Computing Intersection Multiplicity via Triangular Decomposition

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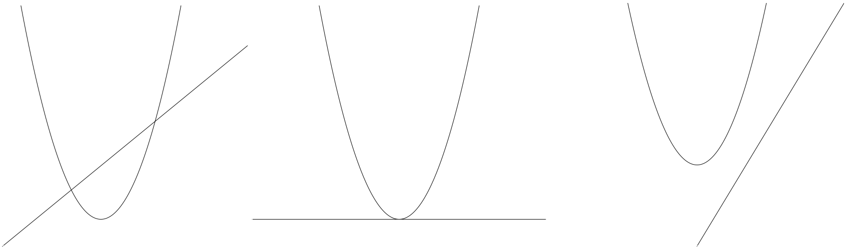
Contributions

1. Devised an algorithm to calculate the Intersection Multiplicity in (generically) all cases, and also (out of necessity)
2. Produced an algorithm to efficiently calculate the Tangent Cone of a curve at a point.

For a parabola (degree 2) and a line (degree 1) we expect the Bézout summand to be two for all possible intersections.

Example



Intersection Multiplicity

The intersection multiplicity is an invariant of algebraic geometry which weighs points of algebraic varieties according to their importance (measured by the dimension of their corresponding tangent spaces). It is a useful invariant whose definition is tailored to satisfy

$$\sum_{p \in \mathbf{V}(\mathbf{h})} \mathrm{im}(p; \mathbf{h}) = \prod_{h \in \mathbf{h}} \deg(h),$$

which implies the number of solutions of a system of polynomials $\mathbf{h}$ is equal to the product of the total degrees among $\mathbf{h}$.

Related Works.

Fulton's Algebraic Curves (more to come).

Cheng and Gao in 2014 wrote "Multiplicity Preserving Triangular Set Decomposition of Two Polynomials" where they give an algorithm which works only for two-polynomials.

Li, Xia, and Zhang in 2010 wrote "Zero Decomposition with Multiplicity of Zero-Dimensional Polynomial Systems" only works at zero-dimensional ideals.

Mora (in 1982) gave an algorithm for calculating standard bases using normal forms which can be used to calculate the intersection multiplicity via its classical definition. (This method manipulates ideals rather than zero sets.)

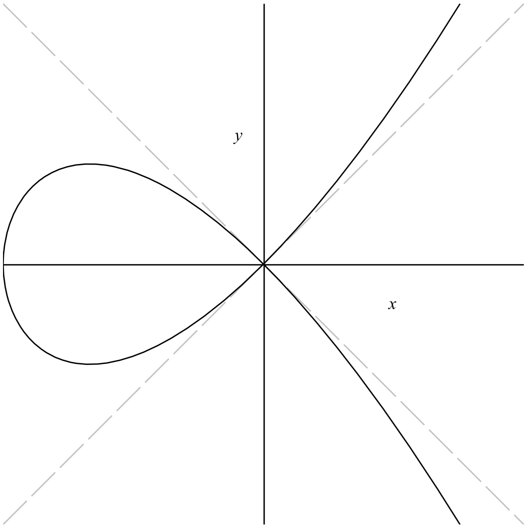
Related Works (in CAS)

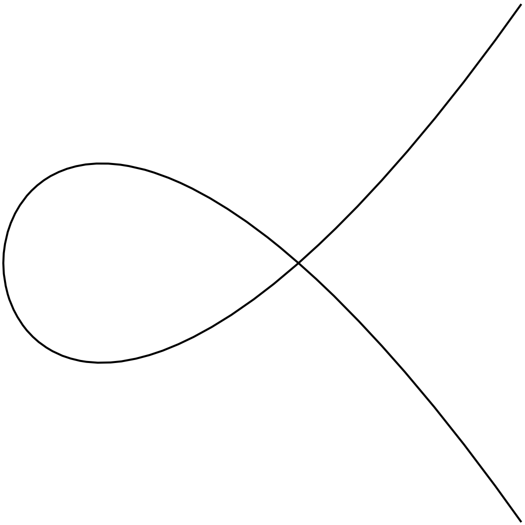
MAGMA provides `IntersectionNumber` and SINGULAR has `iMult`.

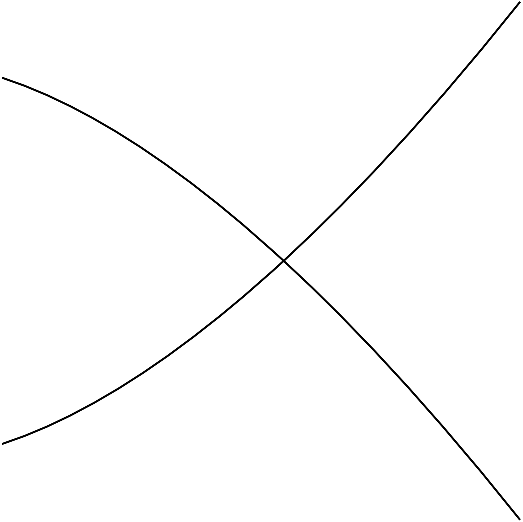
In both cases only the sum of the intersection multiplicities are counted and in fact some tangent lines may be counted twice, leading to over-counting.

