Lines, Planes, and Hyperplanes.

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April 22, 2015

Question

What defines a line?

Answer

Question

What defines a line?

Answer

1. two points,

Question

What defines a line?

Answer

1. two points,
2. point and direction/slope.

Definition (Vector Equations of a Line)

Let $\mathbf{r}_0$ (position) and $\mathbf{v}$ (direction/slope) be vectors of $\mathbb{R}^n$ and $t \in \mathbb{R}$ a scalar. Then

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

is the vector equation of a line.

This is also called the parametric form of the line because the line's points are parameterized by t.

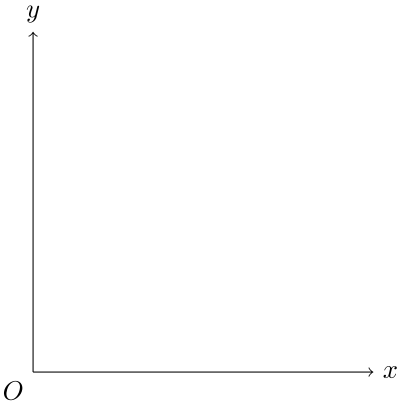
Definition (Direction Numbers)

When $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ the components of the line $\mathbf{v}$ are called the direction numbers of L.

Note any vector parallel to $\mathbf{v}$ could be used to define the same line.

Example

The line L as described by two vectors (i.e. a point $\mathbf{r}_0$ and direction $\mathbf{v}$).



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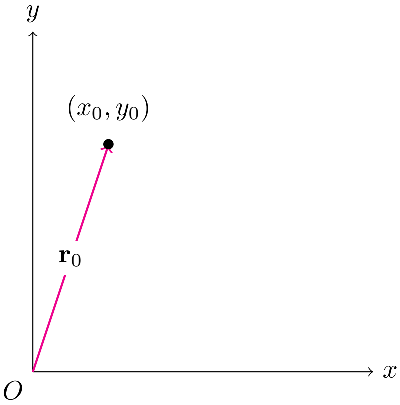

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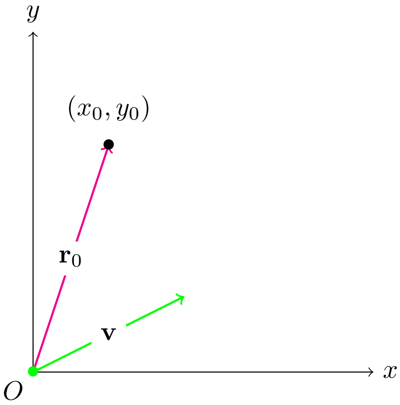

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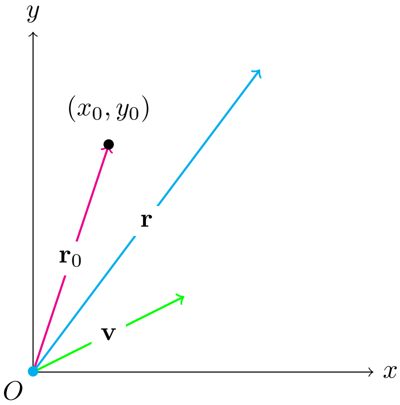

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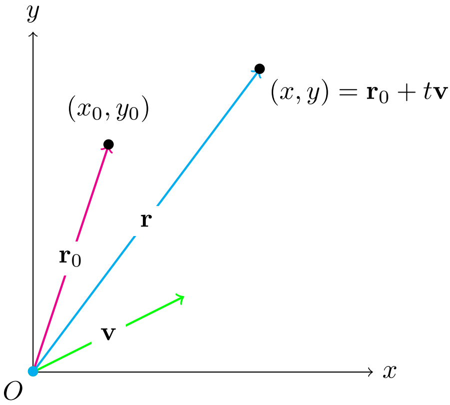

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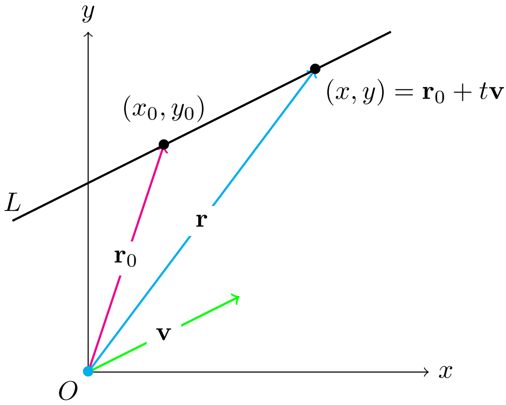

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What is the parametric equation for the line with slope 2 going through $(3, 2)$?

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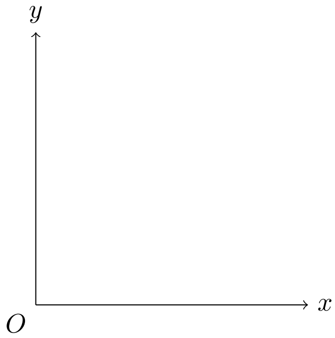
$$\mathbf{r} = \langle 3, 2 \rangle + t\langle 1, 2 \rangle$$



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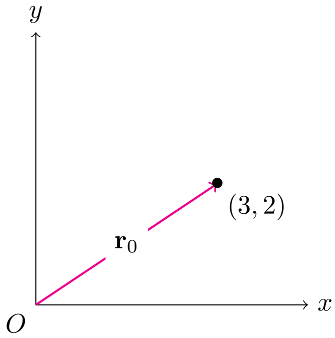

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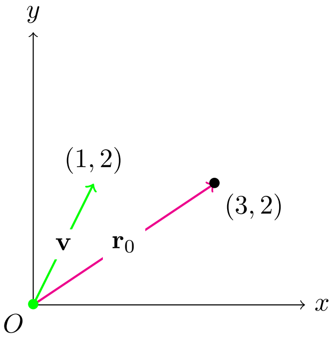

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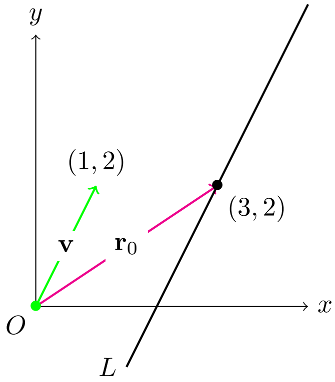
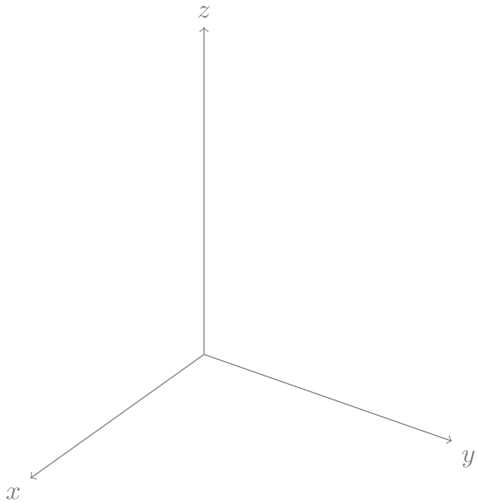


