# Quantum Mechanics Numericals

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**Problem 1.** Calculate the de Broglie wavelength for a proton moving with a speed of $10^6$m/s.

**Problem 2.** Calculate the de Broglie wavelength of an electron accelerated with a voltage of 1000 V.

**Problem 3.** Determine de Broglie wavelength of thermal neutrons at a temperature of 27°C.

**Problem 4.** An ultraviolet light of wavelength 100 nm is incident on a metal whose work function is 0.8 eV. Find out de Broglie wavelength of the ejected electron from the metal due to photoelectric effect.

**Problem 5.** An excited state of an atom has lifetime of $10^{-8}$s and emits light of wavelength 680nm. Calculate and hence show that it has more broadening in its emitted spectral line than a state with lifetime of $10^{-3}$s emitting same wavelength of light.

**Problem 6.** An electron in an excited energy level at 6eV and lifetime of $10^{-8}$s comes down to a lower level at 4.2eV. What will be the broadening of the emitted spectral line.

**Problem 7.** If wavelength of 1Å x-ray photon can be determined with a precision of one part in $10^6$ then what is the uncertainty in the photon's position when it is simultaneously measured.

**Problem 8.** Calculate the difference in energy levels, given that the broadening of the emission line spectrum between them is 0.1 Å and lifetime of the higher energy level is 0.1 µs.

Broadening of the spectral line is given by,

$$\Delta\lambda \geq \frac{1}{4\pi c}\frac{\lambda^2}{\Delta t}$$

We need to find out $\lambda$ and hence $E = h\nu = hc/\lambda$ from it. It is given that $\Delta\lambda = 0.1$Å and $\Delta t = 0.1$ µs.

$$\begin{aligned}\therefore\ \ \lambda = \sqrt{4\pi c\,\Delta\lambda\,\Delta t} &= \sqrt{4 \times 3.142 \times 3 \times 10^8 \text{ms}^{-1} \times 0.1 \times 10^{-10}\text{m} \times 0.1 \times 10^{-6}\text{s}} \\ &= 61.378 \times 10^{-6}\ \text{m}\end{aligned}$$

$$\text{and} \qquad E = \frac{hc}{\lambda} = \frac{6.626 \times 10^{-34}\,\text{J s} \times 3 \times 10^8\text{ms}^{-1}}{61.378 \times 10^{-6}\ \text{m}} = 3.2364 \times 10^{-21}\text{J} = 0.0202\text{eV}$$

**Problem 9.** A spectral line is found to be broadened by 0.00001 nm and has a central wavelength of 600 nm. How much time does an ensemble of these excited atoms take to emit $1/e$ times the total output of light.

$$N = N_0 e^{-t/\tau}$$

where $\tau$ is the lifetime. If $N = N_0/e$ then

$$\frac{N_0}{e} = N_0 e^{-t/\tau}$$

$$\text{or} \qquad e = e^{t/\tau}$$

Upon simplifying by taking natural logarithm on both sides gives,

$$t = \tau$$

Which means life time is the time taken for $N_0/e$ number of entities to decay.

**Problem 10.** At time $t = 0$ a particle is represented by the wave function

$$\Psi(x,0) = \begin{cases} \sqrt{(3/b)}(x/a), & 0 \le x \le a, \\ \sqrt{(3/b)}(b-x)/(b-a), & a \le x \le b, \\ 0, & \text{otherwise,} \end{cases}$$

where $a$ and $b$ are (positive) real constants representing two locations on the $x$-axis. What is the probability of finding the particle i) to the left of $a$ and ii) to the right of $a$?

$$P_{\text{left of } a} = \int_0^a \left| \sqrt{\frac{3}{b}} \frac{x}{a} \right|^2 dx = \int_0^a \frac{3}{ba^2} x^2 \, dx = \frac{a}{b}$$

$$P_{\text{right of } a} = 1 - P_{\text{left of } a} = 1 - \frac{a}{b} = \frac{(b-a)}{b}$$

Or,

$$P_{\text{right of } a} = \int_a^b \left| \sqrt{\frac{3}{b}} \frac{(b-x)}{(b-a)} \right|^2 dx = \int_a^b \frac{3}{b(b-a)^2} (b-x)^2 \, dx$$

$$= \left[ \frac{3}{b(b-a)^2} \frac{(b-x)^3}{(-1)(3)} \right]_a^b = \frac{(b-a)^3}{b(b-a)^2} = \frac{(b-a)}{b} = 1 - \frac{a}{b}$$

