A Volumetric Proof of the Log-Concavity of the Characteristic Polynomial of Matroids

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Chapter 1

Introduction

This thesis will require us to take a tour of mathematics that have been developing for close to a century. The main result synthesizes modern work, from about ten years right up to last year, about a conjecture, posed in seventies, on a mathematical object first formalized in 1935. This threads through work of our friends and mentors, Fields Medal winners, and a host of well-known mathematicians from across the last hundred years. While we think the results alone are quite interesting on their own, much of what has made this project so interesting to us is its broad connections to these various places. We hope, through the more leisurely pace we are allowed to take in a thesis, to show off this side of the math as well.

1.1 What Are We Doing?

The key players in this work are **matroids**, a combinatorial object devised to generalize the notion of "independence". Matroids are interesting for a multitude of reasons, but of note to us is that, although they are combinatorial objects, they can be alternatively studied as geometric objects, known as **Bergman fans**, and algebraic objects called **Chow rings**. In the early 1970's, a conjecture about the **characteristic polynomial** of matroids was posed. The **Heron-Rota-Welsh conjecture** was, in essence a combinatorial question, and would remain unresolved for almost 50 years. It was through viewing the problem from the algebro-geometric side of things that Adiprasito, Huh, and Katz were finally able to prove the conjecture true in 2015. They did this by importing complex machinery from algebraic geometry, known as *Hodge theory*, into the combinatorial world of matroids. It is an impressive work that, in part, won author June Huh a Fields medal.

In this thesis we wish to offer an alternative proof of the Heron-Rota-Welsh conjecture. To do this, we too will tackle the problem from a geometric perspective. In 2021, Nathanson and Ross developed a correspondence between the volume of objects generated from the Bergman fans of matroids, called **normal complexes**, and the evaluation of degree maps on the Chow Ring. This opened the avenue of using the geometric picture of matroids to show characteristics of its algebraic representation. Using our recent work with Nowak and Ross, we show this correspondence and certain volumetric properties of normal complexes is sufficient to prove the Heron-Rota-Welsh conjecture.

1.2 Why Are We Doing This?

Where some would ask why, we much prefer to ask "why not?". More seriously, while a proof of the Heron-Rolta-Welsh conjecture is not new, having a new viewpoint on something is valuable even just in comparing it to the original.

This is an exciting starting application of the theory of normal complexes. Compared to combinatorial Hodge theory, normal complexes have a much lower barrier of entry. That they can prove the same, famously difficult, problem is surprising at least. We hope to see some of these techniques and tools expanded and applied elsewhere.

1.3 Who Is This For?

By this point we've already introduced quite a few words we don't expect every reader to know offhand. Our primary goal is that anyone with a few graduate level courses in mathematics under their belt could read this thesis from start to finish and come out with a comprehensive picture of both the setting and the conclusion. To that end we will be providing context for every word in bold appearing above, linking each of them to the overall picture.

However, we also have some secondary goals in terms of readership. First, we want this to be of at least some interest to someone already knowledgeable in the field. While we are confident that any math of real substance in this thesis will be developed elsewhere, if it's going to appear here it might as well at least be useful to a practitioner. Second, and in somewhat of a contradiction, we want this work to be inviting to a curious non-mathematican. We believe there is a good opportunity here to allow a layperson to follow along with math they may not be otherwise usually exposed too.

In the true spirit of compromise then, we expect no one to be totally happy with the pacing. In general, the intention is the complexity of the material will start somewhat low and increase as we go on. But, there will be technical points interjected in otherwise easy material, and we will attempt to include high level overviews even in sections that really do require a solid mathematical background. We say this largely to give the reader permission to skip the bits that simply don't interest them.

Chapter 2

Matroids

The underpinning of all our work are mathematical objects known as matroids. Though, as we've noted, they've been around since the 1930's, they're not, yet, household objects every mathematician knows. This is the shallowest scratch into the world of matroids, slanted heavily towards what's necessary for our problem at hand. There are many full books on matroids, for those curious to dig into more depth. We are partial to the treatment by Oxley's *Matroid Theory* [Oxl11].

We will build up to matroids by developing some intuition from more familiar, motivating mathematical objects. Then we will introduce the definition(s) of matroids and introduce the characteristic polynomial. We will wrap up this chapter by stating the Heron-Rota-Welsh conjecture.

2.1 Linear Algebra Done Hastily

When the vague notion of independence is mentioned in a mathematical context, we expect that minds wander to *linear* independence. A central concept to the field of linear algebra, this is likely the vast majority's first introduction to the topic. Happily, this mirrors, closely enough, the initial development of matroids. The patterns that emerge viewing the independence of collections of vectors will, quite directly, inspire the first of our definitions of a matroid.

2.1.1 Linear Independence

First, let us recall the definition of linear independence.

Definition 2.1 (Linear Independence)**.** Given a finite set of vectors $\{v_1, v_2, \ldots, v_k\} \subseteq F^n$, for some field F, the set of vectors is called *linearly independent* if the only solution to the linear combination

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$$

is $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$. Otherwise, we say the set is *linearly dependent.*

While this is a familiar definition to many of us, it will be illustrative to all to take a more concrete example. We'll define the vectors $a = (1, 0, 0), b = (0, 1, 0), c = (0, 0, 1)$, and $d = (1, 1, 0)$. Then we have the set $E = \{a, b, c, d\} \subseteq \mathbb{R}^3$.

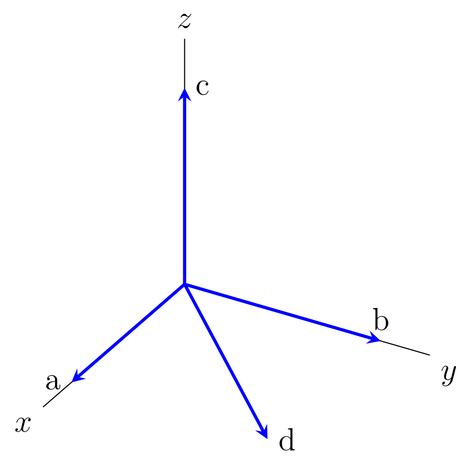


Figure 2.1: The collection of vectors in E

The observant will note that this of course cannot be linearly independent, and indeed we can confirm by showing the linear combination

$$1a + 1b + 0c + (-1)d = 0.$$

But now, a fun little game we could play, at least by our personal reckoning of fun, is to find all subsets of E that *are* linearly independent. For example, consider $\{c, d\} \subseteq E$.

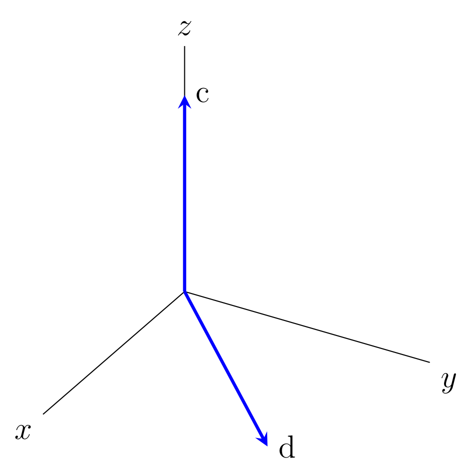


Figure 2.2: A linearly independent subset of E

Take a look to confirm there is no nonzero linear combination of our elements that gives us the 0-vector. Given the relatively small number of elements, it would not take too long to identify every possible subset of E_V that is linearly independent; for the impatient however, they are precisely

$$\big\{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\big\}.$$

For the impatient *and* untrusting, we suggest that the only thing really necessary to check here is that each 3-element set is linearly independent and that there are no other possible 3-element sets in E that are linearly independent.

As a point of pure notation, the above list is ugly. We are going to be working with sets of this form so much in this paper that, in order to avoid a shortage of curly brackets, we will introduce a more tidy notation. Going forward, we will write the elements of the internal sets adjacent to each other to represent the set containing them; for example we will write the set $\big\{\{a, b\}, \{a, b, d\}\big\}$ as $\{ab, abd\}$. Thus, we will more compactly identify the linearly independent subsets of E as

$$\{\emptyset, a, b, c, d, ab, ac, ad, bc, bd, cd, abc, acd, bcd\}.$$

Now that we have this collection, this leads to our next totally fun and normal activity. Namely, looking for patterns amongst these independent sets.

2.1.2 Noteworthy Properties of Linearly Independent Subsets

We suspect that those with some knowledge of linear algebra will immediately be ready to note that the largest independent subsets of E_V have 3 elements. And, sure enough, that is true! But this is more a property of the vector space, $\mathbb{R}^3$ in this case, that we're pulling the vectors from than some intrinsic relationship or property of the subsets. We'd like to call attention to some properties that may be less obvious (or so obvious one forgets they're even there). First, something entirely uninteresting.

Property 2.1 (The empty set is an independent subset)**.** For any finite collection of vectors, E, in vector space,

$$\emptyset \subseteq E$$

and $\emptyset$ is linearly independent.

That $\emptyset$ is linearly independent is what we call vacuously true. That is to say, it's true mostly as a quirk of how we define linear independence. Since we can't form a non-zero linear combination that gives the 0-vector, because there are *no* elements at all, it can't be linearly dependent. But then if it's not linearly dependent, it has to be independent. Proof by being pedantic, really the heart of mathematics if one thinks about it. Next, a property that will surprise no one who has taken a linear algebra class, but is worth making explicit.

Property 2.2 (Any subset of a linearly independent set is itself linearly independent)**.** For any linearly independent set of vectors, I, in vector space, if

$$I' \subseteq I,$$

then I' is linearly independent.

Recall we suggested that in order to check that our list of independent subsets of E_V was correct, it was sufficient to just check the subsets with the most elements. This property tells us that if we've figured out the maximal subsets, then filling in the rest is just a matter of taking subsets of those. One may even begin to see the specter of combinatorics lurking. This property falls out easily from our definition. If no non-zero linear combination of vectors in a set gives us the 0-vector, then using fewer vectors isn't going to change that. Finally, we have a more subtle property.

Property 2.3 (The "independence augmentation" property)**.** Let $I = \{v_1, v_2, \ldots, v_m\}$ and $J = \{u_1, u_2, \ldots, u_n\}$ be linearly independent sets in a vector space, such that $m < n$. Then there exists a $k \in [n]$ such that the set

$$I \cup u_k = \{v_1, v_2, \ldots, v_m, u_k\}$$

is linearly independent.

In other words, we can always find an element of a larger independent set to include in a smaller one that will leave the (new, augmented) set independent. Going back to our running example, consider the sets acd and ab. Then $c \in acd$ is such an element, and we confirm that $ab \cup c = abc$ is indeed linearly independent. This property is not immediately obvious, though may be believable to those who have done a proof based linear algebra class.

These are the three properties of linearly independent sets we wish to highlight here. We could use these properties alone to motivate the first definition of a matroid. However, we have one more detour before we get to matroids proper. There is another area where independence arises quite naturally, and it will be useful to know going forward.

2.2 Graphic Content

The next place our intuition building journey takes us is the world of graphs. Graph theory was the other motivator of matroids, so we too shall delve in. While we tried to not assume too much, we did, secretly, expect the average reader would feel comfortable enough with linear algebra. Graphs, on the other hand we will quickly build up from scratch and develop a notion of independence. Luckily, this is actually a fairly short process.

2.2.1 What a Graph Is

Not to be confused with the graph of a function or whatever it is business analysts put in shareholder presentations, graphs for us are essentially a collection of points, called vertices, and lines between them, called edges. There are quite a few definitions of graphs, each allowing for slightly different properties, but for our purposes, we can use a rather basic definition.

Definition 2.2 (Graph)**.** A *graph* is a pair of sets $G = (V, E)$, where V is a set of objects known as vertices and

$$E \subseteq \big\{\{x, y\} \mid x, y \in V\big\}$$

is a set of edges.

A brief aside for our friends who actually care about graphs; the definition here is for an *undirected simple graph permitting loops.* The treatment of graphs and their relation to matroids in this paper extends easily enough for most other graphs because, as far as matroids are concerned, this kind of graph carries pretty much all the information necessary. Indeed, we will see soon enough that even allowing loops is an unnecessary flourish. We again recommend Oxley [Oxl11] for the serious graph theorist's entry into matroids.

For the rest of us, this definition may feel rather opaque. Here, an example and corresponding picture should help immensely. Let $V = \{v_1, v_2, v_3, v_4\}$ be a vertex set. Now we must define edges between vertices. For later convenience, we will name these edges. Let $a = \{v_1, v_2\},\ b = \{v_2, v_3\},\ c = \{v_3, v_4\}$, and $d = \{v_1, v_3\}$,; then let $E = \{a, b, c, d, e\}$ be our edge set. Recall that, for example, c represents an edge, or connection, between the vertex v_3 and the vertex v_4. With both those pieces, we have the graph $G = (E, V)$. The corresponding picture of our graph is below.

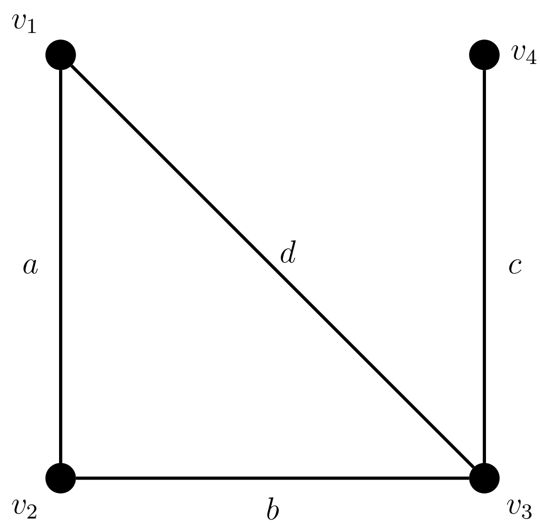


Figure 2.3: Our example graph G

Now that we know what a graph is, it's time to figure out what "independence" could possibly mean.

2.2.2 Independence in the Realm of Graphs

The first thing to note is that we will define independence on the set of edges of a graph; that is, for some graph $G = (V, E)$, an independent set will be some subset $I \subseteq E$, meeting some criteria we'll discuss below. What then would it mean for a set of edges to be independent? Well, if we take some subset of the edges, we restrict which vertices are accessible via those edges. But there might still be redundant edges. Could we remove additional edges from our set and still be able to reach all the same vertices? The answer to that determines if a set of edges is independent, when we can't make our collection of edges any smaller without disconnecting a vertex, or dependent, when we can.

To formalize this we will need to learn a few graph theoretic terms. First, we need the notion of a walk.

Definition 2.3 (Walk)**.** Given a graph $G = (V, E)$, a *walk* is an alternating sequence of vertices and edges

$$(v_1, e_1, v_2, e_2, v_3, \ldots, e_{k-1}, v_k),$$

where each $v_i \in V$, $e_j \in E$ and $v_i \in e_i$ and $v_{i+1} \in e_i$

Intuitively, a walk starts at some vertex and then follows an edge to another, connected vertex then continues to follow edges to vertices until ending at some vertex. If we put our finger on a vertex and trace along edges to another vertex, we've defined a walk. Now that we have a walk, we may define a cycle.

Definition 2.4 (Cycle)**.** A *cycle* is a walk

$$(v_1, e_1, v_2, e_2, v_3, \ldots, e_{k-1}, v_k),$$

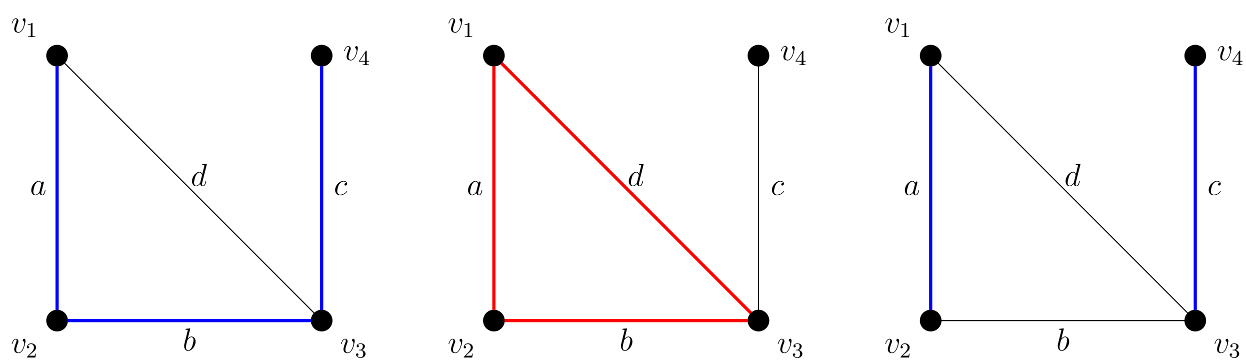
where $v_1 = v_k$ and $v_i \neq v_j$ when $i \neq j$ otherwise.

Further, we say a set of edges *contains a cycle* if there is a cycle whose edges are contained in the set.

That is, a cycle is a walk that starts and ends at the same place and otherwise passes through unique vertices. Given the notion of independence we began to motivate above, hopefully the utility of defining a cycle is apparent. Any subset of edges that contains a cycle must be dependent, as we can always remove the last edge from the walk and still have all the same vertices connected. With this, our definition of independence can finally be formalized.

Definition 2.5 (Independence (of edges of a graph))**.** Let $G = (V, E)$ be a graph. Then a subset of edges $I \subseteq E$ is *independent* if it does not contain a cycle.

Let us immediately take to our example for this section to consider some possible sets of edges.



(a) Independent: the set of edges $\{a, b, c\}$ does not contain a cycle

(b) Dependent: The edges in $\{a, b, d\}$ contain a cycle

(c) Independent: The edge set $\{a, c\}$ does not contain a cycle. It's not necessary for all vertices to be connected

Figure 2.4: Some examples of independent/dependent sets of edges

Now that we've got some practice under our belt, it's time to play our favorite game again. Given our example graph, $G = (V, E)$, we want to identify the set of all possible independent vectors. A few moments of tracing paths along the graph, hunting for cycles, will reveal that from our set of edges $E = \{a, b, c, d\}$, the independent subsets are precisely

$$\{\emptyset, a, b, c, d, ab, ac, ad, bc, bd, cd, abc, acd, bcd\}.$$

This should look familiar! Suspiciously so, even. The independent subsets are the same as those we found in our collection of vectors. Clearly then all the properties of linearly independent subsets we showed above also hold in this example. Indeed, this is not just a quirk of our example. Given any graph, the independent subsets of the edge set will obey the same properties as the linearly independent subsets of a set of vectors. It was the reoccurrence of these properties across different mathematical objects that inspired the creation of matroids.

2.3 Matroids, Finally

Matroids were initially developed by Hassler Whitney in the paper *On the Abstract Properties of Linear Dependence* [Whi35]. The introduction of Whitney's paper parallels our journey so far, covering, much more succinctly, shared properties of linear independence and independence of graph edges. He then goes on to introduce several equivalent definitions of a matroid.

An interesting feature of matroids is just how many definitions exist. Plenty more have been added since the several introduced by Whitney, and any one of these definitions can be taken axiomatically and from them any other definition may be derived. However, it can be extremely non-obvious that a given definition is equivalent to some other. The path between the various axiomatizations can be so difficult to see that they have been affectionately called *cryptomorphic* to one another.

We will primarily be concerned with two axiomatizations, one based on the notion of independent sets and another based on what are called *flats*. The first definition follows closely from the background we've developed so far. This allows us to more easily define the terms and properties of matroids that we will need in the second definition. It is this second definition that will be of key importance for the following chapters, so it is important to develop it here.

2.3.1 Independent Set Axioms

The first definition of matroids should, again, look very familiar.

Definition 2.6 (Matroid — Independent Set Axioms)**.** A *matroid* is a pair $\mathcal{M} = (E, \mathcal{I})$, where E is a finite set, called the *ground set*, and $\mathcal{I} \subseteq 2^E$ is a collection of subsets of E, called the *independent sets*, with the following properties:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(I3) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| \leq |I_2|$, then there exists some $e \in I_2 \setminus I_1$ such that $I_1 \cup e \in \mathcal{I}$.

They correspond precisely to the properties we identified in linearly independent subsets and that we saw again in independent edge sets. We can take this opportunity to define our now familiar examples as a matroid.

Example 2.1. We let the ground set $E = \{a, b, c, d\}$, and pick the independent sets to be

$$\mathcal{I} = \{\emptyset, a, b, c, d, ab, ac, ad, bc, bd, cd, abc, acd, bcd\}.$$

Coming as probably no surprise, this has the same independence relations as both our vector example and our graph example. We should confirm that $\mathcal{I}$ obeys the properties (I1)-(I3), but we already know that particular set must.

2.3.1.1 Aside: Representable Matroids

Given that we've already seen the example "matroid" arise twice in other contexts, it is natural to ask if we've gained anything new with matroids. If every matroid could just be studied as a finite collection of vectors and its independent subsets, we don't really have to go through the trouble defining a whole new object.

It turns out that this is not the case. A matroid that can arise from a finite set of vectors, like our example, is called *representable*. However, there are *unrepresentable* matroids. A lot of them in fact.

The distinction between representable and unrepresentable matroids has no bearing on the results of this thesis, but it's worth noting here. Our examples are representable, as it allows us to leverage some visual intuition, but everything we say here holds for all matroids.

2.3.2 The Uphill Path to Flats

A benefit of introducing the independence axioms first, we feel, is that they are readily interpretable. At least after developing a bit of intuition in the realm of linear independence. For much of the rest of our paper however, we won't be thinking of matroids in this form. We will need a formulation of matroids that use something called *flats*.

To get to this new definition of matroids, or even state what a flat is, we will have to build up our vocabulary surrounding matroids. Our goal here is to develop everything necessary to define a flat. The path there may seem rather wandering, we will introduce quite a few definitions here. But there are no shortcuts; each new definition builds on the last, until we have a nice tower of terms with which to use.

Given their history, matroids borrow a lot of terminology from linear algebra and graph theory. For the most part, their meaning is related to that in the original context, so it can be a useful starting point. Still, it is not necessary to have heard of them before; these definitions exist perfectly fine on their own in the world of matroids, as we shall see.

We use the independent set axioms to these terms and state properties, but we could have started with any of the axioms and developed all these terms. It's actually quite a fun exercise to develop parallel definitions from different starting axioms.

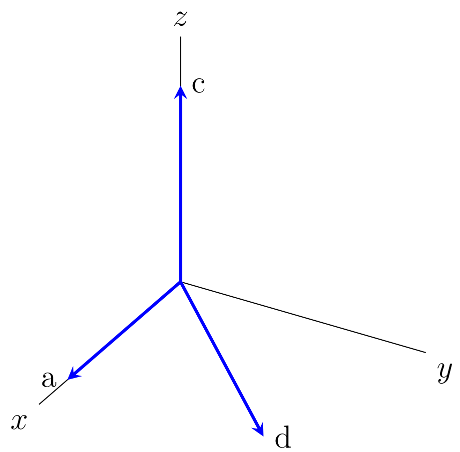
2.3.2.1 All Your Bases Belong to Matroid

First, we will finally address a pattern we've noted earlier, that the largest independent sets all seem to have the same number of elements, or, as we like to say in the business, the same *cardinality*. To do so we'll introduce the notion of a basis of a matroid.

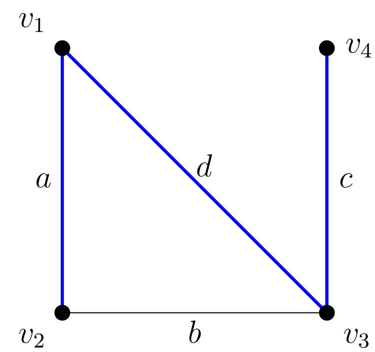
Definition 2.7 (Basis)**.** Given a matroid $\mathcal{M} = (E, \mathcal{I})$, an independent set $B \in \mathcal{I}$ is a *basis* of $\mathcal{M}$ if

$$B \cup e \notin \mathcal{I}$$

for all $e \in E \setminus B$. That is to say, a basis B is a maximally independent subset of E with respect to set inclusion.



(a) A basis in linear algebra is a minimal spanning set



(b) A basis of graph is a spanning tree

Figure 2.5: The set $\{acd\}$ is a basis of M, which we can view in the vector and graph setting

For those recalling their linear algebra, yes, this does have the very useful property we expect from something called a basis.

Proposition 2.1. *All bases of a matroid contain the same number of elements.*

Proof. Let B_1 and B_2 be two bases of $\mathcal{M}$. It must be the case that $|B_1| < |B_2|$, $|B_1| > |B_2|$, or $|B_1| = |B_2|$. Let's assume that $|B_1| < |B_2|$. Then since $B_1, B_2 \in \mathcal{I}$, we may use the property (I3) of matroids. There exists some $b \in B_2 \setminus B_1$ such that $B_1 \cup b \in \mathcal{I}$. This is a contradiction with our definition of a basis, since adding any element not already in B_1 should make it dependent.

We have then that $|B_1| \geq |B_2|$, but assuming the case $|B_1| > |B_2|$, we will arrive at a contradiction by the same steps as above. Thus, $|B_1| = |B_2|$, and we conclude that all bases of $\mathcal{M}$ have the same number of elements. $\square$

As we see in the examples in figure 2.5, a basis has a very literal interpretation in the context of vector spaces and graphs. If pressed for an intuition of a basis in the more general matroid setting, we'd say that they give us an idea of "how much" (in)dependence is going on amongst the elements ground set; likely accompanied by us literally waving our hands through the air. If our matroid has 1000 elements in its ground set, but its bases only have size 3, then there must be a lot of dependence amongst all those elements of the ground set. However vague the idea, it would be very useful to be able to quantify "how much" independence is going on in any subset $X \subseteq E$ of a matroids ground set.

2.3.2.2 Rank and Closure

Indeed, this is an important enough property to get its own name, the *rank*. The rank of any subset of ground elements is simply the size of the largest independent subset.

