Variance Reduction in Equation-free Newton-Krylov-Methods

July 12, 2016

Fokker-Planck equation

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial(\mu(x)\rho(x,t))}{\partial x} = \frac{\sigma^2}{2}\frac{\partial^2 \rho(x,t)}{\partial x^2}$$

Simulate an ensemble of N particles evolving according to the corresponding SDE

$$\mathrm{d}X = \mu f(X)\mathrm{d}t + \sigma \mathrm{d}W_t.$$

Coarse time stepper

$$\boldsymbol{\rho}(t + n\Delta t) = \boldsymbol{\Phi}_T(\boldsymbol{\rho}) = (\mathcal{R} \circ \mathcal{E}(n\Delta t, \omega) \circ \mathcal{L}(\omega))(\boldsymbol{\rho}(t))$$

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- Evolution $\mathcal{E}$:
  Simulation of the SDE for N particles over n timesteps.

$$\mathbf{X}^{n+1}=\mathbf{X}^n+\mu(\mathbf{X}^n)\Delta t+\sqrt{\sigma\Delta t}\cdot\boldsymbol{\xi}^n$$

with

$$\xi^n(\omega)\sim\mathcal{N}(0,1)$$

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- Restriction $\mathcal{R} : \mathbf{x} \to \boldsymbol{\rho}$

$$\frac{1}{N} \sum_{i=1}^{N} w_i \cdot \chi_{\Delta_j}(X_i) = \rho_j$$

Evaluating the bias on Jacobian-vector products

$$\mathbf{Bias}(\hat{\mathbf{J}}\mathbf{v}, \mathbf{J}\mathbf{v}_{FP}) = \hat{\mathbb{E}}[\hat{\mathbf{J}}\mathbf{v}] - \mathbf{J}\mathbf{v}_{FP}$$

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$$\mathbf{J}\mathbf{v}_{FP} \approx \frac{\boldsymbol{\rho}^{\varepsilon}(t + n\Delta t) - \boldsymbol{\rho}(t + n\Delta t)}{\varepsilon}$$

and $\boldsymbol{\rho}$ calculated by explicitly solving

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial(\mu(x)\rho(x,t))}{\partial x} = D\frac{\partial^2 \rho(x,t)}{\partial x^2}$$

using

$$\rho_i^{n+1} = \rho_i^n + \Delta t\left(\frac{D}{\Delta x^2}\left(\rho_{i+1}^n - 2\rho_i^n + \rho_{i-1}^n\right) - \frac{a(x)}{\Delta x}(\rho_i^n - \rho_{i-1}^n)\right)$$

for the value of ρ at position $x = i\Delta x$ and time $t = (n+1)\Delta t$

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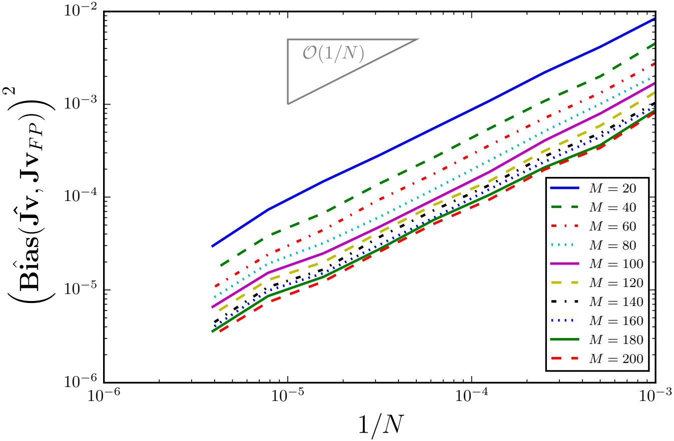

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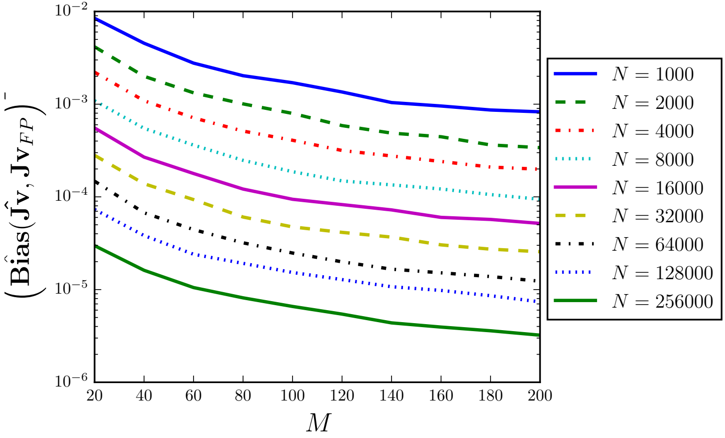

