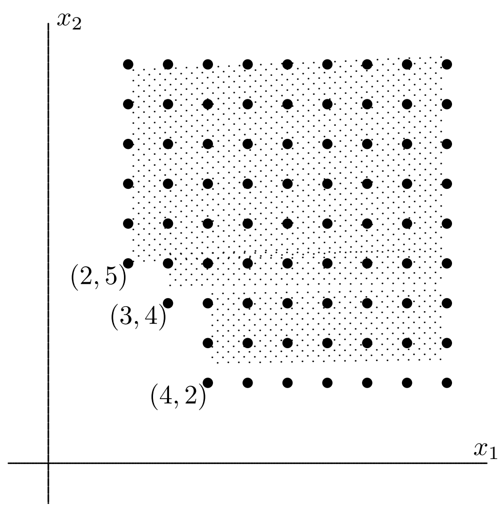
Chapter 2 §4. Monomial Ideals and Dickson's Lemma. An ideal $I \subset k[x_1, \ldots, x_n]$ is a *monomial ideal* if it is generated by a set of monomials $\{x^\alpha \colon \alpha \in A\}$. In this case we will denote I by $\langle x^\alpha \colon \alpha \in A\rangle$. It is clear that the set

$$\{f \in k[x_1, \ldots, x_n] \colon \text{ every term of } f \text{ is divisible by some } x^\alpha,\ \alpha \in A\} \tag{2.4.0.1}$$

is equal to $\langle x^\alpha \colon \alpha \in A\rangle$, because the set (2.4.0.1) is an ideal both contained in and containing $\langle x^\alpha \colon \alpha \in A\rangle$. In particular, a monomial x^β is in $\langle x^\alpha \colon \alpha \in A\rangle$ if and only if it is divisible by one of the generating monomials x^α of I. If I is such a monomial ideal, the *signature* of I is the subset

$$\text{signature}(I) = \{\beta \in \mathbf{Z}^n_{\geq 0} \colon x^\beta \in I\}. \tag{2.4.0.2}$$

It is the union of the sets $\beta + \mathbf{Z}^n_{\geq 0}$, where $x^\beta \in I$, and so has a very special form which we illustrate below for $n = 2$ and $A = \{(2,5), (3,4), (4,2)\}$.



A polynomial $f = \sum_{i=1}^m a_i x^{\beta(i)}$ with $a_i \in k$ is in the ideal $I = \langle x^\alpha \colon \alpha \in A\rangle$ if and only if

$$\{\beta(1), \ldots, \beta(m)\} \subset \text{signature}(I). \tag{2.4.0.3}$$

Notation. For $\gamma \in \mathbf{Z}^n_{\geq 0}$ let $Q_\gamma = \gamma + \mathbf{Z}^m_{\geq 0}$. Then the above discussion has shown that

$$\text{signature}\,(\langle x^\alpha \colon \alpha \in A\rangle) = \bigcup_{\alpha \in A} Q_\alpha = \bigcup_{\beta \in \text{signature}(I)} Q_\beta, \qquad \text{where } I = \langle x^\alpha \colon \alpha \in A\rangle. \tag{2.4.0.4}$$

The main result of this section is

Theorem 2.4.0.5. (Dickson's Lemma). A monomial ideal has a finite basis. In particular, if $I = \langle x^\alpha \colon \alpha \in A\rangle$, there is a finite subset $\{\alpha(1), \ldots, \alpha(s)\} \subset A$ for which

$$\langle x^{\alpha(1)}, \ldots, x^{\alpha(s)}\rangle = \langle x^\alpha \colon \alpha \in A\rangle. \tag{2.4.0.6}$$

Comment. Since monomial ideals are completely determined by their signatures, we could prove this result by showing that

$$Q_{\alpha(1)} \cup \cdots \cup Q_{\alpha(s)} = \bigcup_{\alpha \in A} Q_\alpha \tag{2.4.0.7}$$

for some $\{\alpha(1), \ldots, \alpha(s)\} \subset A$, where $Q_\beta = \beta + \mathbf{Z}^n_{\geq 0}$. The proof we give, however, will be more closely tied to the language of ideals.

Proof. We begin the proof of Dickson's Lemma by showing that every monomial ideal I of $k[x_1,\ldots,x_n]$ is finitely generated.

The proof is by induction on the number of variables n. If $n=1$, then I is the ideal in $k[x_1]$ generated by the monomials x_1^α, where $\alpha \in A \subset \mathbf{Z}_{\geq 0}$. Let $\alpha_0 = \inf A$. Since A is a subset of nonnegative integers, it is well ordered and $\alpha_0 \in A$. $x_1^{\alpha_0} \in I$ and divides each x_1^α with $\alpha \in A$; so $I = \langle x_1^{\alpha_0} \rangle$ and Dickson's Lemma is established when $n=1$.

Assume $n > 1$ and for notation write the variables as $x_1,\ldots,x_{n-1},y$ and the exponents as (α, j), where $\alpha \in \mathbf{Z}_{\geq 0}^{n-1}$ and $j \in \mathbf{Z}$. The monomials in $k[x_1,\ldots,x_{n-1},y]$ can be written in the form $x^\alpha y^j$.

