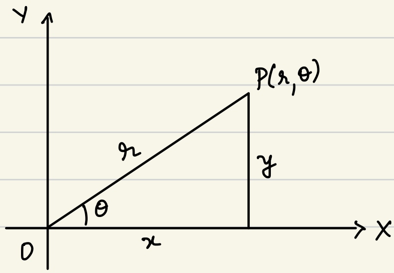
Polar curves

Polar coordinates:

Consider a point P in the xy-plane. Join the point O (origin) and P. Let r be the length of OP and θ be the angle which OP makes with positive axis. (r, θ) are called the polar coordinates of point P and we write $P = (r, \theta)$ or $P(r, \theta)$.



Here r is called the radial distance and θ is called polar angle. O is called the pole and OP is called the radius vector.

Let (x, y) be the cartesian coordinates of the point P. Then we find that

$$r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \rightarrow (1)$$

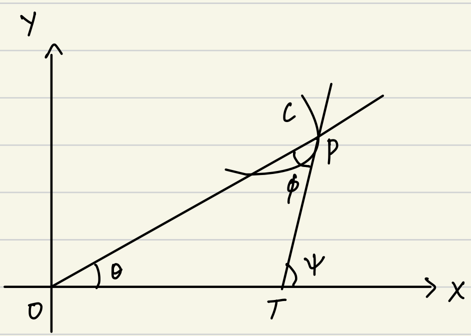
$$x = r\cos\theta, \quad y = r\sin\theta \quad \rightarrow (2)$$

Relation (1) enables us to find the polar coordinates (r, θ) when the cartesian coordinates (x, y) are known. Conversely, relation (2) enables us to find the cartesian coordinates when the polar coordinates are known.

If P is a variable point on a plane curve C, then the equation of the curve in the cartesian form is a relationship of the form $y = f(x)$. Similarly the equation of C in the polar form (or the polar equation of C) is a relationship of the form $r = f(\theta)$, referred to as a polar curve.

Angle between radius vector and tangent

Let $P(r, \theta)$ be a point on a polar curve $r = f(\theta)$ and PT be the tangent to the curve at P meeting the x-axis at the point T. Let ψ be the angle which PT makes with positive x-axis and ϕ be the angle between the radius vector OP and the tangent PT.



If (x, y) are the cartesian coordinates of the point P,

$$x = r\cos\theta, \quad y = r\sin\theta \quad \text{and} \qquad \rightarrow (1)$$

$$\frac{dy}{dx} = \text{slope of } PT = \tan\psi = \tan(\theta + \phi) \qquad \rightarrow (2)$$

because $\psi = \theta + \phi$

From ①, we have

$$\frac{dx}{d\theta} = -r\sin\theta + \frac{dr}{d\theta}\cos\theta$$

$$\frac{dy}{d\theta} = r\cos\theta + \frac{dr}{d\theta}\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta}$$

Dividing both numerator and denominator on the R.H.S. by $(dr/d\theta)\cos\theta$, we get

$$\frac{dy}{dx} = \frac{\tan\theta + r\left(\frac{d\theta}{dr}\right)}{1 - r\left(\frac{d\theta}{dr}\right)\tan\theta} \rightarrow ③$$

From ②, we have

$$\frac{dy}{dx} = \tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} \rightarrow ④$$

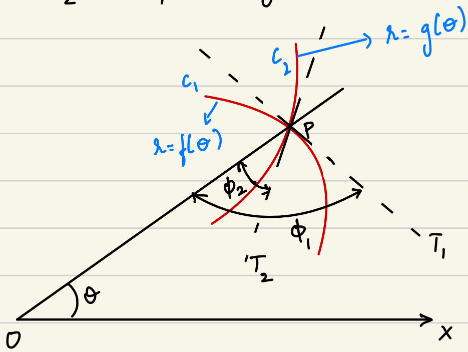
Comparing ③ and ④ we get

$$\tan\phi = r\cdot\frac{d\theta}{dr} \quad \text{or} \quad \cot\phi = \frac{1}{r}\frac{dr}{d\theta}$$

This is the formula for the angle ϕ between the radius vector OP and tangent PT.

Angle between polar curves or angle of intersection of 2 polar curves

The angle of intersection of 2 curves is the angle between their tangents at that point. Let ϕ_1 and ϕ_2 be the angle between the common radius vector OP and tangents PT_1 and PT_2 respectively.



This angle is determined by using the formula

$$\tan|\phi_1 - \phi_2| = |\tan(\phi_1 - \phi_2)|$$

$$= \left| \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \cdot \tan\phi_2} \right|$$

Thus if ϕ_1 and ϕ_2 are known, the angle between C_1 and C_2 at the point of intersection P is determined.

Orthogonal curves: The curves are said to be orthogonal if the angle between the curves C_1 and C_2 at P is $\frac{\pi}{2}$

$$\Rightarrow |\phi_1 - \phi_2| = \frac{\pi}{2} \quad \text{or equivalently,}$$

$\tan\phi_1 \cdot \tan\phi_2 = -1$.

1) Find the angle between radius vector and tangent for the curve $r = a(1-\cos\theta)$

Sol:- Given $r = a(1-\cos\theta)$

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$= -\frac{a(1+\cos\theta)}{a\sin\theta}$$

$$= -\frac{2\cos^2\theta/2}{2\sin\theta/2\cos\theta/2}$$

Differentiating w.r.t. θ we get

$$\frac{dr}{d\theta} = a\sin\theta$$

The angle ϕ b/w the radius vector and the tangent is given by $\tan\phi = r\frac{d\theta}{dr} = r \cdot \frac{1}{a\sin\theta}$

$$\tan\phi = -\cot\theta/2.$$

$$= \frac{a(1-\cos\theta)}{a\sin\theta} = \frac{2\sin^2\theta/2}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{1}{\tan}$$

$$= \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$\therefore \phi = \frac{\pi}{2}+\frac{\theta}{2}.$$

$$\tan\phi = \tan\frac{\theta}{2}$$

$$\boxed{\therefore \phi = \frac{\theta}{2}}$$

2) Find the angle b/w the radius vector and the tangent for the following curve. Also find the slope of tangents at the given point.

