Test for consistency of the given system of equations and solve by Gauss elimination

$$x + y + z = 6$$
$$x - y + 2z = 5$$
$$3x + y + z = 8$$

Sol:- For the given system

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix}$$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & -1 & 2 & : & 5 \\ 3 & 1 & 1 & : & 8 \end{bmatrix}$$ is the augmented matrix

$$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - 3R_1$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & -2 & -2 & : & -10 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & 0 & -3 & : & -9 \end{bmatrix} \to (*)$$

$\therefore$ rank(A) = rank$([A:B]) = 3$

Also n, no of unknowns $= 3$

$\therefore$ The given system of equations is consistent and will have unique solution.

(*) can be written as

$$x + y + z = 6 \rightarrow (1)$$
$$-2y + z = -1 \rightarrow (2)$$
$$-3z = -9 \rightarrow (3) \qquad \Rightarrow \boxed{z = 3}$$

(2) $\Rightarrow$ $-2y + 3 = -1 \Rightarrow -2y = -4 \Rightarrow \boxed{y = 2}$

(1) $\Rightarrow$ $x + 2 + 3 = 6 \Rightarrow \boxed{x = 6 - 5 = 1}$

2)
$$x - 4y + 7z = 14$$
$$3x + 8y - 2z = 13$$
$$7x - 8y + 26z = 5$$

Sol:

$$[A : B] = \begin{bmatrix} 1 & -4 & 7 & : & 14 \\ 3 & 8 & -2 & : & 13 \\ 7 & -8 & 26 & : & 5 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 3R_1, \quad R_3 \rightarrow R_3 - 7R_1$$

$$[A : B] = \begin{bmatrix} 1 & -4 & 7 & : & 14 \\ 0 & 20 & -23 & : & -29 \\ 0 & 20 & -23 & : & -93 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & -4 & 7 & 14 \\ 0 & 20 & -23 & -29 \\ 0 & 0 & 0 & -64 \end{array}\right]$$

$\text{rank}(A) = 2$, $\text{rank}\,[A:B] = 3$

Here $\text{rank}(A) \neq \text{rank}([A:B])$

Hence the system is inconsistent (no solution)

3)
$$x + 2y + 3z = 14$$
$$4x + 5y + 7z = 35$$
$$3x + 3y + 4z = 21$$

Sol:- $[A:B] = \left[\begin{array}{ccc:c} 1 & 2 & 3 & 14 \\ 4 & 5 & 7 & 35 \\ 3 & 3 & 4 & 21 \end{array}\right]$

$$R_2 \to R_2 - 4R_1 \quad , \quad R_3 \to R_3 - 3R_1$$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 2 & 3 & 14 \\ 0 & -3 & -5 & -21 \\ 0 & -3 & -5 & -21 \end{array}\right]$$

$$R_3 \to R_3 - R_2$$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 0 & -3 & -5 & : & -21 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \rightarrow (*)$$

$\text{rank}(A) = \text{rank}([A:B]) = 2 < 3 \quad (r < n)$

The system is consistent and will have infinite solutions. The system has $n-r = 3-2 = 1$ free variable (can take arbitrary value)

$(*)$ can be written as

$$x + 2y + 3z = 14 \rightarrow (1)$$

$$-3y - 5z = -21 \Rightarrow 3y + 5z = 21 \rightarrow (2)$$

Let $z = K$ be arbitrary

$(2) \Rightarrow 3y + 5K = 21$

$$\therefore y = \frac{21 - 5K}{3} = 7 - \frac{5K}{3}$$

$(1) \Rightarrow x + 2\left[7 - \frac{5K}{3}\right] + 3K = 14$

$x + \cancel{14} - \frac{10K}{3} + 3K = \cancel{14}$

$x = -3K + \frac{10K}{3}$

$\therefore x = \frac{-9K + 10K}{3} = 1$

$\therefore x = K/3$

$\therefore x = \frac{K}{3}, \; y = 7 - \frac{5K}{3}, \; z = K$ represent infinite solutions, since K is arbitrary.