Suppose $I \subset k[x_1,\ldots,x_{n-1},y]$ is a monomial ideal. Let J be the ideal in $k[x_1,\ldots,x_{n-1}]$ generated by the monomials x^α for which $x^\alpha y^m \in I$ for some $m \in \mathbf{Z}_{\geq 0}$. J is a monomial ideal in $k[x_1,\ldots,x_{n-1}]$; so by the inductive hypothesis, $J = \langle x^{\alpha(1)},\ldots,x^{\alpha(s)}\rangle$ for some choice of $\alpha(i)$, where for each i there is an $x^{\alpha(i)}y^{m_i} \in I$. Choose such m_i's and let $m = \max_{1\leq i\leq s} m_i$.

Now for each q with $0 \leq q \leq m$ let J_q be the ideal in $k[x_1,\ldots,x_{n-1}]$ generated by the x^β for which $x^\beta y^q \in I$. J_q has a finite generating set by the inductive assumption, say $J_q = \langle x^{\alpha_q(1)},\ldots,x^{\alpha_q(s_k)}\rangle$.

I now claim that the monomials in the sets

$$\begin{array}{ll}
\{x^{\alpha_0(1)},\ldots,x^{\alpha_0(s_0)}\}, & \text{from } J_0 \\
\{x^{\alpha_1(1)}y,\ldots,x^{\alpha_1(s_1)}y\}, & \text{from } J_1 y \\
\{x^{\alpha_2(1)}y^2,\ldots,x^{\alpha_2(s_2)}y^2\}, & \text{from } J_2 y^2 \\
\cdots & \\
\{x^{\alpha_{m-1}(1)}y^{m-1},\ldots,x^{\alpha_{m-1}(s_{m-1})}y^{m-1}\} & \text{from } J_{m-1}y^{m-1}, \text{ together with} \\
\{x^{\alpha(1)}y^m,\ldots,x^{\alpha(s)}y^m\} & \text{from } Jy^m
\end{array} \tag{2.4.0.8}$$

generate I. Let L be the ideal they generate. These monomials clearly belong to I; so $L \subset I$. To establish that $I \subset L$ it suffices to show that each monomial $x^\beta y^j$ of I is in L. There are two cases: $j < m$ and $j \geq m$.

Case I: Suppose $x^\beta y^j \in I$ and $j < m$. Then $x^\beta \in J_j$ and is consequently in the ideal generated by $\{x^{\alpha_j(1)},\ldots,x^{\alpha_j(s_j)}\}$. This means that $x^\beta = \sum_{k=1}^{s_j} f_k(\mathbf{x})x^{\alpha_j(k)}$ for a suitable choice of $f_k(\mathbf{x}) \in k[x_1,\ldots,x_{n-1}]$ (here $\mathbf{x} = (x_1,\ldots,x_{n-1})$). But then $x^\beta y^j = \sum_{k=1}^{s_j} f_k(\mathbf{x})\big(x^{\alpha_j(k)}y^j\big) \in L$ is in the ideal generated by the monomials (2.4.0.8).

Case II: If $x^\beta y^j \in I$ and $j \geq m$, then $x^\beta \in J$; so x^β is in the ideal generated by $\{x^{\alpha(1)},\ldots,x^{\alpha(s)}\}$ and, following an argument like that in Case I, $x^\beta y^m$ is in the ideal generated by $\{x^{\alpha(1)}y^m,\ldots,x^{\alpha(s)}y^m\}$ and hence in the ideal generated by the terms (2.4.0.8). Thus every monomial in I is in L; so $I \subset L$ as we desired to show.

We have shown that I is finitely generated. To complete the proof it remains to show that we can choose a finite set of generators whose exponents lie in the original list A. First, switch back to calling the variables $x_1,x_2,\ldots,x_n$. We have produced a finite set $\{x^\gamma \colon \gamma \in W\}$, with W finite, which generates I. Now (2.4.0.1) states that every term of any $f \in I$ is divisible by some x^α with $\alpha \in A$. In particular, each x^γ is so divisible, say by $x^{\alpha(\gamma)}$. But then the set $\{\alpha(\gamma)\colon \gamma \in W\}$ will satisfy the role required of $\{\alpha(1),\ldots,\alpha(s)\}$ in the statement of the theorem . ∎

Example 2.4.0.9.0. Suppose in $\mathbb{R}[x,y]$ we order the variables $\{x,y\}$ and then order the monomials by using the weight matrix $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$. That is $x^a y^b > x^c y^d$ if and only if either (i) $-a+b > -c+d$ or if (ii) $-a+b = -c+d$ and $a > c$. With this ordering $\Sigma^2_{\geq 0}$ is linearly ordered and the ordering is stable in the sense that if $(a,b) > (c,d)$ and $(p,q) \in \Sigma^2_{\geq 0}$, then $(a+p,b+q) > (c+p,b+q)$. This ordering is not a well ordering, however, because $1 > x > x^2 > x^3 > \cdots$.

With respect to Example 2.4.0.9.0, there is a corollary of Dickson's lemma:

Corollary 2.4.0.9. Suppose $>$ is a relation on $\mathbf{Z}^n_{\geq 0}$ satisfying:

(i) $>$ is a linear ordering on $\mathbf{Z}^n_{\geq 0}$.
(ii) $>$ is stable for additon. That is, if $\alpha < \beta$ then $\alpha + \gamma < \beta + \gamma$ for all $\gamma \in \mathbf{Z}^n_{\geq 0}$.

In this case $>$ well orders $\mathbf{Z}^n_{\geq 0}$ if and only if $\alpha \geq 0$ for all $\alpha \in \mathbf{Z}^n_{\geq 0}$.