$$\frac{2a}{r} = 1-\cos\theta \quad \text{at} \quad \theta = \frac{2\pi}{3}$$

Sol:- Take log on both sides

$$\log\left(\frac{2a}{r}\right) = \log(1-\cos\theta)$$

$$\log 2a - \log r = \log(1-\cos\theta)$$

Differentiate wrt θ

$$0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1-\cos\theta}(\sin\theta)$$

$$-\cot\phi = \frac{\cancel{2}\sin\cancel{\theta/2}\cos\theta/2}{\cancel{2}\sin^2\theta/2} = \cot\theta/2$$

$$\cot\phi = -\cot\frac{\theta}{2}$$

$$\Rightarrow \phi = -\frac{\theta}{2} \qquad (\because \cot(-\theta) = -\cot\theta)$$

At $\theta = \frac{2\pi}{3}$,

$$\phi = -\frac{1}{2}\left(\frac{2\pi}{3}\right) = -\frac{\pi}{3} = -60^\circ$$

$\therefore$ Slope of the tangent $= \tan\psi = \tan(\theta+\phi) = \tan\left(\frac{2\pi}{3} - \frac{\pi}{3}\right) =$

$$\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

3) Find the angle b/w the radius vector and the tangent for the following curves.

a) $r = a\,e^{\theta\cot\alpha}$, where α is a constant

sol:-

Differentiating wrt. θ,

$$\frac{dr}{d\theta} = ae^{\theta\cot\alpha} \cdot \cot\alpha$$

$$\tan\phi = r\frac{d\theta}{dr} = \frac{ae^{\theta\cot\alpha}}{ae^{\theta\cot\alpha}\cdot\cot\alpha}$$

$$\tan\phi = \frac{1}{\cot\alpha} = \tan\alpha$$

$$\therefore \phi = \alpha$$

b) $r^m = a^m(\cos m\theta + \sin m\theta)$

sol: Taking log on b.s.

$$m \log r = \log a^m + \log(\cos m\theta + \sin m\theta)$$

Differentiating wrt θ

$$m \cdot \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos m\theta + \sin m\theta}(-m\sin m\theta + m\cos m\theta)$$

$$\cancel{m} \cot\phi = \frac{\cancel{m}(\cos m\theta - \sin m\theta)}{\cos m\theta + \sin m\theta}$$

$$\cot\phi = \frac{\cancel{\cos m\theta}(1 - \tan m\theta)}{\cancel{\cos m\theta}(1 + \tan m\theta)} = \frac{\tan \pi/4 - \tan m\theta}{1 + \tan \pi/4 \cdot \tan m\theta}$$

$$\cot\phi = \tan\left(\frac{\pi}{4} - m\theta\right)$$

$$\cot\phi = \cot\left[\frac{\pi}{2} - \left(\frac{\pi}{4} - m\theta\right)\right] = \cot\left(\frac{\pi}{4} + m\theta\right)$$

$$\therefore \phi = \frac{\pi}{4} + m\theta$$

c) $r\cos^2\frac{\theta}{2} = a$

Take log on b.s.

$$\log r + 2\log\cos\frac{\theta}{2} = \log a$$

Differentiating wrt θ

$$\frac{1}{r}\frac{dr}{d\theta} + \cancel{2}\cdot\frac{1}{\cos\frac{\theta}{2}} \times -\sin\frac{\theta}{2}\cdot\frac{1}{\cancel{2}} = 0$$

$$\frac{1}{r}\frac{dr}{d\theta} = \tan\frac{\theta}{2}$$

$$\cot\phi = \tan\frac{\theta}{2} = \cot\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$\therefore \phi = \frac{\pi}{2} - \frac{\theta}{2}$$

d) $r^2 = a^2 \sin 2\theta$

sol:- Take log on b.s.

$$2\log r = 2\log a + \log\sin 2\theta$$

Differentiating wrt θ

$$\frac{2}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{1}{\sin 2\theta}\cdot 2\cos 2\theta$$

$$\cancel{2}\cot\phi = \cancel{2}\cot 2\theta$$

$$\therefore \phi = 2\theta$$

or

$$\cancel{2}r\frac{dr}{d\theta} = \cancel{2}a^2\cos 2\theta$$

$$\frac{dr}{d\theta} = \frac{a^2\cos 2\theta}{r}$$

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r\cdot r}{a^2\cos 2\theta} = \frac{r^2}{a^2\cos 2\theta} = \frac{\cancel{a^2}\sin 2\theta}{\cancel{a^2}\cos 2\theta}$$

$$\tan\phi = \tan 2\theta$$

$$\therefore \phi = 2\theta$$

c) Determine angle b/w radius vector and tangent to the curve $\theta = \frac{1}{a}\sqrt{r^2-a^2} - \cos^{-1}\left(\frac{a}{r}\right)$, $r \neq 0 \neq a$

sol:
$$\frac{d\theta}{dr} = \frac{1}{a}\frac{2r}{2\sqrt{r^2-a^2}} + \frac{1}{\sqrt{1-\left(\frac{a^2}{r^2}\right)}}\left(\frac{-a}{r^2}\right) \qquad \Bigg| \; \because \frac{d}{dx}\left(\cos^{-1}x\right) = \frac{-1}{\sqrt{1-x^2}}$$

$$= \frac{r}{a}\cdot\frac{1}{\sqrt{r^2-a^2}} - \frac{r}{\sqrt{r^2-a^2}}\left(\frac{a}{r^2}\right)$$

$$= \frac{r}{a}\cdot\frac{1}{\sqrt{r^2-a^2}} - \frac{a}{r}\cdot\frac{1}{\sqrt{r^2-a^2}}$$

$$\frac{d\theta}{dr} = \frac{1}{\sqrt{r^2-a^2}}\left(\frac{r}{a} - \frac{a}{r}\right) = \frac{r^2-a^2}{\sqrt{r^2-a^2}\,(ar)} = \frac{\sqrt{r^2-a^2}}{ar}$$

$$\therefore \tan\phi = r\frac{d\theta}{dr} = r\cdot\frac{\sqrt{r^2-a^2}}{ar} = \frac{\sqrt{r^2-a^2}}{a}$$

$$\therefore \phi = \tan^{-1}\left(\frac{\sqrt{r^2-a^2}}{a}\right)$$

4) S.T. for the curve $\log(x^2+y^2) = k\tan^{-1}\left(\frac{y}{x}\right)$ the angle b/w radius vector and tangent is the same at all points of curve.