Extending to $\mathbb{R}^3$



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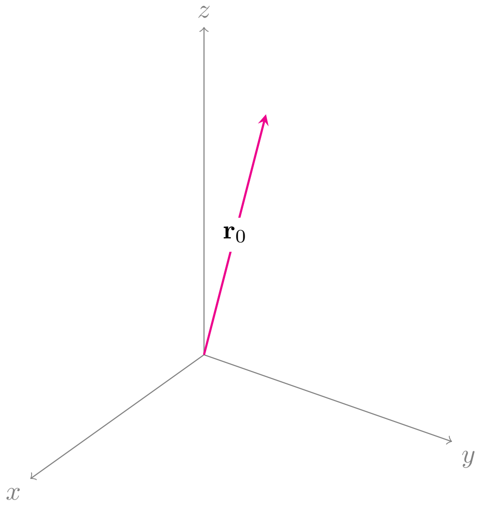

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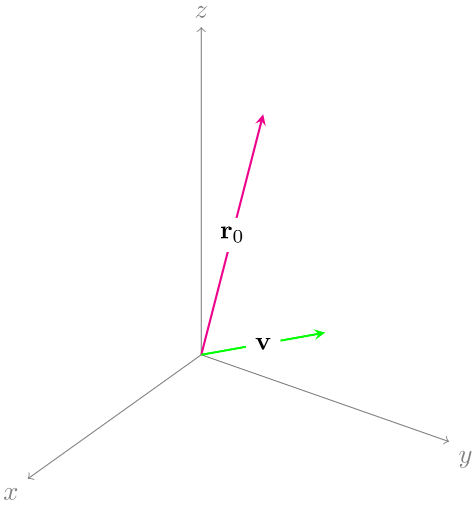

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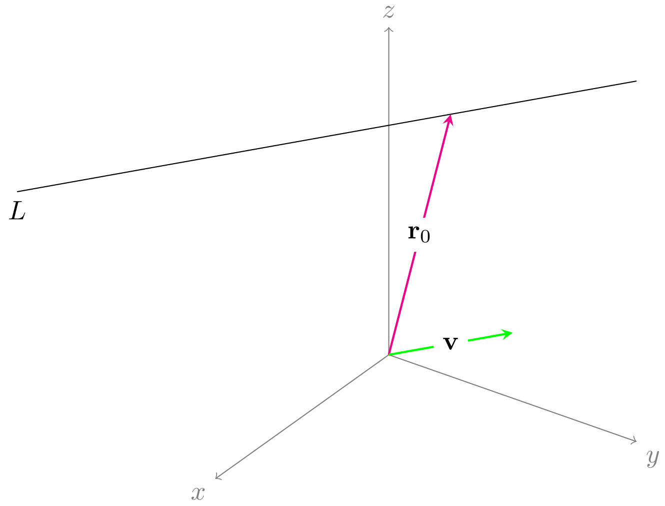

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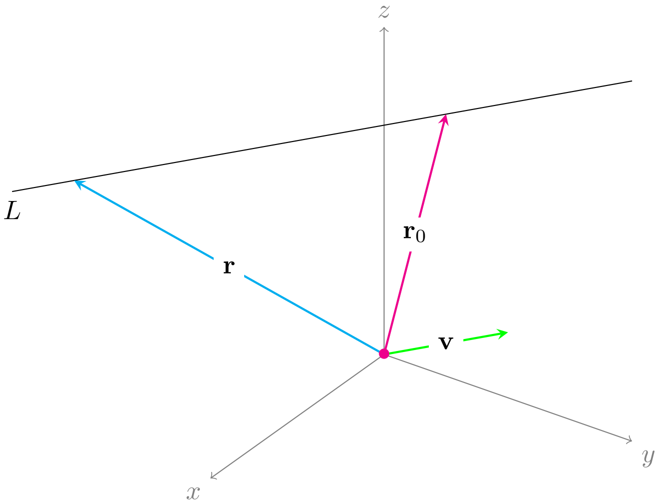

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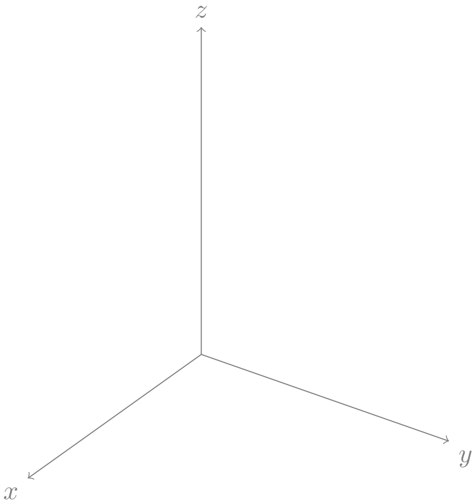

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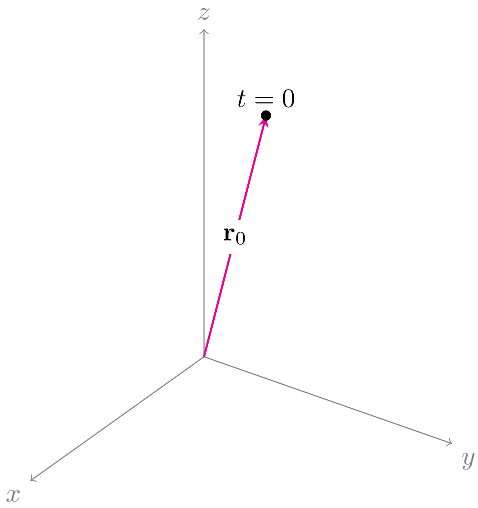

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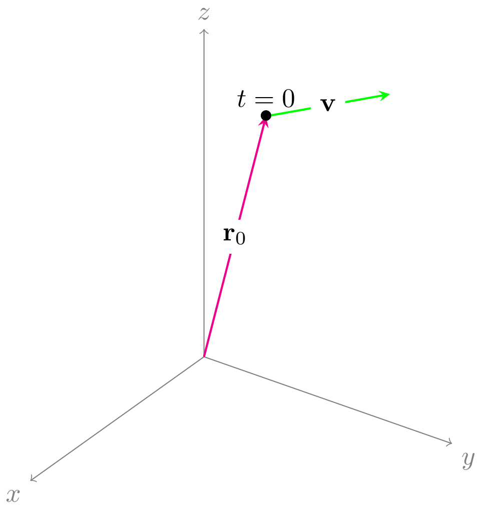

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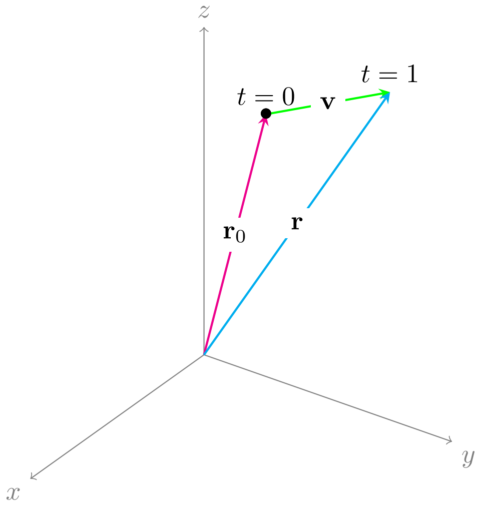

