

Alternative definition:

Scalar point function and scalar field: If to each point (x, y, z) of a region R in space there corresponds a definite scalar denoted by $\phi(x, y, z)$, then $\phi(x, y, z)$ is called scalar point function in R . The region R so defined is called a scalar field.

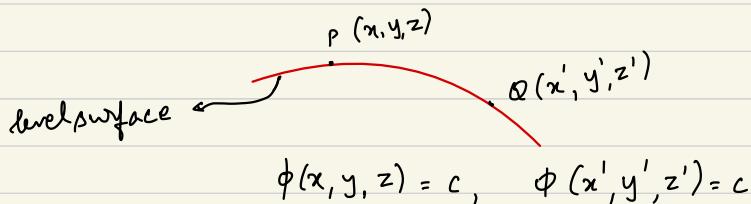
ex: Temperature $\phi(x, y, z)$ at any point P of unevenly heated body is scalar point function. The medium itself is scalar field.

vector point function and vector field: If to each point (x, y, z) of a region R in space there corresponds a definite vector denoted by $\vec{f}(x, y, z)$, then $\vec{f}(x, y, z)$ is called vector point function in R . The region R so defined is called a vector field.

ex: The velocity of moving body in a certain region at time t is a vector point function.

level surface: $\phi(x, y, z) = c$

represents family of surfaces in scalar field. If at each point on surface $\phi(x, y, z) = c$ has same value then surface is called level surface



Vector differentiation

Vector functions: Functions whose values are vectors depending on points P in space.

$$v = v(P) = v_1(P)\hat{i} + v_2(P)\hat{j} + v_3(P)\hat{k} \text{ or } [v_1(P), v_2(P), v_3(P)]$$

(Domains may be 3-D or surface or a curve in space)

Ex: velocity vector function of a rotating body.

Scalar functions: Functions whose values are scalars depending on the points P in space.

$$f = f(P)$$

Ex: The distance $f(P)$ of any point P from a fixed point P_0 in space is a scalar function

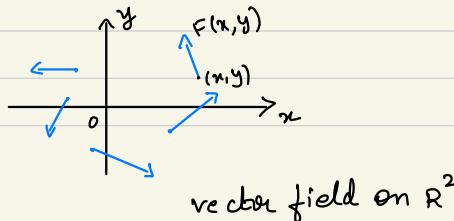
$$f(P) = f(x, y, z) = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

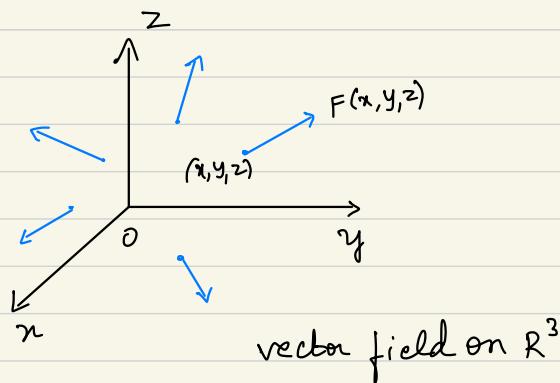
A vector function defines a vector field and a scalar function defines a scalar field in the domain.

In general,

1) Vector field is a function whose domain is a set of points in R^2 (or R^3) whose range is a set of vectors in v_2 ($\text{or } v_3$) or

Let D be a set in R^2 (a plane region). A vector field on R^2 is a function F that assigns to each point (x, y) in D a 2-dimensional vector $F(x, y)$





ex: 1) A magnetic field \mathbf{B} in a region of space

$$\mathbf{B} = B_1 \hat{i} + B_2 \hat{j} + B_3 \hat{k}$$

2) Velocity field of water flowing in a pipe, $\mathbf{v}(x, y, z)$

Scalar field: let $D \subseteq \mathbb{R}^2$ (a plane region)

A scalar field on \mathbb{R}^2 is a function ϕ that assigns a scalar value $\phi(x, y)$ to each point (x, y) of D .

ex: Temperature, concentration, density etc.

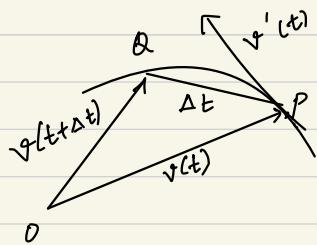
Derivative of vector function: A vector function is said

to be differentiable at a point t if the following limit exists.

$$\mathbf{v}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t}$$

$\mathbf{v}'(t)$ is called derivative of $\mathbf{v}(t)$

$\mathbf{v}'(t)$ is obtained by differentiating each component separately



Triangle law
of addⁿ of vectors

In components wst a given cartesian coordinate system

$$v'(t) = [v'_1(t), v'_2(t), v'_3(t)]$$

Velocity and Acceleration:

If $\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is the position vector of a particle moving along a smooth curve in space, then $v(t) = \frac{d\vec{r}}{dt}$ is the particle's **velocity vector**, tangent to the curve. At any time t , the direction of $v(t)$ is the **direction of motion**, the magnitude of $v(t)$ is the particle's **speed**, and the derivative $a(t) = \frac{dv}{dt}$, when it exists, is the particle's **acceleration vector**.