Sol: let $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore x^2+y^2 = r^2 \quad \text{and} \quad \tan\theta = \left(\frac{y}{x}\right)$$

The given curve reduces to

$$\log r^2 = K\theta$$

$$\therefore 2\log r = K\cdot\theta \Rightarrow \log r = \frac{1}{2}K\theta$$

Differentiate wrt θ on b.s.

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{2}K(1) = \frac{K}{2}$$

$$\cot\phi = \frac{K}{2}$$

$$\therefore \phi = \cot^{-1}\left(\frac{K}{2}\right) = \text{a constant}$$

5) Find the angle of intersection of each of the following pairs of curves:

a) $r = 2\sin\theta$, $r = \sin\theta + \cos\theta$

Sol:- Let ϕ_1 be the angle b/w radius vector and tangent for the first curve and ϕ_2 be the corresponding angle for the second curve.

For the curve $r = 2\sin\theta$,

$$\frac{dr}{d\theta} = 2\cos\theta$$

$$\tan\phi_1 = r\frac{d\theta}{dr} = \frac{2\sin\theta}{2\cos\theta} = \tan\theta$$

$$\therefore \phi_1 = \theta$$

For the curve $r = \sin\theta + \cos\theta$, we have

$$\frac{dr}{d\theta} = \cos\theta - \sin\theta$$

$$\tan\phi_2 = r\cdot\frac{d\theta}{dr} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}$$

$$\tan\phi_2 = \frac{\cancel{\cos\theta}(1+\tan\theta)}{\cancel{\cos\theta}(1-\tan\theta)} = \tan\left(\pi/4 + \theta\right)$$

$$\therefore \phi_2 = \frac{\pi}{4} + \theta$$

$\therefore$ The angle b/w the curves is $|\phi_1 - \phi_2| = \left|\cancel{\theta} - \frac{\pi}{4} - \cancel{\theta}\right| = \frac{\pi}{4}$

b) $r = a(1-\cos\theta)$, $r = 2a\cos\theta$

sol:- For the curve $r = a(1-\cos\theta)$

$$\frac{dr}{d\theta} = a\sin\theta$$

$$\tan\phi_1 = r\frac{d\theta}{dr} = \frac{a(1-\cos\theta)}{a\sin\theta} = \frac{2\sin^2\theta/2}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$$\therefore \phi_1 = \frac{\theta}{2}$$

For the curve $r = 2a\cos\theta$

$$\frac{dr}{d\theta} = 2a(-\sin\theta)$$

$$\tan\phi_2 = r\frac{d\theta}{dr} = \frac{2a\cos\theta}{-2a\sin\theta} = -\cot\theta = \tan\left(\frac{\pi}{2}+\theta\right)$$

$$\therefore \phi_2 = \frac{\pi}{2} + \theta$$

$$\Rightarrow |\phi_1 - \phi_2| = \left|\frac{\pi}{2} + \theta - \frac{\theta}{2}\right| = \frac{\pi}{2} + \frac{\theta}{2} \rightarrow (1)$$

At the point of intersection of the given curves, both the equations $r = a(1-\cos\theta)$ and $r = 2a\cos\theta$ hold. Thus at this point

$$1 - \cos\theta = 2\cos\theta$$

$$\cos\theta = \frac{1}{3}$$

$$\therefore \theta = \cos^{-1}\left(\frac{1}{3}\right)$$

① becomes

$$|\phi_1 - \phi_2| = \frac{\pi}{2} + \frac{1}{2}\cos^{-1}\left(\frac{1}{3}\right)$$

This is the angle b/w the given curves, which is evidently obtuse.

c) $r = a\log\theta$, $r = \dfrac{a}{\log\theta}$

sol: For the curve $r = a\log\theta$,

$$\frac{dr}{d\theta} = \frac{a}{\theta}$$

$$\tan\phi_1 = r\cdot\frac{d\theta}{dr} = \cancel{a}\log\theta\cdot\frac{\theta}{\cancel{a}} = \theta\log\theta$$

For the curve $r = \dfrac{a}{\log\theta}$, we have

$$\frac{dr}{d\theta} = \frac{-a}{(\log\theta)^2}\cdot\frac{1}{\theta}$$

$$\tan\phi_2 = r\frac{d\theta}{dr}$$

$$= \frac{a}{\log\theta}\cdot\left(-\frac{\theta(\log\theta)^2}{a}\right)$$

$$\tan\phi_2 = -\theta(\log\theta)$$

$$\log r = \log\left(\frac{a}{\log\theta}\right)$$

$$\frac{1}{r}\frac{dr}{d\theta} = \log a - \log\left(\frac{1}{\log\theta}\right)$$

$$\frac{1}{r}\frac{dr}{d\theta} = 0 - \frac{1}{\log\theta}\cdot\frac{1}{\theta}$$

$$\cot\phi_2 = -\frac{1}{\log\theta}\,\frac{1}{\theta}$$

$$\tan\phi_2 = -\theta\log\theta$$

Angle of intersection :

$$\tan|\phi_1 - \phi_2| = \left|\frac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1\cdot\tan\phi_2}\right|$$

$$= \left| \frac{2\theta \log\theta}{1-\theta^2(\log\theta)^2} \right| \rightarrow (1)$$

To find θ: At the point of intersection, $\frac{\alpha}{\log\theta} = \alpha \log\theta$

$$\Rightarrow (\log\theta)^2 = 1$$

$$\Rightarrow \log_e \theta = 1$$

$$\therefore \theta = e$$

$$(1) \Rightarrow \tan|\phi_1 - \phi_2| = \frac{2e}{1-e^2}$$

$$\therefore |\phi_1 - \phi_2| = \tan^{-1}\left(\frac{2e}{1-e^2}\right) = 2\tan^{-1}(e)$$

