10.4: The Divergence and Integral Tests

Theorem 10.9: Divergence Test
If $\sum a_k$ converges, then $\lim_{k\to\infty} a_k = 0$. Equivalently, if $\lim_{k\to\infty} a_k \neq 0$, then the series diverges.

Example. If $\lim_{k\to\infty} a_k = 1$, what can we conclude about $\sum_{k=1}^{\infty} a_k$?

Example. If $\sum_{k=1}^{\infty} a_k = 42$, what can we conclude about $\lim_{k\to\infty} a_k$?

Example. If $\lim_{k\to\infty} a_k = 0$, what can we conclude about $\sum_{k=1}^{\infty} a_k$?

Example. Determine which of the following series diverge by the divergence test.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}}$$

$$\sum_{k=1}^{\infty} \frac{k^3+100}{3k^3+k+1}$$

$$\sum_{k=1}^{\infty} \frac{e^k}{k^2}$$

Table 1 Series Convergence				
Scenario	Sequence of Terms $\{a_1, a_2, a_3, \ldots\}$	Sequence of Partial Sums $\{s_1, s_2, s_3, \ldots\}$	Series $\sum_{n=1}^{\infty} a_n$	Possible or Impossible?
A	Converges	Diverges	Diverges	
B	Converges	Diverges	Converges	
C	Converges	Converges	Diverges	
D	Converges	Converges	Converges	
E	Diverges	Converges	Diverges	
F	Diverges	Converges	Converges	
G	Diverges	Diverges	Diverges	
H	Diverges	Diverges	Converges	

Theorem 10.10: Harmonic Series
The harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$ diverges—even though the terms of the series approach zero.

Theorem 10.11: Integral Test
Suppose f is a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Then
$$\sum_{k=1}^{\infty} a_k \text{ and } \int_1^{\infty} f(x)\, dx$$
either both converge or both diverge. In the case of convergence, the value of the integral is *not* equal to the value of the series.

Example. Which of the following series below satisfy all the conditions to use the Integral Test?

$$\sum_{k=1}^{\infty} \arctan(k)$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{e^k}$$

Example. Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

Use the integral test to show that the Harmonic Series diverges. For what values of p does this series converge?

Theorem 10.12: Convergence of the p-series

The p**-series** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

Example. Determine if the following p-series converge or diverge.

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$\sum_{k=1}^{\infty} k^{-1/3}$$

$$\sum_{k=1}^{\infty} \frac{k^2}{k^{\pi}}$$

$$\sum_{k=1}^{\infty} \frac{2}{k}$$

$$\sum_{k=1}^{\infty} \frac{-3}{\sqrt[3]{k^4}}$$

$$\sum_{k=1}^{\infty} \frac{k^3+1}{k^5}$$

Example. Apply the Integral Test to determine if the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}}$ converges or diverges.

Theorem 10.13: Estimating Series with Positive Terms
Let f be a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Let $S = \displaystyle\sum_{k=1}^{\infty} a_k$ be a convergent series and let $S_n = \displaystyle\sum_{k=1}^{n} a_k$ be the sum of the first n terms of the series. The remainder $R_n = S - S_n$ satisfies

$$R_n < \int_n^{\infty} f(x)\,dx.$$

Furthermore, the exact value of the series is bounded as follows:

$$L_n = S_n + \int_{n+1}^{\infty} f(x)\,dx < \sum_{k=1}^{\infty} a_k < S_n + \int_n^{\infty} f(x)\,dx = U_n.$$

Example. How many terms of the convergent p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ must be summed to obtain an approximation that is within 10^{-3} of the exact value of the series?