Differentiation in HKDSE

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1 Differentiation and Derivatives

1.1 Derivatives

Derivative is a measure of rate of change of a function. The derivative of a function at certain point can be visualised as the slope of tangent of the function at that point.

1.2 Differentiation

The process of finding derivatives is called differentiation, we can treat it as an operation. To denote this operation, we use the following symbols

$$\frac{df}{dx} = \frac{df(x)}{dx} = \frac{d}{dx}f(x) = f'(x) = f'$$

These are all symbols representing differentiating the function $f(x)$ **with respect to** x. Please take note that we are finding how sensitive does the function value $f(x)$ changes as x changes.

Sometimes, we may want to know the derivative of a function at a **particular point**, let's say $(x, f(x)$ we denote this particular derivative as

$$\frac{d}{dx}\Big|_{x=x_0} f(x) \text{ or } f'(x_0)$$

But you should NEVER write in the following form

$$\cancel{\frac{d}{dx}f(x_0)}$$

$$\frac{d}{dx}f(x_0) \neq \frac{d}{dx}\Big|_{x=x_0} f(x)$$

This is because $\frac{d}{dx}f(x_0)$ is finding the derivative of the function $f(x_0)$ with respect to x. For example, if $f(x) = x^2 + x$, then $f(x_0) = x_0^2 + x_0$, and in fact $x_0^2 + x_0$ is a constant function of x, differentiating $x_0^2 + x_0$ with respect to x is not difference to differentiating a constant, which gives zero.

If we differentiate a function twice, also called **second derivative**, we denote them as

$$\frac{d^2 f}{dx^2} = \frac{d^2 f(x)}{dx^2} = \frac{d^2}{dx^2}f(x) = f\text{''}(x) = f\text{''}$$

Similarly, NEVER write in the form

$$\cancel{\frac{d}{dx}\frac{d}{dx}f(x)} \text{ or } \cancel{\left(\frac{d}{dx}\right)^2 f(x)}$$

Similarly, the second derivative of a function at a particular point $x = x_0$ can be denoted by

$$\frac{d^2}{dx^2}\Big|_{x=0} f(x) \text{ or } f\text{''}(x_0)$$

1.3 First principle

As mentioned in the chapter *Limit and e*, we can use the concept of limit to find the derivative of a function, this approach is called the **First Principle**.

$$\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

To find the derivative of a function at a particular point $x = x_0$, we can use the form

$$\begin{aligned}\left.\frac{d}{dx}\right|_{x=x_0} f(x) &= \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} \quad \text{(More common)}\\ &= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\end{aligned}$$

In HKDSE, if you are told to perform differentiation from first principles, you have to use them, making use of other differentiation rules will score you no mark.

2 Differentiation rules

2.1 Differentiation of common functions

To start with, let's take a look at what are the derivatives of some fundamental functions in mathematics. In HKDSE, most of the time, when the question does not specify the use of first principle, you only need to directly apply differentiation rules to solve the problem.

2.1.1 Constant functions

$$\frac{d}{dx}C = 0$$

Here C is a constant, which means it does not vary as x changes.
Proof

$$\begin{aligned}\frac{d}{dx}C &= \lim_{h\to 0}\frac{C-C}{h}\\ &= \lim_{h\to 0} 0\\ &= 0\end{aligned}$$

2.1.2 Power rule

$$\frac{d}{dx}x^n = nx^{n-1}$$

You can easily memorise the rule by *"bring down the power and subtract one from power"*. Also, please note that n is a constant (does not vary with x)
Proof (for $n \in \mathbb{Z}^+$)

$$\begin{aligned}\frac{d}{dx}x^n &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}\\ &= \lim_{h\to 0}\frac{\left(\cancel{x^n} + C_1^n hx^{n-1} + C_2^n h^2x^{n-2} + \cdots + h^n\right)\cancel{-x^n}}{h} && \text{(Binomial Theorem)}\\ &= \lim_{h\to 0}\frac{nhx^{n-1} + C_2^n h^2x^{n-2} + \cdots + h^n}{h}\\ &= \lim_{h\to 0} nx^{n-1} + C_2^n hx^{n-2} + \cdots + h^{n-1}\\ &= nx^{n-1} + 0 + 0 + \cdots + 0\\ &= nx^{n-1}\end{aligned}$$

This rule is applicable to all $n \in \mathbb{R}$, but the proof for non-integral values of n is not shown here because it involves the use of advanced techniques.