Proof. ($\Rightarrow$): Suppose $>$ is a well ordering and let α be the smallest element of $\mathbf{Z}^n_{\geq 0}$. If $0 > \alpha$, property (ii) with γ replaced by α, implies that $\alpha = \alpha + 0 > \alpha + \alpha = 2\alpha$. Thus 2α is even smaller than the would be smallest element α and this contradiciton shows that there is no α for which $0 > \alpha$.

($\Leftarrow$): Let A be a nonempty subset of $\mathbf{Z}^n_{\geq 0}$. To show that $>$ is a well ordering we must show that any such A contains a smallest element. Let I be the monomial ideal $\langle x^\alpha \colon \alpha \in A\rangle$ and using Dickson's lemma (which doesn't require any ordering in its proof) let $\{\alpha(1), \ldots, \alpha(s)\}$ be a finite subset of A for which $\langle x^{\alpha(1)}, \ldots, x^{\alpha(s)}\rangle = I$. Since $>$ totally orders $\mathbf{Z}^n_{\geq 0}$, we can,by relabeling if necessary, assume that $\alpha(1) < \alpha(2) < \cdots < \alpha(s)$. Let $\alpha \in A$. Since x^α is a monomial in I, it is divisible by some element $x^{\alpha(i)}$ in the set of generating monomials $\{x^{\alpha(1)}, \ldots, x^{\alpha(s)}\}$, say $x^{\alpha(i)}$. This means that $\alpha = \alpha(i) + \gamma$ for some $\gamma \in \mathbf{Z}^n_{\geq 0}$. But then

$$\gamma \geq 0 \Rightarrow \alpha = \alpha(i) + \gamma \geq \alpha(i) + 0 = \alpha(i) > \alpha(1)$$

shows that $\alpha > \alpha(1)$. Since $\alpha \in A$ was arbitrary this shows that $\alpha(1)$ is the minimal element of A. ∎

Exercises for Chapter 2 §4

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§2.4.1.

Let $I \subset k[x_1, \ldots, x_n]$ be an ideal with the property that for every $f = \sum_\alpha c_\alpha x^\alpha \in I$, every monomial x^α appearing in f is also in I. Show that I is a monomial ideal.

Solution. Let J be the ideal generated by those monomials x^α which appear in the above manner in some $f \in I$. It is always true that $I \subseteq J$. The above mentioned property of I guarantees that $J \subseteq I$; so $J = I$ and I is a monomial ideal, the ideal generated by those monomials which appear in any of the expansions $f = \sum_\alpha c_\alpha x^\alpha$, $f \in I$.

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§2.4.2.

Complete the proof of

Lemma 3. Let I be a monomial ideal, and let $f \in k[x_1, \ldots, x_n]$. Then the following are equivalent:

(i) $f \in I$.
(ii) Every term of f lies in I.
(iii) f is a k-linear combination of the monomials in I.

Proof. It is trivial that $(iii) \Rightarrow (ii) \Rightarrow (i)$. To establish that $(i) \Rightarrow (iii)$ it suffices to show that if $f = \sum_\alpha c_\alpha x^\alpha \in I$ then $c_\alpha \neq 0 \Rightarrow x^\alpha \in I$. We know the following (easily established) fact: As a monomial ideal, the ideal $I = \langle x^\beta \colon \beta \in B\rangle$ for some $B \subset \mathbf{Z}^n_{\geq 0}$ consists of those $f = \sum_\alpha c_\alpha x^\alpha$ for which $c_\alpha \neq 0 \Rightarrow x^\alpha$ is divisible by some x^β, $\beta \in B$. An immediate consequence is that if $f = \sum_\alpha c_\alpha x^\alpha \in I$ then each such $x^\alpha = x^{\alpha-\beta} \cdot x^\beta \in I$ fo some $\beta \in B$. As a consequence f is a k-linear combination of the monomials in I.

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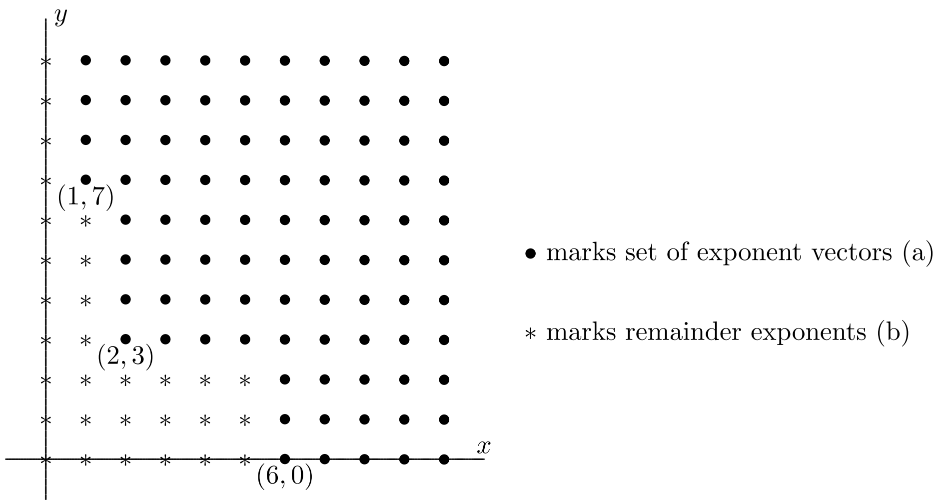
§2.4.3.

Let $I = \langle x^6, x^2y^3, xy^7\rangle \subset k[x, y]$.

