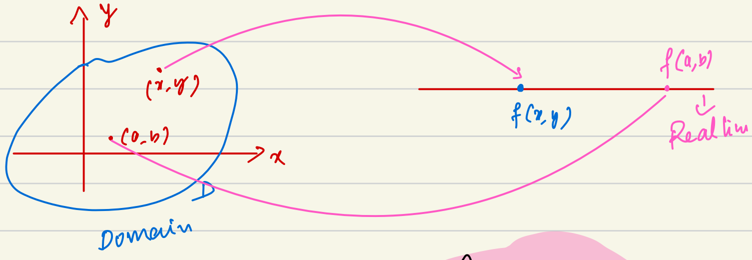
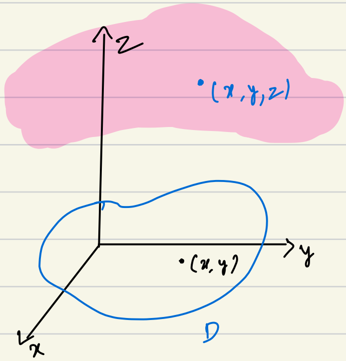
Functions of two variables



A function of two variables is a rule that assigns to each ordered pair of real numbers (x,y) in a set D a unique real number denoted by $f(x,y)$.



If f is a function of two variables with domain D, then the graph of f is the set of all points (x,y,z) in $\mathbb{R}^3$ s.t $z = f(x,y)$ and $(x,y) \in D$.

Definition: Let $z = f(x,y)$ be a function of 2 variables x and y.

The first order partial derivatives of z wrt x and y are given by

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = z_x = f_x = p = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

$\Rightarrow \frac{\partial z}{\partial x}$ is partial derivative of z wrt x, treating y as constant.

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = z_y = f_y = q = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

$\Rightarrow \frac{\partial z}{\partial y}$ is partial derivative of z wrt y, treating x as constant

Second ordered partial derivatives wrt x and y:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = z_{xx} = f_{xx} = r$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = z_{yy} = f_{yy} = t$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = z_{xy} = \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = s$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = z_{yx} = \frac{\partial^2 f}{\partial y \partial x} = f_{yx} = s$$

Note: $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$

1) Evaluate $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if

a) $z = x^2 y - x \sin xy$ b) $x + y + z = \log z$

Sol: a) Differentiating z partially wrt x

$$\frac{\partial z}{\partial x} = (2x)y - (x \cdot \cos(xy) \cdot y + \sin(xy)(1))$$

$$\therefore z_x = xy(2 - \cos xy) - \sin xy$$

Differentiating z partially wrt y,

$$\frac{\partial z}{\partial y} = x^2(1) - x\cos(xy)x = x^2 - x^2 \cos xy$$

$$\therefore z_y = x^2(1 - \cos xy)$$

b) $\log z - z = x + y \rightarrow (1)$

Differentiating (1) partially wrt x,

$$\frac{1}{z}\frac{\partial z}{\partial x} - \frac{\partial z}{\partial x} = 1 + 0$$

$$\Rightarrow \frac{\partial z}{\partial x}\left(\frac{1}{z} - 1\right) = 1$$

i.e. $\frac{\partial z}{\partial x}\left(\frac{1-z}{z}\right) = 1$

$$\frac{\partial z}{\partial x} = \frac{z}{1-z}$$

Differentiating ① partially wrt y,

$$\frac{1}{z}\frac{\partial z}{\partial y} - \frac{\partial z}{\partial y} = 1+0$$

$$\frac{\partial z}{\partial y}\left(\frac{1}{z}-1\right) = 1 \Rightarrow \frac{\partial z}{\partial y} = \frac{z}{1-z}$$

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = \frac{z}{1-z}$$

2) If $\theta = t^n e^{-r^2/4t}$, find value of n such that $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$.

