Introduction to Logic

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Basic Logic Crash Course

1. T,
2. F,
3. or ,
4. and ,
5. $\neg$,
6. DeMorgan's Laws,
7. $\in$,
8. $\implies$,
9. $\exists$,
10. $\mathcal{P}$,
11. $\forall$,
12. contrapositive.

Definition (Or)

The logical statement 'or' is used to express that at least one of two statements is true and is false only when both are false.

$$P \vee Q.$$

$$P \text{ or } Q.$$

or	T	F
T	T	T
F	T	F

Definition (And)

The logical statement 'and' is used to express that both of two statements is true and is false if either is false.

$$P \wedge Q.$$

$$P \text{ and } Q.$$

	T	F
T	T	F
F	F	F

Example

Evaluate the following

$$3 > 7 \text{ or } 7 > 3$$

$$3 > 7 \text{ and } 7 > 3.$$

Definition (Not)

The logical statement '$\neg$' is the negation of logical statement:

$$\neg \mathsf{T} = \mathsf{F}$$

and

$$\neg \mathsf{F} = \mathsf{T}.$$

Definition (Implies)

The logical statement 'implies' is used to express that some conclusion (Q) follows from a premise (P) and is false only in the $\mathsf{T} \implies \mathsf{F}$ case.

$$P \text{ implies } Q.$$

$$\text{Premise} \implies \text{Conclusion}$$

$$P \implies Q$$

	T	F
T	T	F
F	T	T

Question

Express $\implies$

	T	F
T	T	F
F	T	T

using only or , and , and $\neg$.

Question

Express $\implies$

	T	F
T	T	F
F	T	T

using only or , and , and $\neg$.

Answer

$P \implies Q \equiv \neg P$ or Q.

Example

Suppose we know that when it is rainy it is cloudy:

$$\text{rainy} \implies \text{cloudy}.$$

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Question

What can we deduce if it is not cloudy ? What can we deduce if it is not rainy ?

Answer

Example

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Question

What can we deduce if it is not cloudy ? What can we deduce if it is not rainy ?

Answer

Not windy.

Example

Suppose we know that when it is rainy it is cloudy:

$$\text{rainy} \implies \text{cloudy}.$$

Question

What can we deduce if it is not cloudy ? What can we deduce if it is not rainy ?

Answer

Not windy. Nothing.

Definition (Bi-Implies)

The logical statement 'biimplies' is used to express two logical statements are equivalent.

$$P \text{ bi-implies } Q.$$

$$P \iff Q$$

$$P \implies Q \text{ and } Q \implies P$$

	T	F
T	T	F
F	F	T

Definition (Set)

A set is a finte or infinite collection of unordered and distinct objects.

We denote sets using curly braces '{ }' which enclose the set's elements.

A set's cardinality (denoted by '| |' or '#') is the number of such elements the set contains.

Example (Set)

A set of cardinality 6.

$$
\begin{aligned}
A &= \{3, 8, 9, 10, 42, -3\}, \\
|A| &= \#A = 6.
\end{aligned}
$$

Standard sets.

Natural numbers

$\mathbb{N} = \{0, 1, 2, \ldots\}$.

Integers

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ ($\mathbb{Z}$ because the the German word for 'number' is 'Zahlen').

Rational numbers

$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$.

Definition (Predicate)

A predicate is an operator of logic that returns true (T) or false (F) for a specified universe (e.g. $\mathbb{Z}$, days of the week, reserved words).

Example

A predicate test for prime testing:

$$P : \mathbb{Z} \to \{\mathsf{T}, \mathsf{F}\}$$

$$P : x \mapsto \begin{cases} \mathsf{T} & \text{if } x \text{ is prime} \\ \mathsf{F} & \text{otherwise} \end{cases}$$

evaluates to true only when x is a prime number:

$$P(7) = \mathsf{T} \qquad P(8) = \mathsf{F} \qquad P(101) = \mathsf{T}.$$

Definition (Element)

The symbol $\in$ read "is an element of" is a predicate with definition

$$\in : (\text{element}, \text{Universe}) \to \{\mathsf{T}, \mathsf{F}\}$$

$$\in : (e, U) \mapsto e \text{ is an element of } U$$

(Note: The symbol ε, which denotes the empty word, should not be used for set inclusion.)

