THIS IS A TITLE

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Chapter 1

Introduction

Welcome to this thesis.

1.1 Why are we doing this?

TO DO: Introduce the Heron-Rota-Welsh Conjecture. Should I not call it that now that we know it's not just conjecture?

TO DO: I should probably have H-R-W papers as references. Go find those.

TO DO: Why should you care? Let me tell you, briefly.

1.2 What have other people been doing?

TO DO: Talk about A-H-K. Maybe some other people working in this area.

1.3 What are we going to do here?

TO DO: Lay out the general game plan and what we want to show. Maybe outline the trade-offs of this version of the proof vs. A-H-K to motivate why this is worth doing.

Chapter 2

Matroids

Much of our concern is around matroids, so best we get a clear idea of what one is.

TO DO: Literally any other introductory sentence to this chapter.

2.1 What is a Matroid?

Matroids are, in a broad sense, a generalization of the notion of independence among elements of sets. There are several axiomatizations of matroids, and while all are equivalent proving this is nontrivial enough for the relations to affectionaitly be called *cryptomorphisms*. We will primarily be concerned with two axiomatizations, one based on the notion of independent sets and another based on what are called flats.

TO DO: We should mention the graph and matrix as natural jumping of points into matroids.

TO DO: Introduce a graph and a set of vectors that we can use as the generator of some(/the same?) matroids. Then we can use them as examples as we introduce definitions and whatnot

2.1.1 Independent Set Axioms

This formalization is both a natural starting point and closest to the original[1].

Definition 2.1 (Matroid — Indegendent Set Axioms)**.** A *matroid* is an ordered pair $\mathsf{M} = (E, \mathcal{I})$, where E is a finite set called the *ground set* and $\mathcal{I} \subseteq 2^E$, with the properties:

i. $\emptyset \in \mathcal{I}$.

ii. If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

iii. If $I_1, I_2 \in \mathcal{I}$ and $|I_1| \leq |I_2|$, then there exists some $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

TO DO: Do I cite definitions like this? I could cite Oxley or something

Those with some background in linear algebra may be able to see how this holds for some finite collection of vectors. And, if you consider a matrix is just a nice way of holding on to a finite set of vectors, you may begin to see where our matroids got their name.

TO DO: Work through our example graph and vectors to turn them into a ground set and independent set.

2.1.2 The Flat Axioms

The second axiomatization of matroids we will need is, perhaps, slightly less intuitive.

Definition 2.2 (Matroid — Flat Axioms)**.** A *matroid* is an ordered pair $\mathsf{M} = (E, \mathcal{F})$, where E is a finite set of elements called the ground set and $F \subseteq 2^E$ is a collection of *flats* such that

i. $E \in \mathcal{F}$.

ii. If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$

iii. If $F \in \mathcal{F}$ and $F_1, F_2, \ldots, F_k \in \mathcal{F}$ are the minimal, with respect to inclusion, flats containing F, then the sets $F_1 \setminus F$, $F_2 \setminus F$, ..., $F_k \setminus F$ partition $E \setminus F$.

Definition 2.3 (Closure)**.**

TO DO: Define closure in terms of flats? Or closure such that flats are closed...

Definition 2.4 (Rank)**.** Let $\mathsf{M} = (E, \mathcal{I})$ be a matroid and $X \subseteq E$

TO DO: If we define closure with respect just to flats then we can define rank based on closure. Or if we go the other way around, we can define closure based on rank (and flats based on closure?)

TO DO: This feels a bit backwards when we start by introducing the independent set axioms first. Oxley, for example, first defines a basis of a matroid, then the restriction of a matroid to a subset of the ground set, then closure, then defines rank as the cardinality of the basis of the restriction (also there's a proof that bases are all the same size). Should I ease the transition between definitions that way? It would take up space, but I have space to take up I suppose.

