# MEEN 621 Notes

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## Contents

# 1. Mathematics

## 1.1. Integral Tables

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### Basic Forms

$$\int x^n dx = \frac{1}{n+1}x^{n+1} \tag{1}$$

$$\int \frac{1}{x} dx = \ln|x| \tag{2}$$

$$\int udv = uv - \int vdu \tag{3}$$

$$\int \frac{1}{ax+b} dx = \frac{1}{a}\ln|ax+b| \tag{4}$$

### Integrals of Rational Functions

$$\int \frac{1}{(x+a)^2} dx = -\frac{1}{x+a} \tag{5}$$

$$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1}, n \neq -1 \tag{6}$$

$$\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)} \tag{7}$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x \tag{8}$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} \tag{9}$$

$$\int \frac{x}{a^2+x^2} dx = \frac{1}{2}\ln|a^2+x^2| \tag{10}$$

$$\int \frac{x^2}{a^2+x^2} dx = x - a\tan^{-1}\frac{x}{a} \tag{11}$$

$$\int \frac{x^3}{a^2+x^2} dx = \frac{1}{2}x^2 - \frac{1}{2}a^2\ln|a^2+x^2| \tag{12}$$

$$\int \frac{1}{ax^2+bx+c} dx = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}} \tag{13}$$

$$\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a}\ln\frac{a+x}{b+x},\ a \neq b \tag{14}$$

$$\int \frac{x}{(x+a)^2} dx = \frac{a}{a+x} + \ln|a+x| \tag{15}$$

$$\int \frac{x}{ax^2+bx+c} dx = \frac{1}{2a}\ln|ax^2+bx+c| - \frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}} \tag{16}$$

### Integrals with Roots

$$\int \sqrt{x-a}\,dx = \frac{2}{3}(x-a)^{3/2} \tag{17}$$

$$\int \frac{1}{\sqrt{x\pm a}} dx = 2\sqrt{x\pm a} \tag{18}$$

$$\int \frac{1}{\sqrt{a-x}} dx = -2\sqrt{a-x} \tag{19}$$

$$\int x\sqrt{x-a}\,dx = \frac{2}{3}a(x-a)^{3/2} + \frac{2}{5}(x-a)^{5/2} \tag{20}$$

$$\int \sqrt{ax+b}\,dx = \left(\frac{2b}{3a} + \frac{2x}{3}\right)\sqrt{ax+b} \tag{21}$$

$$\int (ax+b)^{3/2} dx = \frac{2}{5a}(ax+b)^{5/2} \tag{22}$$

$$\int \frac{x}{\sqrt{x\pm a}} dx = \frac{2}{3}(x\mp 2a)\sqrt{x\pm a} \tag{23}$$

$$\int \sqrt{\frac{x}{a-x}} dx = -\sqrt{x(a-x)} - a\tan^{-1}\frac{\sqrt{x(a-x)}}{x-a} \tag{24}$$

$$\int \sqrt{\frac{x}{a+x}} dx = \sqrt{x(a+x)} - a\ln\left[\sqrt{x}+\sqrt{x+a}\right] \tag{25}$$

$$\int x\sqrt{ax+b}dx = \frac{2}{15a^2}(-2b^2+abx+3a^2x^2)\sqrt{ax+b} \tag{26}$$

$$\int \sqrt{x(ax+b)}dx = \frac{1}{4a^{3/2}}\left[(2ax+b)\sqrt{ax(ax+b)} -b^2\ln\left|a\sqrt{x}+\sqrt{a(ax+b)}\right|\right] \tag{27}$$

$$\int \sqrt{x^3(ax+b)}dx = \left[\frac{b}{12a}-\frac{b^2}{8a^2x}+\frac{x}{3}\right]\sqrt{x^3(ax+b)} + \frac{b^3}{8a^{5/2}}\ln\left|a\sqrt{x}+\sqrt{a(ax+b)}\right| \tag{28}$$

$$\int \sqrt{x^2\pm a^2}dx = \frac{1}{2}x\sqrt{x^2\pm a^2}\pm\frac{1}{2}a^2\ln\left|x+\sqrt{x^2\pm a^2}\right| \tag{29}$$

$$\int \sqrt{a^2-x^2}dx = \frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\tan^{-1}\frac{x}{\sqrt{a^2-x^2}} \tag{30}$$

