# Improved bounds for randomly colouring simple hypergraphs

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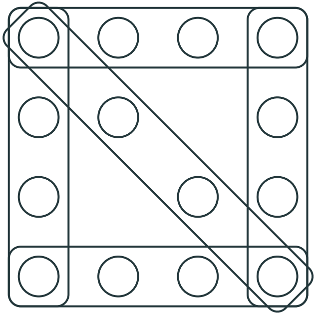
# Hypergraph (proper) colouring

Classical combinatorial/computational problem!



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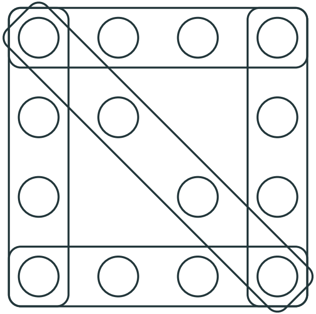

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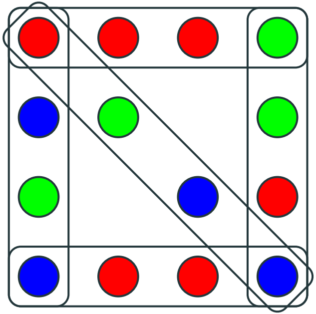
- Hypergraph $(V, \mathcal{E})$
  - Hyperedge $e \in \mathcal{E} : e \subseteq V$
- Proper colouring
  - Forbidding monochromatic hyperedges



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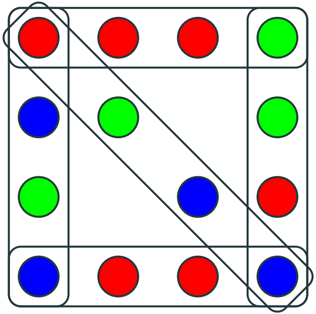

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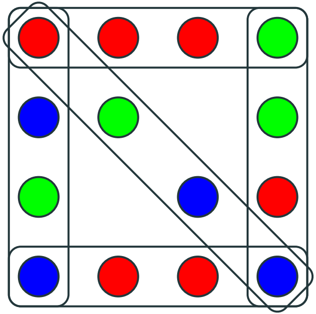

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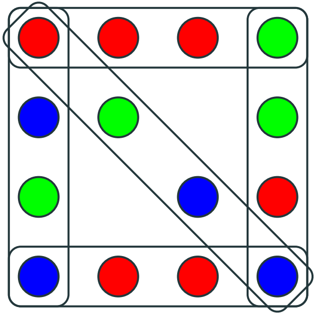

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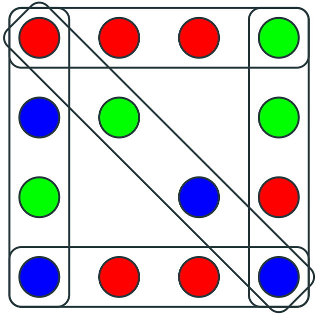

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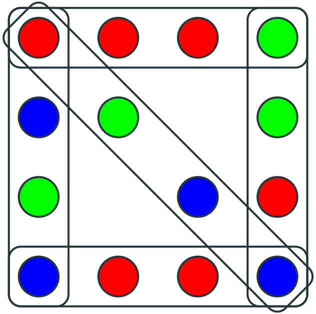

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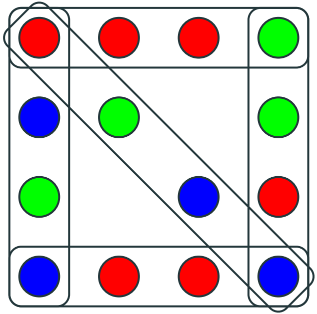

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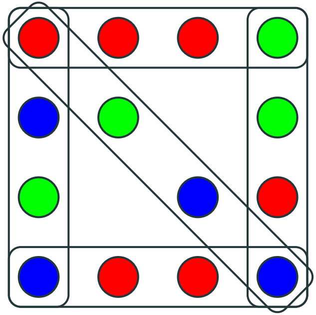

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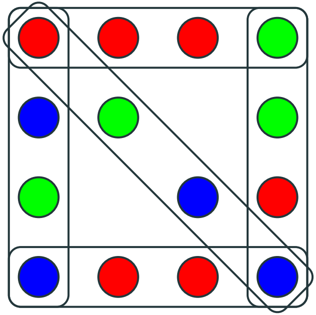

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Posing restrictions to input instances?

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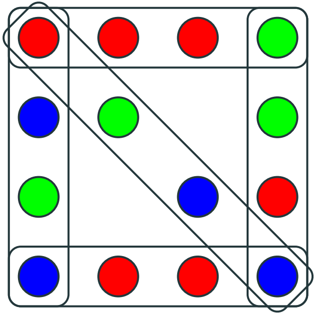

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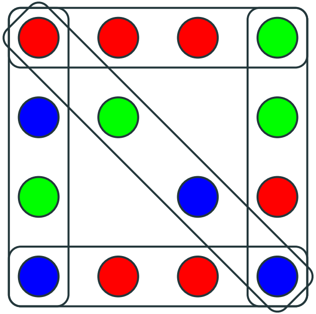

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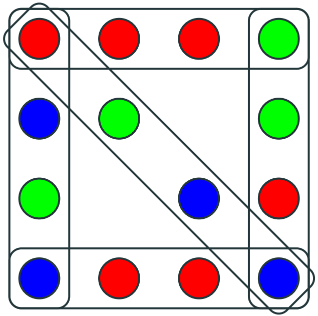

