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Abstract

We investigate the emergence of coarse equilibrium states in a stochastic system of p system of diffusion processes have a stabilizing force acting on each of them, corresponding to a bistable potential.

We develop a Newton-Krylov method that is able to compute accurately and efficiently coarse fixed points when the underlying fine-scale dynamics is stochastic. The main novelty of the algorithm is in the elimination of the noise that is generated when estimating Jacobian-vector products using time-integration of perturbed initial conditions. We present numerical results that demonstrate

1 Problem

1.1 Model problem

We look for solutions of partial differential equations of advection-diffusion-type

$$\frac{\partial \rho(x,t)}{\partial t} + \mu \frac{\partial (f(x)\rho(x,t))}{\partial x} = \frac{\sigma^2}{2} \frac{\partial^2 \rho(x,t)}{\partial x^2} \tag{1}$$

in two ways:

- By explicitly solving eq. (1) using the discretization scheme[1]

$$\rho_i^{n+1} = \rho_i^n + \Delta t \left(\frac{\sigma^2}{2\Delta x^2} \left(\rho_{i+1}^n - 2\rho_i^n + \rho_{i-1}^n \right) - \mu \frac{f(x)}{\Delta x} (\rho_i^n - \rho_{i-1}^n) \right) \tag{2}$$

  for the value of the probabilty density ρ at position $x = i\Delta x$ and time $t = (n+1)\Delta t$.

- By simulating an ensemble of N particles evolving according to the corresponding stochastic differential equation

$$\mathrm{d}x = \mu f(x)\mathrm{d}t + \sigma \mathrm{d}W_t. \tag{3}$$

  The position x^{n+1} of each particle at time $t = (n+1)\Delta t$ is simulated using the Euler-Maruyama scheme

$$x^{n+1} = x^n + \mu f(x^n)\Delta t + \sigma \sqrt{\Delta t} \cdot \xi^n \tag{4}$$

  with $\xi^n \sim \mathcal{N}(0,1)$

[1] First-order upwind scheme for the advective part combined with the Forward-Time Central-Space-method for the diffusive part

1.2 Discretization

2 Method

2.1 Newton-Krylov solver

If we want to compute steady states for the density $\boldsymbol{\rho}_*$ without direct simulation, we can find them by solving the non-linear system

$$F(\boldsymbol{\rho}_*) = \boldsymbol{\rho}_* - \boldsymbol{\Phi}_T(\boldsymbol{\rho}_*) = 0. \tag{5}$$

To find the steady state $\boldsymbol{\rho}_*$, we apply Newton's method to eq. 5 . Starting from an initial state $\boldsymbol{\rho}^0$, we iterate

$$\begin{cases} \texttt{Solve } J(\boldsymbol{\rho}^n)\boldsymbol{\delta_n} = -F(\boldsymbol{\rho}^n) \\ \texttt{Set } \boldsymbol{\rho}^{n+1} = \boldsymbol{\rho}^n + \boldsymbol{\delta_n} \end{cases} \tag{6}$$

until convergence. $J(\boldsymbol{\rho}^n) = F'(\boldsymbol{\rho}^n)$ denotes the system Jacobian. Each Newton iteration n thus involves evaluating the Jacobian of the timestepper $J(\boldsymbol{\Phi}_T(\boldsymbol{\rho}))$. Since we do not have an explicit formula for $J(\boldsymbol{\Phi}_T)$, we are forced to use an iterative method, such as GMRES, that only requires Jacobian-vector products. [1]. The Jacobian $J(\boldsymbol{\Phi}_T)$ applied to a vector $\mathbf{v}$ (with unit norm) will be estimated by a finite difference approximation

$$\begin{aligned} J(\boldsymbol{\Phi}_T)\cdot\mathbf{v} &\approx \frac{\boldsymbol{\Phi}_T(\boldsymbol{\rho}+\varepsilon\mathbf{v},\boldsymbol{\omega_1}) - \boldsymbol{\Phi}_T(\boldsymbol{\rho},\boldsymbol{\omega_2})}{\varepsilon} \\ &\approx \frac{\boldsymbol{\Phi}_T(\boldsymbol{\rho},\boldsymbol{\omega_1}) + \varepsilon D(\boldsymbol{\Phi}_T)(\boldsymbol{\rho},\boldsymbol{\omega_1})\cdot\mathbf{v} - \boldsymbol{\Phi}_T(\boldsymbol{\rho},\boldsymbol{\omega_2})}{\varepsilon}. \end{aligned} \tag{7}$$

