Can you link up with treewidth?

(to prove tight lower bounds for subgraph problems)

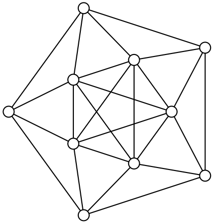
Jiaheng Wang *(Regensburg)*

with

Radu Curticapean *(ITU Copenhagen, Regensburg)*, Simon Döring *(Saarland)* and Daniel Neuen *(MPI)*

LFCS Seminar, 19 Nov 2024

A canonical problem

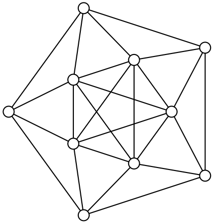


A canonical problem

CLIQUE

Input A graph G and an integer k

Output Does it have a size-k clique?



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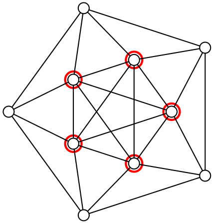

A canonically hard problem

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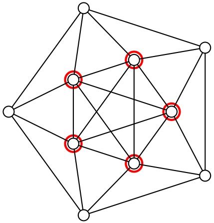

Theorem

[Karp'72]

CLIQUE is **NP**-complete.

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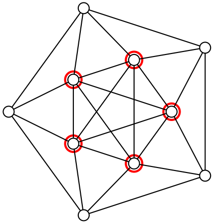
~~poly(n) algorithm~~

unless **P** = **NP**



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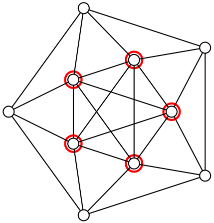

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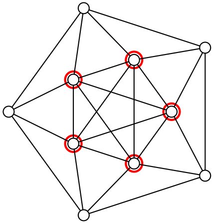

Trivial $O(n^k)$ alg

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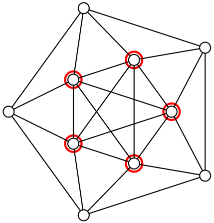

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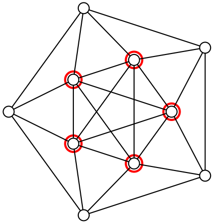

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~~$f(k) \cdot \text{poly}(n)$ alg~~

unless **FPT** = **W[1]**

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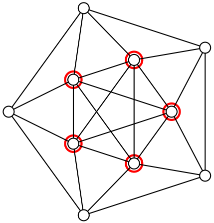
$n^{\log n}$
$2^{\sqrt{n}}$ algorithm?



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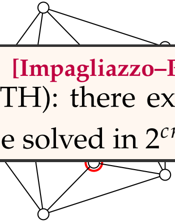

Trivial $O(n^k)$ alg.

unless **FPT** = **W[1]**

Conjecture

[Impagliazzo–Paturi'99, Impagliazzo–Paturi–Zane'01]

Exponential-Time Hypothesis (ETH): there exists an absolute constant c such that 3-SAT on n variables cannot be solved in 2^{cn} time.

Theorem

[Karp'72]

CLIQUE is **NP**-complete.

Theorem

[Downey–Fellows'95]

pCLIQUE is **W[1]**-complete.

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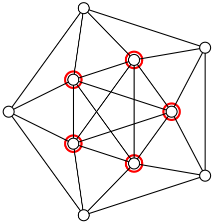
~~$2^{o(n)}$ algorithm~~
unless ETH fails



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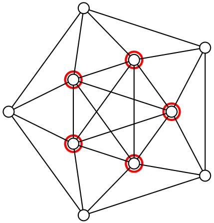

Trivial $O(n^k)$ alg

$f(k) \cdot n^{\sqrt{k}}$ alg?

Theorem [Karp'72]

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A fine-grained canonically hard parameterised problem

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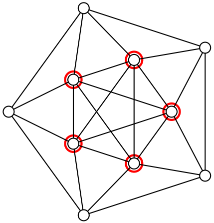

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$\cancel{f(k) \cdot n^{o(k)}}$ ~~alg~~
unless ETH fails

Theorem

[Chen–Huang–Kanj–Xia'06]

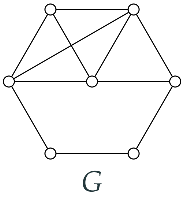
pCLIQUE-k cannot be solved in time $f(k)n^{o(k)}$ unless ETH fails.

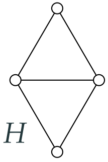
Subgraph detection problem

SUB(H)

Input A graph G

Output Does it have a subgraph (isomorphic to) H?





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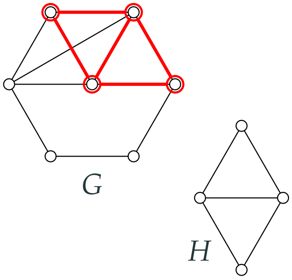

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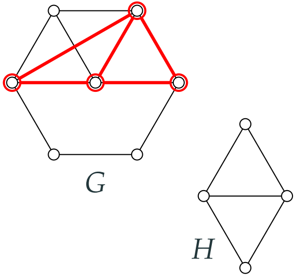

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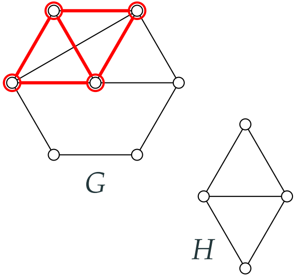

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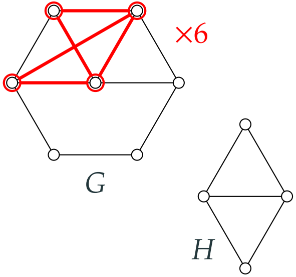

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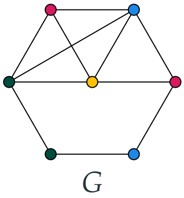
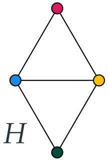
COLSUB(H)

Input A vertex-coloured graph G

Output Does it have a colourful copy of H?





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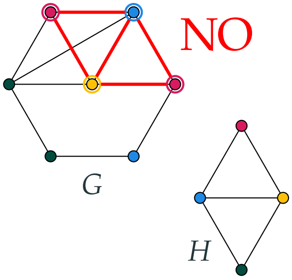

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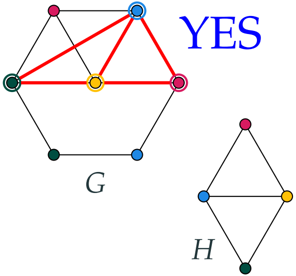

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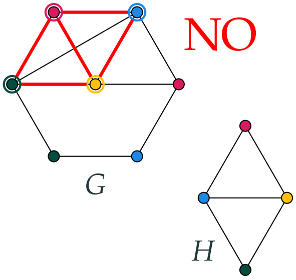

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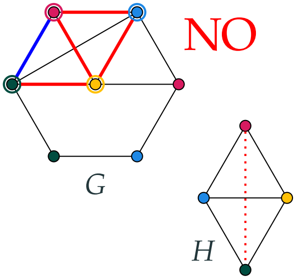

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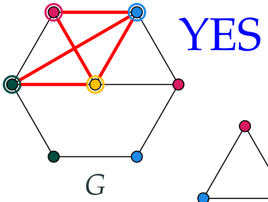
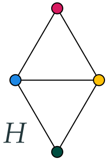




Subgraph detection problem

$\#\textsc{Sub}(H)$

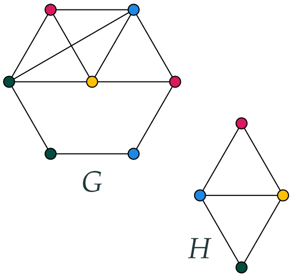
Input A graph G

Output Number of subgraphs (isomorphic to) H

$\#\textsc{ColSub}(H)$

Input A vertex-coloured graph G

Output Number of colourful copies of H



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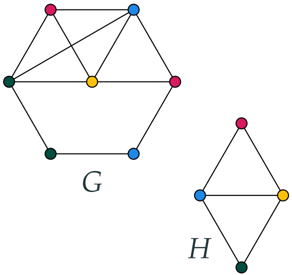
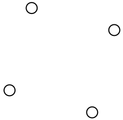


Indset



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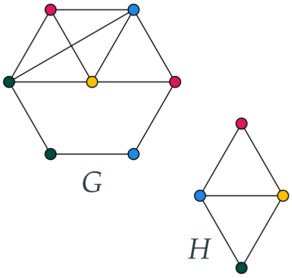

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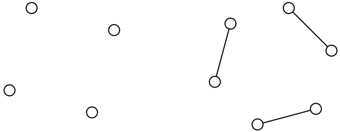
Matching



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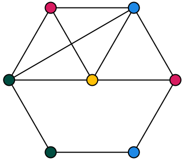

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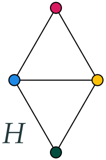
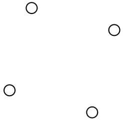
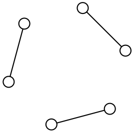
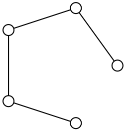


H

Indset	Matching	Path







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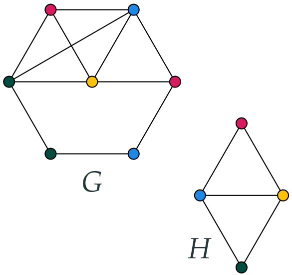

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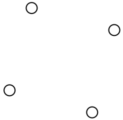

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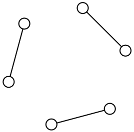

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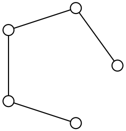
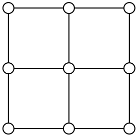


Grid



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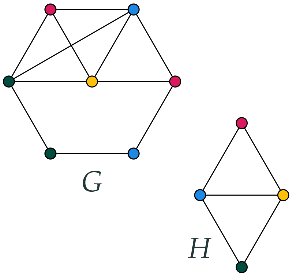
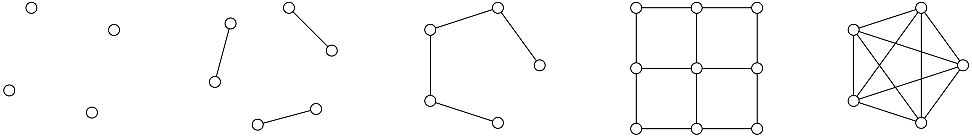


Indset	Matching	Path	Grid	Clique



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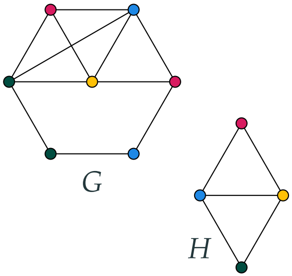

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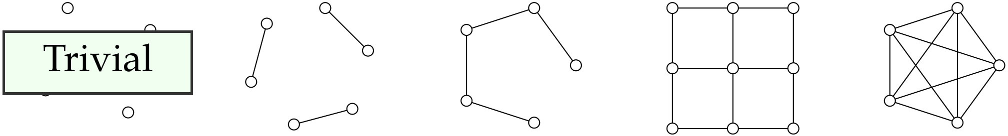

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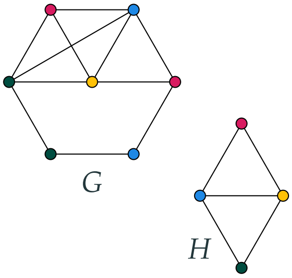