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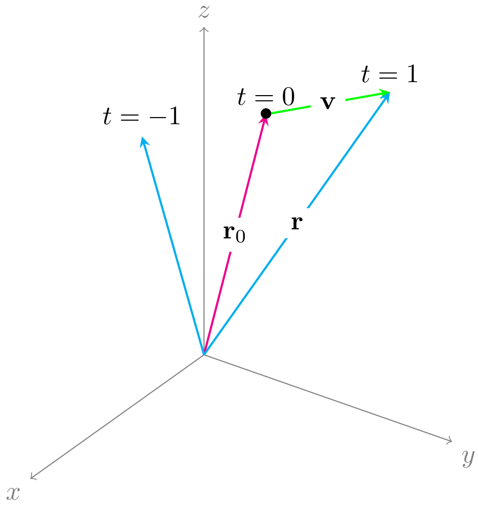

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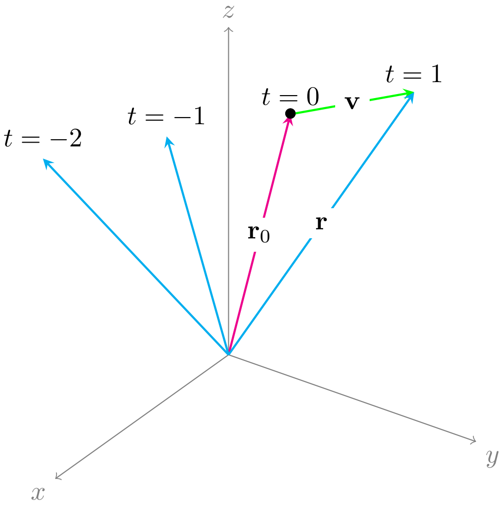

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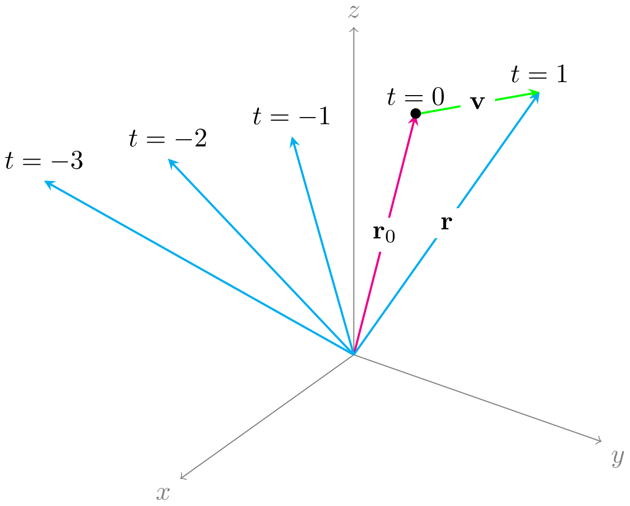

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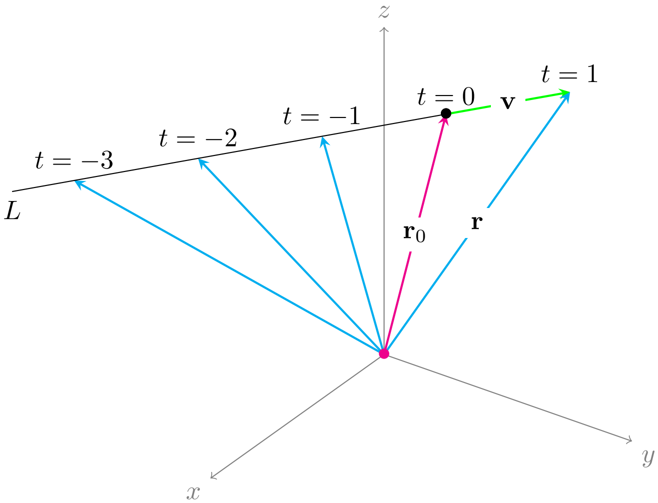

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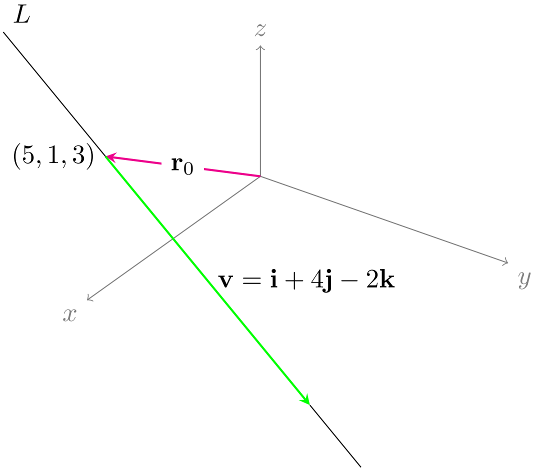
Question

Find a vector equation and parametric equations for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

Find two other points on the line.

Example



Answer

Here $\mathbf{r}_0 = \langle 5, 1, 3 \rangle$ and $v = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k} = \langle 1, 4, -2 \rangle$.

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

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Here $\mathbf{r}_0 = \langle 5, 1, 3 \rangle$ and $v = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k} = \langle 1, 4, -2 \rangle$.

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$$\begin{aligned} \mathbf{r} &= \mathbf{r}_0 + t\mathbf{v} \\ &= \langle 5, 1, 3 \rangle + t\langle 1, 4, -2 \rangle \end{aligned}$$

Two other points are given by $t = -1$ and $t = 1$:

$$\langle 4, -3, 5 \rangle \qquad \langle 6, 5, 1 \rangle.$$

Symmetric Equation

Notice $\mathbf{r} = \langle 5, 1, 3 \rangle + t\langle 1, 4, -2 \rangle$ is equivalent to

$$\langle x, y, z \rangle = \langle 5, 1, 3 \rangle + t\langle 1, 4, -2 \rangle$$

which means we have the set of equations

$$x = 5 + 2t \qquad y = 1 + 8t \qquad z = 3 - 4t.$$

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Solving for t gives

$$t = \frac{x-5}{2} \qquad t = \frac{y-1}{8} \qquad t = \frac{z-3}{-4}.$$

and therefore

$$\frac{x-5}{2} = \frac{y-1}{8} = \frac{z-3}{-4}$$

is another description of the line.

In three space, when $a, b, c \neq 0$,

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

defines a line through (x_0, y_0, z_0) with slope $\langle a, b, c \rangle$.

