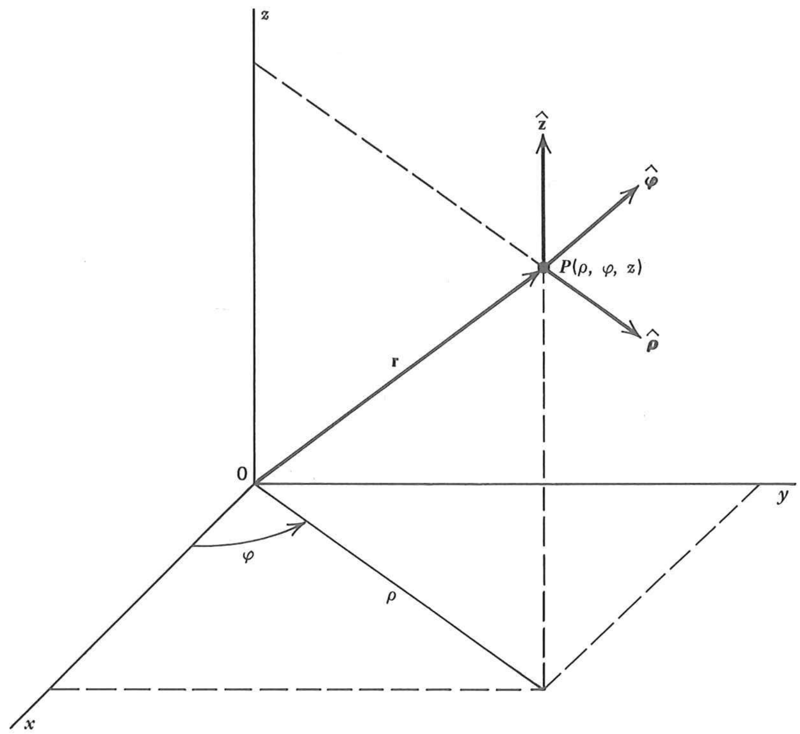

Figure 1-37. Definition of cylindrical coordinates.

1-37; this figure also shows the position vector **r** of the point along with three new unit vectors that we will define shortly. We see that when **r** is projected onto the xy plane, ρ is the length of this projection while φ is the angle that it makes with the positive x axis; z is the same as the corresponding rectangular coordinate. The relation between the cylindrical and rectangular coordinates of P is seen from the figure to be

$$x=\rho\cos\varphi \qquad y=\rho\sin\varphi \qquad z=z \tag{1-74}$$

so that

$$\rho=(x^2+y^2)^{1/2} \qquad \tan\varphi=\frac{y}{x} \tag{1-75}$$

We can now define a set of three mutually perpendicular unit vectors as follows: first, $\hat{\mathbf{z}}$ is the same as the rectangular $\hat{\mathbf{z}}$; second, $\hat{\boldsymbol{\rho}}$ is chosen to be in the direction of increasing ρ and is perpendicular to $\hat{\mathbf{z}}$ so that $\hat{\boldsymbol{\rho}}$ is parallel to the xy plane; finally, $\hat{\boldsymbol{\varphi}}$ is defined to be perpendicular to both of these and in the direction shown. We see that $\hat{\boldsymbol{\varphi}}$ is perpendicular to the semiinfinite plane $\varphi=$ const., and its direction is therefore in the sense of increasing φ. These unit vectors are shown at the location of P in order to emphasize that they are functions of P in the sense that if P is displaced, both $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\varphi}}$ change their directions, although $\hat{\mathbf{z}}$ does not change. Thus, these unit vectors are not all constants, in contrast to $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$.

