Chapter 3 Elimination Theory

§1. The Elimination and Extension Theorems

We start with a definition.

Definition 3.0.1. (Definition 1) Given $I = \langle f_1, \ldots, f_s \rangle \subset k[x_1, \ldots, x_n]$, the ℓ-th *elimination ideal* I_ℓ is the ideal of $k[x_{\ell+1}, \ldots, x_n]$ defined by

$$I_\ell = I \cap k[x_{\ell+1}, \ldots, x_n].$$

This ideal I_ℓ consists of all consequences of $f_1 = f_2 = \cdots = f_s = 0$ which eliminate the variables $x_1, \ldots, x_\ell$. In solving the system of polynomial equations $f_1 = f_2 = \cdots = f_s = 0$, a *solution of the Elimination Step* means giving a systematic procedure for finding elements of the ℓ-th elimination ideal I_ℓ. With proper term ordering Gröbner bases allow us to do this instantly.

Theorem 3.0.2.(Theorem 2) (The Elimination Theorem). Let $I \subset k[x_1, \ldots, x_n]$ be an ideal and let G be a Gröbner basis of I with respect to lex order with $x_1 > x_2 > \cdots > x_n$. Then for every $0 \le \ell \le n$, the set

$$G_\ell = G \cap k[x_{\ell+1}, \ldots, x_n]$$

is a Gröbner basis of the ℓ-th elimination ideal I_ℓ.

Proof. Fix ℓ between 0 and n. Note that $G \subset I$ implies that $G_\ell \subset I_\ell$. Thus to prove theorem 2 it suffices to show that

$$\langle \mathrm{LT}\,(I_\ell) \rangle = \langle \mathrm{LT}\,(G_\ell) \rangle,$$

because by definition a finite set $G_\ell \subset k[\mathbf{x}]$ is a Gröbner basis for the ideal I_ℓ if and only if (i) $G_\ell \subset I_\ell$ and (ii) $\mathrm{LT}\,(I_\ell) \subset \langle \mathrm{LT}\,(G_\ell) \rangle$. Now the inclusion $\supset$ follows from $I \supset G$; so it suffices to establish that $\langle \mathrm{LT}\,(I_\ell) \rangle \subset \langle \mathrm{LT}\,(G_\ell) \rangle$. To do this we we need only show that for an arbitrary $f \in I_\ell$ there is a $g \in G_\ell$ such that the leading term of f is divisible by $\mathrm{LT}\,(g)$. Suppose $f \in I_\ell$. Then, in particular, $f \in I$ and $\mathrm{LT}\,(f)$ is divisible by some $\mathrm{LT}\,(g)$, $g \in G$, because G is a Gröbner basis for I. Since $f \in I_\ell$, the leading term $\mathrm{LT}\,(f)$ involves only the variables $x_{\ell+1}, \ldots, x_n$. But then its divisor $\mathrm{LT}\,(g)$ involves only the variables $x_{\ell+1}, \ldots, x_n$ and because $x_1 > x_2 > \cdots > x_n$ this means that g involves only the variables $x_{\ell+1}, \ldots, x_n$. At the risk of being redundent, the reason for this is that since we are using lex order with $x_1 > \cdots > x_n$, any monomial involving $x_1, \ldots, x_\ell$ is greater than all the monomials in $k[x_{\ell+1}, \ldots, x_n]$, so that $\mathrm{LT}\,(g) \in k[x_{\ell+1}, \ldots, x_n]$ by itself implies that $g \in k[x_{\ell+1}, \ldots, x_n]$. This shows that $g \in G_\ell$ and finishes the proof. ∎

An Illustration. To solve

$$x^2 + y + z = 1,$$
$$x + y^2 + z = 1,$$
$$x + y + z^2 = 1,$$

First compute a Gröbner basis for the ideal $I = \langle x^2 + y + z - 1,\ x + y^2 + z - 1,\ x + y + z^2 - 1 \rangle$. Mathematica's Groebnerbasis[{x^2+y+z-1,x+y^2+z-1,x+y+z^2-1},{x,y,z}] yields

$$\{-z^2 + 4\,z^3 - 4\,z^4 + z^6, -z^2 + 2\,y\,z^2 + z^4, -y + y^2 + z - z^2, -1 + x + y + z^2\}.$$

