A Volumetric Proof of the Log-Concavity of the Characteristic Polynomial of Matroids

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ToDo

Chapter 1

Introduction

1.Add Citations in Introduction

This thesis will require us to take a tour of mathematics that have been developing for close to a century. The main result synthesizes modern work, from about ten years right up to last year, about a conjecture, posed in seventies, on a mathematical object first formalized in 1935. This threads through work of our friends and mentors, Fields Medal winners, and a host of well-known mathematicians from across the last hundred years. While we think the results alone are quite interesting on their own, much of what has made this project so interesting to us is its broad connections to these various places. We hope, through the more leisurely pace we are allowed to take in a thesis, to show off this side of the math as well.

1.1 What Are We Doing?

The key players in this work are **matroids**, a combinatorial object devised to generalize the notion of "independence". Matroids are interesting for a multitude of reasons, but of note to us is that, although they are combinatorial objects, they can be alternatively studied as geometric objects, known as **Bergman fans**, and algebraic objects called **Chow rings**. In the early 1970's, a conjecture about the **characteristic polynomial** of matroids was posed. The **Heron-Rota-Welsh conjecture** was, in essence a combinatorial question, and would remain unresolved for almost 50 years. It was through viewing the problem from the algebro-geometric side of things that Adiprasito, Huh, and Katz were finally able to prove the conjecture true in 2017. They did this by importing complex machinery from algebraic geometry, known as *Hodge theory*, into the combinatorial world of matroids. It is an impressive work that, in part, won author June Huh a Fields medal.

In this thesis we wish to offer an alternative proof of the Heron-Rota-Welsh conjecture. To do this, we too will tackle the problem from a geometric perspective. In 2021, Nathanson and Ross developed a correspondence between the volume of objects generated from the Bergman fans of matroids, called **normal complexes**, and the evaluation of degree maps on the Chow Ring. This opened the avenue of using the geometric picture of matroids to show characteristics of its algebraic representation. Using the recent work of Nowak, O**2.It feels weird to use my name in the 3rd person?**, and Ross, we show this correspondence and certain volumetric properties of normal complexes is sufficient to prove the Heron-Rota-Welsh conjecture.

1.2 Why Are We Doing This?

Where some would ask why, we much prefer to ask "why not?". More seriously, while *a* proof of the Heron-Rolta-Welsh conjecture is not new, having a new viewpoint on something is valuable even just in comparing it to the original.

This is an exciting starting application of the theory of normal complexes. Compared to combinatorial Hodge theory, normal complexes have a much lower barrier of entry. That they can prove the same, famously difficult, problem is surprising at least. We hope to see some of these techniques and tools expanded and applied elsewhere.

1.3 Who Is This For?

By this point we've already introduced quite a few words we don't expect every reader to know offhand. Our primary goal is that anyone with a few graduate level courses in mathematics under their belt could read this thesis from start to finish and come out with a comprehensive picture of both the setting and the conclusion. To that end we will be providing context for every word in bold appearing above, linking each of them to the overall picture.

However, we also have some secondary goals in terms of readership. First, we want this to be of at least some interest to someone already knowledgeable in the field. While we are confident that any math of real substance in this thesis will be developed elsewhere, if it's going to appear here it might as well at least be useful to a practitioner. Second, and in somewhat of a contradiction, we want this work to be inviting to a curious non-mathematican. We believe there is a good opportunity here to allow a layperson to follow along with math they may not be otherwise usually exposed too.

In the true spirit of compromise then, we expect no one to be totally happy with the pacing. In general, the intention is the complexity of the material will start somewhat low and increase as we go on. But, there will be technical points interjected in otherwise easy material, and we will attempt to include high level overviews even in sections that really do require a solid mathematical background. We say this largely to give the reader permission to skip the bits that simply don't interest them.

Chapter 2

Matroids

The underpinning of all our work are mathematical objects known as matroids. Though, as we've noted, they've been around since the 1930's, they're not, yet, household objects every mathematician knows. This is the shallowest scratch into the world of matroids, slanted heavily towards what's necessary for our problem at hand. There are many full books on matroids, for those curious to dig into more depth. We are partial to the treatment by Oxley's *Matroid Theory* [5].

We will build up to matroids by developing some intuition from more familiar, motivating mathematical objects. Then we will introduce the definition(s) of matroids and introduce the characteristic polynomial. We will wrap up this chapter by stating the Heron-Rota-Welsh conjecture.

2.1 Linear Algebra Done Hastily

When the vague notion of independence is mentioned in a mathematical context, we expect that minds wander to *linear* independence. A central concept to the field of linear algebra, this is likely the vast majority's first introduction to the topic. Happily, this mirrors, closely enough, the initial development of matroids. The patterns that emerge viewing the independence of collections of vectors will, quite directly, inspire the first of our definitions of a matroid.

2.1.1 Linear Independence

First, let us recall the definition of linear independence.

Definition 2.1 (Linear Independence)**.** Given a finite set of vectors $\{v_1, v_2, \ldots, v_k\} \subseteq F^n$, for some field F, the set of vectors is called *linearly independent* if the only solution to the linear combination

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$$

is $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$. Otherwise, we say the set is *linearly dependent.*

While this is a familiar definition to many of us, it will be illustrative to all to take a more concrete example. We'll define the vectors $a = (1, 0, 0), b = (0, 1, 0), c = (0, 0, 1)$, and $d = (1, 1, 0)$. Then we have the set $E = \{a, b, c, d\} \subseteq \mathbb{R}^3$.

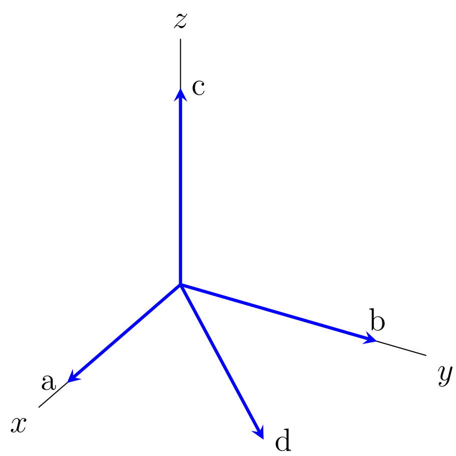


Figure 2.1: The collection of vectors in E

The observant will note that this of course cannot be linearly independent, and indeed we can confirm by showing the linear combination

$$1a + 1b + 0c + (-1)d = 0.$$

But now, a fun little game we could play, at least by our personal reckoning of fun, is to find all subsets of E_V that *are* linearly independent. For example, consider $\{c, d\} \subseteq E_v$.

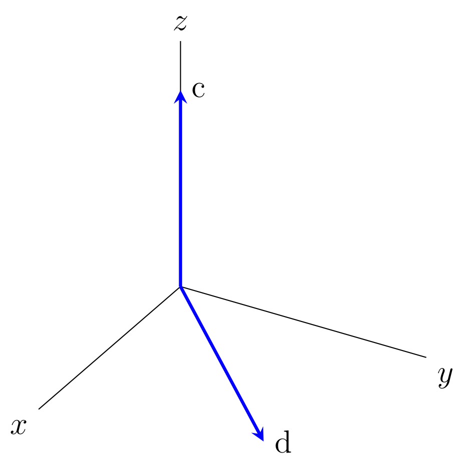


Figure 2.2: A linearly independent subset of E

Take a look to confirm there is no nonzero linear combination of our elements that gives us the 0-vector. Given the relatively small number of elements, it would not take too long to identify every possible subset of E_V that is linearly independent; for the impatient however, they are precisely

$$\big\{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\big\}.$$

For the impatient *and* untrusting, we suggest that the only thing really necessary to check here is that each 3-element set is linearly independent and that there are no other possible 3-element sets in E_V that are linearly independent.

As a point of pure notation, the above list is ugly. We are going to be working with sets of this form so much in this paper that, in order to avoid a shortage of curly brackets, we will introduce a more tidy notation. Going forward, we will write the elements of the internal sets adjacent to each other to represent the set containing them; e.x. we will write the set $\big\{\{a,b\},\{a,b,d\}\big\}$ as $\{ab, abd\}$. Thus, we will more compactly identify the linearly independent subsets of E_V as

$$\{\emptyset, a, b, c, d, ab, ac, ad, bc, bd, cd, abc, acd, bcd\}.$$

Now that we have this collection, this leads to our next totally fun and normal activity. Namely, looking for patterns amongst these independent sets.

2.1.2 Noteworthy Properties of Linearly Independent Subsets

We suspect that those with some knowledge of linear algebra will immediately be ready to note that the largest independent subsets of E_V have 3 elements. And, sure enough, that is true! But this is more a property of the vector space, $\mathbb{R}^3$ in this case, that we're pulling the vectors from than some intrinsic relationship or property of the subsets. We'd like to call attention to some properties that may be less obvious (or so obvious one forgets they're even there). First, something entirely uninteresting.

Property 2.1 (The empty set is an independent subset)**.** For any finite collection of vectors, E, in vector space,

$$\emptyset \subseteq E$$

and $\emptyset$ is linearly independent.

That $\emptyset$ is linearly independent is what we call vacuously true. That is to say, it's true mostly as a quirk of how we define linear independence. Since we can't form a non-zero linear combination that gives the 0-vector, because there are *no* elements at all, it can't be linearly dependent. But then if it's not linearly dependent, it has to be independent. Proof by being pedantic, really the heart of mathematics if one thinks about it. Next, a property that will surprise no one who has taken a linear algebra class, but is worth making explicit.

Property 2.2 (Any subset of a linearly independent set is itself linearly independent)**.** For any linearly independent set of vectors, I, in vector space, if

$$I' \subseteq I,$$

then I' is linearly independent.

Recall we suggested that in order to check that our list of independent subsets of E_V was correct, it was sufficient to just check the subsets with the most elements. This property tells us that if we've figured out the maximal subsets, then filling in the rest is just a matter of taking subsets of those. One may even begin to see the specter of combinatorics lurking. This property falls out easily from our definition. If no non-zero linear combination of vectors in a set gives us the 0-vector, then using fewer vectors isn't going to change that. Finally, we have a more subtle property.

Property 2.3 (The "independence augmentation" property)**.** Let $I = \{v_1, v_2, \ldots, v_m\}$ and $J = \{u_1, u_2, \ldots, u_n\}$ be linearly independent sets in a vector space, such that $m < n$. Then there exists a $k \in [n]$ such that the set

$$I \cup u_k = \{v_1, v_2, \ldots, v_m, u_k\}$$

is linearly independent.

In other words, we can always find an element of a larger independent set to include in a smaller one that will leave the (new, augmented) set independent. Going back to our running example, consider the sets acd and ab. Then $c \in acd$ is such an element, and we confirm that $ab \cup c = abc$ is indeed linearly independent. This property is not immediately obvious, though may be believable to those who have done a proof based linear algebra class.

These are the three properties of linearly independent sets we wish to highlight here. We could use these properties alone motivate the first definition of a matroid. However, we have one more detour before we get to matroids proper. There is another area where independence arises quite naturally, and it will be useful to know going forward.

2.2 Graphic Content

The next place our intuition building journey takes us is the world of graphs. Graph theory was the other motivator of matroids, so we too shall delve in. While we tried to not assume too much, we did, secretly, expect the average reader would feel comfortable enough with linear algebra. Graphs, on the other hand we will quickly build up from scratch and develop a notion of independence. Luckily, this is actually a fairly short process.

2.2.1 What a Graph Is

Not to be confused with the graph of a function or whatever it is business analysts put in shareholder presentations, graphs for us are essentially a collection of points, called vertices, and lines between them, called edges. There are quite a few definitions of graphs, each allowing for slightly different properties, but for our purposes, we can use a rather basic definition.

Definition 2.2 (Graph)**.** A *graph* is a tuple of sets $G = (V, E)$, where V is a set of objects known as vertices and

$$E \subseteq \big\{\{x, y\} \mid x, y \in V\big\}$$

is a set of edges.