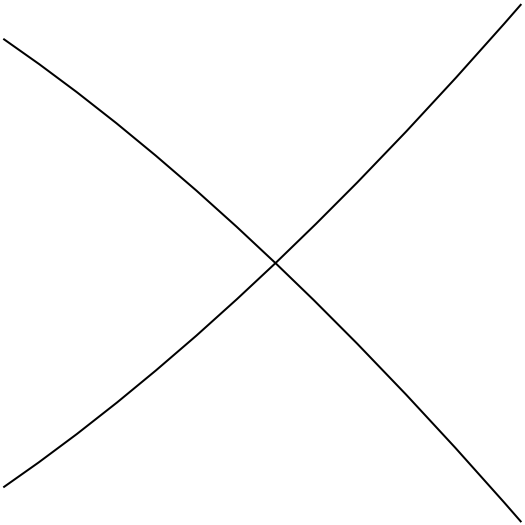
Tangent Cone

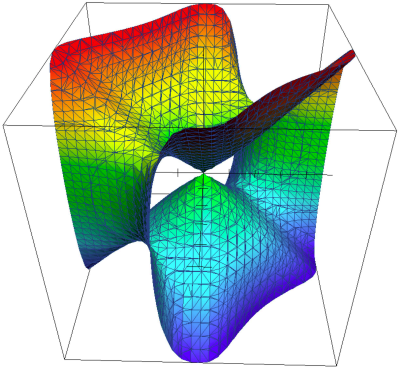
The tangent cone of the fish: $y^2 - x^2(x+1)$ at the origin is $(x+y)(x-y)$.

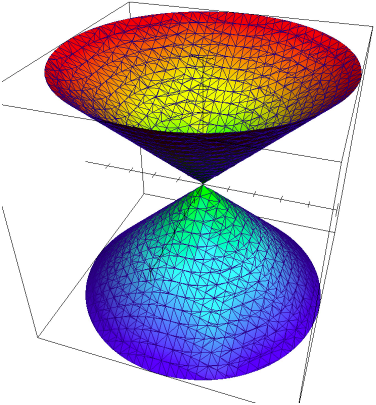












Why we need tangent cones.

In order to reduce the calculation of the intersection multiplicity in $\ell + 1$ variables to ℓ we (sometimes) need to check tranversality at singular points.

Related Works

Actually, Mora's original goal was to compute equations of tangent cones. Recall that he does that through "Gröbner basis like" calculations. — these methods are not practical.

Definition (Polynomial Ring)

Let $\mathbb{Q}[\mathbf{x}]$ be the ring of polynomials with rational coefficients and variables $\mathbf{x} = x_0, \ldots, x_\ell$.

Definition (Variety)

Let $\mathbf{f}$ be a finite collection of polynomials $\{f_0, \ldots, f_s\}$ and $\mathbf{V}(f_0, \ldots, f_s)$ be the set of their common zeros:

$$\mathbf{V}(\mathbf{f}) = \mathbf{V}(f_0, \ldots, f_s) := \{p \in \mathbb{Q} \times \cdots \times \mathbb{Q} : f_0(p) = \cdots = f_s(p) = 0\}.$$

Definition (Ideal Brackets)

Let $\langle\, \mathbf{f} \,\rangle$ be the ideal defined by $\mathbf{f}$ given by

$$\langle\, \mathbf{f} \,\rangle := \{ \sum_{f \in \mathbf{f}} c_f f \,:\, c_f \in \mathbb{Q}[\mathbf{x}] \,\}.$$

(This technical definition is really only necessary for proofs because there is a constructive definition given by Fulton.)

Definition (Bézout's Intersection Multiplicity)

The intersection multiplicity of $\mathbf{f} \subseteq \mathbb{Q}[\mathbf{x}]$ at $p \in \mathbf{V}(\mathbf{f})$ is

$$\operatorname{im}(p;\, \mathbf{f}) := \dim_{\mathbf{vec}}\big(\mathcal{O}_{\mathbb{A}^{\ell+1}(\mathbb{Q}),\, p} / \langle\, \mathbf{f}\, \rangle\big),$$

where

$$\begin{aligned} \mathcal{O}_{\mathbb{A}^{\ell+1}(\mathbb{Q}),\, p} &:= \left\{ \frac{f}{g} \,:\, f, g \in \mathcal{R}[\mathbf{x}],\ g(p) \neq 0 \right\} \\ &= \mathbb{Q}[[\mathbf{x}]]/\langle\, \mathbf{f}\, \rangle. \end{aligned}$$

Theorem (Fulton's Properties)

(In practice, only for planar case and rational points.)

Let two plane curves be given by $h_0,\, h_1 \in \mathbb{Q}[x,\, y]$ *and let* $p \in \mathbb{A}^2\big(\overline{\mathbb{Q}}\big)$.

(2-1) $\operatorname{im}(p;\, h_0,\, h_1) = \infty \iff p \in \mathbf{V}(\gcd(h_0,\, h_1))$,

(2-2) $\operatorname{im}(p;\, h_0,\, h_1) = 0 \iff p \notin \mathbf{V}(h_0) \cap \mathbf{V}(h_1)$,

(2-3) $\operatorname{im}(p;\, h_0,\, h_1)$ *is invariant to affine change of coordinates on* $\mathbb{A}^2(\mathbb{Q})$,

(2-4) $\operatorname{im}(p;\, h_0,\, h_1) = \operatorname{im}(p;\, h_1,\, h_0)$,

(2-5) $\pi_p(h_0) \pitchfork \pi_p(h_1) \implies \operatorname{im}(p;\, h_0,\, h_1) = m_p(h_0) \cdot m_p(h_1)$,

(2-6) $\forall\, g \in \mathbb{Q}[\mathbf{x}];\ \operatorname{im}(p;\, h_0,\, h_1) = \operatorname{im}(p;\, h_0,\, h_1 g) - \operatorname{im}(p;\, h_0,\, g)$, *and*

(2-7) $\forall\, g \in \mathbb{Q}[\mathbf{x}];\ \operatorname{im}(p;\, h_0,\, h_1) = \operatorname{im}(p;\, h_0,\, h_1 + h_0 g)$.

What we did.

1. Extended Fulton's algorithm to work at points in the rational closure.
2. Extended Fulton's properties to arbitrary dimension.

