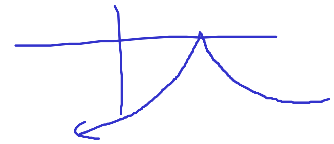
14.2: Calculus of Vector-Valued Functions



Definition. (Derivative and Tangent Vector)
Let $\mathbf{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$, where f,g, and h are differentiable functions on (a, b). Then $\mathbf{r}$ has a **derivative** (or is **differentiable**) on (a, b) and

$$\mathbf{r}'(t) = f'(t)\boldsymbol{i} + g'(t)\boldsymbol{j} + h'(t)\boldsymbol{k}. \quad = \langle f(t), g(t), h(t) \rangle$$

Provided $\mathbf{r}'(t) \neq \mathbf{0}$, $\mathbf{r}'(t)$ is a **tangent vector** at the point corresponding to $\mathbf{r}(t)$.

Example. For the following functions below, find $\mathbf{r}'(t)$ $\quad \log_2(t) = \frac{\ln(t)}{\ln(2)}$

a) $\mathbf{r}(t) = \left\langle e^{-t^2}, \log_2(t-4), \sin(t) \right\rangle$

$\vec{r}'(t) = \left\langle -2t e^{-t^2}, \frac{1}{\ln(2)(t-4)}, \cos(t) \right\rangle$

b) $\mathbf{r}(t) = 3\boldsymbol{i} - 2\tan(t)\boldsymbol{j} + e^t\boldsymbol{k}$

$\vec{r}'(t) = 0\hat{i} - 2\sec^2(t)\hat{j} + e^t\hat{k}$

Example. For $\mathbf{r}(t) = \langle 3t, \sec(2t), \cos(t) \rangle$ compute $\mathbf{r}'(t)$ at $t = \frac{\pi}{4}$.

$\vec{r}'(t) = \langle 3, 2\sec(2t)\tan(2t), -\sin(t) \rangle$

$\vec{r}'(\frac{\pi}{4}) = \langle 3, 2(1)$ [illegible] $, -\frac{\sqrt{2}}{2} \rangle$

$\tan(\frac{\pi}{2})$ DNE

$2c \quad 2\sqrt{2}$

Definition. (Unit Tangent Vector)
Let $\mathbf{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$ be a smooth parameterized curve, for $a \le t \le b$. The **unit tangent vector** for a particular value of t is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Example. For $\mathbf{r}(t) = \langle 3\sin(t),\, -2\cos(2t),\, 3\cos(t)\rangle$, find the unit tangent vector.

$$\vec{r}\,'(t) = \langle 3\cos(t), 4\sin(2t), -3\sin(t)\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{9\cos^2(t) + 16\sin^2(2t) + 9\sin^2(t)} = \sqrt{9 + 16\sin^2(2t)}$$

$$\vec{T}(t) = \frac{\langle 3\cos(t), 4\sin(2t), -3\sin(t)\rangle}{\sqrt{9 + 16\sin^2(2t)}}$$

Example. For $\mathbf{r}(t) = \langle \sin(6t),\, 3t,\, \cos(3t)\rangle$, compute $\mathbf{T}\left(\frac{\pi}{3}\right)$.

$$T(t) = \frac{\langle 6\cos(6t), 3, -3\sin(3t)\rangle}{\sqrt{36\cos^2(6t) + 9 + 9\sin^2(3t)}}$$

$$T(\pi/3) = \frac{\langle 6(1), 3, -3(0)\rangle}{3\sqrt{4(1) + 1 + 0}} = \frac{\langle 6, 3, 0\rangle}{3\sqrt{5}}$$

$$= \frac{\langle 2, 1, 0\rangle}{\sqrt{5}}$$

Derivative Rules
Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector-valued functions, and let f be a differentiable scalar-valued function, all at a point t. Let $\mathbf{c}$ be a constant vector. The following rules apply.

