NEUTRON RADIOGRAPHY USING A 400 keV COCKCROFT-WALTON ELECTROSTATIC ACCELERATOR

Clay Michael Donges

Physics 450--Independent Study

Dr. Robert W. Beyer--Dr. Albert J. Frasca

Wittenberg University

1970-1971

PREFACE

The purpose of this paper is to acquaint the reader with the principles, methods, and possible applications of neutron radiography. The subject matter herein is divided into four parts to make a complete and logical discussion. The first section is devoted to a discussion of Wittenberg University's 400 keV Cockcroft-Walton electrostatic accelerator used to provide a source of neutrons. The second section presents a conceptual discussion of the nuclear physics applicable to neutron radiography. A subsequent section deals explicitly with principles of neutron radiography. The final topic contains experimental procedures, results, and conclusions. This section is further subdivided into thermal (slow) neutron work done with 2.4 MeV neutrons produced by a $D(d,n)He^3$ reaction and fast and thermal neutron work done with 14.3 MeV neutrons produced by a $T(d,n)He^4$ reaction.

I wish to give thanks here to my independent study supervisors Dr. Robert W. Beyer and Dr. Albert J. Frasca, for surely without their patience, guidance, and persistent sarcasm this work would not have been able to be completed. I must also thank Wittenberg University and COSIP for the purchase of the 400 keV Cockcroft-Walton electrostatic accelerator from Accelerators Incorporated.

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SECTION 1: THE 400 keV COCKCROFT-WALTON ELECTROSTATIC ACCELERATOR

A. INTRODUCTION:

The heart of Wittenberg University's nuclear laboratory is its 400 keV Cockcroft-Walton electrostatic accelerator. Emphasis in this section is placed on the operation of the accelerator and on the concepts involved in producing an accelerated beam of charged particles. The first topic involves the actual operation of the accelerator from the control console, subsequent topics providing an explanation of the electrostatic accelerator. Here the electrostatic accelerator is broken down into its consituent parts--accelerating tube, high-voltage terminal, and Cockcroft-Walton high-voltage generator. Special attention is focused on the Cockcroft-Walton high-voltage generator in a detailed mathematical treatment.

B. CONTROL CONSOLE:

The accelerator is operated completely from a control console in the equipment area, shielded from the electrostatic accelerator itself in the experimental area by a Two ~~three~~-foot thick concrete wall for protection against X-rays, gamma-rays, and neutrons generated by the accelerator. The actual operation of the accelerator requires a careful check of starting procedures. After inspecting the previous operator's description of machine operation in the accelerator log, a new entry is made discussing the experiment and machine requirements. Before the accelerator is actually "run" a number of procedures must be carried out in the experimental area. The pressurized dome, shrouding the accelerator tube and filled with sulfur hexaflouride to insulate the terminal from ground, must be checked to insure a pressure of approximately 20 pounds per square inch. The freon pump must be turned on to cool the terminal electronics during operation, and the pressure through the gas line from the gas bottle must be maintained at approximately 20 pounds per square inch. Water flow must also be checked to provide adequate cooling with a steady stream of water through the target, diffusion pump, and freon heat exchanger. As a means of connecting the equipment area with the experimental area, radiation monitors and phones are turned on.

At this point the accelerator operator is ready to attempt development of a beam of nuclear particles in the accelerator from the control console. The motor start button is activated to start the 800 cycle per second motor generator functioning as a source of voltage for the large transformer. Upon activation of the high-voltage on button the Cockcroft-Walton high-voltage generator is capable of producing a maximum voltage of 400,000 volts D. C. After the voltage is turned up to a minimum of 100,000 volts the gas used as a primary source of nuclei is bled into the ion source from a control panel valve. All adjustments made in the accelerator are performed at the console with the aid of small electric motors. Upon reaching a satisfactory amount of gas in the ion source the radio-frequency (R-F) oscillator is turned on at 350 watts and 100 megahertz to ionize the gas. At this frequency and power, in the case of hydrogen gas, the energy density is sufficient enough to remove one electron from the hydrogen atom, leaving positively charged nuclei in the ion source. After a few minutes of warm-up time the beam control is turned on, producing the beam of nuclear particles.

Beam current, Cockcroft-Walton transformer current, and voltage are all monitored from the control console. Gas and R-F power adjustments can be made from the console in addition to the focus for constricting the nuclei as they exit from the ion source by means of a magnetic coil. Probe adjustments can be made for charged nuclei ejection by varying potential from 0 to 10,000 volts, and the focus may be used for electrostatically focusing the beam with the einzel lens. Drawings of the Wittenberg University Accelerator Laboratory, the 400 keV Cockcroft-Walton electrostatic accelerator, and accelerator specifications from Accelerators Incorporated are presented as figures 1, 2, and 3.

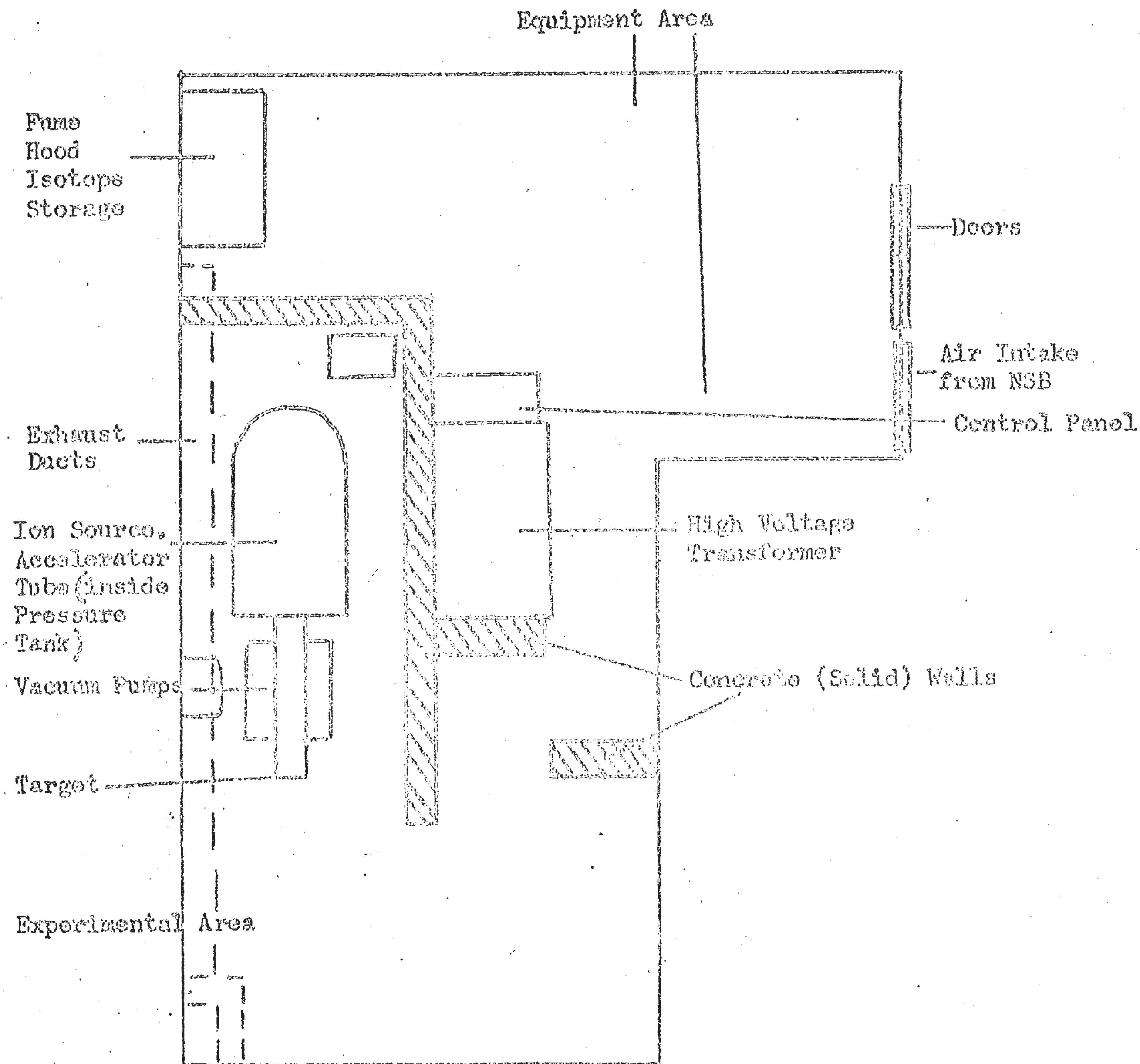
C. <u>ELECTROSTATIC ACCELERATOR</u>:

To observe the excited states of most nuclei one cannot depend upon a PARENT radioactive nucleus decaying to its excited states. In some way excess energy must be imparted to a stable nucleus in its ground state so that upon reaching its excited state the nucleus will decay. The technique commonly used is to accelerate charged particles at some target, imparting excess energy to target nuclei. This process of creating radioactive target nuclei is not one that naturally occurs in nature but the technique of

Experimenters ______ Date ____4____

Reaction ______ Energy ______

WITTENBERG NUCLEAR ACCELERATOR LABORATORY



Indicate nature of radiation detected, amount of radiation and location.

FIGURE 1.

WITTENBERG ACCELERATOR LABORATORY

Cockroft Walton Type Electrostatic Accelerator

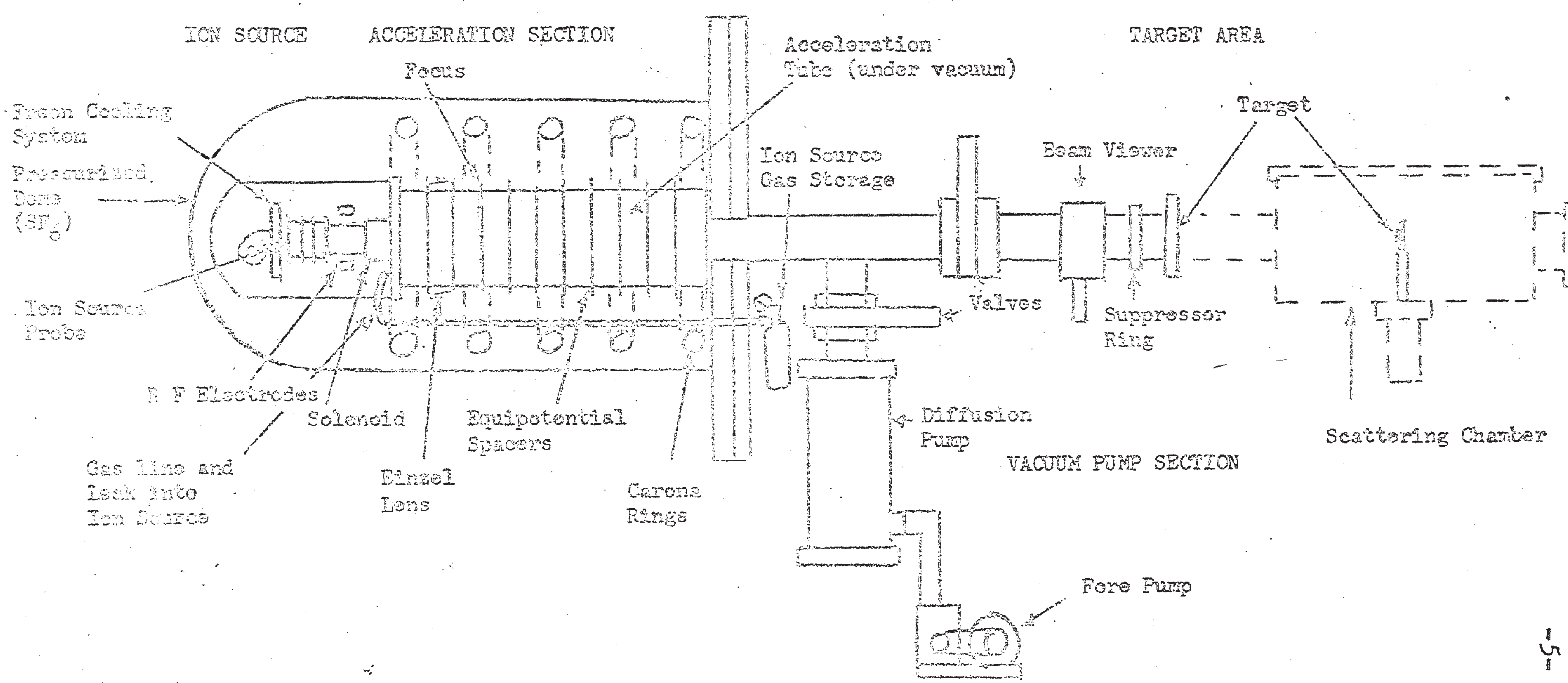


FIGURE 2.

ACCELERATORS INCORPORATED.

PHONE 512 444-3639 • 212 INDUSTRIAL BLVD. • P. O. BOX 3293 AUSTIN, TEXAS 78745

ACCELERATOR 400T

Specifications

Accelerating Voltage:	10 - 400 keV (continuously variable)
Beam Current:	0 to 1.0 MA hydrogen
Beam Focus:	1/8" to 1" continuously variable
Pulsing (optional)	Pre- and post-acceleration
Amplitude:	0 - 1.0 MA (continuously variable)
Pulse Width:	.5 - 10^5 μsec (continuously variable)
Repetition Rate:	1 - 10^5 PPS (continuously variable) plus single pulse control from console
Rise Time (10% - 90%):	Less than 100 nsec
Decay Time:	Less than 250 nsec
Residual:	Less than 10^{-6}

Components

The Accelerator 400T system includes the following components:

I. Accelerator Assembly

II. Insulating Vessel

III. Control Console (with 33 feet of interconnecting cable)

IV. 400 kV High Voltage Power Supply (with isolation transformer)

I. Accelerator Assembly:

1. Accelerator high voltage terminal includes:

   a. 100 mc R-F oscillator

   b. R-F oscillator power supply, output: 1000 V @ 350 MA

FIGURE 3.

ACCELERATOR 400T
Components 2

c. R-F ion source extraction power supply, 0 - 10 kV output continuously adjustable from console

d. Gap lens power supply, 0 to -22 kV output

e. Focus supply, 0 to -5 kV continuously variable from console

f. Ion source: A unique R-F ion source is provided. Ion source provides ions of any non-corrosive gas.

g. Mechanical leak valve

h. Hydrogen lecture bottle

2. Accelerating column:

a. Special dual electrode gap lens with collimator

b. Variable focus electrode

c. Twenty accelerating electrodes

d. Secondary electron suppressor

e. Control rods for focus, extraction probe, mechanical leak, and beam on-off switch

NOTE: Column is designed for air operation at accelerating voltages up to 225 keV.

f. Vacuum system:

A special 680 liter/second oil diffusion pump is supplied. Pump is equipped with builtin baffle which eliminates necessity of external liquid nitrogen traps. No measurable backstreaming when run on a continuous basis; 5×10^{-6} mg/cm^2/min when measured under worst-case start-stop conditions. Start up time to full pumping capacity: 3 minutes. Cool down time to vent: 1 minute. Oil reservoir cooling system operates continuously to prevent oil decomposition. Stainless steel body and jet assembly. Wall of water cooling--no copper tubing. Cooling water requirement at 20° C: 0.25 GPM. Oil capacity: 420 cc. Heater voltage: 230. Heater power: 1200 watts.