Example

For the universe of even numbers $E = \{2, 4, 6, \ldots\}$

$$\begin{aligned} 2 \in E &\iff 2 \text{ is an element of } \{2, 4, 6, \ldots\} \\ &\iff \mathsf{T} \end{aligned}$$

and

$$\begin{aligned} 17 \in E &\iff 17 \text{ is an element of } \{2, 4, 6, \ldots\} \\ &\iff \mathsf{F} \end{aligned}$$

(here we say $17 \notin E$).

Definition (Existence)

The logical statement 'there exists' (alternatively 'there is') is denoted by $\exists$. It is used to express that there is some element of the predicate's universe for which the predicate is true.

Intuitively, for predicate P and universe $U = \{u_0, u_1, u_2, \ldots\}$

$$\exists x \in U : P(x) \iff P(u_0) \text{ or } P(u_1) \text{ or } P(u_2) \cdots$$

Example

1. $\exists x \in \{a, b, c\} : x = a$
2. $\neg \exists x \in \{1, 2, 3\} : x > 7$
3. $\exists x \in \mathbb{Z} : x$ is prime

Example

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2. $\neg\exists x \in \{1, 2, 3\} : x > 7$
3. $\exists x \in \mathbb{Z} : x$ is prime

Question

$$\exists a, b \in \mathbb{Z} \left(\frac{a}{b} = \sqrt{2} \right) \ ?$$

Definition (Every)

The symbol $\forall$ denotes 'for all' (equivalently 'for every', 'for each') and is called the universal quantifier. $\forall$ is used to make/test assertions 'universally' over an entire set.

Intuitively, for predicate P and universe $U = \{u_0, u_1, u_2, \ldots\}$

$$\forall x \in U\, [P(x)] \iff P(u_0) \text{ and } P(u_1) \text{ and } P(u_2) \cdots$$

Question

Let: $Q(x) \iff 2 \mid x \iff \exists y \in \mathbb{Z} : 2y = x$.

It is true that $\forall x \in \mathbb{Z}; Q(x)$?

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Answer

No, because every integer isn't even.

Question

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It is true that $\forall x \in \mathbb{Z}; Q(x)$?

Answer

No, because every integer isn't even.

Question

Let: $R(x) \iff 2 \mid x - 1$.

Is it true that $\forall x \, [Q(x) \text{ or } R(x)] \equiv \mathsf{T}$?

Question

Let: $Q(x) \iff 2 \mid x \iff \exists y \in \mathbb{Z} : 2y = x$.

It is true that $\forall x \in \mathbb{Z}; Q(x)$?

Answer

No, because every integer isn't even.

Question

Let: $R(x) \iff 2 \mid x - 1$.

Is it true that $\forall x \, [Q(x) \text{ or } R(x)] \equiv \mathsf{T}$?

Answer

Yes, because every integer is either even or odd.

Order Matters

Let $P(x, y) : \mathbb{N} \times \mathbb{N} \to \{\mathsf{T}, \mathsf{F}\}$ be a predicate given by

$$P(x, y) = x > y.$$

So, for example $P(1, 2) = \mathsf{F}$ and $P(2, 1) = \mathsf{T}$.

Question

$\forall x; \exists y : P(x, y)$?

Question

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Answer

Yes. Let $y = x + 1$.

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Answer

Yes. Let $y = x + 1$.

Question

$\exists x : \forall y; P(x, y)$?

Answer

No. There is no single number larger than every number. (Note $\infty \notin \mathbb{N}$).

Subset and the Empty Set

Definition

$A \subseteq B$ states that A "is a subset of" B.

$$A \subseteq B := \forall z \, [z \in A \implies z \in B].$$

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Example

$$\{1,2,3\} \subseteq \{1,2,3,4\} \qquad\qquad \neg[\{1,2,3,4\} \subseteq \{1,2,3\}].$$

Definition

$A \subset B$ states that A 'is a proper subset' of B.

$$A \subset B := [A \subseteq B \text{ and } \neg\,(B \subseteq A)]$$

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Example

$$\{1,2,3\} \subset \{1,2,3,4\} \qquad\qquad \neg[\{1,2,3\} \subset \{1,2,3\}].$$

Definition

Some extra notation for negation:

1. $A \neq B := \neg\,(A = B)$,
2. $A \notin B := \neg\,(A \in B)$,
3. $\neg[P \implies Q] \equiv P \not\Longrightarrow Q$.

