Chapter 1, Geometry, Algebra, and Algorithms

§2. Affine Varieties.

Definition 1.2.1. Let k be a field and $f_1, \dots, f_s$ polynomials in $k[x_1, \dots, x_n]$. We set

$$\mathbf{V}(f_1, \dots, f_s) = \{(a_1, \dots, a_n) \in k^n : f_i(a_1, \dots, a_n) = 0,\ 1 \le i \le s\}$$

and call $\mathbf{V}(f_1, \dots, f_s)$ the *affine variety* defined by $f_1, \dots, f_s$.

Lemma 1.2.1. If V and W are affine varieties, then so are $V \cup W$ and $V \cap W$.

Proof. If $V = \mathbf{V}(f_1, \dots, f_s)$ and $W = \mathbf{V}(g_1, \dots, g_t)$, then we claim that

$$V \cap W = \mathbf{V}(f_1, \dots, f_s, g_1, \dots, g_t); \tag{1.2.1.1}$$

$$V \cup W = \mathbf{V}(f_i g_j : 1 \le i \le s,\ 1 \le j \le t). \tag{1.2.1.2}$$

(1.2.1.1) is trivial to prove; so we only prove the second. ($\subset$): If $\mathbf{p} \in V \cup W$ then either (i) every f_i vanishes at $\mathbf{p}$ or (ii) every g_j vanishes at $\mathbf{p}$. In either case every $f_i g_j$ vanishes at $\mathbf{p}$. ($\supset$): If every $f_i g_j$ vanishes at $\mathbf{p}$ then either (i) every f_i vanishes at $\mathbf{p}$ or (ii) every g_j vanishes at $\mathbf{p}$, Suppose otherwise, say $f_1(\mathbf{p}) \ne 0$ and $g_1(\mathbf{p}) \ne 0$. Then $(f_1 g_1)(\mathbf{p}) \ne 0$ contradicting the assumption that each of the products $f_i g_j$ vanishes at $\mathbf{p}$. ∎

---

§1.2.1.

Sketch the following affine varieties in $\mathbf{R}^2$.

(a) $\mathbf{V}(x^2 + 4y^2 + 2x - 16y + 1)$.

Solution. $x^2 + 4y^2 + 2x - 16y + 1 = (x+1)^2 + 4(y-2)^2 - 16$; so the relation $x^2 + 4y^2 + 2x - 16y + 1 = 0$ is equivalent to $\frac{(x+1)^2}{4^2} + \frac{(y-2)^2}{2^2} = 1$. The variety is an ellipse with axes parallel to the coordinate axes; center at $(-1, 2)$; horizontal semimajor axis of length 4; vertical semijor axis of length 2.

(b) $\mathbf{V}(x^2 - y^2)$.

Solution. This variety consists of the the union of the two lines with equations $y = x$ and $y = -x$.

(c) $\mathbf{V}(2x + y - 1,\ 3x - y + 2)$.

Solution. This variety consists of the single point with coordinates $\left(-\frac{1}{5}, \frac{7}{5}\right)$.

---

§1.2.2.

In $\mathbf{R}^2$, sketch $\mathbf{V}(y^2 - x(x-1)(x-2))$. Hint: For which x's is it possible to solve for y? How many y's correspond to each x? What symmetry does the curve have?

Solution. $x(x-1)(x-2) \ge 0$ only when $0 \le x \le 1$ or $2 \le x$. The curve is symmetric w.r.t. the x axis and consists of two disconnected pieces: (i) A closed loop when $0 \le x \le 1$ and (ii) a "sort" of "parabola-like" section opening to the right with "vertex" at $(2, 0)$ and symmetry axis the line $y = 0$ when $2 \le x$.

---

§1.2.3.

In the plane $\mathbf{R}^2$, draw a picture to illustrate

$$\mathbf{V}(x^2 + y^2 - 4) \cap \mathbf{V}(xy - 1) = \mathbf{V}(x^2 + y^2 - 4,\ xy - 1),$$

and determine the points of intersection. Note that this is a special case of Lemma 2.

Solution. The circle and hyperbola intersect at the points with coordinates $\left(t, \frac{1}{t}\right)$ where $t^2 + \frac{1}{t^2} - 4 = 0$ or $(t^2)^2 - 4t^2 + 1 = 0$. There are 4 such points: $(\sqrt{2+\sqrt{3}}, \sqrt{2-\sqrt{3}})$; its image $(\sqrt{2-\sqrt{3}}, \sqrt{2+\sqrt{3}})$ under reflection in the line $y = x$; and the rflections of these two points in the origin.

$\boxed{\S\mathbf{1.2.4.}}$

Sketch the following affine varieties in $\mathbf{R}^3$:

(a) $\mathbf{V}(x^2 + y^2 + z^2 - 1)$.