Since $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\varphi}}$, and $\hat{\mathbf{z}}$ are mutually perpendicular unit vectors, they satisfy relations analogous to (1-18), (1-19), and (1-25):

$$\begin{aligned}
&\hat{\boldsymbol{\rho}}\cdot\hat{\boldsymbol{\rho}}=\hat{\boldsymbol{\varphi}}\cdot\hat{\boldsymbol{\varphi}}=\hat{\mathbf{z}}\cdot\hat{\mathbf{z}}=1\\
&\hat{\boldsymbol{\rho}}\cdot\hat{\boldsymbol{\varphi}}=\hat{\boldsymbol{\varphi}}\cdot\hat{\mathbf{z}}=\hat{\mathbf{z}}\cdot\hat{\boldsymbol{\rho}}=0\\
&\hat{\boldsymbol{\rho}}\times\hat{\boldsymbol{\varphi}}=\hat{\mathbf{z}}\qquad \hat{\boldsymbol{\varphi}}\times\hat{\mathbf{z}}=\hat{\boldsymbol{\rho}}\qquad \hat{\mathbf{z}}\times\hat{\boldsymbol{\rho}}=\hat{\boldsymbol{\varphi}}
\end{aligned}\tag{1-76}$$

The rectangular components of $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\varphi}}$ are found from inspecting Figure 1-37; it is helpful to imagine them projected onto the xy plane, and the results are

$$\hat{\boldsymbol{\rho}}=\cos\varphi\,\hat{\mathbf{x}}+\sin\varphi\,\hat{\mathbf{y}}\qquad \hat{\boldsymbol{\varphi}}=-\sin\varphi\,\hat{\mathbf{x}}+\cos\varphi\,\hat{\mathbf{y}}\tag{1-77}$$

These equations can be solved for $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ to give their components in cylindrical coordinates:

$$\hat{\mathbf{x}}=\cos\varphi\,\hat{\boldsymbol{\rho}}-\sin\varphi\,\hat{\boldsymbol{\varphi}}\qquad \hat{\mathbf{y}}=\sin\varphi\,\hat{\boldsymbol{\rho}}+\cos\varphi\,\hat{\boldsymbol{\varphi}}\tag{1-78}$$

By differentiating (1-77), we can find explicitly how $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\varphi}}$ vary as P is displaced:

$$\frac{d\hat{\boldsymbol{\rho}}}{d\varphi}=\hat{\boldsymbol{\varphi}}\qquad\text{and}\qquad \frac{d\hat{\boldsymbol{\varphi}}}{d\varphi}=-\hat{\boldsymbol{\rho}}\tag{1-79}$$

Since $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\varphi}}$, and $\hat{\mathbf{z}}$ are mutually perpendicular, we can express any vector $\mathbf{A}$ in terms of its components along these directions; by analogy with (1-5), we write $\mathbf{A}$ in the form

$$\mathbf{A}=A_\rho\hat{\boldsymbol{\rho}}+A_\varphi\hat{\boldsymbol{\varphi}}+A_z\hat{\mathbf{z}}\tag{1-80}$$

For the particular case of the position vector $\mathbf{r}$, we see from Figure 1-37 that

$$\mathbf{r}=\rho\hat{\boldsymbol{\rho}}+z\hat{\mathbf{z}}\tag{1-81}$$

which can also be obtained from (1-11) by substitution of (1-74) and the use of (1-77). We can also find the differential of $d\mathbf{r}$ from (1-81) and (1-79):

$$d\mathbf{r}=d\rho\,\hat{\boldsymbol{\rho}}+\rho\,d\hat{\boldsymbol{\rho}}+dz\,\hat{\mathbf{z}}=d\rho\,\hat{\boldsymbol{\rho}}+\rho\,d\varphi\,\hat{\boldsymbol{\varphi}}+dz\,\hat{\mathbf{z}}\tag{1-82}$$

so that its components in the directions of increasing ρ, φ, and z are $d\rho$, $\rho\,d\varphi$, and dz, respectively. These component displacements are shown in Figure 1-38, and we see that they correspond to the distance moved by P resulting from the change of any one coordinate while the other two are held fixed. The shaded volume element has as sides just the components of $d\mathbf{r}$ given by (1-82). Therefore, the element of volume in cylindrical coordinates is given by

$$d\tau=\rho\,d\rho\,d\varphi\,dz\tag{1-83}$$

We also see from this figure that the areas perpendicular to the unit vectors are $\rho\,d\varphi\,dz$, $d\rho\,dz$, and $\rho\,d\rho\,d\varphi$, so that the components of an element of area $d\mathbf{a}$ are