Forepump included with system.

FIGURE 3 (CONTINUED)

ACCELERATOR 400T
Components 3

II. Insulating Vessel:

A grounded insulating vessel encloses the entire high voltage section of the accelerator. It consists of a cylindrical steel tank with a hemispherical top. Over-all length 70", maximum diameter 34-3/8". Tank is supplied with separate caster mounted dolly to facilitate ease of movement. Assembly or removal of the vessel is accomplished by unbolting and detaching it from base plate. Insulation is required for operating voltages which exceed 225 kV. Below these voltages tank may be removed and the machine operated in air.

SF_6 insulating gas is admitted into the vessel through a 5/8" intake valve. Internal pressure conditions are monitored by means of a 0 to 30" (vacuum) gauge and a 0 - 80 psi pressure gauge. Relief valve is set for 70 psi.

III. Control Console:

A desk-type control console is provided for remote operation of the accelerator. The following controls and meters are provided:

1. HV Control - coarse and fine controls provided, coarse adjust variable 0 to 400 kV, fine adjust variable in 0 - 40 kV increments over 0 to 400 kV range
2. High Voltage Meter
3. High Voltage Current Meter
4. High Voltage Protective Device Indicators
5. Discharge Gauge provided to monitor system vacuum during operation
6. Beam Current Meter, two ranges: 0 to 100 $_{u}$a and 0 to 1.0 MA (2% accuracy)
7. Main Power Key Switch
8. Ion Source Gas Pressure - raise-lower
9. Beam Extraction Voltage - raise-lower
10. Focusing Voltage - raise-lower
11. Beam Viewer - in-out
12. Master Parameter Control Variac for isolation transformer

FIGURE 3 (CONTINUED)

ACCELERATOR 400T
Components 4

13. Source Parameter Meter provided for position reading of source parameter control rods

14. Cables: heavy duty cable for 220 VAC to console, 33' 48-conductor cable from console to accelerator, H.V. supply to accelerator-two 16' H.V. cables

IV. High Voltage Power Supply:

1. Output - continuously variable 0 - 400 kV

2. Output current - rated at 1.0 MA at 400 keV

3. RMS ripple - 3% at full voltage and full load

4. Isolation transformer power - 1 kVA

5. Voltage fine control sensitivity - provides fine adjust over 40 kV increments

6. Input voltage - 117 VAC, 60 cycle, single phase

7. The high voltage power supply and isolation transformer are contained within a single tank, approximately 4' wide by 6' long by 7' high. Insulation: SF_6.

8. High voltage protective devices:

   The high voltage power supply is equipped with the following protective devices: automatic shorting system, external interlock, zero start interlock, overload relay, spark gaps, surge resistor, and meter protection. Indicator lights for location of malfunction are provided.

FIGURE 3 (CONTINUED)

"artificially" induced radioactivity that was produced in 1930 with a Cockcroft-Walton type accelerator, bombarding a lithium target with accelerated protons to create beryllium nuclei ($_3Li^7 + {}_1H^1 \longrightarrow {}_4Be^8$). The 400 keV Cockcroft-Walton electrostatic accelerator at Wittenberg University is a prototype of this first Cockcroft-Walton accelerator used to artificially induce radioactivity.

The beam of charged particles produced by the accelerator is developed under an accelerating potential of 0 to 400,000 volts D. C. applied as the particles exit the port. This potential accelerates the particles along the length of the accelerating tube until they collide with the target placed along the path to produce the desired nuclear phenomenon. The fuctioning of the electrostatic accelerator, the simplest type of accelerator, may be further examined in greater detail.

An electrostatic accelerator essentially consists of two parts--the high-voltage generator and the accelerating tube. The accelerating tube consists essentially of an evacuated tube with two electrodes, one at each end. One electrode is maintained at ground potential as the target, and the other is kept at high-voltage, the source of ions being placed inside of it (ion source). The kinetic energy with which the particle hits the target is equal to the voltage drop times the charge of the particle, or $T = Ve$. Therefore, the goal is to obtain the highest voltage possible. But the voltage is limited by discharges that may occur between the electrodes, between the high-voltage electrode and the ground, or between the walls of the dome which shroud the accelerator. For this reason the tube is made as long as possible and kept away from the walls. The high-voltages required are supplied by either a Cockcroft-Walton or Van deGraaff generator. The advantages of the electrostatic accelerator include high beam intenstiy, high energy stability, D. C. operation, and good beam collimation.[1]

(1) <u>ACCELERATING TUBE</u>:

While different types of electrostatic accelerators have different types of generators, they all have essentially the same accelerating tubes. The accelerating tube usually consists of a series of glass (insulating material) cylinders connected to metallic electrodes in vacuum-tight seals. These electrodes are connected to a chain of resistors (of high resistance) to maintain a uniform distribution of potential along the tube. The resistors ~~also protect the tube walls from the beam~~, reducing the possibility of surface discharge, limit charge accumulation on insulation walls responsible for the production of electrostatic beam deflection, and are used to focus the beam, ~~while increasing the mechanical strength of the accelerating tube~~. A schematic diagram for the structure of an accelerator tube is shown in figure 4.

In current design, however, the length of intermediate electrodes is reduced and their number is increased, practically reducing them to metallic rings. To maintain a uniform distribution of potential advantage is taken of the corona current, which flows from one ring to the next, instead of

