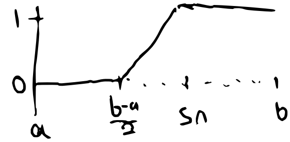
1) Consider the normed space $(X, \|\cdot\|)$ s.t. $X = C[a,b]$, $\|\cdot\| = L^2[a,b]$, $a,b \in \mathbb{R}$

Consider the sequence $x_n(t) = \begin{cases} 0, & t \in [a, \frac{b-a}{2}] \\ (s_n - \frac{b-a}{2})^{-1} t + \frac{b-a}{2}, & t \in (\frac{b-a}{2}, s_n) \\ 1, & t \in [s_n, b] \end{cases}$

s.t. $s_n = \frac{b-a}{2} + 1/n$



Suppose $n \geq m \Rightarrow 1/n \leq 1/m \Rightarrow s_n \leq s_m$ WLOG

$$\Rightarrow \|x_n - x_m\|^2 = \int_a^b (x_n - x_m)^2 dt = \int_{\frac{b-a}{2}}^{s_m} (x_n - x_m)^2 dt \leq \int_{\frac{b-a}{2}}^{s_m} dt = \frac{b-a}{2} + 1/m - \frac{b-a}{2} = 1/m \xrightarrow{m,n \to \infty} 0$$

since $x_n, x_m \leq 1$ for $t \leq b$

$\|x_n - x_m\| \leq \|x_n - x_m\|^2 \xrightarrow{m,n \to \infty} 0 \Rightarrow \forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|x_n - x_m| < \delta \Rightarrow \|x_n - x_m\| < \varepsilon$

Taking $\delta = 1/\varepsilon^2 \Rightarrow \|x_n - x_m\| < \varepsilon \Rightarrow x_n$ is a Cauchy sequence in $(X, \|\cdot\|)$

Consider $\varphi(t) = \begin{cases} 0, & t \in [a, \frac{b-a}{2}) \\ 1, & t \in (\frac{b-a}{2}, b] \end{cases} \Rightarrow \varphi \notin C[a,b] \Rightarrow \varphi \notin (X, \|\cdot\|)$

$$\|x_n - \varphi\|^2 = \int_a^b (x_n - \varphi)^2 dt = \int_{\frac{b-a}{2}}^{s_n} (x_n - \varphi)^2 dt \leq \int_{\frac{b-a}{2}}^{s_n} dt = 1/n \xrightarrow{n \to \infty} 0 \Rightarrow \|x_n - \varphi\| \xrightarrow{n \to \infty} 0$$

$\Rightarrow x_n$ is Cauchy over $(X, \|\cdot\|)$ and converges to $\varphi \notin (X, \|\cdot\|)$

Thus, $(X, \|\cdot\|)$ is incomplete and hence not Banach ■

2) N1: $\|x_1\|_1, \|x_2\|_2 \geq 0 \quad \forall x_1 \in X_1, x_2 \in X_2 \Rightarrow \|x\| = \max(\|x_1\|_1, \|x_2\|_2) \geq 0 \quad \forall x \in X$

N2: $\|x\| = \max(\|x_1\|_1, \|x_2\|_2) = 0 \Rightarrow \|x_1\|_1, \|x_2\| \leq 0 \Rightarrow \|x_1\|_1 = \|x_2\|_2 = 0$

$\Rightarrow x_1 = x_2 = 0 \Rightarrow x = 0$

$x = 0 \Rightarrow x_1 = x_2 = 0 \Rightarrow \|x_1\|_1 = \|x_2\|_2 = 0 \Rightarrow \|x\| = \max(\|x_1\|_1, \|x_2\|_2) = 0$

So $\|x\| = 0 \Leftrightarrow x = 0$

N3: Consider $\alpha \in \mathbb{C}$

$\|\alpha x\| = \max(\|\alpha x_1\|_1, \|\alpha x_2\|_2) = \max(|\alpha| \|x_1\|_1, |\alpha| \|x_2\|_2) = |\alpha| \max(\|x_1\|_1, \|x_2\|_2) = |\alpha| \|x\|$

N4: Consider $y = (y_1, y_2), z = (z_1, z_2) \in X$

$\|x + y\| = \max(\|x_1 + y_1\|_1, \|x_2 + y_2\|_2)$

$\leq \max(\|x_1\|_1 + \|y_1\|_1, \|x_2\|_2 + \|y_2\|_2)$

$\leq \max(\|x_1\|_1, \|x_2\|_2) + \max(\|y_1\|_1, \|y_2\|_2) = \|x\| + \|y\|$

Proof: $\|x_1\|_1 + \|y_1\|_1 \leq \max\{\|x_1\|_1, \|x_2\|_2\} + \|y_1\|_1 \leq \max\{\|x_1\|_1, \|x_2\|_2\} + \max\{\|y_1\|_1, \|y_2\|_2\}$

