

# MEEN 621 Notes

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## 1. Mathematics

### 1.1. First-Order Linear ODE

Equations of the form  $\frac{dy}{dx} + \alpha(x)y = \beta(x)$  can be solved by multiplying both sides by an integrating factor  $e^{\int \alpha(x)dx}$ . Then the left-hand side becomes  $\frac{d(e^{\int \alpha(x)dx}y)}{dx}$  and the right hand side becomes  $e^{\int \alpha(x)dx}\beta(x)$ . The left and right hand sides can then be integrated directly.

## 2. Continuum Flow

### 2.1. Mass Conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \mathbf{v}_j) = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_j)}{\partial x_j} = 0$$

### 2.2. Momentum Conservation

$$\rho \left[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right] = \rho \vec{f} - \vec{\nabla} p + \mu \nabla^2 \vec{v}$$

#### Cartesian Coordinates

$$\rho \left[ \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3} \right] = \rho f_1 - \frac{\partial p}{\partial x_1} + \mu \left[ \frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_3^2} \right]$$

$$\rho \left[ \frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} + v_3 \frac{\partial v_2}{\partial x_3} \right] = \rho f_2 - \frac{\partial p}{\partial x_2} + \mu \left[ \frac{\partial^2 v_2}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_2^2} + \frac{\partial^2 v_2}{\partial x_3^2} \right]$$

$$\rho \left[ \frac{\partial v_3}{\partial t} + v_1 \frac{\partial v_3}{\partial x_1} + v_2 \frac{\partial v_3}{\partial x_2} + v_3 \frac{\partial v_3}{\partial x_3} \right] = \rho f_3 - \frac{\partial p}{\partial x_3} + \mu \left[ \frac{\partial^2 v_3}{\partial x_1^2} + \frac{\partial^2 v_3}{\partial x_2^2} + \frac{\partial^2 v_3}{\partial x_3^2} \right]$$

### 3. Vorticity Dynamics

$$\text{Circulation } \Gamma = \oint_C \vec{v} \cdot d\vec{l} = \int_S \vec{\omega} \cdot \hat{n} dA \text{ (Flux of Vorticity across surface } S)$$

$$\text{Average angular velocity } \bar{\Omega} = \frac{\bar{u}_\theta}{a} = \frac{\oint_C \vec{v} \cdot d\vec{l}}{2\pi a^2} = \frac{\Gamma}{2\pi a^2} \text{ (On a circle of radius } a) = \frac{\omega_j n_j}{2}$$

For Irrotational Flow,  $\Gamma = 0$ ,  $\vec{\omega} = 0$

A vector field  $\vec{\alpha}$  is considered solenoidal if  $\vec{\nabla} \cdot \vec{\alpha} = 0$

$$\vec{\nabla} \cdot \vec{\omega} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) = 0$$

#### 3.1. Streamlines and Vortex Lines

$$\text{Equations for Streamlines (Cartesian Coordinates)} \quad \frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z}$$

$$\text{Equations for Streamlines (Cylindrical Polar Coordinates)} \quad \frac{dR}{u_R} = \frac{Rd\phi}{u_\phi} = \frac{dz}{u_z}$$

$$\text{Equations for Vortex Lines (Cartesian Coordinates)} \quad \frac{dx}{\omega_x} = \frac{dy}{\omega_y} = \frac{dz}{\omega_z}$$

#### 3.2. Terms

- **Barotropic:** Density is a function of pressure only.  $\frac{1}{\rho^2} \vec{\nabla} \rho \times \vec{\nabla} p = 0$
- **Inviscid:** Viscosity can be neglected.  $\nu \nabla^2 \vec{\omega} = 0$
- **Baroclinic:** Measure of how misaligned the gradient of pressure is from the gradient of density in a fluid  $\vec{\nabla} \rho \times \vec{\nabla} p$
- **Isobars:** Constant pressure lines
- **Isopycals:** Constant density lines

### 3.3. Vorticity Transport Equation

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \vec{\nabla})\vec{v} + \frac{1}{\rho^2} \vec{\nabla}\rho \times \vec{\nabla}p + \nu \nabla^2 \vec{\omega}$$

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{\omega} = (\vec{\omega} \cdot \vec{\nabla})\vec{v} + \frac{1}{\rho^2} \vec{\nabla}\rho \times \vec{\nabla}p + \nu \nabla^2 \vec{\omega}$$

#### Legend

- $\frac{D\vec{\omega}}{Dt}$ : Total rate of change of vorticity of a fluid particle
- $(\vec{\omega} \cdot \vec{\nabla})\vec{v}$ : Vorticity production due to stretching/tilting of vortex lines
- $\frac{1}{\rho^2} \vec{\nabla}\rho \times \vec{\nabla}p$ : Vorticity production due to baroclinic effects
- $\nu \nabla^2 \vec{\omega}$ : Viscous diffusion of vorticity which dissipates or redistributes vorticity

### 3.4. Helmholtz's Vortex Theorems for Inviscid Flows

#### Assumptions

- Inviscid
- Barotropic
- Conservative Body Forces  $\vec{F} = -\vec{\nabla}\psi$

#### Statement

- Fluid particles/element originally free of vorticity remain free of vorticity (non-rotating)
- Vortex lines (tubes) move with the fluid for inviscid flows. Vortex line is always comprised of the same fluid particles (Vorticity is frozen to the flow for inviscid flow)
- The strength of the vortex tube (circulation) does not vary with time during the fluid motion. Vortex tubes must be closed, go to infinity, or end on solid boundaries.