In summary,

- Velocity is the derivative of position vector: $v(t) = \frac{d\vec{r}}{dt}$
- Speed is the magnitude of velocity: $speed = |v(t)|$
- Acceleration is the derivative of velocity: $a(t) = \frac{dv}{dt} = \frac{d^2\vec{r}}{dt^2}$
- Unit Tangent vector $\hat{T} = \frac{v(t)}{|v(t)|}$ is the direction of motion at time t .
- Component of velocity along a given vector \vec{C} is $v(t) \cdot \hat{C}$
- Component of acceleration along a given vector \vec{C} is $a(t) \cdot \hat{C}$

Differentiation rules for vector functions:

$$(1) \frac{d}{dt}[\vec{a}(t) + \vec{b}(t)] = \frac{d}{dt}\vec{a}(t) + \frac{d}{dt}\vec{b}(t)$$

$$(2) \frac{d}{dt}[c \vec{a}(t)] = c \left[\frac{d}{dt} \vec{a}(t) \right], \text{ where } c \text{ is constant}$$

$$(3) \frac{d}{dt}[\vec{a}(t) \cdot \vec{b}(t)] = \left[\frac{d}{dt} \vec{a}(t) \right] \cdot \vec{b}(t) + \vec{a}(t) \cdot \left[\frac{d}{dt} \vec{b}(t) \right]$$

$$(4) \frac{d}{dt}[\vec{a}(t) \times \vec{b}(t)] = \left[\frac{d}{dt} \vec{a}(t) \right] \times \vec{b}(t) + \vec{a}(t) \times \left[\frac{d}{dt} \vec{b}(t) \right]$$

1) Find velocity, speed and acceleration of a particle whose motion in space is given by $\vec{r}(t) = 2\cos t \hat{i} + 2\sin t \hat{j} + 5\cos^2 t \hat{k}$.

Sol:

$$\vec{v}(t) = \vec{r}'(t) = -2\sin t \hat{i} + 2\cos t \hat{j} - 10\cos t \sin t \hat{k}$$

$$\vec{v}(t) = -2\sin t \hat{i} + 2\cos t \hat{j} - 5\sin 2t \hat{k}$$

$$\vec{a}(t) = \vec{r}''(t) = -2\cos t \hat{i} - 2\sin t \hat{j} - 10\cos 2t \hat{k}$$

$$\text{speed is } |\vec{v}(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (-5\sin 2t)^2}$$

$$\text{speed} = \sqrt{4 + 25\sin^2 t}$$

2)

A particle moves such that its position vector at time t is $\vec{r} = e^{-t}\hat{i} + 2\cos 3t \hat{j} + 3\sin 3t \hat{k}$. Determine its velocity, acceleration and their magnitude, direction at time $t = 0$.

Solution: velocity : $\vec{v} = \frac{d\vec{r}}{dt} = -e^{-t}\hat{i} - 6\sin 3t \hat{j} + 9\cos 3t \hat{k}$

$$\vec{v}(0) = -\hat{i} + 9\hat{k}, \text{ magnitude} = \sqrt{82}, \text{ direction is } \frac{1}{\sqrt{82}}(-\hat{i} + 9\hat{k})$$

acceleration: $\vec{a} = \frac{d^2\vec{r}}{dt^2} = e^{-t}\hat{i} - 18\cos 3t \hat{j} - 27\sin 3t \hat{k}, \quad \vec{a}(0) = \hat{i} - 18\hat{j}, \quad \text{magnitude} = \sqrt{325},$

direction is $\frac{1}{\sqrt{325}}(\hat{i} - 18\hat{j})$.

2. For the curves whose equations are given below, find the unit tangent vectors:

(i) $x = t^2 + 1, y = 4t - 3, z = 2(t^2 - 3t)$ at $t = 0$.

(ii) $\vec{r} = \cos 3t \hat{i} + \sin 3t \hat{j} + 4at \hat{k}$ at $t = \frac{\pi}{4}$

Solution: (i) In the vector form equation of the given curve is

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = (t^2 + 1)\vec{i} + (4t - 3)\vec{j} + 2(t^2 - 3t)\vec{k}$$

$$\Rightarrow \frac{\vec{d}r}{dt} = 2t\vec{i} + 4\vec{j} + 2(2t - 3)\vec{k}$$

$$\left| \frac{\vec{d}r}{dt} \right| = \sqrt{4t^2 + 16 + (4t - 6)^2} = \sqrt{20t^2 + 52 - 48t} = 2\sqrt{5t^2 + 13 - 12t}$$