Same argument for $\|x_2\|_2 + \|y_2\|_2$

Thus, $(X, \max(\|\cdot\|_1, \|\cdot\|_2))$ is a normed space

3) Consider lin. ops. $T: X \to Y$, $S: Y \to Z$ and suppose their composition $TS: X \to Z$ exists

i: T is linear $\Rightarrow$ $\mathrm{dom}(T) = X = \mathrm{dom}(TS)$ is a V.S. and $\mathrm{ran}(T)$ is a V.S. over same field

S is linear $\Rightarrow$ $\mathrm{ran}(S) = Z = \mathrm{ran}(TS)$ is a V.S. over the same field as $\mathrm{dom}(S)$

TS exists $\Rightarrow$ $\mathrm{ran}(T) \subset \mathrm{codom}(T) = \mathrm{dom}(S)$

$\Rightarrow$ $\mathrm{ran}(TS)$ is a V.S. over the same field as $\mathrm{dom}(TS)$

ii: Consider $x, y \in X$, $\alpha \in \mathbb{C}$

$TS(x+y) = S \circ (T(x) + T(y)) = S \circ T(x) + S \circ T(y) = TS(x) + TS(y)$ since $\mathrm{ran}(T) \subset \mathrm{dom}(S)$ by assumption

$TS(\alpha x) = S \circ (T(\alpha x)) = S \circ (\alpha T(x)) = \alpha\, S \circ T(x) = \alpha\, TS(x)$

Thus, TS exists $\Rightarrow$ TS is linear ∎

4) T is linear $\Rightarrow \dim(\ker(T)) = \dim(\mathrm{dom}(T)) - \dim(\mathrm{ran}(T)) = n - n = 0 \Rightarrow T$ is injective

$\Rightarrow \exists T^{-1}: Y \to X$ which is linear by Th'm 2.6-10ab

$\exists T^{-1}: Y \to X \Rightarrow T^{-1}$ is Linear by Th'm 2.6-10b $\Rightarrow (T^{-1})^{-1} = T$ is linear by above

$\Rightarrow T^{-1}$ is a bijection $\Rightarrow \mathrm{dom}(T^{-1}) = \mathrm{ran}(T) = Y$

Thus, $\mathrm{ran}(T) = Y \Leftrightarrow T^{-1}$ exists $\blacksquare$

5) Consider x s.t. $T(x) = 0$ and suppose $\exists b > 0$ s.t. $\|T(x)\| \geq b\|x\|$ $\forall x \in X$

$T(x) = 0 \Rightarrow 0 = \|T(x)\| \geq b\|x\| \geq 0$ since $b, \|x\| \geq 0$ $\Rightarrow \|x\| = 0$ since $b > 0$

$\Rightarrow T^{-1}$ exists by Th'm 2.6-10a $\Rightarrow T^{-1}(y) = x$ $\forall y \in Y$ since T invertible iff T bijective

$\Rightarrow \|T^{-1}(y)\| = \|x\| \leq \|T(x)\|/b = \|y\|/b < \infty$

6) $f: C[a,b] \to \mathbb{R}: x \mapsto \max_{[a,b]} x(t)$ $\Rightarrow$ f is a functional

$f(x+y) = \max_{[a,b]} (x+y)\circ t = \max_{[a,b]} x(t) + y(t) = \max_{[a,b]} x(t) + \max_{[a,b]} y(t) = f(x) + f(y)$

$f(\alpha x) = \max_{[a,b]} (\alpha x)\circ t = \max_{[a,b]} \alpha x(t) = \alpha \max_{[a,b]} x(t) = \alpha f(x)$ $\Rightarrow$ f is Linear

$|f(x)| = |\max_{[a,b]} x(t)| \leq |\sup_{[a,b]} x(t)| \leq \sup_{[a,b]} |x(t)| = \|x\|$ $\Rightarrow$ f is bounded

Thus, f is a bounded Linear functional on $(C[a,b], \|\cdot\|_\infty)$

7) $\operatorname{ran}(\|\varphi\|)$ is a unit sphere in $\mathbb{R}$ $\Rightarrow$ $\operatorname{dom}(\|\varphi\|)$ is bounded $\Rightarrow$ $\|\varphi\| < \infty$

