# CMFD in 2D

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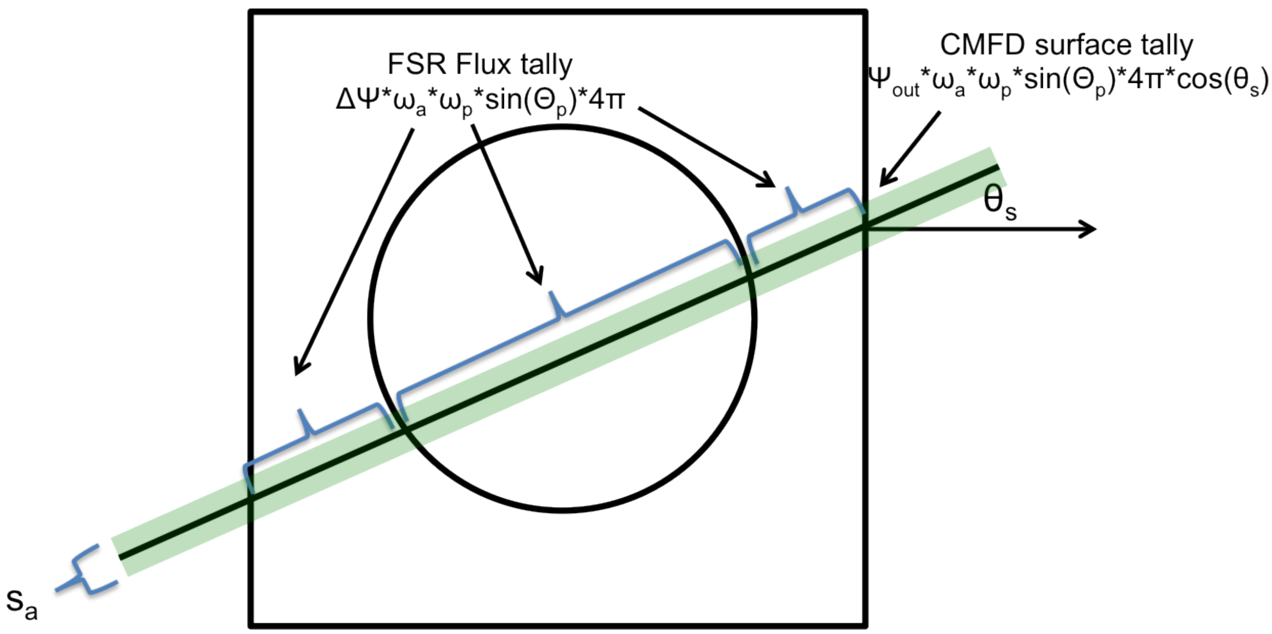
**Code**

*Steps in going from MOC to CMFD and back*

1. **Begin fixed source iteration.**

2. **Track across geometry and tally the scalar flux contribution in each Flat Source Region. For each segment that crosses a CMFD mesh surface, tally the weighted partial current that crosses this boundary:**




Where:

$$\omega_a = \frac{\Delta\psi_a s_a}{\pi} \qquad s_a = \text{track spacing} \tag{1}$$

Partial current is defined as:

$$J^+ = \int_{\vec{n}\cdot\vec{\Omega}>0} d\Omega \vec{n}\cdot\vec{\Omega}\psi(\vec{r}, E, \vec{\Omega}) \tag{2}$$

As the track crosses each surface, this integral becomes:

$$J^+ = \sum_{crossings} \psi_{out}\ \omega_a\ \omega_p\ sin(\Theta_p)\ 2\ \pi\ cos(\Theta_a - \Theta_{normal}) \tag{3}$$

The angular current has an associated azimuthal ($\omega_a$) and polar ($\omega_p$) weight. The azimuthal weight accounts for the fact that every azimuthal angle has a different spacing between tracks to ensure a cyclic track layout and for the finite width over which the neutrons are propagating. The polar weight is defined by the quadrature set which the property that:

$$\sum_p \omega_p = 1 \tag{4}$$

3. **Perform group-wise energy condensation and volume averaging of cross section, flux, and diffusion coefficients for each mesh cell.**