Newton-Krylov solver

Computing steady states $\boldsymbol{\rho}^*$ by solving the non-linear system

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$$\begin{cases} \texttt{Solve} \;\; J(\boldsymbol{\rho}^k)\boldsymbol{\delta_k} = -F(\boldsymbol{\rho}^k) \\ \texttt{Set} \;\; \boldsymbol{\rho}^{k+1} = \boldsymbol{\rho}^k + \boldsymbol{\delta_k} \end{cases}$$

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Problem

$\texttt{Var}(\hat{\mathbf{J}}\mathbf{v}) \sim \mathcal{O}(1/(\varepsilon^2 N)) \Rightarrow$ Numerical noise for $\varepsilon \ll 1$

Variance reduction of Jacobian-vector products

$$\mathbf{J}\mathbf{v} = D(\mathbf{\Phi}_T) \cdot \mathbf{v} \approx \frac{\mathbf{\Phi}_T(\boldsymbol{\rho} + \varepsilon \mathbf{v}, \boldsymbol{\omega}_1) - \mathbf{\Phi}_T(\boldsymbol{\rho}, \boldsymbol{\omega}_2)}{\varepsilon}$$

$$\approx \frac{\mathbf{\Phi}_T(\boldsymbol{\rho}, \boldsymbol{\omega}_1) + \varepsilon D(\mathbf{\Phi}_T)(\boldsymbol{\rho}, \boldsymbol{\omega}_1) \cdot \mathbf{v} - \mathbf{\Phi}_T(\boldsymbol{\rho}, \boldsymbol{\omega}_2)}{\varepsilon}$$

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Solution:
Perturbations on the density $\rightarrow$ perturbations in the weights

$$\frac{1}{N} \sum_{i=1}^{N} w_i \cdot \chi_{\Delta_j}(x_i) = \rho_j$$

$$\frac{1}{N} \sum_{i=1}^{N} w_\varepsilon^i \cdot \chi_{\Delta_j}(x^i) = \rho_j + \varepsilon v_j.$$

For the perturbated density: compute the weight per bin as $w_\varepsilon^j = 1 + \frac{\varepsilon v_j}{\rho_j}$ and assign this value to each particle in Δ_j.

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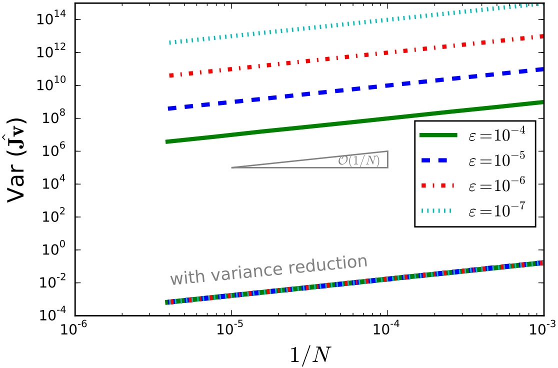
$$\mathtt{Var}(\hat{\mathbf{J}}\mathbf{v}) = \hat{\mathbb{E}}\left[\left(\hat{\mathbf{J}}\mathbf{v} - \hat{\mathbb{E}}[\hat{\mathbf{J}}\mathbf{v}]\right)^2\right] \sim \mathcal{O}(1/N)$$



Mean Field Model

Interaction between components

- Adding *mean field interaction* to model: each particle feels an attractive force towards the mean state (each agent tends to follow the state of the majority)

$$\mathrm{d}X_j = \mu f(X_j)\mathrm{d}t + \sigma \mathrm{d}W_j + \alpha(\bar{X} - X_j)\mathrm{d}t$$

