A Volumetric Proof of the Log-Concavity of the Characteristic Polynomial of Matroids

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by

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CERTIFICATION OF APPROVAL

I certify that I have read *A Volumetric Proof of the Log-Concavity of the Characteristic Polynomial of Matroids* by Patrick O'Melveny and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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This thesis presents a novel proof of the Heron–Rota–Welsh conjecture using a volume theoretic approach. Originating in the 1950's, the conjecture proposes that the coefficients of the characteristic polynomial of a matroid exhibit the property of log-concavity. A complete proof for all matroids was found only in 2018, when Jun Huh, in collaboration with Karim Adiprasito and Eric Katz, achieved this milestone by developing the theory of combinatorial Hodge theory.

We review the link between the combinatorial data of matroids, algebraic objects known as Chow rings, and geometric objects called Bergman fans, and then outline the recent work of Dustin Ross, Anastasia Nathanson, Lauren Nowak, and the author on the theory of normal complexes of fans and their volumetric properties.

Our main result stems from showing that the Bergman fans of matroids meet criteria such that the (extended) mixed volumes of their normal complexes obey the Alexandrov–Fenchel inequality, yielding log-concave sequences. We hope this demonstrates that the theory of normal complexes is a tool able to tackle modern problems in mathematics.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

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Introduction

This thesis will take us on a tour of mathematics that have been developing for close to a century. The main result synthesizes modern work, from about ten years right up to last year, about a conjecture, posed in the seventies, on a mathematical object first formalized in 1935. This threads through work of our friends and mentors, Fields Medal winners, and a host of well-known mathematicians from across the last hundred years. While we think the results alone are quite interesting on their own, much of what has made this project so interesting to us is its broad connections to these various places. We hope, through the more leisurely pace we are allowed to take in a thesis, to show off this side of the math as well.

1.1 What Are We Doing?

The key players in this work are **matroids**, a combinatorial object devised to generalize the notion of "independence". Matroids are interesting for a multitude of reasons, but of note to us is that, although they are combinatorial objects, they can be alternatively studied through associated geometric objects, known as **Bergman fans**, and algebraic objects called **Chow rings**. In the early 1970's, a conjecture about the **characteristic polynomial** of matroids was posed. The **Heron–Rota–Welsh conjecture** was, in essence a combinatorial question, and would remain unresolved for almost 50 years. It was through viewing the problem from the algebro-geometric side of things that Adiprasito, Huh, and Katz were finally able to prove the conjecture true in 2015. They did this by importing complex machinery from algebraic geometry, known as *Hodge theory*, into the combinatorial world of matroids. It is an impressive work that, in part, won June Huh a Fields medal.

In this thesis we wish to outline an alternative proof of the Heron–Rota–Welsh conjecture. This work can be view as a companion to the paper of Lauren Nowak, Dustin Ross, and the author [NOR23]. The paper furthers the work of Anastasia Nathanson and Ross which developed a correspondence between the volume of objects generated from fans, called **normal complexes**, and the evaluation of degree maps on the Chow ring under certain conditions. Our thesis here, then, looks at using the tools we have developed and applying them to matroid-theoretic ideas, taking a more leisurely pace. We highly reccomend Lauren Nowak's thesis, [Now22], as yet another companion that provides a deep-dive into the volume-theoretic aspects of normal complexes.

1.2 Why Are We Doing This?

Where some would ask why, we much prefer to ask "why not?". More seriously, while a proof of the Heron-Rolta-Welsh conjecture is not new, having a new viewpoint on something is valuable even just in comparing it to the original.

This is a starting application of the theory of normal complexes, and shows that they can be applied to current problems in mathematics. We hope to see some of these techniques and tools expanded and applied elsewhere.

1.3 How Will We Do This?

By this point we have already introduced quite a few words we don't expect every reader to know offhand. We will essentially be providing context for every word in bold appearing above. In doing so, we will provide a plethora of definitions, propositions, and theorems across several fields. Why we need to cover quite such a large range will hopefully be apparent as we go on, but we fully acknowledge that this can be a little overwhelming.

We will try to be systematic about what we define, cite and prove. Outside a few cases where it is illustrative for a larger point, we will take linear algebra and basic abstract algebra for granted. For the more specialized objects at play, we will define them in the text. As we introduce already known propositions and theorems, we will cite back to either textbooks on the subject or the results of papers. Occasionally, a proposition will just seem to be "common knowledge" or use a construction very different from the one we use. When feasible, we will provide a proof for these ourselves.

1.4 Who Is This For?

Our primary goal is that anyone with a few graduate level courses in mathematics under their belt could read this thesis from start to finish and come out with a comprehensive picture of both the setting and the conclusion. Besides the stated, we also have some secondary goals in terms of readership. First, we want this to be of at least some interest to someone already knowledgeable in the field. While we are confident that any math of real substance in this thesis will be developed elsewhere, if it is going to appear here it might as well at least be useful to a practitioner. Second, and in somewhat of a contradiction, we want this work to be inviting to a curious non-mathematican. We believe there is a good opportunity here to allow a layperson to follow along with math they may not be otherwise usually exposed too.

In the true spirit of compromise then, we expect no one to be totally happy with the pacing. In general, the intention is that the complexity of the material will start somewhat low and increase as we go on. But, there will be technical points interjected in otherwise easy material, and we will attempt to include high level overviews even in sections that really do require a solid mathematical background.

We say this largely to give the reader permission to skip the bits that simply don't interest them.

Chapter 2

Matroids

The underpinning of all our work are mathematical objects known as matroids. Though, as we have noted, they've been around since the 1930's, they are not, yet, household objects every mathematician knows. This is the shallowest scratch into the world of matroids, slanted heavily towards what's necessary for our problem at hand. There are many books on matroids, for those curious to dig into more depth. We are partial to the treatment by Oxley's *Matroid Theory* [Oxl11].

We will build up to matroids by developing some intuition from more familiar, motivating mathematical objects. Then we will introduce the definition(s) of matroids and introduce the characteristic polynomial. We will wrap up this chapter by stating the Heron–Rota–Welsh conjecture.

2.1 Linear Algebra Done Hastily

When the vague notion of independence is mentioned in a mathematical context, we expect that minds wander to *linear* independence. A central concept to the field of linear algebra, this is likely the vast majority's first introduction to the topic. Happily, this mirrors, closely enough, the initial development of matroids. The patterns that emerge viewing the independence of collections of vectors will, quite directly, inspire the first of our definitions of a matroid.

2.1.1 Linear Independence

First, let us recall the definition of linear independence.

Definition 2.1 (Linear Independence)**.** Given a finite set of vectors $\{v_1, v_2, \ldots, v_k\} \subseteq F^n$, for some field F, the set of vectors is called *linearly independent* if the only solution to the linear combination

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$$

is $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$. Otherwise, we say the set is *linearly dependent*.

While this is a familiar definition to many of us, it will be illustrative to all to take a more concrete example. We will define the vectors $a = (1, 0, 0), b = (0, 1, 0), c = (0, 0, 1)$, and $d = (1, 1, 0)$. Then we have the set $E = \{a, b, c, d\} \subseteq \mathbb{R}^3$.

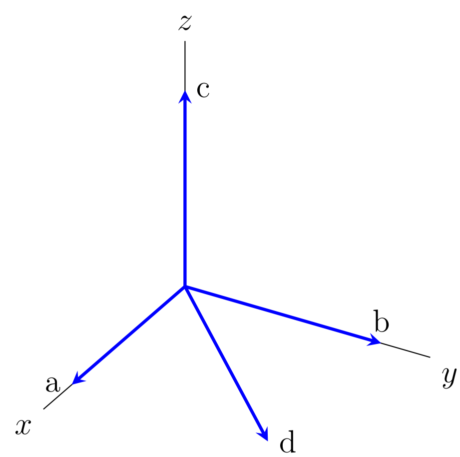


Figure 2.1: The collection of vectors in E.

The observant will note that E of course cannot be linearly independent, and indeed we can confirm by showing the linear combination

$$1a + 1b + 0c + (-1)d = 0.$$

But now, a fun little game we could play, at least by our personal reckoning of fun, is to find all subsets of E that *are* linearly independent. For example, consider $\{c, d\} \subseteq E$.

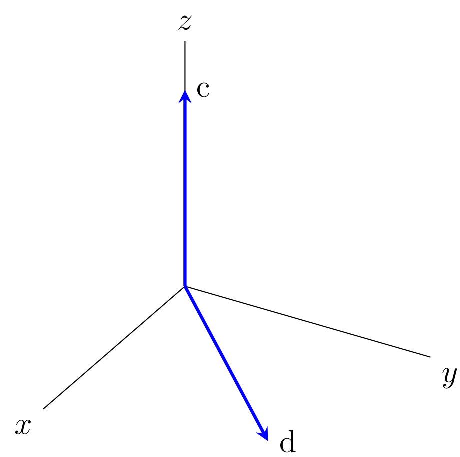


Figure 2.2: A linearly independent subset of E.

Take a look to confirm there is no nonzero linear combination of our elements that gives us the 0-vector. Given the relatively small number of elements, it would not take too long to identify every possible subset of E that is linearly independent; for the impatient however, they are precisely

$$\big\{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}\big\}.$$

For the impatient *and* untrusting, we suggest that the only thing really necessary to check here is that each 3-element set is linearly independent, that the other sets comprise all subsets of those 3-element sets, and that there are no other possible 3-element sets in E that are linearly independent.

As a point of pure notation, the above list is ugly. We are going to be working with sets of this form so much in this paper that, in order to avoid a shortage of curly brackets, we will introduce a more tidy notation. Going forward, we will write the elements of the internal sets adjacent to each other to represent the set containing them; for example we will write the set $\big\{\{a, b\}, \{a, b, d\}\big\}$ as $\{ab, abd\}$. Thus, we will more compactly identify the linearly independent subsets of E as

$$\{\emptyset, a, b, c, d, ab, ac, ad, bc, bd, cd, abc, acd, bcd\}.$$

Now that we have this collection, this leads to our next totally fun and normal activity. Namely, looking for patterns amongst these independent sets.

2.1.2 Noteworthy Properties of Linearly Independent Subsets

We suspect that those with some knowledge of linear algebra will immediately be ready to note that the largest independent subsets of E have 3 elements. And, sure enough, that is true! But this is more a property of the vector space, $\mathbb{R}^3$ in this case, that we are pulling the vectors from than some intrinsic relationship or property of the subsets. We'd like to call attention to some properties that may be less obvious (or so obvious one forgets they are even there). First, something entirely uninteresting.

Property 2.1 (The empty set is an independent subset)**.** For any finite collection of vectors, E,

$$\emptyset \subseteq E$$

and $\emptyset$ is linearly independent.

That $\emptyset$ is linearly independent is what we call vacuously true. That is to say, it is true mostly as a quirk of how we define linear independence. Since we can't form a non-zero linear combination that gives the 0-vector, because there are *no* elements at all, it can't be linearly dependent. But then if it is not linearly dependent, it has to be independent. Proof by being pedantic, really the heart of mathematics if one thinks about it. Next, a property that will surprise no one who has taken a linear algebra class, but is worth making explicit.

Property 2.2 (Any subset of a linearly independent set is itself linearly independent)**.** For any linearly independent set of vectors, I, if

$$I' \subseteq I,$$

then I' is linearly independent.

Recall we suggested that in order to check that our list of independent subsets of E was correct, it was sufficient to just check the subsets with the most elements. This property tells us that if we have figured out the maximal subsets, then filling in the rest is just a matter of taking subsets of those. One may even begin to see the specter of combinatorics lurking. Property 2.2 falls out easily from our definition. If no non-zero linear combination of vectors in a set gives us the 0-vector, then using fewer vectors isn't going to change that. Finally, we have a more subtle property.

Property 2.3 (The "independence augmentation" property)**.** Let $I = \{v_1, v_2, \ldots, v_m\}$ and $J = \{u_1, u_2, \ldots, u_n\}$ be linearly independent sets, such that $m < n$. Then there exists a $k \in [n]$ such that the set

$$I \cup u_k = \{v_1, v_2, \ldots, v_m, u_k\}$$

is linearly independent.

In other words, we can always find an element of a larger independent set to include in a smaller one that will leave the (new, augmented) set independent. Going back to our running example, consider the sets acd and ab. Then $c \in acd$ is such an element, and we confirm that $ab \cup c = abc$ is indeed linearly independent. This property is not immediately obvious, though may be believable to those who have done a proof based linear algebra class.

These are the three properties of linearly independent sets we wish to highlight here. We could use these properties alone to motivate the first definition of a matroid. However, we have one more detour before we get to matroids proper. There is another area where independence arises quite naturally, and it will be useful to know going forward.

2.2 Graphic Content

The next place our intuition building journey takes us is the world of graphs. Graph theory was the other motivator of matroids, so we too shall delve in. While we tried to not assume too much, we did, secretly, expect the average reader would feel comfortable enough with linear algebra. Graphs, on the other hand we will quickly build up from scratch and develop a notion of independence. Luckily, this is actually a fairly short process.

2.2.1 What a Graph Is

Not to be confused with the graph of a function or whatever it is business analysts put in shareholder presentations, graphs for us are essentially a collection of points, called vertices, and lines between them, called edges. There are quite a few definitions of graphs, each allowing for slightly different properties, but for our purposes, we can use a rather basic definition.

Definition 2.2 (Graph)**.** A *graph* is a pair of sets $G = (V, E)$, where V is a set of objects known as vertices, and E is a multiset of edges $\{x, y\}$, for $x, y \in V$.

A brief aside for our friends who actually care about graphs; the definition here is for an *undirected multigraph permitting loops*. We again recommend [Oxl11] for the serious graph theorist's entry into matroids.

For the rest of us, this definition may feel rather opaque. Here, an example and corresponding picture should help immensely. Let $V = \{v_1, v_2, v_3, v_4\}$ be a vertex set. Now we must define edges between vertices. For later convenience, we will name these edges. Let $a = \{v_1, v_2\},\ b = \{v_2, v_3\},\ c = \{v_3, v_4\}$, and $d = \{v_1, v_3\}$; then, let $E = \{a, b, c, d\}$ be our edge set. Recall that, for example, c represents an edge, or connection, between the vertex v_3 and the vertex v_4. With both those pieces, we have the graph $G = (E, V)$. The corresponding picture of our graph is below.

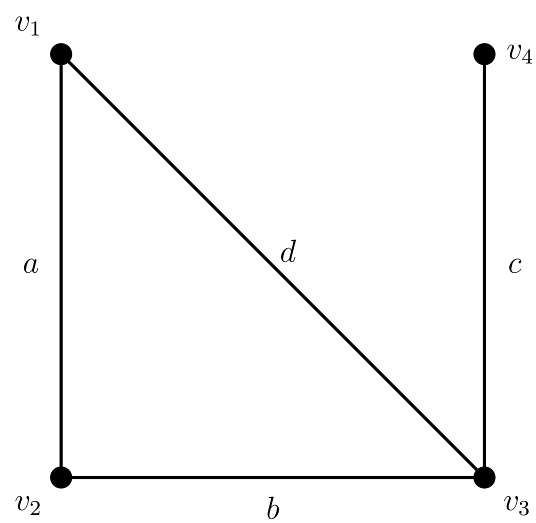


Figure 2.3: Our example graph G.

Now that we know what a graph is, it is time to figure out what "independence" could possibly mean.

2.2.2 Independence in the Realm of Graphs

The first thing to note is that we will define independence on the set of edges of a graph; that is, for some graph $G = (V, E)$, an independent set will be some subset $I \subseteq E$, meeting some criteria we will discuss below. What then would it mean for a set of edges to be independent? Well, if we take some subset of the edges, we restrict which vertices are accessible via those edges. But there might still be redundant edges. Could we remove additional edges from our set and still be able to reach all the same vertices? The answer to that determines if a set of edges is independent, when we can't make our collection of edges any smaller without disconnecting a vertex, or dependent, when we can.

To formalize this we will need to learn a few graph theoretic terms. First, we need the notion of a walk.

Definition 2.3 (Walk)**.** Given a graph $G = (V, E)$, a *walk* is an alternating sequence of vertices and edges

$$(v_1, e_1, v_2, e_2, v_3, \ldots, e_{k-1}, v_k),$$

where each $v_i \in V$, $e_j \in E$ and $v_i \in e_i$ and $v_{i+1} \in e_i$.

Intuitively, a walk starts at some vertex and then follows an edge to another, connected vertex then continues to follow edges to vertices until ending at some vertex. If we put our finger on a vertex and trace along edges to another vertex, we have defined a walk. Now that we have a walk, we may define a cycle.

Definition 2.4 (Cycle)**.** A *cycle* is a walk

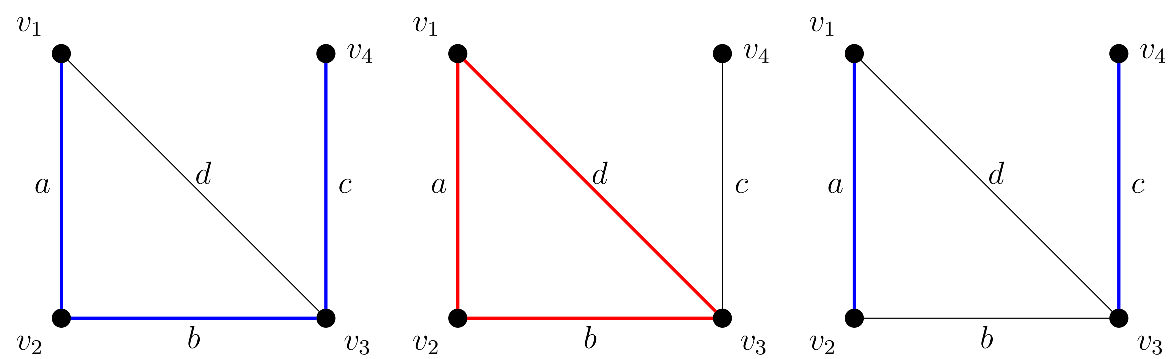
$$(v_1, e_1, v_2, e_2, v_3, \ldots, e_{k-1}, v_k),$$

where $v_1 = v_k$ and $v_i \neq v_j$ when $i \neq j$ otherwise. Further, we say a set of edges *contains a cycle* if there is a cycle whose edges are contained in the set.

That is, a cycle is a walk that starts and ends at the same place and otherwise passes through unique vertices. Given the notion of independence we began to motivate above, hopefully the utility of defining a cycle is apparent. Any subset of edges that contains a cycle must be dependent, as we can always remove the last edge from the walk and still have all the same vertices connected. With this, our definition of independence can finally be formalized.

Definition 2.5 (Independence (of edges of a graph))**.** Let $G = (V, E)$ be a graph. Then a subset of edges $I \subseteq E$ is *independent* if it does not contain a cycle.

Let us immediately take to our example for this section to consider some possible sets of edges.



(a) Independent: the set of edges $\{a, b, c\}$ does not contain a cycle

(b) Dependent: The edges in $\{a, b, d\}$ contain a cycle

(c) Independent: The edge set $\{a, c\}$ does not contain a cycle. It is not necessary for all vertices to be connected

Figure 2.4: Some examples of independent/dependent sets of edges.

Now that we have got some practice under our belt, it is time to play our favorite game again. Given our example graph, $G = (V, E)$, we want to identify the set of all possible independent vectors. A few moments of tracing paths along the graph, hunting for cycles, will reveal that from our set of edges $E = \{a, b, c, d\}$, the independent subsets are precisely

$$\{\emptyset, a, b, c, d, ab, ac, ad, bc, bd, cd, abc, acd, bcd\}.$$

This should look familiar! Suspiciously so, even. The independent subsets are the same as those we found in our collection of vectors. Clearly then all the properties of linearly independent subsets we showed above also hold in this example. Indeed, this is not just a quirk of our example. Given any graph, the independent subsets of the edge set will obey the same properties as the linearly independent subsets of a set of vectors. It was the reoccurrence of these properties across different mathematical objects that inspired the creation of matroids.

2.3 Matroids, Finally

Matroids were developed independently by Hassler Whitney, in the paper *On the Abstract Properties of Linear Dependence* [Whi35], and Takeo Nakasawa, in the series of papers *Zur Axiomatik der linearen Abhängigkeit*[1] [Nak35; Nak36b; Nak36a]. The introduction of Whitney's paper parallels our journey so far, covering, much more succinctly, shared properties of linear independence and independence of graph edges. He then goes on to introduce several equivalent definitions of a matroid.

An interesting feature of matroids is just how many definitions exist. Plenty more have been added since the several introduced by Whitney, and any one of these definitions can be taken axiomatically and from them any other definition may be derived. However, it can be extremely non-obvious that a given definition is equivalent to some other. The path between the various axiomatizations can be so difficult to see that they have been affectionately called *cryptomorphic* to one another.

We will primarily be concerned with two axiomatizations, one based on the

[1] Roughly translated as *On the Axiomatization of Linear Dependence*.

notion of independent sets and another based on what are called *flats*. The first definition follows closely from the background we have developed so far. This allows us to more easily define the terms and properties of matroids that we will need in the second definition. It is this second definition that will be of key importance for the following chapters, so it is important to develop it here.

2.3.1 Independent Set Axioms

The first definition of matroids should, again, look very familiar.

Definition 2.6 (Matroid — Independent Set Axioms)**.** A *matroid* is a pair $\mathcal{M} = (E, \mathcal{I})$, where E is a finite set, called the *ground set*, and $\mathcal{I} \subseteq 2^E$ is a collection of subsets of E, called the *independent sets*, with the following properties:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(I3) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| \leq |I_2|$, then there exists some $e \in I_2 \setminus I_1$ such that $I_1 \cup e \in \mathcal{I}$.

They correspond precisely to the properties we identified in linearly independent subsets and that we saw again in independent edge sets. We can take this opportunity to define our now familiar examples as a matroid.

We name our matroid $\mathsf{M} = (E, \mathcal{I})$, where $E = \{a, b, c, d\}$ and we pick the independent sets to be

$$\mathcal{I} = \{\emptyset, a, b, c, d, ab, ac, ad, bc, bd, cd, abc, acd, bcd\}.$$

Coming as probably no surprise, this has the same independence relations as both our vector example and our graph example. We should confirm that $\mathcal{I}$ obeys the properties (I1)-(I3), but we already know this particular set must.

2.3.1.1 Aside: Representable Matroids

Given that we have already seen the example "matroid" arise twice in other contexts, it is natural to ask if we have gained anything new with matroids. If every matroid could just be studied as a finite collection of vectors and its independent subsets, we don't really have to go through the trouble defining a whole new object.

It turns out that this is not the case. A matroid that can arise from a finite set of vectors, like our example, is called *representable*. However, there are *unrepresentable* matroids. A lot of them in fact.

The distinction between representable and unrepresentable matroids has no bearing on the results of this thesis, but it is worth noting here. Our examples are representable, as it allows us to leverage some visual intuition, but everything we say here holds for all matroids.

2.3.2 The Uphill Path to Flats

A benefit of introducing the independence axioms first, we feel, is that they are readily interpretable. At least after developing a bit of intuition in the realm of linear independence. For much of the rest of our paper however, we won't be thinking of matroids in this form. We will need a formulation of matroids that uses something called *flats*.

To get to this new definition of matroids, or even state what a flat is, we will have to build up our vocabulary surrounding matroids. Our goal here is to develop everything necessary to define a flat. The path there may seem rather wandering, we will introduce quite a few definitions here. But there are no shortcuts; each new definition builds on the last, until we have a nice tower of terms with which to use.

Given their history, matroids borrow a lot of terminology from linear algebra and graph theory. For the most part, their meaning is related to that in the original context, so it can be a useful starting point. Still, it is not necessary to have heard of them before; these definitions exist perfectly fine on their own in the world of matroids, as we shall see.

We use the independent set axioms to define these terms and state properties, but we could have started with any of the axioms and developed all these terms. It is actually quite a fun exercise to develop parallel definitions from different starting axioms.

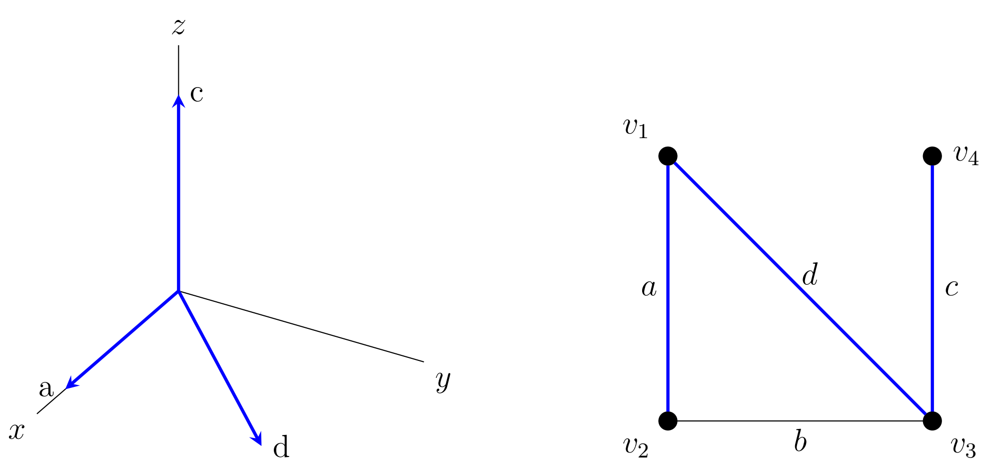
2.3.2.1 All Your Bases Belong to Matroid

First, we will finally address a pattern we have noted earlier, that the largest independent sets all seem to have the same number of elements, or, as we like to say in the business, the same *cardinality*. To do so we will introduce the notion of a basis of a matroid.

Definition 2.7 (Basis)**.** Given a matroid $\mathcal{M} = (E, \mathcal{I})$, an independent set $B \in \mathcal{I}$ is a *basis* of $\mathcal{M}$ if

$$B \cup e \notin \mathcal{I}$$

for all $e \in E \setminus B$. That is to say, a basis B is a maximally independent subset of E with respect to set inclusion.



(a) A basis in linear algebra is a minimal spanning set

(b) A basis of a graph is a spanning tree

Figure 2.5: The set $\{acd\}$ is a basis of our example M, which we can view in the vector and graph setting.

For those recalling their linear algebra, yes, this does have the very useful property we expect from something called a basis [Oxl11, Lemma 1.2.4].

Proposition 2.1. *All bases of a matroid contain the same number of elements.*

As we see in the examples in Figure 2.5, a basis has a very literal interpretation in the context of vector spaces and graphs. If pressed for an intuition of a basis in the more general matroid setting, we'd say that they give us an idea of "how much" (in)dependence is going on amongst the elements in the ground set; likely accompanied by us literally waving our hands through the air. If our matroid has 1000 elements in its ground set, but its bases only have size 3, then there must be a lot of dependence amongst all those elements of the ground set. However vague the idea, it would be very useful to be able to quantify "how much" independence is going on in any subset $X \subseteq E$ of a matroid's ground set.

2.3.2.2 Rank and Closure

Indeed, this is an important enough property to get its own name, the *rank*. The rank of any subset is simply the size of the largest independent subset.

Definition 2.8 (Rank)**.** Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. The *rank function* is the map

$$\begin{aligned} \mathrm{rk}_{\mathcal{M}} \,:\, 2^E &\to \mathbb{Z}_{\geq 0} \\ X &\mapsto |Y| \end{aligned}$$

where $Y \subseteq X$, $Y \in \mathcal{I}$, and there is no $Y \subsetneq Y' \subseteq X$ such that $Y' \in \mathcal{I}$. That is to say, the rank of any subset X is the size of the largest independent set contained in X. We write $\mathrm{rk}_{\mathcal{M}}(E)$ as $\mathrm{rk}_{\mathcal{M}}(\mathcal{M})$, and call it the *rank* of $\mathcal{M}$.

Unless we are in imminent danger of confusion, we will notate $\mathrm{rk}_{\mathcal{M}}(X)$ as just $\mathrm{rk}(X)$. In the land of linear algebra, rank corresponds to the dimension spanned by the vectors. Just as adding more vectors into a linear span won't necessarily increase the dimension spanned, increasing the number of your elements in your subset will not necessarily increase the rank. For instance, in our running example we see that $\mathrm{rk}(ab) = \mathrm{rk}(abd) = 2$. The rank of the matroid itself will be, as we showed above, the size of any basis of the matroid.

This notion that we can add more elements to a subset without changing its rank leads, at last, to the final preliminary definition.

Definition 2.9 (Closure)**.** Given a matroid $\mathcal{M} = (E, \mathcal{I})$, the *closure operator* is a function

$$\begin{aligned} \mathrm{cl}_{\mathcal{M}} \,:\, 2^E &\to 2^E \\ X &\mapsto \{e \in E \mid \mathrm{rk}(X \cup e) = \mathrm{rk}(X)\}. \end{aligned}$$

For any $X \subseteq E$, we call $\mathrm{cl}_{\mathcal{M}}(X)$ the *closure of* X.

Again we will write the closure operator as $\mathrm{cl}(X)$ almost exclusively. Since $\mathrm{cl}(X)$ contains every element of the ground set it can while still being the same rank as X, if we take $\mathrm{cl}\big(\mathrm{cl}(X)\big)$ there won't suddenly be a new element of the ground set we can add without changing rank. This means $\mathrm{cl}\big(\mathrm{cl}(X)\big) = \mathrm{cl}(X)$, a property called *idempotency*. If a basis captures how much "independence" is in a set of elements, the closure of a subset generates a set that is as "dependent" as possible for a given rank (using the elements of that initial set). One might ask if there is anything special about these sets that are as big as they can be with respect to closure. A very insightful question, if we do say so ourselves.

2.3.3 Flats in a Lattice

If you didn't notice our subtle hint above, it may come as a surprise that sets that are as "big" or "dependent" as possible for a given rank are precisely flats.

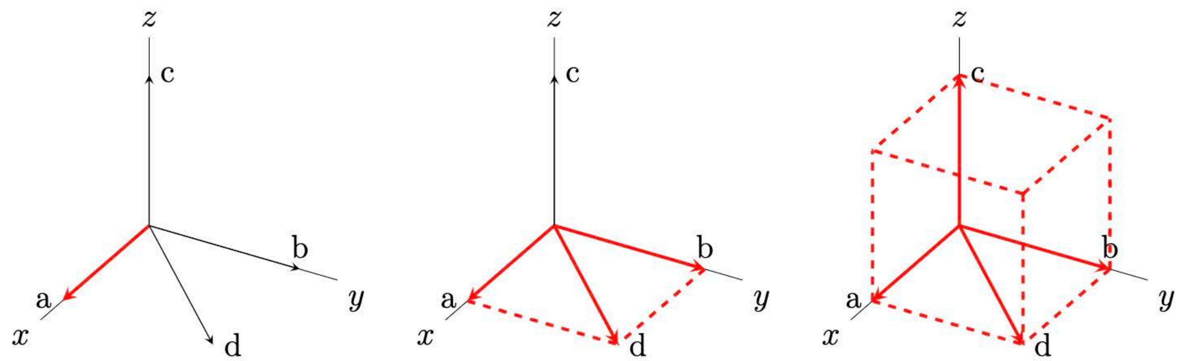
Definition 2.10 (Flat)**.** Given a matroid $\mathcal{M} = (E, \mathcal{I})$ and subset $X \subseteq E$, if

$$X = \mathrm{cl}(X),$$

then X is a *flat* of $\mathcal{M}$.

What if instead of independent sets, we collect all the flats of a matroid. In our running example, we could start applying the closure operator left and right until we collect the set

$$\mathcal{F} = \{\emptyset, a, b, c, d, abd, ac, bc, cd, abcd\}.$$



(a) Rank 1: The single element a is a flat

(b) Rank 2: Note ab is not a flat, but abd is

(c) Rank 3: The matroid itself is rank 3, so the flat of this rank is the whole ground set, $abcd$

Figure 2.6: Examples of flats of rank 1, 2, and 3 in our example matroid M, viewed as vectors.

Since flats are maximal with respect to rank, they naturally divide up by rank; i.e.

$$\mathcal{F}_0 = \{\emptyset\}$$

$$\mathcal{F}_1 = \{a, b, c, d\}$$

$$\mathcal{F}_2 = \{abd, ac, bc, cd\}$$

$$\mathcal{F}_3 = \{abcd\},$$

where $\mathcal{F}_k$ denotes the set of flats of rank k. When laid out like this we may begin to note some interesting patterns. Indeed, just like independent sets have some useful properties, so do the set of flats [Oxl11, pp. 31-32].

Proposition 2.2 (Properties of Flats)**.** *Let $\mathcal{M} = (E, \mathcal{I})$, be a matroid. Then the set of flats*

$$\mathcal{F} = \{X \subseteq E \mid X = \mathrm{cl}(X)\}$$

has the following properties:

(F1) $E \in \mathcal{F}$.

(F2) If $F_1, F_2 \in \mathcal{F}$*, then* $F_1 \cap F_2 \in \mathcal{F}$.

(F3) If $F \in \mathcal{F}$ *and* $F_1, F_2, \ldots, F_k \in \mathcal{F}$ *are the minimal flats such that* $F \subsetneq F_i$*, then the sets* $F_1 \setminus F$, $F_2 \setminus F$, $\ldots$, $F_k \setminus F$ *partition* $E \setminus F$.

