Chapter 2 §2. Orderings on the Monomials in $k[x_1, \dots, x_n]$

Definition 1. A *monomial ordering* on $k[x_1, \dots, x_n]$ is any relation $>$ on $\mathbf{Z}^n_{\geq 0}$, or equivalently, any relation on the set of monomials x^α, $\alpha \in \mathbf{Z}^n_{\geq 0}$, satisfiying:

(i) $>$ is a total (or linear) order on $\mathbf{Z}^n_{\geq 0}$.

(ii) If $\alpha > \beta$ and $\gamma \in \mathbf{Z}^n_{\geq 0}$, then $\alpha + \gamma > \beta + \gamma$.

(iii) $>$ is a well-ordering of $\mathbf{Z}^n_{\geq 0}$.

Note 2.0.1. Every well-ordering is automatically a total order; so condition (iii) implies (i).

Note 2.0.2. The element $\overrightarrow{0} = (0, \dots, 0) \in \mathbf{Z}^n_{\geq 0}$ is necessarily the smallest element in $\mathbf{Z}^n_{\geq 0}$ under any such ordering.

Proof. If $\overrightarrow{0} > \alpha$ then, since $\alpha \in \mathbf{Z}^n_{\geq 0}$, (ii) implies that $\overrightarrow{0} + \alpha > \alpha + \alpha$ or $\alpha > 2\alpha$. We can repeat this argument to conclude that

$$\overrightarrow{0} > \alpha > 2\alpha > 3\alpha > \cdots.$$

But then the set $\{\overrightarrow{0}, \alpha, 2\alpha, \dots\}$ doesn't have a smallest member and the ordering isn't a well-ordering. (iii) is violated. $\blacksquare$

A monomial ordering on $k[x_1, \dots, x_n]$ is used to order the monomials x^α, $\alpha \in \mathbf{Z}^n_{\geq 0}$, in the obvious manner. Namely, $x^\alpha > x^\beta$ if and only if $\alpha > \beta$. To order the monomials in $k[x, y, z]$, for example, we have to agree which variable is x_1, which x_2 and the remaning variable would then be x_3. That is, implicit in any actual ordering of the monomials in $k[x, y, z]$ is a permutation of these variables, say yzx if $y = x_1$, $z = x_2$ and $x = x_3$. Two monomial orderings of $k[x, y, z]$ are *equivalent* if one can be obtained from the other by merely permuting the variables x, y, z. The same holds for monomial orderings on $k[y_1, \dots, y_n]$ and $k[x_1, \dots, x_n]$.

Definition 2. (Lexicographic Order). Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ be in $\mathbf{Z}^n_{\geq 0}$. We say $\alpha >_{lex} \beta$ if $\alpha_i > \beta_i$, where $i = \inf\{j \colon \alpha_j \neq \beta_j\}$.

> In lexicographic order terms are first collected in descending powers of x_1. The resulting coefficients are then collected in descending powers of x_2 and so on.

For later reference: The matrix associated with Lexicographic monomial ordering with four variables x_1, x_2, x_3, x_4 is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As usual we put $|\alpha| = \sum_{k=1}^n |\alpha_k|$ when $\alpha = (\alpha_1, \dots, \alpha_n)$.

Definition 3. (Graded Lex Order). Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ be in $\mathbf{Z}^n_{\geq 0}$. We say $\alpha >_{grlex} \beta$ if either (i) $|\alpha| > |\beta|$ or (ii) $|\alpha| = |\beta|$ and $\alpha_i > \beta_i$, where $i = \inf\{j \colon \alpha_j \neq \beta_j\}$.

> In graded lexicographic order terms are first collected in groups of the same total power and these groups are arranged in order of descending total power. The terms in these groups are then collected in descending powers of x_1. The resulting coefficients are then collected in descending powers of x_2 and so on.

For later reference: The matrix associated with this graded lex monomial ordering or DegreeLexicographic monomial ordering with four variables x_1, x_2, x_3, x_4 is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Definition 4. (Graded Reverse Lex Order). Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be in $\mathbf{Z}^n_{\geq 0}$. We say $\alpha >_{grevlex} \beta$ if either (i) $|\alpha| > |\beta|$ or (ii) $|\alpha| = |\beta|$ and $\beta_i > \alpha_i$, where $i = \sup\{j : \alpha_j \neq \beta_j\}$.

