Computing Intersection Multiplicity via Triangular Decomposition

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Intersection Multiplicity

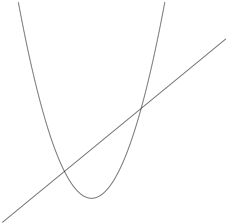
What is the intersection multiplicity of a planar curve at $p \in \mathbb{A}^2(\mathcal{R})$?

Formal The dimension of tangent space of $\mathbf{h}$ at p.

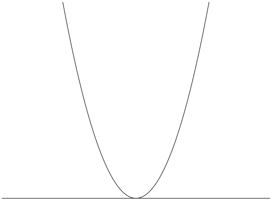
Informal The weights of Bézout's summand.

Constructive A value calculated by Fulton's algorithm.

Example



IM = 1



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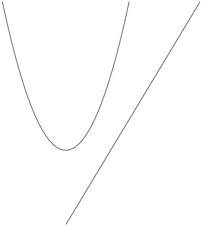


IM = 1

Definition (Bézout's IM for bivariates)

Let $\mathbf{h} \subseteq \mathcal{R}[x,y]^{|<\infty|}$ and $p \in \mathbf{V}(\mathbf{h})$. The INTERSECTION MULTIPLICITY of p in $\mathbf{V}(\mathbf{h})$ is

$$\operatorname{im}(p;\, \mathbf{h}) := \dim_{\text{vec}}\big(\mathcal{O}_{\mathbb{A}^2,\,p}/\langle\, \mathbf{h}\, \rangle\big),$$

where $\mathcal{O}_{\mathbb{A}^2,\,p} := \left\{ \dfrac{f}{g} \,:\, f, g \in \mathcal{R}[x,y],\ g(p) \neq 0 \right\}.$

Definition (Bézout's Summand)

Let $\mathbf{h} \subseteq \mathcal{R}[x, y]^{|<\infty|}$ and $p \in \mathbf{V}(\mathbf{h})$. The INTERSECTION MULTIPLICITY of p in $\mathbf{V}(\mathbf{h})$ satisfies

$$\sum_{p \in \mathbf{V}(\mathbf{h})} \mathrm{im}(p; \mathbf{h}) = \prod_{h \in \mathbf{h}} \deg(h).$$

Fulton's Properties

Let h_0, $h_1 \in \mathcal{R}[x, y]$ and $p \in \mathbb{A}^2(\mathcal{F}[\mathbf{x}])$. The intersection multiplicity of two plane curves satisfies and are uniquely determined by the following.

(2-0) $m_p(h) :=$ MULTIPLICITY of $h \in \mathcal{R}[x, y]$ at p (usually the number of unique tangent lines),

Fulton's Properties

(2-1) $\mathrm{im}(p;\, h_0,\, h_1) = \begin{cases} \infty & \text{if } p \in \mathbf{V}(\gcd(h_0,\, h_1)) \\ n \in \mathbb{N} & \text{otherwise.} \end{cases},$

(2-2) $\mathrm{im}(p;\, h_0,\, h_1) = 0 \iff p \notin \mathbf{V}(h_0) \cap \mathbf{V}(h_1),$

(2-3) $\mathrm{im}(p;\, h_0,\, h_1)$ is invariant to affine change of coordinates on $\mathbb{A}^2(\mathcal{F})$,

Fulton's Properties

(2-5) $\mathrm{im}(p;\, h_0,\, h_1) \geq m_p(h_0) \cdot m_p(h_1)$ with equality occurring if and only if $\mathbf{V}(h_0)$ and $\mathbf{V}(h_1)$ have no tangent lines in common at p. That is:

$$\mathrm{im}(p;\, h_0,\, h_1) = m_p(h_0) \cdot m_p(h_1)$$

$$\iff \pi_p(h_0) \pitchfork \pi_p(h_1),$$

Fulton's Properties

(2-4) $\mathrm{im}(p;\, h_0,\, h_1) = \mathrm{im}(p;\, h_1,\, h_0)$,

(2-6) $\forall\, g \in \mathcal{F}[\mathbf{x}];\ \mathrm{im}(p;\, h_0,\, h_1) = \mathrm{im}(p;\, h_0,\, h_1 g) - \mathrm{im}(p;\, h_0,\, g)$,

and

(2-7) $\forall\, g \in \mathcal{F}[\mathbf{x}];\ \mathrm{im}(p;\, h_0,\, h_1) = \mathrm{im}(p;\, h_0,\, h_1 + g)$.