Solution. This variety is a sphere of radius 1 centered at the origin.

(b) $\mathbf{V}(x^2 + y^2 - 1)$.

Solution. This variety is a cylinder with generators parallel to the z-axis and based on the circle of radius 1 with center at (0,0) in the x, y-plane.

(c) $\mathbf{V}(x + 2, y - 1.5, z)$.

Solution. This variety is the single point $(-2, 1.5, 0)$.

(d) $\mathbf{V}(xz^2 - xy)$. Hint: Factor $xz^2 - xy$.

Solution. $xz^2 - xy = x(z^2 - y)$. This variety consists of the union of the yz-coordinate plane and a parabolic cylinder with generators parallel to the x-coordinate axis based on the parabola $y = z^2$ in the yz-plane.

(e) $\mathbf{V}(x^4 - zx,\, x^3 - yx)$.

Solution. This variety consists of the yz-coordinate plane union the curve with parametric representation $x \mapsto (x, x^2, x^3)$.

(f) $\mathbf{V}(x^2 + y^2 + z^2 - 1,\, x^2 + y^2 + (z-1)^2 - 1)$.

Solution. This variety is a circle. It lies on the plane with equation $z = \frac{1}{2}$; has center at $\left(0, 0, \frac{1}{2}\right)$ and radius $\frac{\sqrt{3}}{2}$.

In each case above does the variety have the dimension you would intuitively expect it to have?

Answers: (a) yes; (b) yes; (c) yes; (d) yes; (e) yes and no; (f) yes.

$\boxed{\S\mathbf{1.2.5.}}$

Use the proof of Lemma 2 to sketch $\mathbf{V}((x-2)(x^2-y),\, y(x^2-y),\, (z+1)(x^2-y))$ in $\mathbf{R}^3$. Hint: This is the union of what two varieties?

Solution. This is $\mathbf{V}(x-2,\, y,\, z+1) \cup \mathbf{V}(x^2 - y)$. It consists of the point (2,0,-1) together with the parabolic cylinder $x^2 = y$.

$\boxed{\S\mathbf{1.2.6.}}$

Show that all finite subsets of k^n are affine varieties.

Proof. Each point is an affine variety and each union of varieties is a variaty; so any finite subset is a variaty.

$\boxed{\S\mathbf{1.2.7.}}$

One of the prettiest examples from polar coordinates is the four-leaved rose whose polar equation is $r = \sin 2\theta$. We will now show that this curve is an affine variety.

(a)$\{(r\cos\theta,\, r\sin\theta)\colon r = \sin 2\theta\} \subseteq \{(x,y)\colon (x^2+y^2)^3 - 4x^2y^2 = 0\}$.

Proof. Suppose $r = \sin 2\theta$. Put $x = r\cos\theta$ and $y = r\sin\theta$ and use the relation $r^2 = x^2 + y^2$. Straightforward computation yields that

$$\begin{aligned}
(x^2+y^2)^3 - 4x^2y^2 &= (r^2)^3 - (2r\cos\theta \cdot r\sin\theta)^2 \\
&= r^4\left(r^2 - \sin^2 2\theta\right) \\
&= r^4(r + \sin 2\theta)(r - \sin 2\theta) \\
&= 0.
\end{aligned}$$

(b) $\{(r\cos\theta,\, r\sin\theta)\colon r = \sin 2\theta\} \supseteq \{(x,y)\colon (x^2+y^2)^3 - 4x^2y^2 = 0\}$.

Proof. Given (x, y) choose r and ψ so that $x = r\cos\psi$ and $y = r\sin\psi$. Another possibility is then $x = (-r)\cos(\psi + \pi)$ and $y = (-r)\sin(\psi + \pi)$. Computation then yields

$$\begin{aligned} 0 &= (x^2 + y^2)^3 - 4x^2y^2 \\ &= \left((x^2 + y^2)^{\frac{3}{2}} - 2xy\right)\left((x^2 + y^2)^{\frac{3}{2}} + 2xy\right) \\ &= r^2\,(r - \sin 2\psi) \cdot r^2\,(r + \sin 2\psi)\,. \end{aligned}$$

There are three cases. CASE I: $r = \sin 2\psi$. In this case we are done because $(x, y) = (r\cos\psi, r\sin\psi)$ with $r = \sin 2\psi$. CASE II: $r = -\sin 2\psi$. In this case $(x, y) = \big((-r)\cos(\psi + \pi), (-r)\sin(\psi + \pi)\big)$ and $(-r) = \sin 2\psi = \sin 2(\psi + \pi)$. So here too (x, y) has the required form with r replaced by $-r$ and θ by $\psi + \pi$. CASE III: $r = 0$. In this case $(0, 0) = (r\cos 0, r\sin 0)$ with $0 = \sin 2 \cdot 0$.

