Fundamentals of Linear Algebra, Calculus and Statistics

(MAT211CT)

UNIT-II

DIFFERENTIAL CALCULUS

Topic Learning Objectives:

Upon Completion of this unit, student will be able to:

- Understand the fundamentals of the differential calculus of functions of one variable.
- Transform the coordinates from rectangular to polar and vice versa.
- Apply concepts of calculus to find angle between polar curves and consequences.
- Visualize Curvature for curves defined in different forms and find circle of curvature.
- Expand the function in power series using Taylor's and Maclaurin's series.
- Simulation using MATLAB.

Recapitulation: Functions of single variable:

The concept of functions is important in calculus because they play a key role in describing the real-world problems in mathematical terms. The temperature at which water boils depends on the elevation above sea level (the boiling point drops as the height increases). The interest paid on a cash investment depends on the length of time the investment is held. The area of a circle depends on the radius of the circle. The distance an object travels from an initial location along a straight-line path depends on its speed. In each case, the value of one variable quantity, which might be called as y, depends on the value of another variable quantity, which might be called x. Since the value of y is completely determined by the value of x, it's said that y is a function of x. Often the value of y is given by a *rule* or formula that says how to calculate it from the variable x. For instance, $A = \pi r^2$, the equation is a rule that calculates the area A of a circle from its radius r.

A symbolic way to say 'y is a function of x' is by writing $y = f(x)$. In this notation, the symbol f represents the function. The letter x, called the independent variable, represents the input value of f, and y, the dependent variable, represents the corresponding output value of f at x.

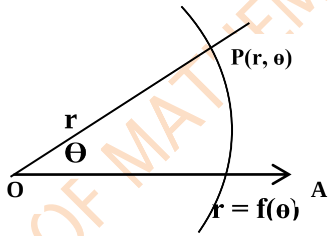
Requirement of new coordinate systems:

There is already a familiarity with Cartesian coordinate system for specifying a point in the XY – plane in two-dimensional geometry and XYZ- space in three- dimensional geometry. The requirement to define any new coordinate system is two-fold. One is based on geometry of the problem of practical situation wherein a more suitable coordinate system has to be chosen. For ex., the study of dispersion of a medicine injected in blood flow requires cylindrical coordinate system as the veins are cylindrical in nature. Use of Cartesian system may not be very suitable as it represents a rectangular channel and the corner effects have to be taken care. The second requirement is more of theoretical in nature. A mathematical expression which cannot be simplified in one coordinate system may be solved in simple way by transforming to other coordinate systems. For ex., log(x+y) cannot be further simplified in Cartesian system whereas it's easier to solve in Polar coordinates.

Basics of polar coordinates and polar curves:

A new coordinate system is introduced to understand the concept of polar curves and their properties.

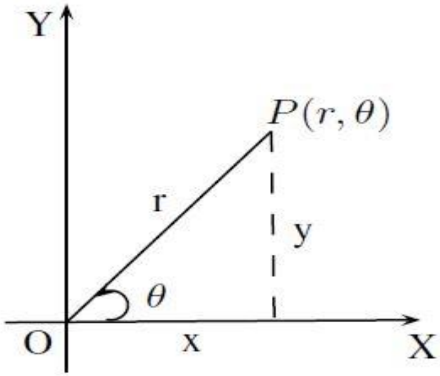
Any point P can be located on a plane with co-ordinates (r,θ) called **polar coordinates** of P where r = **radius vector** OP, (with pole/origin 'O'), θ = projection of OP on the initial line OA.
The equation $r = f(\theta)$ or $\theta = f(r)$ or $f(r,\theta) = c$ are known as a **polar curve**.



Polar curve

Relation between Cartesian and polar coordinates:

Consider a point P in the xy-plane. Join the points O (origin) and P. Let r be the length of OP and θ be the angle which OP makes with the (positive) x-axis. The (r,θ) are called the polar coordinates of the point P, and we write $P = (r,\theta)$, or $P(r,\theta)$. In particular, r is called the radial distance and θ is called the polar angle. Also, O is called the pole, the x-axis is called the inital line and OP is called the radius vector.



Let (x, y) be the Cartesian coordinates of the point P. Then we find that

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \dots(1)$$

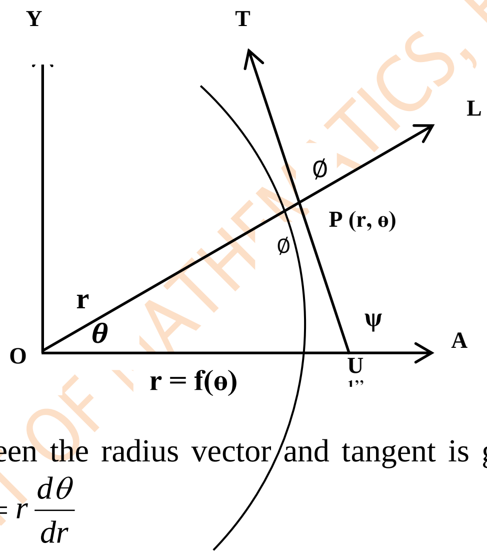
$$x = r\cos\theta\,,$$
$$y = r\sin\theta \qquad \dots(2)$$

Relations (1) enables us to find the polar coordinates (r, θ) when the Cartesian coordinates (x, y) are known. Conversely, relations (2) enable us to find the Cartesian coordinates when the polar coordinates are known. Thus, relations (1) define the transformation from the Cartesian coordinates to polar coordinates and relations (2) defines the inverse transformation.

Angle between the radius vector and the tangent:

With usual notation we can prove that $\tan\phi = r\dfrac{d\theta}{dr}$

Let "ϕ" be the angle between the radius vector OPL and the Tangent TPU at the point 'P' on the polar curve $r = f(\theta)$.



Thus, the angle between the radius vector and tangent is given by the expression: $\tan\phi = r\dfrac{d\theta}{dr}$

Note: (i) $\cot\phi = \left(\dfrac{1}{r}\dfrac{dr}{d\theta}\right)$

(ii) If ϕ_1 and ϕ_2 are the angles between the radius vector and the tangents at the point of intersection of two curves $r = f_1(\theta)$ and $r = f_2(\theta)$ then the angle of intersection of the curves is given by $|\phi_1 - \phi_2|$.

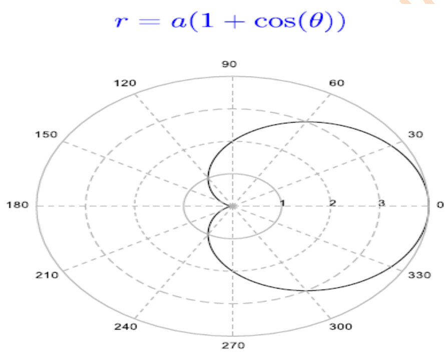
(iii) Suppose we are not able to obtain ϕ_1 and ϕ_2 explicitly then

$$\tan(\phi_1-\phi_2) = \frac{\tan\phi_1-\tan\phi_2}{1+\tan\phi_1\tan\phi_2}$$

(iv) If $\tan\phi_1 . \tan\phi_2 = -1$ then $\tan(\phi_1-\phi_2) = \infty \Rightarrow \phi_1 - \phi_2 = \frac{\pi}{2}$ (condition for the orthogonally of two polar curves)

Examples:

1. **Find the angle between the radius vector and the tangent to the following polar curves:**



[Cardioid $r = a(1-\cos\theta)$ is a curve that is the locus of a point on the circumference of circle rolling round the circumference of a circle of equal radius. Ofcourse the name means 'heart-shaped'. Curve is symmetrical about the initial line.]