Definition 2.8 (Rank)**.** Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. The *rank function* is the map

$$\begin{aligned} \mathrm{rk}_{\mathcal{M}} \,:\, 2^E &\to \mathbb{Z}_{\geq 0} \\ X &\mapsto |Y| \end{aligned}$$

where $Y \subseteq X$, $Y \in \mathcal{I}$, and there is no $Y \subsetneq Y' \subseteq X$ such that $Y' \in \mathcal{I}$. That is to say, the rank of any subset X is the size of the largest independent set contained in X.

We write $\mathrm{rk}_{\mathcal{M}}(E)$ as $\mathrm{rk}_{\mathcal{M}}(\mathcal{M})$, and is called the *rank* of $\mathcal{M}$.

Unless we are in imminent danger of confusion, we will notate $\mathrm{rk}_{\mathcal{M}}(X)$ as just $\mathrm{rk}(X)$. In the land of linear algebra, the rank corresponds to the dimension spanned by the vectors. Just as adding more vectors into a linear span won't necessarily increase the dimension spanned, increasing the number of your elements in your subset will not necessarily increase the rank. For instance, in our running example we see that $\mathrm{rk}(ab) = \mathrm{rk}(abd) = 2$. The rank of the matroid itself will be, as we showed above, the size of any basis of the matroid.

This notion that we can add more elements to a subset without changing its rank leads, at last, to the final preliminary definition.

Definition 2.9 (Closure)**.** Given a matroid $\mathcal{M} = (E, \mathcal{I})$, the *closure operator* is a function

$$\begin{aligned} \mathrm{cl}_{\mathcal{M}} \,:\, 2^E &\to 2^E \\ X &\mapsto \{e \in E \mid \mathrm{rk}(X \cup e) = \mathrm{rk}(X)\}. \end{aligned}$$

For any $X \subseteq E$, we call $\mathrm{cl}(X)$ the closure of X. Adittionally, taking closures is idempotent, so $\mathrm{cl}(X) = \mathrm{cl}\big(\mathrm{cl}(X)\big)$.

Again we will write the closure operator as $\mathrm{cl}(X)$ almost exclusively. If a basis captures how much "independence" is in a set of elements, the closure of a subset generates a set that is as "dependent" as possible for a given rank (using the elements of that initial set). One might ask if there is anything special about these sets that are as big as they can be with respect to closure. A very insightful question, if we do say so ourselves.

2.3.3 Our Flag Means Totally-Ordered Subsets of the Lattice of Flats

If you didn't notice our subtle hint above, it may come as a surprise that sets that are as "big" or "dependent" as possible for a given rank are precisely flats.

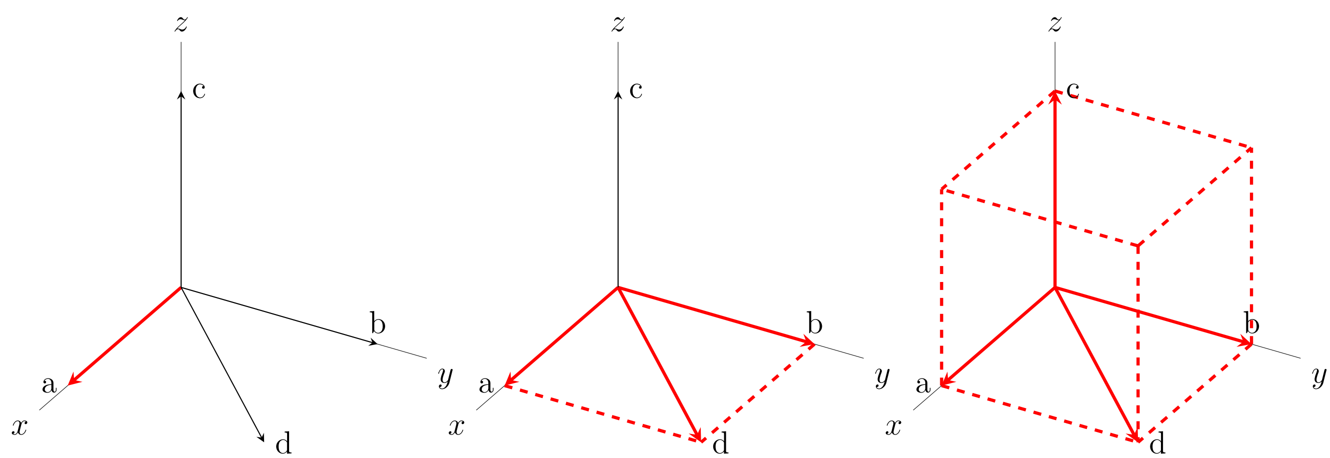
Definition 2.10 (Flat)**.** Given a matroid $\mathcal{M} = (E, \mathcal{I})$ and subset $X \subseteq E$, if

$$X = \mathrm{cl}(X),$$

then X is a *flat* of $\mathcal{M}$.

What if instead of independent sets, we collect all the flats of a matroid. In our running example, we could start applying the closure opreator left and right until we collect the set

$$\mathcal{F} = \{\emptyset, a, b, c, d, abd, ac, bc, cd, abcd\}.$$



(a) Rank 1: The single element a is a flat

(b) Rank 2: Note ab is not a flat, but abd is

(c) Rank 3: The matroid itself is rank 3, so the flat of this rank is the whole ground set, $abcd$

Figure 2.6: Examples of flats of rank 1, 2, and 3 in our example matroid M, viewed as vectors

Since flats are maximal with respect to rank, they naturally divide up by rank; i.e.

$$\begin{aligned}
\mathcal{F}_0 &= \{\emptyset\} \\
\mathcal{F}_1 &= \{a, b, c, d\} \\
\mathcal{F}_2 &= \{abd, ac, bc, cd\} \\
\mathcal{F}_3 &= \{abcd\},
\end{aligned}$$

where everything in $\mathcal{F}_k$ has rank k. When laid out like this we may begin to note some interesting patterns. Indeed, just like independent sets have some useful properties, so do the set of flats.

Proposition 2.2 (Properties of Flats)**.** *Let $\mathcal{M} = (E, \mathcal{I})$, be a matroid. Then the set*

$$\mathcal{F} = \{X \subseteq E \mid X = \mathrm{cl}(X)\}$$

is the set of flats of $\mathcal{M}$, and $\mathcal{F}$ has the following properties:

(F1) $E \in \mathcal{F}$.

(F2) If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$

(F3) If $F \in \mathcal{F}$ and $F_1, F_2, \ldots, F_k \in \mathcal{F}$ are the minimal flats such that each $F_i \subsetneq F$, then the sets $F_1 \setminus F$, $F_2 \setminus F$, $\ldots$, $F_k \setminus F$ partition $E \setminus F$.

Further, let E be any ground set and $\mathcal{F} \subseteq 2^E$ be a collection of subsets of the ground set such that properties (F1)–(F3) hold. Define

$$\mathrm{cl}^* \, : \, 2^E \to 2^E$$

such that $\mathrm{cl}^*(X) = F$ for some flat $F \in \mathcal{F}$ where $X \subseteq F$ and there is no $F' \in \mathcal{F}$ such that $X \subseteq F' \subsetneq F'$. Then $\mathcal{M} = (E, \mathcal{F})$ is a matroid with independent set

$$\mathcal{I} = \{I \subseteq E \mid I_1 \subsetneq I_2 \subseteq I, \, \mathrm{cl}^*(I_1) \neq \mathrm{cl}^*(I_2)\} \, .$$

Let's unpack this proposition, as flats are a bit more difficult than independent sets as a foundation of matroids. Property (F1) says that the ground set, E, is a flat. This follows directly from the fact that the closure of a basis has to be every element of the ground set, since you can't ever get a higher rank than a basis.

The second property (F2) says that the set of flats is closed under intersection; i.e. the elements shared between any two flats is a flat itself. This follows from the properties of closure and a bit of set theory; it's a fun little exercise to prove.

The last property, (F3), looks more intimidating than it is. In essence, if you take a flat, F (with $F \neq E$, since no flats have higher rank than E), then for every element *not* in F you're going to find it in a flat that is one rank higher. This shouldn't be too surprising, since if an element, let's call it x, is not in F, then $\mathrm{cl}(F \cup x)$ will have to have a higher rank than F. That this *partitions* $E \setminus F$ just means that each $e \in E$ that's not in F is going to appear in exactly one flat one rank higher (specifically the flat $\mathrm{cl}(F \cup e)$).

Finally, the proposition asserts that if we start with a ground set and then a collection of subsets of that ground set that meet all three properties (F1)–(F3), then that is sufficient to characterize a matroid. That is, we could take (F1)–(F3) as another axiomatization of a matroid. A recommended exercise would be to reconstruct all the definitions in the preceding section starting with just these axioms.

These properties actually impart a very interesting structure on the set of flats that we will now explore.

2.3.3.1 The Lattice of Flats

First, we recall, or learn here and now, that any collection of subsets of a set form a partially ordered set.

Definition 2.11 (Partially Ordered Set)**.** A *partially ordered set*, often called a poset, is a pair $(P, \preceq)$, where P is a set of elements, and $\preceq$, is a relation between some, but not necessarily all, of the elements of P with the following properties:

i. $a \preceq a$,

ii. if $a \preceq b$ and $b \preceq a$, then $a = b$,

iii. if $a \preceq b$ and $b \preceq c$, then $a \preceq c$, for all $a, b, c \in P$.

With the definition in hand, we can verify that $(\mathcal{F}, \subseteq)$ is a partially ordered set, where $\mathcal{F}$ is the set of all flats of a matroid. But we can do even better than that. Some posets have an even stronger structure, called a lattice.

Definition 2.12 (Lattice)**.** A partially ordered set $(L, \preceq)$ is a *lattice* if there exist binary operations

$$\vee : L \times L \to L,$$

called a *join*, and

$$\wedge : L \times L \to L,$$

called a *meet*, such that for any two elements $a, b \in L$,

i. the join $a \vee b$ is an element of the lattice such that $a \preceq a \vee b$ and $b \preceq a \vee b$, and for any element $c \in L$ such that $a \preceq c$ and $b \preceq c$ it's the case that $a \vee b \preceq c$,

ii. the meet $a \wedge b$ is an element of the lattice such that $a \wedge b \preceq a$ and $a \wedge b \preceq b$, and for any $c \in L$ such that $c \preceq a$ and $c \preceq b$ then we have $c \preceq a \wedge b$.

If you've never seen this definition before, it can be a bit heavy on symbols, but once we ground it in our set of flats it won't be too bad. First though, we must establish that the set of flats does indeed form a lattice.

Proposition 2.3 (The Collection of Flats Form a Lattice)**.** *Let $\mathcal{M}$ be a matroid and $\mathcal{F}$ be the set of all flats of $\mathcal{M}$. Then $(\mathcal{F}, \subseteq)$ is a lattice, with the operations*

$$F_1 \wedge F_2 = F_1 \cap F_2$$
$$F_1 \vee F_2 = \operatorname{cl}(F_1 \cup F_2)$$

for any $F_1, F_2 \in \mathcal{F}$.

Proof. It is sufficient to show that $\mathcal{F}$ is a meet-semilattice and that $\mathcal{F}$ has a maximal element.

To show that F is a meet-semilattice, we must prove that the meet operation is well-defined. Let $F_1, F_2 \in \mathcal{F}$ be flats. Then from property (F2), $F_1 \cap F_2$ is a flat. Naturally, for any $F_3 \in \mathcal{F}$ such that $F_3 \subseteq F_1$ and $F_3 \subseteq F_2$, then $F_3 \subseteq F_1 \cap F_2$. Thus, the meet operation is well-defined.

The maximal element of $\mathcal{F}$ is, trivially, the ground set, E, itself which is a flat by property (F1). A meet-semilattice with a maximal element is a lattice, and so $\mathcal{F}$ forms a lattice. $\square$

Now we can talk about a lattice of flats. To motivate this, let us once again consider our example matroid. It is, if not traditional, convenient to structure a lattice graphically in a *Hasse diagram.*

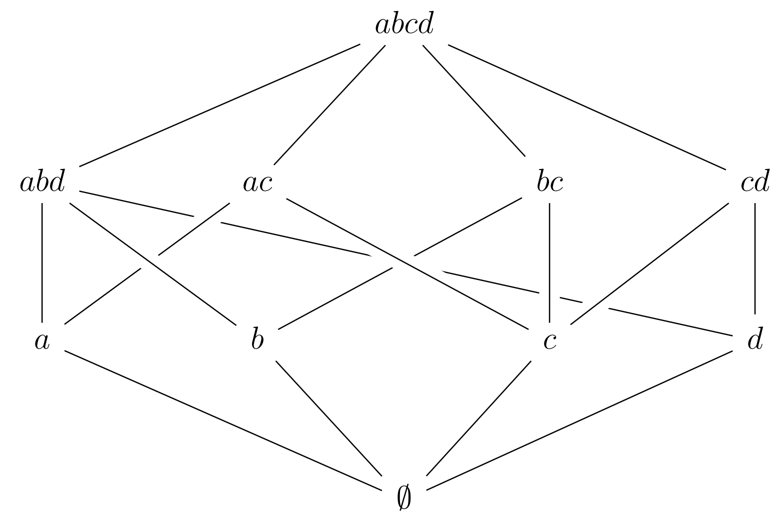


Figure 2.7: The Hasse diagram of flats of our example matroid M

When reading a Hasse diagram, if we have two entries x and y, with a line connecting them and x is higher on the page than y, we say *x covers y*. This corresponds to the relation $y \preceq x$. In the lattice of flats, each level corresponds to a rank, starting at the bottom, which is rank 0. If F_1 covers F_2, then $F_2 \subset F_1$. All of those properties in the definition of the lattice just mean that taking the intersection of two flats, or the closure of the union of two flats, will uniquely identify another element of the lattice (connected by lines to your original two entries).

When considering a matroid in terms of flats, one often sees $\mathcal{M} = (E, \mathcal{L})$ in lieu of $\mathcal{M} = (E, \mathcal{F})$ as a reminder that the set of flats forms a lattice. We will follow that convention going forward as well.

This lattice structure is key to the construction of our objects of interest in the following chapters, as we will soon see. The final definition we need from matroids are called flags, and they are, basically, just reasonable collections of flats.

2.3.3.2 Flags

Given a matroid $\mathcal{M} = (E, \mathcal{L})$, let $\mathcal{L}^*$ be the set of proper flats of $\mathcal{M}$; i.e. all flats with rank greater than 0 and not including E. Since *every* lattice of flats always has 1 element of rank 0 as a minimal element and E as the unique maximal element, $\mathcal{L}^*$ is just the interesting bits of $\mathcal{L}$.

Definition 2.13 (Flag)**.** If $\mathcal{M} = (E, \mathcal{L})$ is a matroid, then a *flag* is a totally ordered subset $\mathscr{F} \subseteq \mathcal{L}^*$ of the proper flats of a matroid,

$$\mathscr{F} = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k\} \subseteq \mathcal{L}^*.$$

If $\mathrm{rk}(\mathcal{M}) = r + 1$, then a flag $\mathscr{F} = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r\}$ is a *maximal* flag of $\mathcal{M}$.

Flags are, then, just collections of flats where you can nest all the flats; a little set theoretic *matryoshka*. On the Hasse diagram, a flag will have at most one element from each rank and there will be a strictly increasing path of lines between all elements of the flag. Maximal flats will be those that take you along a path on the Hasse diagram from rank 1 all the way up to the rank right below that of the matroid itself, including something from every rank in between. One thing to remember is that for every flat F, $\mathscr{F} = \{F\}$ is, indeed, a flag.

2.3.3.3 'Tis the Gift to Be Simple

If a serious matroid theorist is, for some inexplicable reason, subjecting themselves to this section, we feel the need to admit one simplifying assumption we intend to make (and have implicitly made with our example). Since we care primarily about the lattice structure of our matroid, we assume all of our matroids are *simple*.

For the rest of us, the non-serious, a brief explanation. A matroid is simple if it does not have any *loops*, elements in the ground set that have rank 0, or *parallel edges*, sets of elements that share identical independence relations. If this feels overly restrictive, worry not, for Oxley[Oxl11, p. 49] comes to our rescue.

Proposition 2.4 (Simplification Preserves Lattice Structure)**.** *For any matroid $\mathcal{M}$, there exists a unique, up to labeling, matroid* $\mathrm{si}(\mathcal{M})$*, called the* simplification of $\mathcal{M}$ *such that*

i. $\mathrm{si}(\mathcal{M})$ *is simple,*

ii. if $\mathcal{L}$ is the lattice of flats of $\mathcal{M}$ and $\mathcal{L}'$ is the lattice of flats of $\mathrm{si}(\mathcal{M})$*, then*

$$\mathcal{L} \cong \mathcal{L}'.$$

If we care mostly about the lattice of matroids, then we can take any matroid and find a simple matroid with an identical lattice structure. We'll see the main practical benefit of working with simple matroids in the next section. However, we also get convenience, we don't have to keep track of unnecessary letters, and aesthetics, the lattice diagrams look much nicer, as a bonus. If we take our matroid to be simple, then our lattice structure has the following properties.

Proposition 2.5 (Properties of the Lattice of Simple Matroids)**.** *Let $\mathcal{M} = (E, \mathcal{L})$ be a simple matroid. Then*

i. the empty set is the minimal, rank 0, element of $\mathcal{L}$,

ii. for every $e \in E$, there is unique rank 1 flat, F_e, such that $F_e = e$,

iii. for any flat $F \in \mathcal{L}$, if $e \in F$, then $F_e \subseteq F$,

iv. we can write any flat $F \in \mathcal{L}$ as a disjoint union of rank 1 flats; $F = \biguplus_{e \in F} F_e$.

If this seems like a lot, the big takeaway is that this promises that the very bottom of our lattice will always be the empty set, and that the rank 1 flats correspond to the elements of the ground set. For those coming in with lattice knowledge, the second two properties mean the lattice of a simple matroid is *atomic*. We can verify these properties in our example, M, which is a simple matroid.

That admission of simplification done, we have now learned everything we need about the construction of matroids. It's time to learn about some polynomials.

2.4 The Characteristic Polynomial

The conjecture by Heron, Rota, and Welsh, that we promise we are getting to, has to deal with the characteristic polynomial of a matriod. This is some polynomial we can cook up using the structure of a matroid, which is fair enough. But when presented on its own, it feels, at least to us, that it comes out of nowhere. Why anyone would make up this polynomial or why we'd start conjecturing about it is not at all clear.

So first, a little history back in the realm of graphs.

2.4.1 Coloring Graphs and the Chromatic Polynomial

Let us play another game. This time, pick a graph, G, like the one pictured below.

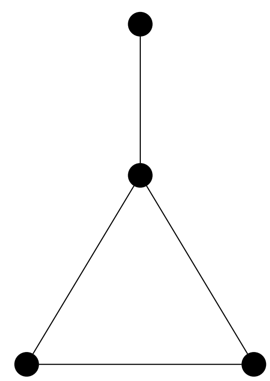


Figure 2.8: An example graph, G

Let's say we have three colors, and we want to color the vertices of the graph so that no two connected vertices have the same color. Such an arrangement of colors would be called a 3-coloring of G. It's not too hard to come up with some colors that work.

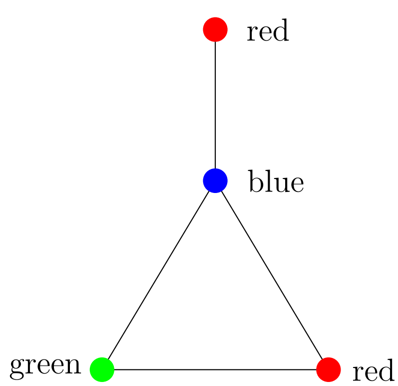


Figure 2.9: A 3-coloring of G

But now, suppose we wanted to know how many unique ways we could use those three colors to color the graph. This isn't too bad. We could just get out our markers and start coloring lots of graphs. Honestly, it sounds relaxing.

But now let's suppose we want to know how many ways we can use 1000 colors to color our little graph, or 10,000, or a billion. Since our set of markers only has 12 distinct colors, we will have to turn to math to solve this one.

The strategy is not too complicated, just pick a vertex and say how many colors we have to choose from, then find a connected vertex that hasn't been assigned a color yet, and say how many colors it is allowed to choose from. Repeat until we're out of vertices to label. Instead of picking a specific number, let's say we have n colors to choose from.

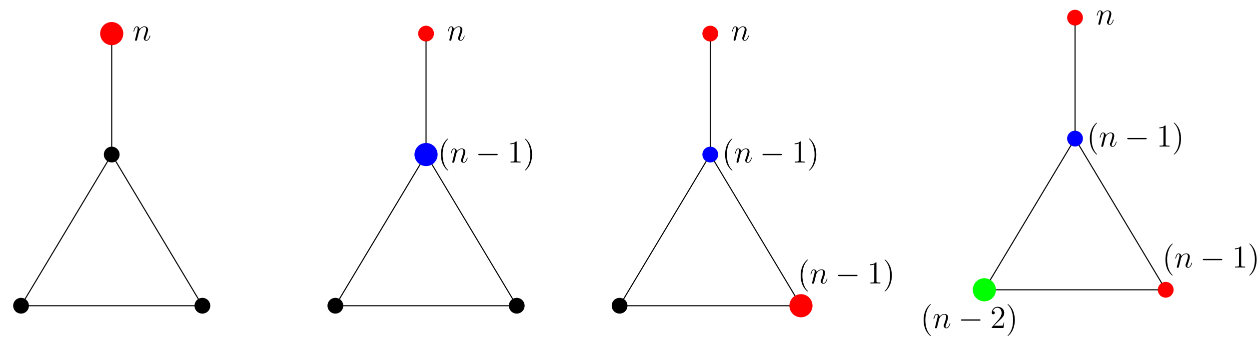


Figure 2.10: The process of figuring out the number of n-colorings of G; the choice of starting vertex doesn't matter, though that's not necessarily obvious

We then just have to multiply the number of possibilities together. For G, if we have n colors to choose from, there are $n(n-1)^2(n-2)$ different ways to arrange those colors on the graph. We've just discovered the *chromatic polynomial* of G. But there's nothing particularly special about our choice of graph, we will get something like this for any graph we come up with.

Proposition 2.6 (Chromatic Polynomial of a Graph). *Let $\chi_G(n)$ be the number of n-colorings of graph G. Then the map*

$$z \mapsto \chi_G(z)$$

is a polynomial with integer coefficients, known as the chromatic polynomial *of G.*

For our purposes we will always expand our polynomials, so for our worked example above we have

$$\chi_G(z) = z^4 - 4z^3 + 5z^2 - 2z.$$

Early work in chromatic polynomials was done by none other than our good friend Whitney [Whi32], and expanded on by the mathematician Tutte in his development of what we now call Tutte polynomials [Tuted].

2.4.2 The Characteristic Polynomial of a Matroid

It was following in this work on chromatic polynomials that Gian-Carlo Rota, who you may recognize as usually sandwiched between Heron and Welsh, extended this concept to matroids [Rot64]. To do this, Rota extended something called the *Möbius function* to lattices (technically any locally finite poset), which for matroids means it uses the lattice structure of the flats. We present an equivalent definition that's easier to state, but with the downside that the relationship to the lattice is obfuscated.

Definition 2.14 (Characteristic Polynomial)**.** Let $\mathcal{M} = (E, \mathcal{L})$ be a matroid. Then the *characteristic polynomial* of $\mathcal{M}$ is given by

$$\chi_{\mathcal{M}}(z) = \sum_{X \subseteq E} (-1)^{|X|} z^{\mathrm{rk}(\mathcal{M}) - \mathrm{rk}(X)}.$$

You may notice that each term of the polynomial will have a power of z between 0 and $\mathrm{rk}(\mathcal{M})$. The Heron-Rota-Welsh conjecture is about the coefficients of this polynomial, specifically once we collect the terms.

Definition 2.15 (Whitney Numbers of the First Kind)**.** Let $\mathcal{M}$ be a matroid with characteristic polynomial

$$\begin{aligned}\chi_{\mathcal{M}}(z) &= \sum_{X \subseteq E} (-1)^{|X|} z^{\mathrm{rk}(\mathcal{M}) - \mathrm{rk}(X)} \\ &= \sum_{k=0}^{\mathrm{rk}(\mathcal{M})} (-1)^k w_k z^{\mathrm{rk}(\mathcal{M}) - k}.\end{aligned}$$

The unsigned portion of the coefficients $w_0, w_1, \ldots, w_{\mathrm{rk}(\mathcal{M})}$ are the *Whitney numbers of the first kind.*

We will return to these numbers very soon. Before that, a few interesting facts about the characteristic polynomial.

Proposition 2.7. *Let $\mathcal{M}$ be a matroid with a loop; that is some element $e \in E$ such that* $\mathrm{rk}(e) = 0$*. Then*

$$\chi_{\mathcal{M}}(z) = 0.$$

Getting to ignore these trivial characteristic polynomials is the concrete benefit for assuming we only deal in simple matroids. A loop makes the characteristic polynomial easy to calculate, just not at all interesting

Finally, we want to wrap up the graph connection. Since any graph can be represented by a matroid, and the characteristic polynomial is in some sense inspired by the chromatic polynomial, it would be natural to ask if there is a relation between them. And there is, in fact, a very nice one.

Proposition 2.8. *Let G be a graph and $\mathcal{M}(G)$ be the matroid that comes from G. Then*

$$\chi_G(z) = z^c \chi_{\mathcal{M}(G)}(z),$$

where c is the number of connected components of G.

For those who want more on the connections between these values, and how they relate to the more general Tutte polynomial, we found the overview given by Ardila [Ard22] to be a great help.

2.5 The Heron-Rota-Welsh Conjecture

We now have all the knowledge of matroids necessary to state the Heron-Rota-Welsh conjecture. Developed and formalized by Heron [Her72], Rota [Rot70], and Welsh [Wel76], this was a conjecture about the coefficients of the characteristic polynomial of matroids. We say "was" because, as noted in the introduction, this has proven by Adiprasito, Huh, and Katz [AHK18]. We're going to keep calling it a conjecture though.