**Problem 11.** A particle in an one dimensional potential well of infinite depth makes two consecutive transitions from energy levels $E_{n+1} \longrightarrow E_n \longrightarrow E_{n-1}$ releasing energy of 3.384 eV and 2.632 eV respectively. Determine $n$.

$$E_{n+1} - E_n = \frac{(n+1)^2 h^2}{8ma^2} - \frac{n^2 h^2}{8ma^2} = (2n+1)\frac{h^2}{8ma^2}$$

$$E_n - E_{n-1} = \frac{n^2 h^2}{8ma^2} - \frac{(n-1)^2 h^2}{8ma^2} = (2n-1)\frac{h^2}{8ma^2}$$

$$\frac{E_{n+1} - E_n}{E_n - E_{n-1}} = \frac{2n+1}{2n-1} = \frac{3.384}{2.632} = 1.286 = f \text{ (say)}$$

$$2n + 1 = 2fn - f$$

or

$$2fn - 2n = f + 1$$

$$\implies \quad n = \frac{f+1}{2f-2} = \frac{1.286+1}{2 \times 1.286 - 2} = 3.9965 \approx 4$$

**Problem 12.** In one dimension, an electron is confined within a potential energy well of infinite depth, and the width of the well is 1Å.

a) What is the least energy of the electron?

The minimum energy state of the electron corresponds to the lowest quantum number $n = 1$, representing the ground state of the electron.

$$\begin{aligned} E_1 &= \frac{h^2}{8ma^2} \\ &= \frac{\left(6.626 \times 10^{-34}\text{J s}\right)^2}{8\left(9.109 \times 10^{-31}\text{kg}\right)\left(1 \times 10^{-10}\text{m}\right)^2} \\ &= 6.025 \times 10^{-18}\text{J} = 37.6\text{eV} \end{aligned}$$

b) What amount of energy is needed to the electron for a quantum leap from its ground state to its second excited state?

The electron jumps from $n = 1$ to $n = 3$ level, the required change in its energy is

$$\begin{aligned} \Delta E = E_3 - E_1 &= \frac{3^2h^2}{8ma^2} - \frac{1^2h^2}{8ma^2} = \frac{h^2}{ma^2} \\ &= \frac{\left(6.626 \times 10^{-34}\text{J s}\right)^2}{\left(9.109 \times 10^{-31}\text{kg}\right)\left(1 \times 10^{-10}\text{m}\right)^2} \\ &= 4.82 \times 10^{-17}\text{J} = 300.8\text{eV} \end{aligned}$$

c) If the electron acquires the energy needed to transition from energy level $E_1$ to $E_3$ through light absorption, what wavelength of light is necessary for this process?

$$
\begin{aligned}
\lambda &= \frac{hc}{\Delta E} \\
&= \frac{\left(6.626 \times 10^{-34}\mathrm{Js}\right)\left(3 \times 10^{8}\mathrm{m/s}\right)}{4.82 \times 10^{-17}\mathrm{J}} = 4.12\,\mathrm{nm}
\end{aligned}
$$

d) After being excited to the second excited state ($n = 3$), what wavelength of light can the electron emit during de-excitation from $n = 3$ to $n = 2$?

The energy difference for the jump is given by

$$
\begin{aligned}
\Delta E = E_3 - E_2 &= \frac{3^2 h^2}{8ma^2} - \frac{2^2 h^2}{8ma^2} = \frac{5h^2}{8ma^2} = 3.015 \times 10^{-17}\mathrm{J} \\
&= \frac{5 \times \left(6.626 \times 10^{-34}\mathrm{J\,s}\right)^2}{8 \times 9.109 \times 10^{-31}\mathrm{kg} \times \left(1 \times 10^{-10}\mathrm{m}\right)^2} \\
&= 3.012 \times 10^{-17}\mathrm{J} = 188.015\,\mathrm{eV}
\end{aligned}
$$

The wavelength of light $\lambda$ emitted during the de-excitation is,

$$
\lambda = \frac{hc}{\Delta E} = \frac{\left(6.626 \times 10^{-34}\mathrm{Js}\right)\left(3 \times 10^{8}\mathrm{m/s}\right)}{3.012 \times 10^{-17}\mathrm{J}} = 6.595\,\mathrm{nm}
$$

**Problem 13.** An electron is trapped in ground state of an one dimensional infinite potential well of width 1Å.

a) What is the probability that the electron can be detected in the left one-third of the well?

