1 17.2: Line Integrals

Definition. (Scalar Line Integral in the Plane)
Suppose the scalar-valued function f is defined on a region containing the smooth curve C given by $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$. The **line integral of f over C** is

$$\int_C f(x(t), y(t))\, ds = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x(t_k^*),\, y(t_k^*))\Delta s_k,$$

provided this limit exists over all partitions of $[a, b]$. When the limit exists, f is said to be **integrable** on C.

Theorem 17.1: Evaluating Scalar Line Integrals in $\mathbb{R}^2$
Let f be continuous on a region containing a smooth curve C: $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$. Then

$$\begin{aligned}\int_C f\, ds &= \int_a^b f(x(t),\, y(t))|\mathbf{r}'(t)|\, dt \\ &= \int_a^b f(x(t),\, y(t))\sqrt{x'(t)^2 + y'(t)^2}\, dt.\end{aligned}$$

Procedure: Evaluating the Line Integral $\int_C f\, ds$

1. Find a parametric description of C in the form $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$.
2. Compute $|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$.
3. Make substitutions for x and y in the integrand and evaluate an ordinary integral:

$$\int_C f\, ds = \int_a^b f(x(t),\, y(t))|\mathbf{r}'(t)|\, dt.$$

Theorem 17.2: Evaluating Scalar Line Integrals in $\mathbb{R}^3$
Let f be continuous on a region containing a smooth curve $C : \mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$, for $a \le t \le b$. Then

$$\begin{aligned}\int_C f\,ds &= \int_a^b f(x(t),\, y(t),\, z(t))|\mathbf{r}'(t)|\,dt \\ &= \int_a^b f(x(t),\, y(t),\, z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt.\end{aligned}$$

Definition. (Line Integral of a Vector Field)
Let $\mathbf{F}$ be a vector field that is continuous on a region containing a smooth oriented curve C parameterized by arc length. Let $\mathbf{T}$ be the unit tangent vector at each point of C consistent with the orientation. The line integral of $\mathbf{F}$ over C is $\displaystyle\int_C \mathbf{F}\cdot\mathbf{T}\,ds$.

Different Forms of Line Integrals of Vector Fields
The line integral $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ may be expressed in the following forms, where $\mathbf{F} = \langle f,\, g,\, h\rangle$ and C has a parameterization $\mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$, for $a \le t \le b$:

$$\begin{aligned}\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt &= \int_a^b (f(t)x'(t) + g(t)y'(t) + h(t)z'(t))\,dt \\ &= \int_C f\,dx + g\,dy + h\,dz \\ &= \int_C \mathbf{F}\cdot d\mathbf{r}.\end{aligned}$$

For line integrals in the plane, we let $\mathbf{F} = \langle f,\, g\rangle$ and assume C is parameterized in the form $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$. Then

$$\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_a^b (f(t)x'(t) + g(t)y'(t))\,dt = \int_C f\,dx + g\,dy = \int_C \mathbf{F}\cdot d\mathbf{r}.$$

Definition. (Work Done in a Force Field)
Let $\mathbf{F}$ be a continuous force field in a region D of $\mathbb{R}^3$. Let

$$C : \mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle,\ \text{for } a \le t \le b,$$

be a smooth curve in D with a unit tangent vector $\mathbf{T}$ consistent with the orientation. The work done in moving an object along C in the positive direction is

$$W = \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt.$$

Definition. (Circulation)
Let $\mathbf{F}$ be a continuous vector field on a region D of $\mathbb{R}^3$, and let C be a closed smooth oriented curve in D. The **circulation** of $\mathbf{F}$ on C is $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$, where $\mathbf{T}$ is the unit vector tangent to C consistent with the orientation.

Definition. (Flux)
Let $\mathbf{F} = \langle f, g\rangle$ be a continuous vector field on a region R of $\mathbb{R}^2$. Let $C : \mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, $a \le t \le b$, be a smooth orientated curve in R that does not intersect itself. The **flux** of the vector field $\mathbf{F}$ across C is

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_a^b (f(t)y'(t) - g(t)x'(t))\,dt,$$

where $\mathbf{n} = \mathbf{T} \times \boldsymbol{k}$ is the unit normal vector and $\mathbf{T}$ is the unit tangent vector consistent with the orientation. If C is a closed curve with counterclockwise orientation, $\mathbf{n}$ is the outward normal vector, and the flux integral gives the **outward flux** across C.