1. $\dfrac{d}{dt}(\mathbf{c}) = \mathbf{0}$ — Constant Rule
2. $\dfrac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$ — Sum Rule
3. $\dfrac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ — Product Rule
4. $\dfrac{d}{dt}(\mathbf{u}(f(t))) = \mathbf{u}'(f(t))f'(t)$ — Chain Rule
5. $\dfrac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ — Dot Product Rule
6. $\dfrac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ — Cross Product Rule

Example. Given $\mathbf{u}(t) = \langle 4t^2, 1, 3t \rangle$ and $\mathbf{v}(t) = \langle e^{-2t}, -2e^t, e^t \rangle$, find $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$.

$$\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$$

$$= \langle 8t, 0, 3\rangle \cdot \langle e^{-2t}, -2e^t, e^t\rangle + \langle 4t^2, 1, 3t\rangle \cdot \langle -2e^{-2t}, -2e^t, e^t\rangle$$

$$= 8te^{-2t} + 3e^t - 8t^2e^{-2t} - 2e^t + 3te^t$$

$$= 8te^{-2t}(1-t) + e^t(1+3t)$$

Definition. (Indefinite Integral of a Vector-Valued Function)
Let $\mathbf{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$ be a vector function, and let $\mathbf{R}(t) = F(t)\boldsymbol{i} + G(t)\boldsymbol{j} + H(t)\boldsymbol{k}$, where F, G, and H are antiderivatives of f, g, and h, respectively. The **indefinate integral** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C},$$

where $\mathbf{C}$ is an arbitrary constant vector. Alternatively, in component form,

$$\int \langle f(t),\, g(t),\, h(t)\rangle\,dt = \langle F(t),\, G(t),\, H(t)\rangle + \langle C_1,\, C_2,\, C_3\rangle.$$

Example. Find $\mathbf{r}(t)$ such that $\mathbf{r}'(t) = \left\langle \frac{t}{t^2+1},\, t^2e^{-t^3},\, \frac{-2t}{\sqrt{t^2+16}} \right\rangle$ and $\mathbf{r}(0) = \left\langle 3,\, \frac{5}{3},\, -5 \right\rangle$.

$\int \frac{t}{t^2+1}\,dt$

$u = t^2+1$
$du = 2t\,dt$
$\frac{1}{2}du = t\,dt$

$= \frac{1}{2}\int u^{-1}\,du$

$= \frac{1}{2}\ln|u| + c_1$

$= \frac{1}{2}\ln|t^2+1| + c_1$

$\int t^2 e^{-t^3}\,dt$

$u = -t^3$
$du = -3t^2\,dt$
$-\frac{1}{3}du = t^2\,dt$

$-\frac{1}{3}\int e^u\,du = -\frac{1}{3}e^{-t^3} + c_2$

$\int \frac{-2t}{\sqrt{t^2+16}}\,dt$

$u = t^2+16$
$du = 2t\,dt$

$-\int u^{-1/2}\,du$

$= -2u^{1/2} + c_3$

$= -2(t^2+16)^{1/2} + c_3$

$\vec{r}(t) = \left\langle \frac{1}{2}\ln(t^2+1),\, -\frac{1}{3}e^{-t^3},\, -2\sqrt{t^2+16} \right\rangle + \langle c_1, c_2, c_3\rangle$

$\langle 3, \frac{5}{3}, -5\rangle = \vec{r}\langle 0\rangle = \langle 0, -\frac{1}{3}, -8\rangle + \langle c_1, c_2, c_3\rangle \longrightarrow$ $c_1 = 3$
$c_2 = 2$
$c_3 = 3$

Definition. (Definite Integral of a Vector-Valued Function)
Let $\mathbf{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$, where f, g, and h are integrable on the interval $[a, b]$. The **definite integral** of $\mathbf{r}$ on $[a, b]$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\boldsymbol{i} + \left(\int_a^b g(t)\,dt\right)\boldsymbol{j} + \left(\int_a^b h(t)\,dt\right)\boldsymbol{k}$$

Example. $\displaystyle\int_{-\pi}^{\pi} \langle \sin(t),\, \cos(t),\, 8t \rangle\,dt$