A brief aside for our friends who actually care about graphs; The definition here is for an *undirected simple graph permitting loops.* The treatment of graphs and their relation to matroids in this paper extends easily enough for most other graphs because, as far as matroids are concerned, this kind of graph carries pretty much all the information necessary. Indeed, we will see soon enough that even allowing loops is an unnecessary flourish. We again recommend Oxley [5] for the serious graph theorist's entry into matroids.

For the rest of us, this definition may feel rather opaque. Here, an example and corresponding picture should help immensely. Let $V = \{v_1, v_2, v_3, v_4\}$ be a vertex set. Now we must define edges between vertices. For later convenience, we will name these edges. Let $a = \{v_1, v_2\}$, $b = \{v_2, v_3\}$, $c = \{v_3, v_4\}$, $d = \{v_1, v_3\}$, and $e = \{v_4, v_4\}$; then let $E = \{a, b, c, d, e\}$ be our edge set. Recall that, for example, c represents an edge, or connection, between the vertex v_3 and the vertex v_4. With both those pieces, we have the graph $G = (E, V)$. The corresponding picture of our graph is below.

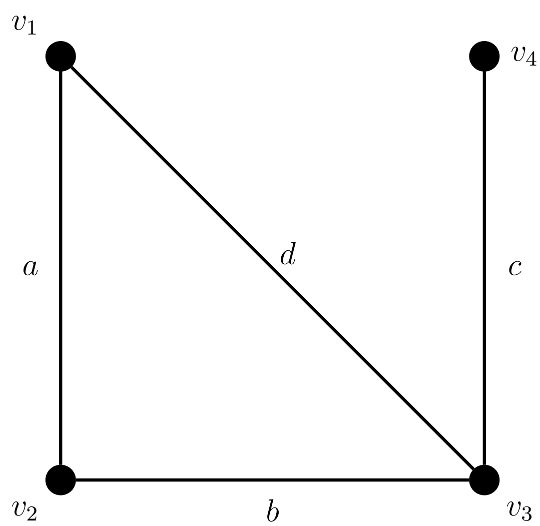


Figure 2.3: Our example graph G

Now that we know what a graph is, it's time to figure out what "independence" could possibly mean.

2.2.2 Independence in the Realm of Graphs

The first thing to note is that we will define independence on the set of edges of a graph; that is, for some graph $G = (V, E)$, an independent set will be some subset $I \subseteq E$, meeting some criteria we'll discuss below. What then would it mean for a set of edges to be independent? Well, if we take some subset of the edges, we restrict which vertices are accessible via those edges. But there might still be redundant edges. Could we remove additional edges from our set and still be able to reach all the same vertices? The answer to that determines if a set of edges is independent, when we can't make our collection of edges any smaller without disconnecting a vertex, or dependent, when we can.

To formalize this we will need to learn a few graph theoretic terms. First, we need the notion of a walk.

Definition 2.3 (Walk)**.** Given a graph $G = (V, E)$, a *walk* is an alternating sequence of vertices and edges

$$(v_1, e_1, v_2, e_2, v_3, \ldots, e_{k-1}, v_k),$$

where each $v_i \in V$, $e_j \in E$ and $v_i \in e_i$ and $v_{i+1} \in e_i$

Intuitively, a walk starts at some vertex and then follows an edge to another, connected vertex then continues to follow edges to vertices until ending at some vertex. If we put our finger on a vertex and trace along edges to another vertex, we've defined a walk. Now that we have a walk, we may define a cycle.

Definition 2.4 (Cycle)**.** A *cycle* is a walk

$$(v_1, e_1, v_2, e_2, v_3, \ldots, e_{k-1}, v_k),$$

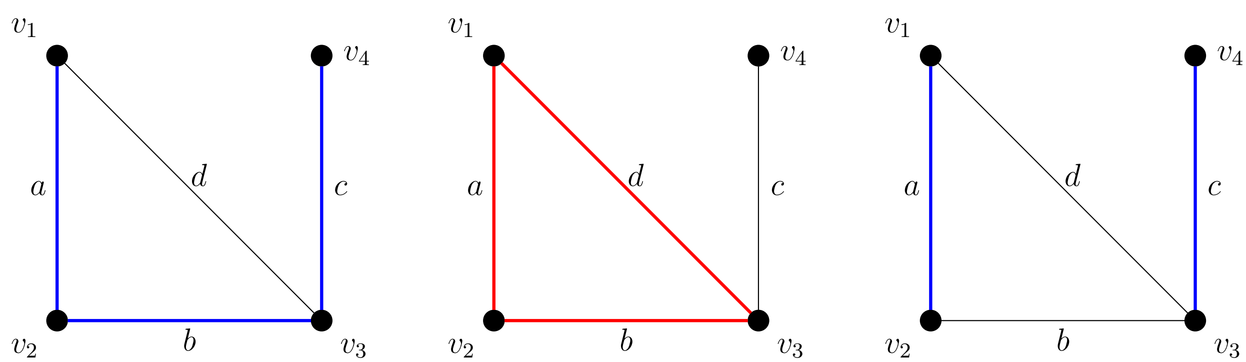
where $v_1 = v_k$ and $v_i \neq v_j$ when $i \neq j$ otherwise.

Further, we say a set of edges *contains a cycle* if any subset of those edges, with the corresponding vertices, is a cycle.

That is, a cycle is a walk that starts and ends at the same place and otherwise passes through unique vertices. Given the notion of independence we began to motivate above, hopefully the utility of defining a cycle is apparent. Any subset of edges that contains a cycle must be dependent, as we can always remove the last edge from the walk and still have all the same vertices connected. With this, our definition of independence can finally be formalized.

Definition 2.5 (Independence (of edges of a graph))**.** Let $G = (V, E)$ be a graph. Then a subset of edges $I \subseteq E$ is *independent* if it does not contain a cycle.

Let us immediately take to our example for this section to consider some possible sets of edges.



(a) Independent: the set of edges $\{a, b, c\}$ does not contain a cycle

(b) Dependent: The edges in $\{a, b, d\}$ contain a cycle

(c) Independent: The edge set $\{a, c\}$ does not contain a cycle. It's not necessary for all vertices to be connected

Figure 2.4: Some examples of independent/dependent sets of edges

Now that we've got some practice under our belt, it's time to play our favorite game again. Given our example graph, $G = (V, E)$, we want to identify the set of all possible independent vectors. A few moments of tracing paths along the graph, hunting for cycles, will reveal that from our set of edges $E = \{a, b, c, d, e\}$, the independent subsets are precisely

$$\{\emptyset, a, b, c, d, ab, ac, ad, bc, bd, cd, abc, acd, bcd\}.$$

This should look familiar! Suspiciously so, even. The independent subsets are the same as those we found in our collection of vectors. Clearly then all the properties of linearly independent subsets we showed above also hold in this example. Indeed, this is not just a quirk of our example. Given any graph, the independent subsets of the edge set will obey the same properties as the linearly independent subsets of a set of vectors. It was the reoccurrence of these properties across different mathematical objects that inspired the creation of matroids.

2.3 Matroids, Finally

Matroids were initially developed by Hassler Whitney in the paper *On the Abstract Properties of Linear Dependence* [10]. The introduction of Whitney's paper parallels our journey so far, covering, much more succinctly, shared properties of linear independence and independence of graph edges. He then goes on to introduce several equivalent definitions of a matroid.

An interesting feature of matroids is just how many definitions exist. Plenty more have been added since the several introduced by Whitney, and any one of these definitions can be taken axiomatically and from them any other definition may be derived. However, it can be extremely non-obvious that a given definition is equivalent to some other. The path between the various axiomatizations can be so difficult to see that they have been affectionately called *cryptomorphic* to one another.

We will primarily be concerned with two axiomatizations, one based on the notion of independent sets and another based on what are called *flats*. The first definition follows closely from the background we've developed so far. This allows us to more easily define the terms and properties of matroids that we will need in the second definition. It is this second definition that will be of key importance for the following chapters, so it is important to develop it here.

2.3.1 Independent Set Axioms

The first definition of matroids should, again, look very familiar.

Definition 2.6 (Matroid — Independent Set Axioms)**.** A *matroid* is a 2-tuple $\mathsf{M} = (E, \mathcal{I})$, where E is a finite set, called the *ground set*, and $\mathcal{I} \subseteq 2^E$ is a collection of subsets of E, called the *independent sets*, with the following properties:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(I3) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| \leq |I_2|$, then there exists some $e \in I_2 \setminus I_1$ such that $I_1 \cup e \in \mathcal{I}$.

They correspond precisely to the properties we identified in linearly independent subsets and that we saw again in independent edge sets. We can take this opportunity to define our now familiar examples as a matroid.

Example 2.1. We let the ground set $E = \{a, b, c, d, e\}$, and the pick the independent sets to be

$$\mathcal{I} = \{\emptyset, a, b, c, d, ab, ac, ad, bc, bd, cd, abc, acd, bcd\}.$$

Coming as probably no surprise, this has the same independence relations as both our vector example and our graph example. We should confirm that $\mathcal{I}$ obeys the properties (I1)-(I3), but we already know that particular set must. We will name this example matroid

$$\mathsf{M} = (E, \mathcal{I}).$$

2.3.1.1 Aside: Representable Matroids

Given that we've already seen the example "matroid" arise twice in other contexts, it is natural to ask if we've gained anything new with matroids. If every matroid could just be studied as a finite collection of vectors and its independent subsets, we don't really have to go thorough the trouble defining a whole new object.

It turns out that this is not the case. A matroid that can arise from a finite set of vectors, like our example, is called *representable*. However, there are *unrepresentable* matroids. A lot of them in fact.

The distinction between representable and unrepresentable matroids has no bearing on the results of this thesis, but it's worth noting here. Our examples are representable, as it allows us to leverage some visual intuition, but everything we say here holds for all matroids.

2.3.2 The Uphill Path to Flats

A benefit of introducing the independence axioms first, we feel, is that they are readily interpretable. At least after developing a bit of intuition in the realm of linear independence. For much of the rest of our paper however, we won't be thinking of matroids in this form. We will need a formulation of matroids that use something called *flats*.

To get to this new definition of matroids, or even state what a flat is, we will have to build up our vocabulary surrounding matroids. Our goal here is to develop everything necessary to define a flat. The path there may seem rather wandering, we will introduce quite a few definitions here. But there are no shortcuts; each new definition builds on the last, until we have a nice tower of terms with which to use.

Given their history, matroids borrow a lot of terminology from linear algebra and graph theory. For the most part, their meaning is related to that in the original context, so it can be a useful starting point. Still, it is not necessary to have heard of them before; these definitions exist perfectly fine on their own in the world of matroids, as we shall see.

We use the independent set axioms to these terms and state properties, but we could have started with any of the axioms and developed all these terms. It's actually quite a fun exercise to develop parallel definitions from different starting axioms.

2.3.2.1 All Your Bases Belong to Matroid

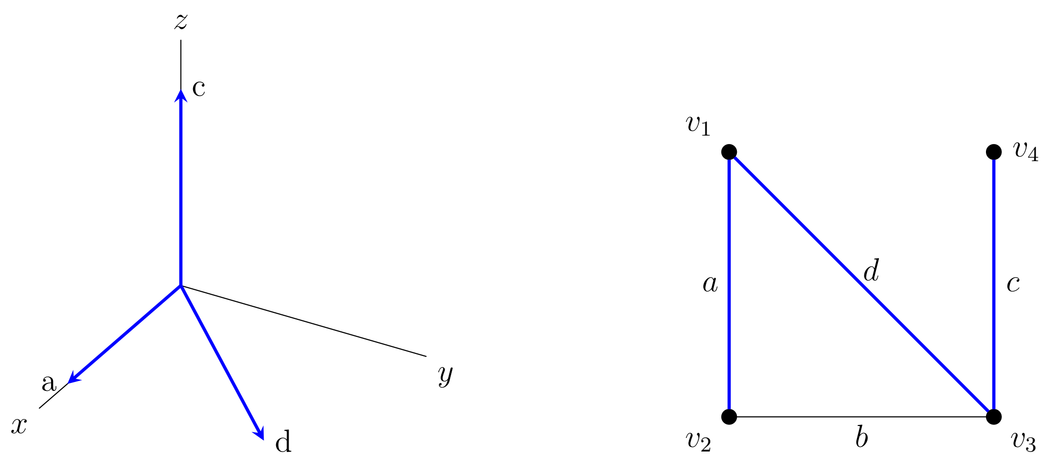
3.change this title

First, we will finally address a pattern we've noted earlier, that the largest independent sets all seem to have the same number of elements, or, as we like to say in the business, the same *cardinality*. To do so we'll introduce the notion of a basis of a matroid.