∴ Unit tangent vector to the given curve at a point ' \hat{n} ' is given by

$$\hat{n} = \frac{\vec{d}r/dt}{|\vec{d}r/dt|} = \frac{2[t\vec{i} + 2\vec{j} + (2t - 3)\vec{k}]}{2\sqrt{5t^2 + 13 - 12t}}$$

$$\text{At } t = 0, \quad \hat{n} = \frac{(2\vec{j} - 3\vec{k})}{\sqrt{13}}$$

(ii) $\vec{r} = a \cos 3t \vec{i} + a \sin 3t \vec{j} + 4at \vec{k}$

$$\frac{\vec{d}r}{dt} = -3a \sin 3t \vec{i} + 3a \cos 3t \vec{j} + 4a \vec{k}$$

$$\left| \frac{\vec{d}r}{dt} \right| = \sqrt{9a^2 \sin^2 3t + 9a^2 \cos^2 3t + 16a^2} = 5a$$

$$\hat{n} = \frac{\vec{d}r/dt}{|\vec{d}r/dt|} = \frac{a[-3\sin 3t \vec{i} + 3\cos 3t \vec{j} + 4 \vec{k}]}{5a}$$

$$\text{At } t = \frac{\pi}{4}, \quad \frac{1}{5} \left[\frac{-3}{\sqrt{2}} \vec{i} - \frac{-3}{\sqrt{2}} \vec{j} + 4 \vec{k} \right] = \frac{1}{5\sqrt{2}} \left[-3\vec{i} - 3\vec{j} + 4\sqrt{2} \vec{k} \right]$$

3. Find the angle between the tangents to the curve $\vec{r} = t^2\mathbf{i} + 2t\mathbf{j} - t^3\mathbf{k}$ at the points $t = \pm 1$.

Solution: $\frac{d\vec{r}}{dt} = 2t\mathbf{i} + 2\mathbf{j} - 3t^2\mathbf{k}$

$$\left| \frac{d\vec{r}}{dt} \right| = \sqrt{4t^2 + 4 + 9t^4}$$

$$\hat{\mathbf{n}} = \frac{\vec{d}\mathbf{r}/dt}{|\vec{d}\mathbf{r}/dt|} = \frac{2t\mathbf{i} + 2\mathbf{j} - 3t^2\mathbf{k}}{\sqrt{4t^2 + 4 + 9t^4}}$$

$$\text{At } t=1, \hat{\mathbf{n}}_1 = \frac{2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}}{\sqrt{17}}$$

$$\text{At } t=-1, \hat{\mathbf{n}}_2 = \frac{-2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}}{\sqrt{17}}$$

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

Angle between unit tangent vectors at the points $t = \pm 1$ is given by

$$\cos \theta = \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 = \frac{9}{17}$$

$$\theta = \cos^{-1}(9/17).$$

4. A particle moves along the curve $x = \cos(t-1)$, $y = \sin(t-1)$, $z = at^3$ where a is a constant. Find a so that acceleration is perpendicular to position vectors at $t=1$.

Solution: At time t position vector of particle is

$$\vec{r} = \cos(t-1)\mathbf{i} + \sin(t-1)\mathbf{j} + at^3\mathbf{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = -\sin(t-1)\mathbf{i} + \cos(t-1)\mathbf{j} + 3at^2\mathbf{k}$$

$$v|_{t=1} = \mathbf{j} + 3a\mathbf{k}$$

$$\text{Acceleration} = \frac{d\vec{v}}{dt} = -\cos(t-1)\mathbf{i} - \sin(t-1)\mathbf{j} + 6at\mathbf{k}$$

$$\frac{d\vec{v}}{dt}|_{t=1} = -\mathbf{i} + 6a\mathbf{k}, \quad \vec{r}|_{t=1} = \mathbf{i} + a\mathbf{k}$$

Given acceleration is perpendicular to position vector,

$$\vec{r} \cdot \frac{d\vec{v}}{dt} = 0$$

$$\Rightarrow -1 + 6a^2 = 0 \Rightarrow a^2 = \frac{1}{6} \Rightarrow a = \pm 1/\sqrt{6}$$

5. A particle moves along the curve $\vec{r} = 2t^2 \mathbf{i} + (t^2 - 4t) \mathbf{j} + (3t - 5) \mathbf{k}$. Find the component of velocity and acceleration in the direction of vector $c = i - 3j + 2k$ at $t=1$.

Solution: Given $\vec{r} = 2t^2 \mathbf{i} + (t^2 - 4t) \mathbf{j} + (3t - 5) \mathbf{k}$

$$\text{Velocity } \vec{v} = \frac{d\vec{r}}{dt} = 4t \mathbf{i} + (2t - 4) \mathbf{j} + 3 \mathbf{k}$$

$$\text{Acceleration} = \frac{d\vec{v}}{dt} = 4 \mathbf{i} + 2 \mathbf{j}$$

$$\text{At } t=1, \vec{v} = 4 \mathbf{i} - 2 \mathbf{j} + 3 \mathbf{k}$$

$$\frac{d\vec{v}}{dt} = 4 \mathbf{i} + 2 \mathbf{j}$$

$$\text{Also } c = i - 3j + 2k$$

$$|c| = \sqrt{14}$$

$$\hat{c} = \frac{\vec{c}}{|c|} = \frac{i - 3j + 2k}{\sqrt{14}}$$

\therefore Component of velocity at $t=1$ along the given vector \vec{c} is,

$$\vec{v} \cdot \hat{c} = \frac{1}{14} (4 + 6 + 6) = \frac{16}{\sqrt{14}}$$

\therefore Component of acceleration at $t=1$ along the given vector \vec{c} is,

$$\frac{d\vec{v}}{dt} \cdot \hat{c} = \frac{1}{14} (4 - 6) = \frac{-2}{\sqrt{14}}.$$

6) The position vector of a particle at time t is
 $\vec{r} = \cos(t-1)\hat{i} + \sinh(t-1)\hat{j} + dt^3\hat{k}$

Find the condition imposed on d by requiring that at time $t=1$, the acceleration is normal to the position vector.

