MATHEMATICS 503 Spring 2006 EXAM 1 Due Monday May 8-th, 2006 by 12 noon

SOLUTIONS

1. **PROVE** or $\boxed{\textbf{DISPROVE}}$ (Circle your choice). If $\{g_1, g_2, g_3\}$ is a Gröbner basis for the ideal I in $k[x_1, x_2, \ldots, x_n]$, then so is $\{g_1,\ g_2 - g_1,\ g_3\}$.

Counterexample. Let $(g_1, g_2, g_3) = (x, y, z)$ with lex order. $\langle g_1, g_2, g_3 \rangle$ is the ideal of polynomials which vanish at $(0, 0, 0)$ whereas $\langle g_1, g_2 - g_1, g_3 \rangle$ is the same ideal, but is not a Gröbner basis in lex order. (there is no element in the basis whose leading term is a power of y.

2. $\boxed{\textbf{PROVE}}$ or **DISPROVE** (Circle your choice). If $\{g_1, g_2, g_3\}$ is a Gröbner basis for the ideal I in $k[x_1, x_2, \ldots, x_n]$ and $h \in I$, then $\{g_1,\ g_2,\ g_3,\ h\}$ is also a Gröbner basis for I.

Since $h \in \langle g_1, g_2, g_3 \rangle$

$$\langle \mathrm{LT}\,(I) \rangle = \langle \mathrm{LT}\,(g_1)\,, \mathrm{LT}\,(g_2)\,, \mathrm{LT}\,(g_3) \rangle = \langle \mathrm{LT}\,(g_1)\,, \mathrm{LT}\,(g_2)\,, \mathrm{LT}\,(g_3)\,, \mathrm{LT}\,(h) \rangle.$$

3. **PROVE** or $\boxed{\textbf{DISPROVE}}$ (Circle your choice). If $\{g_1, g_2, g_3\}$ is a Gröbner basis for the ideal I in $k[x_1, x_2, \ldots, x_n]$ and $h \in I$, then $\{g_1,\ g_1 + hg_2,\ g_3\}$ is also a Gröbner basis for I.

Counterexample. Let $(g_1, g_2, g_3) = (x, y, z)$ with lex order and $h = g_1$.

$$\langle x, y, z \rangle = \langle x, y + xy, z \rangle$$

but $\langle x, y + xy, z \rangle$ is not a Gröbner basis.

4. **PROVE** or $\boxed{\textbf{DISPROVE}}$ (Circle your choice). If $\{g_1, g_2, g_3\}$ is a Gröbner basis for the ideal I in $k[x_1, x_2, \ldots, x_n]$, and $h \in k[x_1, \ldots, x_n]$ then $\{g_1,\ g_2 - hg_1,\ g_3\}$ is also a Gröbner basis for I..

Use the same counterexample as in problem 1 above.

In the problems below: $f_1 = x + y^2$, $f_2 = x^2 + y^2$, $\mathbf{V} = \langle f_1, f_2 \rangle$ and $V_2 = \{f_1, f_2\}$.

In 5 and 6 use the grlex monomial order (with $x > y$)

5. Use the Buchsberger algorithm starting with f_1, f_2 to construct a Gröbner basis for $\mathbf{V}$.

$$\text{Buchsberger Gröbner basis} = \boxed{\{x + y^2,\ x^2 + y^2\}}.$$

Your answer should be a sequence of polynomials (the elements of this basis) and your work should contain evaluations of all $S(f, g)$'s and their remainders $\overline{S(f,g)}^F$'s, where F is an initial segment of your basis.

6. Your answer in the preceding may not be a reduced Gröbner basis. If not explain why not and show how the basis above can be refined to a reduced Gröbner basis. Give below this reduced Gröbner basis. (It may be the same as the basis you found above.) The above Gröbner basis is not reduced because $y^2 = \mathrm{LT}\,(f_1)$ divides the monomial y^2 in f_2. Dividing f_2 by f_1 and using the remainder to replace the f_2 in the preceding basis gives gives

$$\text{Reduced Gröbner basis} = \boxed{\{x + y^2,\ -x + x^2\}}.$$

In 7 and 8 use the lex monomial order (with $x > y$)

7. Use the Buchsberger algorithm starting with f_1, f_2 to construct a Gröbner basis for $\mathbf{V}$. We get

$$\text{Buchsberger Gröbner basis} = \boxed{\{x + y^2,\ x^2 + y^2,\ -y^2 - y^4\}}.$$

Your answer should be a sequence of polynomials (the elements of this basis) and your work should contain evaluations of all $S(f, g)$'s and their remainders $\overline{S(f,g)}^F$'s, where F is an initial segment of your basis.

8. Your answer in the preceding may not be a reduced Gröbner basis. If not explain why not and show how the basis above can be refined to a reduced Gröbner basis. Give below this reduced Gröbner basis. (It may be the same as the basis you found above.) Here $\mathrm{LT}(f_1) = x$ divides $\mathrm{LT}(f_2) = x^2$. Dividing f_2 by f_1 gives a remainder of $y^2 + y^4$ which is f_3 and already in the basis; so f_2 can be dropped from the preceding basis. The result is that

$$\text{Reduced Gröbner basis} = \boxed{\{x + y^2,\, y^2 + y^4\}}.$$

In 9 and 10 use the invlex monomial order (lex order with $y > x$)

9. Use the Buchsberger algorithm starting with f_1, f_2 to construct a Gröbner basis for $\mathbf{V}$.

$$\text{Buchsberger Gröbner basis} = \boxed{\{x + y^2,\, x^2 + y^2,\, -x + x^2\}}.$$

This basis isn't reduced either since $f_2 = 1 \cdot f_1 + (-x + x^2)$. Using this we can remove f_2 from the preceding basis and get

10. Your answer in the preceding may not be a reduced Gröbner basis. If not explain why not and show how the basis above can be refined to a reduced Gröbner basis. Give below this reduced Gröbner basis. (It may be the same as the basis you found above.)

$$\text{Reduced Gröbner basis} = \boxed{\{x + y^2,\, -x + x^2\}}.$$