

# MEEN 621 Notes

Shivanand P

December 12, 2022

## Contents

<b>1</b>	<b>Mathematics</b>	<b>1</b>
1.1	Integral Tables	1
1.1.1	Basic Forms	1
1.1.2	Integrals of Rational Functions	1
1.1.3	Integrals with Roots	1
1.1.4	Integrals with Logarithms	2
1.1.5	Integrals with Exponentials	3
1.1.6	Integrals with Trigonometric Functions	3
1.1.7	Products of Trigonometric Functions and Monomials	4
1.1.8	Products of Trigonometric Functions and Exponentials	4
1.1.9	Integrals of Hyperbolic Functions	5
1.2	First-Order Linear ODE	5
<b>2</b>	<b>Continuum Flow</b>	<b>5</b>
2.1	Mass Conservation	5
2.1.1	Cartesian Coordinates	6
2.1.2	Cylindrical Polar Coordinates	6
2.2	Momentum Conservation (Navier-Stokes Equations)	6
2.2.1	Cartesian Coordinates	6
2.2.2	Cylindrical Polar Coordinates	6
2.3	Flow Conditions	6
2.3.1	Steady State	6
2.3.2	Fully Developed	7
2.3.3	Unidirectional	7
2.3.4	Two-dimensional	7
2.3.5	Free Surface	7
2.3.6	Axisymmetric	7
2.3.7	Simple Couette Flow	7
2.4	Boundary Conditions	7
2.4.1	Fluid-Solid Interface	7
2.4.2	Fluid-Fluid Interface	8
2.4.3	Fluid-Gas Interface	8
<b>3</b>	<b>Vorticity Dynamics</b>	<b>8</b>
3.1	Streamlines and Vortex Lines	8

3.2	Terms . . . . .	9
3.3	Vorticity Transport Equation . . . . .	9
3.3.1	Legend . . . . .	9
3.4	Helmholtz's Vortex Theorems for Inviscid Flows . . . . .	9
3.4.1	Assumptions . . . . .	9
3.4.2	Statement . . . . .	9
3.5	Kelvin's circulation theorem . . . . .	10
3.5.1	Assumptions . . . . .	10
3.5.2	Statement . . . . .	10
3.6	Bernoulli's Equation . . . . .	10
3.6.1	Assumptions . . . . .	10
3.6.2	Statement . . . . .	10
<b>4</b>	<b>Potential Flow . . . . .</b>	<b>11</b>
4.1	Stream Function $\Leftrightarrow$ Potential Function $\Leftrightarrow$ Velocities . . . . .	11
4.2	Cartesian $\Leftrightarrow$ Polar Velocities . . . . .	11
4.3	Complex Potentials . . . . .	11
4.3.1	Legend . . . . .	12
4.4	Infinite Series Expansions . . . . .	13
4.5	Bernoulli's Equation (Irrotational) . . . . .	13
4.6	Forces on a 2D Body . . . . .	13
4.7	Moment on a 2D Body . . . . .	13
4.8	Residue Theorem . . . . .	14
<b>5</b>	<b>Boundary Layer Theory . . . . .</b>	<b>14</b>
5.1	Prandtl Boundary Layer Equations for a Flat Plate . . . . .	14
5.2	Inviscid Equivalence . . . . .	14
5.2.1	Displacement Thickness . . . . .	14
5.2.2	Momentum Thickness . . . . .	15

## 1. Mathematics

### 1.1. Integral Tables

© 2014 From <http://integral-table.com>, last revised June 14, 2014. This material is provided as is without warranty or representation about the accuracy, correctness or suitability of the material for any purpose, and is licensed under the Creative Commons Attribution-Noncommercial-ShareAlike 3.0 United States License. To view a copy of this license, visit <http://creativecommons.org/licenses/by-nc-sa/3.0/> or send a letter to Creative Commons, 171 Second Street, Suite 300, San Francisco, California, 94105, USA.

#### Basic Forms

$$\int x^n dx = \frac{1}{n+1} x^{n+1} \quad (1) \quad \int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a} \ln \frac{a+x}{b+x}, \quad a \neq b \quad (14)$$

$$\int \frac{1}{x} dx = \ln |x| \quad (2) \quad \int \frac{x}{(x+a)^2} dx = \frac{a}{a+x} + \ln |a+x| \quad (15)$$

$$\int u dv = uv - \int v du \quad (3) \quad \int \frac{x}{ax^2 + bx + c} dx = \frac{1}{2a} \ln |ax^2 + bx + c|$$

$$- \frac{b}{a\sqrt{4ac - b^2}} \tan^{-1} \frac{2ax + b}{\sqrt{4ac - b^2}} \quad (16)$$

$$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln |ax+b| \quad (4)$$

#### Integrals of Rational Functions

$$\int \frac{1}{(x+a)^2} dx = -\frac{1}{x+a} \quad (5) \quad \int \sqrt{x-a} dx = \frac{2}{3} (x-a)^{3/2} \quad (17)$$

$$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1}, \quad n \neq -1 \quad (6) \quad \int \frac{1}{\sqrt{x \pm a}} dx = 2\sqrt{x \pm a} \quad (18)$$

