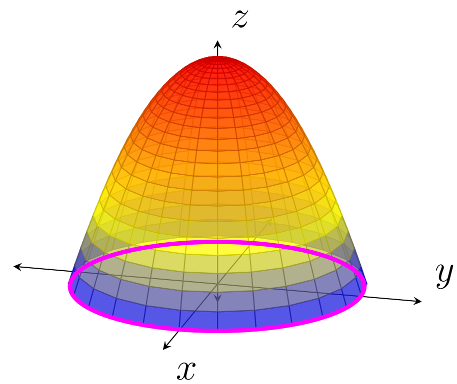
16.3: Double Integrals in Polar Coordinates

Suppose we wish to find the volume bounded by the curve $f(x, y) = 9 - x^2 - y^2$ and the xy-plane. The region of integration would be

$$R = \left\{(x, y) : -3 \le x \le 3, -\sqrt{9 - x^2} \le y \le \sqrt{9 - x^2}\right\}$$



$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 9 - x^2 - y^2 \, dy \, dx$$

$$= \int_{-3}^{3} 9y - x^2 y - \frac{y^3}{3} \Bigg|_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} dx$$

$$= \int_{-3}^{3} 2(9-x^2)\sqrt{9-x^2} - \frac{2(\sqrt{9-x^2})^3}{3} \, dx = \text{ew}$$

Alternatively, we can use polar coordinates where $x = r\cos(\theta)$ and $y = r\sin(\theta)$. The associated region R is called a **polar rectangle**.

$$x^2 + y^2 = r^2$$

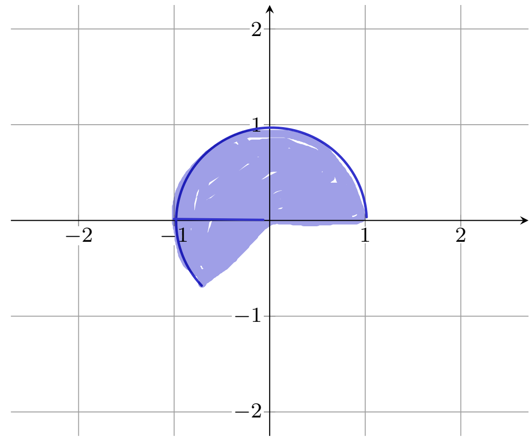
Theorem 16.3: Change of Variables for Double Integrals over Polar Rectangle Regions
Let f be continuous on the region R in the xy-plane expressed in polar coordinates a s$R = \{(r, \theta) : 0 \le a \le r \le b, \alpha \le \theta \le \beta\}$, where $\beta - \alpha = 2\pi$. Then f is integrable over R, and the double integral of f over R is

$$\iint_R f(x, y) \, dA = \int_{\alpha}^{\beta} \int_{a}^{b} f(r\cos\theta, r\sin\theta) \, r \, dr \, d\theta.$$

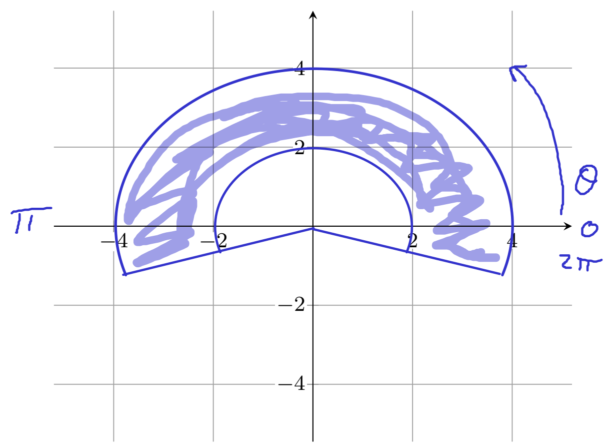
Note: When we convert to polar coordinates, there is an extra factor of r. This is due to the area of the circular segment being $\frac{1}{2}r^2\theta$ (Section 16.7 will elaborate on this).