   In MOC, we typically deal with multi-group flux data in heterogeneous geometries. However, we want to solve the CMFD in one group and over homogenized geometries. Therefore we perform energy condensation and volume averaging of the flux, diffusion coefficient, and cross sections in each mesh cell using the following equations:

$$\Sigma_{x,cell} = \frac{\sum_{FSR}\sum_{g}\Sigma_{x,g}^{FSR}\phi_g^{FSR}V^{FSR}}{\sum_{FSR}\sum_{g}\phi_g^{FSR}V^{FSR}} \tag{5}$$

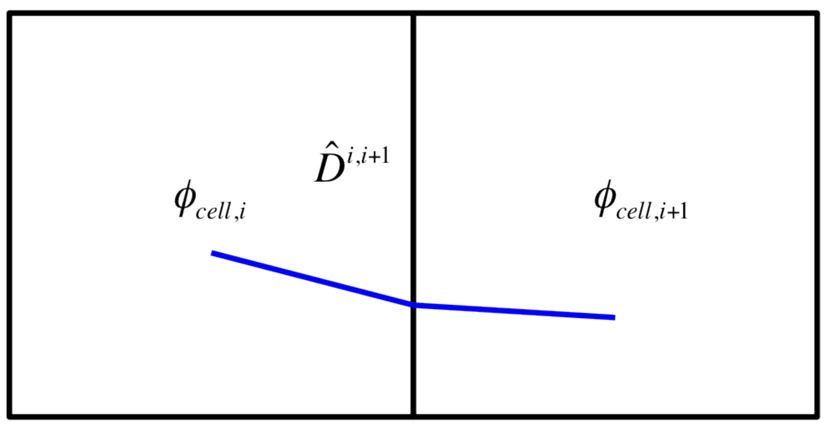
$$D_{cell} = \frac{\sum_{FSR}\sum_{g}\frac{1}{3\Sigma_{tr,g}^{FSR}}\phi_g^{FSR}V^{FSR}}{\sum_{FSR}\sum_{g}\phi_g^{FSR}V^{FSR}} \tag{6}$$

$$\phi_{cell} = \frac{\sum_{FSR}\sum_{g}\phi_g^{FSR}V^{FSR}}{\sum_{FSR}V^{FSR}} \tag{7}$$

4. **Use the cell averaged flux and diffusion coefficients to get surface diffusion coefficients.**

   We can use the diffusion coefficient for each cell to get the diffusion coefficient that links adjacent cells:

$$\hat{D}^{i,i+1} = \frac{2D^iD^{i+1}}{\Delta i(D^i + D^{i+1})} \tag{8}$$




5. **Using surface diffusion coefficients and tallied surface currents, get diffusion coefficient correction factors. This ensures that neutron current is conserved across the boundaries.**

   The partial current across the boundary is written as:

$$J_{i,i+1}^{+} = -\hat{D}^{i,i+1}[\phi^{i+1} - \phi^{i}] \tag{9}$$

   However, this does not guarantee that the net current will be conserved across the surface; Therefore, we add a correction term:

$$J_{i,i+1}^{+} = -\hat{D}^{i,i+1}[\phi^{i+1} - \phi^{i}] - \tilde{D}^{i,i+1}[\phi^{i+1} + \phi^{i}] \tag{10}$$

   Now for the partial current to balance, we set equation 10 equal to equation 2 and solve for $\tilde{D}$:

$$\sum_{crossings}\psi_{out}\,\omega_a\,\omega_p\,sin(\Theta_p)\,2\,\pi\,cos(\Theta_a - \Theta_{normal}) = -\hat{D}^{i,i+1}[\phi^{i+1} - \phi^{i}] - \tilde{D}^{i,i+1}[\phi^{i+1} + \phi^{i}] \tag{11}$$

$$\tilde{D}^{i,i+1} = \frac{\hat{D}^{i,i+1}[\phi^{i+1} - \phi^{i}] - \sum_{crossings} \psi_{out}\ \omega_a\ \omega_p\ sin(\Theta_p)\ 2\ \pi\ cos(\Theta_a - \Theta_{normal})}{[\phi^{i+1} + \phi^{i}]} \tag{12}$$

