Chapter 1, Geometry, Algebra, and Algorithms

§3. Parametrizations of Affine Varieties.

$\boxed{\textbf{§1.3.1.}}$

Parametrize all solutions of the linear equations

$$\begin{aligned} x + 2y - 2z + w &= -1, \\ x + y + z - w &= 2. \end{aligned}$$

Solution. Row reduction leads to the system

$$\begin{aligned} x \quad +4z-3w &= 5 \\ -y+3z-2w &= 3. \end{aligned}$$

From which we can read off the parametrization

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -4 \\ 3 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \end{pmatrix}.$$

$\boxed{\textbf{§1.3.2.}}$

Use a trigonometric identity to show that

$$\begin{aligned} x &= \cos t, \\ y &= \cos 2t \end{aligned}$$

parametrizes a portion of a parabola. Indicate what portion of the parabola is covered.

Solution. $y = \cos^2 t - \sin^2 t = 2\cos^2 t - 1 = 2x^2 - 1$; so the point $(\cos t, \cos 2t)$ is on the parabola $y = 2x^2 - 1$. The portion of this parabola where $-1 \le x \le 1$ is covered. This parabola has the y-axis as axis of symmetry, vertex at $(0, -1)$ and zeros at $x = \pm\frac{1}{\sqrt{2}}$.

$\boxed{\textbf{§1.3.3.}}$

Given $f \in k[x]$, find a parametrization of $\mathbf{V}(y - f(x))$.

Solution. The parametrization $(x, f(x))$ does the job.

$\boxed{\textbf{§1.3.4.}}$

Consider the parametric representation

$$\begin{aligned} x &= \frac{t}{1+t}, \\ y &= 1 - \frac{1}{t^2}, \end{aligned} \qquad t \neq -1,\ t \neq 0. \tag{1.3.4.1}$$

(a) Find the equation of the affine variety determined by the above parametric equations.

(b) Show the above equations parametrize all points of the variety found in part (a) except for (1,1).

Solution. Write the first equation as $x + tx = t$ or $(x-1)(t+1) = -1$ and solve for t to get $t = \frac{x}{1-x}$. This tx-equation describes a hyperbola whose asymptotes are the lines $t = -1$ and $x = 1$. Substituting this in the equation for y yields $y = 1 - \left(\frac{1-x}{x}\right)^2$ or $x^2y - x^2 + (1-x)^2 = 0$. So the variety is $\mathbf{V}(x^2y + 1 - 2x)$. Now we found above that varying t will yield any value for x except for $x = 1$. Thus any point on the curve $x^2y + 1 - 2x = 0$ or $y = \frac{2x-1}{x^2}$ except for $y = 1$ (y is uniquely determined by the value of x). Thus the parametrization covers all points of the variety except for $(1,1)$. Examining the graph shows that $(1,1)$ is the unique local maximum. The curve is asymptotic to $x = 0$ at $y = -\infty$ adn asymptotic to $y = 0$ at $x = \pm\infty$.

$\boxed{\S\mathbf{1.3.5.}}$

This problem will be concerned with the hyperbola $x^2 - y^2 = 1$.

(a) Just as trigonometric functions are used to parametrize the circle, hyperbolic functions are used to parametrize the hyperbola. Show that the point

$$
\begin{aligned}
x &= \cosh t,\\
y &= \sinh t
\end{aligned}
$$

always lies on $x^2 - y^2 = 1$. What portion of the hyperbola is covered?

Solution. $\cosh^2 t - \sinh^2 t = 1$ is a well known identity. Since $\cosh t > 0$ all t, just the "right hand branch of the hyperbola" is covered.

(b) Show that a straight line meets the hyperbola in 0, 1 or 2 points. A non vertical line has equation $y = mx + b$, and intersects the hyperbola at points whose x coordinates satisfy $x^2 - (mx+b)^2 = 1$. This is a quadratic equation in x if $m \neq 1$ and so has 0,1, or 2 roots. If the line is vertical with equation $x = a$, then the points on both the line and the hyperbola have y coordinates satisfying $a^2 - y^2 = 1$. Again there are 0, 1, or 2 possible roots.

(c) Consider nonvertical lines through $(-1,0)$ to get a parametrization of the hyperbola.