$$\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)} \quad (7) \quad \int \frac{1}{\sqrt{a-x}} dx = -2\sqrt{a-x} \quad (19)$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x \quad (8) \quad \int x\sqrt{x-a} dx = \frac{2}{3} a(x-a)^{3/2} + \frac{2}{5} (x-a)^{5/2} \quad (20)$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \quad (9) \quad \int \sqrt{ax+b} dx = \left( \frac{2b}{3a} + \frac{2x}{3} \right) \sqrt{ax+b} \quad (21)$$

$$\int \frac{x}{a^2+x^2} dx = \frac{1}{2} \ln |a^2+x^2| \quad (10) \quad \int (ax+b)^{3/2} dx = \frac{2}{5a} (ax+b)^{5/2} \quad (22)$$

$$\int \frac{x^2}{a^2+x^2} dx = x - a \tan^{-1} \frac{x}{a} \quad (11) \quad \int \frac{x}{\sqrt{x \pm a}} dx = \frac{2}{3} (x \mp 2a) \sqrt{x \pm a} \quad (23)$$

$$\int \frac{x^3}{a^2+x^2} dx = \frac{1}{2} x^2 - \frac{1}{2} a^2 \ln |a^2+x^2| \quad (12) \quad \int \sqrt{\frac{x}{a-x}} dx = -\sqrt{x(a-x)} - a \tan^{-1} \frac{\sqrt{x(a-x)}}{x-a} \quad (24)$$

$$\int \frac{1}{ax^2+bx+c} dx = \frac{2}{\sqrt{4ac-b^2}} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}} \quad (13) \quad \int \sqrt{\frac{x}{a+x}} dx = \sqrt{x(a+x)} - a \ln [\sqrt{x} + \sqrt{x+a}] \quad (25)$$

#### Integrals with Roots

$$\int x\sqrt{ax+b}dx = \frac{2}{15a^2}(-2b^2 + abx + 3a^2x^2)\sqrt{ax+b} \quad (26)$$

$$\int \sqrt{x(ax+b)}dx = \frac{1}{4a^{3/2}} \left[ (2ax+b)\sqrt{ax(ax+b)} - b^2 \ln \left| a\sqrt{x} + \sqrt{a(ax+b)} \right| \right] \quad (27)$$

$$\int \sqrt{x^3(ax+b)}dx = \left[ \frac{b}{12a} - \frac{b^2}{8a^2x} + \frac{x}{3} \right] \sqrt{x^3(ax+b)} + \frac{b^3}{8a^{5/2}} \ln \left| a\sqrt{x} + \sqrt{a(ax+b)} \right| \quad (28)$$

$$\int \sqrt{x^2 \pm a^2}dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \pm \frac{1}{2}a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right| \quad (29)$$

$$\int \sqrt{a^2 - x^2}dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \tan^{-1} \frac{x}{\sqrt{a^2 - x^2}} \quad (30)$$

$$\int x\sqrt{x^2 \pm a^2}dx = \frac{1}{3} (x^2 \pm a^2)^{3/2} \quad (31)$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln \left| x + \sqrt{x^2 \pm a^2} \right| \quad (32)$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} \quad (33)$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2} \quad (34)$$

$$\int \frac{x}{\sqrt{a^2 - x^2}} dx = -\sqrt{a^2 - x^2} \quad (35)$$

$$\int \frac{x^2}{\sqrt{x^2 \pm a^2}} dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \mp \frac{1}{2}a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right| \quad (36)$$

$$\int \sqrt{ax^2 + bx + c}dx = \frac{b+2ax}{4a}\sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8a^{3/2}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx + c)} \right| \quad (37)$$

$$\int x\sqrt{ax^2 + bx + c} = \frac{1}{48a^{5/2}} \left( 2\sqrt{a}\sqrt{ax^2 + bx + c} \times (-3b^2 + 2abx + 8a(c + ax^2)) + 3(b^3 - 4abc) \ln \left| b + 2ax + 2\sqrt{a}\sqrt{ax^2 + bx + c} \right| \right) \quad (38)$$

$$\int \frac{1}{\sqrt{ax^2 + bx + c}} dx = \frac{1}{\sqrt{a}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx + c)} \right| \quad (39)$$

$$\int \frac{x}{\sqrt{ax^2 + bx + c}} dx = \frac{1}{a}\sqrt{ax^2 + bx + c} - \frac{b}{2a^{3/2}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx + c)} \right| \quad (40)$$

$$\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 + x^2}} \quad (41)$$

### Integrals with Logarithms

$$\int \ln ax dx = x \ln ax - x \quad (42)$$

$$\int \frac{\ln ax}{x} dx = \frac{1}{2} (\ln ax)^2 \quad (43)$$

$$\int \ln(ax + b) dx = \left( x + \frac{b}{a} \right) \ln(ax + b) - x, a \neq 0 \quad (44)$$

$$\int \ln(x^2 + a^2) dx = x \ln(x^2 + a^2) + 2a \tan^{-1} \frac{x}{a} - 2x \quad (45)$$

$$\int \ln(x^2 - a^2) dx = x \ln(x^2 - a^2) + a \ln \frac{x+a}{x-a} - 2x \quad (46)$$

$$\int \ln(ax^2 + bx + c) dx = \frac{1}{a}\sqrt{4ac - b^2} \tan^{-1} \frac{2ax + b}{\sqrt{4ac - b^2}} - 2x + \left( \frac{b}{2a} + x \right) \ln(ax^2 + bx + c) \quad (47)$$

$$\int x \ln(ax + b) dx = \frac{bx}{2a} - \frac{1}{4}x^2 + \frac{1}{2} \left( x^2 - \frac{b^2}{a^2} \right) \ln(ax + b) \quad (48)$$

$$\int x \ln(a^2 - b^2x^2) dx = -\frac{1}{2}x^2 + \frac{1}{2} \left( x^2 - \frac{a^2}{b^2} \right) \ln(a^2 - b^2x^2) \quad (49)$$