Sol: Differentiating θ partially wrt r

$$\frac{\partial \theta}{\partial r} = t^n e^{-r^2/4t}\left(-\frac{2r}{4t}\right) = -\frac{r}{2}\, t^{n-1}\, e^{-r^2/4t}$$

$$\therefore r^2\frac{\partial \theta}{\partial r} = -\frac{r^3}{2}\, t^{n-1}\, e^{-r^2/4t}$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = -\frac{3r^2}{2}\, t^{n-1}\, e^{-r^2/4t} + \left(-\frac{r^3}{2}\right) t^{n-1}\, e^{-r^2/4t}\left(-\frac{2r}{4t}\right)$$

$$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \left(-\frac{3}{2}t^{n-1} + \frac{r^2}{4}t^{n-2}\right)e^{-r^2/4t} \rightarrow ①$$

Differentiating θ partially wrt t

$$\frac{\partial \theta}{\partial t} = n t^{n-1} e^{-r^2/4t} + t^n e^{-r^2/4t}\left(\frac{r^2}{4t^2}\right)$$

$$\frac{\partial \theta}{\partial t} = \left(n t^{n-1} + \frac{1}{4} r^2 t^{n-2}\right) e^{-r^2/4t} \rightarrow ②$$

Given : $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$

$$\left(-\frac{3}{2}t^{n-1} + \frac{r^2}{4}t^{n-2}\right)e^{-r^2/4t} = \left(n t^{n-1} + \frac{1}{4}r^2 t^{n-2}\right)e^{-r^2/4t}$$

$$\Rightarrow \left(n+\frac{3}{2}\right)t^{n-1}e^{-r^2/4t} = 0 \Rightarrow n+\frac{3}{2} = 0$$

$$\therefore n = -\frac{3}{2}$$

3) If $z = e^{ax+by}f(ax-by)$, prove that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$

Solution: Consider $z = e^{ax+by}f(ax-by)$ (1)

Differentiating (1) with respect to x using product and chain rules,

$$\frac{\partial z}{\partial x} = \{e^{ax+by}f^1(ax-by)(a)\} + \{f(ax-by)e^{ax+by}(a)\}$$

$$\frac{\partial z}{\partial x} = ae^{ax+by}\{f(ax-by) + f'(ax-by)\} \quad (2)$$

Differentiating (1) with respect to y using product and chain rules,

$$\frac{\partial z}{\partial y} = \{e^{ax+by}f'(ax-by)(-b)\} + \{f(ax-by)e^{ax+by}(b)\}$$

$$\frac{\partial z}{\partial y} = be^{ax+by}\{f(ax-by) - f'(ax-by)\} \quad (3)$$

Multiplying equation (2) by b and equation (3) by a, and adding

$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = abe^{ax+by}[f'(ax-by) + f(ax-by) + f(ax-by) - f'(ax-by)] = abe^{ax+by}[2f(ax-by)] = 2ab[e^{ax+by}f(ax-by)] = 2abz.$

4) If $u = e^{r\cos\theta}\cos(r\sin\theta)$, $v = e^{r\cos\theta}\sin(r\sin\theta)$, P.T.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Sol: Consider $u = e^{r\cos\theta} \cos(r\sin\theta)$

$$\frac{\partial u}{\partial r} = e^{r\cos\theta} \left\{-\sin(r\sin\theta) \cdot \sin\theta\right\} + \cos(r\sin\theta) \cdot e^{r\cos\theta} (\cos\theta)$$

$$= e^{r\cos\theta} \left\{ \cos\theta \cos(r\sin\theta) - \sin\theta \sin(r\sin\theta)\right\}$$

$$\frac{\partial u}{\partial r} = e^{r\cos\theta} \cdot \cos(\theta + r\sin\theta) \rightarrow (1)$$

$$\frac{\partial u}{\partial \theta} = e^{r\cos\theta} \left\{ -\sin(r\sin\theta) \cdot r\cos\theta\right\} + \cos(r\sin\theta) \cdot e^{r\cos\theta} (-r\sin\theta)$$

$$= -r e^{r\cos\theta} \left\{ \sin(r\sin\theta)\cos\theta + \cos(r\sin\theta)\sin\theta\right\}$$

$$\frac{\partial u}{\partial \theta} = -r e^{r\cos\theta} \cdot \sin(r\sin\theta + \theta)$$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = e^{r\cos\theta} \cdot \sin(r\sin\theta + \theta) \rightarrow (2)$$