Example. Graph the following regions:

$$R = \left\{ (r, \theta) : 0 \leq r \leq 1,\ 0 \leq \theta \leq \frac{5\pi}{4} \right\}$$



$$R = \left\{ (r, \theta) : 2 \leq r \leq 4,\ -\frac{\pi}{6} \leq \theta \leq \frac{7\pi}{6} \right\}$$



$x = r\cos(\theta)$ $\quad$ $y = r\sin(\theta)$

Example. Consider the paraboloid given earlier: Find the volume of the solid bounded above by $z = 9 - x^2 - y^2$ and below by the xy-plane.

$$\iint_R f(x,y)\,dA = \int_0^{2\pi}\int_0^3 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta \qquad R = \{(r,\theta) : 0 \le r \le 3,\ 0 \le \theta \le 2\pi\}$$

$$z = f(x,y) = f(r\cos(\theta), r\sin(\theta)) = 9 - r^2\cos^2(\theta) - r^2\sin^2(\theta) = 9 - r^2(\cos^2(\theta) + \sin^2(\theta))$$
$$= 9 - r^2$$

Solve $z = 0 \Rightarrow 0 = 9 - r^2 \Rightarrow r = \pm 3$

$$= \int_0^{2\pi}\int_0^3 (9 - r^2)\,r\,dr\,d\theta$$

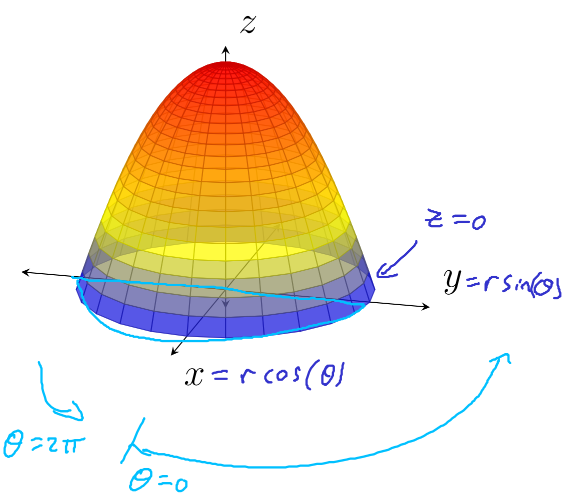
$$= \int_0^{2\pi}\int_0^3 9r - r^3\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{9}{2}r^2 - \frac{1}{4}r^4 \Big|_{r=0}^{r=3}\,d\theta$$

$$= \int_0^{2\pi} \frac{81}{2} - \frac{81}{4}\,d\theta$$

$$= \int_0^{2\pi} \frac{81}{4}\,d\theta$$

$$= \frac{81}{4}\theta\Big|_0^{2\pi} = \boxed{\frac{81\pi}{2}}$$



under f(x,y)
above xy-plane
x positive

$-\frac{\pi}{2} \le \theta \le \pi/2$

$f(x,y) = \left(2 - \sqrt{x^2+y^2}\right) - \left(x^2+y^2\right)$

Example. Find the area of the solid bounded below by the paraboloid $z = \underbrace{x^2 + y^2}_{r^2}$ and bounded above by the ~~cone~~ $z = \underbrace{2 - \sqrt{x^2 + y^2}}_{2-r}$.

$\sqrt{x^2+y^2} = z-2$

$x^2+y^2 = (z-2)^2$

$2 - \sqrt{x^2+y^2} = x^2+y^2$

$z = (z-2)^2$

$0 = z^2 - 5z + 4$

$= (z-4)(z-1) \rightarrow z=4, \; z=1$

$z = r^2 \rightarrow 0 \le r \le 1$

$$\iint_R f(x,y)\,dA = \iint f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \Big((2-r)-(r^2)\Big) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 -r^3 - r^2 + 2r\,dr\,d\theta$$

$$= \int_0^{2\pi} -\frac{r^4}{4} - \frac{r^3}{3} + r^2 \Big|_{r=0}^{r=1} d\theta$$

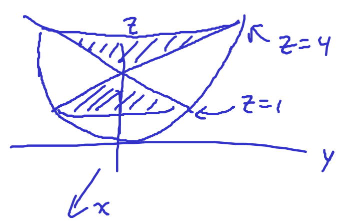
$-\frac{1}{4} - \frac{1}{3} + 1$

$\frac{-3-4+12}{12}$

$$= \int_0^{2\pi} \frac{5}{12}\,d\theta$$

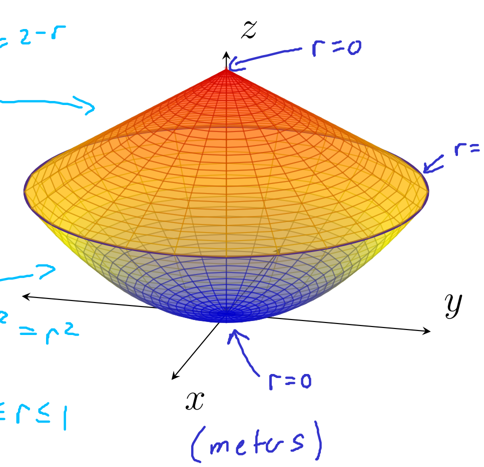
$$= \frac{5}{12}\theta \Big|_{\theta=0}^{\theta=2\pi}$$

