MEEN 621 Notes

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Contents

1. Potential Flow

$$
\begin{aligned}
i &= \sqrt{-1} \\
z &= x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta) \\
\bar{z} &= x - iy = re^{-i\theta} = r(\cos\theta - i\sin\theta) \\
x &= r\cos\theta \\
y &= r\sin\theta \\
r &= \sqrt{x^2 + y^2} = |z| = |z\bar{z}|^{\frac{1}{2}} \\
\theta &= arctan\frac{y}{x}
\end{aligned}
$$

2D Incompressible Potential Flow $\nabla^2\phi = 0 = \nabla^2\psi$

Complex Potential $F(z) = \phi(z) + i\psi(z)$

Complex Velocity $w(z) = u - iv = (v_r - iv_\theta)e^{-i\theta} = \frac{dF(z)}{dz}$

$$\bar{w}(z) = u + iv = (v_r + iv_\theta)e^{i\theta}$$

1.1. Stream Function ⇔ Potential Function ⇔ Velocities

Also called Cauchy Reimann Equations

$$
\begin{aligned}
u &= \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \\
v &= \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}
\end{aligned}
$$

$$
\begin{aligned}
v_r &= \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \\
v_\theta &= \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r}
\end{aligned}
$$

1.2. Cartesian $\Leftrightarrow$ Polar Velocities

$$u = v_r \cos\theta - v_\theta \sin\theta$$
$$v = v_r \sin\theta + v_\theta \cos\theta$$

$$v_r = u\cos\theta + v\sin\theta$$
$$v_\theta = -u\sin\theta + v\cos\theta$$

1.3. Complex Potentials

	F(z)	ϕ	ψ	w(z)	u and v_r	v and v_θ
Uniform	$Ue^{-i\alpha}z$	$U(x\cos\alpha + y\sin\alpha)$	$U(y\cos\alpha - x\sin\alpha)$	$Ue^{-i\alpha}$	$U\cos\alpha$	$U\sin\alpha$
		$Ur\cos(\theta-\alpha)$	$Ur\sin(\theta-\alpha)$		$U\cos(\theta-\alpha)$	$-U\sin(\theta-\alpha)$
Corner	Cz^n			nCz^{n-1}		
		$Cr^n\cos n\theta$	$Cr^n\sin n\theta$		$nCr^{n-1}\cos[(n-1)\theta]$	$-nCr^{n-1}\sin[(n-1)\theta]$
Source/Sink	$\frac{m}{2\pi}\ln(z-z_0)$	$\frac{m}{4\pi}\ln[(x-x_0)^2+(y-y_0)^2]$	$\frac{m}{2\pi}\arctan\frac{y-y_0}{x-x_0}$	$\frac{m}{2\pi(z-z_0)}$	$\frac{m}{2\pi}\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}$	$\frac{m}{2\pi}\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}$
		$\frac{m}{2\pi}\ln r$	$\frac{m}{2\pi}\theta$		$\frac{m}{2\pi r}$	0
Free Vortex	$-\frac{i\Gamma}{2\pi}\ln(z-z_0)$	$\frac{\Gamma}{2\pi}\arctan\frac{y-y_0}{x-x_0}$	$-\frac{\Gamma}{2\pi}\ln\sqrt{(x-x_0)^2+(y-y_0)^2}$	$-\frac{i\Gamma}{2\pi(z-z_0)}$	$-\frac{\Gamma}{2\pi}\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}$	$\frac{\Gamma}{2\pi}\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}$
		$\frac{\Gamma}{2\pi}\theta$	$-\frac{\Gamma}{2\pi}\ln r$		0	$\frac{\Gamma}{2\pi r}$
Dipole	$\frac{\mu}{\pi z}$	$\frac{\mu}{\pi}\frac{x}{x^2+y^2}$	$-\frac{\mu}{\pi}\frac{y}{x^2+y^2}$	$-\frac{\mu}{\pi z^2}$	$\frac{\mu}{\pi}\frac{y^2-x^2}{(x^2+y^2)^2}$	$-\frac{\mu}{\pi}\frac{2xy}{(x^2+y^2)^2}$
		$\frac{\mu}{\pi r}\cos\theta$	$-\frac{\mu}{\pi r}\sin\theta$		$-\frac{\mu}{\pi r^2}\cos\theta$	$-\frac{\mu}{\pi r^2}\sin\theta$

Legend

- Uniform
  - U: Uniform velocity magnitude
  - α: Angle of attack - the angle at which the direction of the uniform velocity is oriented with respect to the horizontal

- Corner
  - C: Indicates the direction. $C > 0$ always
  - n: $\dfrac{\pi}{\text{angle of the corner}}$
- Source/Sink
  - m: Volume flow rate per unit dimension normal to the page. For source, $m > 0$, whereas for sink, $m < 0$
- Dipole/Doublet
  - a: Half distance between the source and sink
  - Q: Volume flow rate per unit dimension normal to the page
  - μ: Qa

1.4. Infinite Series Expansions

- $\ln(1+\epsilon) = \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3} - \dots$ when $|\epsilon| \leq 1$
- $(1+\epsilon)^{-1} = 1 - \epsilon + \epsilon^2 - \epsilon^3 + \dots$ when $|\epsilon| \leq 1$
- $(1+\epsilon)^{\frac{1}{2}} = 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \frac{1}{16}\epsilon^3 - \frac{5}{128}\epsilon^4 + \frac{7}{256}\epsilon^5 - \dots$ when $|\epsilon| \leq 1$
- $(1+\epsilon)^{-\frac{1}{2}} = 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \frac{35}{128}\epsilon^4 - \frac{63}{256}\epsilon^5 + \dots$ when $|\epsilon| \leq 1$
- $e^{\epsilon} = 1 - \frac{\epsilon^2}{2!} + \frac{\epsilon^4}{4!} - \dots$
- $\sin(\epsilon) = \epsilon - \frac{\epsilon^3}{3!} + \frac{\epsilon^5}{5!} - \frac{\epsilon^7}{7!} + \dots$

1.5. Bernoulli's Equation (Irrotational)

$$p_\infty + \frac{\rho v_\infty^2}{2} = p + \frac{\rho|v^2|}{2}$$

1.6. Forces on a 2D Body

All the below forces have the units of Force per unit length normal to the sheet of paper

$$\text{Drag Force} = -\int_C (p - p_\infty)\cos\theta \; ds$$
$$\text{Lift Force} = -\int_C (p - p_\infty)\sin\theta \; ds$$
$$\text{Complex Force } G = D - iL = -i\oint_C p d\bar{z} = -i\oint_C \left[p_\infty + \frac{\rho v_\infty^2}{2} - \frac{\rho|v^2|}{2}\right] d\bar{z}$$
$$= \frac{i\rho}{2}\oint_C |v^2| d\bar{z} = \frac{i\rho}{2}\oint_C w\bar{w} d\bar{z}$$
$$= \frac{i\rho}{2}\oint_C [w(z)]^2 dz \text{ (First Blasius Integral Law)}$$

1.7. Residue Theorem

$$\text{If } F(z) = \sum_i \frac{R_i}{z - z_i}$$
$$\oint_C F(z) dz = 2\pi i \sum_i R_i$$