Definition 2.7 (Basis)**.** Given a matroid $\mathcal{M} = (E, \mathcal{I})$, an independent set $B \in \mathcal{I}$ is a *basis* of M if

$$B \cup e \notin \mathcal{I}$$

for all $e \in E \setminus B$. That is to say, a basis B is a maximally independent subset of E with respect to set inclusion.



(a) A basis in linear algebra is a minimal spanning set (b) A basis of graph is a spanning tree

Figure 2.5: The set $\{acd\}$ is a basis of M, which we can view in the vector and graph setting

For those recalling their linear algebra, yes, this does have the very useful property we expect from something called a basis.

Proposition 2.1. *All bases of a matroid contain the same number of elements.*

Proof. Let B_1 and B_2 be two bases of $\mathcal{M}$. It must be the case that $|B_1| < |B_2|$, $|B_1| > |B_2|$, or $|B_1| = |B_2|$. Let's assume that $|B_1| < |B_2|$. Then since $B_1, B_2 \in \mathcal{I}$, we may use the property (I3) of matroids. There exists some $b \in B_2 \setminus B_1$ such that $B_1 \cup b \in \mathcal{I}$. This is a contradiction with our definition of a basis, since adding any element not already in B_1 should make it dependent.

We have then that $|B_1| \geq |B_2|$, but assuming the case $|B_1| > |B_2|$, we will arrive at a contradiction by the same steps as above. Thus, $|B_1| = |B_2|$, and we conclude that all bases of $\mathcal{M}$ have the same number of elements. $\square$

As we see in the examples in figure 2.5, a basis has a very literal interpretation in the context of vector spaces and graphs. If pressed for an intuition of a basis in the more general matroid setting, we'd say that they give us an idea of "how much" (in)dependence is going on amongst the elements ground set; likely accompanied by us literally waving our hands through the air. If our matroid has 1000 elements in its ground set, but its bases only have size 3, then there must be a lot of dependence amongst all those elements of the ground set. However vague the idea, it would be very useful to be able to quantify "how much" independence is going on in any subset $X \subseteq E$ of a matroids ground set. To get there we'll first need to learn how to make some new matroids.

2.3.2.2 New Matroids From Old

Let's say we already have some matroid $\mathcal{M} = (E, \mathcal{I})$. Then $\mathcal{I}$ already has a notion about which of all possible subsets of E are independent. So if we consider some subset $X \subseteq E$ of the ground set, we should be able to use $\mathcal{M}$'s independent sets to construct independent sets for X as a ground set. This is in fact very easy to do, and we call the resulting matroid a restriction matroid.

Definition 2.8 (Restriction Matroid)**.** Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. Then for any subset $X \subseteq E$, we may define the *restriction matroid*, $\mathcal{M}|X$, as

$$\mathcal{M}|X = (X, \mathcal{I}|X)$$

where $\mathcal{I}|X = \{I \in \mathcal{I} \mid I \subseteq X\}$.

Essentially we just declare X to be the new ground set and just forget about any independent sets of $\mathcal{M}$ that contain any elements not in X.

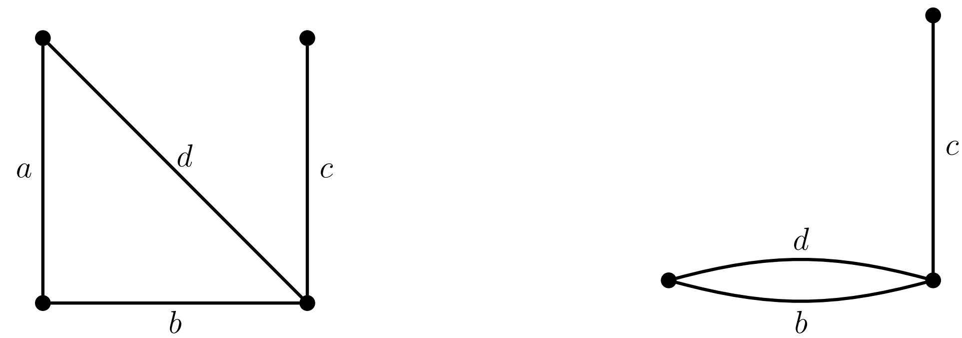
We will need restriction matroids to get to our definition of flats, we would also like to introduce one other way we could make a new matroid from an extant one. These are called contraction matroids, and they are *dual* to restriction matroids. The fact they're dual to restriction matroids isn't really that important here, but as mathematicians we feel compelled to point out duality anytime we see it. While these aren't strictly necessary for flats, it's a surprise tool that will help us later. **4.Should I introduce contraction matroids here? They aren't immediately necessary, but where else would I? 5.I think i need contraction matroids? but if i don't i'll just come back and delete this section**

Definition 2.9 (Contraction Matroids)**.** Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. For any subset $T \subseteq E$ of the ground set, construct the restriction matroid $M|T$. Then let B_T be a basis of $M|T$. The *contraction matroid*, $\mathcal{M}/T$, is defined as

$$\mathcal{M}/T = (E \setminus T, \mathcal{I}/T),$$

where $\mathcal{I}/T = \{I \subseteq (E \setminus T) \mid I \cup B_T \in \mathcal{I}\}$.

This definition is more difficult to explain succinctly, but we can compare it with the restriction matroid to try to get some sense of what this does. The restriction matroid imparts independence on a subset $X \subseteq E$ by saying subsets are independent if they would be independent in the original matroid. The contraction matroid assigns independence on everything *not* in the subset $T \subseteq E$, based on if they'd still be independent if we were to add (a basis of) T back in.



A graph representing a matroid M A graph representing a matroid M'

Figure 2.6: To provide a modicum of explanation as to why these are called *contractions*, we note that M/a and M' are isomorphic

2.3.2.3 Rank and Closure

Before we got sidetracked making new matroids, we were noting that somehow the size of a basis tells us something important about a matroid. Indeed, this is an important enough property to get its own name, the *rank* of a matroid. Better, we can extend the notion of rank to any subset of the ground set of a matroid.

Definition 2.10 (Rank)**.** Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. The *rank function* is the map

$$\begin{aligned} \mathrm{rk}_{\mathcal{M}} \,:\, 2^E &\to \mathbb{Z}_{\geq 0} \\ X &\mapsto |B_X| \end{aligned}$$

where B_X is any basis of the restriction matroid $\mathcal{M}|X$. That is to say the rank of any subset X is the size of the largest independent set contained in X.

Since $\mathcal{M}|E = \mathcal{M}$, we write $\mathrm{rk}_{\mathcal{M}}(E)$ as $\mathrm{rk}_{\mathcal{M}}(\mathcal{M})$, and it is called the *rank* of $\mathcal{M}$.

Unless we are in imminent danger of confusion, we will notate $\mathrm{rk}_{\mathcal{M}}(X)$ as just $\mathrm{rk}(X)$. In the land of linear algebra, the rank corresponds to the dimension spanned by the vectors. Just as adding more vectors into a linear span won't necessarily increase the dimension spanned, increasing the number of your elements in your subset will not necessarily increase the rank. For instance, in our running example we see that $\mathrm{rk}(ab) = \mathrm{rk}(abd) = 2$, and naturally no subset of elements of the matroid will have a rank larger than the basis of the matroid itself.

This notion that we can add more elements to a subset without changing its rank leads, at last, to the final preliminary definition.

Definition 2.11 (Closure)**.** Given a matroid $\mathcal{M} = (E, \mathcal{I})$, the *closure operator* is a function

$$\begin{aligned} \mathrm{cl}_{\mathcal{M}} \,:\, 2^E &\to 2^E \\ X &\mapsto \{e \in E \mid \mathrm{rk}(X \cup e) = \mathrm{rk}(X)\}. \end{aligned}$$

For any $X \subseteq E$, we call $\mathrm{cl}(X)$ the closure of X.

Again we will write the closure operator as cl(X) almost exclusively. If a basis captures how much "independence" is in a set of elements, the closure of a subset generates a set that is as "dependent" as possible for a given rank (using the elements of that initial set). One might ask if there is anything special about these sets that are as big as they can be with respect to closure. A very insightful question, if we do say so ourselves.

2.3.3 Our Flag Means Totally-Ordered Subsets of the Lattice of Flats

6.This title is too pop culture specific, change it

If you didn't notice our subtle hint above, it may come as a surprise that sets that are as "big" or "dependent" as possible for a given rank are precisely flats.

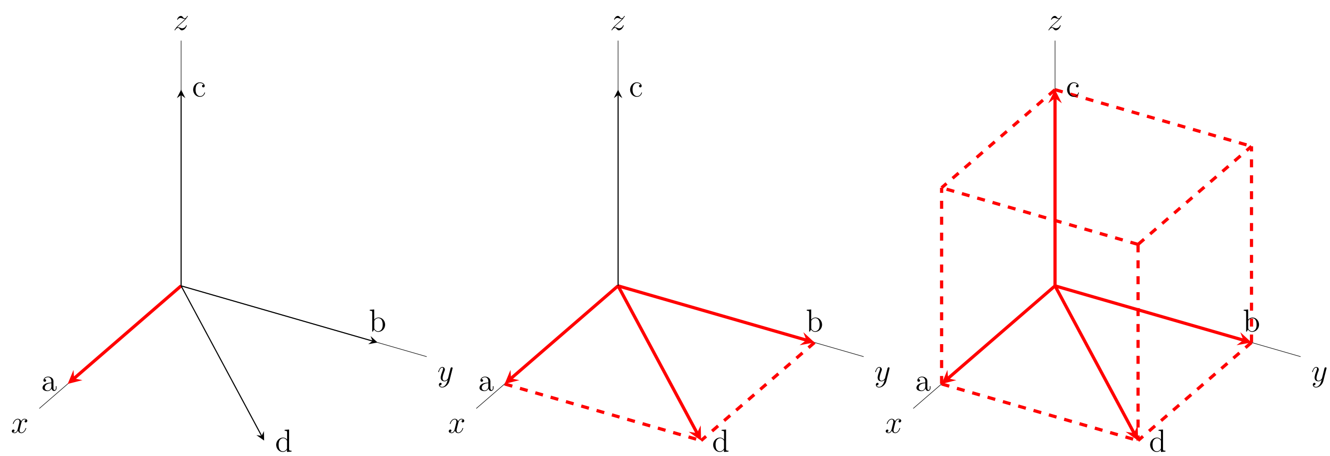
Definition 2.12 (Flat)**.** Given a matroid $\mathcal{M} = (E, \mathcal{I})$ and subset $X \subseteq E$, if

$$X = \text{cl}(X),$$

then X is a *flat* of $\mathcal{M}$.

What if instead of independent sets, we collect all the flats of a matroid. For example, in M, we could start applying the closure opreator left and right until we collect the set

$$\mathcal{F} = \{\emptyset, a, b, c, d, abd, ac, bc, cd, abcd\}.$$



(a) Rank 1: The single element a is a flat

(b) Rank 2: Note ab is not a flat, but abd is

(c) Rank 3: The matroid itself is rank 3, so the flat of this rank is the whole ground set, $abcd$

Figure 2.7: Examples of flats of rank 1, 2, and 3 in our example matroid M, viewed as vectors

Since flats are maximal with respect to rank, they naturally divide up by rank; i.e.

$$\begin{aligned}
\mathcal{F}_0 &= \{\emptyset\} \\
\mathcal{F}_1 &= \{a, b, c, d\} \\
\mathcal{F}_2 &= \{abd, ac, bc, cd\} \\
\mathcal{F}_3 &= \{abcd\},
\end{aligned}$$

where everything in $\mathcal{F}_k$ has rank k. When laid out like this we may begin to note some interesting patterns. Indeed, just like independent sets have some useful properties, so do the set of flats.