6) Prove that the following pairs of curves intersect orthogonally

a) $r = a(1+\cos\theta)$, $r = b(1-\cos\theta)$

Sol: For the curve $r = a(1+\cos\theta)$

$$\log r = \log a + \log(1+\cos\theta)$$

$$\frac{1}{r}\frac{dr}{d\theta} = 0 - \frac{\sin\theta}{1+\cos\theta}$$

$$\cot\phi_1 = -\frac{\cancel{2}\sin\theta/2 \, \cancel{\cos\theta/2}}{\cancel{2}\cos^{\cancel{2}}\theta/2} = -\tan\frac{\theta}{2}$$

$$\cot\phi_1 = \cot\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$$\therefore \phi_1 = \frac{\pi}{2} + \frac{\theta}{2}$$

For the curve $r = b(1-\cos\theta)$

$$\log r = \log b + \log(1-\cos\theta)$$

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1-\cos\theta} \times \sin\theta$$

$$\cot\phi_2 = \frac{\cancel{2}\cancel{\sin\frac{\theta}{2}}\cos\frac{\theta}{2}}{\cancel{2}\sin^{\cancel{2}}\theta/2} = \cot\frac{\theta}{2}$$

$$\therefore \phi_2 = \frac{\theta}{2}$$

$$\therefore |\phi_1 - \phi_2| = \left|\frac{\pi}{2} + \frac{\theta}{2} - \frac{\theta}{2}\right| = \frac{\pi}{2}.$$

b) $r^n = a^n \cos n\theta$, $r^n = b^n \sin n\theta$

Sol: For the curve $r^n = a^n \cos n\theta$

$$n \log r = \log a^n + \log \cos n\theta$$

$$\frac{\cancel{n}}{r}\frac{dr}{d\theta} = 0 + -\frac{\cancel{n}\sin n\theta}{\cos n\theta}$$

$$\cot\phi_1 = -\tan n\theta = \cot\left(\frac{\pi}{2} + n\theta\right)$$

$$\therefore \phi_1 = \frac{\pi}{2} + n\theta$$

For the curve $r^n = b^n \sin n\theta$

$$\frac{\cancel{n}}{r}\frac{dr}{d\theta} = 0 + \frac{\cancel{n}\cos n\theta}{\sin n\theta}$$

$$\cot\phi_2 = \cot n\theta$$

$$\therefore \phi_2 = n\theta$$

$$\therefore |\phi_1 - \phi_2| = \left|\frac{\pi}{2} + n\theta - n\theta\right| = \frac{\pi}{2}$$

c) $r = ae^{\theta}$, $re^{\theta} = b$

sol: For the curve $r = ae^{\theta}$

$$\log r = \log a + \log e^{\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{e^{\theta}} \cdot e^{\theta}$$

$$\cot \phi_1 = 1$$

$$\therefore \phi_1 = \frac{\pi}{4}$$

For the curve $re^{\theta} = b$

$$\log r + \log e^{\theta} = \log b$$

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{1}{e^{\theta}} \cdot e^{\theta} = 0$$

$$\cot \phi_2 = -1 = \cot(-\pi/4)$$

$$\therefore \phi_2 = -\pi/4$$

$\therefore$ Angle of intersection $= |\phi_1 - \phi_2| = \left|\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right|$

$$= \frac{\pi}{2}.$$

Exercise:

1) Find the angle b/w radius vector and tangent for the following curves:

1) $r^2 = a^2 \sin 2\theta$ 2) $\frac{2a}{r} = 1+\cos\theta$ 3) $r = a\log\theta$

Ans: $\phi = 2\theta$ Ans: $\phi = \frac{\pi}{2} - \frac{\theta}{2}$ Ans: $\phi = \tan^{-1}(\theta\log\theta)$

4) $r = a(1+\sin\theta)$

Ans: $\phi = \frac{\pi}{4} + \frac{\theta}{2}$

2) Find the angle of intersection of each of the following pairs of curves

1) $r^2 \sin 2\theta = 4$, $r^2 = 16\sin 2\theta$ 2) $r = a$, $r = 2a\cos\theta$

Ans: $\frac{\pi}{3}$ Ans: $\frac{\pi}{3}$

3) $r = \frac{a}{1+\cos\theta}$, $r = \frac{b}{1-\cos\theta}$ 4) $r = \frac{a\theta}{1+\theta}$, $r = \frac{a}{1+\theta^2}$

Ans: $\frac{\pi}{2}$ Ans: $\tan^{-1}(3)$

3) Prove that the following pairs of curves intersect orthogonally.

1) $r = a\sec^2\left(\frac{\theta}{2}\right)$, $r = b\,\mathrm{cosec}^2\left(\frac{\theta}{2}\right)$ 2) $r = a\theta$, $r = \frac{a}{\theta}$

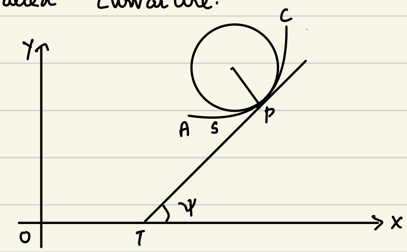
3) $\frac{2a}{r} = 1+\cos\theta$, $\frac{2b}{r} = 1-\cos\theta$ 4) $r^2\sin 2\theta = a^2$, $r^2\cos 2\theta = b^2$

5) $r = a(1+\sin\theta)$, $r = a(1-\sin\theta)$

4) Show that curves $r = a(1+\cos\theta)$, $r^2 = a^2\cos(2\theta)$ intersect at an angle given by $3\sin^{-1}\left[\left(\frac{3}{4}\right)^{1/4}\right]$.

Curvature

The amount of bending of a curve at a given point on it is called curvature.



Let P be any point on the curve C. Draw the tangent at P to the circle. Let this line make an angle ψ with positive x- axis. Then curvature is defined as rate of change of ψ with respect to arc length S.

$$\therefore \text{Curvature at } P = \left| \frac{d\psi}{ds} \right|$$

It is obvious that smaller circle bends more sharply than larger circle and thus smaller circle has a larger curvature.

Radius of curvature is the reciprocal of curvature and it is denoted by ρ.

The circle having the same curvature as the curve at P touching the curve at P, is called circle of curvature. It is also called osculating circle. The centre of circle of curvature is called centre of curvature. The radius of circle of curvature is called radius of curvature

Radius of curvature in cartesian form of the curve $y = f(x)$

$$\boxed{\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}}$$

where $y_1 = \frac{dy}{dx}$, $y_2 = \frac{d^2y}{dx^2}$

Radius of curvature in parametric form

Let $x = f(t)$ and $y = g(t)$ be the parametric equations of a curve.

$$\boxed{\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}}$$

where $x' = \frac{dx}{dt}$, $y' = \frac{dy}{dt}$, $x'' = \frac{d^2x}{dt^2}$, $y'' = \frac{d^2y}{dt^2}$

This is the cartesian form of radius of curvature in parametric form.

