INAPPROXIMABILITY OF COUNTING INDEPENDENT SETS IN LINEAR HYPERGRAPHS

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ABSTRACT. It is shown in this note that approximating the number of independent sets in a k-uniform linear hypergraph with maximum degree at most Δ is **NP**-hard if $\Delta \geq 5 \cdot 2^{k-1} + 1$. This confirms that for the relevant sampling and approximate counting problems, the regimes on the maximum degree where the state-of-the-art algorithms work are tight, up to some small factors. These algorithms include: the approximate sampler and randomised approximation scheme by Hermon, Sly and Zhang (2019), the perfect sampler by Qiu, Wang and Zhang (2022), and the deterministic approximation scheme by Feng, Guo, Wang, Wang and Yin (2022).

1. INTRODUCTION

This note is concerned with the problem of counting independent sets in hypergraphs. We start with the basic definitions. A *hypergraph* $H = (W, \mathcal{E})$ is specified by a set of vertices W and a set of hyperedges, where each hyperedge $e \in \mathcal{E}$ is a subset of W. It is said to be *k-uniform*, if each hyperedge contains exactly k vertices. The degree of a vertex is the number of hyperedges in which it appears, and the *degree* Δ of the hypergraph is the maximum degree of its vertices. A set $I \subset W$ is a (weak) *independent set* if $I \cap e \neq e$ holds for all $e \in \mathcal{E}$.

This problem is naturally parameterised by k and Δ. Unlike many other problems like hypergraph colouring or k-SATs where the existence of a solution is captured by the celebrated Lovász local lemma [EL75] under such parameterisation, even constructing a hypergraph independent set is trivial: the empty set is trivially a solution. However, the story seems no longer diverged when it comes to the computational hardness of the relevant approximate counting problems. Bezáková, Galanis, Goldberg, Guo and Štefankovič prove that approximating the number of independent sets is intractable when $\Delta \geq 5 \cdot 2^{k/2}$ [BGG+19], unless **NP** = **RP**. The exponent $k/2$ coincides with the intractability result for counting hypergraph q-colourings by Galanis, Guo and Wang that $\Delta \geq 5 \cdot q^{k/2}$ [GGW22] when q is an even.

On the other hand, there are several recent breakthroughs from the algorithmic side. Hermon, Sly and Zhang [HSZ19] first give a Markov-chain-based sampler that outputs an independent set almost uniformly at random in polynomial time when $\Delta \leq c2^{k/2}$ for some absolute constant $c > 0$. This also yields a fully-polynomial randomised approximation scheme (FPRAS) for the number of independent sets due to a standard sampling-to-counting reduction [JVV86]. A later work by Qiu, Wang and Zhang [QWZ22] provide a perfect sampler (i.e., the output distribution is unbiased) which runs in expected polynomial time when $\Delta \leq c2^{k/2}/k$ for some absolute constant $c > 0$. Very recently, Feng, Guo, Wang, Wang and Yin [FGW+22a] further derandomise the Markov chain Monte Carlo approach and provide a fully-polynomial deterministic approximation scheme (FPTAS) when $\Delta \leq c2^{k/2}/k^2$ for some absolute constant $c > 0$. All these regimes nearly match the hardness bound.

The notion of *linear* hypergraphs (aka. *simple* hypergraphs) also attracts some attention. We say a hypergraph has *overlap* b, if the intersection of each pair of hyperedges contains at most b vertices. The hypergraph is linear if it has overlap 1. The regimes where the above algorithms work go further when the input hypergraph is restricted to be linear. That is $\Delta \leq c2^k/k^2$ for both FPRAS and perfect sampler, and $\Delta \leq 2^{(1-o(1))k}$ for the FPTAS, established in the same work as above respectively. Are

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these algorithmic regimes asymptotically tight, up to some small factors? As the main claim of this note, we answer the question affirmatively.

Theorem 1.1. *For any $k \geq 2$, $1 \leq b \leq k/2$ and $\Delta \geq 5 \cdot 2^{k-b} + 1$, it is* **NP***-hard to approximate the number of independent sets in k-uniform hypergraphs of maximum degree at most Δ and overlap at most b.*

The hardness for the linear hypergraph is then obtained by plugging in $b = 1$. The above theorem also subsumes the general case in [BGG+19] by setting $b := \lfloor k/2 \rfloor$. In fact, as we will see soon, the reduction there is a special case of ours.

The phenomenon that a more relaxed algorithmic regime (and thus a more restricted hardness regime) exists for linear hypergraphs is also present in the hypergraph q-colouring problem. The up-to-date algorithmic bound is $\Delta \lesssim q^{k/3}$ in the general hypergraph [JPV21, HSW21, FGW+22a], while it goes further to $q^{k/2-o(k)}$ in the linear case [FGW22b]. From the hardness side, the approximate counting problem is known to be **NP**-hard when $\Delta \geq 5 \cdot q^{k/2}$ [GGW22] in the general case where q is an even, but $\Delta \geq 2kq^k \log q + 2q$ in the linear case [GGW22].