First, a few definitions necessary to carefully state the conjecture.

Definition 2.16 (Unimodal). A sequence of numbers $x_0, x_1, \dots, x_k$ is called *unimodal* if there exists an index i such that

$$x_0 \leq x_1 \leq \cdots \leq x_i \geq \cdots \geq x_{k-1} \cdots x_k.$$

The values of a unimodal sequence get larger until a certain point, and after they start to decrease. Such a sequence is also known as *concave*, since the average of any two non-consecutive points in the sequence will be less than a point in between them; i.e. for a sequence $x_0, x_1, \dots, x_k$ and $i < j < k$,

$$2x_j \geq x_i + x_k.$$

We can define an even stronger condition.

Definition 2.17 (Log-Concavity). A sequence of numbers $x_0, x_1, \dots, x_k$ is called *logarithmically concave*, or log-concave, if

$$x_i^2 \geq x_{i-1}x_{i+1}$$

for $0 < i < n$.

When all x_i are positive, log-concavity implies the sequence is also unimodal. This is the last piece of the puzzle.

Theorem 2.9 (Heron-Rota-Welsh Conjecture). *Assume we have a matroid $\mathcal{M} = (E, \mathcal{L})$ of rank $r+1$. If*

$$\chi_{\mathcal{M}}(z) = \sum_{k=0}^{\mathrm{rk}(\mathcal{M})} (-1)^k w_k z^{\mathrm{rk}(\mathcal{M})-k}$$

is the characteristic polynomial of $\mathcal{M}$, where $w_0, w_1, \dots, w_r$ are the Whitney numbers of the first kind, then

$$w_i^2 \geq w_{i-1}w_{i+1}$$

for $0 < i < \mathrm{rk}(M)$.

That is, the absolute values of the coefficients of the characteristic polynomial of the matroid $\mathcal{M}$ are log-concave.

Since we want to show something about the characteristic polynomials of the matroid, we need a way to study it. To do so, we are going to find the characteristic polynomial in some unexpected places, and then leverage properties of those other settings.

Chapter 3

Chow Rings

It is time now to delve into the world of algebra, developing the notion of a Chow ring of a matroid. The primary goal of this section will be to establish the link between the Chow ring and the characteristic polynomial. This will form the first segment of our bridge from combinatorics to geometry.

This section will take some algebraic knowledge for granted; we're not going to define a ring, for example. The main takeaway will be in the last section and should be enough to move on to the next chapter. However, even for those with some basic algebra we don't expect Chow rings to be a familiar object, so our first order of business is to define them.

3.1 What is a Chow Ring?

Broadly, Chow rings are a tool from algebraic geometry for studying the intersections of algebraic varieties. Chow groups of an algebraic variety are equivalence classes of algebraic cycles of its subvarieties, graded by their codimension. They were named after Wei-Liang Chow, who formalized these cycles in [Cho56]. Under certain conditions, a product structure can be induced on these groups to give a ring, which encodes additional information about the intersections of the subvarieties.

For those who don't already have some idea of what a Chow ring is, the above probably invites more questions than it clarifies. Happily, as hinted by the fact that the title of this chapter is not "A Brief Introduction to Algebraic Geometry and Intersection Theory", we will not have to tackle the full theory to get a result here. While there are algebraic varieties hiding in the wings, we will see that the combinatorial data of a matroid allow us to define its corresponding Chow ring quite directly.

3.1.1 The Chow Ring of a Matroid

For our purposes, we will define the Chow ring to be this "short-cut" construction. That this construction corresponds to the idea of the Chow ring presented above comes from the work started by De Cocini and Procesi [DP95] and generalized to the form we will use by Feichtner and Yuzvinsky [FY04].

Definition 3.1 (The Chow Ring of a Matroid)**.** Let $\mathcal{M} = (E, \mathcal{L})$ be a matroid. Associate a polynomial ring with $\mathcal{M}$ given by

$$P_{\mathcal{M}} = \mathbb{R}[x_F \mid F \in \mathcal{L}^*],$$

and let

$$I_{\mathcal{M}} = \langle x_{F_1}x_{F_2} \mid F_1 \not\subseteq F_2 \,\text{and}\, F_2 \not\subseteq F_1 \rangle \,,$$
$$J_{\mathcal{M}} = \left\langle \sum_{e_1 \in F} x_F - \sum_{e_2 \in F} x_F \;\middle|\; e_1, e_2 \in E \right\rangle$$

be ideals of $P_{\mathcal{M}}$.

The *Chow ring of* $\mathcal{M}$ is given by the quotient

$$A^\bullet(\mathcal{M}) = \frac{P_{\mathcal{M}}}{I_{\mathcal{M}} + J_{\mathcal{M}}}.$$

By way of some intuition building, the idea here is to create a polynomial ring with variables corresponding to the proper flats of our matroid, then encode the combinatorial relations of the matroid using a quotient. We could think of the ideal $I_{\mathcal{M}}$ as telling us that any monomial involving flats that don't form a flag are removed. The ideal $J_{\mathcal{M}}$ has a less obvious intuition, but the linear forms that generate it give us useful relations that we will use later. As always, working a small example will help.

3.1.1.1 A Small Chow Ring Example

Recall our ongoing example matroid $\mathcal{M}$, whose ground set is $E = \{a, b, c, d\}$ and with proper flats

$$\mathcal{L}^* = \{a, b, c, d, abd, ac, bc, cd\}.$$

Then we have the polynomial ring

$$P_{\mathcal{M}} = \mathbb{R}[x_a, x_b, x_c, x_d, x_{abd}, x_{ac}, x_{bc}, x_{cd}];$$

i.e. a real polynomial ring in 8 variables. Elements of the ideal $I_{\mathcal{M}}$ will be any multiple of a monomial containing variables corresponding to non-comparable flats, such as

$$x_a x_d \in I_{\mathcal{M}} \quad \text{and} \quad x_c x_{abd} \in I_{\mathcal{M}}.$$

The ideal $J_{\mathcal{M}}$ in turn is generated from differences of sums of all variables that contain a particular ground element. For example,

$$\begin{aligned}\sum_{a \in F} x_F - \sum_{c \in F} x_F &= x_a + x_{abd} + x_{ac} - x_c - x_{ac} - x_{cd} \\ &= x_a + x_{abd} - x_c - x_{cd} \in J_{\mathcal{M}}.\end{aligned}$$

Now, elements in our Chow ring $A^\bullet(\mathcal{M})$ are equivalence classes, as expected for a quotient ring. We see that $J_{\mathcal{M}}$ gives us relations like

$$[x_a + x_{abd}] = [x_c + x_{cd}],$$

or, equivalently,

$$[x_a] = [x_c + x_{cd} - x_{abd}].$$

In continuing to strive for succinct notation, we will drop the square brackets on elements of the Chow ring going forward.

3.2 The Degree Map

Now that we have an idea of what the Chow ring is, we can introduce the next key idea, the degree map. However, to get to the degree map we will need the property that Chow rings are graded.

3.2.1 This Will Be Graded

Our goal here is to show that the Chow ring is a graded ring. Those that feel that this property is obvious can skip to the next section; for everyone else we'll provide a quick summary.

Definition 3.2 (Graded Ring)**.** A ring R is *graded* if the underlying additive group of R can be decomposed into a direct sum

$$R = \bigoplus_{i=0}^{\infty} R_i$$

where each R_i is an abelian group such that $R_i R_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{Z}_{\geq 0}$. Elements of R_i are called *homogeneous of degree* i.

From this definition we can see that any polynomial ring, P is a graded ring

$$P = \bigoplus_{i=0}^{\infty} P_i$$

where each P_i is, naturally, all homogeneous polynomials of degree i.

Additionally, we can define a specific kind of ideal for graded rings, homogeneous ideals.

Definition 3.3 (Homogeneous Ideals)**.** Let R be a graded ring. An ideal I of R is *homogeneous* if

$$I = \langle a_1, a_2, \dots \rangle$$

and each a_i is homogeneous, i.e., $a_i \in R_k$ for some $k \in \mathbb{Z}_{\geq 0}$.

Looking back at our definition, we see that the ideal $I_{\mathcal{M}}$ is generated by quadratic monomials, while the generators of $J_{\mathcal{M}}$ are all linear. Thus, $I_{\mathcal{M}}$ and $J_{\mathcal{M}}$ are homogeneous ideals.

Definitions in place, we can state a small proposition.

Proposition 3.1. *Let R be a graded ring and I a homogeneous ideal. Then the quotient R/I is itself a graded ring. Specifically,*

$$R/I = \bigoplus_{n=0}^{\infty} R_n/I_n$$

where $I_n = I \cap R_n$.

We would say a proof for this could be found in any algebra textbook, but it appears to always be left as an exercise for the reader; a tradition we see no reason to break.

Since the Chow ring is the quotient of a polynomial ring and homogeneous ideals, we have that $A^\bullet(\mathcal{M})$ is a graded ring. Even better, from proposition 5.5 in [AHK18], if $rk(\mathcal{M}) = r+1$ then

$$A^\bullet = \bigoplus_{i=0}^{r} A^i(\mathcal{M}),$$

as $A^k = \{0\}$ for all $k > r$. All the components of $A^\bullet(\mathcal{M})$ correspond to ranks of flats of $\mathcal{M}$ and are empty otherwise.

3.2.2 The Degree Map

We needed that the Chow ring is a graded ring as the degree map is defined only on the top degree components of the ring.

Definition 3.4 (Degree Map)**.** Let $\mathcal{M}$ be a matroid of rank $r+1$. The *degree map* of $\mathcal{M}$ is the linear map

$$\deg: \ A^r(\mathcal{M}) \to \mathbb{Z}$$

such that for any complete flag $\mathscr{F} \subseteq \mathcal{L}$ of $\mathcal{M}$,

$$\deg\left(\prod_{F \in \mathscr{F}} x_F\right) = 1.$$

At first glance, it is not obvious that such a map must exist or that if it does that it would be unique. However, Adiprasito, Huh, and Katz showed in [AHK18] that this map exists for every matroid and is uniquely characterized by this definition.

3.3 Relationship with the Characteristic Polynomial

Right away we are going to have to confess that all of this Chow ring business doesn't actually relate to the characteristic polynomial directly. This section title is a lie. Instead, we have to introduce the actual target of our machinations, the reduced characteristic polynomial. The definition is quite straightforward, but requires a fact about the characteristic polynomial. While this is usually just stated as fact, we wish to take a small tangent for those interested.

3.3.1 Digression: The Möbius Function and the Characteristic Polynomial

Recall we mentioned that Rota defined the characteristic polynomial using the Möbius function, but avoided it for ease of understanding. However, using this alternative definition makes our work in this section quite straightforward. Forgive us for trading clarity for elegance.

Definition 3.5 (Möbius Function)**.** Let $\mathcal{M}$ be a matroid and $\mathcal{L}$ be its lattice of flats. Then the *Möbius function* on $\mathcal{M}$ is the map

$$\mu_{\mathcal{M}} : \mathcal{L} \times \mathcal{L} \to \mathbb{Z}$$

defined recursively for any $F, F' \in \mathcal{L}$ by

$$\mu_{\mathcal{M}}(F, F') = \begin{cases} 1 & \text{if } F = F' \\ -\displaystyle\sum_{F \subseteq X \subsetneq F'} \mu_{\mathcal{M}}(F, X) & \text{when } F \subsetneq F' \\ 0 & \text{otherwise.} \end{cases}$$

Truly, of all the definitions we've presented so far, the Möbius function most deserves some serious contemplation over a cup of coffee. We won't go that far afield here, but we can now use it in an equivalent definition of the characteristic polynomial.

Definition 3.6 (Characteristic Polynomial – via Möbius functions)**.** Let $\mathcal{M} = (E, \mathcal{L})$ be a matroid. Then we may write the characteristic polynomial of $\mathcal{M}$ as

$$\chi_{\mathcal{M}}(z) = \sum_{F \in \mathcal{L}} \mu_{\mathcal{M}}(\emptyset, F) z^{\mathrm{rk}(\mathcal{M}) - \mathrm{rk}(F)}.$$

This definition does make the importance of the lattice structure of the flats much more clear. With this we can present, and prove, our fact about the characteristic polynomial.

Proposition 3.2. *Let $\mathcal{M} = (E, \mathcal{L})$ be any matroid, then*

$$\chi_{\mathcal{M}}(1) = 0.$$

Proof. This follows directly from the definition of the Möbius function, and a little reindexing. We see that

$$\begin{aligned} \chi_{\mathcal{M}}(1) &= \sum_{F \in \mathcal{L}} \mu_{\mathcal{M}}(\emptyset, F) 1^{\mathrm{rk}(\mathcal{M}) - \mathrm{rk}(F)} \\ &= \sum_{F \in \mathcal{L}} \mu_{\mathcal{M}}(\emptyset, F) \\ &= \mu_{\mathcal{M}}(\emptyset, E) + \sum_{\emptyset \subseteq F \subsetneq E} \mu_{\mathcal{M}}(\emptyset, F) \\ &= -\sum_{\emptyset \subseteq F \subsetneq E} \mu_{\mathcal{M}}(\emptyset, F) + \sum_{\emptyset \subseteq F \subsetneq E} \mu_{\mathcal{M}}(\emptyset, F) \\ &= 0, \end{aligned}$$

with the second to last equality coming from the application of the definition. □

From this we have the following corollary and main goal of this section.

Corollary 3.3. *For any matroid $\mathcal{M}$, the characteristic polynomial $\chi_{\mathcal{M}}(z)$ has a factor of $(z - 1)$.*

Digression aside, we can get back to defining the reduced characteristic polynomial.

3.3.2 The Reduced Characteristic Polynomial

As we said, the reduced characteristic polynomial is easy to define in terms of the characteristic polynomial.

Definition 3.7 (Reduced Characteristic Polynomial)**.** Let $\mathcal{M}$ be a matroid of rank $r+1$. The *reduced characteristic polynomial* of $\mathcal{M}$ is

$$\overline{\chi}_{\mathcal{M}}(z) = \frac{\chi_{\mathcal{M}}(z)}{(z-1)}.$$

Further, we may collect the powers of z and write the reduced characteristic polynomial as

$$\overline{\chi}_{\mathcal{M}}(z) = \sum_{k=0}^{r}(-1)^k \overline{w}_k z^{r-k},$$

where the $\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_r$ are the reduced coefficients of $\overline{\chi}_{\mathcal{M}}(z)$.

That this is well-defined for any matroid follows from Corollary 3.3. The following proposition tells us why these coefficients are going to be important.

Lemma 3.4. *Let $\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_r$ be the reduced coefficients of the reduced characteristic polynomial $\overline{\chi}_{\mathcal{M}}(z)$ of some matroid $\mathcal{M}$. If*

$$\{\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_r\}$$

is a log-concave sequence, then the Whitney numbers of the first kind of $\mathcal{M}$,

$$w_0, w_1, \ldots, w_{r+1},$$

also form a log-concave sequence.

Proof. Let $\mathcal{M}$ be a matroid of rank $r+1$ and

$$\{\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_r\}$$

be the reduced coefficients of the reduced characteristic polynomial $\overline{\chi}_{\mathcal{M}}(z)$. Assume that $\{\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_r\}$ is a log-concave sequence.

To recover the characteristic polynomial of $\mathcal{M}$ we can multiply our polynomial by a factor of $(z-1)$. This gives us

$$\begin{aligned}
(z-1)\sum_{k=0}^{r}(-1)^k \overline{w}_k z^{r-k} &= z\sum_{k=0}^{r}(-1)^k \overline{w}_k z^{r-k} - \sum_{k=0}^{r}(-1)^k \overline{w}_k z^{r-k} \\
&= \sum_{k=0}^{r}(-1)^k \overline{w}_k z^{r-k+1} + \sum_{k=0}^{r}(-1)^{k+1} \overline{w}_k z^{r-k} \\
&= \sum_{k=0}^{r}(-1)^k \overline{w}_k z^{r-k+1} + \sum_{k=1}^{r+1}(-1)^k \overline{w}_{k-1} z^{r-k+1} \\
&= \overline{w}_0 z^{r+1} + \sum_{k=1}^{r}(-1)^k (\overline{w}_k + \overline{w}_{k-1}) z^{r-k} + (-1)^{r+1}\overline{w}_r,
\end{aligned}$$

where the last two equalies are just a result of some clever reindexing. For convenience, we'll define $\overline{w}_{-1} = \overline{w}_{r+1} = 0$. We note that $\{\overline{w}_{-1}, \overline{w}_0, \overline{w}_1, \ldots, \overline{w}_r, \overline{w}_{r+1}\}$ remains a log-concave sequence. This lets us write

$$\overline{w}_0 z^{r+1} + \sum_{k=1}^{r}(-1)^k(\overline{w}_k + \overline{w}_{k-1})z^{r-k+1} + (-1)^{r+1}\overline{w}_r = \sum_{k=0}^{r+1}(-1)^k(\overline{w}_k + \overline{w}_{k-1})z^{r-k}.$$

Recalling the definition of the characteristic polynomial and its coefficients, we see that have shown that

$$w_k = \overline{w}_k + \overline{w}_{k-1},$$

for $0 \leq k \leq r+1$.

Now, we wish to show the log-concavity of the coefficients, so consider the expression $w_k^2 - w_{k-1}w_{k+1}$. If we can show this must be non-negative for any $0 < k < r+1$, we'll have that the sequence is log-concave. First let's manipulate the expression a bit to get

$$\begin{aligned}
w_k^2 - w_{k-1}w_{k+1} &= (\overline{w}_k + \overline{w}_{k-1})^2 - (\overline{w}_{k-1} + \overline{w}_{k-2})(\overline{w}_{k+1} + \overline{w}_k) \\
&= \left(\overline{w}_k^2 + 2\overline{w}_k\overline{w}_{k-1} + \overline{w}_{k-1}^2\right) - \left(\overline{w}_{k-1}\overline{w}_{k+1} + \overline{w}_{k-1}\overline{w}_k + \overline{w}_{k+1}\overline{w}_{k-2} + \overline{w}_k\overline{w}_{k-2}\right) \\
&= \left(\overline{w}_k^2 - \overline{w}_{k-1}\overline{w}_{k+1}\right) + \left(\overline{w}_{k-1}^2 - \overline{w}_{k-2}\overline{w}_k\right) + \left(\overline{w}_k\overline{w}_{k-1} - \overline{w}_k\overline{w}_{k-1}\right) + \left(\overline{w}_k\overline{w}_{k-1} - \overline{w}_{k+1}\overline{w}_{k-2}\right) \\
&= \left(\overline{w}_k^2 - \overline{w}_{k-1}\overline{w}_{k+1}\right) + \left(\overline{w}_{k-1}^2 - \overline{w}_{k-2}\overline{w}_k\right) + \left(\overline{w}_k\overline{w}_{k-1} - \overline{w}_{k+1}\overline{w}_{k-2}\right).
\end{aligned}$$

We wish to conclude that this is non-negative. Immediately we have that

$$\begin{aligned}
\overline{w}_k^2 - \overline{w}_{k-1}\overline{w}_{k+1} &\geq 0, \\
\overline{w}_{k-1}^2 - \overline{w}_{k-2}\overline{w}_k &\geq 0
\end{aligned}$$

from the log-concavity of the reduced coefficients. All that's left then is to show that the term

$$\overline{w}_k\overline{w}_{k-1} - \overline{w}_{k+1}\overline{w}_{k-2} \geq 0.$$

If $k = 1$ or $k = r$ then this is immediately true, as $\overline{w}_k\overline{w}_{k-1}$ is certainly non-negative. We'll asuume otherwise, which means $\overline{w}_{k-2}, \overline{w}_{k-1}, \overline{w}_k$, and $\overline{w}_{k+1}$ are all non-negative. Now, consider that it must be that

$$\frac{\overline{w}_{k-1}}{\overline{w}_{k-2}} \geq \frac{\overline{w}_k}{\overline{w}_{k-1}} \geq \frac{\overline{w}_{k+1}}{\overline{w}_k}.$$

This follows directly from the definition of log-concavity and that they're all non-zero. This means

$$\frac{k-1}{\overline{w}_{k-2}} \geq \frac{\overline{w}_{k+1}}{\overline{w}_k}$$

and so

$$\overline{w}_k\overline{w}_{k-1} \geq \overline{w}_{k+1}\overline{w}_{k-2}$$

giving us

$$\overline{w}_k\overline{w}_{k-1} - \overline{w}_{k+1}\overline{w}_{k-2} \geq 0$$

as desired.

Having shown that $w_k^2 - w_{k-1}w_{k+1}$ for all $1 \leq k \leq r$ must be non-negative, we conclude that the sequence $\{w_k\}_{k=0}^{r+1}$ is log-concave.

$\square$

It is actually this sequence, $\{\overline{w}_k\}_{k=0}^r$, that Adiprasito, Huh, and Katz spent most of [AHK18] proving is log-concave. This too is our strategy, so these reduced coefficients really are key players in this story. But we have still yet to link the characteristic polynomial, reduced or not, to the Chow ring.

3.3.3 The Divisors α and β

Right off the bat, let's learn a little jargon.

Definition 3.8 (Divisor)**.** A *divisor* of a Chow ring $A^\bullet(\mathcal{M})$ is any linear term, i.e., an element of $A^1(\mathcal{M})$.

We are working towards introducing a proposition from [AHK18] that links certain divisors to the reduced coefficients, via the degree map. We'll start by introducing these special divisors.

Definition 3.9 (α and β)**.** Let $\mathcal{M}$ be a matroid with ground set E. For every element $e \in E$ we define the divisors

$$\alpha_e = \sum_{e \in F} x_F \quad \text{and} \quad \beta_e = \sum_{e \notin F} x_F.$$

Further, as elements of $A^\bullet(\mathcal{M})$, $[\alpha_e] = [\alpha_{e'}]$ and $[\beta_e] = [\beta_{e'}]$ for any $e, e' \in E$. Going forward we may refer simply to α and β, since the class is independent of a choice of ground element.

With this, we may state the key takeaway from this chapter. Proposition 9.5 of [AHK18] provides the link between α and β and the reduced coefficients.

Proposition 3.5. *Given any matroid $\mathcal{M}$ with reduced coefficients $\overline{w}_0, \ldots, \overline{w}_r$, we have the relationship*

$$\overline{w}_k = \deg(\alpha^{r-k}\beta^k).$$

for all $0 \leq k \leq r$.

Just to confirm our understanding of the degree map, recall it is defined on elements of $A^r(\mathcal{M})$. Since $\alpha, \beta \in A^1(\mathcal{M})$, we'll have $\alpha^{r-k}\beta^k \in A^r(\mathcal{M})$ and so this makes sense as input to the degree map.

This proposition means that if we can prove

$$\deg(\alpha^{r-k}\beta^k)^2 \geq \deg(\alpha^{r-k-1}\beta^{k-1}) \deg(\alpha^{r-k+1}\beta^{k+1})$$

for $0 < k < r$ we'll have shown that the reduced coefficients are log-concave and thus so too are the original coefficients. Here now is where we diverge from the strategy put forth in [AHK18]. They go on to show this relationship in a very algebraic manner, proving that the Chow ring of a matroid has many desirable properties that eventually yield the desired result. We, on the other hand, will now move into the world of geometry and find a different way to generate our log-concave sequence.

Chapter 4

Bergman Fans and their Normal Complexes

As we continue our tour of various branches of mathematics, we arrive at geometry. The primary goal of this chapter is to develop the final segment of our bridge connecting some geometric object back to the Chow ring, and then showing how we can generate log-concave sequences with these objects. To get there we will provide a quick primer on polyhedral geometry and a classic theorem of convex geometry that generates log-concave sequences. Then we'll introduce a geometric object associated to a matroid, the Bergman fan, and show how we can use them to make some new objects called a normal complexes.

4.1 A Little Polyhedral Geometry, as a Treat

Really, the basic building blocks we'll be using are not that weird. It's geometry, we're going to be using some sort of shapes living in some kind of space. We must admit, however, that we personally struggle visualizing the higher dimensional objects at play, and so must fall back on formalism.

This section is a short crash course on basic elements of polyhedral geometry. Our treatment of this topic will often parallel that in Ziegler's "Lectures on Polytopes" [Zie95], which we recommend for those who'd like a little more depth than presented here.

4.1.1 The Cone Zone

We are going to be using two fundamental kinds of convex shapes, polytopes and cones. As a reminder, a convex object is one where if you pick any two points in it, the line connecting those points never leaves the shape. We can, and will, state this formally.

Definition 4.1 (Convexity)**.** Let $K \subseteq \mathbb{R}^n$. We call K *convex* if for every $p, q \in K$, we have

$$[p, q] \subseteq K,$$

where $[p, q] = \{\lambda p + (1 - \lambda)q \mid 0 \leq \lambda \leq 1\}$ is the line segment between p and q.

Figure 4.1: Everyone's first pair of convex and non-convex shapes

While there are generally a few ways one could define polytope and cone, we will use a definition based on construction using some finite collection of points. In brief, a polytope is a *convex hull* of finitely many points and a cone is the *conic combination* of finitely many generating vectors. Let's make this formal.