Theorem

There is a unique set, which we label $\varnothing$, with no members,

$$\exists! \varnothing : \forall x \left(x \notin \varnothing \right).$$

or equivalently

$$\exists! \varnothing : \neg \exists x : x \in \varnothing.$$

Definition (Empty set)

The unique set with no members is $\varnothing$.

Distinguish carefully between $\varnothing$, the empty set, and ϕ/φ, the greek letter 'phi'!

Aside

Theorem (Russell's Paradox)

There is no universal set.

$$\neg\exists A : \forall x \, (x \in A) \, .$$

Defining Sets [1/2]

Explicitly:

$$A = \{2, 3, 2, 4, 1, 2, 3\}$$

Class Abstraction:

$$B = \{x : x \text{ is a prime number}\} = \{2, 3, 5, 7, 11, \ldots\}\,,$$

$$\begin{aligned} C &= \{x : x \text{ is an english word and also a palindrome}\} \\ &= \{\text{a}, \text{dad}, \text{mom}, \ldots\} \end{aligned}$$

Defining Sets [2/2]

Recursively:

1. $2 \in E$,
2. $a, b \in E \implies a \cdot b \in E$.

Question

What is the definition of E using set building notation?

Answer

$\{2^n : n \in \mathbb{N}\}$.

Definition (Power set)

The power set of A, denoted $\mathcal{P}(A)$ or 2^A, is the set of all subsets of A.

$$\mathcal{P}(A) = \{B : B \subseteq A\}$$

Example

If $A = \{1, 2, 3\}$ then the power set of A is

$$\mathcal{P}(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Enumerating $\mathcal{P}(\{1, 2, 3\})$

$$
\begin{aligned}
\texttt{000} &= \{\square, \square, \square\} = \{\} = \varnothing \\
\texttt{001} &= \{\square, \square, 3\} = \{3\} \\
\texttt{010} &= \{\square, 2, \square\} = \{2\} \\
\texttt{011} &= \{\square, 2, 3\} = \{2, 3\} \\
\texttt{100} &= \{1, \square, \square\} = \{1\} \\
\texttt{101} &= \{1, \square, 3\} = \{1, 3\} \\
\texttt{110} &= \{1, 2, \square\} = \{1, 2\} \\
\texttt{111} &= \{1, 2, 3\} = \{1, 2, 3\}\,.
\end{aligned}
$$

Proposition

For any set A, the cardinality of its power set is $2^{|A|}$,

$$|\mathcal{P}(A)| = 2^{|A|}. \tag{1}$$

(This is likely the motivation behind the notation 2^A for power sets.)

Question

What is $\mathcal{P}(\varnothing)$?

Question

What is $\mathcal{P}(\varnothing)$?

Answer

$\mathcal{P}(\varnothing) = \{\varnothing\}$.

Definition (Set difference)

Set difference, which is sometimes read 'A take away B' is denoted $A - B$ and defined by

$$A - B := \{z : z \in A \text{ and } z \notin B\}.$$

Theorem

$z \in A - B \iff z \in A \text{ and } z \notin B.$

Example

$\{1, 2, 3\} - \{3, 4, 5\} = \{1, 2\}.$

Question

$\{3, 4, 5\} - \{1, 2, 3\}?$

Answer

$\{3, 4, 5\} - \{1, 2, 3\} = \{3, 4, 5\}\,.$

Definition (Intersection)

The intersection of A and B, denoted $A \cap B$ is defined

$$A \cap B := \{z : z \in A \text{ and } z \in B\}. \tag{2}$$

Theorem

$z \in A \cap B \iff z \in A$ and $z \in B$.

Example

$\{1, 2, 3, 4, 5\} \cap \{2, 3, 4\} = \{2, 3, 4\}$.

Example

$\{1, 2, 3, 4, 5\} \cap \{2, 3, 4\} = \{2, 3, 4\}$.

Question

What is $C \cap \varnothing$?

Example

$\{1, 2, 3, 4, 5\} \cap \{2, 3, 4\} = \{2, 3, 4\}$.

Question

What is $C \cap \varnothing$?

Answer

$C \cap \varnothing = \varnothing$.

Definition (Disjoint)

We say that A and B are disjoint when

$$A \cap B = \varnothing.$$

Example

$\{1, 2\}$ and $\{3, 4\}$ are disjoint.