$$\int x\sqrt{x^2\pm a^2}dx = \frac{1}{3}\left(x^2\pm a^2\right)^{3/2} \tag{31}$$

$$\int \frac{1}{\sqrt{x^2\pm a^2}}dx = \ln\left|x+\sqrt{x^2\pm a^2}\right| \tag{32}$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} \tag{33}$$

$$\int \frac{x}{\sqrt{x^2\pm a^2}}dx = \sqrt{x^2\pm a^2} \tag{34}$$

$$\int \frac{x}{\sqrt{a^2-x^2}}dx = -\sqrt{a^2-x^2} \tag{35}$$

$$\int \frac{x^2}{\sqrt{x^2\pm a^2}}dx = \frac{1}{2}x\sqrt{x^2\pm a^2}\mp\frac{1}{2}a^2\ln\left|x+\sqrt{x^2\pm a^2}\right| \tag{36}$$

$$\int\sqrt{ax^2+bx+c}dx = \frac{b+2ax}{4a}\sqrt{ax^2+bx+c} + \frac{4ac-b^2}{8a^{3/2}}\ln\left|2ax+b+2\sqrt{a(ax^2+bx^+c)}\right| \tag{37}$$

$$\int x\sqrt{ax^2+bx+c} = \frac{1}{48a^{5/2}}\left(2\sqrt{a}\sqrt{ax^2+bx+c} \times\left(-3b^2+2abx+8a(c+ax^2)\right) +3(b^3-4abc)\ln\left|b+2ax+2\sqrt{a}\sqrt{ax^2+bx+c}\right|\right) \tag{38}$$

$$\int \frac{1}{\sqrt{ax^2+bx+c}}dx = \frac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right| \tag{39}$$

$$\int \frac{x}{\sqrt{ax^2+bx+c}}dx = \frac{1}{a}\sqrt{ax^2+bx+c} - \frac{b}{2a^{3/2}}\ln\left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right| \tag{40}$$

$$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}} \tag{41}$$

## Integrals with Logarithms

$$\int \ln ax\,dx = x\ln ax - x \tag{42}$$

$$\int \frac{\ln ax}{x}dx = \frac{1}{2}(\ln ax)^2 \tag{43}$$

$$\int \ln(ax+b)dx = \left(x+\frac{b}{a}\right)\ln(ax+b)-x, a\neq 0 \tag{44}$$

$$\int \ln(x^2+a^2)\,\mathrm{d}x = x\ln(x^2+a^2)+2a\tan^{-1}\frac{x}{a}-2x \tag{45}$$

$$\int \ln(x^2-a^2)\,\mathrm{d}x = x\ln(x^2-a^2)+a\ln\frac{x+a}{x-a}-2x \tag{46}$$

$$\int \ln\left(ax^2+bx+c\right)dx = \frac{1}{a}\sqrt{4ac-b^2}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}} - 2x+\left(\frac{b}{2a}+x\right)\ln\left(ax^2+bx+c\right) \tag{47}$$

$$\int x\ln(ax+b)dx = \frac{bx}{2a}-\frac{1}{4}x^2 + \frac{1}{2}\left(x^2-\frac{b^2}{a^2}\right)\ln(ax+b) \tag{48}$$

$$\int x\ln\left(a^2-b^2x^2\right)dx = -\frac{1}{2}x^2+ \frac{1}{2}\left(x^2-\frac{a^2}{b^2}\right)\ln\left(a^2-b^2x^2\right) \tag{49}$$

**Integrals with Exponentials**

$$\int e^{ax}dx = \frac{1}{a}e^{ax} \tag{50}$$

$$\int \sqrt{x}e^{ax}dx = \frac{1}{a}\sqrt{x}e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}}\text{erf}\left(i\sqrt{ax}\right), \quad \text{where } \text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \tag{51}$$

$$\int xe^{x}dx = (x-1)e^{x} \tag{52}$$

$$\int xe^{ax}dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax} \tag{53}$$

$$\int x^2e^{x}dx = \left(x^2 - 2x + 2\right)e^{x} \tag{54}$$

$$\int x^2e^{ax}dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)e^{ax} \tag{55}$$

$$\int x^3e^{x}dx = \left(x^3 - 3x^2 + 6x - 6\right)e^{x} \tag{56}$$

$$\int x^ne^{ax}\,\mathrm{d}x = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,\mathrm{d}x \tag{57}$$