2.1.3 Exponential functions

$$\frac{d}{dx}e^x = e^x$$

Proof

$$\begin{aligned}
\frac{d}{dx}e^x &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\
&= e^x \lim_{h\to 0}\left(\frac{e^h-1}{h}\right)\\
&= e^x\cdot 1\\
&= e^x
\end{aligned}$$

2.1.4 Logarithmic functions

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Proof

$$\begin{aligned}
\frac{d}{dx}\ln x &= \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}\\
&= \lim_{h\to 0}\frac{\ln\left(\frac{x+h}{x}\right)}{h}\\
&= \lim_{h\to 0}\ln\left[1+\left(\frac{1}{x}\right)h\right]^{\frac{1}{h}}\\
&= \ln\left[\lim_{h\to 0}\left(1+\left(\frac{1}{x}\right)h\right)^{\frac{1}{h}}\right]\\
&= \ln\left[e^{\frac{1}{x}}\right]\\
&= \frac{1}{x}
\end{aligned}$$

2.1.5 Trigonometric functions

Sine function

$$\frac{d}{dx}\sin x = \cos x$$

Proof

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{2\cos\frac{2x+h}{2}\sin\frac{h}{2}}{h} && \text{(Sum to product)}\\
&= \left[\lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\right]\left[\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right]\\
&= \cos(x+0)\cdot 1\\
&= \cos x
\end{aligned}$$

Cosine function

$$\frac{d}{dx}\cos x = \sin x$$

Proof

$$\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{-2\sin\frac{2x+h}{2}\sin\frac{h}{2}}{h} && \text{(Sum to product)} \\
&= -\left[\lim_{h\to 0}\sin\left(x+\frac{h}{2}\right)\right]\left[\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right] \\
&= -\sin(x+0)\cdot 1 \\
&= -\sin x
\end{aligned}$$

Tangent function

$$\frac{d}{dx}\tan x = \sec^2 x$$

Proof

$$\begin{aligned}
\frac{d}{dx}\tan x &= \lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\cos(x+h)\sin x}{\cos(x+h)\cos x}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right] && \text{(Double angle formula)} \\
&= \left[\lim_{h\to 0}\frac{\sin h}{h}\right]\left[\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}\right] \\
&= 1\cdot\frac{1}{\cos(x+0)\cos x} \\
&= \sec^2 x
\end{aligned}$$

2.1.6 Summary

$$\begin{aligned}
&\frac{d}{dx}C = 0 && \frac{d}{dx}x^n = nx^{n-1} \\
&\frac{d}{dx}e^x = e^x && \frac{d}{dx}\ln x = \frac{1}{x} \\
&\frac{d}{dx}\sin x = \cos x && \frac{d}{dx}\cos x = -\sin x \\
&\frac{d}{dx}\tan x = \sec^2 x
\end{aligned}$$

2.2 Differentiating the sum, product and quotient of functions

In this part, we will look at functions that are constructed from the fundamental functions in 2.1

2.2.1 Addition/subtraction rule

$$\frac{d}{dx}[f(x) \pm g(x) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$
$$\frac{d}{dx}kf(x) = k\frac{d}{dx}f(x)$$

Proof

$$\begin{aligned}
\frac{d}{dx}[f(x) \pm g(x)] &= \lim_{h\to 0} \frac{[f(x+h) \pm g(x+h)] - [f(x) \pm g(x)]}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \pm \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) \\
\frac{d}{dx}kf(x) &= \lim_{h\to 0} \frac{kf(x+h) - kf(x)}{h} \\
&= k \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= k\frac{d}{dx}f(x)
\end{aligned}$$

Example

$$\begin{aligned}
\frac{d}{dx}\left(\sin x + x^2 + \ln x + e^x\right) &= \frac{d}{dx}\sin x + \frac{d}{dx}x^2 + \frac{d}{dx}\ln x + \frac{d}{dx}e^x \\
&= \cos x + 2x + \frac{1}{x} + e^x
\end{aligned}$$

Note: When you have more experiences in differentiation, you will be able to directly compute the derivative.