2.2 Variance reduced Jacobian-vector product

3 Analysis

3.1 Variance

3.2 bias

4 Application: Systemic Risk

Until now, we considered a system of non-interacting particles. We will now extend our model to a mean field model by introducing a third parameter α, which is the degree of interaction or cooporation in the system. A simple form of cooperative behaviour is the case where each agent tends to follow the state of the majority (or, each particle feels an attractive force towards the mean state of the system). To include this cooperation effect in our stochastic simulation, we add this mean reversion term to the SDE (3):

$$\mathrm{d}x = \mu V(x)\mathrm{d}t + \sigma\mathrm{d}W_t + \alpha(\bar{x} - x)\mathrm{d}t, \tag{8}$$

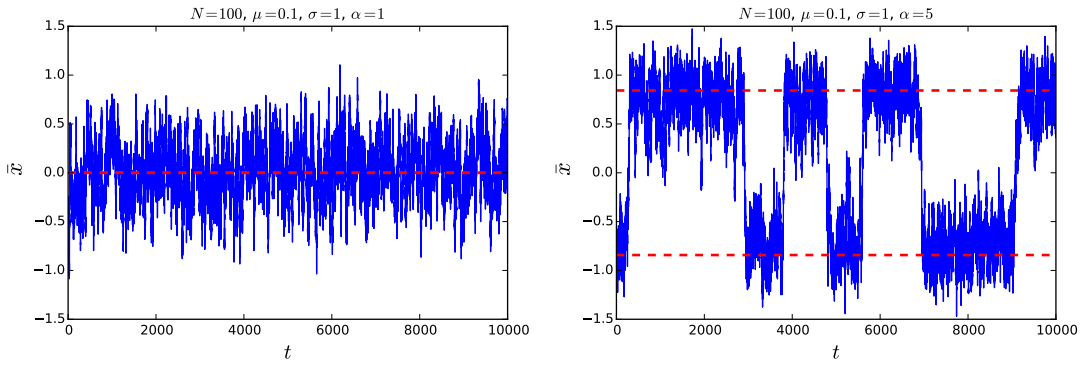

Figure 1: The empirical mean, simulated for different α. *Left*: the system has one single state $\bar{x} = 0$. *Right*: for $\alpha > \alpha_c$ two metastable equilibria emerge. The red dashed lines are approximated analytical solutions for the steady states.

with $\bar{x}(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)$ denoting the empirical mean.

An interesting application of this model to banks and insurance is the emergence of systemic risk. Banks will try to minimize their own individual risk by spreading the risk between each other. However, this may increase the risk that they may all fail: reducing individual risk on a micro-scale can increase systemic risk on a macro-scale. Garnier, Papanicolaou and Yang already made use of the dynamics in eq. (8) to show that interconnectedness between agents indeed affects the stability of the whole system, causing systemic risk [2]. They defined x_i as the state of risk of agent i. The bi-stable-state structure of the potential $V(x)$ ensures that each risk variable stays around -1 (defined as the normal state) or $+1$ (the failed state). A natural measure of systemic risk is then the transition probability of the empirical mean $\bar{x}$ from the normal state to the failed state.

To establish the idea, let us repeat the numerical simulations with eq. 8. The evolution of the system is now characterized by by the initial conditions, the three parameters (μ, σ, α) and by the system size N. Figure 1 illustrates the behavior of the empiral mean $\bar{x}$. The simulations were performed with all agents initially in the normal state. Nevertheless, if randomness dominates the interaction, the agents can move immediately to the other potential well. The system then behaves like N independent difussions, and hence, by the symmetry of the potential, the mean state will be attracted to a single mixed state $\bar{x} = 0$. Upon increasing the interaction parameter α, however, we find two new macroscopic states, suggesting the precence of a pitchfork bifurcation at the macoscopic level. These solutions are no stable steady states, but rather coarse matastable states. Their lifetime is linked to the finite system size.

Table 1: Parameter values

Discretization parameters		SDE
Discretization step	Δx	10^{-2}
Number of discretization steps	n_x	340
Time step	Δt	10^{-2}
Number of timesteps	n	10^6

4.1 Bifurcation study

4.1.1 Calculating fixed points

The steady states are computed with the Newton-Krylov solver described in section 2.1 and compared with the approximated analytical solutions, calculated by Garnier et. al for small h [2].

4.1.2 Continuation

We use a pseudo-arclength contunaution method with secant prediction steps. In fig. **??** we show a bifurcation diagram of the densities.