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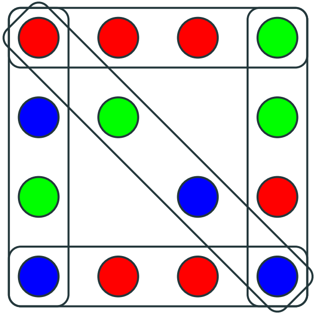

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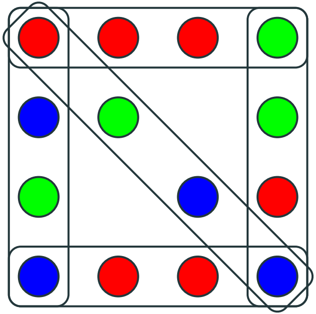

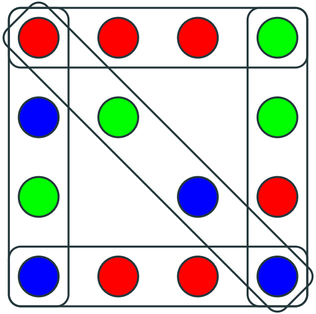

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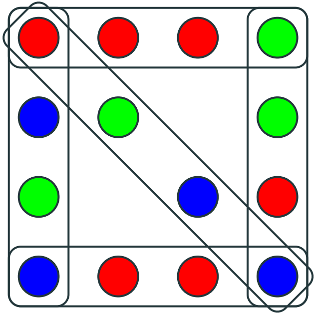

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Open problem: computational threshold for sampling problem

# Simple (aka. linear) hypergraphs

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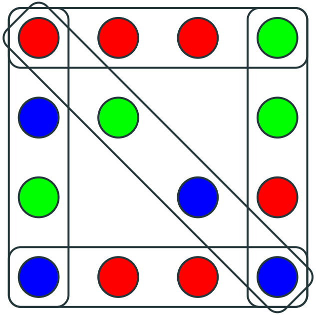
- Overlap of two hyperedges $\leq 1$



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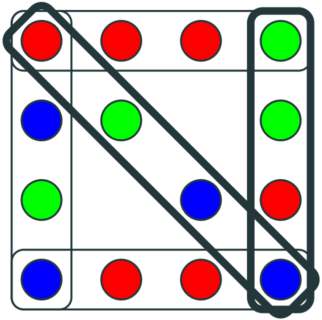

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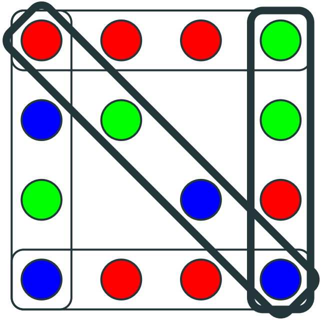

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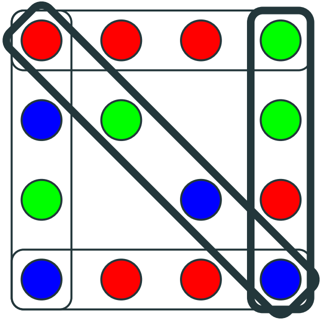

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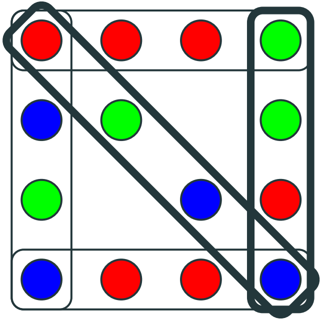

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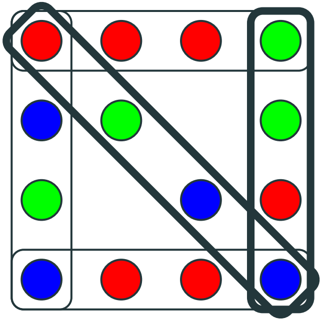

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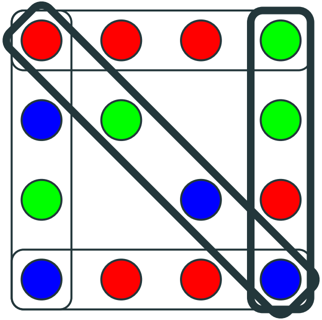

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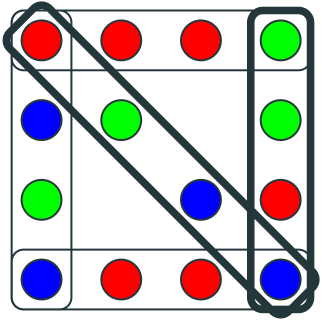

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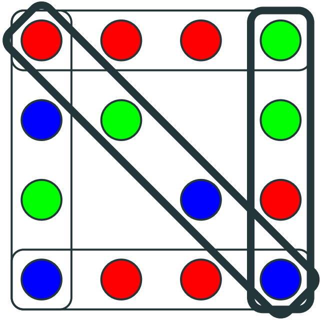

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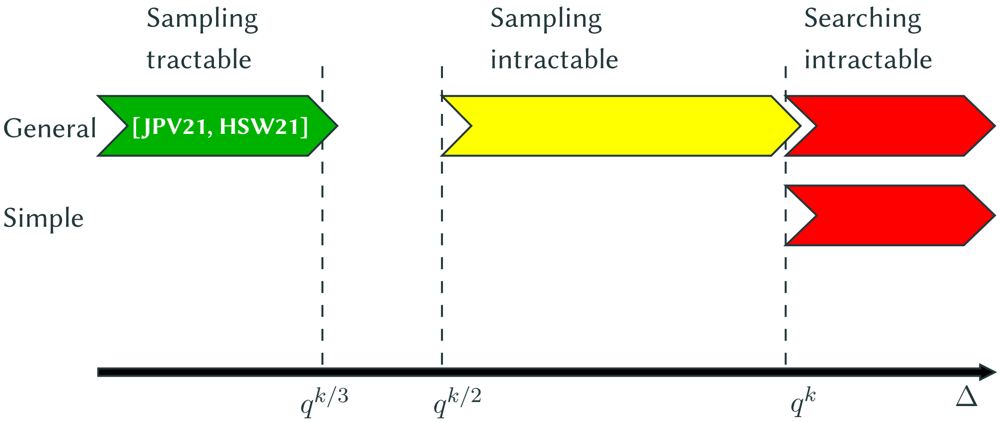
# Our result