2.2.2 Product rule

$$\frac{d}{dx}f(x)g(x) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

Proof

$$
\begin{aligned}
\frac{d}{dx}f(x)g(x) &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)+f(x+h)g(x)-f(x+h)g(x)-f(x)f(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)[g(x+h)-g(x)]+g(x)[f(x+h)-f(x)]}{h} \\
&= \left[\lim_{h\to 0}f(x+h)\right]\left[\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right]+\left[\lim_{h\to 0}g(x)\right]\left[\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right] \\
&= f(x+0)\frac{d}{dx}g(x)+g(x)\frac{d}{dx}f(x) \\
&= f(x)\frac{d}{dx}g(x)+g(x)\frac{d}{dx}f(x)
\end{aligned}
$$

Example

$$
\begin{aligned}
\frac{d}{dx}\sin x\ln x &= \sin x\frac{d}{dx}\ln x+\ln x\frac{d}{dx}\sin x \\
&= \frac{\sin x}{x}+\cos x\ln x
\end{aligned}
$$

2.2.3 Quotient rule

$$
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{[g(x)]^2}
$$

Note: This rule is very messy, I seldom use it when performing differentiation. You are advised to use the product rule in conjunction with negative powers to handle it.

Proof

$$
\begin{aligned}
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{f(x+h)g(x)-f(x)g(x+h)}{g(x+h)g(x)}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{f(x+h)g(x)+f(x)g(x)-f(x)g(x)-f(x)g(x+h)}{g(x+h)g(x)}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{g(x)[f(x+h)-f(x)]-f(x)[g(x+h)-g(x)]}{g(x+h)g(x)}\right] \\
&= \left[\lim_{h\to 0}\frac{1}{g(x+h)g(x)}\right]\left[g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}-f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right] \\
&= \frac{1}{g(x+0)g(x)}\left[g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)\right] \\
&= \frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{[g(x)]^2}
\end{aligned}
$$

Example

$$\begin{aligned}\frac{d}{dx}\left[\frac{\sin x}{\sqrt{x}}\right] &= \frac{\sqrt{x}\frac{d}{dx}\sin x - \sin x\frac{d}{dx}x^{\frac{1}{2}}}{(\sqrt{x})^2} \\ &= \frac{\sqrt{x}\cos x - (\sin x)\left(\frac{1}{2\sqrt{x}}\right)}{x} \\ &= \frac{\cos x}{\sqrt{x}} - \frac{\sin x}{2x\sqrt{x}}\end{aligned}$$

Alternative method:

$$\begin{aligned}\frac{d}{dx}\left[\frac{\sin x}{\sqrt{x}}\right] &= \frac{d}{dx}(\sin x)x^{-\frac{1}{2}} \\ &= \sin x\frac{d}{dx}x^{-\frac{1}{2}} + x^{-\frac{1}{2}}\frac{d}{dx}\sin x \\ &= (\sin x)\left(\frac{-1}{2}x^{-\frac{3}{2}}\right) + x^{-\frac{1}{2}}\cos x \\ &= \frac{\cos x}{\sqrt{x}} - \frac{\sin x}{2x\sqrt{x}}\end{aligned}$$

The alternative method seems more indirect, but it is helpful when dealing with more complex functions. The chances of confusing with + or − sign in quotient rule is reduced.

2.3 Chain rule and composite functions

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$$

This representation may seem abstract to you, recall what we have discussed before, the notation $\frac{dy}{dx}$ means differentiating function y with respect to x. Similarly, $\frac{dy}{du}$ means differentiating function y with respect to u. And $\frac{du}{dx}$ means differentiating u with respect to x.