(a) In the (m, n)-plane, plot the set of exponent vectors (m, n) of monomials $x^m y^n$ appearing in elements of I.

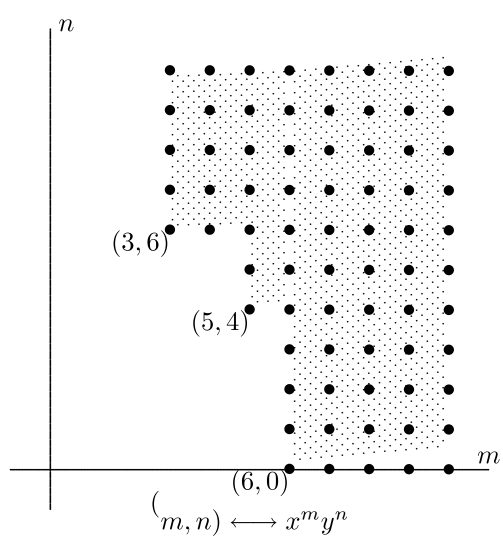
(b) If we apply the division algorithm to an element $f \in k[x, y]$, using the generators of I as divisors, what terms can appear in the remainder?

Solution. We answer both of these questions together. Go back to (2.3.11) for further discussion of the answer. Ideal generated by F; division by F, where $F = (x^6, x^2y^3, xy^7)$.



$\boxed{\S\mathbf{2.4.4.}}$

Let $I \subset k[x, y]$ be the monomial ideal spanned over k by the monomials x^β corresponding to β in the shaded region below:



(a) Use the method given in the proof of Dickson's Lemma to find an ideal basis for I.

Solution. We compute these terms following the definitions at (2.4.0.8).

$$\begin{aligned} J &= x^3k[x] && \text{which is } \langle x^3\rangle;\ m \text{ is } 6. \\ J_0 = J_1 = J_2 = J_3 &= x^6k[x] && \text{which is } \langle x^6\rangle. \\ J_4 = J_5 &= x^5k[x] && \text{which is } \langle x^5\rangle. \end{aligned}$$

From this we get that $I = \langle x^6, x^6y, x^6y^2, x^6y^3, x^5y^4, x^5y^5, x^3y^6\rangle$.

(b) Is your basiss as small as possible, or can some β's be deleted from your basis, yielding a smaller set that generates the same ideal?

Solution. The underlined terms can be deleted in $\{x^6, \underline{x^6 y, x^6 y^2, x^6 y^3}, x^5 y^4, \underline{x^5 y^5}, x^3 y^6\}$.

$\boxed{\S\mathbf{2.4.5.}}$

Suppose that $I = \langle x^\alpha \colon \alpha \in A\rangle$ is a monomial ideal, and let S be the set of all exponents that occur as monomials of I. For any monomial order $>$, prove that the smallest element of S with respect to $>$ must lie in A.

Solution. Let γ be the smallest exponent of any monomial in I taken with respect to the well order $>$. x^γ is known to be divisible by x^α for some $\alpha \in A$. That is $\gamma = \alpha + \tau$ for some $\tau \in \mathbf{Z}^n_{\geq 0}$. It is also known that $0 = (0, \ldots, 0)$ is the $>$-smallest element of $\mathbf{Z}^n_{\geq 0}$; so from $\tau \geq 0$ we get $\alpha + \tau \geq 0 + \alpha = \alpha$. This in turn gives $\gamma = \alpha + \tau \geq \alpha$. Both γ and α are exponents of monomials in I, and since γ is in addition the smallest such exponent we must have $\gamma \leq \alpha$. From this it follows that γ, the smallest element of S, $= \alpha \in A$.

$\boxed{\S\mathbf{2.4.6.}}$

Let $I = \langle x^\alpha \colon \alpha \in A\rangle$ be a monomial ideal, and assume that we have a finite basis $I = \langle x^{\beta(1)}, \ldots, x^{\beta(s)}\rangle$. In the proof of Dickson's Lemma, we observed that each $x^{\beta(j)}$ is divisible by $x^{\alpha(j)}$ for some $\alpha(j) \in A$. Prove that $I = \langle x^{\alpha(1)}, \ldots, x^{\alpha(s)}\rangle$.

Solution. Let $J = \langle x^{\alpha(1)}, \ldots, x^{\alpha(s)}\rangle$. It is clear that $I \subset J$ from the abovementioned division relations. On the otherhand since $\{x^{\alpha(1)}, \ldots, x^{\alpha(s)}\} \subset \{x^\alpha \colon \alpha \in A\}$ it follows that $J \subset I$. It follows that $J = I$.

$\boxed{\S\mathbf{2.4.7.}}$

Prove that Dickson's Lemma is equivalent to the following statement: given a subset $A \subset \mathbf{Z}^n_{\geq 0}$, there are finitely many elements $\alpha(1), \ldots, \alpha(s) \in A$ such that for every $\alpha \in A$, there exists some i and some $\gamma \in \mathbf{Z}^n_{\geq 0}$ such that $\alpha = \alpha(i) + \gamma$.

Solution. Consider the proposition: Dickson's Lemma $\Leftrightarrow$ above statement.

