

Math 2060 Class notes Spring 2021

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Table Of Contents

13.1: Vectors and the Geometry of Space 1

13.2: Vectors in Three Dimensions 6

13.3: Dot Products 13

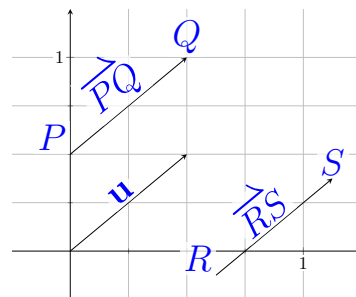
13.4: Cross Products 18

13.1: Vectors and the Geometry of Space

Definition.

- **Vectors**

- Have a direction and magnitude,
- vector \overrightarrow{PQ} has a *tail* at P and a *head* at Q ,
- Can be denoted as \mathbf{u} or \vec{u} ,
- Equal vectors have the same direction and magnitude (not necessarily the same position)



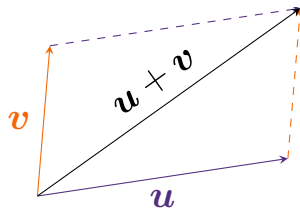
- **Scalars** are quantities with magnitude but no direction (e.g. mass, temperature, price, time, etc.)
- **Zero vector**, denoted $\mathbf{0}$ or $\vec{0}$, has length 0 and no direction

Scalar-vector multiplication:

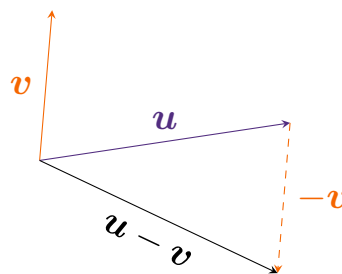
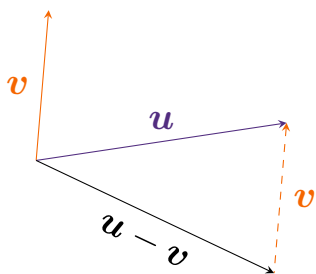
- Denoted $c\mathbf{v}$ or $c\vec{v}$,
- length of vector multiplied by $|c|$,
- $c\mathbf{v}$ has the same direction as \mathbf{v} if $c > 0$, and has the opposite direction as \mathbf{v} if $c < 0$, (what if $c = 0$?)
- \mathbf{u} and \mathbf{v} are **parallel** if $\mathbf{u} = c\mathbf{v}$. (what vectors are parallel to $\mathbf{0}$?)

Vector Addition and Subtraction:

Given two vectors \mathbf{u} and \mathbf{v} , their sum, $\mathbf{u} + \mathbf{v}$, can be represented by the parallelogram (triangle) rule: place the tail of \mathbf{v} at the head of \mathbf{u}

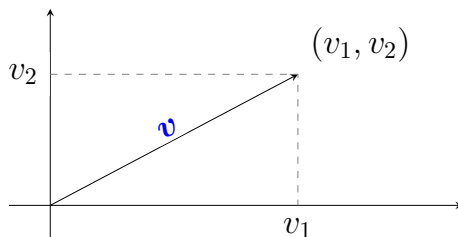


The difference, denoted $\mathbf{u} - \mathbf{v}$, is the sum of $\mathbf{u} + (-\mathbf{v})$:



Vector Components:

A vector \mathbf{v} whose tail is at the origin $(0, 0)$ and head is at (v_1, v_2) is a **position vector** (in **standard position**) and is denoted $\langle v_1, v_2 \rangle$. The real numbers v_1 and v_2 are the x - and y -components of \mathbf{v} .



Vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are equal if and only if $u_1 = v_1$ and $u_2 = v_2$.

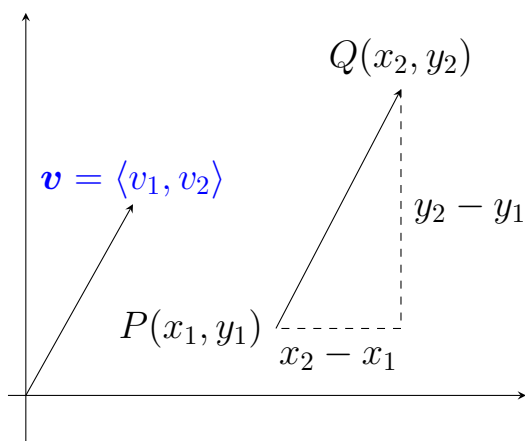
Magnitude:

Given points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the **magnitude**, or **length**, of vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$, denoted $|\overrightarrow{PQ}|$, is the distance between points P and Q .

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The magnitude of position vector $\mathbf{v} = \langle v_1, v_2 \rangle$ is $|\mathbf{v}|$.

(How do $|\overrightarrow{PQ}|$ and $|\overrightarrow{QP}|$ relate to each other?)



