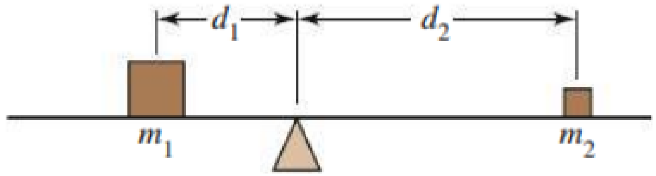
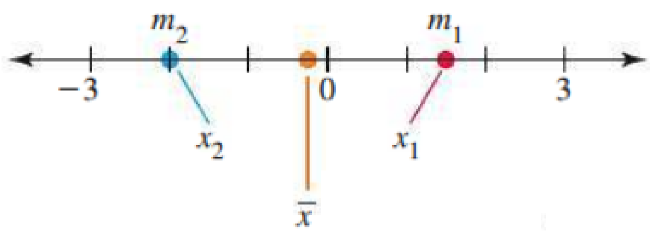
16.6: Integrals for Mass Calculations

Suppose we have two masses m_1 and m_2 on a beam (with no mass) that are distances d_1 and d_2 away from a pivot point. This beam will be balanced when $m_1 d_1 = m_2 d_2$.



This concept can be used to to find the balance point $\bar{x}$ between 2 objects with masses m_1 and m_2:



$$m_1(x_1 - \bar{x}) = m_2(\bar{x} - x_2) \Rightarrow m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) = 0.$$

$$\Rightarrow \bar{x} =$$

Next, we can generalize this to n objects with masses $m_1, \ldots, m_n$:

$$m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) + \cdots + m_n(x_n - \bar{x}) = \sum_{k=1}^{n} m_k(x_k - \bar{x}) = 0.$$

$$\Rightarrow \bar{x} =$$

Definition. (Center of Mass in One Dimension)
Let ρ be an integrable density function on the interval $[a, b]$ (which represents a thin rod or wire). The **center of mass** is located at the point $\bar{x} = \frac{M}{m}$, where the **total moment** M and mass m are

$$M = \int_a^b x\rho(x)\,dx \quad \text{and} \quad m = \int_a^b \rho(x)\,dx.$$

Example. Find the mass and center of mass of the thin rods with the following density functions:

$\rho(x) = 2 + \cos(x)$, for $0 \leq x \leq \pi$

$$\rho(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1 \\ x(2-x) & \text{if } 1 < x \leq 2 \end{cases}$$

Definition. (Center of Mass in Two Dimensions)
Let ρ be an integrable area density function defined over a closed bounded region R in $\mathbb{R}^2$. The coordinates of the center of mass of the object represented by R are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint\limits_R x\rho(x,y)\,dA \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint\limits_R y\rho(x,y)\,dA,$$

where $m = \iint_R \rho(x,y)\,dA$ is the mass, and M_y and M_x are the moments with respect to the y-axis and x-axis, respectively. If ρ is constant, the center of mass is called the **centroid** and is independent of the density.

Example. Find the center of mass of the following plane regions with variable density:

$R = \{(x,y) : 0 \leq x \leq 4,\ 0 \leq y \leq 2\};\ \rho(x,y) = 1 + x/2.$

The quarter disk in the first quadrant bounded by $x^2+y^2=4$ with $\rho(x,y)=1+x^2+y^2$.

Definition. (Center of Mass in Three Dimensions)
Let ρ be an integrable area density function defined over a closed bounded region D in $\mathbb{R}^3$. The coordinates of the center of mass of the region are

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m} \iiint\limits_D x\rho(x,y,z)\,dV$$
$$\bar{y} = \frac{M_{xz}}{m} = \frac{1}{m} \iiint\limits_D y\rho(x,y,z)\,dV$$
$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m} \iiint\limits_D z\rho(x,y,z)\,dV$$

where $m = \iiint_D \rho(x,y,z)\,dA$ is the mass, and M_{yz}, M_{xz}, and M_{xy} are the moments with respect to the coordinate planes.

