

## 15.6: Tangent Planes and Linear Approximation

### **Definition. (Equation of the Tangent Plane for $F(x, y, z) = 0$ )**

Let  $F$  be differentiable at the point  $P_0(a, b, c)$  with  $\nabla F(a, b, c) \neq \mathbf{0}$ . The plane tangent to the surface  $F(x, y, z) = 0$  at  $P_0$ , called the **tangent plane**, is the plane passing through  $P_0$  orthogonal to  $\nabla F(a, b, c)$ . An equation of the tangent plane is

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

**Example.** Consider the ellipsoid

$$F(x, y, z) = \frac{x^2}{9} + \frac{y^2}{25} + z^2 - 1 = 0.$$

a) Find an equation of the plane tangent to the ellipsoid at  $(0, 4, \frac{3}{5})$ .

b) At what points on the ellipsoid is the tangent plane horizontal?

Surfaces of the form  $z = f(x, y)$  are a special case of  $F(x, y, z) = 0$ : Define  $F(x, y, z) = z - f(x, y) = 0$ , then

$$\nabla F(a, b, f(a, b)) = \langle -f_x(a, b), -f_y(a, b), 1 \rangle$$

so the tangent plane is

$$-f_x(a, b)(x - a) - f_y(a, b)(y - b) + 1(z - f(a, b)) = 0$$

**Tangent Plane for  $z = f(x, y)$**

Let  $f$  be differentiable at the point  $(a, b)$ . An equation of the plane tangent to the surface  $z = f(x, y)$  at the point  $(a, b, f(a, b))$  is

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

**Example.** Find an equation of the plane tangent to  $f(x, y) = 4e^{xy^2}$  at  $(3, 0, 4)$  and  $(0, 2, 4)$ .

**Example.** Find an equation of the plane tangent to  $f(x, y) = \tan^{-1}(xy)$  at  $(\sqrt{3}, 1, \frac{\pi}{3})$  and  $(\frac{\sqrt{3}}{3}, 1, \frac{\pi}{6})$ .

**Definition. (Linear Approximation)**

Let  $f$  be differentiable at  $(a, b)$ . The linear approximation to the surface  $z = f(x, y)$  at the point  $(a, b, f(a, b))$  is the tangent plane at that point, given by the equation

$$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b),$$

For a function of three variables, the linear approximation to  $w = f(x, y, z)$  at the point  $(a, b, c, f(a, b, c))$  is given by

$$L(x, y, z) = f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) + f(a, b, c).$$

**Example.** Let  $f(x, y) = \frac{5}{x^2 + y^2}$ . Find the linear approximation to the function at the point  $(-1, 2, 1)$ . Use this to approximate  $f(-1.05, 2.1)$ .

**Example.** Let  $f(x, y) = \sqrt{x^2 + y^2}$ . Find the linear approximation to the function at the point  $(-8, 15, 17)$ . Use this to approximate  $f(-7.91, 14.96)$ .

**Definition. (The differential  $dz$ )**

Let  $f$  be differentiable at the point  $(x, y)$ . The change in  $z = f(x, y)$  as the independent variables change from  $(x, y)$  to  $(x + dx, y + dy)$  is denoted  $\Delta z$  and is approximated by the differential  $dz$ :

$$\Delta z \approx dz = f_x(x, y) dx + f_y(x, y) dy.$$

**Example.** Let  $z = f(x, y) = \frac{5}{x^2 + y^2}$ . Approximate the change in  $z$  when the variables change from  $(-1, 2)$  to  $(-0.93, 1.94)$ .

**Example.** A company manufactures cylindrical aluminum tubes to rigid specifications. The tubes are designed to have an outside radius of  $r = 10 \text{ cm}$ , a height of  $h = 50 \text{ cm}$ , and a thickness of  $t = 0.1 \text{ cm}$ . The manufacturing process produces tubes with a maximum error of  $\pm 0.05 \text{ cm}$  in the radius and height, and a maximum error of  $\pm 0.0005 \text{ cm}$  in the thickness. The volume of the cylindrical tube is  $V(r, h, t) = \pi ht(2r - t)$ . Use differentials to estimate the maximum error in the volume of a tube.