Consider $v = e^{r\cos\theta} \cdot \sin(r\sin\theta)$

$$\frac{\partial v}{\partial r} = e^{r\cos\theta} \cdot \left\{ \cos(r\sin\theta) \cdot \sin\theta\right\} + \sin(r\sin\theta) \cdot e^{r\cos\theta} \cdot (\cos\theta)$$

$$= e^{r\cos\theta} \left\{ \sin\theta \cos(r\sin\theta) + \cos\theta \sin(r\sin\theta)\right\}$$

$$\frac{\partial v}{\partial r} = e^{r\cos\theta} \cdot \sin(\theta + r\sin\theta) \rightarrow (3)$$

$$\frac{\partial v}{\partial \theta} = e^{r\cos\theta}\left\{\cos(r\sin\theta) \cdot r\cos\theta\right\} +$$

$$\sin(r\sin\theta) \cdot e^{r\cos\theta} \cdot (-r\sin\theta)$$

$$= r\, e^{r\cos\theta}\left\{\cos(r\sin\theta) \cdot \cos\theta - \sin(r\sin\theta)\sin\theta\right\}$$

$$\frac{\partial v}{\partial \theta} = r\, e^{r\cos\theta} \cdot \cos(r\sin\theta + \theta)$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = e^{r\cos\theta} \cdot \cos(r\sin\theta + \theta) \rightarrow 4$$

From 1 and 4 $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$

and from 2 and 3 $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$.

5) If $u = \log(x^3+y^3-x^2y-xy^2)$, then S.T.

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = \frac{-4}{(x+y)^2}.$$

Sol: $\frac{\partial u}{\partial x} = \frac{3x^2-2xy-y^2}{x^3+y^3-x^2y-xy^2} \to (1)$, $\frac{\partial u}{\partial y} = \frac{3y^2-x^2-2xy}{x^3+y^3-x^2y-xy^2} \to (2)$

$$(1)+(2) \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{(3x^2-2xy-y^2)+(3y^2-x^2-2xy)}{x^3+y^3-x^2y-xy^2}$$

$$= \frac{3(x^2+y^2)-4xy-(x^2+y^2)}{(x+y)(x^2+y^2-xy)-xy(x+y)} = \frac{2(x^2+y^2)-4xy}{(x+y)(x^2+y^2-xy-xy)}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{2(x^2+y^2-2xy)}{(x+y)(x^2+y^2-2xy)} = \frac{2(x-y)^2}{(x+y)(x-y)^2} = \frac{2}{x+y}$$

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) u$$

Note: $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u \neq \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)^2$

$\frac{\partial^2 u}{\partial x^2} + \frac{2\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} \neq \left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \left(\frac{\partial u}{\partial y}\right)^2$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{2}{x+y}\right)$$

$$= 2\left[\frac{\partial}{\partial x}\left(\frac{1}{x+y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{x+y}\right)\right]$$

$$= 2\left[\frac{-1}{(x+y)^2} - \frac{1}{(x+y)^2}\right] = \frac{-4}{(x+y)^2}$$

6) If $u = e^{xyz}$, S.T. $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1+3xyz+x^2y^2z^2)e^{xyz}$

Sol: $u = e^{xyz}$

$$\frac{\partial u}{\partial x} = e^{xyz} \cdot yz \rightarrow (1)$$

Differentiating (1) partially wrt y

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(yz \cdot e^{xyz}\right) = z\left[y \cdot xze^{xyz} + e^{xyz} \cdot 1\right]$$

$$\therefore \frac{\partial^2 u}{\partial y \partial x} = z\left(xyze^{xyz} + e^{xyz}\right) = \left(xyz^2 + z\right)e^{xyz} \rightarrow (2)$$

Differentiating (2) partially wrt z

$$\frac{\partial}{\partial z}\left(\frac{\partial^2 u}{\partial y \partial x}\right) = \frac{\partial^3 u}{\partial z \partial y \partial x} = \frac{\partial}{\partial z}\left[\left(xyz^2 + z\right)e^{xyz}\right]$$

$$= (2xyz + 1)\, e^{xyz} + \left(xyz^2 + z\right) e^{xyz} \cdot xy$$

$$= e^{xyz}\left[2xyz + 1 + x^2y^2z^2 + xyz\right]$$

$$\therefore \frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz}\left[1 + 3xyz + x^2y^2z^2\right]$$

7) If $u = \log r$, where $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$. S.T.