Further, let E be any set and $\mathcal{F} \subseteq 2^E$ be a collection of subsets of E such that properties (F1)–(F3) hold. Define

$$\mathrm{cl} \, : \, 2^E \to 2^E$$

such that $\mathrm{cl}(X) = F$ *for some flat* $F \in \mathcal{F}$ *where* $X \subseteq F$ *and there is no* $F' \in \mathcal{F}$ *such that* $X \subseteq F' \subsetneq F'$*. Then* $\mathcal{M} = (E, \mathcal{F})$ *is a matroid with independent set*

$$\mathcal{I} = \{I \subseteq E \mid I_1 \subsetneq I_2 \subseteq I, \, \mathrm{cl}(I_1) \neq \mathrm{cl}(I_2)\}.$$

Let's unpack this proposition, as flats are a bit more difficult than independent sets as a foundation of matroids. Property (F1) says that the ground set, E, is a flat. This follows directly from the fact that the closure of a basis has to be every element of the ground set, since you can't ever get a higher rank than a basis.

The second property (F2) says that the set of flats is closed under intersection; i.e. the elements shared between any two flats is a flat itself. This follows from the properties of closure and a bit of set theory; it is a fun little exercise to prove.

The last property, (F3), looks more intimidating than it is. In essence, if you take a flat, F (with $F \neq E$, since no flats have higher rank than E), then for every element *not* in F you're going to find it in a flat that is one rank higher. This shouldn't be too surprising, since if an element, let's call it x, is not in F, then $\mathrm{cl}(F \cup x)$ will have to have a higher rank than F. That this *partitions* $E \setminus F$ just means that each $e \in E$ that's not in F is going to appear in exactly one flat one rank higher (specifically the flat $\mathrm{cl}(F \cup e)$).

Finally, the proposition asserts that if we start with a set and then a collection of subsets that meet all three properties (F1)–(F3), then that is sufficient to characterize a matroid. That is, we could take (F1)–(F3) as another axiomatization of a matroid. A recommended exercise would be to reconstruct all the definitions in the preceding section starting with just these axioms.

These properties actually impart a very interesting structure on the set of flats that we will now explore.

2.3.3.1 The Lattice of Flats

First, we recall, or learn here and now, that any collection of subsets of a set form a partially ordered set.

Definition 2.11 (Partially Ordered Set)**.** A *partially ordered set*, often called a poset, is a pair $(P, \preceq)$, where P is a set of elements, and $\preceq$ is a relation between some, but not necessarily all, of the elements of P with the properties

i. $a \preceq a$,

ii. if $a \preceq b$ and $b \preceq a$, then $a = b$,

iii. if $a \preceq b$ and $b \preceq c$, then $a \preceq c$,

for all $a, b, c \in P$.

With the definition in hand, we can verify that $(\mathcal{F}, \subseteq)$ is a partially ordered set, where $\mathcal{F}$ is the set of all flats of a matroid. But we can do even better than that. Some posets have an even stronger structure, called a lattice.

Definition 2.12 (Lattice)**.** A partially ordered set $(L, \preceq)$ is a *lattice* if there exist binary operations

$$\vee : L \times L \to L,$$

called a *join*, and

$$\wedge : L \times L \to L,$$

called a *meet*, such that for any two elements $a, b \in L$,

i. the join $a \vee b$ is an element of the lattice such that $a \preceq a \vee b$ and $b \preceq a \vee b$, and for any element $c \in L$ such that $a \preceq c$ and $b \preceq c$ it is the case that $a \vee b \preceq c$,

ii. the meet $a \wedge b$ is an element of the lattice such that $a \wedge b \preceq a$ and $a \wedge b \preceq b$, and for any $c \in L$ such that $c \preceq a$ and $c \preceq b$ then we have $c \preceq a \wedge b$.

If you have never seen this definition before, it can be a bit heavy on symbols, but once we ground it in our set of flats it won't be too bad. First though, we must establish how our flats form a lattice [Oxl11, Lemma 1.7.3].

Proposition 2.3 (The Collection of Flats Forms a Lattice)**.** *Let $\mathcal{M}$ be a matroid and $\mathcal{F}$ be the set of all flats of $\mathcal{M}$. Then $(\mathcal{F}, \subseteq)$ is a lattice, with the operations*

$$F_1 \wedge F_2 = F_1 \cap F_2$$
$$F_1 \vee F_2 = \mathrm{cl}(F_1 \cup F_2)$$

for any $F_1, F_2 \in \mathcal{F}$.

This means we can, and so often will, talk about a *lattice of flats*. To motivate this, let us once again consider our example matroid. It is, if not traditional, convenient to structure a lattice graphically in a *Hasse diagram*.

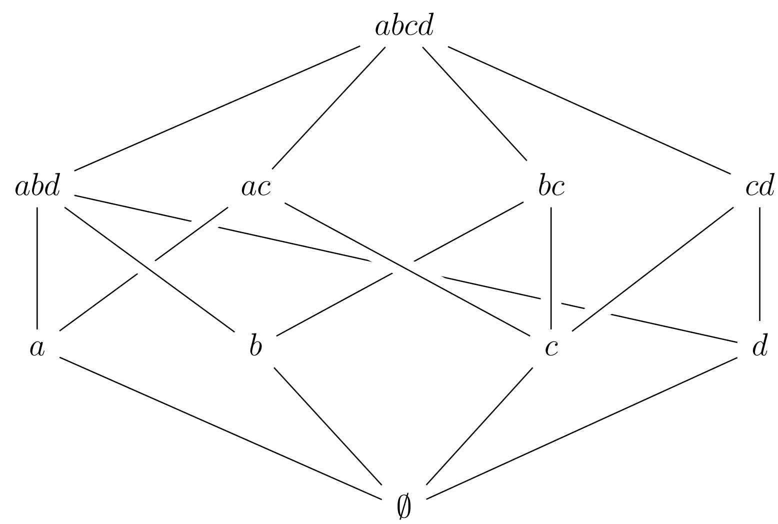


Figure 2.7: The Hasse diagram of flats of our running example matroid.

When reading a Hasse diagram, if we have two entries x and y, with a line connecting them and x is higher on the page than y, we say x *covers* y. This corresponds to the relation $y \preceq x$. In the lattice of flats, each level corresponds to a rank, starting at the bottom, which is rank 0. If F_1 covers F_2, then $F_2 \subset F_1$. All of those properties in the definition of the lattice just mean that taking the intersection of two flats, or the closure of the union of two flats, will uniquely identify another element of the lattice (connected by lines to your original two entries).

When considering a matroid in terms of flats, one often sees $\mathcal{M} = (E, \mathcal{L})$ in lieu of $\mathcal{M} = (E, \mathcal{F})$ as a reminder that the set of flats forms a lattice. We will follow that convention going forward as well.

This lattice structure is key to the construction of our objects of interest in the following chapters, as we will soon see. The final definition we need from matroids are called flags, and they are, basically, just reasonable collections of flats.

2.3.3.2 Our Flag Means Totally-Ordered Subsets of the Lattice of Flats

Given a matroid $\mathcal{M} = (E, \mathcal{L})$, let $\mathcal{L}^*$ be the set of proper flats of $\mathcal{M}$; i.e. all flats with rank greater than 0 and not including E. Since *every* lattice of flats always has 1 element of rank 0 as a minimal element and E as the unique maximal element, $\mathcal{L}^*$ is just the interesting bits of $\mathcal{L}$.

Definition 2.13 (Flag)**.** If $\mathcal{M} = (E, \mathcal{L})$ is a matroid, then a *flag* is a totally ordered subset $\mathscr{F} \subseteq \mathcal{L}^*$ of the proper flats of a matroid,

$$\mathscr{F} = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k\} \subseteq \mathcal{L}^*.$$

If $\mathrm{rk}(\mathcal{M}) = r + 1$, then a flag $\mathscr{F} = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r\}$ is a *maximal* flag of $\mathcal{M}$.

Flags are, then, just collections of flats where you can nest all the flats; a little set theoretic *matryoshka*. On the Hasse diagram, a flag will have at most one element from each rank and there will be a strictly increasing path of lines between all elements of the flag. Maximal flags will be those that take you along a path on the Hasse diagram from rank 1 all the way up to the rank right below that of the matroid itself, including something from every rank in between. One thing to remember is that for every flat F, $\mathscr{F} = \{F\}$ is, indeed, a flag.

2.3.4 New Matroids From Old

Given a matroid, we can make new, smaller matroids called *matroid minors*. These have relations to both the characteristic polynomial and play an important role in some of our later proofs. We will first define them in terms of independent sets, and then return to the relations on flats.

Let's say we already have some matroid $\mathcal{M} = (E, \mathcal{I})$. Then $\mathcal{I}$ already has a notion about which of all possible subsets of E are independent. So if we consider some subset $X \subseteq E$ of the ground set, we should be able to use $\mathcal{M}$'s independent sets to construct independent sets for X as a ground set. This is in fact very easy to do, and we call the resulting matroid a restriction matroid.

Definition 2.14 (Restriction Matroid)**.** Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. Then for any subset $X \subseteq E$, we may define the *restriction matroid*, $\mathcal{M}|X$, as

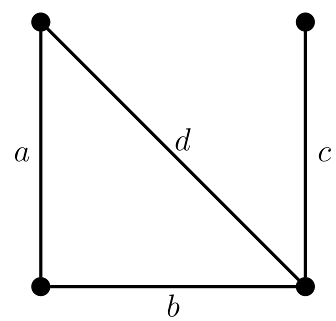
$$\mathcal{M}|X = (X, \mathcal{I}|X)$$

where $\mathcal{I}|X = \{I \in \mathcal{I} \mid I \subseteq X\}$.

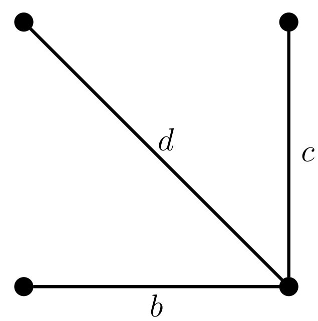
Essentially we just declare X to be the new ground set and just forget about any independent sets of $\mathcal{M}$ that contain any elements not in X. Rather than providing a subset to restrict to, one often finds it useful to specify just the things we want to forget.

Definition 2.15 (Deletion Matroid)**.** Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and $Y \subseteq E$. The matroid that results from the *deletion* of Y from $\mathcal{M}$, sometimes called a *deletion matroid*, is defined as

$$\mathcal{M} \backslash Y = \mathcal{M}|(E \setminus Y).$$



A graph representation of a matroid $\mathcal{M}$



A graph representation of the deletion $\mathcal{M} \setminus a$

Figure 2.8: Deletion matroids generalize deletion in graphs, where we quite literally delete edges.

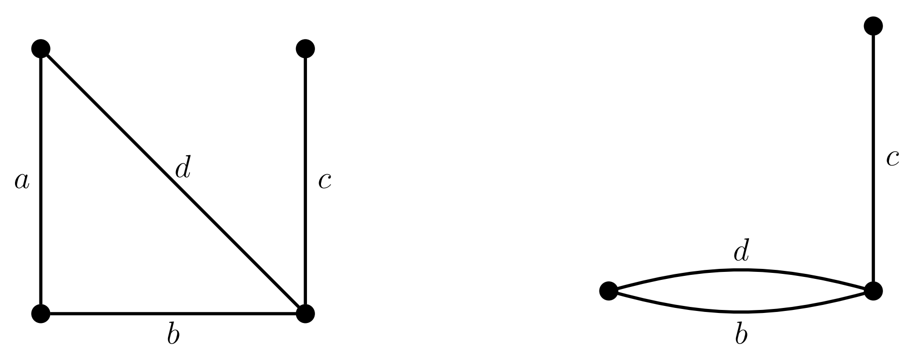
The other way to build a matroid out of an existing one is a little less obvious. These are called contraction matroids, and they are *dual* to restriction matroids. While they are a bit easier to define using duality, we want to avoid introducing all the machinery for that. Still, as mathematicians we feel compelled to point out duality anytime we see it.

Definition 2.16 (Contraction Matroids)**.** Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. For any subset $T \subseteq E$ of the ground set, construct the restriction matroid $\mathcal{M}|T$ and choose a basisB_T of $\mathcal{M}|T$. The *contraction matroid*, $\mathcal{M}/T$, is defined as

$$\mathcal{M}/T = (E \setminus T, \mathcal{I}/T),$$

where $\mathcal{I}/T = \{I \subseteq (E \setminus T) \mid I \cup B_T \in \mathcal{I}\}$.

This definition is more difficult to explain succinctly, but we can compare it with the restriction matroid to try to get some sense of what this does. We can think of restriction matroid as imparting independence on a subset $X \subseteq E$ by saying subsets are independent if they would be independent in the original matroid. The contraction matroid, then, assigns independence on everything *not* in the subset $T \subseteq E$, based on if they'd still be independent if we were to add (a basis of) T back in.



A graph representation of a matroid, $\mathcal{M}$ A graph representing the contraction $\mathcal{M}/a$

Figure 2.9: The term contraction also originates in graph theory, where there is some decent visual intuition.

Importantly, we can combine deletion and contraction, and indeed the resulting matroids are a rather central point of study in matroid theory.

Definition 2.17 (Matroid Minor)**.** A *minor* of a matroid $\mathcal{M}$ is any matroid resulting in any combination of deletions and contractions of $\mathcal{M}$. Since, any series of deletions and contractions can always be rearranged to one deletion and one contraction, any matroid minor is of the form

$$\mathcal{M}\backslash X/Y,$$

where $X, Y \subseteq E$ are disjoint and possibly empty. When $X \cup Y$ is nonempty, we call $\mathcal{M}\backslash X/Y$ a *proper minor* of $\mathcal{M}$.

2.3.4.1 Matroid Minors and Flats

While we have defined restriction and contraction matroids in terms of independent sets, we have clearly established that we are all about flats here. Luckily, we have a very useful property relating the lattice of minors to the lattice of the original matroid. First though, a little notation. If F is a flat of $\mathcal{M}$, we will define

$$\mathcal{M}_{[\emptyset,F]} = \mathcal{M}|F$$

to be the restriction by F, and

$$\mathcal{M}_{[F,E]} = \mathcal{M}/F$$

to be the contraction by F. For any two flats F_1 and F_2 of $\mathcal{M}$, we write

$$\mathcal{M}_{[F_1,F_2]} = \mathcal{M}/F_1 \backslash (E \setminus F_2)$$

to be the minor that results from contracting by F_1 and restricting to F_2. Notation in hand, we can now state a classic result of matroid theory, which can be found, unsurprisingly, in [Oxl11, p. 116].

Proposition 2.4. *Let F_1 and F_2 be flats of a matroid $\mathcal{M} = (E, \mathcal{L})$. Then the lattice of flats of the minor $\mathcal{M}_{[F_1,F_2]}$, $\mathcal{L}_{[F_1,F_2]}$ is isomorphic to the interval of $\mathcal{L}$*

$$[F_1, F_2] = \{F_1 \subseteq F \subseteq F_2 \mid F \in \mathcal{L}\}$$

given by the isomorphism

$$\varphi : \mathcal{L}_{[F_1,F_2]} \to \mathcal{L}$$
$$\varphi(F) = F \cup F_1.$$

This means the lattice of a minor can be "seen" within the lattice structure of our original matroid, just up to some relabeling of the nodes. We note that this proposition only works for flats, not arbitrary subsets of the ground set, but that's more than enough for what we need. If F_1 and F_2 are adjacent to each other in the lattice of flats, then $\mathcal{M}_{[F_1,F_2]}$ is isomorphic to a matroid whose sole flag is

$\{\emptyset \subsetneq F_2 \setminus F_1\}$.

2.3.5 'Tis the Gift to Be Simple

If a serious matroid theorist is, for some inexplicable reason, subjecting themselves to this section, we feel the need to admit one simplifying assumption we intend to make (and have implicitly made with our example). Since we care primarily about the lattice structure of our matroid, we assume all of our matroids are *simple*.

For the rest of us, the non-serious, a brief explanation. A matroid is simple if it does not have any *loops*, elements in the ground set that have rank 0, or *parallel edges*, sets of elements that share identical independence relations. We will briefly return to loops, but in general we will not have to worry about them. If this feels overly restrictive, worry not, for Oxley[Oxl11, p. 49] comes to our rescue.

Proposition 2.5 (Simplification Preserves Lattice Structure)**.** *For any matroid $\mathcal{M}$, there exists a unique, up to labeling, matroid* $\mathrm{si}(\mathcal{M})$*, called the* simplification of $\mathcal{M}$ *such that*

i. $\mathrm{si}(\mathcal{M})$ *is simple,*

ii. if $\mathcal{L}$ is the lattice of flats of $\mathcal{M}$ and $\mathcal{L}'$ is the lattice of flats of $\mathrm{si}(\mathcal{M})$*, then*

$$\mathcal{L} \cong \mathcal{L}'.$$

If we care mostly about the lattice of matroids, then we can take any matroid and find a simple matroid with an identical lattice structure. We will see the main practical benefit of working with simple matroids in the next section. However, we also get convenience, we don't have to keep track of unnecessary letters, and aesthetics, the lattice diagrams look much nicer, as a bonus. If we take our matroid to be simple, then our lattice structure has the following properties.

Proposition 2.6 (Properties of the Lattice of Simple Matroids). *Let $\mathcal{M} = (E, \mathcal{L})$ be a simple matroid. Then*

i. the empty set is the minimal, rank 0, element of $\mathcal{L}$,

ii. for every $e \in E$, there is unique rank 1 flat, F_e, such that $F_e = e$,

iii. for any flat $F \in \mathcal{L}$, if $e \in F$, then $F_e \subseteq F$,

iv. we can write any flat $F \in \mathcal{L}$ as a disjoint union of rank 1 flats; $F = \biguplus_{e \in F} F_e$.

If this seems like a lot, the big takeaway is that this promises that the very bottom of our lattice will always be the empty set, and that the rank 1 flats correspond to the elements of the ground set. For those coming in with lattice knowledge, the second two properties mean the lattice of a simple matroid is *atomic*. We can verify these properties in our example, M, which is a simple matroid.

Our admission of simplification done, we have now learned everything we need about the construction of matroids. It is time to learn about some polynomials.

2.4 The Characteristic Polynomial

The conjecture by Heron, Rota, and Welsh, that we promise we are getting to, has to deal with the characteristic polynomial of a matriod. This is some polynomial we can cook up using the structure of a matroid, which is fair enough. But when presented on its own, it feels, at least to us, that it comes out of nowhere. Why anyone would make up this polynomial or why we'd start conjecturing about it is not at all clear. So first, a little history back in the realm of graphs.

2.4.1 Coloring Graphs and the Chromatic Polynomial

Let us play another game. This time, pick a graph, G, like the one pictured below.

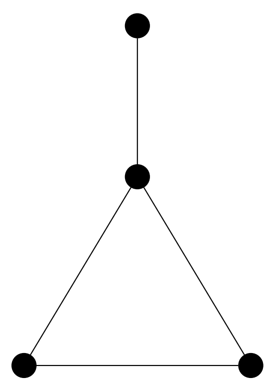


Figure 2.10: An example graph, G.

Let's say we have three colors, and we want to color the vertices of the graph so that no two connected vertices have the same color. Such an arrangement of colors would be called a 3-coloring of G. It is not too hard to come up with some colors that work.

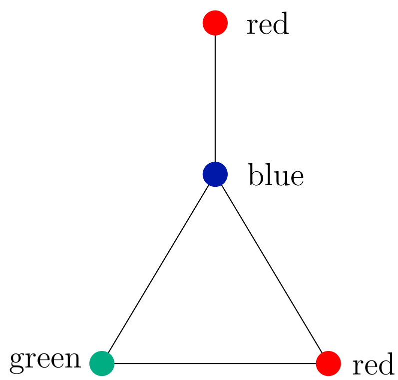


Figure 2.11: A 3-coloring of G.

But now, suppose we wanted to know how many unique ways we could use those three colors to color the graph. This isn't too bad. We could just get out our markers and start coloring lots of graphs. Honestly, it sounds relaxing.

But now let's suppose we want to know how many ways we can use 1000 colors to color our little graph, or 10,000, or a billion. Since our set of markers only has 12 distinct colors, we will have to turn to math to solve this one.

The strategy is not too complicated, just pick a vertex and say how many colors we have to choose from, then find a connected vertex that hasn't been assigned a color yet, and say how many colors it is allowed to choose from. Repeat until we are out of vertices to label. Instead of picking a specific number, let's say we have n colors to choose from.

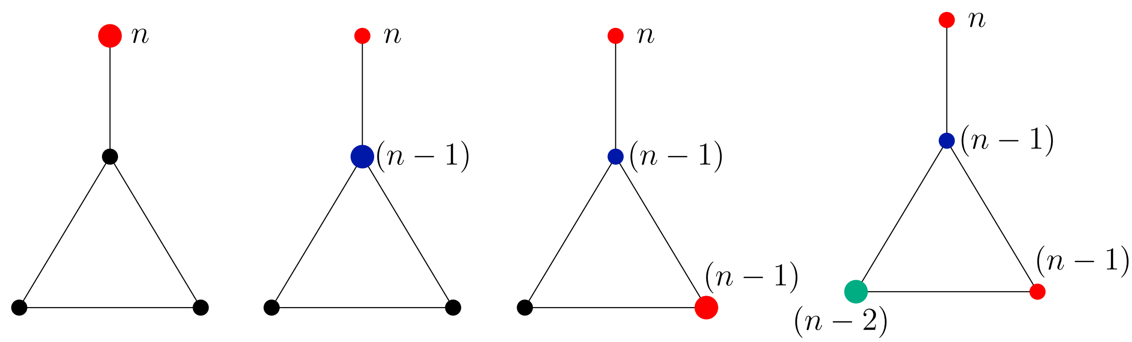


Figure 2.12: The process of figuring out the number of n-colorings of G; the choice of starting vertex doesn't matter, though that's not necessarily obvious.

We then just have to multiply the number of possibilities together. For G, if we have n colors to choose from, there are $n(n-1)^2(n-2)$ different ways to arrange those colors on the graph. We have just discovered the *chromatic polynomial* of G. But there's nothing particularly special about our choice of graph, we will get something like this for any graph we come up with.

Definition 2.18 (Chromatic Polynomial of a Graph)**.** Let $\chi_G(n)$ be the number of n-colorings of graph G. Then the map

$$\begin{aligned} \mathbb{N} &\to \mathbb{N} \\ z &\mapsto \chi_G(z) \end{aligned}$$

is a polynomial with integer coefficients, known as the *chromatic polynomial* of G.

For our purposes we will always expand our polynomials, so for our worked example above we have

$$\chi_G(z) = z^4 - 4z^3 + 5z^2 - 2z.$$

Early work in chromatic polynomials was done by none other than our good friend Whitney [Whi32], and expanded on by the mathematician Tutte in his development of what we now call Tutte polynomials [Tut54].

2.4.2 The Characteristic Polynomial of a Matroid

It was following in this work on chromatic polynomials that Gian-Carlo Rota, who you may recognize as usually sandwiched between Heron and Welsh, extended this concept to matroids [Rot64]. To do this, Rota extended something called the *Möbius function* to lattices (technically any locally finite poset), which for matroids means it uses the lattice structure of the flats. We present an equivalent definition that's easier to state, but with the downside that the relationship to the lattice is obfuscated.

Definition 2.19 (Characteristic Polynomial)**.** Let $\mathcal{M} = (E, \mathcal{L})$ be a matroid. Then the *characteristic polynomial* of $\mathcal{M}$ is given by

$$\chi_{\mathcal{M}}(z) = \sum_{X \subseteq E} (-1)^{|X|} z^{\mathrm{rk}(\mathcal{M}) - \mathrm{rk}(X)}.$$

You may notice that each term of the polynomial will have a power of z between

0 and $\mathrm{rk}(\mathcal{M})$. The Heron–Rota–Welsh conjecture is about the coefficients of this polynomial, specifically once we collect the terms.

Definition 2.20 (Whitney Numbers of the First Kind)**.** Let $\mathcal{M}$ be a matroid with characteristic polynomial

$$\begin{aligned}\chi_{\mathcal{M}}(z) &= \sum_{X\subseteq E} (-1)^{|X|} z^{\mathrm{rk}(\mathcal{M})-\mathrm{rk}(X)} \\ &= \sum_{k=0}^{\mathrm{rk}(\mathcal{M})} (-1)^k w_k z^{\mathrm{rk}(\mathcal{M})-k}.\end{aligned}$$

The unsigned portion of the coefficients, $w_0, w_1, \ldots, w_{\mathrm{rk}(\mathcal{M})}$, are the *Whitney numbers of the first kind*.

We will return to these numbers soon, as they are main players of the Heron–Rota–Welsh conjecture.

2.4.3 Properties of the Characteristic Polynomial

Let's establish some important properties of the characteristic polynomial. We mentioned loops while talking about simple matroids, but we should introduce them, and their dual coloops, officially. While there are many equivalent definitions, we will use the ones that relate to how they interact with closures and rank.

Definition 2.21 (Loop)**.** For a matroid $\mathcal{M}$ with ground set $e \in E$, we say e is a *loop* if for all $X \subseteq E$

$$e \in \mathrm{cl}(X).$$

Equivalently, this means

$$\mathrm{rk}(X) = \mathrm{rk}(X \cup e)$$

for all subsets $X \subseteq E$.

As we said above, loops are elements with rank 0. They appear in every closure, and their addition never changes the rank of a set. They generalize loops in a graph, where the term is particularly apt.

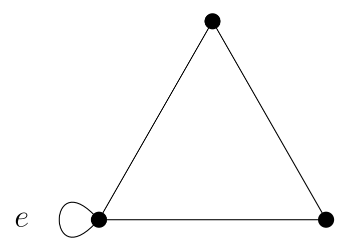


Figure 2.13: The edge e here is a loop; it connects a vertex to itself.

Dual to a loop, is the coloop. It is slightly less intuitive when defined it terms of closure and rank, but has very useful properties.

Definition 2.22 (Coloop)**.** For a matroid $\mathcal{M}$ with ground set $e \in E$, we say e is a *coloop* if

$$e \in \mathrm{cl}(X)$$

implies

$$e \in X$$

for any $X \subseteq E$. Equivalently, for all $X \subseteq E \setminus e$

$$\mathrm{rk}(X \cup e) = \mathrm{rk}(X) + 1.$$

Where a loop is found in every closure, coloops are only found in closures of sets they are already in. This means they'll increase the rank of any set that doesn't already include them by 1. In graph theory, and some older matroid texts, a coloop sometimes goes by the much more evocative term *isthmus*; a narrow strip of land with sea on either side that links two larger areas of land.

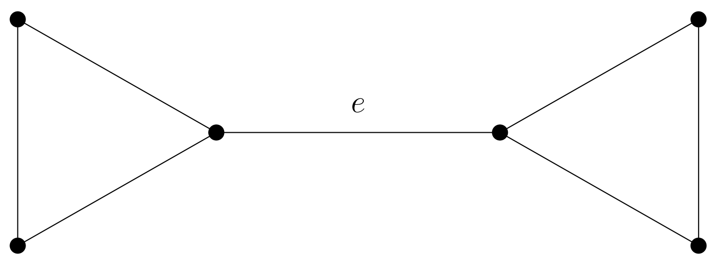


Figure 2.14: The edge e is a coloop; adding it to a cycleless set of edges will never create a cycle.

Now that we know what loops and coloops are, we can state some very useful properties of the characteristic polynomial.

Proposition 2.7. *Let $\mathcal{M}$ be a matroid with ground set E. The characteristic polynomial $\chi_{\mathcal{M}}(z)$ has the following properties:*

(χ1) *If $\mathcal{M}$ contains a loop then*

$$\chi_{\mathcal{M}}(z) = 0.$$

(χ2) *If* $e \in E$ *is a coloop then*

$$\chi_{\mathcal{M}}(z) = (z-1)\chi_{\mathcal{M}\setminus e}(z).$$

(χ3) *If* $e \in E$ *is neither a loop nor a coloop then*

$$\chi_{\mathcal{M}}(z) = \chi_{\mathcal{M}\setminus e}(z) - \chi_{\mathcal{M}/e}(z).$$

Proof. Let $\mathcal{M}$ be a matroid with E as its ground set. We will tackle these properties in order, since they both increase in complexity and the tools we use build on each other.

(χ1) Assume $\mathcal{M}$ contains a loop $\ell \in E$. We will be using the fact that we can write the power set of E as the disjoint union

$$2^E = \{X \mid X \subseteq E \setminus \ell\} \uplus \{X \cup \ell \mid X \subseteq E \setminus \ell\}.$$

This gives us

$$\begin{aligned}
\chi_{\mathcal{M}}(z) &= \sum_{X\subseteq E} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M})-\mathrm{rk}_{\mathcal{M}}(X)} \\
&= \sum_{X\subseteq E\setminus \ell} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M})-\mathrm{rk}_{\mathcal{M}}(X)} + \sum_{X\subseteq E\setminus \ell} (-1)^{|X\cup \ell|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M})-\mathrm{rk}_{\mathcal{M}}(X\cup \ell)} \\
&= \sum_{X\subseteq E\setminus \ell} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M})-\mathrm{rk}_{\mathcal{M}}(X)} - \sum_{X\subseteq E\setminus \ell} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M})-\mathrm{rk}_{\mathcal{M}}(X)} \\
&= 0.
\end{aligned}$$

The third equality comes from two facts. One, since $\ell \notin X$ by construction $|X \cup \ell| = |X| + 1$, and of course $(-1)^{|X|+1} = -(-1)^{|X|}$. Two, from our definition of a loop we have that $\mathrm{rk}_{\mathcal{M}}(X \cup \ell) = \mathrm{rk}_{\mathcal{M}}(X)$. We see then that any matroid with a loop has a characteristic polynomial that is simply 0.

($\chi 2$) Now, let us assume that $e \in E$ is a coloop of $\mathcal{M}$. We will be using the fact that $\mathrm{rk}_{\mathcal{M}}(X \cup e) = \mathrm{rk}_{\mathcal{M}}(X) + 1$ for $X \subseteq E$ whenever $e \notin X$ often in this step. Next, recall from definitions that $\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) = \mathrm{rk}_{\mathcal{M}}(E)$. The deletion matroid $\mathcal{M} \setminus E$ has a ground set $E \setminus e$ and all the independent sets of $\mathcal{M}$ that don't contain e. Thus, the rank functions agree for any subset of E that doesn't contain e; i.e., $\mathrm{rk}_{\mathcal{M}\setminus e}(X) = \mathrm{rk}_{\mathcal{M}}(X)$ for any $X \subseteq E \setminus e$. From the fact e is a coloop, $\mathrm{rk}_{\mathcal{M}}\big((E \setminus e) \cup e\big) = \mathrm{rk}_{\mathcal{M}}(E \setminus e) + 1$, or alternatively

$$\mathrm{rk}_{\mathcal{M}}(E \setminus e) = \mathrm{rk}_{\mathcal{M}}(E) - 1.$$

Since $\mathrm{rk}_{\mathcal{M}}(\mathcal{M})$ is just notational shorthand for $rk_{\mathcal{M}}(E)$, we can instead write

$$\mathrm{rk}_{\mathcal{M}\setminus e}(\mathcal{M} \setminus e) = \mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - 1.$$

With these things established, we see that

$$\begin{aligned}
\chi_{\mathcal{M}}(z) &= \sum_{X \subseteq E} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \mathrm{rk}_{\mathcal{M}}(X)} \\
&= \sum_{X \subseteq E\setminus e} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \mathrm{rk}_{\mathcal{M}}(X)} && + \sum_{X \subseteq E\setminus e} (-1)^{|X \cup e|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \mathrm{rk}_{\mathcal{M}}(X \cup e)} \\
&= \sum_{X \subseteq E\setminus e} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \mathrm{rk}_{\mathcal{M}}(X) + (1-1)} && + \sum_{X \subseteq E\setminus e} (-1)^{|X \cup e|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \mathrm{rk}_{\mathcal{M}}(X \cup e)} \\
&= \sum_{X \subseteq E\setminus e} (-1)^{|X|} z^{\left(\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - 1\right) - \mathrm{rk}_{\mathcal{M}}(X) + 1} && + \sum_{X \subseteq E\setminus e} (-1)^{|X \cup e|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \mathrm{rk}_{\mathcal{M}}(X \cup e)} \\
&= z \sum_{X \subseteq E\setminus e} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}\setminus e}(\mathcal{M}\setminus e) - \mathrm{rk}_{\mathcal{M}\setminus e}(X)} && + \sum_{X \subseteq E\setminus e} (-1)^{|X \cup e|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \mathrm{rk}_{\mathcal{M}}(X \cup e)} \\
&= z\chi_{\mathcal{M}\setminus e}(z) && + \sum_{X \subseteq E\setminus e} (-1)^{|X \cup e|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \mathrm{rk}_{\mathcal{M}}(X \cup e)} \\
&= z\chi_{\mathcal{M}\setminus e}(z) && - \sum_{X \subseteq E\setminus e} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \mathrm{rk}_{\mathcal{M}}(X \cup e)} \\
&= z\chi_{\mathcal{M}\setminus e}(z) && - \sum_{X \subseteq E\setminus e} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \left(\mathrm{rk}_{\mathcal{M}}(X) + 1\right)} \\
&= z\chi_{\mathcal{M}\setminus e}(z) && - \sum_{X \subseteq E\setminus e} (-1)^{|X|} z^{\left(\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - 1\right) - \mathrm{rk}_{\mathcal{M}}(X)} \\
&= z\chi_{\mathcal{M}\setminus e}(z) && - \sum_{X \subseteq E\setminus e} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}\setminus e}(\mathcal{M}\setminus e) - \mathrm{rk}_{\mathcal{M}\setminus e}(X)} \\
&= z\chi_{\mathcal{M}\setminus e}(z) && -\chi_{\mathcal{M}\setminus e}(z) \\
&= (z-1)\chi_{\mathcal{M}\setminus e}(z),
\end{aligned}$$

as desired.