Solution. Each such line has an equation of the form $y = mx + m$. It intersects the hyperbola at the point whose x coordinant satisfies $x^2 - (mx+m)^2 = 1$ or $(1-m^2)x^2 - 2mx - (m^2+1) = (x+1)([1-m^2]x - [1+m^2]) = 0$. Setting the first factor equal to zero gives $x = -1$ and refers to the point $(-1,0)$ on the hyperbola. Setting the second factor equal to zero gives the other point on both the hyperbola and the line and yields the parametrization

$$
x = \frac{1+m^2}{1-m^2}, \qquad y = \frac{2m}{1-m^2}, \qquad\qquad -\infty < m < +\infty,
$$

for the hyperbola $x^2 - y^2 = 1$. This parametrization covers every point on the hyperbola except for the point $(-1,0)$.

$\boxed{\S\mathbf{6\&7.}}$

The "north pole" on the sphereS^{n-1} with equation $x_1^2 + \cdots + x_n^2 = 1$ in $\mathbf{R}^n$ is the point $\overrightarrow{n} = (0,0,\ldots,0,1)$. Let $\overrightarrow{u} = (u_1, u_2, \ldots, u_{n-1}, 0)$ be a point on the hyperplane $x_n = 0$ in $\mathbf{R}^n$. The line joining $\overrightarrow{n}$ and $\overrightarrow{u}$ has the parametrization $t \mapsto \overrightarrow{p}(t) = t\overrightarrow{u} + (1-t)\overrightarrow{n}$. Remembering that $\overrightarrow{u} \cdot \overrightarrow{n} = 0$, the values of t for which this line intersects the sphere S^{n-1} are those which satisfy

$$
\begin{aligned}
1 &= \|\overrightarrow{p}(t)\|^2\\
&= t^2\|\overrightarrow{u}\|^2 + 2t(1-t)\overrightarrow{u}\cdot\overrightarrow{n} + (1-t)^2\\
0 &= t\big[t(\|\overrightarrow{u}\|^2 + 1) - 2\big].
\end{aligned}
$$

They are $t = 0$, in which case $\overrightarrow{p}(0) = \overrightarrow{n}$, and $t = t_s(\overrightarrow{u}) = \frac{2}{\|\overrightarrow{u}\|^2+1}$. This gives

$$
\overrightarrow{p}(t_s(\overrightarrow{u})) = \frac{2}{\|\overrightarrow{u}\|^2+1}\overrightarrow{u} + \frac{\|\overrightarrow{u}\|^2-1}{\|\overrightarrow{u}\|^2+1}\overrightarrow{n}.
$$

Thus the map $\overrightarrow{u} \mapsto \overrightarrow{p}(t_s(\overrightarrow{u}))$ yields a parametrization of $S^{n-1} - \{\overrightarrow{n}\}$ by $\overrightarrow{u} \in \mathbf{R}^{n-1}$.

$\boxed{\S\mathbf{1.3.8.}}$

The curve $y^2 = cx^2 - x^3$ looks sort of like the reflection of the letter "α" in a vertrical line. The curve has a self intersection at the origin when $c > 0$.

(a) The line $y = mx + b$ intersects this curve at points whose x-coordinates satisfy $(mx+b)^2 = cx^2 - x^3$. This last is a cubic equation and hence has either 1, 2 (counting multiplicities), or 3 roots. The line $x = a$ intersects at $(a, \pm\sqrt{ca^2 - a^3})$; thus at either 0, 1, or 2 points. In general a line intersects the curve in either 0, 1, 2, or 3 points.

(b) A non vertical line through the origin has equation $y = mx$ and intersects the curve at points whose x coordinates satisfy $x^3 + (m^2 - c)x^2 = 0$ or $x^2(x - (c - m^2)) = 0$. Other than the origin there is precisely one of these if $c \neq m^2$. If $m^2 = c$ the only intersection is the origin. Replacing x by 0 in the equation $y^2 = cx^2 - x^3$ shows that the y-axis intersects the curve only at the origin. The lines $y = \pm\sqrt{c}\cdot x$ are tangent to the curve at $(0, 0)$.

(c) The line through $(0, 0)$ and $(1, t)$ has equation $y = tx$ and intersects the curve at the origin and at the point whose x-coordinate is $c - t^2$. The y coordinate of this point is t times the x-coordinate or $t(c - t^2)$. So the map $t \mapsto \big(c - t^2,\, t(c - t^2)\big)$ leads to a parametrization of this curve in which the origin is covered twice, once when $t = -c$ and once when $t = c$.