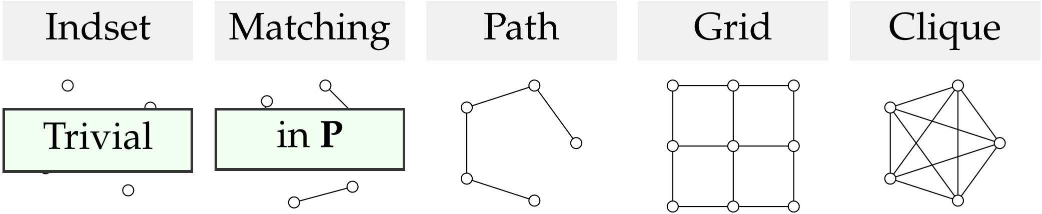

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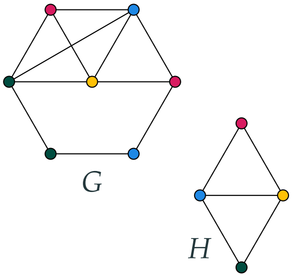

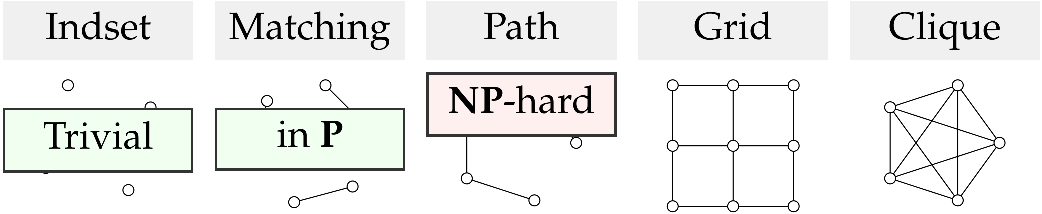

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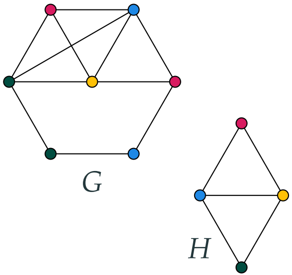
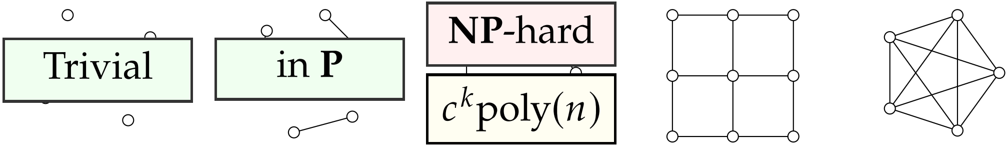


Indset	Matching	Path	Grid	Clique
Trivial	in **P**	**NP**-hard / $c^k\text{poly}(n)$		

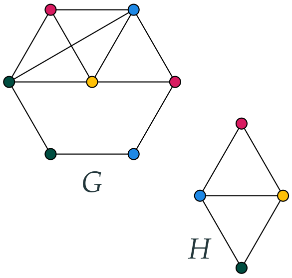


Subgraph detection problem

ColSub(H)

Input A vertex-coloured graph G

Output Does it have a colourful copy of H?



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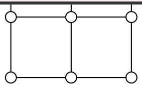

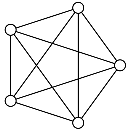

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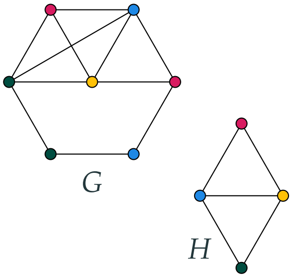

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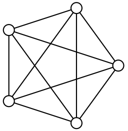
Trivial	in **P**	**NP**-hard	**W[1]**-hard	
		$c^k\text{poly}(n)$	$n^{\sqrt{k}}$	

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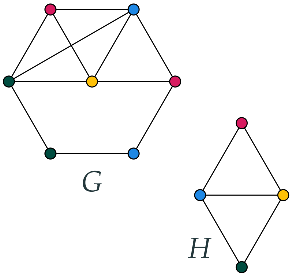

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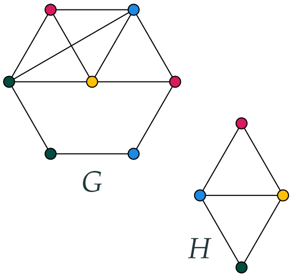

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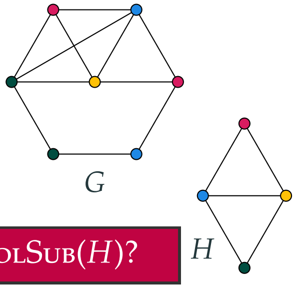

How can we prove tight lower bound for ColSub(H)?

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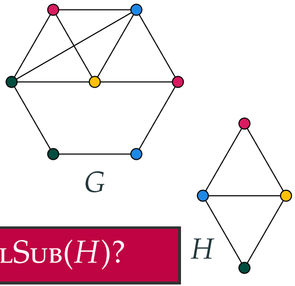

Why **do we need tight lower bound for** ColSub(H)?

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Is pClique a good source of hardness?

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[Chen–Huang–Kanj–Xia'06]

pClique-k cannot be solved in time $f(k)n^{o(k)}$ unless ETH fails.

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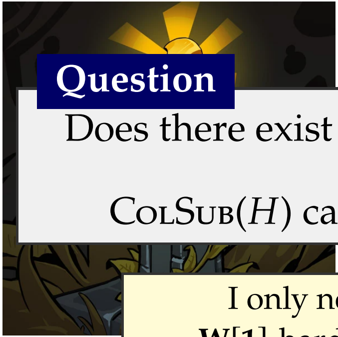

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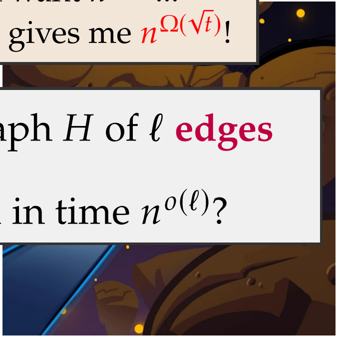

Question

Does there exist any **sparse** graph H of ℓ **edges** such that ColSub(H) cannot be solved in time $n^{o(\ell)}$?

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Hardness of sparse subgraphs

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Unfortunately, we don't know if there is any such H! However...

Almost tight hard instances

Theorem

[Marx'10]

There is a sequence of **degree-3** graphs $H_1, H_2, \cdots$ s.t. H_ℓ has ℓ edges and ColSub(H_ℓ) cannot be solved in time $n^{o(\ell/\log \ell)}$ unless ETH fails.

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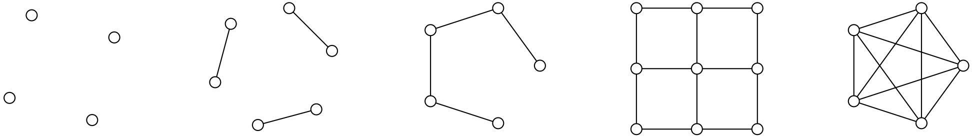

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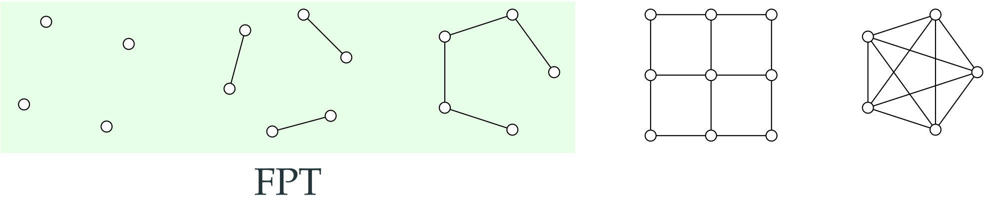

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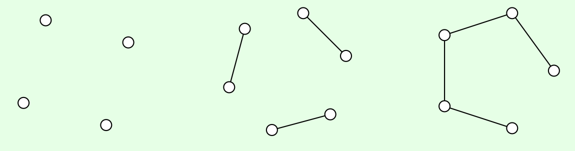
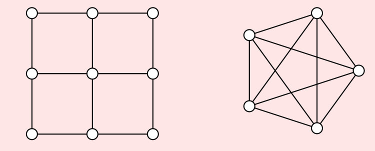


FPT



W[1]-hard

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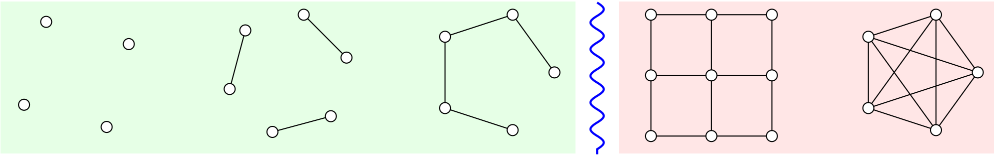

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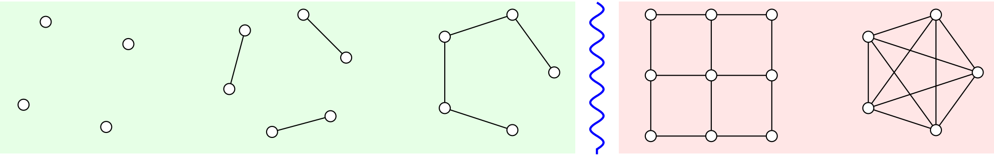

FPT ? **W[1]**-hard

Theorem

[Grohe–Schwentick–Segoufin'01, Grohe'07]

$\textsc{ColSub}(\mathcal{H})$ is tractable if and only if $\mathcal{H}$ is a bounded-**treewidth** graph family, unless **FPT** = **W[1]**.

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www.theoryofcomputing.org

Can You Beat Treewidth?*

Dániel Marx†

Received: September 3, 2008; published: August 27, 2010.

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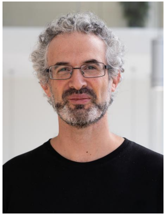
No, you can't. Treewidth is *almost* optimal.



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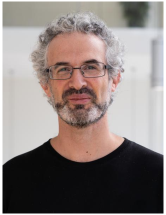

Theorem

[Marx'10]

Let $\mathcal{H}$ be **any** unbounded-treewidth graph family.
Then $\textsc{ColSub}(H)$ for $H \in \mathcal{H}$ cannot be solved in time $f(H)n^{o(\mathrm{tw}(H)/\log \mathrm{tw}(H))}$ unless ETH fails.

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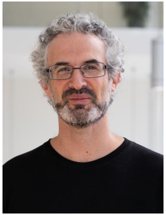

Theorem

[Too many papers]

There is an explicit construction of degree-3 **expanders**.

Theorem

[Grohe–Marx'08]

Expanders has linear treewidth.