### Integrals with Exponentials

$$\int e^{ax} dx = \frac{1}{a} e^{ax} \quad (50)$$

$$\int \sqrt{x} e^{ax} dx = \frac{1}{a} \sqrt{x} e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}} \operatorname{erf}(i\sqrt{ax}),$$

where  $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$  (51)

$$\int x e^x dx = (x - 1) e^x \quad (52)$$

$$\int x e^{ax} dx = \left( \frac{x}{a} - \frac{1}{a^2} \right) e^{ax} \quad (53)$$

$$\int x^2 e^x dx = (x^2 - 2x + 2) e^x \quad (54)$$

$$\int x^2 e^{ax} dx = \left( \frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right) e^{ax} \quad (55)$$

$$\int x^3 e^x dx = (x^3 - 3x^2 + 6x - 6) e^x \quad (56)$$

$$\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx \quad (57)$$

$$\int x^n e^{ax} dx = \frac{(-1)^n}{a^{n+1}} \Gamma[1 + n, -ax],$$

where  $\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt$  (58)

$$\int e^{ax^2} dx = -\frac{i\sqrt{\pi}}{2\sqrt{a}} \operatorname{erf}(ix\sqrt{a}) \quad (59)$$

$$\int e^{-ax^2} dx = \frac{\sqrt{\pi}}{2\sqrt{a}} \operatorname{erf}(x\sqrt{a}) \quad (60)$$

$$\int x e^{-ax^2} dx = -\frac{1}{2a} e^{-ax^2} \quad (61)$$

$$\int x^2 e^{-ax^2} dx = \frac{1}{4} \sqrt{\frac{\pi}{a^3}} \operatorname{erf}(x\sqrt{a}) - \frac{x}{2a} e^{-ax^2} \quad (62)$$

### Integrals with Trigonometric Functions

$$\int \sin ax dx = -\frac{1}{a} \cos ax \quad (63)$$

$$\int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a} \quad (64)$$

$$\int \sin^n ax dx =$$

$$-\frac{1}{a} \cos ax {}_2F_1\left[\frac{1}{2}, \frac{1-n}{2}, \frac{3}{2}, \cos^2 ax\right] \quad (65)$$

$$\int \sin^3 ax dx = -\frac{3 \cos ax}{4a} + \frac{\cos 3ax}{12a} \quad (66)$$

$$\int \cos ax dx = \frac{1}{a} \sin ax \quad (67)$$

$$\int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a} \quad (68)$$

$$\int \cos^p ax dx = -\frac{1}{a(1+p)} \cos^{1+p} ax \times$$

$${}_2F_1\left[\frac{1+p}{2}, \frac{1}{2}, \frac{3+p}{2}, \cos^2 ax\right] \quad (69)$$

$$\int \cos^3 ax dx = \frac{3 \sin ax}{4a} + \frac{\sin 3ax}{12a} \quad (70)$$

$$\int \cos ax \sin bxdx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}, a \neq b \quad (71)$$

$$\int \sin^2 ax \cos bxdx = -\frac{\sin[(2a-b)x]}{4(2a-b)}$$

$$+ \frac{\sin bx}{2b} - \frac{\sin[(2a+b)x]}{4(2a+b)} \quad (72)$$

$$\int \sin^2 x \cos x dx = \frac{1}{3} \sin^3 x \quad (73)$$

$$\int \cos^2 ax \sin bxdx = \frac{\cos[(2a-b)x]}{4(2a-b)} - \frac{\cos bx}{2b}$$

$$- \frac{\cos[(2a+b)x]}{4(2a+b)} \quad (74)$$

$$\int \cos^2 ax \sin ax dx = -\frac{1}{3a} \cos^3 ax \quad (75)$$

$$\int \sin^2 ax \cos^2 bxdx = \frac{x}{4} - \frac{\sin 2ax}{8a} - \frac{\sin[2(a-b)x]}{16(a-b)}$$

$$+ \frac{\sin 2bx}{8b} - \frac{\sin[2(a+b)x]}{16(a+b)} \quad (76)$$

$$\int \sin^2 ax \cos^2 ax dx = \frac{x}{8} - \frac{\sin 4ax}{32a} \quad (77)$$

### Products of Trigonometric Functions and Monomials

$$\int \tan ax dx = -\frac{1}{a} \ln \cos ax \quad (78)$$

$$\int \tan^2 ax dx = -x + \frac{1}{a} \tan ax \quad (79)$$

$$\int \tan^n ax dx = \frac{\tan^{n+1} ax}{a(n+1)} \times {}_2F_1\left(\frac{n+1}{2}, 1, \frac{n+3}{2}, -\tan^2 ax\right) \quad (80)$$