($\chi 3$) Finally let us assume $e \in E$ is neither a loop nor a coloop. We will start with the same general strategy of splitting subsets of E into disjoint sets across the sum,

$$\begin{aligned}
\chi_{\mathcal{M}}(z) &= \sum_{X \subseteq E} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \mathrm{rk}_{\mathcal{M}}(X)} \\
&= \sum_{X \subseteq E \setminus e} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \mathrm{rk}_{\mathcal{M}}(X)} + \sum_{X \subseteq E \setminus e} (-1)^{|X \cup e|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \mathrm{rk}_{\mathcal{M}}(X \cup e)} \\
&= \sum_{X \subseteq E \setminus e} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \mathrm{rk}_{\mathcal{M}}(X)} - \sum_{X \subseteq E \setminus e} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \mathrm{rk}_{\mathcal{M}}(X \cup e)}.
\end{aligned}$$

To prevent this from becoming even more unwieldy, we will look at each sum in this expression individually. Let's look at

$$\sum_{X \subseteq E \setminus e} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \mathrm{rk}_{\mathcal{M}}(X)}$$

first. Since e is not a coloop, $\mathrm{rk}_{\mathcal{M}}(E \setminus e) = \mathrm{rk}_{\mathcal{M}}(E)$, and so $\mathrm{rk}_{\mathcal{M} \setminus e}(\mathcal{M} \setminus e) = \mathrm{rk}_{\mathcal{M}}(\mathcal{M})$. This gives us,

$$\begin{aligned}
\sum_{X \subseteq E \setminus e} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \mathrm{rk}_{\mathcal{M}}(X)} &= \sum_{X \subseteq E \setminus e} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M} \setminus e}(\mathcal{M} \setminus e) - \mathrm{rk}_{\mathcal{M} \setminus e}(X)} \\
&= \chi_{\mathcal{M} \setminus e}(z). \qquad \text{(i)}
\end{aligned}$$

Next we will look at

$$\sum_{X \subseteq E \setminus e} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \mathrm{rk}_{\mathcal{M}}(X \cup e)}.$$

Now we need the fact about contraction minors, namely, $\mathrm{rk}_{\mathcal{M}/e}(X) = \mathrm{rk}_{\mathcal{M}}(X \cup e) - \mathrm{rk}_{\mathcal{M}}(e)$ for $X \subseteq E \setminus e$. Since e is not a loop, this means

$$\mathrm{rk}_{\mathcal{M}/e}(X) = \mathrm{rk}_{\mathcal{M}}(X \cup e) - 1.$$

We can apply this to $E \setminus e \subseteq E$ to get that $\mathrm{rk}_{\mathcal{M}/e}(E \setminus e) = \mathrm{rk}_{\mathcal{M}}(E) - \mathrm{rk}_{\mathcal{M}}(e)$, and so

$$\mathrm{rk}_{\mathcal{M}/e}(\mathcal{M}/e) = \mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - 1.$$

Now we need another way to split up our sum. For any $X \subseteq E \setminus e$ it is either the case that $e \in \mathrm{cl}_{\mathcal{M}}(X)$ or $e \notin \mathrm{cl}_{\mathcal{M}}(X)$. This lets us write

$$\begin{aligned}
\sum_{X \subseteq E \setminus e} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \mathrm{rk}_{\mathcal{M}}(X \cup e)} &= \sum_{\substack{X \subseteq E \setminus e \\ e \in \mathrm{cl}_{\mathcal{M}}(X)}} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \mathrm{rk}_{\mathcal{M}}(X \cup e)} + \sum_{\substack{X \subseteq E \setminus e \\ e \notin \mathrm{cl}_{\mathcal{M}}(X)}} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \mathrm{rk}_{\mathcal{M}}(X)} \\
&= \sum_{\substack{X \subseteq E \setminus e \\ e \in \mathrm{cl}_{\mathcal{M}}(X)}} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - \mathrm{rk}_{\mathcal{M}}(X \cup e)} + \sum_{\substack{X \subseteq E \setminus e \\ e \notin \mathrm{cl}_{\mathcal{M}}(X)}} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - (\mathrm{rk}_{\mathcal{M}}(X) + 1)} \\
&= \sum_{X \subseteq E \setminus e} (-1)^{|X|} z^{\left(\mathrm{rk}_{\mathcal{M}}(\mathcal{M}) - 1\right) - \left(\mathrm{rk}_{\mathcal{M}}(X \cup e) - 1\right)} \\
&= \sum_{X \subseteq E \setminus e} (-1)^{|X|} z^{\mathrm{rk}_{\mathcal{M}/e}(\mathcal{M}/e) - \mathrm{rk}_{\mathcal{M}/e}(X)} \\
&= \chi_{\mathcal{M}/e}(z). && \text{(ii)}
\end{aligned}$$

Our third equality comes from the observation that

$$\big(\mathrm{rk}_{\mathcal{M}}(\mathcal{M})-1\big)-\big(\mathrm{rk}_{\mathcal{M}}(X\cup e)\big)=\begin{cases}\mathrm{rk}_{\mathcal{M}}(\mathcal{M})-\mathrm{rk}_{\mathcal{M}}(X) & e\in \mathrm{cl}_{\mathcal{M}}(X)\\ \mathrm{rk}_{\mathcal{M}}(\mathcal{M})-\big(\mathrm{rk}_{\mathcal{M}}(X)+1\big) & e\notin \mathrm{cl}_{\mathcal{M}}(X).\end{cases}$$

Combining results of (i) and (ii), we have that

$$\chi_{\mathcal{M}}(z)=\chi_{\mathcal{M}\backslash e}(z)-\chi_{\mathcal{M}/e}(z)$$

completing the last portion of our proof. □

Property (χ3) is called the *deletion-contraction* property. This property lets us wrap up the graph connection. Since any graph can be represented by a matroid, and the characteristic polynomial is in some sense inspired by the chromatic polynomial, it would be natural to ask if there is a relation between them. And there is, in fact, a very nice one.

Proposition 2.8. *Let G be a graph and $\mathcal{M}(G)$ be the matroid that comes from G. Then*

$$\chi_G(z)=z^c\chi_{\mathcal{M}(G)}(z),$$

where c is the number of connected components of G.

The difference stems from the fact that the chromatic polynomial of a graph satisfies the more general deletion-contraction property

$$\chi_G(z) = \chi_{G\setminus e}(z) - \chi_{G/e}(z)$$

for any edge e of graph G, regardless of if it is a loop, coloop, or otherwise. For those who want more on the connections between these values, and how they relate to the more general Tutte polynomial, we found the overview given by Ardila, [Ard22], to be a great help.

Finally, we can state some known properties of the Whitney numbers of the first kind. From the work of Gian-Carlo Rota, [Rot64, Theorem 4], we have the following property, which follows from Proposition 2.7.

Proposition 2.9. *Let $\mathcal{M}$ be a simple matroid. Then the Whitney numbers of the first kind of $\chi_{\mathcal{M}}(z)$, $w_0, w_1, \ldots, w_{\mathrm{rk}(\mathcal{M})}$, are strictly positive and $w_0 = 1$.*

Technically, Rota's theorem says they are non-zero and that they alternate in sign. We have defined the Whitney numbers of the first kind to be just the unsigned component of the coefficient, so we adjusted our property accordingly. Rota's paper uses the language of geometric lattices, which are precisely the lattices arising from flats of simple matroids. We found [Zas87] and [Aig87] to be a helpful bridge between Rota's work and how we understand the characteristic polynomial. Speaking of Rota, we may now, at long last, move on to his conjecture.

2.5 The Heron–Rota–Welsh Conjecture

We have all the knowledge of matroids necessary to state the Heron–Rota–Welsh conjecture. Developed and formalized by Heron [Her72], Rota [Rot70], and Welsh [Wel76], this was a conjecture about the coefficients of the characteristic polynomial of matroids. We say "was" because, as noted in the introduction, this has proven by Adiprasito, Huh, and Katz [AHK18]. We are going to keep calling it a conjecture though. First, a few definitions necessary to carefully state the conjecture.

Definition 2.23 (Unimodal)**.** A sequence of numbers $x_0, x_1, \dots, x_k$ is called *unimodal* if there exists an index i such that

$$x_0 \leq x_1 \leq \cdots \leq x_i \geq \cdots \geq x_{k-1} \cdots x_k.$$

The values of a unimodal sequence get larger until a certain point, and after they start to decrease. We can define an even stronger condition.

Definition 2.24 (Log-Concavity)**.** A sequence of numbers $x_0, x_1, \dots, x_k$ is called *logarithmically concave*, or log-concave, if

$$x_i^2 \geq x_{i-1}x_{i+1}$$

for $0 < i < n$.

When all x_i are positive, log-concavity implies the sequence is also unimodal.

This then is what allows us to at last state the conjecture.

Theorem 2.10 (Heron–Rota–Welsh Conjecture)**.** *Let $\mathcal{M}$ be a matroid. If $w_0, w_1, \ldots, w_{\mathrm{rk}(\mathcal{M})}$ are the Whitney numbers of the first kind, then*

$$w_i^2 \geq w_{i-1}w_{i+1}$$

for $0 < i < \mathrm{rk}(\mathcal{M})$. That is, the absolute values of the coefficients of the characteristic polynomial of $\mathcal{M}$ are log-concave.

Since we want to show something about the characteristic polynomials of the matroid, we need a way to study it. To do so, we are going to find the characteristic polynomial in some unexpected places, and then leverage properties of those other settings.

Chapter 3

Chow Rings

It is time now to delve into the world of algebra, developing the notion of a Chow ring of a matroid. The primary goal of this section will be to establish the link between the Chow ring and the characteristic polynomial of a matroid. This will form the first segment of our bridge from combinatorics to geometry.

This section will take some algebraic knowledge for granted; we are not going to define a ring, for example. The main takeaway will be in the last section and should be enough to move on to the next chapter. However, even for those with some basic algebra we don't expect Chow rings to be a familiar object, so our first order of business is to define them.

3.1 What is a Chow Ring?

Broadly, Chow rings are a tool from algebraic geometry for studying the intersections of algebraic varieties. Chow groups of an algebraic variety are equivalence classes of algebraic cycles of its subvarieties, graded by their codimension. They were named after Wei-Liang Chow, who formalized these cycles in [Cho56]. Under certain conditions, a product structure can be induced on these groups to give a ring, which encodes additional information about the intersections of the subvarieties.

For those who don't already have some idea of what a Chow ring is, the above probably invites more questions than it clarifies. Happily, as hinted by the fact that the title of this chapter is not "A Brief Introduction to Algebraic Geometry and Intersection Theory", we will not have to tackle the full theory to get a result here. While there are algebraic varieties hiding in the wings, we will see that the combinatorial data of a matroid allow us to define its corresponding Chow ring quite directly.

3.1.1 The Chow Ring of a Matroid

For our purposes, we will define the Chow ring to be this "short-cut" construction. That this construction corresponds to the idea of the Chow ring presented above, at least for representable matroids, comes from the work started by De Concini and Procesi [DP95] and generalized to the form we will use by Feichtner and Yuzvinsky [FY04].

Definition 3.1 (The Chow Ring of a Matroid)**.** Let $\mathcal{M} = (E, \mathcal{L})$ be a matroid.

Associate a polynomial ring with $\mathcal{M}$ given by

$$P_{\mathcal{M}} = \mathbb{R}[x_F \mid F \in \mathcal{L}^*],$$

and let

$$\begin{aligned} I_{\mathcal{M}} &= \langle x_{F_1} x_{F_2} \mid F_1 \not\subseteq F_2 \,\text{and}\, F_2 \not\subseteq F_1 \rangle\,, \\ J_{\mathcal{M}} &= \left\langle \sum_{e_1 \in F} x_F - \sum_{e_2 \in F} x_F \,\middle|\, e_1, e_2 \in E \right\rangle \end{aligned}$$

be ideals of $P_{\mathcal{M}}$. The *Chow ring of* $\mathcal{M}$ is given by the quotient

$$A^{\bullet}(\mathcal{M}) = \frac{P_{\mathcal{M}}}{I_{\mathcal{M}} + J_{\mathcal{M}}}.$$

By way of some intuition building, the idea here is to create a polynomial ring with variables corresponding to the proper flats of our matroid, then encode the combinatorial relations of the matroid using a quotient. We could think of the ideal $I_{\mathcal{M}}$ as telling us that any monomial involving flats that don't form a flag are removed. The ideal $J_{\mathcal{M}}$ has a less obvious intuition, but the linear forms that generate it give us useful relations that we will use later. As always, working a small example will help.

3.1.1.1 A Small Chow Ring Example

Recall our ongoing example matroid M, whose ground set is $E = \{a, b, c, d\}$ and with proper flats

$$\mathcal{L}^* = \{a, b, c, d, ac, bc, cd, abd\}.$$

Then we have the polynomial ring

$$P_{\mathcal{M}} = \mathbb{R}[x_a, x_b, x_c, x_d, x_{ac}, x_{bc}, x_{cd}, x_{abd}];$$

i.e. a real polynomial ring in 8 variables. Elements of the ideal $I_{\mathcal{M}}$ will be any multiple of a monomial containing variables corresponding to non-comparable flats, such as

$$x_a x_d \in I_{\mathcal{M}} \quad \text{and} \quad x_c x_{abd} \in I_{\mathcal{M}}.$$

The ideal $J_{\mathcal{M}}$ in turn is generated from differences of sums of all variables that contain a particular ground element. For example,

$$\begin{aligned} \sum_{a \in F} x_F - \sum_{c \in F} x_F &= x_a + x_{abd} + x_{ac} - x_c - x_{ac} - x_{cd} \\ &= x_a + x_{abd} - x_c - x_{cd} \in J_{\mathcal{M}}. \end{aligned}$$

Now, elements in our Chow ring $A^\bullet(\mathcal{M})$ are equivalence classes, as expected for a quotient ring. We see that $J_{\mathcal{M}}$ gives us relations like

$$[x_a] + [x_{abd}] = [x_c] + [x_{cd}],$$

or, equivalently,

$$[x_a] = [x_c] + [x_{cd}] - [x_{abd}].$$

In continuing to strive for succinct notation, we will drop the square brackets on elements of the Chow ring going forward when not needed to distinguish them.

3.2 The Degree Map

Now that we have an idea of what the Chow ring is, we can introduce the next key idea, the degree map. However, to get to the degree map we will need the property that Chow rings are graded.

3.2.1 This Will Be Graded

Our goal here is to show that the Chow ring is a graded ring. Those that feel that this property is obvious can skip to the next section; for everyone else we will provide a quick summary.

Definition 3.2 (Graded Ring)**.** A ring R is *graded* if the underlying additive group of R can be decomposed into a direct sum

$$R = \bigoplus_{i=0}^{\infty} R_i$$

where each R_i is an abelian group such that $R_i R_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{Z}_{\geq 0}$. Elements of R_i are called *homogeneous of degree* i.

From this definition we can see that any polynomial ring, P is a graded ring

$$P = \bigoplus_{i=0}^{\infty} P_i$$

where each P_i is, naturally, all homogeneous polynomials of degree i.

Additionally, we can define a specific kind of ideal for graded rings, homogeneous ideals.

Definition 3.3 (Homogeneous Ideals)**.** Let R be a graded ring. An ideal I of R is *homogeneous* if

$$I = \langle a_1, a_2, \dots \rangle$$

where each a_i is homogeneous, i.e., $a_i \in R_k$ for some $k \in \mathbb{Z}_{\geq 0}$.

Looking back at our definition, we see that the ideal $I_{\mathcal{M}}$ is generated by quadratic monomials, while the generators of $J_{\mathcal{M}}$ are all linear. Thus, $I_{\mathcal{M}}$ and $J_{\mathcal{M}}$ are homogeneous ideals.

Definitions in place, we can state a small proposition.

Proposition 3.1. *Let R be a graded ring and I a homogeneous ideal. Then the quotient R/I is itself a graded ring. Specifically,*

$$R/I = \bigoplus_{n=0}^{\infty} R_n/I_n$$

where $I_n = I \cap R_n$.

We would say a proof for this could be found in any algebra textbook, but it appears to always be left as an exercise for the reader; a tradition we see no reason to break.

Since the Chow ring is the quotient of a polynomial ring and homogeneous ideals, we have that $A^\bullet(\mathcal{M})$ is a graded ring. Even better, we get the following from [AHK18, Proposition 5.5].

Proposition 3.2. *Given a matroid $\mathcal{M}$, such that $rk(\mathcal{M}) = r + 1$,*

$$A^\bullet = \bigoplus_{i=0}^{r} A^i(\mathcal{M}),$$

with $A^k = \{0\}$ for all other $k > r$.

All the non-zero components of $A^\bullet(\mathcal{M})$ correspond to ranks of flats of $\mathcal{M}$ and are empty otherwise.

3.2.2 The Degree Map

We needed that the Chow ring is a graded ring as the degree map is defined only on the top graded component of the ring.

Definition 3.4 (Degree Map)**.** Let $\mathcal{M}$ be a matroid of rank $r+1$. The *degree map* of $\mathcal{M}$ is the linear map

$$\deg_{\mathcal{M}} : \ A^r(\mathcal{M}) \to \mathbb{Z}$$

such that for any complete flag $\mathscr{F} \subseteq \mathcal{L}$ of $\mathcal{M}$,

$$\deg_{\mathcal{M}} \left(\prod_{F \in \mathscr{F}} x_F \right) = 1.$$

At first glance, it is not obvious that such a map must exist or that if it does that it would be unique. However, Adiprasito, Huh, and Katz showed in [AHK18, Proposition 5.6] that this map exists for every matroid and is uniquely characterized by this definition. We will briefly return to the degree map in the next chapter to provide a little more context.

3.3 The Reduced Characteristic Polynomial

Right away we are going to have to confess that all of this Chow ring business doesn't actually relate to the characteristic polynomial directly. We have made a slight misdirection. Instead, we have to introduce the actual target of our machinations, the reduced characteristic polynomial. The definition is quite straightforward, we just require one small proposition about the characteristic polynomial.

Proposition 3.3. *For any matroid $\mathcal{M}$, the characteristic polynomial $\chi_{\mathcal{M}}(z)$ has a factor of $(z-1)$.*

Proof. This follows from the recursive definition of the characteristic polynomial we get from Proposition 2.7. Let $\mathcal{M}$ be a matroid. If $\mathcal{M}$ has a loop then, we have that $\chi_{\mathcal{M}}(z) = 0$ which trivially has $(z-1)$ is a factor. Likewise, if $\mathcal{M}$ has any element $e \in E$ such that e is a coloop, we have that

$$\chi_{\mathcal{M}}(z) = (z-1)\chi_{\mathcal{M}\setminus e}(z),$$

which we see has $(z-1)$ as a factor.

So, we should assume that $\mathcal{M}$ has neither a coloop nor a loop. Then we can pick any element $e \in E$ and use the deletion-contraction property to write

$$\chi_{\mathcal{M}}(z) = \chi_{\mathcal{M}\setminus e}(z) - \chi_{\mathcal{M}/e}(z),$$

where each matroid in the sum on the right-hand side has a ground set with one fewer element in its ground set than our original matroid. If either of these minors have a loop or coloop, that part of the sum has a factor of $(z-1)$ and we no longer need to worry about it. Otherwise, we can again use the deletion-contraction property to write these as the characteristic polynomial of matroids with one fewer element.

We can continue this until we have expressed the characteristic polynomial as the sum of characteristic polynomials of matroids that either have a loop, a coloop, or a single element in its ground set.

Up to isomorphism, there are only two matroids with a single element. Both have a ground set $E = \{a\}$, and they have independent sets $\mathcal{I} = \{a\}$ or $\mathcal{I} = \emptyset$. If a is independent, then it is a coloop, since it increases the rank of every subset of $E \setminus a$. If a is not independent, then it is a loop by definition.

So this final sum actually consists only of matroids with coloops or loops, and so each has a factor of $(z - 1)$. We conclude that $\chi_{\mathcal{M}}(z)$ must have a factor of $(z - 1)$. □

Now that we are sure there will always be a factor of $(z - 1)$, the reduced characteristic polynomial is easy to define in terms of the characteristic polynomial.

Definition 3.5 (Reduced Characteristic Polynomial)**.** Let $\mathcal{M}$ be a matroid of rank $r + 1$. The *reduced characteristic polynomial* of $\mathcal{M}$ is

$$\overline{\chi}_{\mathcal{M}}(z) = \frac{\chi_{\mathcal{M}}(z)}{(z-1)}.$$

Collecting the powers of z and writing the reduced characteristic polynomial as

$$\overline{\chi}_{\mathcal{M}}(z) = \sum_{k=0}^{r} (-1)^k \overline{w}_k z^{r-k},$$

we define $\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_r$ as the reduced coefficients of $\overline{\chi}_{\mathcal{M}}(z)$.

That this is well-defined for any matroid follows from Proposition 3.3. The following proposition tells us why these coefficients are going to be important.

Lemma 3.4. *Let $\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_r$ be the reduced coefficients of the reduced characteristic polynomial $\overline{\chi}_{\mathcal{M}}(z)$ of some matroid $\mathcal{M}$. If*

$$\{\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_r\}$$

is a log-concave sequence, then the Whitney numbers of the first kind of $\mathcal{M}$,

$$w_0, w_1, \ldots, w_{r+1},$$

also form a log-concave sequence.

Proof. Let $\mathcal{M}$ be a matroid of rank $r+1$ and

$$\{\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_r\}$$

be the reduced coefficients of the reduced characteristic polynomial $\overline{\chi}_{\mathcal{M}}(z)$. Following from the deletion-contraction property, the reduced characteristic coefficients are positive. Assume that $\{\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_r\}$ is a log-concave sequence.

To recover the characteristic polynomial of $\mathcal{M}$ we can multiply our polynomial by a factor of $(z-1)$. This gives us

$$\begin{aligned}(z-1)\sum_{k=0}^{r}(-1)^k\overline{w}_k z^{r-k} &= z\sum_{k=0}^{r}(-1)^k\overline{w}_k z^{r-k} - \sum_{k=0}^{r}(-1)^k\overline{w}_k z^{r-k} \\ &= \sum_{k=0}^{r}(-1)^k\overline{w}_k z^{r-k+1} + \sum_{k=0}^{r}(-1)^{k+1}\overline{w}_k z^{r-k} \\ &= \sum_{k=0}^{r}(-1)^k\overline{w}_k z^{r-k+1} + \sum_{k=1}^{r+1}(-1)^k\overline{w}_{k-1} z^{r-k+1} \\ &= \overline{w}_0 z^{r+1} + \sum_{k=1}^{r}(-1)^k(\overline{w}_k+\overline{w}_{k-1})z^{r-k} + (-1)^{r+1}\overline{w}_r,\end{aligned}$$

where the last two equalities are just a result of some clever reindexing. For convenience, we will define $\overline{w}_{-1} = \overline{w}_{r+1} = 0$. We note that $\{\overline{w}_{-1}, \overline{w}_0, \overline{w}_1, \ldots, \overline{w}_r, \overline{w}_{r+1}\}$ remains a log-concave sequence. This lets us write

$$\overline{w}_0 z^{r+1} + \sum_{k=1}^{r}(-1)^k(\overline{w}_k+\overline{w}_{k-1})z^{r-k+1} + (-1)^{r+1}\overline{w}_r = \sum_{k=0}^{r+1}(-1)^k(\overline{w}_k+\overline{w}_{k-1})z^{r-k}.$$

Recalling the definition of the characteristic polynomial and its coefficients, we have shown that

$$w_k = \overline{w}_k + \overline{w}_{k-1},$$

for $0 \le k \le r+1$.

Now, we wish to show the log-concavity of the coefficients, so consider the expression $w_k^2 - w_{k-1}w_{k+1}$. If we can show this must be non-negative for any $0 < k < r+1$, we will have that the sequence is log-concave. First let's manipulate the expression a bit to get

$$
\begin{aligned}
w_k^2 - w_{k-1}w_{k+1} &= (\overline{w}_k + \overline{w}_{k-1})^2 - (\overline{w}_{k-1} + \overline{w}_{k-2})(\overline{w}_{k+1} + \overline{w}_k) \\
&= \left(\overline{w}_k^2 + 2\overline{w}_k\overline{w}_{k-1} + \overline{w}_{k-1}^2\right) - \left(\overline{w}_{k-1}\overline{w}_{k+1} + \overline{w}_{k-1}\overline{w}_k + \overline{w}_{k+1}\overline{w}_{k-2} + \overline{w}_k\overline{w}_{k-2}\right) \\
&= (\overline{w}_k^2 - \overline{w}_{k-1}\overline{w}_{k+1}) + (\overline{w}_{k-1}^2 - \overline{w}_{k-2}\overline{w}_k) \\
&\quad + (\overline{w}_k\overline{w}_{k-1} - \overline{w}_k\overline{w}_{k-1}) + (\overline{w}_k\overline{w}_{k-1} - \overline{w}_{k+1}\overline{w}_{k-2}) \\
&= (\overline{w}_k^2 - \overline{w}_{k-1}\overline{w}_{k+1}) + (\overline{w}_{k-1}^2 - \overline{w}_{k-2}\overline{w}_k) + (\overline{w}_k\overline{w}_{k-1} - \overline{w}_{k+1}\overline{w}_{k-2}).
\end{aligned}
$$

It is now this expression we want to show is non-negative. Immediately we have that

$$
\begin{aligned}
\overline{w}_k^2 - \overline{w}_{k-1}\overline{w}_{k+1} &\geq 0, \\
\overline{w}_{k-1}^2 - \overline{w}_{k-2}\overline{w}_k &\geq 0
\end{aligned}
$$

from the assumption of the log-concavity of the reduced coefficients. All that's left then is to show that the term

$$\overline{w}_k\overline{w}_{k-1} - \overline{w}_{k+1}\overline{w}_{k-2} \geq 0.$$

If $k = 1$ or $k = r$ then this is immediately true, as $\overline{w}_{k+1}\overline{w}_{k-2} = 0$ in either of these cases. So, we will go forward and assume that $1 < k < r$. Now, consider that it must be that

$$\frac{\overline{w}_{k-1}}{\overline{w}_{k-2}} \geq \frac{\overline{w}_k}{\overline{w}_{k-1}} \geq \frac{\overline{w}_{k+1}}{\overline{w}_k}.$$

This follows directly from the definition of log-concavity and that they are all positive. This means

$$\frac{\overline{w}_{k-1}}{\overline{w}_{k-2}} \geq \frac{\overline{w}_{k+1}}{\overline{w}_k}$$

and so

$$\overline{w}_k\overline{w}_{k-1} \geq \overline{w}_{k+1}\overline{w}_{k-2}$$

giving us

$$\overline{w}_k\overline{w}_{k-1} - \overline{w}_{k+1}\overline{w}_{k-2} \geq 0$$

as desired.

Having shown that $w_k^2 - w_{k-1}w_{k+1}$ for all $1 \leq k \leq r$ must be non-negative, we conclude that the sequence $\{w_k\}_{k=0}^{r+1}$ is log-concave. □

It is actually this sequence, $\{\overline{w}_k\}_{k=0}^{r}$, that Adiprasito, Huh, and Katz spent most of [AHK18] proving is log-concave. This too is our strategy, so these reduced coefficients really are key players in this story. But we have still yet to link the characteristic polynomial, reduced or not, to the Chow ring.

3.3.1 The Divisors α and β

Right off the bat, let's learn a little jargon.

Definition 3.6 (Divisor)**.** A *divisor* of a Chow ring $A^\bullet(\mathcal{M})$ is any linear term, i.e., an element of $A^1(\mathcal{M})$.

We are working towards introducing a proposition from the Adiprasito, Huh, and Katz paper that links certain divisors to the reduced coefficients, via the degree map. We will start by introducing these special divisors.

Definition 3.7 (α and β)**.** Let $\mathcal{M}$ be a matroid with ground set E. For every element $e \in E$ we define the divisors

$$\alpha_e = \sum_{e \in F} x_F \quad \text{and} \quad \beta_e = \sum_{e \notin F} x_F.$$

Proposition 3.5. *As elements of $A^\bullet(\mathcal{M})$, $\alpha_e = \alpha_{e'}$ and $\beta_e = \beta_{e'}$ for any $e, e' \in E$. Going forward we may refer simply to α and β, since the class is independent of a choice of ground element.*

Proof. Recall that the ideal J of $A^\bullet(\mathcal{M})$ gives us the relation

$$\sum_{e \in F} x_F = \sum_{e' \in F} x_F$$

for any $e, e' \in E$. Thus, we have

$$\begin{aligned} \alpha_e &= \sum_{e \in F} x_F \\ &= \sum_{e' \in F} x_F \\ &= \alpha_{e'} \end{aligned}$$

for any ground elements e, e'. The argument follows identically for β, once we note

$$\beta_e = \sum_{e \notin F} x_F = \sum_{F \in \mathcal{L}^*} x_F - \sum_{e \in F} x_F.$$

So, in the Chow ring, any choice of ground element for α and β is equivalent to any other. □

With this, we may state the key takeaway from this chapter. There is a link between α and β and the reduced coefficients provided by [AHK18, Proposition 9.5].

Proposition 3.6. *Given any matroid $\mathcal{M}$ with reduced coefficients $\overline{w}_0, \dots, \overline{w}_r$, we have the relationship*

$$\overline{w}_k = \deg(\alpha^{r-k}\beta^k).$$

for all $0 \leq k \leq r$.

Just to confirm our understanding of the degree map, recall it is defined on elements of $A^r(\mathcal{M})$. Since $\alpha, \beta \in A^1(\mathcal{M})$, we will have $\alpha^{r-k}\beta^k \in A^r(\mathcal{M})$ and so this makes sense as input to the degree map. An alternative proof of Proposition 3.6 given by Dastidar and Ross, [DR21, Proposition 3.11], shows that the right-hand side satisfies the deletion-contraction property.

Proposition 3.6 means that if we can prove

$$\deg(\alpha^{r-k}\beta^k)^2 \geq \deg(\alpha^{r-k-1}\beta^{k-1})\deg(\alpha^{r-k+1}\beta^{k+1})$$

for $0 < k < r$ we will have shown that the reduced coefficients are log-concave and thus so too are the original coefficients. Here now is where we diverge from the strategy put forth in [AHK18]. They go on to show this relationship in a very algebraic manner, proving that the Chow ring of a matroid has many desirable properties that eventually yield the desired result. We, on the other hand, will now move into the world of geometry and find a different way to generate our log-concave sequence.

Chapter 4

Bergman Fans and their Normal Complexes

As we continue our tour of various branches of mathematics, we arrive at geometry. The primary goal of this chapter is to develop the final segment of our bridge connecting some geometric object back to the Chow ring, and then showing how we can generate log-concave sequences with these objects. To get there we will provide a quick primer on polyhedral geometry and a classic theorem of convex geometry that generates log-concave sequences. Then we will introduce a geometric object associated to a matroid, the Bergman fan, and show how we can use them to make some new objects called normal complexes.

4.1 A Little Polyhedral Geometry, as a Treat

Really, the basic building blocks we will be using are not that weird. It is geometry, we are going to be using some sort of shapes living in some kind of space. We must admit, however, that we personally struggle visualizing the higher dimensional objects at play, and so must fall back on formalism.

This section is a short crash course on basic elements of polyhedral geometry. Our treatment of this topic will often parallel that in Ziegler's "Lectures on Polytopes" [Zie95], which we recommend for those who'd like a little more depth than presented here.

4.1.1 The Cone Zone

We are going to be using two fundamental kinds of convex shapes, polytopes and cones. As a reminder, a convex object is one where if you pick any two points in it, the line connecting those points never leaves the shape. We can, and will, state this formally.

Definition 4.1 (Convexity)**.** Let $K \subseteq \mathbb{R}^n$. We call K *convex* if for every $p, q \in K$, we have

$$[p, q] \subseteq K,$$

where $[p, q] = \{\lambda p + (1 - \lambda)q \mid 0 \leq \lambda \leq 1\}$ is the line segment between p and q.

Figure 4.1: Everyone's first pair of convex and non-convex shapes.

While there are generally a few ways one could define polytope and cone, we will use a definition based on construction using some finite collection of points. In brief, a polytope is a *convex hull* of finitely many points and a cone is the *conic combination* of finitely many generating vectors. Let's make this formal.