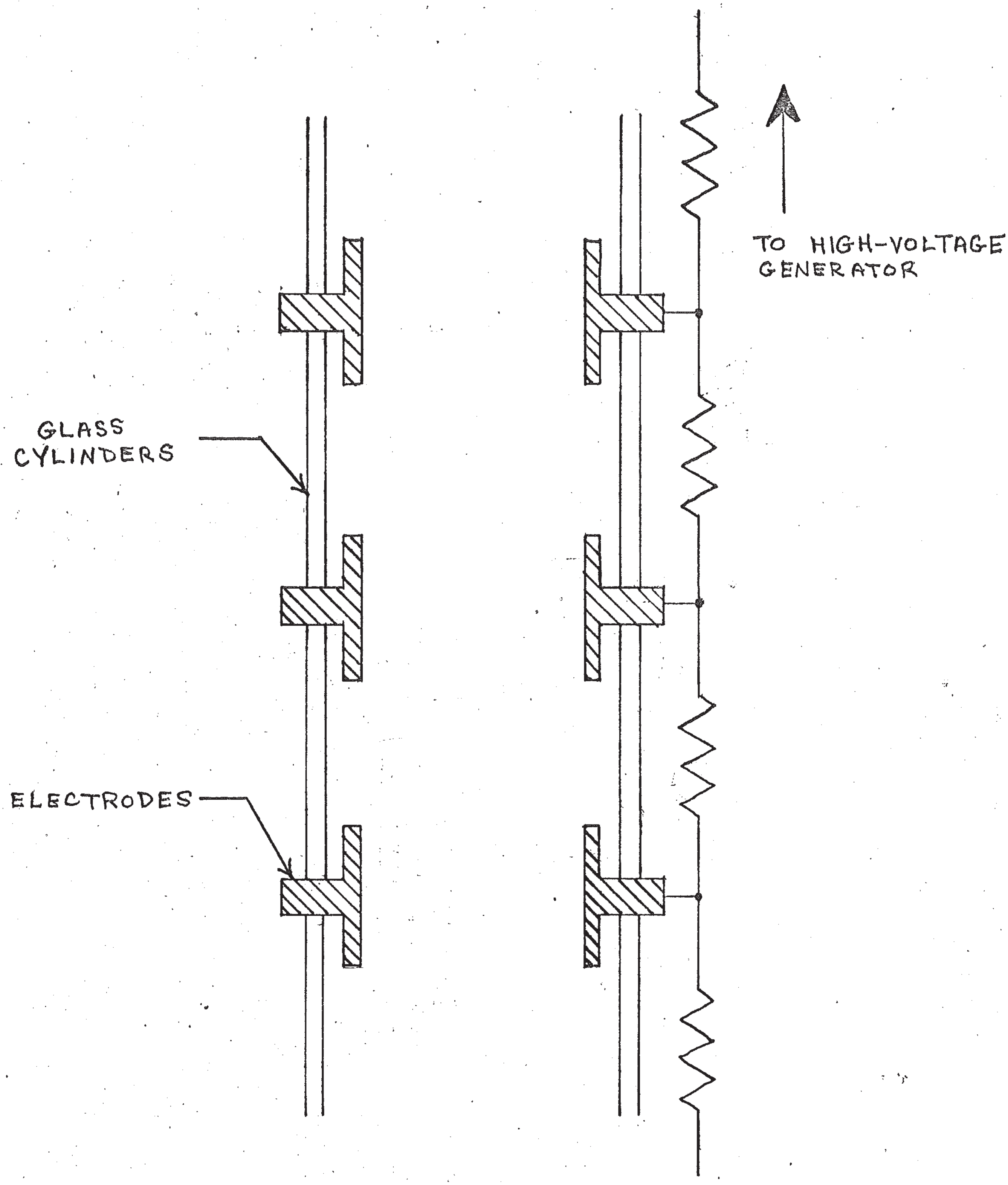


FIGURE 4. SCHEMATIC DIAGRAM OF THE STRUCTURE FOR THE ACCELERATING TUBE.

depending on the current in an external resistive voltage divider. The large inner diameters of these electrodes makes possible high pumping speeds, necessary with modern ion sources of higher intensity, and at the same time a higher gas flux in the tube.[2]

To examine the focusing effect of the electrodes consider two cylindrical electrodes E_1 and E_2 at different potentials with dotted lines representing the lines of force of the electric field as illustrated in figure 5. Suppose that a positively charged particle crosses the gap from E_1 to E_2. This particle will be axially accelerated by the electric field component parallel to the axis and transversely accelerated by the component directed toward the axis in E_1 and away from the axis in E_2. This produces a focusing effect at the gap entrance and defocusing at the exit, but in crossing from E_1 to E_2 the focusing effect that the particle experiences lasts for a longer time and, therefore, is more effective than the defocusing effect.[3]

The study of charged particle trajectories in electrostatic fields provides a basis for "electron optics." This science provides an analogy between the motion of a charged particle in an electrostatic field of potential $V(x,y,z)$ and the path of a ray of light in an inhomogeneous medium with refractive index $n(x,y,z)$. The trajectory of the charged particle and the light ray path will coincide if

$$n(x,y,z) = k\left\{(C-qV)\left[1+(C-qV)/(2mc^2)\right]\right\}^{\frac{1}{2}}, \qquad (1\text{-}1)$$

where C equals the sum of the particle's kinetic and potential

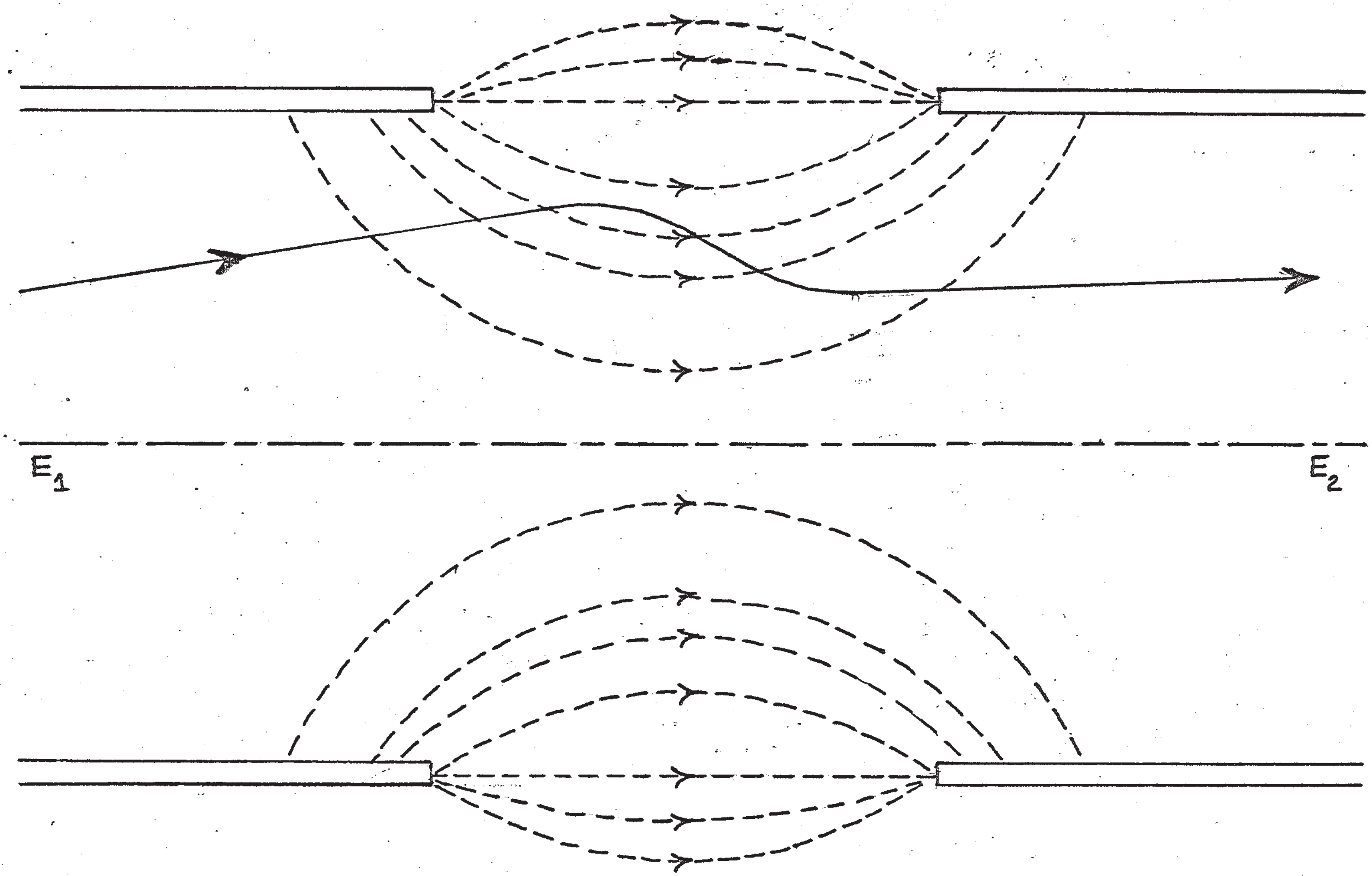


FIGURE 5. THE FOCUSING EFFECT OF AN ELECTROSTATIC LENS.

energy and k is an arbitrary constant. If $(C-qV) \ll mc^2$ or $v^2 \ll c^2$ the relativistic formula can be replaced with the classical one,

$$n(x,y,z) = k(C-qV)^{\frac{1}{2}}, \tag{1-2}$$

where this device represents an electrostatic lens with focal length increasing with the kinetic energy of the particle.[4]

This means that the accelerating tube acts as a system of many lenses, forming the image of the exit hole in the ion source on the target. It is important to note that with increasing kinetic energy of the particles along the tube only the first few lenses are efficient in focusing. The effect of space charge is present if beam intensity is high. Electrostatic repulsion between particles broadens the beam, producing defocusing in the first stages of acceleration. However, when the particle velocity is sufficiently high (relativistic) this repulsion is partially compensated by an electromagnetic attraction, since the particles in motion may be considered as parallel currents.[5] To correct for space charge defocusing the first electrode (probe) is made slightly concave to produce a transverse component of the electric field that moves the particles toward the axis.

