Chapter 1

Bergman Fans and their Normal Complexes

As we continue our tour of various branches of mathematics, we arrive at geometry. The primary goal of this chapter is to develop the final segment of our bridge connecting some geometric object back to the Chow ring, and then showing how we can generate log-concave sequences with these objects. To get there we will provide a quick primer on polyhedral geometry and a classic theorem of convex geometry that generates log-concave sequences. Then we'll introduce a geometric object associated to a matroid, the Bergman fan, and show how we can use them to make some new object called a normal complex.

1.1 A Little Polyhedral Geometry, as a Treat

Really, the basic building blocks we'll be using are not that weird. It's geometry, we're going to be using some sort of shapes living in some kind of space. We must admit, however, that we personally struggle visualizing the higher dimensional objects at play, and so must fall back on formalism.

This section is a short crash course on basic elements of polyhedral geometry. Our treatment of this topic will often parallel that in Ziegler's "Lectures on Polytopes" [Zie95], which we recommend for those who'd like a little more depth than presented here. We'll end this section by stating a classic theorem of convex geometry related to log-concavity.

1.1.1 The Cone Zone

We are going to be using two fundamental kinds of convex shapes, polytopes and cones. As a reminder, a convex object is one where if you pick any two points in it, the line connecting those points never leaves the shape. We can, and will, state this formally.

Definition 1.1 (Convexity)**.** Let $K \subseteq \mathbb{R}^n$. We call K *convex* if for every $p, q \in K$, we have

$$[p, q] \subseteq K,$$

where $[p, q] = \{\lambda p + (1 - \lambda)q \mid 0 \leq \lambda \leq 1\}$ is the line segment between p and q.

1.Non/Convex shape examples here

While there are generally a few ways one could define polytope and cone, we will use a definition based on construction using some finite collection of points. In brief, a polytope is a *convex hull* of finitely many points and a cone is the *conic combination* of finitely many generating vectors. Let's make this formal.

Definition 1.2 (Polytope)**.** Let $P \subseteq \mathbb{R}^n$. We say P is a *polytope* if it is the convex hull of some finite set of points $x_1, x_2, \ldots, x_k$. That is to say P is a polytope if

$$P = \mathrm{conv}(\{x_1, \ldots, x_k\}),$$

where

$$\mathrm{conv}(\{x_1, \ldots, x_k\}) = \left\{ \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k \;\middle|\; \lambda_i \geq 0, \sum_{i=0}^{k} \lambda_i = 1 \right\}$$

is the convex hull of $x_1, x_2, \ldots, x_k$.

An astute reader may notice that our shorthand for line segment above, $[x, y]$, is in fact just $\mathrm{conv}(\{x, y\})$. Towards some intuition, we may think of the convex hull as the smallest convex shape that contains all of its generating points. In 2 dimensions we like to think of this as stretching a rubber band around a bunch of points and letting it constrict around them.

2.Draw some pictures of polytopes

Figure 1.1: Some examples of polytopes with possible generating sets

Likewise, cones are also built of a finite collection of generating vectors.

Definition 1.3 (Cone)**.** Let $C \subseteq \mathbb{R}^n$. We call C a *cone* if it is the conic combination of finitely many vectors $x_1, x_2, \ldots, x_k$. We write this

$$C = \mathrm{cone}(\{x_1, \ldots, x_k\}),$$

where

$$\mathrm{cone}(\{x_1, \ldots, x_k\}) = \{\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k \mid \lambda_i \geq 0\}$$

is the conic combinations of $x_1, x_2, \ldots, x_k$.

Unlike the more familiar notion of cones, these are not pointed cylinders.

3.Draw some cones

Figure 1.2: Some cones

We notice that conic combinations are essentially the span of the generating vectors but taking only non-negative linear combinations. Indeed, one can quickly confirm that $\mathrm{cone}(\{x_1, \ldots, x_k\}) \subseteq \mathrm{span}(\{x_1, \ldots, x_k\})$.

1.1.1.1 Aside: Points vs. Vectors

We have been, and will be going forward, using the words "points" and "vectors" quite interchangeably. Is there a difference? Annoyingly, the answer is kind of yes and kind of no. Points imply elements of an affine space, while vectors, naturally, are elements of a vector space. Affine spaces can be thought of a vector space where the 0-vector is "forgotten" but are otherwise essentially the same collection of "stuff". Mathematicians love to make multiple objects out of the same basic thing by giving (or losing) some extra structure.

We justify skimping on formalizing affine geometry because for the more casual reader it would be mostly obfuscating, a fairly naïve understanding of things like dimension will serve just fine, and for the more advanced it would be mostly tedious. Additionally, for our purposes, the 0-vector is always rather conveniently located, and so we won't, for example, ever have to argue the difference between a "cone" and a "affine cone". For those who would like a more formal treatment, we again recommend [Zie95]. As for word choice, will choose the term point or vector based on their vibe at any given moment. **4.don't use the word vibe, what is wrong with you.**

1.1.2 The Minkowski Sum

Now that we have the basic shapes down we need be able to make new ones out of existing ones. The two general strategies here will be to combine them in to new ones and to break them down. We'll start with learning how we can add shapes together, using what we call the Minkowski sum.

Definition 1.4 (Minkowski Sum)**.** Let $P, Q \subseteq \mathbb{R}^n$. The *Minkowski sum* of P and Q is given by

$$P + Q = \{p + q \mid p \in P, q \in Q\}.$$

Sit with this definition for a few moments to confirm the Minkowski sum does have the nice properties of sums we normally expect. It is commutative, associative, and has an identity in $\{0\}$. An additional feature of the definition is that the empty set has the property that for any $P \subseteq \mathbb{R}^n$,

$$P + \emptyset = \emptyset.$$

A way to think about the Minkowski sum is "smearing" one shape around the other. This makes more sense with a picture.

5.Minkowski sum smearing example

Figure 1.3: A Minkowski sum of a square and a triangle

With the Minkowski sum, we can finally define what the polyhedron is our polyhedral geometry.

Definition 1.5 (Polyhedron)**.** def:polyhedron Let $P \subseteq \mathbb{R}^n$. We call P a *polyhedron* if

$$P = \operatorname{conv}(\{x_1, \ldots, x_k\}) + \operatorname{cone}(\{y_1, \ldots, y_\ell\}),$$

for some finite sets $\{x_1, \ldots, x_k\}, \{y_1, \ldots, y_\ell\} \subseteq \mathbb{R}^n$.

A polyhedron is the result of the Minkowski sum of a polytope and a cone. Clearly every polytope and cone are polyhedron themselves, as $\mathrm{conv}(\{0\}) = \mathrm{cone}(\{0\}) = \{0\}$. Polyhedron are not necessarily bounded, which may seem a bit unusual to those who have seen the word in other contexts. Moreover, all bounded polyhedra are polytopes, which is not necessarily obvious, but useful to keep in mind.

6.Make a polyhedron or two

We will mostly be focused on either polytopes or cones at any one time, but having a more general object that includes both makes our definitions going forward a cleaner.