4) For what values of λ, μ the simultaneous equations $2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$ have

a) unique solution b) no solution c) infinite no of solutions

Sol:-

$$[A:B] = \left[\begin{array}{ccc:c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right]$$

$$R_2 \to R_2 - \frac{7}{2}R_1, \quad R_3 \to R_3 - R_1$$

$$\sim \left[\begin{array}{ccc:c} 2 & 3 & 5 & 9 \\ 0 & \frac{-15}{2} & \frac{-39}{2} & -47/2 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{array}\right]$$

$8 - \frac{7}{2} \times 9 = 8 - \frac{63}{2}$

$3 - \frac{7}{2}(3) = 3 - \frac{21}{2} = \frac{6-21}{2}$

$-2 - \frac{7}{2} \times 5 = -2 - \frac{35}{2} = \frac{-4-35}{2}$

Case 1) Unique solution :- The system will have unique solution if $\lambda - 5 \neq 0 \Rightarrow \lambda \neq 5$ and μ may have any value.

If $\lambda \neq 5$ then $\text{rank}(A) = \text{rank}([A:B]) = 3 =$ no of unknowns.

Case 2) No solution :-

If $\lambda = 5$ and $\mu \neq 9$ then $\text{rank}(A) = 2$ and $\text{rank}([A:B]) = 3$.

Since $\rho(A) \neq \rho([A:B])$ the system of equations will have no solution.

Case 3) Infinite no of solutions:

If $\lambda = 5$ and $\mu = 9$, then $\text{rank}(A) = \text{rank}([A:B]) = 2$ $<$ no of unknowns and system will have infinite no of solutions.

Obtain the solution of the given system by Gauss-Jordan method

1)
$$x_1 - x_2 + 2x_3 = 4$$
$$2x_1 + 3x_2 - x_3 = 1$$
$$3x_1 + 2x_2 + x_3 = 5$$

Sol:-
$$[A:B] = \left[\begin{array}{ccc:c} 1 & -1 & 2 & 4 \\ 2 & 3 & -1 & 1 \\ 3 & 2 & 1 & 5 \end{array}\right]$$

$$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 2 & 4 \\ 0 & 5 & -5 & -7 \\ 0 & 5 & -5 & -7 \end{array}\right]$$

$$R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 2 & 4 \\ 0 & 5 & -5 & -7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 \to \frac{1}{5} R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & -7/5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$4 - \frac{7}{5} = \frac{13}{5}$

$R_1 \to R_1 + R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & : & 13/5 \\ 0 & 1 & -1 & : & -7/5 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \begin{array}{l} \\ \to \text{rref} \\ \to (1) \end{array}$$

rank(A) = rank([A : B]) = 2 < no. of unknowns (3)

$\therefore$ The given system is consistent and will have infinite no. of solutions.

Here, there is $n - r = 3 - 2 = 1$ free variable

(1) $\Rightarrow$ $x_1 + x_3 = 13/5 \to (2)$

$x_2 - x_3 = -7/5 \to (3)$

Let $\boxed{x_3 = K}$ be the free variable

(2) $\Rightarrow$ $\boxed{x_1 = \frac{13}{5} - K}$, (3) $\Rightarrow$ $\boxed{x_2 = \frac{-7}{5} + K}$

$$X = \underbrace{\begin{bmatrix} 13/5 \\ -7/5 \\ 0 \end{bmatrix}}_{\text{particular sol.}} + K \underbrace{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}_{\text{sol. of corres. homogeneous system.}}$$