- Interconnectedness between agents can affect the stability of the whole system

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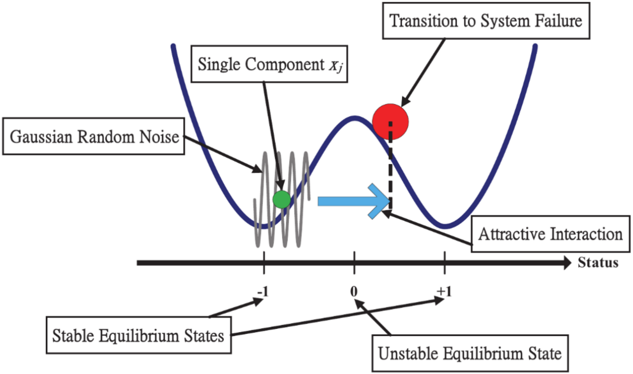
Application: Systemic Risk in Banking Systems

- Banks cooperate. By spreading the risk of credit shocks, they try to minimize their own risk.
- However, this increases the risk that they may all fail

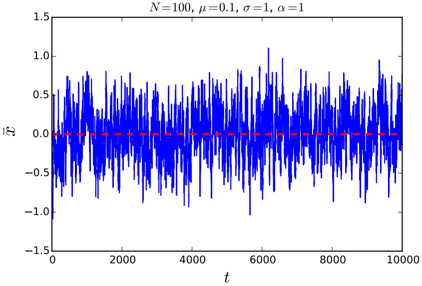
Mathematical Model for Systemic Risk

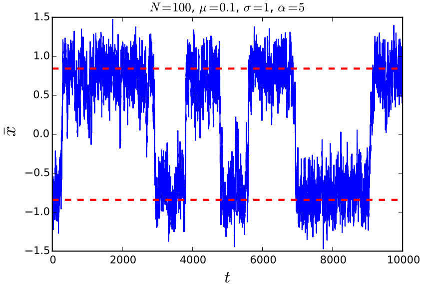
$$\mathrm{d}X_j = \mu f(X_j)\mathrm{d}t + \sigma \mathrm{d}W_j + \alpha(\bar{X} - X_j)\mathrm{d}t$$

- μ The intrinsic stability of each component
- σ The strength of external random perturbations to the system
- α The degree of interconnectedness between agents



Metastable Coarse States





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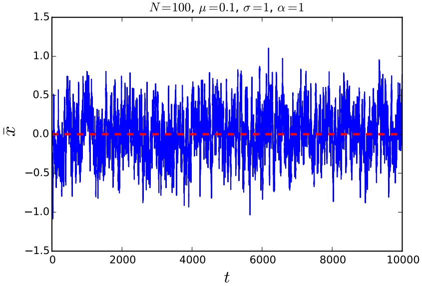

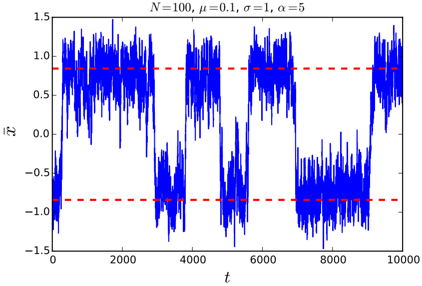

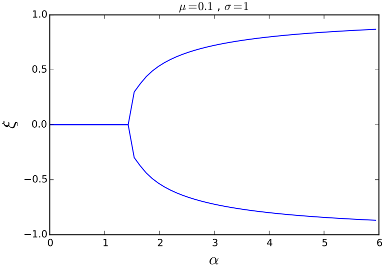

Analytic Solution for Equilibrium Distribution

$$\frac{\partial \rho}{\partial t} = -\mu \frac{\partial (f(x)\rho)}{\partial x} - \alpha \frac{\partial}{\partial x}\left[\left(\int x\rho \mathrm{d}x - x\right)\rho\right] + \frac{\sigma^2}{2}\frac{\partial^2 \rho}{\partial x^2}.$$

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Assuming that $\xi = \lim_{t\to\infty} \int x\rho(x,t)\mathrm{d}x$, an equilibrium solution satisfies

$$-\mu \frac{\partial (f(x)\rho_\xi)}{\partial x} - \alpha \frac{\partial}{\partial x}\left[(\xi - x)\rho_\xi\right] + \frac{\sigma^2}{2}\frac{\partial^2 \rho_\xi}{\partial x^2} = 0$$

