

# MEEN 621 Notes

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## 1. Mathematics

### 1.1. Integral Tables

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#### Basic Forms

$$\int x^n dx = \frac{1}{n+1} x^{n+1} \quad (1) \quad \int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a} \ln \frac{a+x}{b+x}, \quad a \neq b \quad (14)$$

$$\int \frac{1}{x} dx = \ln |x| \quad (2) \quad \int \frac{x}{(x+a)^2} dx = \frac{a}{a+x} + \ln |a+x| \quad (15)$$

$$\int u dv = uv - \int v du \quad (3) \quad \int \frac{x}{ax^2 + bx + c} dx = \frac{1}{2a} \ln |ax^2 + bx + c|$$

$$- \frac{b}{a\sqrt{4ac - b^2}} \tan^{-1} \frac{2ax + b}{\sqrt{4ac - b^2}} \quad (16)$$

$$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln |ax+b| \quad (4)$$

#### Integrals of Rational Functions

$$\int \frac{1}{(x+a)^2} dx = -\frac{1}{x+a} \quad (5) \quad \int \sqrt{x-a} dx = \frac{2}{3} (x-a)^{3/2} \quad (17)$$

$$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1}, \quad n \neq -1 \quad (6) \quad \int \frac{1}{\sqrt{x \pm a}} dx = 2\sqrt{x \pm a} \quad (18)$$

$$\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)} \quad (7) \quad \int \frac{1}{\sqrt{a-x}} dx = -2\sqrt{a-x} \quad (19)$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x \quad (8) \quad \int x\sqrt{x-a} dx = \frac{2}{3} a(x-a)^{3/2} + \frac{2}{5} (x-a)^{5/2} \quad (20)$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \quad (9) \quad \int \sqrt{ax+b} dx = \left( \frac{2b}{3a} + \frac{2x}{3} \right) \sqrt{ax+b} \quad (21)$$

$$\int \frac{x}{a^2+x^2} dx = \frac{1}{2} \ln |a^2+x^2| \quad (10) \quad \int (ax+b)^{3/2} dx = \frac{2}{5a} (ax+b)^{5/2} \quad (22)$$

$$\int \frac{x^2}{a^2+x^2} dx = x - a \tan^{-1} \frac{x}{a} \quad (11) \quad \int \frac{x}{\sqrt{x \pm a}} dx = \frac{2}{3} (x \mp 2a) \sqrt{x \pm a} \quad (23)$$

$$\int \frac{x^3}{a^2+x^2} dx = \frac{1}{2} x^2 - \frac{1}{2} a^2 \ln |a^2+x^2| \quad (12) \quad \int \sqrt{\frac{x}{a-x}} dx = -\sqrt{x(a-x)} - a \tan^{-1} \frac{\sqrt{x(a-x)}}{x-a} \quad (24)$$

$$\int \frac{1}{ax^2+bx+c} dx = \frac{2}{\sqrt{4ac-b^2}} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}} \quad (13) \quad \int \sqrt{\frac{x}{a+x}} dx = \sqrt{x(a+x)} - a \ln [\sqrt{x} + \sqrt{x+a}] \quad (25)$$

#### Integrals with Roots

$$\int x\sqrt{ax+b}dx = \frac{2}{15a^2}(-2b^2 + abx + 3a^2x^2)\sqrt{ax+b} \quad (26)$$

$$\int \sqrt{x(ax+b)}dx = \frac{1}{4a^{3/2}} \left[ (2ax+b)\sqrt{ax(ax+b)} - b^2 \ln \left| a\sqrt{x} + \sqrt{a(ax+b)} \right| \right] \quad (27)$$

$$\int \sqrt{x^3(ax+b)}dx = \left[ \frac{b}{12a} - \frac{b^2}{8a^2x} + \frac{x}{3} \right] \sqrt{x^3(ax+b)} + \frac{b^3}{8a^{5/2}} \ln \left| a\sqrt{x} + \sqrt{a(ax+b)} \right| \quad (28)$$

$$\int \sqrt{x^2 \pm a^2}dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \pm \frac{1}{2}a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right| \quad (29)$$

$$\int \sqrt{a^2 - x^2}dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \tan^{-1} \frac{x}{\sqrt{a^2 - x^2}} \quad (30)$$

$$\int x\sqrt{x^2 \pm a^2}dx = \frac{1}{3} (x^2 \pm a^2)^{3/2} \quad (31)$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln \left| x + \sqrt{x^2 \pm a^2} \right| \quad (32)$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} \quad (33)$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2} \quad (34)$$

$$\int \frac{x}{\sqrt{a^2 - x^2}} dx = -\sqrt{a^2 - x^2} \quad (35)$$

$$\int \frac{x^2}{\sqrt{x^2 \pm a^2}} dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \mp \frac{1}{2}a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right| \quad (36)$$