From this we can read off G_2, G_1, G and putting the ideal generation delimiters around these we get

$$I_2 = \langle -z^2 + 4\,z^3 - 4\,z^4 + z^6 \rangle,$$
$$I_1 = \langle -z^2 + 4\,z^3 - 4\,z^4 + z^6, -z^2 + 2\,y\,z^2 + z^4, -y + y^2 + z - z^2 \rangle,$$
$$I = \langle -z^2 + 4\,z^3 - 4\,z^4 + z^6, -z^2 + 2\,y\,z^2 + z^4, -y + y^2 + z - z^2, -1 + x + y + z^2 \rangle.$$

To solve the system at the begining of the illustration we first solve I_2 finding the points $\mathbf{V}(I_2)$ and then try to extend these to find points in $\mathbf{V}(I_1)$ and so forth. Here we would just substitute the roots of $-z^2 + 4\,z^3 - 4\,z^4 + z^6$ into the basis for I_1 and then solve for y, but in general the possibilities are more complicated. This process is called the *Extension Step*.

Theorem 3.0.3. (Theorem 3) (The Extension Theorem). Let $I = \langle f_1, \ldots, f_s \rangle \subset \mathbb{C}[x_1, \ldots, x_n]$ and let I_1 be the first elimination ideal of I. For each $1 \le i \le s$ write f_i in the form

$$f_i = g_i(x_2, \ldots, x_n)x_1^{N_i} + \text{ terms in } x_1 \text{ of degree } < N_i,$$

where $N_i \ge 0$ and $g_i \in \mathbb{C}[x_2, \ldots, x_n]$ is nonzero. Suppose that we have a partial solution $(a_2, \ldots, a_n) \in \mathbb{V}(I_1)$. If $(a_2, \ldots, a_n) \notin \mathbb{V}(g_1, \ldots, g_s)$, then there exists $a_1 \in \mathbb{C}$ such that $(a_1, a_2, \ldots, a_n) \in \mathbb{V}(I)$.

The proof uses resultants and will be given later in §3.6.

There is a special case of the Extension Theorem which we record as a corollary.

Corollary 3.0.4. (Corollary 4). Let $I = \langle f_1, \ldots, f_s \rangle \subset \mathbb{C}[x_1, \ldots, x_n]$, and suppose that some f_i is of the form

$$f_i = cx_1^N + \text{ terms of lower degree in } x_1,$$

where $c \in \mathbb{C}$ is nonzero and $N > 0$. If I_1 is the first elimination ideal of I and $(a_2, \ldots a_n) \in \mathbb{V}(I_1)$, then there is an $a_1 \in \mathbb{C}$ such that $(a_1, a_2, \ldots, a_n) \in \mathbb{V}(I)$.

Proof. (Modulo the Extension Theorem) Using the notation established in the statement of the Extension Theorem, $g_i = c \ne 0$; so $\mathbb{V}(g_1, \ldots, g_s) = \emptyset$ and there is no $(a_2, \ldots, a_n) \in \mathbb{V}(g_1, \ldots, g_s)$. The existence of an a_1 with the specified properties follows then directly from the Extension Theorem. $\blacksquare$

Chapter 3 Elimination Theory

§2. The Geometry of Elimination.

Let $\pi_\ell \colon \mathbb{C}^n \to \mathbb{C}^{n-\ell}$ be the projection map $(a_1, \ldots, a_n) \mapsto (a_{\ell+1}, \ldots, a_n)$. If I is an ideal of $\mathbb{C}[x_1, \ldots, x_n]$ the *ℓ-th elimination ideal of* I is the ideal $I_\ell = I \cap \mathbb{C}[x_{\ell+1}, \ldots, x_n]$ of $\mathbb{C}[x_{\ell+1}, \ldots, x_n]$, not of $\mathbb{C}[x_1, \ldots, x_n]$. We adjust our notation accordingly; so

$$(\star) \qquad \mathbb{V}(I_\ell) = \{(a_{\ell+1}, \ldots, a_n) \in \mathbb{C}^{n-\ell} \colon I_\ell(a_{\ell+1}, \ldots, a_n) = 0\}.$$