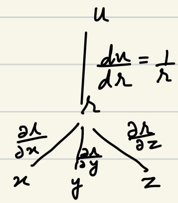
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{r^2}$$

Sol: $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$

Differentiating partially wrt x,

$$2r \frac{\partial r}{\partial x} = 2(x-a) \Rightarrow \frac{\partial r}{\partial x} = \frac{x-a}{r} \rightarrow (1)$$

$$\frac{\partial r}{\partial y} = \frac{y-b}{r} \quad , \quad \frac{\partial r}{\partial z} = \frac{z-c}{r}$$



Now , $u = \log r$

$$\frac{\partial u}{\partial x} = \frac{1}{r}\frac{\partial r}{\partial x} = \frac{1}{r}\left(\frac{x-a}{r}\right)$$

$$\frac{\partial u}{\partial x} = \frac{x-a}{r^2} \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x-a}{r^2}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{r^2(1) - (x-a)\,2r\left(\frac{\partial r}{\partial x}\right)}{r^4} = \frac{r^2 - 2(x-a)^2}{r^4}$$

$|||^{ly}$ $$\frac{\partial^2 u}{\partial y^2} = \frac{r^2 - 2(y-b)^2}{r^4} \quad , \quad \frac{\partial^2 u}{\partial z^2} = \frac{r^2 - 2(z-c)^2}{r^4}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{3r^2 - 2\left[(x-a)^2 + (y-b)^2 + (z-c)^2\right]}{r^4}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{3r^2 - 2r^2}{r^4} = \frac{1}{r^2}$$

Exercise:

1) Evaluate the following at the mentioned points.

1) $f(x,y) = \tan^{-1}(y/x)$; $f_x(2,3)$

Sol: $f_x(2,3) = -\frac{3}{13}$

2) $f(x,y) = x^3 + 2x^2y^2 + y^3$; $f_{yx}(1,2)$

Sol: $f_{yx}(1,2) = 16$

2) Find the first order partial derivatives of foll functions

1) $f(x,y) = x\cos y + ye^x$

$f_x = \cos y + y e^x$

$f_y = -x\sin y + e^x$

2) $f(x,y) = e^{xy}\ln(y)$

$f_x = y e^{xy}\ln(y)$

$f_y = x e^{xy}\ln(y) + \frac{e^{xy}}{y}$

3) $f(x,y) = \frac{2y}{y+\cos x}$

$f_x = \frac{2y\sin x}{(y+\cos x)^2}$, $f_y = \frac{2\cos x}{(y+\cos x)^2}$

3) verify that $\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial y\partial x}$ for the following functions or

verify Clairaut's theorem for the given functions:

1) $u = x^y + y^x$

$u_{xy} = u_{yx} = yx^{y-1}\log x + x^{y-1} + x\cdot y^{-1}\log y + y^{x-1}$

2) $u = \sin(3x+2y)$ $\left(u_{xy} = u_{yx} = -6\sin(3x+2y)\right)$

3) $u(x, y) = \log_e\left(\dfrac{x^2+y^2}{xy}\right)$

$u_{xy} = u_{yx} = \dfrac{-4xy}{(x^2+y^2)^2}$

4) verify if the given functions are harmonic.

1) $f(x, y) = e^x \cos y$ 2) $f(x, y) = x^2 - y^2$

5) If $v = (x^2+y^2+z^2)^{-1/2}$, S.T. $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2} = 0$

6) If $u = \log(x^3+y^3-x^2y-xy^2)$, S.T. $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{-4}{(x+y)^2}$

7) If $u = e^{xyz}$, S.T. $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = (1+3xyz+x^2y^2z^2)e^{xyz}$

8) If $u = \log r$, where $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$, S.T.