$$\int \sqrt{ax^2 + bx + c}dx = \frac{b+2ax}{4a}\sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8a^{3/2}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx + c)} \right| \quad (37)$$

$$\int x\sqrt{ax^2 + bx + c} = \frac{1}{48a^{5/2}} \left( 2\sqrt{a}\sqrt{ax^2 + bx + c} \times (-3b^2 + 2abx + 8a(c + ax^2)) + 3(b^3 - 4abc) \ln \left| b + 2ax + 2\sqrt{a}\sqrt{ax^2 + bx + c} \right| \right) \quad (38)$$

$$\int \frac{1}{\sqrt{ax^2 + bx + c}} dx = \frac{1}{\sqrt{a}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx + c)} \right| \quad (39)$$

$$\int \frac{x}{\sqrt{ax^2 + bx + c}} dx = \frac{1}{a}\sqrt{ax^2 + bx + c} - \frac{b}{2a^{3/2}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx + c)} \right| \quad (40)$$

$$\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 + x^2}} \quad (41)$$

### Integrals with Logarithms

$$\int \ln ax dx = x \ln ax - x \quad (42)$$

$$\int \frac{\ln ax}{x} dx = \frac{1}{2} (\ln ax)^2 \quad (43)$$

$$\int \ln(ax + b) dx = \left( x + \frac{b}{a} \right) \ln(ax + b) - x, a \neq 0 \quad (44)$$

$$\int \ln(x^2 + a^2) dx = x \ln(x^2 + a^2) + 2a \tan^{-1} \frac{x}{a} - 2x \quad (45)$$

$$\int \ln(x^2 - a^2) dx = x \ln(x^2 - a^2) + a \ln \frac{x+a}{x-a} - 2x \quad (46)$$

$$\int \ln(ax^2 + bx + c) dx = \frac{1}{a}\sqrt{4ac - b^2} \tan^{-1} \frac{2ax + b}{\sqrt{4ac - b^2}} - 2x + \left( \frac{b}{2a} + x \right) \ln(ax^2 + bx + c) \quad (47)$$

$$\int x \ln(ax + b) dx = \frac{bx}{2a} - \frac{1}{4}x^2 + \frac{1}{2} \left( x^2 - \frac{b^2}{a^2} \right) \ln(ax + b) \quad (48)$$

$$\int x \ln(a^2 - b^2x^2) dx = -\frac{1}{2}x^2 + \frac{1}{2} \left( x^2 - \frac{a^2}{b^2} \right) \ln(a^2 - b^2x^2) \quad (49)$$

## Integrals with Exponentials

$$\int e^{ax} dx = \frac{1}{a} e^{ax} \quad (50)$$

$$\int \sqrt{x} e^{ax} dx = \frac{1}{a} \sqrt{x} e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}} \operatorname{erf}(i\sqrt{ax}),$$

where  $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$  (51)

$$\int x e^x dx = (x - 1) e^x \quad (52)$$

$$\int x e^{ax} dx = \left( \frac{x}{a} - \frac{1}{a^2} \right) e^{ax} \quad (53)$$

$$\int x^2 e^x dx = (x^2 - 2x + 2) e^x \quad (54)$$

$$\int x^2 e^{ax} dx = \left( \frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right) e^{ax} \quad (55)$$

$$\int x^3 e^x dx = (x^3 - 3x^2 + 6x - 6) e^x \quad (56)$$

$$\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx \quad (57)$$

$$\int x^n e^{ax} dx = \frac{(-1)^n}{a^{n+1}} \Gamma[1 + n, -ax],$$

where  $\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt$  (58)

$$\int e^{ax^2} dx = -\frac{i\sqrt{\pi}}{2\sqrt{a}} \operatorname{erf}(ix\sqrt{a}) \quad (59)$$

$$\int e^{-ax^2} dx = \frac{\sqrt{\pi}}{2\sqrt{a}} \operatorname{erf}(x\sqrt{a}) \quad (60)$$

$$\int x e^{-ax^2} dx = -\frac{1}{2a} e^{-ax^2} \quad (61)$$

$$\int x^2 e^{-ax^2} dx = \frac{1}{4} \sqrt{\frac{\pi}{a^3}} \operatorname{erf}(x\sqrt{a}) - \frac{x}{2a} e^{-ax^2} \quad (62)$$