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Definition (Symmetric Form of a Line)

In $\mathbb{R}^n$ when $\mathbf{x} = \langle x_0, \ldots, x_{n-1} \rangle$ is a vector-valued variable, $\mathbf{p} = \langle p_0, \ldots, p_{n-1} \rangle$ is a fixed, and $\mathbf{a} = \langle a_0, \ldots, a_{n-1} \rangle \in (\mathbb{R} - \{0\})^n$ then

$$\frac{x_0 - p_0}{a_0} = \cdots = \frac{x_{n-1} - p_{n-1}}{a_{n-1}}$$

defines a line through $\mathbf{p}$ with direction $\mathbf{a}$.

Question

Find the parametric and symmetric equations of the line through $(2, 4, -3)$ and $(3, -1, 1)$.

Where does this line intersect the xy-plane?

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Answer

We are not explicitly given a direction vector but notice

$$\mathbf{v} = \langle 3, -1, 1 \rangle - \langle 2, 4, -3 \rangle = \langle 1, -5, 4 \rangle$$

is the direction of the line. We need only pick either $(2, 4, -3)$ or $(3, -1, 1)$ as $\mathbf{r}_0$.

Answer (Continued)

Therefore the parametric equation of the line is given by

$$\langle x, y, z\rangle = \langle 2, 4, -5\rangle + t\langle 1, -5, 4\rangle$$

for $t \in \mathbb{R}$ a parameter.

Answer (Continued)

Therefore the parametric equation of the line is given by

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The symmetric equation is

$$\frac{x-2}{1} = \frac{y-4}{-5} = \frac{z+3}{4}.$$

and thus when in the xy-plane where $z = 0$, x and y are given by

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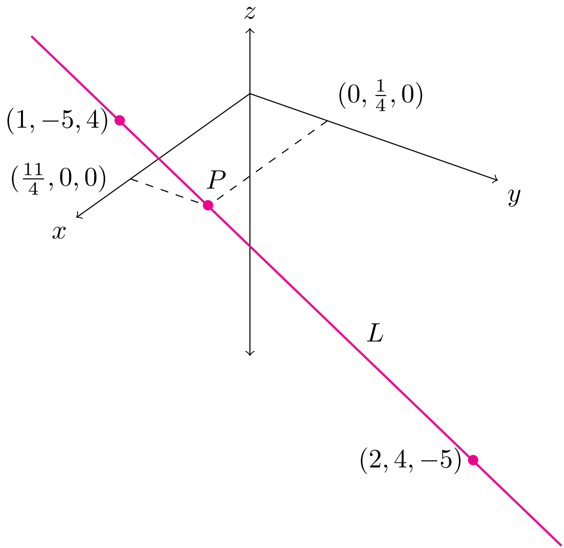
$$\frac{x-2}{1} = \frac{y-4}{-5} = \frac{3}{4}$$

which implies

$$x = \frac{11}{4} \qquad\qquad y = \frac{1}{4}.$$

Answer (Continued)

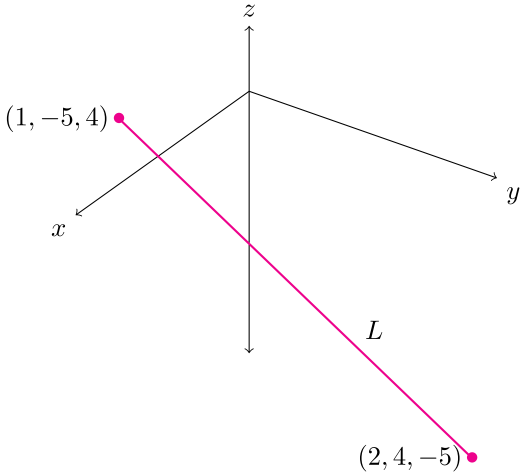
The line intersects the xy axis when $z = 0$.



Line Segments

We can also use parameterized curves to describe line segments:

$$\mathbf{r}(t) = \langle 2 + t,\, 4 - 5t,\, -3 + 4t \rangle \qquad\qquad t \in [0, 1]$$



Line Segments

Proposition

The line through the (tail of the) vectors $\mathbf{r}_0$ and $\mathbf{r}_1$ is given by

$$\mathbf{r}(t) = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0)$$

where the line segment given by $\mathbf{r}_0$ and $\mathbf{r}_1$ is in the interval $t \in [0, 1]$.

Definition (Skew)

Two lines L_0 and L_1 are skew when they do not intersect and are not parallel.

Example

Show that the lines L_0 (parameterized by t) and L_1 (parameterized by s) with the parametric equations

$$x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$

$$x = 2s \qquad y = 3 + s \qquad z = -3 - 4s$$

are skew.

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The corresponding direction vectors for L_0 and L_1 are

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Answer

The corresponding direction vectors for L_0 and L_1 are

$$\langle 1, 3, -1 \rangle \qquad \langle 2, 1, 4 \rangle$$

which are not scalar multiples of one another — thus the lines cannot be parallel.

It remains to show the lines do not intersect.

Answer (Continued)

Towards a contradiction, suppose the lines do have a point of intersection given by

$$\begin{aligned} 1 + t &= 2s \\ -2 + 3t &= 3 + s \\ 4 - t &= -3 + 4s \end{aligned}$$

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Notice substituting the second ($s = -5 + 3t$) into the first gives

$$1 + t = -10 + 6t \implies$$

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$$1 + t = -10 + 6t \implies 5t = 11 \implies$$

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$$1 + t = -10 + 6t \implies 5t = 11 \implies t = \frac{11}{5}$$

which means $s =$

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which means $s = \frac{1+t}{2} = \frac{8}{5}$.

Answer (Continued)

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$$1 + t = 2s$$

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$$1 + t = -10 + 6t \implies 5t = 11 \implies t = \frac{11}{5}$$

which means $s = \frac{1+t}{2} = \frac{8}{5}$.

This implies, by the third equation, that

$$4 - \frac{11}{5} = 2\frac{8}{5} \implies \frac{-39}{20} = 0 \quad \lightning$$

Question

What is defined by

1. three points, or
2. point and two directions?

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Answer

A plane.

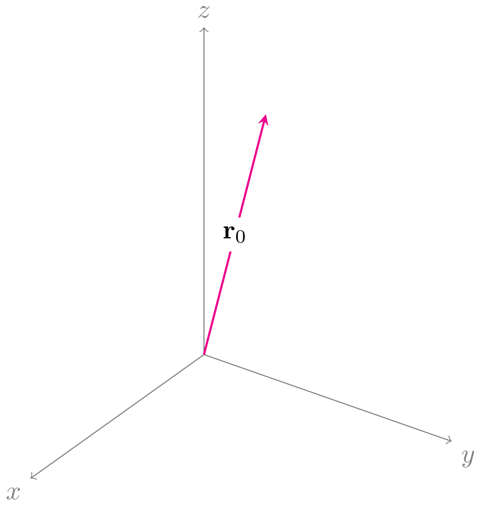
Definition (Parametric Equation of Plane)

Let $\mathbf{r}_0$ (position) and $\mathbf{v}_0, \mathbf{v}$ (direction) be vectors of $\mathbb{R}^n$ and $s, t \in \mathbb{R}$ a scalar. Then

$$\mathbf{r}(s,t) = \mathbf{r}_0 + s\mathbf{v}_0 + t\mathbf{v}_1$$

defines a plane in $\mathbb{R}^n$.