2) $x_1 + x_2 + x_3 + x_4 + x_5 = 2$

$x_1 + x_2 + x_3 + 2x_4 + 2x_5 = 3$

$x_1 + x_2 + x_3 + 2x_4 + 3x_5 = 2$

Sol:- $[A : B] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & : & 2 \\ 1 & 1 & 1 & 2 & 2 & : & 3 \\ 1 & 1 & 1 & 2 & 3 & : & 2 \end{bmatrix}$

$$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_1$$

$$\sim \left[\begin{array}{ccccc:c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

$$R_1 \to R_1 - R_2, \quad R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccccc:c} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$R_2 \to R_2 - R_3$$

$$\sim \left[\begin{array}{ccccc:c} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \to (1)$$

Here rank (A) = rank $([A:B]) = 3 <$ no. of unknowns (5)

$\therefore$ The system is consistent and will have infinite no. of solutions.

No. of free variables $= n - r = 5 - 3 = 2$

$(1) \Rightarrow \quad x_1 + x_2 + x_3 = 1 \quad \to (2)$

$x_4 = 2, \quad x_5 = -1$

Let $x_2 = k_1$ and $x_3 = k_2$ be the free variables

$\therefore \quad x_1 = 1 - K_1 - K_2 \;,\; x_2 = K_1 \;,\; x_3 = K_2 \;,\; x_4 = 2 \;,\; x_5 = -1$ is the solution of the system.

(* The above system is underdetermined system : no of equations is less than unknowns $(m < n)$. It is not possible for an undetermined system to have a unique solution)

3) $x_1 + 2x_2 + x_3 = 1$

$2x_1 - x_2 + x_3 = 2$ by Gauss elimination

$4x_1 + 3x_2 + 3x_3 = 4$

$2x_1 - x_2 + 3x_3 = 5$

Sol:- $[A:B] \sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 2 & -1 & 3 & 5 \end{array}\right]$

$R_2 \to R_2 - 2R_1 \;,\; R_3 \to R_3 - 4R_1 \;,\; R_4 \to R_4 - 2R_1$

$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & 1 & 3 \end{array}\right]$

$$R_3 \to R_3 - R_2, \quad R_4 \to R_4 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 1 \\ 0 & -5 & -1 & : & 0 \\ 0 & 0 & 0 & : & 0 \\ 0 & 0 & 2 & : & 3 \end{bmatrix}$$

$$R_3 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 1 \\ 0 & -5 & -1 & : & 0 \\ 0 & 0 & 2 & : & 3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$R_2 \to -\frac{1}{5} R_2, \quad R_3 \to \frac{1}{2} R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 1 \\ 0 & 1 & 1/5 & : & 0 \\ 0 & 0 & 1 & : & 3/2 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \to (*)$$

Here rank (A) = rank ([A : B]) = 3 = no of unknowns

The system is consistent and will have unique solution

$(*) \Rightarrow$

$$x_1 + 2x_2 + x_3 = 1$$

$$x_2 + \frac{x_3}{5} = 0$$

$$x_3 = 3/2$$

By back substitution

$$x_3 = \frac{3}{2}, \quad x_2 = -\frac{3}{10}, \quad x_1 = 1 - \frac{3}{2} + \frac{6}{10} = \frac{1}{10}$$

(The above system is over determined system: no of equations is greater than no of unknowns)

Homogeneous system of linear equations:

It is of the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0 \end{aligned}\right\} \quad (\$)$$

In matrix form

$$AX = 0$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

i.e. $AX = 0$

unique solution (trivial soln.)
if rank$(A) = n$ (no of unknowns)

Infinite solutions
(non-trivial soln)
if rank$(A) < n$

1) Solve by Gauss-elimination method:

$$x_1 - x_2 + 2x_3 = 0$$
$$2x_1 + 3x_2 - x_3 = 0$$
$$3x_1 + 2x_2 + x_3 = 0$$

Soln: Here $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & -1 \\ 3 & 2 & 1 \end{bmatrix}$ Obtain ref of this matrix.