6. **Set up the A and M matrices and flux vector**

   See below

7. **Solve the eigenvalue problem to get the CMFD $k_{eff}$ and the corresponding eigenvector flux.**

   This eigenvalue problem can be solved using many different methods, but to keep things simple we used a MATLAB-like direct eigenvalue solver (We are using the Armadillo library which is a C++ linear algebra package with MATLAB-like syntax). The function to solve for the eigenvalues and eigenvectors is similar to MATLAB:

$$eig\_gen(eigval, eigvec, inv(A) * M); \tag{13}$$

8. **Update the group-wise MOC FSR scalar flux using the CMFD 1-group mesh cell averaged flux.**

$$\phi^{new}_{FSR,g} = \phi^{old}_{FSR,g} * \frac{\phi^{new}_{cell}}{\phi^{old}_{cell}} \tag{14}$$

*Check for balance*

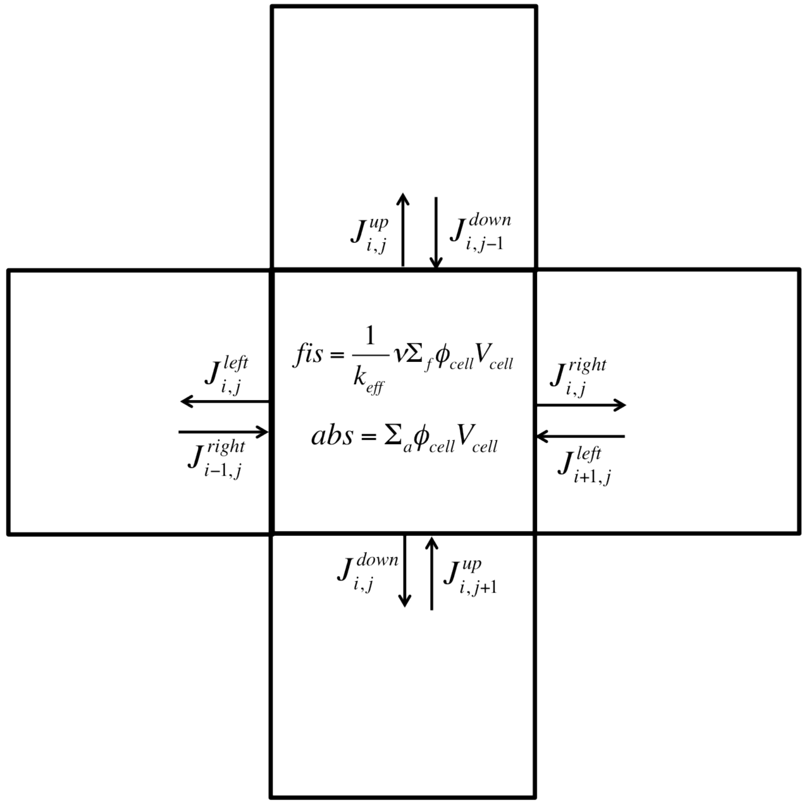
1. **Compute the MOC $k_{eff}$ by summing the fission source and absorption rates.**

$$k_{eff,check} = \frac{\sum_{cells} \nu\Sigma^{cell}_{f}\phi^{cell}_{g}V^{cell}}{\sum_{cells} \Sigma^{cell}_{a}\phi^{cell}V^{cell}} \tag{15}$$

   In running our trials, we find that $k^{MOC}_{eff,check}$ is always the same as the $k_{eff}$ at each iteration in the MOC solver. This should indicate that we are correctly condensing our cross sections.

2. **Compute the neutron balance in each cell by summing the MOC tallied surface leakages, the mesh cell fission source, and the mesh cell absorption rate.**

$$residual = leak + abs - fis; \tag{16}$$




Where:

$$leak = J_{i,j}^{right} + J_{i,j}^{down} + J_{i,j}^{left} + J_{i,j}^{up} - J_{i-1,j}^{right} - J_{i,j-1}^{down} - J_{i+1,j}^{left} - J_{i,j+1}^{up} \tag{17}$$

When we perform this balance on the mesh cells of a heterogeneous, non-uniform geometry, the residual will be a few percent of the scattering (or absorption) source. This error is much greater than we would expect from numerical precision errors and averaging over space and energy, so this lead us to believe the partial surface currents are not being properly tallied.