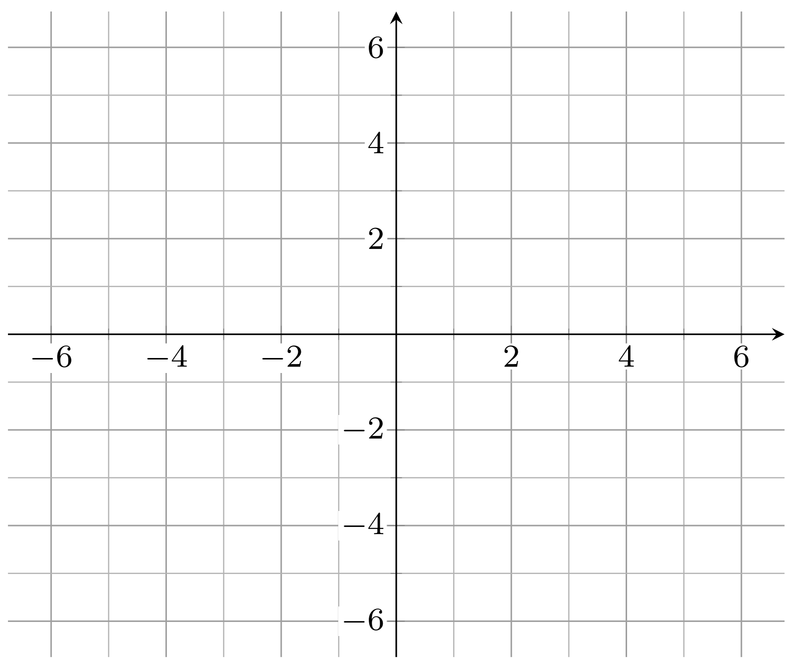
17.1: Vector Fields

Definition. (Vector Fields in Two Dimensions)
Let f and g be defined on a region R of $\mathbb{R}^2$. A **vector field** in $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point in R a vector $\langle f(x,y),\, g(x,y)\rangle$. The vector field is written as

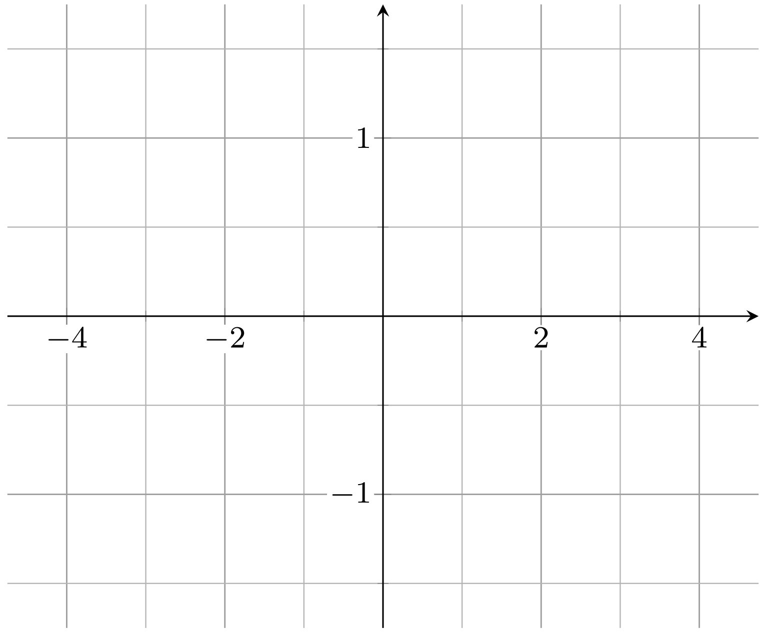
$$\mathbf{F}(x,y) = \langle f(x,y),\, g(x,y)\rangle \quad \text{or}$$
$$\mathbf{F}(x,y) = f(x,y)\boldsymbol{i} + g(x,y)\boldsymbol{j}.$$

A vector field $\mathbf{F} = \langle f, g\rangle$ is continuous or differentiable on a region R of $\mathbb{R}^2$ if f and g are continuous or differentiable on R, respectively.

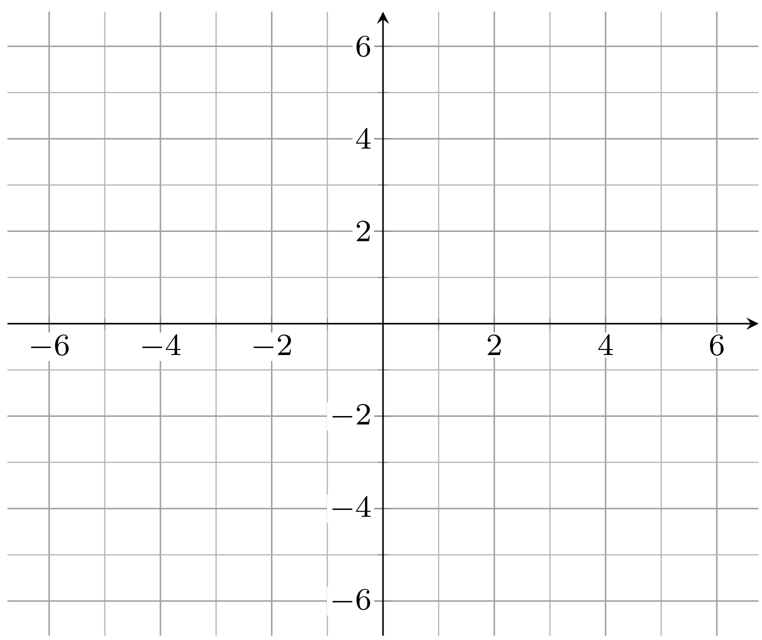
Example. Sketch the vector field $\mathbf{F} = \langle 0, x\rangle$.



