10.8: Choosing a Convergence Test

Example. Consider the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$. Is this series conditionally convergent, absolutely convergent, or divergent? Which test do you use?

abs convergence : $\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges b/c p-series w/ $p = 1 \leq 1$

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ $\Rightarrow$ Not abs convergent

① $a_k = \frac{1}{k}$, $a_{k+1} = \frac{1}{k+1}$; $k < k+1$ ↳ $\frac{1}{k+1} < \frac{1}{k}$ $\Rightarrow$ non increasing

② $\lim_{k \to \infty} a_k = 0$

AST $\Rightarrow$ convergent

$\boxed{\text{Conditionally convergent}}$

Example. Consider the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$. Is this series conditionally convergent, absolutely convergent, or divergent? Which test do you use?

abs convergent

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k^2} \right| = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$\boxed{\text{p-series w/ } p=2>1 \text{ so series converges}}$

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$ AST

1 $k^2 < (k+1)^2 \Rightarrow \frac{1}{(k+1)^2} < \frac{1}{k^2}$

2 $\lim_{k\to\infty} \frac{1}{k^2} = 0$

$\boxed{\text{Converges by AST}}$

2. many choice

Consider the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$. Select all statements that are TRUE about this series.

A. This series is absolutely convergent.

B. This series is conditionally convergent.

C. This series is absolutely convergent by the Alternating Series Test. ← No

D. This series is conditionally convergent by the Alternating Series Test.

E. This series is convergent by the Alternating Series Test.

F. This series is divergent by the Alternating Series Test.

$\left[(-1)^2\right]^k = 1^k = 1$

Example. Which of the following series can be rewritten as a p-series? $\sum_{k=1}^{\infty} \frac{1}{k^p}$

$$\sum_{k=1}^{\infty} \frac{(-1)^{2k}}{k\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$$

p-series w/ $p = 3/2$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^5}$$

Not a p-series

$$\sum_{k=1}^{\infty} \frac{k^2 + k + 1}{k^4 + 2}$$

Not a p-series

Note: can compare to a p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ LCT

$$\sum_{k=1}^{\infty} \frac{3^k}{k^4}$$

Not a p-series

$$\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$$

p-series w/ $p = 3/2$

$k^{1/2} k^{-2} = k^{1/2 - 4/2}$

LC #3

Example. Which test *cannot* be used to determine the convergence of $\sum_{k=1}^{\infty} \frac{k^2\, 2^{k-1}}{(-5)^k}$?

✓ AST

✓ Ratio Test $\lim \left| \frac{(k+1)^2\, 2^k}{(-5)^{k+1}} \cdot \frac{(-5)^k}{k^2\, 2^{k-1}} \right|$

✓ Root Test $\lim \left| \frac{k^2\, 2^{k-1}}{5^k} \right|^{1/k}$

✗ LCT because a_k is negative

$\boxed{\text{LC \#4}}$

Example. For the following series, which test should be used to determine if the series converges or diverges? Use your selected test to show convergence or divergence.

$\boxed{\text{LC \#5, \#6}}$

$$\sum_{k=1}^{\infty} (-1)^k \frac{k}{k+2}$$

✗ AST $\lim_{k\to\infty} a_k = 1$

✓ Divergence Test $\lim_{k\to\infty} a_k = 1 \neq 0 \to$ Diverges

✗ Root $\longrightarrow \rho = 1$

✗ p-Series

✗ Integral Test $\to a_k$ not strictly positive

✗ Ratio $\longrightarrow r = 1$

✗ Geometric Series

$$\sum_{k=1}^{\infty}(-1)^k \frac{k}{k+2}$$

✓ Divergence Test $\lim_{k\to\infty} a_k = 1 \neq 0 \to$ Diverges

$$\sum_{k=1}^{\infty} \frac{k!}{2^k(k+2)!}$$

Ratio Test

$\boxed{LC \# 8}$

$$r = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\overbrace{(k+1)!}^{(k+1)\,k!}}{\underbrace{2^{k+1}}_{2^k\cdot 2}\,\underbrace{(k+3)!}_{(k+3)(k+2)!}} \cdot \frac{2^k\,(k+2)!}{k!}\right|$$

$$= \lim_{k\to\infty}\left|\frac{k+1}{k+3}\,\frac{1}{2}\right| = \frac{1}{2} < 1$$

Converges (absolutely) by the Ratio Test

DCT

$a_k \geq 0,\ b_k \geq 0,$ $\quad \sum b_k$ converges, $\sum a_k$ converges

$a_k \leq b_k$ $\quad \sum a_k$ diverges, $\sum b_k$ diverges

LCT

$0 < L < \infty$, both converge or diverge

$\lim \frac{a_k}{b_k} = L$ $\quad L = \infty$, $\sum a_k$ diverges if $\sum b_k$ diverges