Folklore corollary of ETH

$\exists c > 0$ s.t. 3-Colouring cannot be solved in time $O(2^{cn})$ on degree-4 graphs.

$$\text{3-Colouring}[2^{c_1 n}] \quad \preceq \quad \text{pClique-}k[N^{c_2 k}]$$

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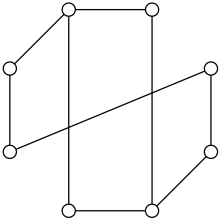

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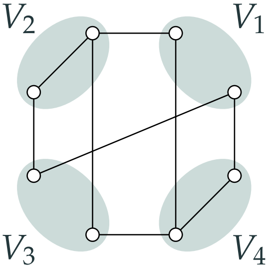

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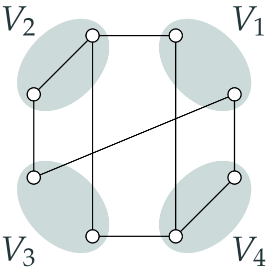

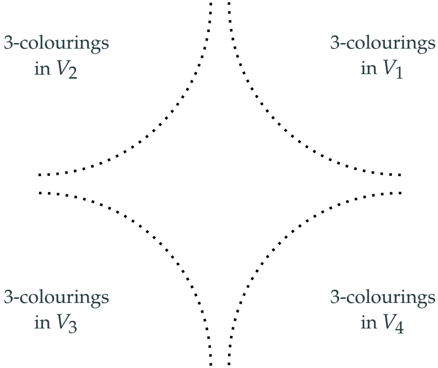

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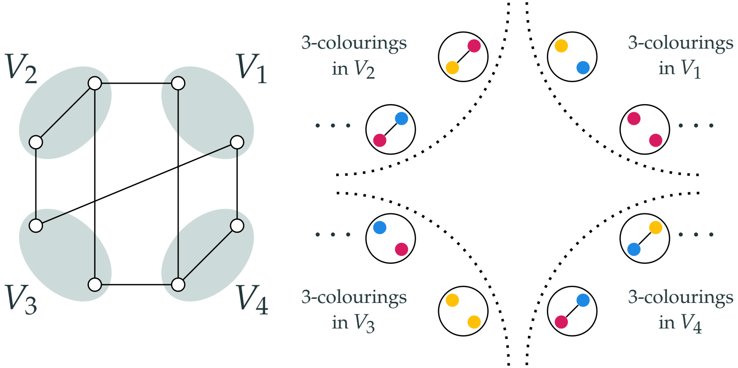

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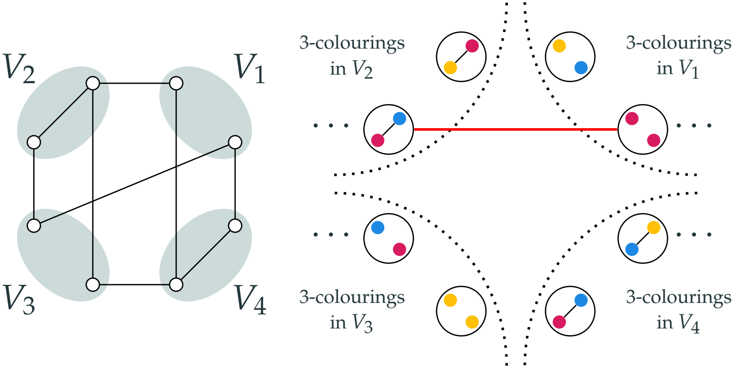

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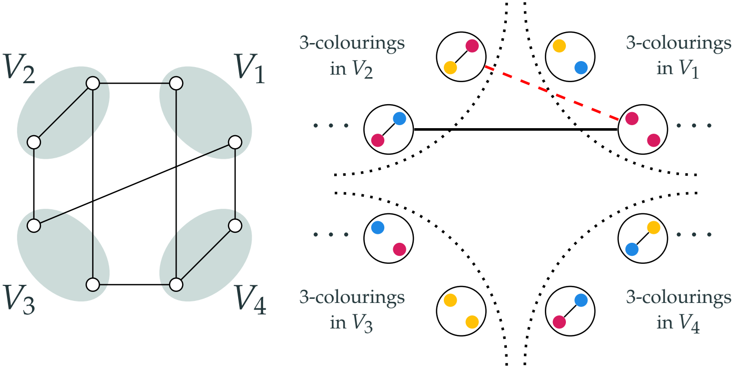

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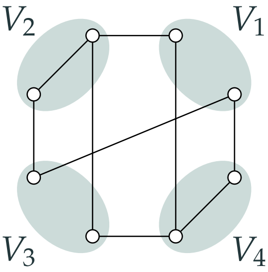

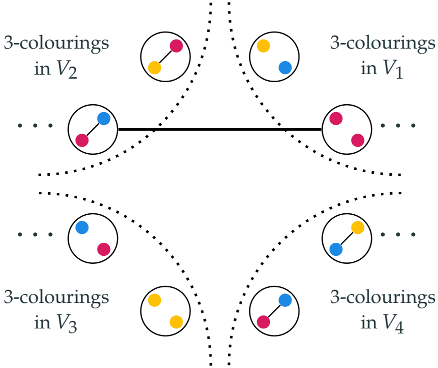

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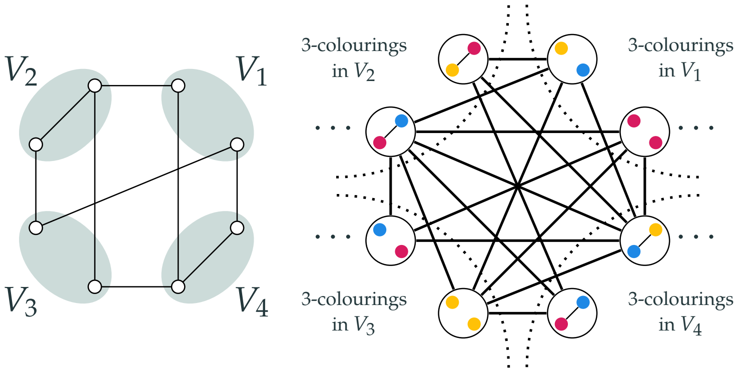

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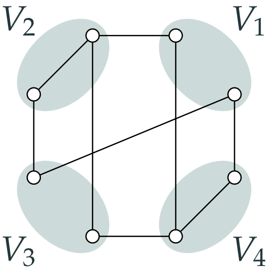

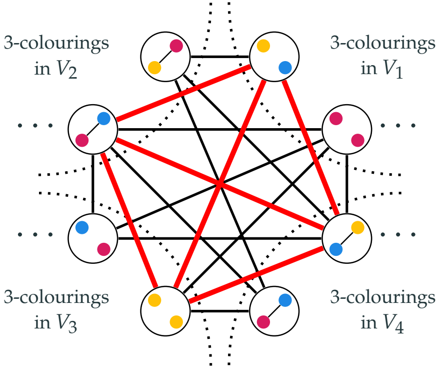

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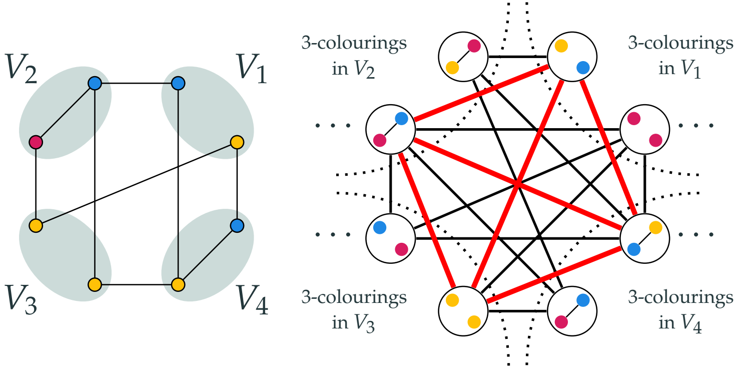

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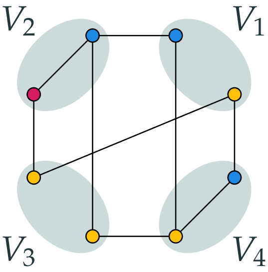
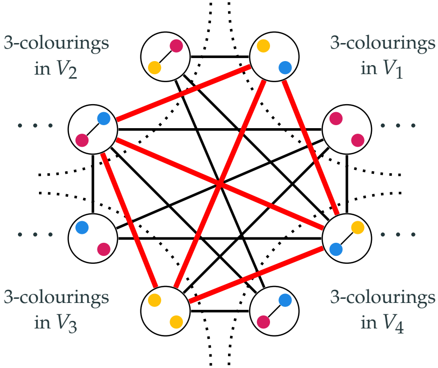
$$\#\text{3-Colouring}(G) \quad = \quad \#\text{pClique-}k(G^*)$$





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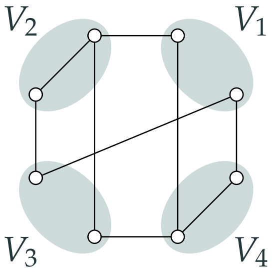
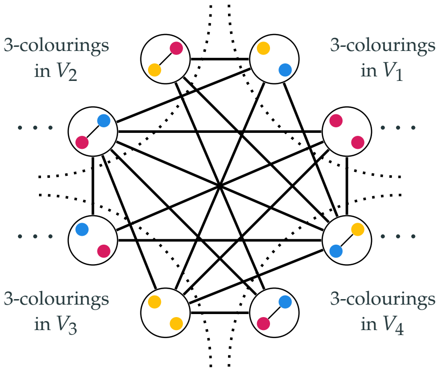
n vertices

k parts

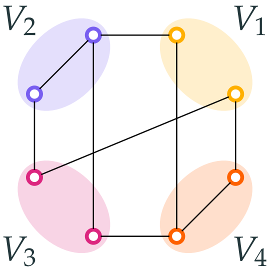


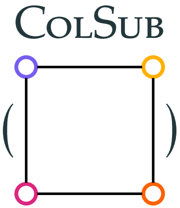


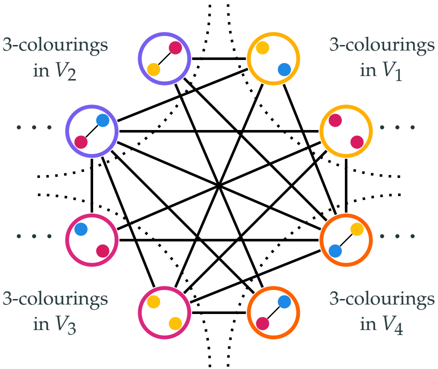
$N = k \cdot 3^{n/k}$ vertices

$$N^{c_2 k} = k^{c_2 k} 2^{(c_2 \log 3/\log 2)\cdot n}$$

A framework of reduction: general patterns







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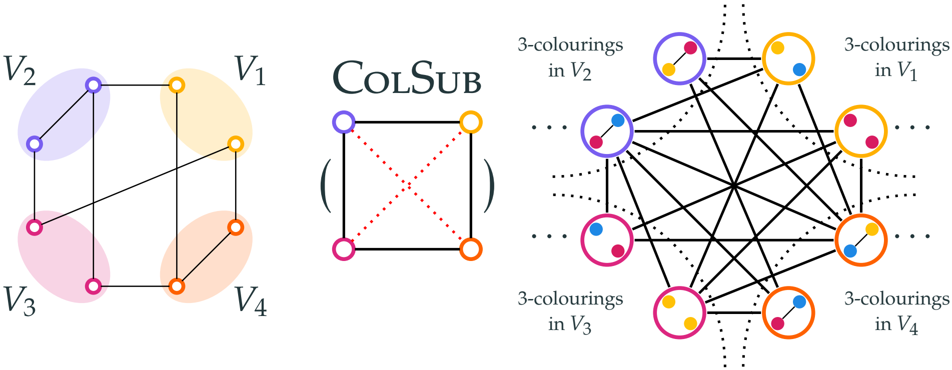