Example. Sketch the vector field $\mathbf{F} = \left\langle 1 - y^2, 0 \right\rangle$ for $|y| \le 1$.



Example. Sketch the vector field $\mathbf{F} = \langle -y, x \rangle$.



Definition. (Radial Vector Fields in $\mathbb{R}^2$)
Let $\mathbf{r} = \langle x, y \rangle$. A vector field of the form $\mathbf{F} = f(x,y)\mathbf{r}$, where f is a scalar valued function, is a **radial vector field**. Of specific interest are the radial vector fields

$$\mathbf{F}(x,y) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y \rangle}{|\mathbf{r}|^p} = \frac{\mathbf{r}}{|\mathbf{r}|}\frac{1}{|\mathbf{r}|^{p-1}},$$

where p is a real number. At every point (expect the origin), the vectors of this field are directed outward from the origin with a magnitude of $|\mathbf{F}| = \dfrac{1}{|\mathbf{r}|^{p-1}}$.

Example. Let C be the circle $x^2 + y^2 = a^2$, where $a > 0$.

a) Show that at each point of C, the radial vector field $\mathbf{F}(x,y) = \dfrac{\mathbf{r}}{|\mathbf{r}|} = \dfrac{\langle x, y \rangle}{\sqrt{x^2+y^2}}$ is orthogonal to the line tangent to C at that point.

b) Show that at each point of C, the rotation vector field $\mathbf{G}(x,y) = \dfrac{\langle -y, x \rangle}{\sqrt{x^2+y^2}}$ is parallel to the line tangent to C at that point.

Definition. (Vector Fields and Radial Vector Fields in $\mathbb{R}^3$)
Let f, g, and h be defined on a region D of $\mathbb{R}^3$. A **vector field** in $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point in D a vector $\langle f(x,y,z),\, g(x,y,z),\, h(x,y,z)\rangle$. The vector field is written as

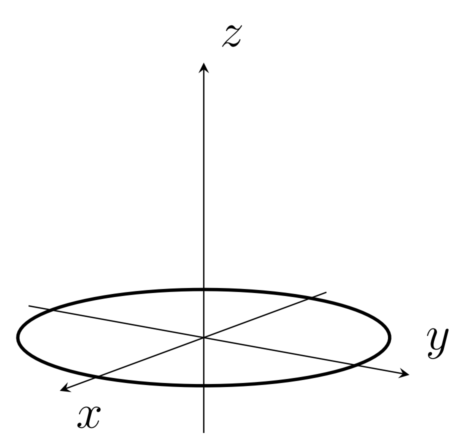
$$\mathbf{F}(x,y,z) = \langle f(x,y,z),\, g(x,y,z),\, h(x,y,z)\rangle \quad \text{or}$$
$$\mathbf{F}(x,y,z) = f(x,y,z)\boldsymbol{i} + g(x,y,z)\boldsymbol{j} + h(x,y,z)\boldsymbol{k}.$$

A vector field $\mathbf{F} = \langle f, g, h\rangle$ is continuous or differentiable on a region D of $\mathbb{R}^3$ if f, g, and h are continuous or differentiable on D, respectively. Of particular importance are the **radial vector fields**

$$\mathbf{F}(x,y,z) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x,y,z\rangle}{|\mathbf{r}|^p},$$

where p is a real number.

Example. Sketch the vector field $\mathbf{F}(x,y,z) = \left\langle 0, 0, 1 - x^2 - y^2\right\rangle$, for $x^2 + y^2 \le 1$.



Definition. (Gradient Fields and Potential Functions)
Let φ be differentiable on a region of $\mathbb{R}^2$ or $\mathbb{R}^3$. The vector field $=\nabla\varphi$ is a **gradient field** and the function φ is a **potential function** for $\mathbf{F}$.

Example. Sketch and interpret the gradient field associated with the temperature function $T = 200 - x^2 - y^2$ on the cirular plane $R = \left\{(x, y) : x^2 + y^2 \leq 25\right\}$.

Example. Sketch and interpret the gradient field associated with the velocity potential $\varphi = \tan^{-1}(xy)$.