$$\int x^ne^{ax}\,\mathrm{d}x = \frac{(-1)^n}{a^{n+1}}\Gamma[1+n, -ax], \quad \text{where } \Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}\,\mathrm{d}t \tag{58}$$

$$\int e^{ax^2}\,\mathrm{d}x = -\frac{i\sqrt{\pi}}{2\sqrt{a}}\text{erf}\left(ix\sqrt{a}\right) \tag{59}$$

$$\int e^{-ax^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt{a}}\text{erf}\left(x\sqrt{a}\right) \tag{60}$$

$$\int xe^{-ax^2}\,\mathrm{d}x = -\frac{1}{2a}e^{-ax^2} \tag{61}$$

$$\int x^2e^{-ax^2}\,\mathrm{d}x = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}\text{erf}(x\sqrt{a}) - \frac{x}{2a}e^{-ax^2} \tag{62}$$

**Integrals with Trigonometric Functions**

$$\int \sin ax dx = -\frac{1}{a}\cos ax \tag{63}$$

$$\int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a} \tag{64}$$

$$\int \sin^n ax dx = -\frac{1}{a}\cos ax \;\, {}_2F_1\left[\frac{1}{2}, \frac{1-n}{2}, \frac{3}{2}, \cos^2 ax\right] \tag{65}$$

$$\int \sin^3 ax dx = -\frac{3\cos ax}{4a} + \frac{\cos 3ax}{12a} \tag{66}$$

$$\int \cos ax dx = \frac{1}{a}\sin ax \tag{67}$$

$$\int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a} \tag{68}$$

$$\int \cos^p ax dx = -\frac{1}{a(1+p)}\cos^{1+p} ax \times \, {}_2F_1\left[\frac{1+p}{2}, \frac{1}{2}, \frac{3+p}{2}, \cos^2 ax\right] \tag{69}$$

$$\int \cos^3 ax dx = \frac{3\sin ax}{4a} + \frac{\sin 3ax}{12a} \tag{70}$$

$$\int \cos ax \sin bx dx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}, a \neq b \tag{71}$$

$$\int \sin^2 ax \cos bx dx = -\frac{\sin[(2a-b)x]}{4(2a-b)} + \frac{\sin bx}{2b} - \frac{\sin[(2a+b)x]}{4(2a+b)} \tag{72}$$

$$\int \sin^2 x \cos x dx = \frac{1}{3}\sin^3 x \tag{73}$$

$$\int \cos^2 ax \sin bx dx = \frac{\cos[(2a-b)x]}{4(2a-b)} - \frac{\cos bx}{2b} - \frac{\cos[(2a+b)x]}{4(2a+b)} \tag{74}$$

$$\int \cos^2 ax \sin ax dx = -\frac{1}{3a}\cos^3 ax \tag{75}$$

$$\int \sin^2 ax \cos^2 bx dx = \frac{x}{4} - \frac{\sin 2ax}{8a} - \frac{\sin[2(a-b)x]}{16(a-b)} + \frac{\sin 2bx}{8b} - \frac{\sin[2(a+b)x]}{16(a+b)} \tag{76}$$

$$\int \sin^2 ax \cos^2 ax dx = \frac{x}{8} - \frac{\sin 4ax}{32a} \tag{77}$$

$$\int \tan ax dx = -\frac{1}{a} \ln \cos ax \tag{78}$$

$$\int \tan^2 ax dx = -x + \frac{1}{a} \tan ax \tag{79}$$

$$\int \tan^n ax dx = \frac{\tan^{n+1} ax}{a(1+n)} \times {}_2F_1\left(\frac{n+1}{2}, 1, \frac{n+3}{2}, -\tan^2 ax\right) \tag{80}$$

$$\int \tan^3 ax dx = \frac{1}{a} \ln \cos ax + \frac{1}{2a} \sec^2 ax \tag{81}$$

$$\int \sec x dx = \ln|\sec x + \tan x| = 2\tanh^{-1}\left(\tan \frac{x}{2}\right) \tag{82}$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax \tag{83}$$

$$\int \sec^3 x \, \mathrm{dx} = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| \tag{84}$$

