MEEN 621 Notes

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Contents

1. Mathematics

2. Continuum Flow

2.1. Mass Conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.(\rho v_j) = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_j)}{\partial x_j} = 0$$

2.2. Momentum Conservation

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\vec{\nabla})\vec{v}\right] = \rho\vec{f} - \vec{\nabla}p + \mu\nabla^2\vec{v}$$

Cartesian Coordinates

$$\rho\left[\frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial x_1} + v_2\frac{\partial v_1}{\partial x_2} + v_3\frac{\partial v_1}{\partial x_3}\right] = \rho f_1 - \frac{\partial p}{\partial x_1} + \mu\left[\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_3^2}\right]$$

$$\rho\left[\frac{\partial v_2}{\partial t} + v_1\frac{\partial v_2}{\partial x_1} + v_2\frac{\partial v_2}{\partial x_2} + v_3\frac{\partial v_2}{\partial x_3}\right] = \rho f_2 - \frac{\partial p}{\partial x_2} + \mu\left[\frac{\partial^2 v_2}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_2^2} + \frac{\partial^2 v_2}{\partial x_3^2}\right]$$

$$\rho\left[\frac{\partial v_3}{\partial t} + v_1\frac{\partial v_3}{\partial x_1} + v_2\frac{\partial v_3}{\partial x_2} + v_3\frac{\partial v_3}{\partial x_3}\right] = \rho f_3 - \frac{\partial p}{\partial x_3} + \mu\left[\frac{\partial^2 v_3}{\partial x_1^2} + \frac{\partial^2 v_3}{\partial x_2^2} + \frac{\partial^2 v_3}{\partial x_3^2}\right]$$

3. Vorticity Dynamics

Circulation $\Gamma = \oint_C \vec{v}.\vec{dl} = \int_S \vec{\omega}.\hat{n}\, dA$ (Flux of Vorticity across surface S)

Average angular velocity $\bar{\Omega} = \frac{\bar{u}_\theta}{a} = \frac{\oint_C \vec{v}.\vec{dl}}{2\pi a^2} = \frac{\Gamma}{2\pi a^2}$ (On a circle of radius a) $= \frac{\omega_j n_j}{2}$

For Irrotational Flow, $\Gamma = 0,\ \vec{\omega} = 0$

3.1. Streamlines and Vortex Lines

Equations for Streamlines (Cartesian Coordinates) $\dfrac{dx}{u_x} = \dfrac{dy}{u_y} = \dfrac{dz}{u_z}$

Equations for Vortex Lines (Cartesian Coordinates) $\dfrac{dx}{\omega_x} = \dfrac{dy}{\omega_y} = \dfrac{dz}{\omega_z}$

4. Potential Flow

$$i = \sqrt{-1}$$
$$z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta)$$
$$\bar{z} = x - iy = re^{-i\theta} = r(\cos\theta - i\sin\theta)$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2} = |z| = |z\bar{z}|^{\frac{1}{2}}$$
$$\theta = arctan\frac{y}{x}$$

2D Incompressible Potential Flow $\nabla^2\phi = 0 = \nabla^2\psi$

Complex Potential $F(z) = \phi(z) + i\psi(z)$

Complex Velocity $w(z) = u - iv = (v_r - iv_\theta)e^{-i\theta} = \dfrac{dF(z)}{dz}$