## Integrals with Trigonometric Functions

$$\int \sin ax dx = -\frac{1}{a} \cos ax \quad (63)$$

$$\int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a} \quad (64)$$

$$\int \sin^n ax dx =$$

$$-\frac{1}{a} \cos ax {}_2F_1\left[\frac{1}{2}, \frac{1-n}{2}, \frac{3}{2}, \cos^2 ax\right] \quad (65)$$

$$\int \sin^3 ax dx = -\frac{3 \cos ax}{4a} + \frac{\cos 3ax}{12a} \quad (66)$$

$$\int \cos ax dx = \frac{1}{a} \sin ax \quad (67)$$

$$\int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a} \quad (68)$$

$$\int \cos^p ax dx = -\frac{1}{a(1+p)} \cos^{1+p} ax \times$$

$${}_2F_1\left[\frac{1+p}{2}, \frac{1}{2}, \frac{3+p}{2}, \cos^2 ax\right] \quad (69)$$

$$\int \cos^3 ax dx = \frac{3 \sin ax}{4a} + \frac{\sin 3ax}{12a} \quad (70)$$

$$\int \cos ax \sin bxdx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}, a \neq b \quad (71)$$

$$\int \sin^2 ax \cos bxdx = -\frac{\sin[(2a-b)x]}{4(2a-b)}$$

$$+ \frac{\sin bx}{2b} - \frac{\sin[(2a+b)x]}{4(2a+b)} \quad (72)$$

$$\int \sin^2 x \cos x dx = \frac{1}{3} \sin^3 x \quad (73)$$

$$\int \cos^2 ax \sin bxdx = \frac{\cos[(2a-b)x]}{4(2a-b)} - \frac{\cos bx}{2b}$$

$$- \frac{\cos[(2a+b)x]}{4(2a+b)} \quad (74)$$

$$\int \cos^2 ax \sin ax dx = -\frac{1}{3a} \cos^3 ax \quad (75)$$

$$\int \sin^2 ax \cos^2 bxdx = \frac{x}{4} - \frac{\sin 2ax}{8a} - \frac{\sin[2(a-b)x]}{16(a-b)}$$

$$+ \frac{\sin 2bx}{8b} - \frac{\sin[2(a+b)x]}{16(a+b)} \quad (76)$$

$$\int \sin^2 ax \cos^2 ax dx = \frac{x}{8} - \frac{\sin 4ax}{32a} \quad (77)$$

### Products of Trigonometric Functions and Monomials

$$\int \tan ax dx = -\frac{1}{a} \ln \cos ax \quad (78)$$

$$\int \tan^2 ax dx = -x + \frac{1}{a} \tan ax \quad (79)$$

$$\int \tan^n ax dx = \frac{\tan^{n+1} ax}{a(n+1)} \times {}_2F_1\left(\frac{n+1}{2}, 1, \frac{n+3}{2}, -\tan^2 ax\right) \quad (80)$$

$$\int \tan^3 ax dx = \frac{1}{a} \ln \cos ax + \frac{1}{2a} \sec^2 ax \quad (81)$$

$$\int \sec x dx = \ln |\sec x + \tan x| = 2 \tanh^{-1} \left( \tan \frac{x}{2} \right) \quad (82)$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax \quad (83)$$

$$\int \sec^3 x dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| \quad (84)$$

$$\int \sec x \tan x dx = \sec x \quad (85)$$

$$\int \sec^2 x \tan x dx = \frac{1}{2} \sec^2 x \quad (86)$$

$$\int \sec^n x \tan x dx = \frac{1}{n} \sec^n x, n \neq 0 \quad (87)$$

$$\int \csc x dx = \ln \left| \tan \frac{x}{2} \right| = \ln |\csc x - \cot x| + C \quad (88)$$

$$\int \csc^2 ax dx = -\frac{1}{a} \cot ax \quad (89)$$

$$\int \csc^3 x dx = -\frac{1}{2} \cot x \csc x + \frac{1}{2} \ln |\csc x - \cot x| \quad (90)$$

$$\int \csc^n x \cot x dx = -\frac{1}{n} \csc^n x, n \neq 0 \quad (91)$$

$$\int \sec x \csc x dx = \ln |\tan x| \quad (92)$$

$$\int x \cos x dx = \cos x + x \sin x \quad (93)$$

$$\int x \cos ax dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax \quad (94)$$

$$\int x^2 \cos x dx = 2x \cos x + (x^2 - 2) \sin x \quad (95)$$

$$\int x^2 \cos ax dx = \frac{2x \cos ax}{a^2} + \frac{a^2 x^2 - 2}{a^3} \sin ax \quad (96)$$

$$\int x^n \cos x dx = -\frac{1}{2} (i)^{n+1} [\Gamma(n+1, -ix) + (-1)^n \Gamma(n+1, ix)] \quad (97)$$

$$\int x^n \cos ax dx = \frac{1}{2} (ia)^{1-n} [(-1)^n \Gamma(n+1, -iax) - \Gamma(n+1, iax)] \quad (98)$$

$$\int x \sin x dx = -x \cos x + \sin x \quad (99)$$

$$\int x \sin ax dx = -\frac{x \cos ax}{a} + \frac{\sin ax}{a^2} \quad (100)$$

$$\int x^2 \sin x dx = (2 - x^2) \cos x + 2x \sin x \quad (101)$$

$$\int x^2 \sin ax dx = \frac{2 - a^2 x^2}{a^3} \cos ax + \frac{2x \sin ax}{a^2} \quad (102)$$

$$\int x^n \sin x dx = -\frac{1}{2} (i)^n [\Gamma(n+1, -ix) - (-1)^n \Gamma(n+1, -ix)] \quad (103)$$

