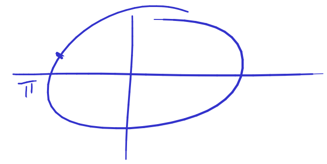
$\sin(\pi) = 0$

$\sin(3) = ?$



11.1: Approximating Functions with Polynomials

A *power series* is an infinite series of the form

$$\sum_{k=0}^{\infty} c_k (x - \overset{\text{center}}{a})^k = \underbrace{\overset{(x-a)^0}{c_0} + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n}_{n\text{th-degree polynomial}} + c_{n-1}(x-a)^{n-1} + \dots,$$

Example. The tangent line of a function $f(x)$ at $x = a$ is a linear function $p_1(x)$ that can approximate $f(x)$ for values of x 'close' to a:

$p_1(a) = f(a)$

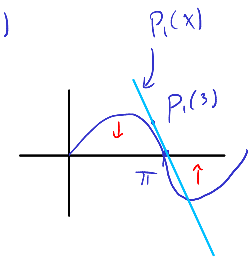
$p_1'(a) = f'(a)$

$$p_1(x) = f(a) + f'(a)(x - a)$$

$f(x) = \sin(x)$ $\quad f(\pi) = 0$

$f'(x) = \cos(x)$ $\quad f'(\pi) = -1$

$p_1(x) = 0 - 1(x - \pi)$



Find a quadratic function $p_2(x)$ that can approximate $f(x)$ near $x = a$,

$p_2(x) = c_0 + c_1(x-a) + c_2(x-a)^2 = f(a) + f'(a)(x-a) + c_2(x-a)^2$

$p_2(a) = f(a)$ $\qquad c_0 = \frac{f(a)}{0!}$

$p_2'(x) = f'(a) + 2c_2(x-a)$ $\qquad p_2'(a) = f'(a)$ $\qquad c_1 = \frac{f'(a)}{1!}$

$p_2''(x) = 2c_2$ $\qquad p_2''(a) = 2c_2 \overset{\text{want}}{=} f''(a) \implies$ $\qquad c_2 = \frac{f''(a)}{2!}$

Find a cubic function $p_3(x)$ that can approximate $f(x)$ near $x = a$,

$p_3(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + c_3(x-a)^3$

$3c_3(x-a)^2$

$3 \cdot 2\, c_3(x-a)$

$3 \cdot 2 \cdot 1\, c_3$

$p_3'''(a) = 3!\, c_3 \overset{\text{want}}{=} f'''(a)$

$\implies c_3 = \frac{f'''(a)}{3!}$

Find an nth degree polynomial $p_n(x)$ that can approximate $f(x)$ near $x = a$.

$$p_n(x) = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Definition. (Taylor Polynomials)
Let f be a function with $f', f'', \ldots,$ and $f^{(n)}$ defined at a. The n**th-order Taylor polynomial** for f with its **center** at a, denoted p_n, has the property that it matches f in value, slope, and all derivatives up to the nth derivative at a; that is,

$$\boxed{p_n(a) = f(a),\ p_n'(a) = f'(a), \ldots,\ \text{and } p_n^{(n)}(a) = f^{(n)}(a).}$$

The nth-order Taylor polynomial centered at a is

$$p_n(x) = \underline{f(a)} + \frac{\underline{f'(a)}}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

More compactly, $p_n(x) = \displaystyle\sum_{k=0}^{\infty} c_k (x-a)^k$, where the **coefficients** are

$$c_k = \frac{f^{(k)}(a)}{k!}, \qquad \text{for } k = 0, 1, 2, \ldots, n.$$

Example (LC 26.1). Suppose $f(4) = 3$, $f'(4) = -1$, $f''(4) = 6$, and $f^{(3)}(4) = 16$. Find the third-order Taylor polynomial $p_3(x)$ for f centered at $a = 4$.