Example



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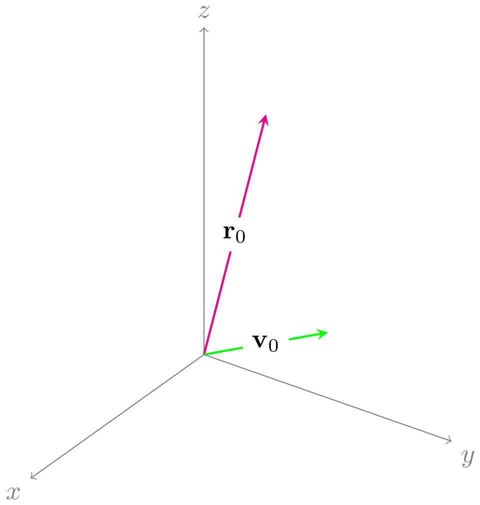

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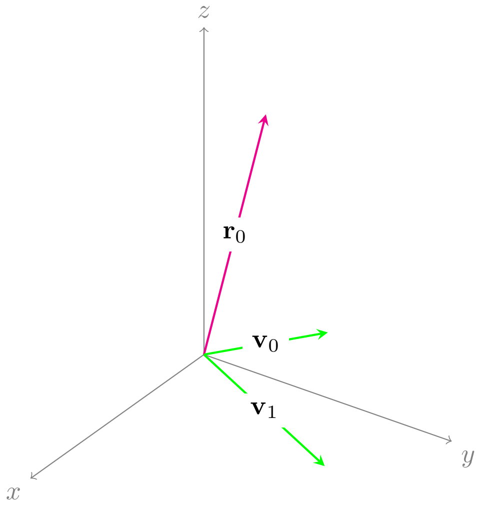

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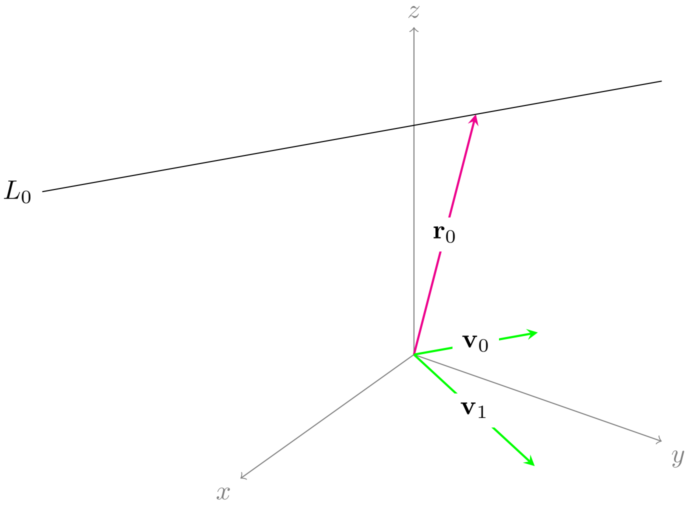

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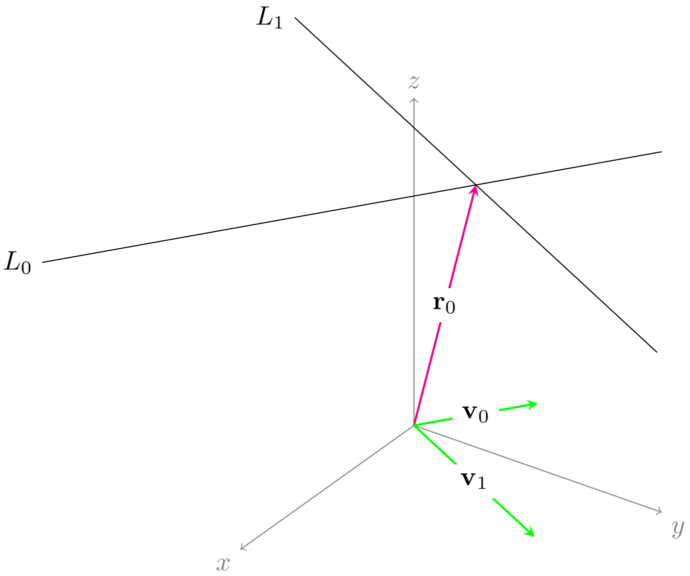

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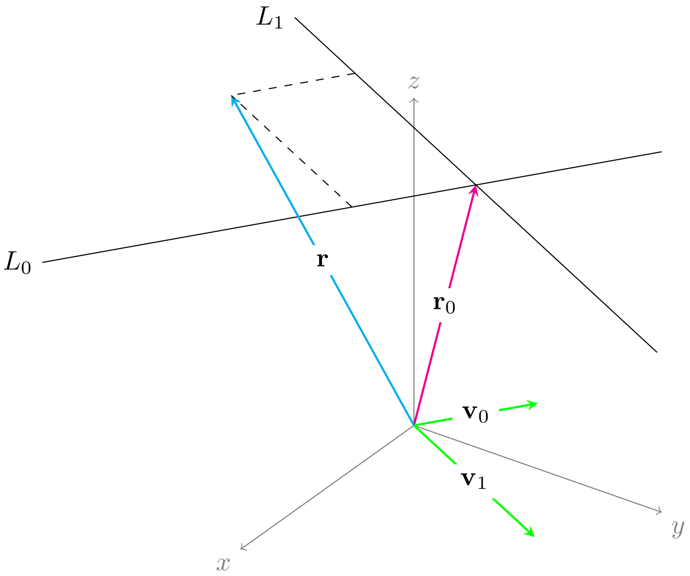
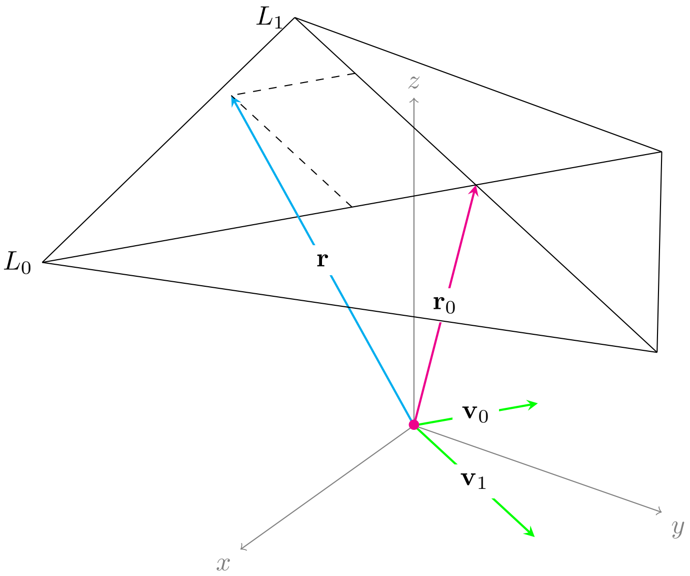


Example



Notice however, that the two vectors $\mathbf{v}_0$ and $\mathbf{v}_1$ uniquely (up to scalar multiple) define a cross product and that this cross product can instead be used to define the vector.