$$= \frac{5\pi}{6} \; m^3$$



$z = 2 - \sqrt{x^2+y^2} = 2-r$

$1 \le z \le 2$



$z = x^2+y^2 = r^2$

$0 \le z \le 1 \rightarrow 0 \le r \le 1$

$z = 2-r$

$z-2 = -r$

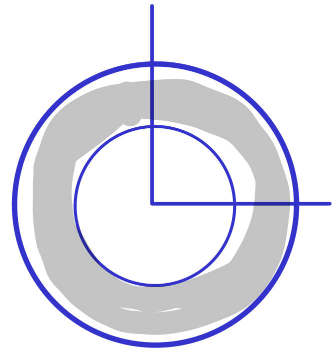
$-1 \le z-2 \le 0$

$\rightarrow -1 \le -r \le 0$

$\rightarrow 0 \le r \le 1$

$$z = (r\cos(\theta))(r\sin(\theta)) + 10$$

Example. Find the volume of the region beneath the surface $z = xy + 10$ and above the annular region $R = \{(r, \theta) : 2 \le r \le 4,\ 0 \le \theta \le 2\pi\}$.



$$\iint_R f(x,y)\,dA = \int_0^{2\pi}\int_2^4 \left(r^2\cos(\theta)\sin(\theta) + 10\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_2^4 \frac{r^3}{2}\sin(2\theta) + 10r\,dr\,d\theta$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

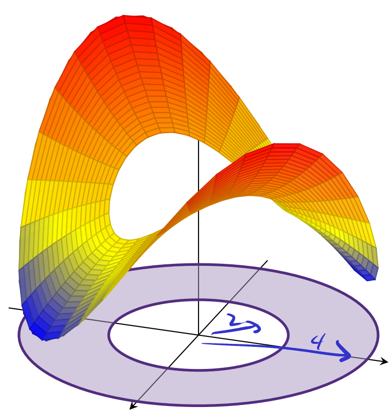
$$= \int_0^{2\pi} \frac{r^4}{8}\sin(2\theta) + 5r^2 \Big|_{r=2}^{r=4}\,d\theta$$

$$= \int_0^{2\pi} (32\sin(2\theta) + 80) - (2\sin(2\theta) + 20)\,d\theta$$

$$= 15\cos(2\theta) + 60\theta \Big|_{\theta=0}^{\theta=2\pi}$$

$$= (15\cos(4\pi) + 120\pi) - (15\cos(0) + 0)$$

$$= 120\pi$$



LC#1 a=120

Theorem 16.4: Change of Variables for Double Integrals over More General Polar Regions
Let f be continuous on the region R in the xy-plane expressed in polar coordinates as

$$R = \{(r, \theta) : 0 \leq g(\theta) \leq r \leq h(\theta),\ \alpha \leq \theta \leq \beta\},$$

where $0 < \beta - \alpha \leq 2\pi$. Then

$$\iint\limits_R f(x, y)\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

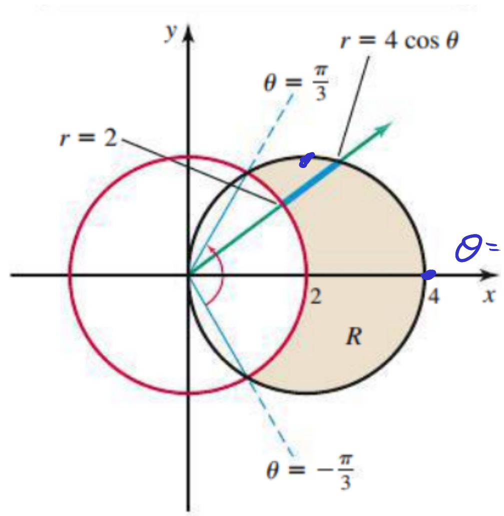
Area of Polar Regions
The area of the polar region $R = \{(r, \theta) : 0 \leq g(\theta) \leq r \leq h(\theta),\ \alpha \leq \theta \leq \beta\}$, where $0 < \beta - \alpha \leq 2\pi$, is

$$A = \iint\limits_R dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} r\, dr\, d\theta.$$

Example. Write an iterated integral in polar coordinates for $\iint_R g(r,\theta)\,dA$ for the region outside the circle $r = 2$ and inside the circle $r = 4\cos(\theta)$.