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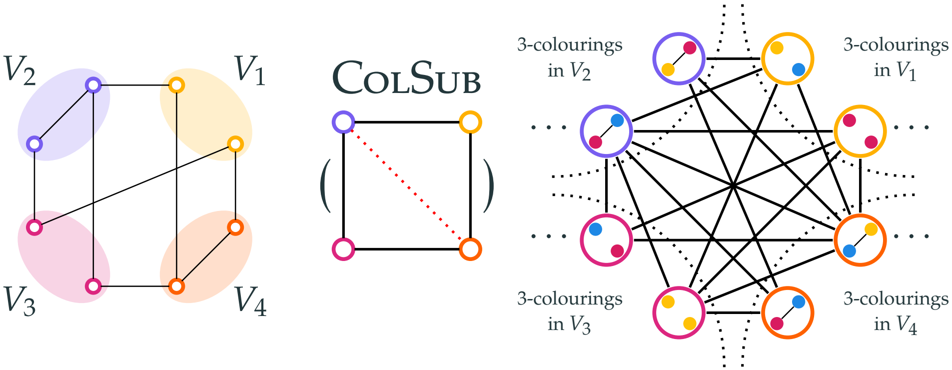

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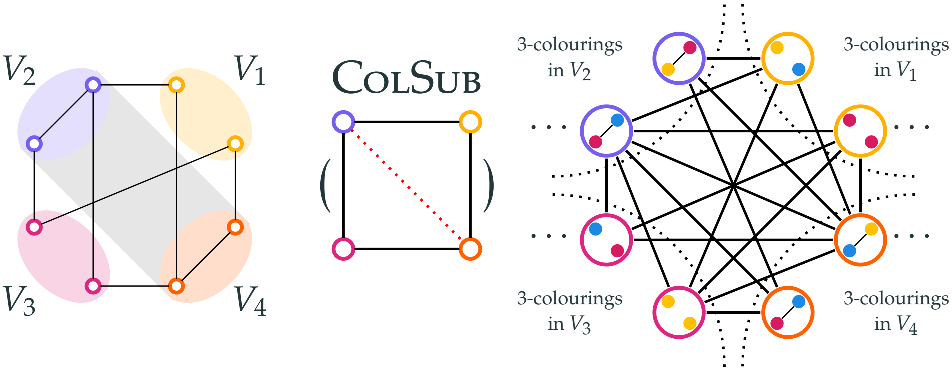

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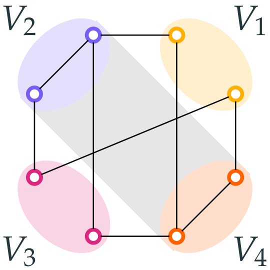

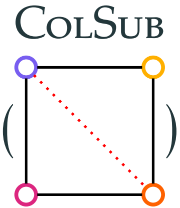

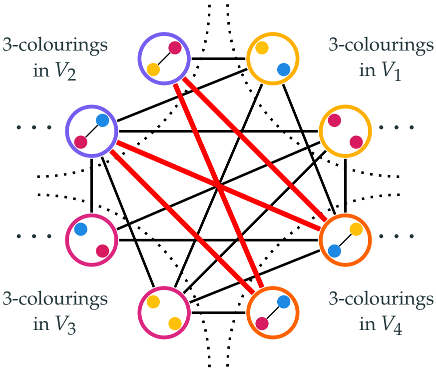

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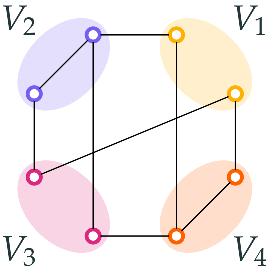

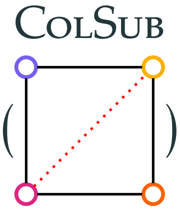

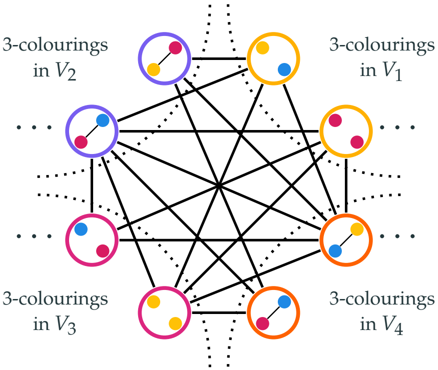

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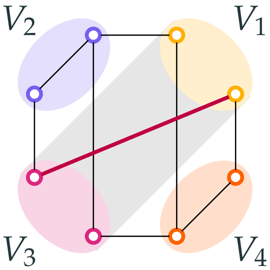
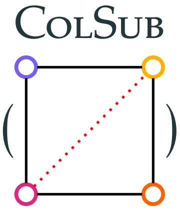
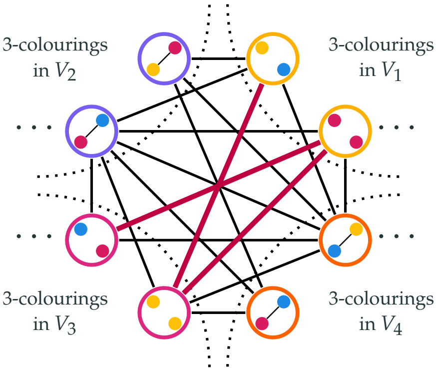
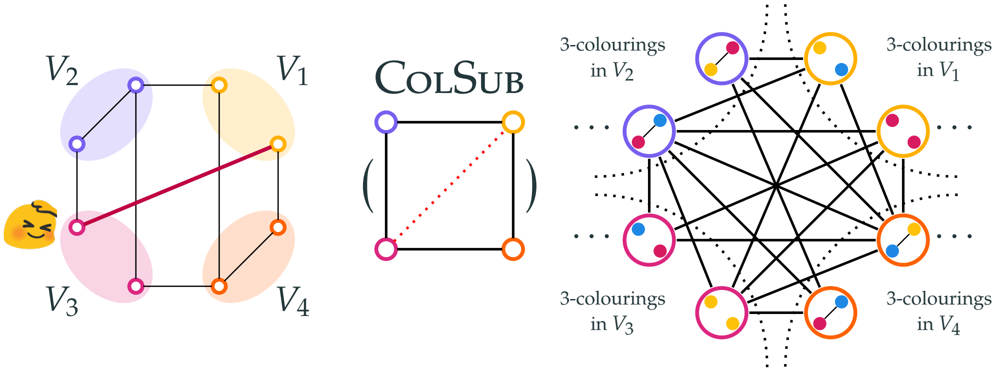




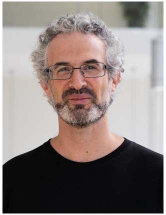




Core problem: Pacify the unhappy edges (Bad partitioning is in general inevitable)

Marx's approach

How do you prove this?



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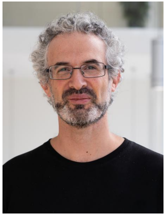

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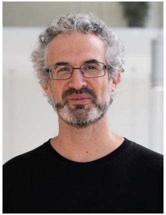

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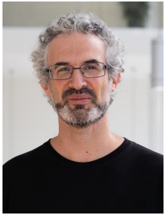
Large treewidth implies the existence of a large set without balance separators.



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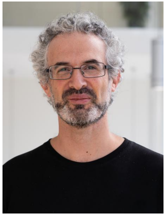

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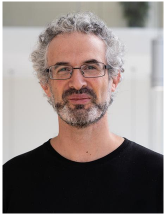

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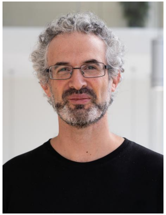

Paths in this concurrent flow can embed a blowup of the line graph of a complete graph.

Bounded-degree input graph can be embedded into this blowup of the complete graph.

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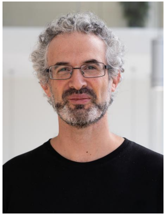
Uh... A bit too complicated. Any simpler proof?

And I only need the existence of hard patterns.



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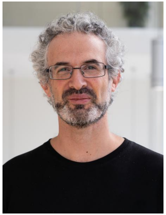

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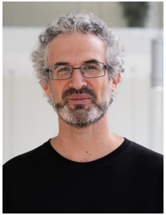

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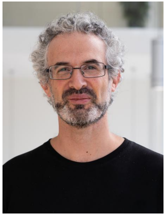

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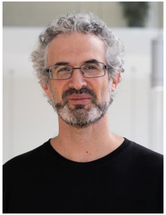

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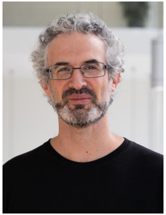

Uh, okay...

Preliminaries of both papers

(Treewidth and brambles) + expanders + graph embedding + (Ramanujan graph / L_1-embeded LP rounding, choose one)

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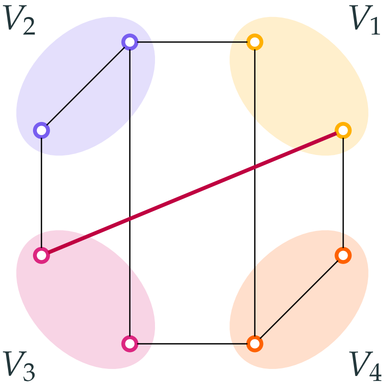
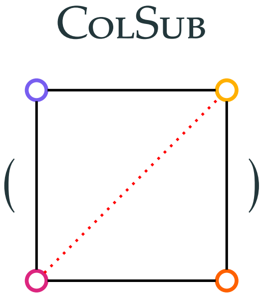
Goal

Can we give a **white-boxed proof** that even **second-year undergraduate students** can understand?!

Our new perspective





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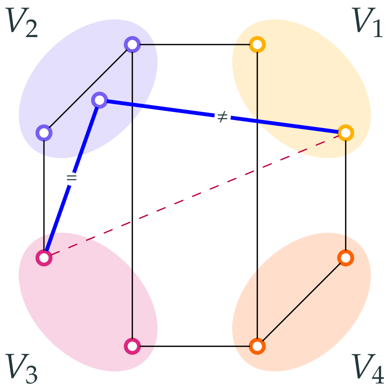

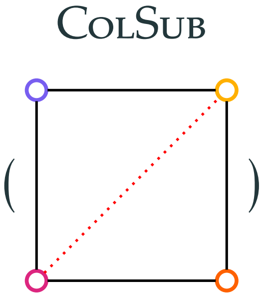

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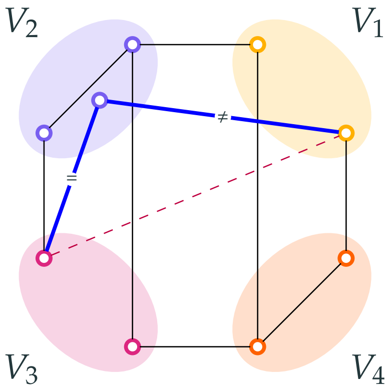
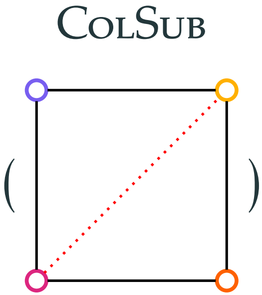
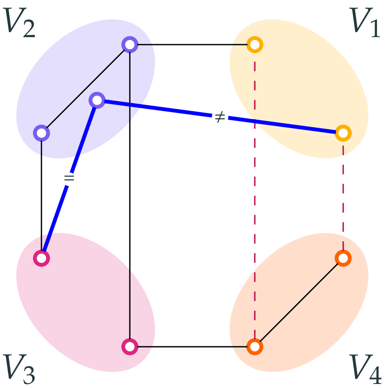




But this costs us something...