Radius of curvature in polar form

$$\boxed{\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}}$$

where $r_1 = \frac{dr}{d\theta}$, $r_2 = \frac{d^2r}{d\theta}$

Find the radius of curvature at any point on the curve $y = a \log \sec\left(\frac{x}{a}\right)$

Sol:
$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$y = a \log \sec\left(\frac{x}{a}\right)$$

$$y_1 = \cancel{a} \times \frac{1}{\sec\left(\frac{x}{a}\right)} \cdot \sec\left(\frac{x}{a}\right) \cdot \tan\left(\frac{x}{a}\right) \cdot \frac{1}{\cancel{a}} = \tan\left(\frac{x}{a}\right)$$

$$y_2 = \sec^2\left(\frac{x}{a}\right) \cdot \frac{1}{a}$$

$$\rho = \frac{\left\{1+\tan^2\left(\frac{x}{a}\right)\right\}^{3/2}}{\frac{1}{a}\sec^2\left(\frac{x}{a}\right)} = \frac{\left\{\sec^2\left(\frac{x}{a}\right)\right\}^{3/2}}{\frac{1}{a}\sec^2\left(\frac{x}{a}\right)} = \frac{a\sec^3\left(\frac{x}{a}\right)}{\sec^2\left(\frac{x}{a}\right)}$$

$$\boxed{\therefore \rho = a\sec\left(\frac{x}{a}\right)}$$

2) Find the radius of curvature at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on $x^3 + y^3 = 3axy$.

Sol:
$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

Given: $x^3 + y^3 = 3axy$

Differentiating wrt x

$$3x^2 + 3y^2 y_1 = 3ay + 3axy_1 = 3a(y + xy_1)$$

$$x^2 + y^2 y_1 = a(xy_1 + y)$$

$$y_1(y^2 - ax) = ay - x^2$$

$$\therefore y_1 = \frac{ay - x^2}{y^2 - ax} \rightarrow (1)$$

Again differentiating wrt x,

$$y_2 = \frac{(y^2 - ax)(ay_1 - 2x) - (ay - x^2)(2yy_1 - a)}{(y^2 - ax)^2} \rightarrow (2)$$

At $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

$$(1) \Rightarrow y_1 = \frac{a\left(\frac{3a}{2}\right) - \left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)} = \frac{6a^2 - 9a^2}{9a^2 - 6a^2}$$

$$y_1 = -1$$

$$(2) \Rightarrow y_2 = \frac{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)(-a - 3a) - \left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)(-3a - a)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2}$$

$$= \frac{-\frac{3a^2}{\cancel{4}} \times \cancel{4}a - \frac{3a^2}{\cancel{4}} \times \cancel{4}a}{\left(\frac{3a^2}{4}\right)^2} = \frac{-6a^3}{\frac{9a^4}{16}}$$

$$\therefore y_2 = \frac{-32}{3a}$$

$$\rho\Big|_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = \frac{\left[1 + (-1)^2\right]^{3/2}}{\left(-\frac{32}{3a}\right)} = -\frac{2\sqrt{2} \times 3a}{32}$$

$$\rho = -\frac{\cancel{2}\sqrt{2} \times 3a}{\cancel{16}_{8} \times 2} = \frac{-3a}{8\sqrt{2}}$$

$\therefore$ Radius of curvature at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ is $\frac{-3a}{8\sqrt{2}}$

3) S.T. the radius of curvature of circle is a constant.

Sol: The general equation of a circle with centre (x_0, y_0) is

$$(x-x_0)^2 + (y-y_0)^2 = a^2 \rightarrow ①$$

The parametric eqns are

$$x - x_0 = a\cos(t), \quad y - y_0 = a\sin(t)$$

$$\left.\begin{aligned} \therefore\ x &= x_0 + a\cos(t) \\ y &= y_0 + a\sin(t) \end{aligned}\right\} \rightarrow ②$$

$$x' = \frac{dx}{dt} = -a\sin t, \qquad y' = \frac{dy}{dt} = a\cos(t)$$

$$x'' = \frac{d^2x}{dt^2} = -a\cos(t), \quad y'' = -a\sin(t)$$

$$\rho = \frac{\left[(x')^2 + (y')^2\right]^{3/2}}{(x'y'' - x''y')} = \frac{[a^2\sin^2 t + a^2\cos^2 t]^{3/2}}{[a^2\sin^2 t + a^2\cos^2 t]}$$

$$\therefore \rho = \frac{(a^2)^{3/2}}{a^2} = \frac{a^3}{a^2} = a, \quad \text{constant}$$

4) S.T. the radius of curvature for an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, at the end of major axis is equal to semi-latus rectum of the ellipse.

Sol: The parametric equations of an ellipse are:

$$x = a\cos(t), \quad y = b\sin(t)$$

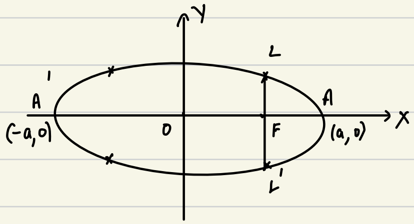
$$x' = \frac{dx}{dt} = -a\sin(t), \qquad y' = \frac{dy}{dt} = b\cos(t)$$

$$x'' = \frac{d^2x}{dt^2} = -a\cos(t), \quad y'' = \frac{d^2y}{dt^2} = -b\sin(t)$$

$$\rho = \frac{\left[(x')^2 + (y')^2\right]^{3/2}}{(x'y'' - x''y')} = \frac{[a^2\sin^2 t + b^2\cos^2 t]^{3/2}}{ab\,[\sin^2 t + \cos^2 t]}$$

$$\rho = \frac{[a^2 \sin^2 t + b^2 \cos^2 t]^{3/2}}{ab}$$

x-axis for an ellipse is called as major axis



$b \rightarrow$ length of semiminor axis
$a \rightarrow$ -11- semimajoraxis
$LL' \rightarrow$ latus rectum.