Sampling tractable
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Searching intractable
General
[JPV21, HSW21]
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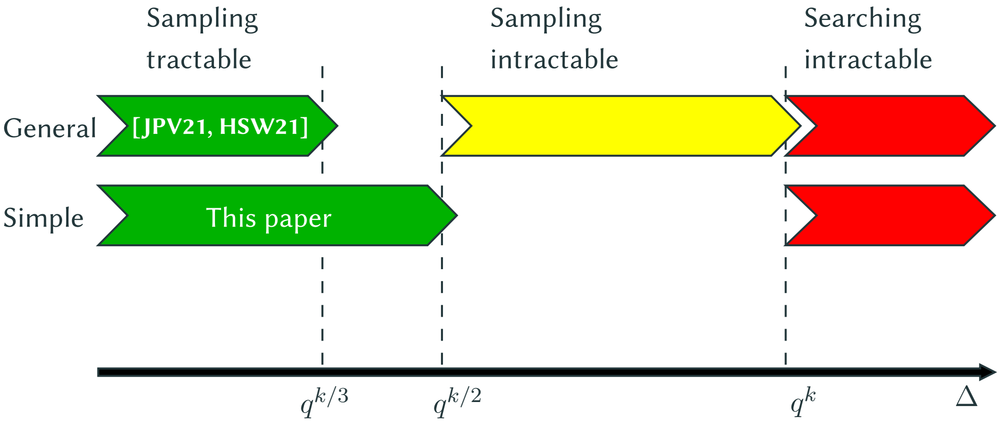
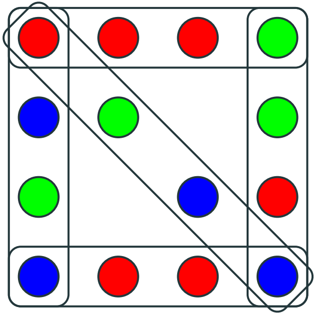



**Theorem**

*There exists an algorithm such that, for any $\delta > 0$, given a $k$-uniform $\Delta$-degree hypergraph as an input, the algorithm outputs an almost uniform random $q$-colouring, if $k \geq 20\left(1+\frac{1}{\delta}\right)$ and $\Delta \leq 0.1^k q^{k/2-(k\delta+1/\delta)}$. The running time is $\tilde{O}(\mathrm{poly}(\Delta k) \cdot n^{1.01})$.*

# Glauber dynamics

Natural approach: Glauber dynamics



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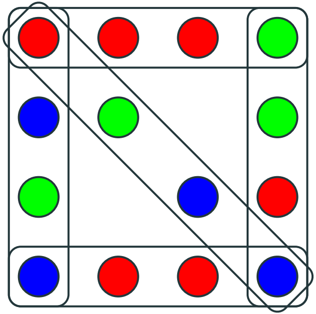

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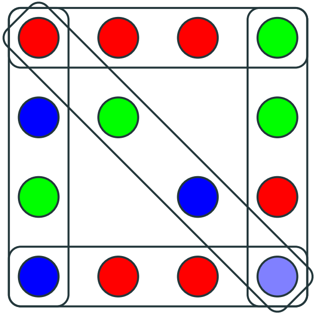

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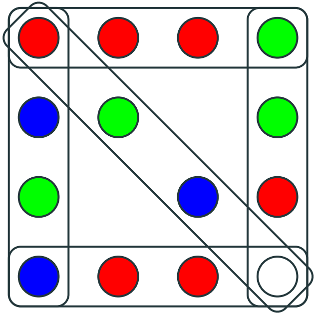

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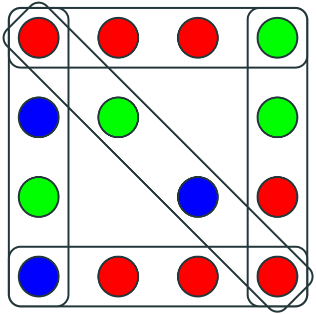
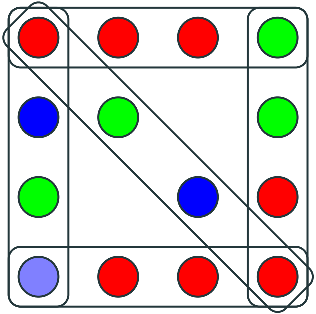


Natural approach: Glauber dynamics

- Uniformly choose one vertex $\Leftarrow$
- Update its value according to its marginal

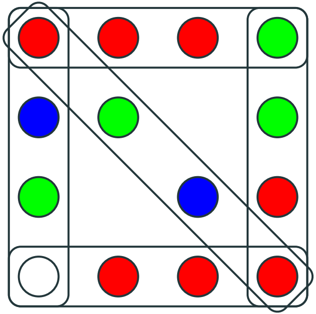


# Glauber dynamics

Natural approach: Glauber dynamics

- Uniformly choose one vertex
- Update its value according to its marginal $\Leftarrow$

$$\Pr[\text{red}] = 0, \Pr[\text{green}] = \frac{1}{2}, \Pr[\text{blue}] = \frac{1}{2}$$