Definition 4.2 (Polytope)**.** Let $P \subseteq \mathbb{R}^n$. We say P is a *polytope* if it is the convex hull of some finite set of points $x_1, x_2, \ldots, x_k$. That is to say P is a polytope if

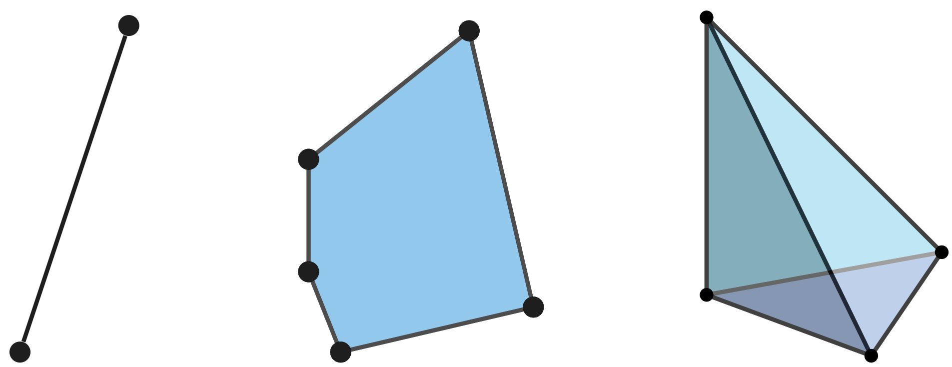
$$P = \operatorname{conv}(\{x_1, \ldots, x_k\}),$$

where

$$\operatorname{conv}(\{x_1, \ldots, x_k\}) = \left\{ \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k \;\middle|\; \lambda_i \geq 0, \sum_{i=0}^{k} \lambda_i = 1 \right\}$$

is the convex hull of $x_1, x_2, \ldots, x_k$.

An astute reader may notice that our shorthand for line segment above, $[x, y]$, is in fact just $\operatorname{conv}(\{x, y\})$. Towards some intuition, we may think of the convex hull as the smallest convex shape that contains all of its generating points. In two dimensions we like to think of this as stretching a rubber band around a bunch of points and letting it constrict around them.



A 1-dimensional polytope A 2-dimensional polytope A 3-dimensional polytope

Figure 4.2: A sampling of polytopes; note that we have highlighted the vertices, which are the minimal set of points that generate the polytope.

We will use a similar definition for cones. They are built out of a finite collection of generating vectors.

Definition 4.3 (Cone)**.** Let $C \subseteq \mathbb{R}^n$. We call C a *cone* if it is the conic combination of finitely many vectors $x_1, x_2, \ldots, x_k$. We write this

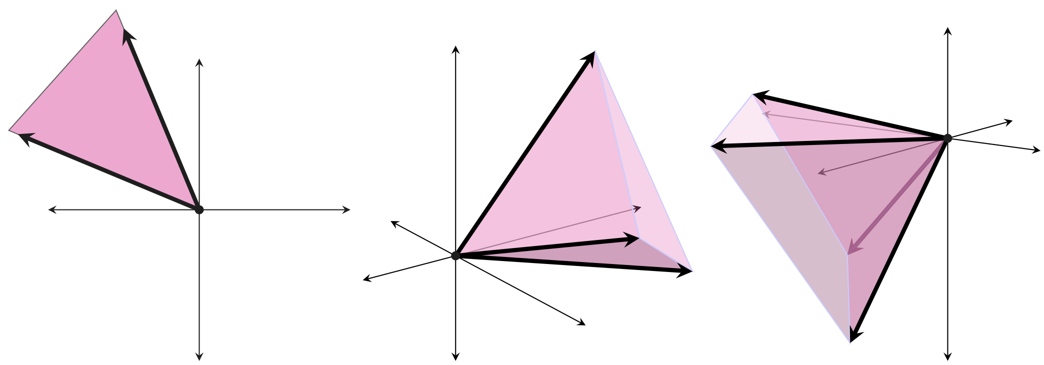
$$C = \operatorname{cone}(\{x_1, \ldots, x_k\}),$$

where

$$\operatorname{cone}(\{x_1, \ldots, x_k\}) = \{\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k \mid \lambda_i \geq 0\}$$

is the conic combinations of $x_1, x_2, \ldots, x_k$.

Unlike the more familiar notion of cones, these are not pointed cylinders.



A 2-dimensional cone   A 3-dimensional cone (generated by 3 vectors)   A 3-dimensional cone (generated by 4 vectors)

Figure 4.3: Some examples of cones.

We notice that conic combinations are essentially the span of the generating vectors but taking only non-negative linear combinations. Indeed, one can quickly confirm that $\text{cone}(\{x_1, \ldots, x_k\}) \subseteq \text{span}(\{x_1, \ldots, x_k\})$.

4.1.2 Points vs. Vectors: An Affine Primer

We have been, and will be going forward, using the words “points” and “vectors” quite interchangeably. Is there a difference? Strictly speaking, yes there is. Points imply elements of an affine space, while vectors, naturally, are elements of a vector space. Affine spaces can be thought of as vector spaces where the 0-vector is “forgotten”, but are otherwise essentially the same collection of “stuff”. Mathematicians love to make multiple objects out of the same basic thing by giving (or losing) some extra structure.

This poses a slight problem since we want to refer to the dimension of our polytopes and cones (and already have been in figures), but a notion of dimension usually relies on a vector space. To settle a notion of dimension here, we first want to define what an affine span (also called an affine hull) is.

Definition 4.4 (Affine Span)**.** Let $S \subseteq \mathbb{R}^d$. The *affine span* of S is the set

$$\operatorname{aff}(S) = \left\{ \sum_{i=1}^{k} \lambda_k x_k \;\middle|\; k > 0,\ x_i \in S,\ \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

The affine span of some set will be something that looks like a linear subspace, but might not include the origin. In fact if $0 \in S$, then the affine span and our more traditional linear span coincide exactly. The insight here then is that we could always translate our affine span so that it includes the origin. Once we have a linear subspace we can just use the linear algebra definition of dimension.

Definition 4.5 (Affine Dimension)**.** Given any set $S \subseteq \mathbb{R}^d$, designate some element $x_0 \in S$. The *dimension* of S is

$$\dim(S) = \dim\big(\operatorname{aff}(S) - x_0\big),$$

where, on the right-hand side, dim is the standard notion of dimension of a subspace in linear algebra.

We are overloading our notation a bit, but we promise this mostly reduces cognitive load. This also goes to explain our switching between "point" or "vector". Since our cones are defined to always include 0, affine and linear terms align, so we are safe to just think in terms of vector spaces. Indeed, for cones we will always just write $\mathrm{span}(\mathcal{C})$ in lieu of $\mathrm{aff}(\mathcal{C})$.

Dimension will mostly align with intuition, but it is good to have the definitions at hand if ever in doubt. Early chapters of [Zie95] and [Grü03] both provide a good treatment of affine spaces.

4.1.3 The Minkowski Sum

Now that we have the basic shapes down we need be able to make new ones out of existing ones. The two general strategies here will be to combine them in to new ones and to break them down. We will start with learning how we can add shapes together, using what we call the Minkowski sum.

Definition 4.6 (Minkowski Sum)**.** Let $P, Q \subseteq \mathbb{R}^n$. The *Minkowski sum* of P and Q is given by

$$P + Q = \{p + q \mid p \in P, q \in Q\}.$$

Sit with this definition for a few moments to confirm the Minkowski sum does have the nice properties of sums we normally expect. It is commutative, associative, and has an identity in $\{0\}$. An additional feature of the definition is that the empty set has the property that for any $P \subseteq \mathbb{R}^n$

$$P + \emptyset = \emptyset.$$

A way to think about the Minkowski sum is "smearing" one shape around the other. You pick some point in either shape, then drag it along the boarder of the other shape in the sum. This makes more sense with a picture.

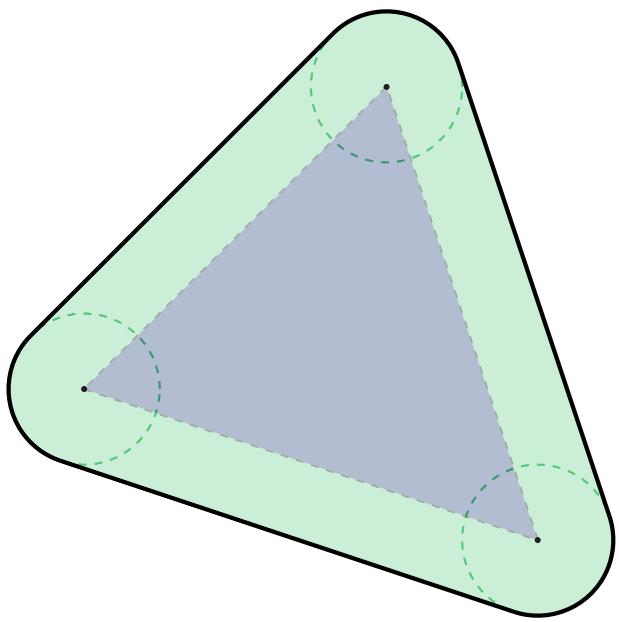


Figure 4.4: A visual representation of a Minkowski sum of a triangle and a circle.

This is a helpful visual intuition, but it may take a moment to be convinced that, up to translation, the resulting shape doesn't depend on either the shape you choose to smear or the choice of point in that shape.

With the Minkowski sum, we can finally define a general polyhedron.

Definition 4.7 (Polyhedron)**.** Let $P \subseteq \mathbb{R}^n$. We call P a *polyhedron* if

$$P = \operatorname{conv}(\{x_1, \ldots, x_k\}) + \operatorname{cone}(\{y_1, \ldots, y_\ell\}),$$

for some finite sets $\{x_1, \ldots, x_k\}, \{y_1, \ldots, y_\ell\} \subseteq \mathbb{R}^n$.

A polyhedron is the result of the Minkowski sum of a polytope and a cone. Clearly every polytope and cone are polyhedra themselves, as $\operatorname{conv}(\{0\}) = \operatorname{cone}(\{0\}) = \{0\}$. Polyhedra are not necessarily bounded, which may seem a bit unusual to those who have seen the word in other contexts. Moreover, all bounded polyhedra are polytopes, which is not necessarily obvious, but useful to keep in mind. We will mostly be focused on either polytopes or cones at any one time, but having a more general object that includes both makes our definitions going forward cleaner.

4.1.4 About Faces

Given a polyhedron P, we also get a whole family of polyhedra, the faces of P. Let's first go back to simpler times. If we were to think of a cube, we would have faces of the cube as the 2 dimensional squares that make up the sides. We'd then call the line segments where any two of those squares meet *edges* and the points where those edges meet *vertices*.

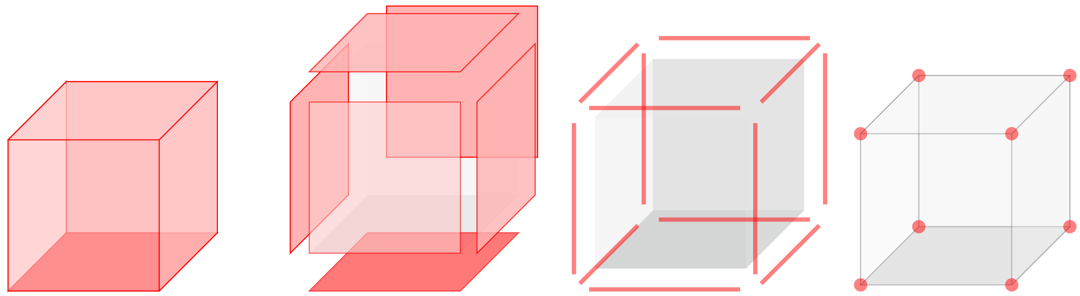


Figure 4.5: The faces of a cube; note that the full cube is a face of itself (as is the empty set, which we drew as accurately as possible).

If this sounds familiar then the intuition for our more general notion of the face of a polyhedron is not far. Back in our world of polyhedral geometry, we know that a cube is a polyhedron and that each of those squares, lines, and points are also themselves polyhedra. Informally, we use the term face to describe all these "sub-polyhedra" that make up the boundary of a polyhedron.

Definition 4.8 (Face)**.** Let $P \subseteq \mathbb{R}^n$ be a polyhedron and $\bullet$ be the standard dot product. The *hyperplane* normal to a vector $x \in \mathbb{R}^n$ at distance $b \in \mathbb{R}$ is given by

$$\mathrm{H}_x(b) = \{v \in \mathbb{R}^n \mid v \bullet x = b\}\,.$$

The lower half-space is given by

$$\mathrm{H}_x^-(b) = \{v \in \mathbb{R}^n \mid v \bullet x \leq b\}\,.$$

Then $F \subseteq P$ is a *face* of P if there exist x, b such that

$$F = P \cap \mathrm{H}_x(b)$$

where

$$P \subseteq \mathrm{H}_x^-(b).$$

Some notable quirks of this definition are that when $x = 0$ and $b \in \mathbb{R}_{\geq 0}$,

$$\mathrm{H}_0(b) \cap P = P.$$

So P is a face of itself by our definition. Likewise, if we pick $x = 0$ and $b < 0$, then

$$\mathrm{H}_x(b) \cap P = \emptyset,$$

which means the empty set too is a face of any polyhedron P. As a note, we do still call the 0-dimensional faces vertices, while the generic term for a $(d-1)$-dimensional face of a d-dimensional polyhedron is a *facet*.

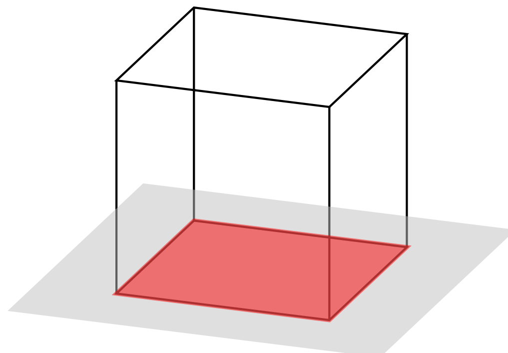


Figure 4.6: Example of specifying a facet of our cube with a plane.

From our cube intuition exercise, there are some things about faces that we would hope to generalize to our broader notion of faces. We will state these without proof, and again refer to the early chapters of [Zie95].

Proposition 4.1. *Given a polyhedron P, then*

- *if $F \subseteq P$ is a face of P, then F is a polyhedron,*
- *if $F \subseteq P$, then any face $F' \subseteq F$ of F is also a face of P,*
- *if $F_1, F_2 \subseteq P$ are two faces of P, then $F_1 \cap F_2$ is a face of P*

It is worth remembering that the intersection of two faces can be empty, but $\emptyset \subseteq P$ is a face of any polyhedron P, so this causes no issues.

We will often write $F \preceq P$ to mean F is a face of P. Indeed, the relation "is a face of" induces a partial ordering on the set of all faces of P. Even stronger, using the propositions above it follows that the relation induces a lattice (where the join of two faces is the smallest face containing both).

4.2 The Geometer's Guide to Generating Log-Concave Sequences

Having developed our shapes, we are going to need to measure them somehow. Specifically, we want their volume. As with the rest of this chapter so far, we are going to take something that most people can intuitively grasp for 3-dimensional shapes and generalize. Not only do we need volume for arbitrary dimensional polytopes, we need something called the mixed volume which gives us some sense of volume of multiple shapes. This work however will allow us to introduce a classic result of convex geometry that relates geometry to log-concave sequences.

From here, the last few background points can no longer be readily found in Ziegler. Instead, we turn to Rolf Schneider's "Convex Bodies" [Sch13] as a comprehensive reference.

4.2.1 Volume Functions

We again need to formalize something that most of us would take for granted. The notion of volume is intuitive enough for 3-dimensional shapes, we however need to generalize this to all dimensions. We will actually only need the volume of polytopes, so we restrict our notion of volume to just them.

Definition 4.9 (Volume Function)**.** A *volume function* is a map

$$\mathrm{Vol}_n : \{\text{polytopes in } \mathbb{R}^n\} \to \mathbb{R}_{\geq 0}$$

such that

1. $\mathrm{Vol}_n(P) > 0$ when $\dim(P) = n$ and $\mathrm{Vol}_n(P) = 0$ when $\dim(P) < n$,

2. $\mathrm{Vol}_n(P) = \mathrm{Vol}_n(P + v)$ for any $v \in \mathbb{R}^n$,

3. when $P \cup Q$ is a polytope,

$$\mathrm{Vol}_n(P \cup Q) = \mathrm{Vol}(P) + \mathrm{Vol}_n(Q) - \mathrm{Vol}(P \cap Q),$$

4. for any linear map $T \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$,

$$\mathrm{Vol}_n(T(P)) = |\det(T)| \, \mathrm{Vol}_n(P).$$

When the dimension is unambiguous, we will write Vol_n as simply Vol. This definition does not uniquely specify a single "volume function", but rather a family of maps that all differ from one another by a constant multiple. We can differentiate different volume maps by their value on a single fixed polytope. For example, in 2-dimensions, our "standard" volume function is the one that takes the unit square to 1.

Figure 4.7: Examples of using our “standard” volume.

Any two volume functions only differ by a constant. Another reasonable choice of volume is *simplicial volume*. Given a vector space with a basis $\{e_1, \ldots, e_n\}$, the associated simplicial volume is the volume map such that

$$\mathrm{Vol}_n\left(\mathrm{conv}(\{0, e_1, e_2, \ldots, e_n\})\right) = 1.$$

Figure 4.8: Example of simplicial volume in $\mathbb{R}^2$; notice it differs from our “standard” by a factor of 2.

4.2.2 Mixed Volume

While volume functions may not be too odd a concept we will use it to define a less widely known function. Given any 2 polytopes in $\mathbb{R}^2$, or 3 polytopes in $\mathbb{R}^3$, or more generally n polytopes in $\mathbb{R}^n$, we want a map from these collections of polytopes to $\mathbb{R}_{\geq 0}$ that is, in some sense, consistent with volume and Minkowski summation. We call this map the mixed volume function.

Definition 4.10 (Mixed Volume – Characterization)**.** The *mixed volume function* is a map MVol_n from ordered multisets $P_1, P_2, \ldots, P_n \subseteq \mathbb{R}^n$ of polytopes to $\mathbb{R}_{\geq 0}$, such that it has the following properties:

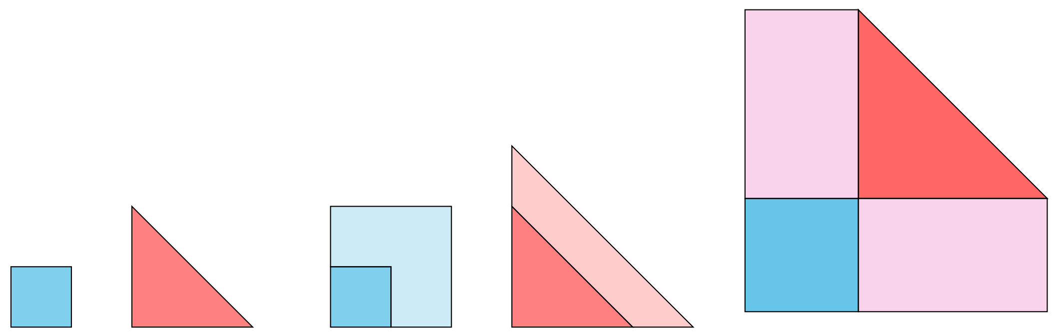
1. $\mathrm{MVol}_n(P, P, \ldots, P) = \mathrm{Vol}_n(P)$, for any polytope $P \subseteq \mathbb{R}^n$,
2. MVol_n is symmetric in all arguments, and
3. MVol_n is multilinear with respect to scaling and Minkowski addition.

Just like volume, we will often just notate this as MVol when safe to do so. The proof that such a function exists and is indeed uniquely defined by these properties can be found in [Sch13]. While this characterization is useful, it goes very little of the way to actually telling us what mixed volumes are.

Consider two polytopes $P, Q \subseteq \mathbb{R}^2$. We could ask ourselves, what is the volume of the Minkowski sum of P and Q. We could be more ambitious and even allow ourselves to scale P and Q by arbitrary values. That is to say, let us consider the volume

$$\mathrm{Vol}_2(\lambda P + \mu Q),$$

for some $\lambda, \mu \in \mathbb{R}$. Answering this question is where mixed volumes appear as something more concrete. While not immediately obvious, this volume can always be expressed as a polynomial in λ and μ, and mixed volumes appear as coefficients of these polynomials. We can consider an example.



$$\mathrm{Vol}(\lambda P + \mu Q) = \mathrm{MVol}(P, P)\lambda^2 + 2\,\mathrm{MVol}(P, Q)\lambda\mu + \mathrm{MVol}(Q, Q)\mu^2$$

Figure 4.9: The mixed volume appears in the coefficients of the volume polynomial; recall that $\mathrm{MVol}(P, P) = \mathrm{Vol}(P)$.

This idea generalizes to any dimension, and can be taken as another definition of mixed volume.

Definition 4.11 (Mixed Volume – As Coefficients)**.** Let $P_1, P_2, \ldots, P_\ell \subseteq \mathbb{R}^n$ be polytopes. The function

$$f(\lambda_1, \lambda_2, \ldots, \lambda_\ell) = \mathrm{Vol}(\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_\ell P_\ell), \quad \lambda_j \geq 0$$

is a homogeneous polynomial of degree n. It can be written symmetrically as

$$f(\lambda_1, \ldots, \lambda_\ell) = \sum_{j_1, j_2, \ldots, j_n = 1}^{\ell} \mathrm{MVol}(P_{j_1}, \ldots, P_{j_n})\lambda_{j_1} \cdots \lambda_{j_n}.$$

The coefficient associated to $\lambda_{j_1} \cdots \lambda_{j_n}$ is the *mixed volume* of $P_{j_1}, \ldots, P_{j_n}$.

Of course, you could start with either definition of mixed volume and derive the other, they are equivalent after all.

At this point, one may begin to wonder why this section should even exist. We seem to have gone rather far afield with our geometry lesson. Remember that our ultimate goal is to show something is a log-concave sequence. Mixed volumes are the key to a method generating log-concave sequences via geometry.

4.2.3 The Alexandrov–Fenchel Inequality

Finally, we conclude with an important classic result in convex geometry. Proved by Alexandr Alexandrov in [Ale37], with a contemporaneous but not quite accurate proof by Werner Fenchel, this theorem gives us a fundamental relationship of mixed volumes of convex bodies. We again restrict ourselves to just polytopes, though the theorem applies more broadly.

Theorem 4.2 (Alexandrov–Fenchel Inequality)**.** *For polytopes $P, Q, K_3, \ldots, K_n$ in $\mathbb{R}^n$,*

$$\operatorname{MVol}(P, Q, K_3, \ldots, K_n)^2 \geq \operatorname{MVol}(P, P, K_3, \ldots, K_n) \operatorname{MVol}(Q, Q, K_3, \ldots, K_n).$$

Remember that mixed volumes are just some non-negative real numbers, so this inequality is exactly what we are looking for in a log-concave sequence. In fact, given any two polytopes, there's a corresponding log-concave sequence.

Corollary 4.3. *For any polytopes $P, Q \subseteq \mathbb{R}^n$, the sequence*

$$\left\{ \mathrm{MVol}(\underbrace{P, \ldots, P}_{n-k}, \underbrace{Q, \ldots, Q}_{k}) \right\}_{k=0}^{n}$$

is log-concave.

This is a promising lead, but all we have is a collection of geometric definitions and a way to generate log concave sequences. None of this actually has any clear relation to matroids. But, just like we could make something algebraic out of the structure of a matroid, so too can we make something geometric.

4.3 Bergman Fans

We are just about ready to introduce one of the main geometric players in this story, the Bergman fan. Before we do though, we need to learn about fans more broadly.

4.3.1 Complexes and Fans

The last geometric structure we need as background consists of particular collections of polyhedra, known as polyhedral complexes.

Definition 4.12 (Polyhedral Complex)**.** A *polyhedral complex* $\mathcal{C}$ is a finite collection of polyhedra in $\mathbb{R}^n$ such that

1. the empty set is in $\mathcal{C}$,
2. for any polyhedron $P \in \mathcal{C}$, all faces of P are also in $\mathcal{C}$,
3. for any two polyhedra $P, Q \in \mathcal{C}$, the intersection $P \cap Q \in \mathcal{C}$ is a face of both P and of Q.

We can think of polyhedral complexes as sets of polyhedra that intersect nicely. We won't be too concerned about complexes of general polyhedra, and instead focus on when our complexes are restricted to either all polytopes or all cones.

Definition 4.13 (Polytopal Complex)**.** A *polytopal complex* $\mathcal{C}$ is a polyhedral complex where every element $P \in \mathcal{C}$ is bounded, i.e., a polytope.

One might guess that next we will define a conic complex or some such thing. In fact, a polyhedral complex of only cones is called a *fan*, we assume mostly to torment anyone trying to do a web-search for information on them.

Definition 4.14 (Fan)**.** A *fan* $\mathbf{\Sigma}$ is a polyhedral complex where every element $\sigma \in \mathbf{\Sigma}$ is a cone.

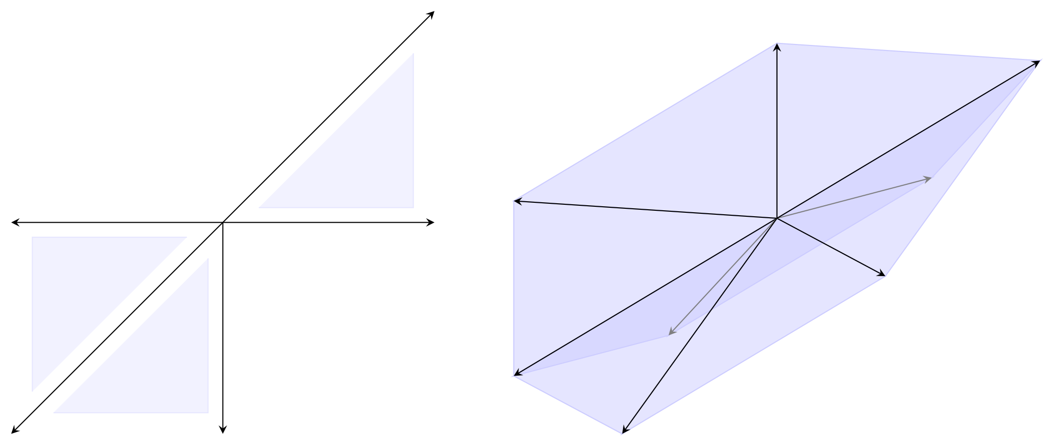


Figure 4.10: Examples of fans; even though they live in different spaces they are both composed only of cones of dimension 2 and less.

From the title of this chapter, it will come as no surprise that fans are quite important. Given that we will be working with fans quite a lot, it is very useful to establish some notational conventions and shorthand. In the definition, we subtly introduced the first convention around fans. While a general polyhedral complex is usually $\mathcal{C}$, fans are given by $\mathbf{\Sigma}$ and cones in the fan are lowercase Greek letters, commonly σ, τ, and ρ.

We will often want to talk just about all cones in a fan of one particular dimension. Often enough that it is useful to have the dedicated notation

$$\mathbf{\Sigma}(d) = \{\sigma \in \mathbf{\Sigma} \mid \dim(\sigma) = d\}$$

to specify the set of d-dimensional cones of a fan $\mathbf{\Sigma}$. This notation turns out to be so convenient that we extend it to cones themselves. If σ is a cone, then

$$\sigma(d) = \{\tau \preceq \sigma \mid \dim(\tau) = d\}$$

is the set of all d-dimensional faces of σ.

As final convention, we exclusively reserve ρ for 1-dimensional cones. Either as a 1-dimensional cone in a fan or a generating ray of a cone itself. This notation here will help us now as we now move on to learn more about some importnat properties fans.

4.3.2 We are Big Fans (of These Properties of Fans)

We would like to introduce several properties of fans. These properties tend to make fans nice to work with as we will see. The first tells us something about the dimension of all the maximal cones in the fan.

Definition 4.15 (Pure). A cone $\sigma \in \mathbf{\Sigma}$ is maximal if it is not the proper face of another cone in $\mathbf{\Sigma}$. A fan is *pure* if every maximal cone in $\mathbf{\Sigma}$ is the same dimension. We say $\mathbf{\Sigma}$ is a d-fan when it is pure of dimension d.

Basically, a fan is pure if the largest cones in it are all of the same dimension. Both fans in figure 4.10 are pure of dimension 2. We can consider a counterexample as well.

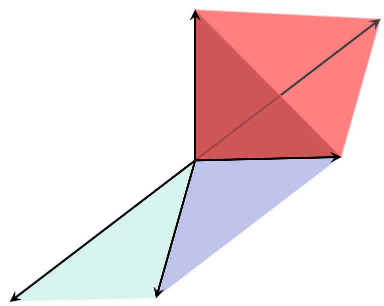


Figure 4.11: This fan is not pure, since it has maximal cones of both dimension 2 and 3.

The next property relates to how many rays are necessary to generate each cone.

Definition 4.16 (Simplicial Fan)**.** A fan Σ is *simplicial* if every cone $\sigma \in \Sigma$ is simplicial. A cone $\sigma \in \Sigma$ is simplicial if

$$\dim(\sigma) = |\sigma(1)|,$$

where $\sigma(1)$ is the set of 1-dimensional faces of σ.

Alternatively, a cone is simplicial if its rays form a basis for the subspace that cone spans. In figure 4.3, the first two cones are simplicial while the third is not. Next, we have something that is not specifically a property of fans but rather some convenient extra information.

Definition 4.17 (Marked Fan)**.** Let $\mathbf{\Sigma}$ be a fan. We say $\mathbf{\Sigma}$ is a *marked fan* if there is a chosen set

$$\{u_\rho \in \rho \mid \rho \in \mathbf{\Sigma}(1)\}$$

such that $\mathrm{cone}(u_\rho) = \rho$.

Given a marked, simplicial d-fan we can layer on yet another property, that of being tropical. Tropical fans come from the realm of tropical geometry, another deep topic we are going to avoid delving into. That tropical fans are in play at all relates back to lurking algebraic varieties that we continue to not fully address. We can, however, at least somewhat easily characterize tropical fans. They are fans that satisfy the weighted balancing condition.

Definition 4.18 (Tropical Fan)**.** Let $\mathbf{\Sigma}$ be a marked, simplicial d-fan. Given a weight function

$$\omega \,:\, \mathbf{\Sigma}(d) \to \mathbb{R}_{>0},$$

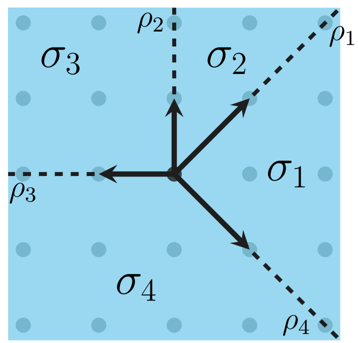
we say the pair $(\mathbf{\Sigma}, \omega)$ is a *tropical fan* if for every $\tau \in \mathbf{\Sigma}(d-1)$

$$\sum_{\substack{\sigma \in \mathbf{\Sigma}(d) \\ \tau \preceq \sigma}} \omega(\sigma) u_{\sigma \setminus \tau} \in \mathrm{span}(\tau),$$

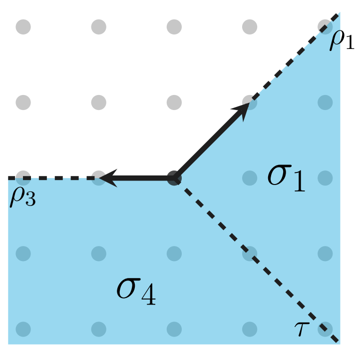
where $u_{\sigma\setminus\tau}$ is shorthand for the vector u_ρ, ρ the single element in $\sigma(1) \setminus \tau(1)$.

A "tropical fan" includes the data of the weight function, and as such when we say $\mathbf{\Sigma}$ is tropical, it carries a weight function implicitly. The exact implications of this definition take a moment to parse, but the main takeaway is that it imposes some relationship between the cones of the fan. We must be able to get back to the span of each $(d-1)$-dimensional cone from the rays that "neighbor" that cone.

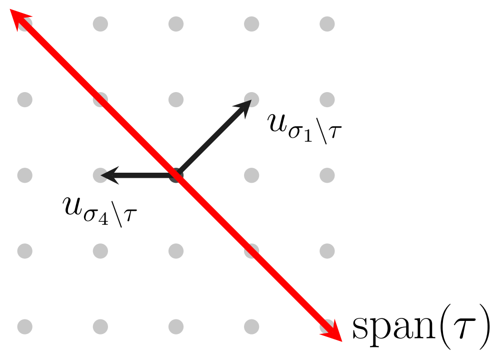
Let's take a look at an example of a tropical fan. Consider the following 2-fan.



We have marked this fan with vectors associated to the first lattice point each ray intersects. For this to be tropical, we need to provide a weight function. We claim that $\omega(\sigma_1) = 1$, $\omega(\sigma_2) = \omega(\sigma_3) = \omega(\sigma_4) = 2$ satisfies the balancing condition. Let's test this by picking the cone $\tau = \rho_4$, a 1-cone of the fan. We can first isolate the top dimensional cones that have τ as a face.