$$p_3(x) = f(4) + \frac{f'(4)}{1!}(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \frac{f^{(3)}(4)}{3!}(x-4)^3$$

$$= 3 - (x-4) + 3(x-4)^2 + \frac{8}{3}(x-4)^3$$

$f(4.1) \approx p_3(4.1)$

$\frac{16}{6}$

$3! = 3 \cdot 2 \cdot 1 = 6$

Example (LC 26.2)**.** For the following functions, find $p_2(x)$, the 2nd degree Taylor polynomial, centered at $a = 0$.

$$P_2(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$y = \sqrt{1+2x}$

$f(x) = (1+2x)^{1/2}$ $\qquad f(0) = 1$

$f'(x) = \frac{1}{2}(1+2x)^{-1/2}(2) = \frac{1}{\sqrt{1+2x}}$ $\qquad f'(0) = 1$

$f''(x) = -\frac{1}{2}(1+2x)^{-3/2}(2) = \frac{-1}{(1+2x)^{3/2}}$ $\qquad f''(0) = -1$

$$\longrightarrow P_2(x) = 1 + \frac{1}{1!}(x-0) + \frac{-1}{2}(x-0)^2$$

$$\sqrt{2} = \sqrt{1+2(\tfrac{1}{2})} = f(\tfrac{1}{2}) \approx p_2(\tfrac{1}{2})$$

$$= 1 + x - \frac{x^2}{2}$$

$y = \dfrac{1}{\sqrt{1+2x}}$

$$P_2(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$f(x) = (1+2x)^{-1/2} = \frac{1}{\sqrt{1+2x}}$ $\qquad f(0) = 1$

$f'(x) = -(1+2x)^{-3/2} = -\frac{1}{(1+2x)^{3/2}}$ $\qquad f'(0) = -1$

$f''(x) = 3(1+2x)^{-5/2} = \frac{3}{(1+2x)^{5/2}}$ $\qquad f''(0) = 3$

$$\Longrightarrow p_2(x) = 1 - \frac{1}{1!}(x-0) + \frac{3}{2!}(x-0)^2$$

$$= 1 - x + \frac{3}{2}x^2$$

$$P_2(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$y = \frac{1}{1+2x} = (1+2x)^{-1}$$

$f(0) = 1$

$$f'(x) = -(1+2x)^{-2}(2) = \frac{-2}{(1+2x)^2}$$

$f'(0) = -2$

$$f''(x) = 2(1+2x)^{-3}(4) = \frac{8}{(1+2x)^3}$$

$f''(0) = 8$

$$\boxed{P_2(x) = 1 - 2x + 4x^2}$$

$$P_2(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$y = \frac{1}{(1+2x)^3} = (1+2x)^{-3}$$

$f(0) = 1$

$$\boxed{P_2(x) = 1 - 6x + 24x^2}$$

$$f'(x) = -3(1+2x)^{-4}(2)$$

$f'(0) = -6$

$$f''(x) = 24(1+2x)^{-5}(2)$$

$f''(0) = 48$

$$P_2(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$y = e^{2x}$

$f'(x) = 2e^{2x}$

$f''(x) = 4e^{2x}$

$f(0) = 1$

$f'(0) = 2$

$f''(0) = 4$

$$\boxed{P_2(x) = 1 + 2x + 2x^2}$$

$$P_2(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$y = e^{-2x}$

$f'(x) = -2e^{-2x}$

$f''(x) = 4e^{-2x}$

$f(0) = 1$

$f'(0) = -2$

$f''(0) = 4$

$$\boxed{P_2(x) = 1 - 2x + 2x^2}$$

Example (LC 26.3)**.** Find the Taylor polynomial $p_3(x)$ centered at $a = \frac{\pi}{4}$ for $f(x) = \sin(x)$.