The probability of finding the electron in the left one-third of the well can be calculated by setting the limits of integration as $x_1 = 0$ and $x_2 = \frac{a}{3}$.

$$
P = \int_{x_1}^{x_2} |\psi(x)|^2 dx = \int_0^{a/3} \left| \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right) \right|^2 dx = \int_0^{a/3} \frac{2}{a} \sin^2\left(\frac{\pi}{a}x\right) dx
$$

With the identity, $\sin^2\theta = (1 - \cos(2\theta))/2$ the above integral can be written as,

$$
P = \frac{2}{a} \int_0^{a/3} \frac{1}{2} \left[1 - \cos\left(\frac{2\pi}{a}x\right)\right] dx
$$

By integrating we get,

$$P = \frac{1}{a}\left[x - \frac{a}{2\pi}\sin\left(\frac{2\pi}{a}x\right)\right]_0^{\frac{a}{3}} = \frac{1}{a}\left[\frac{a}{3} - \frac{a}{2\pi}\sin\left(\frac{2\pi}{a}\frac{a}{3}\right)\right] = \frac{1}{3} - \frac{1}{2\pi}\sin\left(\frac{2\pi}{3}\right)$$
$$= 0.3333 - 0.1378 = 0.1955$$

b) What is the probability that the electron can be detected in the middle one-third of the well?

The probability of detection of electron in the entire well is 1. The probability of the detection in the right one third of the well is also 0.1955 due to symmetry. Thus the probability of the detection of electron in the middle one-third of the well is

$$P = 1 - 0.1955 - 0.1955 = 0.609$$

**Problem 14.** In a one-dimensional infinite potential well with an initial width of $a$ an electron possesses a ground state energy denoted as $E_1$. If the width is altered to a new value, $a'$ resulting in a modified ground state energy of $E_1' = 0.5E_1$, what is the ratio $a/a'$ ?

The minimum energy state of the electron corresponds to the lowest quantum number $n = 1$, representing the ground state of the electron.

$$E_1 = \frac{h^2}{8ma^2} \quad \text{and} \quad E_1' = \frac{h^2}{8ma'^2}$$

$$\therefore \quad \frac{E_1'}{E_1} = \frac{a^2}{a'^2} = 0.5$$

$$\implies \frac{a}{a'} = \sqrt{0.5} = 0.7071$$

**Problem 15.** For a particle in the ground state of an infinite square well with width $a$, calculate the probability of finding the particle in the middle half of the well (between $x_1 = a/4$ and $x_2 = 3a/4$).

The probability of finding the particle in the middle half of the well is,

$$P = \int_{\frac{a}{4}}^{\frac{3a}{4}} \left| \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right) \right|^2 dx$$

With the identity, $\sin^2\theta = (1 - \cos(2\theta))/2$ the above integral can be written as,

$$P = \frac{2}{a} \int_{\frac{a}{4}}^{\frac{3a}{4}} \frac{1}{2} \left[1 - \cos\left(\frac{2\pi}{a}x\right)\right] dx$$

By integrating we get,

$$\begin{aligned}
P &= \frac{1}{a}\left[x - \frac{a}{2\pi}\sin\left(\frac{2\pi}{a}x\right)\right]_{\frac{a}{4}}^{\frac{3a}{4}} \\
&= \frac{1}{a}\left[\frac{3a}{4} - \frac{a}{4} - \frac{a}{2\pi}\sin\left(\frac{2\pi}{a}\frac{3a}{4}\right) + \frac{a}{2\pi}\sin\left(\frac{2\pi}{a}\frac{a}{4}\right)\right] \\
&= \frac{1}{2} - \frac{1}{2\pi}\left[\sin\left(\frac{6\pi}{4}\right) - \sin\left(\frac{2\pi}{4}\right)\right] \\
&= 0.8183
\end{aligned}$$

**Problem 16.** What must be the width of an one-dimensional infinite potential well, if an electron trapped in the $n = 3$ state is to have an energy of 4.7eV?