$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & -5 \\ 0 & 5 & -5 \end{bmatrix}$$

$R_2 \to \frac{1}{5} R_2, \quad R_3 \to \frac{1}{5} R_3$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$ is in ref.

Thus, rank $(A) = 2 < 3$ (no. of unknowns)

System has non trivial solns. Let $\boxed{x_3 = k}$ be free variable.

Equivalent system

$$x_1 - x_2 + 2x_3 = 0$$
$$x_2 - x_3 = 0$$

$\boxed{x_2 = k}$, $\quad x_1 = x_2 - 2x_3$

or $\boxed{x_1 = -k}$

$\therefore$ Soln is $X = \begin{bmatrix} -k \\ k \\ k \end{bmatrix}$

or $X = k \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, k is a scalar.

2) Find x_1, x_2, x_3 and x_4, such that it balance the equation

$$x_1 CO_2 + x_2 H_2O \longrightarrow x_3 O_2 + x_4 C_6 H_{12} O_6$$

Soln : Equation is balanced if

$$x_1 = 6x_4 \qquad C$$

$$2x_1 + x_2 = 2x_3 + 6x_4 \qquad O$$

$$2x_2 = 12x_4 \qquad H$$

$$\Rightarrow \quad x_1 + 0x_2 + 0x_3 - 6x_4 = 0$$

$$2x_1 + x_2 - 2x_3 - 6x_4 = 0$$

$$0x_1 + 2x_2 + 0x_3 - 12x_4 = 0$$

It has non-trivial solns, because rank $(A) \leq 3 < 4$ (no. of unknowns)

where $A = \begin{bmatrix} 1 & 0 & 0 & -6 \\ 2 & 1 & -2 & -6 \\ 0 & 2 & 0 & -12 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & -2 & 6 \\ 0 & 2 & 0 & -12 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 4 & -24 \end{bmatrix}$$

$R_3 \to \frac{1}{4} R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 1 & -6 \end{bmatrix}$$

This is in row echelon form

Equivalent system

$$x_1 - 6x_4 = 0$$
$$x_2 - 2x_3 + 6x_4 = 0$$
$$x_3 - 6x_4 = 0$$

rank $(A) = 3$,

$\therefore$ no. of free variable $= n - r = 4 - 3 = 1$

Let x_4 be a free variable. Let $x_4 = k$

$\boxed{x_3 = 6k}$ $\quad x_2 = 2x_3 - 6x_4$ $\quad x_1 = 6x_4$

$\Rightarrow \boxed{x_2 = 6k}$ $\quad$ or $\boxed{x_1 = 6k}$

Soln $\quad X = \begin{bmatrix} 6k \\ 6k \\ 6k \\ k \end{bmatrix}$ $\quad k$ is a scalar

or $\quad X = k \begin{bmatrix} 6 \\ 6 \\ 6 \\ 1 \end{bmatrix}$

Exercise: Obtain the solution of the given system using Gauss elimination

1) $x+y+z=6$

$x+2y+3z=14$

$x+4y+7z=30$

(system is consistent and has infinite no of solutions)

2) $x_1+2x_2-2x_3=1$

$2x_1+5x_2+x_3=9$

$x_1+3x_2+4x_3=9$

(System has unique soln.)

$x_1=-1,\ x_2=2,\ x_3=1$

3) $x-2y+3t=2$

$2x+y+z+t=-4$

$4x-3y+z+7t=8$

(System has infinite no of solutions)

4) Investigate the values of λ and μ so that the equations $x+y+z=6,\ x+2y+3z=10,\ x+2y+\lambda z=\mu$ have

a) no solution b) unique solution c) infinite no of solutions

Obtain the solution of the given system by Gauss Jordan method

1) $x-y+z=4$

$2x+y-3z=0$

$x+y+z=2$

(System has unique solution $x=2,\ y=-1,\ z=1$)

2) $x-y+z=2$

$x+2y-z=0$

$3x-y+2z=-6$

$x+y+z=3$

(The system is inconsistent)