### Products of Trigonometric Functions and Exponentials

$$\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) \quad (104)$$

$$\int e^{bx} \sin ax dx = \frac{1}{a^2 + b^2} e^{bx} (b \sin ax - a \cos ax) \quad (105)$$

$$\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x) \quad (106)$$

$$\int e^{bx} \cos ax dx = \frac{1}{a^2 + b^2} e^{bx} (a \sin ax + b \cos ax) \quad (107)$$

$$\int x e^x \sin x dx = \frac{1}{2} e^x (\cos x - x \cos x + x \sin x) \quad (108)$$

$$\int x e^x \cos x dx = \frac{1}{2} e^x (x \cos x - \sin x + x \sin x) \quad (109)$$

## Integrals of Hyperbolic Functions

$$\int \cosh ax dx = \frac{1}{a} \sinh ax \quad (110)$$

$$\int e^{ax} \cosh bx dx = \begin{cases} \frac{e^{ax}}{a^2 - b^2} [a \cosh bx - b \sinh bx] & a \neq b \\ \frac{e^{2ax}}{4a} + \frac{x}{2} & a = b \end{cases} \quad (111)$$

$$\int \sinh ax dx = \frac{1}{a} \cosh ax \quad (112)$$

$$\int e^{ax} \sinh bx dx = \begin{cases} \frac{e^{ax}}{a^2 - b^2} [-b \cosh bx + a \sinh bx] & a \neq b \\ \frac{e^{2ax}}{4a} - \frac{x}{2} & a = b \end{cases} \quad (113)$$

$$\int e^{ax} \tanh bx dx = \begin{cases} \frac{e^{(a+2b)x}}{(a+2b)} {}_2F_1 \left[ 1 + \frac{a}{2b}, 1, 2 + \frac{a}{2b}, -e^{2bx} \right] \\ \quad - \frac{1}{a} e^{ax} {}_2F_1 \left[ \frac{a}{2b}, 1, 1E, -e^{2bx} \right] & a \neq b \\ \frac{e^{ax} - 2 \tan^{-1}[e^{ax}]}{a} & a = b \end{cases} \quad (114)$$

$$\int \tanh ax dx = \frac{1}{a} \ln \cosh ax \quad (115)$$

$$\int \cos ax \cosh bx dx = \frac{1}{a^2 + b^2} [a \sin ax \cosh bx + b \cos ax \sinh bx] \quad (116)$$

$$\int \cos ax \sinh bx dx = \frac{1}{a^2 + b^2} [b \cos ax \cosh bx + a \sin ax \sinh bx] \quad (117)$$

$$\int \sin ax \cosh bx dx = \frac{1}{a^2 + b^2} [-a \cos ax \cosh bx + b \sin ax \sinh bx] \quad (118)$$

$$\int \sin ax \sinh bx dx = \frac{1}{a^2 + b^2} [b \cosh bx \sin ax - a \cos ax \sinh bx] \quad (119)$$

$$\int \sinh ax \cosh ax dx = \frac{1}{4a} [-2ax + \sinh 2ax] \quad (120)$$

$$\int \sinh ax \cosh bx dx = \frac{1}{b^2 - a^2} [b \cosh bx \sinh ax - a \cosh ax \sinh bx] \quad (121)$$

## 1.2. First-Order Linear ODE

Equations of the form  $\frac{dy}{dx} + \alpha(x)y = \beta(x)$  can be solved by multiplying both sides by an integrating factor  $e^{\int \alpha(x)dx}$ . Then the left-hand side becomes  $\frac{d(e^{\int \alpha(x)dx} \cdot y)}{dx}$  and the right hand side becomes  $e^{\int \alpha(x)dx} \beta(x)$ . The left and right hand sides can then be integrated directly.

## 2. Continuum Flow

### 2.1. Mass Conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \mathbf{v}_j) = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_j)}{\partial x_j} = 0$$

## Cartesian Coordinates

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_1)}{\partial x_1} + \frac{\partial(\rho v_2)}{\partial x_2} + \frac{\partial(\rho v_3)}{\partial x_3} = 0$$

## Cylindrical Polar Coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r}(r \rho v_r) + \frac{1}{r} \frac{\partial \rho v_\theta}{\partial \theta} + \frac{\partial \rho v_z}{\partial z} = 0$$

## 2.2. Momentum Conservation (Navier-Stokes Equations)

$$\rho \left[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \rho \vec{f} - \nabla p + \mu \nabla^2 \vec{v}$$

