The Dimension of a Monomial Ideal

Paul Vrbik[1]

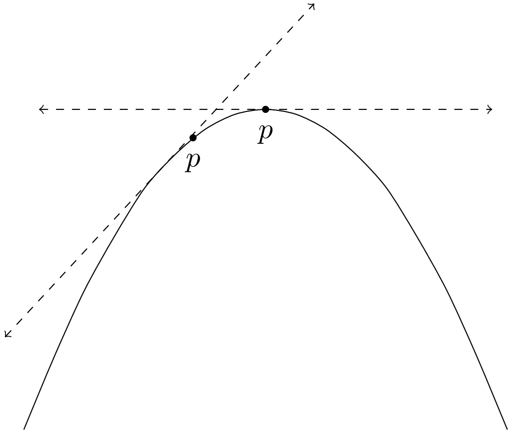
[1]University of Western Ontario

The dimension of an ideal at a point from its variety is equivalent to the vector dimension of the tangent space there.

This is straightforward to calculate as a tangent space is usually a collection of hyperplanes (the exception being singular points where this collection is a tangent cone instead).

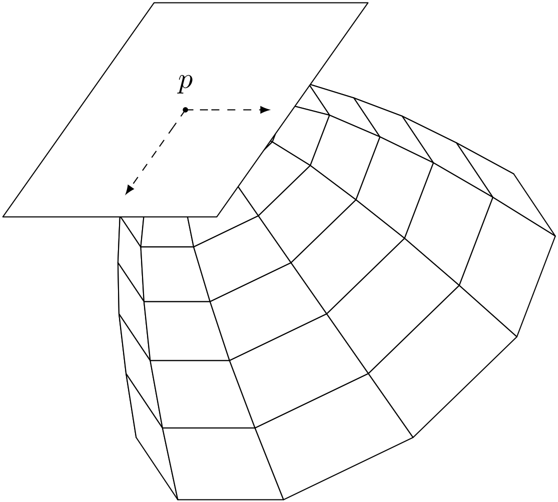
The ideal $\langle\, y - x^2 \,\rangle$ has dimension one at every point

$$p \in \mathbf{V}\big(y - x^2\big) = \big\{\big(p, p^2\big) : p \in \mathbb{R}\big\}.$$



The paraboloid $\langle\, z - y^2 - x^2 \,\rangle$ has dimension two at every point

$$p \in \mathbf{V}\big(z - x^2 - y^2\big)$$



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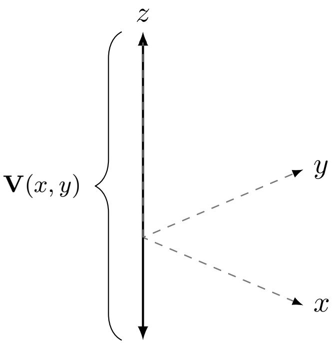
Recall from linear algebra that the dimension of a hyperplane is the number of basis vectors required to span (i.e. capture all points of) the surface.

For our purposes we only need hyperplanes generated by co-ordinate axis, or what are more typically called the x-axis, y-axis, $\ldots$.

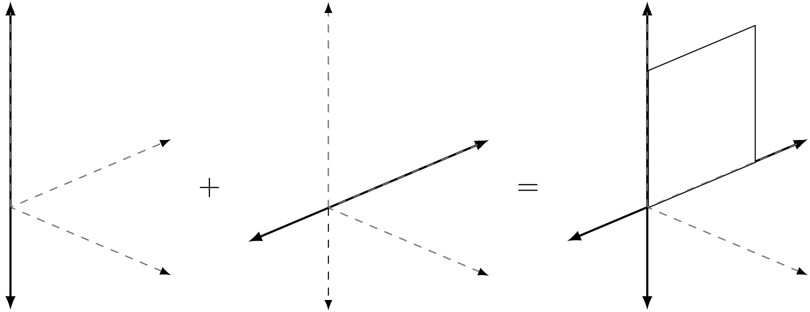
Example

In $\mathbb{A}^3(\mathbb{R}) = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ the z-axis is



Note $\mathbf{V}(x, y) = \mathbf{V}(\{z\}^{\mathrm{c}})$.

