Немного теории.

Пусть $y_{n\times 1}$, $z_{k\times 1}$ — случайные векторы, A — детерминированная матрица (подходящей размерности), b — дет. вектор (подходящей размерности).

1. $\mathbb{E}(Ay+b) = A\mathbb{E}(y)+b$
2. $\underset{n\times k}{\mathrm{cov}}(y,z) = \mathbb{E}(yz') - \mathbb{E}(y)\mathbb{E}(z')$
3. $\underset{n\times n}{\mathrm{cov}}(y,y) = \underset{n\times n}{\mathrm{Var}}(y) = \mathbb{E}(yy') - \mathbb{E}(y)\mathbb{E}(y')$
4. $\underset{n\times k}{\mathrm{cov}}(Ay+b, z) = A\,\mathrm{cov}(y,z)$

   $\mathrm{cov}(y, Az+b) = \mathrm{cov}(y,z)A'$
5. $\mathrm{Var}(Ay+b) = A\,\mathrm{Var}(y)A'$
6. $\mathrm{cov}(y,z) = \mathrm{cov}(z,y)'$
7. $\mathbb{E}(y'Ay) = \mathrm{tr}(A\cdot\mathrm{Var}(y)) + \mathbb{E}(y')A\mathbb{E}(y)$.

След матрицы:

1. $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$
2. $\mathrm{tr}(A\cdot B) = \mathrm{tr}(B\cdot A)$
3. $\mathrm{tr}(\alpha A+\beta B) = \alpha\,\mathrm{tr}(A) + \beta\,\mathrm{tr}(B)$, α,β — константы

Задание 1

$\underset{1\times n}{x'}\underset{n\times n}{A}\underset{n\times 1}{x}$ – скаляр.

$tr(u)=u$, если u – скаляр.

$\mathbb{E}(x'Ax) = \mathbb{E}(tr(x'Ax)) = \mathbb{E}(tr(Axx')) =$

$= tr\,\mathbb{E}(Axx') = tr[A\,\mathbb{E}(xx')] \boxed{=}$

$\circledast$ $Var(x) = \mathbb{E}(xx') - \mathbb{E}(x)\mathbb{E}(x')$

$\boxed{=}\; tr[A(Var(x) + \mathbb{E}(x)\mathbb{E}(x'))] =$

$= tr[A\,Var(x)] + tr[\underbrace{A\mathbb{E}(x)}\,\underbrace{\mathbb{E}(x')}] =$

$= tr(A\,Var(x)] + tr(\mathbb{E}(x')A\mathbb{E}(x))$, ч. т. д.

/ Задание 2.

а) $TSS = (y-\bar{y})'(y-\bar{y}) = (y-\pi y)'(y-\pi y) = ((I-\pi)y)'(I-\pi)y =$
$= y'(I-\pi)y$

$ESS = (\hat{y}-\bar{y})'(\hat{y}-\bar{y}) = (Py-\pi y)'((Py-\pi y)) =$
$= ((P-\pi)y)'((P-\pi)y)) = y'(P-\pi)'(P-\pi)y =$
$= y'(P-\pi)y$

$RSS = (y-\hat{y})'(y-\hat{y}) = (y-Py)'(y-Py) =$
$= ((I-P)y)'((I-P)y) = y'(I-P)'(I-P)y =$
$= y'(I-P)y$

в) $E(x'Ax) = tr(A\,Var(x)) + E(x')AE(x)$ $\quad Var(\varepsilon) = \sigma^2 I_n$

$E(TSS) = E(y'(I_n-\pi)y) = tr((I_n-\pi)\overset{\shortparallel}{\overline{Var(y)}}) + E(y')(I_n-\pi)E(y) =$
$= tr((I_n-\pi)\sigma^2 I_n + (X\beta)'(I_n-\pi)(X\beta) =$
$= \sigma^2 tr(I_n-\pi) + \beta' X'(I_n-\pi)X\beta = (n-1)\sigma^2 + \beta' X'(I_n-\pi)X\beta.$