$$\int \tan^3 ax dx = \frac{1}{a} \ln \cos ax + \frac{1}{2a} \sec^2 ax \quad (81)$$

$$\int \sec x dx = \ln |\sec x + \tan x| = 2 \tanh^{-1} \left( \tan \frac{x}{2} \right) \quad (82)$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax \quad (83)$$

$$\int \sec^3 x dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| \quad (84)$$

$$\int \sec x \tan x dx = \sec x \quad (85)$$

$$\int \sec^2 x \tan x dx = \frac{1}{2} \sec^2 x \quad (86)$$

$$\int \sec^n x \tan x dx = \frac{1}{n} \sec^n x, n \neq 0 \quad (87)$$

$$\int \csc x dx = \ln \left| \tan \frac{x}{2} \right| = \ln |\csc x - \cot x| + C \quad (88)$$

$$\int \csc^2 ax dx = -\frac{1}{a} \cot ax \quad (89)$$

$$\int \csc^3 x dx = -\frac{1}{2} \cot x \csc x + \frac{1}{2} \ln |\csc x - \cot x| \quad (90)$$

$$\int \csc^n x \cot x dx = -\frac{1}{n} \csc^n x, n \neq 0 \quad (91)$$

$$\int \sec x \csc x dx = \ln |\tan x| \quad (92)$$

$$\int x \cos x dx = \cos x + x \sin x \quad (93)$$

$$\int x \cos ax dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax \quad (94)$$

$$\int x^2 \cos x dx = 2x \cos x + (x^2 - 2) \sin x \quad (95)$$

$$\int x^2 \cos ax dx = \frac{2x \cos ax}{a^2} + \frac{a^2 x^2 - 2}{a^3} \sin ax \quad (96)$$

$$\int x^n \cos x dx = -\frac{1}{2} (i)^{n+1} [\Gamma(n+1, -ix) + (-1)^n \Gamma(n+1, ix)] \quad (97)$$

$$\int x^n \cos ax dx = \frac{1}{2} (ia)^{1-n} [(-1)^n \Gamma(n+1, -iax) - \Gamma(n+1, iax)] \quad (98)$$

$$\int x \sin x dx = -x \cos x + \sin x \quad (99)$$

$$\int x \sin ax dx = -\frac{x \cos ax}{a} + \frac{\sin ax}{a^2} \quad (100)$$

$$\int x^2 \sin x dx = (2 - x^2) \cos x + 2x \sin x \quad (101)$$

$$\int x^2 \sin ax dx = \frac{2 - a^2 x^2}{a^3} \cos ax + \frac{2x \sin ax}{a^2} \quad (102)$$

$$\int x^n \sin x dx = -\frac{1}{2} (i)^n [\Gamma(n+1, -ix) - (-1)^n \Gamma(n+1, -ix)] \quad (103)$$

### Products of Trigonometric Functions and Exponentials

$$\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) \quad (104)$$

$$\int e^{bx} \sin ax dx = \frac{1}{a^2 + b^2} e^{bx} (b \sin ax - a \cos ax) \quad (105)$$

$$\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x) \quad (106)$$

$$\int e^{bx} \cos ax dx = \frac{1}{a^2 + b^2} e^{bx} (a \sin ax + b \cos ax) \quad (107)$$

$$\int x e^x \sin x dx = \frac{1}{2} e^x (\cos x - x \cos x + x \sin x) \quad (108)$$

$$\int x e^x \cos x dx = \frac{1}{2} e^x (x \cos x - \sin x + x \sin x) \quad (109)$$

## Integrals of Hyperbolic Functions

$$\int \cosh ax dx = \frac{1}{a} \sinh ax \quad (110)$$

$$\int e^{ax} \cosh bx dx = \begin{cases} \frac{e^{ax}}{a^2 - b^2} [a \cosh bx - b \sinh bx] & a \neq b \\ \frac{e^{2ax}}{4a} + \frac{x}{2} & a = b \end{cases} \quad (111)$$

$$\int \sinh ax dx = \frac{1}{a} \cosh ax \quad (112)$$

$$\int e^{ax} \sinh bx dx = \begin{cases} \frac{e^{ax}}{a^2 - b^2} [-b \cosh bx + a \sinh bx] & a \neq b \\ \frac{e^{2ax}}{4a} - \frac{x}{2} & a = b \end{cases} \quad (113)$$

$$\int e^{ax} \tanh bx dx = \begin{cases} \frac{e^{(a+2b)x}}{(a+2b)} {}_2F_1 \left[ 1 + \frac{a}{2b}, 1, 2 + \frac{a}{2b}, -e^{2bx} \right] \\ \quad - \frac{1}{a} e^{ax} {}_2F_1 \left[ \frac{a}{2b}, 1, 1E, -e^{2bx} \right] & a \neq b \\ \frac{e^{ax} - 2 \tan^{-1}[e^{ax}]}{a} & a = b \end{cases} \quad (114)$$