$$\bar{w}(z) = u + iv = (v_r + iv_\theta)e^{i\theta}$$

4.1. Stream Function ⇔ Potential Function ⇔ Velocities

Also called Cauchy Reimann Equations

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

4.2. Cartesian ⇔ Polar Velocities

$$u = v_r \cos \theta - v_\theta \sin \theta$$

$$v = v_r \sin \theta + v_\theta \cos \theta$$

$$v_r = u \cos \theta + v \sin \theta$$

$$v_\theta = -u \sin \theta + v \cos \theta$$

4.3. Complex Potentials

	F(z)	ϕ	ψ	w(z)	u and v_r	v and v_θ
Uniform	$Ue^{-i\alpha}z$	$U(x\cos\alpha + y\sin\alpha)$	$U(y\cos\alpha - x\sin\alpha)$	$Ue^{-i\alpha}$	$U\cos\alpha$	$U\sin\alpha$
		$Ur\cos(\theta - \alpha)$	$Ur\sin(\theta - \alpha)$		$U\cos(\theta - \alpha)$	$-U\sin(\theta - \alpha)$
Corner	Cz^n			nCz^{n-1}		
		$Cr^n\cos n\theta$	$Cr^n\sin n\theta$		$nCr^{n-1}\cos[(n-1)\theta]$	$-nCr^{n-1}\sin[(n-1)\theta]$
Source/Sink	$\frac{m}{2\pi}\ln(z - z_0)$	$\frac{m}{4\pi}\ln[(x - x_0)^2 + (y - y_0)^2]$	$\frac{m}{2\pi}\arctan\frac{y - y_0}{x - x_0}$	$\frac{m}{2\pi(z - z_0)}$	$\frac{m}{2\pi}\frac{x - x_0}{(x-x_0)^2 + (y-y_0)^2}$	$\frac{m}{2\pi}\frac{y - y_0}{(x-x_0)^2 + (y-y_0)^2}$
		$\frac{m}{2\pi}\ln r$	$\frac{m}{2\pi}\theta$		$\frac{m}{2\pi r}$	0
Free Vortex	$-\frac{i\Gamma}{2\pi}\ln(z - z_0)$	$\frac{\Gamma}{2\pi}\arctan\frac{y - y_0}{x - x_0}$	$-\frac{\Gamma}{2\pi}\ln\sqrt{(x - x_0)^2 + (y - y_0)^2}$	$-\frac{i\Gamma}{2\pi(z - z_0)}$	$-\frac{\Gamma}{2\pi}\frac{y - y_0}{(x-x_0)^2 + (y-y_0)^2}$	$\frac{\Gamma}{2\pi}\frac{x - x_0}{(x-x_0)^2 + (y-y_0)^2}$
		$\frac{\Gamma}{2\pi}\theta$	$-\frac{\Gamma}{2\pi}\ln r$		0	$\frac{\Gamma}{2\pi r}$
Dipole	$\frac{\mu}{\pi z}$	$\frac{\mu}{\pi}\frac{x}{x^2 + y^2}$	$-\frac{\mu}{\pi}\frac{y}{x^2 + y^2}$	$-\frac{\mu}{\pi z^2}$	$\frac{\mu}{\pi}\frac{y^2 - x^2}{(x^2 + y^2)^2}$	$-\frac{\mu}{\pi}\frac{2xy}{(x^2 + y^2)^2}$
		$\frac{\mu}{\pi r}\cos\theta$	$-\frac{\mu}{\pi r}\sin\theta$		$-\frac{\mu}{\pi r^2}\cos\theta$	$-\frac{\mu}{\pi r^2}\sin\theta$

Legend

- Uniform

  - U: Uniform velocity magnitude
  - α: Angle of attack - the angle at which the direction of the uniform velocity is oriented with respect to the horizontal

- Corner

  - C: Indicates the direction. $C > 0$ always
  - n: $\dfrac{\pi}{\text{angle of the corner}}$

- Source/Sink

  - m: Volume flow rate per unit dimension normal to the page. For source, $m > 0$, whereas for sink, $m < 0$

- Dipole/Doublet

  - a: Half distance between the source and sink

- Q: Volume flow rate per unit dimension normal to the page
- μ: Qa

4.4. Infinite Series Expansions

- $\ln(1 + \epsilon) = \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3} - \ldots$ when $|\epsilon| \leq 1$
- $(1 + \epsilon)^{-1} = 1 - \epsilon + \epsilon^2 - \epsilon^3 + \ldots$ when $|\epsilon| \leq 1$
- $(1 + \epsilon)^{\frac{1}{2}} = 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \frac{1}{16}\epsilon^3 - \frac{5}{128}\epsilon^4 + \frac{7}{256}\epsilon^5 - \ldots$ when $|\epsilon| \leq 1$
- $(1 + \epsilon)^{-\frac{1}{2}} = 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \frac{35}{128}\epsilon^4 - \frac{63}{256}\epsilon^5 + \ldots$ when $|\epsilon| \leq 1$
- $e^\epsilon = 1 - \frac{\epsilon^2}{2!} + \frac{\epsilon^4}{4!} - \ldots$
- $\sin(\epsilon) = \epsilon - \frac{\epsilon^3}{3!} + \frac{\epsilon^5}{5!} - \frac{\epsilon^7}{7!} + \ldots$

4.5. Bernoulli's Equation (Irrotational)

$$p_\infty + \frac{\rho v_\infty^2}{2} = p + \frac{\rho |v^2|}{2}$$

4.6. Forces on a 2D Body

All the below forces have the units of Force per unit length normal to the sheet of paper

$$\text{Drag Force} = -\int_C (p - p_\infty) \cos\theta \, ds$$

$$\text{Lift Force} = -\int_C (p - p_\infty) \sin\theta \, ds$$

$$\text{Complex Force } G = D - iL = -i \oint_C p \, d\bar{z} = -i \oint_C \left[p_\infty + \frac{\rho v_\infty^2}{2} - \frac{\rho |v^2|}{2} \right] d\bar{z}$$

$$= \frac{i\rho}{2} \oint_C |v^2| d\bar{z} = \frac{i\rho}{2} \oint_C w\bar{w} \, d\bar{z}$$

$$= \frac{i\rho}{2} \oint_C [w(z)]^2 dz \text{ (First Blasius Integral Law)}$$

4.7. Residue Theorem

If $F(z) = \sum_i \dfrac{R_i}{z - z_i}$

$$\oint_C F(z)\,dz = 2\pi i \sum_i R_i$$