$$\int \sec x \tan x dx = \sec x \tag{85}$$

$$\int \sec^2 x \tan x dx = \frac{1}{2} \sec^2 x \tag{86}$$

$$\int \sec^n x \tan x dx = \frac{1}{n} \sec^n x, n \neq 0 \tag{87}$$

$$\int \csc x dx = \ln\left|\tan \frac{x}{2}\right| = \ln|\csc x - \cot x| + C \tag{88}$$

$$\int \csc^2 ax dx = -\frac{1}{a} \cot ax \tag{89}$$

$$\int \csc^3 x dx = -\frac{1}{2} \cot x \csc x + \frac{1}{2} \ln|\csc x - \cot x| \tag{90}$$

$$\int \csc^n x \cot x dx = -\frac{1}{n} \csc^n x, n \neq 0 \tag{91}$$

$$\int \sec x \csc x dx = \ln|\tan x| \tag{92}$$

**Products of Trigonometric Functions and Monomials**

$$\int x \cos x dx = \cos x + x \sin x \tag{93}$$

$$\int x \cos ax dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax \tag{94}$$

$$\int x^2 \cos x dx = 2x \cos x + \left(x^2 - 2\right) \sin x \tag{95}$$

$$\int x^2 \cos ax dx = \frac{2x \cos ax}{a^2} + \frac{a^2x^2 - 2}{a^3} \sin ax \tag{96}$$

$$\int x^n cos x dx = -\frac{1}{2}(i)^{n+1}\left[\Gamma(n+1, -ix) + (-1)^n \Gamma(n+1, ix)\right] \tag{97}$$

$$\int x^n cos ax dx = \frac{1}{2}(ia)^{1-n}\left[(-1)^n \Gamma(n+1, -iax) - \Gamma(n+1, ixa)\right] \tag{98}$$

$$\int x \sin x dx = -x \cos x + \sin x \tag{99}$$

$$\int x \sin ax dx = -\frac{x \cos ax}{a} + \frac{\sin ax}{a^2} \tag{100}$$

$$\int x^2 \sin x dx = \left(2 - x^2\right) \cos x + 2x \sin x \tag{101}$$

$$\int x^2 \sin ax dx = \frac{2 - a^2x^2}{a^3} \cos ax + \frac{2x \sin ax}{a^2} \tag{102}$$

$$\int x^n \sin x dx = -\frac{1}{2}(i)^n \left[\Gamma(n+1, -ix) - (-1)^n \Gamma(n+1, -ix)\right] \tag{103}$$

**Products of Trigonometric Functions and Exponentials**

$$\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) \tag{104}$$

$$\int e^{bx} \sin ax dx = \frac{1}{a^2 + b^2} e^{bx} (b \sin ax - a \cos ax) \tag{105}$$

$$\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x) \tag{106}$$

$$\int e^{bx} \cos ax dx = \frac{1}{a^2 + b^2} e^{bx} (a \sin ax + b \cos ax) \tag{107}$$

$$\int x e^x \sin x dx = \frac{1}{2} e^x (\cos x - x \cos x + x \sin x) \tag{108}$$

$$\int x e^x \cos x dx = \frac{1}{2} e^x (x \cos x - \sin x + x \sin x) \tag{109}$$

**Integrals of Hyperbolic Functions**

$$\int \cosh ax dx = \frac{1}{a}\sinh ax \tag{110}$$

$$\int e^{ax}\cosh bx dx = \begin{cases} \frac{e^{ax}}{a^2-b^2}[a\cosh bx - b\sinh bx] & a \neq b \\ \frac{e^{2ax}}{4a} + \frac{x}{2} & a = b \end{cases} \tag{111}$$

$$\int \sinh ax dx = \frac{1}{a}\cosh ax \tag{112}$$

$$\int e^{ax}\sinh bx dx = \begin{cases} \frac{e^{ax}}{a^2-b^2}[-b\cosh bx + a\sinh bx] & a \neq b \\ \frac{e^{2ax}}{4a} - \frac{x}{2} & a = b \end{cases} \tag{113}$$

$$\int e^{ax}\tanh bx dx = \begin{cases} \frac{e^{(a+2b)x}}{(a+2b)} {}_2F_1\left[1+\frac{a}{2b}, 1, 2+\frac{a}{2b}, -e^{2bx}\right] \\ \qquad -\frac{1}{a}e^{ax}{}_2F_1\left[\frac{a}{2b}, 1, 1E, -e^{2bx}\right] & a \neq b \\ \frac{e^{ax} - 2\tan^{-1}[e^{ax}]}{a} & a = b \end{cases} \tag{114}$$