$\mathrm{im}(p;\, h_0,\, h_1)$

if $h_0(p),\, h_1(p) \neq 0$ **then**

  return 0;

$r,\, s \leftarrow \deg(h_0(x, p_y),\, h_1(x, p_y))$;

if $r > s$ **then**

  return $\mathrm{im}(p;\, h_1,\, h_0)$;

if $r = -\infty$ **then** /* $(y - p_y) \mid h_0(x, y)$ */

  write $h_1(x, p_y) = (x - p_x)^m (a_m + a_m(x - p_x) + \cdots)$;

  return $m + \mathrm{im}(p;\, \mathrm{quo}(h_0,\, y - p_y),\, h_1)$;

if $r \leq s$ **then**

  $h_1' \leftarrow h_1 - x^{s-r} \dfrac{\mathrm{lc}(h_1(x, p_y))}{\mathrm{lc}(h_0(x, p_y))} h_0$;

  return $\mathrm{im}(p;\, h_1',\, h_0)$;

Algorithm 1: Fulton's Aglorithm

Example

Find the intersection multiplicity of $\mathbf{h} = \{y,\, y - x^2\}$ at the origin.

$$\boxed{\operatorname{im}\left(\mathbf{0};\, y,\, y - x^2\right)}$$

$$\begin{aligned}(r,\, s) &\leftarrow \deg\left(0,\, 0 - x^2\right)\\ &= (-\infty,\, 2)\\ h_1(x,\, 0) &= x^2(-1) \implies m = 2\end{aligned}$$

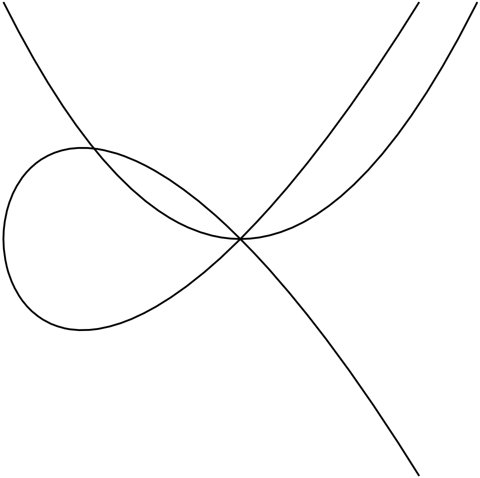
$$\boxed{2 + \operatorname{im}\left(\mathbf{0};\, \operatorname{quo}(y,\, y),\, y - x^2\right)}$$

$$\operatorname{im}\left(\mathbf{0};\, 1,\, y - x^2\right) = 0$$

$$\boxed{2.}$$

Example

Find the intersection multiplicity of $\mathbf{h} = \left\{ y - x^2,\, y^2 - x^3 - x^2 \right\}$ at the origin.



$$\boxed{\operatorname{im}\big(\mathbf{0};\ y - x^2,\ y^2 - x^3 - x^2\big)}$$

$$\begin{aligned}
(r,\ s) &\leftarrow \deg\big(0 - x^2,\ 0 - x^3 - x^2\big) \\
&= (2,\ 3) \\
h_1' &\leftarrow \big(y^2 - x^3 - x^2\big) - x^{3-2}\frac{(-1)}{(-1)}(y - x^2) \\
&= y^2 - x^2 - xy
\end{aligned}$$