1.1.3 About Faces

7.Actually, do we really need this section? Given a polyhedron P, we also get a whole family of polyhedron, the faces of P. Let's first go back to simpler times. If we were to think of a cube, we would have faces of the cube as the 2 dimensional squares that make up the sides. We'd then call the line segments where any two of those squares meet *edges* and the points those edges meet *vertices*.

8.Cube, squares, lines, points

Figure 1.4: Examples of the faces, edges, and vertices of a cube

If this sounds familiar then the intuition for our more general notion of the face of a polyhedron is not far.

Now back in our world of polyhedral geometry, we know that a cube is a polyhedron and that each of those squares, lines, and points are also themselves polyhedron. In general, we use the term face to describe all these "sub-polyhedron" that make up the boundary of a polyhedron. As a note, we do still call the 0-dimensional faces vertices. The generic term for a $(d-1)$-dimensional face of a d-dimensional polyhedron is a *facet*.

It's worth defining face formally as it does only require some linear algebra, but the intuition above will likely suffice.

Definition 1.6 (Face)**.** Let $P \subseteq \mathbb{R}^n$ be a polyhedron and fix inner product $* \in \mathrm{Inn}(\mathbb{R}^n)$. Recall that the hyperplane normal to a vector $x \in \mathbb{R}^n$ at distance $q \in \mathbb{R}$ is given by

$$\mathrm{H}_x(b) = \{v \in \mathbb{R}^n \mid v * x = b\},$$

for some inner product $* \in \mathrm{Inn}(\mathbb{R}^n)$. Additionally, any hyperplane defines an upper and lower half-spaces given by

$$\mathrm{H}_x^+(b) = \{v \in \mathbb{R}^n \mid v * x \geq b\} \quad \text{and} \quad \mathrm{H}_x^-(b) = \{v \in \mathbb{R}^n \mid v * x \leq b\},$$

respectively.

Then $F \subseteq P$ is a *face* of P if there exist x, b such that

$$F = P \cap \mathrm{H}_x(b)$$

such that P lies entirely in the lower half-space (or equivalently upper half-space) of $\mathrm{H}_x(b)$; i.e.,

$$P \subseteq \mathrm{H}_x^-(b)$$
$$\big(\text{or equivalently } \ P \subseteq \mathrm{H}_x^+(b)\big).$$

First, we note that that given any $x \in \mathbb{R}^n$, $F = \mathrm{H}_x(0) \cap P = P$ and so P is face of itself. Likewise, there exist hyperplanes such that $P \cap \mathrm{H}_x(b) \cap P = \emptyset$, which means the empty set too is a face of any polyhedron P.

9.Have a square, make a hyper plane show that it intersects along a face

Figure 1.5: Examples of defining faces of a square with hyperplanes

From our cube intuition exercise, there are some things about faces that we would hope to generalize to our broader notion of faces. We'll state these without proof, and again refer to the early chapters of Ziegler [Zie95].

Proposition 1.1. *Given a polyhedron $P \subseteq R^n$, every face $F \subseteq P$ is a polyhedron. Better, if P is a polytope then F is a polytope, and if P is a cone then F is a cone.*

Proposition 1.2. *Let $P \subseteq \mathbb{R}^n$ be a polyhedron, and $F \subseteq P$ a face of P. From above, we know that F is a polyhedron. Then for any $F' \subseteq F$ a face of F, we have F' is also a face of P.*

Proposition 1.3. *Let $P \subseteq \mathbb{R}^n$ be a polyhedron and $F_1, F_2 \subseteq P$ be two faces of P. Then $F_1 \cap F_2$ is a face of P.*

It's worth remembering that the intersection of two faces can be empty, but $\emptyset \subseteq P$ is a face of any polyhedron P, so this causes no issues.

We will often write $F \preceq P$ to mean F is a face of P. Indeed, the relation "is a face of" induces a partial ordering on the set of all faces of P. Even stronger, using the propositions above it follows that the relation induces a lattice.

1.1.4 Polyhedral Complex

The last geometric structure we need as background consist of particular collections of polyhedra, known as polyhedral complexes.

Definition 1.7 (Polyhedral Complex)**.** A *polyhedral complex* $\mathcal{C}$ is a finite collection of polyhedra in $\mathbb{R}^n$ such that

1. the empty set, a polyhedron by definition, is in $\mathcal{C}$,
2. for any polyhedron $P \in \mathcal{C}$, all faces of P are also in $\mathcal{C}$,
3. for any two polyhedra $P, Q \in \mathcal{C}$, the intersection $P \cap Q \in \mathcal{C}$.

We can think of polyhedral complexes as sets of polyhedra that intersect nicely. We won't be too concerned about complexes of general polyhedra, and instead focus on when our complexes are restricted to either all polytopes or all cones.

Definition 1.8 (Polytopal Complex)**.** A *polytopal complex* $\mathcal{C}$ is a polyhedral complex where every element $P \in \mathcal{C}$ is bounded, i.e., a polytope.

10.random polyhedral complex

Figure 1.6: An example of a polyhedral complex

Definition 1.9 (Fan)**.** A *fan* $\mathbf{\Sigma}$ is a polyhedral complex where every element $\sigma \in \mathbf{\Sigma}$ is a cone.

11.some example fans

Figure 1.7: An example of a fan

We will learn some particular characteristics of fans in the next section, but we've now built up all the background on shapes we will need.

1.1.5 Volume Functions

From here, the last few background points can no longer be readily found in Ziegler, who as a topologist, we suspect without proof, cares little for things like volume. Instead, we swap out our Germans and recommend Rolf Schneider's "Convex Bodies" [Sch13] as a comprehensive reference.

We again need to formalize something that most of us would take for granted. The notion of volume is intuitive enough for 3-dimensional shapes, we however need to generalize this to all dimensions. We will actually only need the volume of polytopes, so we restrict our notion of volume to just them.

Definition 1.10 (Volume Function)**.** A *volume function* is a map

$$\mathrm{Vol}_d : \{\text{polytopes in } \mathbb{R}^n\} \to \mathbb{R}_{\geq 0}$$

such that for any polytopes $P, Q \subseteq \mathbb{R}^n$, Vol_n:

1. Is Non-Negative: $\mathrm{Vol}_n(P) > 0$ when $\dim(P) = n$ and $\mathrm{Vol}_n(P) = 0$ when $\dim(P) < n$,
2. Is Translation Invariant: $\mathrm{Vol}_n(P) = \mathrm{Vol}_n(P + v)$ for any $v \in \mathbb{R}^n$,
3. Respects Inclusion-Exclusion: when $P \cup Q$ is a polytope,
$$\mathrm{Vol}_n(P \cup Q) = \mathrm{Vol}(P) + \mathrm{Vol}_n(Q) - \mathrm{Vol}(P \cap Q),$$
4. Respects Linear Maps: for any $T \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$,
$$\mathrm{Vol}_n(T(P)) = |\det(T)|\,\mathrm{Vol}_n(P).$$

When the dimension is unambiguous, we will write Vol_n as simply Vol. This definition does not uniquely specify a single "volume function", but rather a family of maps that all differ from one another by a constant multiple. We can differentiate different volume maps by which polytope in $\mathbb{R}^n$ they take to 1. For example, in 3-dimensions, our "standard" volume function is the one that takes the unit cube to 1.