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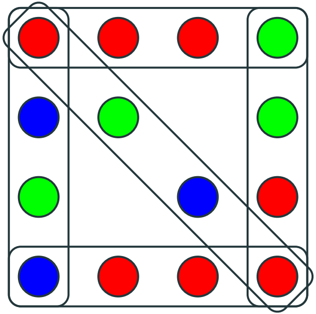

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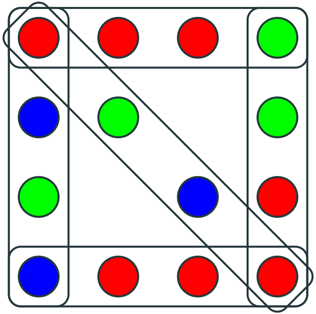

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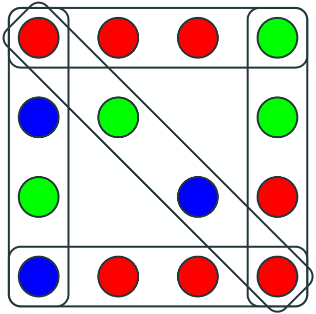

Stationary distribution is uniform (the correct distribution).

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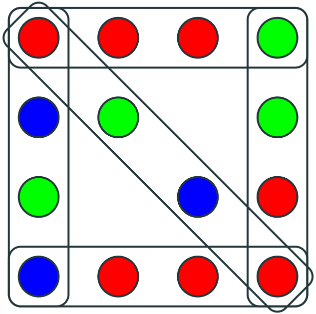
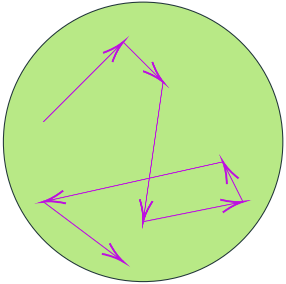


Stationary distribution is uniform (the correct distribution).

Does this chain *mix rapidly* (i.e., converge to stationary distribution quickly)?

# Dystopia of MCMC?



Well connected

Fast mixing

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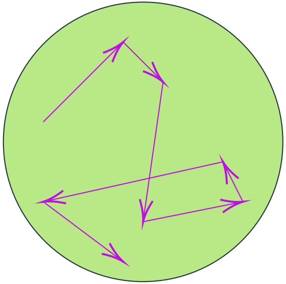

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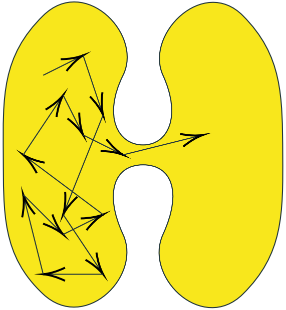

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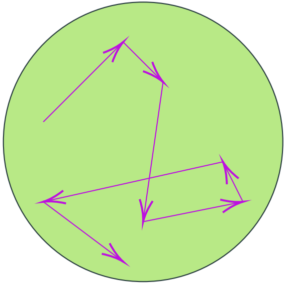

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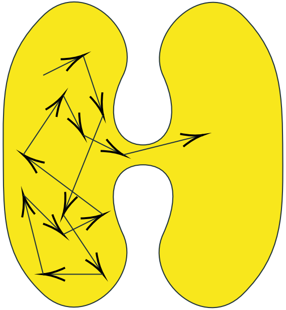

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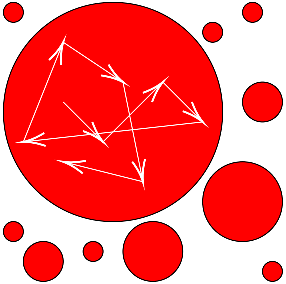

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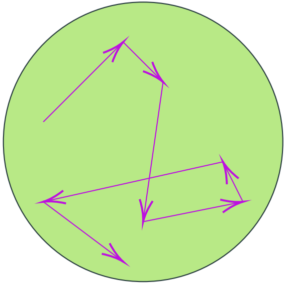

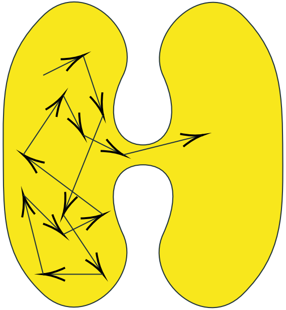

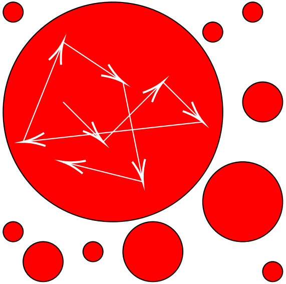

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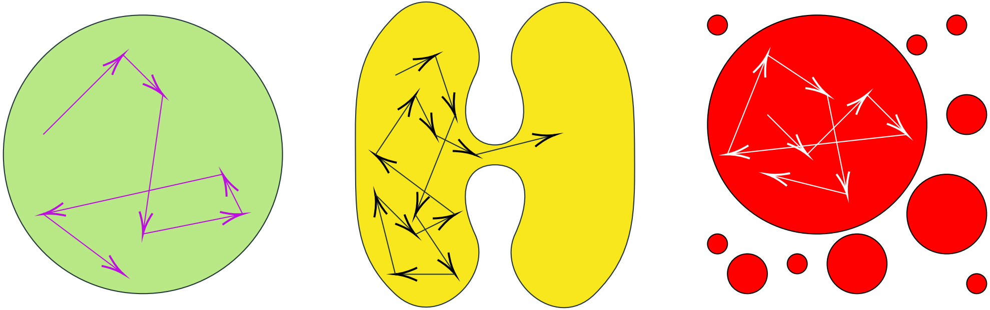

