^3} \, dy \, dx = \int_{y=0}^{1} \int_{x=0}^{y^2} \sqrt{1+y^3} \, dx \, dy$$

$$= \int_{y=0}^{1} \left\{ x \sqrt{1+y^3} \right\}_{x=0}^{y^2} dy = \int_0^1 y^2 \sqrt{1+y^3} \, dy$$

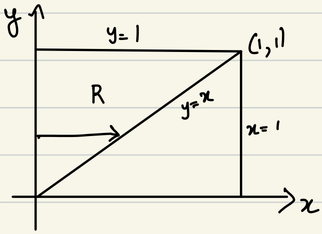
let $1+y^3 = t \Rightarrow 3y^2 \, dy = dt \Rightarrow y^2 \, dy = \frac{dt}{3}$

When $y = 0 \Rightarrow t = 1$, $y = 1 \Rightarrow t = 2$

$$= \frac{1}{3} \int_{t=1}^{2} \sqrt{t} \, dt = \frac{1}{3} \times \frac{2}{3} \left\{ t^{3/2} \right\}_1^2 = \frac{2}{9} \left(2^{3/2} - 1 \right)$$

$$\therefore \int_0^1 \int_{\sqrt{x}}^1 \sqrt{1+y^3} \, dy \, dx = \frac{2}{9} \left(2\sqrt{2} - 1 \right)$$

5) $\int_0^1 \int_x^1 e^{y^2}\, dy\, dx$



$$\int_0^1 \int_x^1 e^{y^2}\, dy\, dx = \int_{y=0}^{1} \int_{x=0}^{y} e^{y^2}\, dx\, dy = \int_{y=0}^{1} x \cdot e^{y^2} \Big|_{x=0}^{y} dy$$

$$= \int_{y=0}^{1} y e^{y^2}\, dy$$

Let $y^2 = t \Rightarrow 2y\, dy = dt \Rightarrow y\, dy = \frac{dt}{2}$

When $y = 0 \Rightarrow t = 0$, $y = 1 \Rightarrow t = 1$

$$\int_0^1 \int_x^1 e^{y^2}\, dy\, dx = \frac{1}{2} \int_{t=0}^{1} e^t\, dt = \frac{1}{2} e^t \Big|_0^1 = \frac{1}{2}(e-1)$$

Evaluation of double integrals by change of variables

Suppose we have $\iint_R f(x,y)\,dx\,dy$ and want to change the variables to u and v given by $x = x(u,v)$ and $y = y(u,v)$. The change of variables formula is

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f(x(u,v), y(u,v))\,|J|\,du\,dv$$

where J is the Jacobian of transformation from (x,y) to (u,v) given by

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$$

and R^* is the new region of integration in the (u,v) plane.

Note: Transforming a double integral into polar coordinates (r, θ).

$$x = r\cos\theta, \quad y = r\sin\theta$$

and Jacobian for polar coordinates is

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\therefore \iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R^*_{r\theta}} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

1) Evaluate $\iint_R e^{-(x^2+y^2)}\,dx\,dy$ where R is the region between 2 circles $x^2+y^2=1$ and $x^2+y^2=4$

Sol: Let $x = r\cos\theta$, $y = r\sin\theta$

$$R = \{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le 2\pi\}$$

$$\iint_R e^{-(x^2+y^2)}\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=1}^{2} e^{-r^2}\, r\,dr\,d\theta$$

Let $r^2 = t \Rightarrow 2r\,dr = dt \Rightarrow r\,dr = \frac{dt}{2}$

When $r=1 \Rightarrow t=1$, $r=2 \Rightarrow t=4$

$$= \frac{1}{2}\int_{\theta=0}^{2\pi}\int_{1}^{4} e^{-t}\,dt\,d\theta = \frac{1}{2}\int_{\theta=0}^{2\pi} -e^{-t}\Big|_{t=1}^{4}\,d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{2\pi} -(e^{-4}-e^{-1})\,d\theta$$

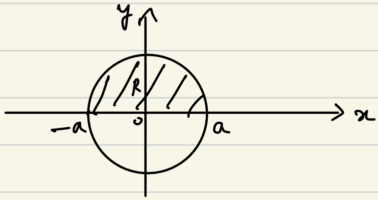
$$= -\frac{1}{2}\{e^{-4}-e^{-1}\}\{\theta\}_0^{2\pi} = -\frac{2\pi}{2}(e^{-4}-e^{-1})$$

$$\therefore \iint_R e^{-(x^2+y^2)}\,dx\,dy = \pi(e^{-1}-e^{-4})$$

2) $$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dy\,dx$$

sol: Let $x = r\cos\theta,\ y = r\sin\theta$, $|J| = r$

Given $R = \left\{ (x, y) \,\middle|\, -a \le x \le a,\ 0 \le y \le \sqrt{a^2 - x^2} \right\}$



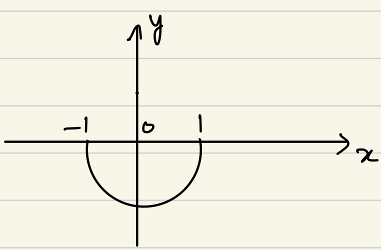
$$\int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx = \int_{\theta=0}^{\pi} \int_{r=0}^{a} r \; r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi} \frac{r^3}{3}\Big|_0^a \, d\theta = \int_{\theta=0}^{\pi} \frac{a^3}{3}\, d\theta$$

$$= \frac{a^3}{3}\, \theta \Big|_0^{\pi} = \frac{\pi a^3}{3}$$

3) Evaluate $\displaystyle\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{0} \cos(x^2+y^2)\, dy\, dx$

sol: Given $R = \left\{ (x, y) \,\middle|\, -1 \le x \le 1,\ -\sqrt{1-x^2} \le y \le 0 \right\}$



Radius = 1
centred at
origin

$$R^* = \{ (r, \theta) \mid 0 \le r \le 1 ,\ \pi \le \theta \le 2\pi \}$$

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{0} \cos(x^2+y^2)\, dy\, dx = \int_{\theta=\pi}^{2\pi} \int_{r=0}^{1} \cos(r^2)\, r\, dr\, d\theta$$

Let $r^2 = t \Rightarrow 2r\,dr = dt \Rightarrow r\,dr = \frac{dt}{2}$

When $r = 0 \Rightarrow t = 0$, $r = 1 \Rightarrow t = 1$

$$= \frac{1}{2} \int_{\theta=\pi}^{2\pi} \int_{0}^{1} \cos t\, dt\, d\theta = \frac{1}{2} \int_{\theta=\pi}^{2\pi} \sin t \Big|_0^1 d\theta$$

$$= \frac{1}{2} \int_{\pi}^{2\pi} \{\sin 1 - \sin 0\} d\theta$$

$$= \frac{1}{2} \sin 1 \{\theta\}_{\pi}^{2\pi} = \frac{\pi}{2} \sin(1)$$

4) Evaluate $\iint_R (x+y)^2\, dx\, dy$ where R is the parallelogram bounded by the lines $x+y=0$, $x+y=1$, $2x-y=0$, $2x-y=3$

Sol:- Let $u = x+y$, $v = 2x-y$

In these new variables, the region R is described by $0 \le u \le 1$, $0 \le v \le 3$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$