The energy of the electron corresponding to quantum number $n = 3$ is

$$\begin{aligned}
E_3 &= \frac{3^2h^2}{8ma^2} \\
4.7 \times 1.602 \times 10^{-19}\text{J} &= \frac{9 \times \left(6.626 \times 10^{-34}\text{Js}\right)^2}{8 \times 9.109 \times 10^{-31}\text{kg} \times a^2}
\end{aligned}$$

By rearranging we get width of the well.

$$a^2 = \frac{9 \times \left(6.626 \times 10^{-34}\text{Js}\right)^2}{8 \times 9.109 \times 10^{-31}\text{kg} \times 4.7 \times 1.602 \times 10^{-19}\text{J}}$$

$$\therefore \quad a = 0.8486\text{nm}$$

**Problem 17.** The normalized wave function for a certain particle is $\psi = \sqrt{\frac{8}{3\pi}} \cos^2 x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and 0 otherwise. Calculate the probability that the particle can be found between $x_1 = 0$ and $x_2 = \frac{\pi}{4}$.

The probability of finding the particle in between two points $x_1$ and $x_2$ is given by

$$
\begin{aligned}
P &= \int_{x_1}^{x_2} |\psi|^2 \, dx = \int_0^{\frac{\pi}{4}} \left| \sqrt{\frac{8}{3\pi}} \cos^2 x \right|^2 dx \\
&= \int_0^{\frac{\pi}{4}} \frac{8}{3\pi} \cos^4 x \, dx = \frac{8}{3\pi} \int_0^{\frac{\pi}{4}} \left(\cos^2 x\right)^2 dx \\
&= \frac{8}{3\pi} \int_0^{\frac{\pi}{4}} \left(\frac{1+\cos(2x)}{2}\right)^2 dx = \frac{2}{3\pi} \int_0^{\frac{\pi}{4}} \left(1 + \cos^2(2x) + 2\cos(2x)\right) dx \\
&= \frac{2}{3\pi} \int_0^{\frac{\pi}{4}} \left(1 + \frac{1+\cos(4x)}{2} + 2\cos(2x)\right) dx = \frac{2}{3\pi} \left[\frac{3}{2}x + \frac{\sin(4x)}{8} + \sin(2x)\right]_0^{\frac{\pi}{4}} \\
&= \frac{2}{3\pi} \left[\frac{3\pi}{8} + \frac{\sin(\pi)}{8} + \sin\left(\frac{\pi}{2}\right)\right] = \frac{1}{4} + \frac{2}{3\pi} = 0.4622
\end{aligned}
$$

**Problem 18.** Find the eigen function for the operator $\left(x + \frac{d}{dx}\right)$

The eigen value equation for the operator $x + \frac{d}{dx}$ is given by

$$\left(x + \frac{d}{dx}\right)\psi = \alpha\psi$$

Where $\psi$ is the eigen function and $\alpha$ is the eigen value. From this equation, we have

$$x\psi + \frac{d\psi}{dx} = \alpha\psi$$

or

$$\frac{d\psi}{dx} = (\alpha - x)\,\psi$$

or

$$\frac{d\psi}{\psi} = (\alpha - x)\,dx$$

Integrating both sides of this equation, we get

$$\ln\psi = \alpha x - \frac{x^2}{2} + \ln c$$

or

$$\ln\frac{\psi}{c} = \alpha x - \frac{x^2}{2}$$

or

$$\psi = ce^{(\alpha x - x^2/2)}$$

**Problem 19.** Find the energy eigenvalue of a particle whose normalized wavefunction is $\psi = Ae^{-ikx}$ and is not subjected to any potential.

The Hamiltonian operator is,

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

and the eigenvalue equation for the operator $\hat{H}$ is

$$\hat{H}\psi = E\psi$$

For the given particle $V = 0$ we have

$$\hat{H}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2(Ae^{-ikx})}{\partial x^2} = -\frac{\hbar^2}{2m}A(-ik)^2e^{-ikx} = \frac{\hbar^2k^2}{2m}\psi$$

By comparing it with the eigenvalue equation the energy eigen value of the particle is,

$$E = \frac{\hbar^2k^2}{2m}$$

**Problem 20.** A particle trapped in an one dimensional well of length $L$ is described by the normalized wave function $\psi = ax$ where $a$ is constant with suitable dimensions. What is the expectation value of the particle's position $\langle x\rangle$.