Our new perspective



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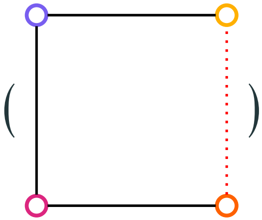
ColSub



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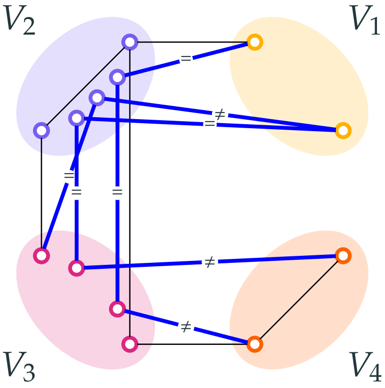

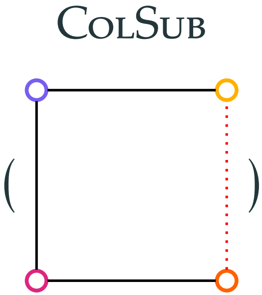

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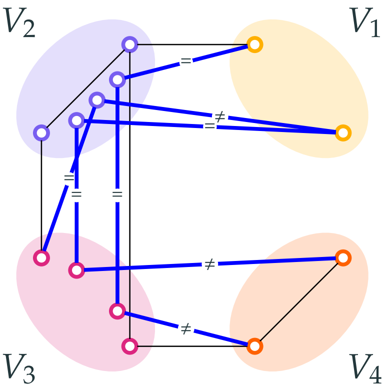
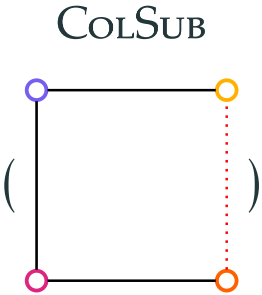




But this costs us something... Too many new vertices in V_2!

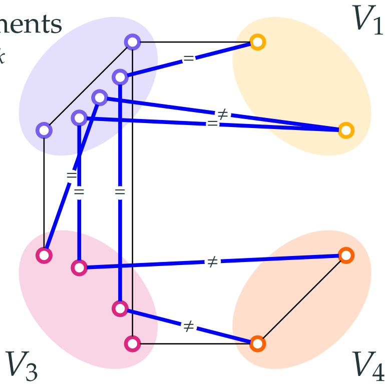
Our new perspective

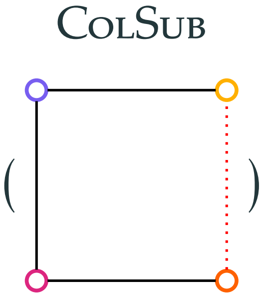
#vertices

$\gg n/k$

#3-assignments

$\gg 3^{n/k}$





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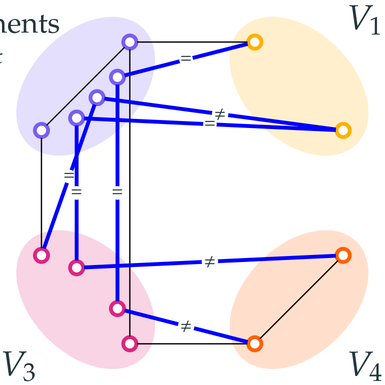
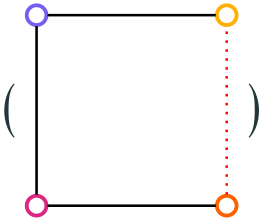


COLSUB



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Routing in paths are highly **congested**!

Our new perspective

#vertices $\gg n/k$

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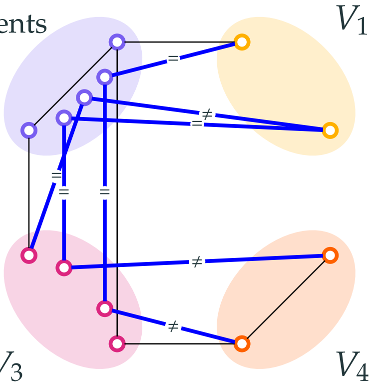

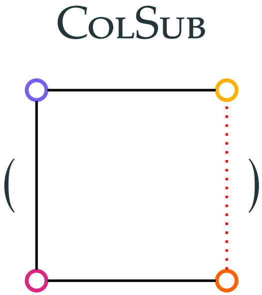

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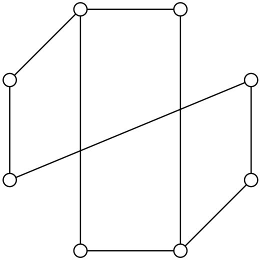
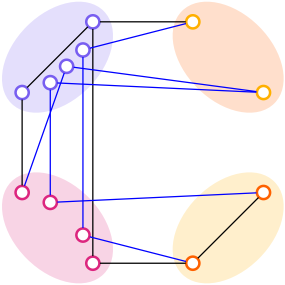
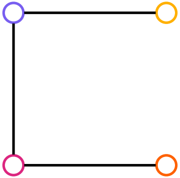
Routing in paths are highly **congested**! Indeed, ColSub(path) is FPT.



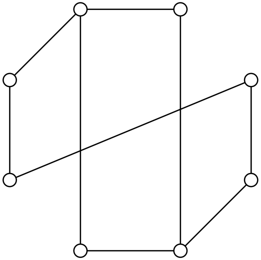
G



G'



H



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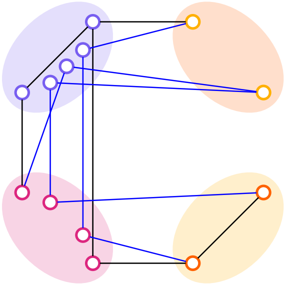

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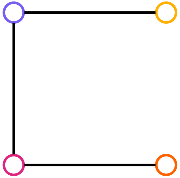
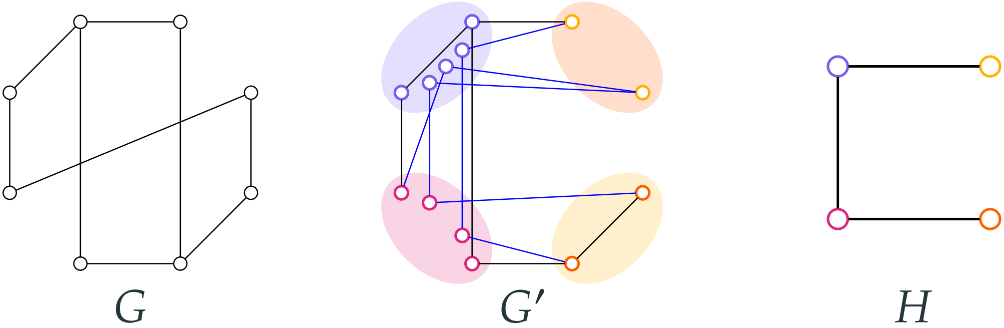


H

Topological minor

G is a **topological minor** of G' if G' is (isomorphic to) an edge-subdivision of G.



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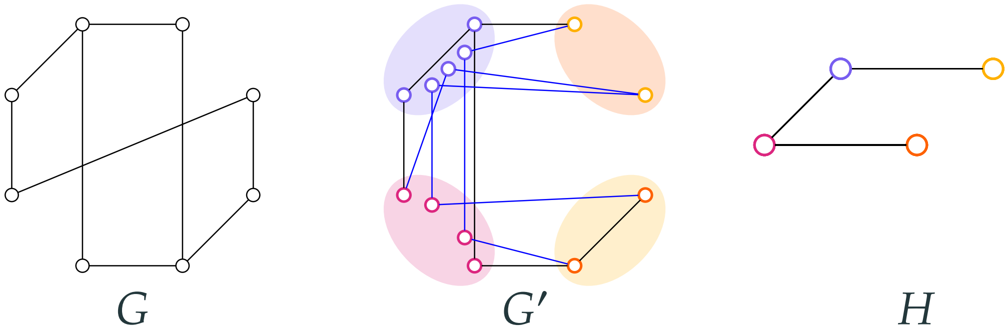

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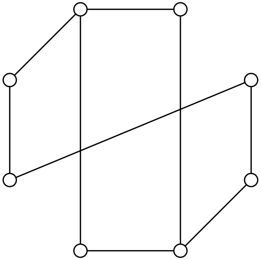

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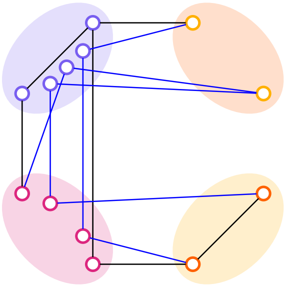
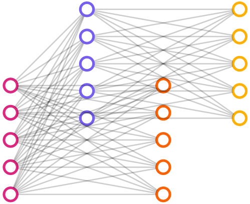


G'



$H \otimes J_5$

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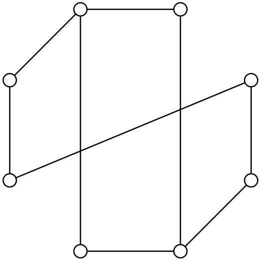

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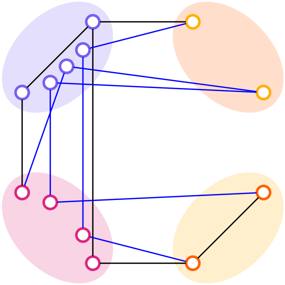

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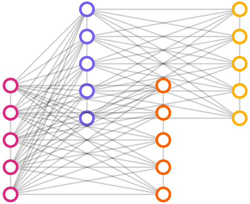
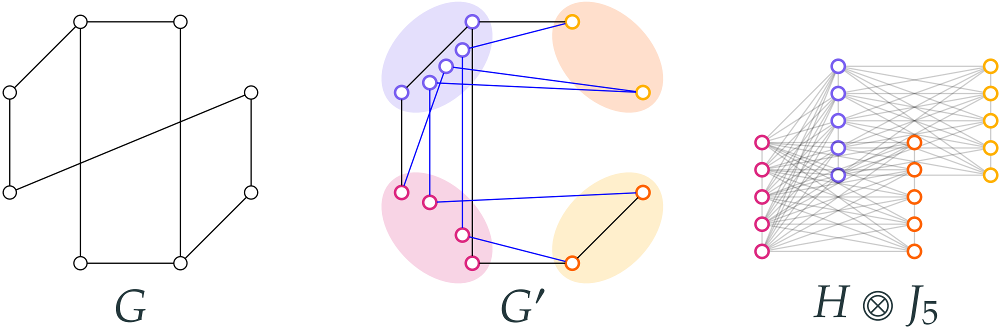


$H \otimes J_5$

Blowup

A t-**blowup graph** is obtained by replacing vertices with K_t and edges with $K_{t,t}$.



Goal: Any degree-4 graph G is a topological minor of $H \otimes J_{t'}$.