(*) $tr(I_n-\pi) = tr(I_n) - tr(\pi) = n-1$
(**) $tr(\pi) = tr\left(\frac{1}{n}\begin{pmatrix}1 & & \\ & \ddots & \\ & & 1\end{pmatrix}\right) = \frac{1}{n}\cdot n = 1$

$E(ESS) = E(y'(P-\pi)y) = tr((P-\pi)Var(y)) + E(y')(P-\pi)E(y) =$
$= \sigma^2 tr(P-\pi) + (X\beta)'(P-\pi)(X\beta) = (k-1)\sigma^2 + \beta' X'(P-\pi)X\beta$

(*) $tr(P-\pi) = tr(P) - tr(\pi) = tr(P) - 1 = (k-1.)$

$tr(P) = tr(\underbrace{X(X'X)^{-1}X'}_{P\text{-идем.}}) = tr(\underset{k\times n}{X'}\underset{n\times k}{X}\underset{k\times k}{(X'X)^{-1}}) = tr(I_k) = k$

Задание 3.

1) $TSS_1 = \sum_{i=1}^{n} (y_i - \bar{y}_1)^2$ – по n наблюдениям,
$\bar{y}_1$ - среднее по n набл.

$$TSS_2 = \sum_{i=1}^{n+1} (y_i - \bar{y}_2)^2 = \sum_{i=1}^{n} (y_i - \bar{y}_2)^2 + (y_{n+1} - \bar{y}_2)^2 =$$
$$= \sum_{i=1}^{n} (y_i - \bar{y}_1 + \bar{y}_1 - \bar{y}_2)^2 + (y_{n+1} - \bar{y}_2)^2 =$$
$$= \underbrace{\sum_{i=1}^{n} (y_i - \bar{y}_1)^2}_{TSS_1} + 2\underbrace{(\bar{y}_1 - \bar{y}_2)}_{\text{const}} \underbrace{\sum_{i=1}^{n} (y_i - \bar{y}_1)}_{=0} + \sum_{i=1}^{n} \underbrace{(\bar{y}_1 - \bar{y}_2)^2}_{\text{const}} +$$
$$+ (y_{n+1} - \bar{y}_2)^2 = TSS_1 + n(\bar{y}_1 - \bar{y}_2)^2 + (y_{n+1} - \bar{y}_2)^2 \boxed{=}$$

$\circledast$ $$\bar{y}_2 = \frac{1}{n+1} \sum_{i=1}^{n+1} y_i = \frac{1}{n+1} \Big(\underbrace{\sum_{i=1}^{n} y_i}_{n\bar{y}_1} + y_{n+1} \Big) =$$
$$= \frac{n}{n+1}\bar{y}_1 + \frac{1}{n+1} y_{n+1},$$

т. е. $$\bar{y}_2 = \frac{n}{n+1}\bar{y}_1 + \frac{1}{n+1} y_{n+1}$$

$$\boxed{=} TSS_1 + n\left(\bar{y}_1 - \frac{n}{n+1}\bar{y}_1 - \frac{1}{n+1} y_{n+1}\right)^2 + \left(y_{n+1} - \frac{n}{n+1}\bar{y}_1 - \frac{1}{n+1} y_{n+1}\right)^2 = TSS_1 + n \cdot \left(\frac{\bar{y}_1 - y_{n+1}}{n+1}\right)^2 + \left(\frac{n y_{n+1} - n\bar{y}_1}{n+1}\right)^2 =$$
$$= TSS_1 + \frac{n}{(n+1)^2} (\bar{y}_1 - y_{n+1})^2 + \frac{n^2}{(n+1)^2} (y_{n+1} - \bar{y}_1)^2 =$$
$$= TSS_1 + (\bar{y}_1 - y_{n+1})^2 \cdot \frac{n + n^2}{(n+1)^2} = TSS_1 + \frac{n}{n+1} (y_{n+1} - \bar{y}_1)^2.$$

$$TSS_2 = TSS_1 + \underbrace{\frac{n}{n+1} (y_{n+1} - \bar{y}_1)^2}_{\geq 0},$$

т. е. TSS либо растёт, либо остаётся неизменным, если $y_{n+1} = \bar{y}_1$.