$$\boxed{\operatorname{im}\big(\mathbf{0};\ y^2 - x^2 - xy,\ y - x^2\big)}$$

$$\boxed{\operatorname{im}\bigl(\mathbf{0};\, y^2 - x^2 - xy,\, y - x^2\bigr)}$$

$$\begin{aligned}
(r,\, s) &\leftarrow \deg\bigl(0 - x^2 - 0,\, 0 - x^2\bigr) \\
&= (2,\, 2) \\
h_1' &\leftarrow \bigl(y^2 - x^2\bigr) - x^{2-2}\frac{(-1)}{(-1)}(y^2 - x^2) \\
&= y - y^2
\end{aligned}$$

$$\boxed{\operatorname{im}\bigl(\mathbf{0};\, y - y^2,\, y^2 - x^2 - xy\bigr)}$$

$$\boxed{\operatorname{im}\big(\mathbf{0};\, y - y^2,\, y^2 - x^2 - xy\big)}$$

$$\begin{aligned}(r,\, s) &\leftarrow \deg\big(0 - 0,\, 0 - x^2 - 0\big)\\ &= (-\infty,\, 2)\\ h_1(x,0) &= x^2(-1) \implies m = 2\end{aligned}$$

$$\boxed{2 + \operatorname{im}\big(\mathbf{0};\, \operatorname{quo}\big(y - y^2,\, y\big),\, y^2 - x^2 - xy\big)}$$

$$\operatorname{im}\big(\mathbf{0};\, 1 - y,\, y^2 - x^2 - xy\big) = 0$$

$$\boxed{2.}$$

$\mathrm{im}_2(\,\mathbf{f}_\triangle \in \mathbb{T}_{\mathrm{reg}}(\mathcal{R}[x,y]);\ h_0,\ h_1)$

if $h_0, h_1 \not\equiv 0, 0 \bmod \langle\, \mathbf{f}_\triangle \,\rangle$ **then**

    return 0;

$r \leftarrow \deg(h_0 \bmod \langle\, \mathbf{f}_\triangle^{\downarrow} \,\rangle)$;

$s \leftarrow \deg(h_1 \bmod \langle\, \mathbf{f}_\triangle^{\downarrow} \,\rangle)$;

if $r > s$ **then**

    return $\mathrm{im}_2(\,\mathbf{f}_\triangle;\ h_1,\ h_0)$;

if $r = -\infty$ **then** /* $\mathbf{f}_\triangle^{\downarrow} \mid h_0$ */

    $m \leftarrow \min\Big(m \in \mathbb{N} : h_1 \not\equiv 0 \bmod \Big\langle\, (\mathbf{f}_\triangle^{\top})^{m+1},\ \mathbf{f}_\triangle^{\downarrow} \,\Big\rangle\Big)$;

    return $m + \mathrm{im}_2(\,\mathbf{f}_\triangle;\ \mathrm{quo}(h_0,\ \mathbf{f}_\triangle^{\downarrow}),\ h_1)$;

if $r \leq s$ **then**

    $h' \leftarrow \mathrm{lc}(h_0 \bmod \langle\, \mathbf{f}_\triangle^{\downarrow} \,\rangle)\, h_1 - x_1^{s-r}\, \mathrm{lc}(h_1 \bmod \langle\, \mathbf{f}_\triangle^{\downarrow} \,\rangle)\, h_0$;

    return $\mathrm{im}_2(\,\mathbf{f}_\triangle;\ h',\ h_0)$;

Algorithm 2: About regular chains (non-splitting).

Example

Calculate the intersection multiplicity of

$$\mathbf{h} = \left\{y - x^2,\, y^2 - x^3 - x^2\right\}$$

at

$$\mathbf{f}_\triangle = \begin{cases} x - y + 1 \\ y^2 - 3y + 1 \end{cases} \quad .$$

$$\boxed{\operatorname{im}\bigl(\mathbf{f}_\triangle;\, y - x^2,\, y^2 - x^3 - x^2\bigr)}$$

