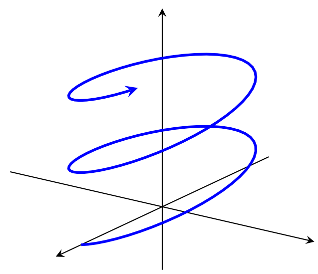
14.1: Vector-Valued Functions

Vector-valued functions are functions of the form $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, where $x(t)$, $y(t)$, and $z(t)$ are parametric equations dependent on t.



Curves in Space

Consider

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k},$$

where f, g, and h are defined for $a \le t \le b$. The **domain** of $\mathbf{r}$ is the largest set of t for which all of f,g, and h are defined.

Example. What plane does the curve $\mathbf{r}(t) = t\boldsymbol{i} + 6t^3\boldsymbol{k}$ lie?

Example (Lines as vector-valued functions)**.** Find a vector function for the line that passes through the points $P(5, 2, -4)$ and $Q(5, 5, -2)$. What about the line segment that connects P and Q?

Example. Find the domain of

$$\mathbf{r}(t) = \sqrt{16 - t^2}\boldsymbol{i} + \sqrt{t}\boldsymbol{j} + \frac{4}{\sqrt{3+t}}\boldsymbol{k}$$

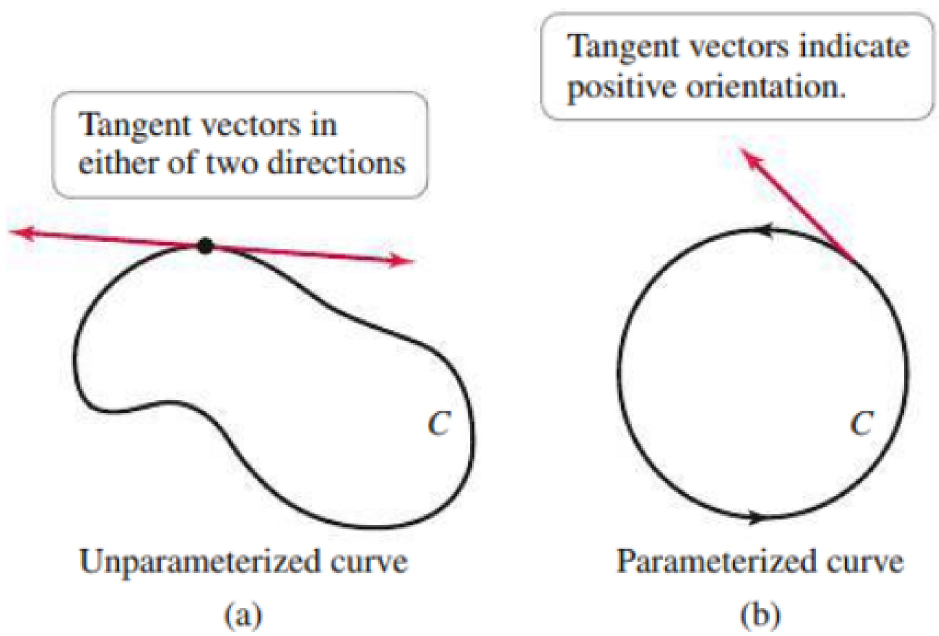
Example. Find the point P on

$$\mathbf{r}(t) = t^2\boldsymbol{i} + 2t\boldsymbol{j} + 2t\boldsymbol{k},$$

closest to $P_0(4, 17, 10)$. What is the distance between P and P_0?

Orientation of Curves

- A **unparameterized curve** is a smooth curve C with no specified direction and the tangent vector can be drawn in two directions.
- A **parameterized curve** is a smooth curve C described by a function $\mathbf{r}(t)$ for $a \le t \le b$ and has a direction referred to as its **orientation**.
- The *positive* orientation is the direction of the curve generated when t increases from a to b.
- The tangent vector of a parameterized curve points in the positive orientation of the curve.



Example. Graph the curve described by the equation

$$\mathbf{r}(t) = 4\cos(t)\boldsymbol{i} + \sin(t)\boldsymbol{j} + \frac{t}{2\pi}\boldsymbol{k},$$

where $0 \leq t \leq 2\pi$. Indicate the positive orientation of this curve.

Limits and Continuity for Vector-Valued Functions

The properties of limits extend to vector-valued functions naturally. In particular, for $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, if

$$\lim_{t \to a} f(t) = L_1, \qquad \lim_{t \to a} g(t) = L_2, \qquad \lim_{t \to a} h(t) = L_3$$

then

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t),\ \lim_{t \to a} g(t),\ \lim_{t \to a} h(t) \right\rangle = \langle L_1, L_2, L_3 \rangle.$$

Definition. (Limit of a Vector-Valued Function)
A vector-valued function $\mathbf{r}$ approaches the limit $\mathbf{L}$ as t approaches a, written $\lim\limits_{x \to a} \mathbf{r}(t) = \mathbf{L}$, provided $\lim\limits_{x \to a} |\mathbf{r}(t) - \mathbf{L}| = 0$.

A function $\mathbf{r}(t)$ is **continuous** at $t = a$, provided $\lim\limits_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$.

Example. Evaluate the following limits:

$$\lim_{t \to \pi} \left(\cos(t)\boldsymbol{i} - 7 \sin\left(-\frac{t}{2}\right)\boldsymbol{j} + \frac{t}{\pi}\boldsymbol{k} \right)$$

$$\lim_{t \to \infty} \left(\frac{t}{t-3}\boldsymbol{i} + \frac{40}{1 + 19e^{-t}}\boldsymbol{j} + \frac{1}{2t}\boldsymbol{k} \right)$$