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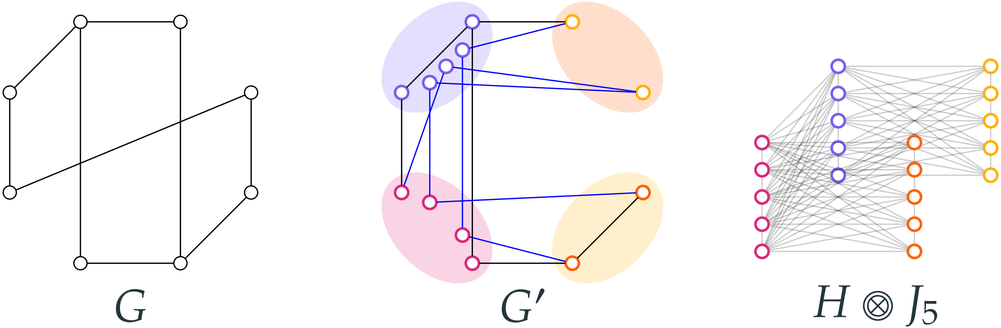
Fitting instance G into Blowups $H \otimes J_t$



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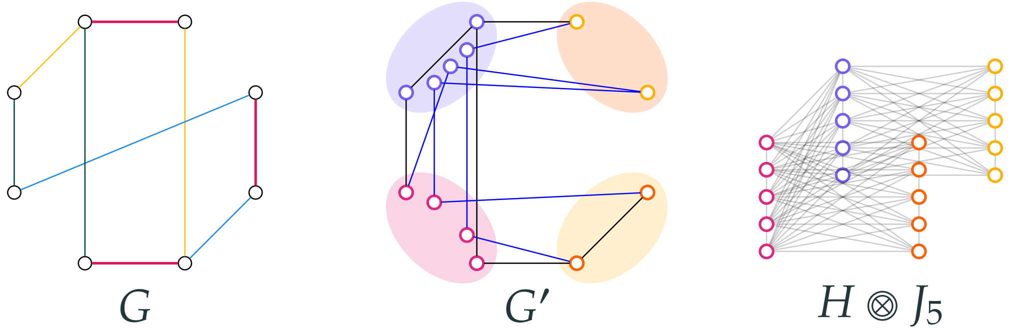
Let $\gamma'(H)$ be the maximum such that the minimum of t' is $n/\gamma'(H)$.

ETH implies an $n^{\Omega(\gamma'(H))}$ lower bound on $\textsc{ColSub}(H)$.

Fitting instance G into Blowups $H \otimes J_t$



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New goal: Any **matching** on n vertices is a topological minor of $H \otimes J_t$.

Vizing's Theorem + definition of blowup: $t' \leq 5t$.

Linkage

H is a small graph (k vertices), but after blowing up, it needs to 'host' a large matching (n vertices)...

Observation: If there are γ vertices in H that can 'host' **any** matching among them (via routing through other vertices), then a $t = n/\gamma$ blowup of H suffices!

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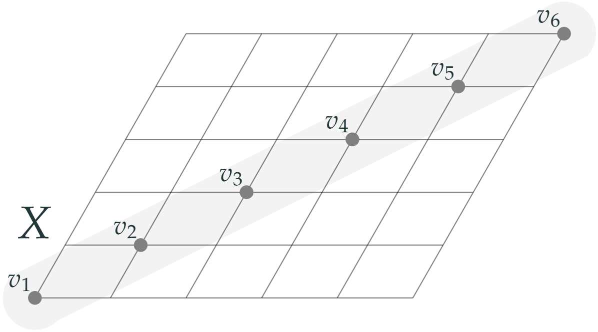

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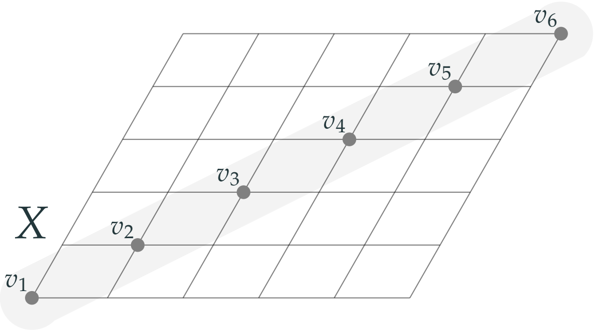

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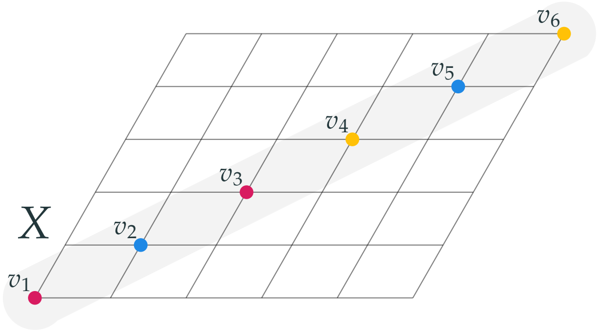

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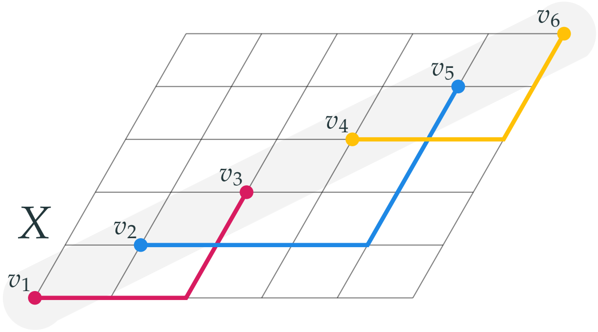

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No vertex can be used twice

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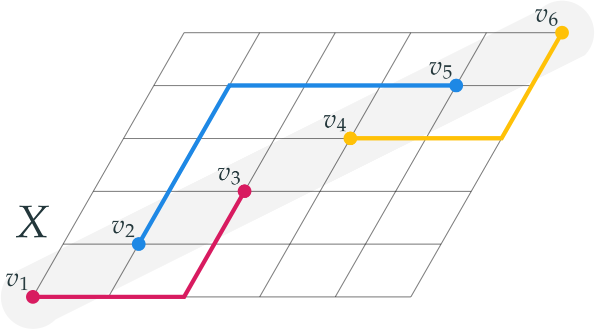

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Congestion-free communication

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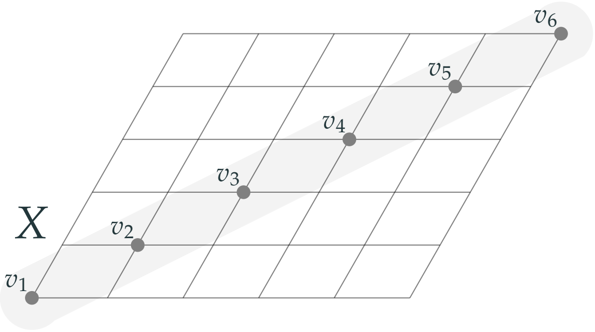

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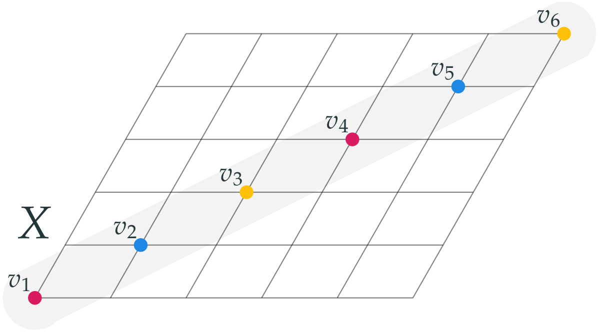

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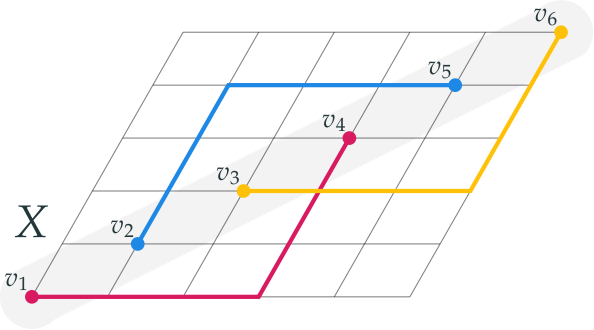

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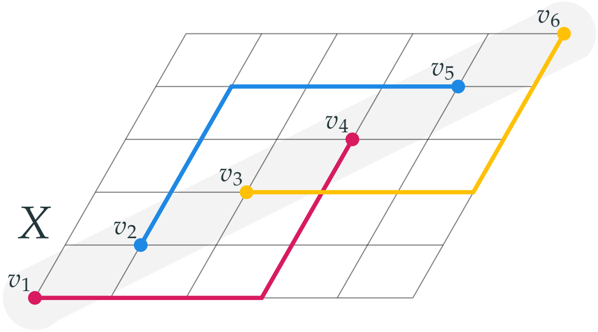

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Theorem

[Marx'10]

There is a sequence of **degree-4** graphs $H_1, H_2, \cdots$ s.t. H_ℓ has ℓ edges and $\textsc{ColSub}(H_\ell)$ cannot be solved in time $n^{o(\ell/\log \ell)}$ unless ETH fails.

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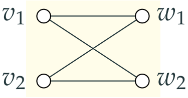
Fun fact: it is NOT an expander.

The Beneš network

... is two butterflies back to back.

$B_1 =$



The Beneš network

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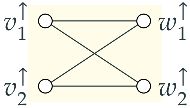

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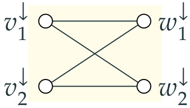

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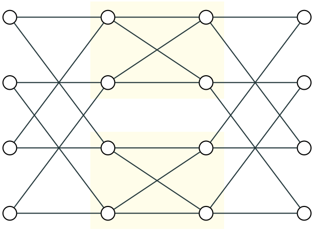
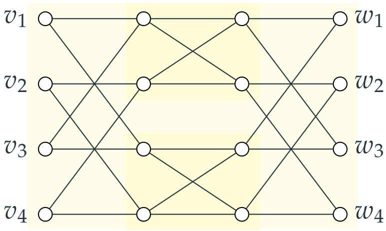


$B_2 =$ 

The Beneš network

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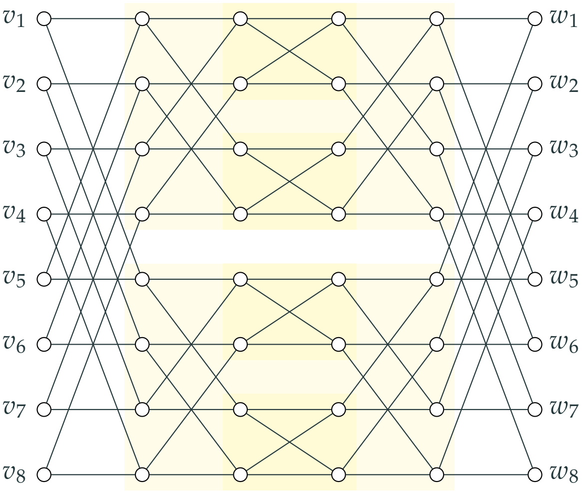
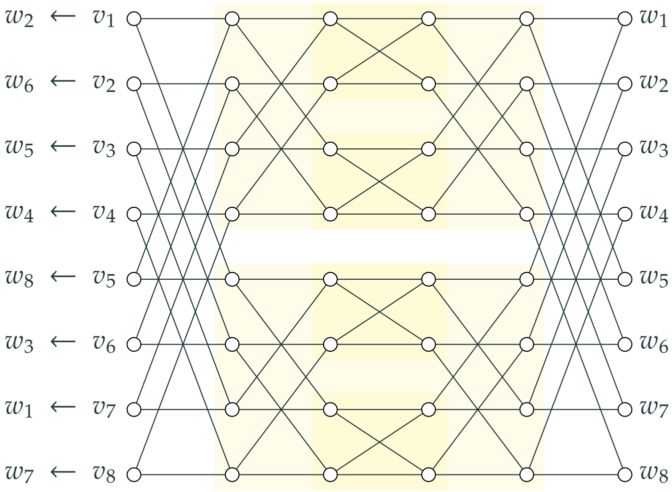


Theorem [Beneš'64]

For any permutation π over $[2^s]$, there is a $\{v_i w_{\pi(i)}\}$-linkage in B_s.