***One group Course Mesh Finite Difference equations in 2-D*** *- Derive the equations for new nodal averaged fluxes in terms of the old nodal averaged fluxes. We will assume all cross sections have already been condensed in space and energy.*

We will begin with the diffusion equation in 2-D:

$$-\nabla \cdot D\nabla\phi + \Sigma_a\phi = \frac{1}{k}\nu\Sigma_f\phi \tag{18}$$

We can expand the diffusing operator to yield:

$$-\frac{\partial}{\partial x}D\frac{\partial\phi}{\partial x} - \frac{\partial}{\partial y}D\frac{\partial\phi}{\partial y} + \Sigma_a\phi = \frac{1}{k}\nu\Sigma_f\phi \tag{19}$$

We can integrate this equation over each dimension of the mesh cell to get:

$$-\int_{j-\frac{1}{2}}^{j+\frac{1}{2}}\int_{i-\frac{1}{2}}^{i+\frac{1}{2}} \frac{\partial}{\partial x}D\frac{\partial\phi}{\partial x}dxdy - \int_{j-\frac{1}{2}}^{j+\frac{1}{2}}\int_{i-\frac{1}{2}}^{i+\frac{1}{2}} \frac{\partial}{\partial y}D\frac{\partial\phi}{\partial y}dxdy + \int_{j-\frac{1}{2}}^{j+\frac{1}{2}}\int_{i-\frac{1}{2}}^{i+\frac{1}{2}} \Sigma_a\phi dxdy = \frac{1}{k}\int_{j-\frac{1}{2}}^{j+\frac{1}{2}}\int_{i-\frac{1}{2}}^{i+\frac{1}{2}} \nu\Sigma_f\phi dxdy \tag{20}$$

Which simplifies to:

$$-\Delta y\int_{i-\frac{1}{2}}^{i+\frac{1}{2}} \frac{\partial}{\partial x}D\frac{\partial\phi}{\partial x}dx - \Delta x\int_{i-\frac{1}{2}}^{i+\frac{1}{2}} \frac{\partial}{\partial x}D\frac{\partial\phi}{\partial x}dx + \Sigma_a\phi\Delta x\Delta y = \frac{1}{k}\nu\Sigma_f\phi\Delta x\Delta y \tag{21}$$

The currents comes out to:

$$-\int_{i-\frac{1}{2}}^{i+\frac{1}{2}} \frac{\partial}{\partial x}D\frac{\partial\phi}{\partial x}dx = J_x^{i+\frac{1}{2}} - J_x^{i-\frac{1}{2}} \tag{22}$$

$$-\int_{j-\frac{1}{2}}^{j+\frac{1}{2}} \frac{\partial}{\partial y}D\frac{\partial\phi}{\partial y}dy = J_y^{j+\frac{1}{2}} - J_y^{j-\frac{1}{2}} \tag{23}$$

The finite difference form of the transport equation is then:

$$\Delta y(J_x^{i+\frac{1}{2}} - J_x^{i-\frac{1}{2}}) + \Delta x(J_y^{j+\frac{1}{2}} - J_y^{j-\frac{1}{2}}) + \Sigma_a\phi\Delta x\Delta y = \frac{1}{k}\nu\Sigma_f\phi\Delta x\Delta y \tag{24}$$

Or:

$$\Delta y(J_x^{+} - J_x^{-}) + \Delta x(J_y^{+} - J_y^{-}) + \Sigma_a\phi\Delta x\Delta y = \frac{1}{k}\nu\Sigma_f\phi\Delta x\Delta y \tag{25}$$

Note that positive $J_n^+$ corresponds to neutrons leaving the cell and positive $J_n^-$ corresponds to neutrons entering the cell.