At $A(a, 0)$, $x = a$, $y = 0$

$\Rightarrow a\cos t = a$, $b\sin t = 0$

$\Rightarrow \cos t = 1$, $\sin t = 0$ as $b \neq 0$

$\therefore t = 0$

$$\therefore \rho = \frac{[0 + b]^{3/2}}{ab} = \frac{b^3}{ab} = \frac{b^2}{a}$$

Total length of latus rectum $= \dfrac{2b^2}{a} = LL'$

$$\therefore \rho = \frac{LL'}{2} = \frac{b^2}{a}.$$

5) Find the radius of curvature of the curve $y = xe^{-x}$ at the point where y is maximum.

Sol: Here $y = xe^{-x}$

$$\therefore y' = x(-e^{-x}) + e^{-x}$$

$$y' = e^{-x}(1 - x)$$

$$y'' = e^{-x}(-1) + (1-x)(-e^{-x})$$

$$y'' = e^{-x}(x - 2)$$

For maximum, $y' = 0 \Rightarrow e^{-x}(1 - x) = 0$

But $e^{-x} \neq 0$ for finite x

$\therefore 1 - x = 0 \Rightarrow x = 1$

At $x = 1$, $y'' = -\frac{1}{e} < 0$

$\therefore$ y is maximum at $x = 1$.

Hence $\rho = \frac{[1 + (y')^2]^{3/2}}{y''} = \frac{[1 + (1-x)^2 e^{-2x}]^{3/2}}{e^{-x}(x-2)}$

Put $x = 1$, $\rho = \frac{[1+0]^{3/2}}{(-1/e)} = -\frac{1}{(1/e)} = -e$

$\therefore$ $\rho = e$ (numerically) ≈ 2.718

6) S.T. the curve $r^n = a^n \cos n\theta$ has the radius of curvature $\frac{a^n}{(n+1)r^{n-1}}$.

sol:

Apply log on both sides

$$n \log r = n \log a + \log \cos n\theta$$

Differentiating wrt θ,

$$\frac{\cancel{n}}{r} \frac{dr}{d\theta} = 0 - \frac{\cancel{n} \sin n\theta}{\cos n\theta}$$

$$r_1 = \frac{dr}{d\theta} = -r \tan n\theta$$

Differentiate wrt θ again

$$r_2 = \frac{d^2 r}{d\theta^2} = -\left\{ r \cdot n \sec^2 n\theta + \tan n\theta \cdot \frac{dr}{d\theta} \right\}$$

$$= -\left\{ n \cdot r \sec^2 n\theta - r \tan^2 n\theta \right\}$$

$$r_2 = r \tan^2 n\theta - n r \sec^2 n\theta$$

Using polar form of ρ,

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$$

$$\rho = \frac{(r^2+r^2\tan^2 n\theta)^{3/2}}{r^2+2(-r\tan n\theta)^2 - r(r\tan^2 n\theta - nr\sec^2 n\theta)}$$

$$\rho = \frac{r^3\sec^3 n\theta}{r^2(1+2\tan^2 n\theta - \tan^2 n\theta + n\sec^2 n\theta)}$$

$$\rho = \frac{r\sec^3 n\theta}{(n+1)\sec^2 n\theta} = \frac{r}{(n+1)\cos n\theta}$$

$$\rho = \frac{r}{(n+1)\left(\frac{r^n}{a^n}\right)} \qquad \left(\because \cos n\theta = \frac{r^n}{a^n}\right)$$

$$\therefore \rho = \frac{a^n}{(n+1)r^{n-1}}$$

7) S.T. for equiangular spiral $r = ae^{\theta\cot\alpha}$ where a and α are constants, $\frac{\rho}{r}$ is constant.

Sol:-

For $r = ae^{\theta\cot\alpha}$

$$\log r = \log a + \log e^{\theta\cot\alpha}$$

$$\log r = \log a + \theta\cdot\cot\alpha$$

Differentiate wrt θ on b.s.

$$\frac{1}{r}\frac{dr}{d\theta} = \cot\alpha$$

$$\frac{dr}{d\theta} = r\cot\alpha$$

$$\Rightarrow r_1 = r\cot\alpha \rightarrow (1)$$

Differentiate wrt θ

$$r_2 = r_1 \cot\alpha \rightarrow (2) = (r\cot\alpha)(\cot\alpha) = r\cot^2\alpha$$

$$r_2 = r\cot^2\alpha \rightarrow (2)$$

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2} = \frac{(r^2+r^2\cot^2\alpha)^{3/2}}{r^2+2r^2\cot^2\alpha - r^2\cot^2\alpha}$$

$$\rho = \frac{r^3\operatorname{cosec}^3\alpha}{r^2+r^2\cot^2\alpha} = \frac{r^3\operatorname{cosec}^3\alpha}{r^2(1+\cot^2\alpha)}$$

$$\rho = \frac{r\operatorname{cosec}^3\alpha}{\operatorname{cosec}^2\alpha} = r\operatorname{cosec}\alpha$$

$$\Rightarrow \quad \frac{\rho}{r} = \operatorname{cosec}\alpha, \text{ a constant}$$

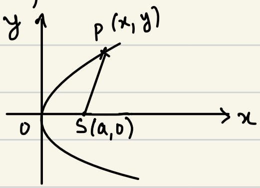
8) For the parabola $y^2 = 4ax$, S.T the square of the radius of curvature at any point varies as the cube of focal distance of the point.

