Chapter 2 §8. First Applications of Gröbner Bases. Let's see how Gröbner bases bear on the following:

> **The Ideal Description Problem:** The Hilbert Basis Theorem will assert that every ideal $I \subseteq k[x_1, \dots, x_n]$ has a finite generating set. That is, $I = \langle f_1, \dots, f_s \rangle$ for some $f_i \in k[x_1, \dots, x_n]$.
>
> **The Ideal Membership Problem:** Given $f \in k[x_1, \dots, x_n]$ and an ideal $I = \langle f_1, \dots, f_s \rangle$, determine if $f \in I$. Geometrically this is closely related to the problem of determining whether f vanishes on $\mathbf{V}(f_1, \dots, f_s)$. The **Hilbert Nullstellensatz** will assert that if f vanishes on $\mathbf{V}(f_1, \dots, f_s)$, then for some integer power m, $f^m \in \langle f_1, \dots, f_s \rangle$.

Here is an **ideal membership algorithm:** Given an ideal $I = \langle f_1, \dots, f_t \rangle$, decide whether a given polynomial f is in I as follows. (Step 1): Find a Gröbner basis $G = \{g_1, \dots, g_s\}$ for I. Then Corollary 2 of §2.6.0 implies that

$$f \in I \iff \overline{f}^G = 0.$$

> **The Problem of solving Polynomial Equations:** Find all common solutions in k^n of a system of polynomial equations

$$f_1(x_1, \dots, x_n) = \cdots = f_s(x_1, \dots, x_n) = 0. \tag{1.0.1}$$

> Of course this is the same as asking for the points of the affine variety $\mathbf{V}(f_1, \dots, f_s)$.

We begin with an example. Example 2.8.0.2. Consider the equations

$$\begin{aligned} x^2 + y^2 + z^2 &= 1, \\ x^2 + z^2 &= y, \\ x &= z \end{aligned}$$

in $\mathbf{C}^3$. These equations determine the ideal $I = \langle x^2 + y^2 + z^2 - 1,\, x^2 + z^2 - y,\, x - z \rangle$ and we want to find all points in $\mathbf{V}(I)$. Replace the above basis for I by a Gröbner basis. Using Mathematica 3.0 we find that the "reduced" Gröbner basis is given by $\{-1 + 2\,z^2 + 4\,z^4, y - 2\,z^2, x - z\}$ which we would normalize to

$$I = \langle z^4 + \frac{1}{2}z^2 - \frac{1}{4},\, y - 2z^2,\, x - z \rangle.$$

The first basis element has 4 roots, $\pm\frac{1}{2}\sqrt{\pm\sqrt{5} - 1}$ and using these in the other basis elements gives 4 points in $\mathbf{V}(I)$, two real and two complex. We will see later why a Gröbner basis computed with lex order $x > y > z$ eliminates the variables, first the x variable, then the y variable, and finally the z variable appears by itself.

> **The Implicitization Problem:** Let V be a subset of k^n given parametrically as

$$\begin{aligned} x_1 &= g_1(t_1, \dots, t_m), \\ &\vdots \\ x_n &= g_n(t_1, \dots, t_m). \end{aligned} \tag{1.0.2}$$

> If the g_i are polynomials (or rational functions) in the variables t_j, V will be (part of) an affine variety. The problem is to find a system of polynomial equations in the x_i that define this variety.

General Procedure: Suppose that the parametric equations

$$\begin{aligned} x_1 &= f_1(t_1, \ldots, t_m), \\ &\vdots \\ x_n &= f_n(t_1, \ldots, t_m) \end{aligned}$$

define a subset of an algebraic variety V in k^n. For instance, this will always be the case if the f_i are rational functions in $t_1, \ldots, t_m$. The problem is to find polynomial equations in the x_i's that define V. We will take this up in general later, but for the moment suppose that the f_i are actually polynomials. (Step 1): Consider the variety W in k^{n+m} given by the equations

$$\begin{aligned} x_1 - f_1(t_1, \ldots, t_m) &= 0, \\ &\vdots \\ x_n - f_n(t_1, \ldots, t_m) &= 0. \end{aligned}$$

We compute a Gröbner basis for the ideal generated by the polynomials on the lefthand sides of the preceding offset in $k[t_1, \ldots, t_m, x_1, \ldots, x_n]$. Use lex order and $t_1 > t_2 > \cdots > t_m > x_1 > \cdots > x_n$. We need to study elimination theory to see just how this works, but for now a few examples will have to suffice.