We can use Fick's equation to describe the current:

$$J_n^+ = -D^n\frac{d\phi^n}{dn}|_{n+} = -D^n\frac{\phi^s - \phi^n}{\frac{\Delta n}{2}} \tag{26}$$

$$J_{n+1}^- = -D^{n+1}\frac{d\phi^{n+1}}{dn}|_{n-} = -D^{n+1}\frac{\phi^{n+1} - \phi^s}{\frac{\Delta n}{2}} \tag{27}$$

We can equate these to get an expression for the flux and current at the boundary:

$$-D^n\frac{\phi^s - \phi^n}{\frac{\Delta n}{2}} = -D^{n+1}\frac{\phi^{n+1} - \phi^s}{\frac{\Delta n}{2}} \tag{28}$$

The interfacial flux is then:

$$\phi^s = \frac{D^{n+1}\phi^{n+1} + D^n\phi^n}{D^n + D^{n+1}} \tag{29}$$

The net current in terms of the mesh fluxes is then:

$$J_n^+ = -\frac{2D^n}{\Delta n}\left[\frac{D^{n+1}\phi^{n+1} + D^n\phi^n}{D^n + D^{n+1}} - \phi^n\right] = \frac{2D^nD^{n+1}}{\Delta n(D^n + D^{n+1})}\phi^{n+1} - \frac{2D^nD^{n+1}}{\Delta n(D^n + D^{n+1})}\phi^n \tag{30}$$

Or:

$$J_n^+ = -\hat{D}^{n,n+1}[\phi^{n+1} - \phi^n] \qquad J_n^- = -\hat{D}^{n-1,n}[\phi^n - \phi^{n-1}] \tag{31}$$

Where:

$$\hat{D}^{n,n+1} = \frac{2D^nD^{n+1}}{\Delta n(D^n + D^{n+1})} \tag{32}$$

However, for our diffusion acceleration scheme we must conserve net current across the mesh cell boundaries; Put another way, the partial surface currents in our diffusion equation must be equal to the weighted partial surface current tallies from our MOC fixed source iteration solver. To make the partial currents agree, we add a diffusion-like coupling coefficient, $\tilde{D}$ :

$$J_n^+ = -\hat{D}^{n,n+1}[\phi^{n+1} - \phi^n] - \tilde{D}^{n,n+1}[\phi^{n+1} + \phi^n] \qquad J_n^- = -\hat{D}^{n-1,n}[\phi^n - \phi^{n-1}] - \tilde{D}^{n-1,n}[\phi^n + \phi^{n-1}] \tag{33}$$

We compute $\tilde{D}$ by plugging in the tallies of directional current across each cell surface, the cell fluxes, and diffusion coefficients:

$$\tilde{D}^{n,n+1} = -\frac{\hat{D}^{n,n+1}[\phi^{n+1} - \phi^n] + J^+_{n,tally}}{\phi^{n+1} + \phi^n} \qquad \tilde{D}^{n-1,n} = -\frac{\hat{D}^{n-1,n}[\phi^n - \phi^{n-1}] + J^-_{n,tally}}{\phi^n + \phi^{n-1}} \tag{34}$$

The finite difference equation is then:

$$\begin{aligned}&\Delta y(\hat{D}^{i-1,i}[\phi_x^{i,j} - \phi_x^{i-1,j}] + \tilde{D}^{i-1,i}[\phi^{i,j} + \phi^{i-1,j}]) - \Delta y(\hat{D}^{i,i+1}[\phi_x^{i+1,j} - \phi_x^{i,j}] + \tilde{D}^{i,i+1}[\phi^{i+1,j} + \phi^{i,j}]) + \\ &\Delta x(\hat{D}^{j-1,j}[\phi_y^{i,j} - \phi_y^{i,j-1}] + \tilde{D}^{j-1,j}[\phi^{i,j} + \phi^{i,j-1}]) - \Delta x(\hat{D}^{j,j+1}[\phi_y^{i,j+1} - \phi_y^{i,j}] + \tilde{D}^{j,j+1}[\phi^{i,j+1} + \phi^{i,j}]) + \Sigma_a\phi^{i,j}\Delta x\Delta y = \frac{1}{k}\nu\Sigma_f\phi^{i,j}\Delta x\Delta y\end{aligned} \tag{35}$$