The D5 Principle

Loosely speaking, any algorithm that works over a field can be made to work over a product of fields defined by special zero-dimensional triangular sets called regular chains.

Original version (Della Dora, Discrescenzo & Duval)

Let $f, g \in \mathbb{Q}[x]$ such that f is square-free. Without using irreducible factorization, one can compute $f_0, \ldots, f_r \in \mathbb{Q}[x]$ such that

- $f = f_0 \cdots f_r$, and
- for each $i = 1, \ldots, r$, either

$$[g \equiv 0 \bmod f_i] \text{ or } [g \text{ is invertible modulo } f_i].$$

Definition (Triangularize)

The triangularization of $\mathbf{h} \subseteq \mathbb{Q}[\mathbf{x}]$ is a mapping from polynomial sets of $\mathbb{Q}[\mathbf{x}]$ into sets of regular chains — this process is called triangular decomposition.

$$\begin{aligned} \triangle : \mathcal{P}(\mathbb{Q}[\mathbf{x}]) &\to \mathcal{P}(\mathbb{T}_{\mathrm{reg}}(\mathbb{Q}[\mathbf{x}])) \\ &\quad \mathbf{h} \mapsto \{\mathbf{f}_{\triangle,\,0},\, \ldots,\, \mathbf{f}_{\triangle,\,r}\} : \mathbf{V}(\mathbf{h}) = \overline{\mathbf{W}(\mathbf{f}_{\triangle,\,0})} \cup \cdots \cup \overline{\mathbf{W}(\mathbf{f}_{\triangle,\,r})} \end{aligned}$$

where $r \in \mathbb{N}$ and

$$\mathbf{W}(\mathbf{f}_\triangle) := \mathbf{V}(\mathbf{f}_\triangle) - \mathbf{V}\big(\prod \mathrm{lcoeff}_{\mathrm{mvar}(f)}(f) : f \in \mathbf{f}_\triangle\big)$$

(i.e. removing points where leading terms vanish).

Example

Let $\mathbf{h} = \left\{x^2 + y + z - 1,\, x + y^2 + z - 1,\, x + y + z^2 - 1\right\} \subseteq \mathbb{Q}[\mathbf{x}]$. A triangular decomposition of $\langle\, \mathbf{h} \,\rangle$ is given by

$$\triangle(\mathbf{h}) = \left\{ \begin{cases} x - z \\ y - z \\ z^2 + 2z - 1 \end{cases}, \begin{cases} x \\ y \\ z - 1 \end{cases}, \begin{cases} x \\ y - 1 \\ z \end{cases}, \begin{cases} x - 1 \\ y \\ z \end{cases} \right\}.$$

A description of $\mathbf{h}$ is a set of tuples

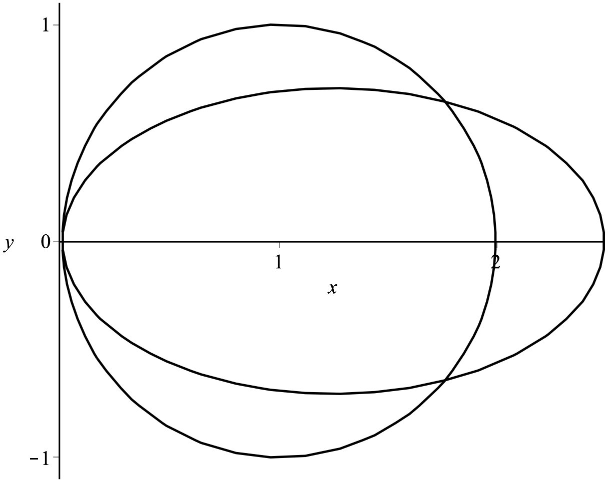
$$\mathbf{D}(\mathbf{h}) = \{(m_0,\, \mathbf{f}_{\triangle,\,0}),\, \ldots,\, (m_r,\, \mathbf{f}_{\triangle,\,r})\}$$

where each $(m_i,\, \mathbf{f}_{\triangle,\,i})$ satisfies $\forall\, p \in \mathbf{V}(\mathbf{f}_{\triangle,\,i})$; $\mathrm{im}(p;\, \mathbf{h}) = m_i$ and $\triangle(\mathbf{h}) = \{\mathbf{f}_{\triangle,\,0}, \ldots, \mathbf{f}_{\triangle,\,r}\}$.

We proved that there is a triangular decomposition of $\mathbf{h}$ such that the regular chains $\mathbf{f}_{\triangle,\,0}$ through $\mathbf{f}_{\triangle,\,r}$ partition the intersection multiplicities as above.

Those regular chains need not be irreducible and this process does not require polynomial factorization.

Example



Example

The circle and ellipse given by

$$\mathbf{h} = \left\{ (x-1)^2 + y^2 - 1,\ \left(\frac{4x}{5} - 1\right)^2 + 2y^2 - 1 \right\} \subseteq \mathbb{Q}[x, y]$$

corresponding to the collection of regular chains

$$\mathbf{f}_{\triangle,\,1} = \begin{cases} x \\ y \end{cases}, \quad \mathbf{f}_{\triangle,\,2} = \begin{cases} 17x - 30 \\ 289y^2 - 120 \end{cases},$$

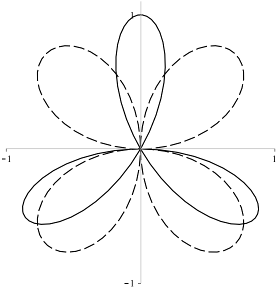
has description $\mathbf{D}(\mathbf{h}) = \{(2,\, \mathbf{f}_{\triangle,\,1}),\, (1,\, \mathbf{f}_{\triangle,\,2})\}$.