### Cartesian Coordinates

$$\begin{aligned} \rho \left[ \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3} \right] &= \rho f_1 - \frac{\partial p}{\partial x_1} + \mu \left[ \frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_3^2} \right] \\ \rho \left[ \frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} + v_3 \frac{\partial v_2}{\partial x_3} \right] &= \rho f_2 - \frac{\partial p}{\partial x_2} + \mu \left[ \frac{\partial^2 v_2}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_2^2} + \frac{\partial^2 v_2}{\partial x_3^2} \right] \\ \rho \left[ \frac{\partial v_3}{\partial t} + v_1 \frac{\partial v_3}{\partial x_1} + v_2 \frac{\partial v_3}{\partial x_2} + v_3 \frac{\partial v_3}{\partial x_3} \right] &= \rho f_3 - \frac{\partial p}{\partial x_3} + \mu \left[ \frac{\partial^2 v_3}{\partial x_1^2} + \frac{\partial^2 v_3}{\partial x_2^2} + \frac{\partial^2 v_3}{\partial x_3^2} \right] \end{aligned}$$

### Cylindrical Polar Coordinates

$$\begin{aligned} \rho \left[ \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} \right] &= \rho f_r - \frac{\partial p}{\partial r} + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r}(r v_r) \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right] \\ \rho \left[ \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \right] &= \rho f_\theta - \frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r}(r v_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right] \\ \rho \left[ \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right] &= \rho f_z - \frac{\partial p}{\partial z} + \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] \end{aligned}$$

## 2.3. Flow Conditions

### Steady State

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= 0, \\ \frac{\partial v_1}{\partial t} &= 0, \quad \frac{\partial v_2}{\partial t} = 0, \quad \frac{\partial v_3}{\partial t} = 0 \end{aligned}$$



### Fully Developed

$$\frac{\partial v_1}{\partial x_1} = 0, \frac{\partial v_2}{\partial x_2} = 0, \frac{\partial v_3}{\partial x_3} = 0$$

### Unidirectional

$$v_{n1} = 0, v_{n2} = 0$$

### Two-dimensional

$$v_n = 0, \\ \frac{\partial v_{t1}}{\partial n} = 0, \frac{\partial v_{t2}}{\partial n} = 0$$

### Free Surface

$$\frac{\partial p}{\partial t} = 0$$

### Axisymmetric

$$v_\theta = 0, \\ \frac{\partial v_r}{\partial \theta} = 0, \frac{\partial v_\theta}{\partial \theta} = 0, \frac{\partial v_z}{\partial \theta} = 0,$$

### Simple Couette Flow

$$\frac{\partial p}{\partial x_1} = 0, \frac{\partial p}{\partial x_2} = 0, \frac{\partial p}{\partial x_3} = 0$$

## 2.4. Boundary Conditions

*Let  $t$  indicate a direction tangent to the boundary*

*Let  $n$  indicate a direction normal to the boundary*

### Fluid-Solid Interface

$$\text{No Slip: } v_t|_{\text{boundary}} = 0$$

$$\text{No Penetration: } v_n|_{\text{boundary}} = 0$$

### Fluid-Fluid Interface

$$\text{Continuity: } \rho_I v_{In} = \rho_{II} v_{II n}$$

$$\text{Jump/Discontinuity: } \rho_I \neq \rho_{II}$$

$$v_{In} \neq v_{II n}$$

$$\text{Shear Stress Continuity: } T_{ij} n_j t_i|_I = T_{ij} n_j t_i|_{II}$$

$$\Rightarrow \mu_I \frac{\partial v_t}{\partial n} \Big|_I = \mu_{II} \frac{\partial v_t}{\partial n} \Big|_{II}$$

### Fluid-Gas Interface

$$\text{Free Surface: } p|_{\text{boundary}} = p_\infty$$

$$\text{Traction-Free: } \frac{\partial v_t}{\partial n} \Big|_{\text{boundary}} = 0$$

## 3. Vorticity Dynamics

$$\text{Circulation } \Gamma = \oint_C \vec{v} \cdot d\vec{l} = \int_S \vec{\omega} \cdot \hat{n} dA$$

$$\begin{aligned} \text{Average angular velocity } \bar{\Omega} &= \frac{\bar{u}_\theta}{a} \\ &= \frac{\oint_C \vec{v} \cdot d\vec{l}}{2\pi a^2} = \frac{\Gamma}{2\pi a^2} \text{ (On a circle of radius } a) \\ &= \frac{\omega_j n_j}{2} \end{aligned}$$

$$\text{For Irrotational Flow, } \Gamma = 0, \vec{\omega} = 0$$

$$\text{A vector field } \vec{\alpha} \text{ is considered solenoidal if } \vec{\nabla} \cdot \vec{\alpha} = 0$$

$$\vec{\nabla} \cdot \vec{\omega} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) = 0$$

### 3.1. Streamlines and Vortex Lines

$$\text{Equations for Streamlines (Cartesian Coordinates)} \quad \frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z}$$

$$\text{Equations for Streamlines (Cylindrical Polar Coordinates)} \quad \frac{dR}{u_R} = \frac{R d\phi}{u_\phi} = \frac{dz}{u_z}$$

$$\text{Equations for Vortex Lines (Cartesian Coordinates)} \quad \frac{dx}{\omega_x} = \frac{dy}{\omega_y} = \frac{dz}{\omega_z}$$