Clarification. Let $f \in \mathbb{C}[x_{\ell+1}, \ldots, x_n]$ and $(a_{\ell+1}, \ldots, a_n) \in \mathbb{C}^n$. What $(\star)$ means is that the homomorphism Ψ of $\mathbb{C}[x_{\ell+1}, \ldots, x_n]$ into $\mathbb{C}$ which is the identity on $\mathbb{C}$ and maps $x_{\ell+1} \mapsto a_{\ell+1}, \ldots, x_n \mapsto a_n$ satisfies $\Psi(f) = 0$.

Lemma 3.2.0.1. (Lemma 1) If $I_\ell = \langle f_1, \ldots, f_s \rangle \cap \mathbf{C}[x_{\ell+1}, \ldots, x_n]$ is the ℓ-th elimination ideal of the ideal $I = \langle f_1, \ldots, f_s \rangle$ and $V = \mathbb{V}(I)$, then $I_\ell \subset \mathbb{C}[x_{\ell+1}, \ldots, x_n])$ and

$$(3.2.0.1.a) \qquad \pi_\ell(V) \subset \mathbb{V}\,(I_\ell)\,.$$

Proof. Fix $f \in I_\ell$ and suppose $\mathbf{a} = (a_1, \ldots, a_n) \in V$. Then f vanishes at $\mathbf{a}$ and involves only $x_{\ell+1}, \ldots, x_n$; so f vanishes on $\pi_\ell(\mathbf{a})$ which means each element f of I_ℓ vanishes on $\pi_\ell(\mathbf{a})$ or, equivilently, $\pi_\ell(\mathbf{a}) \in \mathbb{V}(I_\ell)$. $\blacksquare$

Thus the varieties $\mathbb{V}(I_\ell)$ may be a little bigger than the projections. Just how much bigger we shall now see. Theorem 3.2.0.2 below is called **(THE GEOMETRIC EXTENSION THEOREM)**

Theorem 3.2.0.2. (Theorem 2) Given $V = \mathbb{V}(f_1, \ldots, f_s) \subset \mathbb{C}^n$, with

$$f_i = g_i(x_2, \ldots, x_n)x_1^{N_1} + \text{ lower order terms in } x_1, \qquad 1 \le i \le s.$$

If $I_1 = \langle f_1, \ldots, f_s \rangle \cap \mathbb{C}[x_2, \ldots, x_n]$ is the first elimination ideal of $\langle f_1, \ldots, f_s \rangle$, we have in $\mathbb{C}^{n-1}$ the equality

$$(3.2.0.2.1) \qquad \mathbb{V}(I_1) = \pi_1(V) \cup \big(\mathbb{V}(g_1, \ldots, g_s) \cap \mathbb{V}(I_1)\big),$$

where $\pi_1 \colon \mathbb{C}^n \to \mathbb{C}^{n-1}$ is projection onto the last $n-1$ components.

Parapharased. A point $(b_2, \ldots, b_n) \in \mathbb{V}(I_1)$ either admits an extension $(b_1, b_2, \ldots, b_n) \in V$ or it is a zero of each of the leading coefficients g_i, $1 \le i \le s$.

Proof. From Lemma 1 we know that $\pi_1(V) \subset \mathbb{V}(I_1)$ and this theorem shows that to get all the points in $\mathbb{V}(I_1)$ we need to add those points of $\mathbb{V}(I_1)$ where the polynomials $g_1, \ldots, g_s$ vanish. Suppose $(a_2, \ldots, a_n) \in \mathbb{V}(I_1)$. we need to show that either (i) there is an $a_1 \in \mathbb{C}$ such that $(a_1, a_2, \ldots, a_n) \in V$, in which case $(a_2, \ldots, a_n) \in \pi_1(V)$, or (ii) $(a_2, \ldots, a_n) \in \mathbb{V}(g_1, \ldots, g_s)$. But this is precisely the statement of the Extension Theorem. That is, the extension theorem shows that the lefthand side of (3.2.0.2.1) is contained in the righthand side. The other inclusion is a direct consequence of Lemma 1. ∎

Thus $\pi_1(V)$ fills up the affine variety $\mathbb{V}(I_1)$ except possibly for a part that lies in $\mathbb{V}(g_1, \ldots, g_s)$. Unfortunately this part can be unnaturally large.

