Matrix: It is a rectangular array of numbers arranged in rows and columns. We use capital letters A, B, C and so on to refer matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

In general a_{ij} will denote (i, j) entry of A.

Minor of an element in a matrix:
It is the determinant obtained by deleting the row and column in which the element lies.

ex:- If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then minor of a_{11} is

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Elementary transformations (or operations) on a matrix:

1) Interchange of any 2 rows or columns denoted by $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

2) Multiplication of the elements of a row (column) by a non-zero scalar denoted by $R_i \to KR_i$ or $C_i \to KC_i$

3) Multiplying every element of a row with a non-zero scalar and adding to the corresponding elements of another row denoted by $R_i \rightarrow R_i + kR_j$, $C_i \rightarrow C_i + kC_j$

An elementary transformation is called a row transformation or a column transformation according as it applies to rows or columns.

Rank of a matrix : Let A be a $m \times n$ matrix. The matrix A is said to have rank r if

1) There exists atleast one minor of order r of matrix A, which does not vanish.

2) All minors of order $r+1$ (which can be formed) must vanish.

The rank of the matrix is denoted by $\rho(A)$ or rank (A).

Note :-

1) Rank of null matrix is always 0.

2) Rank of a matrix is ≥ 1 if it is a non-zero matrix

3) $\rho(I_n) = n$ where n is the order of matrix.

4) If A is a matrix of order $m \times n$ then

$\rho(A) \leq \min(m, n)$

5) If A is a matrix of order n and A is non-singular (i.e. $|A| \neq 0$) then $\rho(A) = n$

6) If $|A| = 0$, then $\rho(A) < n$

Ex:- Determine the rank of following matrices:-

1) $A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}$

$$|A| = -1(18-1) + 6(3+30) = 181 \neq 0$$

$$\therefore \rho(A) = 3$$

2) $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

$$|A| = 0$$

$$\therefore \rho(A) < 3$$

Consider a minor of order 2 $= \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2 \neq 0$

$$\therefore \rho(A) = 2$$

3) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}_{3\times 4}$

$$\text{Rank } (A) \leq \min(3,4) \leq 3$$

Consider the minor of order 3

$$\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{vmatrix} = 24 \neq 0$$

$\therefore \rho(A) = 3$

4) $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & 2 & 3 & 2 \end{bmatrix}$

Here $\rho(A) \leq 3$

We observe that minors of order 3 vanish.

Consider minor of order 2,

$$\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} = 3 \neq 0$$

$\therefore \rho(A) = 2$

Rank of matrix can be found by:

1) Minor method
2) Echelon form of matrix
3) Normal form or canonical form

<u>Row echelon form of matrix</u>: A matrix is said to be in row echelon form if it has the following properties:

1) the leading entry (first non-zero entry) of each row is unity

2) all the entries below this leading entry is zero.

3) number of zeros appearing before the leading entry in each row is greater than that in the previous row.

4) zero rows must appear below non-zero rows.

Note: The number of non-zero rows in the echelon form of matrix is called rank of A.

ex: 1) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ✓

2) $\begin{bmatrix} 0 & 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ ✗

This is not in row echelon form.

3) $\begin{bmatrix} 1 & 0 & 0 & 5 & 6 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ ✓

4) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ ✗ This is not in row echelon form

Reduced row echelon form (rref): A matrix is said to be in reduced row echelon form if

1) The matrix is in row echelon form.

2) The first non zero entry in each row is the only non zero entry in its column.

ex:- 1) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in rref

2) $\begin{bmatrix} 0 & 1 & 5 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in rref

3) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ is in rref

4) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is not in rref.

Obtain reduced row echelon form of the matrices and hence find its rank.

1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

$R_2 \to R_2 - R_1$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 2 & 6 & 5 \end{bmatrix}$

$R_3 \to R_3 - 2R_1$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$$R_1 \rightarrow R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \rightarrow \frac{1}{2} R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$ is rref of given matrix.