(2) HIGH-VOLTAGE TERMINAL:

The high-voltage terminal is essentially a part of the high-voltage generator, but since it has its own design problems, such as the problem of insulation, it is commonplace to treat it separately. The high-voltage terminal consists of a spherical or ellipsoidal metal shell with a large center of curvature, supported by at least one insulating column. The terminal dimensions are chosen so that it can be charged to a high-voltage without discharge between it and the walls of its containing chamber. The insulating columns are surrounded by closely spaced equipotential rings to provide a constant electric field along the columns and also prevent discharges. The high-voltage terminal is continuously supplied with electric charge from the high-voltage generator when the machine is running, but it is also discharged by the beam current and by dissipative currents due to different causes. These causes include corona current along the accelerating tube, weak conduction current in the insulators supporting the terminal, and dissipative current from "electron loading." Electron loading accounts for most of the dissipative current, consisting of the extraction of secondary electrons by ions hitting the walls near the end of the accelerating tube.[6] These secondary electrons are focused and then accelerated back toward the ion source, consituting a current that reduces the terminal working voltage. The secondary electrons also produce X-rays as they hit the walls of the tube, and, therefore, safety shielding is needed.

Different currents flowing from the terminal depend on voltage as shown in figure 6. The value of the working voltage V_e is determined by the maximum charging current i_m. The beam (ion) current remains constant, being determined by the nature of the source. Resistive dissipative currents are proportional to voltage, and those due to corona current

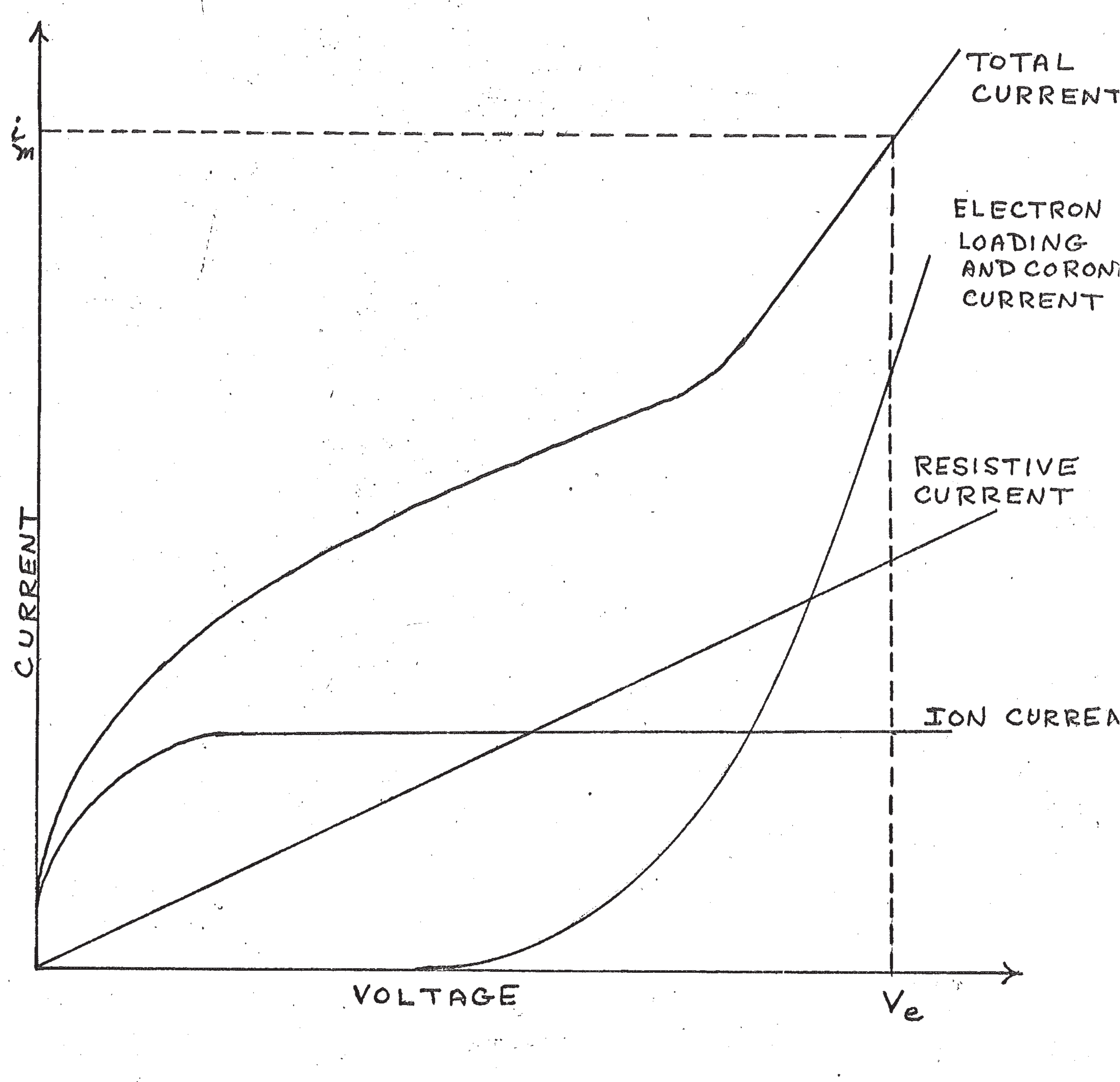


FIGURE 6. DEPENDENCE ON VOLTAGE OF THE LOAD CURRENTS IN AN ELECTROSTATIC ACCELERATOR.

and electron loading are negligible at low voltages, rapidly increasing above a certain voltage. From figure 7, a plot of ion current (beam intensity) against working voltage, it is apparent that the sum of the beam current I and dissipative currents I' are equal to the fixed maximum charging current i_m:

$$I + I' = i_m. \tag{1-3}$$

From figures 6 and 7, the dependence of I' on V_e and I on V_e can be examined. Each point of figure 7 represents a possible working machine condition. In general it is desired to operate at the high-voltage end of the curve, obtaining a voltage V_e^* not much less than V_m and a satisfactory beam current I^*, although this current is much smaller than i_m.[7]

(3) COCKCROFT-WALTON HIGH-VOLTAGE GENERATOR:

The Cockcroft-Walton voltage transformer, named for the British physicists J. D. Cockcroft and E. T. S. Walton and used for the production of some of the first nuclear reactions, is just one type of high-voltage generator used for electrostatic accelerators. This generator is an extension of the idea of the voltage multiplier developed by Greinacher in 1921.[8] The Cockcroft-Walton generator is made up of two columns of capacitors in series, connected by a chain of forward conducting diodes. It is of note that modern generators use solid state rectifiers instead of diodes since they do not require separate power supplies and operate without difficulty in a pressure chamber. The generator is fed by a transformer T, capable of producing an A. C. voltage U(t)