By adding the mean reversion term, the Fokker-Plack equation (1) describing the evolution of the density, now becomes

$$\frac{\partial \rho(x,t)}{\partial t} = -\mu \frac{\partial (V(x)\rho(x,t))}{\partial x} - \alpha \frac{\partial}{\partial x}\left[\left(\int x\rho(x,t)\mathrm{d}x - x\right)\rho(x,t)\right] + \frac{\sigma^2}{2}\frac{\partial^2 \rho(x,t)}{\partial x^2}. \quad (9)$$

Explicit solutions of eq. are not available in general, but we can find equilibrium solutions. Assuming

References

[1] Peter N. Brown and Youcef Saad. Hybrid krylov methods for nonlinear systems of equations. *SIAM J. Sci. Stat. Comput.*, 11(3):450–481, May 1990.

[2] Josselin Garnier, George Papanicolaou, and Tzu-Wei Yang. Large deviations for a mean field model of systemic risk. Papers 1204.3536, arXiv.org, April 2012.

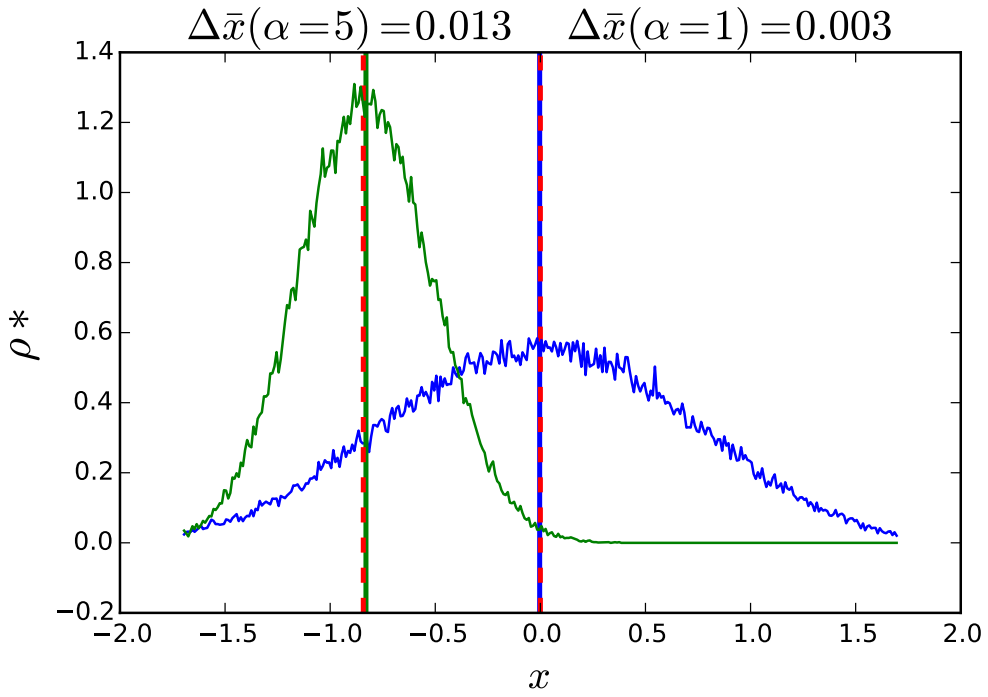

Figure 2

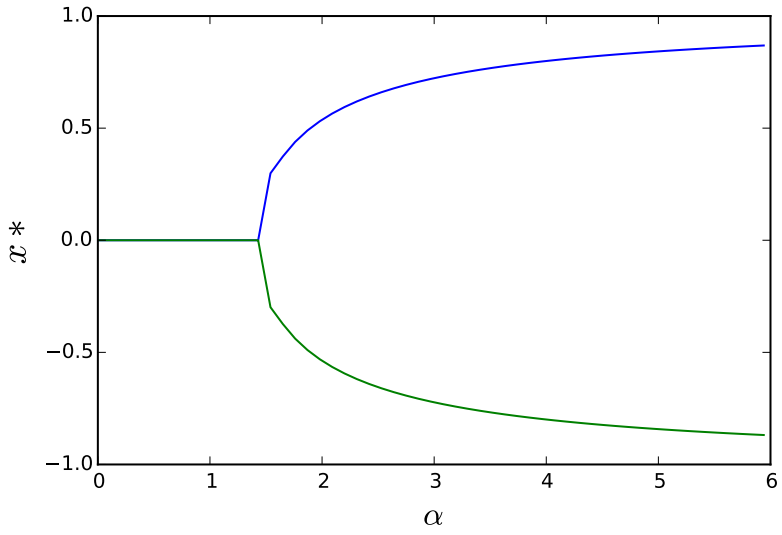

Figure 3