Немного теории.

Пусть $Y_{n\times 1}$, $Z_{k\times 1}$ – случайные векторы, A – детерминированная матрица (подходящей размерности), b – дет. вектор (подходящей размерности).

1. $\mathbb{E}(AY+b) = A\mathbb{E}(Y)+b$
2. $\underset{n\times k}{cov}(Y, Z) = \mathbb{E}(YZ') - \mathbb{E}(Y)\mathbb{E}(Z')$
3. $\underset{n\times n}{cov}(Y, Y) = \underset{n\times n}{Var}(Y) = \mathbb{E}(YY') - \mathbb{E}(Y)\mathbb{E}(Y')$
4. $\underset{n\times k}{cov}(AY+b, Z) = A\,cov(Y, Z)$
   $cov(Y, AZ+b) = cov(Y, Z)A'$
5. $Var(AY+b) = A\,Var(Y)\,A'$
6. $cov(Y, Z) = cov(Z, Y)'$
7. $\mathbb{E}(Y'AY) = tr(A\cdot Var(Y)) + \mathbb{E}(Y')A\mathbb{E}(Y)$.

След матриц:

1. $tr(A+B) = tr(A) + tr(B)$
2. $tr(A\cdot B) = tr(B\cdot A)$
3. $tr(\alpha A+\beta B) = \alpha tr(A) + \beta tr(B)$, α, β – константы

Задание 1

$x'Ax$ – скаляр.
$1 \times m \quad m \times n \quad n \times 1$

$tr(u) = u$, если u – скаляр.

$$E(x'Ax) = E(tr(x'Ax)) = E(tr(Axx')) =$$
$$= tr\,E(Axx') = tr[A\,E(xx')] =$$

(*) $Var(x) = E(xx') - E(x)E(x')$

$$= tr[A(Var(x) + E(x)E(x'))] =$$
$$= tr[A\,Var(x)] + tr[AE(x)E(x')] =$$
$$= tr(A\,Var(x)] + tr(E(x')AE(x)),$$ ч. т. д.

Задание 2.

а) $TSS = (y-\bar{y})'(y-\bar{y}) = (y-\bar{\pi}y)'(y-\bar{\pi}y) = ((I-\bar{\pi})y)'(I-\bar{\pi})y =$

$= y'(I-\bar{\pi})y$

$ESS = (\hat{y}-\bar{y})'(\hat{y}-\bar{y}) = (Py-\bar{\pi}y)'((Py-\bar{\pi}y)) =$

$= ((P-\bar{\pi})y)'((P-\bar{\pi})y)) = y'(P-\bar{\pi})'(P-\bar{\pi})y =$

$= y'(P-\bar{\pi})y$

$RSS = (y-\hat{y})'(y-\hat{y}) = (y-Py)'(y-Py) =$

$= ((I-P)y)'((I-P)y) = y'(I-P)'(I-P)y =$

$= y'(I-P)y$

в) $E(x'Ax) = tr(A\,Var(x)) + E(x')A\,E(x)$ $\quad Var(\varepsilon) = \sigma^2 I_n$

$E(TSS) = E(y'(I_n-\bar{\pi})y) = tr((I_n-\bar{\pi})\underbrace{Var(y)}_{''}) + E(y')(I_n-\bar{\pi})E(y) =$

$= tr((I_n-\bar{\pi})\sigma^2 I_n + (X\beta)'(I_n-\bar{\pi})(X\beta) =$

$= \sigma^2 tr(I_n-\bar{\pi}) + \beta'X'(I_n-\bar{\pi})X\beta = (n-1)\sigma^2 + \beta'X'(I_n-\bar{\pi})X\beta.$

(*) $tr(I_n-\bar{\pi}) = tr(I_n) - tr(\bar{\pi}) = n-1$

(**) $tr(\bar{\pi}) = tr\left(\frac{1}{n}\begin{pmatrix}1 & \cdots & 1\\ \vdots & & \vdots\\ 1 & \cdots & 1\end{pmatrix}\right) = \frac{1}{n}\cdot n = 1$

$E(ESS) = E(y'(P-\bar{\pi})y) = tr((P-\bar{\pi})Var(y)) + E(y')(P-\bar{\pi})E(y) =$

$= \sigma^2 tr(P-\bar{\pi}) + (X\beta)'(P-\bar{\pi})(X\beta) = (k-1)\sigma^2 + \beta'X'(P-\bar{\pi})X\beta$

(*) $tr(P-\bar{\pi}) = tr(P) - tr(\bar{\pi}) = tr(P) - 1 = (k-1.)$

$tr(P) = tr(\underbrace{X(X'X)^{-1}X'}_{P\text{-идемп.}}) = tr(\underset{k\times n}{X'}\underset{n\times k}{X}(\underset{k\times k}{X'X})^{-1}) = tr(I_k) = k.$