Proposition 2.2 (Properties of Flats)**.** *Let $\mathcal{M} = (E, \mathcal{I})$, be a matroid. Then the set*

$$\mathcal{F} = \{X \subseteq E \mid X = \mathrm{cl}(X)\}$$

is the set of flats of $\mathcal{M}$, and $\mathcal{F}$ has the following properties:

(F1) $E \in \mathcal{F}$.

(F2) If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$

(F3) If $F \in \mathcal{F}$ and $F_1, F_2, \ldots, F_k \in \mathcal{F}$ are the minimal flats such that each $F_i \subsetneq F$, then the sets $F_1 \setminus F$, $F_2 \setminus F$, $\ldots$, $F_k \setminus F$ partition $E \setminus F$.

Further, if E' is any ground set and $\mathcal{F}' \subseteq 2^{E'}$ is a collection of subsets of the ground set such that properties (F1)–(F3) hold, then $\mathcal{M}' = (E', \mathcal{F}')$ is a matroid.

Let's unpack this proposition, as flats are a bit more difficult than independent sets as a foundation of matroids. Property (F1) says that the ground set, E, is a flat. This follows directly from the fact that the closure of a basis has to be every element of the ground set, since you can't ever get a higher rank than a basis.

The second property (F2) says that the set of flats in closed under intersection; i.e. the elements shared between any two flats is a flat itself. This follows from the properties of closure and a bit of set theory; it's a fun little exercise to prove.

The last property, (F3), looks more intimidating than it is. In essence, if you take a flat, F (with $F \neq E$, since no flats have higher rank than E), then for every element *not* in F you're going to find it in a flat that is one rank higher. This shouldn't be too surprising, since if an element, let's call it x, is not in F, then $\mathrm{cl}(F \cup x)$ will have to have a higher rank than F. That this *partitions* $E \setminus F$ just means that each $e \in E$ that's not in F is going to appear in exactly one flat one rank higher (specifically the flat $\mathrm{cl}(F \cup e)$).

Finally, the proposition asserts that if we start with a ground set and then a collection of subsets of that ground set that meet all three properties (F1)–(F3), then that is sufficient to characterize a matroid. That is (F1)–(F3) is another axiomatization of a matroid. A recommended exercise would be to reconstruct all the definitions in the preceding section starting with just these axioms, with a goal to reconstruct a notion of an independent set.

These properties actually impart a very interesting structure on the set of flats that we will now explore.

2.3.3.1 The Lattice of Flats

First, we recall, or learn here and now, that any collection of subsets of a set form a partially ordered set.

Definition 2.13. Partially Ordered Set A *partially ordered set*, often called a poset, is a 2-tuple $(P, \preceq)$, where P is a set of elements, and $\preceq$, is a relation between some, but not necessarily all, of the elements of P with the following properties:

i. $a \preceq a$,

ii. if $a \preceq b$ and $b \preceq a$, then $a = b$,

iii. if $a \preceq b$ and $b \preceq c$, then $a \preceq c$,

for all $a, b, c \in P$

With the definition in hand, we can verify that $(\mathcal{F}, \subseteq)$ is a partially ordered set, where $\mathcal{F}$ is the set of all flats of a matroid. But we can do even better than that. Some posets have an even stronger structure, called a lattice.

Definition 2.14 (Lattice)**.** A partially ordered set $(L, \preceq)$ is a *lattice* if there exist binary operations

$$\vee \,:\, L \times L \to L,$$

called a *join*, and

$$\wedge \,:\, L \times L \to L,$$

called a *meet*, such that the following properties hold for all $a, b \in L$:

i. $a \vee b \in L$; $a \preceq a \vee b$ and $b \preceq a \vee b$; for any $c \in L$ such that $a \preceq c$ and $b \preceq c$ then $a \vee b \preceq c$,

ii. $a \wedge b \in L$; $a \wedge b \preceq a$ and $a \wedge b \preceq b$; for any $c \in L$ such that $c \preceq a$ and $c \preceq b$ then $c \preceq a \wedge b$,

If you've never seen this definition before, it can be a bit heavy on symbols, but once we ground it in our set of flats it won't be too bad. First though, we must establish that the set of flats does indeed form a lattice.

Lemma 2.3 (The Collection of Flats Form a Lattice)**.** *Let $\mathcal{M}$ be a matroid and $\mathcal{F}$ be the set of all flats of $\mathcal{M}$. Then $(\mathcal{F}, \subseteq)$ is a lattice, with the operations*

$$F_1 \wedge F_2 = F_1 \cap F_2$$
$$F_1 \vee F_2 = \mathrm{cl}(F_1 \cup F_2)$$

for any $F_1, F_2 \in \mathcal{F}$

Proof. It is sufficient to show that $\mathcal{F}$ is a meet-semilattice and that $\mathcal{F}$ has a maximal element.

To show that F is a meet-semilattice, we must prove that the meet operation is well-defined. Let $F_1, F_2 \in \mathcal{F}$ be flats. Then from property (F2), $F_1 \cap F_2$ is a flat. Naturally, for any $F_3 \in \mathcal{F}$ such that $F_3 \subseteq F_1$ and $F_3 \subseteq F_2$, then $F_3 \subseteq F_1 \cap F_2$. Thus, the meet operation is well-defined.

The maximal element of $\mathcal{F}$ is, trivially, the ground set, E, itself which is a flat by property (F1). A meet-semilattice with a maximal element is a lattice, and so $\mathcal{F}$ forms a lattice. $\square$

Now, finally we can talk about a lattice of flats. To motivate this, let's look at the lattice of our example matroid, M. It is, if not traditional, convenient to structure a lattice graphically in a *Hasse diagram*.

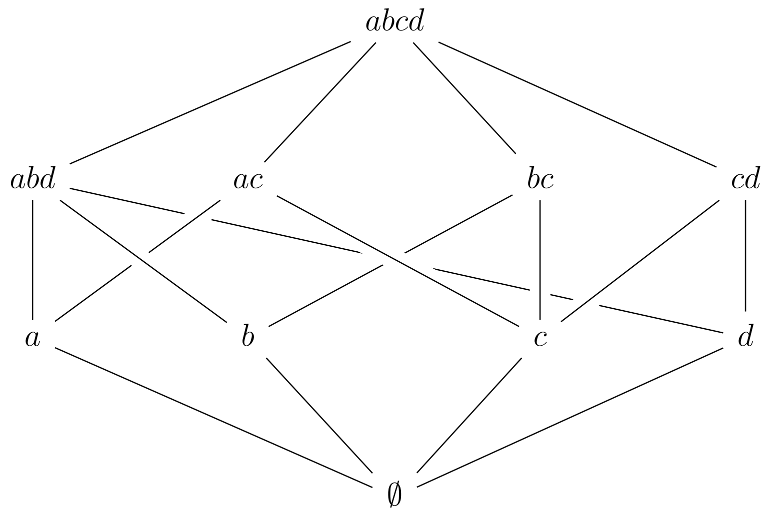


Figure 2.8: The Hasse diagram of flats of our example matroid M

When reading a Hasse diagram, if we have two entries x and y, with a line connecting them and x is higher on the page than y, we say *x covers y*. This corresponds to the relation $y \preceq x$. In the lattice of flats, each level corresponds to a rank, starting at the bottom, which is rank 0. If F_1 covers F_2, then $F_2 \subset F_1$. All of those properties in the definition of the lattice just mean that taking the intersection of two flats, or the closure of the union of two flats, will uniquely identify another element of the lattice (connected by lines to your original two entries).

When considering a matroid in terms of flats, one often sees $\mathcal{M} = (E, \mathcal{L})$ in lieu of $\mathcal{M} = (E, \mathcal{F})$ as a reminder that the set of flats forms a lattice. We will follow that convention going forward as well.

This lattice structure is key to the construction of our objects of interest in the following chapters, as we will soon see. The final definition we need from matroids are called flags, and they are, basically, just reasonable collections of flats.

2.3.3.2 Flags

Given a matroid $\mathcal{M} = (E, \mathcal{L})$, let $\mathcal{L}^*$ be the set of proper flats of $\mathcal{M}$; i.e. all flats with rank greater than 0 and not including E. Since *every* lattice of flats always has 1 element of rank 0 as a minimal element and E as the unique maximal element, $\mathcal{L}^*$ is just the interesting bits of $\mathcal{L}$.

Definition 2.15 (Flag)**.** If $\mathcal{M} = (E, \mathcal{L})$ is a matroid, then a *flag* is a totally ordered subset $\mathscr{F} \subseteq \mathcal{L}^*$ of the proper flats of a matroid,

$$\mathscr{F} = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k\} \subseteq \mathcal{L}^*.$$

If $\mathrm{rk}(\mathcal{M}) = r + 1$, then a flag $\mathscr{F} = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r\}$ is a *maximal* flag of $\mathcal{M}$.

Flags are, then, just collections of flats where you can nest all the flats; a little set theoretic *matryoshka*. On the Hasse diagram, a flag is any collection of entries connected by lines (with the option to skip intermediate flags). Maximal flats will be those that take you along a path on the Hasse diagram from rank 1 all the way up to the rank right below that of the matroid itself, including something from every rank in between. One thing to remember, for every flat F, the single element flag $\mathscr{F} = \{F\}$ is, indeed, a flag.

7.Should i list flags of M?

2.3.3.3 'Tis the Gift to Be Simple

If a serious matroid theorist is, for some inexplicable reason, subjecting themselves to this section, we feel the need to admit one simplifying assumption, pun unintended, we intend to make (and have implicitly made with our example). Since we care primarily about the lattice structure of our matroid, we assume all of our matroids are *simple*.

For the rest of us, the non-serious, a brief explanation. A matroid is simple if it does not have any *loops*, elements in the ground set that have rank 0, or *parallel edges*, sets of elements that share identical (in)dependence relations. If this feels overly restrictive, worry not, for Oxley[5, p. 49] comes to our rescue. **8.get the right page number for oxley**

Proposition 2.4 (Simplification Preserves Lattice Structure)**.** *For any matroid $\mathcal{M}$, there exists a unique, up to labeling, matroid* $\mathrm{si}(\mathcal{M})$*, called the* simplification of $\mathcal{M}$ *such that*

i. $\mathrm{si}(\mathcal{M})$ *is simple,*

ii. if $\mathcal{L}$ is the lattice of flats of $\mathcal{M}$ and $\mathcal{L}'$ is the lattice of flats of $\mathrm{si}(\mathcal{M})$*, then*

$$\mathcal{L} \cong \mathcal{L}'.$$

If we care mostly about the lattice of matroids, then we can take any matroid and find a simple matroid with an identical lattice structure. We'll see the main practical benefit of working with simple matroids in the next section. However, we also get convenience, we don't have to keep track of unnecessary letters, and aesthetics, the lattice diagrams look much nicer, as a bonus. If we take our matroid to be simple, then our lattice structure has the following properties.

Proposition 2.5 (Properties of the Lattice of Simple Matroids)**.** *Let $\mathcal{M} = (E, \mathcal{L})$ be a simple matroid. Then* *i. the empty set is the minimal, rank 0, element of* $\mathcal{L}$*,*

ii. for every $e \in E$*, there is unique rank 1 flat,* F_e*, such that* $F_e = e$*,*

iii. for any flat $F \in \mathcal{L}$*, if* $e \in F$*, then* $F_e \in F$*,*

iv. for any flat $F \in \mathcal{L}$*,* $F = \biguplus_{e \in F} F_e$*.* **9.Do we need the last two properties? If not, maybe don't present them if i'm not going to prove them**

If this seems like a lot, the big takeaway is that this promises that the very bottom of our lattice will always be the empty set, and that the rank 1 flats correspond to the elements of the ground set. For those coming in with lattice knowledge, the second two properties mean the lattice of a simple matroid is *atomic*. We can verify these properties in our example, M, which is a simple matroid.