Simplifying one obtains the linear finite-difference diffusion equation:

$$\begin{aligned}&-\Delta y(\hat{D}^{i-1,i} - \tilde{D}^{i-1,i})\phi^{i-1,j} - \Delta y(\hat{D}^{i,i+1} + \tilde{D}^{i,i+1})\phi^{i+1,j} - \Delta x(\hat{D}^{j-1,j} - \tilde{D}^{j-1,j})\phi^{i,j-1} - \Delta x(\hat{D}^{j,j+1} + \tilde{D}^{j,j+1})\phi^{i,j+1} + \\ &[\Sigma_a^i\Delta x\Delta y + \Delta y(\hat{D}^{i-1,i} + \hat{D}^{i,i+1} + \tilde{D}^{i-1,i} - \tilde{D}^{i,i+1}) + \Delta x(\hat{D}^{j-1,j} + \hat{D}^{j,j+1} + \tilde{D}^{j-1,j} - \tilde{D}^{j,j+1})]\phi^{i,j} = \frac{1}{k}\nu\Sigma_f^i\Delta x\Delta y\phi^{i,j}\end{aligned} \tag{36}$$

Note that for reflective boundary conditions:

$$J^-_{boundary} = J^+_{boundary} \tag{37}$$

For a 3x3 grid the matrix representation will be:

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & 0 & A_{1,4} & 0 & 0 & 0 & 0 & 0 \\ A_{2,1} & A_{2,2} & A_{2,3} & 0 & A_{2,5} & 0 & 0 & 0 & 0 \\ 0 & A_{3,2} & A_{3,3} & A_{3,4} & 0 & A_{3,6} & 0 & 0 & 0 \\ A_{4,1} & 0 & A_{4,3} & A_{4,4} & A_{4,5} & 0 & A_{4,7} & 0 & 0 \\ 0 & A_{5,2} & 0 & A_{5,4} & A_{5,5} & A_{5,6} & 0 & A_{5,8} & 0 \\ 0 & 0 & A_{6,3} & 0 & A_{6,5} & A_{6,6} & A_{6,7} & 0 & A_{6,9} \\ 0 & 0 & 0 & A_{7,4} & 0 & A_{7,6} & A_{7,7} & A_{7,8} & 0 \\ 0 & 0 & 0 & 0 & A_{8,5} & 0 & A_{8,7} & A_{8,8} & A_{8,9} \\ 0 & 0 & 0 & 0 & 0 & A_{9,6} & 0 & A_{9,8} & A_{9,9} \end{bmatrix}$$

Where the diagonals are:

$$A_{1,1} = \Delta x(\hat{D}^{1,4} + \tilde{D}^{1,4}) + \Delta y(\hat{D}^{1,2} - \tilde{D}^{1,2}) + \Sigma_a^1 \Delta x \Delta y \tag{38}$$

$$A_{2,2} = \Delta y(\hat{D}^{2,3} + \hat{D}^{1,2} - \tilde{D}^{2,3} + \tilde{D}^{1,2})) + \Delta x(\hat{D}^{2,5} + \tilde{D}^{2,5}) + \Sigma_a^2 \Delta x \Delta y \tag{39}$$

$$A_{3,3} = \Delta y(\hat{D}^{2,3} + \tilde{D}^{2,3}) + \Delta x(\hat{D}^{3,6} + \tilde{D}^{3,6}) + \Sigma_a^3 \Delta x \Delta y \tag{40}$$

$$A_{4,4} = \Delta x(\hat{D}^{1,4} + \hat{D}^{4,7} - \tilde{D}^{1,4} + \tilde{D}^{4,7}) + \Delta y(\hat{D}^{4,5} - \tilde{D}^{4,5}) + \Sigma_a^4 \Delta x \Delta y \tag{41}$$

$$A_{5,5} = \Delta y(\hat{D}^{4,5} + \hat{D}^{5,6} + \tilde{D}^{4,5} - \tilde{D}^{5,6}) + \Delta x(\hat{D}^{2,5} + \hat{D}^{5,8} - \tilde{D}^{2,5} + \tilde{D}^{5,8}) + \Sigma_a^5 \Delta x \Delta y \tag{42}$$

$$A_{6,6} = \Delta y(\hat{D}^{5,6} - \tilde{D}^{5,6}) + \Delta x(\hat{D}^{3,6} + \hat{D}^{6,9} - \tilde{D}^{3,6} + \tilde{D}^{6,9}) + \Sigma_a^6 \Delta x \Delta y \tag{43}$$