Example 2.8.0.4 Consider the parametric curve V:

$$\begin{aligned} x &= t^4, \\ y &= t^3, \\ z &= t^2 \end{aligned}$$

in $\mathbf{C}^3$. Use Mathematica 3.0 to compute a Gröbner basis G for $I = \langle t^4 - x,\, t^3 - y,\, t^2 - z \rangle$ in $\mathbf{C}[t, x, y, z]$ wrt lex order and $t > x > y > z$. This gives the basis as

$$G = \{y^2 - z^3, x - z^2, -y + t\,z, t\,y - z^2, t^2 - z\},$$

with the leading terms in ascending monomial order. If we write these so that their leading terms appear in descending monomial order we have

$$G = \{t^2 - z,\ ty - z^2,\ tz - y,\ x - z^2,\ y^2 - z^3\}$$

The last two polynomials depend on x, y, z; so they define an affine variety in $\mathbf{C}^3$ and consideration of dimensions leads us to guess that our curve is part of the intersection

$$x - z^2 = 0, \qquad y^2 - z^3 = 0.$$

Question: might there be points on this intersection which aren't on our parametrized curve? The answer is: NO! But we have no way to tell that at the present moment.

Example 2.8.0.5. Consider the tangent surface of the twisted cubic in $\mathbf{R}^3$. This surface is parametrized by

$$\begin{aligned} x &= t + u, \\ y &= t^2 + 2tu, \\ z &= t^3 + 3t^2u. \end{aligned}$$

To eliminate t, u compute a Gröbner basis G for the associated ideal relative to lex order with the variables ordered by $t > u > x > y > z$. Using Mathematica 3.0 the result is

$$\begin{aligned} G = \{&-3\,x^2\,y^2 + 4\,y^3 + 4\,x^3\,z - 6\,x\,y\,z + z^2, \\ &2\,u\,y^3 + x\,y^3 - 4\,x^2\,y\,z + 5\,y^2\,z - 2\,u\,z^2 - 2\,x\,z^2, \\ &-2\,u\,y^2 - x\,y^2 + 2\,u\,x\,z + 2\,x^2\,z - y\,z, \\ &u\,x\,y - x^2\,y + 2\,y^2 - u\,z - x\,z, \\ &2\,u\,x^2 - 2\,x^3 - 2\,u\,y + 3\,x\,y - z, \\ &u^2 - x^2 + y, t + u - x\} \end{aligned}$$

Only one of these Gröbner basis elements doesn't involve t or u. Setting it equal to zero gives

$$-3\,x^2\,y^2 + 4\,y^3 + 4\,x^3\,z - 6\,x\,y\,z + z^2 = 0 \tag{2.8.0.6}$$

which is the equation of a surface containing the tangent surface to the twisted cubic. Of course as far as we know at present there may be points on (2.8.0.6) which aren't on the tangent surface to the twisted cubic.

Exercises for §2.8

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§2.8.1.

Determine whether $f = xy^3 - z^2 + y^5 - z^3$ is in the ideal $I = \langle -x^3 + y,\, x^2y - z\rangle$.

Solution. Using lex order with $x > y > z$ we get the "reduced" Gröbner basis for I as

$$G = \{y^5 - z^3, -y^2 + x\,z, x\,y^3 - z^2, x^2\,y - z, x^3 - y\}.$$

We see easily that $f = g_1 + g_3$; so YES! $f \in I$.

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§2.8.2.

Determine whether $f = x^3z - 2y^2$ is in the ideal $I = \langle xz - y,\, xy + 2z^2,\, y - z\rangle$

Solution. Using lex order with $x > y > z$ we get the "reduced" Gröbner basis for I as

$$G = \{z + 2\,z^2, y - z, -z + x\,z\}.$$

We see that $\mathrm{LT}(G) = \{z^2, y, xz\}$ whereas $\mathrm{LT}(f) = \{x^3z\}$. Since $\mathrm{LT}(f)$ is divisible by xz; so $\mathrm{LT}(f) \in \langle z^2, y, xz\rangle$ we can't reach a decision ruling out the possibility that $f \in I$ this way. Using the division algorithm we calculate

$$\begin{aligned} x^3z - 2y^2 &= (-1)(z + 2z^2) + (-2y - 2z)(y - z) + (1 + x + x^2)(-z + xz) + 2z \text{ or} \\ \overline{f}^G &= 2z \neq 0; \end{aligned}$$

so the answer is: NO! $f \notin I$.

