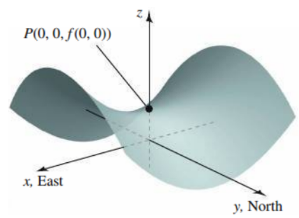
15.3: Partial Derivatives

Recall that for functions with one independent variable, say $y = f(x)$, the derivative measures the change in y with respect to x. For functions with multiple independent variables, we compute derivatives with respect to each variable.



$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Definition. (Partial Derivatives)
The **partial derivative of f with respect to x at the point (a, b) is**

derivative wrt x

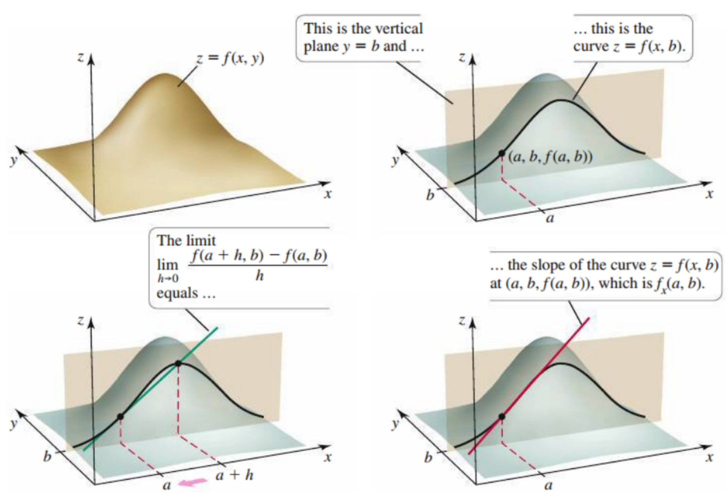
$$f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}.$$

The **partial derivative of f with respect to y at the point (a, b) is**

derivative wrt y

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h},$$

provided these limits exist.



$$\frac{\partial f}{\partial x} = f'(x)$$

When evaluating a partial derivative at a point (a, b), we denote this

$$\frac{\partial f}{\partial x}(a,b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a,b) \text{ and } \frac{\partial f}{\partial y}(a,b) = \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a,b)$$

Example. For the following functions, find the first partial derivatives. If a point is provided, evaluate the partial derivatives.

$$\lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$z = f(x, y) = x^8 + 3y^9 + 8$

$\frac{\partial z}{\partial x}$ $\quad f_x(x,y) = 8x^7 + 0 + 0$

$y = f(x)$

$\frac{\partial z}{\partial y}$ $\quad \frac{\partial f}{\partial y} = 0 + 27y^8 + 0$

$\frac{dy}{dx}$

$(6y^2)x^5 \qquad (2y)x^3$

$g(x, y) = 6x^5y^2 + 2x^3y + 5$

$g_x(x,y) = 30x^4y^2 + 6x^2y + 0$

$g_y(x,y) = 12x^5y + 2x^3 + 0$

$t(x,y,z) = 3x^4yz^2 + 2x^4y^3 + \cos(xy) + 42$

$t_x(x,y,z) = 12x^3yz^2 + 8x^3y^3 - \sin(xy)\,y + 0$

$t_y(x,y,z) = 3x^4z^2 + 6x^4y^2 - \sin(xy)\,x + 0$

$t_z(x,y,z) = 6x^4yz + 0 + 0 + 0$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2}$$

$\boxed{\text{Lo Di Hi; Hi Di Lo; over Lo squared}}$

$$h(s,t) = \frac{s-t}{4s+t} \text{ at } (s,t) = (2,-3)$$

$$h_s(s,t) = \frac{(4s+t)(1) - (s-t)\,4}{(4s+t)^2} = \frac{5t}{(4s+t)^2} \qquad h_s(2,-3) = \frac{-15}{5^2} = \boxed{\frac{-3}{5}}$$

$$h_t(s,t) = \frac{(4s+t)(-1) - (s-t)(1)}{(4s+t)^2} = \frac{-5s}{(4s+t)^2} \qquad h_t(2,-3) = \frac{-10}{25^2} = \boxed{\frac{-2}{5}}$$

$$k(x,y) = \tan^{-1}\left(3x^2y^2\right) \text{ at } (x,y) = (1,1)$$

$$\frac{d}{dx}\left[\int_a^{g(x)} f(u)\,du\right] = f(g(x)) \cdot g'(x)$$

$$G'(u) = g(u)$$

$$\ell(w,v) = \int_v^w g(u)\,du = -\int_w^v g(u)\,du = G(w) - G(v)$$

$$\ell_w(w,v) = g(w)$$

$$\ell_v(w,v) = -g(v)$$

Higher-Order Partial Derivatives:

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$	$(f_x)_x = f_{xx}$	"d squared f dx squared or $f - x - x$"
$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$	$(f_y)_y = f_{yy}$	"d squared f dy squared or $f - y - y$"
derivative wrt y first → $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$	$(f_y)_x = f_{yx}$	"$f - y - x$"
$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$	$(f_x)_y = f_{xy}$	"$f - x - y$"

The order of differentiation is important when finding **mixed partial derivatives** f_{xy} and f_{yx}.