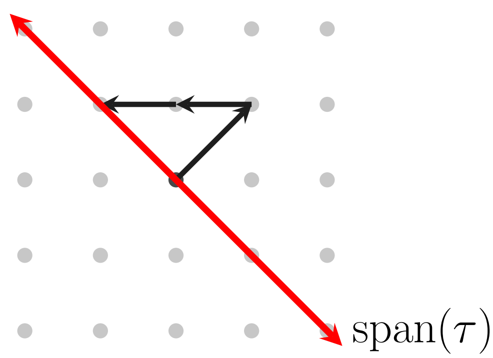
From our definition, we are interested in the rays $\sigma_1 \setminus \tau = \rho_1$ and $\sigma_4 \setminus \tau = \rho_3$, and if the weighted sum of their associated marked points is in the span of τ.



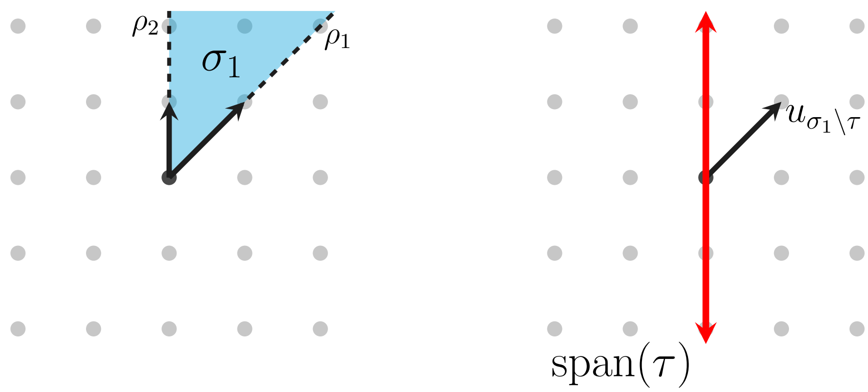
If our weight function is correct, then we should have that

$$\omega(\sigma_1)u_{\sigma_1\backslash\tau} + \omega(\sigma_4)u_{\sigma_4\backslash\tau} = u_{\sigma_1\backslash\tau} + 2u_{\sigma_4\backslash\tau} \in \text{span}(\tau),$$

which we can see is true.



We leave it as an exercise to test that the weight function works for every other ray. Let's briefly look at a counterexample of a non-tropical fan.



We can quickly confirm that no weight function could exist for this fan by picking $\tau = \rho_2$ and following the procedure above. No positive valued scalar could ever get the vector $u_{\sigma_1\setminus\tau}$ to be in the span of τ.

The weight function in our above example is not too complicated, they could be much more involved, but neither is it as simple as it could be. When the weight function of a tropical fan is particularly simple then we have another term.

Definition 4.19 (Balanced Fan)**.** A tropical fan $(\mathbf{\Sigma}, \omega)$ is *balanced* if the weight function ω is the constant function:

$$\omega(\sigma) = 1$$

for all $\sigma \in \mathbf{\Sigma}(d)$.

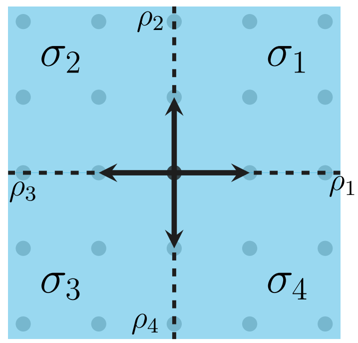


Figure 4.13: An elementary example of a balanced fan.

When a fan is balanced we can omit references to the weight function ω, a very convenient aspect of having a balanced fan.

4.3.3 Products of Fans

Given some existing fans, there is a natural way to use them to produce a new fan.

Definition 4.20 (Product Fan)**.** Let $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$ be fans in vector spaces $\boldsymbol{N}$ and $\boldsymbol{N}'$ respectively. The *product fan* given by $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$ is

$$\mathbf{\Sigma} \times \mathbf{\Sigma}' = \{\sigma \times \sigma' \mid \sigma \in \mathbf{\Sigma}, \sigma' \in \mathbf{\Sigma}'\} \subseteq \boldsymbol{N} \oplus \boldsymbol{N}'.$$

Essentially, when making a product fan, we put the fans in complementary spaces and make cones between all possible combinations of the cones in each fan. We notate cones of the product fan as (σ, σ'), where $\sigma \in \mathbf{\Sigma}$ and $\sigma' \in \mathbf{\Sigma}'$. Rays of the product fan correspond to elements $(\rho, 0)$ or $(0, \rho')$ where ρ and ρ' are rays in their respective fans. Importantly, the product fan can be extended to arbitrary numbers of fans. If we have fans $\mathbf{\Sigma}_1, \ldots, \mathbf{\Sigma}_k$ in vector spaces $\boldsymbol{N}_1, \ldots, \boldsymbol{N}_k$ then

$$\prod_{i=1}^{k} \mathbf{\Sigma}_i \subseteq \bigoplus_{i=1}^{k} \boldsymbol{N}_i$$

is the corresponding product fan, with cones of the form $(\sigma_1, \ldots, \sigma_k)$.

4.3.4 Fan Isomorphisms

We also need a way to determine when two fans are essentially the same. Since fans have both linear and combinatorial components, we need an isomorophism that takes both into account.

Definition 4.21 (Fan Isomorphism)**.** Let $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$ be fans in $\boldsymbol{N}$ and $\boldsymbol{N}'$ respectively. We say that $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$ are *isomorphic* if there exists a linear map

$$\varphi \,:\, \boldsymbol{N} \to \boldsymbol{N}'$$

and an inclusion-preserving bijection

$$\varphi^* \,:\, \mathbf{\Sigma} \to \mathbf{\Sigma}'$$

such that for any $\sigma \in \mathbf{\Sigma}$, the restriction $\varphi|_\sigma$ is a bijective map from σ to $\varphi^*(\sigma)$. We will notate fan isomorphism as $\mathbf{\Sigma} \cong \mathbf{\Sigma}'$.

This is just to say that two fans are isomorphic if we can find a linear map between their respective spaces that preserves the combinatorial data of the fans. As a simple example, we could imagine a fan in $\mathbb{R}^2$ and the exact same fan embedded in $\mathbb{R}^3$. The two vector spaces aren't isomorphic, but these should still be "the same" in terms of fans. Our definition of fan isomorphism can account for these differences in ambient space.

4.3.5 Bergman Fans of Matroids

Much like Chow rings, the general notion of a Bergman fan is broader than what we actually need. Credit goes to, unsurprisingly, George Bergman, who in [Ber71] developed the idea of logarithmic limit-sets of algebraic varieties, which would go on to be called Bergman fans [FS05]. But given we still refuse to carefully define an algebraic variety, the original presentation is not too helpful for us here.

More modern treatments, such as [AK06; HK12], have gone to show us that we can generate the Bergman fan from the combinatorial data of the matroid alone. While it may be more accurate to say we will present the construction of a fan that can be proven to be a Bergman fan, we again simply take it as a definition.

Definition 4.22 (Bergman Fan of a Matroid)**.** Let $\mathcal{M}$ be a matroid with ground set $E = \{e_0, e_1, e_2, \ldots, e_k\}$ which we identify with the basis vectors of $\mathbb{R}^e$, and lattice of flats $\mathcal{L}$. Let

$$\boldsymbol{N}_E = \mathbb{R}^E / \langle e_0 + e_1 + \cdots + e_k \rangle.$$

For any subset $I \subseteq E$ of the ground set, we notate

$$e_I = \sum_{i \in I} e_i$$

as the vector sum of each vector associated to the ground elements in I. The *Bergman fan of* $\mathcal{M}$ is a fan in $\boldsymbol{N}_E$ given by

$$\boldsymbol{\Sigma}_{\mathcal{M}} = \left\{ \operatorname{cone}\bigl(e_F \mid F \in \mathscr{F}\bigr) \;\middle|\; \mathscr{F} \subseteq \mathcal{L}^* \text{ is a flag of } \mathcal{M} \right\}.$$

Let's unpack this definition. First, let's think about what space this fan lives in. Essentially, we can think $\boldsymbol{N}_E = \mathbb{R}^E / \langle e_0 + e_1 + \cdots + e_k \rangle$ as $\mathbb{R}^k$ where we assign all but one ground element of our matroid to the standard basis vectors. This designates one ground element as somewhat special, it doesn't matter which one, but generically we will call it e_0. Then the vector associated to e_0 is the vector of all -1, as the relation in the quotient tells us

$$e_0 = -e_1 - e_2 - \cdots - e_k.$$

Next, let's think about the elements of our fan. They are necessarily cones, and we see that there is one cone per flag in our matroid. This means that there exists a 1-dimensional cone, or *ray*, for each proper flat, as every flat is itself a flag. In general, the length of the flag corresponds to the dimension of the corresponding cone in the fan. As a consequence the largest, by dimension, cones will correspond to complete flags. Similarly, if $F_1, F_2 \in \mathcal{L}$ are non-comparable flats, then there will not be a cone generated by the rays e_{F_1} and e_{F_2}. This is how the fan structure encodes the original combinatorial data of our matroid.

Consistent with our general fan notation, we write $\mathbf{\Sigma}_{\mathcal{M}}(d)$ to be the set of all d-dimensional cones of $\mathbf{\Sigma}$. Using the combinatorial data of a matroid, for any flag

$$\mathscr{F} = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_\ell\}$$

of our matroid we write

$$\sigma_{\mathscr{F}} = \operatorname{cone}\big(e_{F_1}, \dots, e_{F_\ell}\big),$$

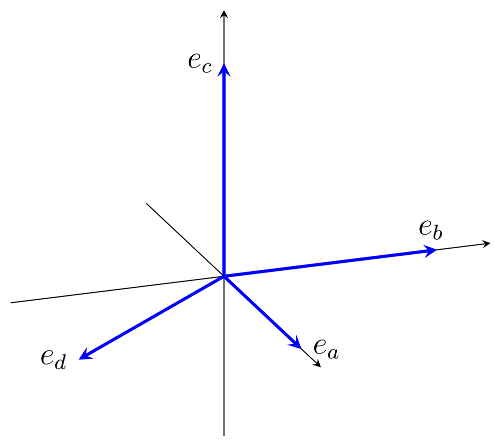
for the cone associated with $\mathscr{F}$. In particular since rays are associated to single element flags, we simply write $\rho_F \in \mathbf{\Sigma}_{\mathcal{M}}(1)$ for flat $F \in \mathcal{L}^*$.

4.3.5.1 An Example Bergman Fan

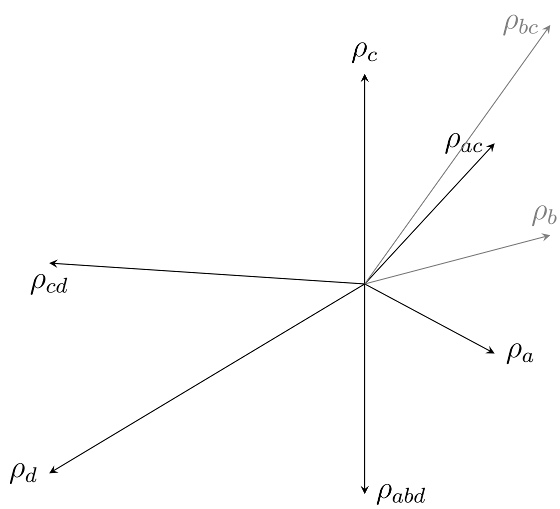
As always, let's turn to our running example and see its corresponding Bergman fan. Recall that

$$E = \{a, b, c, d\} \quad \text{and} \quad \mathcal{L}^* = \{a, b, c, d, ac, bc, cd, abd\}.$$

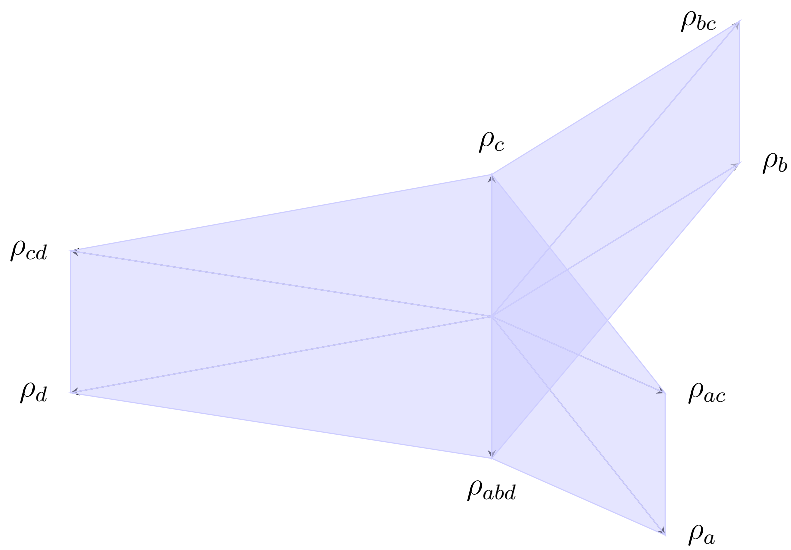
Then we can consider the space $\boldsymbol{N}_E = \mathbb{R}^E / \langle e_a + e_b + e_c + e_d \rangle$. We will designate d as the special element and associate a basis vector of $\mathbb{R}^3$ to the remaining ground elements a, b, c.



Then we can add in all the rays of our fan, corresponding to the flats.



Because our proper flags can only have two elements we will only have at most 2-dimensional cones, since we only have a cone involving the rays ρ_F and $\rho_{F'}$ if $F \subsetneq F'$ or $F' \subsetneq F$.



4.3.6 Properties of the Bergman Fan

Given the properties of fans we introduced earlier, it should come as no surprise that Bergman fans tend to be particularly nice. To start, we have that Bergman fans of matroids are always both pure and simplicial.

Proposition 4.4. *Let $\mathcal{M}$ be a matroid of rank $r+1$. The associated Bergman fan $\mathbf{\Sigma}_{\mathcal{M}}$ is simplicial and pure of dimension r.*

Proof. To prove both of these properties we need one simple fact. By definition, a cone $\sigma \in \mathbf{\Sigma}_{\mathcal{M}}$ is of the form

$$\sigma = \mathrm{cone}(e_{F_1}, \ldots, e_{F_k}),$$

for $F_1 \subsetneq \cdots \subsetneq F_k$ some flag of $\mathcal{M}$. What we need is that $\{e_{F_1}, e_{F_2}, \ldots, e_{F_k}\}$ is linearly independent. To see this, consider the equation

$$\lambda_1 e_{F_1} + \cdots + \lambda_k e_{F_k} = 0.$$

Recall that we defined $e_{F_j} = \sum_{i \in F_j} e_i$ to be the sum of the vectors associated to the ground set, so we can rewrite our equation as

$$\lambda_1 \sum_{i \in F_1} e_i + \cdots + \lambda_k \sum_{i \in F_k} e_i = 0.$$

But now, given that our flats form a flag that have strict inclusion, we may reorder terms to get

$$(\lambda_1 + \lambda_2 + \cdots + \lambda_k) \sum_{i \in F_1} e_i + (\lambda_2 + \cdots + \lambda_k) \sum_{i \in F_2 \setminus F_1} e_i + \cdots + \lambda_k \sum_{i \in F_k \setminus F_{k-1}} e_i = 0.$$

Since each of these sums involves a disjoint set of the vectors associated to the ground set, and any proper subset of these ground set vectors is linearly independent, it must be that $\lambda_1 = \cdots = \lambda_k = 0$, thus showing the set $\{e_{F_1}, e_{F_2}, \ldots, e_{F_k}\}$ is linearly independent.

With this we immediately have that $\mathbf{\Sigma}_{\mathcal{M}}$ is simplicial. For any cone $\sigma_{\mathscr{F}} \in \mathbf{\Sigma}_{\mathcal{M}}$ associated with the flag $\mathscr{F}$, it is generated by $|\mathscr{F}|$ vectors, and because those vectors are linearly independent $\dim(\sigma_{\mathscr{F}}) = |\mathscr{F}|$.

Now, recall that for a d-dimensional simplical cone with generating rays $V = \{v_1, \ldots, v_d\}$, for any subset $V' \subseteq V$, $\text{cone}(V')$ is a face of $\text{cone}(V)$. Since we have shown that every cone of $\mathbf{\Sigma}_{\mathcal{M}}$ is simplicial, we have that any cone associated to a non-complete flag is non-maximal. This follows from the fact that if $\mathscr{F}$ is not a complete flag then there exists another flag $\mathscr{F}'$ such that $\mathscr{F} \subsetneq \mathscr{F}'$. Then we have that $\sigma_{\mathscr{F}}(1) \subsetneq \sigma_{\mathscr{F}'}(1)$ and so $\sigma_{\mathscr{F}}$ is a face of $\sigma_{\mathscr{F}'}$ and by definition non-maximal. Thus, every maximal cone of $\mathbf{\Sigma}_{\mathcal{M}}$ is associated to a complete flag, and any complete flag will have r elements. From above, we may conclude that every maximal cone has dimension r. We have then, that $\mathbf{\Sigma}_{\mathcal{M}}$ is a simplicial and pure of dimension r. □

Now that we know that a Bergman fan must necessarily be simplicial and pure, we can go on to show that the Bergman fan is not only tropical but also balanced.

Proposition 4.5. *Let $\mathcal{M}$ be a matroid. The Bergman fan $\mathbf{\Sigma}_{\mathcal{M}}$ is a balanced tropical fan.*

Proof. Let $\mathcal{M} = (E, \mathcal{L})$ be a matroid of rank $r+1$ and $\mathbf{\Sigma}_{\mathcal{M}} \subseteq \boldsymbol{N}_E$ its Bergman fan. We will mark the rays $\rho \in \mathbf{\Sigma}_{\mathcal{M}}$ such that $u_\rho = e_{F_\rho}$ where $F_\rho \in \mathcal{L}$ is the flat associated to ρ, and we continue to use the convention that

$$e_I = \sum_{i \in I} e_i,$$

for any $I \subseteq E$. Recall that for $\mathbf{\Sigma}_{\mathcal{M}}$ to be balanced it must meet the balancing condition

$$\sum_{\substack{\sigma \in \mathbf{\Sigma}(r) \\ \tau \preceq \sigma}} u_{\sigma \setminus \tau} \in \operatorname{span}(\tau),$$

for every $\tau \in \mathbf{\Sigma}_{\mathcal{M}}(r-1)$.

Let us fix an arbitrary $\tau \in \mathbf{\Sigma}(r-1)$ and consider the flag of flats associated to it

$$\mathscr{F} = \{F_1 \subsetneq \cdots \subsetneq F_{k-1} \subsetneq F_{k+1} \subsetneq \cdots \subsetneq F_r\}.$$

That is, $\mathscr{F}$ is a flag of length $r-1$ with just a single rank k flat missing from being a complete flag. Any maximal cone σ that contains τ as a face will be associated to the same set of flats but with a rank k flat, F_k, added. We can now rephrase our balancing condition in these terms; we wish to show

$$\sum_{\substack{F_k \in \mathcal{L} \\ F_{k-1} \subsetneq F_k \subsetneq F_{k+1}}} e_{F_k} \in \operatorname{span}(e_{F_1}, \ldots, e_{F_{k-1}}, e_{F_{k+1}}, \ldots, e_{F_r}).$$

For convenience, let us call $F_0 = \emptyset$ and $F_{r+1} = E$, as they are the unique rank 0 and r flats, respectively. We will set the convention that $e_{F_0} = \sum_{i \in \emptyset} e_i = 0$. Similarly, we recall that $e_E = 0$. This means we can equivalently show that,

$$\sum_{\substack{F_k \in \mathcal{L} \\ F_{k-1} \subsetneq F_k \subsetneq F_{k+1}}} e_{F_k} \in \operatorname{span}(e_{F_0}, e_{F_1}, \ldots, e_{k-1}, e_{k+1}, \ldots, e_{F_r}, e_{F_{r+1}}).$$

Let's first use our basic definitions to rewrite our sum

$$\begin{aligned}
\sum_{\substack{F_k \in \mathcal{L} \\ F_{k-1} \subsetneq F_k \subsetneq F_{k+1}}} e_{F_k} &= \sum_{\substack{F_k \in \mathcal{L} \\ F_{k-1} \subsetneq F_k \subsetneq F_{k+1}}} \left(\sum_{i \in F_k} e_i \right) \\
&= \sum_{\substack{F_k \in \mathcal{L} \\ F_{k-1} \subsetneq F_k \subsetneq F_{k+1}}} \left(\sum_{i \in F_{k-1}} e_i + \sum_{i \in F_k \setminus F_{k-1}} e_i \right) \\
&= \sum_{\substack{F_k \in \mathcal{L} \\ F_{k-1} \subsetneq F_k \subsetneq F_{k+1}}} \left(e_{F_{k-1}} + e_{F_k \setminus F_{k-1}} \right).
\end{aligned}$$

We note that the term $e_{F_{k-1}}$ in the sum is independent of choice of F_k. We will let $\Lambda > 0$ be the number of complete flags that contain $\mathscr{F}$, letting us write

$$\sum_{\substack{F_k \in \mathcal{L} \\ F_{k-1} \subsetneq F_k \subsetneq F_{k+1}}} \left(e_{F_{k-1}} + e_{F_k \setminus F_{k-1}} \right) = \Lambda e_{F_{k-1}} + \sum_{\substack{F_k \in \mathcal{L} \\ F_{k-1} \subsetneq F_k \subsetneq F_{k+1}}} \left(e_{F_k \setminus F_{k-1}} \right).$$

Now, if we have flats $F_{k-1} \subsetneq F_{k+1}$ of rank $k-1$ and $k+1$ respectively, consider an element of the ground set $i \in F_{k+1} \setminus F_{k-1}$. By definition, the closure $F = \mathrm{cl}(F_{k-1} \cup i)$ is a flat. Since i was not in F_{k-1}, $\mathrm{rk}(F) > \mathrm{rk}(F_{k-1})$, and in particular must be rank k since adding a single element can only increase the rank by at most one. By property (F3) of the properties of flats, we know F is the unique flat such that $F_{k-1} \subsetneq F$ and $i \in F$, so we have that $F_{k-1} \subsetneq F \subsetneq F_{k+1}$. Further, the same property guarantees us that each $i \in F_{k+1} \setminus F_{k-1}$ will appear in exactly one rank k flat, $F_{k,i}$

such that $F_{k-1} \subsetneq F_{k,i} \subsetneq F_{k+1}$. This face lets us write

$$\begin{aligned}
\Lambda e_{F_{k-1}} + \sum_{\substack{F_k \in \mathcal{L} \\ F_{k-1} \subsetneq F_k \subsetneq F_{k+1}}} \left(e_{F_k \setminus F_{k-1}}\right) &= \Lambda e_{F_{k-1}} + e_{F_{k+1} \setminus F_{k-1}} \\
&= (\Lambda - 1) e_{F_{k-1}} + \left(e_{F_{k-1}} + e_{F_{k+1} \setminus F_{k-1}}\right) \\
&= (\Lambda - 1) e_{F_{k-1}} + e_{F_{k+1}}.
\end{aligned}$$

In this form we can clearly see this is in the span of $\{e_{F_0}, e_{F_1}, \ldots, e_{k-1}, e_{k+1}, \ldots, e_{F_r}, e_{F_{r+1}}\}$. With this, we have shown the Bergman fan of $\mathcal{M}$ is balanced. □

Many of the following theorems are about tropical fans more generally, and as such the weight function plays a role. However, since all Bergman fans of matroids are balanced, we will state the theorems just for balanced fans and remove references to the weight function. Simplifying all our lives a little.

We now have a geometric object associated with our matroid, but we still don't yet have a bridge back to the realm of algebra as promised. Towards that goal, it is time we expand the notion of Chow rings a bit.

4.3.7 Chow Rings of Fans and The Degree Map

In the last chapter, we introduced the Chow ring of a matroid. We mentioned we took a bit of a shortcut to defining the Chow ring, using the combinatorial data of the matroid directly. It is perhaps more natural to assign a Chow ring to something geometric in nature. And in fact, a particularly nice geometric object to define a Chow ring on is a fan.

Definition 4.23 (Chow Ring of a Fan)**.** Let $\mathbf{\Sigma} \subseteq \boldsymbol{N}$ be a marked fan and $\boldsymbol{M}$ the dual space of $\boldsymbol{N}$. Associate a polynomial ring with $\mathbf{\Sigma}$ given by

$$P_{\mathbf{\Sigma}} = \mathbb{R}[x_\rho \mid \rho \in \mathbf{\Sigma}(1)],$$

and let

$$\begin{aligned} I_{\mathbf{\Sigma}} &= \langle x_{\rho_1} \cdots x_{\rho_k} \mid \operatorname{cone}(x_{\rho_1}, \ldots, x_{\rho_k}) \notin \mathbf{\Sigma} \rangle \\ J_{\mathbf{\Sigma}} &= \left\langle \sum_{\rho \in \mathbf{\Sigma}(1)} \langle v, u_\rho \rangle x_\rho \;\middle|\; v \in \boldsymbol{M} \right\rangle \end{aligned}$$

be ideals of $P_{\mathbf{\Sigma}}$. The *Chow ring of* $\mathbf{\Sigma}$ is given by the quotient

$$A^\bullet(\mathbf{\Sigma}) = \frac{P_{\mathbf{\Sigma}}}{I_{\mathbf{\Sigma}} + J_{\mathbf{\Sigma}}}.$$

The underlying algebraic geometry here is that to any (sufficiently nice) fan there is an associated algebraic variety, known as a *toric variety.* As mentioned, Chow rings are associated to algebraic varieties, so this gives us the Chow ring of a toric variety arising from a particular fan. For us, what is important is that it doesn't matter if we make the Chow ring from a matroid directly or if we construct it from the Bergman fan of the matroid.

Proposition 4.6. *Let $\mathcal{M}$ be a matroid and $\mathbf{\Sigma}_{\mathcal{M}}$ be the Bergman fan of $\mathcal{M}$. We have that*

$$A^{\bullet}(\mathcal{M}) \cong A^{\bullet}(\mathbf{\Sigma}_{\mathcal{M}}),$$

given by the isomorphism

$$\varphi : A^{\bullet}(\mathcal{M}) \to A^{\bullet}(\mathbf{\Sigma}_{\mathcal{M}})$$
$$x_F \to x_{\rho_F}$$

Proof. Let $\mathcal{M}$ be a matroid with ground set $E = \{e_0, e_1, \ldots, e_n\}$. Let $\mathbf{\Sigma}_{\mathcal{M}} \subseteq \boldsymbol{N}_E$, where we associate $e_1, \ldots, e_n$ to the basis $\{u_{e_1}, \ldots, u_{e_n}\}$ of $\boldsymbol{N}_E$. Then $A^{\bullet}(\mathcal{M})$ and $A^{\bullet}(\mathbf{\Sigma}_{\mathcal{M}})$ are the Chow rings of the matroid and the fan respectively, and we will take φ to be the map between them given in the proposition above. Immediately, we see that φ is an isomorphism of the polynomial rings $P_{\mathcal{M}}$ and $P_{\mathbf{\Sigma}_{\mathcal{M}}}$, as there is exactly one ray in $\mathbf{\Sigma}_{\mathcal{M}}$ for each proper flat of $\mathcal{M}$. What's left to show then is that the ideals used to generate the Chow rings are equivalent.

We will start by looking at the ideal I. Recall that for a matroid we have

$$I_{\mathcal{M}} = \langle x_{F_1} x_{F_2} \mid F_1, F_2 \in \mathcal{L}^*,\ \{F_1, F_2\} \text{ is not a flag} \rangle .$$

We will show this by mutual inclusion; that is, we will show

$$\varphi(I_{\mathcal{M}}) \subseteq I_{\mathbf{\Sigma}_{\mathcal{M}}} \quad \text{and} \quad \varphi^{-1}(I_{\mathbf{\Sigma}_{\mathcal{M}}}) \subseteq I_{\mathcal{M}}.$$

To start let's take $x_F x_{F'} \in I_{\mathcal{M}}$ be an arbitrary generator of $I_{\mathcal{M}}$. Under our map,

$$\varphi(x_F x_{F'}) = x_{\rho_F} x_{\rho_{F'}}$$

and since $\{F, F'\}$ is not a flag of $\mathcal{M}$, it doesn't correspond to a cone of $\mathbf{\Sigma}_{\mathcal{M}}$ by definition. Thus, $x_{\rho_F} x_{\rho_{F'}} \in I_{\mathbf{\Sigma}_{\mathcal{M}}}$ and $\varphi(I_{\mathcal{M}}) \subseteq I_{\mathbf{\Sigma}_{\mathcal{M}}}$. Now, let $x_{\rho_{F_1}} \cdots x_{\rho_{F_k}} \in I_{\mathbf{\Sigma}_{\mathcal{M}}}$ be an arbitrary generator of $I_{\mathbf{\Sigma}_{\mathcal{M}}}$. Since,

$$\mathrm{cone}(x_{\rho_{F_1}}, \ldots, x_{\rho_{F_k}}) \notin \mathbf{\Sigma}_{\mathcal{M}}$$

we have that $\{F_1, \ldots, F_k\}$ is not a flag, and so there are at least two elements F_i, F_j that are incomparable. So, we have

$$\begin{aligned} \varphi^{-1}(x_{\rho_{F_1}} \cdots x_{\rho_{F_k}}) &= x_{F_1} \cdots x_{F_k} \\ &= (x_{F_1} \cdots x_{Fi-1} x_{Fi+1} \cdots x_{Fj-1} x_{Fj+1} \cdots x_{F_k}) x_{F_i} x_{F_j} \in I_{\mathcal{M}}, \end{aligned}$$

showing that $\varphi^{-1}(I_{\mathbf{\Sigma}_{\mathcal{M}}}) \subseteq I_{\mathcal{M}}$.

Turning our attention to the ideal J, we notice that in the fan context, $J_{\mathbf{\Sigma}_{\mathcal{M}}}$ is infinitely generated, while $J_{\mathcal{M}}$ is finitely generated. Our first order of business then is to show that $I_{\mathbf{\Sigma}_{\mathcal{M}}}$ can be finitely generated at all. Recall that $u_{e_0} = -u_{e_1} - \cdots - u_{e_n}$ and that we write

$$u_I = \sum_{e \in I} u_e$$

for any $I \subseteq E$. We will write $\boldsymbol{M}_E$ for the dual space of $\boldsymbol{N}_E$ and take $\{u^{e_1}, \ldots, u^{e_n}\} \subseteq \boldsymbol{M}_E$ to be the dual basis such that

$$\langle u^{e_i}, u_{e_j} \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

In the context of a Bergman fan of a matroid, we can rephrase the ideal as

$$J_{\boldsymbol{\Sigma}_{\mathcal{M}}} = \left\langle \sum_{F \in \mathcal{L}^*} \langle v, u_F \rangle x_{\rho_F} \;\middle|\; v \in \boldsymbol{M}_E \right\rangle.$$

We will show that any single generator of $J_{\boldsymbol{\Sigma}_{\mathcal{M}}}$ can be represented by a finite linear combination. To do so, let us fix a $v \in \boldsymbol{M}_e$, which means it can be represented by the linear combination of dual basis vectors

$$v = \lambda_{e_1} u^{e_1} + \cdots + \lambda_{e_k} u^{e_k}.$$

So, we have that

$$\begin{aligned}
\sum_{F\in\mathcal{L}^*}\langle v,u_F\rangle x_{\rho_F} &= \sum_{F\in\mathcal{L}^*}\left\langle \sum_{i\in[n]}\lambda_{e_i}u^{e_i},u_F\right\rangle x_{\rho_F}\\
&= \sum_{F\in\mathcal{L}^*}\sum_{i\in[n]}\lambda_{e_i}\langle u^{e_i},u_F\rangle x_{\rho_F}\\
&= \sum_{i\in[n]}\lambda_{e_i}\sum_{F\in\mathcal{L}^*}\langle u^{e_i},u_F\rangle x_{\rho_F}\\
&= \sum_{e\in E\setminus e_0}\lambda_e\sum_{F\in\mathcal{L}^*}\langle u^e,u_F\rangle x_{\rho_F},
\end{aligned}$$

following from the bilinearity of $\langle\cdot,\cdot\rangle$ and that $\{e_1,\ldots,e_n\} = E\setminus e_0$. We may conclude here that

$$\left\langle \sum_{F\in\mathcal{L}^*}\langle v,u_F\rangle x_{\rho_F} \;\middle|\; v\in \boldsymbol{M}_E \right\rangle = \left\langle \sum_{F\in\mathcal{L}^*}\langle u^e,u_F\rangle x_{\rho_F} \;\middle|\; e\in E\setminus e_0 \right\rangle$$

and which at least shows that $J_{\mathbf{\Sigma}_\mathcal{M}}$ is finitely generated. Next, we need the observation that for any flat F

$$\langle u^{e_i},u_F\rangle = \begin{cases} 1 & e_i\in F \text{ and } e_0\notin F\\ 0 & e_i\in F \text{ and } e_0\in F\\ 0 & e_i\notin F \text{ and } e_0\notin F\\ -1 & e_i\notin F \text{ and } e_0\in F. \end{cases}$$

This follows again from bilinearity, as we may write

$$\langle u^{e_i}, u_F\rangle = \left\langle u^{e_i}, \sum_{e\in F} u_e \right\rangle$$

and $u_{e_0} = -\sum_{i\in[n]} e_i$. This gives us the direct equality

$$\left\langle \sum_{F\in\mathcal{L}^*} \langle u^e, u_F\rangle x_{\rho_F} \;\middle|\; e \in E\setminus e_0 \right\rangle = \left\langle \sum_{e\in F} x_{\rho_F} - \sum_{e_0\in F} x_{\rho_F} \;\middle|\; e\in E\setminus e_0 \right\rangle$$

from reindexing using the observation above. Nearing the end, we have one last equality to show. We have that

$$\left\langle \sum_{e\in F} x_{\rho_F} - \sum_{e_0\in F} x_{\rho_F} \;\middle|\; e\in E\setminus e_0 \right\rangle \subseteq \left\langle \sum_{e\in F} x_{\rho_F} - \sum_{e'\in F} x_{\rho_F} \;\middle|\; e, e'\in E \right\rangle,$$

since the left-hand side is just the specialization of the right-hand to $e' = e_0$. In the other direction, we see that

$$\left\langle \sum_{e\in F} x_{\rho_F} - \sum_{e_0\in F} x_{\rho_F} \;\middle|\; e\in E\setminus e_0 \right\rangle \supseteq \left\langle \sum_{e\in F} x_{\rho_F} - \sum_{e'\in F} x_{\rho_F} \;\middle|\; e, e'\in E \right\rangle,$$

given that for any $e, e' \in E\setminus e_0$,

$$\sum_{e\in F} x_{\rho_F} - \sum_{e'\in F} x_{\rho_F} = \left(\sum_{e\in F} x_{\rho_F} - \sum_{e_0\in F} x_{\rho_F}\right) - \left(\sum_{e'\in F} x_{\rho_F} - \sum_{e_0\in F} x_{\rho_F}\right),$$

and

$$\sum_{e_0 \in F} x_{\rho_F} - \sum_{e' \in F} x_{\rho_F} = -\left(\sum_{e' \in F} x_{\rho_F} - \sum_{e_0 \in F} x_{\rho_F}\right).$$

All of this together means that we have shown that

$$J_{\mathbf{\Sigma}_{\mathcal{M}}} = \left\langle \sum_{e \in F} x_{\rho_F} - \sum_{e' \in F} x_{\rho_F} \;\middle|\; e, e' \in E \right\rangle$$

and so $\varphi^{-1}(J_{\mathbf{\Sigma}_{\mathcal{M}}})$ is precisely $J_{\mathcal{M}}$. Having shown that the ideals produce equivalent quotients, we conclude that $A^{\bullet}(\mathcal{M}) \cong A^{\bullet}(\mathbf{\Sigma}_{\mathcal{M}})$. □

Now that we know about the Chow rings of fans, we can provide more context to the degree map we defined in the previous chapter. We defined it practically for how we actually need to use it, but we can now give a more full definition.