Example 3.2.0.2a. $\langle xy - 1,\, xz - 1\rangle = \langle (y-z)x^2 + xy - 1,\, (y-z)x^2 + xz - 1\rangle$

Proof. The lefthand side contains $xy - 1 - (xz - 1) = (y - z)x$; so the right hand side is a subset of the lefthand side. On the other hand subtracting the two generating polynomials on the right gives $(y-z)x$; multiplying this by x and subtracting the resulting polynomial $(y-z)x^2$ from each of the generating polynomials of the righthand ideal shows that the lefthand ideal is a subset of the righthand ideal. These ideals must then be equal. ∎

Further Developments. If $I = \langle xy - 1,\, xz - 1\rangle$, then $I_1 = f(y,z)(y - z),\ f(y,z) \in k[y,z]$.

Proof. By definition $I_1 = I \cap k[y,z]$. Suppose $\psi(x,y,z) \in I$. Then

$$\psi(x,y,z) = h(x,y,z)(xy-1) + g(x,y,z)(xz-1) \qquad \text{for some } h, g \in k[x,y,z]. \tag{3.2.0.2b}$$

If $\psi \in I_1$ it must have the form (3.2.0.2b) and not involve the indeterminant x. This means that we can replace x in (3.2.0.2b) by anything which makes algebraic sense without changing it at all. We do this in the following way: Let $k[x,y,z]_{[z]}$ denote the subring of the field of rational functions $k(x,y,z)$ in x,y,z where the only divisors which are allowed are powers of the indeterminant z including 1. $k[x,y,z]$ is imbedded in $k[x,y,z]_{[z]}$ (and in $k(x,y,z)$) as those elements which can be expressed using only 1 for a divisor. It makes algebraic sense then to regard $\psi(x,y,z)$ as an element of $k[x,y,z]_{[z]}$ and there to replace x by $\frac{1}{z}$. Since $\psi(x,y,z)$ doesn't involve x, this won't make any difference. Thus as elements of $k[x,y,z]_{[z]}$ we have

$$\begin{aligned}\psi(x,y,z) &= \psi\left(\frac{1}{z}, y, z\right)\\ &== h\left(\frac{1}{z}, y, z\right)\left(\frac{y}{z} - 1\right) + g\left(\frac{1}{z}, y, z\right)\left(\frac{z}{z} - 1\right)\\ &= \frac{1}{z}\cdot h\left(\frac{1}{z}, y, z\right)(y - z) + 0.\end{aligned}$$

Let $\frac{1}{z}\cdot h\left(\frac{1}{z}, y, z\right) = \frac{r(y,z)}{z^m}$ as an element of $k[x,y,z]_{[z]}$ with $r(y,z) \in k[y,z]$. Then

$$\frac{r(y,z)}{z^m}(y - z) = \frac{1}{z}\cdot h\left(\frac{1}{z}, y, z\right)(y - z) = \psi(x,y,z) \in k[x,y,z].$$

Clearing denominators then yields $r(y,z)(y - z) = z^m\psi(x,y,z)$ and the unique factorization theorem for $k[x,y,z]$ then shows that $y - z$ divides $\psi(x,y,z)$ in $k[x,y,z]$ as asserted. (It is also true of course that, appearances to the contrary, $\frac{1}{z}\cdot h\left(\frac{1}{z}, y, z\right) \in k[y,z]$.) ∎