$$\int \tanh ax\, dx = \frac{1}{a}\ln\cosh ax \tag{115}$$

$$\int \cos ax \cosh bx dx = \frac{1}{a^2+b^2}[a\sin ax \cosh bx + b\cos ax \sinh bx] \tag{116}$$

$$\int \cos ax \sinh bx dx = \frac{1}{a^2+b^2}[b\cos ax \cosh bx + a\sin ax \sinh bx] \tag{117}$$

$$\int \sin ax \cosh bx dx = \frac{1}{a^2+b^2}[-a\cos ax \cosh bx + b\sin ax \sinh bx] \tag{118}$$

$$\int \sin ax \sinh bx dx = \frac{1}{a^2+b^2}[b\cosh bx \sin ax - a\cos ax \sinh bx] \tag{119}$$

$$\int \sinh ax \cosh ax dx = \frac{1}{4a}[-2ax + \sinh 2ax] \tag{120}$$

$$\int \sinh ax \cosh bx dx = \frac{1}{b^2-a^2}[b\cosh bx \sinh ax - a\cosh ax \sinh bx] \tag{121}$$

### 1.2. First-Order Linear ODE

Equations of the form $\frac{dy}{dx} + \alpha(x)y = \beta(x)$ can be solved by multiplying both sides by an integrating factor $e^{\int \alpha(x)dx}$. Then the left-hand side becomes $\frac{d(e^{\int \alpha(x)dx}.y)}{dx}$ and the right hand side becomes $e^{\int \alpha(x)dx}\beta(x)$. The left and right hand sides can then be integrated directly.

## 2. Continuum Flow

### 2.1. Mass Conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.(\rho v_j) = 0$$
$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_j)}{\partial x_j} = 0$$

**Cartesian Coordinates**

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_1)}{\partial x_1} + \frac{\partial(\rho v_2)}{\partial x_2} + \frac{\partial(\rho v_3)}{\partial x_3} = 0$$

**Cylindrical Polar Coordinates**

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial \rho v_\theta}{\partial \theta} + \frac{\partial \rho v_z}{\partial z} = 0$$

## 2.2. Momentum Conservation (Navier-Stokes Equations)

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\vec{\nabla})\vec{v}\right] = \rho\vec{f} - \vec{\nabla}p + \mu\nabla^2\vec{v}$$

**Cartesian Coordinates**

$$\begin{aligned}
\rho\left[\frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial x_1} + v_2\frac{\partial v_1}{\partial x_2} + v_3\frac{\partial v_1}{\partial x_3}\right] &= \rho f_1 - \frac{\partial p}{\partial x_1} + \mu\left[\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_3^2}\right]\\
\rho\left[\frac{\partial v_2}{\partial t} + v_1\frac{\partial v_2}{\partial x_1} + v_2\frac{\partial v_2}{\partial x_2} + v_3\frac{\partial v_2}{\partial x_3}\right] &= \rho f_2 - \frac{\partial p}{\partial x_2} + \mu\left[\frac{\partial^2 v_2}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_2^2} + \frac{\partial^2 v_2}{\partial x_3^2}\right]\\
\rho\left[\frac{\partial v_3}{\partial t} + v_1\frac{\partial v_3}{\partial x_1} + v_2\frac{\partial v_3}{\partial x_2} + v_3\frac{\partial v_3}{\partial x_3}\right] &= \rho f_3 - \frac{\partial p}{\partial x_3} + \mu\left[\frac{\partial^2 v_3}{\partial x_1^2} + \frac{\partial^2 v_3}{\partial x_2^2} + \frac{\partial^2 v_3}{\partial x_3^2}\right]
\end{aligned}$$

**Cylindrical Polar Coordinates**

$$\begin{aligned}
\rho\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{{v^2}_\theta}{r}\right] &= \rho f_r - \frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right]\\
\rho\left[\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right] &= \rho f_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right]\\
\rho\left[\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right] &= \rho f_z - \frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]
\end{aligned}$$