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Not that bad if there is a giant component — start from a random configuration

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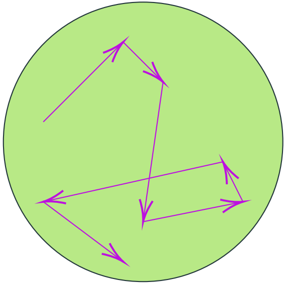

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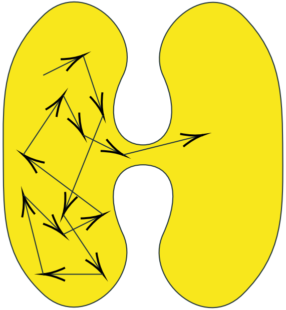

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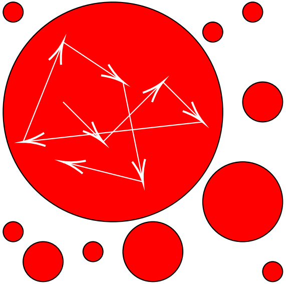

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- Simple hypergraph with $q \geq \max\{\Theta_k(\log n), \Theta_k(\Delta^{\frac{1}{k-1}})\}$ **[Frieze-Anastos'17]**

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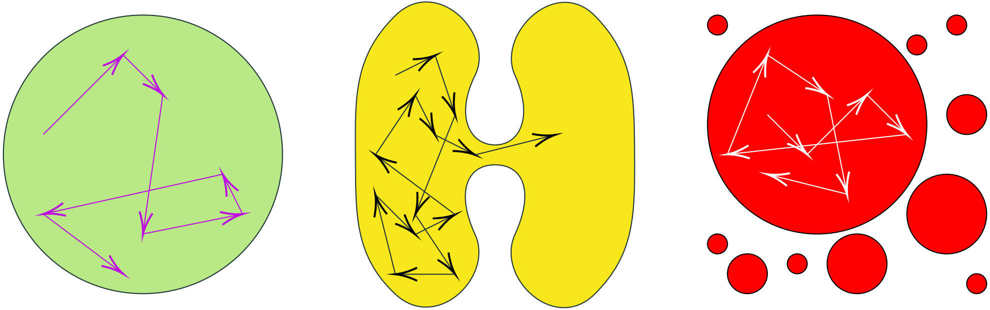

Well connected | Poorly connected | **Not connected**

Fast mixing | Slow mixing | **Not mixing**

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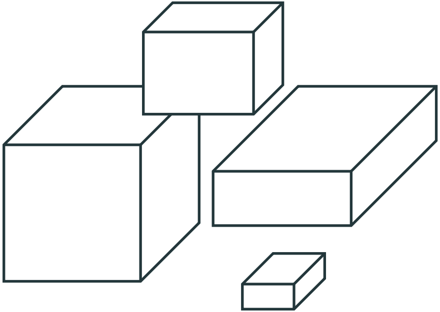
- Simple hypergraph with $q \geq \max\{\Theta_k(\log n), \Theta_k(\Delta^{\frac{1}{k-1}})\}$ **[Frieze-Anastos'17]**
- Constant number of colours?

# Markov chain projection [Feng-Guo-Yin-Zhang'21]



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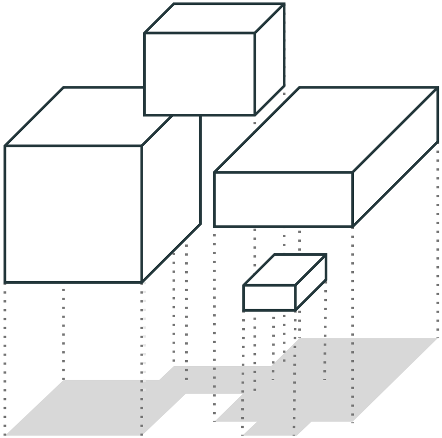

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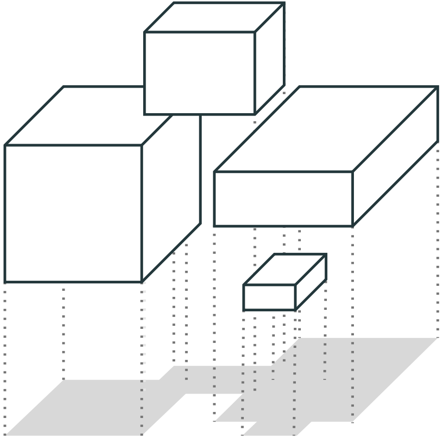

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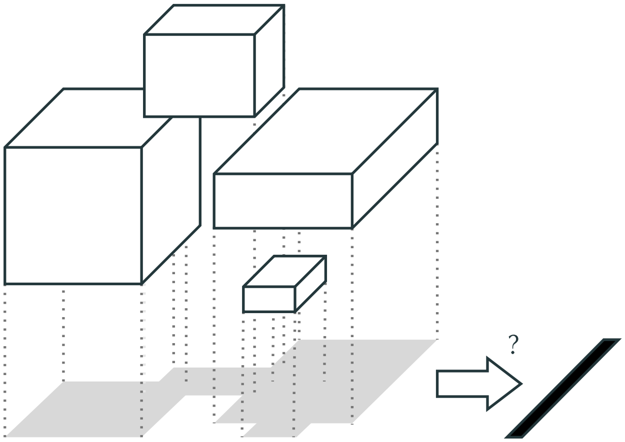