Definition 4.24 (Degree Map (of a Fan))**.** Let $(\mathbf{\Sigma}, \omega)$ be a tropical d-fan. The *degree map* of $\mathbf{\Sigma}$ is the linear map

$$\deg_{(\mathbf{\Sigma},\omega)} \; : \; A^d(\mathbf{\Sigma}) \to \mathbb{Z}$$

such that for any top-dimensional cone $\sigma \in \mathbf{\Sigma}(d)$,

$$\deg_{(\mathbf{\Sigma},\omega)}\left(\prod_{\rho \in \sigma(1)} x_\rho\right) = \omega(\sigma).$$

What [AHK18, Proposition 5.6] more accurately states is that a degree map of a fan is well-defined if and only if there exists a tropical weight function, ω, on the fan. Since we are working with Bergman fans, which we have shown are balanced, we can just use $\omega = 1$ as we do in our original definition.

Worth mentioning here, is that Chow rings of fans play nicely with fan isomorphims and product fans. When two fans are isomorphic, so too are their Chow rings.

Proposition 4.7. *Let $\mathbf{\Sigma}, \mathbf{\Sigma}'$ be fans. If $\mathbf{\Sigma} \cong \mathbf{\Sigma}'$, then*

$$A^\bullet(\mathbf{\Sigma}) \cong A^\bullet(\mathbf{\Sigma}').$$

Recall that in Definition 4.21 a fan isomorphism comes equipped with a bijective map between cones of our fans. This induces the isomorophism between the two Chow rings. When taking the product of fans we have the following relationship of Chow rings.

Proposition 4.8. *Let $\mathbf{\Sigma}, \mathbf{\Sigma}'$ be fans and take $\mathbf{\Sigma} \times \mathbf{\Sigma}'$ to be their product. The relationship between their Chow rings is given by*

$$A^\bullet(\mathbf{\Sigma} \times \mathbf{\Sigma}') \cong A^\bullet(\mathbf{\Sigma}) \otimes A^\bullet(\mathbf{\Sigma}').$$

For those who haven't yet had enough caffeine to deal with tensor products, we may offer an alternative framing. For fans $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$, if their Chow rings are given by

$$A^\bullet(\mathbf{\Sigma}) = \frac{\mathbb{R}[x_1, \ldots, x_k]}{I_{\mathbf{\Sigma}} + J_{\mathbf{\Sigma}}} \quad \text{and} \quad A^\bullet(\mathbf{\Sigma}') = \frac{\mathbb{R}[y_1, \ldots, y_\ell]}{I_{\mathbf{\Sigma}'} + J_{\mathbf{\Sigma}'}}$$

then there is a natural, categorically speaking, isomorophism giving

$$A^\bullet(\mathbf{\Sigma}) \otimes A^\bullet(\mathbf{\Sigma}') \cong \frac{\mathbb{R}[x_1, \ldots, x_k, y_1, \ldots, y_\ell]}{I_{\mathbf{\Sigma}} + J_{\mathbf{\Sigma}} + I_{\mathbf{\Sigma}'} + J_{\mathbf{\Sigma}'}}.$$

This then looks like the much more naïve choice of just smashing the two rings together, though some prodding will show that this is exactly what we want.

Having established this correspondence between the Chow rings of the matriod and its Bergman fan, we may now move on to our final object, normal complexes. These are defined generally for fans, and in particular we will have theorems that relate to the Chow rings of tropical fans. Now we know we can refer to the Chow ring of a Bergman fan of a matroid interchangeably with the Chow ring of the matroid itself, we will be able to leverage these theorems.

4.4 Normal Complexes

The work in this section is by far the most recent, coming from work within the last two years at time of writing. We will provide a brief summary of the work by Nathanson and Ross [NR23] and by Nowak and Ross, jointly with the author [NOR23].

This section will use the Bergman fan of a matroid to create objects called normal complexes. From this we can develop a notion of (mixed) volume, as well as present theorems that relate this volume back to the Chow ring, finally giving us all the components necessary for our main result.

4.4.1 The Normal Complex of a Fan

In brief, our strategy is simply to bound the cones of the fan, so we have a reasonable object to take the volume of. A little more specifically, a normal complex is a truncation of a fan into a polytopal complex using hyperplanes normal to the rays of the fan, thus the name. They were initially developed in [NR23], and the following definitions and propositions come from this work. However, we can't just take any truncation of our fan. For reasons, we will soon see, not any truncation will work. To characterize which ones will, we first need to introduce the idea of cubical and pseudocubical values.

Definition 4.25 (Cubical and Pseudocubical Values)**.** Let $\mathbf{\Sigma} \subseteq \boldsymbol{N}$ be a marked, simplicial d-fan and let $* \in \mathrm{Inn}(\boldsymbol{N})$ be an inner product. Pick a vector $z \in \mathbb{R}^{\mathbf{\Sigma}(1)}$ that associates a real number to each ray of our fan. For each ray $\rho \in \mathbf{\Sigma}(1)$ we have a corresponding hyperplane and half-space

$$\mathrm{H}_{\rho,*}(z) = \{v \in N \mid v * u_\rho = z\} \quad \text{and} \quad \mathrm{H}^-_{\rho,*}(z) = \{v \in N \mid v * u_\rho \leq z\}.$$

For each cone $\sigma \in \mathbf{\Sigma}$, let $w_{\sigma,*}(z)$ be the unique value such that

$$w_{\sigma,*}(z) * u_\rho = z_\rho$$

for each ray $\rho \in \sigma(1)$. We say z is *cubical* if for all $\sigma \in \mathbf{\Sigma}$,

$$w_{\sigma,*}(z) \in \sigma^\circ,$$

where σ° is the relative interior of σ. If we instead require only that

$$w_{\sigma,*}(z) \in \sigma$$

for all $\sigma \in \mathbf{\Sigma}$, we say z is *pseudocubical.*

For a given fan $\mathbf{\Sigma} \subseteq N$ and inner product $* \in \mathrm{Inn}(N)$, we denote cubical values as $\mathrm{Cub}(\mathbf{\Sigma}, *) \subseteq \mathbb{R}^{\mathbf{\Sigma}(1)}$ and the set of pseudocubical values as $\overline{\mathrm{Cub}}(\mathbf{\Sigma}, *) \subseteq \mathbb{R}^{\mathbf{\Sigma}(1)}$. This is all to say that when we select values to generate truncating hyperplanes for each ray, we want the intersections of the hyperplanes to lie within the cones of the fan. We can see an example in 2-dimensions.

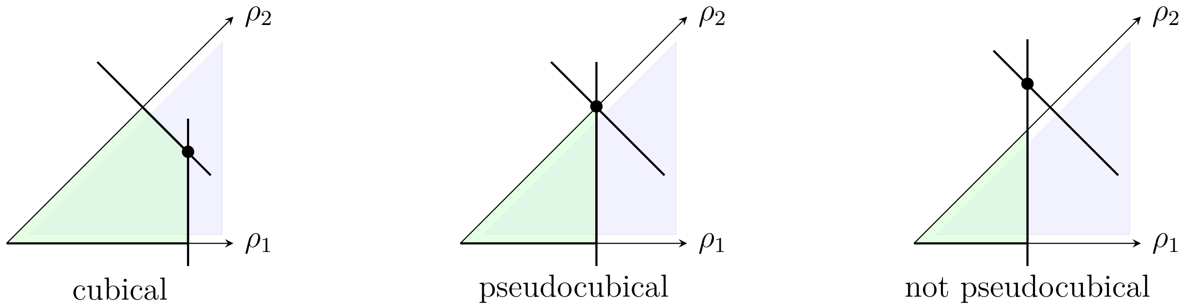


Figure 4.14: Examples of possible normal truncating hyperplane arrangements for a 2-dimensional fan.

With this, we can now define a normal complex.

Definition 4.26 (Normal Complex)**.** Let $\mathbf{\Sigma} \subseteq N$ be a simplicial d-fan, with marked point u_ρ on each ray $\rho \in \mathbf{\Sigma}(1)$. Choose an inner product $* \in \mathrm{Inn}(N)$ and a pseudocubical vector $z \in \overline{\mathrm{Cub}}(\mathbf{\Sigma}, *)$. Recall that for each ray $\rho \in \mathbf{\Sigma}(1)$ we have a corresponding hyperplane and half-space

$$\mathrm{H}_{\rho,*}(z) = \{v \in N \mid v * u_\rho = z\} \quad \text{and} \quad \mathrm{H}^-_{\rho,*}(z) = \{v \in N \mid v * u_\rho \leq z\}.$$

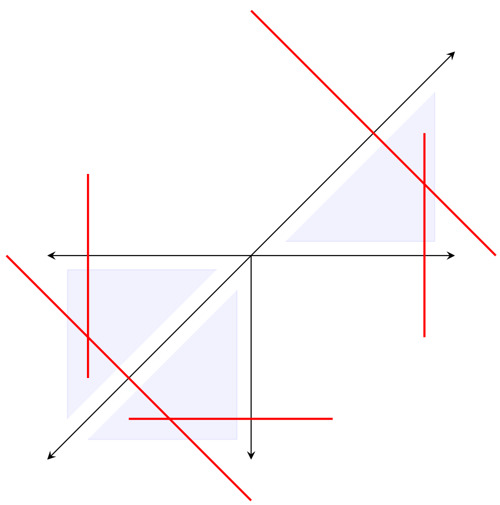
For each cone $\sigma \in \mathbf{\Sigma}$, we define a polytope $P_{\sigma,*}(z)$ given by

$$P_{\sigma,*}(z) = \sigma \cap \left(\bigcap_{\rho \in \sigma(1)} \mathrm{H}^-_{\rho,*}(z) \right).$$

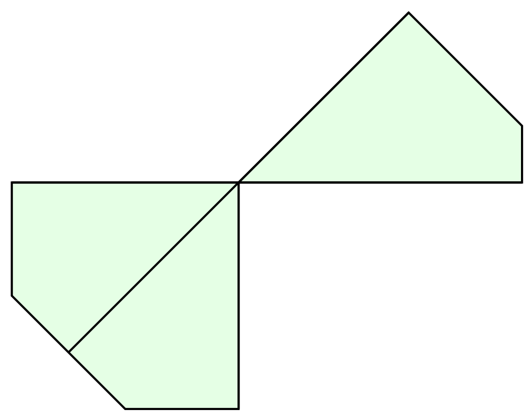
The *normal complex* of $\mathbf{\Sigma}$ is the polytopal complex

$$C_{\mathbf{\Sigma},*}(z) = \bigcup_{\sigma \in \mathbf{\Sigma}} P_{\sigma,*}(z)$$

That these truncations of fans give us well-defined polytopal complex is a result of Proposition 3.7 in [NR23]. The basic idea is that because the pseudocubical condition insures the hyperplanes intersect within each cone exactly at one point, these intersection points define the vertices of each polytope in the complex. Here is another place where an example is worth many words.



(a) A fan with a possible arrangement of hyperplanes given by a cubical value



(b) The resulting normal complex of the fan given the choice of cubical value.

4.4.2 Faces of a Normal Complex

Much of the development of normal complexes comes from inspiration in the polytopal setting. We want an object that, under much stronger conditions, has similar properties to a polytope. Something that will be very useful is having a face structure on normal complexes. These faces are a bit more involved than those of a polytope, so we will need to develop a few preliminary concepts inorder to define them. First, we will consider the neighborhood of a cone in a fan.

Definition 4.27 (Neighborhood)**.** Let $\mathbf{\Sigma}$ be a fan and $\sigma \in \mathbf{\Sigma}$ a cone in the fan. The neighborhood of σ in $\mathbf{\Sigma}$ is

$$\operatorname{nbd}(\sigma, \mathbf{\Sigma}) = \{\tau \mid \tau \preceq \pi,\ \sigma \preceq \pi,\ \text{for some } \pi \in \mathbf{\Sigma}\}\,.$$

The neighborhood of a cone σ is essentially the collection of all other cones in the fan that contain σ as a face. We then take all faces of those cones to insure that $\operatorname{nbd}(\sigma, \mathbf{\Sigma})$ is still a fan. In fact $\operatorname{nbd}(\sigma, \mathbf{\Sigma})$ is still a fan living in the same space as $\mathbf{\Sigma}$. If we additionally quotient out the components of σ from the neighborhood we have what is called the star of σ.

Definition 4.28 (Star)**.** Let $\mathbf{\Sigma} \subseteq \boldsymbol{N}$ be a fan and $\sigma \in \mathbf{\Sigma}$ a cone in the fan. We will write

$$\boldsymbol{N}_\sigma = \operatorname{span}\big(u_\rho \mid \rho \in \sigma(1)\big) \subseteq \boldsymbol{N}$$

to be the span of a cone σ in $\boldsymbol{N}$. The *star* of σ is given by

$$\operatorname{star}(\sigma, \boldsymbol{\Sigma}) = \{\overline{\tau} \mid \tau \in \operatorname{nbd}(\sigma, \boldsymbol{\Sigma})\} \subseteq \boldsymbol{N}/\boldsymbol{N}_\sigma,$$

where $\overline{\tau}$ is the image of τ in the quotient space $\boldsymbol{N}/\boldsymbol{N}_\sigma$.

When provided with an inner product, we can identify the quotient space $\boldsymbol{N}/\boldsymbol{N}_\sigma$ with the orthogonal space, $\boldsymbol{N}_\sigma^\perp$. Given we always specify an inner product when constructing a normal complex, we mostly think of the star as the projection of the neighborhood of σ into $\boldsymbol{N}_\sigma^\perp$. Since the neighborhood was a fan in the original space, the star will be a fan in this orthogonal space, which we notate as $\boldsymbol{\Sigma}^\sigma$.

Perhaps more surprisingly, if we have the necessary data to make a normal complex on some fan $\boldsymbol{\Sigma}$, then for any cone $\sigma \in \boldsymbol{\Sigma}$, we also have everything we need to define a normal complex on $\boldsymbol{\Sigma}^\sigma$. We call the resulting normal complexes faces of the normal complex, in analogy with polytopes, and the details of their construction and existence can be found in section 4 of [NOR23].

Definition 4.29 (Face of Normal Complex)**.** Let $\boldsymbol{\Sigma} \subset \boldsymbol{N}$ be a marked simplicial d-fan and let $* \in \operatorname{Inn}(\boldsymbol{N})$ be an inner product. Given any $z \in \overline{\operatorname{Cub}}(\boldsymbol{\Sigma}, *)$ and cone $\sigma \in \Sigma$, the *face* of $C_{\boldsymbol{\Sigma},*}(z)$ associated to σ is

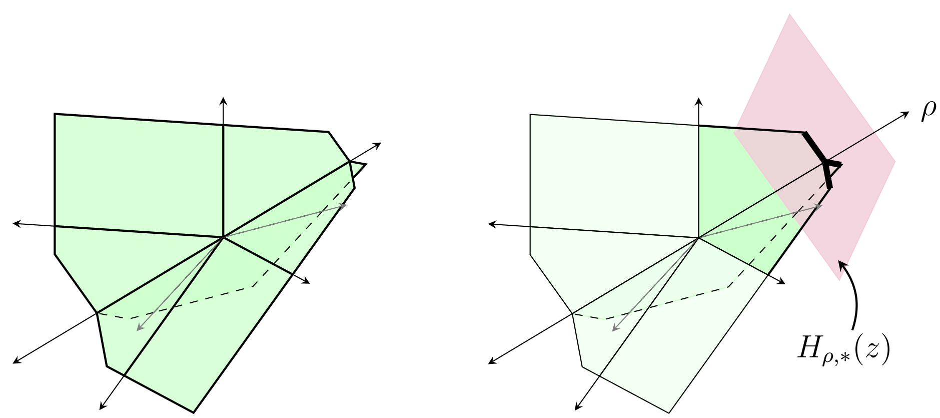
$$F^\sigma\big(C_{\boldsymbol{\Sigma},*}(z)\big) = C_{\boldsymbol{\Sigma}^\sigma,*^\sigma}(z^\sigma),$$

where $*^\sigma$ is the restriction of $*$ to $\boldsymbol{N}_\sigma^\perp$ and $z^\sigma \in \overline{\operatorname{Cub}}(\boldsymbol{\Sigma}^\sigma, *^\sigma)$ is defined component-wise by the rule

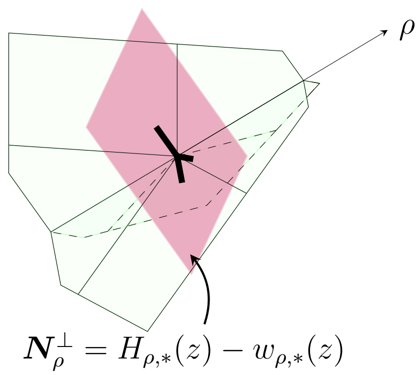
$$z^{\sigma}_{\rho} = z_{\rho} - w_{\sigma,*}(z) * u_{\rho}.$$

A good thing to note here is there is a dimension reversing relationship between the dimension of σ and $\mathbf{\Sigma}^{\sigma}$. If $\mathbf{\Sigma}$ is a d-fan and σ is of dimension k, then $\mathbf{\Sigma}^{\sigma}$ is pure of dimension $d-k$. We would then call $F^{\sigma}\big(C_{\mathbf{\Sigma},*}(z)\big)$ a $(d-k)$-dimensional face of the normal complex $C_{\mathbf{\Sigma},*}(z)$.

Paralleling the faces of polytopes, we specify a face of a normal complex via a hyperplane. Since a normal complex is constructed by hyperplanes, we can specify any face using the hyperplanes associated to each cone in the fan. Below is an example of specifying a face using a hyperplane associated to a ray, with the neighborhood of that ray highlighted.



Since this normal complex is coming from a 2-fan, and we use a hyperplane associated to a ray, we'd expect the result to be a 1-dimensional normal complex. We can see what locally looks similar to a 1-dimensional normal complex in the hyperplane, but this is where faces of normal complexes diverge from faces of polytopes. In the polytopal setting, faces are also polytopes by default, but what we currently have is not a normal complex. This is why we use the star fan in our definition of faces of normal complexes, as the result will be in a space where the complex is centered at the origin.



4.4.3 Volume of a Normal Complex

After so long hinting at the importance of volume, we finally have something related to our matroid we can actually take the volume of. However, there should be something bothering you about taking the volume of these objects. In the construction of normal complexes we have to make several choices, and those choices will directly affect the resulting volume. Recall that the definition of a normal complex requires us to first choose an inner product on the space and then a vector of z-values. We will show that these choices are, for certain fans, not as impactful as one may worry. And in doing so we will find a powerful correspondence between geometry and algebra.

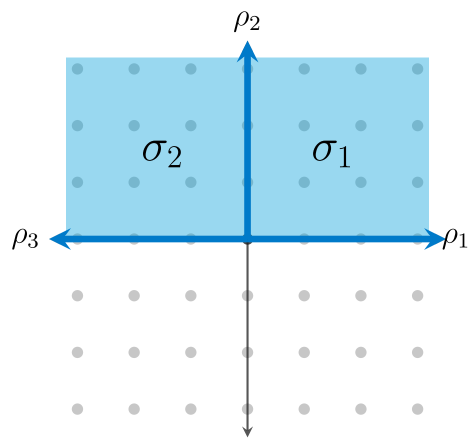
To even begin to discuss volume of normal complexes, we need a working definition. Recall that we only defined volume functions on polytopes, and a normal complex is a polytopal complex. A natural choice here is to just take the volume of each maximal polytope in the complex and sum them together. For now, given a pseudocubical z-value, the volume of the normal complex $C_{\mathbf{\Sigma},*}(z)$ will be given by

$$\mathrm{Vol}_{\mathbf{\Sigma},\bullet}(z) = \sum_{\sigma \in \mathbf{\Sigma}(d)} \mathrm{Vol}_\sigma\left(P_{\sigma,*}(z)\right),$$

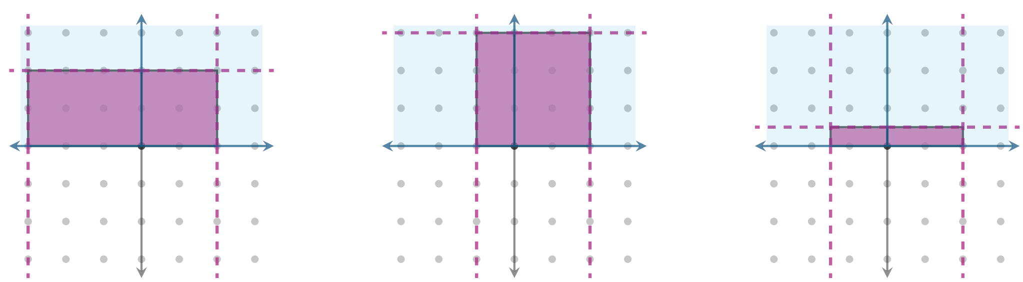
where Vol_σ is some volume function of $\boldsymbol{N}_\sigma$. We don't yet have an opinion on which volume function to use, any choice will do for now. This won't be our final definition of volume, but it uses the same basic idea, and is sufficient for now.

Now it should be clear that once we fix a fan and an inner product, $(\mathbf{\Sigma}, *)$ defines a family of normal complexes, assuming $\overline{\mathrm{Cub}}(\mathbf{\Sigma}, *) \neq \emptyset$. Each choice of $z \in \overline{\mathrm{Cub}}(\mathbf{\Sigma}, *)$ gives rise to a different normal complex. Once we have fixed everything else, the volume $\mathrm{Vol}_{\mathbf{\Sigma},*}(z)$ is determined entirely by the choice of the z-value. In an ideal scenario, we could find an expression for $\mathrm{Vol}_{\mathbf{\Sigma},*}(z)$ in terms of a generic variable z, that, when given a pseudocubcal value, corresponds to the volume.

Let's consider an example where this idea works well. Let $\mathbf{\Sigma} \subseteq \mathbb{R}^2$ be the fan below, and let $* = \bullet$ be the dot product.



We can see that different choices of z yield different complexes and different volumes.



$\mathrm{Vol}_{\mathbf{\Sigma},\boldsymbol{\cdot}}\left((2,2,3)\right) = 10$ $\mathrm{Vol}_{\mathbf{\Sigma},\boldsymbol{\cdot}}\left((2,3,1)\right) = 9$ $\mathrm{Vol}_{\mathbf{\Sigma},\boldsymbol{\cdot}}\left((2,0.5,1.5)\right) = 1.75$

In this particular case, it is not too difficult to see how the volume of the normal complex is a function of z. We can write the volume as the expression

$$\mathrm{Vol}_{\mathbf{\Sigma},\cdot}(z) = z_{\rho_1} z_{\rho_2} + z_{\rho_2} z_{\rho_3}.$$

Written as a polynomial like this, it is defined on any $z \in \mathbb{R}^3$, but it only has a volumetric interpretation when $z \in \overline{\mathrm{Cub}}(\boldsymbol{\Sigma}, \bullet)$. Being able to consider $\mathrm{Vol}_{\boldsymbol{\Sigma},\cdot}(z)$ as a polynomial relieves some guilt of choice, as we can reason more generally about whole families of normal complexes.

So, we can, potentially with a lot of work, generalize over the choice of z by treating it as a variable, but this assumes we have fixed an inner product. If instead we fix a z-value, it seems that the choice of inner product will have a profound effect on the volume of the resulting normal complexes. To learn why this is not always the case, we will need our official definition of volume of a normal complex.

Definition 4.30 (Volume of a Normal Complex)**.** Let $\boldsymbol{N}$ be a real-valued vector space and $\boldsymbol{\Sigma} \subseteq \boldsymbol{N}$ be a simplicial d-fan with marked points u_ρ. Choose an inner product $* \in \mathrm{Inn}(\boldsymbol{N})$ and define

$$\boldsymbol{N}_\sigma = \mathrm{span}\left(u_\rho \mid \rho \in \sigma(1)\right) \subseteq \boldsymbol{N}.$$

Then, let $\boldsymbol{M}_\sigma$ be the vector space dual to $\boldsymbol{N}_\sigma$. Using $\{u_\rho \mid \rho \in \sigma(1)\}$ as the basis of $\boldsymbol{N}_\sigma$, and using our chosen inner product, we may identify the dual basis $\{u^\rho \mid \rho \in \sigma(1)\}$ of $\boldsymbol{M}_\sigma$ as a subset of $\boldsymbol{N}_\sigma$. For each $\sigma \in \boldsymbol{\Sigma}(d)$, we choose the volume function

$$\mathrm{Vol}_\sigma : \{\text{Polytopes in } \boldsymbol{N}_\sigma\} \to \mathbb{R}_{\geq 0}$$

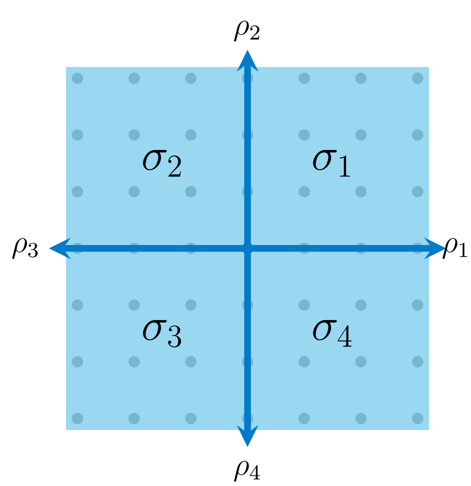
characterized by $\mathrm{Vol}_\sigma\left(\mathrm{conv}(\{0\} \cup \{u^\rho \mid \rho \in \sigma(1)\})\right) = 1$. For pseudocubical value

$z \in \overline{\mathrm{Cub}}(\sigma, *)$, the *volume of the normal complex* $C_{\mathbf{\Sigma},*}(z)$ is

$$\mathrm{Vol}_{\mathbf{\Sigma},*}(z) = \sum_{\sigma \in \mathbf{\Sigma}(d)} \mathrm{Vol}_{\sigma}\left(P_{\sigma,*}(z)\right).$$

The basic idea is the same as we presented above, we are taking the volume of each top-dimensional polytope in the complex and then summing them. Most of the verbosity in the definition comes from the fact that we have to choose the volume function for each component rather carefully. Not only do we use a different volume function for each polytope in the sum, we use the simplical volume of the *dual* space to the span of the cone. This choice of volume has inspiration in the realm of algebraic geometry, and we recommend a deeper dive in [NR23] for those who are interested.

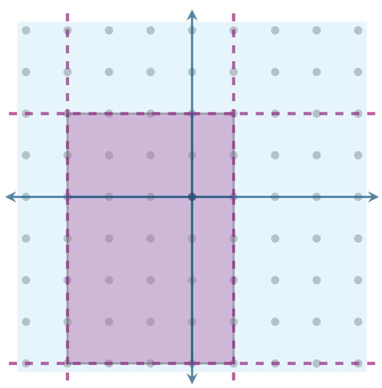
Let's provide a bit of motivation for this definition with a simple example, using the fan below.



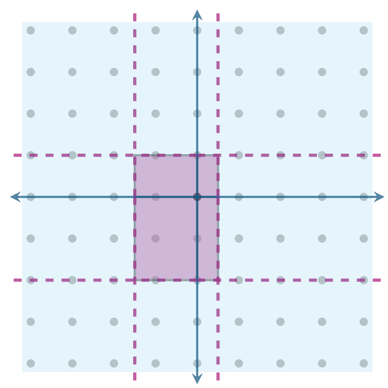
Let $\bullet$ be the standard dot product and $\star$ to be the inner product given by

$$(a, b) \star (c, d) = 2ac + 2bd,$$

a scaled version of the dot product. Now we will fix $z = (1, 2, 3, 4)$ and note, importantly, that $z \in \overline{\text{Cub}}(\mathbf{\Sigma}, \bullet)$ and $z \in \overline{\text{Cub}}(\mathbf{\Sigma}, \star)$. The two inner products give us two normal complexes.

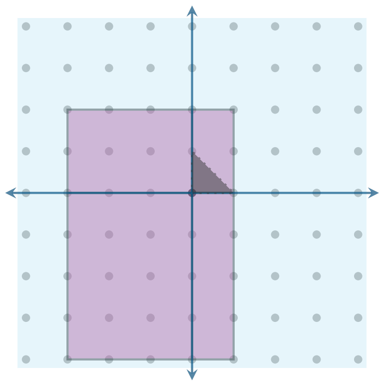


$C_{\mathbf{\Sigma},\bullet}(z)$

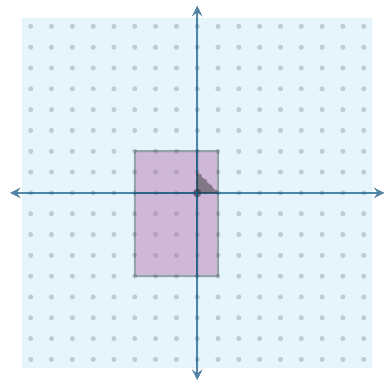


$C_{\mathbf{\Sigma},\star}(z)$

Clearly under our standard notion of volume, these have different volumes. But what if instead we can consider the volume as given in Definition 4.30? We will adjust the grid to represent the integer lattice given by the dual basis, and we can see that the volumes do agree.



$\mathrm{Vol}_{(\mathbf{\Sigma},\bullet)}\big((1,2,3,4)\big) = 48$



$\mathrm{Vol}_{(\mathbf{\Sigma},\star)}\big((1,2,3,4)\big) = 48$

Now, our example fan is particularly nice and the inner products not too out there. So it could be a fluke that the volumes align. But it is in the search of fans whose volumes are independent of choice of inner product that [NR23, Theorem 6.3] was developed.

Theorem 4.9 (Nathanson-Ross 2023)**.** *Let $\mathbf{\Sigma} \subseteq \boldsymbol{N}$ be a balanced d-fan. Choose an inner product $* \in \mathrm{Inn}(\boldsymbol{N})$ and pseudocubical value $z \in \overline{\mathrm{Cub}}(\mathbf{\Sigma}, *)$. We define*

$$D(z) = \sum_{\rho \in \mathbf{\Sigma}(1)} z_\rho x_\rho \in A^1(\mathbf{\Sigma}),$$

a divisor of the Chow ring. Then

$$\mathrm{Vol}_{\mathbf{\Sigma},*}(z) = \deg_{\mathbf{\Sigma}}\big(D(z)^d\big).$$

The term $\deg\big(D(z)^d\big)$ is known as the *volume polynomial.* Despite the name, the volume polynomial was treated as an entirely algebraic idea, having long since been generalized from the original context where it had a relationship to an actual volume. So, this theorem not only gives back a sense of volume to the volume polynomial, it tells us something quite remarkable about the volume of normal complexes of balanced fans. Since $D(z)$ is a linear term, $D(z)^d$ will be a homogeneous polynomial of degree d. Better still, $\deg_{\boldsymbol{\Sigma}}$ has no notion of inner product, so it must be that $\mathrm{Vol}_{\boldsymbol{\Sigma},*}(z)$ is also independent of the inner product. This theorem gives us both a polynomial expression for volume and exculpates us from having to make a choice of inner product.