---

§1.2.8.

It can take some work to show that something is not an affine variety. For example, consider the set

$$X = \{(x, x) \colon x \in \mathbf{R},\ x \neq 1\} \subset \mathbf{R}^2,$$

which is the straight line $y = x$ with the point (1,1) removed. To show that X is not an affine variety, suppose that $X = \mathbf{V}(f_1, \ldots, f_n)$. Then each f_i vanishes on X, and if we can show that f_i also vanishes at $(1, 1)$ we will get the desired contradiction. Thus, here is what you are to prove: if $f \in \mathbf{R}[x, y]$ vanishes on X, then $f(1, 1) = 0$. We give two solutions.

Solution 1. The set $\{(x, y) \colon f(x, y) = 0\}$ is a closed set containing X; so it must contain $(1, 1)$ since this last point is in the closure of X.

Solution 2. Let $g(t) = f(t, t)$. $g(t) \in \mathbf{R}[t]$ is a polynomial and thus is either the zero polynomial or it can have at most a finite number of roots. In this case g vanishes on every point except for 1. It must be the zero polynomial and $0 = g(1) = f(1, 1)$.

---

§1.2.9.

Let $R = \{(x, y) \in \mathbf{R}^2 \colon y > 0\}$ be the upper half plane. Prove that R is not an affine variety.

Proof. Any polynomial which vanishes on the upper half plane must vanish on the entire plane.

---

§1.2.10.

Let $\mathbf{Z}^n \subset \mathbf{C}^n$ consist of the points with integer coordinates. Prove that $\mathbf{Z}^n$ is not an affine variety.

Solution. (By Induction) If $n = 1$ the theorem is true because any nonzero polynomial in one variable has at most a finite number of roots and so $\mathbf{Z}$ is not its set of zeros. Suppose the theorem is true for $n = k$ (k is a positive integer in this solution, not a field.) . Let $f(x_1, \ldots, x_k, x_{k+1}) \in \mathbf{R}[x_1, \ldots, x_{k+1}]$. Suppose f vanishes on $\mathbf{Z}^{k+1}$. Then for each $(a_1, \ldots, a_k) \in \mathbf{Z}^k$, the polynomial $f(a_1, \ldots, a_k, x_{k+1}) \in \mathbf{R}[x_{k+1}]$ vanishes on $\mathbf{Z}$ and is hence the zero polynomial. This means that each of the coefficients of the one variable polynomial $f(a_1, \ldots, a_k, x_{k+1}) \in \mathbf{R}[x_{k+1}]$ is zero. As this must hold for every choice of $(a_1, \ldots, a_k) \in \mathbf{Z}^k$ and these coefficients are polynomials in the a_i's, it follows from the inductive assumption each such coefficient is the zero polynomial in $\mathbf{R}[x_1, \ldots, x_k]$ and hence zero. But then f too is the zero polynomial.

---

§1.2.11.

So far we have discussed varieties over $\mathbf{R}$ or $\mathbf{C}$. It is also possible to consider varieties over the field of rational numbers $\mathbf{Q}$, although the questions here tend to be much harder. For example, let n be a positive integer, and consider the variety $F_n \subset \mathbf{Q}^2$ defined by

$$x^n + y^n = 1.$$

Notice that there are some obvious solutions when x or y is zero. We call these *trivial solutions*. An interesting question is whether or not there are any nontrivial solutions.

(a) Show that F_n has two trivial solutions if n is odd and four trivial solutions if n is even.

Proof. If n is odd $0^n + y^n = 1$ implies that $y = 1$; so there are just the two trivial solutions $(0, 1)$ and $(1, 0)$. If n is even there are the four $(\pm 1, 0)$ and $(0, \pm 1)$.

(b) Show that F_n has a nontrivial solution for some $n \geq 3$ if and only Fermat's Last Theorem is false.

Fermat's Last Theorem states that, for $n \geq 3$, the equation $x^n + y^n = z^n$ has no solutions where x, y, and z are nonzero integers.

Proof. If x, y are non zero integers satisfying $x^n + y^n = 1$, then for any $z \neq 0$, $(xz)^n + (yz)^n = z^n$ which would contradict Fermat's Last Theorem. On the other hand if Fermat's Last Theorem is false and $x^n + y^n = z^n$, then $\left(\frac{x}{z}\right)^n + \left(\frac{y}{n}\right)^n = 1$ and $\left(\frac{x}{z}, \frac{y}{z}\right)$ is a non trivial (rational) solution to $x^n + y^n = 1$

$\boxed{\S\mathbf{1.2.12.}}$