> with(RegularChains):

> with(RegularChains:-AlgebraicGeometryTools):

> $\mathbf{h} := \left[\, \left(x^2+y^2\right)^2 + 3x^2y - y^3,\, \left(x^2+y^2\right)^3 - 4x^2y^2 \,\right]$:

> plots[implicitplot]($\mathbf{h}$, $x = -2..2$, $y = -2..2$);



> $R :=$ PolynomialRing($[x, y]$, 101):

> TriangularizeWithMultiplicity($\mathbf{h}, R$):

$$\left[\left[1, \left\{\begin{array}{l} x-1 \\ y+14 \end{array}\right.\right], \left[1, \left\{\begin{array}{l} x+1 \\ y+14 \end{array}\right.\right], \left[1, \left\{\begin{array}{l} x-47 \\ y-14 \end{array}\right.\right], \left[1, \left\{\begin{array}{l} x+47 \\ y-14 \end{array}\right.\right], \left[14, \left\{\begin{array}{l} x \\ y \end{array}\right.\right]\right]$$

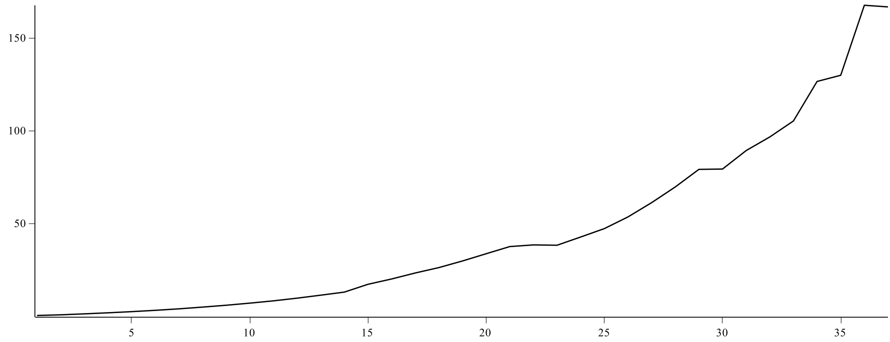
Experimentation

We investigate random homogeneous bivariate polynomials from $\mathbb{Q}[x,y]$ of the form

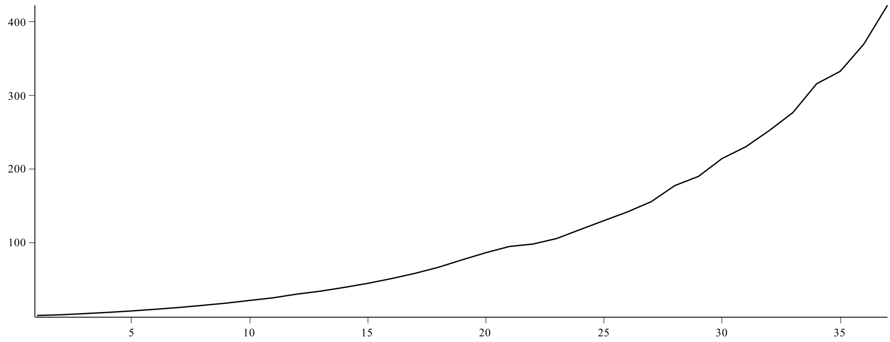
$$c_0 x^{a_0} y^{b_0} + c_1 x^{a_1} y^{b_1} + c_2 x^{a_2} y^{b_2} + c_3 x^{a_3} y^{b_3} + c_4 x^{a_4} y^{b_4}$$

where $a_0 + b_0, \ldots, a_4 + b_4 = d$ for varying $d \in \mathbb{N}^{>1}$ and $c_0, \ldots, c_4 \in \mathbb{Q}$

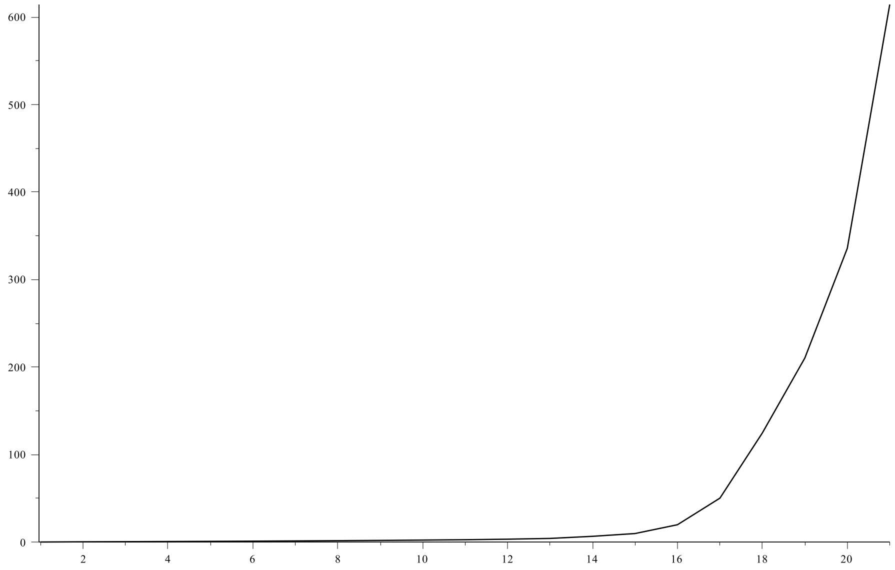
In $\mathbb{Q}_{101}[x, y]$



In $\mathbb{Q}_{962\,592\,769}[x, y]$



In $\mathbb{Q}[x, y]$



What we did.

1. Extended Fulton's algorithm to work at points in the rational closure.
2. Extended Fulton's properties to arbitrary dimension.

Theorem (Extended Fulton's Properties — New stuff!)

