1) a) $(1+L)^2 y_t = (1+2L+L^2) y_t = 1 + 2y_{t-1} + y_{t-2}$

в) $\frac{1}{1+0.4L} y_t = (1 - 0.4L + 0.4^2 L^2 - 0.4^3 L^3 + \ldots) y_t =$

$= y_t - 0.4 y_{t-1} + 0.4^2 y_{t-2} + \ldots$

c) $\frac{1}{1+0.1L^{-1}} y_t = \frac{1}{1+0.1F} y_t = (1 - 0.1F + 0.1^2 F^2 + \ldots) y_t =$

$= y_t - 0.1 y_{t-1} + 0.1^2 y_{t-2} + \ldots)$

Свойства: $L^k y_t = y_{t-k}$ $\quad F^k y_t = y_{t+k}$

$L^k c = c$, где $c = const$ $\quad F^k c = c$

$L^{-1} = F$

2) $y_t = x_t (z_t + z_{t-1})$

a), в) $E(y_t) = E(x_t(z_t + z_{t-1})) = \underbrace{E(x_t) E(z_t + z_{t-1})}_{x_t \text{ и } z_t + z_{t-1} \text{ — незав.}} =$

$= 0$, т.к. $E(x_t) = -1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 0$.

$Var(y_t) = Var(x_t(z_t + z_{t-1})) = E(\underset{=1}{x_t^2} (z_t + z_{t-1})^2) =$

$= E(\underset{= z_t}{z_t^2} + \underset{= z_{t-1}}{z_{t-1}^2} + 2 z_t z_{t-1}) = \{E(z_t) = \frac{1}{2}\} = \frac{1}{2} + \frac{1}{2} + 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{2}$

$\gamma(1) = cov(y_t, y_{t-1}) = E(y_t y_{t-1}) = E(x_t (z_t + z_{t-1}) x_{t-1} (z_{t-1} + z_{t-2})) =$

$= \underset{= 0}{E(x_t)} \cdot E((z_t + z_{t-1}) x_{t-1} (z_{t-1} + z_{t-2})) = 0$.

$\gamma(2) = cov(y_t, y_{t-2}) = 0 \Rightarrow \gamma(k) = 0, k \geq 2$.

y_t: x_t, z_t, z_{t-1} $\quad$ y_{t-2}: $x_{t-2}, z_{t-2}, z_{t-3}$

$\mathbb{E}(y_t) = 0$

$Var(y_t) = \frac{3}{2}$

$\gamma(k) = \begin{cases} 0, & k = 1 \\ 0, & k \geq 2 \end{cases} \Rightarrow \gamma(k) = 0, \ k \geq 1.$ $\Rightarrow$ Процесс y_t стационарный + белый шум.

c) Рассм. на примере:

$$cov(y_t^2, y_{t-1}^2) = \mathbb{E}(y_t^2 y_{t-1}^2) - \underbrace{\mathbb{E}(y_t^2)\mathbb{E}(y_{t-1}^2)}_{\underset{Var(y_t) = \frac{3}{2}}{\shortparallel}} =$$

$$= \mathbb{E}\Big(\underbrace{X_t^2}_{=1}(z_t + z_{t-1})^2 \underbrace{X_{t-1}^2}_{=1}(z_{t-1} + z_{t-2})^2\Big) =$$

$$= \mathbb{E}\left[(z_t + z_{t-1} + 2z_t z_{t-1})(z_{t-1} + z_{t-2} + 2z_{t-1}z_{t-2})\right] =$$

$$= \overset{12}{\frac{1}{4}} + \overset{12}{\frac{1}{4}} + 2\cdot\frac{1}{8} + \overset{14}{\frac{1}{2}} + \overset{12}{\frac{1}{4}} + 2\cdot\overset{12}{\frac{1}{4}} + 2\cdot\overset{12}{\frac{1}{4}} + 2\cdot\frac{1}{8} + 4\cdot\frac{1}{8} = \frac{13}{4}$$

$$= \frac{13}{4} - \frac{9}{4} = \frac{4}{4} = 1 \neq 0.$$

③ $X_1, X_2, X_3, X_4 \sim iid \ N(0,1)$

$L = X_1 + X_2, \quad R = X_2 + X_3, \quad S = X_1 + X_2 + X_3 + X_4$

a) $corr(L, R) = \dfrac{cov(L, R)}{\sqrt{Var(L)\,Var(R)}} = \dfrac{1}{2} = 0.5$

$Var(L) = Var(X_1 + X_2) = Var(X_1) + Var(X_2) = 1 + 1 = 2$

$Var(R) = Var(X_2 + X_3) = 1 + 1 = 2$

$cov(L, R) = cov(X_1 + X_2, X_2 + X_3) = cov(X_2, X_2) = Var(X_2) = 1$

в) $pcorr(L, R; S) = corr(L^*, R^*)$, где $L^* = L - \alpha S$, $R^* = R - \beta S$.