$\therefore$ rank $(A) = 2$

2) $A = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$

$$R_1 \rightarrow -1 \times R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 6R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 5 & 10 & 5 \end{bmatrix}$$

$R_3 \to R_3 - 5R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ is rref of given matrix.

$\therefore$ rank $(A) = 1$

3) $A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$R_3 \to R_3 - \frac{1}{3} R_2$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$-2 - 6x = 0$

$-6x = 2$

$\therefore x = -\frac{1}{3}$

$R_2 \to -\frac{1}{6} \times R_2$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is rref of given matrix.

$\therefore$ rank$(A) = 2$

obtain the echelon form of the matrices and hence find its rank.

4) $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

$$R_2 \rightarrow R_2 - 2R_1, \quad R_3 \rightarrow R_3 - 3R_1, \quad R_4 \rightarrow R_4 - 6R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_4 \rightarrow R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \rightarrow -\frac{1}{4} \times R_2$, $R_3 \rightarrow -\frac{1}{3} \times R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -\frac{3}{4} \\ 0 & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is row echelon form of matrix

$\therefore$ rank $(A) = 3$

5) $A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to \frac{1}{2} R_1$, $R_2 \to \frac{1}{2} R_2$

$$\sim \begin{bmatrix} 1 & 3/2 & 5/2 & 2 \\ 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ rank $(A) = 2$

6) Find the value of b in the matrix $\begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{bmatrix}$

given that its rank is 2.

sol:- Since rank $(A) <$ order of matrix

$\Rightarrow |A| = 0$

$$\begin{vmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{vmatrix} = 0$$

$$1(30-26) - 5(0-2b) + 4(0-3b) = 0$$

$$4 + 10b - 12b = 0$$

$$4 - 2b = 0 \Rightarrow b = 2$$

OR

$R_3 \to R_3 - bR_1$

$$\sim \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ 0 & 13-5b & 10-4b \end{bmatrix}$$

$R_3 \to R_3 - \frac{1}{3}(13-5b)R_2$

$13 - 5b - 3x = 0$

$13 - 5b = 3x$

$\therefore x = \frac{1}{3}(13-5b)$

$$\sim \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ 0 & 0 & \frac{4-2b}{3} \end{bmatrix}$$

$$10-4b-\frac{2}{3}(13-5b)$$

$$\frac{30-12b-26+10b}{3}$$

$$= \frac{4-2b}{3}$$

Since rank(A) = 2

$$\frac{4-2b}{3} = 0 \Rightarrow 4-2b = 0$$

$$\therefore b = 2$$

7) Find value of K such that following matrix A may have rank equal to 2

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & K \\ 1 & 4 & 10 & K^2 \end{bmatrix}$$

sol: $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & K-1 \\ 0 & 3 & 9 & K^2-1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & K-1 \\ 0 & 0 & 0 & K^2-1-3K+3 \end{bmatrix}$$

Since $\text{rank}(A) = 2$

$$K^2 - 3K + 2 = 0$$

$$K^2 - 2K - K + 2 = 0$$

$$K(K-2) - 1(K-2) = 0$$

$$\therefore K = 1, 2$$

8) Obtain rref of the matrix $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$ and its rank

Sol: $R_2 \to R_2 - R_1$

$$\sim \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 4R_1, \quad R_4 \to R_4 - 9R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$R_3 \to R_3 - R_2, \quad R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ rank $(A) = 2$

Exercise :-

obtain the echelon form of the matrices and hence find its rank.

1) $A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$

Ans: rank (A) = 2

2) $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$

Ans:- rank (A) = 3

3) $A = \begin{bmatrix} 2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

Ans:- rank (A) = 3

4) $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Ans:- rank (A) = 2

5) $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

Ans:- rank (A) = 2

6) $A = \begin{bmatrix} 8 & 2 & 1 & 6 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 1 & 3 \\ 5 & 1 & 1 & 4 \end{bmatrix}$

Ans :- rank (A) = 3

System of linear equations :

A set of m linear equations in n unknowns is as follows:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \dots\dots\dots\dots & \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad (*)$$

where a_{ij}'s and b_i's are constants

If $b_1, b_2, \dots b_m$ are all zero, the system $(*)$ is said to be homogeneous.

