MEEN 621 Notes

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Contents

1. Mathematics

1.1. First-Order Linear ODE

Equations of the form $\frac{dy}{dx}+\alpha(x)y=\beta(x)$ can be solved by multiplying both sides by an integrating factor $e^{\int \alpha(x)dx}$. Then the left-hand side becomes $\frac{d(e^{\int \alpha(x)dx}.y)}{dx}$ and the right hand side becomes $e^{\int \alpha(x)dx}\beta(x)$. The left and right hand sides can then be integrated directly.

2. Continuum Flow

2.1. Mass Conservation

$$\frac{\partial \rho}{\partial t}+\vec{\nabla}.(\rho v_j)=0$$
$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v_j)}{\partial x_j}=0$$

2.2. Momentum Conservation

$$\rho\left[\frac{\partial \vec{v}}{\partial t}+(\vec{v}.\vec{\nabla})\vec{v}\right]=\rho\vec{f}-\vec{\nabla}p+\mu\nabla^2\vec{v}$$

Cartesian Coordinates

$$\rho\left[\frac{\partial v_1}{\partial t}+v_1\frac{\partial v_1}{\partial x_1}+v_2\frac{\partial v_1}{\partial x_2}+v_3\frac{\partial v_1}{\partial x_3}\right]=\rho f_1-\frac{\partial p}{\partial x_1}+\mu\left[\frac{\partial^2 v_1}{\partial x_1^2}+\frac{\partial^2 v_1}{\partial x_2^2}+\frac{\partial^2 v_1}{\partial x_3^2}\right]$$
$$\rho\left[\frac{\partial v_2}{\partial t}+v_1\frac{\partial v_2}{\partial x_1}+v_2\frac{\partial v_2}{\partial x_2}+v_3\frac{\partial v_2}{\partial x_3}\right]=\rho f_2-\frac{\partial p}{\partial x_2}+\mu\left[\frac{\partial^2 v_2}{\partial x_1^2}+\frac{\partial^2 v_2}{\partial x_2^2}+\frac{\partial^2 v_2}{\partial x_3^2}\right]$$
$$\rho\left[\frac{\partial v_3}{\partial t}+v_1\frac{\partial v_3}{\partial x_1}+v_2\frac{\partial v_3}{\partial x_2}+v_3\frac{\partial v_3}{\partial x_3}\right]=\rho f_3-\frac{\partial p}{\partial x_3}+\mu\left[\frac{\partial^2 v_3}{\partial x_1^2}+\frac{\partial^2 v_3}{\partial x_2^2}+\frac{\partial^2 v_3}{\partial x_3^2}\right]$$

3. Vorticity Dynamics

$$\text{Circulation } \Gamma = \oint_C \vec{v}.\vec{dl} = \int_S \vec{\omega}.\hat{n}\, dA \text{ (Flux of Vorticity across surface } S\text{)}$$

$$\text{Average angular velocity } \bar{\Omega} = \frac{\bar{u}_\theta}{a} = \frac{\oint_C \vec{v}.\vec{dl}}{2\pi a^2} = \frac{\Gamma}{2\pi a^2} \text{ (On a circle of radius } a\text{)} = \frac{\omega_j n_j}{2}$$

$$\text{For Irrotational Flow, } \Gamma = 0,\ \vec{\omega} = 0$$

$$\text{A vector field } \vec{\alpha} \text{ is considered solenoidal if } \vec{\nabla}.\vec{\alpha} = 0$$

$$\vec{\nabla}.\vec{\omega} = \vec{\nabla}.(\vec{\nabla} \times \vec{v}) = 0$$

3.1. Streamlines and Vortex Lines

$$\text{Equations for Streamlines (Cartesian Coordinates) } \frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z}$$

$$\text{Equations for Streamlines (Cylindrical Polar Coordinates) } \frac{dR}{u_R} = \frac{R d\phi}{u_\phi} = \frac{dz}{u_z}$$

$$\text{Equations for Vortex Lines (Cartesian Coordinates) } \frac{dx}{\omega_x} = \frac{dy}{\omega_y} = \frac{dz}{\omega_z}$$

3.2. Terms

- **Barotropic**: Density is a function of pressure only. $\frac{1}{\rho^2}\vec{\nabla}\rho \times \vec{\nabla}p = 0$
- **Inviscid**: Viscosity can be neglected. $\nu\nabla^2\vec{\omega} = 0$
- **Baroclinic**: Measure of how misaligned the gradient of pressure is from the gradient of density in a fluid $\vec{\nabla}\rho \times \vec{\nabla}p$
- **Isobars**: Constant pressure lines
- **Isopycals**: Constant density lineses

3.3. Vorticity Transport Equation

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega}.\vec{\nabla})\vec{v} + \frac{1}{\rho^2}\vec{\nabla}\rho \times \vec{\nabla}p + \nu\nabla^2\vec{\omega}$$
$$\frac{\partial\vec{\omega}}{\partial t} + (\vec{v}.\vec{\nabla})\vec{\omega} = (\vec{\omega}.\vec{\nabla})\vec{v} + \frac{1}{\rho^2}\vec{\nabla}\rho \times \vec{\nabla}p + \nu\nabla^2\vec{\omega}$$