§2.8.3.

By the method of Examples 2.8.0.2 and 2.8.0.3 find the points in $\mathbf{C}^3$ on the variety

$$V(x^2 + y^2 + z^2 - 1,\, x^2 + y^2 + z^2 - 2x,\, 2x - 3y - z).$$

Solution. We compute a Gröbner basis G for the ideal associated with this variety. Use lex order with $x > y > z$ to get

$$G = \{-23 - 8\,z + 40\,z^2, -1 + 3\,y + z, -1 + 2\,x\}$$

From these equations we see that $x = \frac{1}{2}$, z is a root of the quadratic, and then $y = 1 - z$. So there are just two points on this variety.

$\boxed{\S\mathbf{2.8.4.}}$

Find the points in $\mathbf{C}^3$ on the variety

$$V(x^2y - z^3,\, 2xy - 4z - 1,\, z - y^2,\, x^3 - 4zy).$$

Solution. We compute a Gröbner basis G for the ideal associated with this variety. Use lex order with $x > y > z$ to get

$$G = \{1\}$$

So $V = \emptyset$

$\boxed{\S\mathbf{2.8.5.}}$

Recall from calculus that a *critical point* of a differnetiable function $f(x, y)$ is a point where the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ vanish simultaneously. When $f \in \mathbf{R}[x, y]$ it follows that the critical points can be found by applying our techniques to the system of polynomial equations

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0.$$

To see how this works, consider the function

$$f(x, y) = (x^2 + y^2 - 4)(x^2 + y^2 - 1) + \left(x - \frac{3}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2.$$

(a) Find all the critical points of $f(x, y)$.

Solution.

$$\begin{aligned} f_x &= -3 + 4x(-2 + x^2 + y^2), \\ f_y &= -3 + 4y(-2 + x^2 + y^2). \end{aligned}$$

Let $I = \langle f_x, f_y \rangle$ and let G be a Gröbner basis for I. Mathematica 3.0 shows that

$$G = \{-3 - 8\,y + 8\,y^3, x - y\}.$$

There are three critical points. Their y-coordinates are the roots of the cubic in the Gröbner basis and for each point the x-coordinate is the same as the y-coordinate. The use of Mathematica 3.0's Solve[8y^3+2y-3==0,y] command gave

$$y = -\frac{1}{2};\; y = \frac{1 - \sqrt{13}}{4};\; y = \frac{1 + \sqrt{13}}{4}$$

for these roots.

(b) Classify your critical points as local maxima, local minima, or saddle points. Hint: Use the second derivative test.

Solution.

$$\begin{aligned} f_{xx} &= 4(-2 + x^2 + 3y^2), \\ f_{xy} &= 8xy, \\ f_{yy} &= 4(-2 + y^2 + 3x^2). \end{aligned}$$

The "Discriminant" is $D = f_{xx}f_{yy} - f_{xy}^2$ and the critical point is (i) a local maximum if $D > 0$ and $f_{xx} < 0$; (ii) a saddle point if $D < 0$; and a (iii) local minimum if $D > 0$ and $f_{xx} > 0$. When $x = y = -\frac{1}{2}$ the values are $D = 12$ and $f_{xx} = -4$; so this point is a local maximum. When $x = y = \frac{1-\sqrt{13}}{4}$ the values are $D = 26 - 10\sqrt{13} = -10.0555$ and the point is a saddle point. When $x = y = \frac{1+\sqrt{13}}{4}$ the values are $D = 26 + 10\sqrt{13} = 62.0555$ and $f_{xx} = 2(3 + \sqrt{13}) > 00$; so this point is a local minimum.

$\boxed{\S\mathbf{2.8.6.}}$

Fill in the details of Example 2.8.0.5. In particular, compute the required Gröbner basis, and verify that this gives us (up to a constant multiple) the polynomial appearing on the left-hand side of equation (2.8.0.6)

Solution. This was actually worked out in 2.8.0.

