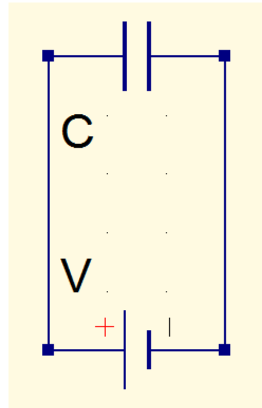
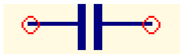
Inductance, Capacitance, and AC circuits.
Beyond resistors:

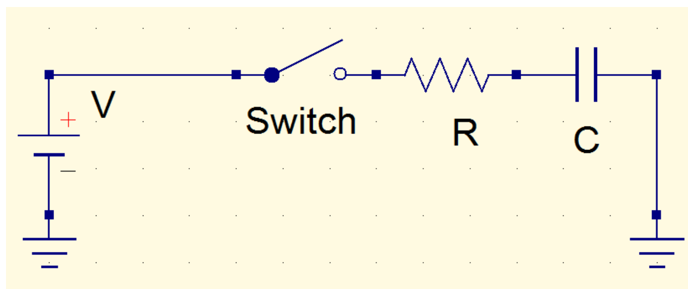


Capacitors: 

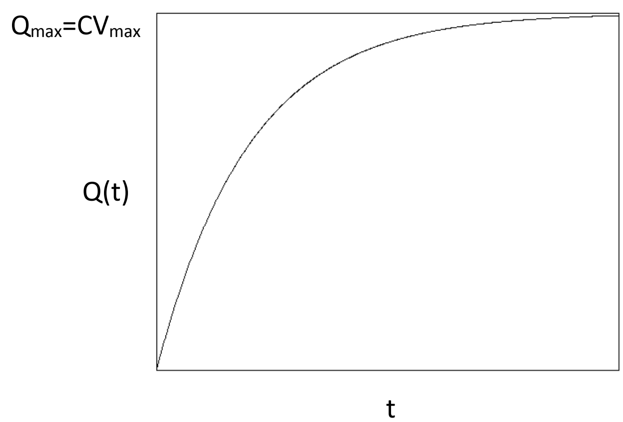
$$C \equiv \frac{Q}{V}$$

Transient response
When a change is made to a system and there's a transition to a new steady state.

Series RC circuit:

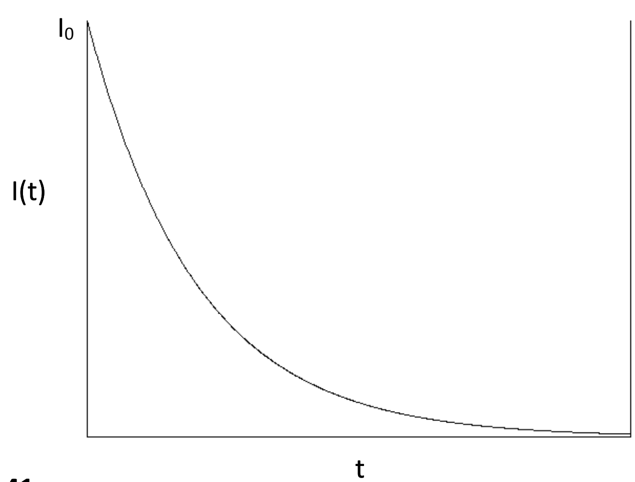


Usually assume Q(t=0)=0 (initial conditions).
Then for charging $Q(t) = Q_{max}(1 - e^{-\frac{t}{\tau}})$, with $\tau = RC$.

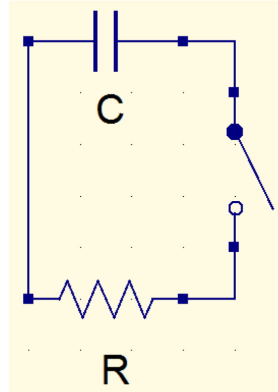


Similar dependence for V_C(t): $V_C(t) = V_{max}(1 - e^{-\frac{t}{\tau}})$

What about the current? Since $I = \frac{dQ}{dt}$, as Q(t) flattens out (approaches horizontal asymptote) I approaches zero. Functionally this is $I(t) = I_0 e^{-t/\tau}$.



Discharging?