Sol. $\frac{d\vec{r}}{dt} = -\sin(t-1)\hat{i} + \cosh(t-1)\hat{j} + 3dt^2\hat{k}$

$$\text{Acceleration} = \frac{d^2\vec{r}}{dt^2} = -\cos(t-1)\hat{i} + \sinh(t-1)\hat{j} + 6dt\hat{k}$$

$$\text{At } t=1, \text{ acceleration} = \frac{d^2\vec{r}}{dt^2} = -\hat{i} + 6\hat{k}$$

$$\vec{r} = \hat{i} + d\hat{k}$$

If \vec{r} and \vec{a} are normal, then dot product is 0.
 $(\hat{i} + 6\hat{k}) \cdot (\hat{i} + d\hat{k}) = 0$
 $-1 + 6d^2 = 0 \Rightarrow d^2 = \frac{1}{6} \Rightarrow d = \pm \frac{1}{\sqrt{6}}$

Practice problems :

1) A particle moves along the curve $x = t^3 + 1$,
 $y = t^2$, $z = 2t + 5$, where t is the time. Find the components of its velocity and acceleration at time $t=1$ in the direction $2\hat{i} + 3\hat{j} + 6\hat{k}$.

Ans: Component of velocity along $2\hat{i} + 3\hat{j} + 6\hat{k} = \frac{24}{7}$

$$-11 - -11 \text{ acceleration} = -11 - -11 = \frac{18}{7}$$

2) A particle moves along the curve $\vec{r} = (t^3 - 4t)\hat{i} + (t^2 + 4t)\hat{j} + (8t^2 - 3t^3)\hat{k}$, where t is the time. Find the magnitude

of tangential components of its acceleration at $t=2$.

Ans: 16

- 3) Find angle between unit tangent vectors drawn to the curve $x = a \cos at$, $y = a \sin at$, $z = at$ at the points $t = \frac{\pi}{6}$ and $t = \frac{\pi}{4}$. Ans: $\theta = \cos^{-1} \left(\frac{2\sqrt{3} + 1}{5} \right)$

The vector differential operator is denoted by ∇ (read as del) and is defined as

$$\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$$

Gradient of a scalar field :

If $\phi(x, y, z)$ is a scalar function then, the gradient of ϕ is denoted by $\text{grad}(\phi)$ or $\nabla \phi$, defined as

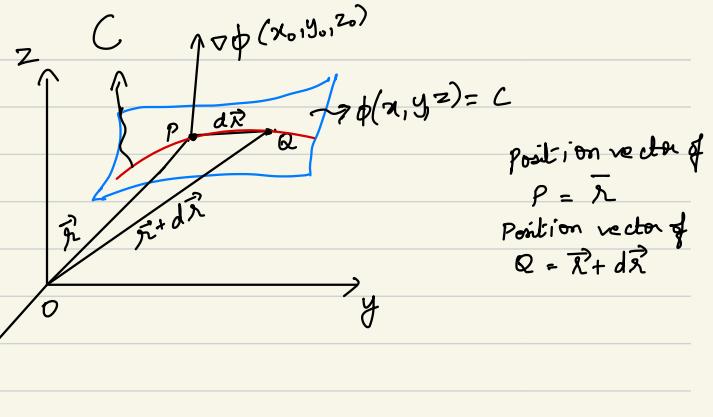
$$\text{grad}(\phi) = \nabla \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$$

Note: The operator gradient is always applied on scalar field and the resultant is a vector i.e. the operator gradient converts a scalar field into a vector field. $\nabla \phi$ is a vector function. Beware ∇ cannot exist alone, it is an operator and must operate on scalar function.

Physical significance: $\nabla \phi$ gives the direction of maximum rate of change of ϕ at a particular point - $\nabla \phi$ points in the direction of most rapid decrease of ϕ .

Geometrical interpretation of gradient: (Function of 3 variables)

Let $\phi(x, y, z) = c$ be the surface. Let $P(x_0, y_0, z_0)$ be a point on the surface. If the differentiable functions $x = f(t)$, $y = g(t)$, $z = h(t)$ are parametric equations of curve C on surface then derivative of $\phi(f(t), g(t), h(t)) = c$ w.r.t t is $\frac{\partial \phi}{\partial x} \frac{dx}{dt} + \frac{\partial \phi}{\partial y} \frac{dy}{dt} + \frac{\partial \phi}{\partial z} \frac{dz}{dt} = 0$



$$\text{or } \left(\frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \right) \cdot \left(\frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j} + \frac{dz}{dt} \hat{k} \right) = 0 \quad \text{①}$$

At $t = t_0$,

$$\nabla F(x_0, y_0, z_0) \cdot r'(t_0) = 0$$

$\Rightarrow \nabla F(x_0, y_0, z_0)$ is orthogonal to vector $r'(t_0)$.