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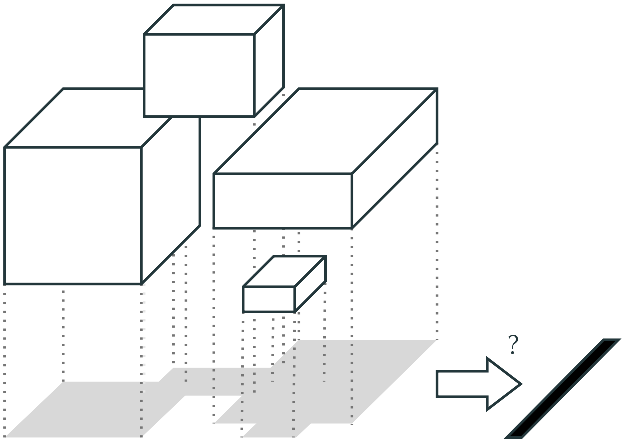


- Disconnected / poorly connected $\xrightarrow{\text{projection}}$ well connected
- Harder to simulate transition / recover a sample

Algorithm:

- Run Glauber dynamics on the projected distribution to get a projected sample $\boldsymbol{Y}$

# The algorithm

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- Sample a proper colouring $\boldsymbol{X}$ conditioned on its projection being $\boldsymbol{Y}$

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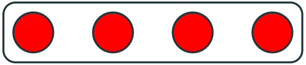
# Bucketing [Feng-He-Yin'21]

- Disconnection arises from hard constraints:



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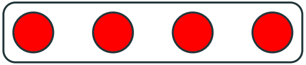
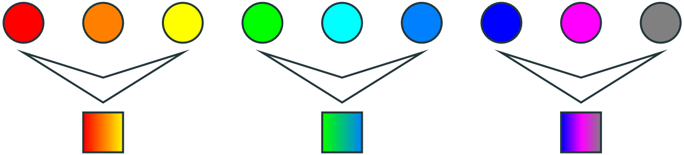


- Projection by bucketing:



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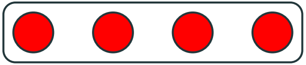

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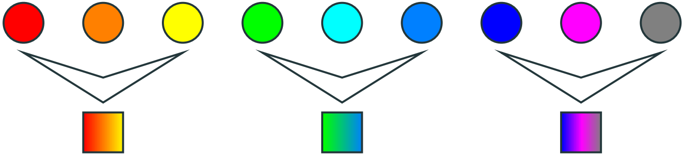
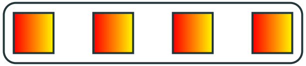


- The constraint is soft (ensured by LLL), i.e.,

$\Pr\Big[$  $\Big] > 0$

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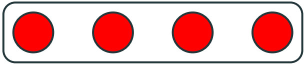

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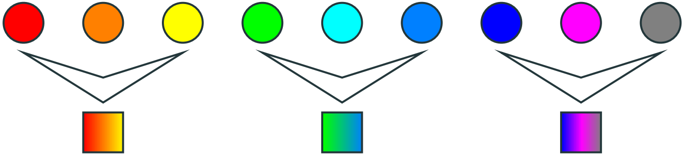

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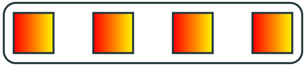
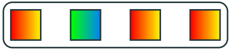
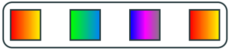
Pr [  ] > 0

despite that

Pr [  ] < Pr [  ] = Pr [  ]

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Things to handle:

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Current $\Delta \lesssim q^{k/3}$ barrier: trade-off between mixing and implementation.

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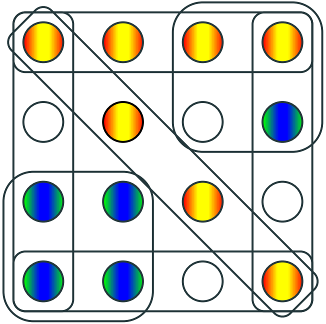
- We focus on implementation in this talk.

# Fast implementation



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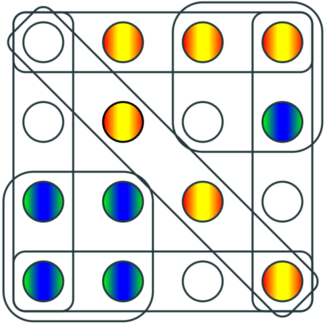

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- How can we know the correct projected distribution?
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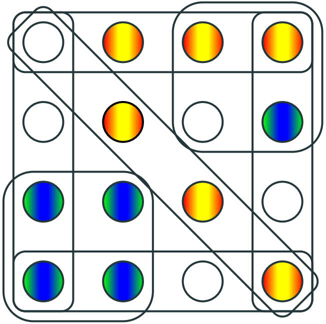

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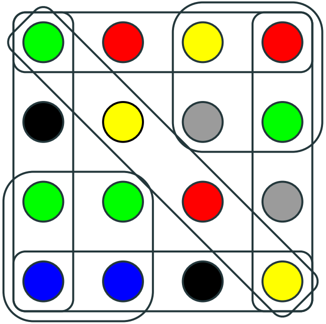

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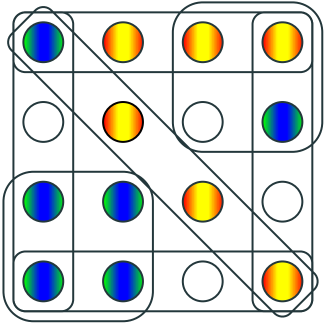

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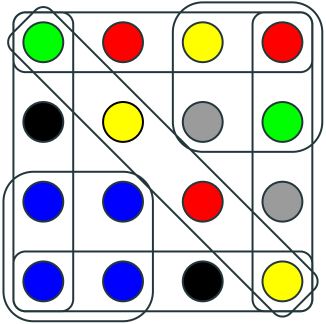