The Picture. $\mathbb{V}(xy - 1) = \left\{\left(t, \frac{1}{t}, z\right) : t, z \in \mathbb{C}\right\}$ and $\mathbb{V}(xz - 1) = \left\{\left(t, y, \frac{1}{t}\right) : t, y \in \mathbb{C}\right\}$; so $\mathbb{V}(I)$ is the hyperbola like curve parametrized by $\left(t, \frac{1}{t}, \frac{1}{t}\right)$, $t \in \mathbb{C}$. $\pi_1(V) = \{(s,s)\colon s \neq 0\}$, and $I_1 = \mathbb{C}[y,z](y - z)$; so $\mathbb{V}(I_1) = \mathbb{V}(y - z) = \{(s,s)\colon s \in \mathbb{C}\}$. If we choose $\{xy - 1,\, xz - 1\}$ as a generating set for I we have in the notation used here $g_1(y,z) = y$ and $g_2(y,z) = z$; so that $\mathbb{V}(g_1, g_2) = \{(0,0)\}$. If, on the otherhand we take $\{(y-z)x^2 + yx - 1, (y-z)x^2 + zx - 1\}$ as the generating set, $\mathbb{V}(g_1, g_2) = \mathbb{V}(y - z) = \{(s,s)\colon s \in \mathbb{C}\}$ and we don't get any information at all about the size of $\pi_1(V)$. Although I, $V = \mathbb{V}(I) \subset \mathbb{C}^n$, $\pi_1(V)$, I_1, and $\mathbb{V}(I_1) \subset \mathbb{C}^{n-1}$ are determined strictly by the ideal I, the variety $\mathbb{V}(g_1, \ldots, g_s) \subset \mathbb{C}^{n-1}$ depends very much on the particular set of generating polynomials $f_1, \ldots, f_s$ used to describe I.

Read the statement of the Closure Theorem (which follows) and then come back to this example. In the terms used there $m = 1$. The variety W of (ii) is $W = \{(0,0)\} = \mathbb{V}(y,z)$. The W_1 and Z_1 of (iii) are, respectively, $W_1 = \mathbb{V}(y-z) = \mathbb{V}(I_1)$ and $Z_1 = \{(0,0)\} = \mathbb{V}(y,z)$.

Theorem 3.2.0.3. (Theorem 3) **(The Closure Theorem).** Let $V = \mathbb{V}(f_1,\dots,f_s) \subset \mathbb{C}^n$ and let I_ℓ be the ℓ-th elimination ideal of $\langle f_1,\dots,f_s\rangle$. That is, $I_\ell = \langle f_1,\dots,f_s\rangle \cap \mathbb{C}[x_{\ell+1},\dots,x_n]$. Then:

(i) $\mathbb{V}(I_\ell)$ is the smallest affine variety containing $\pi_\ell(V) \subset \mathbb{C}^{n-\ell}$.

(ii) When $V \neq \emptyset$, there is an affine variety $W \overset{\subset}{\neq} \mathbb{V}(I_\ell)$ such that $\mathbb{V}(I_\ell) - W \subset \pi_\ell(V)$.

(iii) There are affine varieties $Z_i \subset W_i \subset \mathbb{C}^{n-\ell}$ with $1 \le i \le m$ such that

$$\pi_\ell(V) = \bigcup_{i=1}^{m} (W_i - Z_i).$$

Proof. $\mathbb{V}(I_\ell)$ being smallest means two things
- $\pi_\ell(V) \subset \mathbb{V}(I_\ell)$.
- If Z is any affine variety in $\mathbb{C}^{n-\ell}$ containing $\pi_\ell(V)$, then $\mathbb{V}(I_\ell) \subset Z$.

Remarks and Apologies. When we introduce the Zariski topology (i) will be expressed by saying that $\mathbb{V}(I_\ell)$ is the *Zariski closure* of $\pi_\ell(V)$. This is where the "Closure" in the theorem's name comes from. To prove part (i) we need the Nullstellensatz which we will discuss in Chapter 4 where we finish the proof. We will only prove part (ii) here in the special case when $\ell = 1$. The proof when $\ell > 1$ will be given in §6 of Chapter 5 where we will also prove the stronger version (iii).

Proof. We start with the decomposition

$$\mathbb{V}(I_1) = \pi_1(V) \cup \big(\mathbb{V}(g_1,\dots,g_s) \cap \mathbb{V}(I_1)\big). \tag{3.2.0.2.1}$$

Let $W = \mathbb{V}(g_1,\dots,g_s) \cap \mathbb{V}(I_1)$ and note that W is an affine variety, because if U and V are affine varieties so is $U \cap V$. The decomposition (3.2.0.2.1) implies that $\mathbb{V}(I_1) - W \subset \pi_1(V)$. We are done if $W \neq \mathbb{V}(I_1)$. However, as our example above shows, it can happen that $W = \mathbb{V}(I_1)$.