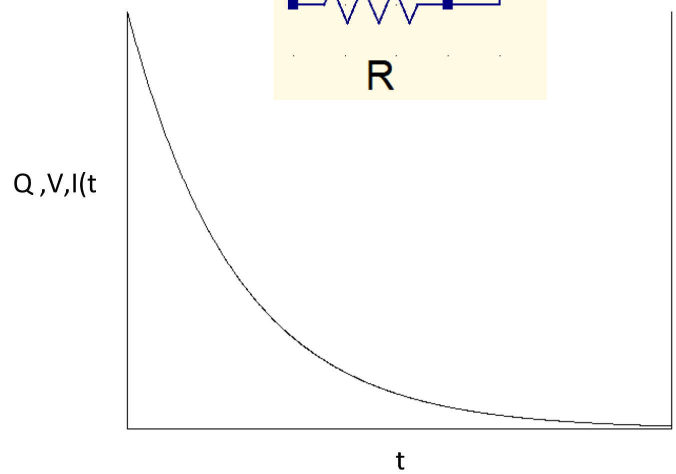


Assuming we start with some Q_0 on the capacitor at t=0:

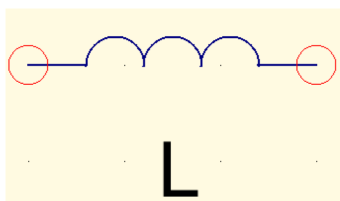
$Q(t) = Q_0 e^{-t/\tau}$

$V(t) = V_0 e^{-t/\tau} = \frac{Q_0}{C} e^{-t/\tau}$

$I(t) = I_0 e^{-t/\tau} = \frac{Q_0}{CR} e^{-t/\tau}$



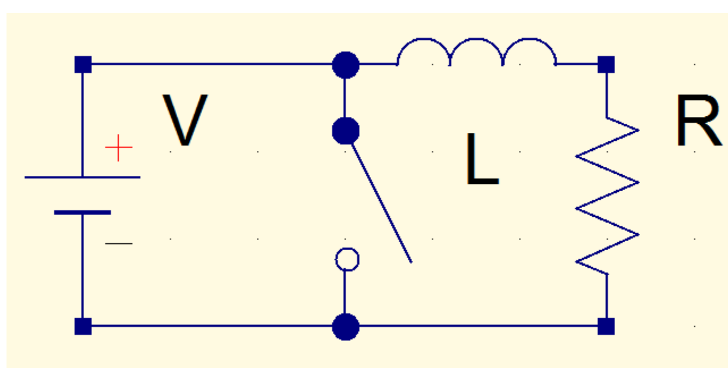
Inductors:



If we think of the ideal solenoid as our proto-type inductor, current through the wire of the coil makes a uniform magnetic field through each loop. We identify the inductance $L = \frac{N\Phi_B}{I}$ where N is the number of turns in the loop, Φ_B is the magnectic flux through each loop and I is the current through the wire.

If the current through the coil is changing, Faraday's law tells us how large a voltage that changing current causes:

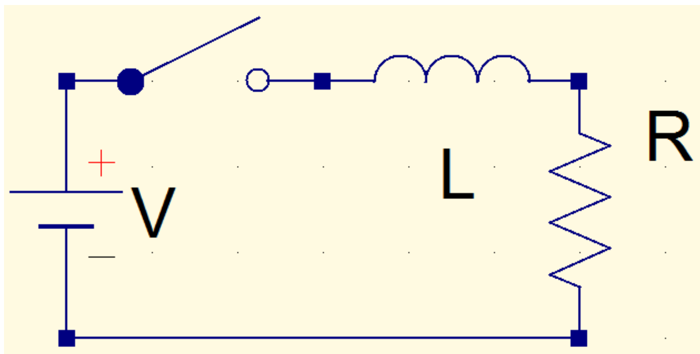
$V_L = L\frac{dI}{dt}$. This is a statement about magnitudes. What about the polarity of the induced voltage? The polarity is such that the induced voltage is in the sense to "oppose" the change in current that caused it.



For example, consider this circuit where the switch has been open for a long time and the current is constant. The current is, of course, I=V/R, clockwise around the outer loop.

If we close the switch, that removes the potential difference from across the LR combination. (In real life, the battery would be shorted and overheat, but that doesn't affect what's going on in the other half of the circuit). The effect then is that we've removed the voltage that was driving the current, but the induced voltage will "try" to maintain the current at what it was before the "change" of closing the switch. So, to keep current moving the same direction (from top to bottom) through R the top of R must be at a voltage more positive than the bottom. This means the induced voltage is such that the right hand side of the coil is at a positive voltage relative to the left hand side. This is a terribly anthropomorphic way of approaching things, but it can help you remember what happens.