Using a process analogous to 'spanning' a hyperplane with unit vectors these axis are extensible to planes



$\mathbf{V}(x, y)$ $\qquad$ $\mathbf{V}(x, z)$ $\qquad$ $\mathbf{V}(x, y) + \mathbf{V}(x, z)$

$$\begin{aligned} \mathbf{V}(xy) + \mathbf{V}(xz) &= \{(s, 0, 0) : s \in \mathbb{R}\} + \{(0, t, 0) : t \in \mathbb{R}\} \\ &= \{(s, t, 0) : s, t \in \mathbb{R}\}. \end{aligned}$$

Definition (Coordinate Axis)

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\begin{aligned}
\mathbb{1}_{x_0} &= (1, 0, \ldots, 0) && \text{“the } x_0\text{-axis”,} \\
\mathbb{1}_{x_1} &= (0, 1, \ldots, 0) && \text{“the } x_1\text{-axis”,} \\
&\vdots && \vdots \\
\mathbb{1}_{x_\ell} &= (0, 0, \ldots, 1) && \text{“the } x_\ell\text{-axis”.}
\end{aligned}
$$

As the position of the unit (i.e. ‘1’) in $\mathbb{1}_{x_i}$ is arbitrary. We write $\mathbb{1}_x$, $\mathbb{1}_y$, $\mathbb{1}_z$ and let the implicit variable ordering assign the ones.

The dimension of a line is one and the dimension of the hyperplane created by removing that line is one less than the ambient space.

Definition

Let $\mathbf{x}$ be a set of variables.

$$\forall\, x \in \mathbf{x};\ \dim(\mathbf{V}(\{x\}^{\mathrm{c}})) := 1,$$

$$\forall\, x \in \mathbf{x};\ \dim(\mathbf{V}(x)) := \ell.$$

($\mathbb{A}^{\ell+1}(\mathbb{R})$ has dimension $\ell + 1$.)

Definition (Span)

Let $\langle\langle\, \mathbb{1}_{x_0}, \ldots, \mathbb{1}_{x_s} \,\rangle\rangle_{\mathbb{R}}$ denote the span of those coordinate axis.

$$\langle\langle\, \mathbb{1}_{x_0}, \ldots, \mathbb{1}_{x_s} \,\rangle\rangle_{\mathbb{R}} = \{c_0 \mathbb{1}_{x_0} + \cdots + c_s \mathbb{1}_{x_s} : c_0, \ldots, c_s \in \mathbb{R}\}.$$

Proposition

Let $x \in \mathbf{x}$.

1. $\mathbf{V}(\{x\}^{\mathrm{c}}) = \langle\, \mathbb{1}_x \,\rangle$, and
2. $\mathbf{V}(x) = \langle\, \mathbb{1}_y : y \in \{x\}^{\mathrm{c}} \,\rangle$.

For principally generated ideals the variety over m (a monomial) decomposes into a union of hyperplanes, each of dimension ℓ:

$$\begin{aligned}\mathbf{V}(m) &= \mathbf{V}\Big(x_0^{d_0}\cdots x_s^{d_s}\Big)\\ &= \mathbf{V}(x_0\cdots x_s)\\ &= \mathbf{V}(x_0)\cup\cdots\cup\mathbf{V}(x_s).\end{aligned}$$

Definition

$$\dim\Big(\mathbf{V}\Big(x_0^{d_0}\cdots x_s^{d_s}\Big)\Big) := \ell,$$

and

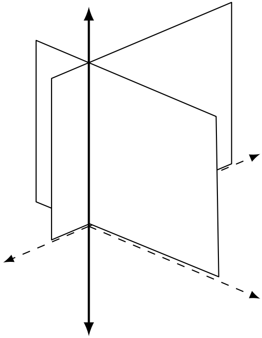
$$\dim(\mathbf{V}(m_0)\cup\cdots\cup\mathbf{V}(m_s)) = \max_{\dim}(\mathbf{V}(m_0), \ldots, \mathbf{V}(m_s)).$$

Seemingly, the dimension of a monomial ideal just requires enumerating a set of names. However, this is only due to the dimension behaving well over unions (in this setting).

In particular, the dimension can never decrease by “unioning” another hyperplane whereas for intersections this is typical.

Example

Consider $\mathbf{V}(x, y)$, the intersection of the planes $\langle\langle\, \mathbb{1}_y, \mathbb{1}_z \,\rangle\rangle_{\mathbb{R}}$ and $\langle\langle\, \mathbb{1}_x, \mathbb{1}_z \,\rangle\rangle_{\mathbb{R}}$.



Although both $\mathbf{V}(x)$, $\mathbf{V}(y)$ have dimension two the dimension of the intersection, $\mathbf{V}(x, y) = \mathbb{1}_z$, is one.

Intersections of hyperplanes are called coordinate subspaces for they inhabit spaces spanned by coordinate axis.

Definition (Coordinate Subspace)

When $\widetilde{\mathbf{x}} \subseteq \mathbf{x}$,

$$\mathbf{V}(\widetilde{\mathbf{x}}) = \bigcap_{y \in \widetilde{\mathbf{x}}} \mathbf{V}(y)$$

is a coordinate subspace.

Our goal is to write these coordinate subspaces using unions rather than intersections so as to pick out the hyperplane of largest dimension.