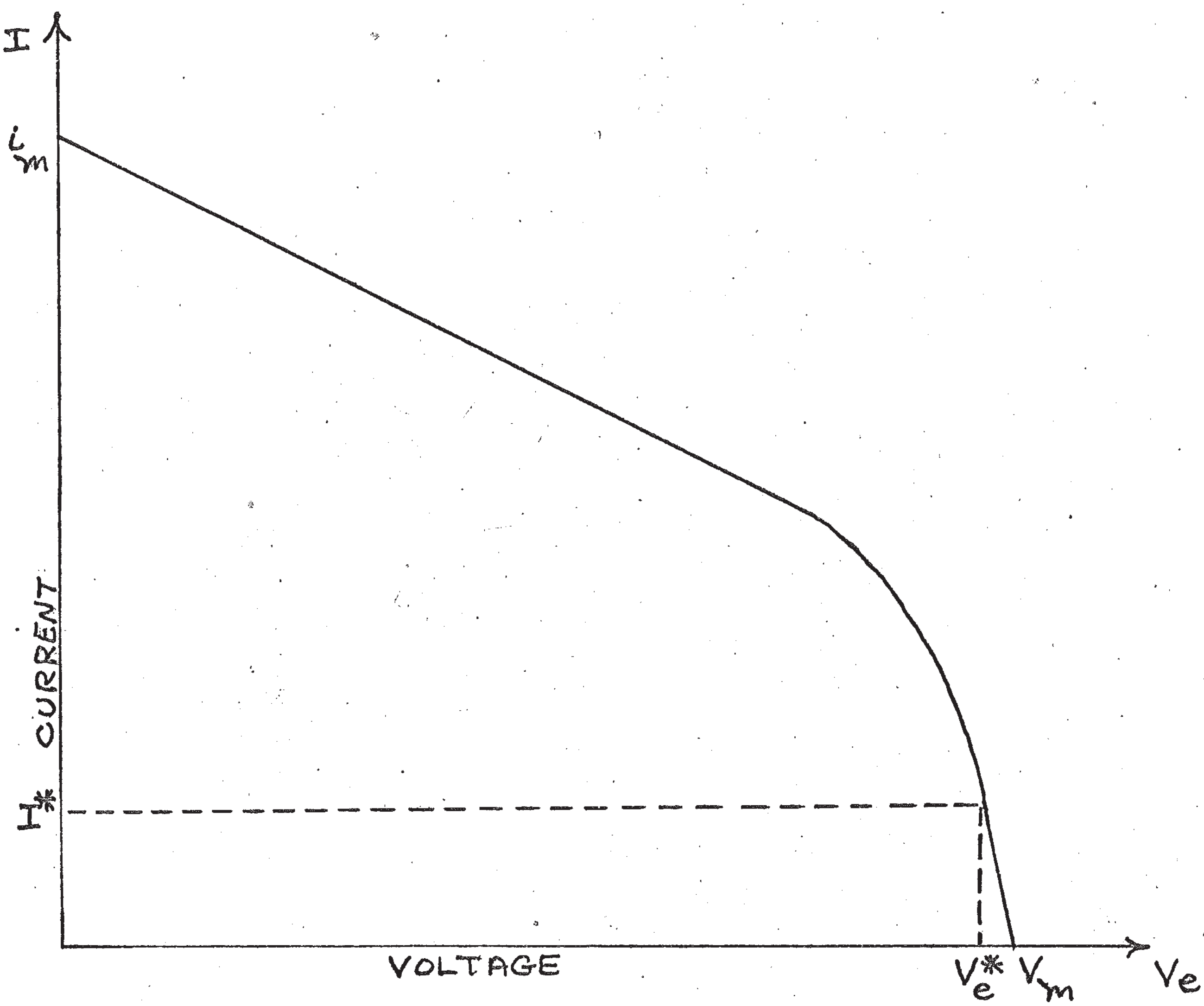


FIGURE 7. DEPENDENCE OF BEAM CURRENT ON THE WORKING VOLTAGE FOR A GIVEN GENERATOR AND ACCELERATING TUBE.

with a peak value U at about 100 keV. When no current is drawn from the high-voltage terminal a stationary state is reached after some initial time period where all the capacitors except the first one are charged to a voltage 2U. Here the potentials of the capacitors on the right are fixed and those on the left oscillate between the fixed limits as indicated in figure 8. Under these conditions the diodes do not conduct current, but if the state is altered the diodes will conduct for some period of time, tending to bring the system back to the stationary state. During machine operation a constant current i is drawn from the high-voltage terminal, the potentials slightly decreased according to a law which may be derived.[9]

Let n denote the number of capacitors in each column, and the capacitances on the right and on the left respectively beginning at the top by C_r , C_r' , where $r = 1,2,\ldots,n$. Let $u_r(t)$ be the voltage to which C_r is charged, and call the diodes conducting from the column on the right d_1 and from the left d_2. The figure shows simultaneous plots of $u_r(t)$ and $U(t)$ against time. Since the capacitors are charged to a voltage slightly less than in the absence of current, when $U(t)$ approaches its maximum value, the upper plates of capacitors C_r' reach higher potentials than those of capacitors C_r. Therefore, for a short time about the instant t_2 , diodes d_2 carry a current while $u_r(t)$ increases. The column on the right is discharged during the next half-period with a high time constant on the accelerating tube and exponentially decreasing $u_r(t)$. At instant t_1 near the minimum of $U(t)$

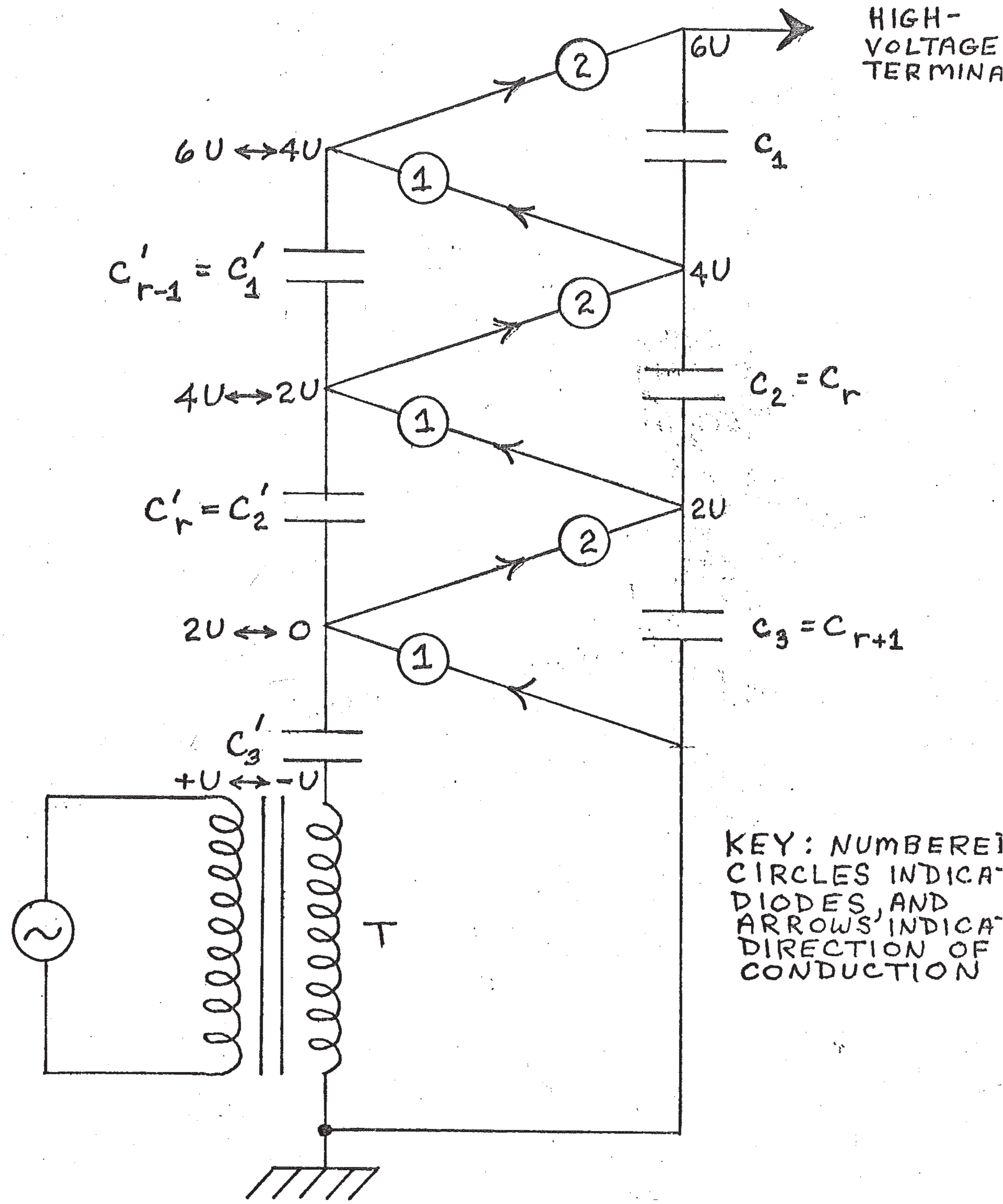


FIGURE 8. CIRCUIT OF A THREE-STAGE COCKCROFT-WALTON ELECTROSTATIC GENERATOR.