This cross product is called the normal of the plane given by $\mathbf{v}_0$ and $\mathbf{v}_1$. We denote by

$$\mathbf{n} = \mathbf{v}_0 \times \mathbf{v}_1.$$

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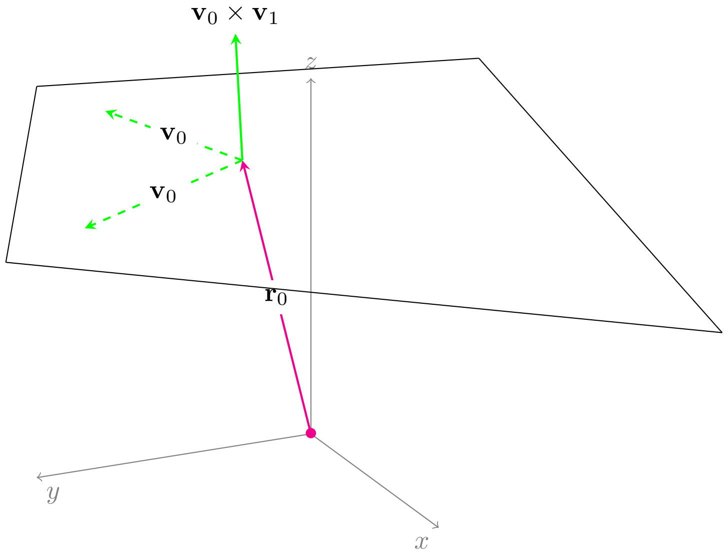
Note every vector in the plane is orthogonal to this normal vector.

Example



Definition (Vector Equation of the Plane)

Let $\mathbf{n}$ be a vector and $\mathbf{r}_0$ be a fixed position vector. The plane through $\mathbf{r}_0$ with normal $\mathbf{n}$ is given by

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

for vector-valued variable $\mathbf{r}$.

Equivalently we may also write

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

for the plane.

Scalar Equation of the Plane

To obtain a scalar equation for the plane write

$$\mathbf{n} = \langle a, b, c \rangle \qquad \mathbf{r} = \langle x, y, z \rangle \qquad \mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$$

and recall the vector equation of the plane is given by $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0)$ or

$$\langle a, b, c \rangle \cdot \mathbf{r} = \langle x - x_0, y - y_0, z - z_0 \rangle.$$

Scalar Equation of the Plane

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$$\mathbf{n} = \langle a,\, b,\, c\rangle \qquad\qquad \mathbf{r} = \langle x,\, y,\, z\rangle \qquad\qquad \mathbf{r}_0 = \langle x_0,\, y_0,\, z_0\rangle$$

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$$\langle a,\, b,\, c\rangle \cdot \mathbf{r} = \langle x - x_0,\, y - y_0,\, z - z_0\rangle.$$

Expanding yields

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

as the equation of the plane in $\mathbb{R}^3$ — a kind of "point-normal" analogue of the point-slope equation for the line.

Definition (Scalar Equation of the Plane)

The scalar equation of the plane through (x_0, y_0, z_0) with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is given by

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

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Definition (Linear Equation of the Plane)

Let a, b, c, d be reals and x, y, z real valued variables. Then

$$ax + by + cz + d = 0$$

defines the plane in three-space.

Question

Consider the plane given by

$$2(x-1) - 3(y-2) + 7(z) = 0.$$

1. What is this planes normal?

Question

Consider the plane given by

$$2(x-1)-3(y-2)+7(z)=0.$$

1. What is this planes normal? $\langle 2,-3,7\rangle$

Question

Consider the plane given by

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1. What is this planes normal? $\langle 2, -3, 7 \rangle$
2. Give a point where this plane passes through.

Question

Consider the plane given by

$$2(x-1)-3(y-2)+7(z)=0.$$

1. What is this planes normal? $\langle 2,-3,7\rangle$
2. Give a point where this plane passes through. $(1,2,0)$.

Question

Find an equation of the plane through the point $(2, 4, -1)$ with normal $\mathbf{n} = \langle 2, 3, 4 \rangle$. Find where this plane intersects the x, y, and z axis and sketch the plane.

Question

Find an equation of the plane through the point $(2, 4, -1)$ with normal $\mathbf{n} = \langle 2, 3, 4 \rangle$. Find where this plane intersects the x, y, and z axis and sketch the plane.

Answer

Trivially, the plane is given by $\mathbf{n} \cdot \langle x-2, y-4, z+1 \rangle$ or equivalently

$$2(x-2) + 3(y-4) + 4(z-1) = 0$$

$$\implies 2x + 3y + 4z = 12.$$

Answer (Continued)

The x-intercept is found by setting $y = z = 0$ (and so on). Doing so yields

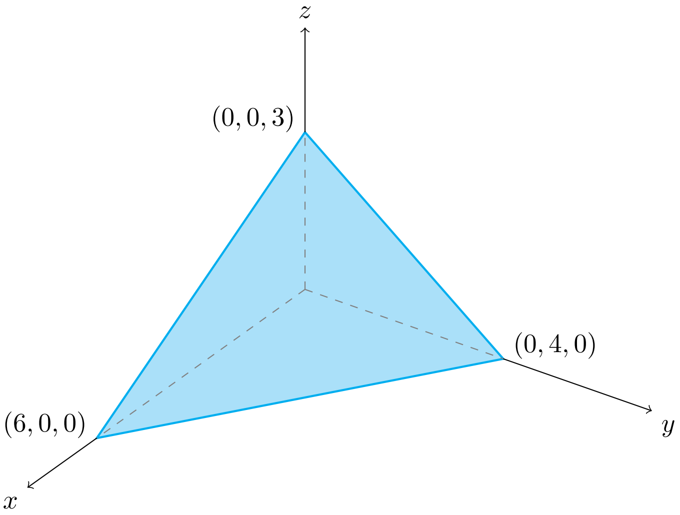
$$x = 6 \qquad\qquad y = 4 \qquad\qquad z = 3.$$

So we have the plane passes through

$$(6, 0, 0) \qquad\qquad (0, 4, 0) \qquad\qquad (0, 0, 3)$$

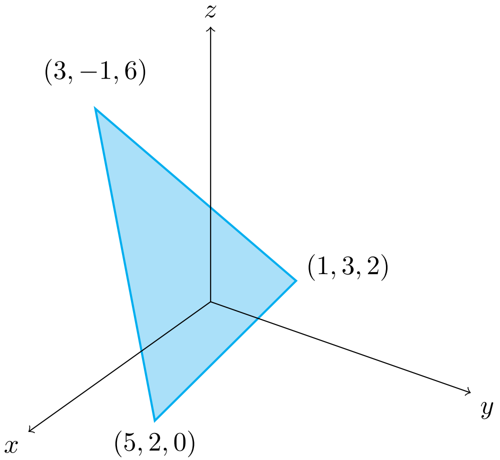
which we can sketch.