So once closed, the induced voltage "tries" to keep up the current, but the energy dissipated in the resistor is no longer replaced by the energy from the power supply, so the current starts decaying: $I(t) = I_0 e^{-t/\tau}$ with $\tau = {}^{L}/_{R}$. $V_L \propto \frac{dI}{dt} \propto \frac{-I_0}{\tau} e^{-t/\tau}$. Note since I(0)=I, steady state V_R must be same which means V_L is supplying emf equal to V (power supply voltage) at t=0.



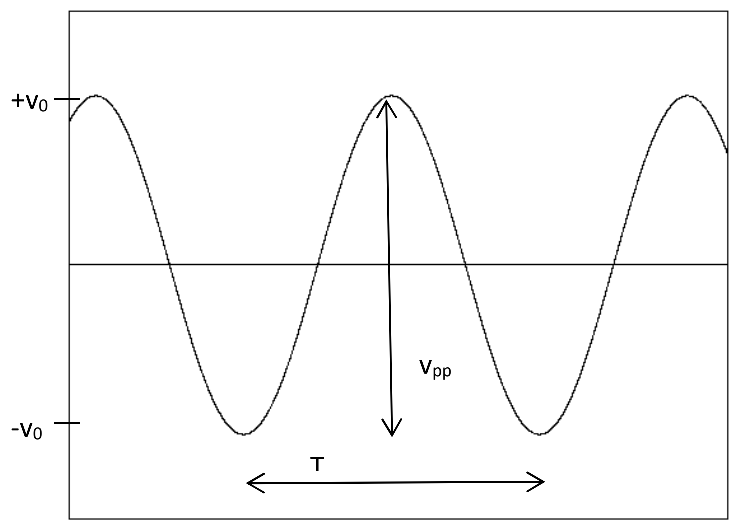
The complementary circuit is shown at right where the switch has been open for a long time and the current is zero everywhere. When we close the switch, V "tries" to drive a current clockwise around the loop, but the induced emf opposes that change ("tries" to keep the current at zero). So the current initially is zero, with the induced emf equal to V and opposite in polarity (left side of inductor more positive than right side). dI/dt is not zero, so eventually the current builds up to the limit of I=V/R. Functionally, this is $I(t) = I_{max}\left(1 - e^{-\frac{t}{\tau}}\right) = \frac{V}{R}(1 - e^{-\frac{t}{\tau}})$.

All of the above treatments were the transient behaviors for DC circuits. What about the AC response of the circuit?

$v(t) = v_0 \cos(\omega t + \varphi_v)$
$i(t) = i_0 \cos(\omega t + \varphi_i)$
Since this represent v or i at any time, we call this the current and voltage in the time domain. Later we will introduce the frequency domain.

The v_0 and i_0 in front of the trig functions are call the amplitude. We will also see "peak to peak" (pp) values are of use (for sinusoids, pp is twice the amplitude). ω is the angular frequency (rad/sec) and other relationship of interest are:

$$\omega = 2\pi f$$
$$f = \frac{1}{T}$$



with f the frequency and T the period of the sinusoid. The φ's in the arguments of the trig functions are the phases of the corresponding quantities. Recall that the phases offset the i(t), v(t) graphs horizontally.

While the amplitude, frequency, and phase fully describe a sinusoidal signal, many of our instruments do not directly tell us those quantities. This is a practical issue largely related to expense and/or (particularly in the days before LCD screens) size.

We might think the average of a quantity is useful but, of course, the average of a sinusoid over any period is zero. More helpful and common is the root mean square or RMS value. This appears a lot in physics, particularly in quantities whose straight average is zero. The RMS is the square root one of the average of the square of the quantity over one period:

$$v_{RMS} = \sqrt{\langle v(t)^2 \rangle}$$

where the angle brackets refer to averaging the time-dependent quantity over one period. Formally this is $\langle v(t)^2 \rangle = \frac{\int_{t_1}^{t_1+T} v(t)^2 dt}{T}$, but we will rarely if ever need to deal with it in this form.

Let's get back to the response of the circuit elements to sinusoidal AC signals.