Найдем α и β из условия ортогональности остатков и S:

$cov(L^*, S) = 0$

$$cov(L-\alpha S,\ S) = 0$$

$$cov(L,S) = \alpha\, cov(S,S)$$

$$\alpha = \frac{cov(L,S)}{cov(S,S)} = \frac{cov(L,S)}{Var(S)},$$

где $Var(S) = 4 \cdot 1 = 4,$

$cov(L,S) = cov(X_1 + X_2, X_1 + X_2 + X_3 + X_4) = cov(X_1, X_2) = Var(X_2 + X_1) = 1+1 = 2$

$$\Rightarrow \alpha = \frac{1}{2}. \Rightarrow \beta = \frac{1}{2}.$$

$$L^* = L - \frac{1}{2} S$$

$$R^* = R - \frac{1}{2} S.$$

$$pcorr(L, R; S) = corr(L^*, R^*) = \frac{cov(L^*, R^*)}{\sqrt{Var(L^*)Var(R^*)}},$$

где $Var(L^*) = Var(X_1 \cdot \frac{1}{2} + \frac{1}{2}X_2 - \frac{1}{2}X_3 - \frac{1}{2}X_4) = \frac{1}{4} \cdot 4 = 1.$

$Var(R^*) = 1,$

$$cov(L^*, R^*) = cov\left(\frac{1}{2}X_1 + \frac{1}{2}X_2 - \frac{1}{2}X_3 - \frac{1}{2}X_4;\ -\frac{1}{2}X_1 + \frac{1}{2}X_2 + \frac{1}{2}X_3 - \frac{1}{2}X_4\right) =$$

$= 0. \Rightarrow$ $corr(L, R) = 0.5$

$pcorr(L, R; S) = 0.$

4. y_t – MA(1) процесс с ур-ем:

$$y_t = \varepsilon_t + 0.5\varepsilon_{t-1}, \quad \sigma_\varepsilon^2 = 4. \qquad (\mathbb{E}(y_t) = 0+0=0).$$

a) Да, y_t – процесс стационарный (любой MA(q) стационарен).

в) Составим хар-ий полином:

$\varepsilon_t = \lambda^t$

$\lambda^t + 0.5\lambda^{t-1} = 0 \quad |: \lambda^{t-1}$

$\lambda + 0.5 = 0$

$|\lambda| = |-0.5| < 1 \Rightarrow$ обратимая запись.

Пример необратимой записи:

$$y_t = \varepsilon_t + 2\varepsilon_{t-1}, \quad \sigma_\varepsilon^2 = 1. \quad \text{(Это пример на паре).}$$

c) $\gamma(0) = Var(y_t) = Var(\varepsilon_t + 2\varepsilon_{t-1}) = 1 + 4\cdot 1 = 5$

$\gamma(1) = cov(y_t, y_{t-1}) = \mathbb{E}(y_t \cdot y_{t-1}) = cov(\varepsilon_t + 2\varepsilon_{t-1}, \varepsilon_{t-1} + 2\varepsilon_{t-2}) =$

$= 2cov(\varepsilon_{t-1}, \varepsilon_{t-1}) = 2Var(\varepsilon_{t-1}) = 2\cdot 1 = 2$

$\gamma(2) = cov(y_t, y_{t-2}) = cov(\varepsilon_t +$

$\gamma(1) = cov(y_t, y_{t-1}) = cov(\varepsilon_t + 0.5\varepsilon_{t-1}, \varepsilon_{t-1} + 0.5\varepsilon_{t-2}) =$

$= 0.5Var(\varepsilon_{t-1}) = 0.5\cdot 4 = 2$

$\gamma(2) = cov(y_t, y_{t-2}) = cov(\varepsilon_t + 0.5\varepsilon_{t-1}, \varepsilon_{t-2} + 0.5\varepsilon_{t-3}) = 0$

$\gamma(k) = 0, \ k \geq 2.$

ACF: $\rho(1) = \dfrac{\gamma(1)}{\gamma(0)} = \dfrac{2}{5} = 0.4$

$\rho(k) = 0, \ k \geq 2.$

PACF:

φ_{kk} – k-й член частной автокорр. ф-ции.

$$\varphi_{kk} = pcorr(y_t, y_{t-k}; y_{t-1}, y_{t-2}, \dots, y_{t-k+1})$$

$$\varphi_{11} = \rho(1) = 0.4$$

Найдём φ_{22}:

выпишем вспомогательное уравнение:

$$y_t = \alpha + \varphi_{21} y_{t-1} + \varphi_{22} y_{t-2} + u_t \quad \left| \begin{array}{l} \cdot y_{t-1}, E(\dots) \\ \cdot y_{t-2}, E(\dots) \end{array} \right.$$

Заметим, что $E(y_t) = 0 \Rightarrow E(y_t \cdot y_{t-k}) = cov(y_t, y_{t-k}) = \gamma(k)$

$$\begin{cases} \gamma(1) = \varphi_{21}\gamma(0) + \varphi_{22}\gamma(1) \\ \gamma(2) = \varphi_{21}\gamma(1) + \varphi_{22}\gamma(0) \end{cases}$$

, т.к. $E(u_t y_{t-1}) = cov(u_t, y_{t-1}) = 0$

Метод Крамера:

$$\varphi_{22} = \frac{\begin{vmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(2) \end{vmatrix}}{\begin{vmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{vmatrix}} = \frac{\gamma(0)\gamma(2) - \gamma^2(1)}{\gamma^2(0) - \gamma^2(1)} =$$

$$= \frac{5 \cdot 0 - 2^2}{5^2 - 2^2} = -\frac{4}{21}.$$

φ_{33} считается аналогично, но нужно 3-е ур-е.