Answer (Continued)



Question

Find an equation of the plane that passes through the points $(1, 3, 2)$, $(3, -1, 6)$, and $(5, 2, 0)$.



Answer

We can get two (arbitrary) direction vectors from these three points. (The vectors should be given from the same tail.)

$$\mathbf{v}_0 = \langle 3, -1, 6\rangle - \langle 1, 3, 2\rangle = \langle 2, -4, 4\rangle$$

$$\mathbf{v}_1 = \langle 5, 2, 0\rangle - \langle 1, 3, 2\rangle = \langle 4, -1, -2\rangle$$

Answer (Continued)

The normal to the plane is then $\langle 2,-4,4\rangle \times \langle 4,-1,-2\rangle$ which we compute by

$$\begin{aligned}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} &= \begin{vmatrix} -4 & 4 \\ -1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 4 \\ 4 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -4 \\ 4 & -1 \end{vmatrix}\mathbf{k} \\ &= \langle 12, 20, 14\rangle.\end{aligned}$$

Answer (Continued)

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Choosing the point $(1, 3, 2)$ (though the other two are fine as well) we can give the equation of the plane as

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Choosing the point $(1, 3, 2)$ (though the other two are fine as well) we can give the equation of the plane as

$$12(x-1) + 20(y-3) + 12(z-2) = 0$$

Answer (Continued)

The normal to the plane is then $\langle 2, -4, 4\rangle \times \langle 4, -1, -2\rangle$ which we compute by

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Choosing the point $(1, 3, 2)$ (though the other two are fine as well) we can give the equation of the plane as

$$12(x-1) + 20(y-3) + 12(z-2) = 0$$

which simplifies to the linear equation

$$6x + 10y + 7z = 50.$$

Question

Find the point at which the line

$$x = 2 + 3t \qquad y = -4t \qquad z = 5 + t$$

intersects the plane $4x + 5y - 2z = 18$.

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Note

What are the types of intersections we can get here?

Answer

The point or line of intersection must be those points (x, y, z) satisfying both equations simultaneously. Thus we substitute the points of the line into the equation of the plane

$$4(2+3t)+5(-4t)-2(5+t)=18$$

and solve for t

$$t=-2.$$

Answer

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$$4(2+3t)+5(-4t)-2(5+t)=18$$

and solve for t

$$t=-2.$$

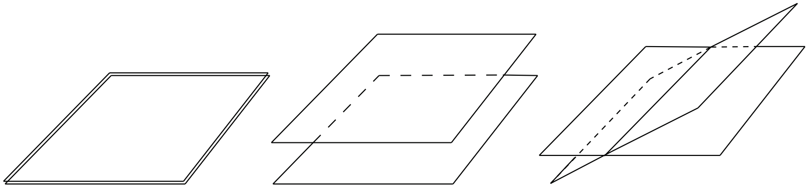
Thus the point or intersection is

$$(2+3(-2),\ -4(-2),\ 5+(-2))=(-4,\ 8,\ 3).$$

Definition

Tow planes are parallel if their normal vectors are parallel. (Note this does not preclude that the planes are identical.)

In fact, the only types of intersection the plane can have are:



Question

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$$x + 2y - 3z = 4 \qquad\qquad 2x + 4y - 6z = 3$$

parallel?

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Answer

Yes.

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Are the two planes given by

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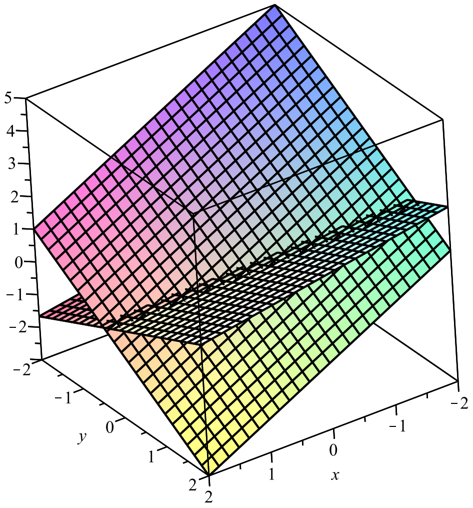
parallel?

Answer

Yes. The normals are $\langle 1, 2, -3\rangle$ and $\langle 2, 4, -6\rangle$ respectively — which are clearly parallel because they only differ by the scalar multiple 2.

Question

Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$ and then give the line of intersection as a symmetric equation.



(Recall: $x + y + z = 1$ and $x - 2y + 3z = 1$)

Answer

The normal vector of these planes are

$$\mathbf{n}_0 = \langle 1, 1, 1 \rangle \qquad\qquad \mathbf{n}_1 = \langle 1, -2, 3 \rangle$$

Notice the angle between the planes is the same as the angle between the normals which is given by

$$\begin{aligned}\cos\theta &= \frac{\mathbf{n}_0 \cdot \mathbf{n}_1}{|\mathbf{n}_0||\mathbf{n}_1|} \\ &= \frac{1(1) + 1(-2) + 1(3)}{\sqrt{1+1+1}\sqrt{1+4+9}} \\ &= \frac{2}{\sqrt{42}} \implies \theta \approx 72^\circ.\end{aligned}$$

Answer (Continued)

Now for the line of intersection.

Answer (Continued)

Now for the line of intersection.

Remember, every line of a plane is perpendicular to the plane's normal.

Thus a line in two planes must be perpendicular to both normals.

Answer (Continued)

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Remember, every line of a plane is perpendicular to the plane's normal. Thus a line in two planes must be perpendicular to both normals.

However, there is a unique (up to scalar multiple) vector $\mathbf{v}$ perpendicular to $\mathbf{n}_0$ and $\mathbf{n}_1$ and that is

$$\mathbf{v} = \mathbf{n}_0 \times \mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \langle 5, -2, -3 \rangle.$$

That is, this is the direction vector for the line! All we need now is a point.

(Recall: $x+y+z=1$ and $x-2y+3z=1$)

Answer (Continued)

As any point on both planes will do let us solve for when $z=0$ in both equations (i.e. a solution in the xy-plane). That is, we want a solution of

$$x+y-1=0 \qquad\qquad x-2y-1=0$$

(Recall: $x+y+z=1$ and $x-2y+3z=1$)

Answer (Continued)

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Subtracting the equations gives

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Answer (Continued)

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Subtracting the equations gives

$$3y = 0$$

and thus

$$(1, 0, 0)$$

is a point on both planes. (Check.)

Answer (Continued)

The line of intersection is given by

$$\langle x, y, z \rangle = \langle 1, 0, 0 \rangle + t \langle 5, -2, -3 \rangle$$

which corresponds to the symmetric equation

Answer (Continued)

The line of intersection is given by

$$\langle x, y, z \rangle = \langle 1, 0, 0 \rangle + t\langle 5, -2, -3 \rangle$$

which corresponds to the symmetric equation

$$\frac{x-1}{5} = \frac{y-0}{-2} = \frac{z-0}{-3}.$$

Alternate Definition of a Line

It stands to reason that we can define lines in three-space by the intersection of two planes.