$\Rightarrow \nabla F(x_0, y_0, z_0)$ is normal (perpendicular) to surface at P.

differential (actual change in r)

Note: $\nabla \phi \cdot d\vec{r} = \left(\frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \right) \cdot (dx \hat{i} + dy \hat{j} + dz \hat{k})$

$$\nabla \phi \cdot dr = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = d\phi$$

total differential of ϕ

Properties: If f and g are continuous and differentiable scalar point functions then

1) $\text{grad}(f \pm g) = \text{grad } f \pm \text{grad } g$

or $\nabla(f \pm g) = \nabla f \pm \nabla g$

2) $\text{grad}\left(\frac{f}{g}\right) = \nabla\left(\frac{f}{g}\right) = \frac{g \nabla f - f \nabla g}{g^2}$

$$3) \nabla(fg) = g \nabla f + f \nabla g$$

Unit normal vector : Since $\nabla\phi$ is normal vector

to surface $\phi(x, y, z) = c$ then unit vector is denoted by \hat{n} and is defined as $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\nabla\phi}{|\nabla\phi|}$

where $\vec{n} = \nabla\phi$ = normal vector.

Note. The angle b/w the normals to the 2 surfaces ϕ_1 and ϕ_2 is $\cos\theta = \hat{n}_1 \cdot \hat{n}_2$ where $\hat{n}_1 = \frac{\nabla\phi_1}{|\nabla\phi_1|}$, $\hat{n}_2 = \frac{\nabla\phi_2}{|\nabla\phi_2|}$

Directional derivative :

From the definition of gradient it is observed the partial derivatives rate of change of ϕ along coordinate axes. The idea of extending this to arbitrary direction makes sense and lead to concept of directional derivative. Directional derivative measures variation of a function along a given direction

Definition: If f is a differentiable function of x and y , then the directional derivative of f in the direction of any unit vector, $\hat{u} = \langle a, b \rangle$ is

$$D_u f(x, y) = f_x(x, y) a + f_y(x, y) b = \nabla f \cdot \hat{u}$$

↓
nothing but projection of grad f
on unit vector \hat{u} .

Note: 1) If the unit vector \hat{u} makes an angle θ with positive x axis then we write $\hat{u} = \langle \cos\theta, \sin\theta \rangle$ and $D_u \phi(x, y) = \phi_x(x, y) \cos\theta + \phi_y(x, y) \sin\theta$

2) Maximum value of directional derivative :

$$D_u f = \nabla f \cdot \hat{u} = |\nabla f| |\hat{u}| \cos\theta = |\nabla f| \cos\theta \quad (\because |\hat{u}| = 1)$$

The range of $\cos\theta$ is $[-1, 1]$ i.e. $-1 \leq \cos\theta \leq 1$

$$\Rightarrow -|\nabla f| \leq D_u f \leq |\nabla f|$$

In other words,

The maximum value of directional derivative is $|\nabla f|$ and it occurs when \hat{u} has same direction as ∇f (when $\cos\theta = 1$, θ is angle b/w \hat{u} and ∇f).

The minimum value of directional derivative is $-|\nabla f|$ and it occurs when \hat{u} and f have opposite directions (when $\cos\theta = -1$).

ex: If you are standing in a hot room and thinking to move in a direction along which temperature change is quickest or maximum. Then gradient of temperature field calculated at your position points to the direction you need.

Directional derivative $D_u T$ measures the variation of temperature in that direction.

Problems:

- 1) Find gradient of scalar field $\phi(x, y, z) = xy^2 + 2x^2yz - 3z^2$

Sol:

$$\nabla \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$$

$$\nabla \phi = (y^2 + 4xyz) \hat{i} + (2xy + 2x^2z - 3z^2) \hat{j} + (2x^2y - 6yz) \hat{k}$$

- 2) If $\phi = x^2yz^3 + 2y^2z^2$, determine grad ϕ at $P = (1, 3, 2)$

Sol:

$$\nabla \phi = (2xyz^3 + y^2z^2) \hat{i} + (x^2z^3 + 2xyz^2) \hat{j} + (3x^2yz^2 + 2y^2z) \hat{k}$$

At $P = (1, 3, 2)$ we get

$$\nabla \phi = 84 \hat{i} + 32 \hat{j} + 72 \hat{k}$$

- 3) Determine the directional derivative $D_u \phi(x, y)$ if $\phi(x, y) = x^3 - 3xy + 4y^2$ and \hat{u} is the unit vector making angle $\theta = \frac{\pi}{6}$ with x -axis. What is $D_u \phi(1, 2)$?

Sol:

$$D_u \phi(x, y) = \phi_x(x, y) \cos \frac{\pi}{6} + \phi_y(x, y) \sin \frac{\pi}{6}$$

$$= (3x^2 - 3y) \frac{\sqrt{3}}{2} + (-3x + 8y) \frac{1}{2}$$

$$= \frac{1}{2} [3\sqrt{3}x^2 - 3x + (8 - 3\sqrt{3})y]$$

$$D_u \phi(1, 2) = \frac{1}{2} [3\sqrt{3} \cdot 1^2 - 3(1) + (8 - 3\sqrt{3})2]$$

$$\therefore D_u \phi(1, 2) = \frac{13 - 3\sqrt{3}}{2}$$

4) Obtain the directional derivative of a function $\phi(x, y, z) = x^2z + 2xy^2 + yz^2$ at the point $P(1, 2, -1)$ in the direction of vector $2\hat{i} + 3\hat{j} - 4\hat{k}$.