The reduction here is inspired by the general case [BGG+19]. The argument therein reduces from the hard-core model (counting weighted independent sets) on graphs, by replacing each vertex in the graph with $k/2$ copies in the hypergraph. This naturally requires large overlaps in the result hypergraph. In our case, linearity (or the requirement of small overlaps) is ensured by controlling the number of copies created, followed by filling up each hyperedge to k vertices. This boils down to a general anti-ferromagnetic 2-spin system, instead of merely the hard-core model. The main complicacy is to establish the so-called “non-uniqueness”, before which we can invoke a theorem by Sly and Sun [SS14], and show the inapproximability of this 2-spin system. We remark that the hardness of linear hypergraph colourings is handled separately in [GGW22]. However, the approach there is based on the hardness of the searching problem, and thus not applicable in our case because, again, constructing an independent set is trivial.

An open problem is to locate the computational phase transition completely for the linear case, as there is still an $O(k^2)$ gap in-between. Moreover, the so-called uniqueness threshold for independent sets on the hypertree is $\Delta \leq 2^k/(2k)$ [BGG+19, Lemma 60]. However, it is not obvious which among these three, if not none, would be the ground truth for the computational phase transition point.

2. Reduction from 2-spin systems

Our reduction is made from the hardness of approximating the partition function of the 2-*spin system* on graphs. A 2-spin system on a graph $G = (V, E)$ is specified by an interaction matrix $\boldsymbol{B}$ and a vector $\boldsymbol{h}$ for the external field:

$$\boldsymbol{B} = \begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix}, \qquad \boldsymbol{h} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix}, \tag{1}$$

where $\beta, \gamma, \lambda \geq 0$. The system is called *anti-ferromagnetic*, if $\beta\gamma < 1$. A configuration $\sigma : V \to \{0, 1\}$ assigns each vertex $v \in V$ with a spin either 0 or 1. The *weight* of a configuration σ is defined by

$$\mathrm{wt}(\sigma) := \lambda^{n_0(\sigma)} \beta^{m_{00}(\sigma)} \gamma^{m_{11}(\sigma)}$$

where $n_0(\sigma)$ is the number of vertices assigned 0 under σ, and $m_{00}(\sigma)$ (resp. $m_{11}(\sigma)$) is the number of edges whose both endpoints are assigned 0 (resp. 1) under σ. The *partition function* is defined by

$$Z_{\beta,\gamma,\lambda}(G) := \sum_{\sigma} \mathrm{wt}(\sigma).$$

The 2-spin system we are interested in is specified by the following choices of parameters:

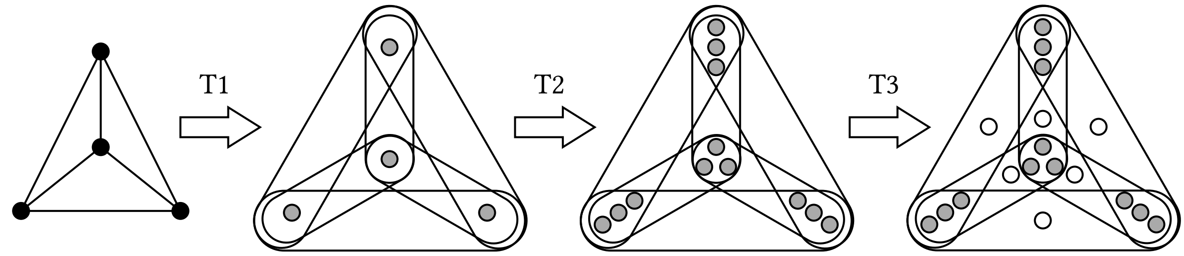
$$\beta = 1, \qquad \gamma = 1 - \frac{1}{2^{k-2b}}, \qquad \lambda = 2^b - 1.$$

The subscription in $Z_{\beta,\gamma,\lambda}$ is thus omitted as the parameters are now fixed.

We now state the reduction. For any given Δ-regular graph $G = (V, E)$, construct the hypergraph H_G according to the following steps.

(T1) Interpret the graph as a 2-uniform hypergraph.
(T2) Replace each vertex with b vertices.
(T3) For each hyperedge, insert another $k - 2b$ vertices independently.

Below is an example illustrating the reduction where $k = 7$, $b = 3$ and $\Delta = 3$.



It is immediate to verify that H_G is k-uniform, has overlap b and maximum degree Δ. Let $\mathcal{I}(H_G)$ be the set of independent sets of H_G.