$\boxed{\S\mathbf{1.3.9.}}$

The *strophoid* is a curve that was studied by various mathematicians, including Isaac Barrow (1630-1677), Jean Bernoulli (1667-1748), and Maria Agnesi (1718-1799). A trigonometric parametrization is given by

$$\begin{aligned} x &= a\sin(t), \\ y &= a\tan(t)\big(1 + \sin(t)\big) = a\frac{\sin t\cos t}{1 - \sin t}, \end{aligned} \tag{1.3.9.1}$$

where a is a constant. Its graph looks sort of like the greek letter α with the singularity at the origin.

(a) Find the equation in x and y that describes the strophoid. Hint: If you are sloppy you will get the equation $(a^2 - x^2)y^2 = x^2(a + x)^2$. This is not correct because the line $x = -a$ lies on this locus but not on the strophoid.

Solution. Suppose first that $a = 1$ and (x, y) lies on the strophoid, i.e. satisfies (1.3.9.1) for some t. If $\sin t = 1$, then $\cos t = 0$ and $\tan t = \pm\infty$. This requires that $y = \pm\infty$. Since the y of the point (x, y) is not infinite we can assume $\sin t < 1$. If $\sin t = -1$, then $\cos t = 0$ and $y = 0$. So the point $(-1, 0)$ is on the strophoid. Otherwise we can as well assume that $-1 < \sin t < 1$ or that $-1 < x < 1$. Then there are two cases:

Case I. $\cos t > 0$:

$$x = \sin t; \qquad \text{so } \tan t = \frac{x}{\sqrt{1 - x^2}} \text{ and } y = \frac{x(1 + x)}{\sqrt{1 - x^2}} = x\sqrt{\frac{1 + x}{1 - x}}.$$

Case II. $\cos t < 0$:

$$x = \sin t; \qquad \text{so } \tan t = \frac{x}{-\sqrt{1 - x^2}} \text{ and } y = \frac{x(1 + x)}{-\sqrt{1 - x^2}} = -x\sqrt{\frac{1 + x}{1 - x}}.$$

Note that $\sin t = \sin(\pi - t)$ whereas $\tan t = -\tan(\pi - t)$; so both the plus and minus signs in the last offset equation give (x, y)-points on the strophoid. Thus both $y = x\sqrt{\frac{1+x}{1-x}}$ and $y = -x\sqrt{\frac{1+x}{1-x}}$ describe parts of the strophoid. Squaring this last offset relation then yields $y^2(1 - x) = x^2(1 + x)$. Replacing y by $\frac{y}{a}$ and x by $\frac{x}{a}$ gives the desired equation

$$y^2(a - x) = x^2(a + x).$$

Remark. (With $a = 1$.) If we take the second equation of (1.3.9.1) written as $y = \frac{\pm x\sqrt{1-x^2}}{1-x}$ and square it we get $y^2 = \frac{x^2(1-x^2)}{(1-x)^2} = \frac{x^2(1+x)}{1-x}$ when $x \neq 1$. This leads to $y^2(1 - x) = x^2(1 + x)$ too.

(b) Find an algebraic parametrization of the strophoid.

Solution. Consider the intersection of the strophoid $y^2(a-x)-x^2(a+x)=0$ with the line $y=mx$. substituting $y=mx$ in $y^2(a-x)-x^2(a+x)=0$ yields $m^2x^2(a-x)-x^2(a+x)=0$ or $m^2(a-x)-(a+x)=0$. Solving this for x gives $x=\frac{a(1-m^2)}{-1-m^2}$ or $x=-a\frac{1-m^2}{1+m^2}$. An algebraic parametrization of the strophoid is given by

$$x=-a\frac{1-m^2}{1+m^2},$$
$$y=-am\frac{1-m^2}{1+m^2}.$$

The point $(0,0)$ is covered twice: once by $m=1$ and once by $m=-1$.

$\boxed{\S\mathbf{1.3.10.}}$

Around 180 B.C. Diocles wrote the book *On Burning-Glasses*, and one of the curves he considered was the *cissoid*. He used this curve to solve the problem of the duplication of the cube (see part c below). The cissoid has the equation $y^2(a+x)=(a-x)^3$, where a is a constant.

