Applying Newton-Krylov Solver on a Model for Systemic Risk

June 22, 2016

Newton-Krylov solver

Computing steady states $\boldsymbol{\rho}^*$ by solving the non-linear system

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- These are estimated by a finite difference approximation

$$J(\boldsymbol{\Phi}_T) \cdot \mathbf{v} \approx \frac{\boldsymbol{\Phi}_T(\boldsymbol{\rho} + \varepsilon \mathbf{v}, \boldsymbol{\omega_1}) - \boldsymbol{\Phi}_T(\boldsymbol{\rho}, \boldsymbol{\omega_2})}{\varepsilon}$$

Variance reduction of Jacobian-vector products

$$
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Solution:
Perturbations on the density $\rightarrow$ perturbations in the weights

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N} w_i\cdot\chi_{\Delta_j}(x_i) \quad &= \quad \rho_j \\ \frac{1}{N}\sum_{i=1}^{N} w_\varepsilon^i\cdot\chi_{\Delta_j}(x^i) \quad &= \quad \rho_j+\varepsilon v_j.\end{aligned}$$

For the perturbated density: compute the weight per bin as $w_\varepsilon^j = 1+\frac{\varepsilon v_j}{\rho_j}$ and assign this value to each particle in Δ_j.

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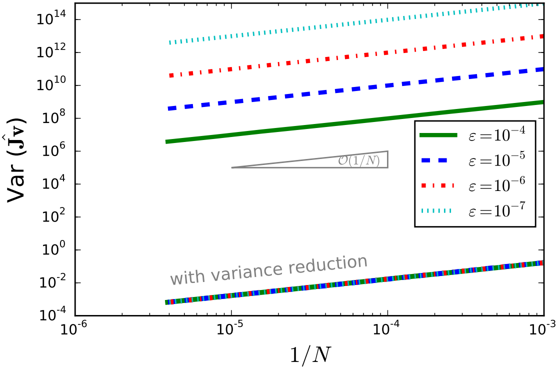
$$\mathtt{Var}(\hat{\mathbf{J}}\mathbf{v}) = \hat{\mathbb{E}}\left[\left(\hat{\mathbf{J}}\mathbf{v} - \hat{\mathbb{E}}[\hat{\mathbf{J}}\mathbf{v}]\right)^2\right] \sim \mathcal{O}(1/N)$$



Mean Field Model

Interaction between components

- Adding *mean field interaction* to model: each particle feels an attractive force towards the mean state (each agent tends to follow the state of the majority)

$$\mathrm{d}X_j = \mu f(X_j)\mathrm{d}t + \sigma \mathrm{d}W_j + \alpha(\bar{X} - X_j)\mathrm{d}t$$

- Interconnectedness between agents can affect the stability of the whole system

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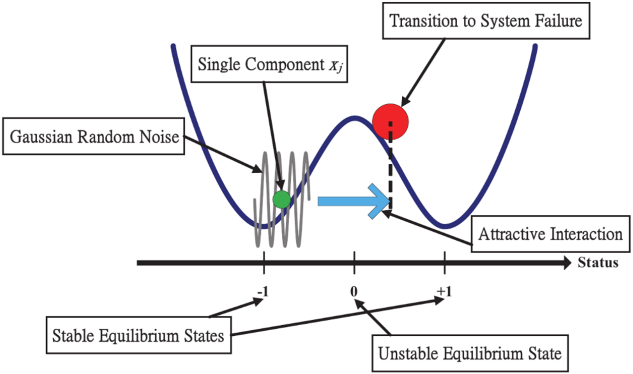
Application: Systemic Risk in Banking Systems

- Banks cooperate. By spreading the risk of credit shocks, they try to minimize their own risk.
- However, this increases the risk that they may all fail

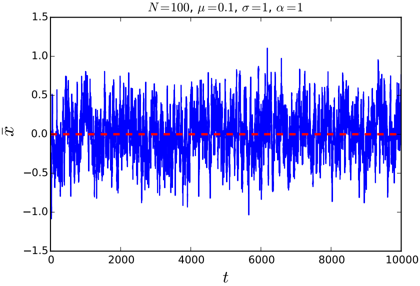
Mathematical Model for Systemic Risk

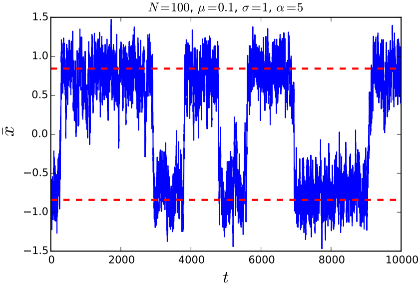
$$\mathrm{d}X_j = \mu f(X_j)\mathrm{d}t + \sigma \mathrm{d}W_j + \alpha(\bar{X} - X_j)\mathrm{d}t$$