$$\int \tanh ax dx = \frac{1}{a} \ln \cosh ax \quad (115)$$

$$\int \cos ax \cosh bx dx = \frac{1}{a^2 + b^2} [a \sin ax \cosh bx + b \cos ax \sinh bx] \quad (116)$$

$$\int \cos ax \sinh bx dx = \frac{1}{a^2 + b^2} [b \cos ax \cosh bx + a \sin ax \sinh bx] \quad (117)$$

$$\int \sin ax \cosh bx dx = \frac{1}{a^2 + b^2} [-a \cos ax \cosh bx + b \sin ax \sinh bx] \quad (118)$$

$$\int \sin ax \sinh bx dx = \frac{1}{a^2 + b^2} [b \cosh bx \sin ax - a \cos ax \sinh bx] \quad (119)$$

$$\int \sinh ax \cosh ax dx = \frac{1}{4a} [-2ax + \sinh 2ax] \quad (120)$$

$$\int \sinh ax \cosh bx dx = \frac{1}{b^2 - a^2} [b \cosh bx \sinh ax - a \cosh ax \sinh bx] \quad (121)$$

## 1.2. First-Order Linear ODE

Equations of the form  $\frac{dy}{dx} + \alpha(x)y = \beta(x)$  can be solved by multiplying both sides by an integrating factor  $e^{\int \alpha(x)dx}$ . Then the left-hand side becomes  $\frac{d(e^{\int \alpha(x)dx} \cdot y)}{dx}$  and the right hand side becomes  $e^{\int \alpha(x)dx} \beta(x)$ . The left and right hand sides can then be integrated directly.

## 2. Continuum Flow

### 2.1. Mass Conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \mathbf{v}_j) = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_j)}{\partial x_j} = 0$$

## Cartesian Coordinates

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_1)}{\partial x_1} + \frac{\partial(\rho v_2)}{\partial x_2} + \frac{\partial(\rho v_3)}{\partial x_3} = 0$$

## Cylindrical Polar Coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r}(r \rho v_r) + \frac{1}{r} \frac{\partial \rho v_\theta}{\partial \theta} + \frac{\partial \rho v_z}{\partial z} = 0$$

## 2.2. Momentum Conservation (Navier-Stokes Equations)

$$\rho \left[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \rho \vec{f} - \nabla p + \mu \nabla^2 \vec{v}$$

### Cartesian Coordinates

$$\begin{aligned} \rho \left[ \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3} \right] &= \rho f_1 - \frac{\partial p}{\partial x_1} + \mu \left[ \frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_3^2} \right] \\ \rho \left[ \frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} + v_3 \frac{\partial v_2}{\partial x_3} \right] &= \rho f_2 - \frac{\partial p}{\partial x_2} + \mu \left[ \frac{\partial^2 v_2}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_2^2} + \frac{\partial^2 v_2}{\partial x_3^2} \right] \\ \rho \left[ \frac{\partial v_3}{\partial t} + v_1 \frac{\partial v_3}{\partial x_1} + v_2 \frac{\partial v_3}{\partial x_2} + v_3 \frac{\partial v_3}{\partial x_3} \right] &= \rho f_3 - \frac{\partial p}{\partial x_3} + \mu \left[ \frac{\partial^2 v_3}{\partial x_1^2} + \frac{\partial^2 v_3}{\partial x_2^2} + \frac{\partial^2 v_3}{\partial x_3^2} \right] \end{aligned}$$

### Cylindrical Polar Coordinates

$$\begin{aligned} \rho \left[ \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} \right] &= \rho f_r - \frac{\partial p}{\partial r} + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r}(r v_r) \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right] \\ \rho \left[ \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \right] &= \rho f_\theta - \frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r}(r v_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right] \\ \rho \left[ \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right] &= \rho f_z - \frac{\partial p}{\partial z} + \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] \end{aligned}$$

## 2.3. Flow Conditions

### Steady State

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= 0, \\ \frac{\partial v_1}{\partial t} &= 0, \quad \frac{\partial v_2}{\partial t} = 0, \quad \frac{\partial v_3}{\partial t} = 0 \end{aligned}$$



### Fully Developed

$$\frac{\partial v_1}{\partial x_1} = 0, \frac{\partial v_2}{\partial x_2} = 0, \frac{\partial v_3}{\partial x_3} = 0$$

### Unidirectional

$$v_{n1} = 0, v_{n2} = 0$$

### Two-dimensional

$$v_n = 0, \\ \frac{\partial v_{t1}}{\partial n} = 0, \frac{\partial v_{t2}}{\partial n} = 0$$

### Free Surface

$$\frac{\partial p}{\partial t} = 0$$

### Axisymmetric

$$v_\theta = 0, \\ \frac{\partial v_r}{\partial \theta} = 0, \frac{\partial v_\theta}{\partial \theta} = 0, \frac{\partial v_z}{\partial \theta} = 0,$$

### Simple Couette Flow

$$\frac{\partial p}{\partial x_1} = 0, \frac{\partial p}{\partial x_2} = 0, \frac{\partial p}{\partial x_3} = 0$$