# 3. Vorticity Dynamics

$$\text{Circulation } \Gamma = \oint_C \vec{v}.\vec{dl} = \int_S \vec{\omega}.\hat{n}\ dA \text{ (Flux of Vorticity across surface } S\text{)}$$
$$\text{Average angular velocity } \bar{\Omega} = \frac{\bar{u}_\theta}{a} = \frac{\oint_C \vec{v}.\vec{dl}}{2\pi a^2} = \frac{\Gamma}{2\pi a^2} \text{ (On a circle of radius } a\text{)} = \frac{\omega_j n_j}{2}$$
$$\text{For Irrotational Flow, } \Gamma = 0,\ \vec{\omega} = 0$$

A vector field $\vec{\alpha}$ is considered solenoidal if $\vec{\nabla}.\vec{\alpha} = 0$

$$\vec{\nabla}.\vec{\omega} = \vec{\nabla}.(\vec{\nabla} \times \vec{v}) = 0$$

## 3.1. Streamlines and Vortex Lines

$$\text{Equations for Streamlines (Cartesian Coordinates) } \frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z}$$
$$\text{Equations for Streamlines (Cylindrical Polar Coordinates) } \frac{dR}{u_R} = \frac{Rd\phi}{u_\phi} = \frac{dz}{u_z}$$
$$\text{Equations for Vortex Lines (Cartesian Coordinates) } \frac{dx}{\omega_x} = \frac{dy}{\omega_y} = \frac{dz}{\omega_z}$$

## 3.2. Terms

- **Barotropic**: Density is a function of pressure only. $\frac{1}{\rho^2}\vec{\nabla}\rho \times \vec{\nabla}p = 0$
- **Inviscid**: Viscosity can be neglected. $\nu\nabla^2\vec{\omega} = 0$
- **Baroclinic**: Measure of how misaligned the gradient of pressure is from the gradient of density in a fluid $\vec{\nabla}\rho \times \vec{\nabla}p$
- **Isobars**: Constant pressure lines
- **Isopycals**: Constant density lineses

## 3.3. Vorticity Transport Equation

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega}.\vec{\nabla})\vec{v} + \frac{1}{\rho^2}\vec{\nabla}\rho \times \vec{\nabla}p + \nu\nabla^2\vec{\omega}$$
$$\frac{\partial\vec{\omega}}{\partial t} + (\vec{v}.\vec{\nabla})\vec{\omega} = (\vec{\omega}.\vec{\nabla})\vec{v} + \frac{1}{\rho^2}\vec{\nabla}\rho \times \vec{\nabla}p + \nu\nabla^2\vec{\omega}$$

**Legend**

- $\frac{D\vec{\omega}}{Dt}$: Total rate of change of vorticity of a fluid particle
- $(\vec{\omega}.\vec{\nabla})\vec{v}$: Vorticity production due to stretching/tilting of vortex lines

- $\frac{1}{\rho^2}\vec{\nabla}\rho \times \vec{\nabla}p$: Vorticity production due to baroclinic effects
- $\nu\nabla^2\vec{\omega}$: Viscous diffusion of vorticity which dissipates or redistributes vorticity

### 3.4. Helmholtz's Vortex Theorems for Inviscid Flows

**Assumptions**

- Inviscid
- Barotropic
- Conservative Body Forces $\vec{F} = -\vec{\nabla}\psi$

**Statement**

- Fluid particles/element orignially free of vorticity remain free of vorticity (non-rotating)
- Vortex lines (tubes) move with the fluid for inviscid flows. Vortex line is always comprised of the same fluid particles (Vorticity is frozen to the flow for inviscid flow)
- The strength of the vortex tube (circulation) does not vary with time during the fluid motion. Vortex tubes must be closed, go to infinity, or end on solid boundaries.

### 3.5. Kelvin's circulation theorem

**Assumptions**

- Inviscid
- Barotropic
- Incompressible
- Conservative Body Forces $\vec{F} = -\vec{\nabla}\psi$

**Statement**

The circulation (strength of the vortex) around a closed curve moving with the fluid will remain constant.

$$\frac{\partial \Gamma_C}{\partial t} + v_j\frac{\partial \Gamma_C}{\partial x_j} = \oint_C \nu\nabla^2\vec{v}.\vec{dx}$$

For inviscid flow, $\frac{\partial \Gamma_C}{\partial t} + v_j\frac{\partial \Gamma_C}{\partial x_j} = 0$

### 3.6. Bernoulli's Equation

**Assumptions**

- Inviscid $\nabla^2 v = 0 \rightarrow$ Euler
- Only gravitational body forces (Conservative) $\rightarrow f_i = -\frac{d\psi}{dx_i}$ where $\psi = gz$

