Задача 1.

Let Z is a random variable with mean $\mu = E(Z)$ and variance $\sigma^2 = V(Z)$. Let we have a sample of size 2, $z_1 = 0$ and $z_2 = 4$ from this distribution.

We are interested in the parameter $\theta = \mu^2$

(a) Let the estimator is $\hat{\theta} = \bar{z}^2 = \left(\dfrac{z_1 + z_2}{2}\right)^2 = \dfrac{1}{4}(z_1 + z_2)^2 = 4$. Find bias of this estimator.

(b) Explain how you would find a bootstrap-bias-corrected estimator, $\hat{\theta}_{Boot}$ and calculate its bias.

(c) Formulate the condition when the bias of $\hat{\theta}_{Boot}$ is in absolute value smaller than the bias of $\hat{\theta}$.

parameters.

Решение:

(a) $E(\hat{\theta}) = \dfrac{1}{4} E\left(Z_1^2 + Z_2^2 + 2Z_1Z_2\right) = \dfrac{1}{4} E\left(2(\mu^2 + \sigma^2) + 2\mu^2\right) = \mu^2 + \dfrac{1}{2}\sigma^2$,

$Bias(\hat{\theta}) = E(\hat{\theta}) - \mu^2 = \dfrac{1}{2}\sigma^2$

(b) $\hat{\theta}_{Boot} = 2\hat{\theta} - \bar{\hat{\theta}}^*$;

$\bar{\hat{\theta}}^* = \dfrac{1}{B}\sum_{b=1}^{B} \hat{\theta}_b^*$; $E\left(\bar{\hat{\theta}}^*\right) = E\left(\hat{\theta}_1^*\right)$, where $\hat{\theta}_1^* = \dfrac{1}{4}\left(Z_1^{*2} + Z_2^{*2}\right)$ has a discrete distribution:

$\hat{\theta}_1^* = \begin{cases} 0, & p = 0.25, \\ 4, & p = 0.50, \\ 16, & p = 0.25. \end{cases}$ Thus, $E\left(\bar{\hat{\theta}}^*\right) = E\left(\hat{\theta}_1^*\right) = 0 \cdot 0.25 + 4 \cdot 0.50 + 16 \cdot 0.25 = 6$.

Mathematical expectation of the bootstrap-bias-corrected estimator is

$E\left(\hat{\theta}_{Boot}\right) = E\left(2\hat{\theta} - \bar{\hat{\theta}}^*\right) = 2\left(\mu^2 + \dfrac{1}{2}\sigma^2\right) - 6 = 2\mu^2 + \sigma^2 - 6$, and its bias is $Bias\left(\hat{\theta}_{Boot}\right) = \mu^2 + \sigma^2 - 6$.

(c) The condition is $\left|\mu^2 + \sigma^2 - 6\right| < \dfrac{1}{2}\sigma^2$, or $-\dfrac{1}{2}\sigma^2 < \mu^2 + \sigma^2 - 6 < \dfrac{1}{2}\sigma^2$, or

$$\max\left\{0,\ 6 - \frac{3}{2}\sigma^2\right\} < \mu^2 < \max\left\{0,\ 6 - \frac{1}{2}\sigma^2\right\}$$

Задача 2.

Данные по пропускам занятий в 2018 г.

calc1	calculus, 1st year (0–100)
stat1	statistics, 1st year (0–100)
mid2	оценка по мидтерму 2 (2-го года) (0–100)
na1	число пропусков семинаров до mid1
na2	число пропусков семинаров от mid1 до mid2

na12= na1 + na2

lcalc1 = ln(1+calc1)

lstat1 = ln(1+stat1)

lmid2 = ln(1+mid2)

Ниже приведены оценки нескольких моделей.

(a) По модели 1 найдите прогноз оценки по второму мидтерму студента с calc1=55, na12=7 и 95%-ный доверительный интервал прогноза. Как изменится оценка при уменьшении количества пропусков на 1?