Proposition

In general, the equation of a line given by

$$\frac{x - a_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

can be regarded to be the line of intersection of the two planes

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Question

Find a formula for the distance D from point $P = (x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

Question

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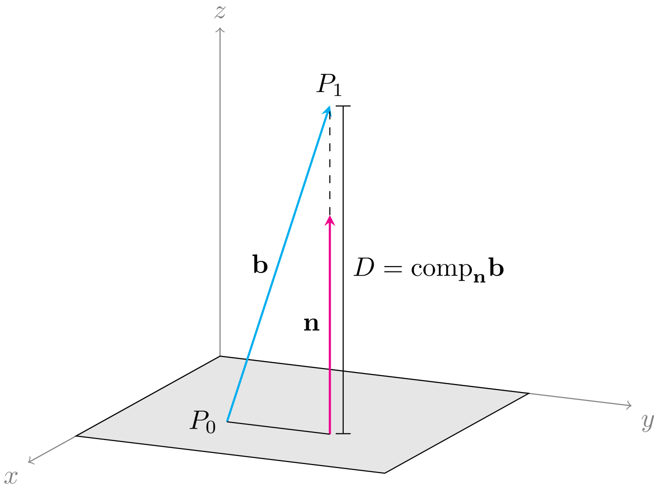
Answer

Let (x_0, y_0, z_0) be a point from the plane and let

$$\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

The shortest distance to the plane is given by the projection of the vector $\mathbf{b}$ into the normal $\mathbf{n} = \langle a, b, c \rangle$ of the plane.

Answer (Continued)



Answer (Continued.)

So, we need only calculate the component (i.e. the length of the projection) of $\mathbf{b} = \langle x_1 - x_0,\, y_1 - y_0,\, z_1 - z_0 \rangle$ onto $\mathbf{n} = \langle a,\, b,\, c \rangle$:

$$\begin{aligned} D &= |\mathrm{comp}_{\mathbf{n}} \mathbf{b}| \\ &= \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} && \text{By definition.} \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}. \end{aligned}$$

Answer (Continued)

We have calculated the distance from P_1 to any point P_0 on the plane with normal $\langle a, b, c \rangle$ is

$$D = \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}.$$

Answer (Continued)

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However, as we know (x_0, y_0, z_0) is on the plane we must have

$$ax_0 + by_0 + cz_0 + d = 0$$

and thereby $ax_0 + by_0 + cz_0 = -d$. (Notice that the point (x_0, y_0, z_0) has been eliminated from the equation!)

Answer (Continued)

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and thereby $ax_0 + by_0 + cz_0 = -d$. (Notice that the point (x_0, y_0, z_0) has been eliminated from the equation!)

Thus, this is the "distance to the plane" is

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Again with alegbra

Let us repeat this answer using the algebra rules instead of using explicit points. That is, we calculate the distance from arbitrary point $\mathbf{x}_1$ to the plane given by $ax + by + cz + d = 0$ with normal $\mathbf{n} = \langle a, b, c\rangle$

Let $\mathbf{x}_0$ lie on the plane (thus $\mathbf{n} \cdot \mathbf{x}_0 = -d$) and let $\mathbf{b} = \mathbf{x}_0 - \mathbf{x}_1$

$$
\begin{aligned}
D = |\mathrm{comp}_{\mathbf{n}}\mathbf{b}| &= \frac{|\mathbf{n}\cdot\mathbf{b}|}{|\mathbf{n}|} \\
&= \frac{\mathbf{n}\mathbf{x}_0 - \mathbf{n}\mathbf{x}_1}{|\mathbf{n}|} && \text{distribution} \\
&= \frac{|-d-\mathbf{n}\mathbf{x}_1|}{|\mathbf{n}|} \\
&= \frac{|\mathbf{n}\mathbf{x}_1 + d|}{|\mathbf{n}|}.
\end{aligned}
$$

Question

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Answer

First notice the normals are $\langle 10, 2, -2 \rangle$ and $\langle 5, 1, -1 \rangle$ which indeed correspond to parallel planes.

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We need only calculate the distance from any point on the first plane to the second plane.

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Answer

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We need only calculate the distance from any point on the first plane to the second plane.

We just devised a formula for this.

Answer (Continued.)

So, let us pick an arbitrary point on

$$10x + 2y - 2z = 5,$$

say $(\frac{1}{2}, 0, 0)$, and find its distance to the plane

$$5x + 1y - 1z = 1.$$

Answer (Continued.)

So, let us pick an arbitrary point on

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say $(\frac{1}{2}, 0, 0)$, and find its distance to the plane

$$5x + 1y - 1z = 1.$$

$$D = \frac{|(5)(\frac{1}{2}) + (1)(0) + (-1)(0) + (-1)|}{|\langle 5, 1, -z \rangle|} = \frac{3/2}{3\sqrt{3}} = \frac{\sqrt{3}}{6}.$$

Question

We previously showed the lines

$$
\begin{array}{llll}
L_0: & x = 1 + t & y = -2 + 3t & z = 4 - t \\
L_1: & x = 2s & y = 3 + s & z = -3 + 4s
\end{array}
$$

were skew. What then, is the distance between them?

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were skew. What then, is the distance between them?

Answer

If the lines L_0 and L_1 are skew then they be viewed as laying on two separate parallel planes P_0 and P_1. The distance between the lines is the same as the distance between the planes.

Answer (Continued.)

The normal of these planes, for them to be parallel, should be the cross-product of the line's direction vectors:

$$\langle 1, 3, -1 \rangle \times \langle 2, 1, 4 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 2 & 1 & 4 \end{vmatrix} = \langle 13, -6, -5 \rangle.$$

We're now free to choose a point from one line and calculate the distance to the plane given by the other line.

Answer (Continued.)

Recall the two lines are

$$\begin{array}{llll} L_0: & x = 1 + t & y = -2 + 3t & z = 4 - t \\ L_1: & x = 2s & y = 3 + s & z = -3 + 4s \end{array}$$

Setting $s = t = 0$ we see the point $(1, -2, 4)$ lies on L_0 and $(0, 3, -3)$ on L_1.

Answer (Continued.)

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Setting $s = t = 0$ we see the point $(1, -2, 4)$ lies on L_0 and $(0, 3, -3)$ on L_1.

The plane defined by L_1 is given by

$$13(x - 0) - 6(y - 3) - 5(z + 3) = 0$$

$$\implies 13x - 6y - 5z + 3 = 0$$

Answer (Continued.)

By our equation, the distance from the point $(1, -2, 4)$ to the plane $13x - 6y - 5z + 3 = 0$ is

$$D = \frac{|13(1) - 6(-2) - 5(4) + 3|}{|\langle 13, -6, -5 \rangle|} = \frac{8}{\sqrt{230}}$$

Next week.

Next week

- Midterm examination.

Week after next

- Matrices.
- Linear system solving.