Sol: Given: $\phi = x^2z + 2xy^2 + yz^2$, $\vec{v} = 2\hat{i} + 3\hat{j} - 4\hat{k}$
 WKT $D_u \phi(x, y, z) = \nabla \phi \cdot \hat{u} \rightarrow ①$

$$\nabla \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$$

$$\nabla \phi = (2xz + 2y^2) \hat{i} + (4xy + z^2) \hat{j} + (x^2 + 2yz) \hat{k}$$

$$\nabla \phi_{(1, 2, -1)} = 6\hat{i} + 9\hat{j} - 3\hat{k}$$

Here \vec{v} is not a unit vector.

$$\text{unit vector, } \hat{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{2\hat{i} + 3\hat{j} - 4\hat{k}}{|2\hat{i} + 3\hat{j} - 4\hat{k}|} = \frac{2}{\sqrt{29}} \hat{i} + \frac{3}{\sqrt{29}} \hat{j} - \frac{4}{\sqrt{29}} \hat{k}$$

$$\begin{aligned} ① \Rightarrow D_u \phi(x, y, z) &= (6\hat{i} + 9\hat{j} - 3\hat{k}) \cdot \left(\frac{2}{\sqrt{29}} \hat{i} + \frac{3}{\sqrt{29}} \hat{j} - \frac{4}{\sqrt{29}} \hat{k} \right) \\ &= \frac{2}{\sqrt{29}} (6) + \frac{3}{\sqrt{29}} (9) + \frac{-4}{\sqrt{29}} (-3) \\ &= \frac{51}{\sqrt{29}} \approx 9.4704 \end{aligned}$$

5) Find unit normal to the surface $xyz = 2$ at $(2, 1, 1)$

Sol. $xyz - 2 = 0$

$$\det \Phi = xyz - 2$$

$$\nabla \phi = yz\hat{i} + xz\hat{j} + xy\hat{k}$$

$$N(2, 1, 1) : \nabla \phi = \hat{i} + 2\hat{j} + 2\hat{k}$$

$$\therefore |\nabla \phi| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$$

Unit vector normal to surface ϕ is

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{3}$$

6) Find the directional derivative of $\phi(x, y, z) = \frac{1}{x^2 + y^2 + z^2}$ at

the point $(1, 2, -3)$ in the direction of $2\hat{i} - 3\hat{j} + \hat{k}$

sol:

$$\nabla \phi = \frac{-2x}{(x^2 + y^2 + z^2)^2} \hat{i} - \frac{2y}{(x^2 + y^2 + z^2)^2} \hat{j} - \frac{2z}{(x^2 + y^2 + z^2)^2} \hat{k}$$

At $(1, 2, -3) \Rightarrow \nabla \phi = -\frac{2}{(14)^2} \hat{i} - \frac{4}{(14)^2} \hat{j} + \frac{6}{(14)^2} \hat{k}$

$$\nabla \phi = -\frac{2}{14^2} [\hat{i} + 2\hat{j} - 3\hat{k}]$$

$$\vec{v} = 2\hat{i} - 3\hat{j} + \hat{k} \Rightarrow |\vec{v}| = \sqrt{14}$$

Unit vector, $\hat{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{14}} (2\hat{i} - 3\hat{j} + \hat{k})$

The directional derivative is $\nabla \phi \cdot \hat{u} = \frac{-2}{(14)^2} (\hat{i} + 2\hat{j} - 3\hat{k}) \cdot \frac{1}{\sqrt{14}} (2\hat{i} - 3\hat{j} + \hat{k})$

$$= \frac{-2}{(14^2) \sqrt{14}} (2 - 6 - 3) = \frac{14}{14^2 \sqrt{14}} = \frac{1}{14 \sqrt{14}}$$

7) Find the angle b/w the normals to the surface
 $xy^3z^2 = 4$ at the points $(-1, -1, 2)$ and $(4, 1, 1)$

Sol: $\det \phi = xy^3z^2 - 4$

$$\nabla \phi = y^3 z^2 \hat{i} + 3x y^2 z^2 \hat{j} + 2x y^3 z \hat{k}$$

Normal at $(-1, -1, 2)$: $n_1 = \nabla \phi \Big|_{(-1, -1, 2)} = -4\hat{i} - 12\hat{j} + 4\hat{k}$

Normal at $(4, 1, -1)$: $n_2 = \nabla \phi \Big|_{(4, 1, -1)} = \hat{i} + 12\hat{j} - 8\hat{k}$

Let θ be angle b/w 2 normals.