Sol:-

$$y^2 = 4ax$$

$$2yy_1 = 4a$$

$$y_1 = \frac{2a}{y}$$



$$y_2 = -\frac{2a}{y^2}\cdot y_1 = -\frac{2a}{4ax}\cdot\frac{2a}{y} = -\frac{a}{xy}$$

The radius of curvature at any point $P(x, y)$ is given by

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\frac{4a^2}{y^2}\right)^{3/2}}{y_2}$$

$$\rho = \frac{1}{y^3}\frac{(y^2+4a^2)^{3/2}}{\left(-\frac{a}{xy}\right)} = -\frac{xy}{a}\cdot\frac{1}{y^3}\left[4ax+4a^2\right]^{3/2}$$

$$= -\frac{x}{ay^2}(4ax+4a^2)^{3/2} = \frac{-x}{a(4ax)}\left[4ax+4a^2\right]^{3/2}$$

$$\rho = -\frac{(4a)^{3/2}}{4a^2}(x+a)^{3/2} = -\frac{4^{1/2}}{a^{1/2}}(x+a)^{3/2}$$

Squaring both sides

$$\rho^2 = \frac{4}{a}(x+a)^3 \rightarrow (1)$$

Using distance formula

$$SP^2 = (x-a)^2 + (y-0)^2$$

$$SP^2 = (x-a)^2 + y^2 = x^2 - 2ax + a^2 + 4ax$$

$$\therefore \quad SP^2 = x^2 + 2ax + a^2 = (x+a)^2$$

$\therefore$ $SP = (x+a)$ = Focal distance of a point.

$$(1) \Rightarrow \quad \rho^2 = \frac{4}{a}(SP)^3$$

9) Find the radius of curvature of $y^2 = \frac{a^2(a-x)}{x}$, at the point $(a, 0)$.

Sol:- Differentiating wrt. $x \Rightarrow 2yy' = \frac{a^2\left[x(0-1) - (a-x)1\right]}{x^2} = -\frac{a^3}{x^2}$

$$\therefore \quad y' = \frac{-a^3}{2x^2y}, \quad \text{at } (a,0) \ y' \text{ does not exist}$$

Hence, $\frac{dx}{dy} = -\frac{2x^2y}{a^3}$

$$\frac{d^2x}{dy^2} = -\frac{2}{a^3}\left[x^2(1) + y\cdot 2x\cdot\frac{dx}{dy}\right]$$

At the point $(a, 0)$, $\frac{dx}{dy} = 0$, $\frac{d^2x}{dy^2} = -\frac{2}{a}$

$$\therefore \rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\left(\frac{d^2x}{dy^2}\right)} = \frac{(1+0)^{3/2}}{(-2/a)} = -\frac{a}{2}$$

$$\therefore \rho = \frac{a}{2} \text{ (numerically)}$$

Exercise:

1) Find the radius of curvature at any point $P(x, y)$ on the curve $y = c \cdot \cosh\left(\frac{x}{c}\right)$, $c > 0$. Ans: $\rho = \frac{y^2}{c}$

2) S.T. the radius of curvature for the cycloid $x = a(t + \sin t)$, $y = a(1-\cos t)$ is given by $\rho = 4a\cos\left(\frac{t}{2}\right)$

3) Find the radius of curvature of the cardioid $r = a(1+\cos\theta)$ at any point (r, θ) on it. P.T. $\frac{\rho^2}{r}$ is a constant.

4) Find the radius of curvature at $(1, -1)$ on the curve $y = x^2 - 3x + 1$ Ans: $\rho\Big|_{(1,-1)} = \sqrt{2}$

5) Find the radius of curvature of $b^2x^2 + a^2y^2 = a^2b^2$ at its point of intersection with the y-axis.

Ans: $\rho\Big|_{(0,b)} = -\frac{a^2}{b}$, $\rho\Big|_{(0,-b)} = \frac{a^2}{b}$

6) If ρ_1 and ρ_2 are the radii of curvature at

the extremities of the cardiod $r = a(1+\cos\theta)$ which passes through the pole. P.T. $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$

7) P.T. the radius of curvature at any point of astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is three times the length of perpendicular from the origin to the tangent at that point.

8) If ρ_1 and ρ_2 are the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax$ then S.T. $\rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}$.

9) S.T. the radius of curvature at any point of the cardioid $r = a(1-\cos\theta)$ varies as $\sqrt{r}$.

10) Find the radius of curvature of the curve $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$, $y = a\sin t$ Ans: $\rho = a\cot t$

sol:

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan\frac{t}{2}} \cdot \sec^2\frac{t}{2} \cdot \frac{1}{2}\right)$$

$$= a\left(-\sin t + \frac{1}{\frac{\sin t/2}{\cancel{\cos t/2}}} \cdot \frac{1}{\cos^{\cancel{2}} t/2} \cdot \frac{1}{2}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right) = a\left(\frac{1-\sin^2 t}{\sin t}\right) = a\frac{\cos^2 t}{\sin t}$$

$$\therefore \frac{dx}{dt} = a\cos t \cot t.$$

$$\frac{dy}{dt} = a\cos t.$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{a\cos t\cot t} = \tan t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}$$

$$= \sec^2 t \cdot \frac{1}{a \cos t \cot t} = \frac{1}{a}\frac{\sec^3 t}{\cos t} \sin t$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{a} \sec^4 t \sin t.$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\tan^2 t)^{3/2}}{\frac{1}{a}\sec^4 t \sin t} = a\frac{\sec^3 t}{\sec^4 t \sin t}$$

$$\therefore \rho = \frac{a}{\sec t \sin t} = a \cdot \frac{\cos t}{\sin t} = a \cot t$$

Centre of curvature

The coordinates of centre of curvature is $C = (\bar{x}, \bar{y})$ given by

$$\boxed{\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}} \quad , \quad \boxed{\bar{y} = y + \frac{(1+y_1^2)}{y_2}}$$

Circle of curvature

The equation of the circle of curvature at $P(x, y)$ is $\boxed{(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2}$.

1) Find the circle of curvature at (3, 4) on $xy = 12$

Sol: Here $y = \frac{12}{x} \rightarrow (1)$

$$\therefore y_1 = \frac{dy}{dx} = \frac{-12}{x^2} \quad \text{and} \quad y_2 = \frac{d^2y}{dx^2} = \frac{24}{x^3}$$

At (3,4), $y_1 = -\frac{12}{9} = -\frac{4}{3}$, $y_2 = \frac{24}{27} = \frac{8}{9}$

The coordinates of centre of curvature is given by

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = 3 - \frac{(-4/3)(1+\frac{16}{9})}{(8/9)}$$

$$\bar{x} = \frac{43}{6}$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2} = \frac{4 + (1+\frac{16}{9})}{(8/9)} = \frac{57}{8}$$

$$\therefore (\bar{x}, \bar{y}) = \left(\frac{43}{6}, \frac{57}{8}\right)$$

$$\therefore \rho = \frac{[1+y_1^2]^{3/2}}{y_2} = \frac{\left(1+\frac{16}{9}\right)^{3/2}}{(8/9)} = \frac{125}{24}$$

The circle of curvature is

$$(x-43/6)^2 + (y-57/8)^2 = \left(\frac{125}{24}\right)^2$$

2) Find the centre of curvature and circle of curvature on the curve $y = e^x$ at a point where the curve crosses the y-axis.