Let $\mathbf{h} = \{h_0, \ldots, h_\ell\} \subseteq \mathbb{Q}[\mathbf{x}]$ *so that* $\langle\, \mathbf{h} \,\rangle$ *is zero dimensional,* $p := (p_0, \ldots, p_\ell) \in \mathbb{A}^{\ell+1}\big(\overline{\mathcal{F}}\big)$, *and let* $\mathbf{h} = \{h_\ell\} \cup \mathbf{h}^{\downarrow}$.

(n-1) $\operatorname{im}(p;\, \mathbf{h}) \in \mathbb{N}$,

(n-2) $\operatorname{im}(p;\, \mathbf{h}) = 0 \iff p \notin \mathbf{V}(\mathbf{h})$,

(n-3) $\operatorname{im}(p;\, \mathbf{h})$ *is invariant to affine change of coordinates on* $\mathbb{A}^{\ell+1}(\mathbb{Q})$,

(n-4) $\operatorname{im}(p;\, \mathbf{h}) = \operatorname{im}(p;\, \sigma(\mathbf{h}))$ *for any permutation* $\sigma(\mathbf{h})$ *of the elements of* $\mathbf{h}$,

(n-5) $\operatorname{im}(p;\, (x_0 - p_0)^{m_0}, \ldots, (x_\ell - p_\ell)^{m_\ell}) = m_0 \cdots m_\ell$,

(n-6) *provided* $\mathbf{h}^{\downarrow}$, $gh = h_\ell$ *is a regular sequence (and thus dim* $\langle\, \mathbf{h}^{\downarrow}, gh \,\rangle = 0$*)*

$$\operatorname{im}(p;\, \mathbf{h}^{\downarrow},\, gh) = \operatorname{im}(p;\, \mathbf{h}^{\downarrow},\, g) + \operatorname{im}(p;\, \mathbf{h}^{\downarrow},\, h),$$

(n-7) $\forall\, g \in \langle\, \mathbf{h}^{\downarrow} \rangle$; $\operatorname{im}(p;\, \mathbf{h}^{\downarrow},\, h_\ell) = \operatorname{im}(p;\, \mathbf{h}^{\downarrow},\, h_\ell + g)$.

Caveat

The Extended Fulton's Properties, unlike the planar case, do not immediately yield an algorithm.

Because, in general, an arbitrary $\mathbb{Q}[\mathbf{x}]$ is not a principal ideal domain, we are not guaranteed (unlike in the bivariate case) a "Euclid like" step from (n-6) and (n-7).

In order to reduce the bivariate case an additional criterion for reducing the $\ell + 1$-variate case to the ℓ-variate one is required.

Transversally intersecting curves are those curves that only intersect at a single point. That is to say, two surfaces cannot have transverse intersection if their common component has nonzero dimension.

Definition (Transverse)

Let h_0, $h_1 \in \mathbb{Q}[\mathbf{x}]$. Two varieties $\mathbf{V}(h_0)$ and $\mathbf{V}(h_1)$ in $\mathbb{A}^{\ell+1}\big(\overline{\mathcal{F}}[\mathbf{x}]\big)$ transversally intersect at $p \in \mathbf{V}(h_0,\, h_1)$ when their tangent cones intersect at $\{p\}$ only or not at all.

$$\mathbf{V}(h_0) \pitchfork \mathbf{V}(h_1) \overset{\text{Defn.}}{\iff} \kappa_p(h_0) \cap \kappa_p(h_1) \in \{\varnothing,\, \{p\}\}.$$

(Note at non-singular points the tangent cone is simply the tangent plane.)

Proposition

Let $h_0, \ldots, h_{\ell-1}, h_\ell \in \mathbb{Q}[\mathbf{x}]$ such that $p \in \mathbb{A}^{\ell+1}(\mathbb{Q})$ is an isolated point of $\mathbf{V}(\mathbf{h})$ and let $\mathbf{h}^{\downarrow} := \{h_0, \ldots, h_{\ell-1}\}$.

Suppose h_ℓ at p is non-singular and transverse to the tangent cone of $\mathbf{V}(\mathbf{h}^{\downarrow})$.

Let π be the tangent hyperplane to $\mathbf{V}(h_\ell)$ at p.

In this setting, the intersection multiplicities of $\{\mathbf{h}^{\downarrow}, h_\ell\}$ and $\{h_0, \ldots, h_{\ell-1}, \pi\}$ at p coincide:

$$\pi \pitchfork \kappa_p(\mathbf{h}^{\downarrow}) \implies \mathrm{im}_{\ell+1}(p; \mathbf{h}^{\downarrow}, h_\ell) = \mathrm{im}_{\ell+1}(p; \mathbf{h}^{\downarrow}, \pi).$$

Caveat

Tangent Cones are sometimes prohibitively expensive to compute.

Tangent Cone

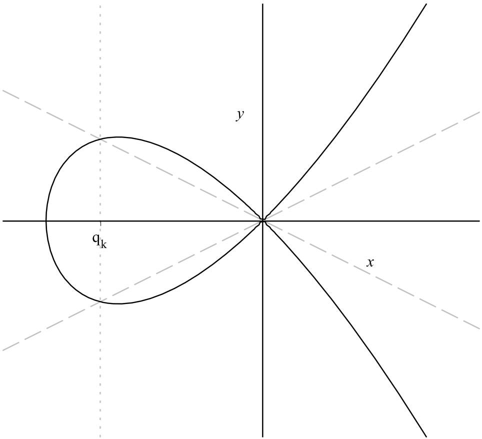
One can compute a graded Gröbner basis $\mathbf{G}$ of $\mathbf{H}$ (the homogenization of $\mathbf{h}$) such that the dehomogenization of $\mathbf{G}$ is $\langle\, g_0, \ldots, g_s \,\rangle$ and

$$\kappa_0(\mathbf{h}) = \langle\, \textsc{hc}_{\mathbf{0}}(g_0; \min), \ldots, \textsc{hc}_{\mathbf{0}}(g_s; \min) \,\rangle.$$

However Gröbner Basis can be expensive.