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The non-zero solutions $\pm\xi$ are

$$\xi = \pm\sqrt{1 - 3\frac{\sigma^2}{2\alpha}}\left(1 + \ \mu\frac{6}{\sigma^2}\left(\frac{\sigma^2}{2\alpha}\right)^2 \frac{1 - 2\frac{\sigma^2}{2\alpha}}{1 - 3\frac{\sigma^2}{2\alpha}}\right) + \mathcal{O}(\mu^2)$$

Calculate fixed points by applying variance reduced Newton-Krylov-solver

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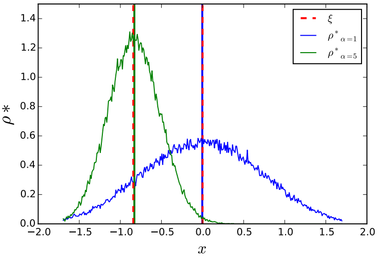
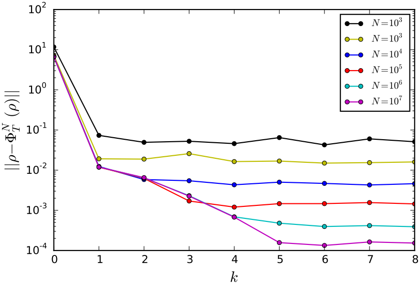
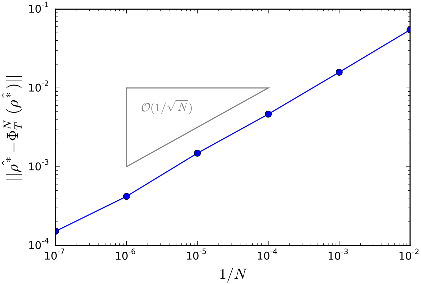
- How to choose the Newton tolerance?
- Which time window to choose for the coarse time stepper $\boldsymbol{\Phi}_T$?



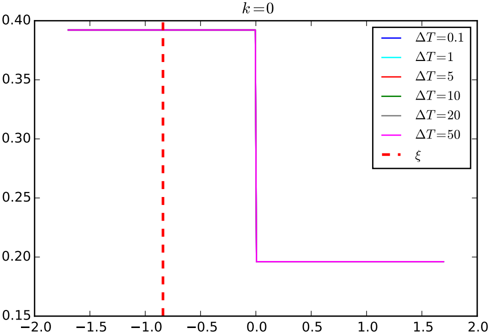
Estimating Stopping Criterion for the Non-linear Solver

- The accuracy on the Newton-Krylov solution is inevitably limited by the noise on the stochastic coarse-time-stepper
- When the Newton-Krylov solution is converged, it stays oscillating around the true solution with a standard deviation depending on the number of particles.





Estimating ΔT



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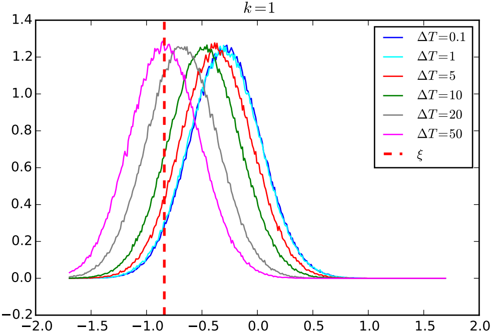

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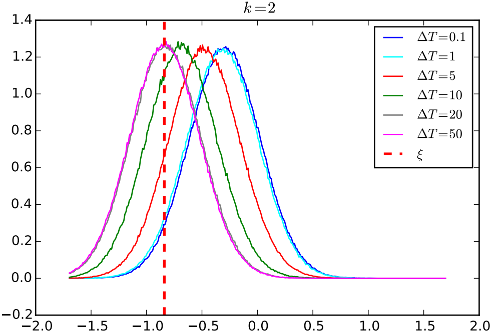

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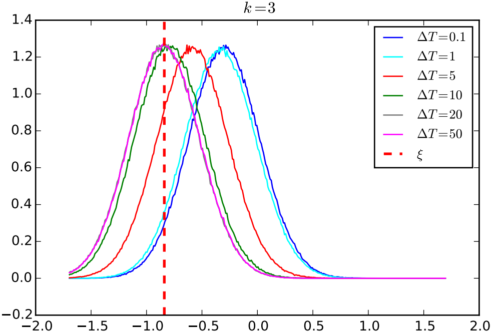