3) $x_1+2x_2-2x_3=1$

$2x_1+5x_2+x_3=9$

$x_1+3x_2+4x_3=9$

(system has unique solution: $x_1=-1,\ x_2=2,\ x_3=1$)

Obtain the solution of homogeneous system of equations by Gauss elimination

1) $x + 2y + 3z = 0$

$3x + 4y + 4z = 0$

$7x + 10y + 12z = 0$

(system has trivial solution: $x = y = z = 0$)

2) $4x + 2y + z + 3w = 0$

$6x + 3y + 4z + 7w = 0$

$2x + y + w = 0$

(system has infinite no of non-trivial solutions)

Find the value of λ for which the following system has a non-trivial solution.

$$2x - y + 2z = 0$$
$$3x + y - z = 0$$
$$\lambda x - 2y + z = 0$$

(Ans: $\lambda = -11$)

Find the value of K for which the system of equations $(3K-8)x + 3y + 3z = 0$, $3x + (3K-8)y + 3z = 0$, $3x + 3y + (3K-8)z = 0$ has a non-trivial solution. (Ans: $K = \frac{2}{3}, \frac{11}{3}, \frac{11}{3}$)

Elementary matrices: An elementary matrix is that, which is obtained from a Identity matrix, by subjecting it to any of the elementary row operations (Exactly once).

Ex:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i) $R_2 \Leftrightarrow R_3$

$$R_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \checkmark$$

ii) $R_2 \to kR_2$

$$kR_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

iii) $R_1 \to R_1 + pR_2$,

$$\begin{bmatrix} 1 & p & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

Thm: Elementary row transformation of a matrix A can be obtained by pre-multiplying A by the corresponding elementary matrices

Ex:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$R_2 \Leftrightarrow R_3$

$$\sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Consider

$$R_{23} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Operation $R_2 \Leftrightarrow R_3$ on A is same as pre-multiplying A by the elementary matrix R_{23}, obtained by interchaning R_2 and R_3 in I_3.

Gauss-Jordan method of finding the inverse: The elementary row transformations which reduce a given matrix A to the identity matrix, reduce the identity matrix to inverse of A.

Let the successive row transformations which reduce A to I result from pre-multiplication by the elementary matrices $R_1, R_2, \ldots, R_i$ so that

$$R_i R_{i-1} \cdots R_2 R_1 A = I \quad \text{——— (1)}$$

$$R_i R_{i-1} \cdots R_2 R_1 A A^{-1} = I A^{-1}$$

or $$R_i R_{i-1} \cdots R_2 R_1 I = A^{-1} \quad (\because A A^{-1} = I, \text{ a identity matrix})$$

(2)

From (1) and (2), it is clear that the $\operatorname{rref}([A : I])$ is $[I : A^{-1}]$.

Ex: Use Gauss-Jordan method to find the inverse of

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

Soln: Consider

$$[A : I]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 + 2R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right]$$

$R_2 \to \frac{1}{2} R_2$

$R_3 \to -\frac{1}{4} R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

$R_1 \to R_1 - R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

$R_1 \to R_1 - 6R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & \frac{3}{2} \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

$R_2 \to R_2 + 3R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & 0 & -5/4 & -1/4 & -3/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right]$$ is the rref of $[A : I]$.

Thus, $$A^{-1} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

Gauss - Seidel method to solve system of linear equations

(iterative method or modified Jacobi's method)

This is an iterative method for solving system of linear equations. In an iterative method, we start from an initial guess of the true solution, which is known as initial approximation. In the succeeding steps we obtain better solutions. This has to be repeated until we get solution of desired accuracy.