That admission of simplification done, we have now learned everything we need about the construction of matroids. It's time to learn about some polynomials.

2.4 The Characteristic Polynomial

The conjecture by Heron, Rota, and Welsh, that we promise we are getting to, has to deal with the characteristic polynomial of a matriod. This is some polynomial we can cook up using the structure of a matroid, which is fair enough. But when presented on its own, it feels, at least to us, that it comes out of nowhere. Why anyone would make up this polynomial or why we'd start conjecturing about it can is not at all clear.

So first, a little history back in the realm of graphs.

2.4.1 Coloring Graphs and the Chromatic Polynomial

Let us play another game. This time, pick a graph, G, like the one pictured below.

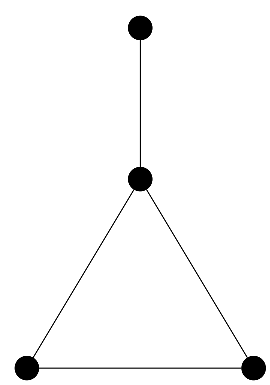


Figure 2.9: An example graph, G

Let's say we have three colors, and we want to color the vertices of the graph so that no two connected vertices have the same color. Such an arrangement of colors would be called a 3-coloring of G. It's not too hard to come up with some colors that work.

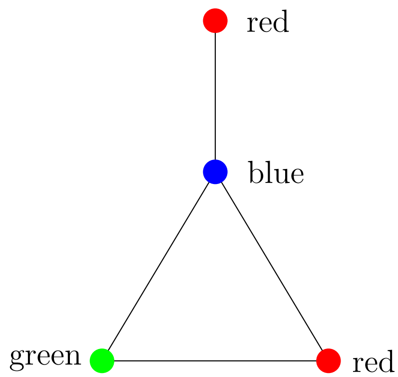


Figure 2.10: A 3-coloring of G

But now, suppose we wanted to know how many unique ways we could use those three colors to color the graph. This isn't too bad. We could just get out our markers and start coloring lots of graphs. Honestly, it sounds relaxing.

But now let's suppose we want to know how many ways we can use 1000 colors to color our little graph, or 10,000, or a billion. Since our set of markers only has 12 distinct colors, we will have to turn to math to solve this one, sadly.

The strategy is not too complicated, just pick a vertex and say how many colors we have to choose from, then find a connected vertex that hasn't been assigned a color yet, and say how many colors it is allowed to choose from. Repeat until we're out of vertices to label. Instead of picking a specific number, let's say we have n colors to choose from.

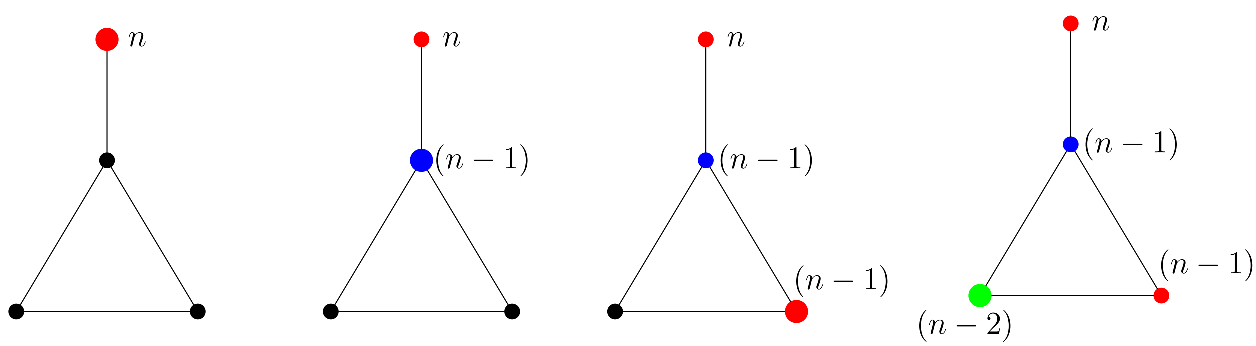


Figure 2.11: The process of figuring out the number of n-colorings of G; the choice of starting vertex doesn't matter, though that's not necessarily obvious

We then just have to multiply the number of possibilities together. For G, if we have n colors to choose from, there are $n(n-1)^2(n-2)$ different ways to arrange those colors on the graph. We've just discovered the *chromatic polynomial* of G. But there's nothing particularly special about our choice of graph, we will get some, we will get something like this for any graph we come up with.

Proposition 2.6 (Chromatic Polynomial of a Graph). *Let $\chi_G(n)$ be the number of n-colorings of graph G. Then the map*

$$z \mapsto \chi_G(z)$$

is a polynomial with integer coefficients, known as the chromatic polynomial *of G.*

For our purposes we will always expand our polynomials, so for our worked example above we have

$$\chi_G(z) = z^4 - 4z^3 + 5z^2 - 2z.$$

Early work in chromatic polynomials was done by none other than our good friend Whitney [11], and expanded on by the mathematician Tutte in his development of what we now call Tutte polynomials [8].

2.4.2 The Characteristic Polynomial of a Matroid

It was following in this work on chromatic polynomials that Gian-Carlo Rota, who you may recognize as usually sandwiched between Heron and Welsh, extended this concept to matroids [6]. To do this, Rota extended something called the *Möbius function* to lattices (technically any locally finite poset), which for matroids means it uses the lattice structure of the flats. We present an equivalent definition that's easier to state, but with the downside that the relationship to the lattice is obfuscated.

Definition 2.16 (Characteristic Polynomial)**.** Let $\mathcal{M} = (E, \mathcal{L})$ be a matroid. Then the *characteristic polynomial* of $\mathcal{M}$ is given by

$$\chi_{\mathcal{M}}(z) = \sum_{X \subseteq E} (-1)^{|X|} z^{\mathrm{rk}(\mathcal{M}) - \mathrm{rk}(X)}.$$

You may notice that each term of the polynomial will have a power of z between 0 and $\mathrm{rk}(\mathcal{M})$. The Heron-Rota-Welsh conjecture is about the coefficients of this polynomial, specifically once we collect the terms.

Definition 2.17 (Whitney Numbers of the Fisrt Kind)**.** Let $\mathcal{M}$ be a matroid with characteristic polynomial

$$\begin{aligned}\chi_{\mathcal{M}}(z) &= \sum_{X \subseteq E} (-1)^{|X|} z^{\mathrm{rk}(\mathcal{M}) - \mathrm{rk}(X)} \\ &= \sum_{k=0}^{\mathrm{rk}(\mathcal{M})} w_k z^{\mathrm{rk}(\mathcal{M}) - k}.\end{aligned}$$

The coefficients $w_0, w_1, \ldots, w_{\mathrm{rk}(\mathcal{M})}$ are the *Whitney numbers of the first kind.*

We will return to these numbers very soon. Before that, a few interesting facts about the characteristic polynomial.

Proposition 2.7. *Let $\mathcal{M}$ be a matroid with a loop; that is some element $e \in E$ such that* $\mathrm{rk}(e) = 0$*. Then*

$$\chi_{\mathcal{M}}(z) = 0.$$

Getting to ignore these trivial characteristic polynomials is the concrete benefit for assuming we only deal in simple matroids. A loop makes the characteristic polynomial easy to calculate, just not at all interesting

Finally, we want to wrap up the graph connection. Since any graph can be represented by a matroid, and the characteristic polynomial is in some sense inspired by the chromatic polynomial, it would be natural to ask if there is a relation between them. And there is, in fact, a very nice one.

Proposition 2.8. *Let G be a graph and $\mathcal{M}(G)$ be the matroid that comes from G. Then*

$$\chi_G(z) = z^c \chi_{\mathcal{M}(G)}(z),$$

where c is the number of connected components of G.

For those who want more on the connections between these values, and how they relate to the more general Tutte polynomial, we found the overview given by Ardila [2] to be a great help.

2.5 The Heron-Rota-Welsh Conjecture

We now have all the knowledge of matroids necessary to state the Heron-Rota-Welsh conjecture. Developed and formalized by Heron [3], Rota [7], and Welsh [9], this was a conjecture about the coefficients of the characteristic polynomial of matroids. We say "was" because, as noted in the introduction, this has proven by Adiprasito, Huh, and Katz [1]. We're going to keep calling it a conjecture though.

First, a few definitions necessary to carefully state the conjecture.

Definition 2.18 (Unimodal)**.** A sequence of numbers $x_0, x_1, \dots, x_k$ is called *unimodal* if there exists an index i such that

$$x_0 \leq x_1 \leq \cdots \leq x_i \geq \cdots \geq x_{k-1} \cdots x_k.$$

The values of a unimodal sequence get larger until a certain point, and after they start to decrease. Such a sequence is also known as *concave*, since the average of any two non-consecutive points in the sequence will be less than a point in between them; i.e. for a sequence $x_0, x_1, \dots, x_k$ and $i < j < k$,

$$2x_j \geq x_i + x_k.$$

We can define an even stronger condition.

Definition 2.19 (Log-Concavity)**.** A sequence of numbers $x_0, x_1, \dots, x_k$ is called *logarithmically concave*, or log-concave, if

$$x_i^2 \geq x_{i-1}x_{i+1}$$

for $0 < i < n$.

When all x_i are positive, log-concavity implies the sequence is also unimodal. This is the last piece of the puzzle.

Theorem 2.9 (Heron-Rota-Welsh Conjecture)**.** *Assume we have a matroid $\mathcal{M} = (E, \mathcal{L})$ of rank $r+1$. If*

$$\chi_{\mathcal{M}}(z) = \sum_{k=0}^{\mathrm{rk}(\mathcal{M})} w_k z^{\mathrm{rk}(\mathcal{M})-k}$$

is the characteristic polynomial of $\mathcal{M}$, where $w_0, w_1, \ldots, w_r$ are the Whitney numbers of the first kind, then

$$w_i^2 \geq w_{i-1} w_{i+1}$$

for $0 < i < \mathrm{rk}(M)$.

That is, the coefficients of the characteristic polynomial of the matroid $\mathcal{M}$ are log-concave.

Since we want to show something about the characteristic polynomials of the matroid, we need a way to study it. To do so, we are going to find the characteristic polynomial in some unexpected places, and then leverage properties of those other settings.

Chapter 3

Chow Rings

It is time now to delve into the world of algebra, developing the notion of a Chow ring of a matroid. Much of section will be presenting results of Adiprasito-Huh-Katz that establish the link between the Chow ring and the characteristic polynomial.

This is going to be a section that will be challenging to parse without some background in abstract algebra. Even with a basic background, Chow rings are somewhat specialized, a development of intersection theory in algebraic geometry. Not something everyone has seen for sure. Luckily, we can exploit the structure of matroids to define a Chow ring without having to go the long way through intersection theory.

3.1 Defining a Chow Ring

A Chow ring is "a generalization of cohomology for alegbraic geometry" **10.Introduce the Chow Ring of a matroid. From lattice of flats to quotient ring 11.Use our example matroid and construct its Chow Ring**

3.1.1 Properties of Chow Rings

12.Use words like *homogeneous polynomial*, *graded ring*, etc. . .

3.1.2 The Degree Map

13.How do I explain this? I guess I can at least say it's linear and sends terms of full degree to 1. Maybe I'll understand it this time around

3.2 Relationship with the Characteristic Polynomial

14.Come up with a nice way of relating the reduced characteristic polynomial with our ring (and therefore fan) 15.Define α and β. Here or in a subsection? Or should it be up when we introduce the ring itself?