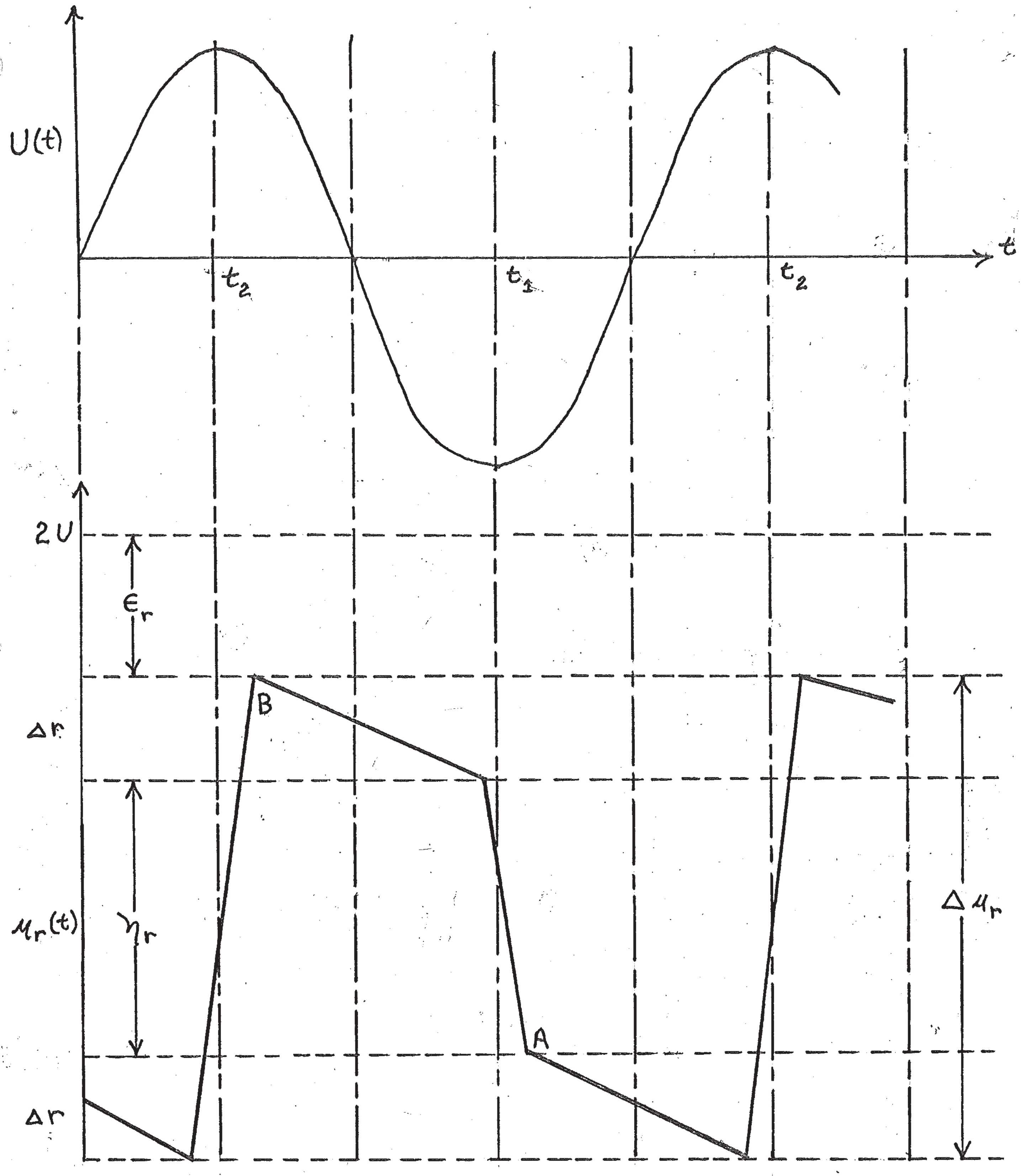


FIGURE 9. DIAGRAM SHOWING THE SINUSOIDAL TRANSFORMER VOLTAGE WAVEFORM AND THE CORRESPONDING VOLTAGE ON CAPACITOR C_r.

the upper plate of C_r is at a potential higher than C_{r-1}' , and the diodes d_1 carry a current while $u_r(t)$ rapidly decreases. In the next half-period the right column supplies current to the accelerating tube, and the voltage drops until instant t_2 where it rises to its initial value.

Therefore $u_r(t)$ fluctuates with amplitude Δu_r around a value less than 2U. Letting ϵ_r be the reduction of the maximum $u_r(t)$ with respect to 2U, Δ_r the decrease of $u_r(t)$ in a half-period, and η_r the voltage drop near t_1 ,

$$\Delta u_r = \eta_r + 2\Delta_r. \qquad (1\text{-}4)$$

For the steady state the charge q supplied by the capacitors in time T is $q = iT$, so

$$\Delta_r = q/2C_r = iT/2C_r. \qquad (1\text{-}5)$$

The net charge on the electrode of C_{r+1} and C_r is $C_{r+1}u_{r+1} - C_r u_r$. A recurrence relation is obtained, since the change in charge at t_1 , $C_{r+1}u_{r+1} - C_r u_r$, is equal to the charge q leaving the electrode and passing through the diode d_1 in the left column,

$$C_{r+1}u_{r+1} - C_r u_r = q, \text{ where } r = 1,2,\ldots,n. \qquad (1\text{-}6)$$

With $\eta_1 = 0$ (since C_1 does not charge any capacitor on the left column), the first s equations 1-6 with $r = 1,2,\ldots,s$ can be added term by term to obtain

$$C_s\eta_s = (s-1)q,$$
$$\eta_s = (s-1)q/C_s, \text{ where } s = 1,2,\ldots,n. \qquad (1\text{-}7)$$

Substituting equations 1-7 and 1-5 into equation 1-4 we obtain,

$$\Delta u_s = sq/C_s, \text{ where } s = 1,2,\ldots n. \qquad (1\text{-}8)$$

The fluctuation in the high-voltage ternimal is therefore

$$\Delta u = \sum_{s=1}^{n} \Delta u_s = q\sum_{s=1}^{n} s/C_s, \qquad (1\text{-}9)$$

and if all capacitances are equal,

$$\Delta u = \frac{n(n+1)}{2}\frac{q}{C} = \frac{n(n+1)}{2}\frac{i}{C\nu}, \text{ where } \nu = 1/T. \qquad (1\text{-}10)$$

Therefore, fluctuation increases as n^2 and is strongly influenced by capacitors near the transformer.[10]