> In graded reverse lexicographic order, terms are first collected in groups of the same total power arranged in descending order. The terms in these groupings are then collected in ascending powers of x_n. The resulting coefficients are collected further in ascending powers of x_{n-1} and so on.

For later reference: The matrix associated with this graded reverse lex monomial ordering with four variables x_1, x_2, x_3, x_4 is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

It is an easily established fact that each of the three above orderings (induces) a a (monomial) ordering of the monomials in $k[x_1, \ldots, x_n]$. What is less obvious is that no two such orderings are equivalent. We proceed to establish this.

Proof. It is clear that the only possibility for equivalence is between the graded lex order and the graded reverse lex order. Consider the terms of total degree 2 in 3 variables. Representing $x_1^{i_1} x_2^{i_2} x_3^{i_3}$ by $i_1 i_2 i_3$ we find that

$$200 >_{grlex} 110 >_{grlex} 101 >_{grlex} 020 >_{grlex} 011 >_{grlex} 002,$$

while

$$200 >_{grevlex} 110 >_{grevlex} 020 >_{grevlex} 101 >_{grevlex} 011 >_{grevlex} 002.$$

Permutations of the variable names produces two orbits, namely $\{200, 020, 002\}$ and $\{110, 101, 011\}$. It is clear from this fact that no permutation of variable names will change the *grlex* order into the *grevlex* order because, taking the terms in descending order, the largest term in the *grlex* order is followed by two terms without interruption from the other orbit while in the *grevlex* order the largest term is followed by just one term from the other orbit before the sequence reverts back to the orbit of the largest term again. ∎

Terminology. Let $f = \sum_\alpha a_\alpha x^\alpha$ be a non zero polynomial in $k[x_1, \ldots, x_n]$ and let $>$ be a monomial order on $k[x_1, \ldots, x_n]$.

(i) The *multidegree* of f is given by

$$\text{multidegree}(f) = \max\{\alpha : a_\alpha \neq 0\}.$$

(ii) The *leading coefficient* of f is given by

$$\text{LC}(f) = a_{\text{multidegree}(f)}.$$

(iii) The *leading monomial* of f is given by

$$\text{LM}(f) = x^{\text{multidegree}(f)}.$$

(iv) The *leading term* of f is given by

$$\text{LT}(f) = \text{LC}(f) \cdot \text{LM}(f).$$

Identity 2.0.3. Suppose $f, g \in k[x_1, \ldots, x_n]$. Then

$$\text{multidegree}(fg) = \text{multidegree}(f) + \text{multidegree}(g). \tag{2.0.3}$$

Proof. If $\alpha > \alpha'$ and $\beta > \beta'$ then we have both (a) $\alpha + \beta > \alpha' + \beta$ and $\alpha' + \beta > \alpha' + \beta'$. Putting these together gives (b) $\alpha + \beta > \alpha' + \beta'$. Thus the (necessarily nonzero) product of the two leading terms of f and g respectively has a higher multidegree than the product of any other two terms. ∎

Identity 2.0.4. Suppose $f, g \in k[x_1, \ldots, x_n]$. If $f + g \neq 0$ then

$$\text{multidegree}(f+g) \leq \max\{\text{multidegree}(f), \text{multidegree}(g)\} \tag{2.0.4}$$

with equality holding in (2.0.3) when (but not only when) multidegree$(f) \neq$ multidegree(g).

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§2.2.1.

Rewrite each of the following polynomials, ordering the terms using the lex order, the grlex order, and the grevlex order, giving LM(f), LT(f), and the multidegree in each case.

(a)$f(x, y, z) = 2x + 3y + z + x^2 - z^2 + x^3$.

Solution. Taking $x_1 = x$, $x_2 = y$ and $x_3 = z$ we find

(lex) $x^3 + x^2 + 2x + 3y - z^2 + z$
(grlex) $x^3 + x^2 - z^2 + 2x + 3y + z$
(grevlex) $x^3 + x^2 - z^2 + 2x + 3y + z$

(b) $f(x, y, z) = 2x^2y^8 - 3x^5yz^4 + xyz^3 - xy^4$.