$L = 0$ $\quad \sum a_k$ converges if $\sum b_k$ converges

Comparison	$\sum_{k=1}^{\infty} a_k$	$0 < a_k \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges	$0 < b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges
Limit Comparison	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \frac{a_k}{b_k} = c$, $0 \leq c < \infty$, $\sum_{k=1}^{\infty} b_k$ converges	$\lim_{k\to\infty} \frac{a_k}{b_k} = c$, $0 < c \leq \infty$, $\sum_{k=1}^{\infty} b_k$ diverges

$$\sum_{k=1}^{\infty} \frac{|\sin(2k)|}{1+2^k}$$

LC #9

$$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$$ convergent Geometric series w/ $r = 1/2$

Converges by DCT

$$\frac{|\sin(2k)|}{1+2^k} \leq \underbrace{\frac{1}{1+2^k} \leq \frac{1}{2^k}}_{2^k \leq 1+2^k}$$

$$\sum_{k=\cancel{1}}^{\infty} \frac{(-1)^{k-1}}{\sqrt{k}-1}$$

2

AST

$\cos(k\pi)$

$\cancel{\cos(k)}$

1 $\quad \frac{1}{\sqrt{k+1}-1} < \frac{1}{\sqrt{k}-1}$

Converges by AST

2 $\quad \lim_{k\to\infty} \frac{1}{\sqrt{k}-1} = 0$

LC #11 & 12

$$\sum_{k=2}^{\infty} \frac{1}{k\sqrt{\ln(k)}}$$

$f(x) = \frac{1}{x\sqrt{\ln(x)}} > 0$ for $x \geq 2$

$f'(x) = -\left(x\sqrt{\ln(x)}\right)^{-2}\left(\sqrt{\ln(x)} - \frac{1}{2\sqrt{\ln(x)}}\right) = -\frac{\left(\sqrt{\ln(x)} - \frac{1}{2\sqrt{\ln(x)}}\right)}{x^2 \ln(x)} < 0$

$u = \ln(x)$

$du = \frac{1}{x}dx$

$\boxed{\text{diverges via Integral Test}}$

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln(x)}}\,dx = \lim_{b\to\infty} \int_{\ln(2)}^{\ln(b)} \frac{1}{\sqrt{u}}\,du = \lim_{b\to\infty} 2\sqrt{u}\Big|_{\ln(2)}^{\ln(b)}$$

$$= \lim_{b\to\infty} \underbrace{2\sqrt{\ln(b)}}_{\infty} - 2\sqrt{\ln(2)} = \infty \Rightarrow \text{series diverges}$$

$$\sum_{k=1}^{\infty} \left(2^{1/k} - 1\right)^k$$

Root test, converge

$$\rho = \lim_{k\to\infty} \left|\left(2^{1/k} - 1\right)^k\right|^{1/k} = \lim_{k\to\infty} \left(2^{1/k} - 1\right) = 1 - 1 = 0 < 1$$

converges

$$\sum_{k=3}^{\infty} \frac{1}{k^{2/5} \ln(k)}$$

$$\sum_{k=1}^{\infty} \frac{8(3k)!}{(k!)^3}$$

$$\sum_{k=1}^{\infty} \sin\left(\frac{9}{k^{12}}\right)$$

Series or Test	**Form of Series**	**Condition for Convergence**	**Condition for Divergence**	**Comments**
Geometric series	$\sum_{k=0}^{\infty} ar^k$, $a \neq 0$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	If $\lvert r\rvert < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$.
Divergence Test	$\sum_{k=1}^{\infty} a_k$	Does not apply	$\lim_{k\to\infty} a_k \neq 0$	Cannot be used to prove convergence.
Integral Test	$\sum_{k=1}^{\infty} a_k$, where $a_k = f(k)$ and f is continuous, positive, and decreasing.	$\int_1^{\infty} f(x)\,dx$ converges.	$\int_1^{\infty} f(x)\,dx$ diverges.	The value of the integral is not the value of the series.
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	Useful for comparison tests.
Ratio Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert < 1$	$\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert > 1$	Inconclusive if $\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert = 1$
Root Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} < 1$	$\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} > 1$	Inconclusive if $\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} = 1$
Comparison Test (DCT)	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$a \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges.	$b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Limit Comparison Test (LCT)	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$, $b_k > 0$	$0 \leq \lim_{k\to\infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges.	$\lim_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Alternating Series Test (AST)	$\sum_{k=1}^{\infty} (-1)^k a_k$, where $a_k > 0$	$\lim_{k\to\infty} a_k$ and $0 < a_{k+1} \leq a_k$	$\lim_{k\to\infty} a_k \neq 0$	Remainder R_n satisfies $\lvert R_n\rvert \leq a_{n+1}$
Absolute Convergence	$\sum_{k=1}^{\infty} a_k$, a_k arbitrary	$\sum_{k=1}^{\infty} \lvert a_k\rvert$ converges.		Applies to arbitrary series