Theorem (Cox, Little, O'Shea)

Let $\mathbf{h} \subseteq \mathbb{Q}[\mathbf{x}]$. *A line* L *through* $p \in \mathbf{V}(\mathbf{h})$ *lies in the tangent cone* $\kappa_p(\mathbf{h})$ *if and only if there is a sequence of points* q_k *from* $\mathbf{V}(\mathbf{h}) - \{p\}$ *converging to* p *where the secant lines* L_k *containing* p *and* q_k *become* L *in the limit.*



$$L \in \kappa_p(\mathbf{h}) \iff$$

$$\exists \{q_k : k \in \mathbb{N}\} \subseteq \mathbf{V}(\mathbf{h}) - \{p\} : \lim_{k\to\infty} q_k = p \text{ and } \lim_{k\to\infty} L_k = L.$$

We calculate a vector of the instantaneous slope

$$\left(\frac{\partial x}{\partial x'} : x \in \mathbf{x} \right)$$

for fixed $x' \in \mathbf{x}$ which reduces transversality checking to a dot product (modulo a regular chain), once the slopes have been calculated.

Proposition (Tangent Cone)

Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{m}$ be sets of variables ordered

$$m_\ell \succ \cdots \succ m_0 \succ x_\ell \succ \cdots \succ x_1 \succ y_\ell \succ \cdots \succ y_0 \succ x_0.$$

The tangent cone of $\{h_0, \ldots, h_{\ell-1}\}$ at $p \in \mathbf{V}(h_0, \ldots, h_{\ell-1})$ can be recovered by triangularizing (the slope system):

$$M = \begin{cases} (x_\ell - y_\ell)\, m_0 = x_0 - y_0 \\ \qquad \vdots \\ (x_\ell - y_\ell)\, m_\ell = x_\ell - y_\ell \\ h_0 \cap \cdots \cap h_{\ell-1} \\ \mathbf{y} = \mathbf{f}_\triangle \end{cases} .$$

using Puiseux-series expansions to account for the fact each $x_\ell - y_\ell = 0$.

Example

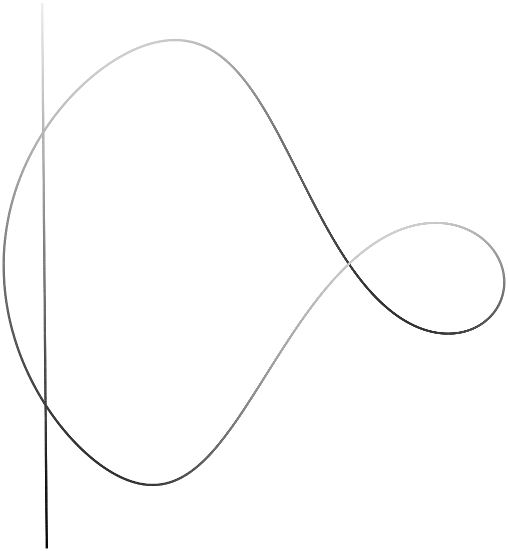
Consider secants along the the curve

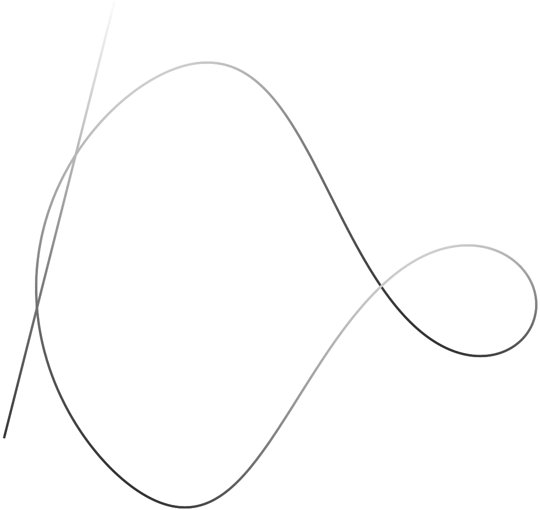
$$\mathbf{h} = \{x^2 + y^2 + z^2 - 1,\, x^2 - y^2 - z\} \subseteq \mathbb{Q}[x,\, y,\, z]$$

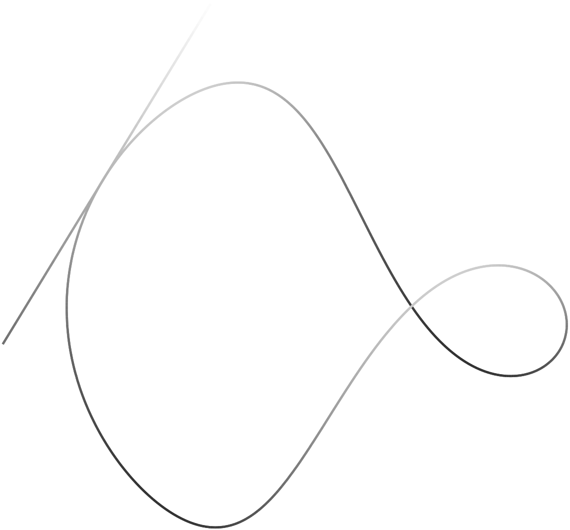
limiting to

$$\mathbf{V}\big(x + y,\, 2y^2 - 1,\, z\big).$$

(Note there are algebraic points encoded here!)