Worked Example 1

Let $y = \sin 2x$, find $\frac{dy}{dx}$

Solution

We may set $u = 2x$ and see what happens

$$\begin{aligned}\frac{dy}{dx} &= \frac{d(\sin 2x)}{dx} \\ &= \frac{d(\sin u)}{dx}\end{aligned}$$

Remember, we cannot apply the rule $\frac{d}{dx}\sin x = \cos x$ directly, this is because we the variable in the function is u but we are differentiating with respect to x, therefore, we can apply the chain rule.

$$\frac{d(\sin u)}{dx} = \frac{d(\sin u)}{du} \cdot \frac{du}{dx}$$

Now, we can apply the rule $\frac{d}{du} \sin u = \cos u$

$$\frac{d(\sin u)}{dx} = \cos u \cdot \frac{du}{dx}$$

So what is $\dfrac{du}{dx}$? Recall $u = 2x$, so $\dfrac{du}{dx} = \dfrac{d(2x)}{dx}$.

$$\begin{aligned} \frac{d}{dx} \sin 2x &= \cos 2x \cdot 2 \\ &= 2 \cos 2x \end{aligned}$$

In practice, the chain can be quite long, and in fact you can make it as long as you like. The essence of chain rule is it helps you differentiate the right function with respect to the right variable.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dh} \cdot \frac{dh}{dk} \cdots \frac{dn}{dx}$$

From time to time, you may find the use of substitution annoying and time consuming, you may want to adopt a more straight forward approach.

$$\frac{d}{dx} f[g(x)] = f'[g(x)] \frac{d}{dx} g(x)$$

What we are doing here is first find out the derivative of $f(x)$, which is $f'(x)$ then put $x = g(x)$ to $f'(x)$, that gives $f'[g(x)]$. Then differentiate $g(x)$ with respect to x.

Worked Example 2

Find $\dfrac{d}{dx} \sin 2x$

Solution

First, we identify $g(x) = 2x$, $f(x) = \sin x$, therefore $f[g(x)] = \sin 2x$.

Then, we ask ourselves what is $f'(x)$, obviously, $f'(x) = \dfrac{d}{dx} \sin x = \cos x$.

We then know $f'[g(x)] = \cos 2x$

Lastly, we want to know what is $\dfrac{d}{dx} g(x)$, obviously, $\dfrac{d}{dx} 2x = 2$

Assemble what we have, we see

$$\begin{aligned} \frac{d}{dx} \sin 2x &= \cos 2x \cdot 2 \\ &= 2 \cos 2x \end{aligned}$$

Worked Example 3 Many candidates are confused with what to differentiate first and the relationship among functions. I will suggest them to spend some time to dissect what composes the given function. For example, consider

$$y = \sqrt{1 + \sin(x^2 + x + 1)}$$

First of all, we see the innermost function is

$$x^2 + x + 1$$

Then we put it into a sine function and add it by 1

$$1 + \sin\left(x^2 + x + 1\right)$$

Finally we take square root of it

$$\sqrt{1 + \sin\left(x^2 + x + 1\right)}$$

So if we count from the outermost function, the order is $\surd \to 1{+}\sin \to x^2{+}x{+}1$ Therefore, when performing chain rule, the order of differentiation is $\surd \to 1 + \sin \to x^2 + x + 1$.

$$\begin{aligned}\frac{d}{dx}\sqrt{1 + \sin\left(x^2 + x + 1\right)} &= \underbrace{\frac{1}{2\sqrt{1 + \sin\left(x^2 + x + 1\right)}}}_{\text{differentiating } \surd} \cdot \underbrace{\cos\left(x^2 + x + 1\right)}_{\text{differentiating } 1+\sin} \cdot \underbrace{(2x + 1)}_{\text{differentiating } x^2+x+1} \\ &= \frac{(2x + 1)\cos\left(x^2 + x + 1\right)}{2\sqrt{1 + \sin\left(x^2 + x + 1\right)}}\end{aligned}$$

Concept check (This type of question never appeared in HKCEE / HKALE / HKDSE)
The following table gives some information about $f(x)$ and $g(x)$.