The expectation value of the particle's position $\langle x\rangle$ when the particle is described by a normalized wavefunction is given by

$$\langle x\rangle = \int_{-\infty}^{\infty} \psi^* x \psi dx$$

Hence, we have

$$\langle x\rangle = \int_0^L |ax|^2 x dx = \int_0^L a^2 x^3 dx = a^2 \left[\frac{x^4}{4}\right]_0^L = \frac{1}{4} a^2 L^4$$

**Problem 21.** The normalized wavefunction $\psi = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right)$ describes a particle in an one dimensional potential well of length $a$ and infinite depth. What is the expectation value of the particle's position $\langle x\rangle$.

The expectation value of the particle's position $\langle x\rangle$ is given by

$$\langle x\rangle = \int_{-\infty}^{\infty} \psi^* x \psi dx$$

Hence we have

$$\begin{aligned}
\langle x\rangle &= \int_0^a \left|\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right)\right|^2 x\, dx \\
&= \frac{2}{a} \int_0^a \sin^2\left(\frac{n\pi}{a} x\right) x\, dx = \frac{2}{a} \int_0^a x \frac{1}{2} \left[1 - \cos\left(2\frac{n\pi}{a} x\right)\right] dx \\
&= \frac{1}{a} \int_0^a \left[x - x\cos\left(\frac{2n\pi}{a} x\right)\right] dx \\
&= \frac{1}{a} \left[\frac{x^2}{2} - x\frac{a}{2n\pi} \sin\left(\frac{2n\pi}{a} x\right) - \frac{a^2}{(2n\pi)^2} \cos\left(\frac{2n\pi}{a} x\right)\right]_0^a \\
&= \frac{1}{a} \left[\frac{a^2}{2} - \frac{a^2}{(2n\pi)^2} \cos(2n\pi) + \frac{a^2}{(2n\pi)^2} \cos(0)\right] = \frac{a}{2}
\end{aligned}$$

**Problem 22.** The normalized wavefunction $\psi = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right)$ describes a particle in an one dimensional potential well of length $a$ and infinite depth. What is the expectation value of the particle's momentum $\langle p\rangle$.

The expectation value of the particle's momentum $\langle p \rangle$ is given by

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^* \left( -i\hbar \frac{\partial}{\partial x} \right) \psi \, dx$$

Hence we have

$$\begin{aligned}
\langle p \rangle &= \int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) \left(-i\hbar \frac{\partial}{\partial x}\right) \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) \, dx \\
&= \frac{2}{a}(-i\hbar) \int_0^a \sin\left(\frac{n\pi}{a}x\right) d\left[\sin\left(\frac{n\pi}{a}x\right)\right] \\
&= \frac{2}{a}(-i\hbar) \left[\frac{1}{2}\sin^2\left(\frac{n\pi}{a}x\right)\right]_0^a \\
&= \frac{1}{a}(-i\hbar) \left[\sin^2(n\pi) - \sin^2 0\right] = 0
\end{aligned}$$

**Problem 23.** The normalized wavefunction $\psi = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$ describes a particle in an one dimensional potential well of length $a$ and infinite depth. What is the expectation value of the particle's kinetic energy $\langle T \rangle$.

The expectation value of the particle's kinetic energy $\langle T \rangle$ is given by

$$\langle T \rangle = \int_{-\infty}^{\infty} \psi^* \left( \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \psi \, dx$$

Hence we have

$$
\begin{aligned}
\langle T\rangle &= \int_0^a \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)\,dx \\
&= \frac{-\hbar^2}{2m}\left(\frac{2}{a}\right)\left(\frac{n\pi}{a}\right)^2\int_0^a -\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x\right)dx \\
&= \frac{n^2h^2}{4ma^3}\int_0^a \frac{1}{2}\left[1-\cos\left(2\frac{n\pi}{a}x\right)\right]dx \\
&= \frac{n^2h^2}{8ma^3}\left[x-\frac{a}{2n\pi}\sin\left(2\frac{n\pi}{a}x\right)\right]_0^a \\
&= \frac{n^2h^2}{8ma^3}\left[a-\frac{a}{2n\pi}\sin(2n\pi)+\frac{a}{2n\pi}\sin(0)\right] \\
&= \frac{n^2h^2}{8ma^3}a = \frac{n^2h^2}{8ma^2} \qquad \text{with } n = 1, 2, 3, \ldots
\end{aligned}
$$

**Problem 24.** The first excited state wave function of a particle in an infinite well is $B\sin(10^9\pi x)$. Calculate $B$.