### 3.2. Terms

- **Barotropic:** Density is a function of pressure only.  $\frac{1}{\rho^2} \vec{\nabla} \rho \times \vec{\nabla} p = 0$
- **Inviscid:** Viscosity can be neglected.  $\nu \nabla^2 \vec{\omega} = 0$
- **Baroclinic:** Measure of how misaligned the gradient of pressure is from the gradient of density in a fluid  $\vec{\nabla} \rho \times \vec{\nabla} p$
- **Isobars:** Constant pressure lines
- **Isopycals:** Constant density lines

### 3.3. Vorticity Transport Equation

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \vec{\nabla}) \vec{v} + \frac{1}{\rho^2} \vec{\nabla} \rho \times \vec{\nabla} p + \nu \nabla^2 \vec{\omega}$$

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{\omega} = (\vec{\omega} \cdot \vec{\nabla}) \vec{v} + \frac{1}{\rho^2} \vec{\nabla} \rho \times \vec{\nabla} p + \nu \nabla^2 \vec{\omega}$$

#### Legend

- $\frac{D\vec{\omega}}{Dt}$ : Total rate of change of vorticity of a fluid particle
- $(\vec{\omega} \cdot \vec{\nabla}) \vec{v}$ : Vorticity production due to stretching/tilting of vortex lines
- $\frac{1}{\rho^2} \vec{\nabla} \rho \times \vec{\nabla} p$ : Vorticity production due to baroclinic effects
- $\nu \nabla^2 \vec{\omega}$ : Viscous diffusion of vorticity which dissipates or redistributes vorticity

### 3.4. Helmholtz's Vortex Theorems for Inviscid Flows

#### Assumptions

- Inviscid
- Barotropic
- Conservative Body Forces  $\vec{F} = -\vec{\nabla} \psi$

#### Statement

- Fluid particles/element originally free of vorticity remain free of vorticity (non-rotating)
- Vortex lines (tubes) move with the fluid for inviscid flows. Vortex line is always comprised of the same fluid particles (Vorticity is frozen to the flow for inviscid flow)
- The strength of the vortex tube (circulation) does not vary with time during the fluid motion. Vortex tubes must be closed, go to infinity, or end on solid boundaries.

### 3.5. Kelvin's circulation theorem

#### Assumptions

- Inviscid
- Barotropic
- Incompressible
- Conservative Body Forces  $\vec{F} = -\nabla\psi$

#### Statement

The circulation (strength of the vortex) around a closed curve moving with the fluid will remain constant.

$$\frac{\partial \Gamma_C}{\partial t} + v_j \frac{\partial \Gamma_C}{\partial x_j} = \oint_C \nu \nabla^2 \vec{v} \cdot d\vec{x}$$

For inviscid flow,  $\frac{\partial \Gamma_C}{\partial t} + v_j \frac{\partial \Gamma_C}{\partial x_j} = 0$

### 3.6. Bernoulli's Equation

#### Assumptions

- Inviscid  $\nabla^2 v = 0 \rightarrow$  Euler
- Only gravitational body forces (Conservative)  $\rightarrow f_i = -\frac{d\psi}{dx_i}$  where  $\psi = gz$
- Barotropic  $\rightarrow \rho = \rho(p)$
- Steady flow  $\rightarrow \frac{\partial}{\partial t} = 0$

Note: No restriction on the compressibility effects

#### Statement

$$\nabla \left[ \frac{\vec{v} \cdot \vec{v}}{2} + gz + \int \frac{dp}{\rho} \right] = -\vec{\omega} \times \vec{v}$$

Along streamline or vortex line  $\frac{\vec{v} \cdot \vec{v}}{2} + gz + \int \frac{dp}{\rho} = \text{constant}$

Unsteady flow  $\frac{\partial \phi}{\partial t} + \frac{\vec{v} \cdot \vec{v}}{2} + gz + \int \frac{dp}{\rho} = F(t)$

Steady Potential flow  $\frac{\vec{v} \cdot \vec{v}}{2} + gz + \int \frac{dp}{\rho} = \text{constant throughout the flow}$