In general, since any two volume functions only differ by constant, we can choose the volume function that makes our equations easiest to read. We will be using *simplicial volume* going forward, the volume defined such that

$$\mathrm{Vol}_n\left(\mathrm{conv}(\{0, e_1, e_2, \dots, e_n\})\right) = 1$$

where $e_1, \dots, e_n$ are the basis vectors of our n-dimensional vector space.

1.1.6 Mixed Volume

While the volume function may not be too odd a concept we will use it to define a less widely known function. Given any 2 polytopes in $\mathbb{R}^2$, or 3 polytopes in $\mathbb{R}^3$, or more generally n polytopes in $\mathbb{R}^n$, we want a map from these collections of polytopes to $\mathbb{R}_{\geq 0}$ that is, in some sense, consistent with volume. We call this map the mixed volume function.

Definition 1.11 (Mixed Volume – Characterization)**.** The *mixed volume function* is a map MVol_n from an ordered multiset $P_1, P_2, \dots, P_n \subseteq \mathbb{R}^n$ of polytopes to $\mathbb{R}_{\geq 0}$, such that it has the following properties:

1. $\mathrm{MVol}_n(P, P, \dots, P) = \mathrm{Vol}_n(P)$, for any polytope $P \subseteq R^n$,
2. MVol_n is symmetric in all arguments, and
3. MVol_n is multilinear with respect to scaling and Minkowski addition.

Just like volume we'll often just notate this as MVol when safe to do so. The proof that such a function exists and is indeed uniquely defined by these properties can be found in [Sch13]. While this characterization is useful, it goes very little of the way to actually telling us what the mixed volume is.

Consider two polytopes $P, Q \subseteq \mathbb{R}^2$. We could ask ourselves, what is the general equation for the volume of the Minkowski sum of P and Q. We could be more ambitious and even allow ourselves to scale P and Q by arbitrary values. That is to say, let us consider the volume of

$$\mathrm{Vol}_2(\lambda P + \mu Q),$$

for some $\lambda, \mu \in \mathbb{R}$. Answering this question is where mixed volumes appear as something more concrete. While not immediately obvious, this volume can always be expressed as a polynomial, the mixed volume appears as coefficients of these polynomials. We can consider an example.

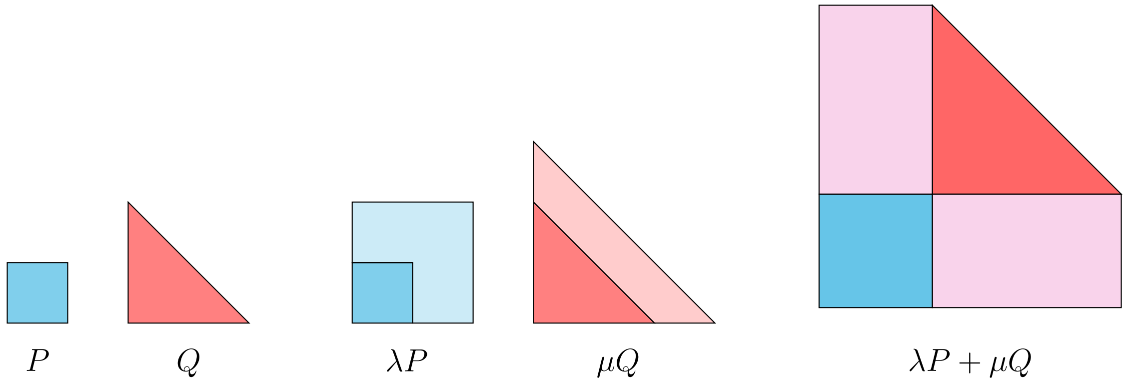


Figure 1.8: $\mathrm{Vol}(\lambda P + \mu Q) = \mathrm{Vol}(P)\lambda^2 + 2\,\mathrm{MVol}(P,Q)\lambda\mu + \mathrm{Vol}(Q)\mu^2$

12.Add side lengths to diagram

Our example in the figure is nice because it is already symmetrical. In general this will not be the case and the mixed volume and the coefficients will need to be rewritten to be symmetric, though this can always be done. This idea generalizes to any dimension, and provides another definition of mixed volume.

Definition 1.12 (Mixed Volume – As Coefficients of Volume Polynomial)**.** Let $P_1, P_2, \ldots, P_\ell \subseteq \mathbb{R}^n$ be polytopes. The function

$$f(\lambda_1, \lambda_2, \ldots, \lambda_\ell) = \mathrm{Vol}(\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_\ell P_\ell), \quad \lambda_j \geq 0$$

is symmetrically a homogeneous polynomial of degree n. It can be written as

$$f(\lambda_1, \ldots, \lambda_\ell) = \sum_{j_1, j_2, \ldots, j_n = 1}^{\ell} \mathrm{MVol}(P_{j_1}, \ldots, P_{j_n}) \lambda_{j_1} \cdots \lambda_{j_n}.$$

The coefficient associated to $\lambda_{j_1} \cdots \lambda_{j_n}$ is the *mixed volume* of $P_{j_1}, \ldots, P_{j_n}$.

Of course, you could start with either definition of mixed volume and derive the other, they are equivalent after all.

At this point, one may begin to wonder why this section should even exist. We seem to have gone rather far afield with our geometry lesson. Remember that our ultimate goal is to show something is a log-concave sequence. Mixed volumes are the key to a method generating log-concave sequences via geometry.

1.1.7 The Alexandrov–Fenchel Inequality

Finally, we conclude with an important classic result in convex geometry. Proved by Aleksandr Aleksandrov in [Ale37], with a contemporaneous but not quite accurate proof by Werner Fenchel, this theorem gives us a fundamental relationship of mixed volumes of convex bodies. We again restrict ourselves to just polytopes, though the theorem applies more broadly.

Theorem 1.4 (Alexandrov–Fenchel Inequality [Alexandrov 1937])**.** *For polytopes $P, Q, K_3, \dots, K_n$ in $\mathbb{R}^n$,*

$$\mathrm{MVol}(P, Q, K_3, \dots, K_n)^2 \geq \mathrm{MVol}(P, P, K_3, \dots, K_n)\,\mathrm{MVol}(Q, Q, K_3, \dots, K_n).$$

Remember that the mixed volume is just some non-negative real number, so this inequality is exactly what we are looking for in a log-concave sequence. In fact, given any two polytopes, there's a corresponding log-concave sequence.

Corollary 1.5. *For any polytopes $P, Q \subseteq \mathbb{R}^n$, the sequence*

$$\left\{ \mathrm{MVol}(\underbrace{P, \dots, P}_{n-k}, \underbrace{Q, \dots, Q}_{k}) \right\}_{k=0}^{n}$$

is log-concave.