## 2.4. Boundary Conditions

*Let  $t$  indicate a direction tangent to the boundary*

*Let  $n$  indicate a direction normal to the boundary*

### Fluid-Solid Interface

$$\text{No Slip: } v_t|_{\text{boundary}} = 0$$

$$\text{No Penetration: } v_n|_{\text{boundary}} = 0$$

### Fluid-Fluid Interface

$$\text{Continuity: } \rho_I v_{In} = \rho_{II} v_{II n}$$

$$\text{Jump/Discontinuity: } \rho_I \neq \rho_{II}$$

$$v_{In} \neq v_{II n}$$

$$\text{Shear Stress Continuity: } T_{ij} n_j t_i|_I = T_{ij} n_j t_i|_{II}$$

$$\Rightarrow \mu_I \frac{\partial v_t}{\partial n} \Big|_I = \mu_{II} \frac{\partial v_t}{\partial n} \Big|_{II}$$

### Fluid-Gas Interface

$$\text{Free Surface: } p|_{\text{boundary}} = p_\infty$$

$$\text{Traction-Free: } \frac{\partial v_t}{\partial n} \Big|_{\text{boundary}} = 0$$

## 3. Vorticity Dynamics

$$\text{Circulation } \Gamma = \oint_C \vec{v} \cdot d\vec{l} = \int_S \vec{\omega} \cdot \hat{n} dA$$

$$\begin{aligned} \text{Average angular velocity } \bar{\Omega} &= \frac{\bar{u}_\theta}{a} \\ &= \frac{\oint_C \vec{v} \cdot d\vec{l}}{2\pi a^2} = \frac{\Gamma}{2\pi a^2} \text{ (On a circle of radius } a) \\ &= \frac{\omega_j n_j}{2} \end{aligned}$$

$$\text{For Irrotational Flow, } \Gamma = 0, \vec{\omega} = 0$$

$$\text{A vector field } \vec{\alpha} \text{ is considered solenoidal if } \vec{\nabla} \cdot \vec{\alpha} = 0$$

$$\vec{\nabla} \cdot \vec{\omega} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) = 0$$

### 3.1. Streamlines and Vortex Lines

$$\text{Equations for Streamlines (Cartesian Coordinates)} \quad \frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z}$$

$$\text{Equations for Streamlines (Cylindrical Polar Coordinates)} \quad \frac{dR}{u_R} = \frac{R d\phi}{u_\phi} = \frac{dz}{u_z}$$

$$\text{Equations for Vortex Lines (Cartesian Coordinates)} \quad \frac{dx}{\omega_x} = \frac{dy}{\omega_y} = \frac{dz}{\omega_z}$$

### 3.2. Terms

- **Barotropic:** Density is a function of pressure only.  $\frac{1}{\rho^2} \vec{\nabla} \rho \times \vec{\nabla} p = 0$
- **Inviscid:** Viscosity can be neglected.  $\nu \nabla^2 \vec{\omega} = 0$
- **Baroclinic:** Measure of how misaligned the gradient of pressure is from the gradient of density in a fluid  $\vec{\nabla} \rho \times \vec{\nabla} p$
- **Isobars:** Constant pressure lines
- **Isopycals:** Constant density lines

### 3.3. Vorticity Transport Equation

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \vec{\nabla}) \vec{v} + \frac{1}{\rho^2} \vec{\nabla} \rho \times \vec{\nabla} p + \nu \nabla^2 \vec{\omega}$$

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{\omega} = (\vec{\omega} \cdot \vec{\nabla}) \vec{v} + \frac{1}{\rho^2} \vec{\nabla} \rho \times \vec{\nabla} p + \nu \nabla^2 \vec{\omega}$$

#### Legend

- $\frac{D\vec{\omega}}{Dt}$ : Total rate of change of vorticity of a fluid particle
- $(\vec{\omega} \cdot \vec{\nabla}) \vec{v}$ : Vorticity production due to stretching/tilting of vortex lines
- $\frac{1}{\rho^2} \vec{\nabla} \rho \times \vec{\nabla} p$ : Vorticity production due to baroclinic effects
- $\nu \nabla^2 \vec{\omega}$ : Viscous diffusion of vorticity which dissipates or redistributes vorticity

### 3.4. Helmholtz's Vortex Theorems for Inviscid Flows

#### Assumptions

- Inviscid
- Barotropic
- Conservative Body Forces  $\vec{F} = -\vec{\nabla} \psi$

#### Statement

- Fluid particles/element originally free of vorticity remain free of vorticity (non-rotating)
- Vortex lines (tubes) move with the fluid for inviscid flows. Vortex line is always comprised of the same fluid particles (Vorticity is frozen to the flow for inviscid flow)
- The strength of the vortex tube (circulation) does not vary with time during the fluid motion. Vortex tubes must be closed, go to infinity, or end on solid boundaries.