$$\begin{aligned}
(r,\, s) &\leftarrow \partial_\top\bigl(x^2 - y,\, x^3 + x^2 - y^2 \bmod \left\langle\, y^2 - 3y + 1 \,\right\rangle\bigr) \\
&= \partial_\top\bigl(y - x^2,\, x^3 + x^2 - 3y + 1\bigr) \\
&= (2,\, 3) \\
h_1' &\leftarrow h_1 - \operatorname{pivot}(\mathbf{f}_\triangle;\, \mathbf{h}) \\
&= \bigl(y^2 - x^3 - x^2\bigr) - \bigl(-x^{3-2}\bigr)(y - x^2) \\
&= x^2 + xy - y^2
\end{aligned}$$

$$\boxed{\operatorname{im}\big(\mathbf{f}_\triangle;\, x^2 + xy - y^2,\, x^2 - y\big)}$$

$$\begin{aligned}
(r,\, s) &\leftarrow \partial_\top\big(x^2 + xy - y^2,\, x^2 - y \bmod \langle\, \mathbf{f}_\triangle^{\downarrow}\,\rangle\big) \\
&= \partial_\top\big(x^2 + xy - 3y + 1,\, x^2 - y\big) \\
&= (2,\, 2) \\
h_1' &\leftarrow h_1 - \operatorname{pivot}(\mathbf{f}_\triangle;\, \mathbf{h}) \\
&= \big(x^2 - y\big) - \big(-x^{2-2}\big)(x^2 + xy - y^2) \\
&= y^2 - xy - y
\end{aligned}$$

$$\boxed{\operatorname{im}\bigl(\mathbf{f}_{\triangle};\, y^2 - xy - y,\, x^2 + xy - y^2\bigr)}$$

$$\begin{aligned}
(r,\, s) &\leftarrow \partial_{\top}\bigl(y^2 - xy - y,\, x^2 + xy - y^2 \bmod \langle\, \mathbf{f}_{\triangle}^{\downarrow} \,\rangle\bigr) \\
&= \partial_{\top}\bigl(2y - xy - 1,\, x^2 + xy - 3y + 1\bigr) \\
&= (1,\, 2) \\
h_1' &\leftarrow (x^2 + xy - y^2) - (-2 + y)\bigl(x^{2-1}\bigr)\bigl(y^2 - xy - y\bigr) \\
&= \bigl(y^2 - 3y + 1\bigr)x^2 + \bigl(-y^3 + 4y^2 - 2y\bigr)x - y^2
\end{aligned}$$

$$\boxed{\operatorname{im}\big(\mathbf{f}_\triangle;\ h_1',\ y^2 - xy - y\big)}$$

$$\begin{aligned}
(r,\ s) &\leftarrow \partial_\top\big(h_0,\ y^2 - xy - y \bmod \langle\, \mathbf{f}_\triangle^{\downarrow}\,\rangle\big)\\
&= \partial_\top(2xy - x - 3y + 1,\ -xy + 2y - 1)\\
&= (1,\ 1)\\
h_1' &\leftarrow \big(y^2 - xy - y\big) - (2y+5)\big(x^{1-1}\big)h_0\\
&= \big(2y^4 - 11y^3 + 17y^2 - 5y\big)x^2\\
&\quad + \big(-2y^5 + 13y^4 - 24y^3 + 10y^2 - y\big)x\\
&\quad + \big(-2y^4 + 5y^3 + y^2 - y\big)
\end{aligned}$$

$$\boxed{\mathrm{im}\big(\mathbf{f}_{\triangle};\, h_1',\, h_0\big)}$$

$$\begin{aligned}
(r,\, s) &\leftarrow \partial_{\top}(h_0,\, h_1 \bmod \langle\, \mathbf{f}_{\triangle}^{\downarrow}\, \rangle) \\
&= \partial_{\top}(0,\, 2xy - x - 3y + 1) \\
&= (-\infty,\, 1) \\
m &\leftarrow \text{Tailing degree of} \\
&\qquad\qquad h_1 = (\mathbf{f}_{\triangle}^{\downarrow} x + 2y - 1)\mathbf{f}_{\triangle}^{\top} + \mathbf{f}_{\triangle}^{\downarrow} \\
&\qquad \text{in } \mathcal{R}[\mathbf{f}_{\triangle}^{\top}]/\langle\, \mathbf{f}_{\triangle}^{\downarrow}\, \rangle. \\
&= 1
\end{aligned}$$

$$\boxed{1 + \mathrm{im}(\mathbf{f}_\triangle;\, \mathrm{quo}(h_0,\, \mathbf{f}_\triangle^{\downarrow}),\, h_1)}$$