We solve M and get the slopes

$$\begin{cases} m_1 - 1 \\ m_2 \\ m_3 \end{cases} \quad \cup \quad \begin{cases} 2x^2 - 1 \\ 2y^2 - 1 \\ z \end{cases}$$

corresponding to the equations

$$\left\{ z \pm \frac{4x}{\sqrt{2}} + 2,\, y - x \pm \frac{2}{\sqrt{2}} \right\}.$$

Notice the slope for four points are encoded here. In particular the points

$$\left\{ \left(\frac{1}{\pm\sqrt{2}}, \frac{1}{\pm\sqrt{2}}, 0 \right), \left(-\frac{1}{\pm\sqrt{2}}, \frac{1}{\mp\sqrt{2}}, 0 \right) \right\}$$

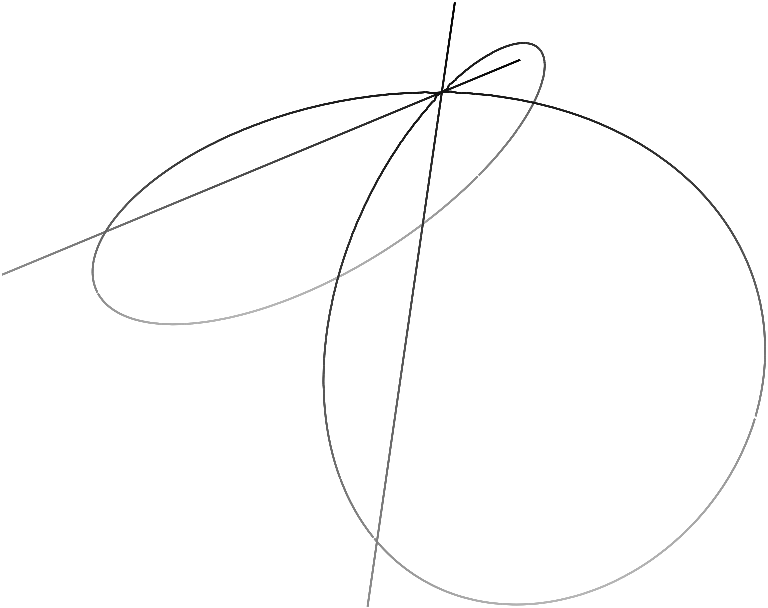
have slope $(\, 1,\, 0,\, 0\,)$.

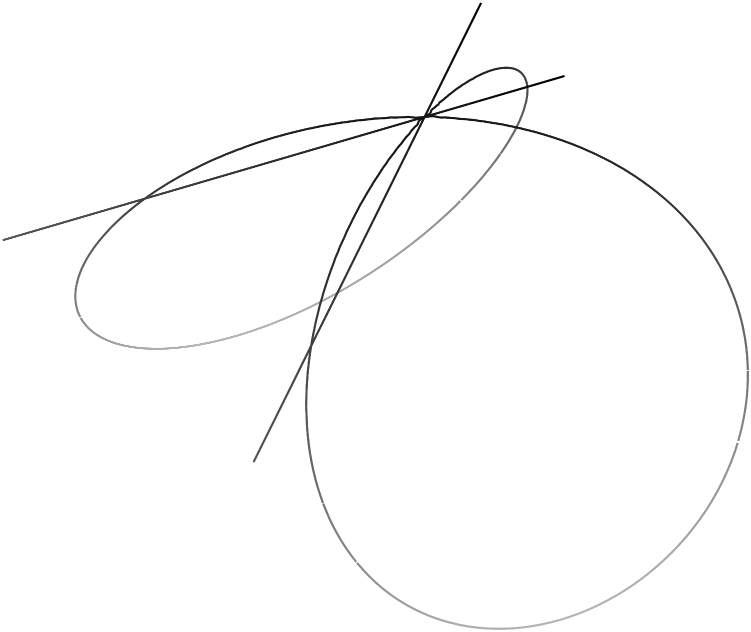
Example

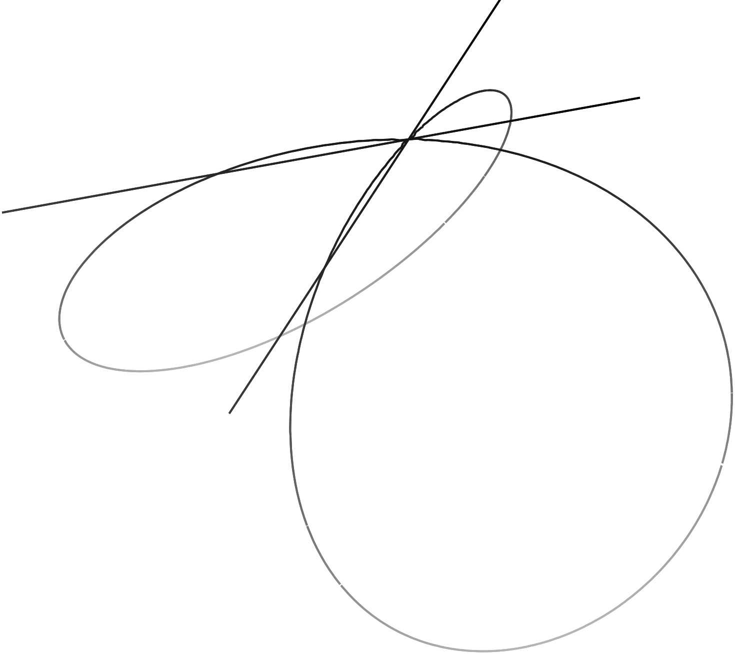
Consider secants along the curve

$$\mathbf{h} = \{x^2 + y^2 + z^2 - 1,\ x^2 - y^2 - z(z-1)\} \subseteq \mathbb{Q}[x,\ y,\ z]$$

limiting to $(0,\ 0,\ 1)$.