This is a promising lead, but all we have is a collection of geometric definitions and a way to generate log concave sequences. None of this actually has any clear relation to matroids. But, just like we could make something algebraic out of the structure of a matroid, so too can we make something geometric.

1.2 Bergman Fans

Much like Chow rings, the general notion of a Bergman fan is broader than what we actually need. Credit goes to, unsurprisingly, George Bergman, who in [Ber71] developed the idea of logarithmic limit-sets of algebraic varieties, which would go on to be called Bergman fans [FS05]. But given we still refuse to carefully define an algebraic variety, the original presentation not too helpful for us here. More modern treatments, such as [AK06; HK12], have gone to show us that we can generate the a Bergman fan from the combinatorial data of the matroid alone.

1.2.1 Bergman Fans of Matroids

While it may be more accurate to say we'll present the construction of a fan that can be proven to be a Bergman fan, we again simply take it as a definition.

Definition 1.13 (Bergman Fan of a Matroid)**.** Let $\mathcal{M}$ be a matroid with a ground set $E = \{e_0, e_1, e_2, \dots, e_k\}$ and lattice of flats $\mathcal{L}$. Let

$$N = \mathbb{R}^E / \langle e_0 + e_1 + \cdots + e_k \rangle$$

be a real-valued vector space that identifies $e_1, \dots, e_k$ with the standard basis vectors. For any subset $I \subseteq E$ of the ground set, we notate

$$e_I = \sum_{i \in I} e_i$$

as the vector sum of each vector associated to the ground elements in I.

The *Bergman fan of* $\mathcal{M}$ is a fan in N given by

$$\mathbf{\Sigma}_{\mathcal{M}} = \left\{\operatorname{cone}\bigl(e_F \mid F \in \mathscr{F}\bigr) \;\middle|\; \mathscr{F} \subseteq \mathcal{L}^* \text{ is a flag of } \mathcal{M}.\right\}$$

Let's unpack this definition. First, let's think about what space this fan lives in. Essentially, we can think $N = \mathbb{R}^E/\langle e_0 + e_1 + \cdots + e_k\rangle$ as $\mathbb{R}^k$ where we assign all but one ground element of our matroid to the standard basis vectors. This designates one ground element as somewhat special, it doesn't matter which one, but generically we'll designate it e_0. Then the vector associated to e_0 is the vector of all -1, as the relation in the quotient tells us

$$e_0 = -e_1 - e_2 - \cdots - e_k.$$

Next, let's think about the elements of our fan. They are necessarily cones, and we see that there is one cone per flag in our matroid. This means that there exists a 1-dimensional cone, often called a *ray*, for each proper flat, as every flat is itself a flag. In general, the length of the flat corresponds to the dimension of the corresponding cone in the fan. As a consequence the biggest, by dimension, cones will correspond to proper flats. Similarly, if $F_1, F_2 \in \mathcal{L}$ are non-comparable flats, then there will not be a cone generated by the rays e_{F_1} and e_{F_2}. This is how the fan structure encodes the original combinatorial data of our matroid.

A bit of notation before moving on. We will let $\mathbf{\Sigma}_{\mathcal{M}}(d)$ be the set of all d-dimensional cones in $\mathbf{\Sigma}$. We notate the rays of our Bergman fan as $\rho_F \in \mathbf{\Sigma}(1)$ for flat $F \in \mathcal{L}^*$. More generally, for any flag $\mathscr{F} = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_\ell\}$ of our matroid, we write

$$\sigma_{\mathscr{F}} = \operatorname{cone}\bigl(e_{F_1}, \ldots, e_{F_\ell}\bigr),$$

for the cone associated with the flag.

1.2.1.1 An Example Bergman Fan

As always, let's turn to our running example and see its corresponding Bergman fan. Recall that

$$E = \{a, b, c, d\} \quad \text{and} \quad \mathcal{L}^* = \{a, b, c, d, abd, ac, bc, cd\}.$$

Then we can consider the space $N = \mathbb{R}^E/\langle e_a + e_b + e_c + e_d$. We will designate d as the special element and associate a basis vector of $\mathbb{R}^3$ to the remaining ground elements a, b, c.

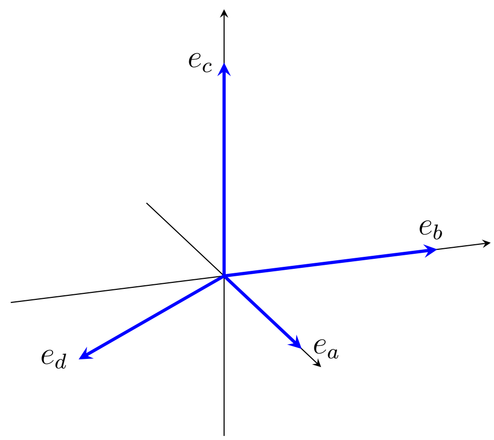


Figure 1.9: The vector space N with the vectors associated to the ground set

Then we can add in all the rays of our fan, corresponding to the flats.

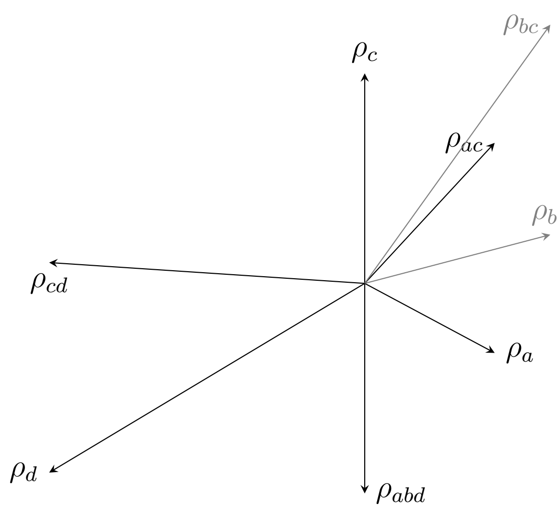


Figure 1.10: The rays $\mathbf{\Sigma}_{\mathcal{M}}(1)$

Since our proper flags can only have two elements, we need only have at most 2-dimensional cones. We can see that we only have a cone involving the rays ρ_F and ρ'_F if $F \subsetneq F'$ or $F' \subsetneq F$.