Gauss - Seidel method is used to solve a linear system of 'n' equations in 'n' unknowns where the system is strictly diagonally dominant

Consider

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \rightarrow (*)$$

<u>Strictly diagonally dominant system:</u> The system is said to be strictly diagonally dominant if the absolute value of diagonal element in each row is greater than the sum of absolute values of all other non-diagonal entries in that row.

i.e. $|a_{11}| > |a_{12}| + |a_{13}|$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{33}|$$

$$\Rightarrow |a_{ii}| > \sum_{\substack{j \neq i \\ j=1}}^{n} |a_{ij}| \quad \forall i$$

Working procedure:

Step 1: From (*) write $x_1 = \frac{1}{a_{11}} (b_1 - a_{12} x_2 - a_{13} x_3) \rightarrow (1)$

$$x_2 = \frac{1}{a_{22}} (b_2 - a_{21} x_1 - a_{23} x_3) \rightarrow (2)$$

$$x_3 = \frac{1}{a_{33}} (b_3 - a_{31} x_1 - a_{32} x_2) \rightarrow (3)$$

Step 2: Choose the initial approximations for x_1, x_2 and x_3. We denote it by $x_1^{(0)}$, $x_2^{(0)}$ and $x_3^{(0)}$ respectively.

Step 3: Find successive approximations of x_1, x_2 and x_3 by substituting the last obtained approximate value of the relevant unknowns in eqns (1), (2) and (3).

Step 4: Repeat Step 3 for the next set of approximations until we get the desired accuracy.
i.e. for each $k \geq 1$, generate the components $x_i^{(k)}$ by

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} (a_{ij} x_j^{(k)}) - \sum_{j=i+1}^{n} (a_{ij} x_j^{(k-1)}) \right], \text{ for } i = 1, 2, \ldots n$$

k here is the iteration number.

<u>Matrix form of Gauss-Seidel method:</u>

$$x_1^{(k)} = \frac{1}{a_{11}} (b_1 - a_{12} x_2^{(k-1)} - a_{13} x_3^{(k-1)}) \rightarrow (4)$$

$$x_2^{(k)} = \frac{1}{a_{22}} (b_2 - a_{21} x_1^{(k)} - a_{23} x_3^{(k-1)}) \rightarrow (5)$$

$$x_3^{(k)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)}\right) \rightarrow 6$$

$$4 \Rightarrow a_{11}x_1^{(k)} = b_1 - a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)}$$

$$5 \Rightarrow a_{22}x_2^{(k)} + a_{21}x_1^{(k)} = b_2 - a_{23}x_3^{(k-1)}$$

$$6 \Rightarrow a_{31}x_1^{(k)} + a_{32}x_2^{(k)} + a_{33}x_3^{(k)} = b_3$$

$$(L+D)\,X^{(k)} = B - U\,X^{(k-1)}$$

$$\therefore X^{(k)} = (L+D)^{-1}B - (L+D)^{-1}U\,X^{(k-1)}$$

where $L = \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix}$, $D = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$, $U = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$

lower triangular matrix   diagonal matrix   upper triangular

Absolute relative error: (Absolute relative percentage error)

$$|\varepsilon_a| = \left| \frac{x_i^{new} - x_i^{old}}{x_i^{new}} \right| \times 100$$

Solve the following system by Gauss - Seidel iterative method.

1) $12x_1 + 3x_2 - 5x_3 = 1$

$x_1 + 5x_2 + 3x_3 = 28$

$3x_1 + 7x_2 + 13x_3 = 76$

Sol:- $|12| > |3| + |5|$

$|5| > |1| + |3|$

$|13| > |3| + |7|$

The system is strictly diagonally dominant.

The Gauss - Seidel iteration for the system is

$$x_1^{(k)} = \frac{1}{12}\left[1 - 3x_2^{(k-1)} + 5x_3^{(k-1)}\right],$$

$$x_2^{(k)} = \frac{1}{5}\left[28 - x_1^{(k)} - 3x_3^{(k-1)}\right],$$

$$x_3^{(k)} = \frac{1}{13}\left[76 - 3x_1^{(k)} - 7x_2^{(k)}\right]$$

Let: $x_1^{(0)} = 1$, $x_2^{(0)} = 0$, $x_3^{(0)} = 1$ be the initial approximation.