We solve M and get the slopes

$$\begin{cases} m_1 + m_2 \\ 2m_2^2 - 6m_2 + 3 \\ m_3 \end{cases} \cup \quad \begin{cases} x \\ y \\ z - 1 \end{cases}$$

corresponding to the equations

$$\left\{z - 1,\, y^2 - 3x^2\right\}.$$

Notice the values of the slopes here are in the algebraic closure of the coefficient ring. In particular, they are

$$\left\{\left(\tfrac{3}{2} + \sqrt{6},\, \tfrac{3}{2} + \sqrt{6},\, 0\right),\, \left(\tfrac{3}{2} - \sqrt{6},\, \tfrac{3}{2} - \sqrt{6},\, 0\right)\right\}.$$

Cylindrification

It is simple to devise a degenerate system which does not satisfy transversality. Take, for instance Ojika2:

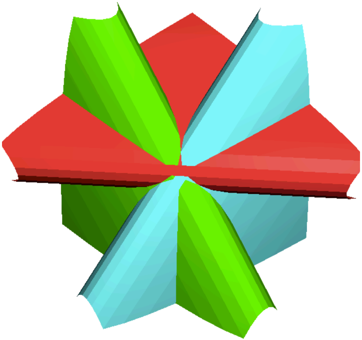
$$\left\{x^2 + y + z - 1,\, x + y^2 + z - 1,\, x + y + z^2 - 1\right\} \subseteq \mathbb{Q}[x,\, y,\, z]$$

at any of the coordinates $(1, 0, 0)$, $(0, 1, 0)$, or $(0, 0, 1)$.

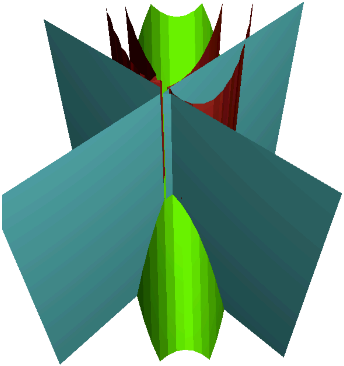
Notice though, that if one uses $x^2 + y + z - 1$ to eliminate z we obtain:

$$h_0' = x + y^2 - x^2 - y \quad \text{and} \quad h_1' = x - y + x^4 + 2x^2y - 2x^2 + y^2$$

independent of z. Consequently, the curve given by $\mathbf{V}(h_0',\, h_1')$ does not depend on z as well — in other words, it is a cylinder with base $\mathbf{V}(h_0',\, h_1')$.



Ojika



Cylindrified Ojika

$>$ with(RegularChains):

$>$ with(RegularChains:-AlgebraicGeometryTools):

$>$ $\mathbf{h} := \left[\, x^2+y+z-1,\, x+y^2+z-1,\, x+y+z^2-1 \,\right]$:

$>$ $R :=$ PolynomialRing($[x, y, z]$, 101):

$>$ TriangularizeWithMultiplicity($\mathbf{h}, R$):

$$\left[\left[1, \begin{cases} x-z \\ y-z \\ z^2+2z-1 \end{cases}\right], \left[2, \begin{cases} x \\ y \\ z-1 \end{cases}\right], \left[1, \begin{cases} x \\ y-1 \\ z \end{cases}\right], \left[2, \begin{cases} x-1 \\ y \\ z \end{cases}\right]\right]$$

Experimentation

The Jacobean trick

There is a (very good) trick which can be applied:

$$\mathrm{Jac}(\mathbf{h}, \mathbf{x}) \text{ at } p \text{ is invertible} \iff \mathrm{im}(p; \mathbf{h}) = 1.$$

We report timings using both optimized and unoptimized versions as the intersection multiplicity is typically one.

$\mathbf{h} = \text{ojika2} \quad p = 962\,592\,769.$

$\mathrm{im}(\mathbf{f}_\triangle;\, \mathbf{h})$	$\lvert\mathbf{f}_\triangle\rvert$	Bézout Weight	Cones	Total	Optimized
2	1	2	0.796	1.460	1.360
2	1	2	0.408	0.636	1.300
1	1	1	0.208	0.264	0.024
1	1	1	0.212	0.348	0.028
2	1	2	0.792	1.180	1.264
		8	2.416	3.888	3.976

$\mathbf{h} = \text{eco5} \quad p = 962\,592\,769.$

$\text{im}(\mathbf{f}_\triangle;\, \mathbf{h})$	$\lvert\mathbf{f}_\triangle\rvert$	Bézout Weight	Cones	Total	Optimized
1	3	3	5.728	8.730	0.928
1	3	3	5.929	8.910	0.956
1	1	1	1.464	2.710	0.352
1	1	1	1.996	2.970	0.352
		8	15.117	23.321	2.588

$\mathbf{h}$ = Arnborg-Lazard-rev $p = 962\,592\,769$.

$\mathrm{im}(\mathbf{f}_\triangle;\, \mathbf{h})$	$\lvert\mathbf{f}_\triangle\rvert$	Bézout Weight	Cones	Total	Optimized
1	6	6	25.310	26.000	0.296
1	6	6	27.302	28.100	0.372
1	6	6	16.861	17.700	0.332
1	2	2	7.876	8.480	0.308
		20	77.349	80.321	1.308

A Word on Implementation

Our implementation is unique in the sense that we first compute a triangular decomposition (using any method) without trying to preserve intersection multiplicity information.

Instead we work locally to obtain the intersection multiplicity for each regular chain.

Summary of Work

1. Generalized Fulton's algorithm (with some generic assumptions) to the rational closure and to ℓ variables.
2. Gave a standard-basis free method (i.e. practically efficient method) for calculating tangent cones at points on curves. This, in itself, is an important contribution as there was no efficient method for calculating tangent cones before.
3. Implemented these algorithms into the RegularChains library as the package AlgebraicGeometryTools.