(a) Find an algebraic parametrization of the cissoid.

Solution. Try intersecting the cissoid with the line $y=m(a-x)$. Substituting $y=m(a-x)$ in $y^2(a+x)=(a-x)^3$ gives $m^2(a-x)^2(a+x)-(a-x)^3=0$ or $(a-x)^2\big(m^2(a+x)-(a-x)\big)=0$. Thus $x=a$ or $x=a\frac{1-m^2}{1+m^2}$. The desired algebraic parametrization is

$$x=a\frac{1-m^2}{1+m^2},$$
$$y=m\left(a-a\frac{1-m^2}{1+m^2}\right)=a\frac{2m^3}{1+m^2}.$$

(b) Diocles described the cissoid using the following geometric construction. Given a circle of radius a (which we take as centered at the origin), pick x between a and $-a$, and draw the line L connecting $(a,0)$ to the point $P=(-x,\sqrt{a^2-x^2})$ on the circle. The point $Q=(x,y)$ on L with first coordinate x is on the cissoid. Prove that the cissoid is the locus of all such points Q.

Solution. The line joining $(a,0)$ to $P=(-x,\sqrt{a^2-x^2})$ consists of those points (X,Y) satisfying

$$Y=\frac{-\sqrt{a^2-x^2}}{a+x}(X-a)=-\sqrt{\frac{a-x}{a+x}}(X-a).$$

Putting $X=x$ gives the y-coordinate as satisfying $y=\frac{(a-x)^{\frac{3}{2}}}{\sqrt{a+x}}$. Actually the statement above is not quite correct. The cissoid also contains the reflection of these points in the x-axis. These satisfy $y=-\frac{(a-x)^{\frac{3}{2}}}{\sqrt{a+x}}$ and consist of those points on the the line L' connecting $(a,0)$ to the point $P'=(-x,-\sqrt{a^2-x^2})$ on the circle. Thus the equation of the cissoid is

$$0=\left(y-\frac{(a-x)^{\frac{3}{2}}}{\sqrt{a+x}}\right)\left(y+\frac{(a-x)^{\frac{3}{2}}}{\sqrt{a+x}}\right)=y^2-\frac{(a-x)^3}{a+x}.$$

(c) The duplication of the cube is the classical Greek problem of trying to construct $\sqrt[3]{2}$ using ruler and compass. It is known that this is impossible given just a ruler and compass. Diocles showed that if in addition, you allow the use of the cissoid, then one can construct $\sqrt[3]{2}$. Here is how it works. Draw the line J connecting $(-a,0)$ to $\left(0,\frac{a}{2}\right)$. This line will meet the cissoid at a point (x,y). Then prove that $2=\left(\frac{a-x}{y}\right)^3$, which shows how to construct $\sqrt[3]{2}$ using ruler, compass and cissoid.

Solution. The line J has equation $y = \frac{1}{2}(x+a)$. Thus

$$\begin{aligned} y^2 &= \frac{(a-x)^3}{a+x}, \\ y &= \frac{1}{2}(x+a) \end{aligned} \quad \text{imply that} \quad \left(\frac{(a-x}{y}\right)^3 = \frac{(a-x)^3}{y^2 \cdot \frac{1}{2}(a+x)} = 2\frac{a+x}{a+x} = 2.$$

$\boxed{\S\mathbf{1.3.11.}}$

In this problem we will derive the parametrization

$$\begin{aligned} x &= t(u^2 - t^2), \\ y &= u, \\ z &= u^2 - t^2, \end{aligned} \tag{1.3.11.1}$$

of the surface $x^2 - y^2z^2 + z^3 = 0$.

(a) From part (d) of Exercise 1.3.8 we know that the curve $x^2 = cz^2 - z^3$ is parametrized by

$$\begin{aligned} z &= c - t^2, \\ x &= t(c - t^2). \end{aligned}$$

(b) Replace the c in part (a) by y^2. This gives the parametrization of the curve $x^2 = y^2z^2 - z^3$, y-fixed. Letting $y = u$ then yields the parametrization in part (1.3.11.1) above. Thus all the t, u points of (1.3.11.1) lie on $\mathbf{V}(x^2 - y^2z^2 + z^3)$. This can of course be checked directly, viz

$$\begin{aligned} x^2 &= t^2(u^2-t^2)^2 = t^2u^4 - 2t^4u^2 + t^6, \\ y^2z^2 &= u^2(u^2-t^2)^2 = u^6 - 2u^4t^2 + u^2t^4, \\ z^3 &= (u^2-t^2)^3 = u^6 - 3u^4t^2 + 3u^2t^4 - t^6. \end{aligned}$$

(c) Explain why this parametrization covers the entire surface $\mathbf{V}(x^2, y^2z^2 + z^3)$.

