Domain Theory

A *partially ordered set* is a pair $D = (U, \sqsubseteq_D)$ where

$\forall x \in U,\ x \sqsubseteq_D x$ *(Reflexivity)*

$\forall x,y \in U,\ x \sqsubseteq_D y$ and $y \sqsubseteq_D x$ implies $x = y$ *(Antisymmetry)*

$\forall x,y,z \in U,\ x \sqsubseteq_D y$ and $y \sqsubseteq_D z$ implies $x \sqsubseteq_D z$ *(Transitivity)*

Examples

Three partially ordered sets:

- The set of numbers with ≤, (N, ≤).
- The set of sets with $\subseteq$, (S, $\subseteq$)
- The set of Haskell values is partially ordered by the approximation relation: $\sqsubseteq$

x $\sqsubseteq$ y if x is an approximation of y

Bottom

- $\perp$ is used to represent the bottom element of the partially ordered set of Haskell values and most partially ordered sets.
- $\forall x \in U, \perp \sqsubseteq x$
- In $(N, \leq)$, $\perp = 0$, for natural numbers
- In $(S, \subseteq)$, $\perp$ is the empty set
- In Haskell, $\perp$ represents an incomplete evaluation. $\perp$ is an approximation of every value.

Haskell Approximation

Value A approximates value B, $A \sqsubseteq B$, if A is $\perp$ or A and B have the same form with each part of A approximating the corresponding part of B.

$1:2:3:\perp \sqsubseteq [1,2,3,4,5]$ *(because $\perp \sqsubseteq [4,5]$)*

$\perp \sqsubseteq (\perp, \perp) \sqsubseteq (\text{'a'}, \perp) \sqsubseteq (\text{'a'}, \text{'b'})$

But not: $(\perp, \text{'b'}) \sqsubseteq (\text{'a'}, \perp)$ because $\text{'b'} \not\sqsubseteq \perp$

Join

The *join* of elements x and y of a partially ordered set D is $x \sqcup y \in D$ where

$x \sqsubseteq x \sqcup y$ and $y \sqsubseteq x \sqcup y$

$\forall z \in D$, $x \sqsubseteq z$ and $y \sqsubseteq z$ implies $x \sqcup y \sqsubseteq z$

For sets, the join is usually the union.

For numbers, the join is the max of x and y.

Least Upper Bound

For a subset X of a partially ordered set D, the least upper bound is $\bigsqcup X \in D$ where

$\forall x \in X,\ x \sqsubseteq \bigsqcup X$

$\forall y \in D$, if $\forall x \in X,\ x \sqsubseteq y$ then $\bigsqcup X \sqsubseteq y$

Chains

A chain is a sequence of approximations, such as:

$\perp \sqsubseteq \perp : \perp : \perp \sqsubseteq 1 : \perp : \perp \sqsubseteq [1, 2, 3]$

A nonempty set C is a *chain* if

$\forall x,y \in C,\ x \sqsubseteq y$ or $y \sqsubseteq x$

For numbers, all sets are chains.

For sets, a chain is like a Russian doll.

Complete Partial Order

A partially ordered set is *complete* if every chain has a least upper bound.

A complete partial order may also require a bottom element, depending on who you ask.

The set of numbers is not a complete partial order for $\leq$, but any finite subset is.

For any set S, power set of S is a complete partial order for $\subseteq$.

Monotonicity

For any two partially ordered sets, D_1 and D_2, $f : D_1 \rightarrow D_2$ is *monotonic* if

$\forall x,y \in D_1$, $x \sqsubseteq y$ implies $f(x) \sqsubseteq f(y)$

Continuity

For any two complete partial orders, D_1 and D_2, $f : D_1 \rightarrow D_2$ is *continuous* if for all chains of D_1,

$$f(\bigsqcup C) = \bigsqcup \{f(c) \mid c \in C\}$$

Fixpoint

Let D be a partially ordered set

Let f be a function from D to D

d is a *fixpoint* of f if $d \in$ D and $f(d) = d$

d is the *least fixpoint* if

$$\forall d' \in \mathrm{D}[f(d') = d' \Rightarrow d \sqsubseteq d']$$

Fixpoint and Continuity

For any continuous function f, the least fixpoint is fix(f), where

$$\text{fix}(f) = \bigsqcup \{ f^{i}(\bot) \mid i \geq 0 \}$$

$$f^{0}(x) = x \text{ and } f^{i}(x) = f(f^{i-1}(x))$$