- μ The intrinsic stability of each component
- σ The strength of external random perturbations to the system
- α The degree of interconnectedness between agents



Metastable Coarse States





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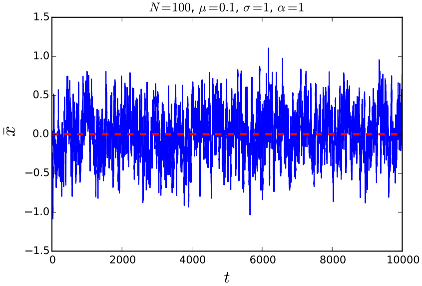

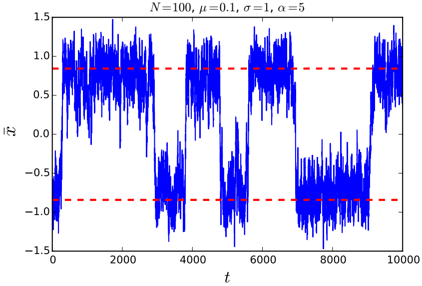

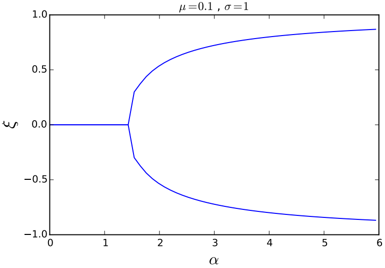

Analytic Solution for Equilibrium Distribution

$$\frac{\partial \rho}{\partial t} = -\mu \frac{\partial (f(x)\rho)}{\partial x} - \alpha \frac{\partial}{\partial x}\left[\left(\int x\rho \mathrm{d}x - x\right)\rho\right] + \frac{\sigma^2}{2}\frac{\partial^2 \rho}{\partial x^2}.$$

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Assuming that $\xi = \lim_{t\to\infty}\int x\rho(x,t)\mathrm{d}x$, an equilibrium solution satisfies

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The non-zero solutions $\pm\xi$ are

$$\xi = \pm\sqrt{1 - 3\frac{\sigma^2}{2\alpha}}\left(1 + \ \mu\frac{6}{\sigma^2}\left(\frac{\sigma^2}{2\alpha}\right)^2 \frac{1 - 2\frac{\sigma^2}{2\alpha}}{1 - 3\frac{\sigma^2}{2\alpha}}\right) + \mathcal{O}(\mu^2)$$

Calculate fixed points by applying variance reduced Newton-Krylov-solver

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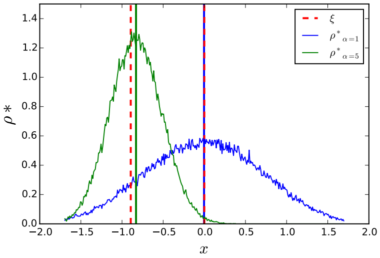
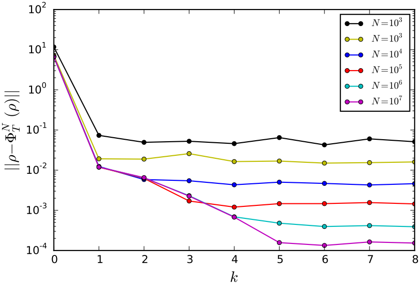
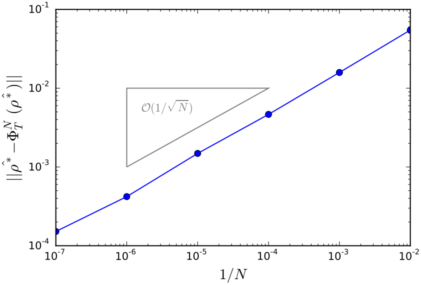
- How to choose the Newton tolerance?
- Which time window to choose for the coarse time stepper $\boldsymbol{\Phi}_T$?



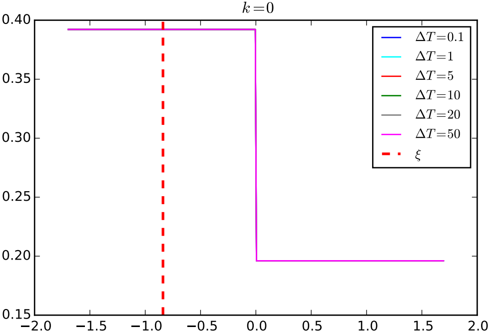
Estimating Stopping Criterion for the Non-linear Solver

- The accuracy on the Newton-Krylov solution is inevitably limited by the noise on the stochastic coarse-time-stepper
- When the Newton-Krylov solution is converged, it stays oscillating around the true solution with a standard deviation depending on the number of particles.