$f(x) = \sin(x) \qquad f(\pi/4) = \sqrt{2}/2$

$f'(x) = \cos(x) \qquad f'(\pi/4) = \sqrt{2}/2$

$f''(x) = -\sin(x) \qquad f''(\pi/4) = -\sqrt{2}/2$

$f^{(3)}(x) = -\cos(x) \qquad f'''(\pi/4) = -\sqrt{2}/2$

$$p_3(x) = f(\pi/4) + \frac{f'(\frac{\pi}{4})}{1!}(x-\pi/4) + \frac{f''(\pi/4)}{2!}(x-\pi/4)^2 + \frac{f^{(3)}(\pi/4)}{3!}(x-\pi/4)^3$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3$$

$-\frac{\sqrt{2}}{2}\cdot\frac{1}{6}$

Example (LC 26.4). Use the 4th degree Taylor polynomial of $y = \ln(x)$ centered at $a = 1$ to approximate $\ln(1.1)$.

$f^{(0)}(x)$

$$f(x) = \ln(x) \qquad f(1) = 0$$

$$f'(x) = 1/x \qquad f'(1) = 1 \qquad (-1)^0 0!$$

$$f''(x) = -1/x^2 \qquad f''(1) = -1 \qquad (-1)^1 1!$$

$$f^{(3)}(x) = 2/x^3 \qquad f^{(3)}(1) = 2 \qquad (-1)^2 2!$$

$$f^{(4)}(x) = \frac{-6}{x^4} \qquad f^{(4)}(1) = -6 \qquad (-1)^3 3!$$

$$P_4(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4$$

$$= 0 + \frac{0!}{1!}(x-1) - \frac{1!}{2!}(x-1)^2 + \frac{2!}{3!}(x-1)^3 - \frac{3!}{4!}(x-1)^4$$

$$= 0 + 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

$$= \boxed{(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}}$$

$$\ln(1.1) = f(1.1) \approx P_4(1.1) = 1\left(\frac{1}{10}\right) - \frac{1}{2}\left(\frac{1}{100}\right) + \frac{1}{3}\left(\frac{1}{1000}\right) - \frac{1}{4}\left(\frac{1}{10000}\right)$$

$$\approx \boxed{0.095308}$$

$$\ln(1.1) \approx 0.09531\ldots$$

Definition. (Remainder in a Taylor Polynomial)
Let p_n be the Taylor polynomial of order n for f. The **remainder** in using p_n to approximate f at the point x is

$$R_n(x) = f(x) - p_n(x).$$

true function

approximate polynomial

Theorem 11.1: Taylor's Theorem (Remainder Theorem)
Let f have continuous derivatives up to $f^{(n+1)}$ on an open interval I containing a. For all x in I,

$$f(x) = p_n(x) + R_n(x),$$

$P_n(x) = f(a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

where p_n is the nth-order Taylor polynomial for f centered at a and the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

c is unknown

for some point c between x and a.

$x \leq c \leq a$ or $a \leq c \leq x$

Theorem 11.2: Estimate of the Remainder
Let n be a fixed positive integer. Suppose there exists a number M such that $\boxed{\left|f^{(n+1)}(c)\right| \leq M}$, for all c between a and x inclusive. The remainder in the nth-order Taylor polynomial for f centered at a satisfies

$$|R_n(x)| = |f(x) - p_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}.$$

$|f^{(n+1)}(x)| = |\cos(x)| \leq 1$

M

Example (LC 27.1-27.2)**.** The third-order Taylor polynomial centered at $a = 1$ for $f(x) = x\ln(x)$ is

$$p_3(x) = (x-1) + \frac{(x-1)^2}{2} - \frac{(x-1)^3}{6}.$$

Find the smallest number M such that $\left|f^{(4)}(x)\right| \le M$ for $\frac{1}{2} \le x \le \frac{3}{2}$.