Proposition

Coordinate subspaces are spanned by coordinate axis. That is, when $\widetilde{\mathbf{x}} \subseteq \mathbf{x}$

$$\mathbf{V}(\widetilde{\mathbf{x}}) = \langle\!\langle\, \mathbb{1}_v : v \in \widetilde{\mathbf{x}}^{\mathrm{c}} \,\rangle\!\rangle_{\mathbb{R}}$$

Proof.

$$\mathbf{V}(\widetilde{\mathbf{x}}) = \bigcap_{y \in \widetilde{\mathbf{x}}} \mathbf{V}(y) = \bigcap_{y \in \widetilde{\mathbf{x}}} \langle\!\langle\, \mathbb{1}_v : v \in \{y\}^{\mathrm{c}} \,\rangle\!\rangle_{\mathbb{R}} = \langle\!\langle\, \mathbb{1}_v : v \in \bigcap_{y \in \widetilde{\mathbf{x}}} \{y\}^{\mathrm{c}} \,\rangle\!\rangle_{\mathbb{R}}$$

$$= \langle\!\langle\, \mathbb{1}_v : v \in (\bigcup_{y \in \widetilde{\mathbf{x}}} \{y\})^{\mathrm{c}} \,\rangle\!\rangle_{\mathbb{R}} = \langle\!\langle\, \mathbb{1}_v : v \in \widetilde{\mathbf{x}}^{\mathrm{c}} \,\rangle\!\rangle_{\mathbb{R}}.$$

□

We demonstrated $\mathbf{V}(\widetilde{\mathbf{x}})$ is spanned by $|\widetilde{\mathbf{x}}^{\mathrm{c}}|$ many coordinate axis; thus

$$\dim(\mathbf{V}(\widetilde{\mathbf{x}})) := (\ell + 1) - |\widetilde{\mathbf{x}}|.$$

(Note: $|\widetilde{\mathbf{x}}^{\mathrm{c}}| = |\mathbf{x}| - |\widetilde{\mathbf{x}}| = \ell + 1 - |\widetilde{\mathbf{x}}|$.)

Dimension of a Monomial Ideal

Intuitively, the dimension of an arbitrary monomial ideal $\langle\, \mathbf{m} \,\rangle$ is the largest subspace (i.e. $\langle\, \mathbb{1}_v : v \in \widetilde{\mathbf{x}} \,\rangle$ with largest $\widetilde{\mathbf{x}}$) embedded in $\langle\, \mathbf{m} \,\rangle$. Extracting this information from unions of the form

$$\mathbf{V}(\widetilde{\mathbf{x}}_0) \cup \cdots \cup \mathbf{V}(\widetilde{\mathbf{x}}_s)$$

is merely a matter of calculating the dimension of the individual hyperplanes among the union.

Unfortunately then, is that the “natural” expansion of $\mathbf{V}(\mathbf{m})$ is into intersections of coordinate subspaces:

$$\mathbf{V}(\mathbf{m}) = \bigcap_{m \in \mathbf{m}} \mathbf{V}(m) = \bigcap_{m \in \mathbf{m}} \bigcup_{x \in \mathrm{indets}(\mathbf{m})} \mathbf{V}(x).$$

However, we can convert between Conjunctive normal forms into Disjunctive normal forms to take a disjunction of coordinate subspaces, or

Intersections over unions of hyperplanes,

to a conjunction of coordinate subspaces, or

Unions over intersections of hyperplanes.

Example (CNF to DNF conversion)

Let $\mathbf{V}_t := \mathbf{V}(t)$ for any variable $t \in [x, y, z]$

$$\begin{aligned}
\mathbf{V}(xz, yz) &= \mathbf{V}_{xz} \cap \mathbf{V}_{yz} \\
&= (\mathbf{V}_x \cup \mathbf{V}_z) \cap (\mathbf{V}_y \cup \mathbf{V}_z) \\
&= (\mathbf{V}_x \cap \mathbf{V}_y) \cup (\mathbf{V}_x \cap \mathbf{V}_z) \cup (\mathbf{V}_z \cap \mathbf{V}_y) \cup (\mathbf{V}_z \cap \mathbf{V}_z) \\
&= \mathbf{V}(x, y) \cup \mathbf{V}(x, z) \cup \mathbf{V}(y, z) \cup \mathbf{V}(z).
\end{aligned}$$

The dimensions of $\mathbf{V}(x, y)$, $\mathbf{V}(x, z)$, $\mathbf{V}(y, z)$, and $\mathbf{V}(z)$ are 1, 1, 1, and 2 (resp.); thus $\dim(\mathbf{V}(xz, yz)) = 2$.