$$A_{7,7} = \Delta y(\hat{D}^{7,8} - \tilde{D}^{7,8}) + \Delta x(\hat{D}^{4,7} - \tilde{D}^{4,7}) + \Sigma_a^7 \Delta x \Delta y \tag{44}$$

$$A_{8,8} = \Delta y(\hat{D}^{7,8} + \hat{D}^{8,9} + \tilde{D}^{7,8} - \tilde{D}^{8,9}) + \Delta x(\hat{D}^{5,8} - \tilde{D}^{5,8}) + \Sigma_a^8 \Delta x \Delta y \tag{45}$$

$$A_{9,9} = \Delta y(\hat{D}^{8,9} + \tilde{D}^{8,9}) + \Delta x(\hat{D}^{6,9} - \tilde{D}^{6,9}) + \Sigma_a^9 \Delta x \Delta y \tag{46}$$

The uppers are:

$$A_{1,2} = -\Delta y(\hat{D}^{1,2} + \tilde{D}^{1,2}) \tag{47}$$

$$A_{2,3} = -\Delta y(\hat{D}^{2,3} + \tilde{D}^{2,3}) \tag{48}$$

$$A_{3,4} = 0 \tag{49}$$

$$A_{4,5} = -\Delta y(\hat{D}^{4,5} + \tilde{D}^{4,5}) \tag{50}$$

$$A_{5,6} = -\Delta y(\hat{D}^{5,6} + \tilde{D}^{5,6}) \tag{51}$$

$$A_{6,7} = 0 \tag{52}$$

$$A_{7,8} = -\Delta y(\hat{D}^{7,8} + \tilde{D}^{7,8}) \tag{53}$$

$$A_{8,9} = -\Delta y(\hat{D}^{8,9} + \tilde{D}^{8,9}) \tag{54}$$

The 2-D uppers are:

$$A_{1,4} = -\Delta x(\hat{D}^{1,4} + \tilde{D}^{1,4}) \tag{55}$$

$$A_{2,5} = -\Delta x(\hat{D}^{2,5} + \tilde{D}^{2,5}) \tag{56}$$

$$A_{3,6} = -\Delta x(\hat{D}^{3,6} + \tilde{D}^{3,6}) \tag{57}$$

$$A_{4,7} = -\Delta x(\hat{D}^{4,7} + \tilde{D}^{4,7}) \tag{58}$$

$$A_{5,8} = -\Delta x(\hat{D}^{5,8} + \tilde{D}^{5,8}) \tag{59}$$

$$A_{6,9} = -\Delta x(\hat{D}^{6,9} + \tilde{D}^{6,9}) \tag{60}$$

The lowers are:

$$A_{2,1} = -\Delta y(\hat{D}^{2,1} - \tilde{D}^{2,1}) \tag{61}$$

$$A_{3,2} = -\Delta y(\hat{D}^{3,2} - \tilde{D}^{3,2}) \tag{62}$$

$$A_{4,3} = 0 \tag{63}$$

$$A_{5,4} = -\Delta y(\hat{D}^{5,4} - \tilde{D}^{5,4}) \tag{64}$$

$$A_{6,5} = -\Delta y(\hat{D}^{6,5} - \tilde{D}^{6,5}) \tag{65}$$

$$A_{7,7} = 0 \tag{66}$$

$$A_{8,7} = -\Delta y(\hat{D}^{8,7} - \tilde{D}^{8,7}) \tag{67}$$

$$A_{9,8} = -\Delta y(\hat{D}^{9,8} - \tilde{D}^{9,8}) \tag{68}$$

The 2-D lowers are:

$$A_{4,1} = -\Delta x(\hat{D}^{4,1} - \tilde{D}^{4,1}) \tag{69}$$

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$$A_{7,4} = -\Delta x(\hat{D}^{7,4} - \tilde{D}^{7,4}) \tag{72}$$

$$A_{8,5} = -\Delta x(\hat{D}^{8,5} - \tilde{D}^{8,5}) \tag{73}$$

$$A_{9,6} = -\Delta x(\hat{D}^{9,6} - \tilde{D}^{9,6}) \tag{74}$$