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- How can we know the correct projected distribution?
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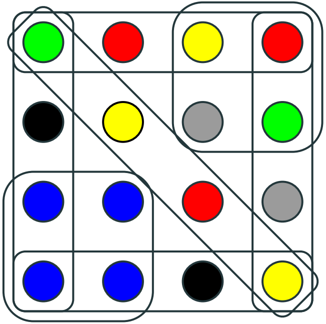

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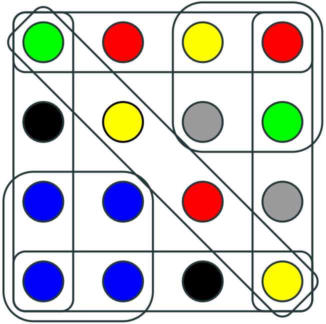

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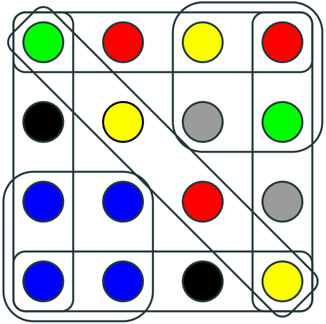

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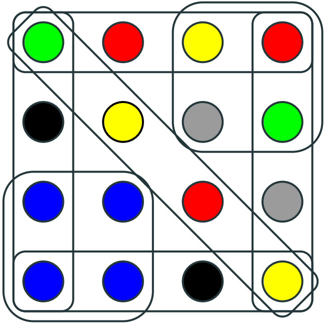

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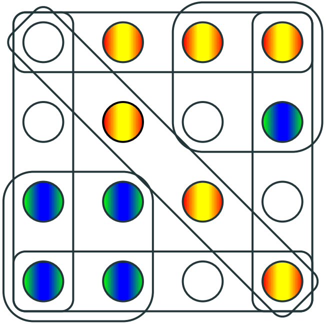

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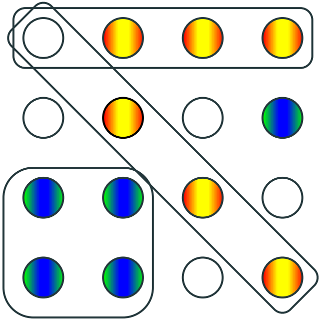

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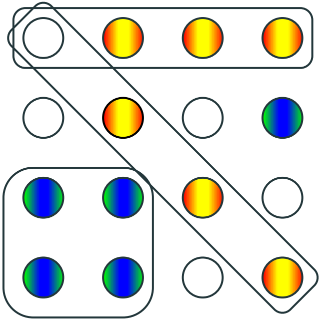
- Disconnected components affect nothing.



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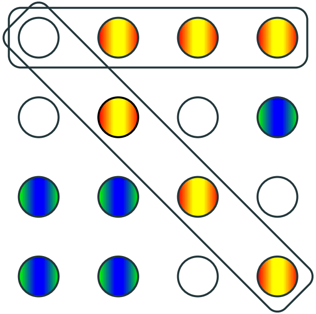

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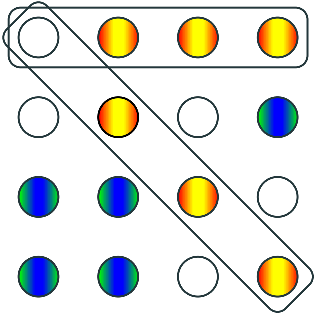
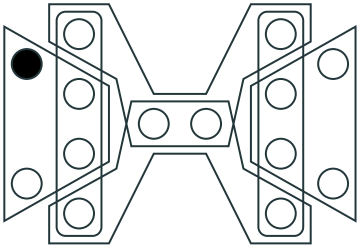
- Disconnected components affect nothing.
- We are done if number of hyperedges is $O(\log n)$ w.h.p.



# Connected component

Union bound over all possible size-$\alpha$(#edges) components:



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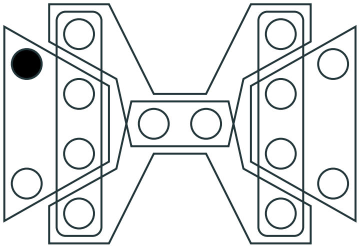

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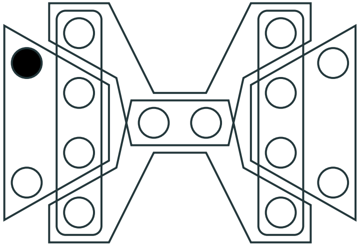

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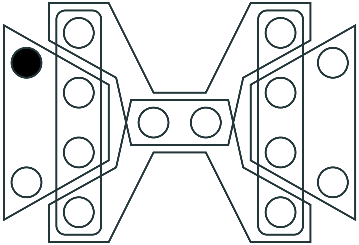

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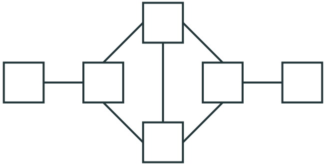

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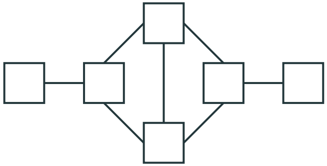

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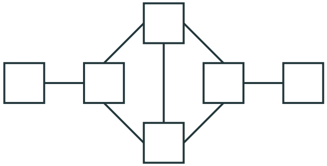

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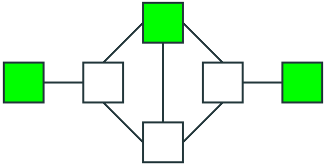

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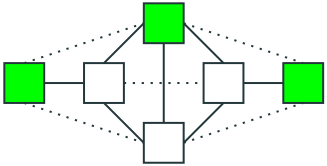

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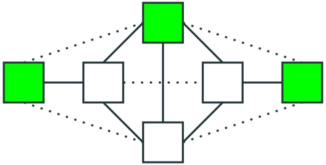

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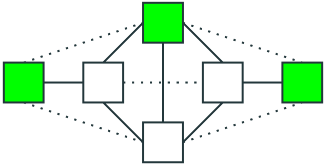
- Union bound over all 2-trees instead.