$h_0,\, h_1 \not\equiv \mathbf{0} \bmod \langle\, \mathbf{f}_\triangle \,\rangle$

$= 0$

$$\boxed{\mathrm{im}(\mathbf{f}_\triangle;\, h_0,\, h_1) = 1.}$$

$\mathbf{h}$ = randpoly([x,y], homogeneous, deg=d) mod 962 592 769.

d	Max Time	Min Time	Average
2	0.206	0.160	0.222
3	1.712	.404	1.174
4	3.545	2.048	2.771
5	5.416	3.376	4.456
6	7.457	5.113	6.412
7	10.216	6.768	8.684
8	12.833	8.825	11.130
9	15.901	9.021	13.988
10	19.145	14.009	17.049
11	24.062	15.457	20.742
12	28.19	19.201	24.342
13	33.414	22.477	29.372
14	38.819	24.73	33.396
15	45.106	28.57	38.475
16	55.375	33.738	44.004
17	58.612	33.298	50.474
18	65.812	41.927	57.648
19	81.237	51.727	66.026
20	89.282	53.596	76.124
21	100.21	64.956	85.827
22	109.747	68.308	94.293

Extending Fulton's Properties

Definition (Bézout's Intersection Multiplicity)

Let $\mathbf{h} \subseteq \mathcal{F}[\mathbf{x}]^{|<\infty|}$ and $p \in \mathbf{V}(\mathbf{h})$. The INTERSECTION MULTIPLICITY of p in $\mathbf{V}(\mathbf{h})$ is

$$\operatorname{im}(p; \mathbf{h}) := \dim_{\text{vec}}(\mathcal{O}_{\mathbb{A}^{\ell+1},\, p} / \langle\, \mathbf{h} \,\rangle).$$

$$\mathcal{O}_{\mathbb{A}^{\ell+1},\, p} := \left\{ \frac{f}{g} \,:\, f, g \in \mathcal{F}[\mathbf{x}],\; g(p) \neq 0 \right\}.$$

Example

Locally at the origin the system

$$\mathbf{h} = \left\{x,\, x - y^2 - z^2,\, y - z^3\right\} \subseteq \mathcal{F}[x,\, y,\, z]$$

is $x,\, y = z^3,\, z^2\bigl(z^4 + 1\bigr)$. Near $\mathbf{0}$ we have

$$\begin{aligned}
\mathcal{F}[\mathbf{x}]/\langle\, \mathbf{h}\, \rangle &= \mathcal{F}[\mathbf{x}]/\left\langle\, x,\, y - z^3,\, z^2\, \right\rangle \\
&= \mathcal{F}[\mathbf{x}]/\left\langle\, x\, y,\, z^2\, \right\rangle \\
&= \{a + bz : a,\, b \in \mathcal{F}\} \\
&= \langle\langle\, 1,\, z\, \rangle\rangle_{\mathcal{F}}
\end{aligned}$$

where $\langle\langle\, 1,\, z\, \rangle\rangle_{\mathcal{F}}$ is a $\mathcal{F}$-vector space. Thus $\mathrm{im}(\mathbf{0};\, \mathbf{h}) = 2$.

Extended Fulton's Properties

Let $\mathbf{h} \subseteq \mathcal{R}[\mathbf{x}]^{|<\infty|}$ so that $\dim \langle\, \mathbf{h} \,\rangle = 0$, let

$$p := (p_0,\, \ldots, p_\ell) \in \mathbb{A}^{\ell+1}\big(\overline{\mathcal{F}}\big),$$

and recall $\mathbf{h}^{\downarrow}$ denotes the removal of some top element $\mathbf{h}^{\top} \in \mathbf{h}$ (i.e. $\mathbf{h} = \{\mathbf{h}^{\top}\} \cup \mathbf{h}^{\downarrow}$).