Definition 4.2 (Polytope)**.** Let $P \subseteq \mathbb{R}^n$. We say P is a *polytope* if it is the convex hull of some finite set of points $x_1, x_2, \ldots, x_k$. That is to say P is a polytope if

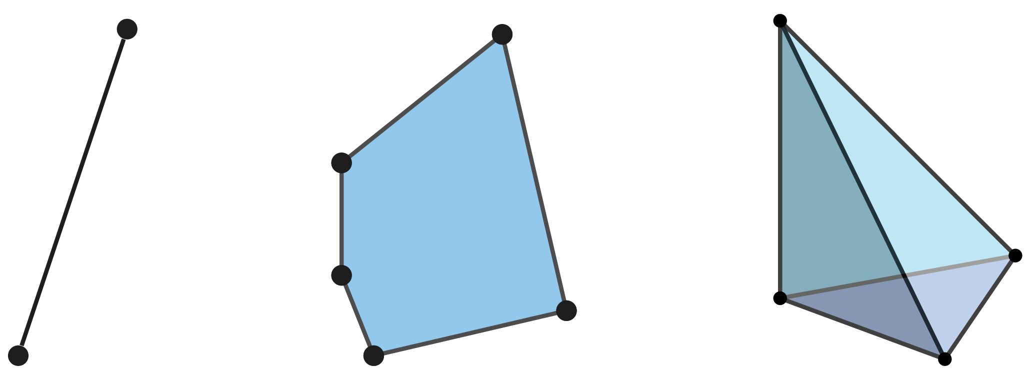
$$P = \operatorname{conv}(\{x_1, \ldots, x_k\}),$$

where

$$\operatorname{conv}(\{x_1, \ldots, x_k\}) = \left\{ \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k \;\middle|\; \lambda_i \geq 0, \sum_{i=0}^{k} \lambda_i = 1 \right\}$$

is the convex hull of $x_1, x_2, \ldots, x_k$.

An astute reader may notice that our shorthand for line segment above, $[x, y]$, is in fact just $\operatorname{conv}(\{x, y\})$. Towards some intuition, we may think of the convex hull as the smallest convex shape that contains all of its generating points. In two dimensions we like to think of this as stretching a rubber band around a bunch of points and letting it constrict around them.



A 1-dimensional polytope A 2-dimensional polytope A 3-dimensional polytope

A 1-dimensional polytope A 2-dimensional polytope

Figure 4.2: A sampling of polytopes; note that we've only highlighted the vertices, which are the minimal set of points that generate the polytope

We will use a similar definition for cones. They are built out of a finite collection of generating vectors.

Definition 4.3 (Cone)**.** Let $C \subseteq \mathbb{R}^n$. We call C a *cone* if it is the conic combination of finitely many vectors $x_1, x_2, \ldots, x_k$. We write this

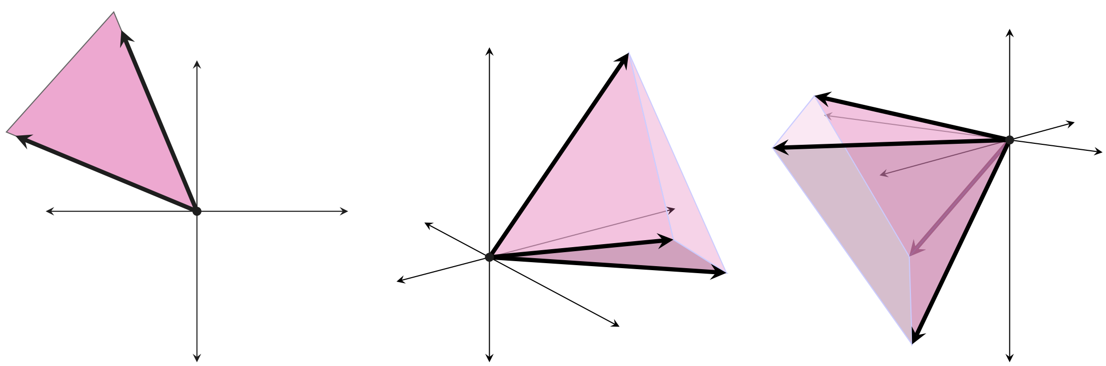
$$C = \operatorname{cone}(\{x_1, \ldots, x_k\}),$$

where

$$\operatorname{cone}(\{x_1, \ldots, x_k\}) = \{\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k \mid \lambda_i \geq 0\}$$

is the conic combinations of $x_1, x_2, \ldots, x_k$.

Unlike the more familiar notion of cones, these are not pointed cylinders.



A 2-dimensional cone   A 3-dimensional cone (generated by 3 vectors)   A 3-dimensional cone (generated by 4 vectors)

Figure 4.3: Some examples of cones

We notice that conic combinations are essentially the span of the generating vectors but taking only non-negative linear combinations. Indeed, one can quickly confirm that $\operatorname{cone}(\{x_1, \ldots, x_k\}) \subseteq \operatorname{span}(\{x_1, \ldots, x_k\})$.

4.1.2 Points vs. Vectors: An Affine Primer

We have been, and will be going forward, using the words “points” and “vectors” quite interchangeably. Is there a difference? Strictly speaking, yes there is. Points imply elements of an affine space, while vectors, naturally, are elements of a vector space. Affine spaces can be thought of as vector spaces where the 0-vector is “forgotten”, but are otherwise essentially the same collection of “stuff”. Mathematicians love to make multiple objects out of the same basic thing by giving (or losing) some extra structure.

This poses a slight problem since we want to refer to the dimension of our polytopes and cones (and already have been in figures), but a notion of dimension usually relies on a vector space. To settle a notion of dimension here, we first want to define what an affine span (also called an affine hull) is.

Definition 4.4 (Affine Span)**.** Let $S \subseteq \mathbb{R}^d$. The *affine span* of S is the set

$$\operatorname{aff}(S) = \left\{ \sum_{i=1}^{k} \lambda_k x_k \;\middle|\; k > 0,\, x_i \in S,\, \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

The affine span of some set will be something that looks like a linear subspace, but might not include the origin. In fact if $0 \in S$, then the affine span and our more traditional linear span coincide exactly. The insight here then is that we could always translate our affine span so that it includes the origin. Once we have a linear subspace we can just use the linear algebra definition of dimension.

Definition 4.5 (Affine Dimension)**.** Given any set $S \subseteq \mathbb{R}^d$, designate some element $x_0 \in S$. The *dimension* of S is

$$\dim(S) = \dim\bigl(\operatorname{aff}(S) - x_0\bigr),$$

where, on the right-hand side, dim is the standard notion of dimension of a subspace in linear algebra.

We're overloading our notation a bit, but we promise this mostly reduces cognitive load. This also goes to explain our switching between "point" or "vector". Since our cones are defined to always include 0, affine and linear terms mostly line up and we are mostly safe to just think in terms of vector spaces. Indeed, for cones we will always just write $\operatorname{span}(\mathcal{C})$ in lieu of $\operatorname{aff}(\mathcal{C})$.

Dimension will mostly align with intuition, but it's good to have the definitions at hand if ever in doubt. Early chapters of [Zie95] and [Grü03] both provide a good treatment of affine spaces.

4.1.3 The Minkowski Sum

Now that we have the basic shapes down we need be able to make new ones out of existing ones. The two general strategies here will be to combine them in to new ones and to break them down. We'll start with learning how we can add shapes together, using what we call the Minkowski sum.

Definition 4.6 (Minkowski Sum)**.** Let $P, Q \subseteq \mathbb{R}^n$. The *Minkowski sum* of P and Q is given by

$$P + Q = \{p + q \mid p \in P, q \in Q\}.$$

Sit with this definition for a few moments to confirm the Minkowski sum does have the nice properties of sums we normally expect. It is commutative, associative, and has an identity in $\{0\}$. An additional feature of the definition is that the empty set has the property that for any $P \subseteq \mathbb{R}^n$

$$P + \emptyset = \emptyset.$$

A way to think about the Minkowski sum is "smearing" one shape around the other. You pick some point in either shape, then drag it along the boarder of the other shape in the sum. This makes more sense with a picture.

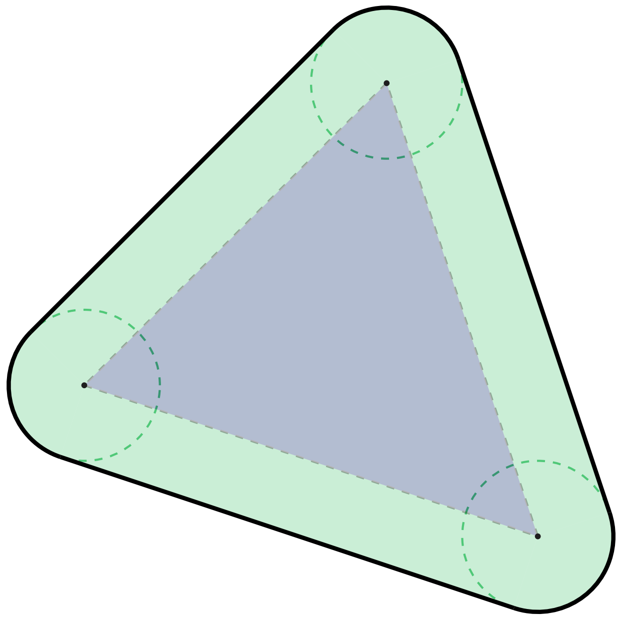


Figure 4.4: A visual representation of a Minkowski sum of a triangle and a circle

This is a helpful visual intuition, but it may take a moment to be convinced that, up to translation, the resulting shape doesn't depend on either the shape you choose to smear or the choice of point in that shape.

With the Minkowski sum, we can finally define a general polyhedron.

Definition 4.7 (Polyhedron)**.** Let $P \subseteq \mathbb{R}^n$. We call P a *polyhedron* if

$$P = \operatorname{conv}(\{x_1, \dots, x_k\}) + \operatorname{cone}(\{y_1, \dots, y_\ell\}),$$

for some finite sets $\{x_1, \dots, x_k\}, \{y_1, \dots, y_\ell\} \subseteq \mathbb{R}^n$.

A polyhedron is the result of the Minkowski sum of a polytope and a cone. Clearly every polytope and cone are polyhedra themselves, as $\operatorname{conv}(\{0\}) = \operatorname{cone}(\{0\}) = \{0\}$. Polyhedra are not necessarily bounded, which may seem a bit unusual to those who have seen the word in other contexts. Moreover, all bounded polyhedra are polytopes, which is not necessarily obvious, but useful to keep in mind. We will mostly be focused on either polytopes or cones at any one time, but having a more general object that includes both makes our definitions going forward cleaner.

4.1.4 About Faces

Given a polyhedron P, we also get a whole family of polyhedron, the faces of P. Let's first go back to simpler times. If we were to think of a cube, we would have faces of the cube as the 2 dimensional squares that make up the sides. We'd then call the line segments where any two of those squares meet *edges* and the points those edges meet *vertices*.

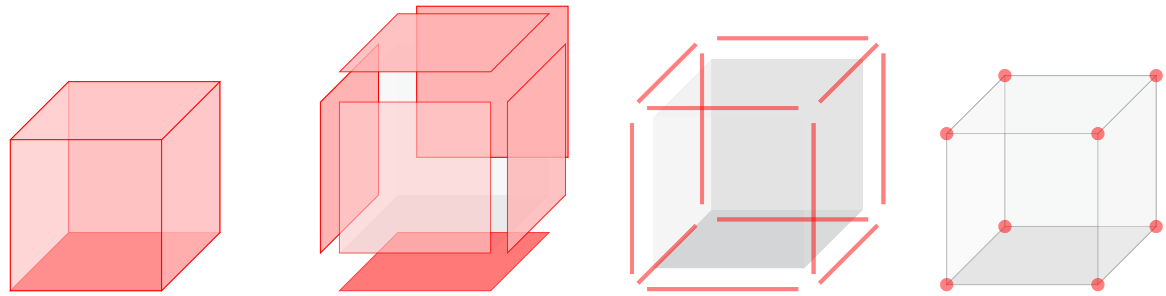


Figure 4.5: The faces of a cube; note that the full cube is a face of itself (as is the empty set, which we drew as accurately as possible)

If this sounds familiar then the intuition for our more general notion of the face of a polyhedron is not far. Back in our world of polyhedral geometry, we know that a cube is a polyhedron and that each of those squares, lines, and points are also themselves polyhedron. Informally, we use the term face to describe all these "sub-polyhedra" that make up the boundary of a polyhedron.

Definition 4.8 (Face)**.** Let $P \subseteq \mathbb{R}^n$ be a polyhedron and fix an inner product $* \in \mathrm{Inn}(\mathbb{R}^n)$. Recall that the hyperplane normal to a vector $x \in \mathbb{R}^n$ at distance $q \in \mathbb{R}$ is given by

$$\mathrm{H}_x(b) = \{v \in \mathbb{R}^n \mid v * x = b\}\,.$$

Additionally, any hyperplane defines a lower half-spaces given by

$$\mathrm{H}_x^-(b) = \{v \in \mathbb{R}^n \mid v * x \leq b\}\,,$$

respectively.

Then $F \subseteq P$ is a *face* of P if there exist x, b such that

$$F = P \cap \mathrm{H}_x(b)$$

such that P lies entirely in the lower half-space of $\mathrm{H}_x(b)$; i.e.,

$$P \subseteq \mathrm{H}_x^-(b).$$

Some notable quirks of this definition are that given any $x \in \mathbb{R}^n$, $F = \mathrm{H}_x(0) \cap P = P$ and so P is face of itself. Likewise, there exist infinitely many hyperplanes such that $\mathrm{H}_x(b) \cap P = \emptyset$, which means the empty set too is a face of any polyhedron P. As a note, we do still call the 0-dimensional faces vertices, while the generic term for a $(d-1)$-dimensional face of a d-dimensional polyhedron is a *facet*.

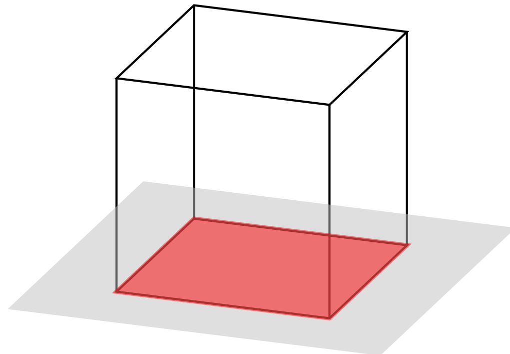


Figure 4.6: Example of specifying a facet of our cube with a plane

From our cube intuition exercise, there are some things about faces that we would hope to generalize to our broader notion of faces. We'll state these without proof, and again refer to the early chapters of [Zie95].

Proposition 4.1. *Given a polyhedron $P \subseteq R^n$, every face $F \subseteq P$ is a polyhedron. Better, if P is a polytope then F is a polytope, and if P is a cone then F is a cone.*

Proposition 4.2. *Let $P \subseteq \mathbb{R}^n$ be a polyhedron, and $F \subseteq P$ a face of P. From above, we know that F is a polyhedron. Then for any $F' \subseteq F$ a face of F, we have F' is also a face of P.*

Proposition 4.3. *Let $P \subseteq \mathbb{R}^n$ be a polyhedron and $F_1, F_2 \subseteq P$ be two faces of P. Then $F_1 \cap F_2$ is a face of P.*

It's worth remembering that the intersection of two faces can be empty, but $\emptyset \subseteq P$ is a face of any polyhedron P, so this causes no issues.

We will often write $F \preceq P$ to mean F is a face of P. Indeed, the relation "is a face of" induces a partial ordering on the set of all faces of P. Even stronger, using the propositions above it follows that the relation induces a lattice.

4.1.5 Polyhedral Complex

The last geometric structure we need as background consist of particular collections of polyhedra, known as polyhedral complexes.

Definition 4.9 (Polyhedral Complex)**.** A *polyhedral complex* $\mathcal{C}$ is a finite collection of polyhedra in $\mathbb{R}^n$ such that

1. the empty set is in $\mathcal{C}$,
2. for any polyhedron $P \in \mathcal{C}$, all faces of P are also in $\mathcal{C}$,
3. for any two polyhedra $P, Q \in \mathcal{C}$, the intersection $P \cap Q \in \mathcal{C}$.

We can think of polyhedral complexes as sets of polyhedra that intersect nicely. We won't be too concerned about complexes of general polyhedra, and instead focus on when our complexes are restricted to either all polytopes or all cones.

Definition 4.10 (Polytopal Complex)**.** A *polytopal complex* $\mathcal{C}$ is a polyhedral complex where every element $P \in \mathcal{C}$ is bounded, i.e., a polytope.

One might guess that next we'll define a conic complex or some such thing. In fact, a polyhedral complex of only cones is called a *fan*, we assume mostly to torment anyone trying to do a web-search for information on them.

Definition 4.11 (Fan)**.** A *fan* $\mathbf{\Sigma}$ is a polyhedral complex where every element $\sigma \in \mathbf{\Sigma}$ is a cone.

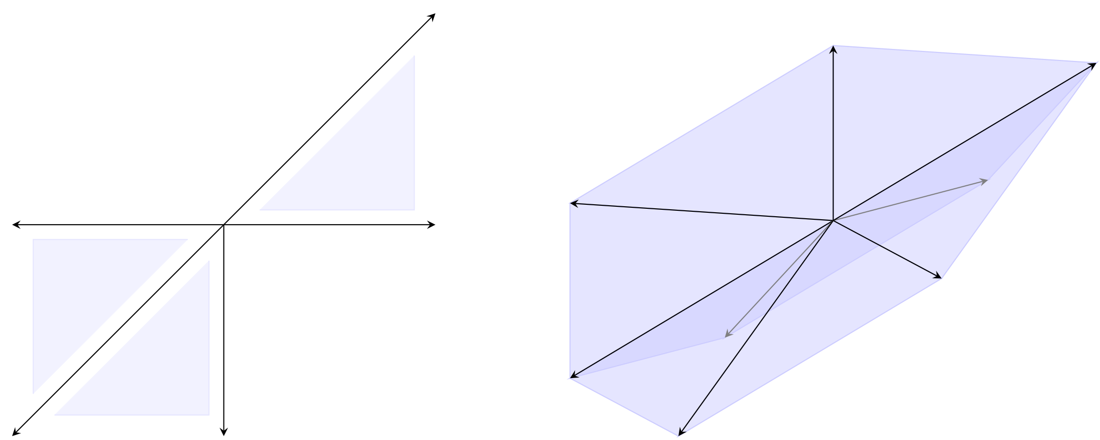


Figure 4.7: Examples of fans; even though they live in different spaces they are both composed only of fans of dimension 2 and less

From the title of this chapter, it will come as no surprise that fans are quite important. As such there's a few properties of fans that we want to introduce.

4.1.6 We're Big Fans (of These Properties of Fans)

We would like to introduce three properties of fans. These properties tend to make fans nice to work with as we will see. The first tells us something about the dimension of all the maximal cones in the fan.

Definition 4.12 (Pure)**.** A cone $\sigma \in \mathbf{\Sigma}$ is maximal if it is not the proper face of another cone in $\mathbf{\Sigma}$. A fan is *pure* if every maximal cone in $\mathbf{\Sigma}$ is the same dimension.

We say $\mathbf{\Sigma}$ is a d-fan when it is pure of dimension d.

Basically, a fan is pure if the largest cones in it are all of the same dimension. Both fans in figure 4.7 are pure of dimension 2. We can consider a counterexample as well.

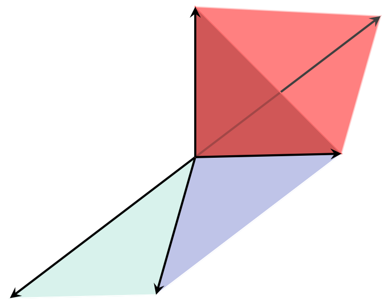


Figure 4.8: This fan is not pure, since it has maximal cones of both dimension 2 and 3

The next property relates to how many rays are necessary to generate each cone.

Definition 4.13 (Simplicial Fan)**.** A fan Σ is *simplicial* if every cone $\sigma \in \Sigma$ is simplicial. A cone $\sigma \in \Sigma$ is simplicial if

$$\dim(\sigma) = |\sigma(1)|,$$

where $\sigma(1)$ is the set of 1-dimensional faces of σ.

Alternatively, a cone is simplicial if its rays form a basis for the subspace that cone spans. In figure 4.3, the first two cones are simplicial while the third is not.

Next, we have something that is not specifically a property of fan but rather some convenient extra information.

Definition 4.14 (Marked Fan)**.** Let $\mathbf{\Sigma}$ be a fan. We say $\mathbf{\Sigma}$ is a *marked fan* if there is a chosen set

$$\{u_\rho \in \rho \mid \rho \in \mathbf{\Sigma}(1)\}$$

such that $\operatorname{cone}(u_\rho) = \rho$.

We may also say that $\mathbf{\Sigma}$ is marked with the points $\{u_\rho \in \rho \mid \rho \in \mathbf{\Sigma}(1)\}$. Since there will always exist some choice of points that mark a fan we may just say the fan is marked, and not worry about specific values. Otherwise, we'll provide the marking explicitly if necessary.

Given a simplicial d-fan, it can have yet another property, that of being tropical. Tropical fans come from the realm of tropical geometry, another deep topic we are going to avoid delving into. That tropical fans are in play at all relates back to lurking algebraic varieties that we continue to not fully address. We can, however, at least somewhat easily characterize tropical fans. They are fans that satisfy the weighted balancing condition.

Definition 4.15 (Tropical Fan)**.** Let $\mathbf{\Sigma}$ be a marked, simplicial d-fan. Given a weight function

$$\omega \,:\, \mathbf{\Sigma}(d) \to \mathbb{R}_{>0},$$

we say the pair $(\mathbf{\Sigma}, \omega)$ is a *tropical fan* if for every $\tau \in \mathbf{\Sigma}(d-1)$

$$\sum_{\substack{\sigma \in \mathbf{\Sigma}(d) \\ \tau \preceq \sigma}} \omega(\sigma) u_{\sigma \setminus \tau} \in \operatorname{span}(\tau),$$

where $\sigma \setminus \tau$ is shorthand for the single element in $\sigma(1) \setminus \tau(1)$.

We will go on to immediately abuse notation and say a fan $\mathbf{\Sigma}$ is itself tropical, as long there exists a weight function ω that satisfies the condition. The exact implications of this definition takes a moment to parse, but the main takeaway is that it imposes some relationship between the cones of the fan. If the weight function is particularly simple then we have another term.

Definition 4.16 (Balanced Fan)**.** A tropical fan $(\mathbf{\Sigma}, \omega)$ is *balanced* if the weight function ω is the constant function

$$\omega(\sigma) = 1$$

for all $\sigma \in \mathbf{\Sigma}(d)$.

When a fan is balanced we can omit references to the weight function ω, a very convenient aspect of having a balanced fan. We've now built up all the background on shapes we'll need.

4.2 The Geometer's Guide to Generating Log-Concave Sequences

Having developed our shapes, we are going to need to measure them somehow. Specifically, we want their volume. As with the rest of this chapter so far, we're going to take something that most people can intuitively grasp for 3-dimensional shapes and generalize the hell out of it. Not only do we need volume for arbitrary dimensional polyhedra, we need something called the mixed volume which gives us some sense of volume of multiple shapes. This work however will allow us to introduce a classic result of convex geometry that relates geometry to log-concave sequences.

From here, the last few background points can no longer be readily found in Ziegler, who as a topologist, we suspect without proof, cares little for things like volume. Instead, we swap out our Germans and recommend Rolf Schneider's "Convex Bodies" [Sch13] as a comprehensive reference.

4.2.1 Volume Functions

We again need to formalize something that most of us would take for granted. The notion of volume is intuitive enough for 3-dimensional shapes, we however need to generalize this to all dimensions. We will actually only need the volume of polytopes, so we restrict our notion of volume to just them.

Definition 4.17 (Volume Function)**.** A *volume function* is a map

$$\mathrm{Vol}_n : \{\text{polytopes in } \mathbb{R}^n\} \to \mathbb{R}_{\geq 0}$$

such that for any polytopes $P, Q \subseteq \mathbb{R}^n$, Vol_n

1. is non-negative: $\mathrm{Vol}_n(P) > 0$ when $\dim(P) = n$ and $\mathrm{Vol}_n(P) = 0$ when $\dim(P) < n$,
2. is translation invariant: $\mathrm{Vol}_n(P) = \mathrm{Vol}_n(P + v)$ for any $v \in \mathbb{R}^n$,
3. respects inclusion-exclusion: when $P \cup Q$ is a polytope,

$$\mathrm{Vol}_n(P \cup Q) = \mathrm{Vol}(P) + \mathrm{Vol}_n(Q) - \mathrm{Vol}(P \cap Q),$$

4. respects linear maps: for any $T \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$,

$$\mathrm{Vol}_n(T(P)) = |\det(T)|\,\mathrm{Vol}_n(P).$$

When the dimension is unambiguous, we will write Vol_n as simply Vol. This definition does not uniquely specify a single "volume function", but rather a family of maps that all differ from one another by a constant multiple. We can differentiate different volume maps by which polytope in $\mathbb{R}^n$ they take to 1. For example, in 3-dimensions, our "standard" volume function is the one that takes the unit cube to 1.

In general, since any two volume functions only differ by constant, we can choose the volume function that makes our equations easiest to read. For example, we will be using *simplicial volume* quite a bit. This is the volume defined such that

$$\mathrm{Vol}_n\left(\mathrm{conv}(\{0, e_1, e_2, \ldots, e_n\})\right) = 1$$

where $e_1, \ldots, e_n$ are the basis vectors of our n-dimensional vector space.

4.2.2 Mixed Volume

While the volume function may not be too odd a concept we will use it to define a less widely known function. Given any 2 polytopes in $\mathbb{R}^2$, or 3 polytopes in $\mathbb{R}^3$, or more generally n polytopes in $\mathbb{R}^n$, we want a map from these collections of polytopes to $\mathbb{R}_{\geq 0}$ that is, in some sense, consistent with volume. We call this map the mixed volume function.