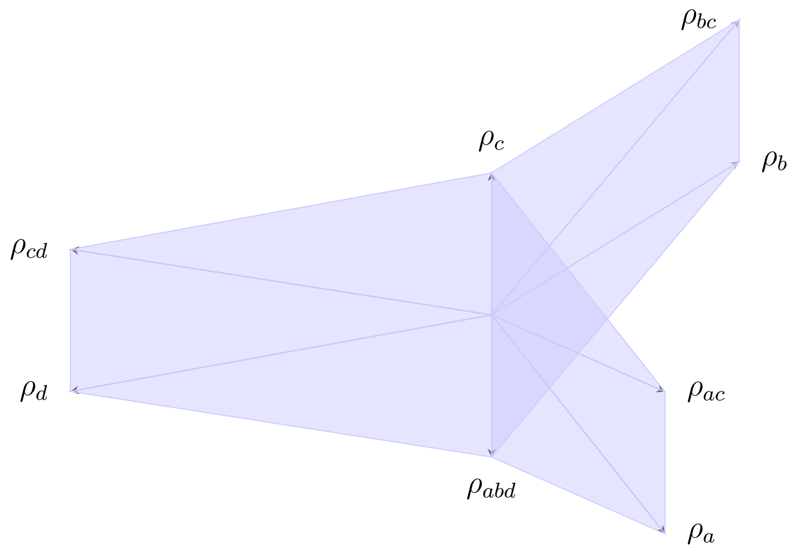


Figure 1.11: The Bergman fan of $\mathcal{M}$, $\mathbf{\Sigma}_{\mathcal{M}}$

We often think of these 2-dimensional Bergman fans look like dart fletching. This provides no mathematical insight, it's just a polyhedral rorschach test.

1.2.2 Properties of the Bergman Fan

Bergman fans of matroids have some nice properties that will be important later. These properties of fans that deal with the kinds of cones in them.

Definition 1.14 (Pure)**.** A cone $\sigma \in \mathbf{\Sigma}$ is maximal if it is not the proper face of another cone in $\mathbf{\Sigma}$. If every maximal cone in a fan $\mathbf{\Sigma}$ is the same dimension, we say the fan is *pure*. We $\mathbf{\Sigma}$ is a d-fan when it is pure of dimension d.

Definition 1.15 (Simplicial)**.** A fan Σ is *simplicial* if for every cone $\sigma \in \Sigma$,

$$\dim(\sigma) = |\sigma(1)|,$$

where $\sigma(1)$ is the set of 1-dimensional faces of σ.

Of course, as stated, the Bergman fan of a matroid is both pure and simplicial.

Proposition 1.6. *Let $\mathcal{M}$ be a matroid of rank $r+1$. The associated Bergman fan $\mathbf{\Sigma}_{\mathcal{M}}$ is a simplicial r-fan.*

Proof. To prove both of these properties we need one simple fact. By definition, a cone $\sigma \in \mathbf{\Sigma}$ is of the form

$$\sigma = \operatorname{cone}(e_{F_1}, \dots, e_{F_k}),$$

for $F_1 \subsetneq \cdots \subsetneq F_k$ some flag of $\mathcal{M}$. What we need is that $\{e_{F_1}, e_{F_2}, \dots, e_{F_k}\}$ is linearly independent. To see this, consider the equation

$$\lambda_1 e_{F_1} + \cdots + \lambda_k e_{F_k} = 0.$$

Recall that we defined $e_{F_j} = \sum_{i\in F_j} e_i$ to be the sum of the vectors associated to the ground set, so we can rewrite our equation as

$$\lambda_1 \sum_{i\in F_1} e_i + \cdots + \lambda_k \sum_{i\in F_k} e_i = 0.$$

But now, given that our flats form a flag that have strict inclusion, we may reorder terms to get

$$(\lambda_1 + \lambda_2 + \cdots + \lambda_k) \sum_{i\in F_1} e_i + (\lambda_2 + \cdots + \lambda_k) \sum_{i\in F(_2\setminus F_1} e_i + \cdots + \lambda_k \sum_{i\in F_k\setminus F_{k-1}} e_i = 0.$$

Since each of these sums involves a disjoint set of the vectors associated to the ground set, and any proper subset of these ground set vectors is linearly independent, it must be that $\lambda_1 = \cdots = \lambda_k = 0$, thus showing the set $\{e_{F_1}, e_{F_2}, \ldots, e_{F_k}\}$ is linearly independent.

With this we immediately have that $\mathbf{\Sigma}_{\mathcal{M}}$ is simplicial. For any cone $\sigma_{\mathscr{F}} \in \mathbf{\Sigma}_{\mathcal{M}}$ associated the flag $\mathscr{F}$, it is generated by $|\mathscr{F}|$ vectors, and because those vectors are linearly independent $\dim(\sigma_{\mathscr{F}}) = |\mathscr{F}|$.

Now, recall that for a d-dimensional simplical cone with generating rays $V = \{v_1, \ldots, v_d\}$, for any subset $V' \subseteq V$, $\text{cone}(V')$ is a face of $\text{cone}(V)$. Since we've shown that every cone of $\mathbf{\Sigma}_{\mathcal{M}}$ is simplicial, we have that any cone associated to a non-complete flag is non-maximal. This follows from the fact that if $\mathscr{F}$ is not a complete flag then there exists another flag $\mathscr{F}'$ such that $\mathscr{F} \subsetneq \mathscr{F}'$. Then we have that $\sigma_{\mathscr{F}}(1) \subsetneq \sigma_{\mathscr{F}'}(1)$ and so $\sigma_{\mathscr{F}}$ is a face of $\sigma_{\mathscr{F}'}$ and by definition non-maximal. Thus, every maximal cone of $\mathbf{\Sigma}_{\mathcal{M}}$ is associated to a complete flag, and any complete flag will have r elements. From above, we may conclude that every maximal flag then has dimension r.

With that we've shown that $\mathbf{\Sigma}_{\mathcal{M}}$ is a simplical r-fan.

□

The other property we want is that the Bergman fan of a matroid is balanced.

Definition 1.16. Balanced Let $bSigma$ be a simplicial d-fan. We say $\mathbf{\Sigma}$ is *balanced* if for every $\tau \in \mathbf{\Sigma}(d-1)$,

$$\sum_{\substack{\sigma\in\mathbf{\Sigma}(d)\\ \tau\preceq\sigma}} e_{\sigma(1)\setminus\tau(1)} \in \text{span}(\tau).$$

Proposition 1.7. *Let $\mathcal{M}$ be a matroid. The Bergman fan $\mathbf{\Sigma}_{\mathcal{M}}$ is a balanced fan.*

13.Should I also prove that it's a balanced fan? I should I'll come back and do this A balanced fan is a type of broader category of fans known as tropical fans. We won't delve too far into tropical fans broadly, but what is important to note is that if $\mathbf{\Sigma}$ is a tropical fan, then it has an associated Chow ring $A^\bullet(\mathbf{\Sigma})$. This is built analogously to how we developed the Chow ring of a matroid, starting with a polynomial ring with one variable per ray in $\mathbf{\Sigma}(1)$. Notably, if $\mathbf{\Sigma}$ is the Bergman fan of a matroid $\mathcal{M}$, then $A^\bullet(\mathbf{\Sigma}) = A^\bullet(\mathcal{M})$.

We now have a geometric object associated with our matroid, but we still don't yet have a bridge back to the realm of algebra as promised. Also, we seemed to hint volume was going to be useful, but there's no obvious way to take the volume of a fan. To wrap this up, we move on to our final geometric object.

1.3 Normal Complexes

The work in this section is by far the most recent, coming from work within the last two years at time of writing. We will provide a brief summary of the work by Nathanson and Ross [NR23] and by Nowak and Ross, jointly with the authors [NOR23].