$u_{xx} + u_{yy} + u_{zz} = \dfrac{1}{r^2}$.

∗ Differential or Total differential or Exact differential

For a differentiable function of one variable, $y = f(x)$ we define the differential dx to be an independent variable, i.e. dx can be given the value of any real no. The differential of y is defined as

$$dy = f'(x)\,dx$$

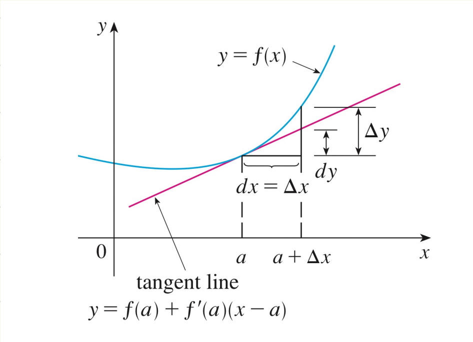


Fig: 1

Figure 1 shows the relationship b/w the increment Δy and differential dy. Δy represents the change in height of curve $y = f(x)$ and dy represents the change in height of the tangent line when x changes by an amount $dx = \Delta x$.

For a function of 2 variables, $z = f(x, y)$ we define dx and dy to be independent variables, i.e. they can be given any values. Then the differential dz, also called the total differential, is defined by

$$\boxed{dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy}$$

If $w = f(x, y, z)$ is a function of 3 variables then the differential dw is given by

$$\boxed{dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz}$$

Chain rule :

Recall that chain rule for functions of a single variable gives the rule for differentiating a composite function. If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions then y is indirectly a differentiable function of t and

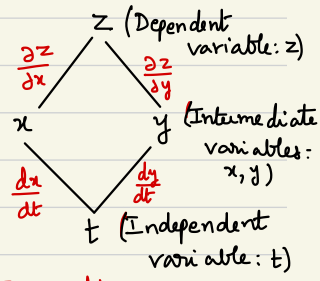
$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

For functions of more than one variable, the Chain rule has several versions, each of them giving a rule for differentiating a composite function.

☆ **Chain rule (case 1) :** If $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are functions of t. Then z is a differentiable function of t called as total derivative of z given by

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

(To remember the chain rule it's helpful to draw tree diagram)



Tree diagram

※ Chain rule (Case 2): If $z = f(x, y)$ is a differentiable function of x and y where $x = g(s, t)$ and $y = h(s, t)$ are functions of s and t then

$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \& \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}}$$

Tree diagram for the above case:

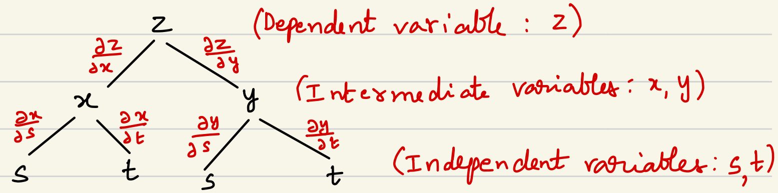


Fig: 2

resembles the one-dimensional Chain Rule in Equation 1.

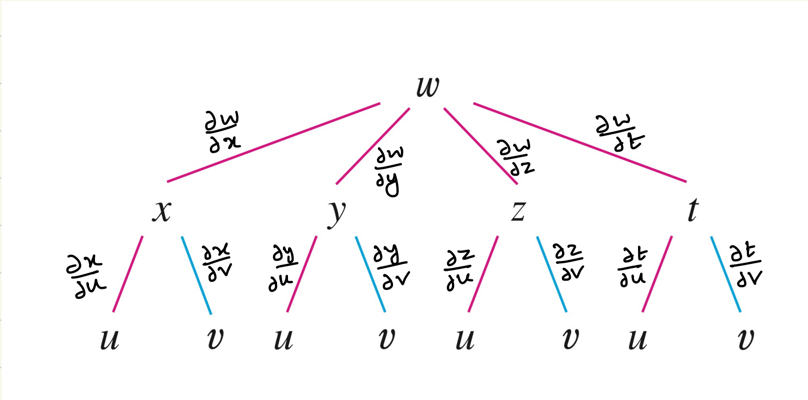
To remember the Chain Rule, it's helpful to draw the **tree diagram** in Figure 2. We draw branches from the dependent variable z to the intermediate variables x and y to indicate that z is a function of x and y. Then we draw branches from x and y to the independent variables s and t. On each branch we write the corresponding partial derivative. To find $\partial z/\partial s$, we find the product of the partial derivatives along each path from z to s and then add these products:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Similarly, we find $\partial z/\partial t$ by using the paths from z to t.