Legend

- $\frac{D\vec{\omega}}{Dt}$: Total rate of change of vorticity of a fluid particle
- $(\vec{\omega}.\vec{\nabla})\vec{v}$: Vorticity production due to stretching/tilting of vortex lines
- $\frac{1}{\rho^2}\vec{\nabla}\rho \times \vec{\nabla}p$: Vorticity production due to baroclinic effects
- $\nu\nabla^2\vec{\omega}$: Viscous diffusion of vorticity which dissipates or redistributes vorticity

3.4. Helmholtz's Vortex Theorems for Inviscid Flows

Assumptions

- Inviscid
- Barotropic
- Conservative Body Forces $\vec{F} = -\vec{\nabla}\psi$

Statement

- Fluid particles/element orignially free of vorticity remain free of vorticity (non-rotating)
- Vortex lines (tubes) move with the fluid for inviscid flows. Vortex line is always comprised of the same fluid particles (Vorticity is frozen to the flow for inviscid flow)
- The strength of the vortex tube (circulation) does not vary with time during the fluid motion. Vortex tubes must be closed, go to infinity, or end on solid boundaries.

3.5. Kelvin's circulation theorem

Assumptions

- Inviscid
- Barotropic
- Incompressible
- Conservative Body Forces $\vec{F} = -\nabla\vec{\psi}$

Statement

The circulation (strength of the vortex) around a closed curve moving with the fluid will remain constant.

$$\frac{\partial \Gamma_C}{\partial t} + v_j \frac{\partial \Gamma_C}{\partial x_j} = \oint_C \nu \nabla^2 \vec{v}.\vec{dx}$$

For inviscid flow, $\dfrac{\partial \Gamma_C}{\partial t} + v_j \dfrac{\partial \Gamma_C}{\partial x_j} = 0$

3.6. Bernoulli's Equation

Assumptions

- Inviscid $\nabla^2 v = 0 \rightarrow$ Euler
- Only gravitational body forces (Conservative) $\rightarrow f_i = -\dfrac{d\psi}{dx_i}$ where $\psi = gz$
- Barotropic $\rightarrow \rho = \rho(p)$
- Steady flow $\rightarrow \dfrac{\partial}{\partial t} = 0$

Note: No restriction on the compressiblity effects

Statement

$$\nabla\left[\frac{\vec{v}.\vec{v}}{2}+gz+\int\frac{dp}{\rho}\right]=-\vec{\omega}\times\vec{v}$$

$$\text{Along streamline or vortex line } \frac{\vec{v}.\vec{v}}{2}+gz+\int\frac{dp}{\rho}=\text{constant}$$

$$\text{Unsteady flow } \frac{\partial\phi}{\partial t}+\frac{\vec{v}.\vec{v}}{2}+gz+\int\frac{dp}{\rho}=F(t)$$

$$\text{Steady Potential flow } \frac{\vec{v}.\vec{v}}{2}+gz+\int\frac{dp}{\rho}=\text{constant throughout the flow}$$

4. Potential Flow

$$i=\sqrt{-1}$$

$$z=x+iy=re^{i\theta}=r(\cos\theta+i\sin\theta)$$

$$\bar{z}=x-iy=re^{-i\theta}=r(\cos\theta-i\sin\theta)$$

$$x=r\cos\theta$$

$$y=r\sin\theta$$

$$r=\sqrt{x^2+y^2}=|z|=|z\bar{z}|^{\frac{1}{2}}$$

$$\theta=arctan\frac{y}{x}$$

$$\text{2D Incompressible Potential Flow } \nabla^2\phi=0=\nabla^2\psi$$

$$\text{Complex Potential } F(z)=\phi(z)+i\psi(z)$$

$$\text{Complex Velocity } w(z)=u-iv=(v_r-iv_\theta)e^{-i\theta}=\frac{dF(z)}{dz}$$

$$\bar{w}(z)=u+iv=(v_r+iv_\theta)e^{i\theta}$$

4.1. Stream Function ⇔ Potential Function ⇔ Velocities

Also called Cauchy Reimann Equations

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$
$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$
$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

4.2. Cartesian ⇔ Polar Velocities

$$u = v_r \cos\theta - v_\theta \sin\theta$$
$$v = v_r \sin\theta + v_\theta \cos\theta$$