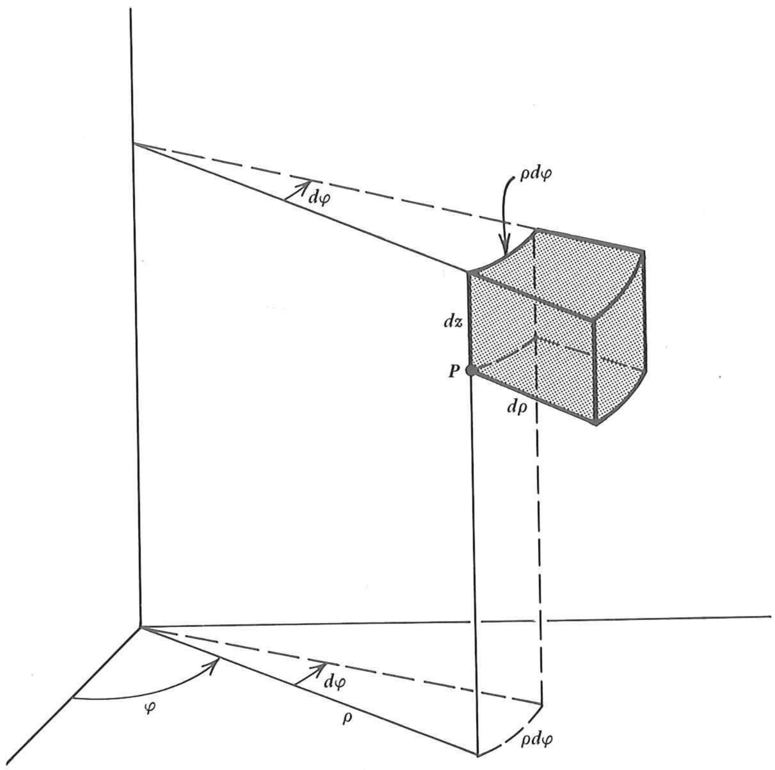

Figure 1-38. Volume element in terms of cylindrical coordinates.

given by

$$da_\rho = \pm \rho\, d\varphi\, dz \qquad da_\varphi = \pm\, d\rho\, dz \qquad da_z = \pm \rho\, d\rho\, d\varphi \tag{1-84}$$

where the proper sign is to be chosen in the same manner as for (1-55).

Now we can go on to find what our differential operators become when expressed in this system. If $u = u(\rho, \varphi, z)$, then

$$du = \frac{\partial u}{\partial \rho}\, d\rho + \frac{\partial u}{\partial \varphi}\, d\varphi + \frac{\partial u}{\partial z}\, dz$$

and, on comparing this with (1-82) and the general definition of the gradient given in (1-38), and noting that $d\mathbf{s} = d\mathbf{r}$ in this case, we find that the expression for the gradient in cylindrical coordinates is

$$\nabla u = \hat{\boldsymbol{\rho}} \frac{\partial u}{\partial \rho} + \hat{\boldsymbol{\varphi}} \frac{1}{\rho} \frac{\partial u}{\partial \varphi} + \hat{\mathbf{z}} \frac{\partial u}{\partial z} \tag{1-85}$$

so that the del operator can be written as

$$\nabla = \hat{\boldsymbol{\rho}} \frac{\partial}{\partial \rho} + \hat{\boldsymbol{\varphi}} \frac{1}{\rho} \frac{\partial}{\partial \varphi} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \tag{1-86}$$