Lemma 2.1. *For any Δ-regular graph $G = (V, E)$ and the constructed hypergraph H_G, it holds that $|\mathcal{I}(H_G)| = 2^{|E|(k-2b)} Z(G)$.*

Proof. We define the following partition over all the independent sets $\mathcal{I}(H_G) = \biguplus_\sigma \mathcal{S}(\sigma)$ in the hypergraph H_G, where σ rolls over all configurations of the 2-spin system. For any vertex $v \in V$, let B_v be the set of constructed vertices in H_G corresponding to v as in step (T2) of the construction. Given an independent set $I \in \mathcal{I}(H_G)$, the part $\mathcal{S}(\sigma)$ that I falls into is given by, for any $v \in V$,

- $\sigma(v) = 0$, if $|B_v \cap I| \leq b - 1$;
- $\sigma(v) = 1$, if $|B_v \cap I| = b$ (namely, $B_v \subseteq I$).

Apparently this is a partition because each $I \in \mathcal{I}(H_G)$ falls into exactly one part. We then show that $|S(\sigma)| = 2^{|E|(k-2b)} \mathrm{wt}(\sigma)$, and then the lemma follows immediately.

- Consider the vertices constructed in (T2).
  - For each $v \in V$ such that $\sigma(v) = 0$, there are $2^b - 1$ feasible partial configurations of B_v.
  - For each $v \in V$ such that $\sigma(v) = 1$, there is just one feasible partial configuration of B_v.
- Consider the vertices constructed in (T3).
  - For each edge e such that both its endpoints take spin 1, the rest $k - 2b$ vertices of the corresponding hyperedge cannot be in an independent set together, so there are $2^{k-2b} - 1$ feasible partial configurations.
  - For any other edge, the corresponding $k - 2b$ vertices are free to be included in an independent set, so there are 2^{k-2b} feasible partial configurations.

In all, this gives

$$|S(\sigma)| = \left(2^{k-2b} - 1\right)^{m_{11}(\sigma)} \left(2^{k-2b}\right)^{|E|-m_{11}(\sigma)} \left(2^b - 1\right)^{n_0(\sigma)} = 2^{|E|(k-2b)} \mathrm{wt}(\sigma). \qquad \square$$

Our goal then boils down to showing the inapproximability of the constructed 2-spin system. To establish this, we invoke the following celebrated result by Sly and Sun [SS14], which connects the so-called non-uniqueness property of any general anti-ferromagnetic 2-spin system with computational hardness. Denote by $\mathbb{T}_\Delta$ the infinite Δ-regular tree, and by $\hat{\mathbb{T}}_\Delta$ the infinite $(\Delta - 1)$-ary tree.

Theorem 2.2 ([SS14])**.** *For any nondegenerate homogeneous anti-ferromagnetic 2-spin system with interaction matrix $\boldsymbol{B}$ on Δ-regular graphs that lies in the $\mathbb{T}_\Delta$ non-uniqueness region, the partition function is* **NP***-hard to approximate, even within a factor of 2^{cn} for some constant $c(\boldsymbol{B}, \Delta) > 0$.*

We remark that uniqueness/non-uniqueness regions for $\mathbb{T}_\Delta$ coincide with those for $\hat{\mathbb{T}}_\Delta$. However, $(\Delta - 1)$-ary trees are more convenient to handle, so we move to $\hat{\mathbb{T}}_\Delta$ onwards. It is known that $\hat{\mathbb{T}}_\Delta$ (non-)uniqueness corresponds to the solutions of the standard tree recursion on the ratio of the Gibbs measure, namely $\mu_v(0)/\mu_v(1)$. The following lemma, originally due to Martinelli, Sinclair and Weitz [MSW07, Section 6.2], characterises these solutions.

Lemma 2.3 ([GŠV16, Lemma 7])**.** *For $\Delta \geq 3$ and an anti-ferromagnetic 2-spin system specified by (1), consider the system of equations*

$$x = \lambda \left(\frac{\beta y + 1}{y + \gamma} \right)^{\Delta - 1}, \qquad y = \lambda \left(\frac{\beta x + 1}{x + \gamma} \right)^{\Delta - 1}$$

where $x, y \geq 0$. Then,

- *in the $\hat{\mathbb{T}}_\Delta$ uniqueness region, the system has a unique solution $(Q^\times, Q^\times)$;*
- *in the $\hat{\mathbb{T}}_\Delta$ non-uniqueness region, the system has three solutions $(Q^+, Q^-), (Q^\times, Q^\times), (Q^-, Q^+)$ where $Q^+ > Q^\times > Q^-$.*

Let $d := \Delta - 1$. Using the above lemma, it suffices to show that the two-step recursion has three fixed points $Q^+ > Q^\times > Q^-$ in order to establish non-uniqueness. Equivalently, we are to show that the following function has 3 distinct zeros on $(0, +\infty)$ in the regime of parameters in Theorem 1.1.

$$f(z) := (2^b - 1) \left(1 + \frac{1}{2^{k-2b}(2^b - 1)\left(1 + \frac{1}{2^{k-2b}z + 2^{k-2b} - 1}\right)^d + 2^{k-2b} - 1} \right)^d - z. \tag{2}$$

Because the solution $Q^\times$ is the unique fixed point of the one-step recursion, it also helps to consider the function

$$g(z) := (2^b - 1) \left(1 + \frac{1}{2^{k-2b}z + 2^{k-2b} - 1} \right)^d - z. \tag{3}$$

The following lemma is sufficient to derive our main theorem.