Solution. Let (x, y, z) satisfy $x^2 - y^2z^2 + z^3 = 0$. According to Exercise 1.3.8, There are values of t for which $x = (y^2 - t^2)$ and $z = t(y^2 - t^2)$. Thus there is a value of u for which (1.3.11.1), namely $u = y$ holds, and the parametrization covers the variety.

$\boxed{\S\mathbf{1.3.12.}}$

Consider the variety $V = \mathbf{V}(y - x^2,\, z - x^4) \subset \mathbb{R}^3$.

(a) Draw a picture of V.

Solution. This is the curve $x \mapsto (x, x^2, x^4)$. Its projection on the xy-plane is the parabola $y = x^2$ and its projection on the xz-plane is the quartic $z = x^4$.

(b) Parametrize V by t as $t \mapsto (t, t^2, t^4)$.

(c) Parametrize the tangent surface of V.

Solution. Using the parametrization of (b), the tangent vector at (t, t^2, t^4) is $\overrightarrow{v} = (1, 2t, 4t^3)$; so the line "along" this tangent vector is $u \mapsto (t, t^2, t^4) + u(1, 2t, 4t^3)$ and this leads to the parametrization

$$(t, u) \mapsto (t + u,\, t^2 + 2tu,\, t^4 + 4t^3u).$$

This is the desired parametrization of the tangent surface.

$\boxed{\S\mathbf{1.3.13.}}$

The general problem of finding the equation of a parametrized surface will be studied in Chapters 2 and 3. However, when the surface is a plane, methods from calculus or linear algebra can be used. For example, consideer the plane $\mathbf{R}^3$ parametrized by

$$\begin{aligned} x &= 1+u-v, \\ y &= u+2v, \\ z &= -1-u+v. \end{aligned}$$

Find the equations of the plane determined this way. Hint: Let the equation of the plane be $ax+by+cz=d$. Then substitute in the above parametrization to obtain a system of equations for a, b, c, d. Another way to solve the problem would be to write the parametrization in vector form as $(1,0,-1)+u(1,1,-1),+v(-1,2,1)$. Then one can get a quick solution using the cross product.

Solution. Following the second hint, the plane parametrized by $\overrightarrow{A}+u\overrightarrow{B}+v\overrightarrow{C}$ has equation

$$(\overrightarrow{B}\times\overrightarrow{C})\cdot(\overrightarrow{R}-\overrightarrow{A})=0, \qquad \text{where } \overrightarrow{R}=(x,y,z).$$

Here $\overrightarrow{B}=(1,1,-1)$, $\overrightarrow{C}=(-1,2,1)$ and $\overrightarrow{A}=(1,0,-1)$; so

$$\mathbf{B}\times\overrightarrow{C}=\begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{vmatrix}=(3,0,3).$$

The equation is $3(x-1)+3(z+1)=0$ or $x+z=0$.

$\boxed{\S\mathbf{1.3.14.}}$

This problem deals with convex sets and will be used in the next exercise to show that a Bézier cubic lies within its control polygon. A subset $C\subset\mathbb{R}^2$ is *convex* if for all P, $Q\in C$, the line segment joining P to Q also lies in C.

(a) That is, if $P=\binom{x}{y}$ and $Q=\binom{Z}{w}$ lie in a convex set C then

$$t\begin{pmatrix} x \\ y \end{pmatrix}+(1-t)\begin{pmatrix} z \\ w \end{pmatrix}\in C, \qquad \text{when } 0\le t\le 1.$$

(b) It is also well known that if the points $P_i=\binom{x_i}{y_j}$ lie in a convex set C for $1\le i\le n$, then

$$\sum_{i=1}^{n} t_i\begin{pmatrix} x_i \\ y_j \end{pmatrix}\in C$$

whenever $t_1,\dots,t_n$ are nonnegative numbers for which $\sum_{i=1}^{n} t_i=1$.

