MEEN 621 Notes

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Contents

1. Potential Flow

$$
\begin{aligned}
i &= \sqrt{-1} \\
z &= x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta) \\
\bar{z} &= x - iy = re^{-i\theta} = r(\cos\theta - i\sin\theta) \\
x &= r\cos\theta \\
y &= r\sin\theta \\
r &= \sqrt{x^2 + y^2} = |z| = |z\bar{z}|^{\frac{1}{2}} \\
\theta &= arctan\frac{y}{x}
\end{aligned}
$$

Complex Potential $F(z) = \phi(z) + i\psi(z)$

Complex Velocity $w(z) = u - iv = (v_r - iv_\theta)e^{-i\theta} = \dfrac{dF(z)}{dz}$

$$\bar{w}(z) = u + iv = (v_r + iv_\theta)e^{i\theta}$$

1.1. Cartesian ⇔ Polar Velocities

$$
\begin{aligned}
u &= v_r\cos\theta - v_\theta\sin\theta \\
v &= v_r\sin\theta + v_\theta\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
v_r &= u\cos\theta + v\sin\theta \\
v_\theta &= -u\sin\theta + v\cos\theta
\end{aligned}
$$

1.2. Stream Function ⇔ Potential Function ⇔ Velocities

Also called Cauchy Reimann Equations

$$
\begin{aligned}
u &= \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \\
v &= \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}
\end{aligned}
$$

$$
\begin{aligned}
v_r &= \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \\
v_\theta &= \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r}
\end{aligned}
$$

1.3. Complex Potentials in Cartesian Coordinates

	ϕ	ψ	u	v
Uniform	$U(x\cos\alpha + y\sin\alpha)$	$U(y\cos\alpha - x\sin\alpha)$	$U\cos\alpha$	$U\sin\alpha$
Corner				
Source	$\frac{m}{4\pi}\ln[(x-x_0)^2+(y-y_0)^2]$	$\frac{m}{2\pi}\arctan\frac{y-y_0}{x-x_0}$	$\frac{m}{2\pi}(\frac{x}{x^2+y^2})$	$\frac{m}{2\pi}(\frac{y}{x^2+y^2})$
Free Vortex	$\frac{m}{4\pi}\ln[(x-x_0)^2+(y-y_0)^2]$	$\frac{m}{2\pi}\arctan\frac{y-y_0}{x-x_0}$	$\frac{m}{2\pi}(\frac{x}{x^2+y^2})$	$\frac{m}{2\pi}(\frac{y}{x^2+y^2})$

1.4. Complex Potentials in Polar Coordinates

	ϕ	ψ	v_r	v_θ
Uniform	$Ur\cos(\theta-\alpha)$	$Ur\sin(\theta-\alpha)$	$U\cos(\theta-\alpha)$	$-U\sin(\theta-\alpha)$
Corner	$Cr^n\cos n\theta$	$Cr^n\sin n\theta$	$nCr^{n-1}\cos[(n-1)\theta]$	$-nCr^{n-1}\sin[(n-1)\theta]$
Source	$\frac{m}{2\pi}\ln r$	$\frac{m}{2\pi}\theta$	$\frac{m}{2\pi r}$	0
Free Vortex	$\frac{m}{4\pi}\ln[(x-x_0)^2+(y-y_0)^2]$	$\frac{m}{2\pi}\arctan\frac{y-y_0}{x-x_0}$	$\frac{m}{2\pi}(\frac{x}{x^2+y^2})$	$\frac{m}{2\pi}(\frac{y}{x^2+y^2})$

Legend

- Uniform
  - U: Uniform velocity magnitude
  - α: Angle of attack - the angle at which the direction of the uniform velocity is oriented with respect to the horizontal
- Corner
  - C: Indicates the direction. $C > 0$ always
  - n: $\dfrac{\pi}{\text{angle of the corner}}$
- Source/Sink
  - m: Volume flow rate per unit dimension normal to the page. For source, $m > 0$, whereas for sink, $m < 0$