Capacitor:

$i_{RMS} = \frac{v_{RMS}}{X_C}$ where X_C is a quantity called the capacitive reactance which in some sense behaves like a generalized resistance in so much as it's relating the RMS voltage across the capacitor with the RMS current through the capacitor. (We often say the "current through" the capacitor in the context of AC circuits, but remember no charge moves between the plates of the capacitor, it's just that as the charge accumulates on one plate and later is removed during one period of the sinusoidal signal, the RMS value is greater than zero).

The capacitive reactance is frequency dependent according t$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f}$o.

Inductor:
$i_{RMS} = \frac{v_{RMS}}{X_L}$ wher X_L is now the inductive reactance: $X_L = \omega L = 2\pi f L$.

While that appears to make capacitors and inductors in AC circuits look simple, there is a complicating factor: phases. The above are just relations for RMS values. There are a number of ways to deal with the phases involved in AC circuits. Some of them are easier, some of them are harder. Some of them are easy for simple circuits but not really workable with more complicated circuits. We will be using the latter also known as the phasor diagram method.

What are the phase relationships between voltage and current in our various components? For resistors, voltage and current are in phase. For inductors, voltage leads current by 90° (that is one quarter of the cycle). For capacitors, currently this voltage by 90°.

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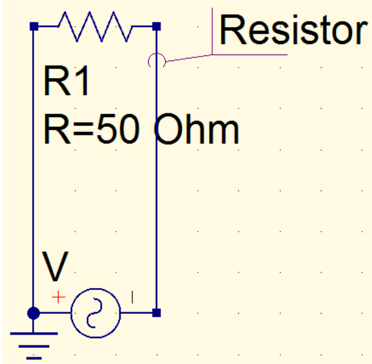
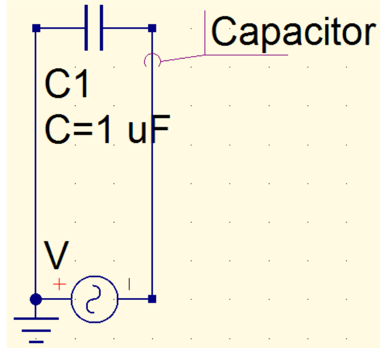
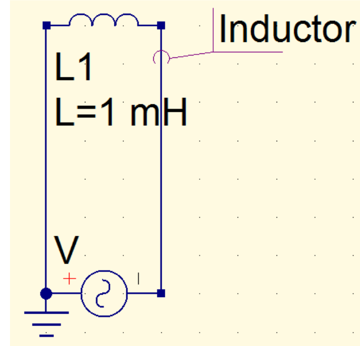
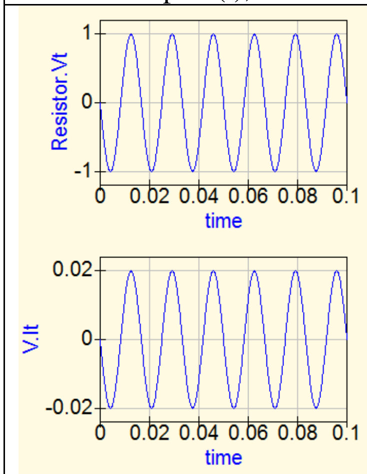
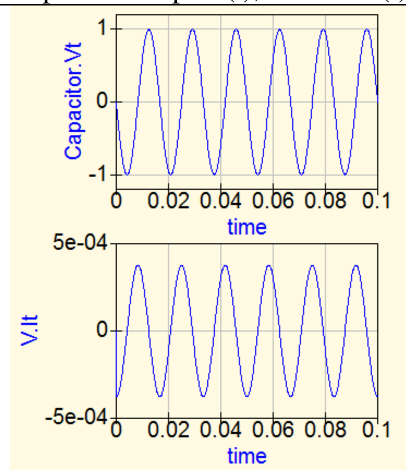
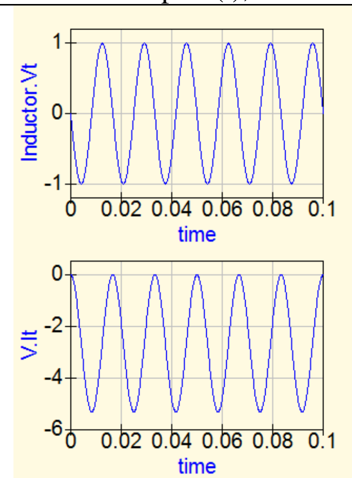
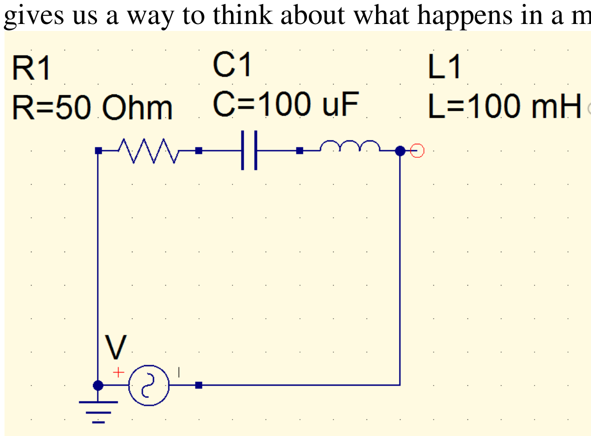
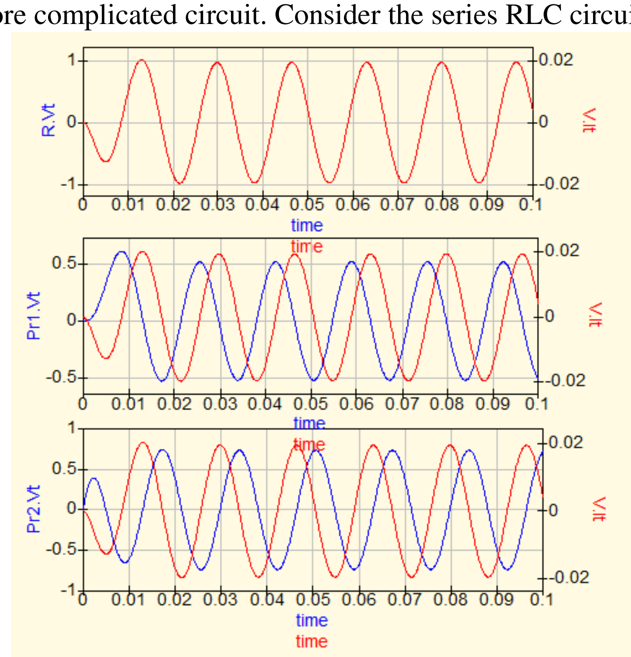
Physics 313