SubTheorem. If $W = \mathbb{V}(I_1)$, then $V = \mathbb{V}(f_1,\dots,f_s,g_1,\dots,g_s)$.

Proof of subtheorem. It is obvious that $\mathbb{V}(f_1,\dots,f_s,g_1,\dots,g_s) \subset \mathbb{V}(f_1,\dots,f_s) = V$. To get the opposite inclusion, let $\mathbb{a} = (a_1,\dots,a_n) \in V$. Certainly each f_i vanishes at $\mathbb{a}$, and since $(a_2,\dots,a_n) \in \pi_1(V) \subset \mathbb{V}(I_1)$ which $= W$ by hypothesis, it follows that each g_j vanishes at $\mathbb{a}$ too. Thus $V \subset \mathbb{V}(f_1,\dots,f_s,g_1,\dots,g_s)$ as stated in the subtheorem. ∎

Let $I = \langle f_1,\dots,f_s\rangle$ and $\tilde{I} = \langle f_1,\dots,f_s,g_1,\dots,g_s\rangle$. Notice that $\mathbb{V}(I) = \mathbb{V}(\tilde{I})$ but it still may be that $I \overset{\subset}{\neq} \tilde{I}$. However, using conclusion (i) of the closure theorem, since $\mathbb{V}(I_1)$ and $\mathbb{V}(\tilde{I}_1)$ are both the smallest variety containing $\pi_1(V)$, it follows that $\mathbb{V}(I_1) = \mathbb{V}(\tilde{I}_1)$.

The next step is to find a better basis for $\tilde{I}$. First, recall that the g_j's are defined by writing

$$f_i = g_i(x_2,\dots,x_n)x_1^{N_1} + \text{ terms of lower degree in } x_1,$$

where $N_i \ge 0$ and $g_i \in \mathbb{C}[x_2,\dots,x_n]$. (Note: If $N_i = 0$ then $f_i = g_i$.) Now set

$$\tilde{f}_i = \begin{cases} f_i - g_i x_1^{N_i}, & \text{if } N_i > 0; \\ 0, & \text{if } N_i = 0. \end{cases}$$

Note that for each i, $\tilde{f}_i$ is either zero or has strictly smaller degree in x_1 than f_i.

SubTheorem. $\langle \tilde{f}_1,\dots,\tilde{f}_s,g_1,\dots,g_s\rangle = \langle f_1,\dots,f_s,g_1,\dots,g_s\rangle = \tilde{I}$.

Proof of subtheorem. ($\subset$): Each $\tilde{f}_i \in \tilde{I}$. ($\supset$): Given f_i there are two cases to consider. Case $N_i > 0$: In this case $f_i = \tilde{f}_i + g_i x_1^{N_i}$ which is certainly in the ideal $\langle \tilde{f}_1, \ldots, \tilde{f}_s, g_1, \ldots, g_s \rangle$. Case $N_i = 0$: In this case $f_i = g_i$ which is also in this ideal. ∎

Now apply the Geometric Extension Theorem to $V = \mathbb{V}(\tilde{f}_1, \ldots, \tilde{f}_s, g_1, \ldots, g_s)$. The leading coefficients (and the number) of the generators are different when we use $\tilde{f}_1, \ldots, \tilde{f}_s, g_1, \ldots, g_s$ rather than $f_1, \ldots, f_s$ to define the variety V, so that we get a different decomposition