$$
\psi_2 = \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi}{a}x\right) = B\sin(10^9\pi x) \qquad (n = 2 \text{ for first excited state})
$$

$$
k = \frac{n\pi}{a} = \frac{2\pi}{a} = 10^9\pi\text{m}^{-1} \qquad \therefore\ a = 2\times10^{-9}\text{m}
$$

$$
B = \sqrt{\frac{2}{a}} = \sqrt{\frac{2}{2\times10^{-9}\text{m}}} = 10^{9/2}\text{m}^{-1/2} = 31\,622.78\,\text{m}^{-1/2}
$$

**Problem 25.** An electron is confined to move between two rigid wall separated by 20Å. Find the de Broglie wavelength and the corresponding energy eigen values for the first two allowed energy states.

$$
E = \frac{n^2h^2}{8ma^2} \qquad \text{with } n = 1, 2, 3, \ldots
$$

$$\therefore \lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{2m\frac{n^2h^2}{8ma^2}}} = \frac{h}{\sqrt{\frac{n^2h^2}{4a^2}}} = \frac{2a}{n}$$

For $n = 1$

$$\lambda_1 = \frac{2a}{n} = \frac{2 \times 20\text{Å}}{1} = 40\text{Å}$$

$$E_1 = \frac{n^2h^2}{8ma^2} = \frac{1^2 \times \left(6.626 \times 10^{-34}\text{J s}\right)^2}{8 \times 9.109 \times 10^{-31}\text{kg} \times (20 \times 10^{-10})^2} = 1.506 \times 10^{-20}\text{J} = 0.094\,\text{eV}$$

For $n = 2$

$$\lambda_2 = \frac{2a}{n} = \frac{2 \times 20\text{Å}}{2} = 20\text{Å}$$

$$E_2 = \frac{n^2h^2}{8ma^2} = \frac{2^2 \times \left(6.626 \times 10^{-34}\text{J s}\right)^2}{8 \times 9.109 \times 10^{-31}\text{kg} \times (20 \times 10^{-10})^2} = 6.025 \times 10^{-20}\text{J} = 0.376\,\text{eV}$$

**Problem 26.** An infinite well between 0 to $a$ is shifted to the new position $-0.5a$ to $0.5a$. Will the eigenvalues and eigenfunctions change for this new configuration? Explain why.

Energy of the particle in the potential well is inversely dependent on square of the well width.

$$E = \frac{n^2h^2}{8ma^2}$$

As the well width $= 0.5a - (-0.5a) = a$ does n't change due to shifting the energy eigenvalues remain same.

However the wave functions will change to account for the new symmetry of the well.

**Problem 27.** An infinite well between 0 to $a$ is shifted to the new position $-a/2$ to $a/2$. What will be the eigenfunctions for this new configuration? Hint: Make use of the eigenfunction for the potential well between 0 to $a$.

Due to the shifting the new position coordinates $x'$ will be related to old coordinates $x$ by the relation $x' = x - a/2$ . Therefore, by substituting

$$x \longrightarrow x + a/2$$

in the eigenfunctions of the potential well from 0 to $a$,

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right)$$

we get new wave function as,

$$\begin{aligned}
\rightarrow \psi_n(x) &= \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}(x + a/2)\right) \\
&= \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x + \frac{n\pi}{a}\frac{a}{2}\right) \\
&= \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x + \frac{n\pi}{2}\right)
\end{aligned}$$

$$\therefore \quad \psi_n(x) = \begin{cases} (-1)^{n/2}\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right) & \text{for } n \text{ even} \\ (-1)^{(n-1)/2}\sqrt{\frac{2}{a}} \cos\left(\frac{n\pi}{a} x\right) & \text{for } n \text{ odd} \end{cases}$$

**Problem 28.** Use the expression relating momentum of a particle to the wavelength of its equivalent wave to arrive at the expression for the energy of a particle of mass $m$ in the ground state of an infinite well of width $a$.

For particle in ground state of an infinite square well, $\lambda = 2a$ (wave function is a half sine wave) where $a$ is the width of the well. Using de Broglie's wavelength, its momentum is given by,

$$p = \frac{h}{\lambda}$$

Therefore, energy of this particle is,

$$E = \frac{p^2}{2m} = \frac{1}{2m}\frac{h^2}{\lambda^2} = \frac{h^2}{2m(2a)^2} = \frac{h^2}{8ma^2}$$