Solution: Consider $r = a(1+\cos\theta)$,

Differentiating with respect to θ

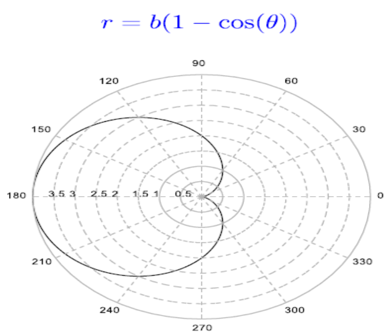
$$\frac{dr}{d\theta} = -a\sin\theta$$

$$r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta}$$

$$tan\phi = -\frac{2\cos^2\left(\frac{\theta}{2}\right)}{2sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)} = -cot\left(\frac{\theta}{2}\right)$$

$$tan\phi = tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right) \Rightarrow \phi = \left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

(ii) Cardioid $r = b(1 - cos\theta)$n other orientation]



Solution:

Consider $r = b(1 - cos\theta)$Differentiating with respect to θ

$$\frac{dr}{d\theta} = b\sin\theta$$

$$r\frac{d\theta}{dr} = \frac{b(1-\cos\theta)}{b\sin\theta}$$

$$tan\phi = \frac{2\sin^2\left(\frac{\theta}{2}\right)}{2sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)} = tan\left(\frac{\theta}{2}\right)$$

$$tan\phi = tan\left(\frac{\theta}{2}\right) \Rightarrow \phi = \left(\frac{\theta}{2}\right)$$

(iii) Circle: $r = sin\theta + cos\theta$

$\left[\text{This is a circle centered at } \left(\frac{1}{2},\frac{1}{2}\right) \text{ with a radius of} \sqrt{\frac{1}{2}}\right]$

Solution: Consider $r = \sin\theta + \cos\theta$,
Differentiating with respect to θ

$$\frac{dr}{d\theta} = \cos\theta - \sin\theta$$

$$r\frac{d\theta}{dr} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}$$

$$\tan\phi = \frac{\tan\theta + 1}{1 - \tan\theta} \quad \text{(Divide numerator and denominator by } \cos\theta\text{)}$$

$$tan\phi = \frac{tan(\theta) + 1}{1 - tan(\theta)} = tan\left(\frac{\pi}{4} + \theta\right)$$

$$\phi = \frac{\pi}{4} + \theta.$$

(iv) $r = 16\sec^2\left(\frac{\theta}{2}\right)$

Solution:

Consider $r = 16\sec^2\left(\frac{\theta}{2}\right)$

Differentiating with respect to θ

$$\frac{dr}{d\theta} = 32\sec^2\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right).\left(\frac{1}{2}\right) = 16\sec^2\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right)$$

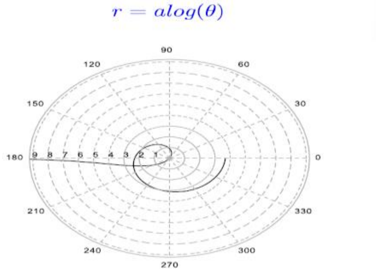
$$r\frac{d\theta}{dr} = \frac{16\sec^2\left(\frac{\theta}{2}\right)}{16\sec^2\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right)}$$

$$tan\phi = cot\left(\frac{\theta}{2}\right) = tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \Rightarrow \phi = \left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

2. Find the angle between two curves for the following:

(i) $r = a\log\theta$ and $r = \frac{a}{log\theta}$

[$r = a\log\theta$ some kind of 'logarithmic spiral'. The graph comes from negative x-infinity, goes through the origin at θ = 1, and then spirals outwards. It looks like it is heading to a definite limit of the radius but this is an illusion



Solution: Consider $\frac{dr}{d\theta} = \frac{a}{\theta}$

$$r\frac{d\theta}{dr} = a\log\theta\left(\frac{\theta}{a}\right)$$

$$tan\phi_1 = \theta log\theta$$

Consider $r = \frac{a}{log\theta}$

$$logr = loga - \log(log\theta) \quad \frac{1}{r}\frac{dr}{d\theta} = -\frac{1}{log\theta.\theta}$$

$$cot\phi_2 = -\frac{1}{\theta log\theta}$$

$$tan\phi_2 = -\theta log\theta$$

Now consider $\tan(\phi_1 - \phi_2) = \frac{2\theta log\theta}{1-(\theta log\theta)^2}$

We have to solve for θ by solving the pair of curves,

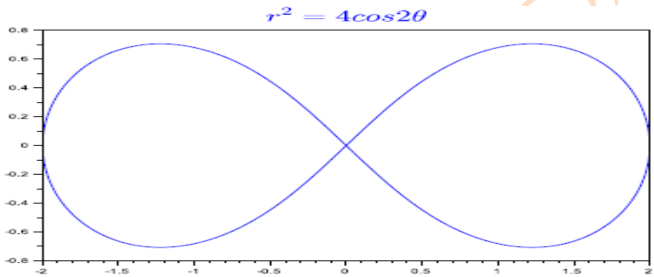
$r = a\log\theta$ and $r = \frac{a}{log\theta}$

$(\log\theta)^2 = 1$, $\log\theta = \pm 1$, $\theta = e$ or $\frac{1}{e}$

$\Longrightarrow \tan(\phi_1 - \phi_2) = \frac{2e}{1-e^2}$.

(ii) $r = 2(1 + cos\theta)$ and $r^2 = 4cos2\theta$

[$r^2 = 4cos2\theta$ is Lemniscate. Curve is symmetrical about both the axis]



$logr = log2 + \log(1 + cos\theta)$

$\frac{1}{r}\frac{dr}{d\theta} = \frac{-sin\theta}{1+cos\theta}$

$$cot\phi_1 = \frac{-2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2cos^2\left(\frac{\theta}{2}\right)}$$

$$cot\phi_1 = -\tan\left(\frac{\theta}{2}\right) = \cot\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$\phi_1 = \frac{\pi}{2} + \frac{\theta}{2}$.

Consider $r^2 = 4cos2\theta$

$$\frac{dr}{d\theta} = -\frac{4sin2\theta}{r}$$

$tan\phi_2 = -\tan\left(\frac{\pi}{2} + 2\theta\right)$

$\phi_2 = \frac{\pi}{2} + 2\theta$.