Solution. Taking $x_1 = x$, $x_2 = y$ and $x_3 = z$ we find

(lex) $-3x^5yz^4 + 2x^2y^8 - xy^4 + xyz^3$
(grlex) $-3x^5yz^4 + 2x^2y^8 - xy^4 + xyz^3$
(grevlex) $2x^2y^8 - 3x^5yz^4 - xy^4 + xyz^3$

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§2.2.2.

Each of the following polynomials is written with its monomials ordered according to (exactly) one of lex, grlex, or grevlex order. Determine which monomials order was used in each case. Assume $x_1 = x$, $x_2 = y$, $x_3 = z$.

(a) $f(x, y, z) = 7x^2y^4z - 2xy^6 + x^2y^2$.

Solution. This is grlex order.

(b) $f(x, y, z) = xy^3z + xy^2z^2 + x^2z^3$.

Solution. This is grevlex order.

(c) $f(x, y, z) = x^4y^5z + 2x^3y^2z - 4xy^2z^4$.

Solution. This is lex order.

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§2.2.3.

Rewrite each of the following polynomials, ordering the terms using the lex order, the grlex order, and the grevlex order, giving LM(f), LT(f), and the multidegree in each case. Order the variables $z > y > x$.

(a)$f(x, y, z) = 2x + 3y + z + x^2 - z^2 + x^3$.

Solution. Taking $x_1 = z$, $x_2 = y$ and $x_3 = x$ we find

(lex) $-z^2 + z + 3y + x^3 + x^2 + 2x$
(grlex) $x^3 - z^2 + x^2 + z + 3y + 2x$
(grevlex) $x^3 - z^2 + x^2 + z + 3y + 2x$

(b) $f(x, y, z) = 2x^2y^8 - 3x^5yz^4 + xyz^3 - xy^4$.

Solution. Taking $x_1 = z$, $x_2 = y$ and $x_3 = x$ we find

(lex)	$-3x^5yz^4 + xyz^3 + 2x^2y^8 - xy^4$
(grlex)	$-3x^5yz^4 + 2x^2y^8 + xyz^3 - xy^4$
(grevlex)	$2x^2y^8 - 3x^5yz^4 + xyz^3 - xy^4$

$\boxed{\S\mathbf{2.2.4.}}$

Show that grlex is a monomial order.

$\boxed{\S\mathbf{2.2.5.}}$

Show that grevlex is a monomial order.

Solution. Each of these orderings has ordinal type ω and is stable for multiplication of monomials.

$\boxed{\S\mathbf{2.2.6.}}$

Another monomial order is the **inverse Lexicographic** or **invlex** order defined as follows: $\alpha >_{invlex} \beta$ if and only if $\alpha_i > \beta_i$, where $i = \max\{j \colon \alpha_j \neq \beta_j\}$. Show that invlex is equivalent to the lex order with the variables permuted in a certain way. (Which permutation?)

Solution. The lex order on the monomials taken with respect to the primitive ordering $x_n > x_{n-1} > \cdots > x_2 > x_1$, is just the invlex order on the monomials taken with respect to the usual primitive ordering $x_1 > x_2 > \cdots > x_n$.

$\boxed{\S\mathbf{2.2.7.}}$

Let $>$ be any monomial order.

(a) Show that $\alpha \geq 0$ for all $\alpha \in \mathbf{Z}^n_{\geq 0}$.

Solution. We did this earlier. If $\alpha < 0$, then $2\alpha = \alpha + \alpha < 0 + \alpha = \alpha$ and we get (after a trivial extension of this argument) that $0 > \alpha > 2\alpha > 3\alpha > \cdots$. This is an infinite decreasing sequence and violates the central property that $>$ is a well-ordering.

(b) Show that if x^α divides x^β, then $\alpha \leq \beta$. Is the converse true?

