MMM Math

1) $R = k \prod_i \left(\frac{1+\frac{S_i}{C_i}}{1+\frac{S_i}{M_i}} \right)^{\gamma_i}$,

Where R is revenue, S is media spend, k is the reg constant, I is an index over variables. γ is the regression parameter indexed by i. 0 < C < M, are hyperparameters.

For MMM we can use (1) and fit the parameters, γ, in log space, using linear techniques.

1a) $$\log(R) = \log\left(k \prod_i \left(\frac{1+\frac{S_i}{C_i}}{1+\frac{S_i}{M_i}} \right)^{\gamma_i} \right),$$

1b) $$\log(R) = k + \sum_i \gamma_i \log\left(\frac{1+\frac{S_i}{C_i}}{1+\frac{S_i}{M_i}} \right),$$

To understand (1), we take the derivative an then calculate the elasticity.

Derivative done by hand and by derivative-calculator.net

1) $\frac{\partial R}{\partial S_i} = \frac{\gamma_i * R * (M_i - C_i)}{(S_i + C_i)(S_i + M_i)}$.

Elasticity is then,

2) $\frac{\partial R}{\partial S_i} \frac{S_i}{R} = \frac{\gamma_i * S_i * (M_i - C_i)}{(S_i + C_i)(S_i + M_i)}$,

3) $\frac{\partial R}{\partial S}\frac{S_i}{R} = \gamma_i * \frac{M_i - C_i}{M_i} * \frac{S_i}{S_i + C_i} * \frac{M_i}{S_i + M_i}$.

In (4), the factors are:

a. γ, the fit parameter
b. A constant
c. $\lim_{S_i \to 0} \frac{S_i}{S_i + C_i} = 0$
d. $\lim_{S_i \to \infty} \frac{M_i}{S_i + M_i} = 0$.

This shows that the elasticity is 0, at both 0 and inf, and, has the sign of gamma in between.

4) Some definitions:
   a. $\overline{S_i} \stackrel{\text{def}}{=} mean\left(S_i \middle| S_{i,t} > 0\right)$. This is the average value of the media channel when it is used. Loosely, it is some central value of S determine either by data or as an input. We'll scale S, M and C by this number for each channel.
   b. $x_i \stackrel{\text{def}}{=} \frac{S_i}{\overline{S_i}}$,
   c. $c_i \stackrel{\text{def}}{=} \frac{C_i}{\overline{S_i}}$ (lower case "c"),
   d. $m_i \stackrel{\text{def}}{=} \frac{M_i}{\overline{S_i}}$ (lower case "m").

We can now rewrite (1) (and dropping subscripts) as,

5) $R = k \prod_i \left(\frac{1 + \frac{x * \overline{S}}{c * \overline{S}}}{1 + \frac{x * \overline{S}}{m * \overline{S}}} \right)^{\gamma}$

6) $R = k \prod_i \left(\frac{1 + \frac{x}{c}}{1 + \frac{x}{m}} \right)^{\gamma}$,

Showing the marketing response function (1) is unchanged with the transformed variables.

Substituting into (2) we get the partial derivatives,

7) $\frac{\partial R}{\partial x_i} = \frac{\gamma_i * R * (m_i - c_i)}{(x_i + c_i)(x_i + m_i)`}$.

For optimization we want the partial derivatives relative to spend, not scaled spend. To get this we can use the chain rule,

8) $\frac{\partial R}{\partial S_i} = \frac{\partial R}{\partial x_i} * \frac{\partial x_i}{\partial S_i}$,

9) $\frac{\partial R}{\partial S_i} = \frac{\gamma_i * R * (m_i - c_i)}{(x_i + c_i)(x_i + m_i)} * \frac{1}{\overline{\overline{S_i}}}$.

At equilibrium,

10) $$\frac{\partial R}{\partial S_i}\left(\overline{S_i}\right) = \frac{\partial R}{\partial S_j}\left(\overline{S_j}\right),$$

Which says that the marginal return for any two media variables, i and j, should be equal at their level spend, when $S_i = \overline{S_i}$.

By definition 5b,

11) $$x_i = \frac{\overline{S_i}}{\overline{\overline{S_i}}} = 1$$

12) $$\frac{\gamma_i * R * (m_i - c_i)}{(1 + c_i)(x_i + m_i)} * \frac{1}{\overline{\overline{S_i}}} = \frac{\gamma_j * R * (m_j - c_j)}{(1 + c_j)(1 + m_j)} * \frac{1}{\overline{\overline{S_j}}}.$$

R, revenue, cancels. If we assume, $c = c_i\,; m = m_i, \forall i$ then c and m also cancel and (13) becomes,

13) $\frac{\gamma_i}{\overline{S_i}} = \frac{\gamma_j}{\overline{S_j}}$.

Equation (14) says that the coefficients of the scaled variables in proportion to the average spend implies equal marginal return.

Next we'll tweak the response function so the coefficients themselves are directly comparable. That is, equal coefficients imply equal marginal return at spend level S_bar.

14) $\delta = \frac{\gamma}{\bar{S}}$,

Equation (13) is then,

13a) $\delta_i = \delta_j$

Substituting back into (6),

15) $R = k \prod_i \left(\frac{1+\frac{x}{c}}{1+\frac{x}{m}}\right)^{\delta * \overline{S}}$,

In log space,

16) $\log(R) = k + \sum_i \delta * \overline{S} * \log\left(\frac{1+\frac{x}{c}}{1+\frac{x}{m}}\right)$,