

Linear ordinary differential equations of higher order

Differential equation: An equation containing the derivatives of one or more dependent variables (or unknown functions) with respect to one or more independent variables is called a differential equation.

$$\text{ex: } 1) \frac{dy}{dx} + 5y = 0 \quad 2) \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 2y = 0$$

$$3) \frac{dy}{dx} + 4x \sin y = e^x \quad 4) x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0$$

$$5) u_{xx} + u_{yy} = 2xy \quad 6) \frac{\partial u}{\partial t} = c \frac{\partial^2 u}{\partial x^2}$$

Ordinary differential equation (ODE): A differential equation involving ordinary derivatives of one or more dependent variables w.r.t a single independent variable is called an ODE. Examples (1) - (3) are ODEs.

The alternative (more than one independent variable) is called a partial differential equation (PDE).

Examples (4) - (6) are PDEs.

Order of differential eq: It is defined as highest derivative appearing in the equation.

Degree of differential eq: It is defined as the positive integral power of the highest ordered derivative appearing in the equation.

Note: In order to obtain degree of differential equation, the equation must be expressed in a form free from the radicals.

and fractional powers as far as derivatives are concerned.

ex: 1) $\frac{d^2y}{dx^2} + 16 \frac{dy}{dx} + 2y = 0$: Order - 2, Degree - 1

2) $\left[\frac{d^2y}{dx^2} + 2y \right]^{\frac{1}{2}} = \frac{dy}{dx} + 2$: Order - 2, Degree - 3

3) $\frac{d^4y}{dx^4} + \sin y''' = 0$: Order - 4, Degree - not defined

Linear differential equation (LDE):

A differential equation is said to be linear if the dependent variable and all its derivatives are of first degree and they are not multiplied together.

Otherwise, the D.E. is said to be non-linear.

Examples:- 1) $\frac{dy}{dx} + 2y = e^x$ → linear

2) $\frac{d^2y}{dx^2} + 3 \frac{dy}{dx} + 5y = \cos x$ → linear

3) $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y = 0$ → Non-linear

4) $y''' + y'y'' + 2y \tan x = 0$ → Non-linear

5) $y'' + 2xy' = \sin y$ → Non-linear

because $\sin y$ involves powers of y .

Initial value problem (IVP):

Differential equation subject to the condition prescribed at one point is called an IVP.

ex: 1) $\frac{dy}{dx} + x^2 y = e^{2x}$; $y(0) = 1$

2) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 6y = \sin x$; $y(1) = 1$, $y'(1) = 2$

Boundary value problem (BVP): Differential equation subject to the condition prescribed at more than one point is called boundary value problem.

ex: 1) $\frac{d^2y}{dx^2} + y = \cos 2x$; $y(0) = 1$, $y(\pi) = 0$

2) $x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + 6y = 2$; $y(0) = 1$, $y(2) = \pi$

Linear ODE of higher order

A linear ODE of n^{th} order is of the form:

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = g(x)$$

$$\text{or } (a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n) y = f(x) \rightarrow ①$$

Where a_0, a_1, \dots, a_n are constants or functions of x alone,

$$D = \frac{d}{dx} \text{ and } a_0 \neq 0.$$

Let $F(D) = a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n$ be a differential operator then ① can be written as

$$F(D)y = g(x) \rightarrow ②$$

If $g(x) = 0$, then ① is said to be homogeneous otherwise it is said to be non-homogeneous.

If y_1, y_2, \dots, y_n are n linearly independent solutions of

n^{th} order homogeneous differential eq. then $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ where c_1, c_2, \dots, c_n are arbitrary constants is also a solution of homogeneous equation.

If y_p is a particular function which satisfies non-homogeneous linear differential eq. then the general solution of ② is

$$y = y_c + y_p$$

where $y_c = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ is called as complementary function (CF) of corresponding homogeneous D.E. and y_p is called particular integral of ② involving no arbitrary constants

Homogeneous LDE with constant coefficients

Consider a n^{th} order homogeneous ODE,

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n) y = 0 \rightarrow \textcircled{*}$$

where a_i 's are constants.

Method to find solution of $\textcircled{*}$

Consider 2nd order homogeneous LDE,

$$a_0 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0 \rightarrow \textcircled{1}$$

Let $y = e^{mx}$ be a trial solution of ①

$$\text{Then } \textcircled{1} \Rightarrow a_0 m^2 e^{mx} + a_1 m e^{mx} + a_2 e^{mx} = 0$$

$$\Rightarrow (a_0 m^2 + a_1 m + a_2) e^{mx} = 0$$

$$\Rightarrow a_0 m^2 + a_1 m + a_2 = 0 \rightarrow \textcircled{2} \quad (\because e^{mx} \neq 0, \forall x)$$

② is called Auxiliary equation (A.E) or characteristic equation of ①

The independent solutions of ① depend on the nature of the roots of above quadratic equation.

