Problem 1

Let m be the Lebesgue measure on $\mathbb{R}$, and m^* denotes the Lebesgue outer measure. Prove that $B \subset \mathbb{R}$ is not Lebesgue measurable if and only if there exists $\epsilon > 0$ such that for every Lebesgue measurable $A \subset B$, $m^*(B \setminus A) \geq \epsilon$.

This follows immediately from the following lemma: a set E is measurable iff for all $\epsilon > 0$, there exists a closed set $F \subset E$ such that $m^*(E \setminus F) < \epsilon$. This is a standard theorem which characterizes measurable sets, and is often given as the definition; you can also look at my solution for the August 2015 exam #3 for a proof.

Now, if B is measurable then for all $\epsilon > 0$ there exists $F \subset B$, F closed (hence measurable) with $m^*(B \setminus F) < \epsilon$. This proves the reverse direction by contrapositive. Similarly, for the other direction, we take contrapositive. Suppose that for all $\epsilon > 0$, there exists a measurable set $A \subset B$ with $m^*(B \setminus A) < \epsilon$. Let $\epsilon > 0$ be arbitrary and choose $A \subset B$ with $m^*(B \setminus A) < \frac{\epsilon}{2}$. Since A is measurable, the lemma implies that there exists F closed, $F \subset A$ and $m^*(A \setminus F) < \frac{\epsilon}{2}$. Therefore $F \subset A$ and $m^*(B \setminus A) < \epsilon$, so the lemma implies that B is measurable.

Problem 2

Let $\Omega \subset \mathbb{C}$ be a domain and $\{z_1, \ldots, z_{2n}\}$ and even number of points in the same connected component of $\mathbb{C} - \bar{\Omega}$. Show that there exists a holomorphic function $f(z)$ on Ω such that

$$f^2(z) = (z - z_1) \cdots (z - z_{2n})$$

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Problem 3

Assume that f is a monotone increasing function on $[0,1]$. Prove that the following two statements are equivalent.
(a)f is absolutely continuous.
(b) For every absolutely continuous function g on $[0,1]$ and for every $x \in [0,1]$,

$$\int_0^x f(t)g'(t)\ dt + \int_0^x f'(t)g(t)\ dt = f(x)g(x) - f(0)g(0)$$

First suppose f is absolutely continuous. Then since g is absolutely continuous, so is fg. Indeed, both are continuous on $[0,1]$ and so attain max. values, say $|f(x)| \leq M_f, |g(x)| \leq M_g$ on $[0,1]$. Let $\epsilon > 0$ and take $\{a_k, b_k\}_{k=1}^N \subset [0,1]$ pairwise disjoint with $\sum_{k=1}^N (b_k - a_k) < \min\{\delta_f, \delta_g\}$, where δ_f corresponds to the $\frac{\epsilon}{2M_g}$-challenge for absolute continuity of f, and δ_g corresponds to the $\frac{\epsilon}{2M_f}$-challenge for absolute continuity of g. Then

$$\begin{aligned}
\sum_{k=1}^N |f(b_k)g(b_k) - f(a_k)g(b_k)| &= \sum_{k=1}^N |f(b_k)g(b_k) - f(b_k)g(a_k) + f(b_k)g(a_k) - f(a_k)g(a_k)| \\
&\leq \sum_{k=1}^N |f(b_k)[g(b_k) - g(a_k)]| + \sum_{k=1}^N |g(a_k)[f(b_k) - f(a_k)]| \\
&\leq M_f \sum_{k=1}^N |g(b_k) - g(a_k)| + M_g \sum_{k=1}^N |f(b_k) - f(a_k)| \\
&< \epsilon
\end{aligned}$$

Since fg is absolutely continuous, $(fg)'$ exists a.e. and $(fg)(x) = (fg)(0) + \int_0^x (fg)'(t)\ dt$. Expanding gives

$$f(x)g(x) = f(0)g(0) + \int_0^x f'(t)g(t)\ dt + \int_0^x g'(t)f(t)\ dt$$

as desired. Therefore $(a) \Rightarrow (b)$. For the other direction, suppose (b) holds. The function $g(x) \equiv 1$ is absolutely continuous on $[0,1]$ so (b) implies

$$\int_0^x f(t)g'(t)\ dt + \int_0^x f'(t)g(t)\ dt = f(x)g(x) - f(0)g(0) \Rightarrow \int_0^x f'(t)\ dt = f(x) - f(0)$$

Since f is monotone increasing on $[0,1]$, it's of bounded variation (its variation is simply $f(1) - f(0)$. Therefore f' exists a.e. and we have $\int_0^1 f'(t)\ dt \leq f(1) - f(0)$. In other words, $f' \in L^1[0,1]$. Since f is the integral of an L^1 function, it's absolutely continuous.