We have to solve for θ by solving the pair of curves,

$r = 2(1 + cos\theta)$ and $r^2 = 4cos2\theta$

Solving for θ, we get

$\theta = 1 + \sqrt{3}$

$\phi_1 - \phi_2 = \frac{3(1+\sqrt{3})}{2}$

(iii) $r = \frac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$

Solution: Consider $r = \frac{a\theta}{1+\theta}$

$\frac{1}{r} = \frac{1+\theta}{a\theta} = \frac{1}{a}\left(\frac{1}{\theta} + 1\right)$

$2\theta = -\frac{a}{r^2}\frac{dr}{d\theta}$

$\frac{1}{r}\frac{dr}{d\theta} = \frac{r}{a\theta^2}$

$r\frac{d\theta}{dr} = \frac{a\theta^2}{r}$

$tan\phi = \frac{a\theta^2}{\frac{a\theta}{(1+\theta)}}$

$\therefore \tan\phi = \theta(1+\theta)$

Consider $r = \frac{a}{1+\theta^2}$

$logr = loga - \log(1 + \theta^2)$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-2\theta}{1+\theta^2}$$
$$cot\phi = \frac{-2\theta}{1+\theta^2}$$
$$tan\phi = \frac{1+\theta^2}{-2\theta}.$$

We have to solve for θ by solving the pair of curves,

$$r = \frac{a\theta}{1+\theta} \quad \text{and} \quad r = \frac{a}{1+\theta^2}$$
$$\theta^3 = 1, \quad \theta = 1$$
$$\Longrightarrow \tan(\phi_1 - \phi_2) = \frac{2-(-1)}{1+(-2)}.$$
$$\phi_1 - \phi_2 = tan^{-1}(-3)$$

Exercise:

1. Find the angle between radius vector and tangent to the following polar curves:

i.	$r = a\sin^3\left(\frac{\theta}{3}\right)$	**Ans**: $\frac{\theta}{2}$
ii.	$r^m = a^m(\cos m\theta + \sin m\theta)$	**Ans:** $\frac{\pi}{4} + m\theta$
iii.	$r^n = a^n \sin n\theta$	**Ans:** $n\theta$
iv.	$r = 2a\cos^2\left(\frac{\theta}{2}\right)$	**Ans:** $\frac{\pi}{2} + \frac{\theta}{2}$
v.	$\frac{2a}{r} = 1 - \cos\theta$	**Ans:** $\pi - \frac{\theta}{2}$

2. Find the slope of the following curves:

i.	$r^2\cos 2\theta = a^2$ at $\theta = \frac{\pi}{12}$	**Ans:** $2 + \sqrt{3}$
ii.	$r = a\sin 2\theta$ at $\theta = \frac{\pi}{4}$	**Ans:** -1
iii.	$r = a(1 + \sin\theta)$ at $\theta = \frac{\pi}{2}$	**Ans:** 0

3. Prove that the following pairs of polar curves intersect orthogonally

i. $r = a\sec^2\left(\frac{\theta}{2}\right)$ and $r = b\, cosec^2\left(\frac{\theta}{2}\right)$

ii. $r^n = a^n \cos n\theta$ and $r^n = b^n \sin n\theta$

iii. $re^\theta = a$ and $r = be^\theta$

4. Find the angle of intersection for each of the following pairs of curves:

i. $r = a\cos\theta$ and $2r = a$ **Ans:** $\frac{\pi}{3}$

ii. $r = a\,(1-\cos\theta)$ and $r = 2a\cos\theta$ **Ans:** $\frac{\pi}{2} + \frac{\cos^{-1}\left(\frac{1}{3}\right)}{2}$

iii. $r = 2(1+\cos\theta)$ and $r = 6\cos\theta$ **Ans:** $\frac{\pi}{2}$

Curvature and Radius of Curvature:

Curvature is the amount by which a geometric object deviates from being *flat*, or *straight* in the case of a line, but this is defined in different ways depending on the context. In geometry, the **radius of curvature**, *R*, of a curve at a point is a measure of the radius of the circular arc which best approximates the curve at that point. It is the reciprocal of the curvature. The distance from the center of a circle or sphere to its surface is its radius. For other curved lines or surfaces, the **radius of curvature** at a given point is the radius of a circle that mathematically best fits the curve at that point. In the case of a surface, the radius of curvature is the radius of a circle that best fits a *normal section*.

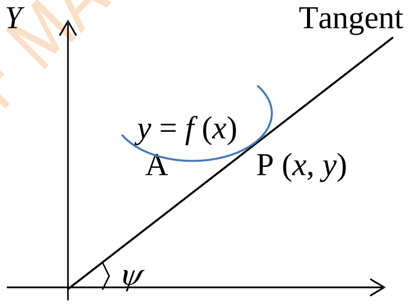
Imagine driving a car on a curvy road on a completely flat plain (so that the geographic plain is a geometric plane). At any one point along the way, lock the steering wheel in its position, so that the car thereafter follows a perfect circle. The car will, of course, deviate from the road, unless the road is also a perfect circle. The circle that the car makes is the circle of curvature, radius and the centre of the circle are radius of curvature and centre of curvature of the curvy road at the point at which the steering wheel was locked.

The more sharply curved the road is at the point you locked the steering wheel, the smaller the radius of curvature.

Some of the Applications:

- Radius of curvature is applied to measurements of the stress in the semiconductor structures.
- When engineers design trains track, they need to ensure the curvature of the track to be safe and provide a comfortable ride for the given speed of the trains.

Let P be a point on the curve $y = f(x)$ at the length 's' from a fixed-point A on it. Let the tangent at 'P' makes are angle ψ with positive direction of x – axis. As the point 'P' moves along curve, both s and ψ vary.



The rate of change ψ w.r.t s, $\frac{d\psi}{ds}$ is called the Curvature of the curve at 'P'.

The reciprocal of the Curvature at P is called the radius of curvature at P and is denoted by ρ.

$$\rho = \frac{ds}{d\psi}$$

Radius of curvature for Cartesian curve *y* = *f*(*x*):

If the curve is given in Cartesian coordinates as *y(x)*, then the radius of curvature is:

$$\rho = \left|\frac{(1+y'^2)^{\frac{3}{2}}}{y''}\right|,$$

where $y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}$

Radius of curvature for parametric equations $x = f(t)$, $y = \phi(t)$:

If the curve is given parametrically by functions $x(t)$ and $y(t)$, then the radius of curvature is

$$\rho = \frac{ds}{d\psi} = \left|\frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}\right|,$$

where $\dot{x} = \frac{dx}{dt}, \ddot{x} = \frac{d^2x}{dt^2}$ and

$\dot{y} = \frac{dy}{dt},\ \ddot{y} = \frac{d^2y}{dt^2}$

Radius of curvature for Polar curve $\boldsymbol{r = f(\theta)}$:

The radius of curvature of a polar curve $\boldsymbol{r = f(\theta)}$ is given by:

$$\rho = \frac{(r^2 + {r_1}^2)^{\frac{3}{2}}}{r^2 + 2{r_1}^2 - rr_2}$$

Examples:

1. Find the curvature at any point on the curve $y = x^3$.

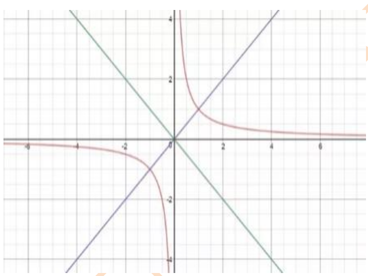
Solution:

$$\text{curvature } k = \frac{y_2}{\left(1+y_1^2\right)^{\frac{3}{2}}}$$

here $\quad y = x^3 \Rightarrow \quad y_1 = 3x^2$ and $\quad y_2 = 6x$

$$\therefore \quad k = \frac{6x}{\left(1+9x^4\right)^{\frac{3}{2}}}$$

2. Find the curvature at any point on the rectangular hyperbola $xy = c^2$



Solution: curvature $k = \dfrac{y_2}{\left(1+y_1^2\right)^{\frac{3}{2}}}$

$$\therefore \; y_1 = -\frac{c^2}{x^2} \quad \text{and} \quad y_2 = \frac{2c^2}{x^3}$$

here $xy = c^2 \Rightarrow \quad y = \dfrac{c^2}{x}$

$$k = \frac{\dfrac{2c^2}{x^3}}{\left\{1+\left(-\dfrac{c^2}{x^2}\right)^2\right\}^{\frac{3}{2}}} = \frac{2c^2 \times x^6}{x^3\left\{x^4+c^4\right\}^{\frac{3}{2}}}$$

$$= \frac{2c^2x^3}{\left(x^4+x^2y^2\right)^{\frac{3}{2}}} \qquad \because \quad xy = c^2$$

$$= \frac{2c^2}{\left(x^2+y^2\right)^{\frac{3}{2}}}$$

3. Find the radius of curvature at the origin on $y = x\left(x-a\right)^2$.

Solution:

Radius of curvature $\rho = \dfrac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y^2}$ it is required to find $\rho\left(0,0\right)$

Here $y = x\left(x-a\right)^2$

$x\left(x^2-2ax+a^2\right) = x^3-2ax^2+a^2x$

$\therefore\ y_1 = 3x^2-4ax+a^2$

$y_2 = 6x-4a$

Now $y_1\left(0,0\right) = a^2$

$y_2\left(0,0\right) = -4a$

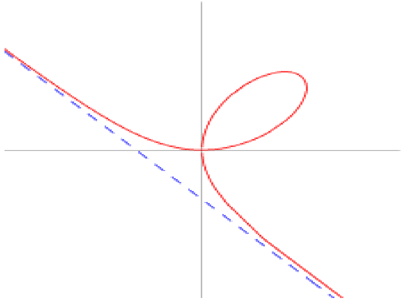
$$\therefore\ \rho\left(0,0\right) = \frac{\left(1+a^4\right)^{\frac{3}{2}}}{-4a} = -\frac{\left(1+a^4\right)^{\frac{3}{2}}}{4a}$$

Taking the magnitude of $\rho\left(0,0\right) = \dfrac{\left(1+a^4\right)^{\frac{3}{2}}}{4a}$

4. Find the radius of curvature at $\left(\dfrac{3a}{2},\dfrac{3a}{2}\right)$ on $x^3+y^3 = 3axy$

[Name of the curve is Folium Descartes; it is symmetrical about the line y = x.

It is not symmetric about any other line, nor it is symmetric about the origin.]



Solution:

It is required to find $\rho\left(\frac{3a}{2},\frac{3a}{2}\right)$ on $x^3+y^3=3axy$

Differentiating with respect to x, we get

$$3x^2+3y^2y_1=3a\left(xy_1+y\right)$$

$$\Rightarrow\ 3\left(y^2-ax\right)y_1=3\left(ay-x^2\right)$$

$$\Rightarrow\ y_1=\frac{ay-x^2}{y^2-ax} \qquad (1)$$

Differentiating with respect to x, we get

$$y_2=\frac{\left(y^2-ax\right)\left(ay_1-2x\right)-\left(ay-x^2\right)\left(2yy_1-a\right)}{\left(y^2-ax\right)^2} \qquad (2)$$

Now, from (1) $\quad y_{1\left(\frac{3a}{2},\frac{3a}{2}\right)}=\left\{-\frac{\left(x^2-ay\right)}{\left(y^2-ax\right)}\right\}_{\left(\frac{3a}{2},\frac{3a}{2}\right)}=-1$

(observe that $x=y$ at the point)

and, from (2) $y_{2\left(\frac{3a}{2},\frac{3a}{2}\right)} = \dfrac{\left(\frac{9a^2}{4}-\frac{3a^2}{2}\right)(-a-3a)-\left(\frac{3a^2}{2}-\frac{9a^2}{4}\right)(-3a-a)}{\left(y^2-ax\right)^2}$

$$y_{2\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{-\frac{3}{4}a^2\times 4a-\frac{3a^2}{4}\times 4a}{\left(\frac{3a^2}{4}\right)^2} = -\frac{6a^3}{\left(\frac{9a^4}{16}\right)} = -\frac{32}{3a}$$

Using these, $\rho_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \dfrac{\left\{1+(-1)^2\right\}^{\frac{3}{2}}}{\left(-\frac{32}{3a}\right)} = -\dfrac{2\sqrt{2}\times 3a}{32} = -\dfrac{3a}{8\sqrt{2}}$

$\therefore$ Radius of curvature at $\left(\frac{3a}{2},\frac{3a}{2}\right)$ is $\dfrac{3a}{8\sqrt{2}}$

5. Find the radius of curvature at $b^2x^2+a^2y^2=a^2b^2$ at its point of intersection with $y-$axis

Solution:

It is required to find $\rho = \dfrac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y^2}$ at $x=0$ on $b^2x^2+a^2y^2=a^2b^2$

when $x=0$, $b^2x^2+a^2y^2=a^2b^2$, reduces to $a^2y^2=a^2b^2 \Rightarrow y=\pm b$

That is, the point is $(0,b)$ or $(0,-b)$

The curve is $b^2x^2+a^2y^2=a^2b^2$

Differentiating with respect to x,

$2b^2x+2a^2yy_1=0$

$$y_1 = -\frac{b^2x}{a^2y}$$

Differentiating with respect to x,

$$y_2 = -\frac{b^2}{a^2}\left(\frac{y - xy_1}{y^2}\right)$$

Now, at $(0,b)$, $y_1 = -\frac{b^2}{a^2} \times \frac{0}{b} = 0$

$$y_2 = -\frac{b^2}{a^2}\left(\frac{b-0}{b^2}\right) = -\frac{b}{a^2}$$

$$\therefore \ \rho_{(0,b)} = \frac{(1+0)^{\frac{3}{2}}}{\left(-\frac{b}{a^2}\right)} = -\frac{a^2}{b}$$

i.e., Radius of curvature at $(0,b)$ is $\frac{a^2}{b}$

Next consider at $(0,-b)$, $y_1 = -\frac{b^2}{a^2} \times \frac{0}{-b} = 0$

$$y_2 = -\frac{b^2}{a^2}\left(\frac{-b-0}{b^2}\right) = \frac{b}{a^2}$$

$$\rho_{(0,-b)} = \frac{(1+0)^{\frac{3}{2}}}{\left(\frac{b}{a^2}\right)} = \frac{a^2}{b}$$

i.e., Radius of curvature at $(0,-b)$ is $\frac{a^2}{b}$

6. Show that the radius of curvature at any point (x, y) on $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is $3a^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}$.