Additionally, we have the first link back to the Chow ring. By carefully taking volumes of the normal complex, we can evaluate top degrees of divisors under the degree map. This isn't the full story, as we want to evaluate terms of the form $\alpha^{d-k}\beta^k$, which is not a single divisor raised to the top power. These are called mixed degrees of divisors, and perhaps that is a good hint as to what we need to develop next.

4.4.3.1 An Example By Way of Lemma

This is a good opportunity to introduce a lemma that we will be useful later on. This gives a direct application of Theorem 4.9, and the proof is a nice example of how it works in practice. It is also an opportunity to think a bit more about the degree map. We would like to mention that this proposition was first proven by Anastasia

Nathanson in her master's thesis [Nat21], and without the aid of Theorem 4.9.

Lemma 4.10. *Let $\mathcal{M} = (E, \mathcal{L})$ be a rank 3 matroid and $\mathbf{\Sigma}_{\mathcal{M}}$ its Bergman fan. Then for any $* \in \mathrm{Inn}(\boldsymbol{N}_E)$ and $z \in \overline{\mathrm{Cub}}(\mathbf{\Sigma}_{\mathcal{M}}, *)$,*

$$\mathrm{Vol}_{\mathbf{\Sigma}_{\mathcal{M}},*}(z) = 2\sum_{\substack{F \subsetneq G \\ F,G \in \mathcal{L}^*}} z_F z_G - \sum_{\substack{G \in \mathcal{L}^* \\ \mathrm{rk}(G)=2}} z_G^2 - \sum_{\substack{F \in \mathcal{L}^* \\ \mathrm{rk}(F)=1}} (\mathcal{L}^\sharp(F) - 1) z_F^2,$$

where z_F is the component of z associated to the ray ρ_F and $\mathcal{L}^\sharp(F)$ is the number of minimal flats containing F.

Proof. As above, we let $\mathcal{M} = (E, \mathcal{L})$ be a rank 3 matroid and $\mathbf{\Sigma}_{\mathcal{M}}$ its associated Bergman fan. Pick an arbitrary $* \in \mathrm{Inn}(\boldsymbol{N}_E)$ and $z \in \overline{\mathrm{Cub}}(\mathbf{\Sigma}_{\mathcal{M}})$. Since $\mathcal{M}$ is a rank 3 matroid, $\mathbf{\Sigma}_{\mathcal{M}}$ is a balanced 2-fan. By Theorem 4.9 we have

$$\mathrm{Vol}_{\mathbf{\Sigma}_{\mathcal{M}},*}(z) = \deg_{\mathbf{\Sigma}_{\mathcal{M}}}\big(D(z)^2\big).$$

Now by Proposition 4.6 we know that the degree maps of the matroid Chow ring and the Bergman fan Chow ring agree, so

$$\deg_{\mathbf{\Sigma}_{\mathcal{M}}}\big(D(z)^2\big) = \deg_{\mathcal{M}}\big(D(z)^2\big).$$

This allows us to reframe the divisor in terms of flats,

$$D(z) = \sum_{F \in \mathcal{L}^*} z_F x_F,$$

which when squared gives us

$$D(z)^2 = \sum_{F,G \in \mathcal{L}^*} z_F z_G x_F x_G.$$

Since the degree map is linear this gives us

$$\begin{aligned} \deg_{\mathcal{M}}\left(D(z)^2\right) &= \sum_{F,G \in \mathcal{L}^*} z_F z_G \deg(x_F x_G) \\ &= 2 \sum_{F \neq G} z_F z_G \deg(x_F x_G) + \sum_{F \in \mathcal{L}^*} z_F^2 \deg(x_F^2), \end{aligned}$$

Now, in the Chow ring, the ideal $I_{\mathcal{M}}$ tells us that if $\{F, G\}$ is not a flat, then $x_F x_G = 0$. On the other hand, if $\{F, G\}$ is a flag, then the definition of the degree map tells us $\deg_{\mathcal{M}}(x_F x_G) = 1$. This gives us

$$2 \sum_{F \neq G} z_F z_G \deg(x_F x_G) + \sum_{F \in \mathcal{L}^*} z_F^2 \deg(x_F^2) = 2 \sum_{F \subsetneq G} z_F z_G + \sum_{F \in \mathcal{L}^*} z_F^2 \deg(x_F^2),$$

giving us part of our sum just in terms of z.

We now need to look at the term $\sum_{F \in \mathcal{L}^*} z_F^2 \deg_{\mathcal{M}}(x_F^2)$, which needs to be addressed depending on the rank of the flat and use the relations given by $J_{\mathcal{M}}$. First consider $G \in L^*$ such that $\mathrm{rk}(G) = 2$. We will take $a \in G$ to be any ground element in G. Likewise since G only has rank 2, there must be a choice of $b \in E$ such that $b \notin G$. The relations in $J_{\mathcal{M}}$ give us

$$x_G = -\sum_{\substack{a \in F \\ F \neq G}} x_F + \sum_{b \in F} x_F,$$

and so

$$x_G^2 = -\sum_{\substack{a \in F \\ F \neq G}} x_G x_F + \sum_{b \in F} x_G x_F.$$

Since $b \notin G$, whenever $b \in F$, F and G are incomparable, and so $\sum_{b \in F} x_G x_F = 0$. On the other hand, there is only one rank 1 flat that contains a, and a rank 2 flat is incomparable with all other rank 2 flats, so

$$\sum_{\substack{a \in F \\ F \neq G}} x_G x_F = -x_a x_G,$$

giving us $\deg(x_G^2) = \deg(-x_a x_G) = -1$ for any rank 2 flat G.

Now let $F \in \mathcal{L}$ be a rank 1 flat. Let a be the unique element in F and take $b \in E \setminus a$. Writing $x_a = x_F$, we have

$$x_a^2 = -\sum_{\substack{a \in G \\ G \neq F}} x_a x_G + \sum_{b \in G} x_a x_G.$$

Here, the set of flats $\{G \in \mathcal{L}^* \,|\, a \in G,\, G \neq \{a\}\}$ is just precisely the set of rank 2 flats that contain a, so any flat in it will form a flag with F, giving us

$$\deg\left(-\sum_{\substack{a\in G\\ G\neq F}} x_a x_G\right) = -\mathcal{L}^\sharp(F)$$

where, again, $\mathcal{L}^\sharp(F)$ just counts the minimal flats that cover F. Since different rank 1 flats are incomparable, when considering $\sum_{b\in G} x_a x_G$ we only have to pay attention to the rank 2 flats. However, recall that property ((F3)) of flats tells us b can appear in only one of the rank 2 flats that cover F, so all the rest are incomparable. This means $\deg(\sum_{b\in G} x_a x_G) = 1$, which combined means

$$\deg(x_F^2) = -\mathcal{L}^\sharp(F) + 1$$

when F is a rank 1 flat.

Putting these components together, we can finish our equality

$$\begin{aligned} 2\sum_{F\subsetneq G} z_F z_G + \sum_{F\in\mathcal{L}^*} z_F^2 \deg(x_F^2) &= 2\sum_{\substack{F\subsetneq G\\ F,G\in\mathcal{L}^*}} z_F z_G - \sum_{\substack{G\in\mathcal{L}^*\\ \mathrm{rk}(G)=2}} z_G^2 \deg(x_G^2) - \sum_{\substack{F\in\mathcal{L}^*\\ \mathrm{rk}(F)=1}} z_F^2 \deg(x_F^2) \\ &= 2\sum_{\substack{F\subsetneq G\\ F,G\in\mathcal{L}^*}} z_F z_G - \sum_{\substack{G\in\mathcal{L}^*\\ \mathrm{rk}(G)=2}} z_G^2 - \sum_{\substack{F\in\mathcal{L}^*\\ \mathrm{rk}(F)=1}} (\mathcal{L}^\sharp(F) - 1) z_F^2 \end{aligned}$$

as expected. $\square$

4.4.4 Mixed Volumes of Normal Complexes

Here we finally can justify introducing the mixed volume function earlier. We can't define the mixed volume of the full normal complexes directly, as the Minkowski sum of two normal complexes would certainly not be a normal complex itself in general. But like volume we could define it component wise.

A distinction from the original mixed volume function here is that we can't take the mixed volume of an arbitrary collection of normal complexes. We can only take the mixed volumes of normal complexes that have the same underlying fan.

Definition 4.31 (Mixed Volume of Normal Complexes)**.** Let $\boldsymbol{\Sigma} \subseteq \boldsymbol{N}$ be a simplicial d-fan, $* \in \mathrm{Inn}(\boldsymbol{N})$ be an inner product, and pseudocubical values $z_1, \ldots, z_d \in \overline{\mathrm{Cub}}(\boldsymbol{\Sigma}, *)$. The mixed volume of $C_{\boldsymbol{\Sigma},*}(z_1), \ldots, C_{\boldsymbol{\Sigma},*}(z_d)$, written $\mathrm{MVol}_{\boldsymbol{\Sigma},*}(z_1, \ldots, z_d)$ is given by

$$\mathrm{MVol}_{\boldsymbol{\Sigma},*}(z_1, \ldots, z_d) = \sum_{\sigma \in \boldsymbol{\Sigma}(d)} \mathrm{MVol}_\sigma \left(P_{\sigma,*}(z_1), \ldots, P_{\sigma,*}(z_d)\right),$$

where MVol_σ is the mixed volume of polytopes given by Definition 4.10 using the volume function Vol_σ for the cone σ as in Definition 4.30.

We are taking the mixed volume of members of the family of normal complexes associated to a particular pair $(\boldsymbol{\Sigma}, *)$. The basic reason we can get away with this is that taking the Minkowski sum of things in a cone will still be in the same cone. This means our cone-wise Minkowski additions will yield another truncation of the same cone.

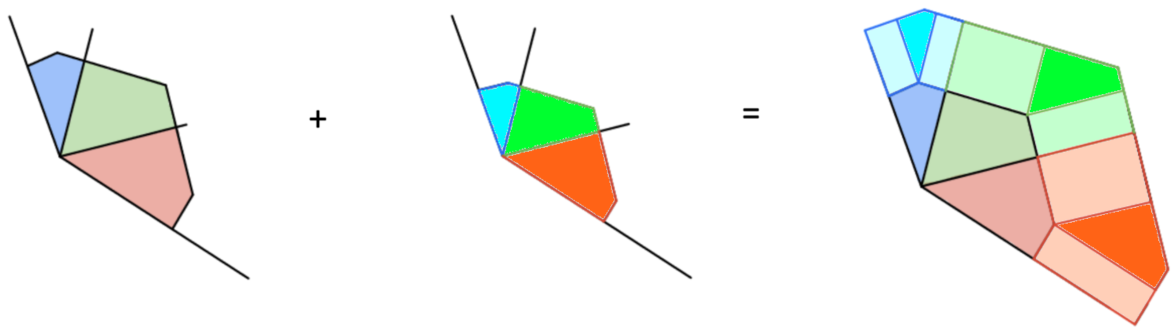


Figure 4.20: Component wise Minkowski sums of two normal complexes of the same fan; note each polytope in the sum is still in its correct cone.

However, it is far from obvious that that this newly defined function will have any of the same nice properties of the original mixed volumes. The development and proof of the following proposition was originally the work of Lauren Nowak in her master's thesis [Now22], which goes much deeper into the volume theory that we are utilising. The results were incorporated into our work with Nowak and Ross, [NOR23, Proposition 3.1], which gives us the following guarantee.

Proposition 4.11. *Let $\boldsymbol{\Sigma} \subset \boldsymbol{N}$ be a marked, simplicial d-fan, $* \in \mathrm{Inn}(\boldsymbol{N})$ an inner product, and pseudocubical values $z_1, \ldots, z_n \in \overline{\mathrm{Cub}}(\boldsymbol{\Sigma}, *)$. The function*

$$\mathrm{MVol}_{\boldsymbol{\Sigma},*} \,:\, \overline{\mathrm{Cub}}(\boldsymbol{\Sigma}, *)^d \to \mathbb{R}_{\geq 0}$$

as defined above has the following properties:

1. $\mathrm{MVol}_{\mathbf{\Sigma},*}(z, z, \ldots, z) = \mathrm{Vol}_{\mathbf{\Sigma},*}(z)$,

2. $\mathrm{MVol}_{\mathbf{\Sigma},*}$ *is symmetric in all arguments,*

3. $\mathrm{MVol}_{\mathbf{\Sigma},*}$ *is multilinear with respect to Minkowski addition in each maximal cone.*

Further, any function $\overline{\mathrm{Cub}}(\mathbf{\Sigma}, *)^d \to \mathbb{R}_{\geq 0}$ *satisfying properties 1–3 must be* $\mathrm{MVol}_{\sigma,*}$.

Our new mixed volume function then is well-defined and is uniquely characterized by essentially the same properties as the original mixed volume function. Further, [NOR23, Theorem 3.6] provides an extension to Theorem 4.9 that links mixed volumes to the evaluation of mixed degrees of divisors.

Theorem 4.12. *Let* $\mathbf{\Sigma} \subset \boldsymbol{N}$ *be a balanced d-fan. Choose an inner product* $* \in \mathrm{Inn}(\boldsymbol{N})$ *and pseudocubical values* $z_1, \ldots, z_d \in \overline{\mathrm{Cub}}(\mathbf{\Sigma}, *)$. *Then*

$$\mathrm{MVol}_{\mathbf{\Sigma},*}(z_1, \ldots, z_d) = \deg\big(D(z_1) \cdots D(z_d)\big).$$

This is a successful bridge from the realm of geometry back to algebra. We are now so close to having all the necessary components to prove our main result. Not only do we have the link between geometry and algebra, it uses the concept of mixed volumes, which are closely related to log-concave sequences. However, the Alexandrov–Fenchel inequality is, classically, very dependent on convexity, and normal complexes are, in general, decidedly non-convex.

4.4.5 Amazing AF Fans

If we take a fan at random, odds are high that we wouldn't see the mixed volumes of its normal complexes obey the Alexandrov–Fenchel inequality. Unlike the convex setting, it is just generally not true. However, one can find examples of fans such that the mixed volumes of their normal complex do obey the inequality. This is in fact a property of the fan itself.

Definition 4.32 (Alexandrov–Fenchel Property)**.** Let $\mathbf{\Sigma} \subseteq \boldsymbol{N}$ be a marked simplicial d-fan and $* \in \mathrm{Inn}(\boldsymbol{N})$ an inner product. We say that $(\mathbf{\Sigma}, *)$ is *Alexandrov–Fenchel*, or just *AF*, if $\mathrm{Cub}(\mathbf{\Sigma}, *) \neq \emptyset$ and

$$\mathrm{MVol}_{\mathbf{\Sigma},*}(z_1, z_2, z_3, \dots, z_d)^2 \geq \mathrm{MVol}_{\mathbf{\Sigma},*}(z_1, z_1, z_3, \dots, z_d)\,\mathrm{MVol}_{\mathbf{\Sigma},*}(z_2, z_2, z_3 \dots, z_d)$$

for all $z_1, z_2, z_3, \dots, z_d \in \mathrm{Cub}(\mathbf{\Sigma}, *)$.

If a fan is AF, then taking the mixed volumes of any normal complexes built from it will give rise to a log-concave sequence. Our goal now is then to show that Bergman fans of matroids are AF, but we will need some way to prove that in general. It is not feasible to check the Bergman fan of every matroid one by one.

Luckily, there is a way to more generally determine if a fan is AF. Inspired by a strategy from a modern proof of the Alexandrov–Fenchel inequality, our work with Nowak and Ross showed a theorem, [NOR23, Theorem 5.1], that gives us criteria to check. We reproduce the theorem here.

Theorem 4.13 (Nowak-O-Ross 2023)**.** *Let* $\mathbf{\Sigma} \subseteq \boldsymbol{N}$ *be a marked d-fan, and* $* \in \mathrm{Inn}(\boldsymbol{N})$ *an inner product such that* $\mathrm{Cub}(\mathbf{\Sigma}, *) \neq \emptyset$*. Then* $(\mathbf{\Sigma}, *)$ *is AF if*

i. for every cone $\sigma \in \mathbf{\Sigma}(k)$*, with* $k \leq d-2$*,*

$$\mathbf{\Sigma}^\sigma \setminus \{0\}$$

is connected and,

ii. for every cone $\sigma \in \mathbf{\Sigma}(d-2)$*, the volume polynomial of the star fan* Σ^σ *is a quadratic form whose associated matrix has exactly one positive eigenvalue.*

We can provide a little insight into these two criteria. In the first, connectedness is the classic topological definition of connected. But, for those without a topological background, practically we can think of this as being able to draw a path between any two points in the star fans. In this case, however they must remain connected when the origin is removed. The effect of this criterion is that it tells us our fans can't have "pinch" points. If the cones of the fan share a non-0-dimensional face that is not a facet, then when taking stars within the fan we may get the case that the cones in the star fan meet only at the origin. Once we then remove the origin, this will disconnect the star fan, violating the criterion.

For the second condition, let's quickly review quadratic forms. A quadratic form is just a homogeneous polynomial of degree 2. If $p(x_1, \ldots, x_n)$ is a quadratic form on n variables, then there exists an $n \times n$ matrix A such that

$$p(x_1, \ldots, x_n) = \boldsymbol{x}^T A \boldsymbol{x}$$

where $\boldsymbol{x}^T = \begin{bmatrix} x_1 \cdots x_n \end{bmatrix}$. Since we are looking at stars of $(d-2)$-dimensional cones, the corresponding star fan will be a 2-fan. As we have seen, its volume polynomial $\deg_{\boldsymbol{\Sigma}^\sigma}(D(z))$ will be a homogeneous polynomial of degree 2 in z. This second criterion is then a question of finding the signs of the eigenvalues of the matrix associated to each of these.

The inspiration for this proof comes from the paper, "One more proof of the Alexandrov–Fenchel inequality" [Cor+19] which offers a proof of the Alexandrov–Fenchel inequalities in the classic convex setting. This proof uses a dimension-reducing inductive argument on the volume of faces of polytopes. This is why it was necessary to develop a notion of faces on normal complexes. The first criterion of Theorem 4.13 ensures that the faces of normal complexes meet a connectedness assumption that is needed in the inductive argument. The second criterion is the base case, which in the convex setting is just always true but here must be shown per fan.

We now finally have laid out every piece of existing work we will need to prove our main result. All that's left is to put them together.

Chapter 5

A Proof of the Heron–Rota–Welsh Conjecture

We have now all of our dominoes lined up, and we are just about ready to start knocking them down. The bulk of this section will go to proving that Bergman fans are AF. Once we have that, the Heron–Rota–Welsh conjecture is easily in sight, just requiring us to address a few lingering points. Then, working our way back from the realm of geometry we follow a quick series of implications back to matroids and the the characteristic polynomial.

5.1 Stars and the Products of Minors

Before we can prove that Bergman fans are AF, we would like to establish a relationship between the stars of fans of matroids and the fans of that matroid's minors. We are going to show that there is an isomorphism between star fans of a matroid and the product of fans of the minors. This will let us use knowledge of matroids in proving facts about stars, rather than treating them as purely geometric objects. To get there we need the first ingredient of a fan isomorphism, a linear map.

Lemma 5.1. *Let $\boldsymbol{\Sigma}_{\mathcal{M}} \subset \boldsymbol{N}_E$ be the Bergman fan of a matroid,*

$$\mathscr{F} = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k\} \subseteq \mathcal{L}^*$$

be a flag of flats, and define $F_0 = \emptyset$ and $F_{k+1} = E$. The map

$$\begin{aligned} \varphi :\ \boldsymbol{N}_E / \operatorname{span}(\sigma_{\mathscr{F}}) &\to \bigoplus_{i=1}^{k+1} \boldsymbol{N}_{F_i \setminus F_{i-1}} \\ [e_I] &\mapsto \bigoplus_{i=1}^{k+1} e_{I \cap (F_i \setminus F_{i-1})} \end{aligned}$$

is a linear isomorphism.

Proof. Consider the very similar map

$$\begin{aligned} \varphi' :\ \boldsymbol{N}_E &\to \bigoplus_{i=1}^{k+1} \boldsymbol{N}_{F_i \setminus F_{i-1}} \\ e_I &\mapsto \bigoplus_{i=1}^{k+1} e_{I \cap (F_i \setminus F_{i-1})}. \end{aligned}$$

We will show φ' is a surjective linear map and that its kernel is exactly $\operatorname{span}(\sigma_{\mathscr{F}})$. With that, we can leverage the first isomorphism theorem, which will give us that φ is an isomorphism.

Our map is linear by its definition, so let's first show that it is surjective. Consider some vector $w \in \bigoplus_{i=1}^{k+1} \boldsymbol{N}_{F_i \setminus F_{i-1}}$ of the form

$$w = \bigoplus_{i=1}^{k+1} \left(\sum_{j \in (F_i \setminus F_{i-1})} \lambda_j e_j \right).$$

Then w is a linear combination of the elements in some $F_i \setminus F_{i-1}$. This of course is the image of the vector

$$v = \sum_{i=1}^{k+1} \left(\sum_{j \in (F_i \setminus F_{i-1})} \lambda_j e_j \right) \in \boldsymbol{N}_E,$$

and so we have surjectivity.

Next we want to show that any element in $\operatorname{span}(\sigma_{\mathscr{F}})$ is in the kernel. From construction, anything in $\operatorname{span}(e_{F_i \setminus F_{i-1}})$ is sent to 0 in $\boldsymbol{N}_{F_i \setminus F_{i-1}}$, in particular $e_{F_i \setminus F_{i-1}}$ itself. Given some $v \in \operatorname{span}(\sigma_{\mathscr{F}})$ of the form

$$v = \lambda_1 e_{F_1} + \lambda_2 e_{F_2} + \cdots + \lambda_k e_{F_k},$$

we have

$$
\begin{aligned}
\varphi'(v) &= \lambda_1\varphi'(e_{F_1}) + \lambda_2\varphi'(e_{F_2}) + \cdots + \lambda_k\varphi'(e_{F_k}) \\
&= \lambda_1\left(e_{F_1}\right) + \lambda_2\left(e_{F_1} \oplus e_{F_2\setminus F_1}\right) + \cdots + \lambda_k\left(e_{F_1} \oplus \lambda_k e_{F_2\setminus F_1} \oplus \cdots \oplus \lambda_k e_{F_k\setminus F_{k-1}}\right) \\
&= (\lambda_1 + \lambda_2 + \cdots + \lambda_k)e_{F_1} \oplus (\lambda_2 + \lambda_3 + \cdots + \lambda_k)e_{F_2\setminus F_1} \oplus \cdots \oplus \lambda_k e_{F_k\setminus F_{k-1}} \\
&= (\lambda_1 + \lambda_2 + \cdots + \lambda_k)0 \oplus (\lambda_2 + \lambda_3 + \cdots + \lambda_k)0 \oplus \cdots \oplus \lambda_k 0 \\
&= 0
\end{aligned}
$$

showing that $\operatorname{span}(\sigma_{\mathscr{F}}) \subseteq \ker(\varphi')$.

Now we just need to count some dimensions. By definition, $\dim(\boldsymbol{N}_E) = |E| - 1$. Likewise, each $\boldsymbol{N}_{F_i\setminus F_{i-1}}$ has dimension $|F_i \setminus F_{i-1}| - 1$. This gives us

$$
\begin{aligned}
\dim\left(\bigoplus_{i=1}^{k+1} \boldsymbol{N}_{F_i\setminus F_{i-1}}\right) &= \sum_{i=1}^{k+1} |F_i \setminus F_{i-1}| - 1 \\
&= \left(\sum_{i=1}^{k+1} |F_i \setminus F_{i-1}|\right) - (k+1) \\
&= |E| - (k+1),
\end{aligned}
$$

as $\{F_i \setminus F_{i-1}\}_{i=1}^{k+1}$ is just a partition of E. Since φ' is surjective, this means

$$
\dim\big(\operatorname{Im}(\varphi')\big) = |E| - (k+1)
$$

as well. We have that $\mathscr{F}$ is a flag of k flats, so $\dim(\sigma_{\mathscr{F}}) = k$, which thanks to the inclusion we found above means $\dim\big(\ker(\varphi')\big) \geq k$. But by the rank-nullity theorem

$$\begin{aligned}\dim\big(\ker(\varphi')\big) &= \dim(\boldsymbol{N}_E) - \dim\big(\operatorname{Im}(\varphi')\big)\\ &= |E| - 1 - (|E| - (k+1))\\ &= k\end{aligned}$$

giving us that $\ker(\varphi')$ is exactly $\operatorname{span}(\sigma_{\mathscr{F}})$. With that we have

$$\boldsymbol{N}_E/\operatorname{span}(\sigma_{\mathscr{F}}) \cong \bigoplus_{i=1}^{k+1} \boldsymbol{N}_{F_i \setminus F_{i-1}}$$

by the first isomorphism theorem and that our map φ is the given isomorphism between these spaces. □

With an isomorphism between these two spaces in hand, we can show that the stars of a matroid's Bergman fan have a local structure equivalent to the product of the Bergman fans of its minors.

Lemma 5.2. *Let $\boldsymbol{\Sigma}_{\mathcal{M}} \subseteq \boldsymbol{N}_E$ be the Bergman fan of a matroid, and $\sigma_{\mathscr{F}} \in \boldsymbol{\Sigma}(k)$ be a cone with ray generators corresponding to the flag $\mathscr{F} = \{F_1, \ldots, F_k\}$. Then the star fan associated to $\sigma_{\mathscr{F}}$ is isomorphic to the product fan of minors given by the* *intervals of $\mathscr{F}$. That is to say*

$$\operatorname{star}(\sigma_{\mathscr{F}}, \mathbf{\Sigma}_{\mathcal{M}}) \cong \prod_{i=1}^{k+1} \mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}}.$$

Proof. Let $\mathbf{\Sigma}_{\mathcal{M}} \subset \boldsymbol{N}_E$ be the Bergman fan of a matroid,

$$\mathscr{F} = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k\} \subseteq \mathcal{L}^*$$

be a flag of flats, and again define $F_0 = \emptyset$ and $F_{k+1} = E$. Take φ to be the isomorphism given in Lemma 5.1. As per Definition 4.21, a fan isomorphism still requires an additional component, a bijective map

$$\varphi^* : \operatorname{star}(\sigma_{\mathscr{F}}, \mathbf{\Sigma}_{\mathcal{M}}) \to \prod_{i=1}^{k+1} \mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}}$$

between cones of the fan. We will show that we can use φ to induce this map between cones. Without loss of generality let us assume $\sigma_{\mathscr{F}}$ is not a maximal cone, as 0-dimensional fans are trivially isomorphic.

Consider a cone of the star fan $\overline{\tau} \in \operatorname{star}(\sigma_{\mathscr{F}}, \mathbf{\Sigma}_{\mathcal{M}})$. From our definitions, we know that $\overline{\tau}$ corresponds to some cone in the neighborhood of $\sigma_{\mathscr{F}}$, $\tau \in \operatorname{nbd}(\sigma_{\mathscr{F}}, \mathbf{\Sigma}_{\mathcal{M}})$. Further then, we know that τ is the face of some cone π such that $\pi \in \mathbf{\Sigma}_{\mathcal{M}}(r)$ and $\sigma_{\mathscr{F}} \preceq \tau$. Since π is a maximal cone, it must be associated to some complete flag $\mathscr{F}'$, and since $\sigma_{\mathscr{F}} \preceq \pi$ it must be that $\mathscr{F} \subseteq \mathscr{F}'$.

Let $\mathscr{F}' = \{F_1, \dots, F_k, G_1, \dots, G_\ell\}$, where $G_i \in \mathcal{L}^*$ and $\{F_1, \dots, F_k\} \cap \{G_1, \dots, G_\ell\} = \emptyset$. As $\tau \subseteq \pi$, we know that

$$\tau = \operatorname{cone}(e_{F_{i_1}}, \dots, e_{F_{i_m}}, e_{G_{j_1}}, \dots, e_{G_{j_n}}) \in \boldsymbol{N}_E$$

giving us

$$\overline{\tau} = \operatorname{cone}([e_{G_{j_1}}], \dots, [e_{G_{j_n}}]) \in \operatorname{star}(\sigma_{\mathscr{F}}, \boldsymbol{\Sigma}_{\mathcal{M}}).$$

Because flats are totally ordered, for each G_j there must be some $1 \le i \le k+1$ such that

$$F_{i-1} \subsetneq G_j \subsetneq F_i.$$

Then for each G_j we have that

$$\begin{aligned}
\varphi([e_{G_j}]) &= 0 \oplus \cdots \oplus e_{G_j \cap (F_i \setminus F_{i-1})} \oplus \cdots \oplus 0 \\
&= e_{G_j \cap (F_i \setminus F_{i-1})} \\
&= e_{G_j \setminus F_{i-1}},
\end{aligned}$$

where Proposition 2.4 tells us that $G_j \setminus F_{i-1}$ is a flat of the matroid minor $\mathcal{M}_{[F_{i-1}, F_i]}$ and so $\varphi([G_j])$ is a ray in product fan. If we have the case

$$F_{i-1} \subsetneq G_j \subsetneq G_j' \subsetneq F_i$$

then $G_j \subsetneq G'_j$ implies $G_j \setminus F_{i-1} \subsetneq G'_j \setminus F_{i-1}$ and so

$$\mathrm{cone}\big(\varphi([e_{G_j \setminus F_{i-1}}]), \varphi([e_{G'_j \setminus F_{i-1}}])\big) \in \mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}}.$$

Otherwise, the cone between rays of different fans in the product are in the product fan by definition. This means that we have that

$$\mathrm{cone}\big(\varphi([e_{G_{j_1}}]), \ldots, \varphi([e_{G_{j_n}}])\big)$$

is an element of the product fan.

Consider the map

$$\varphi^* = \mathrm{star}(\sigma_{\mathscr{F}}, \mathbf{\Sigma}_{\mathcal{M}}) \to \prod_{i=1}^{k+1} \mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}}$$
$$\varphi^*\big(\mathrm{cone}\big([e_{G_{j_1}}], \ldots, [e_{G_{j_n}}]\big)\big) = \mathrm{cone}\big(\varphi([e_{G_{j_1}}]), \ldots, \varphi([e_{G_{j_n}}])\big)$$

as our bijection between cones, given that we can write any cone in the star

$$\overline{\tau} = \mathrm{cone}\big([e_{G_{j_1}}], \ldots, [e_{G_{j_n}}]\big)$$

as above. Also from above, we see that φ^* is injective, since φ maps rays of the star uniquely to rays of the product. To confirm it is surjective, consider an arbitrary element

$$(\tau_1, \ldots, \tau_{k+1}) \in \prod_{i=1}^{k+1} \mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}}.$$

Each τ_i is a cone in $\mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}}$ corresponding to a flag $\{G_{i,1} \subsetneq \cdots \subsetneq G_{i,m}\}$ of $\mathcal{M}_{[F_{i-1},F_i]}$. We can again use Proposition 2.4 which tells us that each $G_{i,j} \cup F_{i-1}$ is a flat of $\mathcal{M}$. Further, since each $G_{i,j} \cup F_{i-1}$ can be totally ordered with respect to both themselves and elements of $\mathscr{F}$ they correspond to a flag. This flag, $\mathscr{F}'$, contains $\mathscr{F}$, and so the cone $\tau_{\mathscr{F}'}$ is in the neighborhood of $\sigma_{\mathscr{F}}$. From the work above we can then conclude that

$$\varphi^*(\overline{\tau}_{\mathscr{F}'}) = (\tau_1, \ldots, \tau_m)$$

giving us surjectivity. This gives us that φ^* is a bijection of cones.

The condition that for any cone $\overline{\tau} \in \mathrm{star}(\sigma_{\mathscr{F}}, \mathbf{\Sigma}_{\mathcal{M}})$ the restriction $\varphi|_{\overline{\tau}}$ is a bijection between $\overline{\tau}$ and $\varphi^*(\overline{\tau})$ follows immediately from the definition of φ^* and that φ is already a bijection. We may conclude that our fans are isomorphic. □

5.2 Bergman Fans of Matroids are AF

With everything in the last section at our disposal, we may finally prove a key theorem.

Theorem 5.3. *For any matroid $\mathcal{M}$, its Bergman fan $\mathbf{\Sigma}_{\mathcal{M}}$ is AF.*

We need only show that the conditions of Theorem 4.13 are met specifically for

Bergman fans of matroids. We will tackle this one condition at a time, starting with the connectedness condition.

