8.2: Integration by Parts

Integraton by Parts

Suppose u and v are differentiable functions. Then

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

A good mnemonic is ILATE.

$$\int \frac{d}{dx}[u\,v]\,dx = \int \frac{du}{dx}\,v\,dx + \int u\,\frac{dv}{dx}\,dx \longrightarrow uv = \int v\,du + \int u\,dv$$

I nverse trig
L ogarithmic
A lgebraic
T rig
E xponential

precedence of taking the derivative

Example. Evaluate $\int \overset{\text{algebraic}}{x} \overset{\text{expo}}{e^{-\frac{x}{2}}}\, dx$.

Inverse trig
Logarithmic
→ Algebraic
Trig
→ Exponential

$$\int \boxed{u}\,\boxed{dv} = uv - \int v\, du$$

$\boxed{u =}\ x \qquad v = -2e^{-x/2}$

$du = dx \qquad \boxed{dv =}\ e^{-x/2}\, dx$

$$\int x e^{-x/2}\, dx = -2xe^{-x/2} - \int -2e^{-x/2}\, dx$$

$$= -2xe^{-x/2} + 2\int e^{-x/2}\, dx$$

$$= -2xe^{-x/2} - 4e^{-x/2} + C$$

$$\int u\,dv = uv - \int v\,du$$

Integration by Parts for Definite Integrals
Let u and v be differentiable. Then

$$\int_a^b u(x)\underbrace{v'(x)\,dx}_{dv} = u(x)v(x)\Big|_a^b - \int_a^b v(x)\underbrace{u'(x)\,dx}_{du}$$

Example. Find the area of the region between the x-axis and $f(x) = \dfrac{\ln(x)}{x^2}$ on $[1, e]$.

$$\int_1^e \frac{\ln(x)}{x^2}\,dx$$

$u = \ln(x)$ $\quad v = -x^{-1}$

$du = \frac{1}{x}\,dx$ $\quad dv = x^{-2}\,dx$

I nverse trig
→ L ogarithmic
→ A lgebraic
T rig
E xponential

$$= \left[-\frac{\ln(x)}{x}\right]_1^e - \int_1^e -\frac{1}{x}(x^{-1})\,dx$$

$$= \left[-\frac{1}{e} - \left(-\frac{0}{1}\right)\right] + \int_1^e x^{-2}\,dx$$

$$= -\frac{1}{e} - \left[x^{-1}\right]_1^e$$

$$= -\frac{1}{e} - \left(\frac{1}{e} - 1\right) = \boxed{1 - 2/e} = \boxed{1 - 2e^{-1}}$$

Example. Evaluate $\int x^2 \cos(2x)\,dx$.

algebraic trig

$$\int u\,dv = uv - \int v\,du$$

$u = x^2$ $\quad v = \frac{\sin(2x)}{2}$

$du = 2x\,dx$ $\quad dv = \cos(2x)\,dx$

I nverse trig
L ogarithmic
A lgebraic
T rig
E xponential

$$= \frac{x^2 \sin(2x)}{2} - \int \frac{2x \sin(2x)}{2}\,dx$$

$u = x$ $\quad v = \frac{-\cos(2x)}{2}$

$du = dx$ $\quad dv = \sin(2x)\,dx$

$$= \frac{x^2 \sin(2x)}{2} - \left[-\frac{x\cos(2x)}{2} - \int -\frac{\cos(2x)}{2}\,dx \right]$$

$$= \frac{x^2 \sin(2x)}{2} + \frac{x\cos(2x)}{2} - \int \frac{\cos(2x)}{2}\,dx$$

$$= \boxed{\frac{x^2 \sin(2x)}{2} + \frac{x\cos(2x)}{2} - \frac{\sin(2x)}{4} + C}$$

$\int u\,dv = uv - \int v\,du$

Example. Evaluate $\int \overset{\text{Expo}}{e^{-x}} \overset{\text{trig}}{\sin(3x)}\,dx$.