Solution: We have $\rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2}$

The equation of the curve is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Differentiating with respect to x,

$$\frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}y_1=0$$

$$\Rightarrow \quad y_1=-\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}}=-\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

Differentiating again,

$$y_2=\frac{-\frac{1}{3}\left\{-x^{\frac{1}{3}}y^{-\frac{2}{3}}\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}-y^{\frac{1}{3}}x^{-\frac{2}{3}}\right\}}{x^{\frac{2}{3}}}=\frac{1}{3}\frac{\left\{\frac{1}{y^{\frac{1}{3}}}+\frac{y^{\frac{1}{3}}}{x^{\frac{2}{3}}}\right\}}{x^{\frac{2}{3}}}=\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{3x^{\frac{4}{3}}y^{\frac{1}{3}}}=\frac{a^{\frac{2}{3}}}{3x^{\frac{4}{3}}y^{\frac{1}{3}}}$$

Now, $\quad 1+y_1^2=1+\dfrac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}=\dfrac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{2}{3}}}=\dfrac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}$

$$\therefore \quad \rho=\frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2}=\frac{\left(\dfrac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right)^{\frac{3}{2}}}{\left(\dfrac{a^{\frac{2}{3}}}{3x^{\frac{4}{3}}y^{\frac{1}{3}}}\right)}=\frac{a\times 3x^{\frac{4}{3}}y^{\frac{1}{3}}}{x\times a^{\frac{2}{3}}}=3a^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}$$

7. Show that for ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1,\ \rho=\dfrac{a^2b^2}{p^3}$ where p is the length of the perpendicular from the center upon the tangent at (x, y) to the ellipse.

Solution: The ellipse is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1.$

$$\Rightarrow \quad \frac{2x}{a^2}+\frac{2y\,y_1}{b^2}=0 \qquad \Rightarrow y_1=-\frac{b^2}{a^2}\frac{x}{y}$$

Differentiating again,

$$y_2 = -\frac{b^2}{a^2}\left\{\frac{y\times 1 - x\times y_1}{y^2}\right\} = -\frac{b^2}{a^2}\left\{\frac{y + x\frac{b^2}{a^2}\frac{x}{y}}{y^2}\right\}$$

$$= -\frac{b^2}{a^2y^3}\left\{\frac{y^2}{b^2}+\frac{x^2}{a^2}\right\} = -\frac{b^2}{a^2y^3} \qquad \because \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Now,

$$\rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2} = \frac{\left(1+\frac{b^4x^2}{a^4y^2}\right)^{\frac{3}{2}}}{\left(-\frac{b^4}{a^2y^3}\right)} = -\frac{\left(a^4y^2+b^4x^2\right)^{\frac{3}{2}}}{a^6y^3}\times\frac{a^2y^3}{b^4} = -\frac{\left(a^4y^2+b^4x^2\right)^{\frac{3}{2}}}{a^4b^4}$$

Taking the magnitude

$$\rho = \frac{\left(a^4y^2+b^4x^2\right)^{\frac{3}{2}}}{a^4b^4}$$

The tangent at (x_0, y_0) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$$

Length of perpendicular from $(0,0)$ upon this tangent $= \frac{1}{\sqrt{\left(\frac{x_0}{a^2}\right)^2+\left(\frac{y_0}{b^2}\right)^2}} = \frac{a^2b^2}{\sqrt{a^4y_0^2+b^4x_0^2}}$

So, the length of perpendicular from the origin upon the tangent at *(x,y)* is

$$p = \frac{a^2b^2}{\sqrt{a^4y^2+b^4x^2}} \qquad \text{(by replacing } x_0 \text{ by } x \ \& \ y_0 \text{ by } y\text{)}$$

$$\Rightarrow \quad \frac{1}{p^3} = \frac{\left(a^4y^2 + b^4x^2\right)^{\frac{3}{2}}}{a^6b^6} = \frac{\left(a^4y^2 + b^4x^2\right)^{\frac{3}{2}}}{a^4b^4} \times \frac{1}{a^2b^2}$$

$$= \frac{\rho}{a^2b^2}$$

$$\Rightarrow \quad \rho = \frac{a^2b^2}{p^3}$$

8. Find the radius of curvature at the point θ on the curve

$$x = a \log \sec\theta$$
$$y = \mathrm{a}(\tan\theta - \theta)$$

Solution:

$$x = a\log\ \sec\theta \quad \Rightarrow \frac{dx}{d\theta} = a \times \frac{1}{\sec\theta} \times \sec\theta\tan\theta = a\tan\theta$$

$$y = a(\tan\theta - \theta) \Rightarrow \quad \frac{dy}{d\theta} = a(\sec^2\theta - 1) = a\tan^2\theta$$

$$\therefore \quad y_1 = \frac{dy}{dx} = \frac{\left(\dfrac{dy}{d\theta}\right)}{\left(\dfrac{dx}{d\theta}\right)} = \frac{a\tan^2\theta}{a\tan\theta} = \tan\theta$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}(\tan\theta)$$

$$= \frac{d}{d\theta}(\tan\theta) \times \frac{d\theta}{dx} = \sec^2\theta \times \frac{1}{a\tan\theta} = \frac{\sec^2\theta}{a\tan\theta} = asec\theta tan\theta$$

Now, $$\rho = \frac{\left(1 + y_1^2\right)^{\frac{3}{2}}}{y_2} = \frac{\left(1 + \tan^2\theta\right)^{\frac{3}{2}}}{\left(\dfrac{\sec^2\theta}{a\tan\theta}\right)} = \frac{\sec^3\theta}{\sec^2\theta} \times a\tan\theta$$

$$\boldsymbol{\rho = a\sec\theta\tan\theta}$$

9. Show that the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ is $4a\cos(\theta/2)$.

Solution:

$$\frac{dx}{d\theta} = a(1+\cos\theta), \frac{dy}{d\theta} = a\sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} + \frac{dx}{d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\theta/2\,\cos\theta/2}{2\cos^2\theta/2} = \tan\theta/2$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\cdot\frac{d\theta}{dx} = \frac{1}{2}\sec^2\frac{\theta}{2}\cdot\frac{1}{a(1+\cos\theta)}$$

$$= \frac{1}{2}\sec^2\frac{\theta}{2}\cdot\frac{1}{2a\cos^2\theta/2} = \frac{1}{4a}\sec^4\frac{\theta}{2}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{4a\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)^{3/2}}{\sec^4\left(\frac{\theta}{2}\right)}$$

$$= 4\mathrm{a}\cdot\left(\sec^2\left(\frac{\theta}{2}\right)\right)^{3/2}\cdot\cos^4\left(\frac{\theta}{2}\right)$$

$$= 4a\cos\left(\frac{\theta}{2}\right)$$

10. Find radius of curvature at a point for the curves $x = 6t^2 - 3t^4$, $y = 8t^3$.

Solution: $\dot{x} = \dfrac{dx}{dt} = 12t - 12t^3 = 12t(1-t^2)$; $\qquad \dot{y} = 24t^2$

$$\ddot{x} = \frac{dx}{dt} = 12 - 36t^2 = 12(1-3t^2); \qquad \ddot{y} = 48t$$

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} = \frac{\left\{\left[12t^2(1-t^2)\right]^2 + (24t^2)^2\right\}^{3/2}}{12t(1-t^2)48t - 24t^2(12-36t^2)}$$

$$= \frac{\left\{144t^4(1-2t^2+t^4) + 24^2 t^4\right\}^{3/2}}{48\times 12t^2 - 12\times 48t^3 - 24\times 12t^2 - 24\times 36t^4}$$

$$= \frac{\left\{(1+t^2)^2\right\}^{3/2}\times 6t}{1+t^2} = \frac{\left\{(1+t^2)^3\right\}^3\times 6t}{1+t^2} = 6t(1+t^2)^2$$