Proof. □($\Rightarrow$): Let by Dickson's Lemma $\{\alpha(1), \ldots, \alpha(s)\}$ be a finite subset of A such that

$$\langle x^{\alpha(1)}, \ldots, x^{\alpha(s)}\rangle = \langle x^\alpha \colon \alpha \in A\rangle = I.$$

If $\alpha \in A$ we know that x^α is a monomial in $\langle x^{\alpha(1)}, \ldots, x^{\alpha(s)}\rangle$ and hence divisible by some $x^{\alpha(j)}$. That is, $\alpha = \alpha(j) + \gamma$ for some $\gamma \in \mathbf{Z}^n_{\geq 0}$ which proves the statement above.

($\Leftarrow$): Suppose the above statement is true and that $A \subset \mathbf{Z}^n_{\geq 0}$. Suppose that $\alpha(1), \ldots, \alpha(s)$ are chosen to satisfy the condions of the above statement. I claim that

$$\langle x^{\alpha(1)}, \ldots, x^{\alpha(s)}\rangle = \langle x^\alpha \colon \alpha \in A\rangle.$$

Indeed, if x^β is any monomial in $\langle x^\alpha \colon \alpha \in A\rangle$ then x^β is known to be divisible by some x^α, $\alpha \in A$. This means that $\beta = \alpha + \delta$ for some $\delta \in \mathbf{Z}^n_{\geq 0}$. Then if i and γ are chosen to match the conditions in the statement, $\beta = \alpha + \delta = \alpha(i) + (\gamma + \delta)$ and since $\delta + \gamma \in \mathbf{Z}^n_{\geq 0}$ it follows that $x^{\alpha(i)}$ divides x^β. This means that x^β is in the ideal generated by the $x^{\alpha(j)}$'s and these latter terms form a finite set of monomial generators for $\langle x^\alpha \colon \alpha \in A\rangle$ as well as being members of A. The existence of such a finite set of $\alpha(i)$'s is the statement of Dickson's Lemma.

$\boxed{\S\mathbf{2.4.8.}}$

A basis $\{x^{\alpha(1)}, \ldots, x^{\alpha(s)}\}$ for a monomial ideal I is *minimal* if no $x^{\alpha(i)}$ divides another $x^{\alpha(j)}$ for $i \neq j$.

(a) Prove that every monomial ideal has a minimal basis.
(b) Show that every monomial ideal has a *unique* minimal basis.

Solution. First use Dickson's Lemma to choose a finite basis $\{x^{\alpha(1)}, \ldots, x^{\alpha(s)}\}$ and then delete any monomial in this basis which is a multiple of another monomial in the basis. The result is still a basis and it is minimal. Next consider two minimal bases for the monomial ideal I:

$$\{x^{\alpha(1)}, \ldots, x^{\alpha(s)}\} \text{ and } \{x^{\delta(1)}, \ldots, x^{\delta(t)}\}.$$

Since $x^{\alpha(1)}$ is a monomial in I it follows that it must be divisible by (at least) one monomial from the basis $\{x^{\delta(1)}, \ldots, x^{\delta(t)}\}$, say $x^{\delta(1)}$. For the same reason $x^{\delta(1)}$ is divisible by one of the $x^{\alpha(i)}$'s, but because $\{x^{\alpha(1)}, \ldots, x^{\alpha(s)}\}$ is a minimal basis there is only one candidate: $x^{\delta(1)}$ is divisible by $x^{\alpha(1)}$. It follows that $x^{\alpha(1)} = x^{\delta(1)}$. Taking up $x^{\alpha(2)}$ we can argue similarly that it is equal to precisely one of the $x^{\delta(i)}$'s, say $x^{\delta(2)}$ and continuing in this manner it follows that these two minimal bases for I are identical.

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§**2.4.9.**

If $I = \langle x^{\alpha(1)}, \ldots, x^{\alpha(s)} \rangle$ is a monomial ideal, prove that a polynomial f is in I if and only if the remainder of f on division by $(x^{\alpha(1)}, \ldots, x^{\alpha(s)})$ is zero.

Proof. (⇐): Division yields an expression $f = a_1 x^{\alpha(1)} + \cdots + a_s x^{\alpha(s)} + r$ where no term in the remainder r is a multiple of the leading term in any of the divisors. In this case this amounts to saying that none of the $x^{\alpha(i)}$'s divides any of the terms in r. In particular, if the remainder is zero, f is clearly a $k[x_1, \ldots, x_n]$ linear combination of the $x^{\alpha(i)}$'s. That is, $f \in I$.

(⇒): If f is in I it follows from the division expression that $r \in I$. But since none of the terms in r is divisible by any of the $x^{\alpha(i)}$'s, it must be that $r = 0$. ∎

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§**2.4.10.**

Suppose we have the polynomial ring $k[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Let us define a monomial order $>_{mixed}$ on this ring that mixes lex order for $x_1, \ldots, x_n$ with grlex order for $y_1, \ldots, y_m$. If we write monomials in the $n + m$ variables as $x^\alpha y^\beta$, where $\alpha \in \mathbf{Z}^n_{\geq 0}$ and $\beta \in \mathbf{Z}^m_{\geq 0}$, then we define

$$x^\alpha y^\beta >_{mixed} x^\gamma y^\delta \Longleftrightarrow x^\alpha >_{lex} y^\gamma \text{ or } x^\alpha = x^\gamma \text{ and } y^\beta >_{grlex} y^\delta.$$

Use the corollary to Dickson's Lemma to prove that $>_{mixed}$ is a monomial order. This is an example of what is called a *product order*. It is clear that many other monomial orders can be created by this method.