Lemma 2.4. *Assume integers $k \geq 2$, $1 \leq b \leq k/2$ and $d = 5 \cdot 2^{k-b}$. Define $z^* := d/2^{k-2b} = 5 \cdot 2^b$. Then $f(z^*) > 0$ and $g(z^*) < 0$.*

Proof of Theorem 1.1. Note that $g(0) > 0$ and $\lim_{z \to +\infty} g(z) = -\infty$. By $g(z^*) < 0$, we know that the unique zero of g, which is $Q^\times$, is smaller than z^*. On the other hand, by $f(z^*) > 0$ and $\lim_{z \to +\infty} f(z) = -\infty$, there is a zero of f on $(z^*, +\infty)$, and it cannot be $Q^\times$. This establishes non-uniqueness due to Lemma 2.3. **NP**-hardness then follows after Theorem 2.2. □

In the proof of Lemma 2.4, the following standard inequality is useful.

$$\exp\{x\} > \left(1 + \frac{x}{y}\right)^y > \exp\left\{\frac{xy}{x + y}\right\} \qquad \text{for all } x, y > 0. \tag{4}$$

Proof of Lemma 2.4. The $g(z^*)$ part is due to a straightforward estimation:

$$g(z^*) \leq (2^b - 1)\left(1 + \frac{1}{5 \cdot 2^{k-b}}\right)^{5 \cdot 2^{k-b}} - 5 \cdot 2^b < (2^b - 1)\mathrm{e} - 5 \cdot 2^b < 0.$$

For the $f(z^*)$ part, first assume $k \geq 3$. Then

$$\begin{aligned}
f(z^*) =& (2^b - 1) \left(1 + \frac{1}{2^{k-2b}(2^b - 1)\left(1 + \frac{1}{5 \cdot 2^{k-b} + 2^{k-2b} - 1}\right)^{5 \cdot 2^{k-b}} + 2^{k-2b} - 1} \right)^{5 \cdot 2^{k-b}} - 5 \cdot 2^b \\
\geq& (2^b - 1) \left(1 + \frac{1}{2^{k-2b}(2^b - 1)\left(1 + \frac{1}{5 \cdot 2^{k-b}}\right)^{5 \cdot 2^{k-b}} + 2^{k-2b} - 1} \right)^{5 \cdot 2^{k-b}} - 5 \cdot 2^b \\
>& (2^b - 1) \left(1 + \frac{1}{2^{k-2b}(2^b - 1)\mathrm{e} + 2^{k-2b} - 1} \right)^{5 \cdot 2^{k-b}} - 5 \cdot 2^b
\end{aligned}$$

$$
\begin{aligned}
&> (2^b - 1)\left(1 + \frac{1}{2^{k-2b}(2^b-1)\mathrm{e} + 2^{k-2b}}\right)^{5\cdot 2^{k-b}} - 5\cdot 2^b \\
\text{(By (4))} \qquad &> (2^b - 1)\exp\left\{\frac{5\cdot 2^{b+k}}{2^k + 2^{2b} + 2^k(2^b-1)\mathrm{e}}\right\} - 5\cdot 2^b.
\end{aligned}
$$

Note that the fraction in $\exp\{\cdot\}$ is monotone increasing with respect to k. Discuss further analysis by two cases.

- In the case that $b \geq 2$, the whole term minimises at $k = 2b$. Plug this in, we further get

$$
f(z^*) > (2^b - 1)\exp\left\{\frac{5\cdot 2^b}{2 + (2^b-1)\mathrm{e}}\right\} - 5\cdot 2^b =: h(b).
$$

  Now it suffices to show $h(b) > 0$ for $b \geq 2$. If $b = 2$, we have $h(2) > 1.5$. If $b \geq 3$, then

$$
h(b) > (2^b - 1)\exp\left\{\frac{5\cdot 2^b}{\mathrm{e}\cdot 2^b}\right\} - 5\cdot 2^b > 1.29\cdot 2^b - 6.29 > 0.
$$

- In the case that $b = 1$, the whole term minimises at $k = 3$, which is at least 0.7.

Finally, if $k = 2$, then b can only take 1, in which case $f(z^*) > 16.0$. □

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