- Barotropic $\rightarrow \rho = \rho(p)$
- Steady flow $\rightarrow \dfrac{\partial}{\partial t} = 0$

Note: No restriction on the compressiblity effects

**Statement**

$$\nabla\left[\frac{\vec{v}.\vec{v}}{2} + gz + \int \frac{dp}{\rho}\right] = -\vec{\omega} \times \vec{v}$$

$$\text{Along streamline or vortex line } \frac{\vec{v}.\vec{v}}{2} + gz + \int \frac{dp}{\rho} = \text{constant}$$

$$\text{Unsteady flow } \frac{\partial \phi}{\partial t} + \frac{\vec{v}.\vec{v}}{2} + gz + \int \frac{dp}{\rho} = F(t)$$

$$\text{Steady Potential flow } \frac{\vec{v}.\vec{v}}{2} + gz + \int \frac{dp}{\rho} = \text{constant throughout the flow}$$

## 4. Potential Flow

$$i = \sqrt{-1}$$
$$z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta)$$
$$\bar{z} = x - iy = re^{-i\theta} = r(\cos\theta - i\sin\theta)$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2} = |z| = |z\bar{z}|^{\frac{1}{2}}$$
$$\theta = arctan\frac{y}{x}$$
$$\text{2D Incompressible Potential Flow } \nabla^2\phi = 0 = \nabla^2\psi$$
$$\text{Complex Potential } F(z) = \phi(z) + i\psi(z)$$
$$\text{Complex Velocity } w(z) = u - iv = (v_r - iv_\theta)e^{-i\theta} = \frac{dF(z)}{dz}$$
$$\bar{w}(z) = u + iv = (v_r + iv_\theta)e^{i\theta}$$

### 4.1. Stream Function ⇔ Potential Function ⇔ Velocities

Also called Cauchy Reimann Equations

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$
$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$
$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

### 4.2. Cartesian ⇔ Polar Velocities

$$u = v_r \cos\theta - v_\theta \sin\theta$$
$$v = v_r \sin\theta + v_\theta \cos\theta$$

$$v_r = u\cos\theta + v\sin\theta$$
$$v_\theta = -u\sin\theta + v\cos\theta$$

### 4.3. Complex Potentials

| | F(z) | $\phi$ | $\psi$ | w(z) | $u$ and $v_r$ | $v$ and $v_\theta$ |
|---|---|---|---|---|---|---|
| Uniform | $Ue^{-i\alpha}z$ | $U(x\cos\alpha + y\sin\alpha)$ | $U(y\cos\alpha - x\sin\alpha)$ | $Ue^{-i\alpha}$ | $U\cos\alpha$ | $U\sin\alpha$ |
| | | $Ur\cos(\theta-\alpha)$ | $Ur\sin(\theta-\alpha)$ | | $U\cos(\theta-\alpha)$ | $-U\sin(\theta-\alpha)$ |
| Corner | $Cz^n$ | | | $nCz^{n-1}$ | | |
| | | $Cr^n\cos n\theta$ | $Cr^n\sin n\theta$ | | $nCr^{n-1}\cos[(n-1)\theta]$ | $-nCr^{n-1}\sin[(n-1)\theta]$ |
| Source/Sink | $\frac{m}{2\pi}\ln(z-z_0)$ | $\frac{m}{4\pi}\ln[x^2+y^2]$ | $\frac{m}{2\pi}\arctan\frac{y}{x}$ | $\frac{m}{2\pi(z-z_0)}$ | $\frac{m}{2\pi}\frac{x}{x^2+y^2}$ | $\frac{m}{2\pi}\frac{y}{x^2+y^2}$ |
| | | $\frac{m}{2\pi}\ln r$ | $\frac{m}{2\pi}\theta$ | | $\frac{m}{2\pi r}$ | $0$ |
| Free Vortex | $-\frac{i\Gamma}{2\pi}\ln(z-z_0)$ | $\frac{\Gamma}{2\pi}\arctan\frac{y}{x}$ | $-\frac{\Gamma}{4\pi}\ln[x^2+y^2]$ | $-\frac{i\Gamma}{2\pi(z-z_0)}$ | $-\frac{\Gamma}{2\pi}\frac{y}{x^2+y^2}$ | $\frac{\Gamma}{2\pi}\frac{x}{x^2+y^2}$ |
| | | $\frac{\Gamma}{2\pi}\theta$ | $-\frac{\Gamma}{2\pi}\ln r$ | | $0$ | $\frac{\Gamma}{2\pi r}$ |
| Dipole | $\frac{\mu}{\pi z}$ | $\frac{\mu}{\pi}\frac{x}{x^2+y^2}$ | $-\frac{\mu}{\pi}\frac{y}{x^2+y^2}$ | $-\frac{\mu}{\pi z^2}$ | $\frac{\mu}{\pi}\frac{y^2-x^2}{(x^2+y^2)^2}$ | $-\frac{\mu}{\pi}\frac{2xy}{(x^2+y^2)^2}$ |
| | | $\frac{\mu}{\pi r}\cos\theta$ | $-\frac{\mu}{\pi r}\sin\theta$ | | $-\frac{\mu}{\pi r^2}\cos\theta$ | $-\frac{\mu}{\pi r^2}\sin\theta$ |