$$\therefore \cos \theta = \frac{n_1 \cdot n_2}{|n_1| |n_2|} = \frac{(-4\hat{i} - 12\hat{j} + 4\hat{k}) \cdot (\hat{i} + 12\hat{j} - 8\hat{k})}{\sqrt{16+144+16} \sqrt{1+144+64}}$$

$$\text{or } \cos \theta = \frac{\hat{n}_1 \cdot \hat{n}_2}{|\nabla \phi_{(-1, -1, 2)}|}$$

where $\hat{n}_1 = \frac{\nabla \phi_{(-1, -1, 2)}}{|\nabla \phi_{(-1, -1, 2)}|}$

$$\cos \theta = \frac{-180}{\sqrt{176} \sqrt{209}}$$

$$\therefore \theta = \cos^{-1} \left(\frac{-180}{\sqrt{176} \sqrt{209}} \right) = 159.765^\circ$$

8) Find the angle b/w 2 surfaces $x^2+y^2+z^2=9$ and $x^2+y^2-z=3$ at $(2, -1, 2)$

Sol: The angle b/w 2 surfaces at common point is angle b/w the normals drawn to the surfaces at that point.

Given: $\phi_1 = x^2 + y^2 + z^2 - 9$, $\phi_2 = x^2 + y^2 - z - 3$

$$\nabla \phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}, \quad \nabla \phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\nabla \phi_1 \Big|_{(2, -1, 2)} = 4\hat{i} - 2\hat{j} + 4\hat{k}, \quad \nabla \phi_2 \Big|_{(2, -1, 2)} = 4\hat{i} - 2\hat{j} - \hat{k}$$

$$\cos \theta = \frac{\nabla \phi_1 \cdot \nabla \phi_2}{|\nabla \phi_1| |\nabla \phi_2|} = \frac{16+4-4}{\sqrt{36} \sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\therefore \theta = \cos^{-1} \left(\frac{8}{3\sqrt{21}} \right) = 54.41^\circ$$

q) Find the maximum directional derivative of $\phi(x, y, z) = x^3 y^2 z$ at $(1, -2, 3)$.

sol:

$$\nabla \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} = 3x^2 y^2 z \hat{i} + 2x^3 y z \hat{j} + x^3 y^2 \hat{k}$$

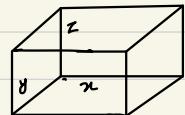
$$\text{at } (1, -2, 3) \Rightarrow \nabla \phi = 36 \hat{i} - 12 \hat{j} + 4 \hat{k}$$

$$\text{The maximum directional derivative} = |\nabla \phi| = \sqrt{36^2 + (-12)^2 + 4^2} = 4\sqrt{91}$$

10) The temperature in a rectangular box is approximated by $T(x, y, z) = xyz(1-x)(2-y)(3-z)$, $0 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq 3$. If a mosquito is located at $(\frac{1}{2}, 1, 1)$ in which direction should it fly to cool off as rapidly as possible?

$$\begin{aligned} \text{sol: } \nabla T(x, y, z) &= yz(2-y)(3-z)(1-2x) \hat{i} + xz(1-x)(3-z)(2-2y) \hat{j} \\ &\quad + xy(1-x)(2-y)(3-2z) \hat{k} \end{aligned}$$

$$\nabla T \Big|_{(\frac{1}{2}, 1, 1)} = \frac{1}{4} \hat{k}$$



To cool off most rapidly, the mosquito should fly in the direction of $-\frac{1}{4} \hat{k}$

1) Find constants a and b so that surface
 $3x^2 - 2y^2 - 3z^2 + 8 = 0$ is orthogonal to surface
 $ax^2 + y^2 = bz$ at point $(-1, 2, 1)$

Sol:

$$\det \phi_1(x, y, z) = 3x^2 - 2y^2 - 3z^2 + 8$$

$$\Rightarrow \nabla \phi_1 = 6x\hat{i} - 4y\hat{j} - 6z\hat{k}$$

$$\text{At } (-1, 2, 1), \nabla \phi_1 = -6\hat{i} - 8\hat{j} - 6\hat{k}$$

$$\Rightarrow |\nabla \phi_1| = \sqrt{6^2 + 8^2 + 6^2} = \sqrt{136} = 2\sqrt{34}$$

$$\hat{n}_1 = \frac{\nabla \phi_1}{|\nabla \phi_1|} = \frac{1}{2\sqrt{34}} (-6\hat{i} - 8\hat{j} - 6\hat{k}) = -\frac{1}{\sqrt{34}} (3\hat{i} + 4\hat{j} + 3\hat{k})$$

$$\det \phi_2(x, y, z) = ax^2 + y^2 - bz \Rightarrow \nabla \phi_2 = 2ax\hat{i} + 2y\hat{j} - b\hat{k}$$

$$\text{At } (-1, 2, 1) \Rightarrow \nabla \phi_2 = -2a\hat{i} + 4\hat{j} - b\hat{k}$$

$$\Rightarrow |\nabla \phi_2| = \sqrt{4a^2 + 16 + b^2}$$

$$\hat{n}_2 = \frac{\nabla \phi_2}{|\nabla \phi_2|} = \frac{-2a\hat{i} + 4\hat{j} - b\hat{k}}{\sqrt{4a^2 + b^2 + 16}}$$