## 4. Potential Flow

$$i = \sqrt{-1}$$

$$z = x + iy = re^{i\theta} = r(\cos \theta + i \sin \theta)$$

$$\bar{z} = x - iy = re^{-i\theta} = r(\cos \theta - i \sin \theta)$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$r = \sqrt{x^2 + y^2} = |z| = |z\bar{z}|^{\frac{1}{2}}$$

$$\theta = \arctan \frac{y}{x}$$

$$\text{2D Incompressible Potential Flow } \nabla^2 \phi = 0 = \nabla^2 \psi$$

$$\text{Complex Potential } F(z) = \phi(z) + i\psi(z)$$

$$\text{Complex Velocity } w(z) = u - iv = (v_r - iv_\theta)e^{-i\theta} = \frac{dF(z)}{dz}$$

$$\bar{w}(z) = u + iv = (v_r + iv_\theta)e^{i\theta}$$

### 4.1. Stream Function $\Leftrightarrow$ Potential Function $\Leftrightarrow$ Velocities

Also called Cauchy Reimann Equations

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$$

$$v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

### 4.2. Cartesian $\Leftrightarrow$ Polar Velocities

$$u = v_r \cos \theta - v_\theta \sin \theta$$

$$v = v_r \sin \theta + v_\theta \cos \theta$$

$$v_r = u \cos \theta + v \sin \theta$$

$$v_\theta = -u \sin \theta + v \cos \theta$$

### 4.3. Complex Potentials

	F(z)	$\phi$	$\psi$	w(z)	u and $v_r$	v and $v_\theta$
Uniform	$Ue^{-i\alpha}z$	$U(x \cos \alpha + y \sin \alpha)$	$U(y \cos \alpha - x \sin \alpha)$	$Ue^{-i\alpha}$	$U \cos \alpha$	$U \sin \alpha$
		$Ur \cos(\theta - \alpha)$	$Ur \sin(\theta - \alpha)$		$U \cos(\theta - \alpha)$	$-U \sin(\theta - \alpha)$
Corner	$Cz^n$			$nCz^{n-1}$		
		$Cr^n \cos n\theta$	$Cr^n \sin n\theta$		$nCr^{n-1} \cos[(n-1)\theta]$	$-nC r^{n-1} \sin[(n-1)\theta]$
Source/Sink	$\frac{m}{2\pi} \ln(z - z_0)$	$\frac{m}{4\pi} \ln[x^2 + y^2]$	$\frac{m}{2\pi} \arctan \frac{y}{x}$	$\frac{m}{2\pi(z-z_0)}$	$\frac{m}{2\pi} \frac{x}{x^2+y^2}$	$\frac{m}{2\pi} \frac{y}{x^2+y^2}$
		$\frac{m}{2\pi} \ln r$	$\frac{m}{2\pi} \theta$		$\frac{m}{2\pi r}$	0
Free Vortex	$-\frac{i\Gamma}{2\pi} \ln(z - z_0)$	$\frac{\Gamma}{2\pi} \arctan \frac{y}{x}$	$-\frac{\Gamma}{4\pi} \ln[x^2 + y^2]$	$-\frac{i\Gamma}{2\pi(z-z_0)}$	$-\frac{\Gamma}{2\pi} \frac{y}{x^2+y^2}$	$\frac{\Gamma}{2\pi} \frac{x}{x^2+y^2}$
		$\frac{\Gamma}{2\pi} \theta$	$-\frac{\Gamma}{2\pi} \ln r$		0	$\frac{\Gamma}{2\pi r}$
Dipole	$\frac{\mu}{\pi z}$	$\frac{\mu}{\pi} \frac{x}{x^2+y^2}$	$-\frac{\mu}{\pi} \frac{y}{x^2+y^2}$	$-\frac{\mu}{\pi z^2}$	$\frac{\mu}{\pi} \frac{y^2-x^2}{(x^2+y^2)^2}$	$-\frac{\mu}{\pi} \frac{2xy}{(x^2+y^2)^2}$
		$\frac{\mu}{\pi r} \cos \theta$	$-\frac{\mu}{\pi r} \sin \theta$		$-\frac{\mu}{\pi r^2} \cos \theta$	$-\frac{\mu}{\pi r^2} \sin \theta$

### Legend

- Uniform
  - U: Uniform velocity magnitude
  - $\alpha$ : Angle of attack - the angle at which the direction of the uniform velocity is oriented with respect to the horizontal
- Corner
  - C: Indicates the direction.  $C > 0$  always
  - $n$ :  $\frac{\text{angle of the corner}}{\pi}$
- Source/Sink
  - m: Volume flow rate per unit dimension normal to the page. For source,  $m > 0$ , whereas for sink,  $m < 0$
- Dipole/Doublet
  - a: Half distance between the source and sink
  - Q: Volume flow rate per unit dimension normal to the page
  - $\mu$ :  $Qa$

#### 4.4. Infinite Series Expansions

- $\ln(1 + \epsilon) = \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3} - \dots$  when  $|\epsilon| \leq 1$
- $(1 + \epsilon)^{-1} = 1 - \epsilon + \epsilon^2 - \epsilon^3 + \dots$  when  $|\epsilon| \leq 1$
- $(1 + \epsilon)^{\frac{1}{2}} = 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \frac{1}{16}\epsilon^3 - \frac{5}{128}\epsilon^4 + \frac{7}{256}\epsilon^5 - \dots$  when  $|\epsilon| \leq 1$
- $(1 + \epsilon)^{-\frac{1}{2}} = 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \frac{35}{128}\epsilon^4 - \frac{63}{256}\epsilon^5 + \dots$  when  $|\epsilon| \leq 1$
- $e^\epsilon = 1 + \frac{\epsilon}{1!} + \frac{\epsilon^2}{2!} + \frac{\epsilon^3}{3!} + \dots$
- $\sin(\epsilon) = \epsilon - \frac{\epsilon^3}{3!} + \frac{\epsilon^5}{5!} - \frac{\epsilon^7}{7!} + \dots$