Ex: Write out the chain rule for the case $w = f(x, y, z, t)$ and $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ and $t = t(u, v)$.

Sol:

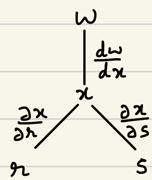


$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$

If w is a function of x alone i.e. $w = f(x)$ and $x = \phi(u, v)$ then our equations are even simpler.

$$\boxed{\frac{\partial w}{\partial r} = \frac{dw}{dx} \cdot \frac{\partial x}{\partial r} \quad \& \quad \frac{\partial w}{\partial s} = \frac{dw}{dx}\,\frac{\partial x}{\partial s}}$$



∗ Implicit differentiation or differentiation of implicit functions

The chain rule can be used for the process of implicit differentiation. An implicit function with x as an independent variable and y as the dependent variable is generally of the form $z = f(x, y) = 0$.

This gives $\frac{dz}{dx} = 0.$

Using case ① of chain rule, take $t = x$

$$\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$0 = \frac{\partial f}{\partial x}\cdot 1 + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

$$\boxed{\therefore\ \frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{-f_x}{f_y}}, \text{ if } \frac{\partial f}{\partial y} \neq 0.$$

ex: Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy.$

Sol: $f(x, y) = x^3 + y^3 - 6xy = 0$

$f_x = 3x^2 - 6y$, $f_y = 3y^2 - 6x$

$$\therefore\ \frac{dy}{dx} = \frac{-f_x}{f_y} = -\frac{(3x^2 - 6y)}{3y^2 - 6x} = -\frac{(x^2 - 2y)}{y^2 - 2x}$$

Consider $x^3 + y^3 - 6xy = 0$

$3x^2 + 3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} - 6y = 0$

$\Rightarrow \frac{dy}{dx}(3y^2 - 6x) = -3x^2 + 6y$

$\therefore\ \frac{dy}{dx} = \frac{-3x^2 + 6y}{3y^2 - 6x} = \frac{-x^2 + 2y}{y^2 - 2x}$

Now, if z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0.$

We can use chain rule to differentiate the equation $F(x, y, z) = 0$ as follows:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0$$

But $\frac{\partial}{\partial x}(x) = 1$ and $\frac{\partial}{\partial x}(y) = 0$

$$\therefore\ \frac{\partial F}{\partial x}(1) + 0 + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\therefore\ \frac{\partial z}{\partial x} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = \frac{-F_x}{F_z}, \quad |||^{ly}\ \frac{\partial z}{\partial y} = \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = \frac{-F_y}{F_z}$$

Ex: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3+y^3+z^3+6xyz=1$.

Sol: Let $F(x,y,z) = x^3+y^3+z^3+6xyz-1=0$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(3x^2+6yz)}{3z^2+6xy} = \frac{-(x^2+2yz)}{z^2+2xy} \rightarrow 1$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{(3y^2+6xz)}{3z^2+6xy} = -\frac{(y^2+2xz)}{z^2+2xy}$$

or

Differentiating the given eq. partially wrt x

$$3x^2+3z^2\frac{\partial z}{\partial x}+6yz+6xy\frac{\partial z}{\partial x}=0$$

$$\frac{\partial z}{\partial x}(3z^2+6xy) = -(3x^2+6yz)$$

$$\therefore \frac{\partial z}{\partial x} = -\frac{(3x^2+6yz)}{3z^2+6xy} = -\frac{(x^2+2yz)}{z^2+2xy} \rightarrow 2$$

1 and 2 are equal but evaluating $\frac{\partial z}{\partial x}$ by implicit differentiation formula is easier.