$u = \sin(3x)$ $\quad v = -e^{-x}$

$du = 3\cos(3x)\,dx$ $\quad dv = e^{-x}\,dx$

Inverse trig
Logarithmic
Algebraic
Trig
Exponential

$$= -e^{-x}\sin(3x) - \int -3e^{-x}\cos(3x)\,dx$$

$$= -e^{-x}\sin(3x) + 3\int e^{-x}\cos(3x)\,dx$$

$u = \cos(3x)$ $\quad v = -e^{-x}$

$du = -3\sin(3x)\,dx$ $\quad dv = e^{-x}\,dx$

$$= -e^{-x}\sin(3x) + 3\left[-e^{-x}\cos(3x) - \int 3e^{-x}\sin(3x)\,dx\right]$$

$$\int e^{-x}\sin(3x)\,dx = -e^{-x}\sin(3x) - 3e^{-x}\cos(3x) - 9\int e^{-x}\sin(3x)\,dx$$

$$10\int e^{-x}\sin(3x)\,dx = -e^{-x}\sin(3x) - 3e^{-x}\cos(3x) + C$$

$$\int e^{-x}\sin(3x)\,dx = \boxed{\frac{1}{10}\left(-e^{-x}\sin(3x) - 3e^{-x}\cos(3x)\right) + C}$$

Example. Evaluate $\int e^{4x}\cos(3x)\,dx$.

$\int u\,dv = uv - \int v\,du$

$u = \cos(3x)$ $\quad v = e^{4x}/4$

$du = -3\sin(3x)\,dx$ $\quad dv = e^{4x}\,dx$

I nverse trig
L ogarithmic
A lgebraic
T rig
E xponential

$$\int e^{4x}\cos(3x)\,dx = \frac{e^{4x}\cos(3x)}{4} - \int -\frac{3}{4}e^{4x}\sin(3x)\,dx$$

$$= \frac{e^{4x}\cos(3x)}{4} + \frac{3}{4}\int e^{4x}\sin(3x)\,dx$$

$u = \sin(3x)$ $\quad v = e^{4x}/4$

$u = 3\cos(3x)\,dx$ $\quad dv = e^{4x}\,dx$

$$= \frac{e^{4x}\cos(3x)}{4} + \frac{3}{4}\left[\frac{e^{4x}\sin(3x)}{4} - \int \frac{3}{4}e^{4x}\cos(3x)\,dx\right]$$

$$\int e^{4x}\cos(3x)\,dx = \frac{e^{4x}\cos(3x)}{4} + \frac{3}{16}e^{4x}\sin(3x) - \frac{9}{16}\int e^{4x}\cos(3x)\,dx$$

$$\int e^{4x}\cos(3x)\,dx = \frac{8}{5}\left[\frac{e^{4x}\cos(3x)}{4} + \frac{3}{16}e^{4x}\sin(3x)\right] + C$$

$\int u\,dv = uv - \int v\,du$

Example. Derive the integral formula

algebraic

log

$$\int 1 \cdot \ln(x)\,dx = x\ln(x) - x + C$$

Inverse trig
Logarithmic
Algebraic
Trig
Exponential

$u = \ln(x)$ $\qquad v = x$

$du = \frac{1}{x}\,dx$ $\qquad dv = 1\,dx$

$$\int \ln(x)\,dx = x\ln(x) - \int \underbrace{\frac{x}{x}}_{1}\,dx$$

$$= \boxed{x\ln(x) - x + C}$$

~~algebraic~~ ~~trig~~

Example. Evaluate $\int 10\cos(\sqrt{x})\,dx$

~~$u =$~~ ~~$v =$~~

~~$du =$~~ ~~$dv =$~~

Inverse trig
Logarithmic
Algebraic
Trig
Exponential

$t = \sqrt{x}$

$dt = \frac{1}{2\sqrt{x}}\,dx$

$20\,dt = \frac{10}{\sqrt{x}}\,dx$

$20\sqrt{x}\,dt = 10\,dx$

$20t\,dt = 10\,dx$

algebraic

trig

$$\int 20t\cos(t)\,dt$$

$$\int \ln(x)\,dx = x\ln(x) - x + C$$

Example. Evaluate $\displaystyle\int_1^e \ln(2x)\,dx$.

$u = \ln(2x)$ $\quad v = x$

$du = \frac{1}{x}\,dx$ $\quad dv = dx$

I nverse trig
L ogarithmic
A lgebraic
T rig
E xponential

$$\int_1^e \ln(2x)\,dx = x\ln(2x)\Big|_1^e - \int_1^e \frac{x}{x}\,dx$$

$$= \Big(e\underbrace{\ln(2e)}_{\ln(2)+\ln(e)} - \ln(2)\Big) - x\Big|_1^e$$

$$= (e-1)\ln(2) + e - (e-1)$$

$$= \boxed{(e-1)\ln(2) + 1}$$