$$\mathbb{V}(I_1) = \mathbb{V}(\tilde{I}_1) = \pi_1(V) \cup \tilde{W},$$

where $\tilde{W}$ consists of those zeros of I_1 where the leading x_1-coefficients of the $\tilde{f}_1, \ldots, \tilde{f}_s, g_1, \ldots, g_s$ vanish. Unfortunately, in the general case, there is nothing to guarantee that $\tilde{W}$ will be strictly smaller than W. (It is certainly true, however, that the highest power of x_1 will be strictly lower in this new set of generators.) So it can still happen that $\tilde{W} = \mathbb{V}(I_1)$. (If $\tilde{W} \overset{\subset}{\neq} W = \mathbb{V}(I_1)$, we are done.) If this is the case we repeat the process getting $\tilde{\tilde{W}}$ and continue in this manner until we do get something smaller or until V is expressed as $V = \mathbb{V}(h_1, \ldots, h_t)$ where for each j the $h_j \in \mathbb{C}[x_2, \ldots, x_n]$. If this last happens we will have arrived at the situation where $V = \mathbb{V}(J)$ with $J = \langle h_1, \ldots, h_t \rangle$ and $\mathbb{V}(J_1) = \mathbb{V}(I_1)$.

Subtheorem. In this case $\mathbb{V}(I_1) = \pi_1(V)$.

Proof of subtheorem. Each of the following lines implies the next. (Remember that $h_1, \ldots, h_t \in \mathbb{C}[x_2, \ldots, x_n]$.)

- $(a_2, \ldots, a_n) \in \mathbb{V}(I_1)$;
- $(a_2, \ldots, a_n) \in \mathbb{V}(J_1)$;
- $(a_2, \ldots, a_n)$ is a zero of $\langle h_1, \ldots, h_t \rangle \cap \mathbb{C}[x_2, \ldots, x_n]$;
- for each $c \in \mathbb{C}$, and each j, $h_j(c, a_2, \ldots, a_n) = 0$;
- for each $c \in \mathbb{C}, (c, a_2, \ldots, a_n) \in \mathbb{V}(h_1, \ldots, h_t) = V$;
- $(a_2, \ldots, a_n) \in \pi_1(V)$.

As a consequence $\pi_1(V) \supset \mathbb{V}(I_1)$ and since we already know from part (i) of the Closure Theorem that $\pi_1(V) \subset \mathbb{V}(I_1)$, it follows that $\pi_1(V) = \mathbb{V}(I_1)$. ∎

The proof of those parts of the Closure Theorem which we have untertaken is finished. ∎

Here is a Corollary to the special case of the Closure Theorem whose proof we have actually given modulo the Extension Theorem. It is a consequence of the fact that in this case the g_j's can never simultaneously vanish, so that $\mathbb{V}(g_1, \ldots, g_s) = \emptyset$.

Corollary 3.2.0.4.(Corollary 4) Let $V = \mathbb{V}(f_1, \ldots, f_s) \subset \mathbb{C}^n$, and assume that for some i, f_i is of the form

$$f_i = c x_1^N + \text{ terms of lower degree in } x_1,$$

where $N > 0$ and $c \in \mathbb{C}$ is a constant. If $I_1 = \langle f_1, \ldots, f_s \rangle \cap \mathbb{C}[x_2, \ldots, x_n]$, then

$$\pi_1(V) = \mathbb{V}(I_1) \qquad \text{in } \mathbb{C}^{n-1},$$

where π_1 is the projection $(a_1, a_2, \ldots, a_n) \mapsto (a_2, \ldots, a_n)$.

An Example to show that $\tilde{W}$ may be smaller that W. References are to the preceding proof. Let $I = \langle (y-z)x^2 + xy - 1,\ (y-z)x^2 + xz - 1 \rangle = \langle xy - 1,\ xz - 1 \rangle$. A Gröbner basis for I is $\{y - z,\ -1 + xz\}$; so $I_1 = \langle y - z \rangle$ and $g_1 = g_2 = y - z$ with $N_1 = N_2 = 0$. Thus $W = \mathbb{V}(g_1, g_2) \cap \mathbb{V}(I_1) = \mathbb{V}(I_1)$ in this case. Now

$$\tilde{I} = \langle (y-z)x^2 + xy - 1,\ (y-z)x^2 + xz - 1,\ y - z \rangle = \langle xy - 1,\ xz - 1,\ y - z \rangle.$$

Applying the Geometric Extension Theorem to $\tilde{I}$ the new set of g_i's is $\{y,\ z,\ 0\}$ and the set where these vanish simultaneously is $\tilde{W} = \{(0, 0)\}$ which is strictly smaller than $W = \{(z, z) \colon z \in \mathbb{C}\}$.
