1 Newton Convergence

We show some numerical experiments that show how the Newton method converges to the fixed point.

1.1 Maximal accuracy

The accuracy on the Newton-Krylov solution is inevitably limited by the noise on the stochastic coarse-time-stepper, which appears in the evaluation of the residual and in the evaluation of the Jacobian-vector-product. The variance on the coarse-time-stepper is of order $\mathcal{O}(\frac{1}{N})$, as shown in fig. 1. If we use weighted restriction, the variance on the Jacobian-vector-product converges in the same way. Therefore, the accuracy on the Newton-Krylov solution will depend on the number of particles used to calculate the coarse time-stepper. This is confirmed in fig. 2, which shows the convergence of the Newton-residuals. After the Newton-Krylov solution is converged, it stays oscillating around the true solution with a standard deviation depending on the number of particles. Because the solution only depends on the previous density and on the random choices in the coarse time stepper, we can interpret this as a Markov chain. The residual averaged out over the converged Newton steps is a good indication of the best tolerance we can achieve. It converges as $\mathcal{O}(\frac{1}{\sqrt{N}})$. Given the number of particles N, we will use this tolerance as the stopping criterion for the Newton-method.

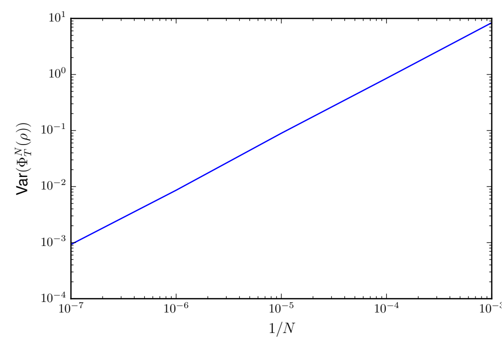

Figure 1: The variance of the coarse time-stepper is of $\mathcal{O}(\frac{1}{N})$

1.2 Continuation

We start from a known solution to the steady-state problem, increment the continuation parameter and solve a new problem using the previous solution as an initial guess. We chose $\Delta\sigma = 0.05$ as continuation step seize. Performing a bifurcation with smaller steps is pointless, because the difference in densities will become too small in comparison with the noise on the residual.

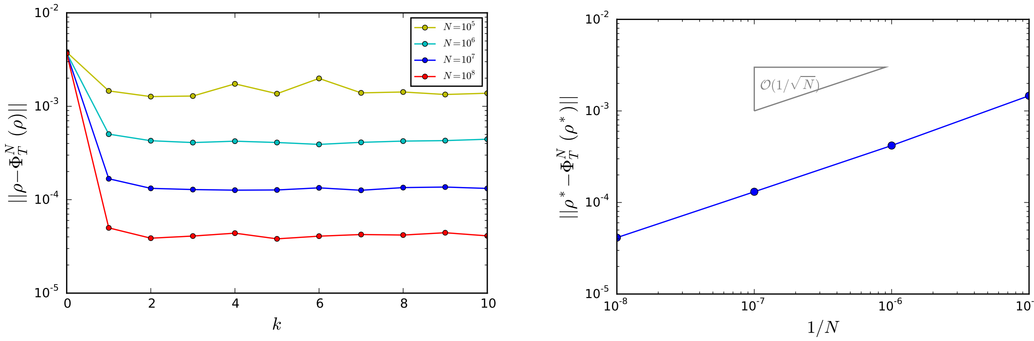

Figure 2: The non-linear solver converges after 1 or 2 Newton iterations, up to a tolerance which depends on the number of particles N (*left*). This best achieved tolerance converges to zero with $\mathcal{O}(\frac{1}{\sqrt{N}})$ (*right*). Parameter values: $\Delta t = 10^{-3}, \Delta T = 10^{-1}, \epsilon_{\texttt{GMRES}} = 10^{-7}$

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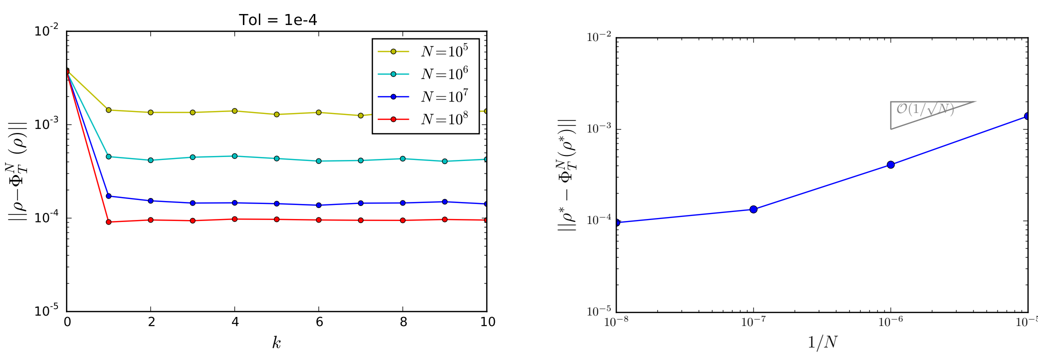