Solution. We must show (i) that $>_{mixed}$ is a total order, i.e. orders $\mathbf{Z}^n_{\geq 0} \times \mathbf{Z}^m_{\geq 0}$ as a chain. (ii) that it is stable for addition. (iii) that $(0_n, 0_m)$ is the minimal element of $\mathbf{Z}^n_{\geq 0} \times \mathbf{Z}^m_{\geq 0}$ in the $>_{mixed}$ order.

(i) That it is a total order is easy. Either $x^\alpha y^\beta >_{mixed} x^\gamma y^\delta$ or $x^\alpha y^\beta <_{mixed} x^\gamma y^\delta$ or $x^\alpha y^\beta = x^\gamma y^\delta$.

(ii) That it is stable under addition is equally easy to establish and really obvious because both lex and grlex are stable under addition which in this case is componentwise addition.

(iii) Comparing $x^\alpha y^\beta$ with $x^{0_n} y^{0_m}$, there are three possibilities: (1) either $x^\alpha >_{lex} x^{0_n}$ and then necessarily $x^\alpha y^\beta >_{mixed} x^{0_n} y^{0_m}$, or (2) $\alpha = 0_n$ and $y^\beta >_{grlex} y^{0_m}$ from which $x^\alpha y^\beta >_{mixed} x^{0_n} y^{0_m}$, or (3) $\alpha = 0_n$ and $\beta = 0_m$ and then $x^\alpha y^\beta = x^{0_n} y^{0_m}$. In any case it is true that $(0_n, 0_m)$ is the minimal element of $\mathbf{Z}^n_{\geq 0} \times \mathbf{Z}^m_{\geq 0}$ in the $>_{mixed}$ order.

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§**2.4.11.**

In this exercise we will investigate a special case of a *weight order*. Let $\mathbf{u} = (u_1, \ldots, u_n)$ be a vector in $\mathbf{R}^n$ such that $u_1, \ldots, u_n$ are positive and linearly independent over $\mathbf{Q}$. We say that $\mathbf{u}$ is an *independent weight vector*. Then, for $\alpha, \beta \in \mathbf{Z}^n_{\geq 0}$, define

$$\alpha >_{\mathbf{u}} \beta \Longleftrightarrow \mathbf{u} \cdot \alpha > \mathbf{u} \cdot \beta,$$

where the centered dot is the usual dot product of vectors. We call $>_{\mathbf{u}}$ the *weight order* determined by $\mathbf{u}$.

(a) Use the corollary to Dickson's Lemma to prove that $>_{\mathbf{u}}$ is a monomial order. Hint: Where does your argument use the linear independence of $u_1, \dots, u_n$?

Solution. If $\mathbf{u}\cdot\alpha = \mathbf{u}\cdot\beta$, then $\mathbf{u}\cdot(\alpha-\beta) = 0$ or $u_1(\alpha_1-\beta_1)+\cdots+u_n(\alpha_n-\beta_n) = 0$. The linear independence of the u_i's over $\mathbf{Z}$ then implies that $\alpha_i - \beta_i = 0$, $1 \le i \le n$. For this reason the map $\alpha \mapsto \mathbf{u}\cdot\alpha$ of $\mathbf{Z}^n_{\geq 0}$ into $\mathbf{R}_{\geq 0}$ is an injection. The $>_{\mathbf{u}}$-order is obtained by "lifting" the usual order on $\mathbf{R}_{\geq 0}$ back to $\mathbf{Z}^n_{\geq 0}$. This establishes that (i) $>_{\mathbf{u}}$ is a total ordering of $\mathbf{Z}^n_{\geq 0}$. (iii) Since every component of $\mathbf{u}$ is positive it follows that $\mathbf{u}\cdot\alpha \geq 0$ for all $\alpha \in \mathbf{Z}^n_{\geq 0}$ and thus that 0_n is the minimum element in $\mathbf{Z}^n_{\geq 0}$ for the $>_{\mathbf{u}}$-order. (ii) Finally, this lifting also preserves translation: If $\gamma \in \mathbf{Z}^n_{\geq 0}$, then

$$\alpha >_{\mathbf{u}} \beta \Leftrightarrow \mathbf{u}\cdot\alpha > \mathbf{u}\cdot\beta \Rightarrow \mathbf{u}\cdot(\alpha+\gamma) > \mathbf{u}\cdot(\beta+\gamma) \Leftrightarrow \alpha+\gamma >_{\mathbf{u}} \beta+\gamma.$$

Using the corollary to Dickson's Lemma these facts establish that $>_{\mathbf{u}}$ is a monomial order.

(b) Show that $\mathbf{u} = (1, \sqrt{2})$ is an independent weight vector, so that $>_{\mathbf{u}}$ is a weight order on $\mathbf{Z}^2_{\geq 0}$.

Solution. This is a simple consequence of the algebraic independence of 1 and $\sqrt{2}$.

(c) Show that $\mathbf{u} = (1, \sqrt{2}, \sqrt{3})$ is an independent weight vector, so that $>_{\mathbf{u}}$ is a weight order on $\mathbf{Z}^3_{\geq 0}$.

Solution. Here again this is a simple consequence of the algebraic independence of 1, $\sqrt{2}$, and $\sqrt{3}$.

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§2.4.12.