$\{1, 2, 3\}$ and $\{3, 4\}$ are not disjoint as $\{1, 2, 3\} \cap \{3, 4\} = \{3\} \neq \varnothing$.

Question

Are A and $\varnothing$ disjoint?

Question

Are A and $\varnothing$ disjoint?

Answer

Yes! Any set is disjoint with the empty set:

$$\begin{aligned} z \in A \cap \varnothing &\iff z \in A \text{ and } z \in \varnothing \\ &\iff z \in A \text{ and } \mathsf{F} \\ &\iff \mathsf{F} \end{aligned}$$

Thus $A \cap \varnothing = \varnothing$.

Definition (Union)

The union of A and B, denoted $A \cup B$ is defined

$$A \cup B := \{z : z \in A \text{ or } z \in B\}.$$

Theorem

$z \in A \cup B \iff (z \in A \text{ or } z \in B)$.

Example

If $A = \{a, b, c\}$ and $B = \{b, c, d, e\}$ then $A \cup B = \{a, b, c, d, e\}$.

Theorem (De Morgan's Laws)

For sets A, B, and C

1. $A \cap (B \cap C) = (A \cap B) \cup (A \cap C)$,
2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
3. $A - (B \cap C) = (A - B) \cap (A - C)$, *and*
4. $A - (B \cap C) = (A - B) \cap (A - C)$.

Definition (Ordered Pair)

An ordered pair is denoted (x, y) and is defined by

$$(x, y) = \{\{x\}, \{x, y\}\}$$

Theorem

$(x, y) = (A, B) \iff x = A$ and $y = B$.

Definition

In general, an n-tuple is written $(x_0, \ldots, x_{n-1})$.

Example (Ordered tuples)

- $(2, 3) \neq (3, 2)$
- $(2, 2, 2) \neq (2, 2)$
- (1)

Definition (Cartesian product)

The Cartesian product of sets A and B is denoted $A \times B$ and is given by

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Example

$$\{2, 3\} \times \{x, y, z\} = \{(2, x), (2, y), (2, z), (3, x), (3, y), (3, z)\}$$

Notation

When can write A^2 for $A \times A$.

Definition (Sequence)

A sequence is an ordered-list of elements. Sequences are denoted

$$S = (x_0, x_1, \ldots, x_{n-1}) \tag{3}$$

and have this short form: $S_i = x_i$ (like array indexing).

Two sequences, A and B are equal when

$$\forall i;\ A_i = B_i.$$

Defining infinite sequences.

Example (Fibonacci sequence)

Explicitly $F = (0, 1, 1, 2, 3, 5, 8, 13, 34, 5, 89, 144, \ldots)$

Recursively $F_{n-2} = F_n - F_{n-1}$ with $F_0 = 0$, $F_1 = 1$.

Closed Form Let $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$

$$F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi} = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

Side Note

$\varphi = \frac{1+\sqrt{5}}{2}$ or the golden ratio is pervasive in industrial and everyday design. For instance the pantheon and any credit card share the same relative dimensions because of the golden ratio.

Definition (Limit)

Let $f : \mathbb{R} \to \mathbb{R}$ be a function, $c \in \mathbb{R}$, $L \in \mathbb{R}$.

$$\lim_{x \to c} f(x) = L$$

$$\iff (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in D)(0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon)$$

Proof Techniques

Prove the following:

1. $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$,
2. $\neg\exists a, b \in \mathbb{N} : \frac{a}{b} = \sqrt{2}$,
3. a chess board can be tiled with L-shaped dominoes so that only one square remains uncoverd,
4. 2 among 5 points in a unit square are $\frac{1}{\sqrt{2}}$ close to each other.

Direct Proof

The misbehaving (and still unrealized genius of number theory) Gauß (1777-1855), was exiled to the corner of his classroom and told not to return until he had calculated the sum of the first 100 numbers.

Gauß returned with the correct answer shortly afterward.

Proposition

The sum of the first 100 numbers is $5\,050$.

Proof.

$$
\begin{aligned}
&1 + 2 + 3 + \cdots + 50 + 51 + \cdots + 98 + 99 + 100 \\
&\qquad = (1 + 100) + (2 + 99) + (3 + 98) + \cdots + (50 + 51) \\
&\qquad = (101) + (101) + (101) + \cdots + (101) \\
&\qquad = \frac{100}{2} \cdot 101 \\
&\qquad = 5\,050.
\end{aligned}
$$

$\square$

PMI

The principle of mathematical induction states $P(m)$ is true for all $m \in \mathbb{N}$ if

1. $P(m) \implies P(m+1)$ for any $m \in \mathbb{N}$; and
2. $P(0)$.