Case 1: Roots are real and distinct ($m_1 \neq m_2$, real)

If m_1 and m_2 are 2 distinct real roots, then the linearly independent solutions are $e^{m_1 x}$ and $e^{m_2 x}$

The general solution of ① is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Case 2: Roots are real and equal ($m_1 = m_2$)

If the roots of auxiliary eq. are real and repeated say $m_1 = m_2 = m$ then the general solution of ① is

$$y = c_1 e^{mx} + c_2 x e^{mx} = (c_1 + c_2 x) e^{mx}$$

Case 3: Roots are complex ($m_1 = a+ib$, $m_2 = a-ib$)

The general solution of ① is

$$y = e^{ax} (c_1 \cos bx + c_2 \sin bx)$$

Note: If the ODE is 4th order and roots of auxiliary eq. are complex repeated say $m = a \pm ib$, $m = a \pm ib$ then the general solution of ① is

$$y = e^{ax} [(c_1 + c_2 x) \cos bx + (c_3 + c_4 x) \sin bx]$$

Solve the foll D.Es:

$$1) \frac{d^2y}{dx^2} - 5 \frac{dy}{dx} - 3y = 0$$

Sol: The auxiliary eq. is $2m^2 - 5m - 3 = 0$

$$2m^2 - 6m + m - 3 = 0$$

$$2m(m-3) + 1(m-3) = 0$$

$$\begin{array}{c} -6m \\ \diagup \\ -6m \end{array} \quad m$$

$$\therefore m = 3, -\frac{1}{2}$$

Roots are real and distinct

\therefore The general solution is $y = c_1 e^{3x} + c_2 e^{-\frac{x}{2}}$

$$2) y'' - 10y' + 25y = 0$$

Sol: The auxiliary eq. is $m^2 - 10m + 25 = 0$
 $\Rightarrow (m-5)^2 = 0$

$$\therefore m = 5, 5$$

Roots are real and equal

\therefore G.S. is $y = (c_1 + c_2 x) e^{5x}$

$$3) y'' + 4y' + 7y = 0$$

Sol: The auxiliary eq. is $m^2 + 4m + 7 = 0$

$$m = \frac{-4 \pm \sqrt{16-28}}{2} = \frac{-4 \pm \sqrt{-12}}{2} = \frac{-4 \pm 2\sqrt{3}i}{2}$$

$$\therefore m = -2 \pm \sqrt{3}i$$

$$\Rightarrow m_1 = -2 + \sqrt{3}i, m_2 = -2 - \sqrt{3}i$$

The general solution is

$$y = e^{-2x} (c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x)$$

$$4) (D^3 + 1)y = 0$$

Sol: AE : $m^3 + 1 = 0$

$$(m+1)(m^2 + 1 - m) = 0$$

$$m+1=0, \quad m^2 - m + 1 = 0$$

$$\Rightarrow m = -1, \quad m = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

The general solution is

$$y = c_1 e^{-x} + e^{\frac{1}{2}x} \left[c_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}x\right) \right]$$

$$5) (D^4 + 8D^2 + 16)y = 0$$

Sol: AE : $m^4 + 8m^2 + 16 = 0$

$$(m^2 + 4)^2 = 0$$

$$\Rightarrow (m^2 + 4)(m^2 + 4) = 0$$

$$\Rightarrow m = \pm 2i, \pm 2i$$

The general solution is

$$y = (c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x$$

$$6) y''' - 3y'' + 4y = 0$$

Sol: AE : $m^3 - 3m^2 + 4 = 0$

By inspection $m = -1$ is a root

$$\begin{array}{c|cccc} -1 & 1 & -3 & 0 & 4 \\ \hline & 0 & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & \underline{10} \end{array}$$

A.E : $(m+1)(m^2 - 4m + 4) = 0$

$$(m+1) = 0, \quad (m-2)^2 = 0$$

$$m_1 = -1, \quad m_2 = m_3 = 2$$

\therefore The general solution is

$$y = c_1 e^{-x} + (c_2 + c_3 x) e^{2x}$$

7) Solve IVP $\frac{d^2x}{dt^2} + 5 \frac{dx}{dt} + 6x = 0$, given $x(0) = 0, \frac{dx}{dt}(0) = 15$

sol: AE: $m^2 + 5m + 6 = 0$

$$(m+3)(m+2) = 0$$

$$\therefore m = -3, -2$$

The general sol. is $x = c_1 e^{-3t} + c_2 e^{-2t} \rightarrow ①$

Using the initial conditions $x(0) = 0$

$$① \Rightarrow 0 = c_1 + c_2 \rightarrow ②$$

$$x' = \frac{dx}{dt} = -3c_1 e^{-3t} - 2c_2 e^{-2t}$$

$$x'(0) = 15 \Rightarrow 15 = -3c_1 - 2c_2 \rightarrow ③$$

$$② \times 3 \Rightarrow 3c_1 + 3c_2 = 0 \rightarrow ④$$

$$③ + ④ \Rightarrow c_2 = 15$$

$$\therefore c_1 = -15$$

$$① \Rightarrow x = -15 e^{-3t} + 15 e^{-2t} = 15 (e^{-2t} - e^{-3t})$$

8) Solve BVP $y'' - 2y' + 2y = 0$, given $y(0) = 1, \underbrace{y'(\pi) = 1}_{\text{boundary conditions}}$

Sol: AE : $m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$$\therefore G.S. \text{ is } y = e^x (c_1 \cos x + c_2 \sin x) \rightarrow ①$$

Given: $y(0) = 1$

$$① \Rightarrow 1 = e^0 (c_1 \cos 0 + c_2 \sin 0) = 1(c_1 + 0)$$

$$\therefore c_1 = 1$$

$$① \Rightarrow y' = e^x (-c_1 \sin x + c_2 \cos x) + (c_1 \cos x + c_2 \sin x) e^x \rightarrow ②$$

Given: $y'(\pi) = 1$

$$② \Rightarrow 1 = e^\pi (c_2 \cos \pi) + c_1 \cos \pi e^\pi$$

$$1 = -c_2 e^\pi - e^\pi$$

$$1 + e^\pi = -c_2 e^\pi$$

$$\therefore c_2 = -\left(\frac{1+e^\pi}{e^\pi}\right)$$

$$① \Rightarrow y = e^x \left[\cos x - \left(\frac{1+e^\pi}{e^\pi}\right) \sin x \right]$$

Exercise:

1) Solve $(D^3 - 13D + 12)y = 0$ Ans: $y = c_1 e^x + c_2 e^{-4x} + c_3 e^{3x}$

2) Solve $(D^4 - D^3 - 9D^2 - 11D - 4)y = 0$ Ans: $c_1 e^{4x} + (c_2 + c_3 x + c_4 x^2) e^{-x}$

3) $(D^2 + 1)^2 + (D - 1)^2 y = 0$ Ans: $(c_1 + c_2 x) \sin x + (c_3 + c_4 x) \cos x + (c_5 + c_6 x) e^x$

4) Solve IVP $\frac{d^2y}{dx^2} + 8y = 0$, given $y(0) = 1$, $y'(0) = 2\sqrt{2}$
 Ans: $y = c_0 \cos 2\sqrt{2}x + \beta \sin 2\sqrt{2}x$

Non-homogeneous linear ODE with constant coefficients - Inverse differential operator

The general soln of non-homogeneous ODE has 2 parts - complementary function and particular integral

The complementary function is solution of homogeneous ODE.

Method of finding particular integral

Consider $F(D)y = g(x)$

$$\text{then } y_p = \frac{1}{F(D)}g(x) \rightarrow \textcircled{1}$$

i.e. P.I. = $\frac{1}{F(D)}g(x)$ where $\frac{1}{F(D)}$ is called

'inverse differential operator.'

Rules for finding particular integral:

1) Let $g(x) = K e^{ax}$ where $a \in \mathbb{R}$ then (1) becomes

$$y_p = \frac{K e^{ax}}{F(D)} = \frac{K e^{ax}}{F(a)}, \text{ provided } F(a) \neq 0$$

If $F(a) = 0$ then $y_p = \frac{K x e^{ax}}{F'(a)}$, provided $F'(a) \neq 0$

If $F'(a) = 0$ then $y_p = \frac{K x^2 e^{ax}}{F''(a)}$, provided $F''(a) \neq 0$ and

so on.

$$\left| \begin{array}{l} \text{Ex:} \\ \text{P.I.} = \frac{e^{-x}}{D^3+1} = \frac{x e^{-x}}{3D^2} \\ = \frac{x e^{-x}}{3} \end{array} \right.$$

2) Let $g(x) = \sin(ax+b)$ or $\cos(ax+b)$ then (1) becomes

$$y_p = \frac{1}{F(D)} \sin(ax+b) \text{ or } \frac{1}{F(D)} \cos(ax+b)$$

We have 3 cases:

Case 1: Suppose $F(D)$ contains only even powers of D
i.e. $F(D) = F(D^2)$

$$\text{Then } y_p = \frac{1}{F(D^2)} \sin(2x) \text{ or } \frac{1}{F(D^2)} \cos(2x)$$

It can be shown that

$$y_p = \frac{1}{F(-a^2)} \sin ax \text{ or } \frac{1}{F(-a^2)} \cos ax, \text{ provided } F(-a^2) \neq 0$$

$$\begin{aligned} \text{ex: } \frac{1}{D^4+D^2+1} \sin 2x &= \frac{\sin 2x}{(D^2)^2+D^2+1} = \frac{\sin 2x}{(-4)^2+(-4)+1} = \frac{\sin 2x}{16-4+1} \\ &= \frac{\sin 2x}{13} \end{aligned}$$

If $F(-a^2) = 0$, we proceed as follows:

$$\begin{aligned} \text{ex: } i) \quad \frac{1}{D^2+4} \sin 2x &= \frac{x}{2D} \sin 2x = \frac{x}{2} \left\{ \frac{1}{D} \sin 2x \right\} \\ &= \frac{x}{2} \left\{ -\frac{\cos 2x}{2} \right\} = -\frac{x}{4} \cos 2x \end{aligned}$$

$$\begin{aligned} ii) \quad \frac{1}{D^2+9} \cos 3x &= \frac{x}{2D} \cos 3x = \frac{x}{2} \left\{ \frac{1}{D} \cos 3x \right\} \\ &= \frac{x}{2} \left\{ \frac{1}{3} \sin 3x \right\} = \frac{x}{6} \sin 3x \end{aligned}$$

Case 2: Suppose $F(D)$ also contains odd powered terms.

Then we proceed as in the full example:

$$\begin{aligned} \text{ex: } y_p &= \frac{1}{D^2+2D+1} \sin 3x = \frac{1}{-9+2D+1} \sin 3x = \frac{\sin 3x}{2D-8} \\ &= \frac{2D+8}{(2D-8)(2D+8)} \sin 3x \\ &= \frac{2D+8}{4D^2-64} \sin 3x \end{aligned}$$

$$= \frac{(2D+8)}{-100} \sin 3x$$

$$= \frac{6 \cos 3x}{-100} - \frac{8 \sin 3x}{100}$$

$$= -\frac{1}{100} \left\{ 6 \cos 3x + 8 \sin 3x \right\}$$

$$\therefore y_p = -\frac{1}{50} \left\{ 3 \cos 3x + 4 \sin 3x \right\}$$

Case 3: If $F(D) = D^2 + a^2$

Then $F(D)y = \sin ax$ or $\cos ax$

$$\Rightarrow y_p = \frac{1}{D^2 + a^2} \sin ax \text{ or } \frac{1}{D^2 + a^2} \cos ax$$

Now if we replace D^2 by $(-a^2)$ then the denominator becomes zero, so in this case we cannot replace D^2 by $-a^2$

It can be shown that

$$\frac{1}{D^2 + a^2} \sin ax = -\frac{x}{2a} \cos ax$$

$$\frac{1}{D^2 + a^2} \cos ax = \frac{x}{2a} \sin ax$$

3) If $g(x) = x^n$ then $y_p = \frac{1}{F(D)} x^n$

In order to evaluate P.I., we write $\frac{1}{F(D)}$ as $[F(D)]^{-1}$ and expand it in ascending powers of D and then each term in R.H.S. is operated on x^n to get the required P.I.