This section will use the Bergman fan of a matroid to make an object called a normal complex. From this we can develop a notion of volume, as well as present theorems that relate this volume back to the Chow ring, finally giving us all the components necessary for our main result.

1.3.1 The Normal Complex of a Fan

In essence, the normal complex of a fan is simply a truncation of a fan into a polytopal complex using hyperplanes normal to the rays of the fan, thus the name. They were initially developed in [NR23], and the following definitions and propositions come from this work. First, we can't just take any truncation of our fan. We need a way to specify truncations that work well, and so we introduce the idea of cubical and pseudocubical values.

Definition 1.17 (Cubical Values)**.** Let $\mathbf{\Sigma} \subseteq N$ be a simplicial d-fan. For each ray $\rho \in \mathbf{\Sigma}(1)$ we choose u_ρ to be a non-zero vector in ρ. Choose an inner product $* \in \mathrm{Inn}(N)$ and a vector $z \in \mathbb{R}^{\mathbf{\Sigma}(1)}$ that associates a real number to each ray of our fan. For each ray $\rho \in \mathbf{\Sigma}(1)$ we have a corresponding hyperplane and half-space

$$\mathrm{H}_{\rho,*}(z) = \{v \in N \mid v * u_\rho = z\} \quad \text{and} \quad \mathrm{H}^-_{\rho,*}(z) = \{v \in N \mid v * u_\rho \leq z\}.$$

For each cone $\sigma \in \mathbf{\Sigma}$, let $w_{\sigma,*}(z)$ be the unique value such that

$$w_{\sigma,*}(z) * u_\rho = z_\rho$$

for each ray $\rho \in \sigma(1)$.

We say z is *cubical* if for all $\sigma \in \mathbf{\Sigma}$,

$$w_{\sigma,*}(z) \in \sigma^\circ,$$

where σ° is the interior of σ.

Definition 1.18 (Pseudocubical)**.** Given everything exactly as the previous definition, if we instead require only that

$$w_{\sigma,*}(z) \in \sigma$$

for all $\sigma \in \mathbf{\Sigma}$, we say z is *pseudocubical*.

For a given fan $\mathbf{\Sigma} \subseteq N$ and inner product $* \in \mathrm{Inn}(N)$, we denote cubical values as $\mathrm{Cub}(\mathbf{\Sigma}, *) \subseteq \mathbb{R}^{\mathbf{\Sigma}(1)}$ and the set of pseudocubical values as $\overline{\mathrm{Cub}}(\mathbf{\Sigma}, *) \subseteq \mathbb{R}^{\mathbf{\Sigma}(1)}$. This is all to say that when we select values to generate truncating hyperplanes for each ray, we want the intersections of the hyperplanes to lie within the cones of the fan. We can see an example in 2-dimensions.

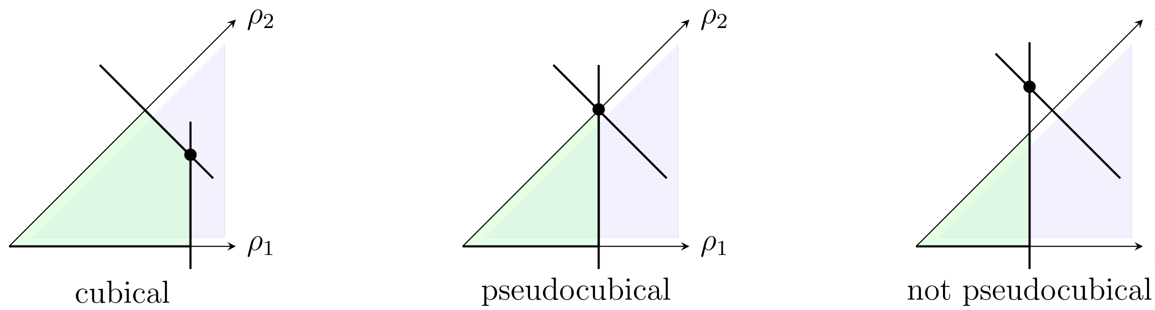


Figure 1.12: Examples of possible normal truncating hyperplane arrangements for a simple 2-dimensional fan

With this, we can now define a normal complex.

Definition 1.19 (Normal Complex)**.** Let $\mathbf{\Sigma} \subseteq N$ be a simplicial d-fan, with marked point u_ρ on each ray $\rho \in \mathbf{\Sigma}(1)$. Choose an inner product $* \in \mathrm{Inn}(N)$ and a pseudocubical vector $z \in pCub(\mathbf{\Sigma}, *)$. Recall that for each ray $\rho \in \mathbf{\Sigma}(1)$ we have a corresponding hyperplane and half-space

$$\mathrm{H}_{\rho,*}(z) = \{v \in N \mid v * u_\rho = z\} \quad \text{and} \quad \mathrm{H}^-_{\rho,*}(z) = \{v \in N \mid v * u_\rho \leq z\}.$$

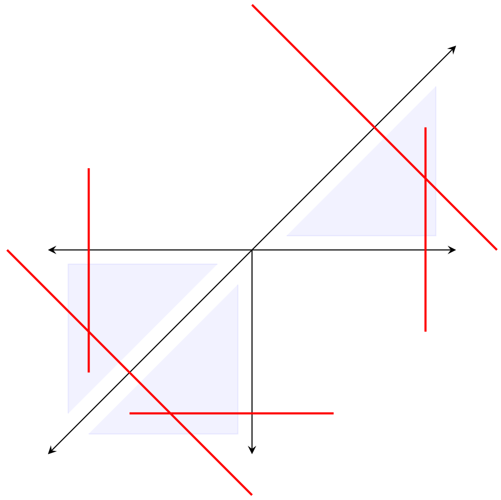
For each cone $\sigma \in \mathbf{\Sigma}$, we define a polytope $P_{\sigma,*}(z)$ given by

$$P_{\sigma,*}(z) = \sigma \cap \left(\bigcap_{\rho \in \sigma(1)} \mathrm{H}^-_{\rho,*}(z) \right).$$

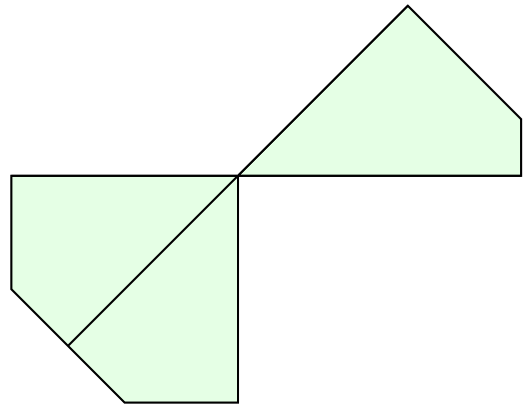
The *normal complex* of $\mathbf{\Sigma}$ is the polytopal complex

$$C_{\mathbf{\Sigma},*}(z) = \bigcup_{\sigma \in \mathbf{\Sigma}} P_{\sigma,*}(z)$$

That these truncations of fans give us well-defined polytopal complex is a result of Propositon 3.7 in [NR23]. Here is another place where an example is worth many words. Let's start with one in 2-dimensions.