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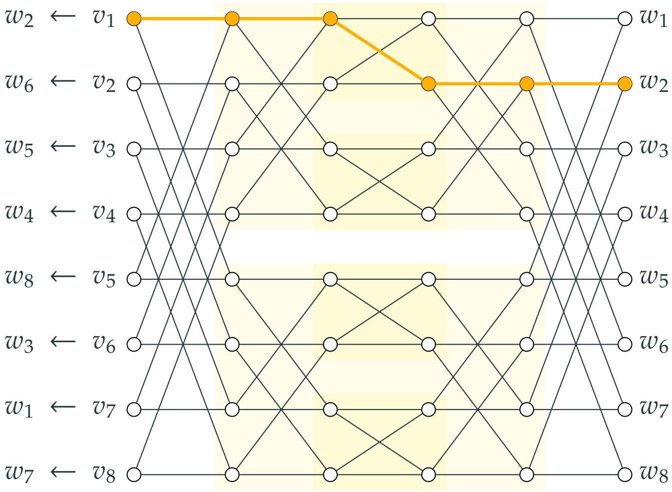

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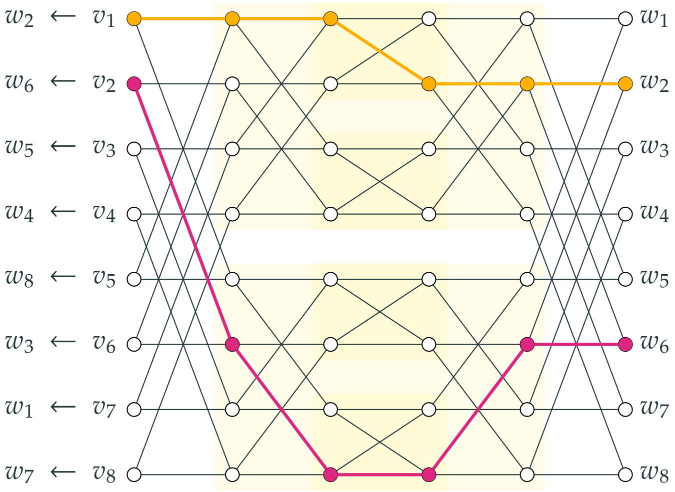

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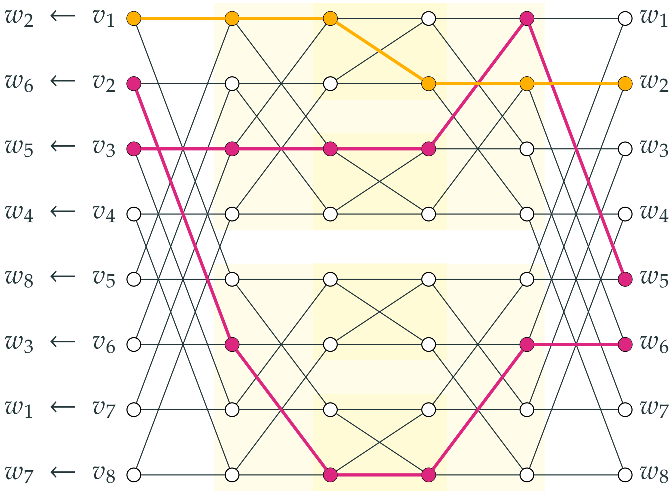

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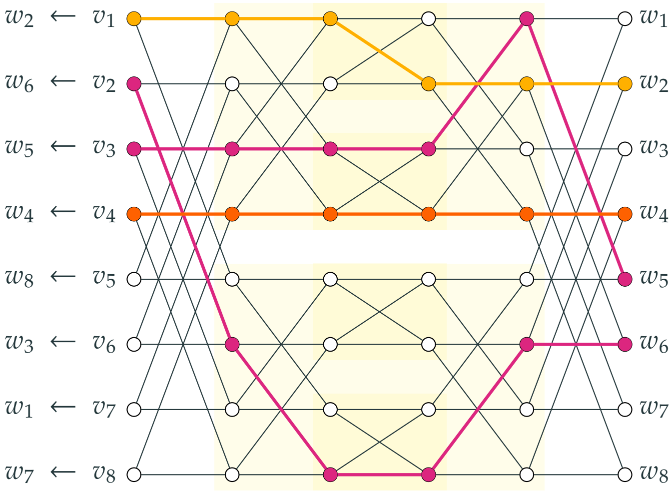

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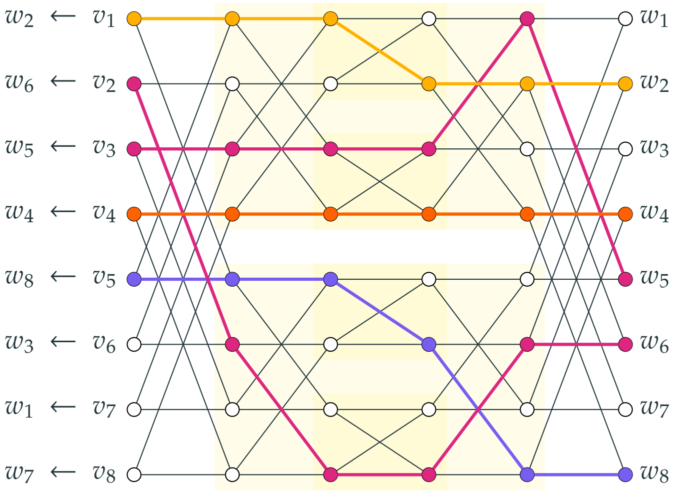

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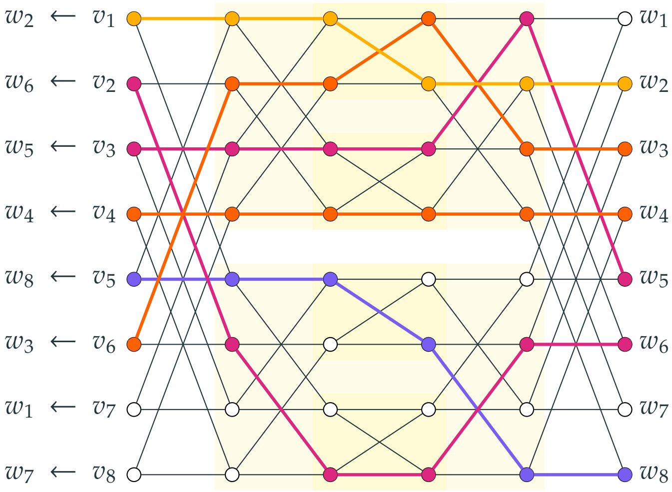

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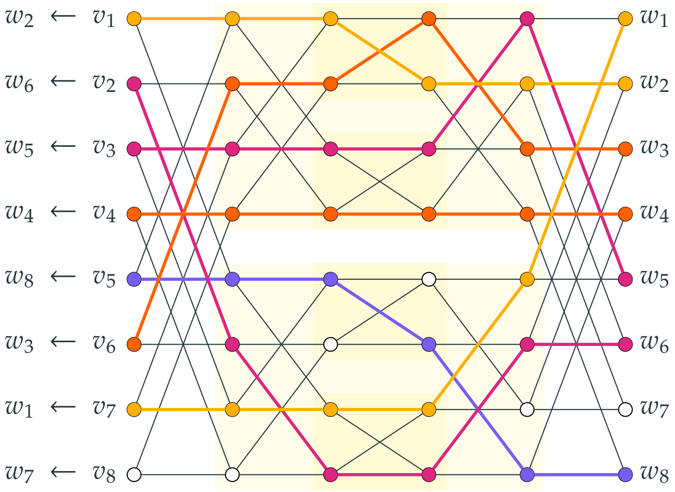

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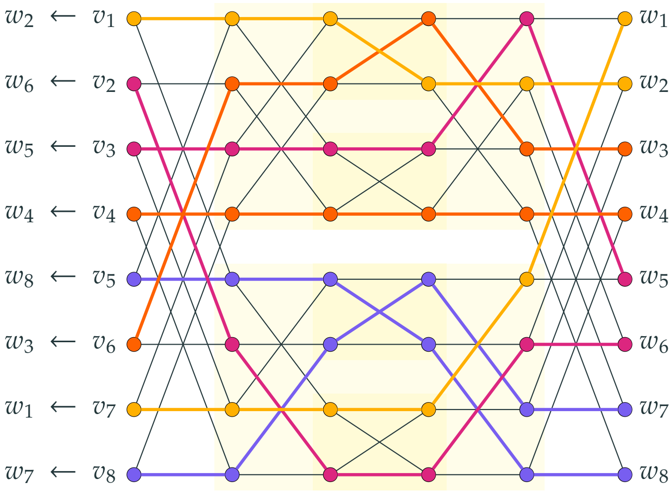
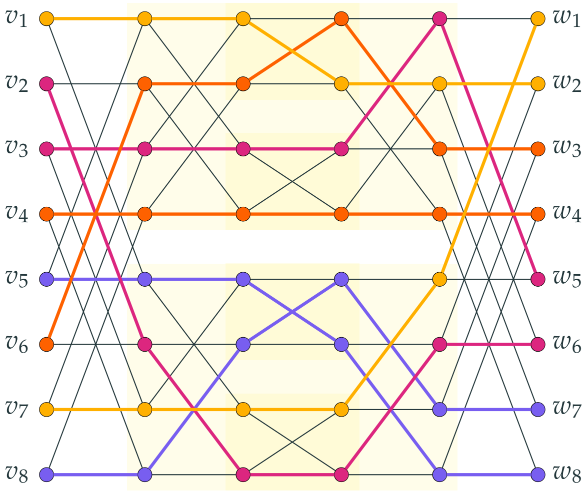


M-linkage for $M = \{v_1v_7,\ v_2v_3,\ v_4v_6,\ v_5v_8\}$?