Lemma 5.4 (Connectedness)**.** *Let $\mathbf{\Sigma}_{\mathcal{M}}$ be the Bergman fan of a matroid of rank $r+1$. For every cone $\sigma \in \mathbf{\Sigma}_{\mathcal{M}}(k)$, with $k \leq r-2$,*

$$\operatorname{star}(\tau, \mathbf{\Sigma}_{\mathcal{M}}) \setminus \{0\}$$

is connected.

Proof. To prove this, start by picking and arbitrary$\sigma \in \mathbf{\Sigma}_{\mathcal{M}}(k)$, where $k \leq r-2$. It is sufficient to show that between any two rays $\rho_1, \rho_k \in \operatorname{star}(\tau, \mathbf{\Sigma}_{\mathcal{M}})$ there exists a series of faces

$$\rho_1 \prec \tau_1 \succ \rho_2 \prec \cdots \succ \rho_{k-1} \prec \tau_{k-1} \succ \rho_k$$

where each ρ_i is a ray and each τ_i is a 2-dimensional face. Since any point in a fan is connected to a ray, specifically one of the generating rays of the cone the point lives in, a path like this between arbitrary rays is enough to show our fan is connected without the origin.

For some notational convenience, we will write

$$\rho_1 \sim \tau_1 \sim \rho_2 \sim \cdots \sim \rho_{k-1} \sim \tau_{k-1} \sim \rho_k$$

and let the reader interpret the correct face inclusions. Additionally, in keeping with our general convention, we will write e_F for the ray generated by $\sum_{i\in F} e_{\{i\}}$ and τ_{F_1,F_2} for the cone generated by rays e_{F_1} and e_{F_2}. Finally, we assume $\mathcal{M}$ is at least a rank 3 matroid, and so $\mathbf{\Sigma}_{\mathcal{M}}$ has maximal cones of dimension at least 2, as otherwise the lemma is vacuously true.

Now, we will consider this in two steps. First we will look at $\mathbf{\Sigma}_{\mathcal{M}} \setminus \{0\}$ itself. Let $e_F, e_{F'} \in \mathbf{\Sigma}_{\mathcal{M}}(1)$ be two arbitrary rays of our fan. If there exists some ground element $i \in F \cap F'$, then we have the sequence

$$e_F \sim \tau_{\{F,\{i\}\}} \sim e_{\{i\}} \sim \tau_{\{\{i\},F'\}} \sim e_{F'}.$$

Let us now consider that $F \cap F' = \emptyset$. Let $a \in F$ be an element of F and $b \in F'$. To start, we have the sequence

$$e_F \sim \tau_{\{F,\{a\}\}} \sim e_{\{a\}}.$$

Now, recall the properties of the flats of a matroid, specifically property (F3). This tells us that the flats of rank 2 partition $E \setminus \{a\}$ and so there must be a rank 2 flat, $\widehat{F}$, such that $\{a, b\} \subseteq \widehat{F}$. Then we have

$$e_{\{a\}} \sim \tau_{\left\{\{a\},\widehat{F}\right\}} \sim e_{\widehat{F}} \sim \tau_{\left\{\widehat{F},b\right\}} \sim e_{\{b\}} \sim \tau_{\{\{b\},F'\}} \sim e_{F'},$$

showing a sequence from e_F to e'_F, as desired. With that we have shown there is always a path along 2 dimensional faces between any two rays of the Bergman fan of a matroid, and so is connected even without the origin.

Next, we will turn to the stars of our matroid. Let $k \leq r-2$ and $\sigma_{\mathscr{F}} \in \mathbf{\Sigma}_{\mathcal{M}}(K)$ be a k-dimensional cone. We have that $\mathrm{star}(\sigma_{\mathscr{F}}, \mathbf{\Sigma}_{\mathcal{M}})$ is a fan with maximal cones of dimension $r-k$, which specifically means they are at least 2-dimensional. From Lemma 5.2, we know that, given $\mathscr{F} = \{F_1 \subsetneq \cdots \subsetneq F_k\}$, we have that $\mathrm{star}(\sigma_{\mathscr{F}}, \mathbf{\Sigma}_{\mathcal{M}})$ is in bijection with the product fan

$$\prod_{i=1}^{k+1} \mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}}.$$

So, it is sufficient to show this product fan is connected after removing the origin. Recall that cones of the product fan are of the form

$$(\sigma_1, \sigma_2, \ldots, \sigma_k) \in \mathbf{\Sigma}_{\mathcal{M}_{[\emptyset,F_1]}} \times \mathbf{\Sigma}_{\mathcal{M}_{[F_1,F_2]}} \times \cdots \times \mathbf{\Sigma}_{\mathcal{M}_{[F_{k-1},F_k]}} \times \mathbf{\Sigma}_{\mathcal{M}_{[F_k,E]}},$$

and that the dimension of the cone $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ is the sum of the dimensions of each cone σ_i. Rays of the product fan are then of the form $(0, 0, \ldots, 0, \rho, 0, \ldots, 0)$ where ρ is a ray of the corresponding fan in the product.

Now, to show our product fan is connected, we will consider two cases. We will omit irrelevant zeros from our notation going forward for ease of reading, but we can add as many zeros in other positions as necessary without changing any part of the proof. In the first case, our two rays come from different fans in the product. If

$(\rho_1, 0)$ and $(0, \rho_2)$ are rays, then there exists the path

$$(\rho_1, 0) \sim (\rho_1, \rho_2) \sim (0, \rho_2)$$

connecting them. The more nuanced case is if we have two rays from the same fan.

Consider two rays $\rho_1, \rho_2 \in \mathbf{\Sigma}(1)$. If the minor that generates $\mathbf{\Sigma}$ is at least rank 3, then our work above shows there must exist a path between them using only the cones of $\mathbf{\Sigma}$. Where this breaks down, however, is if the minor has rank 2; i.e., the Bergman fan has only 1-dimensional cones. We can't, after removing the origin, get between rays solely within this fan. Recall though that our star fan must be pure of at least dimension 2. This means if this $\mathbf{\Sigma}$ has only 1-dimensional cones, then there is at least one other nonzero fan, which we will call $\mathbf{\Sigma'}$, in the product that has at least one ray. Let $\eta \in \mathbf{\Sigma'}$ be said ray. Then we have the path

$$(\rho_1, 0) \sim (\rho_1, \eta) \sim (0, \eta) \sim (\rho_1, \eta) \sim (\rho_1, 0)$$

connecting the two rays of $\mathbf{\Sigma}$.

With this we have shown that any possible star of a Bergman fan of a matroid is connected without the origin. $\square$

Next we may turn to the second criterion, that the quadratic form that determines the volume of the 2-dimensional faces of the normal complex corresponds to a matrix with exactly one positive eigenvalue. For this, we will be using a classic result of linear algebra known as Sylvester's Law of Inertia [Syl52].

Proposition 5.5 (Sylvester's Law of Inertia). *Two symmetric square matrices, A and B, of the same size have the same number of positive, negative, and zero eigenvalues if and only if*

$$B = SAS^T$$

for some non-singular matrix S.

Our general strategy will be to first find the volume polynomials of star fans and then show that we can find an invertible change of basis that lets us express it as a sum of squares. A quadratic form that consists only of squares corresponds to a diagonal matrix, so we can immediately read off the signs of the eigenvalues from the signs of the squared terms. Sylvester's law of inertia then promises us that the signs of the eigenvalues are not particular to our choice of basis, rather invariant under any choice.

Lemma 5.6 (Volume Quadratic Form Has One Positive Eigenvalue). *Let $\mathcal{M}$ be a matroid of rank $r+1$ and $\mathbf{\Sigma}_{\mathcal{M}}$ be the Bergman fan associated to the matroid, with $* \in \mathrm{Inn}(\boldsymbol{N}_E)$ an inner product. For every $\sigma \in \mathbf{\Sigma}_{\mathcal{M}}(r-2)$, the quadratic form associated to the volume polynomial of $\mathrm{star}(\sigma, \mathbf{\Sigma}_{\mathcal{M}})$ has exactly one positive eigenvalue.*

Proof. Consider some $\tau_{\mathscr{F}} \in \mathbf{\Sigma}_{\mathcal{M}}(r-2)$, where $\mathscr{F} = \{F_1 \subsetneq \cdots \subsetneq F_k\}$. From the dimension reversing relationship of cones and their stars, we know that $\mathrm{star}(\tau_{\mathscr{F}}, \mathbf{\Sigma}_{\mathcal{M}})$

is a 2-fan. Once again using Lemma 5.2, we have that

$$\operatorname{star}(\tau_{\mathscr{F}}, \mathbf{\Sigma}_{\mathcal{M}}) \cong \prod_{i=0}^{r-2} \mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}}.$$

Recall that F_{i-1} and F_i are adjacent to each other in the lattice of flats, $\mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}}$ is just $\{0\}$ and can be ignored as an element of the product. Since flags of $\mathbf{\Sigma}_{\mathcal{M}}$ have r elements and $|\mathscr{F}| = r-2$, there are only two possible structures for our fan, either

$$\mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}} \times \mathbf{\Sigma}_{\mathcal{M}_{[F_{j-1},F_j]}} \qquad \text{or} \qquad \mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}}.$$

Where in the first case the missing ranks of the flag are between two different pairs of flats and in the second the missing ranks are adjacent. We will split our proof into cases based on the two possible structures of our star fan.

In the first case, let us assume that $\operatorname{star}(\tau_{\mathscr{F}}, \mathbf{\Sigma}_{\mathcal{M}}) = \mathbf{\Sigma} \times \mathbf{\Sigma}'$ where $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$ are both pure of dimension 1. This means that the 2-dimensional cones in the product fan are exactly

$$\{\operatorname{cone}(\rho, \rho') \mid \rho \in \mathbf{\Sigma}, \rho' \in \mathbf{\Sigma}\}.$$

The volume polynomial is given by

$$\deg_{\mathbf{\Sigma}\times\mathbf{\Sigma}'}\left(D(z)^2\right) = \sum_{\rho_1,\rho_2 \in \mathbf{\Sigma}\times\mathbf{\Sigma}'(1)} z_{\rho_1} z_{\rho_2} \deg_{\mathbf{\Sigma}\times\mathbf{\Sigma}'}(x_{\rho_1} x_{\rho_2}).$$

Since each fan only has rank one flats there are no cones in the fan between rays of an individual fan. Conversely, in a product fan there is a cone between every pair of rays of different fans in the product. In our fairly simple case then, we see that this gives us

$$\deg_{\mathbf{\Sigma}\times\mathbf{\Sigma}'}(x_{\rho_1}x_{\rho_2}) = \begin{cases} 0 & \rho_1,\rho_2 \in \mathbf{\Sigma} \text{ or } \rho_1,\rho_2 \in \mathbf{\Sigma}' \\ 1 & \rho_1 \in \mathbf{\Sigma}, \rho_2 \in \mathbf{\Sigma}' \text{ or } \rho_2 \in \mathbf{\Sigma}, \rho_1 \in \mathbf{\Sigma}'. \end{cases}$$

We may use this to get the volume polynomial as

$$\sum_{\rho_1,\rho_2\in\mathbf{\Sigma}\times\mathbf{\Sigma}'(1)} z_{\rho_1}z_{\rho_2}\deg_{\mathbf{\Sigma}\times\mathbf{\Sigma}'}(x_{\rho_1}x_{\rho_2}) = 2\sum_{\rho\in\mathbf{\Sigma},\rho'\in\mathbf{\Sigma}'} z_\rho z_{\rho'},$$

accounting for the fact that $z_\rho z_{\rho'} = z_{\rho'}z_\rho$.

As per our strategy, we want to write this polynomial as a sum of squared terms. Some prodding will reveal that

$$2\sum_{\rho\in\mathbf{\Sigma},\rho'\in\mathbf{\Sigma}'} z_\rho z_{\rho'} = \frac{1}{2}\left(\sum_{\rho\in\mathbf{\Sigma}} z_\rho + \sum_{\rho'\in\mathbf{\Sigma}} z_\rho'\right)^2 - \frac{1}{2}\left(\sum_{\rho\in\mathbf{\Sigma}} z_\rho - \sum_{\rho'\in\mathbf{\Sigma}} z_\rho'\right)^2$$

is such a way to write the volume. The key idea is that in the expression

$$-\left(\sum_{\rho\in\mathbf{\Sigma}} z_\rho - \sum_{\rho'\in\mathbf{\Sigma}} z_\rho'\right)^2$$

only terms of the form $z_\rho z_{\rho'}$ are positive, with the rest going to cancel out the unwanted terms in the first expression. After adjusting for some slight over counting we see the polynomials agree. Since this is the sum of squared terms, we may read the signs of the eigenvalues of the associated matrix from the signs of the sums themselves, showing we have exactly one positive eigenvalue. By Sylvester's law of inertia then any matrix congruent to the one associated to the quadric form will have one positive eigenvalue, as desired.

With the first case done, we have to turn to the second, when our face is the normal complex associated to a rank 3 minor. Let $\mathbf{\Sigma} = \mathrm{star}(\tau, \mathbf{\Sigma}_{\mathcal{M}})$ and recall that as a Bergman fan of a rank 3 matroid, Lemma 4.10 gives us the volume polynomial as

$$\deg_{\mathbf{\Sigma}}(D(z)^2) = 2\sum_{\substack{F \subsetneq G \\ F, G \in \mathcal{L}^*}} z_F z_G - \sum_{\substack{G \in \mathcal{L}^* \\ \mathrm{rk}(G)=2}} z_G^2 - \sum_{\substack{F \in \mathcal{L}^* \\ \mathrm{rk}(F)=1}} (\mathcal{L}^\sharp(F) - 1) z_F^2,$$

where $\mathcal{L}^\sharp(F)$ is the number of minimal flats containing F. We propose that we can write this instead as

$$\deg_{\mathbf{\Sigma}}(z) = \left(\sum_{\mathrm{rk}(F)=1} z_F \right)^2 - \sum_{\mathrm{rk}(G)=2} \left(z_G - \sum_{F \subsetneq G} z_F \right)^2,$$

which gives us the volume in terms of sums of squares. We again see this would have one positive eigenvalue.

To see that these are equal, we will break this new expression down. First, we note that

$$\left(\sum_{\mathrm{rk}(F)=1} z_F\right)^2 = \sum_{\mathrm{rk}(F)=1} z_F^2 + 2\sum_{\substack{\mathrm{rk}(F_1)=1,\\ \mathrm{rk}(F_2)=1}} z_{F_1} z_{F_2}. \tag{5.1}$$

Then, let's look at the internal part of the second expression to see for a fixed rank 2 flat $\widehat{G}$ we have

$$\left(z_{\widehat{G}} - \sum_{F\subsetneq\widehat{G}} z_F\right)^2 = z_{\widehat{G}}^2 - 2\sum_{F\subsetneq\widehat{G}} z_F z_{\widehat{G}} + 2\sum_{F_1,F_2\subsetneq\widehat{G}} z_{F_1} z_{F_2} + \sum_{F\subsetneq\widehat{G}} z_F^2. \tag{5.2}$$

What's important to note here is that if $F_1, F_2 \subseteq G$, then there cannot be another rank 2 flat G' that contains both. This is a consequence of property ((F3)) of flats. But when we let the outer sum range over all possible rank 2 flats, we will get $\mathcal{L}^\sharp(F)$ copies of each z_F^2, and so we have

$$\sum_{\mathrm{rk}(G)=2}\left(z_G - \sum_{F\subsetneq G} z_F\right)^2 = \sum_{\mathrm{rk}(G)=2} z_G^2 - \sum_{F\subsetneq G} 2z_F z_G + 2\sum_{\substack{\mathrm{rk}(F_1)=1,\\ \mathrm{rk}(F_2)=1}} z_{F_1} z_{F_2} + \sum_{\mathrm{rk}(F)=1} \mathcal{L}^\sharp(F) z_F^2. \tag{5.3}$$

If we subtract our result in (1.3) from the one in (1.1), we are left with

$$2\sum_{\substack{F\subsetneq G\\ F,G\in\mathcal{L}^*}} z_F z_G - \sum_{\substack{G\in\mathcal{L}^*\\ \mathrm{rk}(G)=2}} z_G^2 - \sum_{\substack{F\in\mathcal{L}^*\\ \mathrm{rk}(F)=1}} (\mathcal{L}^\sharp(F) - 1) z_F^2,$$

recovering our original volume polynomial. We may again appeal to Sylvester's law of inertia to conclude that the matrix associated to the volume polynomial will have one positive eigenvalue. Having covered both cases we conclude that the 2-stars of a Bergman fans of matroids have quadratic forms with one positive eigenvalue. □

With Lemma 5.4 and Lemma 5.6, we see that Bergman fans of matroids will always satisfy the criteria of Theorem 4.13.

5.3 Tying Up Loose Ends

We now know that Bergman fans of matroids are AF, but this will only help us if we can find a pseudocubical value associated to the divisors α and β. An immediate problem to address is that it could be that no cubical values exist at all. Luckily, Proposition 7.4 of [NR23] gives us the following guarantee.

Proposition 5.7. *Let $\mathcal{M} = (E, \mathcal{L})$ be a matroid with ground set $E = \{e_0, e_1, \ldots, e_n\}$. Take $\boldsymbol{\Sigma}_{\mathcal{M}}$ to be the Bergman fan of $\mathcal{M}$ in $\boldsymbol{N}_E$ such that we associate $(e_1, e_2, \ldots, e_n)$ to the standard basis vectors and $*$ to be the standard inner product on this basis. Then*

$$\mathrm{Cub}(\boldsymbol{\Sigma}_{\mathcal{M}}, *) \neq \emptyset.$$

Going forward in this section, we will take for granted that a matroid $\mathcal{M} = (E, \mathcal{L})$ has ground set $E = \{e_0, e_1, \ldots, e_n\}$, and that $\boldsymbol{\Sigma}_{\mathcal{M}} \subseteq \boldsymbol{N}_E$ associates $\{e_1, e_2, \ldots, e_n\}$ to the standard basis vectors. This makes e_0 our distinguished element, such that

$$e_0 = -\sum_{i=1}^{n} e_i.$$

Additionally, we will take $*$ to be the standard inner product. These assumptions allow us to invoke the above proposition.

Another important element from [NR23], is that $\overline{\mathrm{Cub}}(\mathbf{\Sigma}_{\mathcal{M}}, *)$ is a cone in the space $\mathbb{R}^{\mathbf{\Sigma}_{\mathcal{M}}(1)}$. Specifically, purely cubical values are elements of the relative interior of this cone and pseudocubical values include elements on the boundary. If $\mathrm{Cub}(\mathbf{\Sigma}_{\mathcal{M}}, *)$ is non-empty then so is $\overline{\mathrm{Cub}}(\mathbf{\Sigma}_{\mathcal{M}}, *)$. We now define the z-values associated to α and β and show that they are indeed pseudocubical.

Recall that the definition of α and β are independent of choice of ground element. This means we are free to pick, and going forward we will work with the assumption that

$$\alpha = \sum_{e_0 \in F} x_F \quad \text{and} \quad \beta = \sum_{e_0 \notin F} x_F.$$

We stated that the divisor associated to a z-value is given by

$$D(z) = \sum_{\rho \in \mathbf{\Sigma}_{\mathcal{M}}} z_\rho x_\rho,$$

but since our fan is the Bergman fan of a matroid, and rays are associated to flats, we may rephrase this as

$$D(z) = \sum_{F \in \mathcal{L}^*} z_F x_F.$$

We define $z^\alpha = (z^\alpha_F \,|\, F \in \mathcal{L}^*)$ and $z^\beta = (z^\beta_F \,|\, F \in \mathcal{L}^*)$ where

$$z^\alpha_F = \begin{cases} 1 & e_0 \in F \\ 0 & e_0 \notin F \end{cases} \qquad \text{and} \qquad z^\beta_F = \begin{cases} 0 & e_0 \in F \\ 1 & e_0 \notin F \end{cases}$$

determines the components. A quick inspection shows that $D(z^\alpha) = \alpha$ and $D(z^\beta) = \beta$ as desired. Now we just need to show that z^α and z^β are pseudocubical.

Proposition 5.8. *The z-values z^α and z^β lie in the pseudocubical cone:*

$$z^\alpha, z^\beta \in \overline{\mathrm{Cub}}(\boldsymbol{\Sigma}_{\mathcal{M}}, *).$$

Proof. Let $\boldsymbol{\Sigma}_{\mathcal{M}} \subseteq \boldsymbol{N}_E$ be the Bergman fan of $\mathcal{M}$ and take $*$ to be the standard inner product. To avoid confusion, let $E = \{e_0, e_1, \ldots, e_n\}$ be the set of ground elements of $\mathcal{M}$ and $\{u_{e_1}, \ldots, u_{e_n}\}$ to be the standard basis vectors of $\boldsymbol{N}_E$ associated to the ground elements. As usual, we will write $u_I = \sum_{i \in I} u_i$ and recall that $u_{e_0} = -u_{e_1} - \cdots - u_{e_n}$. For any $z \in \mathbb{R}^{\boldsymbol{\Sigma}_{\mathcal{M}}}$, we will define $w_\sigma(z)$ to be the unique element in the intersection

$$\mathrm{span}(\sigma) \cap \{v \in \boldsymbol{N}_E \mid v * u_F = z_F \text{ for all flats } F \text{ associated to rays in } \sigma(1)\}\,.$$

That this intersection always yields a single element comes from the fact that our cones are simplicial.

In order to prove that $z^\alpha, z^\beta \in \mathbb{R}^{\mathbf{\Sigma}_{\mathcal{M}}(1)}$ are pseudocubical, we must show that for all cones $\sigma \in \mathbf{\Sigma}_{\mathcal{M}}$,

$$w_\sigma(z^\alpha) \in \sigma \quad \text{and} \quad w_\sigma(z^\beta) \in \sigma.$$

Take $\sigma_{\mathscr{F}} \in \mathbf{\Sigma}_{\mathcal{M}}$ to be an arbitrary cone and $\mathscr{F} = \{F_1 \subsetneq \cdots \subsetneq F_k\}$ to be its associated flag. We will prove that

$$w_{\sigma_{\mathscr{F}}}(z^\alpha) = \begin{cases} \frac{1}{|F_k^{\mathsf{c}}|} u_{F_k} & e_0 \in F_k \\ 0 & e_0 \notin F_k \end{cases}$$

$$w_{\sigma_{\mathscr{F}}}(z^\beta) = \begin{cases} 0 & e_0 \in F_1 \\ \frac{1}{|F_1|} u_{F_1} & e_0 \notin F_1 \end{cases}$$

with F_k^{c} being the set complement, $F_k^{\mathsf{c}} = E \setminus F_k$. Since these are in a ray of $\sigma_{\mathscr{F}}$, they would of course be in $\sigma_{\mathscr{F}}$. This is sufficient to show that $z^\alpha, z^\beta \in \overline{\mathrm{Cub}}(\mathbf{\Sigma}_{\mathcal{M}}, *)$.

Before we go on, we will need a fact about the inner product in our space. For subsets $I \subseteq E$ and $J \subseteq E$,

$$u_I * u_J = \begin{cases} \ \ |I \cap J| & e_0 \notin I \text{ and } e_0 \notin J \\ -|I \cap J^{\mathsf{c}}| & e_0 \notin I \text{ and } e_0 \in J \\ \ \ |I^{\mathsf{c}} \cap J^{\mathsf{c}}| & e_0 \in I \text{ and } e_0 \in J. \end{cases}$$

The first case follows simply from the fact that the inner product multiplies the entries pairwise, and the 1 associated to u_{e_j} will only contribute to the sum if it is in both vectors. The second case follows from the fact that if $e_0 \in J$, $u_J = -u_{E \setminus J}$, thanks to u_{e_0} being a vector with every entry set to -1. The final case follows from the previous two, and that $(-1) \cdot (-1) = 1$.

We will start with $w_{\sigma_{\mathscr{F}}}(z^\alpha)$. Since it is uniquely characterized by the condition

$$w_{\sigma_{\mathscr{F}}}(z^\alpha) * u_{F_j} = z^\alpha_{F_j}$$

for all $1 \leq j \leq k$, we just need to show that this equality holds. First, let's assume that $e_0 \in F_k$. Then

$$\begin{aligned} w_{\sigma_{\mathscr{F}}}(z^\alpha) * u_{F_j} &= \frac{1}{|F_k^{\mathsf{c}}|} u_{F_k} * u_{F_j} \\ &= \frac{1}{|F_k^{\mathsf{c}}|} (u_{F_k} * u_{F_j}), \end{aligned}$$

thanks to inner products being bilinear. From above, we know

$$u_{F_k} * u_{F_j} = \begin{cases} |F_k^{\mathsf{c}} \cap F_j| & e_0 \notin F_j \\ |F_k^{\mathsf{c}} \cap F_j^{\mathsf{c}}| & e_0 \in F_j \end{cases}$$

and since $F_j \subseteq F_k$ we can see that

$$u_{F_k} * u_{F_j} = \begin{cases} 0 & e_0 \notin F_j \\ |F_k^{\mathsf{c}}| & e_0 \in F_j \end{cases}$$

by some elementary set theory. And so when we incorporate the scalar back in, we get

$$w_{\sigma_{\mathscr{F}}}(z^{\alpha}) * u_{F_j} = \begin{cases} 0 & e_0 \notin F_j \\ 1 & e_0 \in F_j \end{cases}$$

which matches the value of $z^{\alpha}_{F_j}$. Turning to the other case, let's assume that $e_0 \notin F_k$. Then by definition, $z^{\alpha}_{F_j} = 0$ for all $1 \leq j \leq k$, and since $0 * u_{F_j} = 0$ we again have

$$w_{\sigma_{\mathscr{F}}}(z^{\alpha}) * u_{F_j} = z^{\alpha}_j.$$

We conclude then that $w_{\sigma_{\mathscr{F}}}(z^{\alpha})$ is in the cone $\sigma_{\mathscr{F}}$, and so $z^a lpha$ is pseudocubical.

The proof that w^{β}_z is pseudocubical follows analogously. We want to show that

$$w_{\sigma_{\mathscr{F}}}(z^{\beta}) * u_{F_j} = z^{\beta}_{F_j}$$

for all $1 \leq j \leq k$. This time we start with the case that $e_0 \notin F_1$. Then as defined, we have

$$w_{\sigma_{\mathscr{F}}}(z^{\beta}) * u_{F_j} = \frac{1}{|F_k|}(u_{F_k} * u_{F_j}).$$

Using some set theory we get

$$u_{F_1} * u_{F_j} = \begin{cases} |F_1 \cap F_j| = |F_1| & e_0 \notin F_j \\ \\ |F_1 \cap F_j^{\mathsf{c}}| = 0 & e_0 \in F_j \end{cases}$$

and so

$$w_{\sigma_{\mathscr{F}}}(z^\beta) * u_{F_j} = \begin{cases} 1 & e_0 \notin F_j \\ \\ 0 & e_0 \in F_j \end{cases}$$

again matching how we have defined the component $z^\beta_{F_j}$. On the other hand, if $e_0 \in F_1$, then $z^\beta_{F_j} = 0$ for each $1 \leq j \leq k$. Since $w_{\sigma_{\mathscr{F}}}(z^\beta) = 0$ this again works just fine.

With that we have shown that both z^α and z^β are pseudocubical values in $\overline{\mathrm{Cub}}(\mathbf{\Sigma}_{\mathcal{M}}, *)$. □

Finally, we have one last problem to address. From the proof above we can see that $z^\alpha, z^\beta \notin \mathrm{Cub}(\mathbf{\Sigma}_{\mathcal{M}}, *)$. They are on the boundary of $\overline{\mathrm{Cub}}(\mathbf{\Sigma}_{\mathcal{M}}, *)$, however the definition of AF fans, Definition 4.32, is only for strictly cubical z-values. We turn, in desperation, to a bit of analysis to help us here.

Lemma 5.9. *Let $\mathbf{\Sigma} \subseteq \boldsymbol{N}$ be a fan and $* \in \mathrm{Inn}(\boldsymbol{N}_E)$ an inner product such that* $\mathrm{Cub}(\mathbf{\Sigma}, *)$ *is nonempty. Then for any $z \in \overline{\mathrm{Cub}}(\mathbf{\Sigma}, *)$, there exists values $z_t \in$*

$\mathrm{Cub}(\mathbf{\Sigma}_{\mathcal{M}}, *)$, $t \in (0,1]$, *such that*

$$\lim_{t\to 0} z_t = z.$$

Proof. This actually follows from convex geometry. Take any $z \in \overline{\mathrm{Cub}}(\mathbf{\Sigma}, *)$, and recall that $\overline{\mathrm{Cub}}(\mathbf{\Sigma}, *)$ is a cone in $\mathbb{R}^{\mathbf{\Sigma}(1)}$. Since we have assumed $\mathrm{Cub}(\mathbf{\Sigma}, *)$ is nonempty, let $z_1 \in \mathrm{Cub}(\mathbf{\Sigma}, *)$ be a point in the interior of the cone. Then define

$$z_t = tz_1 + (1-t)z.$$

Then $\lim_{t\to 0} z_t = z$ and when t is restricted to the interval $[0,1]$ this is just the convex hull of z and z_1. Since this line segment must be contained in the cone and z_1 is in the interior, all points must be in the interior except maybe z. Thus for all $t \in (0,1)$ we have $z_t \in \mathrm{Cub}(\mathbf{\Sigma}, *)$. $\square$

This allows us to approximate any pseudocubical value using only cubical values. We now, truly, have everything we need to get to prove our final theorem.

5.4 Putting It All Together

Main Result. For any matroid $\mathcal{M}$, the Heron–Rota–Welsh conjecture is true.

Proof. Let $\mathcal{M} = (E, \mathcal{L})$ be a matroid with ground set $E = \{e_0, e_1, \ldots, e_n\}$. Take $\mathbf{\Sigma}_{\mathcal{M}} \subseteq \boldsymbol{N}_E$ to be the Bergman fan of $\mathcal{M}$ and associate $(e_1, e_2, \ldots, e_n)$ to the standard basis vectors and choose $* \in \mathrm{Inn}(\boldsymbol{N}_E)$ to be the standard inner product on this basis.

By Proposition 3.6, we know that the reduced characteristic coefficients of $\mathcal{M}$ can be associated to mixed degrees of divisors under the degree map,

$$\overline{w}_k = \deg(\alpha^{r-k}\beta^k).$$

And by Theorem 4.12, we have that

$$\mathrm{MVol}_{\boldsymbol{\Sigma},*}(z_1, \ldots, z_d) = \deg\big(D(z_1)\cdots D(z_d)\big).$$

So, by using the z-values defined in the previous section, we can identify reduced characteristic coefficients with mixed volumes of normal complexes, given by

$$\overline{w}_k = \mathrm{MVol}_{\Sigma,*}(\underbrace{z^\alpha, \ldots, z^\alpha}_{r-k}, \underbrace{z^\beta, \ldots, z^\beta}_{k}).$$

By Proposition 5.7 we know that $\mathrm{Cub}(\boldsymbol{\Sigma}, *)$ is non-empty, and so we may use Lemma 5.9 to get sequences of cubical values z_t^α and z_t^β such that

$$\lim_{t\to 0} z_t^\alpha = z^\alpha \quad \text{and} \quad \lim_{t\to 0} z_t^\beta = z^\beta.$$

Define

$$\overline{w}_{k,t} = \mathrm{MVol}_{\Sigma,*}(\underbrace{z_t^\alpha, \ldots, z_t^\alpha}_{r-k}, \underbrace{z_t^\beta, \ldots, z_t^\beta}_{k})$$

which means $\lim_{t\to 0} \overline{w}_{k,t} = \overline{w}_k$. Because $z_t^\alpha, z_t^\beta \in \mathrm{Cub}(\boldsymbol{\Sigma}_\mathcal{M}, *)$ and $\boldsymbol{\Sigma}_\mathcal{M}$ is AF by Theorem 5.3, we have that

$$\mathrm{MVol}_{\Sigma,*}(\underbrace{z_t^\alpha, \dots, z_t^\alpha}_{r-k}, \underbrace{z_t^\beta, \dots, z_t^\beta}_{k})^2 \geq \mathrm{MVol}_{\Sigma,*}(\underbrace{z_t^\alpha, \dots, z_t^\alpha}_{r-k-1}, \underbrace{z_t^\beta, \dots, z_t^\beta}_{k-1})\,\mathrm{MVol}_{\Sigma,*}(\underbrace{z_t^\alpha, \dots, z_t^\alpha}_{r-k+1}, \underbrace{z_t^\beta, \dots, z_t^\beta}_{k+1})$$

for $1 < k < r$, as per Definition 4.32, which in turn means

$$\overline{w}_{k,t}^2 \geq \overline{w}_{k-1,t}\overline{w}_{k+1,t}.$$

Since $\mathrm{MVol}_{\Sigma,*}$ is always non-negative, this makes the sequence $\{\overline{w}_{1,t}, \dots, \overline{w}_{k-1,t}\}$ log-concave for any $t \in (0,1]$. This implies that under the limit as $t \to 0$ we have that $\{\overline{w}_1, \dots, \overline{w}_{k-1}\}$ is a log-concave sequence. As the reduced characteristic coefficients are log-concave, Lemma 3.4 tells us we may conclude that the Whitney numbers of the first kind are log-concave, proving the Heron–Rota–Welsh conjecture. $\square$

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