	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
$x=0$	1	2	1	1
$x=1$	2	3	0	2
$x=2$	3	4	1	3

Find $\left.\frac{d}{dx}\right|_{x=1} f[g(x)]$
Solution
By chain rule, we see

$$\begin{aligned}\frac{d}{dx} f[g(x)] &= f'[g(x)]g'(x) \\ \left.\frac{d}{dx}\right|_{x=1} f[g(x)] &= f'[g(1)]g'(1) \\ &= f'(0)\cdot 2 \\ &= 2\cdot 2 = 4\end{aligned}$$

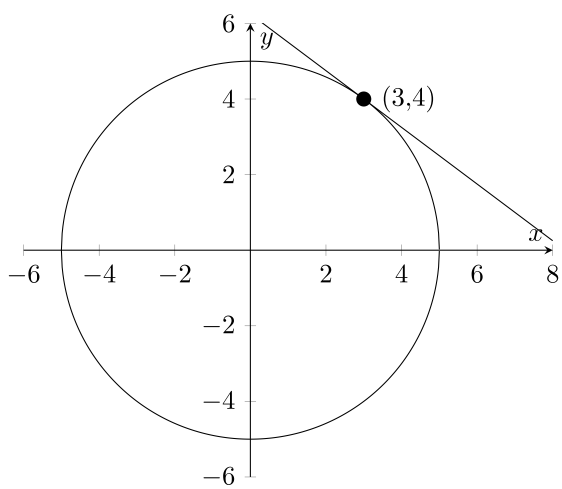
3 Advanced Differentiation Techniques

3.1 Implicit differentiation

Previously, we often see equations in the form $y = f(x)$, y can be expressed in terms of x easily. However, some relationships between x and y can hardly (or impossible) to be represented in this way, for example

$$e^{xy} + \sin y = 1$$

Therefore, we apply the concept of chain rule to find the derivatives of the function. It may seem abstract to you, let us start with some examples.
Example 1
Find $\frac{dy}{dx}$ if $x^2 + y^2 = 25$
(Obviously it is the equation of circle which origin lies at (0,0) with radius 5)
Hence, find $\left.\frac{dy}{dx}\right|_{x=3,y=4}$



Solution

$$\begin{aligned}
x^2 + y^2 &= 25 \\
\frac{d}{dx}\left(x^2 + y^2\right) &= \frac{d}{dx}25 \quad \text{(differentiating both sides w.r.t. } x\text{)} \\
2x + 2y\left(\frac{dy}{dx}\right) &= 0 \qquad\qquad\qquad\qquad \text{(apply chain rule)} \\
\frac{dy}{dx} &= -\frac{x}{y}
\end{aligned}$$

Hence, we see $\left.\frac{dy}{dx}\right|_{x=3,y=4} = -\frac{3}{4}$

Example 2 Find $\dfrac{dy}{dx}$ if $e^{xy} + \sin y = 1$.
Solution

$$\begin{aligned} e^{xy} + \sin y &= 1 \\ e^{xy}\left(x\frac{dy}{dx} + y\right) + \cos y\left(\frac{dy}{dx}\right) &= 0 \\ xe^{xy}\left(\frac{dy}{dx}\right) + ye^{xy} + \cos y\left(\frac{dy}{dx}\right) &= 0 \\ (xe^{xy} + \cos y)\frac{dy}{dx} &= -ye^{xy} \\ \frac{dy}{dx} &= -\frac{ye^{xy}}{xe^{xy} + \cos y} \end{aligned}$$

3.2 Differentiation of inverse functions

Inverse functions (denoted as $f^{-1}(x)$ for $f(x)$ means we can find back what value of x gives a particular function value $f(x)$. i.e. if $f(a) = b, f^{-1}(b) = a$. Or we can refer to a common example

$$\begin{aligned} \sin\left(\frac{\pi}{6}\right) &= \frac{1}{2} \\ \sin^{-1}\left(\frac{1}{2}\right) &= \frac{\pi}{6} \end{aligned}$$