Another important weight order is constructed as follows. Let $\mathbf{u} = (u_1, \dots, u_n)$ be in $\mathbf{Z}^n_{\geq 0}$, and fix a monomial order $>_\sigma$ (such as $>_{lex}$ or $>_{grevlex}$) on $\mathbf{Z}^n_{\geq 0}$. Then, for $\alpha, \beta \in \mathbf{Z}^n_{\geq 0}$, define $\alpha >_{\mathbf{u},\sigma} \beta$ if and only if

$$\mathbf{u}\cdot\alpha > \mathbf{u}\cdot\beta \text{ or } \mathbf{u}\cdot\alpha = \mathbf{u}\cdot\beta \text{ and } \alpha >_\sigma \beta.$$

We call $>_{\mathbf{u},\sigma}$ the *weight order* determined by $\mathbf{u}$ and σ.

(a) Use the corollary to Dickson's Lemma to show that $>_{\mathbf{u},\sigma}$ is a monomial order.

Solution. (i) That it is a total order is clear. $>_{\mathbf{u},\sigma}$ uses the ordinary $>_{\mathbf{u}}$-order but since the components of $\mathbf{u}$ are nonnegative integers we no longer have that they are algebraically independent and there may be ties. To break these ties we appeal to the monomial order $>_\sigma$. (iii) It is also clear that 0_n is the smallest element of $\mathbf{Z}^n_{\geq 0}$ in this order. (ii) That it is stable under addition is also easy to establish: If $\gamma \in \mathbf{Z}_{\geq 0}+^n$ and $\alpha >_{\mathbf{u},\sigma} \beta$, there are two cases.

Case 1: $\mathbf{u}\cdot\alpha > \mathbf{u}\cdot\beta$. In this case $\mathbf{u}\cdot(\alpha+\gamma) > \mathbf{u}\cdot(\beta+\gamma)$ and, accordingly, $\alpha+\gamma >_{\mathbf{u},\sigma} \beta+\gamma$.

Case 2: $\mathbf{u}\cdot\alpha = \mathbf{u}\cdot\beta$ and $\alpha >_\sigma \beta$. In this case $\mathbf{u}\cdot(\alpha+\gamma) = \mathbf{u}\cdot(\beta+\gamma)$ and $\alpha+\gamma >_\sigma \beta+\gamma$. Accordingly, $\alpha+\gamma >_{\mathbf{u},\sigma} \beta+\gamma$.

(b) Find $\mathbf{u} \in \mathbf{Z}^n_{\geq 0}$ so that the weight order $>_{\mathbf{u},lex}$ is the gradlex order $>_{gradlex}$.

Solution. I don't know what the gradlex order is. If $\mathbf{u} = (1, 1, \dots, 1)$, then $>_{\mathbf{u},lex}$ is the grlex order. Perhaps this is what was intended.

(c) In the definition of $>_{\mathbf{u},\sigma}$, the order $>_\sigma$ is used to break ties, and it turns out that ties will *always* occur in this case. More precisely, prove that given $\mathbf{u} \in \mathbf{Z}^n_{\geq 0}$, there are $\alpha \neq \beta$ in $\mathbf{Z}^n_{\geq 0}$ such that $\mathbf{u}\cdot\alpha = \mathbf{u}\cdot\beta$. Hint: Consider the linear equation $u_1a_1 + \cdots + u_na_n = 0$ over $\mathbf{Q}$. Show that there is a nonzero integer solution $(a_1, \dots, a_n)$, and then show that $(a_1, \dots, a_n) = \alpha - \beta$ for some $\alpha, \beta \in \mathbf{Z}^n_{\geq 0}$.

Solution. The hint really steals the problem here. Too bad they gave it! If $\mathbf{u}\cdot\alpha = \mathbf{u}\cdot\beta$, then $\overrightarrow{a} = (a_1, \dots, a_n) = \alpha - \beta$ satisfies $\mathbf{u}\cdot\overrightarrow{a} = 0$. Call this last equation (#). If $u_1 = 0$ then $\overrightarrow{a} = (1, 0, 0, \dots, 0)$ is a solution to (#). More generally, if $u_i = 0$ for some i, then $a_j = [j = i]$ or δ_{ij} is a solution to (#). If no $u_i = 0$, then $\overrightarrow{a} = (u_2, -u_1, 0, \dots, 0)$ is a solution to (#). It is then easy to find a $\beta, \alpha = \beta - \overrightarrow{a} \in \mathbf{Z}^n_{\geq 0}$ whose $>_{\mathbf{u},\sigma}$ comparison requires "the breaking of a tie".

(d) A useful example of a weight order is the *elimination order* introduced by Bayer and Stillman (1987b). Fix an integer $1 \le i \le n$ and let $u_j = [j \le i]$, that is, $\mathbf{u} = (1, \ldots, 1, 0, \ldots, 0)$ where there are i 1's and $n - i$ 0's. Then the *i-th elimination order* $>_i$ is the weight order $>_{\mathbf{u},grevlex}$. Prove that $>_i$ has the following property: If x^α is a monomial in which one of $x_1, \ldots, x_i$ appears, then $x^\alpha >_i x^\beta$ for any monomial involving only $x_{i+1}, \ldots, x_n$. Elimination orders play an important role in elimination theory, which we will study in the next chapter.

Solution. Suppose x^α is a monomial in which one of $x_1, \ldots, x_i$ appears, and that x^β is a monomial involving only $x_{i+1}, \ldots, x_n$. Then $\alpha \cdot \mathbf{u}_i \ge 1$ whereas $\beta \cdot \mathbf{u}_i = 0$ Thus there is no tie in the first test of $>_i$ and $x^\alpha > x^\beta$.