$\boxed{\S\mathbf{2.8.7.}}$

Let the surface S in $\mathbf{R}^3$ be formed by taking the union of the straight lines joining pairs of points on the lines

$$\left\{\begin{matrix} x = 7 \\ y = 0 \\ z = 1 \end{matrix}\right\}, \qquad \left\{\begin{matrix} x = 0 \\ y = 1 \\ z = t \end{matrix}\right\}$$

with the same parameter (i.e., t) value. (this is a special example of a class of surfaces called *ruled surfaces*).

(a) Show that the surface S can be given in the parametric form

$$\begin{aligned} x &= ut, \\ y &= 1 - u, \\ z &= u + t - ut. \end{aligned} \tag{2.8.7.a}$$

Solution. These points lie on the lines joining $(t, 0, 1)$ and $(0, 1, t)$; so (think of starting at the second point and heading towards the first) they have the form $(x, y, z) = u[(t, 0, 1) - (0, 1, t)] + (0, 1, t)$ which is exactly the set of equations above.

(b) Using the method of Examples 2.8.4 and 2.8.5, find an (implicit) equation of a variety V containing the surface S.

Solution. Consider the ideal $J = \langle ut - x,\, 1 - u - y,\, u + t - ut - z\rangle$ and find a Gröbner basis G relative to lex order with $t > u > x > y > z$. The result is

$$G = \{1 - 2\,y + x\,y + y^2 - z + y\,z, -1 + u + y, 1 + t - x - y - z\}$$

The surface S lies on the variety $V(1 - 2\,y + x\,y + y^2 - z + y\,z)$ and every one of its points satisfies

$$1 - 2\,y + x\,y + y^2 - z + y\,z = 0 \tag{2.8.7.b}$$

(c) Show $V = S$ (that is, show that every point of the variety V can be obtained by substituting some values for t, u in the equations of part (a).) Hint: Try to "solve" the implicit equation of V for one variable as a function of the other two.

Solution. The problem is to show that every (x, y, z) which satisfies (2.8.7.b) can be written in the form (2.8.7.a) for suitable choice of t, u. So suppose (x, y, z) do indeed satisfy (2.8.7.b). First, notice that if $y = 1$, then (2.8.7.b) reads $x = 0$ and z is unrestricted.

(Case I:) $y \neq 1$. Put

$$\begin{aligned} u &= 1 - y, \\ t &= \frac{x}{1 - y}. \end{aligned} \tag{2.8.7.c}$$

(2.8.7.a) then says z must be

$$z = (1 - y) + \frac{x}{1 - y} - x = \frac{(1 - y)^2 + x - x(1 - y)}{1 - y} = \frac{1 + xy - 2y + y^2}{1 - y}$$

in exact agreement with (2.8.7.b).

(Case II:) $y = 1$. In this case put $u = 0$ and $t = z$. If $y = 1$ the points on (2.8.7.b) have the form $(0, 1, z)$ and can be obtained form (2.8.7.a) with $u = 0$, $t = z$. The conclusion is that every point whose coordinates satisfy (2.8.7.b) can be obtained in the form (2.8.7.a).

Remark. It is pretty easy to eliminate t, u from (2.8.7.a) directly without using Gröbner bases.

$\boxed{\S\mathbf{2.8.8.}}$

Some parametric curves and surfaces are algebraic varieties even when the given parametrizations involve transcendental functions such as sin and cos. In this problem we will see that the parametric surface T,

$$\begin{aligned} x &= (2 + \cos t)\cos u, \\ y &= (2 + \cos t)\sin u, \\ z &= \sin t \end{aligned}$$

lies on an affine variety in $\mathbf{R}^3$.

(a) Draw a picture of T. Hint: Use cylindrical coordinates.