11. Show that the radius of curvature at any point of the Cardiod $\boldsymbol{r = a(1 - cos\,\theta)}$ varies as $\sqrt{\boldsymbol{r}}$

Solution: Differentiating w.r.t $\boldsymbol{\theta}$ we get:

$$r_1 = a\sin\theta\,, r_2 = a\cos\theta$$

$$(r^2 + {r_1}^2)^{\frac{3}{2}} = [a^2(1-\cos\theta)^2 + a^2 sin^2\theta]^{\frac{3}{2}}$$

$$r^2 + 2{r_1}^2 - rr_2 = a^2(1-\cos\theta)^2 - a^2(1-\cos\theta)\cos\theta + 2a^2 sin^2\theta$$
$$= 3a^2(1-\cos\theta)$$

$$Now, \qquad \rho = \frac{(r^2 + {r_1}^2)^{\frac{3}{2}}}{r^2 + 2{r_1}^2 - rr_2} = \frac{a^3 2\sqrt{2(1-\cos\theta)^{\frac{3}{2}}}}{3a^2(1-\cos\theta)}$$

$$\frac{2\sqrt{2a(1-\cos\theta)^{\frac{1}{2}}}}{3} = \frac{2a\sqrt{2}}{3}\left(\frac{r}{a}\right)^{\frac{1}{2}} \alpha \sqrt{\boldsymbol{r}}$$

12. Find the radius of curvature at any point for the curve

$$r^n = a^n \cos n\theta$$

Solution: Taking log on both sides

$$n \log r = n \log a + \log \cos n\theta$$

Differentiating w.r.t $\boldsymbol{\theta}$ we get:

$$\frac{n}{r}\frac{dr}{d\theta} = -\frac{\sin n\theta}{\cos n\theta}.n$$

$$r_1 = \frac{dr}{d\theta} = -r \tan n\theta$$

Differentiating again w.r.t $\boldsymbol{\theta}$ we get:

$$r_2 = \frac{d^2r}{d\theta^2} = r\,tan^2 n\theta - nr\,sec^2 n\theta$$

$$Now, \quad \rho = \frac{(r^2 + {r_1}^2)^{\frac{3}{2}}}{r^2 + 2{r_1}^2 - rr_2}$$

$$= \frac{(r^2 + r^2 tan^2 n\theta)^{\frac{3}{2}}}{r^2 + 2r^2 tan^2 n\theta - r^2 tan^2 n\theta + nr^2 sec^2 n\theta}$$

$$= \frac{(r^3 sec^3 n\theta)^{\frac{3}{2}}}{(n+1) r^2 sec^2 n\theta} = \frac{r}{(n+1)\cos n\theta}$$

$$= \frac{r.a^n}{(n+1) r^n} = \frac{a^n r^{1-n}}{(n+1)}$$

Exercise:

1. For the curve $y = c\log\left[\sec(x/c)\right]$ find the radius of curvature at a point (x, y).

2. Find radius of curvature at a point for the curves $x = a[\cos t + \log(\tan t/2)]$ and $y = a\sin t$.

3. For the curve $y = 4\sin x - \sin 2x$ find the radius of curvature at a point $x = \frac{\pi}{2}$.

4. Find the radius of curvature at the point $(a, 0)$on $xy^2 = a^3 - x^2$.

5. Find the radius of curvature at any point on $y = a\cosh\left(\frac{x}{a}\right)$.

6. Find the radius of curvature of the curve $r^2 = a^2 \sin 2\theta$ at any point (r, θ)

7. For the curve $\frac{2a}{r} = 1 + \cos\theta$ find the radius of curvature at a point (r, θ).

Answers:

1.$c\ sec\left(\frac{x}{c}\right)$

2. $a\cot t$

3. $\frac{5\sqrt{5}}{4}$

4.$\frac{3a}{2}$

5.$\frac{y^2}{a}$

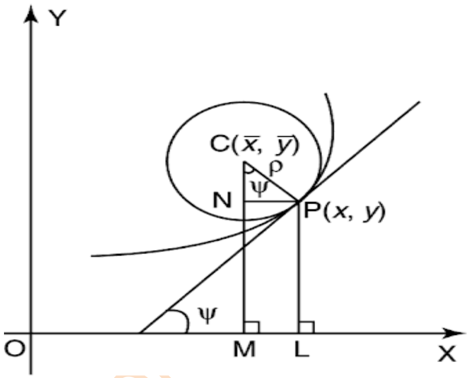
6. $\frac{a^2}{3r}$

7. $2\sqrt{\frac{r^3}{a}}$

Centre of curvature:

In geometry, the **center of curvature** of a curve is found at a point that is at a distance from the curve equal to the radius of curvature lying on the normal vector. It is the point at infinity if the curvature is zero. The osculating circle to the curve is centered at the Centre of curvature. Cauchy defined the Centre of curvature *C* as the intersection point of two infinitely close normal lines to the curve.

Centre of curvature at any point P(*x*, *y*) on the curve *y* = *f*(*x*) is given by: $\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$, $\bar{y} = y + \frac{(1+y_1^2)}{y_2}$



Let C($\bar{x}, \bar{y}$) be the centre of curvature and ρ be the radius of curvature at P(*x*, *y*). Draw PL and CM perpendicular to OX and PN perpendicular to CM. Let the tangent at P makes an angle ψ with the *x*-axis. Then ∠NCP = 90 – ∠NPC = ψ

$\bar{x}$= OM = OL –ML = OL-NP = $\rho sin\psi$ $= x - \frac{y_1(1+y_1^2)}{y_2}$

Similarly, $\bar{y}$= MC = MN + NC = LP $+\rho cos\psi$ $= y + \frac{(1+y_1^2)}{y_2}$

Note: Equation of the circle of curvature at P(x, y) is $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

Examples:

1. Find the coordinates of the centre of curvature at any point of the parabola
$y^2 = 4ax$.

Solution: Given $y^2 = 4ax$,

We have $y_1 = \frac{2a}{y}$ and $y_2 = -\frac{4a^2}{y^3}$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}, \qquad \bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$\bar{x} = 3x + 2a, \quad \bar{y} = -\frac{2x^{\frac{3}{2}}}{\sqrt{a}}$$

2. Find circle of curvature at $(1,0)$ on $y = x^3 - x^2$.

Solution: At (x_0, y_0) the equation of the circle of curvature is $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

$$\frac{dy}{dx} = 3x^2 - 2x; \qquad y_{1(1,0)} = 1$$

$$\frac{d^2y}{dx^2} = 6x - 2; \qquad y_{2(1,0)} = 4$$

$$(1+y_1^{\,2})_{(1,0)} = 2,$$

$$\bar{x}_{(1,0)} = \frac{1}{2}; \qquad \bar{y}_{(1,0)} = \frac{1}{2}; \qquad \rho_{(1,0)} = \frac{1}{\sqrt{2}}$$

$$\left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{1}{2} \Rightarrow x^2 + y^2 - x - y = 0.$$

Exercise:

1. Find circle of curvature of $x + y = ax^2 + by^2 + cx^3$ at the origin.

2. Find circle of curvature of $x^3 + y^3 = 3xy$ at $\left(\frac{3}{2}, \frac{3}{2}\right)$.

Answers:

1. $(a+b)(x^2+y^2) = 2(x+y)$.
2. $x^2+y^2-\frac{21}{8}(x+y)+\frac{432}{128}=0$.