Problem 4

Let $f(z)$ be a holomorphic function on the disk $|z| < R$. Suppose that $|f(z)| \leq M$ for all $|z| < R$, and that

$$f(0) = f'(0) = \ldots = f^{(n)}(0) = 0$$

for some integer $n \geq 0$. Show that

$$|f(z)| \leq M \left(\frac{|z|}{R}\right)^{n+1}$$

for all $|z| < R$. Moreover, if equality holds for some point, then

$$f(z) = \alpha \cdot M \left(\frac{z}{R}\right)^{n+1}$$

for some complex number α with $|\alpha| = 1$ and all $|z| < R$.

This is a generalization of the Schwarz Lemma; the proof will be very similar. Since $f(z)$ is holomorphic with a zero of order $(n+1)$ at 0, we can write $f(z) = z^{n+1}g(z)$ where g is holomorphic and $g(0) \neq 0$. Note also that $f(z)$ has a power series expansion $f(z) = \sum_{n=1}^{\infty} a_k z^k = \sum_{k=n+1}^{\infty} a_k z^k$.

Define $h(z) =$

$$\begin{cases} \dfrac{f(z)}{z^{n+1}} & z \neq 0 \\ \dfrac{f^{(n+1)}(0)}{n!} & z = 0 \end{cases}$$

Observe that $h(z)$ is analytic in $D_R = \{|z| < R\}$. Let $0 < r < R$. Then the Maximum Modulus Principle implies that $|h(z)| = \left|\dfrac{f(z)}{z^{n+1}}\right| \leq \dfrac{M}{r^{n+1}}$ on D_R. Letting $r \to R$ gives the result.

If equality holds for some $0 \neq z_0 \in D_R$, then $|h(z)| = \left|\dfrac{f(z)}{z^{n+1}}\right| = \dfrac{M}{R^{n+1}}$ and MMP implies that $|h|$ is constant on the interior i.e. $\left|\dfrac{f(z)}{z^{n+1}}\right| = \alpha$ for some $\alpha \in \mathbb{C}$, $|\alpha| = \dfrac{M}{R^{n+1}}$. Alternatively, apply the Schwarz Lemma above to $\dfrac{f(Rz)}{M} : D \to D$.

Problem 5

Let m be the Lebesgue measure on $\mathbb{R}$ and consider a sequence $\{f_n\}_{n\geq 1}$ of nonnegative Lebesgue measurable functions on $\mathbb{R}$. Suppose there exist $M > 0$ and positive convergent series $\sum_{n\geq 1} \alpha_n < \infty$ and $\sum_{n\geq 1} \beta_n < \infty$ such that

$$\int_{f_n \leq M} f_n \, dm \leq \alpha_n, \text{ and } m(\{f_n > M\}) \leq \beta_n \ \forall n \geq 1$$

Prove that $\sum_{n\geq 1} f_n(x) < \infty$ a.e.

Problem 6

Evaluate the integral: $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)} dx$

We seek $\displaystyle\int_{-\infty}^{\infty} \Re(f(z))\ dz$ where $f(z) = \dfrac{e^{iz}}{z^2+1}$. For $R > 0$ consider the semicircular contour $\gamma_R = [-R, R] \cup C_R$, where C_R is the arc $\{Re^{it} : t \in [0, \pi]\}$. Then when $R > 1$, f is holomorphic in γ_R except for a simple pole at $z = i$. Therefore the residue theorem gives

$$\int_{\gamma_R} f(z)\ dz = 2\pi i \mathrm{Res}(i) = 2\pi i \lim_{z\to i} \frac{e^{iz}}{z+i} = 2\pi i \left(\frac{e^{-1}}{2i}\right) = \frac{\pi}{e}$$

Along C_R, we have $z = Re^{i\theta}$ with $\theta \in [0, \pi]$ and so

$$\begin{aligned}
\left|\int_{C_R} f(z)\ dz\right| \le \int_{C_R} |f(z)|\ dz &= \int_0^{\pi} \left|\frac{e^{iRe^{i\theta}}}{(Re^{i\theta})^2+1}\right|\ d\theta \\
&= \int_0^{\pi} \frac{e^{-R\sin\theta}}{|R^2e^{2i\theta}+1|}\ \ d\theta \\
&\le \int_0^{\pi} \frac{1}{R^2-1}\ d\theta \\
&= \frac{\pi}{R^2-1} \to 0
\end{aligned}$$

The last inequality follows since $R\sin\theta > 0$, combined with the reverse triangle inequality for the denominator. Putting this all together,

$$\frac{\pi}{e} = \int_{\gamma_R} f(z)\ dz = \int_{-R}^{R} f(z)\ dz + \int_{C_R} f(z)\ dz$$

and letting $R \to \infty$ we have

$$\frac{\pi}{e} = \int_{-\infty}^{\infty} f(z)\ dz$$

Finally,

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\ dx = \int_{-\infty}^{\infty} \Re(f(z))\ dz = \Re\left(\int_{\infty}^{\infty} f(z)\ dz\right) = \Re\left(\frac{\pi}{e}\right) = \frac{\pi}{e}$$