If φ is a linear functional, then $|\varphi(x)| \leq \|\varphi\| \|x\| < \infty$ by 2.8-2

N1: $|\cdot| \geq 0 \Rightarrow \|\varphi\| = \sup_{\|x\|=1} |\varphi(x)| \geq 0$

N2: $\varphi = 0 \Rightarrow |\varphi(x)| = 0 \ \forall x \in X \Rightarrow \sup_{\|x\|=1} |\varphi(x)| = \|\varphi\| = 0$

$\sim(\varphi = 0) \Rightarrow \exists x \in X$ s.t. $|\varphi(x)| \neq 0 \Rightarrow \sup_{\|x\|=1} |\varphi(x)| = \|\varphi\| \neq 0$

N3: $\|\alpha\varphi\| = \sup_{\|x\|=1} |\alpha\varphi \circ x| = \sup_{\|x\|=1} |\alpha \cdot \varphi(x)| = |\alpha| \sup_{\|x\|=1} |\varphi(x)| = |\alpha| \|\varphi\|$

N4: $\|\varphi + \psi\| = \sup_{\|x\|=1} |(\varphi + \psi) \circ x| = \sup_{\|x\|=1} |\varphi(x) + \psi(x)|$

$\leq \sup_{\|x\|=1} (|\varphi(x)| + |\psi(x)|)$

$\leq \sup_{\|x\|=1} |\varphi(x)| + \sup_{\|x\|=1} |\psi(x)| = \|\varphi\| + \|\psi\|$

Proof: Same argument as Q2-N4

Since the above holds for bounded linear functionals, $\|\varphi\|$ defines a norm on X^*

8) $|\langle x, 0\rangle| = |\langle 0, y\rangle| = 0$ by definition of inner product

$\|x\|\|0\| = \|0\|\|y\| = 0$ by definition of norm $\Rightarrow |\langle x,y\rangle| \leq \|x\|\|y\|$ if $y=0$

Suppose $y \neq 0$

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 = \|x\|^2\|y\|^2 - \langle y,x\rangle\langle x,y\rangle$$

$$= \langle x,x\rangle\langle y,y\rangle - \langle y,x\rangle\langle x,y\rangle + \left(\langle x,y\rangle\langle y,x\rangle - \langle x,y\rangle\langle y,x\rangle\right)$$

$$= \langle x, \|y\|^2 x\rangle - \langle x, \langle x,y\rangle y\rangle - \langle \langle x,y\rangle y, x\rangle + \langle x,y\rangle\langle y,x\rangle$$

$$\Rightarrow \|y\|^2\left(\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2\right)$$

$$= \langle \|y\|^2 x, \|y\|^2 x\rangle - \langle \|y\|^2 x, \langle x,y\rangle y\rangle - \langle \langle x,y\rangle y, \|y\|^2 x\rangle + \|y\|^2\langle x,y\rangle\langle y,x\rangle$$

$$= \langle \|y\|^2 x - \langle x,y\rangle y, \|y\|^2 x - \langle x,y\rangle y\rangle = \|\,\|y\|^2 x - \langle x,y\rangle y\|^2 \geq 0$$

$$\Rightarrow \|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 = \|\,\|y\|^2 x - \langle x,y\rangle y\|^2 / \|y\|^2 \geq 0$$

$$\Rightarrow |\langle x,y\rangle|^2 \leq \|x\|^2\|y\|^2 \Rightarrow |\langle x,y\rangle| \leq \|x\|\|y\|$$

$$y = \alpha x \Rightarrow \|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 = \|\,|\alpha|\langle x,x\rangle x - \alpha\langle x,x\rangle x\|^2 / |\alpha|\langle x,x\rangle = 0$$

$$\Rightarrow |\langle x,y\rangle|^2 = \|x\|^2\|y\|^2 \Rightarrow |\langle x,y\rangle| = \|x\|\|y\|$$

$$|\langle x,y\rangle| = \|x\|\|y\| \Rightarrow |\langle x,y\rangle|^2 = \|x\|^2\|y\|^2 \Rightarrow \|\,\|y\|^2 x - \langle x,y\rangle y\|^2 = 0$$

$$\Rightarrow \langle y,y\rangle x = \langle x,y\rangle y \Rightarrow x = \frac{\langle x,y\rangle}{\|y\|^2} y \Rightarrow y \in \operatorname{span}(x) \blacksquare$$