By applying (1-86) to (1-80), we can find corresponding expressions for the divergence and curl. In doing this, however, we must remember that the unit vectors are not constant and we have to take (1-79) into account; in addition, we need to use (1-76). We illustrate this process for $\nabla \cdot \mathbf{A}$:

$$\begin{aligned}
\nabla \cdot \mathbf{A} &= \left(\hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho} + \hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial}{\partial \varphi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\cdot\left(A_\rho\hat{\boldsymbol{\rho}} + A_\varphi\hat{\boldsymbol{\varphi}} + A_z\hat{\mathbf{z}}\right)\\
&= \hat{\boldsymbol{\rho}}\cdot\left(\frac{\partial A_\rho}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{\partial A_\varphi}{\partial \rho}\hat{\boldsymbol{\varphi}} + \frac{\partial A_z}{\partial \rho}\hat{\mathbf{z}}\right)\\
&\quad + \hat{\boldsymbol{\varphi}}\cdot\frac{1}{\rho}\left(\frac{\partial A_\rho}{\partial \varphi}\hat{\boldsymbol{\rho}} + A_\rho\frac{\partial \hat{\boldsymbol{\rho}}}{\partial \varphi} + \frac{\partial A_\varphi}{\partial \varphi}\hat{\boldsymbol{\varphi}} + A_\varphi\frac{\partial \hat{\boldsymbol{\varphi}}}{\partial \varphi} + \frac{\partial A_z}{\partial \varphi}\hat{\mathbf{z}}\right)\\
&\quad + \hat{\mathbf{z}}\cdot\left(\frac{\partial A_\rho}{\partial z}\hat{\boldsymbol{\rho}} + \frac{\partial A_\varphi}{\partial z}\hat{\boldsymbol{\varphi}} + \frac{\partial A_z}{\partial z}\hat{\mathbf{z}}\right)\\
&= \frac{\partial A_\rho}{\partial \rho} + \frac{A_\rho}{\rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}
\end{aligned}$$

which is usually written as

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z} \tag{1-87}$$

In the same manner, the expression for the curl becomes

$$\nabla \times \mathbf{A} = \hat{\boldsymbol{\rho}}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right) + \hat{\boldsymbol{\varphi}}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) + \hat{\mathbf{z}}\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\varphi) - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \varphi}\right] \tag{1-88}$$

while $\nabla \cdot \nabla u$ turns out to be

$$\nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} \tag{1-89}$$

where u is a scalar function of position.

It is very important to remember that (1-86), (1-87), (1-88), and (1-89) *cannot* be obtained from the corresponding expressions in rectangular coordinates as given by (1-41), (1-42), (1-43), and (1-46) by the simple replacement of x, y, z by ρ, φ, z. Similarly, (1-44) and (1-47) can only be used for rectangular coordinates; see (1-122) for the definition of $\nabla^2\mathbf{A}$ for other coordinate systems. You would be surprised at how often these mistakes are made.

1-17 Spherical Coordinates

In this system, the location of a point P is specified by the three quantities r, θ, φ shown in Figure 1-39. We see that r is the distance from the origin and thus the magnitude of the position vector $\mathbf{r}$, θ is the angle made by $\mathbf{r}$ with the positive z axis, while φ is again the angle made with the positive x axis by the projection of $\mathbf{r}$ onto the xy plane. The relations between the rectangular and spherical coordi-