Solution. From the first two equations we find easily that $x^2+y^2 = (2+\cos t)^2$. From this we see immediately that T lies on a surface of revolution about the z-axis which we will call S. Since $\cos t = \pm\sqrt{1-z^2}$, any point of T satisfies either

$$x^2 + y^2 = (2 + \sqrt{1 - z^2})^2$$

or

$$x^2 + y^2 = (2 - \sqrt{1 - z^2})^2$$

Since we are working with a surface of revolution we can restrict our plot to the case where $x > 0$ and $y = 0$. Then the points of the trace of S on the part of the xz-plane where $x > 0$ satisfy either (i) $x = 2 + \sqrt{1 - z^2}$ or (ii) $x = 2 - \sqrt{1 - z^2}$. The union of the arcs (i) and (ii) is the locus $(x - 2)^2 = 1 - z^2$. It is a circle of radius 1 centered at the point $x = 2$, $z = 0$ on the xz-plane. The upshot of all this is that T lies on the surface of a torus, the torus obtained by rotating the circle mentioned in the preceding paragraph about the z-axis.

Once we have made the observation about T being on a torus it is easy to see where this parametrization comes from and that in fact the parametrization traces over this entire torus infinitely many times. u is just the usual cylindrical, or spherical, θ. To describe t let $\overrightarrow{P}$ be the position vector to a point on T. Let $\overrightarrow{H}$ be the projection of $\overrightarrow{P}$ onto the xy-plane. Let $\overrightarrow{C} = 2 \cdot \frac{\overrightarrow{H}}{\|\overrightarrow{H}\|}$ Then t is the angle between $\overrightarrow{C}$ or, equivalently $\overrightarrow{H} - \overrightarrow{C}$ and $\overrightarrow{P} - \overrightarrow{C}$.

(b) Let $a = \cos t$, $b = \sin t$, $c = \cos u$, $d = \sin u$, and rewrite the above equations as polynomial equations in a, b, c, d, x, y, z.

Solution.

$$\begin{aligned} (2 + a)c - x &= 0, \\ (2 + a)d - y &= 0, \\ b - z &= 0. \end{aligned}$$

(c) The pairs a, b and c, d are not independent since there are additional polynomial identities

$$a^2 + b^2 - 1 = 0, \qquad\qquad c^2 + d^2 - 1 = 0$$

stemming from the basic trigonometric identity. Form a system of five equations by adjoining the above equations to those from part (b) and compute a Gröbner basis for the corresponding ideal. Use the lex ordering with $a > b > c > d > x > y > z$. There should be exactly one polynomial in your basis that depends only on x, y, z. Setting this equal to zero gives the equation of a variety containing T.

Solution. Consider the ideal $I = \langle (2+a)c - x,\, (2+a)d - y,\, b - z,\, a^2 + b^2 - 1,\, c^2 + d^2 - 1 \rangle$. We get a "reduced" Gröbner basis G for I relative to lex order with variables ordered $a > b > c > d > x > y > z$ using Mathematica 3.01 as

$$\begin{aligned} G = \{ & 9 - 10\,x^2 + x^4 - 10\,y^2 + 2\,x^2\,y^2 + y^4 + 6\,z^2 + 2\,x^2\,z^2 + 2\,y^2\,z^2 + z^4, \\ & 12\,d - 13\,y + x^2\,y + y^3 + 4\,d\,z^2 + y\,z^2, \\ & 4\,d\,x^2 - 3\,y - x^2\,y + 4\,d\,y^2 - y^3 - y\,z^2, \\ & 12\,c - 13\,x + x^3 + x\,y^2 + 4\,c\,z^2 + x\,z^2, \\ & -(d\,x) + c\,y, \\ & -3 + 4\,c\,x - x^2 + 4\,d\,y - y^2 - z^2, \\ & -1 + c^2 + d^2, b - z, \\ & 5 + 4\,a - x^2 - y^2 - z^2 \} \end{aligned}$$

T lies on the variety

$$9 - 10\,x^2 + x^4 - 10\,y^2 + 2\,x^2\,y^2 + y^4 + 6\,z^2 + 2\,x^2\,z^2 + 2\,y^2\,z^2 + z^4 = 0.$$

which can be rewritten in the form

$$(x^2 + y^2 + z^2 + 3)^2 = 16(x^2 + y^2). \tag{2.8.8.c}$$

The equation of the torus in the language at the end of part (b) is $\|\overrightarrow{P} - \overrightarrow{C}\| = 1$ which with $\overrightarrow{P} = (x, y, z)$ becomes

$$\left(x - 2\frac{x}{\sqrt{x^2+y^2}} \right)^2 + \left(y - 2\frac{y}{\sqrt{x^2+y^2}} \right)^2 + z^2 = 1. \tag{2.8.8.c1}$$

It is easy to check that (2.8.8.c1) and (2.8.8.c) are equivalent equations verifying that indeed (2.8.8.c) is the equation of the asserted torus.