Proposition

Let $\mathbb{Y} = \{\widetilde{\mathbf{y}}_0, \ldots, \widetilde{\mathbf{y}}_s\} \in \mathcal{P}(\mathcal{P}(x_0, \ldots, x_\ell))$ then

$$\exists\, \mathbb{X} : \bigcup_{\widetilde{\mathbf{x}} \in \mathbb{X}} \mathbf{V}(\widetilde{\mathbf{x}}) = \bigcap_{\widetilde{\mathbf{y}} \in \mathbb{Y}} \mathbf{V}(\widetilde{\mathbf{y}}).$$

And amazingly, there is an explicit writing for this conversion.

$$\mathbb{X} = \Big\{ \{\widetilde{y}_0, \ldots, \widetilde{y}_s\} : (y_0, \ldots, y_s) \in \widetilde{\mathbf{y}}_0 \times \cdots \times \widetilde{\mathbf{y}}_s \Big\} \qquad (1)$$

Proof.

$$
\begin{aligned}
& p \in \bigcup_{\widetilde{\mathbf{x}} \in \mathbb{X}} \mathbf{V}(\widetilde{\mathbf{x}}) \\
& \iff p \in \bigcup_{\widetilde{\mathbf{x}} \in \mathbb{X}} \bigcap_{x \in \widetilde{\mathbf{x}}} \mathbf{V}(x) \\
& \iff \exists\, \widetilde{\mathbf{x}} \in \mathbb{X} : \forall\, x \in \widetilde{\mathbf{x}};\, p \in \mathbf{V}(x) \\
& \iff \exists\, \widetilde{\mathbf{x}} \in \left\{ \{y_0, \ldots, y_s\} : (y_0, \ldots, y_s) \in \widetilde{\mathbf{y}}_0 \times \cdots \times \widetilde{\mathbf{y}}_s \right\} : \forall\, x \in \widetilde{\mathbf{x}};\, p \in \mathbf{V}(x) \\
& \iff \exists\, (y_0, \ldots, y_s) \in \widetilde{\mathbf{y}}_0 \times \cdots \times \widetilde{\mathbf{y}}_s : \forall\, x \in \widetilde{\mathbf{x}};\, p \in \mathbf{V}(x) \\
& \iff \exists\, (y_0, \ldots, y_s) \in \widetilde{\mathbf{y}}_0 \times \cdots \times \widetilde{\mathbf{y}}_s : p \in \mathbf{V}(y_0) \cap \cdots \cap \mathbf{V}(y_s) \\
& \iff \forall\, \widetilde{\mathbf{y}} \in \mathbb{Y};\, \exists\, y \in \widetilde{\mathbf{y}} : p \in \mathbf{V}(y) \\
& \iff p \in \bigcap_{\widetilde{\mathbf{y}} \in \mathbb{Y}} \bigcup_{y \in \widetilde{\mathbf{y}}} \mathbf{V}(y) \\
& \iff p \in \bigcap_{\widetilde{\mathbf{y}} \in \mathbb{Y}} \mathbf{V}(\widetilde{\mathbf{y}}).
\end{aligned}
$$

Example

Let $\mathbb{Y} = \{\mathrm{indets}(xz), \mathrm{indets}(yz)\} = \{\{x,z\},\{y,z\}\}$ so that

$$\mathbf{V}(xz, yz) = \bigcap_{\widetilde{\mathbf{y}} \in \mathbb{Y}} \mathbf{V}(\widetilde{\mathbf{y}}).$$

$$\begin{aligned} \exists\, \mathbb{X} &= \{(\widetilde{y}_0, \widetilde{y}_1) : (y_0, y_1) \in \{x,z\} \times \{y,z\}\} \\ &= \{\{x,y\},\{x,z\},\{z,y\},\{z,z\}\} \\ &= \{\{x,y\},\{x,z\},\{y,z\},\{z\}\} \end{aligned}$$

so that $\mathbf{V}(xz, yz) = \bigcup_{\widetilde{\mathbf{x}} \in \mathbb{X}} \mathbf{V}(\widetilde{\mathbf{x}})$ and thus

$$\mathbf{V}(xy, yz) = \mathbf{V}(x,y) \cup \mathbf{V}(x,z) \cup \mathbf{V}(y,z) \cup \mathbf{V}(z).$$

Theorem

Any monomial variety can be decomposed into a union of coordinate subspaces.

$$\forall\, m_0, \ldots, m_s \in [\mathbf{x}];\ \exists\, n_0, \ldots, n_t \in [\mathbf{x}] :$$
$$\mathbf{V}(m_0, \ldots, m_s) = \mathbf{V}(n_0) \cup \cdots \cup \mathbf{V}(n_t).$$

Proof.

Let $\widetilde{\mathbf{y}}_i = \mathrm{indets}(m_i)$ and $\{n_0, \ldots, n_t\} = \left\{ \prod_{x \in \widetilde{\mathbf{x}}_i} x : \widetilde{\mathbf{x}} \in \mathbb{X} \right\}$ in last Proposition. $\square$