(It is implicit that $P(0)$ means $P(0) \equiv \mathsf{T}$.)

Theorem (The Principle of Mathematical Induction)

For any predicate $P : \mathbb{N} \to \{\mathsf{T}, \mathsf{F}\}$

$$\{P(0) \text{ and } \forall m \in \mathbb{N}\,[P(m) \implies P(m+1)]\} \implies \{\forall m \in \mathbb{N}, P(m)\}$$

(we use superfluous bracketing to express something more meaningful).

Induction proofs contain the bizarre statement 'assume $P(n) \equiv \mathsf{T}$' which is seemingly what requires proof.

However, there is a huge difference between

$$[\forall n P(n)] \implies P(n+1)$$

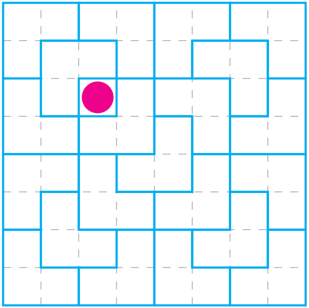
and

$$\forall n \, [P(n) \implies P(n+1)] \, .$$

We are not assuming $\forall n P(n)$, rather we letting n be arbitrary (and fixed) to show that when $P(n) = \mathsf{T}$ that $P(n+1) = \mathsf{T}$ as well.

A standard chess board can be filled with 'corner-tiles':

so that only one space is left uncovered and no tiles overlaps. Here is one such tiling:



where the disc shows the (arbitrary) location of the uncovered square.

Proposition

Any $2^n \times 2^n$ board with $n \in \mathbb{N}$ can be tiled so that only one square is left uncovered.

Proof of Tiling

Base Case ($n = 0$).

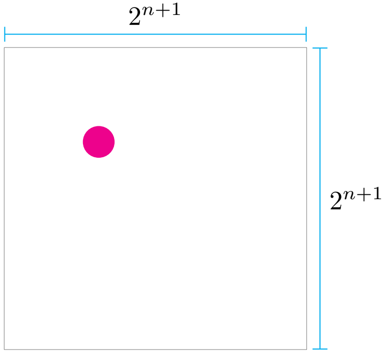
We can cover a $2^0 \times 2^0 = 1 \times 1$ board with zero tiles:

Induction Hypothesis

Assume a $2^n \times 2^n$ board can be tiled.

Proof of Tiling

(Demonstrate, given our assumption that a $2^n \times 2^n$ board can be tiled, that we can tile a $2^{n+1} \times 2^{n+1}$ board.)



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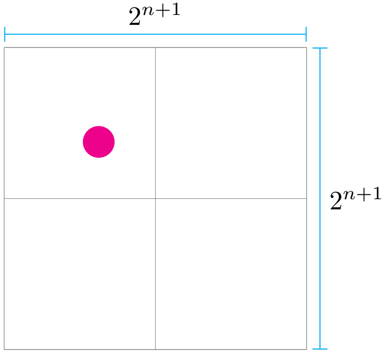

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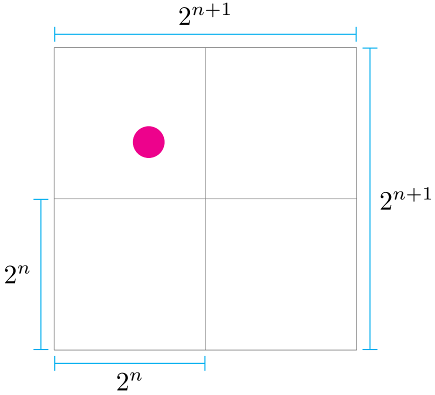

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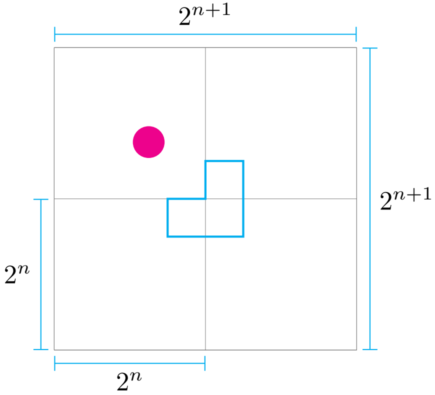