Solution. If x^α divides x^β then $x^\beta = x^\alpha \cdot f(x)$ for some $f \in k[x_1, \ldots, x_n]$. Suppose $f(x) = \sum_\gamma a_\gamma x^\gamma$. Then

$$x^\beta = \sum_\gamma a_\gamma x^{\gamma+\alpha}.$$

Comparing terms it follows that $a_\gamma \neq 0$ for precisely one value of γ, say γ', and $\gamma' + \alpha = \beta$, $a_{\gamma'} = 1$. Now $0 \leq \gamma'$; so by the second property of monomial orderings $0 + \alpha \leq \gamma' + \alpha$ which $= \beta$ giving that $\alpha \leq \beta$. In general the converse is not true. For example using lex order in $k[x, y]$ with $x > y$, we have $x > y$; so in multiindex language $(1, 0) > (0, 1)$, but it is not true that y divides x.

(c) Show that if $\alpha \in \mathbf{Z}^n_{\geq 0}$, then α is the smallest element of $\alpha + \mathbf{Z}^n_{\geq 0}$.

Solution. Suppose $\beta = \gamma + \alpha \in \alpha + \mathbf{Z}^n_{\geq 0}$ and $\beta < \alpha$, then $\gamma < 0$ which is impossible; so by contradiction α is the smallest element of $\alpha + \mathbf{Z}^n_{\geq 0}$.

$\boxed{\S\mathbf{2.2.8.}}$

Write a precise definition of what it means for a system of linear equations to be in echelon form, using the ordering $x_1 > x_2 > \cdots > x_n$.

Solution. If the system is $\sum_{j=1}^n +1 A_{ij} x_j = 0$ with $x_{n+1} = 1$, there are two conditions:

(i) The function $m(i) = \inf\{j\colon A_{ij} \neq 0\}$ is strictly increasing in the sense that $a < b$ implies $m(a) < m(b)$. (Here we agree that $\inf \emptyset = +\infty$ and it is true that $+\infty < +\infty$.)

(ii) When $m(i) < +\infty$ we have $A_{im(i)} = 1$. (In this case system is in rowreduced echelon form.)

$\boxed{\S\mathbf{2.2.9.}}$

In this exercise we study grevlex in more detail. Let $>_{invlex}$ be the order given in Exercise 2.2.6, and define $>_{rinvlex}$ to be the reversal of this ordering, i.e.,

$$\alpha >_{rinvlex} \beta \Longleftrightarrow \beta >_{invlex} \alpha, \qquad \alpha, \beta \in \mathbf{Z}^n_{\geq 0}.$$

Notice that rinvlex is a "double reversal" of lex in the sense that we first reverse the order of the variables and then we reverse the ordering itself.

(a) Show that $\alpha >_{grevlex} \beta$ if and only if $|\alpha| > |\beta|$, or $|\alpha| = |\beta|$ and $\alpha >_{rinvlex} \beta$.

Solution. Suppose $|\alpha| = |\beta|$ and

$$\begin{aligned} \alpha &= (\alpha_1, \alpha_2, \ldots, \alpha_n); \\ \beta &= (\beta_1, \beta_2, \ldots, \beta_n); \\ \alpha - \beta &= (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n). \end{aligned}$$

Let k be the position of the last nonzero entry of $\alpha - \beta$. Then in this case (where $|\alpha| = |\beta|$)

$$\begin{aligned} &\alpha >_{grevlex} \beta \text{ iff } \alpha_k - \beta_k < 0 \text{ iff } \alpha_k < \beta_k. \\ &\alpha >_{rinvlex} \beta \text{ iff } \beta >_{invlex} \alpha \text{ iff } \beta_k - \alpha_k > 0 \text{ iff } \alpha_k < \beta_k. \end{aligned}$$

(b) Is rinvlex a monomial ordering according to definition 1? I so, prove it; if not, say which properties fail.

Solution. In the invlex order $x_1 <_{invlex} x_1^2 <_{invlex} x_1^3 <_{invlex} \cdots <_{invlex} x_2$; so when this is reversed in the rinvlex order $x_1 >_{rinvlex} x_1^2 >_{rinvlex} x_1^3 >_{rinvlex} \cdots >_{rinvlex} x_2$ and rinvlex is not a well ordering. Of course if we only compare monomials of the same total degree it is a well ordering.