Lemma 2.1 (The Collection of Flats Form a Lattice)**.**

TO DO: This seems important

2.2 Matroids from Graphs; Matroids from Matrices

2.2.1 Graphical Matroids

TO DO: Maybe here we make a matroid from our graph? Or we do it in subsubsections as we define matroids

2.2.2 Representable Matroids

TO DO: If we didn't do it above, we do it here

2.2.3 Unrepresentable Matroids

TO DO: Fun fact about how most matroids don't come from a matrix.

2.3 The Characteristic Polynomial

2.3.1 The Chromatic Polynomial of a Graph

Some motivation.

TO DO: Work out the chromatic polynomial of the example graph we've been using

2.3.2 The Characteristic Polynomial of a Matroid

TO DO: Define the Characteristic Polynomial

Relation between Characteristic Polynomial (of a Matroid of a Graph) and Chromatic Polynomial (of a Graph)

TO DO: Give the basic little translation

2.3.3 The Reduced Characteristic Polynomial

TO DO: Show what reduced means. Segway into Chow Rings. Chapter end?

Chapter 3

Chow Rings and Bergmann Fans

3.1 Chow Rings

TO DO: Introduce the Chow Ring of a matroid. From lattice of flats to quotient ring

TO DO: Use our example matroid and construct its Chow Ring

3.1.1 Properties of Chow Rings

TO DO: Use words like *homogeneous polynomial*, *graded ring*, etc. . .

3.1.2 The Degree Map

TO DO: How do I explain this? I guess I can at least say it's linear and sends terms of full degree to 1. Maybe I'll understand it this time around

3.2 Bergmann Fans

TO DO: Quick definition of a fan?

3.2.1 How to make a Bergmann Fan

TO DO: Show definition from Chow Ring to Bergmann Fan

TO DO: Work our small example into a fan

3.2.2 Properties of Bergmann Fans

TO DO: Figure out their important properties. They're unimodal, so we'll include that. Oh, they're balanced as well (and so tropical). What am I missing?

TO DO: Define star of a fan. Reference A-H-K; star of a Bergmann fan is again a Bergmann fan? Or something like that

3.3 Relationship with the Characteristic Polynomial

TO DO: Come up with a nice way of relating the reduced characteristic polynomial with our ring (and therefore fan)

TO DO: Define α and β. Here or in a subsection? Or should it be up when we introduce the ring itself?

3.3.1 How to show Log-Concavity

TO DO: Offer a self-contained proof that log-concavity is equivalent to showing that one particular relationship between α and β found in A-H-K. They have the proof, but it requires digging through citations. Should be able to consolidate it.

Chapter 4

Normal Complexes

4.1 What the Hell is a Normal Complex

4.2 Analogues to Polytopes

4.2.1 Oh no, We Can Define the Faces of a Normal Complex

TO DO: Define facets and show they're normal complexes themselves. Show the analogues of polytope faces (mostly?) hold

4.3 Computing Volumes of Normal Complexes

4.3.1 Recursive Definition of Volume (via Facets)

4.3.2 Volume as Degree Map Evaluation

4.4 Mixed Volumes of Normal Complexes

4.4.1 What is the Standard Mixed Volume

4.4.2 Extending Mixed Volume to Normal Complexes

4.4.3 Some Nice Properties of Mixed Volumes

TO DO: Cite Lauren. A-F Inequalities.

Chapter 5

A Volumetric Approach to Heron-Rota-Welsh Conjecture

5.1 α and β have Pseudo-Cubical Interpretations

TO DO: Show α and β are pseudo-cubical

5.1.1 A (potentially) fun relation to the simplex

TO DO: Check the thing about α living in the negative simplex and whatnot. I think it maybe does work out!

5.1.2 What else do I need to show???

TO DO: I'm sure more stuff goes here. Figure out what it is

Bibliography

[1] Hassler Whitney. “On the Abstract Properties of Linear Dependence”. In: *American Journal of Mathematics* 57.3 (1935), pp. 509–533. ISSN: 0002-9327. DOI: 10.2307/2371182. JSTOR: 2371182.