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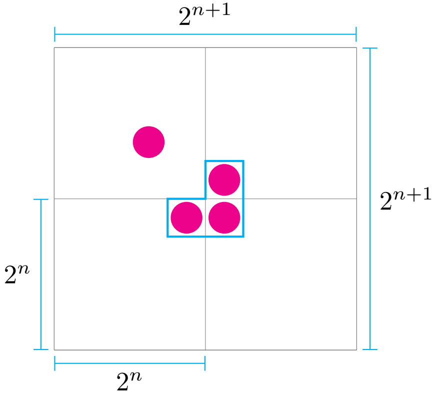

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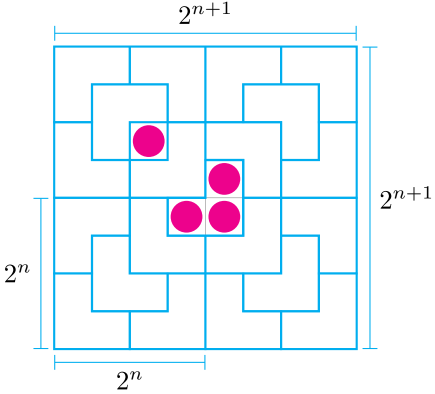


Contradiction

Contradiction is a proof technique where, in order to show some predicate P true, we assume $\neg P$ and deduce F. (That is, we show that an invalid P has an absurd consequence).

Theorem (Proof by Contradiction)

For any predicate P

$$[\neg P \implies \mathsf{F}] \implies P.$$

Proposition

$\sqrt{2}$ can not be expressed as a fraction (i.e. $\sqrt{2}$ is an irrational number)

Proof of Irrationality of $\sqrt{2}$

Towards a contradiction, suppose $\sqrt{2}$ can be expressed as the fraction

$$\sqrt{2} = \frac{a}{b} \tag{4}$$

with $a, b \in \mathbb{N}$.

Assume further that $\frac{a}{b}$ is a reduced fraction so that $\gcd(a, b) = 1$.

Squaring yields

$$2 = \frac{a^2}{b^2} \implies 2b^2 = a^2.$$

Trivially $2 \,\middle|\, 2b^2$ and so $2 \,\middle|\, a^2$.

But 2 cannot be decomposed (it is prime) so it must be that $2 \mid a$ and thus $4 \mid a^2$. Similarly (applying this argument in the reverse direction) $4 \mid 2b^2 \implies 2 \mid b^2$ and thus $2 \mid b$.

However, if both a and b are divisible by 2 it must be the case that $\frac{a}{b}$ is not reduced, i.e. $\gcd(a, b) = 2 \neq 1$. $\lightning$

The Pigeonhole Principle

The pigeonhole principle is the mathematical formalization of the statement:

> *If you put $n+1$ pigeons into n holes there is (at least) one hole with two pigeons.*

Proposition

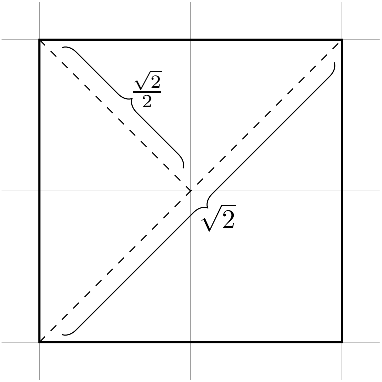
Recall the Euclidean distance between two points in the plane:

$$\left|\overline{(x_1, y_1)(x_2, y_2)}\right| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Prove if 5 points are drawn within the boundaries of a unit square (i.e. a planar square of side length 1), then there are two points that have Euclidean distance $< \frac{1}{\sqrt{2}}$.

Proof

The diagonal of a unit square has length $\sqrt{2}$ (Pythagoras' theorem) and a subsquare $\frac{1}{4}$ the area has diagonal length $\frac{\sqrt{2}}{2}$,



Therefore, any points within the same subsquare can be at most $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ units apart.

Proof

By PHP, if five points are drawn within the interior of a square, then two points must be in the same subsquare (there are only four such subsquares). Thus there are two points with Euclidean distance less than $\frac{1}{\sqrt{2}}$.

Next Week

1. Number systems.
2. Rings, groups, and fields.
3. Complex Numbers.