$\boxed{\S\mathbf{2.8.10.}}$

Use the method of Lagrange Multipliers to find the point(s) on the surface $x^4 + y^2 + z^2 - 1 = 0$ closest to the point $(1, 1, 1)$ in $\mathbf{R}^3$.

Solution. To find the minimum (and maximum) points of $(x-1)^2 + (y-1)^2 + (z-1)^2$ subject to the constraint $x^4 + y^2 + z^2 - 1 = 0$ we first find the critical point(s) of

$$F(\lambda, x, y, z) = (x-1)^2 + (y-1)^2 + (z-1)^2 - \lambda \left(x^4 + y^2 + z^2 - 1 \right).$$

These are the solution(s) of

$$\begin{aligned} F_\lambda &= x^4 + y^2 + z^2 - 1 = 0, \\ F_x &= 2(x-1) - 4\lambda x^3 = 0, \\ F_y &= 2(y-1) - 2\lambda y = 0, \\ F_z &= 2(z-1) - 2\lambda z = 0. \end{aligned}$$

To "eliminate" λ from these equations we find a Gröbner basis for the ideal

$$I = \langle x^4 + y^2 + z^2 - 1,\, 2(x-1) - 4\lambda x^3,\, 2(y-1) - 2\lambda y,\, 2(z-1) - 2\lambda z \rangle$$

relative to lex order with $\lambda > x > y > z$. This Gröbner basis is

$$
\begin{aligned}
&-8+32\,z+4\,z^2-172\,z^3+176\,z^4+232\,z^5-471\,z^6+96\,z^7+304\,z^8-256\,z^9+64\,z^{10},\\
&y-z,\\
&-68+4\,x+156\,z+252\,z^2-840\,z^3+54\,z^4+1285\,z^5-896\,z^6-400\,z^7+640\,z^8-192\,z^9,\\
&4+8\,\lambda+4\,z-172\,z^2+176\,z^3+232\,z^4-471\,z^5+96\,z^6+304\,z^7-256\,z^8+64\,z^9.
\end{aligned}
$$

The first three of these polynomials don't involve λ; so setting them equal to zero gives the critical points of F. That is the critical points of F are the solutions of

$$
\begin{aligned}
-8+32\,z+4\,z^2-172\,z^3+176\,z^4+232\,z^5-471\,z^6+96\,z^7+304\,z^8-256\,z^9+64\,z^{10}&=0,\\
y-z&=0,\\
-68+4\,x+156\,z+252\,z^2-840\,z^3+54\,z^4+1285\,z^5-896\,z^6-400\,z^7+640\,z^8-192\,z^9&=0.
\end{aligned}
$$

There are just two real roots (found numerically using Mathematica's NSolve command). They are

$$
\begin{gathered}
(-0.686761\ldots,\ -0.62352\ldots,\ -0.62352\ldots)\\
(0.663676\ldots,\ 0.634819\ldots,\ 0.634819\ldots)
\end{gathered}
$$

The second of these is clearly the closest to (1,1,1) yielding a distance of $0.616302\ldots$.

Alternative Solution. Using the fact that $x^4+y^2+z^2=1$ is a surface of revolution about the x-axis, and that the distance from $(1,1,1)$ to the x-axis is $\sqrt{2}$, this problem is seen to be equivalent to the ordinary calculus problem: Find the distance of the curve with equation $x^4+y^2=1$ from the point $(1,\sqrt{2})$. To solve this we set up the condition that the normal line to $y=\sqrt{1-x^4}$ at the point $(x,\sqrt{1-x^4})$ passes through the point $(1,\sqrt{2})$ and use Mathematica 3.01 to search for a solution between 0.4 and 1.0 starting at 0.5. This gives rise to the Mathematica input
FindRoot[{(Sqrt[2]-Sqrt[1-x^4])/(1-x)==(Sqrt[1-x^4])/(2*x^3)},{x,0.5,0.4,1.0}]
and produces the output $(x->0.663676\ldots)$ checking the value for x obtained previously.