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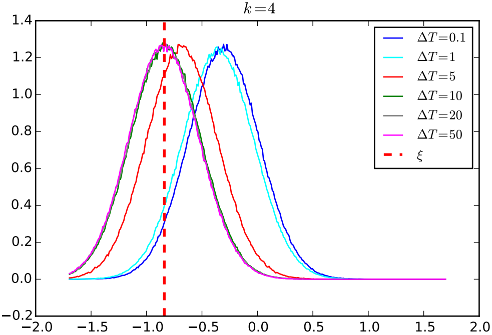

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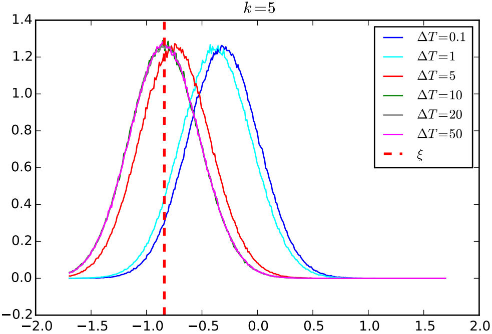

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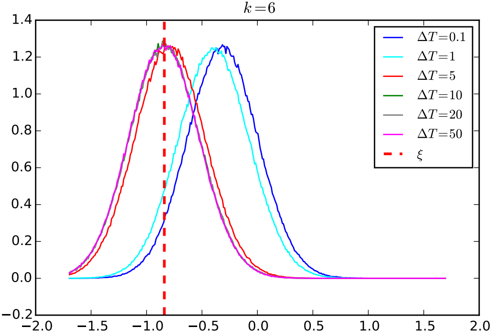

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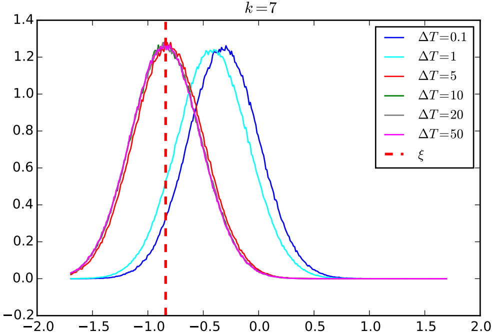

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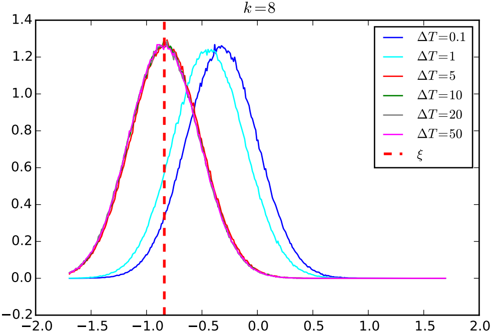

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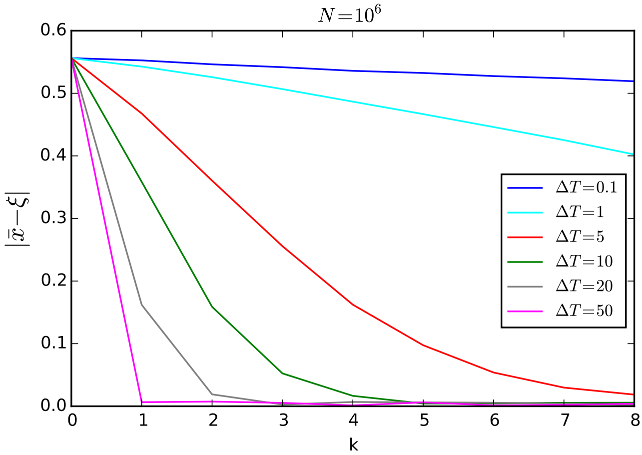