Mean Value Theorem:

Taylor's Mean Value Theorem: (English Mathematician Brook Taylor 1685-1731)

Suppose a function $f(x)$ satisfies the following two conditions:

(i) $f(x)$ and it's first (n-1) derivatives are continuous in a closed interval $[a,b]$

(ii) $f^{(n-1)}(x)$ is differentiable in the open interval (a,b)

Then there exists at least one point c in the open interval (a,b) such that

$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{\lfloor 2}f''(a)+\frac{(b-a)^3}{\lfloor 3}f'''(a)+\ldots..$$

$$\ldots..+\frac{(b-a)^{n-1}}{\lfloor n-1}f^{(n-1)}(a)+\frac{(b-a)^n}{\lfloor n}f^{(n)}(c)\rightarrow(1)$$

Taking $b=a+h$ and for $0<\theta<1$, the above expression (1) can be rewritten as

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a)+\frac{h^3}{\lfloor 3}f'''(a)+\ldots.+\frac{h^{n-1}}{\lfloor n-1}f^{(n-1)}(a)+\frac{h^n}{\lfloor n}f^{(n)}(a+\theta h)\rightarrow(2)$$

Taking b=x in (1) we may write

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{\lfloor 2}f''(a)+\frac{(x-a)^3}{\lfloor 3}f'''(a)+\ldots+\frac{(x-a)^{n-1}}{\lfloor n-1}f^{(n-1)}(a)+R_n\rightarrow(3)$$

$$Where \quad R_n=\frac{(x-a)^n}{\lfloor n}f^{(n)}(c)\rightarrow \text{Re}\,mainder\ term\ after\ n\ t$$

When $n \to \infty$, we can show that $|R_n| \to 0$, thus we can write the **Taylor's series** as

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{\underline{|2}} f''(a) + \ldots + \frac{(x-a)^{n-1}}{\underline{|n-1}} f^{(n-1)}(a) + \ldots.$$

$$= f(a) + \sum_{n=1}^{\infty} \frac{(x-a)^n}{\underline{|n}} f^{(n)}(a) \to (4)$$

Using (4) we can write a Taylor's series expansion for the given function f(x) in powers of (x-a) or about the point 'a'.

Maclaurin's series: (Scottish Mathematician Colin Maclaurin 1698-1746)

When a = 0, expression (4) reduces to a Maclaurin's expansion given by:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\underline{|2}} f''(0) + \ldots + \frac{x^{n-1}}{\underline{|n-1}} f^{(n-1)}(0) + \ldots.$$

$$= f(0) + \sum_{n=1}^{\infty} \frac{x^n}{\underline{|n}} f^{(n)}(0) \to (5)$$

Advantages of using Taylor's series and Maclaurin's series

- Taylor series are studied because polynomial functions are easy and if one could find a way to represent complicated functions as series (infinite polynomials) then one can easily study the properties of difficult functions.

- Evaluating definite Integrals: Some functions have no antiderivative which can be expressed in terms of familiar functions. This makes evaluating definite integrals for some functions difficult because the Fundamental Theorem of Calculus cannot be used. If we have a polynomial representation of a function, we can oftentimes use that to evaluate a definite integral.

- Understanding asymptotic behavior: Sometimes, a Taylor series can tell us useful information about how a function behaves in an important part of its domain.

- Understanding the growth of functions
- Solving differential equations

Examples:

1. Obtain a Taylor's expansion for $f(x) = \sin x$ in the ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to the fourth-degree term.

Solution: The Taylor's expansion for $f(x)$ about $\frac{\pi}{4}$ is

$$f(x) = f\left(\frac{\pi}{4}\right) + \left(x - \frac{\pi}{4}\right) f'\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{\lfloor 2} f''\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^3}{\lfloor 3} f'''\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^4}{\lfloor 4} f^{(iv)}\left(\frac{\pi}{4}\right) \ldots . \rightarrow (1)$$

$$f(x) = \sin x \Rightarrow f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad ;$$

$$f'(x) = \cos x \Rightarrow f'\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$f''(x) = -\sin x \Rightarrow f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$f'''(x) = -\cos x \Rightarrow f'''\left(\frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$f^{(iv)}(x) = \sin x \Rightarrow f^{(iv)}\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Substituting these in (1) we obtain the required Taylor's series in the form

$$f(x) = \frac{1}{\sqrt{2}} + \left(x - \frac{\pi}{4}\right)\left(\frac{1}{\sqrt{2}}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{\lfloor 2}\left(-\frac{1}{\sqrt{2}}\right) + \frac{\left(x-\frac{\pi}{4}\right)^3}{\lfloor 3}\left(-\frac{1}{\sqrt{2}}\right) + \frac{\left(x-\frac{\pi}{4}\right)^4}{\lfloor 4}\left(\frac{1}{\sqrt{2}}\right) \ldots .$$

$$f(x)=\frac{1}{\sqrt{2}}\left[1+\left(x-\frac{\pi}{4}\right)-\frac{\left(x-\frac{\pi}{4}\right)^2}{\lfloor 2}+\frac{\left(x-\frac{\pi}{4}\right)^3}{\lfloor 3}-\frac{\left(x-\frac{\pi}{4}\right)^4}{\lfloor 4}+\ldots\right]$$

2. Obtain a Taylor's expansion for $f(x)=\log_e x$ up to the term containing $(x-1)^4$ and hence find $\log_e(1.1)$.

Solution: The Taylor's series for $f(x)$ about the point 1 is

$$f(x)=f(1)+(x-1)f'(1)+\frac{(x-1)^2}{\lfloor 2}f''(1)+\frac{(x-1)^3}{\lfloor 3}f'''(1)+\frac{(x-1)^4}{\lfloor 4}f^{(iv)}(1)\ldots.\rightarrow(1)$$

Here, $f(x)=\log_e x \Rightarrow f(1)=\log 1=0$; $\quad f'(x)=\frac{1}{x}\Rightarrow f'(1)=1$

$f''(x)=-\frac{1}{x^2}\Rightarrow f''(1)=-1;\quad f'''(x)=\frac{2}{x^3}\Rightarrow f'''(1)=2$