Sol:

$$y = e^x \Rightarrow y_1 = y_2 = e^x$$

The eq of y axis is $x = 0$.

$\therefore y = e^0 = 1$ $\quad \therefore$ The point is $(0, 1)$

Now $y_1 = y_2 = 1$ at $(0, 1)$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = 2^{3/2} = 2\sqrt{2}$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = 0 - \frac{1(1+1^2)}{1} = -2$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2} = 1 + \frac{(1+1)}{1} = 3$$

$\therefore$ Centre of curvature is $(\bar{x}, \bar{y}) = (-2, 3)$

Circle of curvature is

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$\therefore (x+2)^2 + (y-3)^2 = 8$$

3) Find the coordinates of centre of curvature at any point of the parabola $y^2 = 4ax$

Folium of Descartes $x^3+y^3 = 3axy$.
cycloid
Application of ρ

Sol: Differentiating $y^2 = 4ax$ wrt x

$$2yy_1 = 4a \quad \text{i.e.} \quad y_1 = \frac{2a}{y}$$

$$y_2 = -\frac{2a}{y^2}\cdot y_1 = -\frac{4a^2}{y^3}$$

The coordinates of centre of curvature are,

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = x - \frac{(2a/y)(1+4a^2/y^2)}{-4a^2/y^3}$$

$$= x + \frac{2a}{\cancel{y}} \times \left(\frac{y^2+4a^2}{\cancel{y^2}}\right) \times \frac{1}{4a^2}\cancel{y^3}$$

$$= x + \frac{y^2+4a^2}{2a} = x + \frac{4ax+4a^2}{2a} = \frac{6ax+4a^2}{2a}$$

$$\therefore \bar{x} = 3x+2a$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = y + \frac{1+4a^2/y^2}{-4a^2/y^3} = y - \left(\frac{y^2+4a^2}{\cancel{y^2}\times 4a^2}\times \cancel{y^3}\right)$$

$$= y - \frac{y(y^2+4a^2)}{4a^2} = \frac{\cancel{4a^2y} - y^3 - \cancel{4a^2y}}{4a^2}$$

$$\bar{y} = \frac{-y^3}{4a^2} = -\frac{\cancel{4}ax\cdot y}{\cancel{4}a^2} = -\frac{\cancel{a}x\,2\sqrt{ax}}{a^{\cancel{2}}}$$

$$\bar{y} = -\frac{2x^{3/2}}{\sqrt{a}}$$

$$\therefore \text{Centre of curvature} = (\bar{x},\bar{y}) = \left(3x+2a,\ -\frac{2x^{3/2}}{\sqrt{a}}\right)$$

4) S.T. the circle of curvature at the origin of the curve $x+y = ax^2 + by^2 + cx^3$ is $(a+b)(x^2+y^2) = 2(x+y)$

Sol:- Differentiating the curve wrt x

$$1+y_1 = 2ax + 2byy_1 + 3cx^2$$

$$y_1 = 2(ax+byy_1)+3cx^2-1$$

$$y_2 = 2[a+b(yy_2+y_1^2)]+6cx$$

$$y_2(1-2by) = 2(a+by_1^2)+6cx$$

$$\therefore y_2 = \frac{2(a+by_1^2)+6cx}{(1-2by)}$$

At origin, $y_1 = -1$, $y_2 = \frac{2(a+b)}{1} = 2(a+b)$

$$\therefore \rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{2(a+b)} = \frac{2\sqrt{2}}{2(a+b)} = \frac{\sqrt{2}}{(a+b)}$$

Coordinates of centre of curvature at $(0,0)$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = 0 + \frac{(1+1)}{2(a+b)} = 0 + \frac{1}{a+b}$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2} = 0 + \frac{1+1}{2(a+b)} = \frac{1}{a+b}$$

$\therefore$ Circle of curvature :

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$\left(x - \frac{1}{a+b}\right)^2 + \left(y - \frac{1}{a+b}\right)^2 = \frac{2}{(a+b)^2}$$

$$x^2 + \cancel{\frac{1}{(a+b)^2}} - 2x\cdot\frac{1}{a+b} + y^2 + \cancel{\frac{1}{(a+b)^2}} - 2y\cdot\frac{1}{a+b} = \cancel{\frac{2}{(a+b)^2}}$$

$$x^2+y^2 - \frac{2}{a+b}(x+y) = 0$$

$$\Rightarrow x^2+y^2 = \frac{2}{a+b}(x+y)$$

$$\therefore (a+b)(x^2+y^2) = 2(x+y)$$

Exercise.

1) Find the circle of curvature at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$ of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$

Ans: $\left(x-\frac{3a}{4}\right)^2+\left(y+\frac{3a}{4}\right)^2=\frac{a^2}{2}$

2) Find the centre of curvature of the curve $a^2y=x^2$.

Ans: $\bar{x}=\frac{x}{2}\left(1-\frac{9x^4}{a^4}\right)$, $\bar{y}=\frac{5x^3}{2a^2}+\frac{a^2}{6x}$

3) Find the circle of curvature at $(1,0)$ on $y=x^3-x^2$

Ans: $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{2} \Rightarrow x^2+y^2-x-y=0.$

4) Find the circle of curvature at the point $(2,3)$ on $\frac{x^2}{4}+\frac{y^2}{9}=2$

Ans: $\left(x+\frac{5}{4}\right)^2+\left(y-\frac{5}{6}\right)^2=\frac{13^2}{12^2}$

5) Find the centre of curvature of $y=x^2$ at $\left(\frac{1}{2}, \frac{1}{4}\right)$

Ans: $(\bar{x}, \bar{y})=\left(-\frac{1}{2}, \frac{5}{4}\right)$.

6) S.T. the circle of curvature at origin for the curve $y=mx+\frac{x^2}{a}$ is $x^2+y^2=a(1+m^2)(y-mx)$.