**Legend**

- Uniform
  - $U$: Uniform velocity magnitude
  - $\alpha$: Angle of attack - the angle at which the direction of the uniform velocity is oriented with respect to the horizontal
- Corner
  - $C$: Indicates the direction. $C > 0$ always
  - $n$: $\frac{\pi}{\text{angle of the corner}}$
- Source/Sink
  - $m$: Volume flow rate per unit dimension normal to the page. For source, $m > 0$, whereas for sink, $m < 0$
- Dipole/Doublet
  - $a$: Half distance between the source and sink
  - $Q$: Volume flow rate per unit dimension normal to the page
  - $\mu$: $Qa$

### 4.4. Infinite Series Expansions

- $\ln(1+\epsilon) = \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3} - \dots$ when $|\epsilon| \leq 1$
- $(1+\epsilon)^{-1} = 1 - \epsilon + \epsilon^2 - \epsilon^3 + \dots$ when $|\epsilon| \leq 1$
- $(1+\epsilon)^{\frac{1}{2}} = 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \frac{1}{16}\epsilon^3 - \frac{5}{128}\epsilon^4 + \frac{7}{256}\epsilon^5 - \dots$ when $|\epsilon| \leq 1$
- $(1+\epsilon)^{-\frac{1}{2}} = 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \frac{35}{128}\epsilon^4 - \frac{63}{256}\epsilon^5 + \dots$ when $|\epsilon| \leq 1$
- $e^{\epsilon} = 1 - \frac{\epsilon^2}{2!} + \frac{\epsilon^4}{4!} - \dots$
- $\sin(\epsilon) = \epsilon - \frac{\epsilon^3}{3!} + \frac{\epsilon^5}{5!} - \frac{\epsilon^7}{7!} + \dots$

### 4.5. Bernoulli's Equation (Irrotational)

$$p_\infty + \frac{\rho v_\infty^2}{2} = p + \frac{\rho |v^2|}{2}$$

### 4.6. Forces on a 2D Body

*All the below forces have the units of Force per unit length normal to the sheet of paper*

$$\begin{aligned}
\text{Drag Force} &= -\int_C (p - p_\infty)\cos\theta \, ds \\
\text{Lift Force} &= -\int_C (p - p_\infty)\sin\theta \, ds \\
\text{Complex Force } G = D - iL &= -i\oint_C p d\bar{z} = -i\oint_C \left[p_\infty + \frac{\rho v_\infty^2}{2} - \frac{\rho|v^2|}{2}\right] d\bar{z} \\
&= \frac{i\rho}{2}\oint_C |v^2| d\bar{z} = \frac{i\rho}{2}\oint_C w\bar{w} d\bar{z} \\
&= \frac{i\rho}{2}\oint_C [w(z)]^2 dz \text{ (First Blasius Integral Law)}
\end{aligned}$$

### 4.7. Moment on a 2D Body

$$\text{Moment } M = -\frac{\rho}{2} Re\left\{\oint_C z w^2 dz\right\} \text{(Second Blasius Integral Law)}$$

### 4.8. Residue Theorem

$$\text{If } F(z) = \sum_j \frac{B_j}{z - z_j}$$

$$R_k = \sum B$$

$$\oint_C F(z)dz = 2\pi i \sum_k R_k$$