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§2.8.11.

Suppose we have numbers a,b,c which satisfy the equations

$$
\begin{aligned}
a+b+c&=3,\\
a^2+b^2+c^2&=5,\\
a^3+b^3+c^3&=7.
\end{aligned}
$$

(a) Prove that $a^4+b^4+c^4=9$. Hint: Regard a,b,c as variables and show carefully that

$$a^4+b^4+c^4-9\in\langle a+b+c-3,\ a^2+b^2+c^2-5,\ a^3+b^3+c^3-7\rangle.$$

Solution. The command sequence
H=GroebnerBasis[{a^3+b^3+c^3-7,a^2+b^2+c^2-5,a+b+c-3}]
PolynomialReduce[a^4+b^4+c^4-9,H,{a,b,c}]
in Mathematica 3.01 produces the output

$$
\begin{aligned}
&\{\{3+\frac{a}{3}+\frac{b}{3}+\frac{c}{3},14+3\,a+a^2+3\,b+b^2-6\,c-a\,c-b\,c,\\
&\frac{43}{3}+7\,a+3\,a^2+a^3-7\,b-3\,a\,b-a^2\,b-7\,c-3\,a\,c-a^2\,c+3\,b\,c+a\,b\,c\},0\}
\end{aligned}
$$

From which it follows that the value of $a^4+b^4+c^4-9$ is zero.

(b) Show that $a^5+b^5+c^5\neq 11$.

Solution. The command sequence
H=GroebnerBasis[{a^3+b^3+c^3-7,a^2+b^2+c^2-5,a+b+c-3}]
PolynomialReduce[a^5+b^5+c^5,H,{a,b,c}]
in Mathematica 3.01 produces the output

$$\begin{aligned}&\{\{7+a+\frac{a^2}{3}+b+\frac{b^2}{3}+c+\frac{c^2}{3},\\ &\frac{86}{3}+7\,a+3\,a^2+a^3+7\,b+3\,b^2+b^3-14\,c-3\,a\,c-a^2\,c-3\,b\,c-b^2\,c,\\ &27+\frac{43\,a}{3}+7\,a^2+3\,a^3+a^4-\frac{43\,b}{3}-7\,a\,b-3\,a^2\,b-a^3\,b-\frac{43\,c}{3}-7\,a\,c-3\,a^2\,c\\ &-a^3\,c+7\,b\,c+3\,a\,b\,c+a^2\,b\,c\},\\ &\frac{29}{3}\}\end{aligned}$$

From which it follows that the value of $a^5+b^5+c^5$ is $\frac{29}{3}$.

(c) What are $a^5+b^5+c^5$ and $a^6+b^6+c^6$? Hint: Compute remainders.

Solution. The command sequence
H=GroebnerBasis[{a^3+b^3+c^3-7,a^2+b^2+c^2-5,a+b+c-3}]
PolynomialReduce[a^6+b^6+c^6,H,{a,b,c}]
in Mathematica 3.01 produces the output

$$\begin{aligned}&\{\{\frac{43}{3}+\frac{7\,a}{3}+a^2+\frac{a^3}{3}+\frac{7\,b}{3}+b^2+\frac{b^3}{3}+\frac{7\,c}{3}+c^2+\frac{c^3}{3},\\ &54+\frac{43\,a}{3}+7\,a^2+3\,a^3+a^4+\frac{43\,b}{3}+7\,b^2+3\,b^3+b^4-\frac{86\,c}{3}-7\,a\,c-3\,a^2\,c-a^3\,c-7\,b\,c-3\,b^2\,c-b^3\,c,\\ &\frac{143}{3}+27\,a+\frac{43\,a^2}{3}+7\,a^3+3\,a^4+a^5-27\,b-\frac{43\,a\,b}{3}-7\,a^2\,b-3\,a^3\,b-a^4\,b-27\,c-\frac{43\,a\,c}{3}-7\,a^2\,c\\ &\qquad-3\,a^3\,c-a^4\,c+\frac{43\,b\,c}{3}+7\,a\,b\,c+3\,a^2\,b\,c+a^3\,b\,c\},\\ &\frac{19}{3}\}\end{aligned}$$

From which it follows that the value of $a^6+b^6+c^6$ is $\frac{19}{3}$.