K = 1

$$\Rightarrow x_1^{(1)} = \frac{6}{12} = 0.5, \quad x_2^{(1)} = 4.9, \quad x_3^{(1)} = 3.0923$$

$$|\varepsilon_a|_1 = \left|\frac{0.5 - 1}{0.5}\right| \times 100 = 100\%, \quad |\varepsilon_a|_2 = \left|\frac{4.9 - 0}{4.9}\right| \times 100 = 100\%$$

$$|\varepsilon_a|_3 = \left|\frac{3.0923 - 1}{3.0923}\right| \times 100 = 67.662\%$$

$\underline{K=2}$

$$\Rightarrow x_1^{(2)} = 0.14679, \quad x_2^{(2)} = 3.7153, \quad x_3^{(2)} = 3.8118$$

$\underline{K=3}$

$$\Rightarrow x_1^{(3)} = 0.74275, \quad x_2^{(3)} = 3.1644, \quad x_3^{(3)} = 3.9708$$

$\underline{K=4}$

$$\Rightarrow x_1^{(4)} = 0.94675, \quad x_2^{(4)} = 3.0281, \quad x_3^{(4)} = 3.9971$$

$$|\varepsilon_a|_1 = 22\%, \quad |\varepsilon_a|_2 = 5\%, \quad |\varepsilon_a|_3 = 0.65\%$$

$\underline{K=5}$

$$\Rightarrow x_1^{(5)} = \frac{1 - 3x_2^{(4)} + 5x_3^{(4)}}{12} = \frac{1 - 3(3.0281) + 5(3.9971)}{12} = 0.9918,$$

$$x_2^{(5)} = \frac{28 - x_1^{(5)} - 3x_3^{(4)}}{5} = \frac{28 - 0.9918 - 3(3.9971)}{5} = 3.0034$$

$$x_3^{(5)} = \frac{76 - 3x_1^{(5)} - 7x_2^{(5)}}{13} = \frac{76 - 3(0.9918) - 7(3.0034)}{13} = 4.0001$$

$\underline{K=6}$

$$x_1^{(6)} = 0.9991, \quad x_2^{(6)} = 3.0001, \quad x_3^{(6)} = 4.0001$$

$$|\varepsilon_a|_1 = 0.743\%, \quad |\varepsilon_a|_2 = 0.1\%, \quad |\varepsilon_a|_3 = 0.001\%$$

$$\Rightarrow x_1 = 1, \quad x_2 = 3, \quad x_3 = 4.$$

2) $2x_1 + 8x_2 - x_3 = 11$

$5x_1 - x_2 + x_3 = 10$

$-x_1 + x_2 + 4x_3 = 3$

Sol: The given system is not strictly diagonally dominant. So, we interchange the first and second equations.

$$\left.\begin{aligned} 5x_1 - x_2 + x_3 &= 10 \\ 2x_1 + 8x_2 - x_3 &= 11 \\ -x_1 + x_2 + 4x_3 &= 3 \end{aligned}\right\} \rightarrow (*)$$

The system (*) now is diagonally dominant.

$$x_1^{(k)} = \frac{1}{5}\left[10 + x_2^{(k-1)} - x_3^{(k-1)}\right]$$

$$x_2^{(k)} = \frac{1}{8}\left[11 - 2x_1^{(k)} + x_3^{(k-1)}\right]$$

$$x_3^{(k)} = \frac{1}{4}\left[3 + x_1^{(k)} - x_2^{(k)}\right]$$

Let $x_1^{(0)} = 0$, $x_2^{(0)} = 0$, $x_3^{(0)} = 0$ be the initial approximation