Chapter 4

Bringing It All Together

16.I'm not sure what context we'll already have from the last chapter. probably most basic things will be covered. I'm just going to put weird filler text between theorems for now

4.1 An Important Theorem

The authors work with Lauren Nowak and Dustin Ross has lead to the following theorem:

Theorem 4.1 (Nowak-O-Ross). **17.restate this theorem better** **18.I should add the weight function if i'm going to state this for a general fan** *Let* $\mathbf{\Sigma}$ *be a simplicial* **19.tropical?** *fan, pure of dimension* r*, and* $* \in \mathrm{Inn}(N_{\mathbb{R}})$ *an inner product. Then* $(\mathbf{\Sigma}_{\mathcal{M}}, *)$ *is AF if*

i. for every cone $\sigma \in \mathbf{\Sigma}(k)$*, with* $k \leq r-2$*,*

$$\mathrm{star}(\tau, \mathbf{\Sigma}_{\mathcal{M}}) \setminus \{0\}$$

is connected,

ii. for each 2-dimensional face of the associated normal complex, $C(\mathbf{\Sigma}, z)$*, for some cubical* $z \in \mathrm{Cub}(\mathbf{\Sigma}, *)$*, the quadratic form associated to the volume polynomial has exactly one positive eigenvalue.*

This is the key theorem that will allow us to pull everything together to prove log-concavity of characteristic polynomials.

4.2 Assorted Lemmas

To prove the above theorem holds the Bergman fan of *any* matroid, we will need a few helping facts. Some of these are classic results and some are classic-ish, with a short proof provided. First a classical result from linear algebra. **20.actually, maybe i should introduce this just before I try to prove property ii? This feels a little shoehorned in here.**

Lemma 4.2 (Sylvester's Law of Inertia)**.** *Two symmetric square matrices, A and B, of the same size have the same number of positive, negative and zero eigenvalues if and only if they are congruent; that is if*

$$B = SAS^T$$

where S is non-singular.

Hopefully the utility of this is fairly clear. If we're going to be hunting eigenvalues, matrices are somewhere near. This lemma means we won't lose information about the sign of eigenvalues if we manipulate the matrix, just so long as we do it in an invertible way.

4.2.1 Surprise Auxiliary Matroid Theory

We lied about having all the information on matroids necessary after the first chapter, apologies. We need a little more to get us to the end. First, we need a way to chop matroids up into smaller bits.

4.2.1.1 New Matroids From Old

Let's say we already have some matroid $\mathcal{M} = (E, \mathcal{I})$. Then $\mathcal{I}$ already has a notion about which of all possible subsets of E are independent. So if we consider some subset $X \subseteq E$ of the ground set, we should be able to use $\mathcal{M}$'s independent sets to construct independent sets for X as a ground set. This is in fact very easy to do, and we call the resulting matroid a restriction matroid.

Definition 4.1 (Restriction Matroid)**.** Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. Then for any subset $X \subseteq E$, we may define the *restriction matroid*, $\mathcal{M}|X$, as

$$\mathcal{M}|X = (X, \mathcal{I}|X)$$

where $\mathcal{I}|X = \{I \in \mathcal{I} \mid I \subseteq X\}$.

Essentially we just declare X to be the new ground set and just forget about any independent sets of $\mathcal{M}$ that contain any elements not in X. Rather than providing a subset to restrict to, one often finds it useful to specify just the things we want to forget.

Definition 4.2 (Deletion Matroid)**.** Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and $Y \subseteq E$. The matroid that results from the *deletion* of Y from $\mathcal{M}$, sometimes called a *deletion matroid*, is defined as

$$\mathcal{M} \backslash Y = \mathcal{M}|(E \setminus Y)$$

Clearly if $X = (E \setminus Y)$, then $\mathcal{M}|X = \mathcal{M}\backslash Y$. The choice of deletion or restriction is just a matter of what one wants to emphasize, what we keep or what we remove.

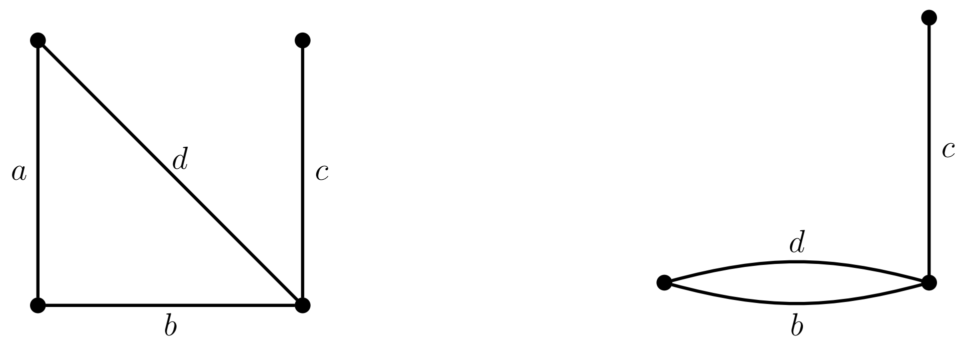
The other way to build a matroid out of an existing one is a little less obvious. These are called contraction matroids, and they are *dual* to restriction matroids. While they're a bit easier to define using duality, we want to avoid introducing all the machinery for that. Still, as mathematicians we feel compelled to point out duality anytime we see it.

Definition 4.3 (Contraction Matroids)**.** Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. For any subset $T \subseteq E$ of the ground set, construct the restriction matroid $M|T$. Then let B_T be a basis of $M|T$. The *contraction matroid*, $\mathcal{M}/T$, is defined as

$$\mathcal{M}/T = (E \setminus T, \mathcal{I}/T),$$

where $\mathcal{I}/T = \{I \subseteq (E \setminus T) \mid I \cup B_T \in \mathcal{I}\}$.

This definition is more difficult to explain succinctly, but we can compare it with the restriction matroid to try to get some sense of what this does. We can think of restriction matroid as imparting independence on a subset $X \subseteq E$ by saying subsets are independent if they would be independent in the original matroid. The contraction matroid, then, assigns independence on everything *not* in the subset $T \subseteq E$, based on if they'd still be independent if we were to add (a basis of) T back in.



A graph representing a matroid M A graph representing a matroid M'

Figure 4.1: To provide a modicum of explanation as to why these are called *contractions*, we note that M/a and M' are isomorphic

Importantly, we can combine deletion and contraction, and indeed the resulting matroids are a rather central point of study in matroid theory.

Definition 4.4 (Matroid Minor)**.** A *minor* of a matroid $\mathcal{M}$ is any matroid resulting in any combination of deletions and contractions of $\mathcal{M}$.

Further, any series of deletions and contractions can always be rearranged to, and so any matroid minor is of the form,

$$\mathcal{M} \backslash X / Y,$$

where $X, Y \subseteq E$ are disjoint and possibly empty.

When $X \cap Y$ is nonempty, we call $\mathcal{M} \backslash X/Y$ a *proper minor* of $\mathcal{M}$.

4.2.1.2 Matroid Minors and Flats

The careful reader may have noted that the definitions of restriction and contraction matroids are given in terms of independent sets, but we've clearly established we are all about flats here. Luckily, we have a very useful property relating the lattice of minors and to the lattice of the original matroid. First though, a little notation. If F is a flat of $\mathcal{M}$, we will define

$$\mathcal{M}_{[\emptyset, F]} = \mathcal{M}|F$$

to be the restriction by F, and

$$\mathcal{M}_{[F,E]} = \mathcal{M}/F$$

to be the contraction by F. For any two flats F_1 and F_2 of $\mathcal{M}$, we write

$$\mathcal{M}_{[F_1,F_2]} = \mathcal{M}/F_1 \backslash (E \setminus F_2)$$

to be the minor that results from contracting by F_1 and restricting to F_2. Notation in hand, we can now state a classic result of matroid theory [5, p. 116]. **21.ahh how do i cite things well?**

Proposition 4.3. *Let F_1 and F_2 be flats of a matroid $\mathcal{M} = (E, \mathcal{L})$. Then the lattice of flats of the minor $\mathcal{M}_{[F_1,F_2]}$ is isomorphic to the interval of $\mathcal{L}$*

$$[F_1, F_2] = \{F_1 \preceq F \preceq F_1 \mid F \in \mathcal{L}\}.$$

This means the lattice of a minor can be "seen" within the lattice structure of our original matroid, just up to some relabeling of the nodes. We note that this proposition only works for flats, not arbitrary subsets of the ground set, but that's more than enough for what we need.

Of note then is that if F_1 and F_2 are adjacent to each other in the lattice of flats, then $\mathcal{M}_{[F_1,F_2]}$ is isomorphic to a matroid whose sole flag is $\{\emptyset \subsetneq E\}$. The Bergman fan of such a matroid is just the point $\{0\}$, living in a 0-dimnesional vector space.

4.2.2 Quotients, Products, and Stars

Finally, we are going to see that quotienting the vector spaces that Bergman fans live in by cones of the fan will split up both the space and the fan itself in very nice ways.

Proposition 4.4. *Let $\mathbf{\Sigma}_{\mathcal{M}} \subset \boldsymbol{N}_E$ be the Bergman fan of a matroid, and*

$$\mathscr{F} = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k\} \subseteq \mathcal{L}^*$$

be a proper flag of flats. Define, for convenience, $F_0 = \emptyset$ and $F_{k+1} = E$. Then the map

$$\begin{aligned} \varphi : \boldsymbol{N}_E/\langle \sigma_{\mathscr{F}} \rangle &\to \bigoplus_{i=1}^{k+1} \boldsymbol{N}_{F_i \backslash F_{i-1}} \\ e_I &\mapsto \bigoplus_{i=1}^{k+1} e_{I \cap (F_i \backslash F_{i-1})} \end{aligned}$$

is an isomorphism.

Proof. Consider the very similar map

$$\begin{aligned} \varphi' : \boldsymbol{N}_E &\to \bigoplus_{i=1}^{k+1} \boldsymbol{N}_{F_i \backslash F_{i-1}} \\ e_I &\mapsto \bigoplus_{i=1}^{k+1} e_{I \cap (F_i \backslash F_{i-1})}. \end{aligned}$$

We will show φ' is a surjective linear map and that its kernel is exactly $\mathrm{span}(\sigma_{\mathscr{F}})$. With that, we can leverage Noether's first isomorphism theorem, and we will have that φ is an isomorphism.

Our map is linear by its definition, so let's first show that it is surjective. This isn't too bad. Consider some vector $w \in \bigoplus_{i=1}^{k+1} \boldsymbol{N}_{F_i \setminus F_{i-1}}$, is of the form

$$w = \bigoplus_{i=1}^{k+1} \left(\sum_{j \in (F_i \setminus F_{i-1})} \lambda_j e_j \right);$$

i.e.; each is a linear combination of the elements in some $F_i \setminus F_{i-1}$. This of course is the image of the vector

$$v = \sum_{i=1}^{k+1} \left(\sum_{j \in (F_i \setminus F_{i-1})} \lambda_j e_j \right),$$

so we have surjectivity.

Next we want to show that any element in the $\mathrm{span}(\sigma_{\mathscr{F}})$ in the kernel. We can figure this out seeing what gets mapped to 0 in any one of the $\boldsymbol{N}_{F_i \setminus F i-1}$. From construction of the space, anything in $\mathrm{span}(e_{F_i \setminus F i-1})$ is 0 in $\boldsymbol{N}_{F_i \setminus F_{i-1}}$ in particular, the ray $e_{F_i \setminus F_{i-1}}$ itself. So we can see that some $v \in \mathrm{span}(\sigma)$ is of the form

$$v = \lambda_1 e_{F_1} + \lambda_2 e_{F_2} + \cdots + \lambda_k e_{F_k},$$

and under our map will be

$$\begin{aligned}
\varphi'(v) &= \lambda_1 \varphi'(e_{F_1}) + \lambda_2 \varphi'(e_{F_2}) + \cdots + \lambda_k \varphi'(e_{F_k}) \\
&= \lambda_1 \left(e_{F_1}\right) + \lambda_2 \left(e_{F_1} \oplus e_{F_2 \setminus F_1}\right) + \cdots + \lambda_k \left(e_{F_1} \oplus \lambda_k e_{F_2 \setminus F_1} \oplus \cdots \oplus \lambda_k e_{F_k \setminus F_{k-1}}\right) \\
&= (\lambda_1 + \lambda_2 + \cdots + \lambda_k) e_{F_1} \oplus (\lambda_2 + \lambda_3 + \cdots + \lambda_k) e_{F_2 \setminus F_1} \oplus \cdots \oplus \lambda_k e_{F_k \setminus F_{k-1}} \\
&= (\lambda_1 + \lambda_2 + \cdots + \lambda_k) 0 \oplus (\lambda_2 + \lambda_3 + \cdots + \lambda_k) 0 \oplus \cdots \oplus \lambda_k 0 \\
&= 0
\end{aligned}$$

Finally we note that the only other thing mapped to 0 would be an element of the form $\lambda e_{E \setminus F_k}$ which is 0 when mapped to $\boldsymbol{N}_{F_{k+1} \setminus F_k}$. It is not immediately obvious that this is in $\mathrm{span}(\sigma_{\mathscr{F}})$. However, we recall that in $\boldsymbol{N}_E$, $e_E = 0$, which means $e_{E \setminus F_k} = -e_{F_k}$, which clearly is in the span of $\sigma_{\mathscr{F}}$. We may conclude then that $\ker(\varphi') = \mathrm{span}(\sigma_{\mathscr{F}})$.