$\boxed{\S\mathbf{1.3.15.}}$

Let a Bézier cubic be given by

$$\begin{aligned} x &= (1-t)^3x_0+3t(1-t)^2x_1+3t^2(1-t)x_2+t^3x_3, \\ y &= (1-t)^3y_0+3t(1-t)^2y_1+3t^2(1-t)y_2+t^3y_3. \end{aligned}$$

(a) In vector form these equations can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix}=(1-t)^3\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}+3t(1-t)^2\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}+3t^2(1-t)\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}+t^3\begin{pmatrix} x_3 \\ y_3 \end{pmatrix}. \tag{1.3.15.a}$$

(b) The sum of the coefficients in (1.3.15.a) is $\big(t+(1-t)\big)^3$; so the point (1.3.15.a) lies inside its control polygon, i.e., the smallest convex set containing the points $P_i=\binom{x_i}{y_i}$, $0\le i\le 3$.

$\boxed{\textbf{§1.3.16.}}$

One disadvantage of Bézier cubics is that curves like circles and hyperbolas cannot be described exactly by cubics. In this exercise, we will discuss a method for parametrizing conic sections. Our treatment is based on Ball (1987).

A conic section is a curve in the plane defined by a second degree polynomial equation of the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$. Now consider the curve parametrized by

$$
\begin{aligned}
x &= \frac{(1-t)^2 x_1 + 2t(1-t)wx_2 + t^2 x_3}{(1-t)^2 + 2t(1-t)w + t^2}, \\
y &= \frac{(1-t)^2 y_1 + 2t(1-t)wy_2 + t^2 y_3}{(1-t)^2 + 2t(1-t)w + t^2}
\end{aligned} \tag{1.3.16.1}
$$

for $0 \le t \le 1$. The constants $w, x_1, y_1, x_2, y_2, x_3, y_3$ are specified by the design engineer, and we will assume that $w \ge 0$. In Chapter 3 we will show that the equations (1.3.16.1) parametrize a conic section. The goal of this exercise is to give a geometric interpretation for the quantities $w, x_1, y_1, x_2, y_2, x_3, y_3$.

(a) Show that the assumption $w \ge 0$ implies that the denominator in the above formulas never vanishes.

Solution. For $0 < t < 1$ the term $2t(1-t) > 0$. So $(1-t)^2 + 2t(1-t)w + t^2 = 0$ would imply that $w = -\frac{(1-t)^2+t^2}{2t(1-t)} < 0$. If $t = 0$ or $t = 1$ $(1-t)^2 + 2t(1-t)w + t^2 = 1 \ne 0$. Thus the denominator doesn't vanish for $0 \le t \le 1$. It certainly does vanish for some values of t outside this interval.

(b) Evaluate the formulas (1.3.16.1) at $t = 0$ and $t = 1$.

Solution. Evaluation gives $\big(x(0),\, y(0)\big) = (x_1,\, y_1)$ and $\big(x(1),\, y(1)\big) = (x_3,\, y_3)$.

(c) Now compute $\big(x'(0),\, y'(0)\big)$ and $\big(x'(1),\, y'(1)\big)$. Use this to show that (x_2, y_2) is the intersection is the intersection of the tangent lines at the start and end of the curve. Explain why $(x_1,\, y_1)$, $(x_2,\, y_2)$ and $(x_3,\, y_3)$ are called *control points* of the curve.

Solution. To better display these computations we let

$$
\begin{aligned}
N(t) &= (1-t)^2 x_1 + 2t(1-t)wx_2 + t^2 x_3, \\
N'(t) &= -2(1-t)x_1 + 2(1-t)wx_2 - 2twx_2 + 2tx_3, \\
D(t) &= (1-t)^2 + 2t(1-t)w + t^2, \\
D'(t) &= -2(1-t) + 2(1-t)w - 2tw + 2t,
\end{aligned}
$$

and note that

$$
\begin{aligned}
N(0) &= x_1, & N(1) &= x_3, \\
N'(0) &= -2x_1 + 2wx_2, & N'(1) &= -2wx_2 + 2x_3, \\
D(0) &= 1, & D(1) &= 1, \\
D'(0) &= -2 + 2w, & D'(1) &= -2w + 2.
\end{aligned}
$$