$(x, y) = (r\cos\theta, r\sin\theta)$

θ	(x,y)
0	$(4,0)$
$\pi/4$	$(2,2)$



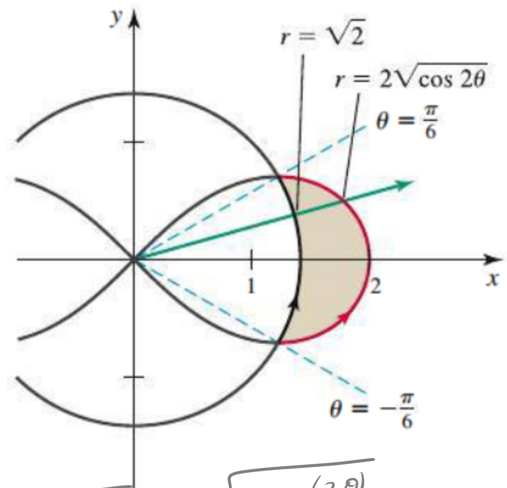
$\theta = 0$

$2 = 4\cos\theta$

$\frac{1}{2} = \cos(\theta) \to \theta = \pm\frac{\pi}{3}$

$$\iint_R g(r,\theta)\,dA = \int_{-\pi/3}^{\pi/3}\int_2^{4\cos\theta} g(r,\theta)\, r\,dr\,d\theta$$

Example. Compute the area of the region in the first and fourth quadrants outside the circle $r = \sqrt{2}$ and inside the lemniscate $r^2 = 4\cos(2\theta)$.



$$\text{area} = \iint_R dA = \int_{-\pi/6}^{\pi/6} \int_{\sqrt{2}}^{2\sqrt{\cos(2\theta)}} r \, dr \, d\theta$$

$$= \int_{-\pi/6}^{\pi/6} \frac{r^2}{2} \Big|_{r=\sqrt{2}}^{r=2\sqrt{\cos(2\theta)}} d\theta$$

$$= \int_{-\pi/6}^{\pi/6} 2\cos(2\theta) - 1 \, d\theta$$

$$= \sin(2\theta) - \theta \Big|_{-\pi/6}^{\pi/6}$$

$$= \left(\sin\left(\pi/3\right) - \pi/6\right) - \left(\sin\left(-\pi/3\right) + \pi/6\right)$$

$$= \frac{\sqrt{3}}{2} - \frac{\pi}{6} - \left(-\frac{\sqrt{3}}{2}\right) - \pi/6$$

$$= \sqrt{3} - \pi/3$$

$$\sqrt{2} = 2\sqrt{\cos(2\theta)}$$

$$\frac{\sqrt{2}}{2} = \sqrt{\cos(2\theta)}$$

$$\frac{1}{2} = \cos(2\theta) \rightarrow 2\theta = \pm\frac{\pi}{3}$$

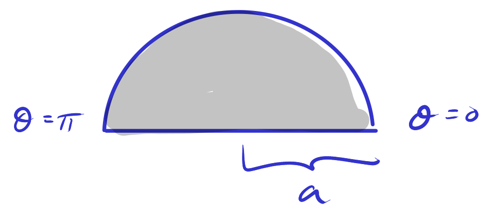
$$\theta = \pm\frac{\pi}{6}$$

Lc # 2

a = 3

Example. Find the average value of the y-coordinates of the points in the semicircular disk of radius a given by $R = \{(r, \theta) : 0 \le r \le a,\ 0 \le \theta \le \pi\}$.

$$\bar{f} = \frac{1}{\text{area } R} \iint_R f(x,y)\, dA$$



$$f(x,y) = y = r\sin(\theta)$$

$$\bar{y} = \frac{1}{\pi a^2/2} \int_0^{\pi}\int_0^{a} (r\sin\theta) r\, dr\, d\theta$$

$$= \frac{2}{\pi a^2} \int_0^{\pi} \frac{r^3}{3}\sin\theta \Big|_{r=0}^{r=a} d\theta$$

$$= \frac{2}{\pi a^2} \int_0^{\pi} \frac{a^3}{3}\sin\theta\, d\theta$$

$$= \frac{-2a}{3\pi} \cos\theta \Big|_{\theta=0}^{\theta=\pi}$$

$$= \frac{-2a}{3\pi}\left[-1 - 1\right] = \frac{4}{3}\frac{a}{\pi} \approx 0.42$$

LC #3

$c = 4/3$

$$\iint_R f(x,y)\, dA = \iint f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$