(б) По модели 2 найдите прогноз оценки по второму мидтерму студента с calc1=55, na12=7. Как изменится оценка при уменьшении количества пропусков на 1?
(в) По модели 3 найдите вероятность того что студент с calc1=55, na12=7 получит оценку больше 35 (проходной балл). Как изменится эта вероятность при уменьшении количества пропусков на 1?
(г) По модели 4 найдите вероятность того что студент с calc1=55, na12=7 получит оценку больше 35 (проходной балл). Как изменится эта вероятность при уменьшении количества пропусков на 1?
(д) Интерпретируйте результаты модели 5.

дескриптивные статистики

.sum calc1 stat1 mid2 na1 na2 na12 lcalc1 lstat1 lmid2

```
Variable      Obs    Mean    Std. dev.     Min    Max
calc1  222    54.57207      16.07249      21.4   97.2
stat1  222    53.46847      18.87363      16     91
mid2   214    39.41121      20.52021      0      95
na1    222    2.162162      2.259382      0      7
na2    222    3.306306      2.652729      0      7
na12   222    5.468468      4.504035      0      14
lcalc1 222    3.971025      .3174502      3.11   4.59
lstat1 222    3.931668      .3759945      2.83   4.52
lmid2  214    3.473327      .8605712      0      4.56
```

Модель 1
. reg mid2 na12 calc1

```
      Source |       SS           df       MS      Number of obs   =       214
-------------+----------------------------------   F(2, 211)       =    159.56
       Model |  53991.0084         2  26995.5042   Prob > F        =    0.0000
    Residual |  35698.8047       211  169.188648   R-squared       =    0.6020
-------------+----------------------------------   Adj R-squared   =    0.5982
       Total |  89689.8131       213  421.078935   Root MSE        =    13.007

------------------------------------------------------------------------------
        mid2 | Coefficient  Std. err.      t    P>|t|     [95% conf. interval]
-------------+----------------------------------------------------------------
        na12 |  -1.185705   .2274389    -5.21   0.000    -1.634049   -.7373611
       calc1 |   .8001184   .0634011    12.62   0.000     .6751377     .925099
       _cons |   1.490929   4.297359     0.35   0.729    -6.980329    9.962187
------------------------------------------------------------------------------
```

(матрица ковариаций оценок)
. matrix list e(V)

```
symmetric e(V)[3,3]
             na12       calc1       _cons
 na12   .05172848
calc1   .00678865    .0040197
_cons  -.64858158  -.25792686   18.467295
```

(а) Прогноз оценки при calc1=55, na12=7

$$\hat{m}2 = 1.49 - 1.186 * 7 + 0.80 * 55 = 37.2$$

$V(\hat{m}2) = V(-1.18*b1 + 0.80b2 + 1.19*b3) =$

$= 1.18^2 V(b1) + 0.80^2 V(b2) + 1.19^2 V(b3) -$

$-2*1.18*0.80*Cov(b1,b2) - 2*1.18*1.19*Cov(b1,b3) + 2*0.80*1.19*Cov(b2,b3) = 42.8 = 6.54^2$

$CI = 37.2 \pm 1.96 \cdot 6.54 = (34.4, 50.0)$

Эффект от na12 → na12-1 +1.19 к оценке

Модель 2

. reg lmid2 na12 lcalc1

```
      Source |       SS           df       MS      Number of obs   =       214
-------------+----------------------------------   F(2, 211)       =     77.99
       Model |  67.0495207         2  33.5247604   Prob > F        =    0.0000
    Residual |  90.6946277       211  .429832359   R-squared       =    0.4251
-------------+----------------------------------   Adj R-squared   =    0.4196
       Total |  157.744148       213  .740582857   Root MSE        =    .65562

------------------------------------------------------------------------------
       lmid2 | Coefficient  Std. err.      t    P>|t|     [95% conf. interval]
-------------+----------------------------------------------------------------
        na12 |  -.0534786   .0114449    -4.67   0.000    -.0760397   -.0309176
      lcalc1 |   1.306943   .1623836     8.05   0.000     .9868405    1.627045
       _cons |   -1.45109   .6789243    -2.14   0.034    -2.789433   -.1127461
------------------------------------------------------------------------------
```