### 3.5. Kelvin's circulation theorem

#### Assumptions

- Inviscid
- Barotropic
- Incompressible
- Conservative Body Forces  $\vec{F} = -\nabla\psi$

#### Statement

The circulation (strength of the vortex) around a closed curve moving with the fluid will remain constant.

$$\frac{\partial \Gamma_C}{\partial t} + v_j \frac{\partial \Gamma_C}{\partial x_j} = \oint_C \nu \nabla^2 \vec{v} \cdot d\vec{x}$$

For inviscid flow,  $\frac{\partial \Gamma_C}{\partial t} + v_j \frac{\partial \Gamma_C}{\partial x_j} = 0$

### 3.6. Bernoulli's Equation

#### Assumptions

- Inviscid  $\nabla^2 v = 0 \rightarrow$  Euler
- Only gravitational body forces (Conservative)  $\rightarrow f_i = -\frac{d\psi}{dx_i}$  where  $\psi = gz$
- Barotropic  $\rightarrow \rho = \rho(p)$
- Steady flow  $\rightarrow \frac{\partial}{\partial t} = 0$

Note: No restriction on the compressibility effects

#### Statement

$$\nabla \left[ \frac{\vec{v} \cdot \vec{v}}{2} + gz + \int \frac{dp}{\rho} \right] = -\vec{\omega} \times \vec{v}$$

Along streamline or vortex line  $\frac{\vec{v} \cdot \vec{v}}{2} + gz + \int \frac{dp}{\rho} = \text{constant}$

Unsteady flow  $\frac{\partial \phi}{\partial t} + \frac{\vec{v} \cdot \vec{v}}{2} + gz + \int \frac{dp}{\rho} = F(t)$

Steady Potential flow  $\frac{\vec{v} \cdot \vec{v}}{2} + gz + \int \frac{dp}{\rho} = \text{constant throughout the flow}$

## 4. Potential Flow

$$i = \sqrt{-1}$$

$$z = x + iy = re^{i\theta} = r(\cos \theta + i \sin \theta)$$

$$\bar{z} = x - iy = re^{-i\theta} = r(\cos \theta - i \sin \theta)$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$r = \sqrt{x^2 + y^2} = |z| = |z\bar{z}|^{\frac{1}{2}}$$

$$\theta = \arctan \frac{y}{x}$$

$$\text{2D Incompressible Potential Flow } \nabla^2 \phi = 0 = \nabla^2 \psi$$

$$\text{Complex Potential } F(z) = \phi(z) + i\psi(z)$$

$$\text{Complex Velocity } w(z) = u - iv = (v_r - iv_\theta)e^{-i\theta} = \frac{dF(z)}{dz}$$

$$\bar{w}(z) = u + iv = (v_r + iv_\theta)e^{i\theta}$$

### 4.1. Stream Function $\Leftrightarrow$ Potential Function $\Leftrightarrow$ Velocities

Also called Cauchy Reimann Equations

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$$

$$v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

### 4.2. Cartesian $\Leftrightarrow$ Polar Velocities

$$u = v_r \cos \theta - v_\theta \sin \theta$$

$$v = v_r \sin \theta + v_\theta \cos \theta$$

$$v_r = u \cos \theta + v \sin \theta$$

$$v_\theta = -u \sin \theta + v \cos \theta$$

### 4.3. Complex Potentials

	F(z)	$\phi$	$\psi$	w(z)	u and $v_r$	v and $v_\theta$
Uniform	$Ue^{-i\alpha}z$	$U(x \cos \alpha + y \sin \alpha)$	$U(y \cos \alpha - x \sin \alpha)$	$Ue^{-i\alpha}$	$U \cos \alpha$	$U \sin \alpha$
		$Ur \cos(\theta - \alpha)$	$Ur \sin(\theta - \alpha)$		$U \cos(\theta - \alpha)$	$-U \sin(\theta - \alpha)$
Corner	$Cz^n$			$nCz^{n-1}$		
		$C r^n \cos n\theta$	$C r^n \sin n\theta$		$nC r^{n-1} \cos[(n-1)\theta]$	$-nC r^{n-1} \sin[(n-1)\theta]$
Source/Sink	$\frac{m}{2\pi} \ln(z - z_0)$	$\frac{m}{4\pi} \ln[x^2 + y^2]$	$\frac{m}{2\pi} \arctan \frac{y}{x}$	$\frac{m}{2\pi(z-z_0)}$	$\frac{m}{2\pi} \frac{x}{x^2+y^2}$	$\frac{m}{2\pi} \frac{y}{x^2+y^2}$
		$\frac{m}{2\pi} \ln r$	$\frac{m}{2\pi} \theta$		$\frac{m}{2\pi r}$	0
Free Vortex	$-\frac{i\Gamma}{2\pi} \ln(z - z_0)$	$\frac{\Gamma}{2\pi} \arctan \frac{y}{x}$	$-\frac{\Gamma}{4\pi} \ln[x^2 + y^2]$	$-\frac{i\Gamma}{2\pi(z-z_0)}$	$-\frac{\Gamma}{2\pi} \frac{y}{x^2+y^2}$	$\frac{\Gamma}{2\pi} \frac{x}{x^2+y^2}$
		$\frac{\Gamma}{2\pi} \theta$	$-\frac{\Gamma}{2\pi} \ln r$		0	$\frac{\Gamma}{2\pi r}$
Dipole	$\frac{\mu}{\pi z}$	$\frac{\mu}{\pi} \frac{x}{x^2+y^2}$	$-\frac{\mu}{\pi} \frac{y}{x^2+y^2}$	$-\frac{\mu}{\pi z^2}$	$\frac{\mu}{\pi} \frac{y^2-x^2}{(x^2+y^2)^2}$	$-\frac{\mu}{\pi} \frac{2xy}{(x^2+y^2)^2}$
		$\frac{\mu}{\pi r} \cos \theta$	$-\frac{\mu}{\pi r} \sin \theta$		$-\frac{\mu}{\pi r^2} \cos \theta$	$-\frac{\mu}{\pi r^2} \sin \theta$