		
Resistor: Top= v(t), Bottom=i(t)	Capacitor: Top= v(t), Bottom=i(t)	Inductor: Top= v(t), Bottom=i(t)
		

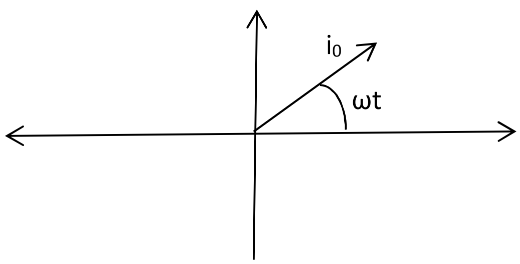
For a simple series combination of circuit elements, the current must be the same in each element. This gives us a way to think about what happens in a more complicated circuit. Consider the series RLC circuit



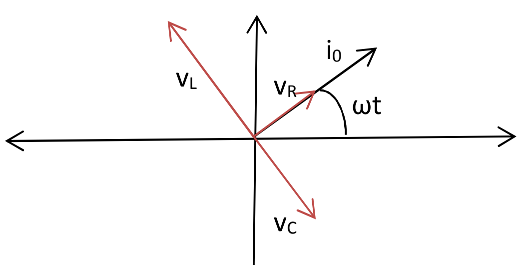


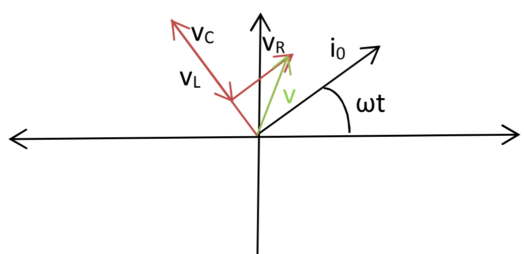
shown, and the resulting voltage and current vs time graphs. In each graph the red line is the current (read on the right hand axis) and the blue curve is the voltage across the resistor, the capacitor, and the inductor from top to bottom respectively. In the top graph the resistor voltage and current lie on top of each other since they are in phase and the graph is auto scaled. Note that for the capacitor to current bridges and extreme value one quarter cycle before the voltage goes. Therefore the current is leading the voltage. On the other hand, the voltage in the inductor gets to an extreme value before the current does. Note that in the simulations that resulted in these graphs the voltage signal began anti-equal zero from there having been no voltage signal applied and therefore there is a small transient response time period at the beginning.