(б) $l\hat{m}2 = -1.45 - 0.053*7 + 1.31*\ln(1+55) = 3.44$

$\hat{m}2 = \exp(3.44) - 1 = 30.0$

Эффект от na12 → na12-1

$l\hat{m}2 \to l\hat{m}2 + 0.053; \quad \hat{m}2 + 1 \to (m2+1)*\exp(0.053) = 31.0*1.05 = 37.5$

+6.5 к оценке

Модель 3

gen pass= (mid2>35) if year==2018

. probit pass na12 lcalc1

```
Probit regression                                       Number of obs =    222
                                                        LR chi2(2)    =  82.98
                                                        Prob > chi2   = 0.0000
Log likelihood = -111.65921                             Pseudo R2     = 0.2709

------------------------------------------------------------------------------
        pass | Coefficient  Std. err.      z    P>|z|     [95% conf. interval]
-------------+----------------------------------------------------------------
        na12 |  -.0633342   .0238393    -2.66   0.008    -.1100585     -.01661
      lcalc1 |   2.335395   .3856257     6.06   0.000     1.579582    3.091207
       _cons |  -8.809419   1.587522    -5.55   0.000    -11.92091   -5.697932
------------------------------------------------------------------------------
```

(в) $P(pass) = \Phi(-0.063*7 + 2.34*\ln(1+55) - 8.81) = \Phi(0.15) = 0.559$

Маргинальный эффект

$P(pass1) = \Phi(-0.063*6 + 2.34*\ln(1+55) - 8.81) = \Phi(0.21) = 0.583$

DeltaP = 0.559 – 0.583 = 0.025 = 2.5%.

Модель 4
gen grade= pass
replace grade=grade+1 if (mid2>50)

. tab grade

```
      grade |      Freq.     Percent        Cum.
------------+-----------------------------------
          0 |        102       45.95       45.95
          1 |         45       20.27       66.22
          2 |         75       33.78      100.00
------------+-----------------------------------
      Total |        222      100.00
```

oprobit grade na12 lcalc1

```
Ordered probit regression                               Number of obs =    222
                                                        LR chi2(2)    =  89.82
                                                        Prob > chi2   = 0.0000
Log likelihood = -187.625                               Pseudo R2     = 0.1931

------------------------------------------------------------------------------
       grade | Coefficient  Std. err.      z    P>|z|     [95% conf. interval]
-------------+----------------------------------------------------------------
        na12 |  -.0511547   .0217157    -2.36   0.018    -.0937167   -.0085928
      lcalc1 |   2.347722   .3502968     6.70   0.000     1.661152    3.034291
-------------+----------------------------------------------------------------
       /cut1 |   8.937546    1.45024                      6.095128    11.77996
       /cut2 |   9.631782   1.464066                      6.762266     12.5013
------------------------------------------------------------------------------
```

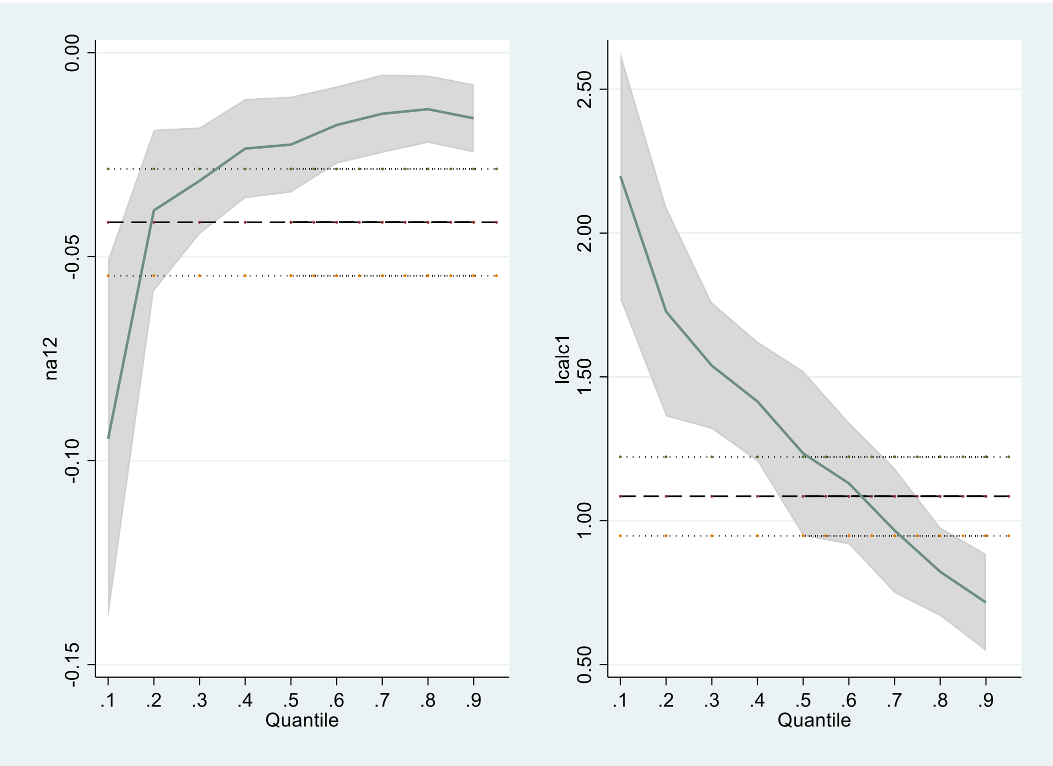
(г) $P(grade=0)=\Phi(\text{cut}1-\text{xb}) == \Phi(-0.155)=0.438=1-0.561$
$P(pass)=0.561$