$f^{(iv)}(x)=-\frac{6}{x^4}\Rightarrow f^{(iv)}(1)=-6$ etc.,

Using all these values in (1) we get

$$f(x)=\log_e x=0+(x-1)(1)+\frac{(x-1)^2}{\lfloor 2}(-1)+\frac{(x-1)^3}{\lfloor 3}(2)+\frac{(x-1)^4}{\lfloor 4}(-6)\ldots.$$

$$\Rightarrow \log_e x=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}\ldots.$$

Taking x=1.1 in the above expansion we get

$$\Rightarrow \log_e(1.1)=(0.1)-\frac{(0.1)^2}{2}+\frac{(0.1)^3}{3}-\frac{(0.1)^4}{4}\ldots.=0.0953$$

3. Using Taylor's Theorem, expand log (sin x) in ascending powers of (x–3).

Solution: $f(x)$ = *log (sinx)*, a = 3 and $f(3)$ = *log (sin3)*

Now $f'(x) = \cot(x)$, $\qquad f'(3) = \cot(3)$

$f''(x) = -\text{cosec}^2(x), f''(3) = -\text{cosec}^2(3)$,

$f'''(x) = 2\text{cosec}^3(x)\cot(x), f'''(3) = 2\text{cosec}^3(3)\cot(3)$,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \frac{(x-a)^3}{6}f'''(a) + \cdots$$

$\log(\sin x) = \log(\sin(3)) + (x-3)\cot(3) - \frac{(x-3)^2}{2}\text{cosec}^2(3) + \frac{(x-3)^3}{3}(\text{cosec}^3(3)\cot(3) + \cdots$

4. Obtain a Maclaurin's series for $f(x) = \sin x$ up to the term containing x^5.

Solution: The Maclaurin's series for $f(x)$ is

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{\lfloor 2}f''(0) + \frac{x^3}{\lfloor 3}f'''(0) + \frac{x^4}{\lfloor 4}f^{(4)}(0) + \frac{x^5}{\lfloor 5}f^{(5)}(0) \ldots . \to (1)$$

Here, $f(x) = \sin x \Rightarrow f(0) = \sin 0 = 0$

$f'(x) = \cos x \Rightarrow f'(0) = \cos 0 = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = -\sin 0 = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -\cos 0 = -1$

$f^{(iv)}(x) = \sin x \Rightarrow f^{(iv)}(0) = \sin 0 = 0$

$f^{(v)}(x) = \cos x \Rightarrow f^{(v)}(0) = \cos 0 = 1$

Substituting these values in (1), we get the Maclaurin's series for $f(x) = Sin\ x$ as

$$f(x) = \sin x = 0 + x(1) + \frac{x^2}{\lfloor 2}(0) + \frac{x^3}{\lfloor 3}(-1) + \frac{x^4}{\lfloor 4}(0) + \frac{x^5}{\lfloor 5}(1) \ldots . \Rightarrow \sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} \ldots .$$

5. Obtain Taylor's expansion of the function $\cos\left(\frac{\pi}{4}+h\right)$ in ascending powers of h up to the terms containing h^4.

Solution: Taylors's expansion of $f(x+h)$ is given by

$$f(x+h)=f(x)+\sum_{n=1}^{\infty}\frac{h^n}{h!}f^{(n)}(x)=f(x)+\frac{h}{1!}f'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\frac{h^4}{4!}f^{iv}(x)+\ldots$$

At $x=\frac{\pi}{4}$

$f(x)=\cos x \Rightarrow \qquad f(\pi/4)=\cos(\pi/4)=\frac{1}{\sqrt{2}}$

$f'(x)=-\sin x \Rightarrow \qquad f'(\pi/4)=-\sin(\pi/4)=-\frac{1}{\sqrt{2}}$

$f''(x)=-\cos x \Rightarrow \qquad f''(\pi/4)=-\cos(\pi/4)=-\frac{1}{\sqrt{2}}$

$f'''(x)=\sin x \Rightarrow \qquad f'''(\pi/4)=\sin(\pi/4)=\frac{1}{\sqrt{2}}$

Then we have

$$\cos(x+\pi/4)=\frac{1}{\sqrt{2}}-\frac{h}{1!}\frac{1}{\sqrt{2}}-\frac{h^2}{2!}\frac{1}{\sqrt{2}}+\frac{h^3}{3!}\frac{1}{\sqrt{2}}+\frac{h^4}{4!}\frac{1}{\sqrt{2}}+\ldots$$

$$=\frac{1}{\sqrt{2}}\left[1-h-\frac{h^2}{2!}+\frac{h^3}{3!}+\frac{h^4}{4!}+\ldots\right]$$

6. Obtain expansion of $f(x)=\frac{x}{\sin x}$ upto the term containing x^4

Solution: Maclaurin's series expansion of $\sin x$ is given by

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots$$

$$\therefore\ f(x)=\frac{x}{\sin x}=x\left\{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\right\}^{-1}=xx^{-1}\left\{1-\frac{x^2}{3!}+\frac{x^4}{5!}-\dots\right\}$$

$$=\left\{1-\left(\frac{x^2}{3!}-\frac{x^4}{5!}-\dots\right)\right\}^{-1}$$

$$=1+\left(\frac{x^2}{3!}-\frac{x^4}{5!}-\dots\right)+\left(\frac{x^2}{3!}-\frac{x^4}{5!}-\dots\right)^2+\dots \qquad \text{By Binomial expansion}$$

$$=1+\frac{x^2}{3!}-\frac{x^4}{5!}+\left(\frac{x^2}{3!}\right)^2, \text{ terms of order } > x^4 \text{ are neglected}$$

$$=1+\frac{x^2}{6}-\frac{x^4}{120}+\frac{x^4}{36}+\dots$$

$$=1+\frac{x^2}{6}-\frac{7x^4}{360}+\dots$$

Note: As done in the above example, we can find the Maclaurin's series for various functions, for example:

$$(i)\quad Cos x=1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}-\frac{x^6}{\lfloor 6}\dots\dots \quad (ii)\quad e^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}+\frac{x^5}{\lfloor 5}+\frac{x^6}{\lfloor 6}\dots\dots$$

$$(iii)\quad (1+x)^m=1+mx+\frac{m(m-1)}{\lfloor 2}x^2+\frac{m(m-1)(m-2)}{\lfloor 3}x^3+\frac{m(m-1)(m-2)(m-3)}{\lfloor 4}x^4\dots\dots$$

Exercise:

1. Expand a^x in ascending powers of x.

2. Expand $\log(secx)$ in ascending powers of x up to and including the term in x^6 and hence deduce the expansion of $tanx$.

3. Show that $\sin^{-1}(3x-4x^3)=3\left[x+\frac{x^3}{6}+3\frac{x^5}{40}+\ldots\right]$

Hint: put $x = sin\theta$

Answers:

1. $1 + xlog_e a + \frac{x^2}{2}(log_e a)^2 + \frac{x^3}{6}(log_e a)^3 + \cdots$

2. $\frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \cdots$

Video Links:

1. **Polar Co-ordinates:** https://youtu.be/aSdaT62ndYE
2. **Curvature & Radius of curvature:** https://youtu.be/VGcJv8tLPTU
3. **Taylor Series:** https://youtu.be/3d6DsjIBzJ4