For a circuit like this, how can I determine what the relative phases of the voltages are with respect to the current (and therefore with respect to each other)? We do this by representing each quantity of interest as a fixed length vector (the length being equal to the amplitude of that quantity) and imagine it rotating counterclockwise around the origin of a Cartesian coordinate system:



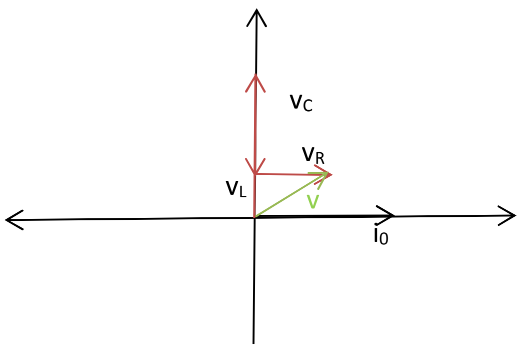
The projection of this rotating vector on the horizontal axis is just $i(t)=i_0\cos(\omega t)$, which behaves like one of our sinusoidal circuit quantities. The phase of one quantity (usually the current in a series circuit) may be arbitrarily set to zero since it is only the relative phases between the different quantities that matter from a physics standpoint. We can represent other quantities in our series RLC circuit according to the leading and lagging rules we laid out above. The voltage across the resistor is in phase with the current so it should be parallel to the current vector. The voltage across the inductor leads current by 90° so it is drawn at 90° further counterclockwise to the current vector. The voltage across the capacitor lags the current by 90° and therefore it is drawn at 90° clockwise to the current vector. The vector sum of these three voltages must be equal to the applied voltage (Kirkoff voltage loop rule). So we used our favorite vector addition techniques such as drawing the vector's tip to tail and find out what the resultant is.

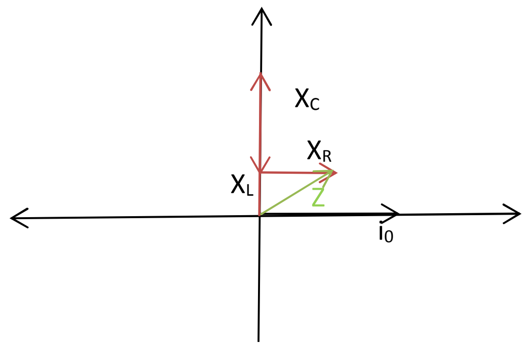




As this is drawn, we show it at some arbitrary time. However the relative phases between the voltages and currents is not time-dependent so we can orient our vectors at some convenient angle in order to evaluate the relative phases. This method of circuit analysis which relies on graphical drawing of current and voltage quantities as rotating around some origin is called the phasor diagram. If we can remember correctly which quantities lead and lag the current then this is fairly straightforward. There is mnemonic for remembering the phases as follows. Since voltage leads current in an inductor and another way to describe a voltage is an EMF (electro-motive force, a somewhat obsolete term) we can write ELI to signify that EMF comes before the current (I) in an inductor (L). Similarly for a capacitor we have ICE signifying that current leads EMF in a capacitor. That may not seem very memorable but the mnemonic is "ELI the ICEman" which makes it somewhat more clear.

Redrawing the voltage vectors so they are lined up horizontally and vertically lets us calculate the phases we may want.





In this orientation we can read off the components of the total voltage (the green vector labeled with the green v). The y components are V_L minus V_C, and the x component is V_R. Since each of these v's is proportional to the current (ac-Ohm's law), we can relabel this in terms of reactances (right-hand figure). The total vector now labeled "Z" is called an impedance, since it is a combination of multiple reactance is. A little geometry and trigonometry will show you that

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

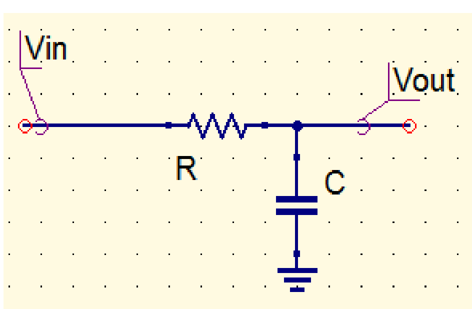
and

$$\varphi_v = tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

Where the phase is the phase of the applied voltage relative to the current that flows. A positive phase means the applied voltage leads the resulting current, a negative phase means the applied voltage lags the resulting current. Similar figures and analysis can be used to figure the phase of any two voltages with respect to each other, for example, or many other quantities in the circuit.