### 3.5. Kelvin's circulation theorem

#### Assumptions

- Inviscid
- Barotropic
- Incompressible
- Conservative Body Forces  $\vec{F} = -\nabla\psi$

#### Statement

The circulation (strength of the vortex) around a closed curve moving with the fluid will remain constant.

$$\frac{\partial \Gamma_C}{\partial t} + v_j \frac{\partial \Gamma_C}{\partial x_j} = \oint_C \nu \nabla^2 \vec{v} \cdot d\vec{x}$$

For inviscid flow,  $\frac{\partial \Gamma_C}{\partial t} + v_j \frac{\partial \Gamma_C}{\partial x_j} = 0$

### 3.6. Bernoulli's Equation

#### Assumptions

- Inviscid  $\nabla^2 v = 0 \rightarrow$  Euler
- Only gravitational body forces (Conservative)  $\rightarrow f_i = -\frac{d\psi}{dx_i}$  where  $\psi = gz$
- Barotropic  $\rightarrow \rho = \rho(p)$
- Steady flow  $\rightarrow \frac{\partial}{\partial t} = 0$

Note: No restriction on the compressibility effects

## Statement

$$\nabla \left[ \frac{\vec{v} \cdot \vec{v}}{2} + gz + \int \frac{dp}{\rho} \right] = -\vec{\omega} \times \vec{v}$$

Along streamline or vortex line  $\frac{\vec{v} \cdot \vec{v}}{2} + gz + \int \frac{dp}{\rho} = \text{constant}$

Unsteady flow  $\frac{\partial \phi}{\partial t} + \frac{\vec{v} \cdot \vec{v}}{2} + gz + \int \frac{dp}{\rho} = F(t)$

Steady Potential flow  $\frac{\vec{v} \cdot \vec{v}}{2} + gz + \int \frac{dp}{\rho} = \text{constant throughout the flow}$

## 4. Potential Flow

$$i = \sqrt{-1}$$

$$z = x + iy = re^{i\theta} = r(\cos \theta + i \sin \theta)$$

$$\bar{z} = x - iy = re^{-i\theta} = r(\cos \theta - i \sin \theta)$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$r = \sqrt{x^2 + y^2} = |z| = |z\bar{z}|^{\frac{1}{2}}$$

$$\theta = \arctan \frac{y}{x}$$

2D Incompressible Potential Flow  $\nabla^2 \phi = 0 = \nabla^2 \psi$

Complex Potential  $F(z) = \phi(z) + i\psi(z)$

Complex Velocity  $w(z) = u - iv = (v_r - iv_\theta)e^{-i\theta} = \frac{dF(z)}{dz}$