Note: Of course $\sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}, \dfrac{5\pi}{6} \cdots$ But in this context we consider $x = \sin^{-1} y$ for $x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ only. The reason behind this assumption is not emphasised here. I just want to give you some fundamental concepts about inverse functions. Back to the topic, how we can differentiate inverse functions like $y = \sin^{-1} x$ with respect to x?
Example 1

$$\begin{aligned} y &= \sin^{-1} x \\ \sin y &= x && (1) \\ \cos y\left(\frac{dy}{dx}\right) &= 1 && \text{(differentiating both sides w.r.t. } x) \\ \frac{dy}{dx} &= \frac{1}{\cos y} \\ \frac{dy}{dx} &= \frac{1}{\sqrt{1 - \sin^2 y}} && (\text{Remember } \sin^2 y + \cos^2 y = 1) \\ \frac{d}{dx}\sin^{-1} x &= \frac{1}{\sqrt{1 - x^2}} && (\text{Using (1) }) \end{aligned}$$

3.3 Logarithmic differentiation

Some functions are hard to be differentiated (especially those involving non-constant powers and multiplication of lengthy functions). For these cases, we may first take natural logarithm to both sides, then differentiate both sides.

Example 1

$$
\begin{aligned}
y &= \left(x^2+x+1\right)^x \\
\ln y &= \ln\left(x^2+x+1\right)^x \\
\ln y &= x\ln\left(x^2+x+1\right) && \text{(remember } \ln a^b = b\ln a\text{)} \\
\frac{1}{y}\left(\frac{dy}{dx}\right) &= \frac{d}{dx}x\ln\left(x^2+x+1\right) && \text{(differentiating both sides w.r.t. } x\text{)} \\
\frac{1}{y}\left(\frac{dy}{dx}\right) &= x\left(\frac{2x+1}{x^2+x+1}\right)+\ln\left(x^2+x+1\right) \\
\frac{dy}{dx} &= y\left[x\left(\frac{2x+1}{x^2+x+1}\right)+\ln\left(x^2+x+1\right)\right] \\
&= \left(x^2+x+1\right)^x\left[\frac{x(2x+1)}{x^2+x+1}+\ln\left(x^2+x+1\right)\right]
\end{aligned}
$$

Example 2

$$y=\frac{(x+1)\left(x^2+1\right)^2}{x-1}$$

In fact, we can use product rules and quotient rules to differentiate the function, but it takes plenty of time. Logarithmic differentiation provides an alternative to us.

$$
\begin{aligned}
y &= \frac{(x+1)\left(x^2+1\right)^2}{x-1} \\
\ln y &= \ln(x+1)+2\ln\left(x^2+1\right)-\ln(x-1) \\
\frac{1}{y}\left(\frac{dy}{dx}\right) &= \frac{1}{x+1}+\frac{2(2x)}{x^2+1}-\frac{1}{x-1} \\
\frac{dy}{dx} &= \frac{(x+1)\left(x^2+1\right)^2}{x-1}\left[\frac{1}{x+1}+\frac{4x}{x^2+1}-\frac{1}{x-1}\right]
\end{aligned}
$$

But remember this method does not provide a simplified form of the answer, further manipulation is required. However, if the question only asks $\left.\frac{dy}{dx}\right|_{x=0}$, we can directly substitute$x=0$ to the result.

$$
\begin{aligned}
\left.\frac{dy}{dx}\right|_{x=0} &= \frac{(0+1)\left(0^2+1\right)^2}{0-1}\left[\frac{1}{0+1}+\frac{4(0)}{0^2+1}-\frac{1}{0-1}\right] \\
&= -1(1+1) \\
&= -2
\end{aligned}
$$

3.4 Differentiation by calculator

If you have not owned fx–3650P II / fx–3650P calculators, purchase one.
The aforementioned calculators allow us to check the derivative of a function at a certain point. The button can be found at the RHS of "Prog". To call the function, press **SHIFT** then $\int$ **dx**.
You should see **d/dX (**, The syntax of the operator is **d/dX(function,value)**.
For example, to calculate

$$\left.\frac{d}{dx}\right|_{x=0} \frac{(x+1)\left(x^2+1\right)^2}{x-1}$$

The output is –2, which means $\left.\frac{d}{dx}\right|_{x=0} \frac{(x+1)\left(x^2+1\right)^2}{x-1} = -2$