$$\mathrm{im}(p;\, \mathbf{h}) \text{ satisfies } (n\text{-}1) \text{ through } (n\text{-}7).$$

(n-1) $\operatorname{im}(p;\, \mathbf{h}) \in \mathbb{N}$,

(n-2) $\operatorname{im}(p;\, \mathbf{h}) = 0 \iff p \notin \mathbf{V}(\mathbf{h})$,

(n-3) $\operatorname{im}(p;\, \mathbf{h})$ is invariant to affine change of coordinates on $\mathbb{A}^{\ell+1}(\mathcal{F})$,

$(n\text{-}5)$ $\mathrm{im}(p;\ (x_0 - p_0)^{m_0},\ \ldots,\ (x_\ell - p_\ell)^{m_\ell}) = m_0 \cdots m_\ell,$

$(n\text{-}6)$ provided $\dim \langle\, \mathbf{h}^{\downarrow},\ gh \,\rangle = 0,$

$$\mathrm{im}(p;\ \mathbf{h}^{\downarrow},\ gh) = \mathrm{im}(p;\ \mathbf{h}^{\downarrow},\ g) + \mathrm{im}(p;\ \mathbf{h}^{\downarrow},\ h),$$

$(n\text{-}7)$ $\forall\, g \in \langle\, \mathbf{h}^{\downarrow} \rangle;\ \mathrm{im}(p;\ \mathbf{h}^{\downarrow},\ \mathbf{h}^{\top}) = \mathrm{im}(p;\ \mathbf{h}^{\downarrow},\ \mathbf{h}^{\top} + g).$

Theorem

For $\mathbf{h} \subseteq \mathcal{R}[\mathbf{x}]^{|<\infty|}$ *and* $p \in \mathbb{A}^{\ell+1}(\mathcal{R})$

$$\mathbf{V}(\pi_p(\mathbf{h}^{\top})) \pitchfork \mathbf{V}(\kappa_p(\mathbf{h}^{\downarrow}))$$

$$\Downarrow$$

$$\mathrm{im}(p;\, \mathbf{h}) = \mathrm{im}(p;\, \mathbf{h}^{\downarrow},\, \pi_p(\mathbf{h}^{\top})).$$

Corollary

For $\mathbf{h} \subseteq \mathcal{R}[\mathbf{x}]^{|<\infty|}$ *and* $p \in \mathbb{A}^{\ell+1}(\mathcal{R})$. *Suppose* $\mathbf{x}^{\top} \in \mathrm{monos}(\pi_p(\mathbf{h}^{\top}))$ *and*

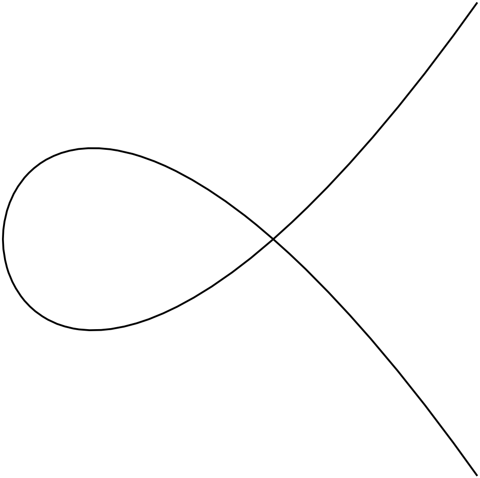
$$\mathbf{V}(\pi_p(\mathbf{h}^{\top})) \pitchfork \mathbf{V}(\kappa_p(\mathbf{h}^{\downarrow}))$$