Definition 4.18 (Mixed Volume – Characterization)**.** The *mixed volume function* is a map MVol_n from an ordered multiset $P_1, P_2, \ldots, P_n \subseteq \mathbb{R}^n$ of polytopes to $\mathbb{R}_{\geq 0}$, such that it has the following properties:

1. $\mathrm{MVol}_n(P, P, \ldots, P) = \mathrm{Vol}_n(P)$, for any polytope $P \subseteq \mathbb{R}^n$,

2. MVol_n is symmetric in all arguments, and

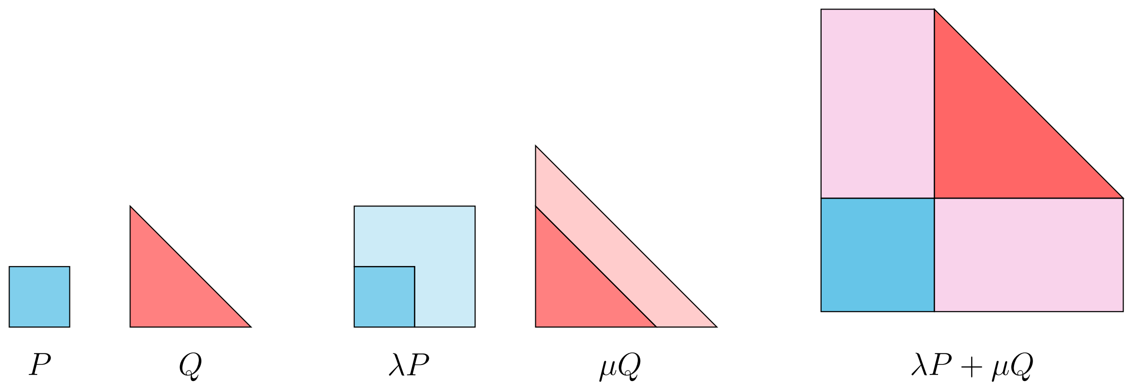
3. MVol_n is multilinear with respect to scaling and Minkowski addition.

Just like volume, we'll often just notate this as MVol when safe to do so. The proof that such a function exists and is indeed uniquely defined by these properties can be found in [Sch13]. While this characterization is useful, it goes very little of the way to actually telling us what the mixed volume is.

Consider two polytopes $P, Q \subseteq \mathbb{R}^2$. We could ask ourselves, what is the volume of the Minkowski sum of P and Q. We could be more ambitious and even allow ourselves to scale P and Q by arbitrary values. That is to say, let us consider the volume of

$$\mathrm{Vol}_2(\lambda P + \mu Q),$$

for some $\lambda, \mu \in \mathbb{R}$. Answering this question is where mixed volumes appear as something more concrete. While not immediately obvious, this volume can always be expressed as a polynomial in λ and μ, and mixed volumes appear as coefficients of these polynomials. We can consider an example.



$$\mathrm{Vol}(\lambda P + \mu Q) = \mathrm{MVol}(P, P)\lambda^2 + 2\,\mathrm{MVol}(P, Q)\lambda\mu + \mathrm{MVol}(Q, Q)\mu^2$$

Figure 4.9: The mixed volume appears in the coefficients of the volume polynomial; recall that $\mathrm{MVol}(P, P) = \mathrm{Vol}(P)$

Our example in the figure is nice because it is already symmetrical. In general this will not be the case and the coefficients will need to be rewritten to be symmetric, though this can always be done. This idea generalizes to any dimension, and can be taken as another definition of mixed volume.

Definition 4.19 (Mixed Volume – As Coefficients)**.** Let $P_1, P_2, \dots, P_\ell \subseteq \mathbb{R}^n$ be polytopes. The function

$$f(\lambda_1, \lambda_2, \dots, \lambda_\ell) = \mathrm{Vol}(\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_\ell P_\ell), \quad \lambda_j \geq 0$$

is a homogeneous polynomial of degree n. It can be written symmetrically as

$$f(\lambda_1, \dots, \lambda_\ell) = \sum_{j_1, j_2, \dots, j_n = 1}^{\ell} \mathrm{MVol}(P_{j_1}, \dots, P_{j_n})\lambda_{j_1} \cdots \lambda_{j_n}.$$

The coefficient associated to $\lambda_{j_1} \cdots \lambda_{j_n}$ is the *mixed volume* of $P_{j_1}, \dots, P_{j_n}$.

Of course, you could start with either definition of mixed volume and derive the other, they are equivalent after all.

At this point, one may begin to wonder why this section should even exist. We seem to have gone rather far afield with our geometry lesson. Remember that our ultimate goal is to show something is a log-concave sequence. Mixed volumes are the key to a method generating log-concave sequences via geometry.

4.2.3 The Alexandrov–Fenchel Inequality

Finally, we conclude with an important classic result in convex geometry. Proved by Alexandr Alexandrov in [Ale37], with a contemporaneous but not quite accurate proof by Werner Fenchel, this theorem gives us a fundamental relationship of mixed volumes of convex bodies. We again restrict ourselves to just polytopes, though the theorem applies more broadly.

Theorem 4.4 (Alexandrov–Fenchel Inequality)**.** *For polytopes $P, Q, K_3, \ldots, K_n$ in $\mathbb{R}^n$,*

$$\mathrm{MVol}(P, Q, K_3, \ldots, K_n)^2 \geq \mathrm{MVol}(P, P, K_3, \ldots, K_n)\,\mathrm{MVol}(Q, Q, K_3, \ldots, K_n).$$

Remember that the mixed volume is just some non-negative real number, so this inequality is exactly what we are looking for in a log-concave sequence. In fact, given any two polytopes, there's a corresponding log-concave sequence.

Corollary 4.5. *For any polytopes $P, Q \subseteq \mathbb{R}^n$, the sequence*

$$\left\{ \mathrm{MVol}(\underbrace{P, \ldots, P}_{n-k}, \underbrace{Q, \ldots, Q}_{k}) \right\}_{k=0}^{n}$$

is log-concave.

This is a promising lead, but all we have is a collection of geometric definitions and a way to generate log concave sequences. None of this actually has any clear relation to matroids. But, just like we could make something algebraic out of the structure of a matroid, so too can we make something geometric.

4.3 Bergman Fans

Much like Chow rings, the general notion of a Bergman fan is broader than what we actually need. Credit goes to, unsurprisingly, George Bergman, who in [Ber71] developed the idea of logarithmic limit-sets of algebraic varieties, which would go on to be called Bergman fans [FS05]. But given we still refuse to carefully define an algebraic variety, the original presentation is not too helpful for us here. More modern treatments, such as [AK06; HK12], have gone to show us that we can generate the Bergman fan from the combinatorial data of the matroid alone.

4.3.1 Bergman Fans of Matroids

While it may be more accurate to say we'll present the construction of a fan that can be proven to be a Bergman fan, we again simply take it as a definition.

Definition 4.20 (Bergman Fan of a Matroid)**.** Let $\mathcal{M}$ be a matroid with ground set $E = \{e_0, e_1, e_2, \ldots, e_k\}$ and lattice of flats $\mathcal{L}$. Let

$$\boldsymbol{N}_E = \mathbb{R}^E / \langle e_0 + e_1 + \cdots + e_k \rangle$$

be a real-valued vector space that identifies $e_1, \ldots, e_k$ with the standard basis vectors. For any subset $I \subseteq E$ of the ground set, we notate

$$e_I = \sum_{i \in I} e_i$$

as the vector sum of each vector associated to the ground elements in I.

The *Bergman fan of $\mathcal{M}$* is a fan in N given by

$$\boldsymbol{\Sigma}_{\mathcal{M}} = \left\{ \mathrm{cone}\big(e_F \mid F \in \mathscr{F}\big) \;\middle|\; \mathscr{F} \subseteq \mathcal{L}^* \text{ is a flag of } \mathcal{M} \right\}.$$

Let's unpack this definition. First, let's think about what space this fan lives in. Essentially, we can think $\boldsymbol{N}_E = \mathbb{R}^E/\langle e_0 + e_1 + \cdots + e_k\rangle$ as $\mathbb{R}^k$ where we assign all but one ground element of our matroid to the standard basis vectors. This designates one ground element as somewhat special, it doesn't matter which one, but generically we'll call it e_0. Then the vector associated to e_0 is the vector of all -1, as the relation in the quotient tells us

$$e_0 = -e_1 - e_2 - \cdots - e_k.$$

Next, let's think about the elements of our fan. They are necessarily cones, and we see that there is one cone per flag in our matroid. This means that there exists a 1-dimensional cone, often called a *ray*, for each proper flat, as every flat is itself a flag. In general, the length of the flat corresponds to the dimension of the corresponding cone in the fan. As a consequence the largest, by dimension, cones will correspond to complete flags. Similarly, if $F_1, F_2 \in \mathcal{L}$ are non-comparable flats, then there will not be a cone generated by the rays e_{F_1} and e_{F_2}. This is how the fan structure encodes the original combinatorial data of our matroid.

A bit of notation before moving on. We will write $\boldsymbol{\Sigma}_{\mathcal{M}}(d)$ to be the set of all d-dimensional cones in $\boldsymbol{\Sigma}_{\mathcal{M}}$. We will reserve ρ to designate the rays of our Bergman fan; that is $\rho_F \in \boldsymbol{\Sigma}_{\mathcal{M}}(1)$ for flat $F \in \mathcal{L}^*$. More generally, for any flag $\mathscr{F} = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_\ell\}$ of our matroid, we write

$$\sigma_{\mathscr{F}} = \mathrm{cone}\big(e_{F_1}, \ldots, e_{F_\ell}\big),$$

for the cone associated with $\mathscr{F}$.

4.3.1.1 An Example Bergman Fan

As always, let's turn to our running example and see its corresponding Bergman fan. Recall that

$$E = \{a, b, c, d\} \quad \text{and} \quad \mathcal{L}^* = \{a, b, c, d, abd, ac, bc, cd\}.$$

Then we can consider the space $\boldsymbol{N}_E = \mathbb{R}^E/\langle e_a + e_b + e_c + e_d\rangle$. We will designate d as the special element and associate a basis vector of $\mathbb{R}^3$ to the remaining ground elements a, b, c.

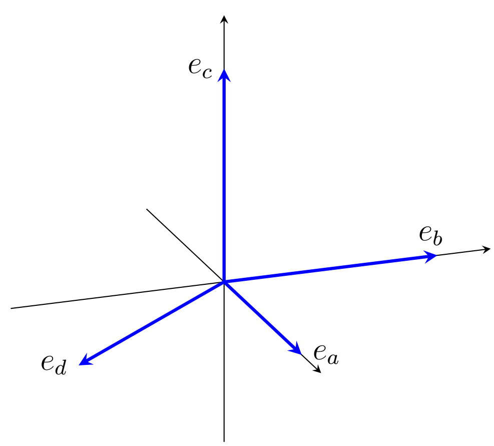


Figure 4.10: The vector space $\boldsymbol{N}_E$ with the vectors associated to the ground set

Then we can add in all the rays of our fan, corresponding to the flats.

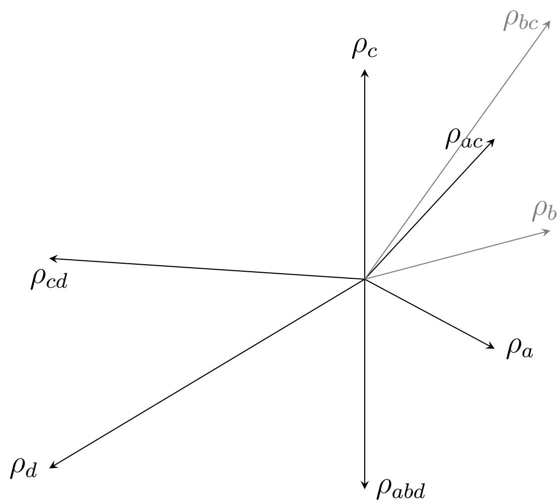


Figure 4.11: The rays $\boldsymbol{\Sigma}_{\mathcal{M}}(1)$

Because our proper flags can only have two elements we will only have at most 2-dimensional cones, since we only have a cone involving the rays ρ_F and $\rho_{F'}$ if $F \subsetneq F'$ or $F' \subsetneq F$.

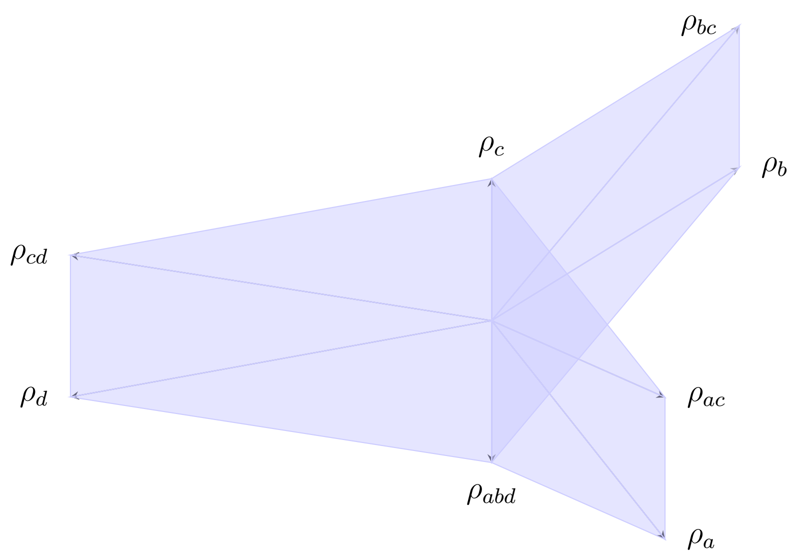


Figure 4.12: The Bergman fan of $\mathcal{M}$ with all of its cones

4.3.2 Properties of the Bergman Fan

Given the properties of fans we introduced earlier, it should come as no surprise that Bergman fans tend to be particularly nice. To start, we have that Bergman fans of matroids are always both pure and simplicial.

Proposition 4.6. *Let $\mathcal{M}$ be a matroid of rank $r+1$. The associated Bergman fan $\mathbf{\Sigma}_{\mathcal{M}}$ is a simplicial r-fan.*

Proof. To prove both of these properties we need one simple fact. By definition, a cone $\sigma \in \mathbf{\Sigma}$ is of the form

$$\sigma = \operatorname{cone}(e_{F_1}, \dots, e_{F_k}),$$

for $F_1 \subsetneq \cdots \subsetneq F_k$ some flag of $\mathcal{M}$. What we need is that $\{e_{F_1}, e_{F_2}, \dots, e_{F_k}\}$ is linearly independent. To see this, consider the equation

$$\lambda_1 e_{F_1} + \cdots + \lambda_k e_{F_k} = 0.$$

Recall that we defined $e_{F_j} = \sum_{i \in F_j} e_i$ to be the sum of the vectors associated to the ground set, so we can rewrite our equation as

$$\lambda_1 \sum_{i \in F_1} e_i + \cdots + \lambda_k \sum_{i \in F_k} e_i = 0.$$

But now, given that our flats form a flag that have strict inclusion, we may reorder terms to get

$$(\lambda_1 + \lambda_2 + \cdots + \lambda_k) \sum_{i \in F_1} e_i + (\lambda_2 + \cdots + \lambda_k) \sum_{i \in F_2 \setminus F_1} e_i + \cdots + \lambda_k \sum_{i \in F_k \setminus F_{k-1}} e_i = 0.$$

Since each of these sums involves a disjoint set of the vectors associated to the ground set, and any proper subset of these ground set vectors is linearly independent, it must be that $\lambda_1 = \cdots = \lambda_k = 0$, thus showing the set $\{e_{F_1}, e_{F_2}, \dots, e_{F_k}\}$ is linearly independent.

With this we immediately have that $\mathbf{\Sigma}_{\mathcal{M}}$ is simplicial. For any cone $\sigma_{\mathscr{F}} \in \mathbf{\Sigma}_{\mathcal{M}}$ associated the flag $\mathscr{F}$, it is generated by $|\mathscr{F}|$ vectors, and because those vectors are linearly independent $\dim(\sigma_{\mathscr{F}}) = |\mathscr{F}|$.

Now, recall that for a d-dimensional simplical cone with generating rays $V = \{v_1, \dots, v_d\}$, for any subset $V' \subseteq V$, $\operatorname{cone}(V')$ is a face of $\operatorname{cone}(V)$. Since we've shown that every cone of $\mathbf{\Sigma}_{\mathcal{M}}$ is simplicial, we have that any cone associated to a non-complete flag is non-maximal. This follows from the fact that if $\mathscr{F}$ is not a complete flag then there exists another flag $\mathscr{F}'$ such that $\mathscr{F} \subsetneq \mathscr{F}'$. Then we have that $\sigma_{\mathscr{F}}(1) \subsetneq \sigma_{\mathscr{F}'}(1)$ and so $\sigma_{\mathscr{F}}$ is a face of $\sigma_{\mathscr{F}'}$ and by definition non-maximal. Thus, every maximal cone of $\mathbf{\Sigma}_{\mathcal{M}}$ is associated to a complete flag, and any complete flag will have r elements. From above, we may conclude that every maximal cone has dimension r.

With that we've shown that $\mathbf{\Sigma}_{\mathcal{M}}$ is a simplicial r-fan.

□

Now that we know that a Bergman fan must necessarily be simplicial and pure, we can go on to show that the Bergman fan is not only tropical but also balanced.

Proposition 4.7. *Let $\mathcal{M}$ be a matroid. The Bergman fan $\mathbf{\Sigma}_{\mathcal{M}}$ is a balanced tropical fan.*

Proof. Let $\mathcal{M} = (E, \mathcal{L})$ be a matroid of rank $r+1$ and $\mathbf{\Sigma}_{\mathcal{M}} \subseteq \boldsymbol{N}_E$ its Bergman fan. We'll mark the rays $\rho \in \mathbf{\Sigma}_{\mathcal{M}}$ such that $u_\rho = e_{F_\rho}$ where $F_\rho \in \mathcal{L}$ is the flat associated to ρ, and we continue to use the convention that

$$e_I = \sum_{i \in I} e_i,$$

for any $I \subseteq E$. Recall that for $\mathbf{\Sigma}_{\mathcal{M}}$ to be balanced it must meet the balancing condition

$$\sum_{\substack{\sigma \in \mathbf{\Sigma}(r) \\ \tau \preceq \sigma}} u_{\sigma \setminus \tau} \in \operatorname{span}(\tau),$$

for every $\tau \in \mathbf{\Sigma}(r-1)$.

Let us fix an arbitrary $\tau \in \mathbf{\Sigma}(r-1)$ and consider the flag of flats associated to it

$$\mathscr{F} = \{F_1 \subsetneq \cdots \subsetneq F_{k-1} \subsetneq F_{k+1} \subsetneq \cdots \subsetneq F_r\}.$$

That is, $\mathscr{F}$ is a flag of length $r-1$ with just a single rank k flat missing from being a complete flag. Any maximal cone σ that contains τ as a face will be associated to the same set of flats but with the rank k flat, F_k, added. We can now rephrase our balancing condition in these terms; we wish to show

$$\sum_{\substack{F_k \in \mathcal{L} \\ F_{k-1} \subsetneq F_k \subsetneq F_{k+1}}} e_{F_k} \in \operatorname{span}(e_{F_1}, \ldots, e_{F_{k-1}}, e_{F_{k+1}}, \ldots, e_{F_r}).$$

For convenience, let us call $F_0 = \emptyset$ and $F_{r+1} = E$, as they are the unique rank 0 and r flats, respectively. We'll set the convention that $e_{F_0} = \sum_{i \in \emptyset} e_i = 0$. Similarly, we recall that $e_E = 0$. This means we can equivalently show that,

$$\sum_{\substack{F_k \in \mathcal{L} \\ F_{k-1} \subsetneq F_k \subsetneq F_{k+1}}} e_{F_k} \in \operatorname{span}(e_{F_0}, e_{F_1}, \ldots, e_{k-1}, e_{k+1}, \ldots, e_{F_r}, e_{F_{r+1}}).$$

Let's first use our basic definitions to rewrite our sum

$$\begin{aligned}
\sum_{\substack{F_k \in \mathcal{L} \\ F_{k-1} \subsetneq F_k \subsetneq F_{k+1}}} e_{F_k} &= \sum_{\substack{F_k \in \mathcal{L} \\ F_{k-1} \subsetneq F_k \subsetneq F_{k+1}}} \left(\sum_{i \in F_k} e_i \right) \\
&= \sum_{\substack{F_k \in \mathcal{L} \\ F_{k-1} \subsetneq F_k \subsetneq F_{k+1}}} \left(\sum_{i \in F_{k-1}} e_i + \sum_{i \in F_k \setminus F_{k-1}} e_i \right) \\
&= \sum_{\substack{F_k \in \mathcal{L} \\ F_{k-1} \subsetneq F_k \subsetneq F_{k+1}}} \left(e_{F_{k-1}} + e_{F_k \setminus F_{k-1}} \right).
\end{aligned}$$

Indexing nightmare aside, we note that the term $e_{F_{k-1}}$ in the sum is indepentent of choice of F_k. We'll let $\Lambda > 0$ be the number of complete flags that contain $\mathscr{F}$, letting us write

$$\sum_{\substack{F_k \in \mathcal{L} \\ F_{k-1} \subsetneq F_k \subsetneq F_{k+1}}} \left(e_{F_{k-1}} + e_{F_k \setminus F_{k-1}} \right) = \Lambda e_{F_{k-1}} + \sum_{\substack{F_k \in \mathcal{L} \\ F_{k-1} \subsetneq F_k \subsetneq F_{k+1}}} \left(e_{F_k \setminus F_{k-1}} \right).$$

Now, if we have flats $F_{k-1} \subsetneq F_{k+1}$ of rank $k-1$ and $k+1$ respectively, consider an element of the ground set $i \in F_{k+1} \setminus F_{k-1}$. By definition, the closure $F = \operatorname{cl}(F_{k-1} \cup i)$ is a flat. Since i was not in F_{k-1}, $\mathrm{rk}(F) > \mathrm{rk}(F_{k-1})$, and in particular must be rank k since adding a single element can only increase the rank by at most one. By property (F3) of the properties of flats, we know F is the unique flat such that $F_{k-1} \subsetneq F$ and $i \in F$, so we have that $F_{k-1} \subsetneq F \subsetneq F_{k+1}$. Further, the same property guarantees us that each $i \in F_{k+1} \setminus F_{k-1}$ will appear in exactly one rank k flat, $F_{k,i}$ such that $F_{k-1} \subsetneq F_{k,i} \subsetneq F_{k+1}$. This face lets us write

$$
\begin{aligned}
\Lambda e_{F_{k-1}} + \sum_{\substack{F_k \in \mathcal{L} \\ F_{k-1} \subsetneq F_k \subsetneq F_{k+1}}} \left(e_{F_k \setminus F_{k-1}}\right) &= \Lambda e_{F_{k-1}} + e_{F_{k+1} \setminus F_{k-1}} \\
&= (\Lambda - 1) e_{F_{k-1}} + \left(e_{F_{k-1}} + e_{F_{k+1} \setminus F_{k-1}}\right) \\
&= (\Lambda - 1) e_{F_{k-1}} + e_{F_{k+1}}.
\end{aligned}
$$

In this form we can clearly see this is in the span of $\{e_{F_0}, e_{F_1}, \ldots, e_{k-1}, e_{k+1}, \ldots, e_{F_r}, e_{F_{r+1}}\}$. With this, we've shown the Bergman fan of $\mathcal{M}$ is balanced. □

Many of the following theorems are about tropical fans more generally, and as such the weight function plays a role. However, since all Bergman fans of matroids are balanced, we will state the theorems just for balanced fans and remove refrences to the weight function. Simplifying all our lives a little bit.

We now have a geometric object associated with our matroid, but we still don't yet have a bridge back to the realm of algebra as promised. Also, we seemed to hint volume was going to be useful, but there's no obvious way to take the volume of a fan. To wrap this up, we move on to our final geometric object.

4.4 Normal Complexes

The work in this section is by far the most recent, coming from work within the last two years at time of writing. We will provide a brief summary of the work by Nathanson and Ross [NR23] and by Nowak and Ross, jointly with the author [NOR23].

This section will use the Bergman fan of a matroid to make an object called a normal complex. From this we can develop a notion of volume, as well as present theorems that relate this volume back to the Chow ring, finally giving us all the components necessary for our main result.

4.4.1 The Normal Complex of a Fan

In essence, a normal complex of a fan is simply a truncation of a fan into a polytopal complex using hyperplanes normal to the rays of the fan, thus the name. They were initially developed in [NR23], and the following definitions and propositions come from this work. However, we can't just take any truncation of our fan. We need a way to specify the truncations that will work, and so we introduce the idea of cubical and pseudocubical values.

Definition 4.21 (Cubical Values)**.** Let $\mathbf{\Sigma} \subseteq \mathbb{R}^n$ be a marked, simplicial d-fan and $* \in \mathrm{Inn}(\mathbb{R}^n)$ an inner product. Pick a vector $z \in \mathbb{R}^{\mathbf{\Sigma}(1)}$ that associates a real number to each ray of our fan. For each ray $\rho \in \mathbf{\Sigma}(1)$ we have a corresponding hyperplane and half-space

$$\mathrm{H}_{\rho,*}(z) = \{v \in N \mid v * u_\rho = z\} \quad \text{and} \quad \mathrm{H}^-_{\rho,*}(z) = \{v \in N \mid v * u_\rho \leq z\}.$$

For each cone $\sigma \in \mathbf{\Sigma}$, let $w_{\sigma,*}(z)$ be the unique value such that

$$w_\sigma(z) * u_\rho = z_\rho$$

for each ray $\rho \in \sigma(1)$.

We say z is *cubical* if for all $\sigma \in \mathbf{\Sigma}$,

$$w_\sigma(z) \in \sigma^\circ,$$

where σ° is the relative interior of σ.