Note: The norm, denoted $\|\mathbf{u}\|$ or $\|\mathbf{u}\|_2$, is equivalent to the magnitude of a vector.

Equation of a Circle:

Definition.

A **circle** centered at (a, b) with radius r is the set of points satisfying the equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

A **disk** centered at (a, b) with radius r is the set of points satisfying the inequality

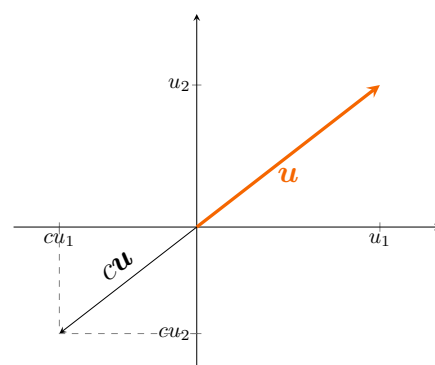
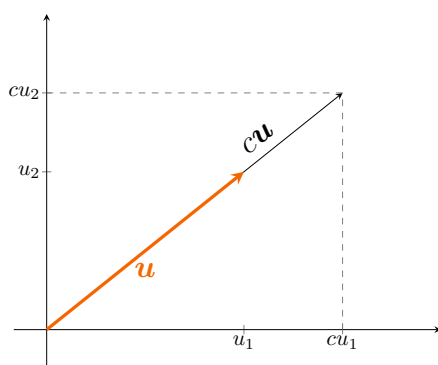
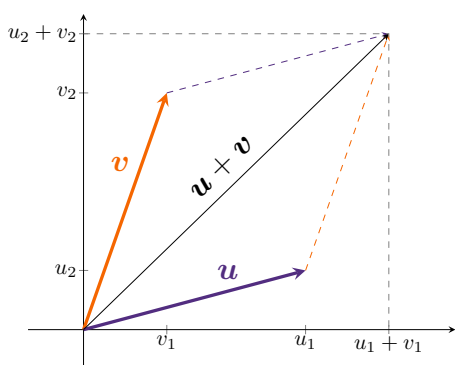
$$(x - a)^2 + (y - b)^2 \leq r^2.$$

Vector Operations in Terms of Components

Definition. (Vector Operations in \mathbb{R}^2)

Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2 \rangle$, and $\mathbf{v} = \langle v_1, v_2 \rangle$.

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= \langle u_1 + v_1, u_2 + v_2 \rangle && \text{Vector addition} \\ \mathbf{u} - \mathbf{v} &= \langle u_1 - v_1, u_2 - v_2 \rangle && \text{Vector subtraction} \\ c\mathbf{u} &= \langle cu_1, cu_2 \rangle && \text{Scalar multiplication}\end{aligned}$$



Example. Let $\mathbf{u} = \langle 1, 2 \rangle$, $\mathbf{v} = \langle -2, 3 \rangle$, $c = 2$, and $d = 3$. Find the following:

$$\mathbf{u} + \mathbf{v}$$

$$c\mathbf{u}$$

$$c\mathbf{u} + d\mathbf{v}$$

$$\mathbf{u} - c\mathbf{v}$$

Definition.

A **unit vector** is any vector with length 1.

In \mathbb{R}^2 , the **coordinate unit vectors** are $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.

Example. Let $\mathbf{u} = \langle -7, 3 \rangle$. Find two unit vectors parallel to \mathbf{u} . Find another vector parallel to \mathbf{u} with a magnitude of 2.

Properties of Vector Operations:

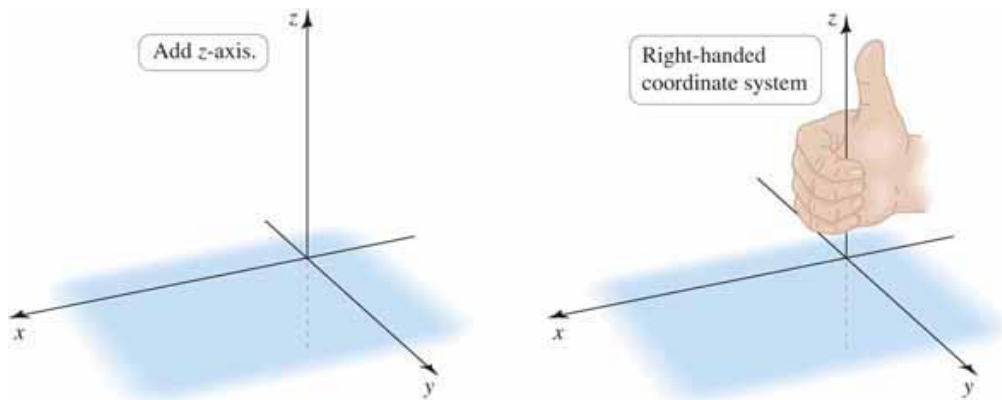
Suppose \mathbf{u} , \mathbf{v} , and \mathbf{w} are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