$\boxed{\S\mathbf{2.2.10.}}$

In $\mathbf{Z}_{\geq 0}$ with the usual ordering, between any two integers, there are only a finite number of other integers. Is this necessarily true in $\mathbf{Z}^n_{\geq 0}$ for a monomial order? Is it true for the grlex order?

Solution. In the usual lex order

$$x_1 >_{lex} \cdots >_{lex} x_2^{100} >_{lex} x_2^{99} >_{lex} \cdots >_{lex} x_2^2 >_{lex} x_2;$$

so it is not true for the lex order. It is, however, true for the grlex order because there are only a finite number of monomials whose total degrees lie between two given bounds.

$\boxed{\S\mathbf{2.2.11.}}$

Let $>$ be a monomial order on $k[x_1, \ldots, x_n]$.

(a) Let $f \in k[x_1, \ldots, x_n]$ and let m be a monomial. Show that $\mathrm{LT}(m \cdot f) = m \cdot \mathrm{LT}(f)$.

Solution. If $\alpha > \alpha'$ and $\beta > \beta'$ then we have both (a) $\alpha + \beta > \alpha' + \beta$ and $\alpha' + \beta > \alpha' + \beta'$. Putting these together gives (b) $\alpha + \beta > \alpha' + \beta'$. Thus the (necessarily nonzero) product of the two leading terms of f and g respectively has a higher multidegree than the product of any other two terms. From this fact it follows that $\mathrm{LT}(f \cdot g) = \mathrm{LT}(f) \cdot \mathrm{LT}(g)$; so of course $\mathrm{LT}(m \cdot f) = m \cdot \mathrm{LT}(f)$

(b) Let $f, g \in k[x_1, \ldots, x_n]$. Is it necessarily true that $\mathrm{LT}(f \cdot g) = \mathrm{LT}(f) \cdot \mathrm{LT}(g)$?

Solution. Yes! See the argument in the solution to (a) above.

(c) If $f_i, g_i \in k[x_1, \ldots, x_n]$. Is it necessarily true that $\mathrm{LT}(\sum_i f_i \cdot g_i) = \mathrm{LT}(f_i) \cdot \mathrm{LT}(g_i)$ for some i?

Solution. No! A trivial counterexample is $f_1(x) = f_2(x) = 1$, $g_1(x) = -g_2(x)$ with $g_1(x) = x_1$, say.

$\boxed{\S\mathbf{2.2.12.}}$

(a) **Lemma 8.** Let $f, g \in k[x_1, \ldots, x_n]$ be nonzero polynomials. Then:

(i) $multidegree(fg) = multidegree(f) + multidegree(g)$.

(ii) If $f + g \neq 0$, then $multidegree(f + g) \leq max(multidegree(f), multidegree(g))$. If, in addition, $multidegree(f) \neq multidegree(g)$, then equality occurs.

Proof. Exercise 2.2.11 showed, among other things, that $\mathrm{LT}\,(fg) = \mathrm{LT}\,(f) \cdot \mathrm{LT}\,(g)$. Since $multidegree(x^\alpha x^\beta) = \alpha + \beta$, this establishes (i).

To establish (ii) suppose $f = ax^\alpha + f'$ and $g = bx^\beta + g'$ where $\mathrm{LT}\,(f) = ax^\alpha$ and $\mathrm{LT}\,(g) = bx^\beta$. If $\alpha > \beta$, then $multidegree(f + g) = \alpha = \max\big(multidegree(f), multidegree(g)\big)$, and by symmetry the same statement holds if $\alpha < \beta$. Otherwise $\alpha = \beta$. In this case 2if $a + b \neq= 0$ we have $multidegree(f + g) = multidegree(f) = multidegree(g)$ and equality still holds in (i). But if $a + b = 0$ it is necessarily true that inequality holds in (i). ∎

(b) An example: Suppose $f(x) = x_1^2 + x_1$ and $g(x) = -x_1^2 + 3x_1$, then

$$(2, 0, \cdots, 0) = \text{multigree}(f) = \text{multidegree}(g), \qquad \text{, but}$$
$$\text{multidegree}(f + g) = (1, 0, \ldots, 0).$$