Definition 4.22 (Pseudocubical)**.** Given everything exactly as in the previous definition, if we instead require only that

$$w_\sigma(z) \in \sigma$$

for all $\sigma \in \mathbf{\Sigma}$, we say z is *pseudocubical.*

For a given fan $\mathbf{\Sigma} \subseteq N$ and inner product $* \in \mathrm{Inn}(N)$, we denote cubical values as $\mathrm{Cub}(\mathbf{\Sigma}, *) \subseteq \mathbb{R}^{\mathbf{\Sigma}(1)}$ and the set of pseudocubical values as $\overline{\mathrm{Cub}}(\mathbf{\Sigma}, *) \subseteq \mathbb{R}^{\mathbf{\Sigma}(1)}$. This is all to say that when we select values to generate truncating hyperplanes for each ray, we want the intersections of the hyperplanes to lie within the cones of the fan. We can see an example in 2-dimensions.

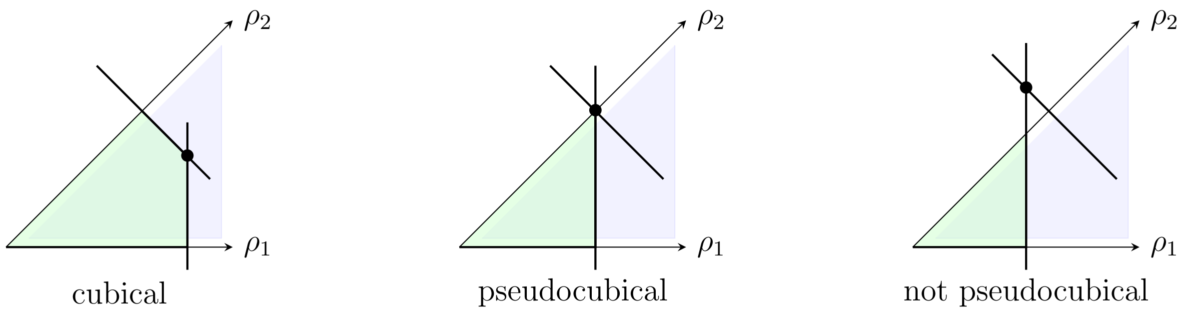


Figure 4.13: Examples of possible normal truncating hyperplane arrangements for a simple 2-dimensional fan

With this, we can now define a normal complex.

Definition 4.23 (Normal Complex)**.** Let $\mathbf{\Sigma} \subseteq N$ be a simplicial d-fan, with marked point u_ρ on each ray $\rho \in \mathbf{\Sigma}(1)$. Choose an inner product $* \in \mathrm{Inn}(N)$ and a pseudocubical vector $z \in \overline{\mathrm{Cub}}(\mathbf{\Sigma}, *)$. Recall that for each ray $\rho \in \mathbf{\Sigma}(1)$ we have a corresponding hyperplane and half-space

$$\mathrm{H}_{\rho,*}(z) = \{v \in N \mid v * u_\rho = z\} \quad \text{and} \quad \mathrm{H}^-_{\rho,*}(z) = \{v \in N \mid v * u_\rho \leq z\}.$$

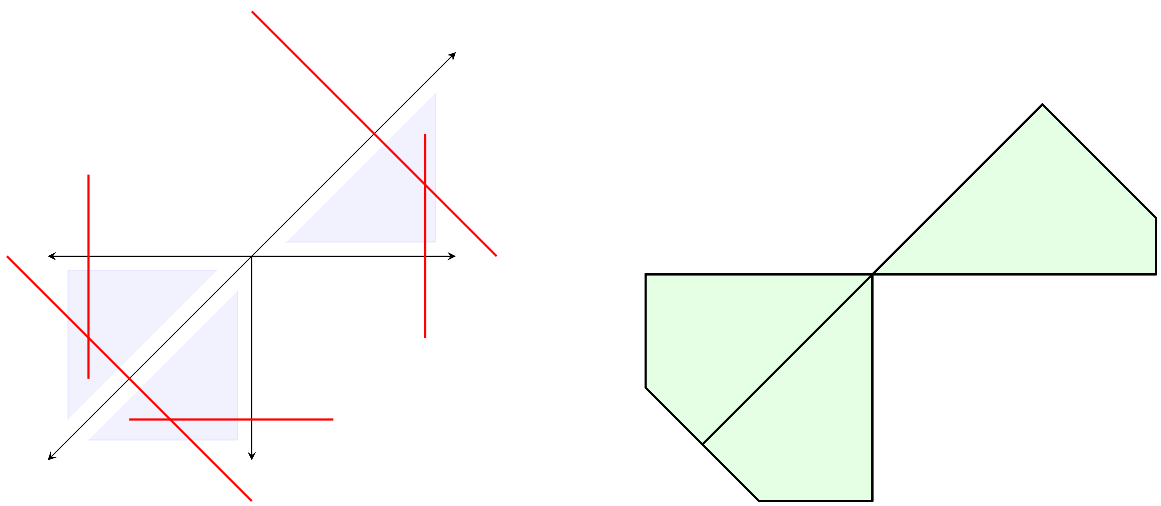
For each cone $\sigma \in \boldsymbol{\Sigma}$, we define a polytope $P_{\sigma,*}(z)$ given by

$$P_{\sigma,*}(z) = \sigma \cap \left(\bigcap_{\rho \in \sigma(1)} \mathrm{H}^{-}_{\rho,*}(z) \right).$$

The *normal complex* of $\boldsymbol{\Sigma}$ is the polytopal complex

$$C_{\boldsymbol{\Sigma},*}(z) = \bigcup_{\sigma \in \boldsymbol{\Sigma}} P_{\sigma,*}(z)$$

That these truncations of fans give us well-defined polytopal complex is a result of Proposition 3.7 in [NR23]. The basic idea is that because the pseudocublical condition insures the hyperplanes intersect within each cone exactly at one point, these intersection points define the vertices of each polytope in the complex. Those with more knowledge of polytopes can also see there is hyperplane description of these polytopes readily available as well. Here is another place where an example is worth many words.



(a) A fan with a possible arrangement of hyperplanes given by a cubical value

(b) The resulting normal complex of the fan given the choice of cubical value

4.4.2 The Volume of a Normal Complex

After so long hinting at the importance of volume, we finally have something related to our matroid we can actually take the volume of. Remember that we, rather lazily, only defined volumes on polytopes. We can extend the definition to normal complexes, though out of necessity how we do so is a bit fussy.

Definition 4.24 (Volume of a Normal Complex)**.** Let $\boldsymbol{N}$ be a real-valued vector space and $\boldsymbol{\Sigma} \subseteq \boldsymbol{N}$ be a simplicial d-fan with marked points u_ρ. Choose an inner product $* \in \mathrm{Inn}(\boldsymbol{N})$ and define

$$\boldsymbol{N}_\sigma = \mathrm{span}\left(u_\rho \mid \rho \in \sigma(1)\right) \subseteq \boldsymbol{N}.$$

Then, let $\boldsymbol{M}_\sigma$ be the vector space dual to $\boldsymbol{N}_\sigma$. Using $\{u_\rho \mid \rho \in \sigma(1)\}$ as the basis of $\boldsymbol{N}_\sigma$, and using our chosen inner product, we may identify the dual basis $\{u^\rho \mid \rho \in \sigma(1)\}$ of $\boldsymbol{M}_\sigma$ as a subset of $\boldsymbol{N}_\sigma$. For each $\sigma \in \boldsymbol{\Sigma}(d)$, we choose the volume function

$$\mathrm{Vol}_\sigma : \{\text{Polytopes in } \boldsymbol{N}_\sigma\} \to \mathbb{R}_{\geq 0}$$

characterized by $\mathrm{Vol}_\sigma\left(\mathrm{conv}(\{0\} \cup \{u^\rho \mid \rho \in \sigma(1)\})\right) = 1$.

For pseudocubical value $z \in \overline{\mathrm{Cub}}(\sigma, *)$, the *volume of the normal complex $C_{\boldsymbol{\Sigma},*}(z)$* is

$$\mathrm{Vol}_{\boldsymbol{\Sigma},*}(z) = \sum_{\sigma \in \boldsymbol{\Sigma}(d)} \mathrm{Vol}_\sigma\left(P_{\sigma,*}(z)\right).$$

The basic idea here is that we are taking the volume of each top-dimensional polytope in the complex and summing them. Nothing too wild there. Most of the verbosity in the definition comes from the fact that we have to choose the volume function for each component rather carefully. The specifics of why this must be done we leave to the reader to explore in [NR23], but suffice to say it is necessary for the main theorem of that paper. This theorem provides the first part of the bridge we are developing between geometry and algebra.

Theorem 4.8 (Nathanson-Ross 2021)**.** *Let $\boldsymbol{\Sigma}$ be a balanced d-fan. Choose an inner product $* \in \mathrm{Inn}(N)$ and pseudocubical value $z \in \overline{\mathrm{Cub}}(\boldsymbol{\Sigma}, *)$. We define*

$$D(z) = \sum_{\rho \in \boldsymbol{\Sigma}(1)} z_\rho x_\rho \in A^1(\boldsymbol{\Sigma}),$$

a divisor of the Chow ring. Then

$$\mathrm{Vol}_{\boldsymbol{\Sigma},*}(z) = \deg\left(D(z)^d\right).$$

Finally, we have the first link back to the Chow ring. By carefully taking volumes of the normal complex, we can evaluate top degrees of divisors under the degree map. While this is very cool, we have a problem. This only allows us to evaluate a single divisor raised to the top power under the degree map. Recall that we want to reason about elements of the form $\alpha^{d-k}\beta^k$. These are called mixed degrees of divisors, and perhaps that is a good hint as to what we need to develop next.

4.4.3 Mixed Volumes of Normal Complexes

Here we finally can justify introducing the mixed volume function earlier. We can't define the mixed volume of the full normal complexes directly, as the Minkowski sum of two normal complexes would certainly not be a normal complex itself in general. But like volume we could define it component wise.

A distinction from the original mixed volume function is that here is that we can't take the mixed volume of an arbitrary collection of normal complexes. We can only take the mixed volumes of normal complexes that have the same underlying fan.

Definition 4.25 (Mixed Volume of Normal Complexes)**.** Let $\mathbf{\Sigma} \subseteq N$ be a simplicial d-fan, $* \in \mathrm{Inn}(N)$ be an inner product, and pseudocubical values $z_1, \ldots, z_d \in \overline{\mathrm{Cub}}(\mathbf{\Sigma}, *)$. The mixed volume of $C_{\mathbf{\Sigma},*}(z_1), \ldots, C_{\mathbf{\Sigma},*}(z_d)$, written $\mathrm{MVol}_{\mathbf{\Sigma},*}(z_1, \ldots, z_d)$ is given by

$$\mathrm{MVol}_{\mathbf{\Sigma},*}(z_1, \ldots, z_d) = \sum_{\sigma \in \mathbf{\Sigma}(d)} \mathrm{MVol}_\sigma \left(P_{\sigma,*}(z_1), \ldots, P_{\sigma,*}(z_d) \right),$$

where MVol_σ is the mixed volume of polytopes as defined above using the volume function Vol_σ.

Here, the basic idea as to why this may work is that taking the Minkowski sum of polytopes in the same cone will produce another polytope in this cone.

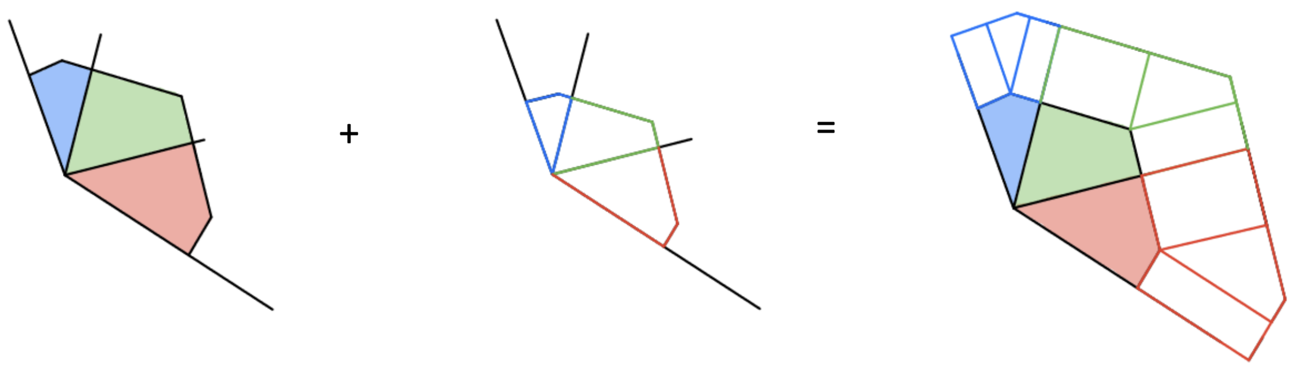


Figure 4.15: Component wise Minkowski sums of two normal complexes of the same fan; note each polytope in the sum is still in its correct cone

That this works in general is not obvious. Nor is it immediate that this newly defined function has all the nice properties of the original mixed volumes. The development and proof of the following proposition was originally the work of Lauren Nowak in her master's thesis [Now22]. This work was then incorperated into [NOR23] from which Proposition 3.1 gives us the following guarantee.

Proposition 4.9. *Let $\mathbf{\Sigma} \subset N$ be a simplicial d-fan, $* \in \mathrm{Inn}(N)$ an inner product on $\mathbb{R}^n$, and pseudocubical values $z_1, \ldots, z_n \in \overline{\mathrm{Cub}}(\mathbf{\Sigma}, *)$.*

The function

$$\mathrm{MVol}_{\mathbf{\Sigma},*} : \overline{\mathrm{Cub}}(\mathbf{\Sigma}, *)^d \to \mathbb{R}_{\geq 0}$$

as defined above has the following properties:

1. $\mathrm{MVol}_{\mathbf{\Sigma},*}(z, z, \ldots, z) = \mathrm{MVol}_{\mathbf{\Sigma},*}(z)$,
2. $\mathrm{MVol}_{\mathbf{\Sigma},*}$ *is symmetric in all arguments,*
3. $\mathrm{MVol}_{\mathbf{\Sigma},*}$ *is multilinear with respect to Minkowski addition in each maximal cone.*

Further, any function $\overline{\mathrm{Cub}}(\mathbf{\Sigma}, *)^d \to \mathbb{R}_{\geq 0}$ satisfying properties 1–3 must be $\mathrm{MVol}_{\sigma,*}$.

Our new mixed volume function then is well-defined and is uniquely characterized by the same properties as the original. Theorem 3.6 of [NOR23] extends Theorem 4.8 to a form that lets us evaluate mixed degrees.

Theorem 4.10. *Let $\mathbf{\Sigma} \subset N$ be a balanced d-fan. Choose an inner product $* \in \mathrm{Inn}(N)$ and pseudocubical values $z_1, \ldots, z_d \in \overline{\mathrm{Cub}}(\mathbf{\Sigma}, *)$. Then*

$$\mathrm{MVol}_{\mathbf{\Sigma},*}(z_1, \ldots, z_d) = \deg\big(D(z_1) \cdots D(z_d)\big).$$

This is a successful bridge from the realm of geometry back to algebra. We are now so close to having all the necessary components to prove our main result. Not only do we have the link between geometry and algebra, it uses the concept of mixed volumes, which are closely related to log-concave sequences. However, the Alexandrov-Fenchel inequalities are, classically, very dependent on convexity, and normal complexes are decidedly non-convex.

4.4.4 Amazing AF Fans

Given our extended mixed volume function on normal complexes, the resulting polynomials will sometimes produce coefficients who obey the Alexandrov-Fenchel inequality. This is in fact a property of the underlying fan itself.

Definition 4.26 (Alexandrov-Fenchel Property)**.** Let $\mathbf{\Sigma} \subseteq \mathbb{R}^n$ be a simplicial d-fan and $* \in \mathrm{Inn}(\mathbb{R}^n)$ an inner product. We say that $(\mathbf{\Sigma}, *)$ is *Alexandrov-Fencel*, or just *AF*, if $\mathrm{Cub}(\mathbf{\Sigma}, *) \neq \emptyset$ and

$$\mathrm{MVol}_{\mathbf{\Sigma},*}(z_1, z_2, z_3, \ldots, z_d)^2 \geq \mathrm{MVol}_{\mathbf{\Sigma},*}(z_1, z_1, z_3, \ldots, z_d)\,\mathrm{MVol}_{\mathbf{\Sigma},*}(z_2, z_2, z_3 \ldots, z_d)$$

for all $z_1, z_2, z_3, \ldots, z_d \in \mathrm{Cub}(\mathbf{\Sigma}, *)$.

If a fan is AF, then taking the mixed volumes of any normal complexes built from it will give rise to a log-concave sequence. This is naturally weaker than the Alexandrov-Finchel inequality in the convex setting, which is just universally true, but that it is still sometimes true in a non-convex setting is Our goal now then is to show that Bergman fans of matroids are AF. Luckily, Theorem 5.1 in [NOR23] gives us exact criteria to use.

However, to state the theorem, we will need to know how to consider smaller components of a fan, and therefore also normal complexes. First, we'll consider the neighborhood of a cone in a fan.

Definition 4.27 (Neighborhood)**.** Let $\mathbf{\Sigma}$ be a fan and $\sigma \in \mathbf{\Sigma}$ a cone in the fan. The neighborhood of σ in $\mathbf{\Sigma}$ is

$$\mathrm{nbd}(\sigma, \mathbf{\Sigma}) = \{\tau \mid \tau \preceq \pi,\, \sigma \preceq \pi\}.$$

The neighborhood of a cone σ is essentially the collection of all other cones in the fan contain σ as a face. We then take all faces of those cones to insure that $\mathrm{nbd}(\sigma, \mathbf{\Sigma})$ is still a fan. In fact $\mathrm{nbd}(\sigma, \mathbf{\Sigma})$ is still a fan living in the same space as $\mathbf{\Sigma}$. If we additionally quotient out the components of σ from the neighborhood we have what is called the star of σ.

Definition 4.28 (Star)**.** Let $\mathbf{\Sigma} \subseteq \mathbb{R}^d$ be a fan and $\sigma \in \mathbf{\Sigma}$ a cone in the fan. The *star* of σ is given by

$$\operatorname{star}(\sigma, \mathbf{\Sigma}) = \{\overline{\tau} \mid \tau \in \operatorname{nbd}(\sigma, \mathbf{\Sigma})\} \subseteq \mathbb{R}^d / \operatorname{span}(\sigma),$$

where $\overline{\tau}$ is the equivalence class of τ in the quotient space $\mathbb{R}^d / \operatorname{span}(\sigma)$.

When provided with an inner product, we can identify the quotient space $\mathbb{R}^d / \operatorname{span}(\sigma)$ with the orthogonal space, $\operatorname{span}(\sigma)^\perp$. Given we are always providing an inner product, we mostly think of the star as the projection of the neighborhood of σ into $\operatorname{span}(\sigma)^\perp$. Since the neighborhood was a fan in the original space, the star will be a fan in this orthogonal space, which we notate as $\mathbf{\Sigma}^\sigma$.

Perhaps more surprisingly, if we have the necessary data to make a normal complex on some fan $\mathbf{\Sigma}$, then for any cone $\sigma \in \mathbf{\Sigma}$, we also have everything we need to define a normal complex on $\mathbf{\Sigma}^\sigma$. We call these faces of the normal complex, in analogy with polytopes, and the details of their construction and existence can be found in section 4 of [NOR23].

Definition 4.29 (Face of Normal Complex)**.** Let $\mathbf{\Sigma} \subset \mathbb{R}^n$ be a simplicial d-fan and $* \in \operatorname{Inn}(\mathbb{R}^n)$ be an inner product. Given any $z \in \overline{\operatorname{Cub}}(\mathbf{\Sigma}, *)$ and cone $\sigma \in \Sigma$, the *face* of $C_{\mathbf{\Sigma},*}(z)$ associated to σ is

$$F^\sigma\big(C_{\mathbf{\Sigma},*}(z)\big) = C_{\mathbf{\Sigma}^\sigma, *^\sigma}(z^\sigma),$$

where $*^\sigma$ is the restriction of $*$ to $\operatorname{span}(\sigma)^\perp$ and $z^\sigma \in \overline{\operatorname{Cub}}(\mathbf{\Sigma}^\sigma, *^\sigma)$ is uniquely determined by $\Sigma, \sigma,$ and z.

A good thing to note here is there is a dimension reversing relationship between the dimension of σ and $\mathbf{\Sigma}^\tau$. If $\mathbf{\Sigma}$ is a d-fan and σ is of dimension k, then $\mathbf{\Sigma}^\sigma$ is pure of dimension $d-k$. We would then call $F^\sigma\big(C_{\mathbf{\Sigma},*}(z)\big)$ a $(d-k)$-dimensional face of the normal complex $C_{\mathbf{\Sigma},*}(z)$. This finally gives us enough background information to state Theorem 5.1 of [NOR23].

Theorem 4.11 (Nowak-O-Ross 2022)**.** *Let $\mathbf{\Sigma}$ be a balanced d-fan, and $* \in \operatorname{Inn}(N_\mathbb{R})$ an inner product such that* $\operatorname{Cub}(\mathbf{\Sigma}, *) \neq \emptyset$*. Then* $(\mathbf{\Sigma}, *)$ *is AF if*

i. for every cone $\sigma \in \mathbf{\Sigma}(k)$, with $k \leq d-2$,

$$\operatorname{star}(\sigma, \mathbf{\Sigma}_\mathcal{M}) \setminus \{0\}$$

is connected and,

ii. for any choice of $z \in \operatorname{Cub}(\mathbf{\Sigma}, *)$ the volume of each 2-dimensional face of the associated normal complex, $C(\mathbf{\Sigma}, z)$, is quadratic form whose associated matrix has exactly one positive eigenvalue.

We can provide a little insight into these two criteria. In the first, connectedness is the classic topological definition of connected, but, for those without a topological background, practically we can think of this as being able draw a path between any two points. In this case with the origin removed. The effect of this criteria is that is it tells us our fans can't have "pinch" points. If the cones of the fan share a non-0-dimensional face that's not a facet, then when taking stars within the fan we may get the case that the cones in the star fan meet only at the origin. Once we then remove the origin, then this will disconnect the star fan, violating the criteria.

For the second condition, first we recall that since we're looking at 2-dimensional faces of normal complexes, these will be polytopal complexes with maximal elements of dimension 2. The volume of the polytopes will depend on the z-values that determine the truncation. Since each polytope's volume will be determined by 2 z-values each, we can write the volume of the whole complex as a polynomial in z_i where each monomial is of degree 2. This is what is called a quadratic form, and there is always a matrix associated to quadratic forms, and it is this matrix's eigenvalues that are of interest to us.

We now finally have laid out every piece of existing work we'll need to prove our main result. All that's left is to put them together.

Chapter 5

Results

We have now all of the dominos lined up, and we're ready to start knocking them down. The bulk of this section will go to proving that our Bergman fans are AF. Once we have that, our goal is in sight. We'll work our way back from the realm of geometry through a series of implications until we arrive back at the characteristic polynomial.

5.1 Some Necessary Tools

Before we can prove that need to develop a few more concepts in order to prove that Bergman fans are AF. This includes introducing a theorem, cramming in some more definitions, and proving a relationship We'll then develop a few lemmas of our own that will give us a useful way to reframe star fans.

5.1.1 A Useful Linear Algebra Theorem

We'll start with a classic theorem from linear algebra [Syl52].

Proposition 5.1 (Sylvester's Law of Inertia)**.** *Two symmetric square matrices, A and B, of the same size have the same number of positive, negative, and zero eigenvalues if and only if*

$$B = SAS^T$$

where for some non-singular matrix S.

Hopefully the utility of this is fairly clear. If we're going to be hunting eigenvalues, matrices are somewhere near. This lemma means we won't lose information about the sign of eigenvalues if we manipulate the matrix, just so long as we do it in an invertible way.

5.1.2 Surprise Auxiliary Matroid Theory

We lied about having all the information on matroids necessary after the first chapter, apologies. We need a little more to get us to the end. First, we need a way to chop matroids up into smaller bits.

5.1.2.1 New Matroids From Old

Let's say we already have some matroid $\mathcal{M} = (E, \mathcal{I})$. Then $\mathcal{I}$ already has a notion about which of all possible subsets of E are independent. So if we consider some subset $X \subseteq E$ of the ground set, we should be able to use $\mathcal{M}$'s independent sets to construct independent sets for X as a ground set. This is in fact very easy to do, and we call the resulting matroid a restriction matroid.

Definition 5.1 (Restriction Matroid)**.** Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. Then for any subset $X \subseteq E$, we may define the *restriction matroid*, $\mathcal{M}|X$, as

$$\mathcal{M}|X = (X, \mathcal{I}|X)$$

where $\mathcal{I}|X = \{I \in \mathcal{I} \mid I \subseteq X\}$.

Essentially we just declare X to be the new ground set and just forget about any independent sets of $\mathcal{M}$ that contain any elements not in X. Rather than providing a subset to restrict to, one often finds it useful to specify just the things we want to forget.

Definition 5.2 (Deletion Matroid)**.** Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and $Y \subseteq E$. The matroid that results from the *deletion* of Y from $\mathcal{M}$, sometimes called a *deletion matroid*, is defined as

$$\mathcal{M} \backslash Y = \mathcal{M}|(E \setminus Y)$$

Clearly if $X = (E \setminus Y)$, then $\mathcal{M}|X = \mathcal{M} \backslash Y$. The choice of deletion or restriction is just a matter of what one wants to emphasize, what we keep or what we remove.

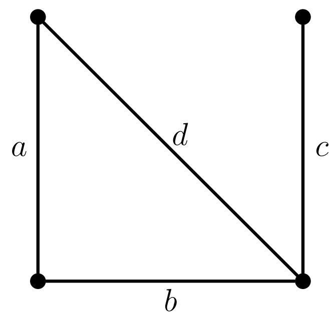
The other way to build a matroid out of an existing one is a little less obvious. These are called contraction matroids, and they are *dual* to restriction matroids. While they're a bit easier to define using duality, we want to avoid introducing all the machinery for that. Still, as mathematicians we feel compelled to point out duality anytime we see it.