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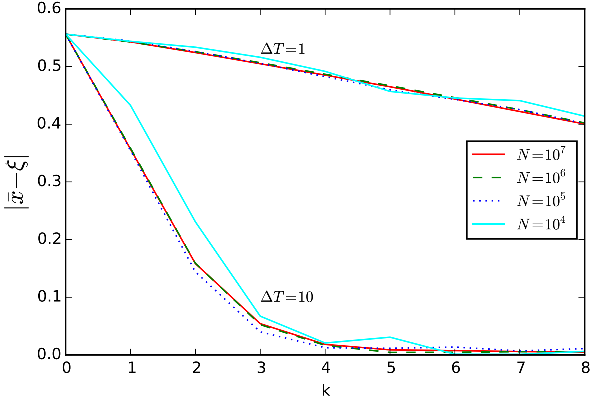
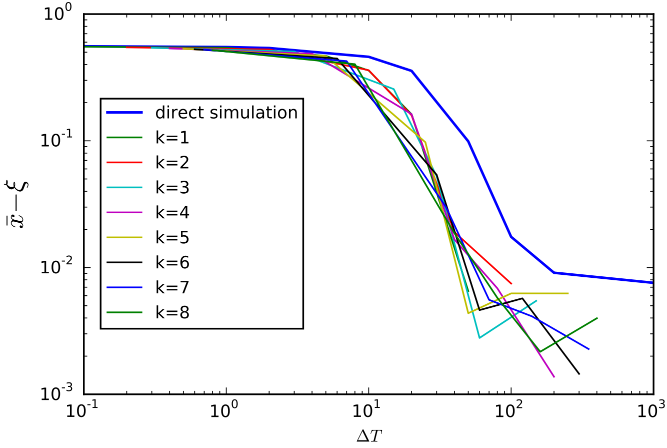
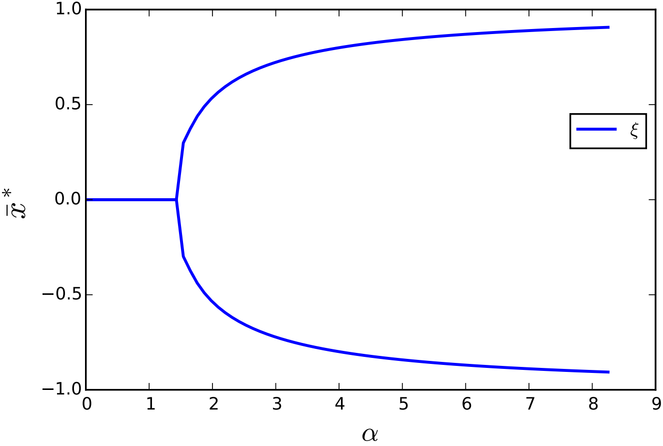


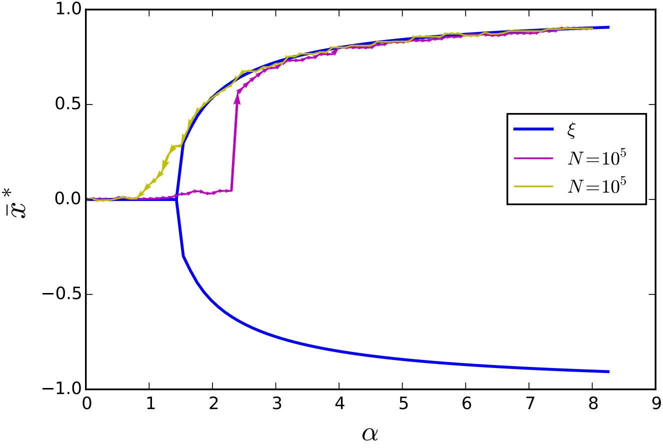
Efficiency compared with direct simulation



Bifurcation diagram



Bifurcation diagram



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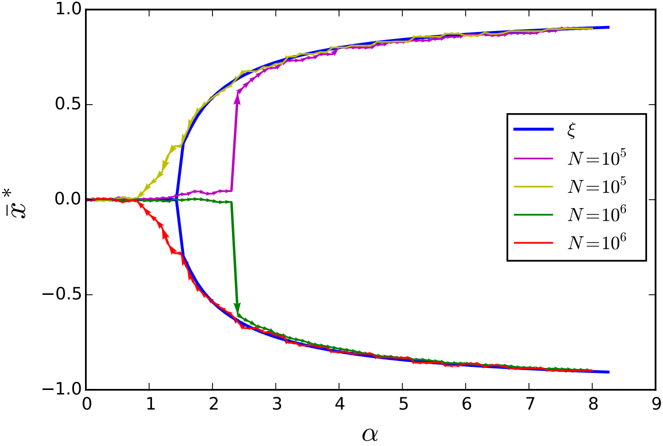

Courses

Contribution to Education

- Exercises Analysis 1
- Exercises Analysis 2

Doctoral Training Programme

- Seminar scientific integrity
- Teacher assistant training
- Science Communication and Outreach

Other courses

- Functional Analysis