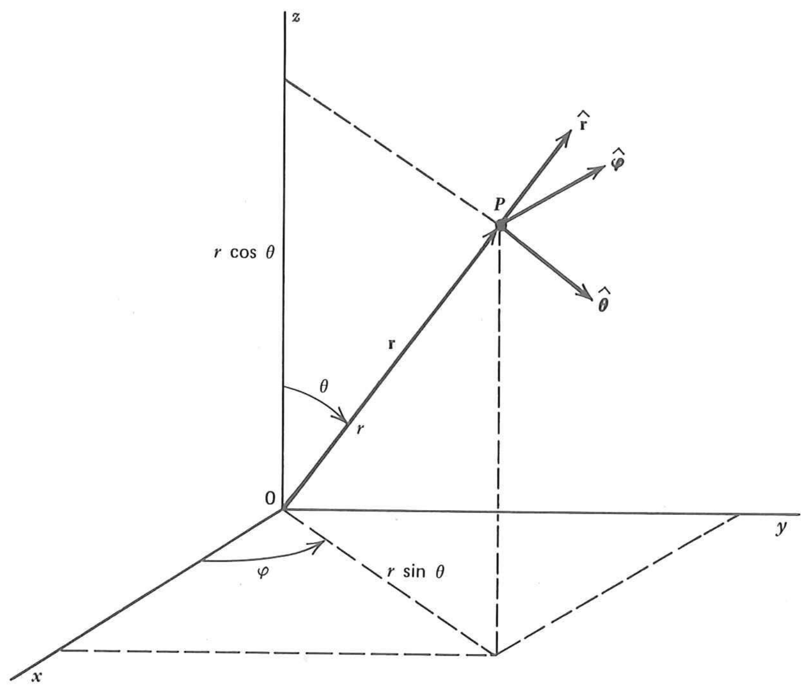

Figure 1-39. Definition of spherical coordinates.

nates are seen to be

$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta \tag{1-90}$$

so that

$$r = (x^2+y^2+z^2)^{1/2} \qquad \tan\theta = \frac{(x^2+y^2)^{1/2}}{z} \qquad \tan\varphi = \frac{y}{x} \tag{1-91}$$

We define a set of mutually perpendicular unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\varphi}}$ in the sense of increasing r, θ, and φ, respectively, as shown in Figure 1-39; we see that as the location of P is changed, all three of these vectors also change. They satisfy relations analogous to (1-76):

$$\begin{gathered}
\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\varphi}}\cdot\hat{\boldsymbol{\varphi}} = 1 \\
\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}\cdot\hat{\mathbf{r}} = 0 \\
\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\varphi}} \qquad \hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\varphi}} = \hat{\mathbf{r}} \qquad \hat{\boldsymbol{\varphi}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}
\end{gathered} \tag{1-92}$$

Their rectangular components are found from inspection of the figure to be

$$\begin{aligned}
\hat{\mathbf{r}} &= \sin\theta\cos\varphi\hat{\mathbf{x}} + \sin\theta\sin\varphi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \\
\hat{\boldsymbol{\theta}} &= \cos\theta\cos\varphi\hat{\mathbf{x}} + \cos\theta\sin\varphi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}} \\
\hat{\boldsymbol{\varphi}} &= -\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}}
\end{aligned} \tag{1-93}$$

and therefore

$$\hat{\mathbf{x}} = \sin\theta\cos\varphi\hat{\mathbf{r}} + \cos\theta\cos\varphi\hat{\boldsymbol{\theta}} - \sin\varphi\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{y}} = \sin\theta\sin\varphi\hat{\mathbf{r}} + \cos\theta\sin\varphi\hat{\boldsymbol{\theta}} + \cos\varphi\hat{\boldsymbol{\varphi}} \tag{1-94}$$

$$\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$$

By differentiating (1-93), we obtain

$$\begin{aligned} \frac{\partial\hat{\mathbf{r}}}{\partial\theta} &= \hat{\boldsymbol{\theta}} & \frac{\partial\hat{\mathbf{r}}}{\partial\varphi} &= \sin\theta\hat{\boldsymbol{\varphi}} \\ \frac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta} &= -\hat{\mathbf{r}} & \frac{\partial\hat{\boldsymbol{\theta}}}{\partial\varphi} &= \cos\theta\hat{\boldsymbol{\varphi}} \\ \frac{\partial\hat{\boldsymbol{\varphi}}}{\partial\theta} &= 0 & \frac{\partial\hat{\boldsymbol{\varphi}}}{\partial\varphi} &= -\sin\theta\hat{\mathbf{r}} - \cos\theta\hat{\boldsymbol{\theta}} \end{aligned} \tag{1-95}$$

A vector **A** will be written in component form as

$$\mathbf{A} = A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\varphi\hat{\boldsymbol{\varphi}} \tag{1-96}$$

The position vector is

$$\mathbf{r} = r\hat{\mathbf{r}} \tag{1-97}$$

and its differential is found with the use of (1-93) and (1-95) to be

$$d\mathbf{r} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\varphi\,\hat{\boldsymbol{\varphi}} \tag{1-98}$$

