Homework 1

AMATH 563, Spring 2023

Due on Apr 14, 2023 at midnight.

Directions, Reminders and Policies

- You must upload a pdf file of your HW to Canvas by the due date.
- Make sure your solutions are well-written, complete, and readable.
- I suggest you use LaTeX(Overleaf is a great option) to prepare your HW and typeset your mathematical equations.
- If you prefer to hand in a handwritten solution then simply scan and upload the pdf.
- Remember you have two extension tokens that you can use for a day extension for your HWs throughout the quarter.
- I encourage collaborations and working with your colleagues to solve HW problems but you should only hand in your own work. We have a zero tolerance policy when it comes to academic misconduct and dishonesty including: Cheating; Falsification; Plagiarism; Engaging in prohibited behavior; Submitting the same work for separate courses without the permission of the instructor(s); Taking deliberate action to destroy or damage another person's academic work. **Such behavior will be reported to the UW Academic Misconduct office without warning**.

PROBLEMS

Hint: read chapters 2 and 3 of Kreyszig.

1. Prove that $C([a,b])$ equipped with the $L^2([a,b])$ norm is not a Banach space.
2. If $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ are normed spaces, show that the (Cartesian) product space $X = X_1 \times X_2$ becomes a normed space with the norm $\|x\| = \max(\|x_1\|_1, \|x_2\|_2)$ where $x \in X$ is defined as the tuple $x = (x_1, x_2)$ with addition and scalar multiplication operations: $(x_1, x_2) + (y_1, y_2) = (x_1 + x_2, y_1 + y_2)$ and $\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$.
3. Show that the product (composition) of two linear operators, if it exists, is a linear operator.
4. Let $T : X \to Y$ be a linear operator and $\dim X = \dim Y = n < +\infty$. Show that $\mathrm{Range}(T) = Y$ if and only if T^{-1} exists.
5. Let T be a bounded linear operator from a normed space X onto a normed space Y. Show that if there is a positive constant b such that $\|Tx\| \geq b\|x\|$ for all $x \in X$ then T^{-1} exists and is bounded.
6. Consider the functional $f(x) = \max_{t \in [a,b]} x(t)$ on $C([a,b])$ equipped with the sup norm. Is this functional linear? is it bounded?
7. Let X be a Banach space and denote its dual as X^*. Show that $\|\varphi\| : \varphi \mapsto \sup_{\|x\|=1} |\varphi(x)|$ is a norm on X^*.
8. Prove the Schwartz inequality on inner product spaces: $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$ for all $x, y \in X$, where equality holds if and only if x, y are linearly dependent.