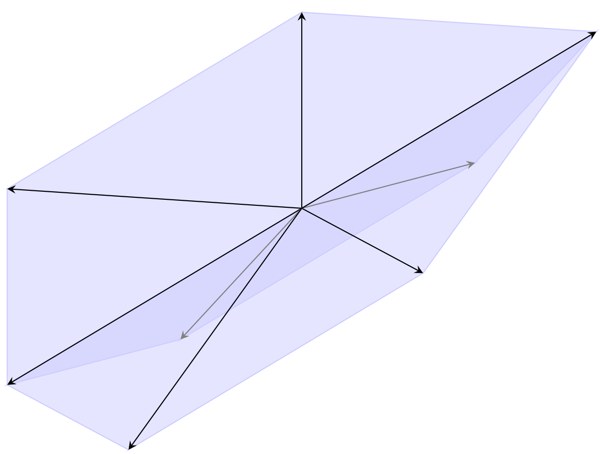


(a) A fan with a possible of arrangements of hyperplanes given by a cubical value

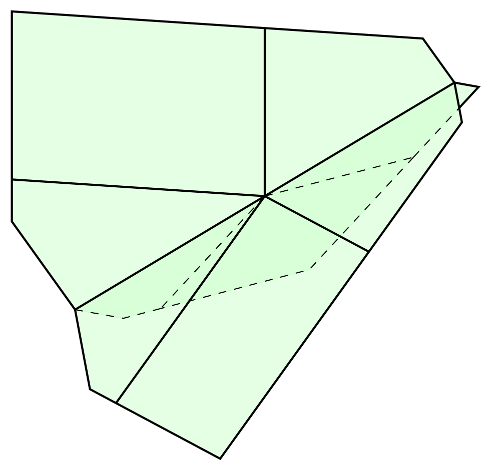


(b) The resulting normal complex of the fan given the choice of cubical value

We've already seen the Bergman fan of our example matroid, for which we can now give an example of a normal complex.



$\mathbf{\Sigma}_{\mathcal{M}}$



$C_{\mathbf{\Sigma}_{\mathcal{M}},*}(z)$

Figure 1.14: The Bergman fan and a normal complex of our example matroid

14.My example matroid is slightly different. I need to make a corresponging normal complex for it

1.3.2 The Volume of a Normal Complex

After so long hinting at the importance of volume, we finally have something related to our matroid we can actually take the volume of. Remember that we, rather lazily, only defined volumes on polytopes. We can extend the definition to normal complexes, though out of necessity how we do so is a bit fussy.

Definition 1.20 (Volume of a Normal Complex)**.** Let $\mathbf{\Sigma} \subset N$ be some simplicial d-fan with marked points u_ρ for each ray $\rho \in \mathbf{\Sigma}(1)$. Choose an inner product $* \in \mathrm{Inn}(N)$ and define

$$N_\sigma = \mathrm{span}(u_\rho \mid \rho \in \sigma(1)) \subseteq N.$$

Then, let M_σ be the vector space dual to N_σ. Picking $\{u_\rho \mid \rho \in \sigma(1)\}$ as the basis of N_σ, and using our chosen inner product, we may identify the dual basis $\{u^\rho \mid \rho \in \sigma(1)\}$ of M_σ. For each $\sigma \in \mathbf{\Sigma}(d)$, we choose the volume function

$$\mathrm{Vol}_\sigma : \{\text{Polytopes in } N_\sigma\} \to \mathbb{R}_{\geq 0}$$

characterized by $\mathrm{Vol}_\sigma\big(\mathrm{conv}(\{0\} \cup \{u^\rho \mid \rho \in \sigma(1)\})\big) = 1$.

For pseudocubical value $z \in \overline{\mathrm{Cub}}(\sigma, *)$, the *volume of the normal complex* $C_{\mathbf{\Sigma},*}(z)$ is

$$\mathrm{Vol}_{\mathbf{\Sigma},*}(z) = \sum_{\sigma \in \mathbf{\Sigma}(d)} \mathrm{Vol}_\sigma\big(P_{\sigma,*}(z)\big).$$

15.check with Dusty this definition is accurate. I'm skipping over the tensor product, but maybe I shouldn't...

The basic idea here is that we are taking the volume of each top-dimensional polytope in the complex and summing them. Nothing too wild there. Most of the verbosity in the definition comes from the fact that we have to choose the volume function for each component rather carefully. The specifics of why this must be done we leave to the reader to explore in [NR23], but it is necessary for the main result of the paper.

Theorem 1.8 (Nathanson-Ross 2021)**.** *Let $\mathbf{\Sigma}$ be a balanced d-fan. Choose an inner product $* \in \mathrm{Inn}(N)$ and pseudocubical value $z \in \overline{\mathrm{Cub}}(\mathbf{\Sigma}, *)$. We define*

$$D(z) = \sum_{\rho \in \mathbf{\Sigma}(1)} z_\rho x_\rho \in A^1(\mathbf{\Sigma}),$$

a divisor of the Chow ring. Then

$$\mathrm{Vol}_{\mathbf{\Sigma},*}(z) = \deg\big(D(z)^d\big).$$

Finally, we have the first link back to the Chow ring. By carefully taking volumes of the normal complex, we can evaluate top degree divisors under the degree map. While this is very cool, we have a problem. This only allows us to find evaluate a single divisor raised to the top power under the degree map. Recall that we want to reason about elements of the form $\alpha^{d-k}\beta^k$. These are called mixed degrees of divisors, and perhaps that is a good hint as to what we need to develop next.

1.3.3 Mixed Volumes of Normal Complexes

Here we finally can justify introducing the mixed volume function earlier. We can't define the mixed volume of on the full normal complexes directly, as the Minkowski sum of two normal complexes would certainly not be a normal complex itself in general. But like volume we could define it component wise.

A distinction from the original mixed volume function is that here is that we can't take the mixed volume of an arbitrary collection of normal complexes. We can only take the mixed volumes of normal complexes that have the same underlying fan.

Definition 1.21 (Mixed Volume of Normal Complexes)**.** Let $\mathbf{\Sigma} \subseteq N$ be a simplicial d-fan, $* \in \operatorname{Inn}(N)$ be an inner product, and pseudocubical values $z_1, \ldots, z_d \in \overline{\operatorname{Cub}}(\mathbf{\Sigma}, *)$. The mixed volume of $C_{\mathbf{\Sigma},*}(z_1), \ldots, C_{\mathbf{\Sigma},*}(z_d)$, written $\operatorname{MVol}_{\mathbf{\Sigma},*}(z_1, \ldots, z_d)$ is given by

$$\operatorname{MVol}_{\mathbf{\Sigma},*}(z_1, \ldots, z_d) = \sum_{\sigma \in \mathbf{\Sigma}(d)} \operatorname{MVol}_\sigma \left(P_{\sigma,*}(z_1), \ldots, P_{\sigma,*}(z_d)\right),$$

where $\operatorname{MVol}_\sigma$ is the mixed volume of polytopes as defined above using the volume function $\operatorname{Vol}_\sigma$.