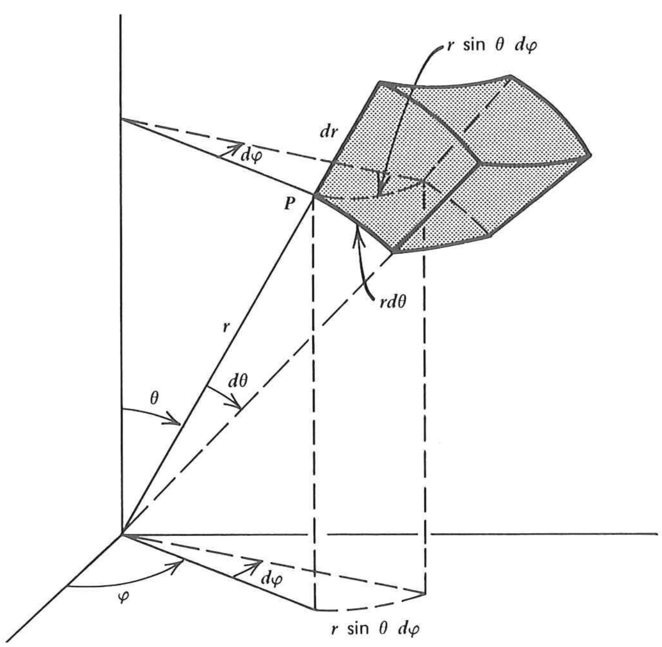

Figure 1-40. Volume element in terms of spherical coordinates.

These component displacements are shown in Figure 1-40. Since the shaded volume element has as sides just the components of (1-98), the volume element will be

$$d\tau = r^2 \sin\theta \, dr \, d\theta \, d\varphi \tag{1-99}$$

while the components of an element of area $d\mathbf{a}$ will be given by

$$da_r = \pm r^2 \sin\theta \, d\theta \, d\varphi \qquad da_\theta = \pm r \sin\theta \, dr \, d\varphi \qquad da_\varphi = \pm r \, dr \, d\theta \tag{1-100}$$

If $u = u(r,\theta,\varphi)$, then

$$du = \frac{\partial u}{\partial r} dr + \frac{\partial u}{\partial \theta} d\theta + \frac{\partial u}{\partial \varphi} d\varphi$$

so that the gradient as obtained from (1-38) and (1-98) is

$$\nabla u = \hat{\mathbf{r}} \frac{\partial u}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial u}{\partial \theta} + \hat{\boldsymbol{\varphi}} \frac{1}{r \sin\theta} \frac{\partial u}{\partial \varphi} \tag{1-101}$$

showing that the del operator is

$$\nabla = \hat{\mathbf{r}} \frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\boldsymbol{\varphi}} \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \tag{1-102}$$

If we now proceed to use (1-102) and (1-96) in the same manner as in the last section, taking account of (1-92) and (1-95), we find:

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r \sin\theta} \frac{\partial A_\varphi}{\partial \varphi} \tag{1-103}$$

$$\begin{aligned} \nabla \times \mathbf{A} = {} & \frac{\hat{\mathbf{r}}}{r \sin\theta} \left[\frac{\partial}{\partial \theta}(\sin\theta A_\varphi) - \frac{\partial A_\theta}{\partial \varphi} \right] + \frac{\hat{\boldsymbol{\theta}}}{r} \left[\frac{1}{\sin\theta} \frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r}(r A_\varphi) \right] \\ & + \frac{\hat{\boldsymbol{\varphi}}}{r} \left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \end{aligned} \tag{1-104}$$

$$\nabla^2 u = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 u}{\partial \varphi^2} \tag{1-105}$$

Remarks similar to those following (1-89) also apply here. You *cannot* get the correct spherical coordinate expressions for (1-102) through (1-105) from the corresponding ones in rectangular coordinates by simply replacing the symbols x,y,z by r,θ,φ.

1-18 The Helmholtz Theorem

We will not prove this theorem at this time but simply quote it as an aid to understanding the motivations for many of the procedures we will be following. We will, in effect, prove it eventually, but piece by piece.