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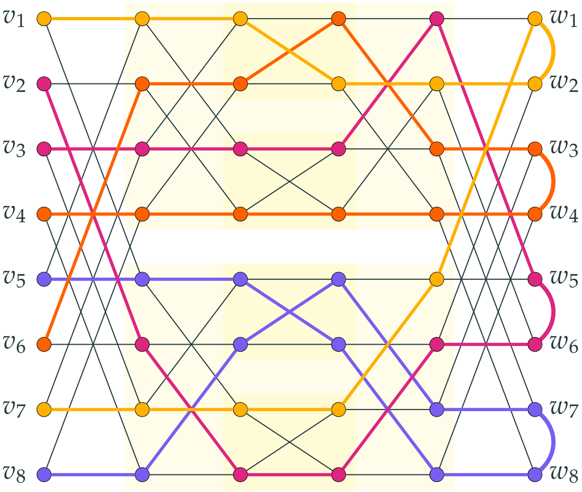
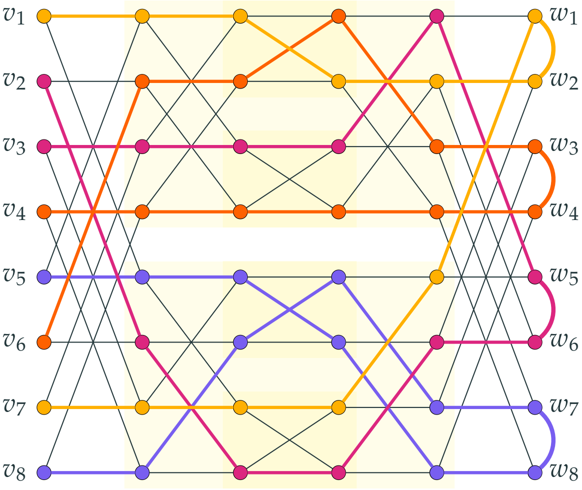
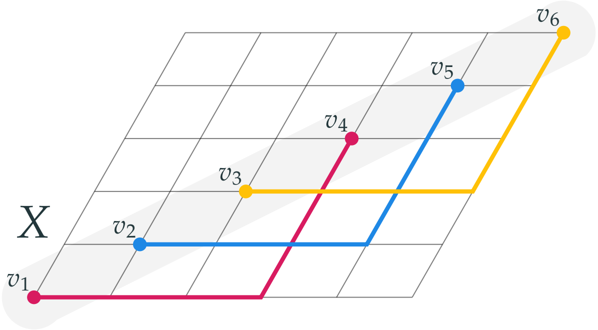


The augmented Beneš network

$\{v_1, \cdots, v_{2^s}\}$ is matching-linked in $\check{B}_s$!





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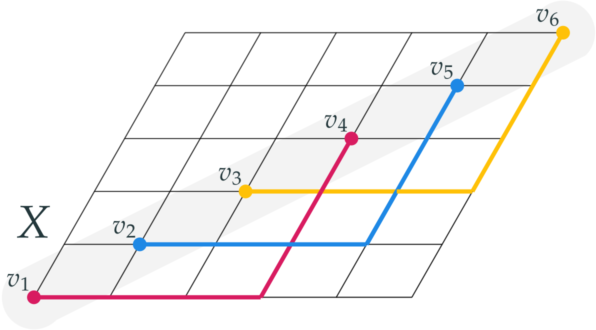

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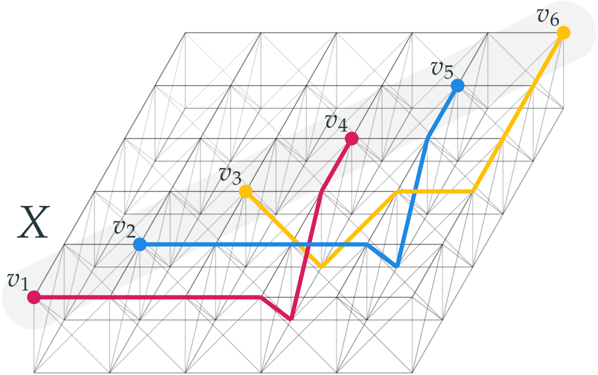

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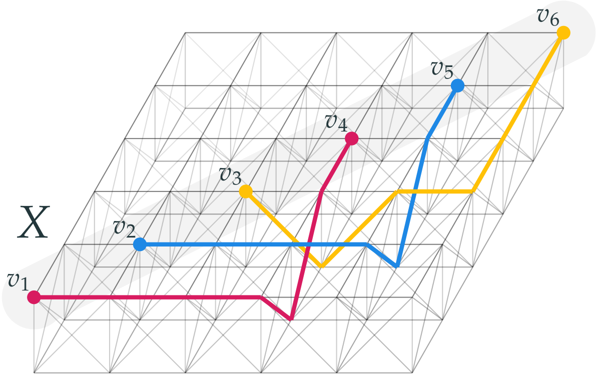

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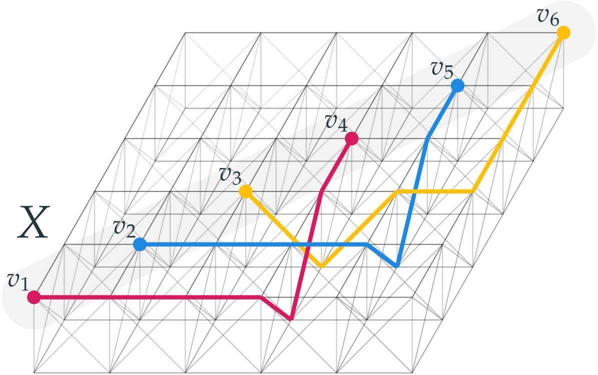
Linkage capacity

The linkage capacity $\gamma(H)$ is the supermum over $c > 0$ such that $H \otimes J_t$ contains a matching-linked set with $|X| = \lfloor ct \rfloor$ for all large t.



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Theorem (main, rough statement)

For **any** graph H, $\textsc{ColSub}(H)$ cannot be solved in time $n^{o(\gamma(H))}$ unless ETH fails.

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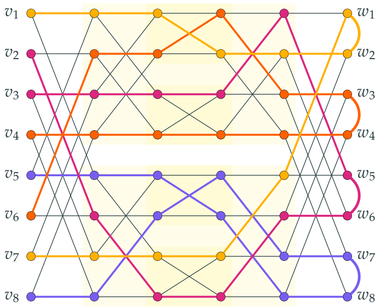
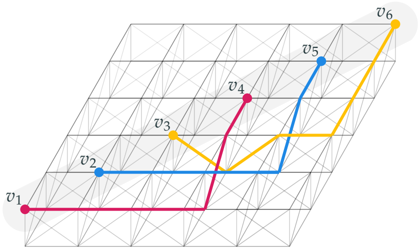
Implications to *induced subgraph counting*.

[Roth–Schmitt–Wellnitz'20, Döring–Marx–Wellnitz'24a,24b, Curticapean–Neuen'24]

Summary

Hardness of subgraph counting via **linkage**.





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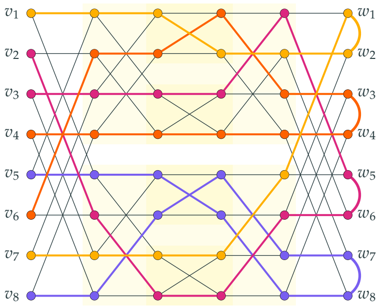
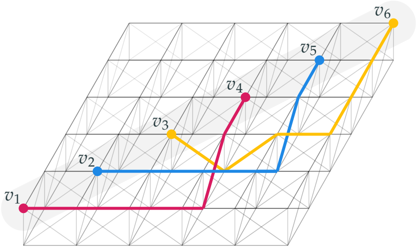
Hardness of subgraph counting via **linkage**.

Beneš network for $n^{\Omega(k/\log k)}$ lower bound.





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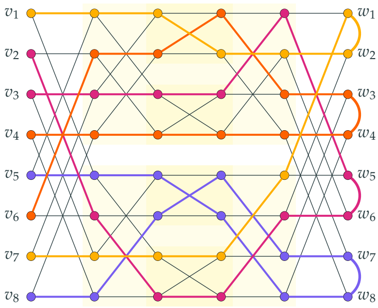

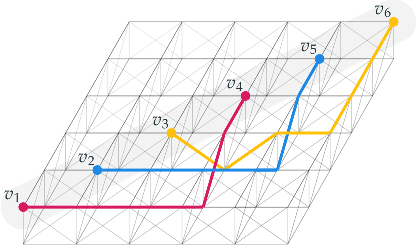

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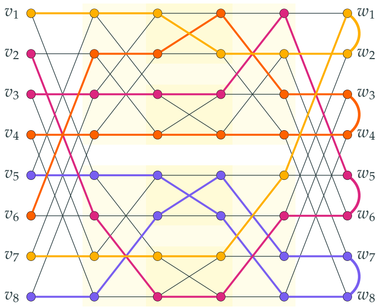
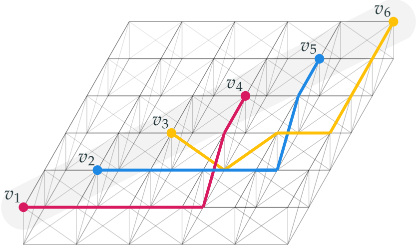


Open questions A:

Close the gap between $n^{\Omega(k/\log k)}$ lower bound and $n^{O(k)}$ algorithms?



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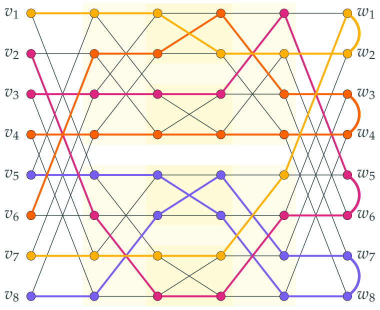

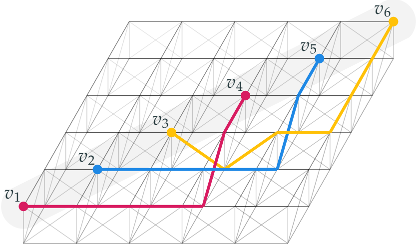

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Can you beat treewidth? ($n^{\Omega(\mathrm{tw}(H))}$ lower bound?)

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Open questions A:

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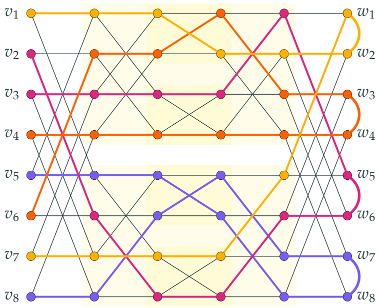
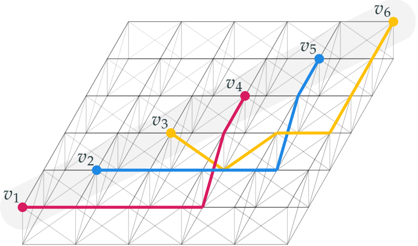
Can you beat treewidth? ($n^{\Omega(\mathrm{tw}(H))}$ lower bound?)

Design algorithms based on linkage capacity? ($n^{O(\gamma(H))}$ algorithm?)





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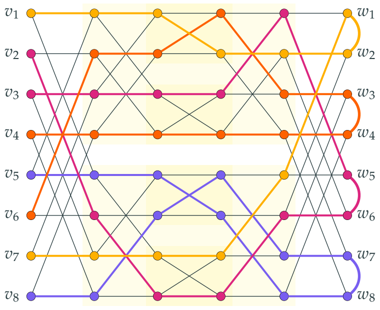

Open questions B:

Novel usages of communication networks?

- This has been used in extension complexity **[Göös–Jain–Watson'18]**, and "fine-grained" hardness of gap amplification (PCP) **[Bafna–Minzer–Vyas'24]**.

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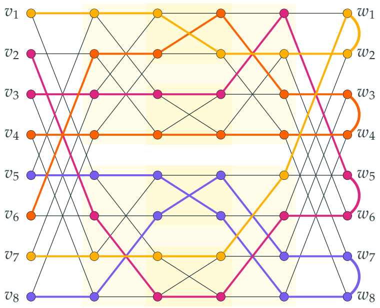

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New proofs of $t/\log t$ lower bounds in other settings that use treewidth-separator duality?

- E.g., $\mathbf{AC}^0$ lower bounds for subgraph isomorphism **[Li–Razborov–Rossman'17]**?

Thank you!

arXiv: 2410.02606