Figure 3: The best achieved tolerance is limited because of the size of the GMRES-tolerance ($\epsilon_{\texttt{GMRES}} = 10^{-4}$)

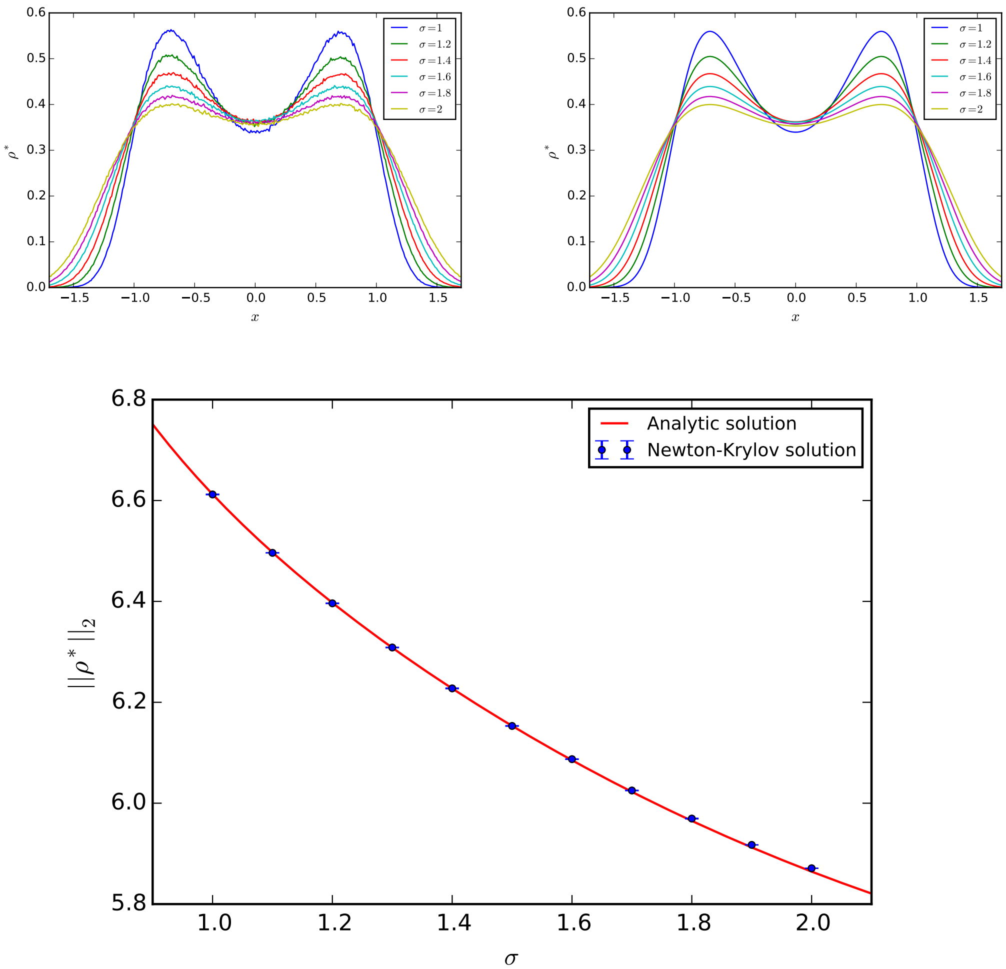

Figure 4: The steady states calculated with the Newton-Krylov-solver for the SDE (*top left*) are compared with the analytical solutions (*top right*). The fixed points are visualized as the 2-norm of the density, and plotted as a function of the continuation parameter σ (*bottom*). The branch of fixed points appears to be in correspondence with the 2-norm of the analytic function $\rho^*(\sigma) = \exp\left[-\frac{2V(x)}{\sigma^2}\right]$. Parameter values: $\Delta\sigma = 0.1$, $N = 10^6$, $M = 10$, $\Delta t = 10^{-3}, \Delta T = 10^{-1}, \Delta x = 10^{-2}, \epsilon_{\texttt{GMRES}} = 10^{-5}, \delta_{\texttt{Newton}} = 45 \cdot 10^{-5}, \varepsilon_J = 10^{-5}$

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