Definition 5.3 (Contraction Matroids)**.** Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. For any subset $T \subseteq E$ of the ground set, construct the restriction matroid $M|T$. Then let B_T be a basis of $M|T$. The *contraction matroid*, $\mathcal{M}/T$, is defined as

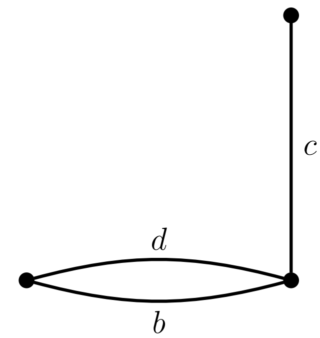
$$\mathcal{M}/T = (E \setminus T, \mathcal{I}/T),$$

where $\mathcal{I}/T = \{I \subseteq (E \setminus T) \mid I \cup B_T \in \mathcal{I}\}$.

This definition is more difficult to explain succinctly, but we can compare it with the restriction matroid to try to get some sense of what this does. We can think of restriction matroid as imparting independence on a subset $X \subseteq E$ by saying subsets are independent if they would be independent in the original matroid. The contraction matroid, then, assigns independence on everything *not* in the subset $T \subseteq E$, based on if they'd still be independent if we were to add (a basis of) T back in.



A graph representing a matroid $\mathcal{M}$



A graph representing a matroid $\mathcal{M}'$

Figure 5.1: To provide a modicum of explanation as to why these are called *contractions*, we note that $\mathcal{M}/a$ and $\mathcal{M}'$ are isomorphic

Importantly, we can combine deletion and contraction, and indeed the resulting matroids are a rather central point of study in matroid theory.

Definition 5.4 (Matroid Minor)**.** A *minor* of a matroid $\mathcal{M}$ is any matroid resulting in any combination of deletions and contractions of $\mathcal{M}$.

Further, any series of deletions and contractions can always be rearranged to, and so any matroid minor is of the form,

$$\mathcal{M}\backslash X/Y,$$

where $X, Y \subseteq E$ are disjoint and possibly empty. When $X \cup Y$ is nonempty, we call $\mathcal{M}\backslash X/Y$ a *proper minor* of $\mathcal{M}$.

5.1.2.2 Matroid Minors and Flats

The careful reader may have noted that the definitions of restriction and contraction matroids are given in terms of independent sets, but we've clearly established that we are all about flats here. Luckily, we have a very useful property relating the lattice of minors and to the lattice of the original matroid. First though, a little notation. If F is a flat of $\mathcal{M}$, we will define

$$\mathcal{M}_{[\emptyset,F]} = \mathcal{M}|F$$

to be the restriction by F, and

$$\mathcal{M}_{[F,E]} = \mathcal{M}/F$$

to be the contraction by F. For any two flats F_1 and F_2 of $\mathcal{M}$, we write

$$\mathcal{M}_{[F_1,F_2]} = \mathcal{M}/F_1\backslash(E \setminus F_2)$$

to be the minor that results from contracting by F_1 and restricting to F_2. Notation in hand, we can now state a classic result of matroid theory, which can be found, unsurprisingly, in Oxley [Oxl11, p. 116].

Proposition 5.2. *Let F_1 and F_2 be flats of a matroid $\mathcal{M} = (E, \mathcal{L})$. Then the lattice of flats of the minor $\mathcal{M}_{[F_1,F_2]}$, $\mathcal{L}_{[F_1,F_2]}$ is isomorphic to the interval of $\mathcal{L}$*

$$[F_1, F_2] = \{F_1 \preceq F \preceq F_2 \mid F \in \mathcal{L}\}$$

given by the isomorphism

$$\varphi : \mathcal{L}_{[F_1,F_2]} \to \mathcal{L}$$
$$\varphi(F) = F \cup F_1.$$

This means the lattice of a minor can be "seen" within the lattice structure of our original matroid, just up to some relabeling of the nodes. We note that this proposition only works for flats, not arbitrary subsets of the ground set, but that's more than enough for what we need.

Importantly, if F_1 and F_2 are adjacent to each other in the lattice of flats, then $\mathcal{M}_{[F_1,F_2]}$ is isomorphic to a matroid whose sole flag is $\{\emptyset \subsetneq E\}$. The Bergman fan of such a matroid is just the point $\{0\}$, living in a 0-dimnesional vector space.

5.1.3 Products and Isomorphisms of Fans

The last tool we're going to need is to develop a relationship between the stars of Bergman fans and the fans of its minors. Let's start by defining the product of a fan.

Definition 5.5 (Product Fan)**.** Let $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$ be fans in vector spaces $\boldsymbol{N}$ and $\boldsymbol{N}'$ respectively. The *product fan* given by $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$ is

$$\mathbf{\Sigma} \times \mathbf{\Sigma}' = \{\sigma \times \sigma' \mid \sigma \in \mathbf{\Sigma}, \sigma' \in \mathbf{\Sigma}'\} \subseteq \boldsymbol{N} \oplus \boldsymbol{N}'.$$

Essentially, when making a product fan, we put the fans in orthogonal spaces and make cones between all possible combinations of the cones in each fan. We notate cones of the product fan as (σ, σ'), where $\sigma \in \mathbf{\Sigma}$ and $\sigma' \in \mathbf{\Sigma}'$. Rays of the product fan correspond to elements $(\rho, 0)$ or $0, \rho'$ where ρ and ρ' are rays in their respective fans. Importantly, the product fan can be extended to arbitrary numbers of fans. If we have fans $\mathbf{\Sigma}_1, \ldots, \mathbf{\Sigma}_k$ in vector spaces $\boldsymbol{N}_1, \ldots, \boldsymbol{N}_k$ then

$$\prod_{i=1}^{k} \mathbf{\Sigma}_i \subseteq \bigoplus_{i=1}^{k} \boldsymbol{N}_i$$

is the corresponding product fan, with cones of the form $(\sigma_1, \ldots, \sigma_k)$.

Now we need to define what it means for two fans to be isomorphic.

Definition 5.6 (Fan Isomorphism)**.** Let $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$ be fans in $\boldsymbol{N}$ and $\boldsymbol{N}'$ respectively. We say that $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$ are *isomorphic* if there exists a linear bijection

$$\varphi : \boldsymbol{N} \to \boldsymbol{N}'$$

that induces a bijective map between cones of $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$, given by

$$\varphi^* : \mathbf{\Sigma} \to \mathbf{\Sigma}'$$
$$\varphi^*(\sigma) = \operatorname{cone}\big(\varphi(\rho) \mid \rho \in \sigma(1)\big).$$

Given that the fans are isomorphic, we will notate this as $\mathbf{\Sigma} \cong \mathbf{\Sigma}'$.

This is just to say that two fans are isomorphic if we can find an isomorphism between their respective spaces that preserves the combinatorial data of the fans. We are going to show that there is an isomorphism between star fans of a matroid and product fans of matroid minors. To get there we need the first ingrediant of a fan isomorphism, a linear bijection.

Proposition 5.3. *Let $\mathbf{\Sigma}_{\mathcal{M}} \subset \boldsymbol{N}_E$ be the Bergman fan of a matroid,*

$$\mathscr{F} = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k\} \subseteq \mathcal{L}^*$$

be a flag of flats, and define $F_0 = \emptyset$ and $F_{k+1} = E$. The map

$$\varphi : \boldsymbol{N}_E / \operatorname{span}(\sigma_{\mathscr{F}}) \to \bigoplus_{i=1}^{k+1} \boldsymbol{N}_{F_i \setminus F_{i-1}}$$

$$[e_I] \mapsto \bigoplus_{i=1}^{k+1} e_{I \cap (F_i \setminus F_{i-1})}$$

is an isomorphism.

Proof. Consider the very similar map

$$\varphi' : \boldsymbol{N}_E \to \bigoplus_{i=1}^{k+1} \boldsymbol{N}_{F_i \setminus F_{i-1}}$$

$$e_I \mapsto \bigoplus_{i=1}^{k+1} e_{I \cap (F_i \setminus F_{i-1})}.$$

We will show φ' is a surjective linear map and that its kernel is exactly $\operatorname{span}(\sigma_{\mathscr{F}})$. With that, we can leverage Noether's first isomorphism theorem, and we will have that φ is an isomorphism.

Our map is linear by its definition, so let's first show that it is surjective. This isn't too bad. Consider some vector $w \in \bigoplus_{i=1}^{k+1} \boldsymbol{N}_{F_i \setminus F_{i-1}}$, is of the form

$$w = \bigoplus_{i=1}^{k+1} \left(\sum_{j \in (F_i \setminus F_{i-1})} \lambda_j e_j \right);$$

i.e.; each is a linear combination of the elements in some $F_i \setminus F_{i-1}$. This of course is the image of the vector

$$v = \sum_{i=1}^{k+1} \left(\sum_{j \in (F_i \setminus F_{i-1})} \lambda_j e_j \right),$$

so we have surjectivity.

Next we want to show that any element in the $\operatorname{span}(\sigma_{\mathscr{F}})$ in the kernel. We can figure this out seeing what gets mapped to 0 in any one of the $\boldsymbol{N}_{F_i \setminus Fi-1}$. From construction of the space, anything in $\operatorname{span}(e_{F_i \setminus Fi-1})$ is sent to 0 in $\boldsymbol{N}_{F_i \setminus F_{i-1}}$, in particular $e_{F_i \setminus F_{i-1}}$ itself. So we can see that some $v \in \operatorname{span}(\sigma)$ is of the form

$$v = \lambda_1 e_{F_1} + \lambda_2 e_{F_2} + \cdots + \lambda_k e_{F_k},$$

and under our map will be

$$\begin{aligned}
\varphi'(v) &= \lambda_1 \varphi'(e_{F_1}) + \lambda_2 \varphi'(e_{F_2}) + \cdots + \lambda_k \varphi'(e_{F_k}) \\
&= \lambda_1 \left(e_{F_1}\right) + \lambda_2 \left(e_{F_1} \oplus e_{F_2 \setminus F_1}\right) + \cdots + \lambda_k \left(e_{F_1} \oplus \lambda_k e_{F_2 \setminus F_1} \oplus \cdots \oplus \lambda_k e_{F_k \setminus F_{k-1}}\right) \\
&= (\lambda_1 + \lambda_2 + \cdots + \lambda_k) e_{F_1} \oplus (\lambda_2 + \lambda_3 + \cdots + \lambda_k) e_{F_2 \setminus F_1} \oplus \cdots \oplus \lambda_k e_{F_k \setminus F_{k-1}} \\
&= (\lambda_1 + \lambda_2 + \cdots + \lambda_k) 0 \oplus (\lambda_2 + \lambda_3 + \cdots + \lambda_k) 0 \oplus \cdots \oplus \lambda_k 0 \\
&= 0
\end{aligned}$$

Finally, we note that the only other thing mapped to 0 would be an element of the form $\lambda e_{E \setminus F_k}$ which is 0 when mapped to $\boldsymbol{N}_{F_{k+1} \setminus F_k}$. It is not immediately obvious that this is in $\operatorname{span}(\sigma_{\mathscr{F}})$. However, we recall that in $\boldsymbol{N}_E$, $e_E = 0$, which means $e_{E \setminus F_k} = -e_{F_k}$, which clearly is in the span of $\sigma_{\mathscr{F}}$. We may conclude then that $\ker(\varphi') = \operatorname{span}(\sigma_{\mathscr{F}})$.

With that we have

$$\boldsymbol{N}_E / \operatorname{span}(\sigma_{\mathscr{F}}) \cong \bigoplus_{i=1}^{k+1} \boldsymbol{N}_{F_i \setminus F_{i-1}}$$

by Noether's first isomorphism theorem and that our map φ is the given isomorphism between these spaces. $\square$

With an isomorphism between these two spaces in hand, we can show that the stars of a matroid's Bergman fan have a local structure equivalent to the product of the Begrman fans of its minors.

Lemma 5.4. *Let $\boldsymbol{\Sigma}_{\mathcal{M}} \subseteq \boldsymbol{N}_E$ be the Bergman fan of a matroid, and $\sigma_{\mathscr{F}} \in \boldsymbol{\Sigma}(k)$ be a cone with ray generators corresponding to the flag $\mathscr{F} = \{F_1, \ldots, F_k\}$. Then the star fan associated to $\sigma_{\mathscr{F}}$ is isomorphic to the product fan of minors given by the intervals of $\mathscr{F}$. That is to say*

$$\operatorname{star}(\sigma_{\mathscr{F}}, \boldsymbol{\Sigma}_{\mathcal{M}}) \cong \prod_{i=1}^{k+1} \boldsymbol{\Sigma}_{\mathcal{M}_{[F_{i-1}, F_i]}}.$$

Proof. Let $\boldsymbol{\Sigma}_{\mathcal{M}} \subset \boldsymbol{N}_E$ be the Bergman fan of a matroid,

$$\mathscr{F} = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k\} \subseteq \mathcal{L}^*$$

be a flag of flats, and again define $F_0 = \emptyset$ and $F_{k+1} = E$. Take φ to be the isomorphism given in Proposition 5.3. As per the definition, to show that the two fans are isomorphic, we just need to show that φ induces a bijection

$$\varphi^* \;:\; \operatorname{star}(\sigma_{\mathscr{F}}, \boldsymbol{\Sigma}_{\mathcal{M}}) \to \prod_{i=1}^{k+1} \boldsymbol{\Sigma}_{\mathcal{M}_{[F_{i-1}, F_i]}}.$$

Thanks to linearity, we check that rays of the star fan are maped to the rays of the product fan and vice versa.

Let's assume $\sigma_{\mathscr{F}}$ is not a maximal cone, as otherwise it's trivially true that the star fan, which is 0-dimensional, is isomorphic to the product fan, the product of several 0-dimensional fans. Consider a ray of the star fan $\overline{\rho} \in \operatorname{star}(\sigma_{\mathscr{F}}, \mathbf{\Sigma}_{\mathcal{M}})$. From our definitions, we know that $\overline{\rho}$ corresponds to some ray in the neighborhood of $\sigma_{\mathscr{F}}$, $\rho \in \operatorname{nbd}(\sigma_{\mathscr{F}}, \mathbf{\Sigma}_{\mathcal{M}})$. Further then, we know that ρ is the face of some cone τ such that $\tau \in \mathbf{\Sigma}_{\mathcal{M}}(r)$ and $\sigma_{\mathscr{F}} \preceq \tau$. Since τ is a maxiamal cone, it must be associated to some complete flag $\mathscr{F}'$, and since $\sigma_{\mathscr{F}} \preceq \tau$ it must be that $\mathscr{F} \subseteq \mathscr{F}'$. Let $\mathscr{F}' = \{F_1, \ldots, F_k, G_1, \ldots, G_\ell\}$, where $G_i \in \mathcal{L}^*$ are proper flats and $\{F_1, \ldots, F_k\} \cap \{G_1, \ldots, G_\ell\} = \emptyset$. As $\rho \subseteq \tau$ and $\overline{\rho} \neq 0$, we know that

$$\rho = \operatorname{cone}(e_{G_j})$$

for some $1 \leq j \leq \ell$. This then means that

$$\overline{\rho} = \operatorname{cone}([e_{G_j}]).$$

Because flats are totally ordered, there must be some $1 \leq i \leq k+1$ such that

$$F_{i-1} \subsetneq G_j \subsetneq F_i.$$

Then we have that

$$\varphi^*(\rho) = \operatorname{cone}(\varphi([e_{G_j}]))$$

and

$$\begin{aligned}
\varphi([e_{G_j}]) &= 0 \oplus \cdots \oplus e_{G_j \cap (F_i \setminus F_{i-1})} \oplus \cdots \oplus 0 \\
&= e_{G_j \cap (F_i \setminus F_{i-1})} \\
&= e_{G_j \setminus F_{i-1}},
\end{aligned}$$

with the last equality coming from the fact that $G_j \subsetneq F_i$. From Proposition 5.2, we know $G_j \setminus F_{i-1}$ corresponds to a flat of the matroid minor $\mathcal{M}_{[F_{i-1}, i]}$. Thus, $\operatorname{cone}(e_{G_j \setminus F_{i-1}})$ is a ray in the product fan of minors. This shows that every ray in the star fan is uniquely mapped to a ray in the product fan.

In the other direction, consider a ray in the product fan, $\rho = (0, \ldots, \rho_i, \ldots, 0)$. We know that ρ_i is a ray of its respective fan in the product, $\mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1}, F_i]}}$. Further, we have that ρ_i is associated to a flat, G, in $\mathcal{M}_{[F_{i-1}, F_i]}$. This means that the ray of the product fan is

$$\rho = \operatorname{cone}(e_G).$$

Taking the inverse of φ^*, we have

$$\begin{aligned}
(\varphi^{-1})^*(\rho) &= \operatorname{cone}\big(\varphi^{-1}(e_G)\big) \\
&= \operatorname{cone}([e_G]) \\
&= \operatorname{cone}([e_G] + [e_{F_{i-1}}]) \\
&= \operatorname{cone}([e_G + e_{F_{i-1}}]) \\
&= \operatorname{cone}([e_{G \cup F_{i-1}}]),
\end{aligned}$$

where we use the fact that $e_{F_{i-1}} \in \operatorname{span}(\sigma_{\mathscr{F}})$ and so $[e_{F_{i-1}}] = [0]$ in the star fan. Again by Proposition 5.2 we have that $G \cup F_{i-1}$ is a flat of $\mathcal{M}$ and is included in a maximal cone that contains $\sigma_{\mathscr{F}}$. This means that $\operatorname{cone}([e_{G\cup F_{i-1}}])$ is a ray in the star fan, as desired.

Since we've shown that the isomorphismφ induces a bijection between cones of $\operatorname{star}(\sigma_{\mathscr{F}}, \mathbf{\Sigma}_{\mathcal{M}})$ and $\prod_{i=1}^{k+1} \mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}}$, we may conclude that the fans are isomorphic. □

This lemma lets us move from star fans, which can be a little hard to reason about, to the product of several Bergman fans of matroids.

5.2 The Bergman Fans of Matroids are AF

With everything in the last section at our disposal, we may finally prove a key theorem.

Theorem 5.5. *For any matroid $\mathcal{M}$, its Bergman fan $\mathbf{\Sigma}_{\mathcal{M}}$ is AF.*

We need only show that the conditions of Theorem 4.11 are met specifically for Bergman fans of matroids. We'll tackle this one condition at a time, starting with the connectedness condition.

Lemma 5.6 (Connectedness)**.** *Let $\mathbf{\Sigma}_{\mathcal{M}}$ be the Bergman fan of a matroid of rank $r+1$. For every cone $\sigma \in \mathbf{\Sigma}_{\mathcal{M}}(k)$, with $k \leq r-2$,*

$$\operatorname{star}(\tau, \mathbf{\Sigma}_{\mathcal{M}}) \setminus \{0\}$$

is connected.

Proof. To prove this, it is sufficient to show that for any two rays in the fan, we may find a series of faces

$$\rho_1 \prec \tau_1 \succ \rho_2 \prec \cdots \succ \rho_{k-1} \prec \tau_{k-1} \succ \rho_k$$

where each ρ_i is a ray and each τ_i is a 2-dimmensional face. Since any point in a fan is connected to a ray, specifically one of the generating rays of the cone the point lives in, a path like this between arbitrary rays is enough to show our fan is connected without the origin.

For some notational convenience, we will write

$$\rho_1 \sim \tau_1 \sim \rho_2 \sim \cdots \sim \rho_{k-1} \sim \tau_{k-1} \sim \rho_k$$

and let the reader interpret the correct face inclusions. If this seems lazy, please take it up with Fields Medal winner June Huh, from whom we lifted this notation. Additionally, in keeping with our general convention, we'll write e_F for the ray generated by $\sum_{i\in F} e_{\{i\}}$ and τ_{F_1,F_2} for the cone generated by rays e_{F_1} and e_{F_2}. Finally, we assume cM is at least a rank 3 matroid, and so $\mathbf{\Sigma}_{\mathcal{M}}$ has maximal cones of dimension at least 2.

Now, we will consider this in two steps. First we'll look at $\mathbf{\Sigma}_{\mathcal{M}} \setminus \{0\}$ itself. Let $e_F, e_{F'} \in \mathbf{\Sigma}_{\mathcal{M}}(1)$ be two arbitrary rays of our fan. If there exists some ground element $i \in F \cap F'$, then, trivially, we have the sequence

$$e_F \sim \tau_{\{F,\{i\}\}} \sim e_{\{i\}} \sim \tau_{\{\{i\},F'\}} \sim e'_F.$$

So, let us consider instead that $F \cap F' = \emptyset$. Let $a \in F$ be an element of F and $b \in F'$. To start, we have the sequence

$$e_F \sim \tau_{\{F,\{a\}\}} \sim e_{\{a\}}.$$

Now, recall the properties of the flats of a matroid, specifically property (F3). This tells us that the flats of rank 2 partition $E \setminus \{a\}$ and so there must be a rank 2 flat, $\widehat{F}$, such that $\{a, b\} \subseteq \widehat{F}$. Then we have

$$e_{\{a\}} \sim \tau_{\left\{\{a\},\widehat{F}\right\}} \sim e_{\widehat{F}} \sim \tau_{\left\{\widehat{F},b\right\}} \sim e_{\{b\}} \sim \tau_{\{\{b\},F'\}} \sim e_{F'},$$

showing a sequence from e_F to e_F', as desired. With that we have shown there is always a path along 2 dimensional faces between any two rays of the Bergman fan of a matroid, and so is connected even without the origin.

Next, we'll turn to the stars of our matroid. Let $k \leq r - 2$ and $\sigma_{\mathscr{F}} \in \mathbf{\Sigma}_{\mathcal{M}}(K)$ be a k-dimensional cone. We have that $\mathrm{star}(\sigma_{\mathscr{F}}, \mathbf{\Sigma}_{\mathcal{M}})$ is a fan with maximal cones of dimension $r - k$, which specifically means they are at least 2-dimensional. From Lemma 5.4, we know that, given $\mathscr{F} = \{F_1 \subsetneq \cdots \subsetneq F_k\}$, we have that $\mathrm{star}(\sigma_{\mathscr{F}}, \mathbf{\Sigma}_{\mathcal{M}})$ is in bijection with the product fan

$$\prod_{i=1}^{k+1} \mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}}.$$

So, it is sufficient to show this product fan is connected. Recall that cones of the product fan are of the form

$$(\sigma_1, \sigma_2, \ldots, \sigma_k) \in \mathbf{\Sigma}_{\mathcal{M}_{[\emptyset,F_1]}} \times \mathbf{\Sigma}_{\mathcal{M}_{[F_1,F_2]}} \times \cdots \times \mathbf{\Sigma}_{\mathcal{M}_{[F_{k-1},F_k]}} \times \mathbf{\Sigma}_{\mathcal{M}_{[F_k,E]}},$$

and that the dimension of the cone $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ is the sum of the dimensions of each cone σ_i. Rays of the product fan are then of the form $(0, 0, \ldots, 0, \rho, 0, \ldots, 0)$ where ρ is a ray of the corresponding fan in the product.

Now, to show our product fan is connected, we'll consider two cases. We will omit irrelevant zeros from our notation going forward for ease of reading, but we can add as many zeros in other positions as necessary without changing any part of the proof. In the first case, our two rays come from different fans in the product If $(\rho_1, 0)$ and $(0, \rho_2)$ are rays, then, trivially, there exists the path

$$(\rho_1, 0) \sim (\rho_1, \rho_2) \sim (0, \rho_2)$$

connecting them. The more nuanced case is if we have two rays from the same fan.

Consider two rays $\rho_1, \rho_2 \in \mathbf{\Sigma}(1)$. If the minor that generates $\mathbf{\Sigma}$ is at least rank 3, then our work above shows there must exist a path between them using only the cones of $\mathbf{\Sigma}$. Where this breaks down, however, is if the minor has rank 2; i.e., the Bergman fan has only 1-dimensional cones. We can't, after removing the origin, get between rays solely within this fan. Recall though that our star fan must be pure of at least dimension 2. This means if this $\mathbf{\Sigma}$ has only 1-dimensional cones, then there is at least one other nonzero fan, which we'll call $\mathbf{\Sigma'}$, in the product that has at least a one ray. Let $\eta \in \mathbf{\Sigma'}$ be said ray. Then we have the path

$$(\rho_1, 0) \sim (\rho_1, \eta) \sim (0, \eta) \sim (\rho_1, \eta)\ (\rho_1, 0)$$

connecting the two rays of $\mathbf{\Sigma}$.

With this we've shown that any possible star of a Bergman fan of a matroid is connected even without the origin. $\square$

Next we may turn to the other requirement, that the quadratic form that determines the volume of the 2-dimensional faces of the normal complex corresponds to a matrix with exactly one positive eigenvalue.

Lemma 5.7 (Volume Quadratic Form Has One Positive Eigenvalue)**.** *Let $\mathcal{M}$ be a matroid of rank $r+1$ and $\mathbf{\Sigma}_{\mathcal{M}}$ be the Bergman fan associated to the matroid, with $* \in \mathrm{Inn}(\boldsymbol{N}_E)$ an inner product.*

For any cubical $z \in \mathrm{Cub}(\mathbf{\Sigma}_{\mathcal{M}}, *)$, the quadratic form associated to the volume polynomial of each 2-dimensional face of the normal complex $C_{\mathbf{\Sigma}_{\mathcal{M}},*}(z)$ has exactly one positive eigenvalue.