For more complicated combinations of circuit elements (parallel, series/parallel), the phasor method does not really work very well and a formalism involving complex numbers is handier. For this course, the above phasor treatment will be adequate.

Let's work out an example using the phasor analysis. Consider an AC voltage divider.



We can start by looking at the relationship of the magnitudes of v_{out} and v_{in}. This is done just as we did in the DC case, using Ohm's law only now it's the AC Ohm's law.

$$v_{out} = v_{in}\frac{X_C}{|Z|}$$

And the only significant difference is that we must determine Z according to the relationship discovered above so we have

$$v_{out} = v_{in}\frac{X_C}{\sqrt{R^2+{X_C}^2}}.$$

Oftentimes you will see discussion off of v_{out} versus v_{in} but rather of the gain (G) of the circuit defined as the ratio of v_{out} to v_{in}:

$\frac{v_{out}}{v_{in}} \equiv G = \frac{X_C}{\sqrt{R^2+{X_C}^2}}$, and of course recalling that $X_C = \frac{1}{\omega C}$, we see that the gain is frequency dependent.

So first consider two limiting cases of frequency:
$X_C>>R$ ("small" frequencies): in this case inside the square root the capacitive reactance is large compared to the resistance and so we can ignore the resistance. Therefore G approaches one.

$X_C<<R$ ("large" frequencies): in this case inside the square root the capacitive reactance is now small and we ignore it compared to the resistance. Since the resistance is constant, and the capacitive reactance in the numerator is small compared to that and we have that the gain is very small (much much less than one).

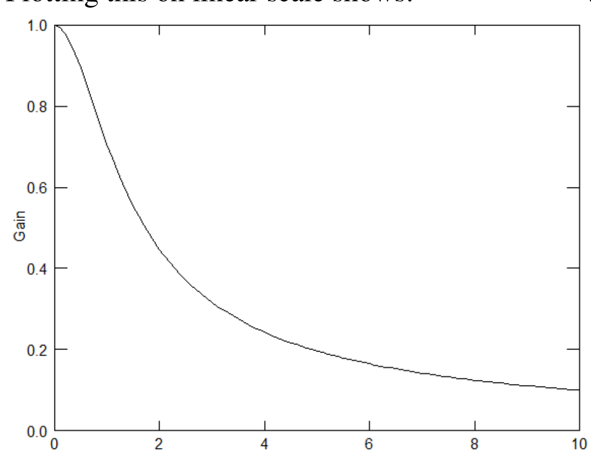
Since v_{out} is large for small frequencies and small for big frequencies, we call this circuit applied in this form a "low pass" filter: low-frequency signals applied at the input appear at the output at close to the same size. Conversely, high-frequency signals appear at the output at a much attenuated (reduced in amplitude) level.

For some frequency we will have the condition that $X_C=R$. We call this the breakpoint frequency or 3dB point. Solving from the definition of X_C we get $\omega_b = \frac{1}{RC}$ or $f_b = \frac{1}{2\pi RC}$.

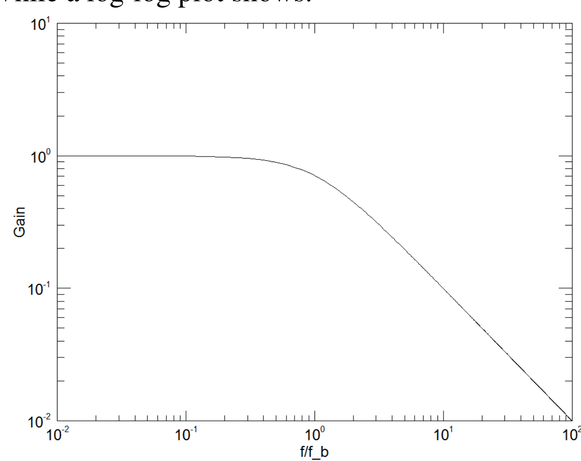
At the breakpoint frequency, you can convince yourself that the gain is $1/\sqrt{2}$

Writing out G with its full frequency dependence: $G = \frac{1}{\sqrt{R^2\omega^2C^2+1}} = \frac{1}{\sqrt{\left(\omega/\omega_b\right)^2+1}}$

Plotting this on linear scale shows:



While a log-log plot shows:



Note the max value on the horizontal scales is different. These gain vs frequency plots are also known as Bode plots for short (and in recognition of Bode).

