

MEEN 621 Notes

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1. Vorticity Dynamics

$$\text{Circulation } \Gamma = \oint_C \vec{v} \cdot d\vec{l} = \int_S \vec{\omega} \cdot \hat{n} dA \text{ (Flux of Vorticity across surface } S)$$

$$\text{Average angular velocity } \bar{\Omega} = \frac{\bar{u}_\theta}{a} = \frac{\oint_C \vec{v} \cdot d\vec{l}}{2\pi a^2} = \frac{\Gamma}{2\pi a^2} \text{ (On a circle of radius } a) = \frac{\omega_j n_j}{2}$$

For Irrotational Flow, $\Gamma = 0$, $\vec{\omega} = 0$

2. Potential Flow

$$i = \sqrt{-1}$$

$$z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$\bar{z} = x - iy = re^{-i\theta} = r(\cos\theta - i\sin\theta)$$

$$x = r \cos\theta$$

$$y = r \sin\theta$$

$$r = \sqrt{x^2 + y^2} = |z| = |z\bar{z}|^{\frac{1}{2}}$$

$$\theta = \arctan \frac{y}{x}$$

$$\text{2D Incompressible Potential Flow } \nabla^2 \phi = 0 = \nabla^2 \psi$$

$$\text{Complex Potential } F(z) = \phi(z) + i\psi(z)$$

$$\text{Complex Velocity } w(z) = u - iv = (v_r - iv_\theta)e^{-i\theta} = \frac{dF(z)}{dz}$$

$$\bar{w}(z) = u + iv = (v_r + iv_\theta)e^{i\theta}$$

2.1. Stream Function \Leftrightarrow Potential Function \Leftrightarrow Velocities

Also called Cauchy Reimann Equations

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$$

$$v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

2.2. Cartesian \Leftrightarrow Polar Velocities

$$u = v_r \cos \theta - v_\theta \sin \theta$$

$$v = v_r \sin \theta + v_\theta \cos \theta$$

$$v_r = u \cos \theta + v \sin \theta$$

$$v_\theta = -u \sin \theta + v \cos \theta$$

2.3. Complex Potentials

	$F(z)$	ϕ	ψ	$w(z)$	u and v_r	v and v_θ
Uniform	$Ue^{-i\alpha}z$	$U(x \cos \alpha + y \sin \alpha)$	$U(y \cos \alpha - x \sin \alpha)$	$Ue^{-i\alpha}$	$U \cos \alpha$	$U \sin \alpha$
		$Ur \cos(\theta - \alpha)$	$Ur \sin(\theta - \alpha)$		$U \cos(\theta - \alpha)$	$-U \sin(\theta - \alpha)$
Corner	Cz^n			nCz^{n-1}		
		$Cr^n \cos n\theta$	$Cr^n \sin n\theta$		$nCr^{n-1} \cos[(n-1)\theta]$	$-nCr^{n-1} \sin[(n-1)\theta]$
Source/Sink	$\frac{m}{2\pi} \ln(z - z_0)$	$\frac{m}{4\pi} \ln[(x - x_0)^2 + (y - y_0)^2]$	$\frac{m}{2\pi} \arctan \frac{y-y_0}{x-x_0}$	$\frac{m}{2\pi(z-z_0)}$	$\frac{m}{2\pi} \frac{x-x_0}{(x-x_0)^2 + (y-y_0)^2}$	$\frac{m}{2\pi} \frac{y-y_0}{(x-x_0)^2 + (y-y_0)^2}$
		$\frac{m}{2\pi} \ln r$	$\frac{m}{2\pi} \theta$		$\frac{m}{2\pi r}$	0
Free Vortex	$-\frac{i\Gamma}{2\pi} \ln(z - z_0)$	$\frac{\Gamma}{2\pi} \arctan \frac{y-y_0}{x-x_0}$	$-\frac{\Gamma}{2\pi} \ln \sqrt{(x - x_0)^2 + (y - y_0)^2}$	$-\frac{i\Gamma}{2\pi(z-z_0)}$	$-\frac{\Gamma}{2\pi} \frac{y-y_0}{(x-x_0)^2 + (y-y_0)^2}$	$\frac{\Gamma}{2\pi} \frac{x-x_0}{(x-x_0)^2 + (y-y_0)^2}$
		$\frac{\Gamma}{2\pi} \theta$	$-\frac{\Gamma}{2\pi} \ln r$		0	$\frac{\Gamma}{2\pi r}$
Dipole	$\frac{\mu}{\pi z}$	$\frac{\mu}{\pi} \frac{x}{x^2 + y^2}$	$-\frac{\mu}{\pi} \frac{y}{x^2 + y^2}$	$-\frac{\mu}{\pi z^2}$	$\frac{\mu}{\pi} \frac{y^2 - x^2}{(x^2 + y^2)^2}$	$-\frac{\mu}{\pi} \frac{2xy}{(x^2 + y^2)^2}$
		$\frac{\mu}{\pi r} \cos \theta$	$-\frac{\mu}{\pi r} \sin \theta$		$-\frac{\mu}{\pi r^2} \cos \theta$	$-\frac{\mu}{\pi r^2} \sin \theta$