Модель 5
. sqreg lmid2 na12 lcalc1, q(.1 .2 .3 .4 .5 .6 .7 .8 .9) reps(200)

```
------------------------------------------------------------------------------
             |              Bootstrap
       lmid2 | Coefficient  std. err.      t    P>|t|     [95% conf. interval]
-------------+----------------------------------------------------------------
q10          |
        na12 |  -.0838806   .0426027    -1.97   0.050     -.167862    .0001007
      lcalc1 |   2.426944   .6571186     3.69   0.000     1.131586    3.722303
       _cons |  -6.487357   2.814974    -2.30   0.022    -12.03643    -.938281
-------------+----------------------------------------------------------------
q20          |
        na12 |  -.0559812   .0177183    -3.16   0.002    -.0909087   -.0210537
      lcalc1 |   1.641716   .3300163     4.97   0.000     .9911649    2.292268
       _cons |  -3.009746   1.396078    -2.16   0.032    -5.761793   -.2576981
-------------+----------------------------------------------------------------
q30          |
        na12 |  -.0354652    .012701    -2.79   0.006    -.0605024   -.0104281
      lcalc1 |    1.49511   .1634331     9.15   0.000     1.172939    1.817281
       _cons |  -2.365492   .6855298    -3.45   0.001    -3.716857   -1.014128
-------------+----------------------------------------------------------------
q40          |
        na12 |  -.0307546   .0084764    -3.63   0.000    -.0474639   -.0140452
      lcalc1 |   1.357278   .1743089     7.79   0.000     1.013668    1.700888
       _cons |  -1.756461   .7341471    -2.39   0.018    -3.203664   -.3092587
-------------+----------------------------------------------------------------
q50          |
        na12 |  -.0339861   .0096406    -3.53   0.001    -.0529902   -.0149819
      lcalc1 |   1.207537   .1586549     7.61   0.000     .8947857    1.520289
```

```
       _cons |  -1.047199   .6704901    -1.56   0.120    -2.368916    .2745188
-------------+----------------------------------------------------------------
q60          |
        na12 |  -.0273181   .0093108    -2.93   0.004    -.0456722    -.008964
      lcalc1 |   1.107011   .1254063     8.83   0.000     .8598014    1.354221
       _cons |  -.5841629   .5314225    -1.10   0.273     -1.63174    .4634146
-------------+----------------------------------------------------------------
q70          |
        na12 |  -.0224138   .0083664    -2.68   0.008    -.0389063   -.0059214
      lcalc1 |   .9942065   .0839169    11.85   0.000     .8287835    1.159629
       _cons |  -.0789144   .3660945    -0.22   0.830    -.8005857    .6427569
-------------+----------------------------------------------------------------
q80          |
        na12 |   -.020736   .0090724    -2.29   0.023    -.0386202   -.0028517
      lcalc1 |    .920427   .0986422     9.33   0.000     .7259765    1.114877
       _cons |    .323837   .4444675     0.73   0.467    -.5523287    1.200003
-------------+----------------------------------------------------------------
q90          |
        na12 |  -.0248994   .0077376    -3.22   0.001    -.0401522   -.0096466
      lcalc1 |   .7533497   .1121907     6.71   0.000     .5321915    .9745079
       _cons |   1.162592   .4726978     2.46   0.015     .2307763    2.094407
------------------------------------------------------------------------------
```