Proof. Recall that faces of a normal complex, $F^{\tau}(C_{\mathbf{\Sigma}_{\mathcal{M}},*}(z))$, correspond to truncations of the star fan for some cone $\tau \preceq \mathbf{\Sigma}_{\mathcal{M}}$. Since we want 2-dimensional faces, we will consider the stars associated to some $\tau_{\mathscr{F}} \in \mathbf{\Sigma}_{\mathcal{M}}(r-2)$, where $\mathscr{F} = \{F_1 \subsetneq \cdots \subsetneq F_k\}$. Once again using Lemma 5.4, we have that $\mathrm{star}(\tau, \mathbf{\Sigma}_{\mathcal{M}})$ is in bijection with the product fan

$$\prod_{i=0}^{k+1} \mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}}.$$

Recall that if F_{i-1} and F_i are adjacent to each other in the lattice of flats then the resulting fan $\mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}} = \{0\}$ and so contributes nothing to the product. Since here we have that $k = r-2$, there are only 2 flats not in $\mathscr{F}$. This means our product fan has two possible forms. Either

$$\mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}} \times \mathbf{\Sigma}_{\mathcal{M}_{[F_{j-1},F_j]}} \qquad \text{or} \qquad \mathbf{\Sigma}_{\mathcal{M}_{[F_{i-1},F_i]}}$$

where the first is the product of two rank 2 minors and the second is a rank 3 minor. This just depends on if the two missing flats are adjacent to each other or not. We will again take this proof in cases, considering the two possible structures of our star fan.

In the first case, let us assume that $\mathrm{star}(\tau_{\mathscr{F}}, \mathbf{\Sigma}_{\mathcal{M}}) = \mathbf{\Sigma} \times \mathbf{\Sigma}'$ where $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$ are both pure of dimension 1. This means that the 2-dimensional cones are exactly

$$\{\mathrm{cone}(\rho, \rho') \mid \rho \in \mathbf{\Sigma}(1), \rho' \in \mathbf{\Sigma}(1)\}.$$

Let $z^{\tau_{\mathscr{F}}}$ be the z-values our face. For convenience, let $(x_1, \ldots, x_k) \subset z^{\tau_{\mathscr{F}}}$ be the z-values associated to rays of $\mathbf{\Sigma}$ and $(y_1, \ldots, y_\ell) \subset z^{\tau_{\mathscr{F}}}$ be the z-values associated to rays of $\mathbf{\Sigma}'$. Recall that the volume of a normal complex is the sum of the volumes of the polytopes that comprise it. Since every ray of $\mathbf{\Sigma}$ is orthogonal to every ray of $\mathbf{\Sigma}'$, this has the straight forward volume:

$$\mathrm{Vol}\left(F^{\tau_{\mathscr{F}}}\left(C_{\mathbf{\Sigma}_{\mathcal{M}},*}(z)\right)\right) = \sum_{\substack{1 \le i \le k \\ 1 \le j \le \ell}} x_i y_j.$$

Thinking of this as a polynomial in variables $(x_1, \ldots, x_k, y_1, \ldots, y_\ell) = (\boldsymbol{x}, \boldsymbol{y})$, we see each monomial is of degree 2, so this is a quadratic form. Thus, there exists a symmetric matrix A such that

$$\begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix} A \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \sum_{\substack{1 \le i \le k \\ 1 \le j \le \ell}} x_i y_j.$$

Taking $(\boldsymbol{x}, \boldsymbol{y})$ as a basis, it is not too difficult to work out that $A = [a_{i,j}]$ where

$$a_{i,j} = \begin{cases} \frac{1}{2} & 1 \le i \le k \text{ and } 1 \le j - k \le \ell \\ 0 & \text{otherwise.} \end{cases}$$

Finding the eigenvalues of this matrix, however, is not so easy.

To get around this, we would like to note another way to write our volume polynomial. We offer,

$$\sum_{\substack{1 \le i \le k \\ 1 \le j \le \ell}} x_i y_j = \frac{1}{4}\big((x_1 + \cdots + x_k + y_1 + \cdots y_\ell)^2 - (x_1 + \cdots + x_k - y_1 - \cdots - y_\ell)^2\big).$$

This works, because in the expression $-(x_1 + \cdots + x_k - y_1 - \cdots - y_\ell)^2$, only terms of the form $2x_i y_j$ are positive, with the rest going to cancel out the unwanted terms in the first expression, ultimitly yielding the same volume as above. Our goal now is to find an invertible change of basis, S, that gives us $\frac{1}{4}(x_1 + \cdots + x_k + y_1 + \cdots y_\ell)$ and $\frac{1}{4}(-x_1 - \cdots - x_k + y_1 + \cdots + y_\ell)$ as the first two basis elements. Then the matrix associated with this quadratic form, in this basis, would trivially have eigenvalues $1, -1, 0, \ldots, 0$. Then we could use Proposition 5.1, Sylvester's Law of Inertia, to get that there is exactly one positive eigenvalue of A as well.

Luckily, a rather naïve change of basis matrix works out here. As a quick intuition building example, if $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$, then our change of basis matrix would be

$$S = \frac{1}{4} \left[\begin{array}{rr|rrr} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Clearly this gives us our basis. In general this matrix, which we'll scale to remove the fraction, $4S = [s_{i,j}]$ will be given by

$$s_{i,j} = \begin{cases} 1, & i = 1 \\ -1, & i = 2 \quad \text{and} \quad 1 \le j \le k \\ 1, & i = 2 \quad \text{and} \quad 1 \le j - k \le \ell \\ 1, & i = j + 1 \quad \text{and} \quad 1 \le j - k \le \ell \\ 0, & \text{otherwise.} \end{cases}$$

While this looks rather complicated, it's just two rows of ± 1 and then just 1 along the subdiagonal of all other rows. This matrix is diagonaizable by elementary row operations and will have no zeros along the diagonal, so we are safely assured that S is invertible. Then the quadratic form

$$B = S^{-1}AS$$

is a diagonal matrix with eigenvalues $1, -1, 0, 0, \ldots$, and so we conclude that A must also have exactly 1 positive eigenvalue.

With the first case done, we have to turn to the second, when our face is the normal complex associated to a rank 3 minor. Then $\operatorname{star}(\tau, \mathbf{\Sigma}_{\mathcal{M}}) = \mathbf{\Sigma}$ is a minor of our matroid with rank 3. For convenience, let $F \subseteq \mathcal{L}$ be all rank 1 flats of $\mathbf{\Sigma}$ and let $G \subseteq$ be all rank 2 flats of $\mathbf{\Sigma}$. The work of Nathanson and Ross in [NR23] gives us that the volume of the normal complex of any rank 3 matroid is given by

$$\operatorname{Vol}\left(C_{\mathbf{\Sigma},*}(z)\right) = 2\sum_{F \subsetneq G} z_F z_G - \sum_G z_G^2 - \sum_F (\mathcal{L}^\sharp(F) - 1) z_F^2,$$

where we take $\mathcal{L}^\sharp(F)$ is the number of minimal flats containing F.

Considering this as a polynomial in $z_{F_1}, \ldots, z_{F_k}, z_{G_1}, \ldots, z_{G_\ell}$, it is not immediately obvious that this would have only 1 positive eigenvalue. Again, we propose a different way to write this that has obvious eigenvalues. We propose that

$$\operatorname{Vol}\left(C_{\mathbf{\Sigma},*}(z)\right) = \left(\sum_F z_F\right)^2 - \sum_G \left(z_G - \sum_{F \subsetneq G} z_F\right)^2.$$

This would have eigenvalues $1, -1, \ldots, -1, 0, \ldots, 0$ as desired. To see that these are equal, We will break this down slightly. First, we note that

$$\left(\sum_F z_F\right)^2 = \sum_F z_F^2 + \sum_{F_1, F_2 \in F} 2 z_{F_1} z_{F_2}. \tag{5.1}$$

Then, let's look at the internal part of the second expression to see for a single $\widehat{G} \in G$ we have

$$\left(z_{\widehat{G}} - \sum_{F \subsetneq \widehat{G}} z_F\right)^2 = z_{\widehat{G}}^2 - \sum_{F \subseteq \widehat{G}} 2 z_F z_{\widehat{G}} + \sum_{F_1, F_2 \subseteq \widehat{G}} 2 z_{F_1} z_{F_2} + \sum_{F \in \widehat{G}} z_F^2. \tag{5.2}$$

What's important to note here is that if $F_1, F_2 \subseteq G$, then there cannot be another rank 2 flat G' that contains both. This is a direct consequence of property (F3) of lattice matroids. But when we let the outer sum range over all possible rank 2 flats, we will get $\mathcal{L}^\sharp(F)$ copies of each z_F^2, and so we have

$$\sum_G \left(z_G - \sum_{F \subsetneq G} z_F\right)^2 = \sum_G z_G^2 - \sum_{F \subseteq G} 2 z_F z_G + \sum_{F_1, F_2 \subseteq F} 2 z_{F_1} z_{F_2} + \sum_F \mathcal{L}^\sharp(F) z_F^2. \tag{5.3}$$

So if we subtract our result in (1.3) from the one in (1.1), we are left with

$$2\sum_{F \subset G} z_F z_G - \sum_G z_G^2 - \sum_F (\mathcal{L}\sharp(F) - 1) z_F,$$

thus concluding our final case and proof.

$\square$

With Lemma 5.6 and Lemma 5.7, we see that Bergman fans of matroids will always satisfy the criteria of ?THM? **??**.

5.3 Tying Up Loose Ends

We now know that Bergman fans of matroids are AF, but this will only help us if we can find a pseudocubcal value associated to the divisors α and β. An immediate problem to address is that it could be that no cubical values exist at all. Luckily, Proposition 7.4 of [NR23] gives us the following guarantee.

Proposition 5.8. *Let $\mathcal{M} = (E, \mathcal{L})$ be a matroid with ground set $E = \{e_0, e_1, \ldots, e_n\}$. Take $\boldsymbol{\Sigma}_{\mathcal{M}}$ to be the Bergman fan of $\mathcal{M}$ in $\boldsymbol{N}^E$ such that we associate $(e_1, e_2, \ldots, e_n)$ to the standard basis vectors and $*$ to be the standard inner product on this basis. Then*

$$\mathrm{Cub}(\boldsymbol{\Sigma}_{\mathcal{M}}, *) \neq \emptyset.$$

Going forward in this section, we'll take for granted that a matroid $\mathcal{M} = (E, \mathcal{L})$ has a ground set $E = \{e_0, e_1, \ldots, e_n\}$, and that $\boldsymbol{\Sigma}_{\mathcal{M}} \subseteq \boldsymbol{N}^E$ associates $\{e_1, e_2, \ldots, e_n\}$ to the standard basis vectors. This makes e_0 our distinguished element, such that

$$e_0 = -\sum_{i=1}^{n} e_i.$$

Additionaly, we'll just take $*$ to be the standard inner product. These assumptions allow us to easily invoke the above proposition.

Another important element from [NR23], is that $\overline{\mathrm{Cub}}(\boldsymbol{\Sigma}_{\mathcal{M}}, *)$ is a cone in the space $\mathbb{R}^{\boldsymbol{\Sigma}_{\mathcal{M}}(1)}$. Specifically, purely cubical values are elements of the relative interior of this cone and pseudocubical values are elements on the boundary. If $\mathrm{Cub}(\boldsymbol{\Sigma}_{\mathcal{M}}, *)$ is non-empty then so is $\overline{\mathrm{Cub}}(\boldsymbol{\Sigma}_{\mathcal{M}}, *)$. Averting that problem, we now have to define the z-values associated to α and β and show that they are indeed pseudocubcal.

Recall that the definition of α and β are independent of choice of ground element. This means we're free to pick, and going forward we'll work with the assumption for flats $F \in \mathcal{L}^*$,

$$\alpha = \sum_{e_0 \in F} x_F \quad \text{and} \quad \beta = \sum_{e_0 \notin F} x_F.$$

We stated that the divisor associated to a z-value is given by

$$D(z) = \sum_{\rho \in \boldsymbol{\Sigma}_{\mathcal{M}}} z_\rho x_\rho,$$

but since our fan is the Bergman fan of a matroid, and rays are associated to flats, we may rephrase this as

$$D(z) = \sum_{F \in \mathcal{L}^*} z_F x_F.$$

We define $z^\alpha = (z^\alpha_F \mid F \in \mathcal{L}^*)$ and $z^\beta = (z^\beta_F \mid F \in \mathcal{L}^*)$ where

$$z^\alpha_F = \begin{cases} 1 & e_0 \in F \\ 0 & e_0 \notin F \end{cases} \qquad \text{and} \qquad z^\beta_F = \begin{cases} 0 & e_0 \in F \\ 1 & e_0 \notin F \end{cases}$$

determines the components. A quick inspection shows that $D(z^\alpha) = \alpha$ and $D(z^\beta) = \beta$ as desired. Now we just need to show that z^α and z^β are pseudocubcal.

Proposition 5.9. *Let $\mathcal{M}$ be a matroid, $\mathbf{\Sigma}_\mathcal{M} \subseteq \boldsymbol{N}_E$ be it's Bergman fan, and $*$ be the standard inner product. The z-values z^α and z^β lie in the pseudocubcal cone,*

$$z^\alpha, z^\beta \in \overline{\mathrm{Cub}}(\mathbf{\Sigma}_\mathcal{M}, *).$$

Proof. Let $\mathbf{\Sigma}_\mathcal{M} \subseteq \boldsymbol{N}_E$ be the Bergman fan of $\mathcal{M}$ and take $*$ to be the standard inner product. To avoid confusion, let $E = \{e_0, e_1, \dots, e_n\}$ be the set of ground elements of $\mathcal{M}$ and $\{u_{e_1}, \dots, u_{e_n}\}$ to be the standard basis vectors of $\boldsymbol{N}_E$ associated to the ground elements. As usual, we'll write $u_I = \sum_{i \in I} u_i$ and recall that $u_{e_0} = -u_{e_1} - \cdots - u_{e_n}$. For any $z \in \mathbb{R}^{\mathbf{\Sigma}_{\mathcal{L}^*}}$, we will define $w_\sigma(z)$ to be the unique element in the intersection

$$\mathrm{span}(\sigma_\mathscr{F}) \cap \{v \in \boldsymbol{N}_E \mid v * u_F = z_F \text{ for all flats } F \text{ associated to rays in } \sigma(1)\} = \{w_\sigma\}.$$

That this intersection always yields a single element comes from the fact that our cones are always simplicial.

In order to prove that $z^\alpha, z^\beta \in \mathbb{R}^{\mathbf{\Sigma}_\mathcal{M}(1)}$ are pseudocubcal, we must show that for all cones $\sigma \in \mathbf{\Sigma}_\mathcal{M}$,

$$w_\sigma(z^\alpha) \in \sigma \quad \text{and} \quad w_\sigma(z^\beta) \in \sigma.$$

Take $\sigma_\mathscr{F} \in \mathbf{\Sigma}_\mathcal{M}$ to be an arbitrary cone and $\mathscr{F} = \{F_1 \subsetneq \cdots \subsetneq F_k\}$ to be its associated flag. We will prove that

$$w_{\sigma_\mathscr{F}}(z^\alpha) = \begin{cases} \frac{1}{|F_k^\mathsf{c}|} u_{F_k} & e_0 \in F_k \\ 0 & e_0 \notin F_k \end{cases}$$
$$w_{\sigma_\mathscr{F}}(z^\beta) = \begin{cases} 0 & e_0 \in F_1 \\ \frac{1}{|F_1|} u_{F_1} & e_0 \notin F_1 \end{cases}$$

with F_k^c being the expected set compliment, $F_k^\mathsf{c} = E \setminus F_k$. Since these are in a ray of $\sigma_\mathscr{F}$, they would of course be in $\sigma_\mathscr{F}$. This is sufficient to show that $z^\alpha, z^\beta \in \overline{\mathrm{Cub}}(\mathbf{\Sigma}_\mathcal{M}, *)$.

Before we go on, we will need a fact about the inner product in our space. For subsets $I \subseteq E$ and $J \subseteq E$,

$$u_I * u_J = \begin{cases} |I \cap J| & e_0 \notin I \text{ and } e_0 \notin J \\ -|I \cap J^\mathsf{c}| & e_0 \notin I \text{ and } e_0 \in J \\ |I^\mathsf{c} \cap J^\mathsf{c}| & e_0 \in I \text{ and } e_0 \in J. \end{cases}$$

The first case follows simply from the fact that the dot product multiplies the entries pairwise, and the 1 associated to u_{e_j} will only contribute to the sum if it's in both vectors. The second case follows from the fact that if $e_0 \in J$, $u_J = -u_{E\setminus J}$, thanks to u_{e_0} being a vector with every entry set to -1. The final case follows from the previous two, and that $(-1) \cdot (-1) = 1$.

We'll start with $w_{\sigma_{\mathscr{F}}}(z^\alpha)$. Since it is uniquely characterized by the condition

$$w_{\sigma_{\mathscr{F}}}(z^\alpha) * u_{F_j} = z^\alpha_{F_j}$$

for all $1 \le j \le k$, we just need to show that this equality holds. First, let's assume that $e_0 \in F_k$. Then

$$\begin{aligned} w_{\sigma_{\mathscr{F}}}(z^\alpha) * u_{F_j} &= \frac{1}{|F_k^{\mathsf{c}}|} u_{F_k} * u_{F_j} \\ &= \frac{1}{|F_k^{\mathsf{c}}|}(u_{F_k} * u_{F_j}), \end{aligned}$$

thanks to inner products being bilinear. From above, we know

$$u_{F_k} * u_{F_j} = \begin{cases} |F_k^{\mathsf{c}} \cap F_j| & e_0 \notin F_j \\ |F_k^{\mathsf{c}} \cap F_j^{\mathsf{c}}| & e_0 \in F_j \end{cases}$$

and since $F_j \subseteq F_k$ we can see that

$$u_{F_k} * u_{F_j} = \begin{cases} 0 & e_0 \notin F_j \\ |F_k^{\mathsf{c}}| & e_0 \in F_j \end{cases}$$

by some elementary set theory. And so when we incorperate back in our scalar, we get

$$w_{\sigma_{\mathscr{F}}}(z^\alpha) * u_{F_j} = \begin{cases} 0 & e_0 \notin F_j \\ 1 & e_0 \in F_j \end{cases}$$

which matches the value of $z^\alpha_{F_j}$. Turning to the other case, let's assume that $e_0 \notin F_k$. Then by definition, $z^\alpha_{F_j} = 0$ for all $1 \le j \le k$, and since $0 * u_{F_j} = 0$ we again have

$$w_{\sigma_{\mathscr{F}}}(z^\alpha) * u_{F_j} = z^\alpha_j.$$

We conclude then that we have correctly defined $w_{\sigma_{\mathscr{F}}}(z^\alpha)$ and that it is in the cone $\sigma_{\mathscr{F}}$, and so pseudocubcal.

The proof that $w_{\sigma_{\mathscr{F}}}(z^\beta)$ is well-defined follows analogously. We want to show that

$$w_{\sigma_{\mathscr{F}}}(z^\beta) * u_{F_j} = z^\beta_{F_j}$$

for all $1 \le j \le k$. This time we start with the case that $e_0 \notin F_1$. Then as defined, we have

$$w_{\sigma_{\mathscr{F}}}(z^\beta) * u_{F_j} = \frac{1}{|F_k|}(u_{F_k} * u_{F_j}).$$

Using some set theory we get

$$u_{F_1} * u_{F_j} = \begin{cases} |F_1 \cap F_j| = |F_1| & e_0 \notin F_j \\ |F_1 \cap F_j^{\mathsf{c}}| = 0 & e_0 \in F_j \end{cases}$$

and so

$$w_{\sigma_{\mathscr{F}}}(z^\beta) * u_{F_j} = \begin{cases} 1 & e_0 \notin F_j \\ 0 & e_0 \in F_j \end{cases}$$

again matching how we have defined the component $z^\beta_{F_j}$. On the other hand, if $e_0 \in F_1$, then $z^\beta_{F_j} = 0$ for each $1 \leq j \leq k$. Since $w_{\sigma_{\mathscr{F}}}(z^\beta) = 0$ this again works just fine.

With that we've shown that both z^α and z^β are pseudocubcal values in $\overline{\mathrm{Cub}}(\mathbf{\Sigma}_{\mathcal{M}}, *)$. □

Finally, we have one last problem to address. From above we can see that $z^\alpha, z^\beta \notin \mathrm{Cub}(\mathbf{\Sigma}_{\mathcal{M}}, *)$. They are on the boundary of $\overline{\mathrm{Cub}}(\mathbf{\Sigma}_{\mathcal{M}}, *)$, however Theorem 4.11 assumes stirctly cubical z-values. We turn, in desperation, to a bit of analysis to help us here.

Lemma 5.10. *Let $\mathcal{M}$ be a matroid and $* \in \mathrm{Inn}(\boldsymbol{N}_E)$ an inner product such that $\mathrm{Cub}(\mathbf{\Sigma}_{\mathcal{M}}, *)$ is nonempty. Then for any $z \in \overline{\mathrm{Cub}}(\mathbf{\Sigma}_{\mathcal{M}}, *)$, there exists values z_t such that*

$$\lim_{t \to 0} z_t = z$$

and every $z_t \in \mathrm{Cub}(\mathbf{\Sigma}_{\mathcal{M}}, *)$.

Proof. This actually follow from convex geometry. Take any $z \in \overline{\mathrm{Cub}}(\mathbf{\Sigma}_{\mathcal{M}}, *)$, and recall that $\mathrm{Cub}(\mathbf{\Sigma}_{\mathcal{M}}, *)$ is a cone in $\mathbb{R}^{\mathbf{\Sigma}_{\mathcal{M}}(1)}$. Since we've assumed $\mathrm{Cub}(\mathbf{\Sigma}_{\mathcal{M}}, *)$ is nonempty, let $z_1 \in \mathrm{Cub}(\mathbf{\Sigma}_{\mathcal{M}}, *)$ be a point in the interior of the cone. Then define

$$z_t = tz_1 + (1-t)z.$$

Clearly $\lim_{t \to 0} z_t = z$ and when t is restricted to the interval $[0, 1]$ this is just the convex hull of z and z_1. Since this line segment must be contained in the cone, all points must be in the interior except maybe z, and so $z_t \in \mathrm{Cub}(\mathbf{\Sigma}_{\mathcal{M}}, *)$. □

This allows us to approximate any pseudocubcal value using only cubical values. We now, truly, have everything we need to get to the final result.

5.4 Putting It All Together

Main Result. For any matroid $\mathcal{M}$, the Heron-Rota-Welsh conjecture is true.

Proof. Let $\mathcal{M} = (E, \mathcal{L})$ be a matroid with ground set $E = \{e_0, e_1, \ldots, e_n\}$. Take $\mathbf{\Sigma}_{\mathcal{M}} \subseteq \boldsymbol{N}_E$ to be the Bergman fan of $\mathcal{M}$ in $\boldsymbol{N}^E$ such that we associate $(e_1, e_2, \ldots, e_n)$ to the standard basis vectors and choose $* \in \mathrm{Inn}(\boldsymbol{N}_E)$ to be the standard inner product on this basis.

By Proposition 3.5, we know that the reduced characteristic coefficients of $\mathcal{M}$ can be associated to mixed degrees of divisors under the degree map,

$$\overline{w}_k = \deg(\alpha^{r-k}\beta^k).$$

And by Theorem 4.10, we have that

$$\mathrm{MVol}_{\mathbf{\Sigma},*}(z_1, \ldots, z_d) = \deg\big(D(z_1) \cdots D(z_d)\big).$$

So, by using the z-values defined in the previous section, we can identify reduced characteristic coefficients with mixed volumes of normal complexes, given by

$$\overline{w}_k = \mathrm{MVol}_{\Sigma,*}(\underbrace{z^\alpha, \ldots, z^\alpha}_{r-k}, \underbrace{z^\beta, \ldots, z^\beta}_{k}).$$

By Proposition 5.8 we know that $\overline{\mathrm{Cub}}(\mathbf{\Sigma}, *)$ is non-empty and so we may use Lemma 5.10 to get z_t^α and z_t^β such that

$$\lim_{t\to 0} z_t^\alpha = z^\alpha \quad \text{and} \quad \lim_{t\to 0} z_t^\beta = z^\beta.$$

Define

$$\overline{w}_{k,t} = \mathrm{MVol}_{\Sigma,*}(\underbrace{z_t^\alpha, \ldots, z_t^\alpha}_{r-k}, \underbrace{z_t^\beta, \ldots, z_t^\beta}_{k})$$

which means $\lim_{t\to 0} \overline{w}_{k,t} = \overline{w}_k$. Because $z_t^\alpha, z_t^\beta \in \mathrm{Cub}(\mathbf{\Sigma}_{\mathcal{M}}, *)$ and $\mathbf{\Sigma}_{\mathcal{M}}$ is AF by Theorem 5.5, we have that

$$\left(\mathrm{MVol}_{\Sigma,*}(\underbrace{z_t^\alpha, \ldots, z_t^\alpha}_{r-k}, \underbrace{z_t^\beta, \ldots, z_t^\beta}_{k})\right)^2 \geq \mathrm{MVol}_{\Sigma,*}(\underbrace{z_t^\alpha, \ldots, z_t^\alpha}_{r-k-1}, \underbrace{z_t^\beta, \ldots, z_t^\beta}_{k-1})\,\mathrm{MVol}_{\Sigma,*}(\underbrace{z_t^\alpha, \ldots, z_t^\alpha}_{r-k+1}, \underbrace{z_t^\beta, \ldots, z_t^\beta}_{k+1})$$

for $1 < k < r$, as per Definition 4.26. This in turn means

$$\overline{w}_{k,t}^2 \geq \overline{w}_{k-1,t}\overline{w}_{k+1,t}.$$

Since $\mathrm{MVol}_{\Sigma,*}$ is always non-negative, this makes the sequence $\{\overline{w}_{1,t}, \ldots, \overline{w}_{k-1,t}\}$ log-concave for any $t \in (0, 1]$. This implies that under the limit as $t \to 0$ we also have $\{\overline{w}_1, \ldots, \overline{w}_{k-1}\}$ is a log-concave sequence. As the reduced characteristic coefficients are log-concave, Lemma 3.4 tells us we may conclude that the Whitney numbers of the first kind are log-concave, proving the Heron-Rota-Welsh conjecture.

□

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