Example. Find the four 2nd-order partial derivatives of the following functions

$$z = 4ye^{3x}$$

$$z_x = \frac{\partial z}{\partial x} = 12ye^{3x}$$

$$z_y = \frac{\partial z}{\partial y} = 4e^{3x}$$

$$z_{xx} = \frac{\partial^2 z}{\partial x^2} = 36ye^{3x} \qquad z_{xy} = \frac{\partial^2 z}{\partial y \partial x} = 12e^{3x}$$

$$z_{yx} = \frac{\partial^2 z}{\partial x \partial y} = 12e^{3x} \qquad z_{yy} = \frac{\partial^2 z}{\partial y^2} = 0$$

$$f(x,y) = \sin^2(x^3y) = (\sin(x^3y))^2$$

$$f_x(x,y) = \underline{2\sin(x^3y)\cos(x^3y)}\,3x^2y = 3x^2y\sin(2x^3y)$$

$$f_y(x,y) = \underline{2\sin(x^3y)\cos(x^3y)}\,x^3 = x^3\sin(2x^3y)$$

← double angle formula

$$f_{xx}(x,y) = 6xy\sin(2x^3y) + 3x^2y\cos(2x^3y)\,6x^2y$$

$$f_{xy}(x,y) = 3x^2\sin(2x^3y) + 3x^2y\cos(2x^3y)\,2x^3$$

$$f_{yx}(x,y) = 3x^2\sin(2x^3y) + x^3\cos(2x^3y)\,6x^2y$$

← equal

$$f_{yy}(x,y) = x^3\sin(2x^3y)\,2x^3 = 2x^6\sin(2x^3y)$$

Theorem 15.4: (Clairut) Equality of Mixed Partial Derivatives Assume f is defined on an open set D of $\mathbb{R}^2$, and that f_{xy} and f_{yx} are continuous throughout D. Then $f_{xy} = f_{yx}$ at all points of D.

Note: Clairut's theorem also extends to higher order derivatives of f. $f_{xyy} = f_{yxy} = f_{yyx}$

Example. Ideal Gas Law: The pressure P, volume V, and temperature T of an ideal gas are related by the equation $PV = kT$, where $k > 0$ is a constant depending on the amount of gas.

Determine the rate of change of the pressure with respect to the volume

Explicit $P = \frac{kT}{V} = kTV^{-1} \longrightarrow P_V = -\frac{kT}{V^2}$

Implicit $\frac{\partial}{\partial V}[PV] = \frac{\partial}{\partial V}[kT]$

$P_V V + P\cdot(1) = 0 \longrightarrow P_V = \frac{-P}{V} = \frac{-\frac{kT}{V}}{V} = -\frac{kT}{V^2}$

Determine the rate of change of the pressure with respect to the temperature

Explicit $P = \frac{kT}{V} \Rightarrow P_T = \frac{k}{V}$

Implicit $\frac{\partial}{\partial T}[PV] = \frac{\partial}{\partial T}[kT]$

$P_T V = k \longrightarrow P_V = \frac{k}{V}$

Definition. (Differentiability)
The function $z = f(x,y)$ is **differentiable at** (a,b) provided $f_x(a,b)$ and $f_y(a,b)$ exist and the change $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$ equals

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y = \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where for fixed a and b, ε_1 and ε_2 are functions that depend only on Δx and δy, with $(\varepsilon_1, \varepsilon_2) \to (0,0)$ as $(\Delta x, \Delta y) \to (0,0)$. A function is **differentiable** on an open set R if it is differentiable at every point of R.

Theorem 15.5: Conditions for Differentiability
Suppose the function f has partial derivatives f_x and f_y defined on an open set containing (a,b), with f_x and f_y continuous at (a,b). Then f is differentiable at (a,b).

Not cont $\Rightarrow$ not diff

Theorem 15.6: Differentiable Implies Continuous
If a function f is differentiable at (a,b), then it is continuous at (a,b).

Example. Why is the function

$$f(x,y) = \begin{cases} \frac{3xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

not continuous at $(x,y) = (0,0)$?