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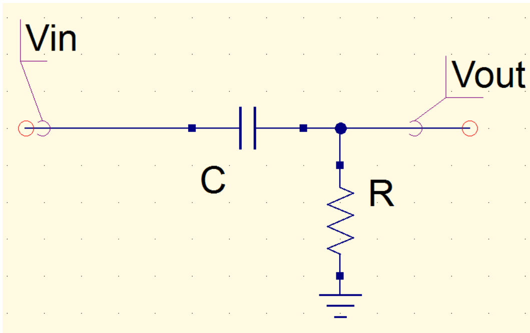
Physics 313

Passive Differentiator /Integrator



$v_{out} = iR$
and
$i = \frac{dQ}{dt}$
so
$v_{out} = \frac{dQ}{dt}R$

<u>but</u>
$Q = Cv_c = C(v_{in} - v_{out})$ (from Kirchoff Voltage Law)
So
$\frac{dQ}{dt} = C\frac{d}{dt}(v_{in} - v_{out})$
Use this in v_{out} equation above to get:
$v_{out} = RC\frac{d}{dt}(v_{in} - v_{out})$.
Now, if $v_{out} << v_{in}$ (more correctly $dv_{out}/dt << dv_{in}/dt$, but this is same for sine waves), this becomes simply
$v_{out} = RC\frac{dv_{in}}{dt}$ →v_{out} is proportional to the time derivative of v_{in}!

When is this condition satisfied? That is, under what conditions does this circuit behave as a differentiator? For $v_{out} << v_{in}$, it must be $v_C >> v_{out}$, which means $X_C >> R$. Putting in the frequency dependence of X_C:

$$\frac{1}{\omega C} \gg R$$

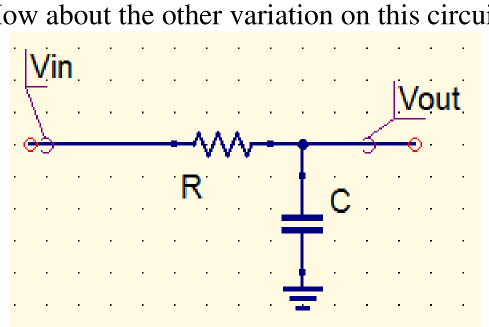
$$\frac{1}{\omega} \gg RC$$

And since f=1/T, we can define $\omega = 1/\tau$ and write this as:

$$\tau \gg \tau_{RC}$$

Or, in another way, if we define $\omega_C = 1/RC$ (this is often called the Characteristic frequency), then the condition is $\omega << \omega_C$. Keep clear that here ω has to do with the signal and ω_C is a characteristic of the circuit.

How about the other variation on this circuit?



In a similar development we now have:

$$v_R = v_{in} - v_{out} = iR$$

and so

$$\frac{v_{in} - v_{out}}{R} = i = \frac{dQ}{dt}$$

finally,

$$\frac{v_{in} - v_{out}}{R} = \frac{d(Cv_C)}{dt} = C\frac{dv_C}{dt} = C\frac{dv_{out}}{dt}$$

More compactly:

$$\frac{v_{in} - v_{out}}{RC} = \frac{dv_{out}}{dt}$$

Again, if $v_{out} << v_{in}$, we then have

$$\frac{v_{in}}{RC} = \frac{dv_{out}}{dt}$$

Integrating both sides with respect to time gives me

$$\frac{1}{RC}\int v_{in}dt = v_{out}$$

So v_{out} is proportional to the time integral of the input signal!
Once more we ask, when is this a good approximation? The condition of $v_{out} \ll v_{in}$ means

$X_C \ll R$

$$\frac{1}{\omega C} \ll R$$

$$\frac{1}{\omega} \ll RC$$

Which, using the same notation as before, is $\omega \gg \omega_C$.