$$\Big\Downarrow$$

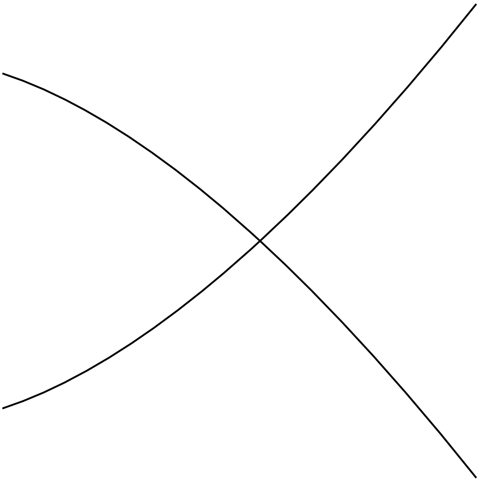
$$\mathrm{im}(p;\, \mathbf{h}) = \mathrm{im}(p^{\downarrow};\, \textsc{prem}(\mathbf{h}^{\downarrow},\, \pi_p(\mathbf{h}^{\top}),\, \mathbf{x}^{\top})\ )$$

where the right-hand-side is a calculation in $\mathcal{R}[\mathbf{x}^{\downarrow}]$.

Tangent Cones



Tangent Cones



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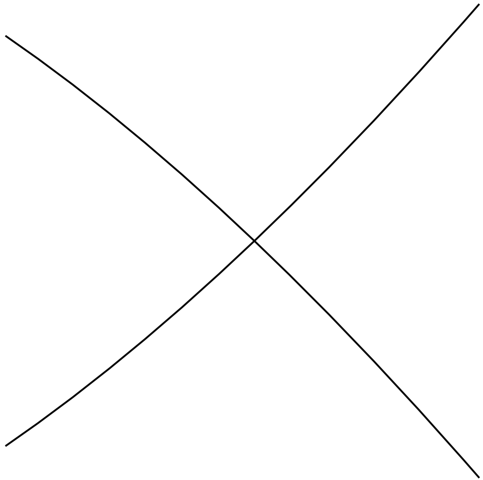
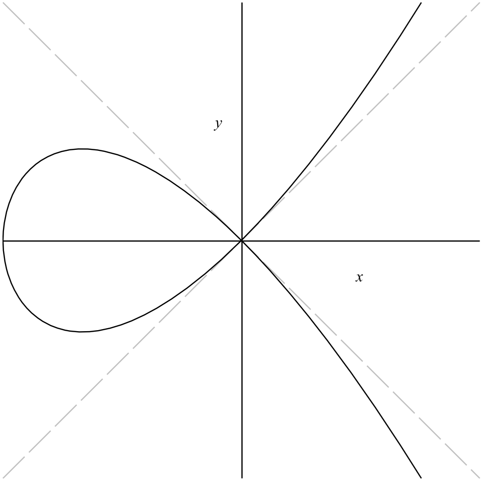
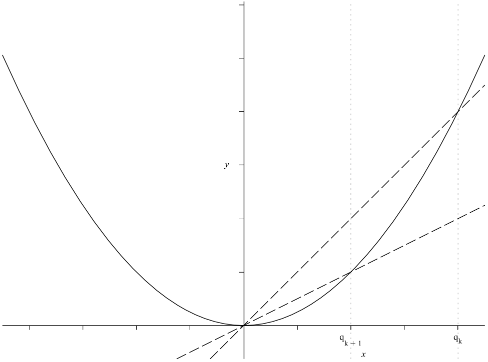


Tangent Cones



Secant Cones



$$\lim_{k\to\infty} q_k = \lim_{k\to\infty} \mathbf{0} + \left\langle\!\!\left\langle\, \frac{1}{k},\ \frac{1}{k^2} \,\right\rangle\!\!\right\rangle_{\mathcal{R}}$$

$$= \mathbf{0} + \langle\!\langle\, 0,\ 0 \,\rangle\!\rangle_{\mathcal{R}}$$

versus

$$\lim_{k\to\infty} \hat{q}_k = \lim_{k\to\infty} \mathbf{0} + \left\langle\!\!\left\langle\, \frac{1/k}{1/k^2\sqrt{k^2+1}},\ \frac{1/k^2}{1/k^2\sqrt{k^2+1}} \,\right\rangle\!\!\right\rangle_{\mathcal{R}}$$

$$= \lim_{k\to\infty} \mathbf{0} + \left\langle\!\!\left\langle\, \frac{k}{\sqrt{k^2+1}},\ \frac{1}{\sqrt{k^2+1}} \,\right\rangle\!\!\right\rangle_{\mathcal{R}}$$