Then

$$
\begin{aligned}
x'(0) &= \frac{N'(0)D(0) - N(0)D'(0)}{\big(D(0)\big)^2} = (-2x_1 + 2wx_2)\cdot 1 - x_1 \cdot (-2 + 2w) = 2w(x_2 - x_1). \\
y'(0) &= 2w(y_2 - y_1). \\
x'(1) &= \frac{N'(1)D(1) - N(1)D'(1)}{\big(D(1)\big)^2} = -2wx_2 + 2x_3 - x_3(-2w + 2) = 2w(x_3 - x_2), \\
y'(1) &= 2w(y_3 - y_2).
\end{aligned}
$$

If $w > 0$, the tangent line to the curve at $\big(x(0),\, y(0)\big)$ has equation

$$
y'(0)(x - x_1) - x'(0)(y - y_1) = 0 \text{ or } 2w\big((y_2 - y_1)(x - x_1) - (x_2 - x_1)(y - y_1)\big) = 0.
$$

So the point $(x_2,\, y_2)$ certainly lies on this tangent line.

Similarly, if $w > 0$, the tangent line to the curve at $\big(x(1),\, y(1)\big)$ has equation

$$y'(1)(x-x_3) - x'(1)(y-y_3) = 0 \text{ or } 2w\big((y_3-y_2)(x-x_3) - (x_2-x_3)(y-y_3)\big) = 0,$$

and the point $(x_2,\, y_2)$ also lies on this tangent line.

If $w = 0$, the "curve" is a line segment joining (x_1, y_1) to (x_3, y_3), and in this case x_2 and y_2 do not enter into the expressions defining the curve.

(d) The *control polygon* in this case is the triangle with vertices $(x_i,\, y_i)$, $1 \le i \le 3$. That is, the convex hull of these three points. Now the equations (1.3.16.1) show that if we weight these points respectively with the positive weights $(1-t)^2$, $2t(1-t)w$ and t^2, then the point $\big(x(t),\, y(t)\big)$ is within this triangle or control polygon. It remains to explain the constant w which is called the *shape factor*. Let the curve (1.3.16.1) be $t \mapsto \overrightarrow{P}(t)$ and $\overrightarrow{P}_i = (x_i, y_i)$, $1 \le i \le 3$. If the parameter t is taken as "time", we have shown that the velocity $\overrightarrow{P}'(t)$ satisfies:

$$\begin{aligned}
\overrightarrow{P}'(0) &= w(\overrightarrow{P}_2 - \overrightarrow{P}_1), \text{ a vector headed from } \overrightarrow{P}_1 \text{ towards } \overrightarrow{P}_2.\\
\overrightarrow{P}'(1) &= w(\overrightarrow{P}_3 - \overrightarrow{P}_2), \text{ a vector headed from } \overrightarrow{P}_2 \text{ towards } \overrightarrow{P}_3.
\end{aligned}$$

So the constant is related to the initial and final "speed" of this "motion". Somehow we might guess that a larger w would "force" the curve closer to $(x_2,\, y_2)$.

(e) Prove that

$$\begin{pmatrix} x\left(\frac{1}{2}\right) \\ y\left(\frac{1}{2}\right) \end{pmatrix} = \frac{1}{2(w+1)}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \frac{1}{2(w+1)}\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} + \frac{w}{w+1}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

and use this formula to show that $\begin{pmatrix} x\left(\frac{1}{2}\right) \\ y\left(\frac{1}{2}\right) \end{pmatrix}$ lies on the line segment connecting (x_2, y_2) to the midpoint of the line between (x_1, y_1) and $x_3, y_3)$.

Solution. According to (1.3.16.1), $x\left(\frac{1}{2}\right) = \frac{x_1+2wx_2+x_3}{1+2w+1}$ which is exactly the relationship asserted above. The expression for $y\left(\frac{1}{2}\right)$ is virtually identical (except for y's instead of x's. When it is written in the form

$$\begin{pmatrix} x\left(\frac{1}{2}\right) \\ y\left(\frac{1}{2}\right) \end{pmatrix} = \frac{1}{w+1}\left[\frac{1}{2}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x_3 \\ y_3 \end{pmatrix}\right] + \frac{w}{w+1}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix},$$

we see that $\left(x\left(\frac{1}{2}\right),\, y\left(\frac{1}{2}\right)\right)$ is a weighted average with positive weights of the coordinates of the midpoint of the line segment joining (x_1, y_1) to (x_3, y_3) and the coordinates of (x_2, y_2). This means that the point $\left(x\left(\frac{1}{2}\right),\, y\left(\frac{1}{2}\right)\right)$ lies on this line segment.