Estimating ΔT



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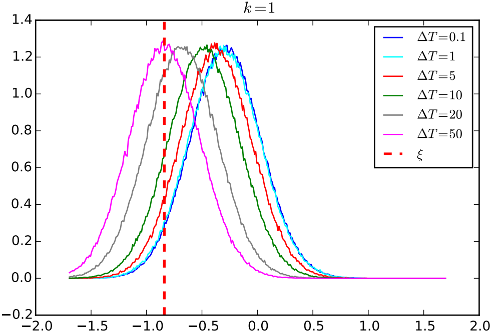

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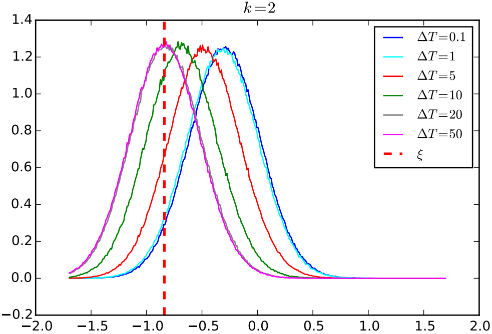

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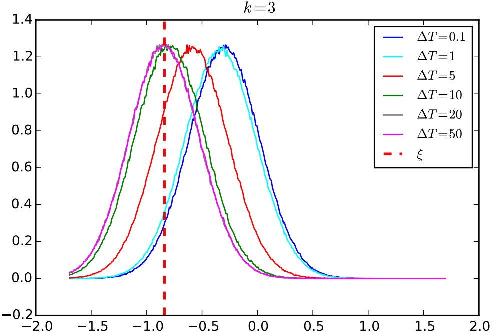

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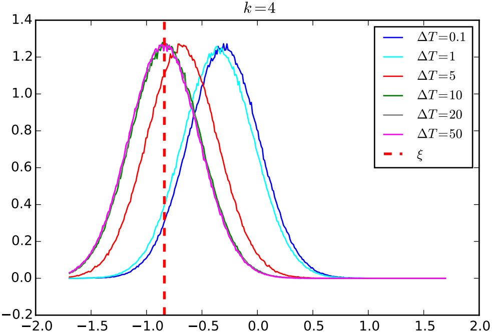

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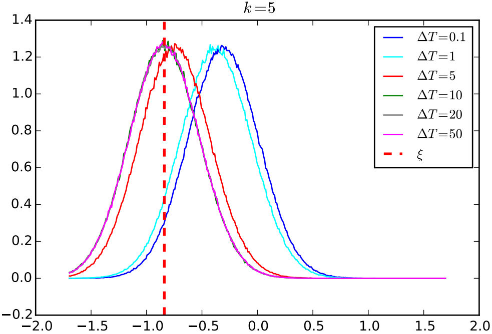

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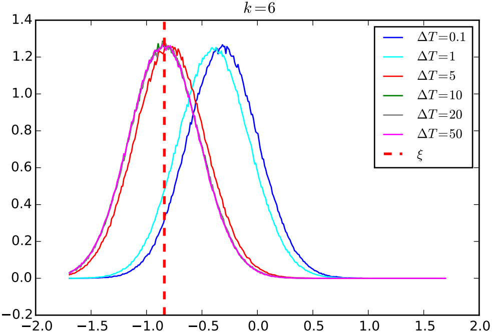

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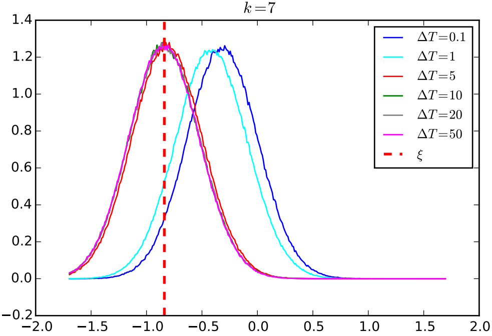

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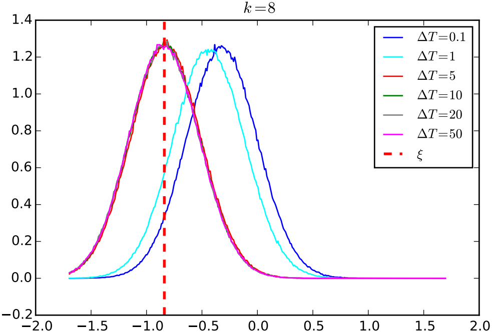