$$= \mathbf{0} + \langle\!\langle\, 1,\ 0 \,\rangle\!\rangle_{\mathcal{R}}$$

Want solutions to slope system

$$M = \begin{cases} m = \frac{x}{y} \\ x^2 - y \\ y \end{cases}$$

however

$$\triangle(M) = \begin{cases} m' \\ \vdots \\ \color{red}{y \neq 0} \end{cases} \quad .$$

$$\operatorname{qlim}\begin{cases} ym - x \\ x^2 - y \\ y \end{cases} = \begin{cases} m' \\ \vdots \\ {\color{red} y = 0} \end{cases} .$$

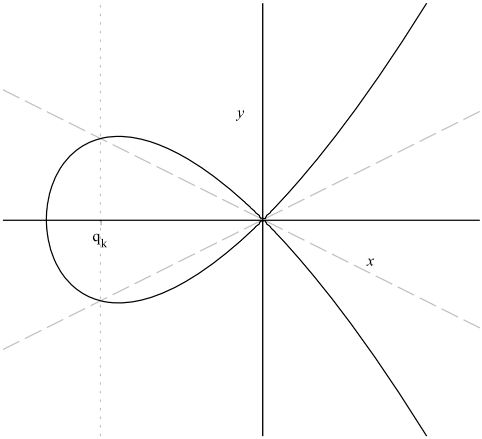
Proposition (Tangent Cone)

The tangent cone of $\mathbf{h} \subseteq \mathcal{R}[\mathbf{x}]^{|<\infty|}$ at $\mathbf{f}_\triangle \in \mathbb{T}_{\mathrm{reg}}(\mathcal{R}[\mathbf{x}])$ is

$$\kappa_{\mathbf{f}_\triangle}(\mathbf{h}^\downarrow) \subseteq \operatorname{qlim}\begin{cases} (\mathbf{x}^\perp - \mathbf{y}^\perp)\mathbf{m} = \mathbf{x} - \mathbf{y} \\ \mathbf{h}^\downarrow \\ [\mathbf{f}_\triangle]_{\mathbf{x}=\mathbf{y}} \end{cases}$$

when computed in $\mathcal{R}[\mathbf{m} \succ \mathbf{x}^\top \succ \mathbf{y} \succ \mathbf{x}^\perp]$.

Secant Cones



Implementation

Clever "Marc-timizations"

1. The Jacobean trick

$$\det(\mathrm{Jac}(\mathbf{h})) \not\equiv \mathbf{0} \bmod \langle\, \mathbf{f}_\triangle \,\rangle \iff \mathrm{im}(\mathbf{f}_\triangle; \mathbf{h}) = 1.$$

2. Cylindrification (alternative recursive step)

$$\mathrm{im}(\mathbf{f}_\triangle; \mathbf{h}) = \mathrm{im}(\mathbf{f}_\triangle; \text{pseudo remainder sequence on } \mathbf{h}).$$

$\mathbf{h} = \text{ojika2} \quad p = 0.$

$\text{im}(\mathbf{f}_\triangle;\, \mathbf{h})$	$\lvert\mathbf{f}_\triangle\rvert$	Bézout Weight	Cones	Total	Optimized
1	2	2	0.192	0.228	0.012
2	1	2	0.564	0.816	0.800
2	1	2	0.560	0.748	0.744
2	1	2	0.560	0.740	0.736
		8	1.876	2.532	2.292

(CASC paper timing = 8.80.)

$\mathbf{h} = \text{ojika3} \quad p = 0.$

$\text{im}(\mathbf{f}_\triangle; \mathbf{h})$	$\lvert\mathbf{f}_\triangle\rvert$	Bézout Weight	Cones	Total	Optimized
1	1	1	0.136	0.156	0.008
1	1	1	0.136	0.152	0.004
1	1	1	0.132	0.152	0.008
1	1	1	0.132	0.236	0.008
		4	0.536	0.696	0.028

(CASC paper timing = ∞, that is, intractable.)