### Legend

- Uniform
  - U: Uniform velocity magnitude
  - $\alpha$ : Angle of attack - the angle at which the direction of the uniform velocity is oriented with respect to the horizontal
- Corner
  - C: Indicates the direction.  $C > 0$  always
  - $n$ :  $\frac{\text{angle of the corner}}{\pi}$
- Source/Sink
  - m: Volume flow rate per unit dimension normal to the page. For source,  $m > 0$ , whereas for sink,  $m < 0$
- Dipole/Doublet
  - a: Half distance between the source and sink
  - Q: Volume flow rate per unit dimension normal to the page
  - $\mu$ :  $Qa$

#### 4.4. Infinite Series Expansions

- $\ln(1 + \epsilon) = \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3} - \dots$  when  $|\epsilon| \leq 1$
- $(1 + \epsilon)^{-1} = 1 - \epsilon + \epsilon^2 - \epsilon^3 + \dots$  when  $|\epsilon| \leq 1$
- $(1 + \epsilon)^{\frac{1}{2}} = 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \frac{1}{16}\epsilon^3 - \frac{5}{128}\epsilon^4 + \frac{7}{256}\epsilon^5 - \dots$  when  $|\epsilon| \leq 1$
- $(1 + \epsilon)^{-\frac{1}{2}} = 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \frac{35}{128}\epsilon^4 - \frac{63}{256}\epsilon^5 + \dots$  when  $|\epsilon| \leq 1$
- $e^\epsilon = 1 + \frac{\epsilon}{1!} + \frac{\epsilon^2}{2!} + \frac{\epsilon^3}{3!} + \dots$
- $\sin(\epsilon) = \epsilon - \frac{\epsilon^3}{3!} + \frac{\epsilon^5}{5!} - \frac{\epsilon^7}{7!} + \dots$

#### 4.5. Bernoulli's Equation (Irrotational)

$$p_\infty + \frac{\rho v_\infty^2}{2} = p + \frac{\rho |v^2|}{2}$$

#### 4.6. Forces on a 2D Body

All the below forces have the units of Force per unit length normal to the sheet of paper

$$\text{Drag Force} = - \int_C (p - p_\infty) \cos \theta \, ds$$

$$\text{Lift Force} = - \int_C (p - p_\infty) \sin \theta \, ds$$

$$\begin{aligned} \text{Complex Force } G = D - iL &= -i \oint_C p d\bar{z} = -i \oint_C \left[ p_\infty + \frac{\rho v_\infty^2}{2} - \frac{\rho |v^2|}{2} \right] d\bar{z} \\ &= \frac{i\rho}{2} \oint_C |v^2| d\bar{z} = \frac{i\rho}{2} \oint_C w \bar{w} d\bar{z} \\ &= \frac{i\rho}{2} \oint_C [w(z)]^2 dz \quad (\text{First Blasius Integral Law}) \end{aligned}$$

#### 4.7. Moment on a 2D Body

$$\text{Moment } M = -\frac{\rho}{2} \text{Re} \left\{ \oint_C z w^2 dz \right\} \quad (\text{Second Blasius Integral Law})$$

#### 4.8. Residue Theorem

$$\text{If } F(z) = \sum_j \frac{B_j}{z - z_j}$$

$$R_k = \sum B$$

$$\oint_C F(z) dz = 2\pi i \sum_k R_k$$

### 5. Boundary Layer Theory

Through dimensional analysis,

$$\frac{\delta}{L} \sim \sqrt{\frac{\nu}{vL}} \sim \frac{1}{\sqrt{Re_L}}$$

#### 5.1. Prandtl Boundary Layer Equations for a Flat Plate

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial p}{\partial y} = 0$$

#### 5.2. Inviscid Equivalence

##### Displacement Thickness

Decrease in flow-rate due to viscous flow vs inviscid flow  
(per unit length perpendicular to the plane)

$$= \int_A u_\infty dy dx - \int_A u dy dx = \int_A (u_\infty - u) dy dx$$

$$= \int_0^\infty (u_\infty - u) dy = u_\infty \delta^*$$

$$\Rightarrow \delta^* = \int_0^\infty \left( 1 - \frac{u}{u_\infty} \right) dy$$



## Momentum Thickness

Decrease in momentum flux due to viscous flow vs inviscid flow  
(per unit length in the z-direction)

$$= \int_A \rho u_\infty u \, dy dx - \int_A \rho u u \, dy dx = \int_A \rho (u_\infty - u) u \, dy dx$$

$$= \int_0^\infty \rho (u_\infty - u) u \, dy = \rho u_\infty^2 \theta$$

$$\Rightarrow \theta = \int_0^\infty \frac{u}{u_\infty} \left( 1 - \frac{u}{u_\infty} \right) dy$$