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§**2.4.13. Monomial Orders on $\mathbf{Z}^n_{\ge 0}$.** The following is detailed in L. Robbiano (1986), *On the theory of graded structures*, J. Symbolic Comp. 2, 139-170.

If $>$ is a monomial order on $\mathbf{Z}^n_{\ge 0}$, then there exists a finite sequence of weight orders with associated vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_\ell$ such that

$$> \text{ is the product order } >_{\mathbf{u}_1,\ldots,\mathbf{u}_\ell} .$$

Thus to test $\alpha > \beta$, first test $\alpha >_{\mathbf{u}_1} \beta$. Break ties by using $>_{\mathbf{u}_2}$. If there is still a tie use $>_{\mathbf{u}_3}$ and so on until a decision is reached. **Caution.** These $\mathbf{u}_j$'s are in $\mathbf{R}^n$. Their components need not be positive. They cannot be arbitrary, however, because with arbitrary components the ordering induced on $\mathbf{Z}^n_{\ge 0}$ might not be a well ordering. I can in fact be more explicit. Let the $\mathbf{u}_i$'s be written out in component form as the rows of a matrix W with real entries which I shall call the *weight matrix* for a particular monomial ordering. The first row of W is $\mathbf{u}_1$, the second $\mathbf{u}_2$, and so on. A $\ell \times n$-matrix W with real entries is the weight matrix for a monomial ordering if it meets three conditions: (i) The first nonzero element in each column must be positive. This guarantees that the ordering is a well ordering. (ii) When the rows are regarded as real valued functions on $\mathbf{R}^n$ through the dot product, they separate the points of $\mathbf{Z}^n_{\ge 0}$. (iii) They are linearly independent. This last condition is only to avoid too many duplicate weightings. Condition (ii) is of course satisfied if the rows span $\mathbf{R}^n$.

The lex order. To implement this using weight vectors in the above manner. Let $\mathbf{w}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with a 1 in the i-th component. Then

$$>_{lex} \text{ is the product order } >_{\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_n} .$$

The grlex order. Let $\mathbf{u} = (1, 1, \ldots, 1)$. Then using the preceding notation

$$>_{grlex} \text{ is the product order } >_{\mathbf{u},\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_{n-1}} .$$

The grevlex order. Using the preceding notation

$$>_{grevlex} \text{ is the product order } >_{\mathbf{u},-\mathbf{w}_n,-\mathbf{w}_{n-1},\ldots,-\mathbf{w}_2} .$$

Implementation in Mathematica 3.01. Mathematica implements these weightings in a number of its commands, viz. **GroebnerBasis, PolynomialReduce, and the simple listing, MonomialList**. There follows a listing of some Mathematica commands and their outputs.

```
Poly=x^3+y^3+z^3+x^2*y+x^2*z+x*y*z+x*y ^2+z*y^2+x*z^2+y*z^2+x^2+y^2
     +z^2+x*y+x*z+y*z+x+y+z+1
```

$$(1 + x + x^2 + x^3 + y + x\,y + x^2\,y + y^2 + x\,y^2 + y^3 + z + x\,z + x^2\,z + y\,z + x\,y\,z + y^2\,z + z^2 + x\,z^2 + y\,z^2 + z^3)$$

```
# grevlex with x > y > z
MonomialList[Poly,{x,y,z},MonomialOrder->{{1,1,1},{0,0,-1},{0,-1,0}}]
```

$$\{x^3, x^2\,y, x\,y^2, y^3, x^2\,z, x\,y\,z, y^2\,z, x\,z^2, y\,z^2, z^3, x^2, x\,y, y^2, x\,z, y\,z, z^2, x, y, z, 1\}$$

grlex with $x > y > z$
MonomialList[Poly,{x,y,z},MonomialOrder->{{1,1,1},{1,0,0},{0,1,0}}]

$$\{x^3, x^2\,y, x\,y^2, y^3, x^2\,z, x\,y\,z, y^2\,z, x\,z^2, y\,z^2, z^3, x^2, x\,y, y^2, x\,z, y\,z, z^2, x, y, z, 1\}$$

lex with $x > y > z$
MonomialList[Poly,{x,y,z},MonomialOrder->{{1,0,0},{0,1,0},{0,0,1}}]

$$\{x^3, x^2\,y, x^2\,z, x^2, x\,y^2, x\,y\,z, x\,y, x\,z^2, x\,z, x, y^3, y^2\,z, y^2, y\,z^2, y\,z, y, z^3, z^2, z, 1\}$$

1-st elimination order of Bayer and Stillman (?) with $x > y > z$
MonomialList[Poly,{x,y,z},MonomialOrder->{{1,0,0},{1,1,1},{0,0,-1}}]

$$\{x^3, x^2\,y, x^2\,z, x^2, x\,y^2, x\,y\,z, x\,z^2, x\,y, x\,z, x, y^3, y^2\,z, y\,z^2, z^3, y^2, y\,z, z^2, y, z, 1\}$$

2-nd elimination order of Bayer and Stillman (?) with $x > y > z$
MonomialList[Poly,{x,y,z},MonomialOrder->{{1,1,0},{1,1,1},{0,-1,0}}]

$$\{x^3, x^2\,y, x\,y^2, y^3, x^2\,z, x\,y\,z, y^2\,z, x^2, x\,y, y^2, x\,z^2, y\,z^2, x\,z, y\,z, x, y, z^3, z^2, z, 1\}$$