Note: 1) WKT $D^k x^n = \frac{d^k}{dx^k} (x^n) = 0$ if $k > n$

So the expansion is carried out only upto the term containing D^n

2) The following series expansions are useful:

$$a) (1+D)^{-1} = 1 - D + D^2 - D^3 + \dots$$

$$b) (1-D)^{-1} = 1 + D + D^2 + D^3 + \dots$$

$$c) (1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \dots$$

$$d) (1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \dots$$

$$\begin{aligned} \text{ex: } & \frac{1}{D-1} \cdot x^3 \\ &= -\frac{1}{(1-D^2)} x^3 \\ &= -(1+D^2+D^4+\dots)x^3 \\ &= -(x^3 + 6x) \end{aligned}$$

4) If $g(x) = e^{ax} V(x)$ where $V(x)$ is a function of x

$$y_p = \frac{1}{F(D)} e^{ax} V(x) = e^{ax} \frac{1}{F(D+a)} V(x)$$

The R.H.S. can be evaluated by earlier methods.

Example:

$$1) \text{ Solve: } (D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$$

Sol: The auxiliary equation is $(m-2)^2 = 0$, the roots

$$\text{are } m = 2, 2$$

$$\therefore y_c = C_1 e^{2x} + C_2 x e^{2x}$$

The particular integral is obtained as follows:

$$y_p = 8 \left[\frac{e^{2x}}{(D-2)^2} + \frac{\sin 2x}{(D-2)^2} + \frac{x^2}{(D-2)^2} \right]$$

$$\frac{1}{(D-2)^2} e^{2x} = \frac{x^2 e^{2x}}{2}$$

$$\frac{1}{(D-2)^2} \sin 2x = \frac{1}{D^2 - 4D + 4} \sin 2x = -\frac{1}{4(D-4)} \sin 2x = -\frac{1}{4D} \sin 2x = \frac{\cos 2x}{8}$$

$$\frac{1}{(D-2)^2} x^2 = \frac{1}{\{-2(1-\frac{D}{2})\}^2} x^2 = \frac{1}{4} \left(1 - \frac{D}{2}\right)^{-2} = \frac{1}{4} \left[\frac{1+2D+3}{2} \left(\frac{D}{2}\right)^2 + \dots\right] x^2$$

$$\frac{1}{(D-2)^2} x^2 = \frac{1}{4} \left\{ x^2 + D(x^2) + \frac{3}{4} D^2(x^2) \right\} = \frac{1}{4} \left\{ x^2 + 2x + \frac{3}{2} \right\}$$

$$\therefore y_p = 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$$

\therefore The general solution is $y = y_c + y_p$

$$\therefore y = c_1 e^{2x} + c_2 x e^{2x} + 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$$

2) $y'' - 4y = \cosh(2x-1) + 3^x$

Sol: The auxiliary equation is: $m^2 - 4 = 0$

$$\Rightarrow m = \pm 2$$

$$\therefore y_c = c_1 e^{2x} + c_2 x^{-2x}$$

The particular integral is:

$$y_p = \frac{1}{(D^2 - 4)} \cosh(2x-1) + \frac{1}{(D^2 - 4)} 3^x$$

$$\frac{1}{(D^2 - 4)} \cosh(2x-1) = \frac{1}{(D^2 - 4)} \left\{ \frac{e^{2x-1} + e^{-2x+1}}{2} \right\} = \frac{x e^{2x-1}}{2 \cdot 2D} + \frac{x e^{-2x+1}}{2 \cdot 2D}$$

$$= \frac{1}{8} \left\{ x e^{2x-1} - x e^{-2x+1} \right\}$$

$$\frac{1}{(D^2 - 4)} 3^x = \frac{1}{(D^2 - 4)} e^{x \ln 3} = \frac{1}{[(\ln 3)^2 - 4]} 3^x$$

\therefore The general solution is $y = y_c + y_p$

$$\Rightarrow y = C_1 e^{2x} + C_2 x e^{-2x} + \frac{1}{8} (x e^{2x-1} - x e^{-2x+1}) + \frac{1}{[(\log 3)^2 - 4]} x^3$$

3) $y'' - 2y' + y = x e^x \sin x$

Sol: The A.E. is $m^2 - 2m + 1 = 0$

$$\Rightarrow (m-1)^2 = 0$$

$$\therefore m = 1, 1$$

$$y_c = C_1 e^x + C_2 x e^x$$

$$y_p = \frac{1}{(D-1)^2} e^x x \sin x = e^x \frac{1}{(D-1+1)^2} x \sin x = e^x \frac{1}{D^2} x \sin x$$

Integrating $x \sin x$ twice using by parts

$$\int x \sin x dx = x(-\cos x) - \int 1 \cdot (-\cos x) dx = -x \cos x + \sin x$$

$$\begin{aligned} \int (-x \cos x + \sin x) dx &= \int (-x \cos x) dx + \int \sin x dx \\ &= - \left\{ x \sin x - \int 1 \cdot \sin x dx \right\} - \cos x \end{aligned}$$

$$= - \left\{ x \sin x + \cos x \right\} - \cos x = -x \sin x - 2 \cos x$$

$$\therefore y_p = -e^x (x \sin x + 2 \cos x)$$

The general solution is $y = y_c + y_p$

$$\therefore y = C_1 e^x + C_2 x e^x - e^x (x \sin x + 2 \cos x)$$

$$4) (D^3 + 3D^2 + 3D + 1)y = e^{-x}$$

Sol: The auxiliary equation is $m^3 + 3m^2 + 3m + 1 = 0$
 $\Rightarrow (m+1)^3 = 0$
 $\therefore m = -1, -1, -1$

$$\therefore y_c = c_1 e^{-x} + c_2 xe^{-x} + c_3 x^2 e^{-x}$$

$$y_p = \frac{1}{(D+1)^3} e^{-x} = \frac{x^3 e^{-x}}{6}$$

The general solution is $y = y_c + y_p$

$$\therefore y = c_1 e^{-x} + c_2 xe^{-x} + c_3 x^2 e^{-x} + \frac{x^3 e^{-x}}{6}$$