The set of values $x_1, x_2, \dots x_n$ which satisfy all the equations simultaneously is called solution of system of equations.

The above system $(*)$ can be written in matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

i.e. $\underset{m\times n}{A}\,\underset{n\times 1}{X} = \underset{m\times 1}{B}$

$AX = [0]$ is the matrix representation of homogeneous system of equations where $[0]$ is the null matrix of order $m \times 1$. Here $x_1 = x_2 = x_3 = \dots = x_n = 0$ is always a solution of homogeneous system of equations and is called <u>trivial solution</u>. If atleast one x_i $(i = 1, 2, \dots n)$ is non-zero then it is called a non-trivial solution. If B is not a zero matrix, that is not all $b_1, b_2, \dots b_m$ are zero, then the system (*) is called non-homogeneous linear system of equations.

Consistency of system of linear equations:

A system of linear equations is said to be consistent if it has atleast one solution.

A system of equations is inconsistent if it has no solution.

Rank of matrix helps us to conclude if the system is consistent or inconsistent.

The system of equations represented by matrix equation $AX = B$ is consistent if $\text{rank}(A) = \text{rank}([A : B])$

$$\text{where} \quad [A : B] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & : & b_2 \\ \vdots & & & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & : & b_m \end{bmatrix}$$

is called augmented matrix.

The system of equations is inconsistent if

$\text{rank}(A) \neq \text{rank}([A:B])$

<u>Conditions for 3 types of solution:</u>

1) Unique solution : $\text{rank}(A) = \text{rank}([A:B]) = r = n$, n being no of unknowns.

2) Infinite solutions: $\text{rank}(A) = \text{rank}([A:B]) = r < n$

3) No solution : $\text{rank}(A) \neq \text{rank}([A:B])$

Gauss - elimination method to solve system of equations (*)

1) Consider, Augmented matrix: $[A \cdot B]$

2) Obtain row echelon form of $[A:B]$ (upper triangular form) (if we obtain rref of $[A:B]$, then method is called Gauss – Jordan method) by employing elementary row operations

3) Determine rank of A and rank of $[A:B]$

4) If $\text{rank}(A) = \text{rank}([A:B])$, then system is consistent (have solution/s). Otherwise, system is inconsistent

5) If $\text{rank}(A) = \text{rank}([A:B]) = n$ (no of unknowns) then the system has unique solution

6) If $\text{rank}(A) = \text{rank}([A:B]) < n$ (no of unknowns) then the system has infinite solutions.

In this case we choose $n-r$ free variables

Here r is $\text{rank}(A)$ (or $\text{rank}([A:B])$)

$$AX = B$$

consistent (has a soln or solns)
if $\text{rank}(A) = \text{rank}([A:B])$ (say r)

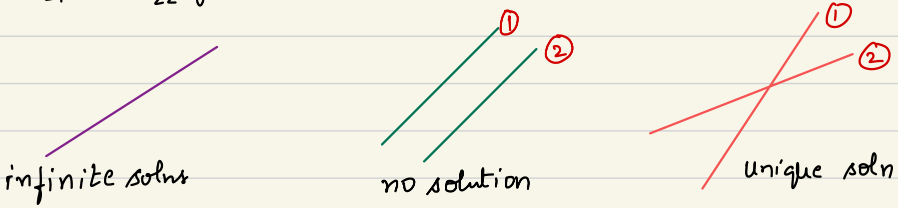
- **Unique soln**
  if $r = n$, no of unknowns
- **Infinite solns**
  if $r < n$, no of unknowns and $n-r$ free variables.

Inconsistent (has no soln)
if $\text{rank}(A) \neq \text{rank}([A:B])$

Geometric explanation:

$$a_{11}x + a_{12}y = b_1 \rightarrow ①$$
$$a_{21}x + a_{22}y = b_2 \rightarrow ②$$

Geometrically ① and ② represent straight lines



* Method of finding solution to the system of equations using reduced row echelon form (diagonal matrix form) is called Gauss - Jordan method