(f) Note, finally, that

$$\begin{pmatrix} x\left(\frac{1}{2}\right) \\ y\left(\frac{1}{2}\right) \end{pmatrix} = \left[\frac{1}{2}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x_3 \\ y_3 \end{pmatrix}\right] + \frac{w}{w+1}\left\{\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - \left[\frac{1}{2}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x_3 \\ y_3 \end{pmatrix}\right]\right\},$$

which shows that $Q = (\,x\left(\frac{1}{2}\right),\quad y\left(\frac{1}{2}\right)\,)$ lies $\frac{w}{w+1}$-th of the way along the line segment from the midpoint to the point with coordinates $(x_2,\, y_2)$. That is, the distance of Q from the midpoint of the line segment joining $\overrightarrow{P}_1$ to $\overrightarrow{P}_3$ is w times its distance from $\overrightarrow{P}_2$. If w is large Q is closer to $\overrightarrow{P}_2$.

$\boxed{\S\mathbf{1.3.17.}}$

Use the formulas of Exercise 1.3.17 to parametrize the arc of the circle $x^2 + y^2 = 1$ from (1,0) to (0,1).

Solution. The formulas being referenced are

$$(1.3.16.1) \qquad \begin{aligned} x &= \frac{(1-t)^2x_1 + 2t(1-t)wx_2 + t^2x_3}{(1-t)^2 + 2t(1-t)w + t^2}, \\ y &= \frac{(1-t)^2y_1 + 2t(1-t)wy_2 + t^2y_3}{(1-t)^2 + 2t(1-t)w + t^2} \end{aligned}$$

for $0 \le t \le 1$, where the arc goes from (x_1, y_1) at $t = 0$ to (x_3, y_3) at $t = 1$. The tangent lines to the arc at these two endpoints meet at the point (x_2, y_2). The point

$$\begin{pmatrix} x\left(\frac{1}{2}\right) \\ y\left(\frac{1}{2}\right) \end{pmatrix} = \left[\frac{1}{2}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x_3 \\ y_3 \end{pmatrix}\right] + \frac{w}{w+1}\left\{\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - \left[\frac{1}{2}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x_3 \\ y_3 \end{pmatrix}\right]\right\}.$$

Call this point P. If L, the line segment joining (x_1, y_1) to (x_3, y_3), has midpoint Q, P is on the line segment J which joins Q to (x_2, y_2). It is positioned $\frac{w}{w+1}$-th of the way along J as measured from Q towards (x_2, y_2). That is,

$$(1.3.17.1) \qquad w = \frac{\text{the length of } \overline{QP}}{\text{the length of } \overline{P(x_2, y_2)}}.$$

Here we put $(x_1, y_1) = (1, 0)$ and $(x_3, y_3) = (0, 1)$. The tangents at these two points meet at $(1, 1) = (x_2, y_2)$. It remains to determine w. We know that the midpoint of L is $Q = \left(\frac{1}{2}, \frac{1}{2}\right)$. $P = \left(x\left(\frac{1}{2}\right), y\left(\frac{1}{2}\right)\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Thus

$$\begin{aligned} \text{the length of } \overline{QP} &= \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}, \\ \text{the length of } \overline{P(x_2, y_2)} &= \sqrt{2}\left(1 - \frac{1}{\sqrt{2}}\right) = \sqrt{2} - 1. \\ \text{therefore } w &= \frac{1}{\sqrt{2}}. \end{aligned}$$

Substituting these values in (1.3.16.1) yields the parametrization

$$\begin{aligned} x(t) &= \frac{\sqrt{2}(1-t)^2 + 2t(1-t)}{2t(1-t) + \sqrt{2}\big((1-t)^2 + t^2\big)}, \\ y(t) &= \frac{\sqrt{2} \cdot t^2 + 2t(1-t)}{2t(1-t) + \sqrt{2}\big((1-t)^2 + t^2\big)}. \end{aligned}$$

It is a straightforward algebraic computation to check that this "strange" parametrization is indeed a parametrization of the circle $x^2 + y^2 = 1$.