. grqreg, ci ols olsci reps(200)



(д) начальные условия особенно важны для слабых студентов. Так же и посещение семинаров.

Задача 3.
Below you can find estimation results from a multinomial logit model. The dependent variable is
status = 0, if if enrolled in school; (College)
= 1, if not in school and not working (Home);
= 2, if working (Work).

Choice 0 (College) is the base choice in the model. Regressors are number of years of education at the current moment (*educ*), experience of employment (*exper*), and its square (*exper*^{2}), also a race dummy *black* = 1, if the person is afro American.

Table 1: Multinomial Logit Estimates of School and Labor Market Decisions

Explanatory Variable	Home (status=1)	Work (status=2)
educ	-0.674	-0.315
	(0.070)	(0.065)
exper	-0.106	0.849
	(0.173)	(0.157)
exper^2	-0.013	-0.077
	(0.025)	(0.023)
black	0.813	0.311
	(0.303)	(0.282)
cons	10.28	5.54
	(1.13)	(1.09)
# of obs	1717	
% correctly predicted	79.6	
Log-likelihood value	-907.86	
Pseudo-R^2	0.243	

(a) (10 points) Give interpretation of the coefficients at *educ* in both columns of the table (Home, Work). How *educ* influence on the probability to be in school?

(b) (10 points) Calculate the marginal effect $\dfrac{\partial P(y=0\,|\,x)}{\partial\, educ}$ for a white person with 12 years of schooling and 10 years of experience.

(c) (5 points) Describe the model for which the property "Independence of irrelevant alternatives" is relevant. What is "Independence of irrelevant alternatives"?

Решение:

(a) *Home*: With additional year of education, odds ratio $P(y=1\,|\,x)/P(y=0\,|\,x)$ decreases since the coefficient at *educ* is negative (–0.674). *Work*: odds ratio $P(y=2\,|\,x)/P(y=0\,|\,x)$ also decreases since the coefficient at *educ* is negative (–0.315). So far as $P(y=2\,|\,x)+P(y=1\,|\,x)+P(y=0\,|\,x)\equiv 1$, we have $\dfrac{P(y=2\,|\,x)}{P(y=0\,|\,x)}+\dfrac{P(y=1\,|\,x)}{P(y=0\,|\,x)}+1\equiv\dfrac{1}{P(y=0\,|\,x)}$, hence, the probability of choice 0, $P(y=0\,|\,x)$ increases.

(b) $$\frac{\partial P(y=0\,|\,x)}{\partial\, educ}=\frac{\partial}{\partial x_1}\frac{1}{1+\sum_{k=1}^{2}\exp(x'\beta_k)}=\frac{-\left(\exp(x'\beta_1)\beta_{11}+\exp(x'\beta_2)\beta_{21}\right)}{\left(1+\sum_{k=1}^{2}\exp(x'\beta_k)\right)^2};$$

$\exp(x'\beta_1)=\exp(-0.168)=0.845$, $\exp(x'\beta_2)=\exp(2.550)=12.807$.

$$\frac{\partial P(y=0\,|\,x)}{\partial\, educ}=-\frac{0.845\cdot(-0.674)+12.807\cdot(-0.315)}{\left(1+0.845+12.807\right)^2}=0.021.$$

(c) Under the multinomial logit model, the probability of selecting choice *i* is $P_j=\dfrac{e^{V_i}}{\sum_{k\in C}e^{V_k}}$, were random utility $U_i=V_i+\varepsilon_i$. Thus $\dfrac{P_i}{P_j}=e^{V_i-V_j}$, the ratio of probabilities depends only on choices *i* and *j*, but not on any other alternatives. This follows from the assumption of i.i.d. ε_i.