# Do $2$-trees suffice?

Assuming bucketing into $\sqrt{q}$ buckets.

$$\sum_{\ell} \Pr[\text{size-}\ell \text{ 2-tree exists}] < 1.$$

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Assuming bucketing into $\sqrt{q}$ buckets.

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$$\text{Number of 2-trees} \times (\sqrt{q})^{1-k} \lesssim 2^{-\ell}$$

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**Lemma (Corollary of [Borgs-Chayes-Kahn-Lovász'13])**

*Let $G$ be a graph with maximum degree $D$ and $v$ is a vertex. Then the number of $2$-trees in $G$ of size $\ell$ containing $v$ is at most $(eD^2)^{\ell-1}/2$.*

$$(e(k\Delta)^2)^{\ell-1} \times (\sqrt{q})^{1-k} \lesssim 2^{-\ell}$$

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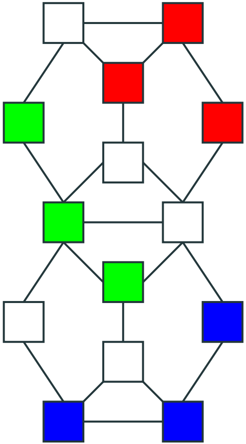
Best we can do using $2$-trees: $\Delta \lesssim q^{k/3}$ [Jain-Pham-Vuong'21].

- $q^{2/3}$ buckets and trade-off with mixing.

Idea: utilising small overlaps!

- Single vertex in 2-tree $\rightarrow$ size-$\theta$ component (block)



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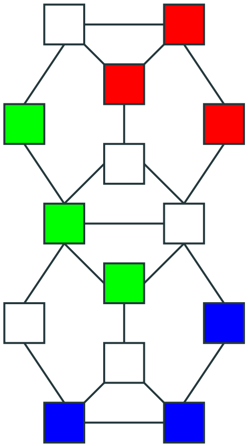

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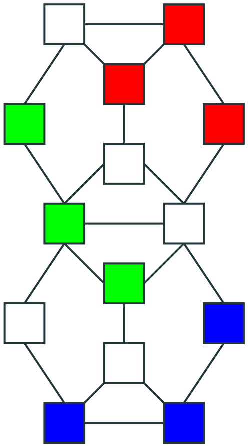

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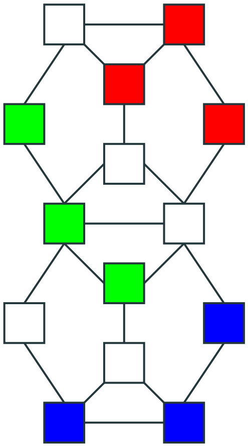

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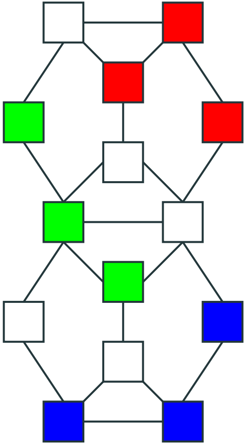

Requires:

$$\Delta \lesssim q^{\frac{k}{2+O(1/\theta)}}$$

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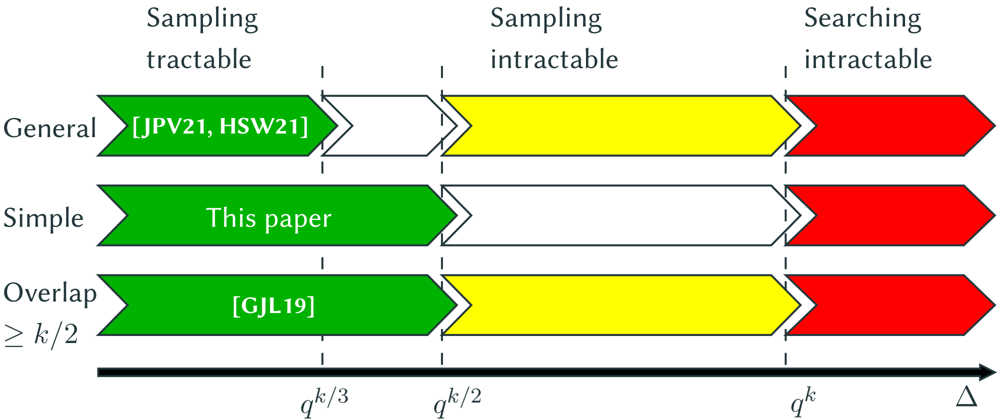
- Utilising overlap information?
  - Partial rejection sampling **[Guo-Jerrum-Liu'19]** gives transition at $\Delta \approx q^{k/2}$ when overlaps are large.


Sampling
tractable
Sampling
intractable
Searching
intractable
General
[JPV21, HSW21]
Simple
This paper
Overlap
$\geq k/2$
[GJL19]
$q^{k/3}$
$q^{k/2}$
$q^k$
$\Delta$

# Thank you!

arXiv: 2202.05554