$f(x) = x\ln(x)$

$f'(x) = \ln(x) + \frac{x}{x} = \ln(x) + 1$

$f''(x) = \frac{1}{x}$

$f^{(3)}(x) = -\frac{1}{x^2}$

$f^{(4)}(x) = \frac{2}{x^3}$

$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

$\left|f^{(4)}(x)\right| = \left|\frac{2}{x^3}\right| \le f^{(4)}\left(\frac{1}{2}\right) = 16 =: M$

x	$f^{(4)}(x)$
$1/2$	16
$3/2$	$16/27$

$f^{(5)}(x) = \frac{-6}{x^4} \leftarrow$ neg $\Rightarrow f^{(4)}(x)$ dec $\Rightarrow f^{(4)}(1/2) > f^{(4)}(3/2)$

$\rightarrow$ No max inside $\left[\frac{1}{2}, \frac{3}{2}\right]$

Compute the upper bound for $|R_3(x)|$.

over $[1/2, 3/2]$

$x \in \left[\frac{1}{2}, \frac{3}{2}\right]$

$a = 1$

$$|R_3(x)| \le \frac{16}{4!}|x-1|^4$$

$$\le \frac{16}{24}\left|\frac{3}{2}-1\right|^4$$

could use $x = 1/2$

$$= \frac{\cancel{16}}{24}\cdot\left|\frac{1}{\cancel{2}}\right|^{\cancel{4}}$$

$$= \boxed{\frac{1}{24}}$$

Example (LC 27.3-27.5)**.** Consider $f(x) = e^x$.

Find the Taylor polynomial $p_4(x)$ centered at $a = 0$.

$f(x) = e^x$ $\quad f(0) = 1$

$f'(x) = e^x$ $\quad f'(0) = 1$

$f''(x) = e^x$ $\quad f''(0) = 1$

$f^{(3)}(x) = e^x$ $\quad f^{(3)}(0) = 1$

$f^{(4)}(x) = e^x$ $\quad f^{(4)}(0) = 1$

$$p_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!}(x-0)^k$$

$$= \frac{1}{0!}(x)^0 + \frac{1}{1!}(x)^1 + \frac{1}{2!}(x)^2 + \frac{1}{3!}(x)^3 + \frac{1}{4!}(x)^4$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

6 $\quad$ 24

What is the smallest *integer* M such that $\left|f^{(5)}(x)\right| \leq M$ for $0 \leq x \leq 1/4$?

$f^{(5)}(x) = e^x$

Inc

Want $\left|f^{(5)}(x)\right| \leq M$ on $0 \leq x \leq 1/4$

$$\left|f^{(5)}(x)\right| = |e^x| \leq e^{1/4} \approx 1.28$$

$$\leq 2$$

M

$e^0 = 1$

$e^1 \approx 2.718\ldots$

Compute the upper bound for $|R_4(x)|$ when $p_4(x)$ is used to compute $e^{1/4}$.

$$|R_4(1/4)| \leq M \frac{|1/4 - 0|^5}{5!} = \frac{2\,(0.25)^5}{120} = \frac{(0.25)^5}{60}$$

Example (LC 27.6-27.7)**.** We want to approximate $\sin(0.2)$ with an absolute error no greater than 10^{-3} by using a nth degree Taylor polynomial for $f(x) = \sin(x)$ centered at $a = 0$. We want to determine the minimum order of the Taylor polynomial that is required to meet this condition.

What is the smallest *integer* number M that bounds $f^{(n+1)}(x)$ on $0 \leq x \leq 0.2$?

$$f(x) = \sin(x)$$
$$f'(x) = \cos(x)$$
$$f''(x) = -\sin(x)$$
$$f'''(x) = -\cos(x)$$
$$\vdots$$

$$\left|f^{(n+1)}(x)\right| \leq \boxed{1 =: M}$$

Apply Taylor's Estimate of the Remainder Theorem to find the minimum value of n such that $|R_n(x)| \leq \frac{1}{10^3}$.

$[0, 0.2]$

$$|R_n(x)| \leq \underset{\substack{\uparrow \\ M}}{1} \frac{|x-0|^{n+1}}{(n+1)!}$$

$$\leq \frac{(0.2)^{n+1}}{(n+1)!}$$

$$n+1 = 4 \implies \frac{(0.2)^4}{4!} \approx 0.000067\ldots$$

$$\implies n = 3$$

	A	B	C	D
1	0	=(0.2)^(A1+1)/FACT(A1+1)		
2	1	0.020000		
3	2	0.001333		
4	3	0.000067		0.001
5	4	0.000003		0.001
6	5	0.000000		