Since surfaces intersect orthogonally

$$\hat{n}_1 \cdot \hat{n}_2 = 0 \Rightarrow -\frac{1}{\sqrt{34}} (3\hat{i} + 4\hat{j} + 3\hat{k}) \cdot \frac{1}{\sqrt{4a^2 + b^2 + 16}} (-2a\hat{i} + 4\hat{j} - b\hat{k}) = 0$$

$$\Rightarrow (3\hat{i} + 4\hat{j} + 3\hat{k}) \cdot (-2a\hat{i} + 4\hat{j} - b\hat{k}) = 0$$

$$\Rightarrow -6a + 16 - 3b = 0 \quad \text{i.e. } 6a + 3b = 16 \rightarrow ①$$

Also the point $(-1, 2, 1)$ lies on surface $ax^2 + y^2 = bz$

$$\Rightarrow a + 4 = b \quad \text{i.e. } a - b = -4 \rightarrow ②$$

$$\text{Solving } ① \text{ & } ② \Rightarrow a = \frac{4}{9} \text{ and } b = \frac{40}{9}.$$

Practice problems :

1. Find the gradient of the given function at indicated point.

I. $\phi(x, y, z) = x^2yz + 4z^2$; $(1, -2, -1)$ Ans.: $8\vec{i} - \vec{j} + 10\vec{k}$

II. $\phi(x, y, z) = x^2z^2 \sin 4y$; $(-2, \frac{\pi}{3}, 1)$

III. $\phi(x, y, z) = x^3yz + xy^2z^3$; $(1, 2, 3)$ Ans.: $126\vec{i} + 111\vec{j} + 110\vec{k}$

2. Find the directional derivative of the given function at the given point in the indicated direction.

I. $\phi(x, y, z) = 4xz^2 + x^2yz$; $(1, -2, 1)$, $2\vec{i} + 3\vec{j} + 4\vec{k}$ Ans.: $\frac{27}{\sqrt{29}}$

II. $\phi(x, y, z) = x^2yz + 4xz^2$; $(1, -2, -1)$, $2\vec{i} - \vec{j} - 2\vec{k}$ Ans.: $\frac{37}{3}$

3. Find the directional derivative of the function $\phi = xe^y + ye^z + ze^x$ at the point $(0, 0, 0)$ in the direction of the vector $\vec{v} = -\vec{i} - 2\vec{j} + 2\vec{k}$. Also find the maximal directional derivative.

Ans.: Directional derivative of ϕ is $\hat{a} \cdot \text{grad } \phi = \frac{-1}{3}$. Maximal directional derivative is $|\nabla \phi| = \sqrt{3}$.

4. The temperature of a point in a space is given by $T(x, y, z) = (x^2 + y^2 + z^2)$. Determine unit vector when a honey bee located at $(2, 1, 2)$ desires to fly in such a direction that it will get heat as soon as possible.

Ans.: $\nabla T(2, 1, 2) = 2\vec{i} + 2\vec{j} - 4\vec{k}$. Unit vector along gradient is $\frac{i+j-2k}{3}$

5) Find the angle b/w normals to the surface
 $z^2 - xy = 0$ at the points $(4, 1, 2)$ and $(3, 3, -3)$

Ans: $\cos^{-1} \left(\frac{1}{\sqrt{22}} \right)$

6) Find the constants a and b such that surface $x^2 + ay z = 3x$ and $b x^2 y + z^3 = (b-8)y$ intersect orthogonally at point $(1, 1, -2)$

Ans: $a = -1, b = 2$

7) Find the angle b/w normals to the surface $x \log z = y^2 - 1$ at points $(1, 1, 1)$ and $(2, 1, 1)$

Ans: $\cos^{-1} \left(\frac{3}{\sqrt{10}} \right)$

8) Find the directional derivative of function $\phi(x, y, z) = x^3 + y^3 + z^3$ at the point $(1, -1, 2)$ in the direction of normal to surface

$x^3 + y^3 + z^3 = 8$. Ans: $9\sqrt{2}$

$$\begin{vmatrix} \nabla \phi \cdot \hat{u} \\ \text{where } \hat{u} = \frac{\nabla \phi_1}{|\nabla \phi_1|} \end{vmatrix}$$

9) If $f(x, y) = xe^y$, find rate of change of f at point $(2, 0)$, in the direction of \vec{PQ} with $Q(\frac{1}{2}, 2)$. In what direction does f have maximum rate of change? What is the maximum rate of change? Ans: Direction is $\nabla f(2, 0) = \langle 1, 2 \rangle$, Max. rate of change is $|\langle 1, 2 \rangle| = \sqrt{5}$

10) Suppose that temperature at a point (x, y, z) in space is given by the following relationship:

$$T(x, y, z) = \frac{80}{1+x^2+2y^2+3z^2}, \quad \text{where } T \text{ is measured in}$$

degree celsius and x, y, z in metres. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is maximum rate of change?

Ans: Direction is $\nabla T(1, 1, -2) = \frac{5}{8} (-\hat{i} - 2\hat{j} + 6\hat{k})$

Max rate of change is $|\nabla T| = \frac{5\sqrt{41}}{8} \approx 4^\circ \text{C/m}$