5) Find the particular integral of $(D^3 + 4D)y = 2 \sin 2x$

Sol: $y_p = \frac{1}{D^3 + 4D} \sin(2x) = \frac{1}{D(D^2 + 4)} \sin 2x$

$$\therefore y_p = \frac{1}{3D^2 + 4} \sin 2x = \frac{x}{3(-4) + 4} \sin 2x = -\frac{x}{8} \sin 2x$$

6) Find the P.I. of $(D^2 + 9)y = x \cos x$

Sol:

$$\begin{aligned} y_p &= \frac{1}{D^2 + 9} \operatorname{Re}(x e^{ix}) = \operatorname{Re} \frac{1}{D^2 + 9} (x e^{ix}) \\ &= \operatorname{Re}(e^{ix}) \frac{1}{(D+i)^2 + 9} (x) = \operatorname{Re}(e^{ix}) \frac{1}{D^2 + i^2 + 2iD + 9} \cdot x \\ &= \operatorname{Re}(e^{ix}) \frac{1}{D^2 + 2iD + 8} x = \operatorname{Re}(e^{ix}) \frac{1}{8(1 + \frac{D^2 + 2iD}{8})} x \end{aligned}$$

$$= \frac{1}{8} \operatorname{Re}(e^{ix}) \left\{ 1 + \frac{D^2 + 2iD}{8} \right\}^{-1} x$$

$$= \frac{1}{8} \operatorname{Re}(e^{ix}) \left\{ 1 - \frac{D^2 + 2iD}{8} + \left(\frac{D^2 + 2iD}{8} \right)^2 - \dots \right\} x$$

$$= \frac{1}{8} \operatorname{Re}(e^{ix}) \left\{ x - \frac{i}{4} \right\} = \frac{1}{8} \operatorname{Re}(\cos x + i \sin x) \left(x - \frac{i}{4} \right)$$

$$= \operatorname{Re} \left\{ \frac{1}{8} (\cos x + i \sin x) \left(\frac{4x - i}{4} \right) \right\} = \frac{1}{32} (4x \cos x + \sin x)$$

7) Find P.I. of $(D^2 - 4D + 4)y = x^2 e^{3x} + \sin^2 x$

Sol: $y_p = \frac{1}{(D-2)^2} (x^2 e^{3x} + \sin^2 x)$

Consider $\frac{1}{(D-2)^2} x^2 e^{3x} = e^{\frac{3x}{2}} \frac{1}{(D+3-2)^2} x^2 = e^{\frac{3x}{2}} \frac{1}{(D+1)^2} x^2$

$$= e^{\frac{3x}{2}} (1+D)^{-2} x^2 = e^{\frac{3x}{2}} \left\{ 1 - 2D + 3D^2 - \dots \right\} x^2$$

$$\frac{1}{(D-2)^2} x^2 e^{3x} = e^{\frac{3x}{2}} \left\{ x^2 - 4x + 6 \right\}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2} x^2$$

$$\frac{1}{(D-2)^2} \sin^2 x = \frac{1}{(D-2)^2} \left\{ \frac{1 - \cos 2x}{2} \right\}$$

$$= \frac{1}{2} \frac{1}{(D^2 - 4D + 4)} (1 - \cos 2x)$$

$$= \frac{1}{2} \left\{ \frac{1}{D^2 - 4D + 4} (1) - \frac{1}{D^2 - 4D + 4} (\cos 2x) \right\}$$

$$= \frac{1}{2} \left\{ \frac{1}{D^2 - 4D + 4} (e^{0x}) - \frac{1}{(D^2 - 4D + 4)} \cos 2x \right\}$$

$$= \frac{1}{2} \left\{ \frac{1}{4} - \frac{1}{-4 - 4D + 4} \cos 2x \right\}$$

$$\frac{1}{(D-2)^2} \sin^2 x = \frac{1}{8} + \frac{1}{2} \cdot \frac{1}{4D} (\cos 2x) = \frac{1}{8} + \frac{1}{8} \frac{\sin 2x}{2}$$

$$\therefore y_p = e^{3x} (x^2 - 4x + 6) + \frac{1}{8} + \frac{1}{16} \sin 2x$$

8) Find P.I. of $(D^2+1)y = \sin x \sin 2x$

Sol: P.I. is

$$y_p = \frac{1}{D^2+1} \frac{1}{2} [\cos(-x) - \cos(3x)]$$

$$= \frac{1}{2} \left[\frac{1}{D^2+1} \cos x - \frac{1}{D^2+1} \cos 3x \right]$$

$$= \frac{1}{2} \left[\frac{x \cos x}{2D} - \frac{1}{(-9+1)} \cos 3x \right]$$

$$\sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$$

$$\sin A \cos B = \frac{1}{2} [\sin(A+B) + \sin(A-B)]$$

$$\cos A \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)]$$

$$\cos A \sin B = \frac{1}{2} [\sin(A+B) - \sin(A-B)]$$

$$\therefore y_p = \frac{1}{2} \left[\frac{x \sin x}{2} + \frac{1}{8} \cos 3x \right]$$