With that we have

$$\boldsymbol{N}_E / \langle \sigma_{\mathscr{F}} \rangle \cong \bigoplus_{i=1}^{k+1} \boldsymbol{N}_{F_i \setminus F_{i-1}}$$

by Noether's first isomorphism theorem and further that φ is an isomorphism between these spaces. $\square$

With our isomorphism in hand, we may use it to show that the stars of a matroid's Bergman fan have a local structure equivalent to the product of the Begrman fans of its minors.

Lemma 4.5. *Let $\mathbf{\Sigma}_{\mathcal{M}} \subseteq \boldsymbol{N}_E$ be the Bergman fan of a matroid, and $\sigma \in \mathbf{\Sigma}(k)$ be a cone with ray generators corresponding to some flag $\mathscr{F}$.*

The isomorphism given in Proposition 4.4 induces a bijection

$$\mathrm{star}(\sigma, \mathbf{\Sigma}_{\mathcal{M}}) \to \prod_{i=1}^{k+1} \mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}}.$$

Proof. 22.Words for this proof. Words bad □

4.3 The Bergman Fan of Matroids are AF

Theorem 4.6. 23.is this more a corollary? maybe? *For any matroid, $\mathcal{M}$, inner product, $* \in \mathrm{Inn}(\boldsymbol{N}_E)$, and cubical value $z \in \mathrm{Cub}(\mathbf{\Sigma}_{\mathcal{M}}, *)$, $(\mathbf{\Sigma}_{\mathcal{M}}, z)$ is AF.*

We need only show that the conditions of Theorem 4.1 are met specifically for Bergman fans of matroids. We'll tackle this one condition at a time.

Lemma 4.7 (Connectedness)**.** *Let $\mathbf{\Sigma}_{\mathcal{M}}$ be the Bergman fan of a matroid of rank $r+1$. For every cone $\sigma \in \mathbf{\Sigma}_{\mathcal{M}}(k)$, with $k \leq r-2$,*

$$\mathrm{star}(\tau, \mathbf{\Sigma}_{\mathcal{M}}) \setminus \{0\}$$

is connected.

Proof. To prove this, it is sufficient to show that for any two rays in the fan, we may find a series of faces

$$\rho_1 \prec \tau_1 \succ \rho_2 \prec \cdots \succ \rho_{k-1} \prec \tau_{k-1} \succ \rho_k$$

where each ρ_i is a ray and each τ_i is a 2-dimmensional face. Since any point in a fan is connected to a ray, specifically one of the generating rays of the cone the point lives in, a path like this between arbitrary rays is enough to show our fan is connected without the origin.

For some notational convenience, we will write

$$\rho_1 \sim \tau_1 \sim \rho_2 \sim \cdots \sim \rho_{k-1} \sim \tau_{k-1} \sim \rho_k$$

and let the reader interpret the correct face inclusions. If this seems lazy, please take it up with Fields Medal winner June Huh, from whom we lifted this notation. Additionally, in keeping with our general convention, we'll write e_F for the ray generated by $\sum_{i\in F} e_{\{i\}}$ and τ_{F_1,F_2} for the cone generated by rays e_{F_1} and e_{F_2}. Finally, we note that $\mathbf{\Sigma}_{\mathcal{M}}$ is at least a rank 3 matroids, and so has maximal cones of dimension at least 2. A 1-dimensional fan is, indeed, not connected if you remove the origin.

Now, we will consider this in two steps. First we'll look at $\mathbf{\Sigma}_{\mathcal{M}} \setminus \{0\}$ itself. Let $e_F, e_{F'} \in \mathbf{\Sigma}_{\mathcal{M}}(1)$ be two arbitrary rays of our fan. If there exists some ground element $i \in F \cap F'$, then, trivially, we have the sequence

$$e_F \sim \tau_{\{F,\{i\}\}} \sim e_{\{i\}} \sim \tau_{\{\{i\},F'\}} \sim e'_F.$$

So, let us consider instead that $F \cap F' = \emptyset$. Let $a \in F$ be an element of F and $b \in F'$. To start, we have the sequence

$$e_F \sim \tau_{\{F,\{a\}\}} \sim e_{\{a\}}.$$

Now, recall the properties of the flats of a matroid, specifically property (F3). This tells us that the flats of rank 2 partition $E \setminus \{a\}$ and so there must be a rank 2 flat, $\widehat{F}$, such that $\{a, b\} \subseteq \widehat{F}$. Then we have

$$e_{\{a\}} \sim \tau_{\left\{\{a\},\widehat{F}\right\}} \sim e_{\widehat{F}} \sim \tau_{\left\{\widehat{F},b\right\}} \sim e_{\{b\}} \sim \tau_{\{\{b\},F'\}} \sim e_{F'},$$

showing a sequence from e_F to e'_F, as desired. With that we have shown there is always a path along 2 dimensional faces between any two rays of the Bergman fan of a matroid, and so is connected even without the origin.

Next, we'll turn to the stars of our matroid. Let $k \leq r-2$ and $\sigma_{\mathscr{F}} \in \mathbf{\Sigma}_{\mathcal{M}}(K)$ be a k-dimensional cone. We have that $\mathrm{star}(\sigma_{\mathscr{F}}, \mathbf{\Sigma}_{\mathcal{M}})$ is a fan with maximal cones of dimension $r-k$, which specifically means they are at least 2-dimensional. From Lemma 4.5, we know that, given $\mathscr{F} = \{F_1 \subsetneq \cdots \subsetneq F_k\}$, we have that $\mathrm{star}(\sigma_{\mathscr{F}}, \mathbf{\Sigma}_{\mathcal{M}})$ is in bijection with the product fan

$$\prod_{i=1}^{k+1} \mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}}.$$

So, it is sufficient to show this product fan is connected. Recall that cones of the product fan are of the form

$$(\sigma_1, \sigma_2, \ldots, \sigma_k) \in \mathbf{\Sigma}_{\mathcal{M}_{[\emptyset,F_1]}} \times \mathbf{\Sigma}_{\mathcal{M}_{[F_1,F_2]}} \times \cdots \times \mathbf{\Sigma}_{\mathcal{M}_{[F_{k-1},F_k]}} \times \mathbf{\Sigma}_{\mathcal{M}_{[F_k,E]}},$$

and that the dimension of the cone $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ is the sum of the dimensions of each cone σ_i. Rays of the product fan are then of the form $(0, 0, \ldots, 0, \rho, 0, \ldots, 0)$ where ρ is a ray of the corresponding fan in the product.

Now, to show our product fan is connected, we'll consider two cases. We will omit irrelevant zeros from our notation going forward for ease of reading, but we can add as many zeros in other positions as necessary without changing any part of the proof. In the first case, our two rays come from different fans in the product If $(\rho_1, 0)$ and $(0, \rho_2)$ are rays, then, trivially, there exists the path

$$(\rho_1, 0) \sim (\rho_1, \rho_2) \sim (0, \rho_2)$$

connecting them. The more nuanced case is if we have two rays from the same fan.

Consider two rays $\rho_1, \rho_2 \in \mathbf{\Sigma}(1)$. If the minor that generates $\mathbf{\Sigma}$ is at least rank 3, then our work above shows there must exist a path between them using only the cones of $\mathbf{\Sigma}$. Where this breaks down, however, is if the minor has rank 2; i.e., the Bergman fan has only 1-dimensional cones. We can't, after removing the origin, get between rays solely within this fan. Recall though that our star fan must be pure of at least dimension 2. This means if this $\mathbf{\Sigma}$ has only 1-dimensional cones, then there is at least one other nonzero fan, which we'll call $\mathbf{\Sigma'}$, in the product that has at least a one ray. Let $\eta \in \mathbf{\Sigma'}$ be said ray. Then we have the path

$$(\rho_1, 0) \sim (\rho_1, \eta) \sim (0, \eta) \sim (\rho_1, \eta)\ (\rho_1, 0)$$

connecting the two rays of $\mathbf{\Sigma}$.

With this we've shown that any possible star of a Bergman fan of a matroid is connected even without the origin. $\square$

Next we may turn to the other requirement, that the quadratic form that determines the volume of the 2-dimensional faces of the normal complex have exactly one negative eigenvalue.

Lemma 4.8 (Volume Quadratic Form Has 1 Positive Eigenvalue)**.** *Let $\mathcal{M}$ be a matroid of rank $r+1$ and $\mathbf{\Sigma}_{\mathcal{M}}$ be the Bergman fan associated to the matroid, with $* \in \operatorname{Inn}(\boldsymbol{N}_E)$ an inner product.*

For any cubical $z \in \operatorname{Cub}(\mathbf{\Sigma}_{\mathcal{M}}, *)$, the quadratic form associated to the volume polynomial of each 2-dimensional face of the normal complex $C_{\mathbf{\Sigma}_{\mathcal{M}},*}(z)$ has exactly one positive eigenvalue.

Proof. Recall that faces of a normal complex, $F^{\tau}(C_{\mathbf{\Sigma}_{\mathcal{M}},*}(z))$, correspond to truncations of the star fan for some cone $\tau \preceq \mathbf{\Sigma}_{\mathcal{M}}$. Since we want 2-dimensional faces, we will consider the stars associated to some $\tau_{\mathscr{F}} \in \mathbf{\Sigma}_{\mathcal{M}}(r-2)$, where $\mathscr{F} = \{F_1 \subsetneq \cdots \subsetneq F_k\}$. Once again using Lemma 4.5, we have that $\operatorname{star}(\tau, \mathbf{\Sigma}_{\mathcal{M}})$ is in bijection with the product fan

$$\prod_{i=0}^{k+1} \mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}}.$$

Recall that if F_{i-1} and F_i are adjacent to each other in the lattice of flats then the resulting fan $\mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}} = \{0\}$ and so contributes nothing to the product. Since here we have that $k = r-2$, there are only 2 flats not in $\mathscr{F}$. This means our product fan has two possible forms. Either

$$\mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}} \times \mathbf{\Sigma}_{\mathcal{M}_{[F_{j-1},F_j]}} \qquad \text{or} \qquad \mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}},$$

where the first is the product of two rank 2 minors and the second is a rank 3 minor. This just depends on if the two missing flats are adjacent to each other or not. We will again take this proof in cases, considering the two possible structures of our star fan.