$$v_r = u\cos\theta + v\sin\theta$$
$$v_\theta = -u\sin\theta + v\cos\theta$$

4.3. Complex Potentials

	F(z)	ϕ	ψ	w(z)	u and v_r	v and v_θ
Uniform	$Ue^{-i\alpha}z$	$U(x\cos\alpha + y\sin\alpha)$	$U(y\cos\alpha - x\sin\alpha)$	$Ue^{-i\alpha}$	$U\cos\alpha$	$U\sin\alpha$
		$Ur\cos(\theta-\alpha)$	$Ur\sin(\theta-\alpha)$		$U\cos(\theta-\alpha)$	$-U\sin(\theta-\alpha)$
Corner	Cz^n			nCz^{n-1}		
		$Cr^n\cos n\theta$	$Cr^n\sin n\theta$		$nCr^{n-1}\cos[(n-1)\theta]$	$-nCr^{n-1}\sin[(n-1)\theta]$
Source/Sink	$\frac{m}{2\pi}\ln(z-z_0)$	$\frac{m}{4\pi}\ln[(x-x_0)^2+(y-y_0)^2]$	$\frac{m}{2\pi}\arctan\frac{y-y_0}{x-x_0}$	$\frac{m}{2\pi(z-z_0)}$	$\frac{m}{2\pi}\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}$	$\frac{m}{2\pi}\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}$
		$\frac{m}{2\pi}\ln r$	$\frac{m}{2\pi}\theta$		$\frac{m}{2\pi r}$	0
Free Vortex	$-\frac{i\Gamma}{2\pi}\ln(z-z_0)$	$\frac{\Gamma}{2\pi}\arctan\frac{y-y_0}{x-x_0}$	$-\frac{\Gamma}{2\pi}\ln\sqrt{(x-x_0)^2+(y-y_0)^2}$	$-\frac{i\Gamma}{2\pi(z-z_0)}$	$-\frac{\Gamma}{2\pi}\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}$	$\frac{\Gamma}{2\pi}\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}$
		$\frac{\Gamma}{2\pi}\theta$	$-\frac{\Gamma}{2\pi}\ln r$		0	$\frac{\Gamma}{2\pi r}$
Dipole	$\frac{\mu}{\pi z}$	$\frac{\mu}{\pi}\frac{x}{x^2+y^2}$	$-\frac{\mu}{\pi}\frac{y}{x^2+y^2}$	$-\frac{\mu}{\pi z^2}$	$\frac{\mu}{\pi}\frac{y^2-x^2}{(x^2+y^2)^2}$	$-\frac{\mu}{\pi}\frac{2xy}{(x^2+y^2)^2}$
		$\frac{\mu}{\pi r}\cos\theta$	$-\frac{\mu}{\pi r}\sin\theta$		$-\frac{\mu}{\pi r^2}\cos\theta$	$-\frac{\mu}{\pi r^2}\sin\theta$

Legend

- Uniform
  - U: Uniform velocity magnitude
  - α: Angle of attack - the angle at which the direction of the uniform velocity is oriented with respect to the horizontal
- Corner
  - C: Indicates the direction. $C > 0$ always
  - n: $\dfrac{\pi}{\text{angle of the corner}}$
- Source/Sink
  - m: Volume flow rate per unit dimension normal to the page. For source, $m > 0$, whereas for sink, $m < 0$
- Dipole/Doublet
  - a: Half distance between the source and sink

- Q: Volume flow rate per unit dimension normal to the page
- μ: Qa

4.4. Infinite Series Expansions

- $\ln(1+\epsilon) = \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3} - \dots$ when $|\epsilon| \leq 1$
- $(1+\epsilon)^{-1} = 1 - \epsilon + \epsilon^2 - \epsilon^3 + \dots$ when $|\epsilon| \leq 1$
- $(1+\epsilon)^{\frac{1}{2}} = 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \frac{1}{16}\epsilon^3 - \frac{5}{128}\epsilon^4 + \frac{7}{256}\epsilon^5 - \dots$ when $|\epsilon| \leq 1$
- $(1+\epsilon)^{-\frac{1}{2}} = 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \frac{35}{128}\epsilon^4 - \frac{63}{256}\epsilon^5 + \dots$ when $|\epsilon| \leq 1$
- $e^{\epsilon} = 1 - \frac{\epsilon^2}{2!} + \frac{\epsilon^4}{4!} - \dots$
- $\sin(\epsilon) = \epsilon - \frac{\epsilon^3}{3!} + \frac{\epsilon^5}{5!} - \frac{\epsilon^7}{7!} + \dots$

4.5. Bernoulli's Equation (Irrotational)

$$p_\infty + \frac{\rho v_\infty^2}{2} = p + \frac{\rho |v^2|}{2}$$

4.6. Forces on a 2D Body

All the below forces have the units of Force per unit length normal to the sheet of paper

$$\begin{aligned}
\text{Drag Force} &= -\int_C (p - p_\infty)\cos\theta \; ds \\
\text{Lift Force} &= -\int_C (p - p_\infty)\sin\theta \; ds \\
\text{Complex Force } G = D - iL &= -i\oint_C p d\bar{z} = -i\oint_C \left[p_\infty + \frac{\rho v_\infty^2}{2} - \frac{\rho|v^2|}{2}\right] d\bar{z} \\
&= \frac{i\rho}{2}\oint_C |v^2| d\bar{z} = \frac{i\rho}{2}\oint_C w\bar{w} d\bar{z} \\
&= \frac{i\rho}{2}\oint_C [w(z)]^2 dz \text{ (First Blasius Integral Law)}
\end{aligned}$$

4.7. Residue Theorem

$$\text{If } F(z) = \sum_i \frac{R_i}{z - z_i}$$

$$\oint_C F(z) dz = 2\pi i \sum_i R_i$$