Exercise Problems: Solve the foll DE's:

$$1) \frac{d^2y}{dx^2} + 3 \frac{dy}{dx} + 2y = e^{2x} \cdot \sin x \quad \text{Ans: } y = c_1 e^{-x} + c_2 e^{-2x} - \frac{e^{2x}}{170} (7 \cos x - 11 \sin x)$$

$$2) (D^3 + D^2 - D - 1)y = \cos 2x \quad \text{Ans: } y = c_1 e^x + (c_2 + c_3 x) e^{-x} - \frac{1}{25} (2 \sin x + \cos 2x)$$

$$3) (D^2 + 1)y = 3 + e^{-x} + 5e^{2x} \quad \text{Ans: } y = c_1 e^x + e^{x/2} \left(c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x \right) + 3 + \frac{x}{3} e^{-x} + \frac{5}{9} e^{2x}$$

$$4) (D^2 - 2D + 5)y = \sin 3x \quad \text{Ans: } y = e^x (c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{26} (3 \cos 3x - 2 \sin 2x)$$

$$5) (D^2 - 6D + 9)y = e^{3x} (x^2 + 7x + 5) \quad \text{Ans: } y = (c_1 + c_2 x) e^{3x} + \frac{e^{3x}}{12} (x^4 + 14x^3 + 30x^2)$$

$$6) \frac{d^2y}{dx^2} - y = 2 + 5x \quad \text{Ans: } y = c_1 e^x + c_2 e^{-x} - 2 - 5x$$

$$7) (D^2 - 3D + 2)y = x^2 e^{3x} \quad \text{Ans: } y = c_1 e^{2x} + c_2 e^x - \frac{3e^{3x}}{4} (2x + 3)$$

$$8) (4D^2 - 1)y = e^{x/2} + 12e^x \quad \text{Ans: } y = c_1 e^{x/2} + c_2 e^{-x/2} + \frac{x}{2} e^{x/2} + 4e^x$$

$$9) (D^2 + 2D + 2)y = 2e^{-x} \sin x \quad \text{Ans: } y = e^{-x} (c_1 \cos x + c_2 \sin x) - x e^{-x} \cos x$$

$$10) (D^2 - 4D + 4)y = e^{-4x} + 5 \cos 3x \quad \text{Ans: } y = (c_1 + c_2 x) e^{2x} + \frac{1}{36} e^{-4x} - \frac{5}{169} (12 \sin x + 5 \cos 3x)$$

$$11) (D^3 + 8)y = x^4 + 2x + 1 \quad \text{Ans: } y = c_1 e^{-2x} + e^x (c_2 \cos \sqrt{3}x + c_3 \sin \sqrt{3}x) + \frac{1}{8} (x^4 - x + 1)$$

Method of variation of parameters: It is useful to find particular integral (P.I) for higher order differential equation (provided C.F. is known)

This method is more general method in which the function $g(x)$ is not restricted to any particular form.

The working rule is:

Consider a second order ODE $a_0(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x)y = g(x) \rightarrow \textcircled{*}$

The C.F. of $\textcircled{*}$ is $y_c = c_1 y_1(x) + c_2 y_2(x)$. $\rightarrow \textcircled{1}$

We replace c_1 and c_2 in $\textcircled{1}$ by respective functions $A(x)$ and $B(x)$ (which will be determined later), we obtain

P.I. = $A(x) y_1(x) + B(x) y_2(x)$ (hence the name variation of parameters.), where

$$A(x) = - \int \frac{y_2(x) g(x)}{a_0(x) W(y_1, y_2)} dx = - \int \frac{y_2 g(x)}{a_0 W} dx$$

$$B(x) = \int \frac{y_1(x) g(x)}{a_0(x) W(y_1, y_2)} dx = \int \frac{y_1 g(x)}{a_0 W} dx$$

with the Wronskian of y_1 and y_2 $W(y_1, y_2) = W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$

Solve the foll ODEs by method of variation of parameters:

i) $\frac{d^2y}{dx^2} + y = \tan x$ ————— ①

Soh: The G.S is

$$y = y_c + y_p$$

To find y_c :

Consider $\frac{d^2y}{dx^2} + y = 0$

A.E: $m^2 + 1 = 0$

$$\begin{cases} (D^2 + 1)y = 0 \\ A.E: m^2 + 1 = 0 \end{cases}$$

Roots are $m = \pm \sqrt{-1}$

$$= \pm i$$

indep. solns are $\cos x, \sin x$.

Thus, $y_c = C_1 \cos x + C_2 \sin x$ ————— ②

To find y_p :

Assume $y_p = A(x) \cos x + B(x) \sin x$ ————— ③

here $y_1(x) = \cos x, y_2(x) = \sin x, a_0(x) = 1$

non homo. term $g(x) = \tan x,$

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$\therefore A(x) = \int -\frac{g(x) y_2(x)}{a_0(x) W(y_1, y_2)}$$

$$= \int -\frac{\tan x \cdot \sin x}{1} dx$$

$$\begin{aligned}
 &= \int -\frac{\sin^2 x}{\cos x} dx \\
 &= \int -\frac{(1-\cos^2 x)}{\cos x} dx \\
 &= \int (-\sec x + \cos x) dx \\
 &= -\log(\sec x + \tan x) + \sin x
 \end{aligned}$$

$$\begin{aligned}
 B(x) &= \int \frac{g(x) y_1(x)}{a_0(x) W(y_1, y_2)} dx \\
 &= \int \frac{\tan x \cos x}{1} dx \\
 &= \int \sin x dx = -\cos x
 \end{aligned}$$