Legend

- Uniform
 - U : Uniform velocity magnitude
 - α : Angle of attack - the angle at which the direction of the uniform velocity is oriented with respect to the horizontal
- Corner
 - C : Indicates the direction. $C > 0$ always
 - n : $\frac{\pi}{\text{angle of the corner}}$
- Source/Sink
 - m : Volume flow rate per unit dimension normal to the page. For source, $m > 0$, whereas for sink, $m < 0$
- Dipole/Doublet
 - a : Half distance between the source and sink

- Q : Volume flow rate per unit dimension normal to the page
- μ : Qa

2.4. Infinite Series Expansions

- $\ln(1 + \epsilon) = \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3} - \dots$ when $|\epsilon| \leq 1$
- $(1 + \epsilon)^{-1} = 1 - \epsilon + \epsilon^2 - \epsilon^3 + \dots$ when $|\epsilon| \leq 1$
- $(1 + \epsilon)^{\frac{1}{2}} = 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \frac{1}{16}\epsilon^3 - \frac{5}{128}\epsilon^4 + \frac{7}{256}\epsilon^5 - \dots$ when $|\epsilon| \leq 1$
- $(1 + \epsilon)^{-\frac{1}{2}} = 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \frac{35}{128}\epsilon^4 - \frac{63}{256}\epsilon^5 + \dots$ when $|\epsilon| \leq 1$
- $e^\epsilon = 1 + \frac{\epsilon}{1!} + \frac{\epsilon^2}{2!} + \frac{\epsilon^3}{3!} + \dots$
- $\sin(\epsilon) = \epsilon - \frac{\epsilon^3}{3!} + \frac{\epsilon^5}{5!} - \frac{\epsilon^7}{7!} + \dots$

2.5. Bernoulli's Equation (Irrotational)

$$p_\infty + \frac{\rho v_\infty^2}{2} = p + \frac{\rho |v^2|}{2}$$

2.6. Forces on a 2D Body

All the below forces have the units of Force per unit length normal to the sheet of paper

$$\text{Drag Force} = - \int_C (p - p_\infty) \cos \theta \, ds$$

$$\text{Lift Force} = - \int_C (p - p_\infty) \sin \theta \, ds$$

$$\begin{aligned} \text{Complex Force } G = D - iL &= -i \oint_C p d\bar{z} = -i \oint_C \left[p_\infty + \frac{\rho v_\infty^2}{2} - \frac{\rho |v^2|}{2} \right] d\bar{z} \\ &= \frac{i\rho}{2} \oint_C |v^2| d\bar{z} = \frac{i\rho}{2} \oint_C w \bar{w} d\bar{z} \\ &= \frac{i\rho}{2} \oint_C [w(z)]^2 dz \quad (\text{First Blasius Integral Law}) \end{aligned}$$

2.7. Residue Theorem

$$\text{If } F(z) = \sum_i \frac{R_i}{z - z_i}$$

$$\oint_C F(z) dz = 2\pi i \sum_i R_i$$