To calculate the voltage reduction ϵ_r consider the figure 10, analogous to figure 9 and showing the voltage $u_r'(t)$ on the capacitor C_r'. At times t_2 when the capacitors C_r are discharged, C_r' are charged. This situation of course reverses itself at time instant t_1 , and the voltage is constant (no current drawn from capacitors) during a half-period. At t_1 diodes d_1 carry a current, C_r and C_{r-1}' connected in parallel to equalize plate voltage when the diodes have finished conducting. Therefore, the ordinates of A and A' are equal such that

$$2U-\epsilon_r-\Delta_r-\eta_r = 2U-\epsilon_{r-1}', \qquad (1\text{-}11)$$

and in a similar fashion for B and B' at t_2,

$$2U-\epsilon_r = 2U-\epsilon_r'-\eta_r'. \qquad (1\text{-}12)$$

Using the method of derivation for equation 1-7 and taking

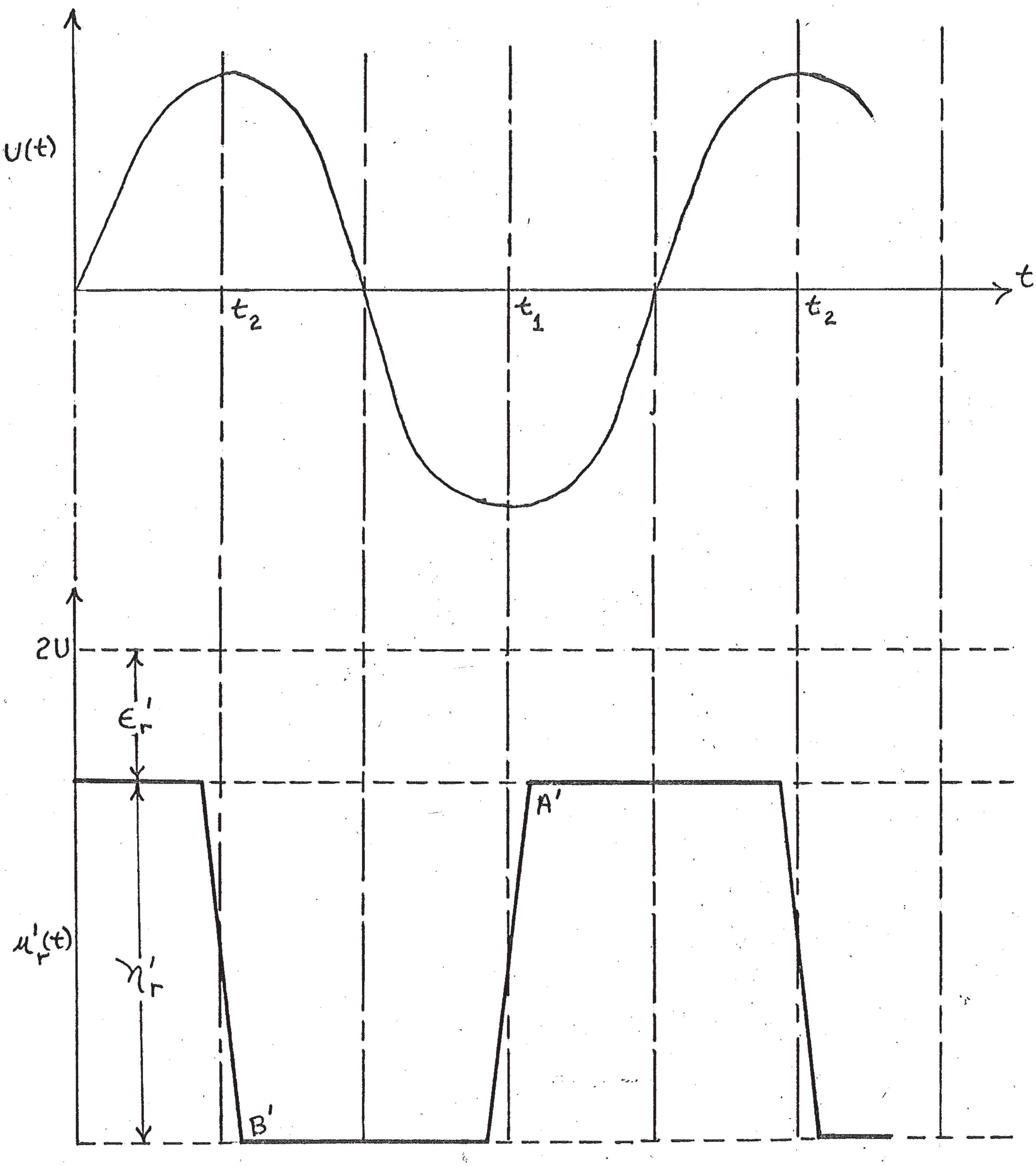


FIGURE 1.0. DIAGRAM SHOWING THE SINUSOIDAL TRANSFORMER VOLTAGE WAVEFORM AND THE CORRESPONDING VOLTAGE ON CAPACITOR C_r'.

into account that $\eta_1' = q/C_1'$,

$$\eta_s' = sq/C_s'. \tag{1-13}$$

Writing equation 1-12 for the index (r-1) and substituting for Δ_r in equation 1-11 from equation 1-5 the following relations are obtained:

$$\epsilon_r = \epsilon_{r-1}' - \eta_r - q/2C_r, \text{ and} \tag{1-14}$$

$$\epsilon_{r-1} = \eta_{r-1}' + \epsilon_{r-1}'.$$

Taking the difference of these two equations and using equations 1-7 and 1-13,

$$\begin{aligned} \epsilon_r - \epsilon_{r-1} &= -\eta_r - \eta_{r-1}' - q/2C_r \\ &= -\frac{(r-1)}{C_r}q - \frac{(r-1)}{C_{r-1}'}q - \frac{q}{2C_r} \\ &= q\left[\frac{1}{2C_r} - \frac{r}{C_r} - \frac{(r-1)}{C_{r-1}'}\right]. \end{aligned} \tag{1-15}$$

Writing this relation for r = 2,3,...,t and adding term by term,

$$\epsilon_t - \epsilon_1 = q \sum_{r=2}^{t} \left[\frac{1}{2C_r} - \frac{r}{C_r} - \frac{(r-1)}{C_{r-1}'}\right]. \tag{1-16}$$

Writing equation 1-16 for t = n and making the subtraction for the original equation 1-16 itself from the result,

$$\epsilon_n - \epsilon_t = q \sum_{r=t+1}^{n} \left[\frac{1}{2C_r} - \frac{r}{C_r} - \frac{(r-1)}{C_{r-1}'}\right]. \tag{1-17}$$

In calculating the voltage reduction ϵ_n of the voltage on the lowest capacitor of the right column C_n, it can be noted that at instant t_2 when diode d_2 has stopped carrying current the voltage on C_n equals the sum of the voltage on C_n' and on the transformer T such that

$$u_n = u_n' + U \text{ and}$$
$$2U - {}_n = U - \eta_n' + U, \tag{1-18}$$

taking into account that C_n' is charged to voltage U and $\epsilon_n' = 0$, since the capacitor is charged directly by the transformer through diode d_1 with no reduction in voltage. Therefore, from equations 1-13 and 1-18,

$$\epsilon_n = \eta_n' = nq/C_n', \tag{1-19}$$

and substituting into equation 1-17,

$$t = q\left[\sum_{r=t+1}^{n} \frac{(r-\frac{1}{2})}{C_r} + \sum_{r=t}^{n} \frac{r}{C_r'}\right]. \tag{1-20}$$

The reduction in terminal voltage ΔV_o is then given as

$$\Delta V_o = \sum_{t=1}^{n} \epsilon_t = q\sum_{t=1}^{n}\left[\sum_{r=t+1}^{n} \frac{(r-\frac{1}{2})}{C_r} + \sum_{r=t}^{n} \frac{r}{C_r'}\right], \tag{1-21}$$

and if the capacitances are equal,

$$\Delta V_o = \frac{q}{C}\left(\frac{2n^3}{3} + \frac{1n^2}{4} + \frac{1}{12}n\right). \tag{1-22}$$

Thus, the reduction in voltage increases approximately as the cube of the number of stages, and it may be seen that it is advantageous to reduce the number of stages and use a transformer of high peak voltage. It is also apparent that as V_o is proportional to q, it is inversely proportional to the supply frequency, so it is convenient to use high frequencies.[11]