Sub for $A(x)$ and $B(x)$ in ③,

$$\begin{aligned}
 y_p &= (-\log(\sec x + \tan x) + \sin x) \cos x - \cos x \sin x \\
 \Rightarrow y_p &= -\log(\sec x + \tan x) \cos x \quad \text{--- (4)}
 \end{aligned}$$

The GS is

$$y = y_c + y_p \quad (\text{sub.})$$

$$\text{ii) } y'' - 4y' + 4y = (x+1)e^{2x} \quad \text{--- ①}$$

Soln The G.S is

$$y = y_c + y_p.$$

To find y_c :

$$\text{Consider } y'' - 4y' + 4y = 0$$

$$AE: m^2 - 4m + 4 = 0$$

Roots are $m = 2, 2$ (repeated)

Indep. solns are e^{2x}, xe^{2x}

$$\therefore y_c = c_1 e^{2x} + c_2 x e^{2x} \quad \text{--- ②}$$

To find y_p :

$$y_p = A(x) e^{2x} + B(x) xe^{2x} \quad \text{--- ③}$$

Here

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + x^2 e^{2x} \end{vmatrix} \\ &= e^{4x} + x^2 e^{4x} - 2x e^{4x} \\ &= e^{4x} \end{aligned}$$

$$\begin{aligned} y_1(x) &= e^{2x} \\ y_2(x) &= xe^{2x} \\ a_0(x) &= 1 \\ g(x) &= (x+1)e^{2x} \end{aligned}$$

$$A(x) = \int -\frac{g(x) y_2(x)}{a_0 W(y_1, y_2)} dx$$

$$= \int -\frac{(x+1)e^{2x} xe^{2x}}{e^{4x}} dx$$

$$= \int (-x^2 - x) dx$$

$$= -\frac{x^3}{3} - \frac{x^2}{2}$$

$$B(x) = \int \frac{g(x) y_1(x)}{W(y_1, y_2)} dx$$

$$= \int \frac{(x+1)e^{2x} e^{2x}}{e^{4x}} dx$$

$$= \int (x+1) dx$$

$$= \frac{x^2}{2} + x$$

Sub for A and B in ③,

$$y_p = \left(-\frac{x^3}{3} - \frac{x^2}{2} \right) e^{2x} + \left(\frac{x^2}{2} + x \right) x e^{2x}.$$

$$\Rightarrow y_p = \left(\frac{1}{6} x^3 + \frac{1}{2} x^2 \right) e^{2x}$$

The C_rs

$$y = y_c + y_p \quad (\text{sub.})$$

Exercise:

Solve the following differential eqns by the method of variation of parameters.

$$1) y'' - 2y' + y = xe^x \sin x$$

Ans: $y = (c_1 + c_2 x)e^x - e^x (x \sin x + 2 \cos x)$.

$$2) y'' - 2y' = e^x \sin x$$

Ans: $y = c_1 + c_2 e^{2x} - \frac{1}{2} e^x \sin x$.

$$3) y'' - 2y' + y = xe^x \log x$$

Ans: $y = c_1 e^x + c_2 xe^x + \frac{1}{6} x^3 e^x \log x - \frac{5}{36} x^3 e^x$.

$$4) y'' - 2y' + 2y = e^x \tan x$$

Ans: $y = e^x (c_1 \cos x + c_2 \sin x) - e^x \cos x \log (\sec x + \tan x)$.

$$5) y'' + 4y = \cot 2x$$

Ans: $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{2} \log(\tan x)$.

$$6) y'' - y = \cosh x$$

Ans: $y = c_1 e^x + c_2 e^{-x} + \frac{1}{2} x \sinh x$.

$$7) 4y'' + 36y = \operatorname{cosec} 3x$$

Ans: $y = c_1 \cos 3x + c_2 \sin 3x - \frac{x}{12} \cos 3x + \frac{1}{36} \log(\sin 3x) \sin 3x$

Cauchy-Euler equations

A. D.E. of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_{n-2} x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_1 y = g(x) \quad (1)$$

where a_i 's are constants and $g(x)$ is a function of x

is called Cauchy-Euler D.E. of order n .

① can be reduced to ODE with constant coefficients by a suitable substitution as follows.

② can be written as

$$(a_n x^n D^n + a_{n-1} x^{n-1} D^{n-1} + \dots + a_1) y = g(x) \rightarrow (2) \text{ when } D = \frac{d}{dx}$$

Put $x = e^z \Rightarrow z = \log_e x$

$$\frac{dz}{dx} = \frac{1}{x}$$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \cdot \frac{dy}{dz}$$

$$\therefore x \frac{dy}{dx} = \frac{dy}{dz} \Rightarrow x D_y = D_1 y \quad \text{where } D_1 = \frac{d}{dz}$$

$$\text{Again, } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{1}{x} \cdot \frac{dy}{dz} \right)$$

$$= \left\{ -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x} \frac{d}{dx} \left(\frac{dy}{dz} \right) \frac{dz}{dx} \right\}$$

$$= -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x^2} \frac{d^2y}{dz^2} = \frac{1}{x^2} \left[\frac{d^2y}{dz^2} - \frac{dy}{dz} \right]$$

$$\therefore x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$\Rightarrow x^2 D^2 y = D_1^2 y - D_1 y = D_1 (D_1 - 1)y \quad \left| \text{ where } D_1^2 = \frac{d^2}{dz^2} \right.$$

III^{ly}

$$x^3 D^3 y = D_1 (D_1 - 1) (D_1 - 2)y \quad \text{and so on.}$$

\therefore ② becomes $F(D_1) y = g_1(z) \Rightarrow$ ③ where $g_1(z)$ is the value of $g(x)$ in terms of z .

③ is now clearly LDE with constant coefficients and can be solved by the methods already discussed.