In the first case, let us assume that $\operatorname{star}(\tau_{\mathscr{F}}, \mathbf{\Sigma}_{\mathcal{M}}) = \mathbf{\Sigma} \times \mathbf{\Sigma}'$ where $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$ are both pure of dimension 1. This means that the 2-dimensional cones are exactly

$$\{\operatorname{cone}(\rho, \rho') \mid \rho \in \mathbf{\Sigma}(1), \rho' \in \mathbf{\Sigma}(1)\}.$$

Let $z^{\tau_{\mathscr{F}}}$ be the z-values our face. For convenience, let $(x_1, \ldots, x_k) \subset z^{\tau_{\mathscr{F}}}$ be the z-values associated to rays of $\mathbf{\Sigma}$ and $(y_1, \ldots, y_\ell) \subset z^{\tau_{\mathscr{F}}}$ be the z-values associated to rays of $\mathbf{\Sigma}'$. Recall that the volume of a normal complex is the sum of the volumes of the polytopes that comprise it. Since every ray of $\mathbf{\Sigma}$ is orthogonal to every ray of $\mathbf{\Sigma}'$, this has the straight forward volume:

$$\operatorname{Vol}\left(F^{\tau_{\mathscr{F}}}\left(C_{\mathbf{\Sigma}_{\mathcal{M}},*}(z)\right)\right) = \sum_{\substack{1 \le i \le k \\ 1 \le j \le \ell}} x_i y_j.$$

Thinking of this as a polynomial in variables $(x_1, \dots, x_k, y_1, \dots, y_\ell) = (\boldsymbol{x}, \boldsymbol{y})$, we see each monomial is of degree 2, so this is a quadratic form. Thus, there exists a symmetric matrix A such that

$$\begin{bmatrix}\boldsymbol{x} & \boldsymbol{y}\end{bmatrix} A \begin{bmatrix}\boldsymbol{x} \\ \boldsymbol{y}\end{bmatrix} = \sum_{\substack{1\le i\le k \\ 1\le j\le \ell}} x_i y_j.$$

Taking $(\boldsymbol{x}, \boldsymbol{y})$ as a basis, it is not too difficult to work out that $A = [a_{i,j}]$ where

$$a_{i,j} = \begin{cases} \frac{1}{2} & 1 \le i \le k \text{ and } 1 \le j-k \le \ell \\ 0 & \text{otherwise.} \end{cases}$$

Finding the eigenvalues of this matrix, however, is not so easy.

To get around this, we would like to note another way to write our volume polynomial. We with some thought, we see that

$$\sum_{\substack{1\le i\le k \\ 1\le j\le \ell}} x_i y_j = \frac{1}{4}\big((x_1 + \cdots + x_k + y_1 + \cdots y_\ell)^2 - (x_1 + \cdots + x_k - y_1 - \cdots - y_\ell)^2\big),$$

since in $-(x_1 + \cdots + x_k - y_1 - \cdots - y_\ell)^2$, only terms of the form $2x_iy_j$ are positive, with the rest going to cancel out the unwanted terms in the first expression. Our goal now is to find an invertible change of basis, S, that gives us $\frac{1}{4}(x_1 + \cdots + x_k + y_1 + \cdots y_\ell)$ and $\frac{1}{4}(-x_1 - \cdots - x_k + y_1 + \cdots + y_\ell)$ as the first two basis elements. Then the matrix associated with this quadratic form, in this basis, would trivially have eigenvalues $1, -1, 0, \dots, 0$. Then we could use Lemma 4.2, Sylvester's Law of Inertia, to get that there is exactly one positive eigenvalue of A as well.

Luckily, a rather naïve change of basis matrix works out here. As a quick intuition building example, if $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$, then our change of basis matrix would be

$$S = \frac{1}{4}\left[\begin{array}{rr|rrr} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Clearly this gives us our basis. In general this matrix, which we'll scale to remove the fraction, $4S = [s_{i,j}]$ will be given by

$$s_{i,j} = \begin{cases} 1, & i = 1 \\ -1, & i = 2 \quad \text{and} \quad 1 \le j \le k \\ 1, & i = 2 \quad \text{and} \quad 1 \le j-k \le \ell \\ 1, & i = j+1 \quad \text{and} \quad 1 \le j-k \le \ell \\ 0, & \text{otherwise.} \end{cases}$$

While this looks rather complicated, it's just two rows of ± 1 and then just 1 along the subdiagonal of all other rows. Clearly this matrix is trivially diagonaizable by elementary row operations and will have no zeros along the diagonal, so we are safely assured that S is invertible. Then the quadratic form

$$B = S^{-1}AS$$

is a diagonal matrix with eigenvalues $1, -1, 0, 0, \ldots$, and so we conclude that A must also have exactly 1 positive eigenvalue.

With the first case done, we have to turn to the second, when our face is the normal complex associated to a rank 3 minor. Then $\operatorname{star}(\tau, \mathbf{\Sigma}_{\mathcal{M}}) = \mathbf{\Sigma}$ is a minor of our matroid with rank 3. For convenience, let $F \subseteq \mathcal{L}$ be all rank 1 flats of $\mathbf{\Sigma}$ and let $G \subseteq$ be all rank 2 flats of $\mathbf{\Sigma}$. The work of Nathanson and Ross in [4] **24.should I actually cite her thesis?** tells us that the volume of the normal complex our (or any) rank 3 matroid is given by

$$\operatorname{Vol}\left(C_{\mathbf{\Sigma},*}(z)\right) = 2\sum_{F \subsetneq G} z_F z_G - \sum_G z_G^2 - \sum_F (\mathcal{L}^\sharp(F) - 1) z_F^2,$$

where $\mathcal{L}^\sharp(F)$ is the number of flats one rank above F that contain F **25.sorry i'm in a hurry, I panic picked notation. i'll think of something better.**

Considering this a polynomial in $z_{F_1}, \ldots, z_{F_k}, z_{G_1}, \ldots, z_{G_\ell}$, it is not immediately obvious that this would have only 1 positive eigenvalue. Again, we propose a different way to write this that has obvious eigenvalues. We propose that

$$\operatorname{Vol}\left(C_{\mathbf{\Sigma},*}(z)\right) = \left(\sum_F z_F\right)^2 - \sum_G \left(z_G - \sum_{F \subsetneq G} z_F\right)^2.$$

This would have eigenvalues $1, -1, \ldots, -1, 0, \ldots, 0$, which is exactly what we want. To see that these are equal, We will break this down slightly. First, we note that

$$\left(\sum_F z_F\right)^2 = \sum_F z_F^2 + \sum_{F_1, F_2 \in F} 2 z_{F_1} z_{F_2}. \tag{4.1}$$

Then, let's look at the internal part of the second expression to see for a single $\widehat{G} \in G$ we have

$$\left(z_{\widehat{G}} - \sum_{F \subsetneq \widehat{G}} z_F\right)^2 = z_{\widehat{G}}^2 - \sum_{F \subseteq \widehat{G}} 2 z_F z_{\widehat{G}} + \sum_{F_1, F_2 \subseteq \widehat{G}} 2 z_{F_1} z_{F_2} + \sum_{F \in \widehat{G}} z_F^2. \tag{4.2}$$

What's important to note here is that if $F_1, F_2 \subseteq G$, then there cannot be another rank 2 flat G' that contains both. This is a direct consequence each property (F3) of lattice matroids. But when we let the outer sum range over all possible rank 2 flats, we will get $\mathcal{L}^\sharp(F)$ copies of each z_F^2, and so we have

$$\sum_G \left(z_G - \sum_{F \subsetneq G} z_F\right)^2 = \sum_G z_G^2 - \sum_{F \subseteq G} 2 z_F z_G + \sum_{F_1, F_2 \subseteq F} 2 z_{F_1} z_{F_2} + \sum_F \mathcal{L}^\sharp(F) z_F^2. \tag{4.3}$$

So if we subtract our result in (1.3) from the one in (1.1), we are left with

$$2\sum_{F\subset G} z_F z_G - \sum_G z_G^2 - \sum_F (\mathcal{L}\sharp(F)-1)z_F.$$

Which is exactly what we want.

□

4.4 The Log-Concavity of Characteristic Polynomials of Matroids

Main Result. **26.type out main result**

From the work of Nathanson and Ross [4], we have that there always exist a cubical value of

Lemma 4.9. *There is always a cubical value of a Bergman fan of a matroid.*

So, we also Theorem 4.1 works for any pseudocubical values as well.

Lemma 4.10. *The values α and β correspond to pseudocubcal z-values.*

Proof. Let $\mathcal{M} = (E, \mathcal{L})$ be a matroid and $A^*(\mathcal{M})$ be its associated Chow ring. First, we will recall the definition of α and β. They are linear elements of the Chow ring, so $\alpha, \beta \in A^1(\mathcal{M})$, defined by

$$\alpha = \sum_{i\in F} \qquad \text{and} \qquad \beta = \sum_{i\notin F}$$

for some element in the ground set $i \in E$ and ranging over all$F \in \mathcal{L}$. We remember that thanks to the equivalence relations in $A^*(\mathcal{M})$ any choice of ground element yields an element of the same class, so we need not distinguish the element. Which is also to say that we are free to choose any element of the ground set when we need.

□

We are done!

Bibliography

[1] Kareem Adiprasito, June Huh, and Eric Katz. "Hodge Theory for Combinatorial Geometries". In: *Annals of Mathematics* 188.2 (Sept. 2018), pp. 381–452. ISSN: 0003-486X, 1939-8980. DOI: 10.4007/annals.2018.188.2.1. URL: https://projecteuclid.org/journals/annals-of-mathematics/volume-188/issue-2/Hodge-theory-for-combinatorial-geometries/10.4007/annals.2018.188.2.1.full (visited on 03/08/2022).

[2] Federico Ardila. "Tutte Polynomials of Hyperplane Arrangements and the Finite Field Method". In: *Handbook of the Tutte Polynomial and Related Topics*. Chapman and Hall/CRC, 2022. ISBN: 978-0-429-16161-2.

[3] AP Heron. "Matroid Polynomials". In: *Combinatorics* 340058.9 (1972), pp. 164–202.

[4] Anastasia Nathanson and Dustin Ross. "Tropical Fans and Normal Complexes". Oct. 16, 2021. arXiv: 2110.08647 [math]. URL: http://arxiv.org/abs/2110.08647 (visited on 03/08/2022).

[5] James Oxley. *Matroid Theory*. 2nd ed. Oxford Graduate Texts in Mathematics. Oxford: Oxford University Press, 2011. 684 pp. ISBN: 978-0-19-856694-6. DOI: 10.1093/acprof:oso/9780198566946.001.0001. URL: https://oxford.universitypressscholarship.com/10.1093/acprof:oso/9780198566946.001.0001/acprof-9780198566946 (visited on 03/08/2022).

[6] Gian -Carlo Rota. "On the Foundations of Combinatorial Theory I. Theory of Möbius Functions". In: *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* 2.4 (Jan. 1, 1964), pp. 340–368. ISSN: 1432-2064. DOI: 10.1007/BF00531932. URL: https://doi.org/10.1007/BF00531932 (visited on 10/20/2022).

[7] Gian-Carlo Rota. "Combinatorial Theory, Old and New". In: Actes Du Congres International Des Mathématiciens (Nice, 1970). Vol. 3. 1970, pp. 229–233.

[8] W. T. Tutte. "A Contribution to the Theory of Chromatic Polynomials". In: *Canadian Journal of Mathematics* 6 (1954/ed), pp. 80–91. ISSN: 0008-414X, 1496-4279. DOI: 10.4153/CJM-1954-010-9. URL: https://www.cambridge.org/core/journals/canadian-journal-of-mathematics/article/contribution-to-the-theory-of-chromatic-polynomials/E1EE1F053B494D08746F0EE23F736CC2 (visited on 10/20/2022).

[9] D. J. A. Welsh. *Matroid Theory*. Academic Press, 1976. 452 pp. ISBN: 978-0-12-744050-7. Google Books: cfXuAAAAMAAJ.

[10] Hassler Whitney. “On the Abstract Properties of Linear Dependence”. In: *American Journal of Mathematics* 57.3 (1935), pp. 509–533. ISSN: 0002-9327. DOI: 10.2307/2371182. JSTOR: 2371182.

[11] Hassler Whitney. “The Coloring of Graphs”. In: *Annals of Mathematics* 33.4 (1932), pp. 688–718. ISSN: 0003-486X. DOI: 10.2307/1968214. JSTOR: 1968214.