#### 4.5. Bernoulli's Equation (Irrotational)

$$p_\infty + \frac{\rho v_\infty^2}{2} = p + \frac{\rho |v^2|}{2}$$

#### 4.6. Forces on a 2D Body

All the below forces have the units of Force per unit length normal to the sheet of paper

$$\text{Drag Force} = - \int_C (p - p_\infty) \cos \theta \, ds$$

$$\text{Lift Force} = - \int_C (p - p_\infty) \sin \theta \, ds$$

$$\begin{aligned} \text{Complex Force } G = D - iL &= -i \oint_C p d\bar{z} = -i \oint_C \left[ p_\infty + \frac{\rho v_\infty^2}{2} - \frac{\rho |v^2|}{2} \right] d\bar{z} \\ &= \frac{i\rho}{2} \oint_C |v^2| d\bar{z} = \frac{i\rho}{2} \oint_C w \bar{w} d\bar{z} \\ &= \frac{i\rho}{2} \oint_C [w(z)]^2 dz \quad (\text{First Blasius Integral Law}) \end{aligned}$$

#### 4.7. Moment on a 2D Body

$$\text{Moment } M = -\frac{\rho}{2} \text{Re} \left\{ \oint_C z w^2 dz \right\} \quad (\text{Second Blasius Integral Law})$$

#### 4.8. Residue Theorem

$$\text{If } F(z) = \sum_j \frac{B_j}{z - z_j}$$

$$R_k = \sum B$$

$$\oint_C F(z) dz = 2\pi i \sum_k R_k$$

### 5. Boundary Layer Theory

Through dimensional analysis,

$$\frac{\delta}{L} \sim \sqrt{\frac{\nu}{vL}} \sim \frac{1}{\sqrt{Re_L}}$$

#### 5.1. Prandtl Boundary Layer Equations for a Flat Plate

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial p}{\partial y} = 0$$

#### 5.2. Inviscid Equivalence

##### Displacement Thickness

Decrease in flow-rate due to viscous flow vs inviscid flow  
(per unit length perpendicular to the plane)

$$= \int_A u_\infty dy dx - \int_A u dy dx = \int_A (u_\infty - u) dy dx$$

$$= \int_0^\infty (u_\infty - u) dy = u_\infty \delta^*$$

$$\Rightarrow \delta^* = \int_0^\infty \left( 1 - \frac{u}{u_\infty} \right) dy$$



## Momentum Thickness

Decrease in momentum flux due to viscous flow vs inviscid flow  
(per unit length in the z-direction)

$$\begin{aligned}
 &= \int_A \rho u_\infty u \, dy dx - \int_A \rho u u \, dy dx = \int_A \rho (u_\infty - u) u \, dy dx \\
 &= \int_0^\infty \rho (u_\infty - u) u \, dy = \rho u_\infty^2 \theta \\
 &\Rightarrow \boxed{\theta = \int_0^\infty \frac{u}{u_\infty} \left(1 - \frac{u}{u_\infty}\right) dy}
 \end{aligned}$$

## Von Karman Momentum Integral Equation

$$\frac{d}{dx}(\tilde{v}^2 \theta) + \tilde{v} \frac{d\tilde{v}}{dx} \delta^* = \frac{\tau_w}{\rho}$$

where,

$\delta^*$  is the displacement thickness

$\theta$  is the momentum thickness

$$\tilde{v}(x) = u|_{\text{boundary layer edge}}$$

$$\tau_w = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0}$$

Special Case: If  $\tilde{v} = u_\infty = \text{constant}$ ,

$$\text{Shear Stress on the plate } \tau_w = \rho u_\infty^2 \frac{d\theta}{dx}$$

$$\text{Force on the plate } F = \int \tau_w dx = \rho u_\infty^2 \theta$$

$$\text{Skin Friction Coefficient } C_f = \frac{\tau_w}{\left(\frac{\rho u_\infty^2}{2}\right)} = 2 \frac{d\theta}{dx}$$

## Momentum Equation at the Cylindrical Surface

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} = -\rho \tilde{v} \frac{\partial \tilde{v}}{\partial x}$$

## Boundary Layer Separation Point

$$\left. \frac{\partial u}{\partial y} \right|_{y=0} = 0$$

## Shape Factor

Higher the shape factor  $\frac{\delta^*}{\theta}$ , the closer a flow is towards separation