Domain Theory

Partially Ordered Set $(\cup, \sqsubseteq)$

A **partially ordered set** is a pair $D = (\cup, \sqsubseteq)$ where

$\forall x \in \cup$, $x \sqsubseteq x$ (Relexivity).
$\forall x \in \cup$, $(x \sqsubseteq y \Rightarrow x = y)$(Antisymmetry).
$\forall x, y, z \in \cup$, $x \sqsubseteq y$ and $y \sqsubseteq z \Rightarrow x \sqsubseteq z$ (Transitivity).

- If x and y are any enumerated data type then

$$(x \sqsubseteq y) \equiv (x = \perp) \vee (x = y)$$

Bottom $\perp$

$\perp$ is used to represent the bottom element of the partially ordered set of Haskell values and most partially ordered sets.

$$(\forall x \in \cup) \Rightarrow (\perp \sqsubseteq x)$$

- $1 : 2 : 3 : \perp \sqsubseteq [1, 2, 3, 4, 5]$
- $\perp \sqsubseteq (\perp, \perp) \sqsubseteq (a, \perp) \sqsubseteq (a, b)$
- But not $(\perp, b) \sqsubseteq (a, \perp)$ because $b \not\sqsubseteq \perp$.

Join $\sqcup$

The **join** of elements x and y of a partially ordered set D is $x \sqcup y \in D$ where

$x \sqsubseteq x \sqcup y$ and $y \sqsubseteq x \sqcup y$
$\forall z \in D$, $x \sqsubseteq z$ and $(y \sqsubseteq z) \Rightarrow (x \sqcup y \sqsubseteq z)$

Least Upper Bound $\bigsqcup$

For a subset X of a partially ordered set D, the least upper bound $\bigsqcup X \in D$ where

$\forall x \in X$, $x \sqsubseteq X$
$\forall y \in D$, if $\forall x \in X$, $(x \sqsubseteq y) \Rightarrow (\bigsqcup X \sqsubseteq y)$

Chains

A **chain** is a sequence of approximations, such as:

$$\perp \sqsubseteq \perp : b\perp : \perp \sqsubseteq 1 : \perp : \perp \sqsubseteq [1, 2, 3]$$

A nonempty set C is a chain if

$$(x, y \in C) \Rightarrow (x \sqsubseteq y \wedge y \sqsubseteq x)$$

Complete Partial Order

A partial order is **complete** if every chain has a least upper bound.

- For any set S, $\texttt{powerset}(S)$ is a complete partial order on $\subseteq$.

Monotonicity

For any two partially ordered sets, D_1 and D_2, $f : D_1 \to D_2$ is **monotonic** if

$$(\forall x, y \in D_1)\,(x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y))$$

Continuity

For any two complete partial orders, D_1 and D_2, $f : D_1 \to D_2$ is **continuous** if for all chains of D_1,

$$f\left(\bigsqcup C\right) = \bigsqcup \{f(c) | c \in C\}$$

Or alternatively in haskell.

$$f(lim_{n\to\infty}) = \lim_{n\to\infty} f(x_n)$$

Fixpoint

Let D be a partially ordered set and $f : D \to D$, d is a **fixpoint** of f if $d \in D$ and $f(d) = d$.

Least Fixpoint

Let D be a partially ordered set and $f : D \to D$, d is a **fixpoint** of f if

$$(\forall d' \in D)(f(d') = d' \Rightarrow d \sqsubseteq d')$$

Fixpoint and Continuity

For any continuous function f, the least fixpoint is $\texttt{fix}(f)$, where

$$\texttt{fix}(f) = \bigsqcup \left\{ f^i(\perp) | i \geq 0 \right\}$$

$$\text{where} \quad \mathrm{f^0(x) = x} \quad \text{and} \quad \mathrm{f^i(x) = f(f^{i-1}(x))}$$