Homework 2

AMATH 563, Spring 2023

Due on Apr 28, 2023 at midnight.

Directions, Reminders and Policies

- You must upload a pdf file of your HW to Canvas by the due date.
- Make sure your solutions are well-written, complete, and readable.
- I suggest you use LaTeX(Overleaf is a great option) to prepare your HW and typeset your mathematical equations.
- If you prefer to hand in a handwritten solution then simply scan and upload the pdf.
- Remember you have two extension tokens that you can use for a day extension for your HWs throughout the quarter.
- I encourage collaborations and working with your colleagues to solve HW problems but you should only hand in your own work. We have a zero tolerance policy when it comes to academic misconduct and dishonesty including: Cheating; Falsification; Plagiarism; Engaging in prohibited behavior; Submitting the same work for separate courses without the permission of the instructor(s); Taking deliberate action to destroy or damage another person's academic work. **Such behavior will be reported to the UW Academic Misconduct office without warning**.

Problems

Theory

1. Suppsoe $\Gamma : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is a PDS kernel. Prove that $\forall x, x' \in \mathcal{X}$ it holds that $|\Gamma(x, x')|^2 \leq \Gamma(x, x)\Gamma(x', x')$.
2. Given a kernel K on $\mathcal{X}$ define its normalized version as

$$\bar{K}(x, x') = \begin{cases} 0 & \text{if} \quad K(x, x) = 0 \text{ or } K(x', x') = 0 \\ \dfrac{K(x, x')}{\sqrt{K(x, x)}\sqrt{K(x', x')}} & \text{Otherwise.} \end{cases}$$

   Show that if K is PDS then so is $\bar{K}$.
3. Show that the following kernels on $\mathbb{R}^d$ are PDS:

   - Polynomial kernel: $K(x, x') = \left(x^T x' + c\right)^\alpha$ for $c > 0$ and $\alpha \in \mathbb{N}$.
   - Exponential kernel: $K(x, x') = \exp(x^T x')$.
   - RBF kernel: $K(x, x') = \exp(-\gamma^2 \|x - x'\|_2^2)$.
4. Let $\Omega \subseteq \mathbb{R}^d$ and let $\{\psi_j\}_{j=1}^n$ be a sequence of continuous functions on Ω and $\{\lambda_j\}_{j=1}^n$ a sequence of non-negative numbers. Show that $K(x, x') = \sum_{j=1}^n \lambda_j \psi_j(x)\psi_j(x')$ is a PDS kernel on Ω.
5. Show that: (i) if K and K' are two reproducing kernels for an RKHS $\mathcal{H}$, then they have to be the same. (ii) the RKHS of a PDS kernel K is unique.

Computation

Download the MNIST training and test .csv files from Canvas and load them on your computer. I suggest you use Python or MATLAB for this excercise.

- Use Principle Component Analysis (PCA) on the training set to reduce the dimension of your input. How many modes do you need to preserve 95% of the variance in the training set?
- Extract the digits 1 and 9 from the training set. Use kernel regression to design and train a classifier to distinguish these digits using three different kernels of your choosing (I suggest RBF, Polynomila, and linear). It is a good idea to use PCA to reduce your input dimensions here. Also, you may use cross validation to tune your kernel/regularization/nugget parameters if you need them. Present the training and test error of your method.
- Repeat the above experiment for the digits (3, 8), (1, 7), and (5, 2).
- Write a report of a maximum of four pages, outlining your results and findings.