$$\bar{w}(z) = u + iv = (v_r + iv_\theta)e^{i\theta}$$

#### 4.1. Stream Function $\Leftrightarrow$ Potential Function $\Leftrightarrow$ Velocities

Also called Cauchy Reimann Equations

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$$

$$v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

#### 4.2. Cartesian $\Leftrightarrow$ Polar Velocities

$$u = v_r \cos \theta - v_\theta \sin \theta$$

$$v = v_r \sin \theta + v_\theta \cos \theta$$

$$v_r = u \cos \theta + v \sin \theta$$

$$v_\theta = -u \sin \theta + v \cos \theta$$



### 4.3. Complex Potentials

	$F(z)$	$\phi$	$\psi$	$w(z)$	$u$ and $v_r$	$v$ and $v_\theta$
Uniform	$Ue^{-i\alpha}z$	$U(x \cos \alpha + y \sin \alpha)$	$U(y \cos \alpha - x \sin \alpha)$	$Ue^{-i\alpha}$	$U \cos \alpha$	$U \sin \alpha$
		$Ur \cos(\theta - \alpha)$	$Ur \sin(\theta - \alpha)$		$U \cos(\theta - \alpha)$	$-U \sin(\theta - \alpha)$
Corner	$Cz^n$			$nCz^{n-1}$		
		$Cr^n \cos n\theta$	$Cr^n \sin n\theta$		$nCr^{n-1} \cos[(n-1)\theta]$	$-nCr^{n-1} \sin[(n-1)\theta]$
Source/Sink	$\frac{m}{2\pi} \ln(z - z_0)$	$\frac{m}{4\pi} \ln[x^2 + y^2]$	$\frac{m}{2\pi} \arctan \frac{y}{x}$	$\frac{m}{2\pi(z-z_0)}$	$\frac{m}{2\pi} \frac{x}{x^2+y^2}$	$\frac{m}{2\pi} \frac{y}{x^2+y^2}$
		$\frac{m}{2\pi} \ln r$	$\frac{m}{2\pi} \theta$		$\frac{m}{2\pi r}$	0
Free Vortex	$-\frac{i\Gamma}{2\pi} \ln(z - z_0)$	$\frac{\Gamma}{2\pi} \arctan \frac{y}{x}$	$-\frac{\Gamma}{4\pi} \ln[x^2 + y^2]$	$-\frac{i\Gamma}{2\pi(z-z_0)}$	$-\frac{\Gamma}{2\pi} \frac{y}{x^2+y^2}$	$\frac{\Gamma}{2\pi} \frac{x}{x^2+y^2}$
		$\frac{\Gamma}{2\pi} \theta$	$-\frac{\Gamma}{2\pi} \ln r$		0	$\frac{\Gamma}{2\pi r}$
Dipole	$\frac{\mu}{\pi z}$	$\frac{\mu}{\pi} \frac{x}{x^2+y^2}$	$-\frac{\mu}{\pi} \frac{y}{x^2+y^2}$	$-\frac{\mu}{\pi z^2}$	$\frac{\mu}{\pi} \frac{y^2-x^2}{(x^2+y^2)^2}$	$-\frac{\mu}{\pi} \frac{2xy}{(x^2+y^2)^2}$
		$\frac{\mu}{\pi r} \cos \theta$	$-\frac{\mu}{\pi r} \sin \theta$		$-\frac{\mu}{\pi r^2} \cos \theta$	$-\frac{\mu}{\pi r^2} \sin \theta$

### Legend

- Uniform
  - $U$ : Uniform velocity magnitude
  - $\alpha$ : Angle of attack - the angle at which the direction of the uniform velocity is oriented with respect to the horizontal
- Corner
  - $C$ : Indicates the direction.  $C > 0$  always
  - $n$ :  $\frac{\pi}{\text{angle of the corner}}$
- Source/Sink

- $m$ : Volume flow rate per unit dimension normal to the page. For source,  $m > 0$ , whereas for sink,  $m < 0$
- Dipole/Doublet
  - $a$ : Half distance between the source and sink
  - $Q$ : Volume flow rate per unit dimension normal to the page
  - $\mu$ :  $Qa$

#### 4.4. Infinite Series Expansions

- $\ln(1 + \epsilon) = \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3} - \dots$  when  $|\epsilon| \leq 1$
- $(1 + \epsilon)^{-1} = 1 - \epsilon + \epsilon^2 - \epsilon^3 + \dots$  when  $|\epsilon| \leq 1$
- $(1 + \epsilon)^{\frac{1}{2}} = 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \frac{1}{16}\epsilon^3 - \frac{5}{128}\epsilon^4 + \frac{7}{256}\epsilon^5 - \dots$  when  $|\epsilon| \leq 1$
- $(1 + \epsilon)^{-\frac{1}{2}} = 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \frac{35}{128}\epsilon^4 - \frac{63}{256}\epsilon^5 + \dots$  when  $|\epsilon| \leq 1$
- $e^\epsilon = 1 + \frac{\epsilon}{1!} + \frac{\epsilon^2}{2!} + \frac{\epsilon^3}{3!} + \dots$
- $\sin(\epsilon) = \epsilon - \frac{\epsilon^3}{3!} + \frac{\epsilon^5}{5!} - \frac{\epsilon^7}{7!} + \dots$

#### 4.5. Bernoulli's Equation (Irrotational)

$$p_\infty + \frac{\rho v_\infty^2}{2} = p + \frac{\rho |v|^2}{2}$$

#### 4.6. Forces on a 2D Body

*All the below forces have the units of Force per unit length normal to the sheet of paper*

$$\text{Drag Force} = - \int_C (p - p_\infty) \cos \theta \, ds$$

$$\text{Lift Force} = - \int_C (p - p_\infty) \sin \theta \, ds$$

$$\begin{aligned} \text{Complex Force } G = D - iL &= -i \oint_C p d\bar{z} = -i \oint_C \left[ p_\infty + \frac{\rho v_\infty^2}{2} - \frac{\rho |v^2|}{2} \right] d\bar{z} \\ &= \frac{i\rho}{2} \oint_C |v^2| d\bar{z} = \frac{i\rho}{2} \oint_C w \bar{w} d\bar{z} \\ &= \frac{i\rho}{2} \oint_C [w(z)]^2 dz \quad (\text{First Blasius Integral Law}) \end{aligned}$$

#### 4.7. Residue Theorem

$$\text{If } F(z) = \sum_j \frac{B_j}{z - z_j}$$

$$R_k = \sum B$$

$$\oint_C F(z) dz = 2\pi i \sum_k R_k$$