1) Solve : $4x^2 \frac{d^2y}{dx^2} + 8x \frac{dy}{dx} + y = 0$

Sol. The given ODE is 2^{nd} order Cauchy-Euler eq.
Let $x = e^z$

$$\Rightarrow x D_y = D_1 y, x^2 D^2 y = D_1 (D_1 - 1)y \quad \text{where } D_1 = \frac{d}{dz}$$

The given eq reduces to

$$4D_1(D_1 - 1)y + 8D_1y + y = 0$$

$$(4D_1^2 - 4D_1 + 8D_1 + 1)y = 0$$

$$\Rightarrow (4D_1^2 + 4D_1 + 1)y = 0$$

which is a LDE with constant coefficients

Auxiliary eq is $4m^2 + 4m + 1 = 0$

$$\Rightarrow (2m+1)^2 = 0$$

$$\therefore m = -\frac{1}{2}, -\frac{1}{2}$$

The general solution is: $y = C_1 e^{-\frac{1}{2}x} + C_2 x e^{-\frac{1}{2}x}$

$$\therefore y = C_1 e^{-\frac{1}{2}\log x} + C_2 \log x e^{-\frac{1}{2}\log x} = \frac{C_1}{\sqrt{x}} + \frac{C_2 \log x}{\sqrt{x}}$$

2) Solve $x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = \log x$

Sol: let $x = e^z \Rightarrow z = \log x$

$$x^2 D^2 y = D_1(D_1 - 1)y, \quad x D y = D_1 y \text{ when } D_1 = \frac{d}{dz}$$

The given eq reduces to

$$(D_1(D_1 - 1) - D_1 + 1)y = z$$

$$(D_1^2 - D_1 - D_1 + 1)y = z$$

$\therefore (D_1^2 - 2D_1 + 1)y = z$ is linear DDE with constant coefficients

$$A.E: m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0 \quad \therefore m = 1, 1$$

$$\therefore y_c = c_1 e^z + c_2 z e^z$$

$$y_p = \frac{1}{(D_1 - 1)^2} z = \frac{1}{(1 - D_1)^2} z = (1 - D_1)^{-2} z = (1 + 2D_1 + 3D_1^2 + \dots) z$$

$$\therefore y_p = z + 2$$

\therefore The complete solution is

$$y = y_c + y_p = c_1 e^z + c_2 z e^z + z + 2 \quad \text{where } z = \log x$$

$$\text{i.e. } y = c_1 x + c_2 \log x \cdot x + \log x + 2$$

$$3) x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + 5y = x^2 \sin(\log x)$$

$$\text{Sol: let } x = e^z \quad \text{i.e. } z = \log x$$

The given eq becomes

$$[D_1(D_1 - 1) - 3D_1 + 5]y = e^{2z} \sin z$$

$$\Rightarrow (D_1^2 - 4D_1 + 5)y = e^{2z} \sin z$$

$$A.E: m^2 - 4m + 5 = 0$$

$$m = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

$$C.F: y_c = e^{2z} (c_1 \cos z + c_2 \sin z)$$

$$P.I: y_p = \frac{1}{D_1^2 - 4D_1 + 5} e^{2z} \sin z = \frac{e^{2z}}{\frac{1}{(D_1 + 2)^2 - 4(D_1 + 2) + 5}} \sin z$$

$$= e^{2z} \frac{1}{D_1^2 + 4 + 4D_1 - 4D_1 - 8 + 5} \sin z = e^{2z} \frac{1}{D_1^2 + 1} \sin z$$

$$\therefore y_p = e^{\frac{az}{2}} \frac{z \sin z}{2D_1} = \frac{e^{\frac{az}{2}}}{2} z \cdot \frac{1}{D_1} \sin z = -\frac{z}{2} e^{\frac{az}{2}} \cos z$$

$$\therefore y = y_c + y_p$$

$$\Rightarrow y = e^{\frac{az}{2}} (c_1 \cos z + c_2 \sin z) - \frac{z}{2} e^{\frac{az}{2}} \cos z$$

$$\therefore y = x^2 \left\{ c_1 \cos(\log x) + c_2 \sin(\log x) \right\} - \frac{x^2}{2} \log x \cos(\log x)$$

Exercise : Solve the following Differential equations:

$$1) x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = e^x \quad \text{Ans: } y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{1}{x^2} e^x$$

$$2) x^3 \frac{d^3 y}{dx^3} + 3x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = x + \log x$$

$$\text{Ans: } \frac{c_1}{x} + \sqrt{x} \left[c_2 \cos \frac{\sqrt{3}}{2} (\log x) + c_3 \sin \frac{\sqrt{3}}{2} (\log x) \right] + \frac{x}{2} + \log x$$

$$3) 4x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} - y = 4x^2 \quad \text{Ans: } c_1 \sqrt{x} + c_2 \frac{1}{\sqrt{x}} + \frac{4}{15} x^2$$

$$4) \frac{d^3 y}{dx^3} - \frac{4}{x} \frac{d^2 y}{dx^2} + \frac{5}{x^2} \frac{dy}{dx} - \frac{2y}{x^3} = 1 \quad (\text{Hint: Multiply throughout by } x^3) \quad \text{Ans: } y = c_1 x^2 + x^{5/2} (c_2 x^{\sqrt{2}/2} + c_3 x^{-\sqrt{2}/2}) - \frac{1}{5} x^3$$

$$5) (x^3 D^3 + 2x^2 D^2 + 2)y = 10 \left(x + \frac{1}{x} \right)$$

$$\text{Ans: } y = \frac{c_1}{x} + x \left[c_2 \cos(\log x) + c_3 \sin(\log x) \right] + 5x + \frac{2}{x} \log x$$