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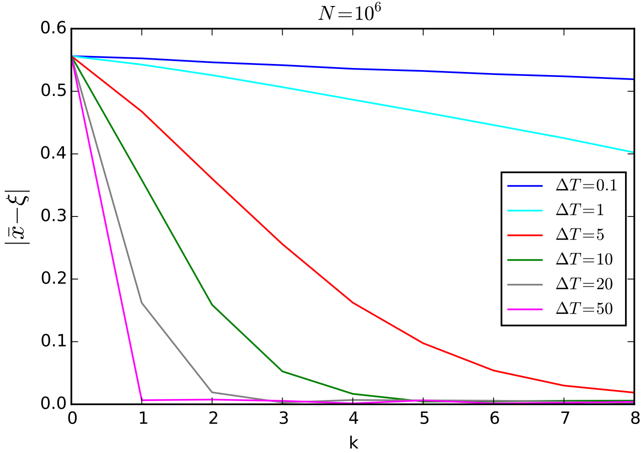

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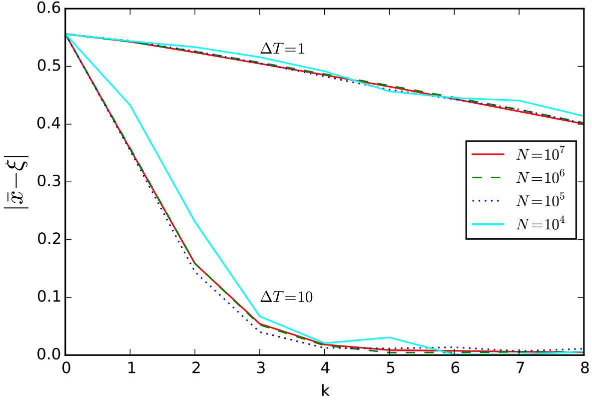
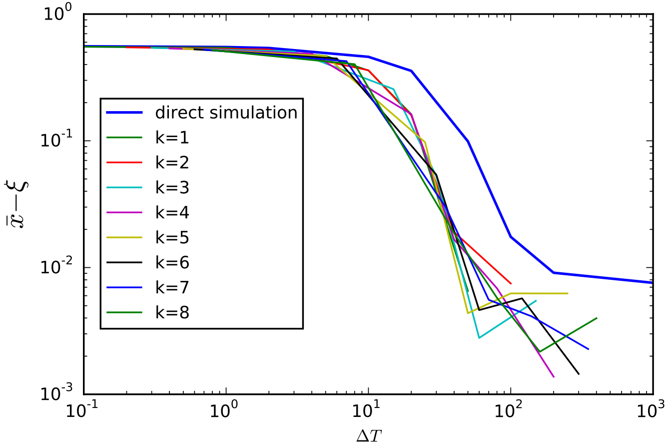
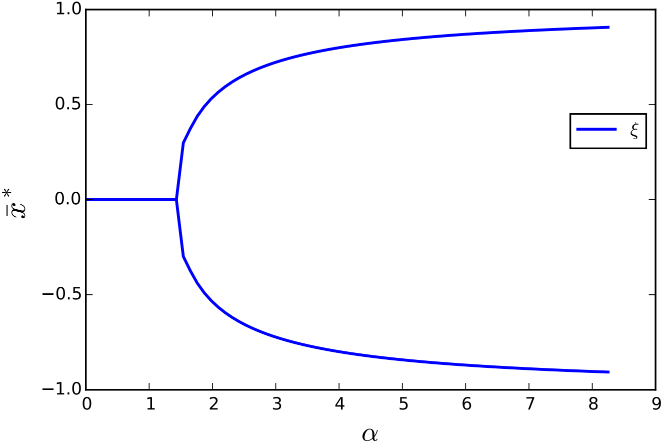


Efficiency compared with direct simulation



Bifurcation diagram



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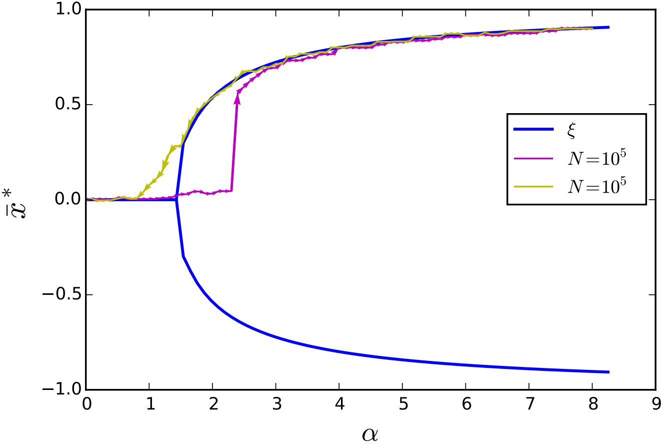

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