$k=1$

$$\Rightarrow x_1^{(1)} = \frac{1}{5}\left[10 + x_2^{(0)} - x_3^{(0)}\right] = \frac{1}{5}\left[10 + 0 - 0\right] = 2$$

$$x_2^{(1)} = \frac{1}{8}\left[11 - 2x_1^{(1)} + x_3^{(0)}\right] = \frac{1}{8}\left[11 - 4\right] = \frac{7}{8} = 0.875$$

$$x_3^{(1)} = \frac{1}{4}\left[3 + x_1^{(1)} - x_2^{(1)}\right] = \frac{1}{4}\left[3 + 2 - \frac{7}{8}\right] = \frac{33}{32} = 1.0313$$

$k=2$

$$x_1^{(2)} = \frac{1}{5}\left[10 + x_2^{(1)} - x_3^{(1)}\right] = \frac{1}{5}\left[10 + 0.875 - 1.0313\right] = 1.9687$$

$$x_2^{(2)} = \frac{1}{8}\left[11 - 2x_1^{(2)} + x_3^{(1)}\right] = \frac{1}{8}\left[11 - 2(1.9687) + 1.0313\right]$$
$$= 1.0117$$

$$x_3^{(2)} = \frac{1}{4}\left[3 + x_1^{(2)} - x_2^{(2)}\right] = \frac{1}{4}\left[3 + 1.9687 - 1.0117\right] = 0.9893$$

<u>K = 3</u>

$$x_1^{(3)} = \frac{1}{5}\left[10 + x_2^{(2)} - x_3^{(2)}\right] = \frac{1}{5}\left[10 + 1.0117 - 0.9893\right]$$

$$= 2.0045$$

$$x_2^{(3)} = \frac{1}{8}\left[11 - 2x_1^{(3)} + x_3^{(2)}\right] = \frac{1}{8}\left[11 - 2(2.0045) + 0.9893\right]$$

$$= 0.9975$$

$$x_3^{(3)} = \frac{1}{4}\left[3 + x_1^{(3)} - x_2^{(3)}\right] = \frac{1}{4}\left[3 + 2.0045 - 0.9975\right]$$

$$= 1.0018$$

<u>K = 4</u>

$$x_1^{(4)} = \frac{1}{5}\left(10 + x_2^{(3)} - x_3^{(3)}\right) = \frac{1}{5}\left(10 + 0.9975 - 1.0018\right)$$

$$= 1.9991$$

$$x_2^{(4)} = \frac{1}{8}\left(11 - 2x_1^{(4)} + x_3^{(3)}\right) = \frac{1}{8}\left(11 - 2(1.9991) + 1.0018\right)$$

$$= 1.0005$$

$$x_3^{(4)} = \frac{1}{4}\left(3 + x_1^{(4)} - x_2^{(4)}\right) = \frac{1}{4}\left(3 + 1.9991 - 1.0005\right)$$

$$= 0.9997$$

$$\therefore \boxed{x_1 = 2, \quad x_2 = 1, \quad x_3 = 1}$$

Obtain the inverse of a matrix by Gauss-Jordan method:

1) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$

2) $\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

Ans:- $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$

Ans:- $\begin{bmatrix} -1/9 & 2/9 & 2/9 \\ 2/9 & -1/9 & 2/9 \\ 2/9 & 2/9 & -1/9 \end{bmatrix}$

Solve the following systems using Gauss-Seidel iterative method

1) $28x + 4y - z = 32$

$x + 3y + 10z = 24$

$2x + 17y + 4z = 35$

Ans:- $x = 0.9936$, $y = 1.5069$, $z = 1.8486$

2) $x + y + 54z = 110$

$27x + 6y - z = 85$

$6x + 15y + 2z = 72$

Ans: $x = 2.425$, $y = 3.573$, $z = 1.926$

3) $4x + 2y + z = 14$

$x + 5y - z = 10$

$x + y + 8z = 20$

Ans: $x = 2$, $y = 1.9999$, $z = 1.9999$