Here, the basic idea as to why this may work is that taking the Minkowski sum of polytopes in the same cone will produce another polytope in this cone.

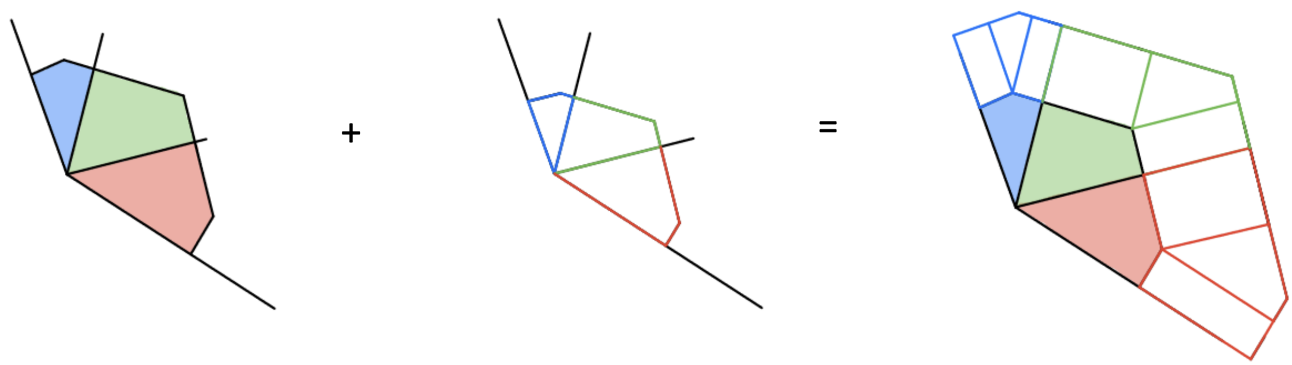


Figure 1.15: Component wise Minkowski sums of two normal complexes of the same fan; note each polytope in the sum is still in its correct cone

That this works in general is not obvious. Nor is it immediate that this newly defined function has all the nice properties of the original mixed volumes. However, from Proposition 3.1 in [NOR23], we have the following guarantee.

Proposition 1.9. *Let $\mathbf{\Sigma} \subset N$ be a simplicial d-fan, $* \in \operatorname{Inn}(N)$ an inner product on $\mathbb{R}^n$, and pseudocubical values $z_1, \ldots, z_n \in \overline{\operatorname{Cub}}(\mathbf{\Sigma}, *)$.*

The function

$$\operatorname{MVol}_{\mathbf{\Sigma},*} : \overline{\operatorname{Cub}}(\mathbf{\Sigma}, *)^d \to \mathbb{R}_{\geq 0}$$

as defined above has the following properties:

1. $\operatorname{MVol}_{\mathbf{\Sigma},*}(z, z, \ldots, z) = \operatorname{MVol}_{\mathbf{\Sigma},*}(z)$,
2. $\operatorname{MVol}_{\mathbf{\Sigma},*}$ *is symmetric in all arguments,*
3. $\operatorname{MVol}_{\mathbf{\Sigma},*}$ *is multilinear with respect to Minkowski addition in each maximal cone.*

Further, any function $\overline{\operatorname{Cub}}(\mathbf{\Sigma}, *)^d \to \mathbb{R}_{\geq 0}$

Our new mixed volume function then is well-defined and is uniquely characterized by the same properties as the original. Theorem 3.6 of [NOR23] extends Theorem 1.8 to a form that lets us evaluate mixed degrees.

Theorem 1.10. *Let $\mathbf{\Sigma} \subset N$ be a balanced d-fan. Choose an inner product $* \in \mathrm{Inn}(N)$ and pseudocubical values $z_1, \ldots, z_d \in \overline{\mathrm{Cub}}(\mathbf{\Sigma}, *)$. Then*

$$\mathrm{MVol}_{\mathbf{\Sigma},*}(z_1, \ldots, z_d) = \deg\big(D(z_1) \cdots D(z_d)\big).$$

This is a successful bridge from the realm of geometry back to algebra. We are now so close to having all the necessary components to prove our main result. Not only do we have the link between geometry and algebra, it uses the concept of mixed volumes, which are closely related to log-concave sequences. However, the Alexandrov-Fenchel inequalities are, classically, very dependent on convexity, and normal complexes are decidedly non-convex.

1.3.4 Amazing AF Fans

16.should I add a bit more about the recursive structure of normal complexes? It feels not strictly necessary, but then i've gone on lots of not strictly necessary tangents Luckily, more recently there have been proofs of the Aleksandrov-Fenchel inequality that make a clearer delineation of the geometric requirements from the rest of the machinery used in the proof. Using this, [NOR23] develops criteria to check which fans will obey inequalities.

Definition 1.22 (Alexandrov-Fenchel)**.** Let $\mathbf{\Sigma} \subset N$ be a simplicial d-fan and $* \in \mathrm{Inn}(N)$ an inner product. We say that $(\mathbf{\Sigma}, *)$ is *Alexandrov-Fencel*, or just *AF*, if $\mathrm{Cub}(\mathbf{\Sigma}, *) \neq \emptyset$ and

$$\mathrm{MVol}_{\mathbf{\Sigma},*}(z_1, z_2, z_3, \ldots, z_d)^2 \geq \mathrm{MVol}_{\mathbf{\Sigma},*}(z_1, z_1, z_3, \ldots, z_d)\,\mathrm{MVol}_{\mathbf{\Sigma},*}(z_2, z_2, z_3 \ldots, z_d)$$

for all $z_1, z_2, z_3, \ldots, z_d \in \mathrm{Cub}(\mathbf{\Sigma}, *)$.

So fans that are AF have exactly the properties we want in generating a log-concave sequence. The downside, of course, is that having left the convex world, we can't expect every fan to have this property. If we want to use this in our proof of the main result, we'll need to show that Bergman fans of matroids are AF. Luckily, we have a theorem from [NOR23] that gives us a short list of properties to check.

Theorem 1.11 (Nowak-O-Ross 2022)**.** *Let $\mathbf{\Sigma}$ be a simplicial d-fan, and $* \in \mathrm{Inn}(N_{\mathbb{R}})$ an inner product. Then $(\mathbf{\Sigma}_{\mathcal{M}}, *)$ is AF if*

i. for every cone $\sigma \in \mathbf{\Sigma}(k)$, with $k \leq r-2$,

$$\mathrm{star}(\tau, \mathbf{\Sigma}_{\mathcal{M}}) \setminus \{0\}$$

is connected,

ii. for each 2-dimensional face of the associated normal complex, $C(\mathbf{\Sigma}, z)$, for some cubical $z \in \mathrm{Cub}(\mathbf{\Sigma}, *)$, the quadratic form associated to the volume polynomial has exactly one positive eigenvalue.

We now finally have laid out every piece of the puzzle we'll need to prove our main result. All that's left is to put them together.