- | | |
|--|---|
| 1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | Commutative property of addition |
| 2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | Associative property of addition |
| 3. $\mathbf{v} + \mathbf{0} = \mathbf{v}$ | Additive identity |
| 4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ | Additive inverse |
| 5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ | Distributive property 1 |
| 6. $(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$ | Distributive property 2 |
| 7. $0\mathbf{v} = \mathbf{0}$ | Multiplication by zero scalar |
| 8. $c\mathbf{0} = \mathbf{0}$ | Multiplication by zero vector |
| 9. $1\mathbf{v} = \mathbf{v}$ | Multiplicative identity |
| 10. $a(c\mathbf{v}) = (ac)\mathbf{v}$ | Associative property of scalar multiplication |

13.2: Vectors in Three Dimensions

The xyz - Coordinate System:

The three-dimensional coordinate system is created by adding the z -axis, which is perpendicular to both the x -axis and the y -axis. When looking at the xy -plane, the positive direction of the z -axis protrudes towards the viewer. This can also be shown using the right-hand rule (Figure 13.25 from Briggs):

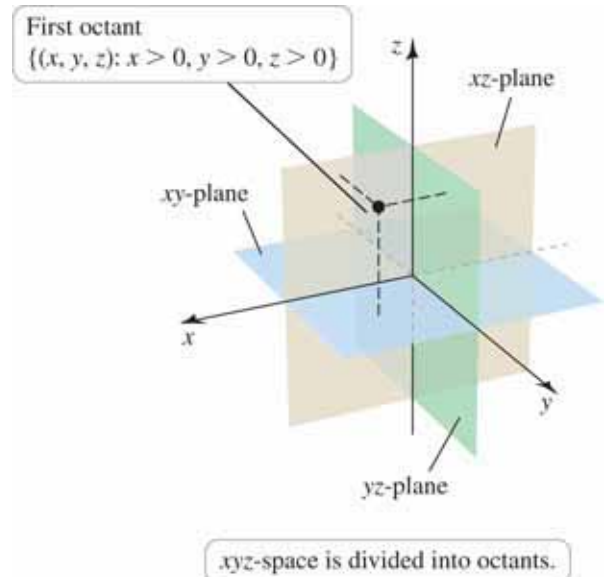


Definition.

This three-dimensional coordinate system is broken up into eight **octants**, which are separated by

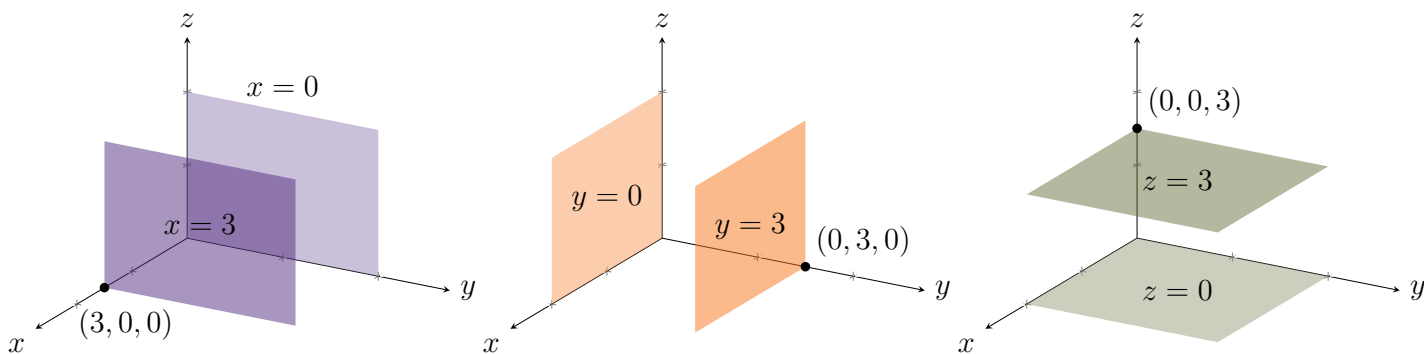
- the xy -**plane** ($z = 0$),
- the xz -**plane** ($y = 0$), and
- the yz -**plane** ($x = 0$).

The **origin** is the location where all three axes intersect.

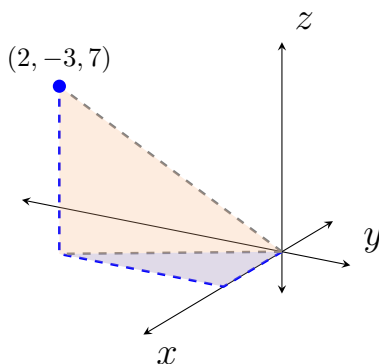


Equations of Simple Planes:

Planes in three-dimensions are analogous to lines in two-dimensions. Below, we see the yz -plane, the xz -plane, and the xy -plane, along with planes that are parallel where x , y , and z are fixed respectively:



Example (Parallel planes). Determine the equation of the plane parallel to the xz -plane passing through the point $(2, -3, 7)$.

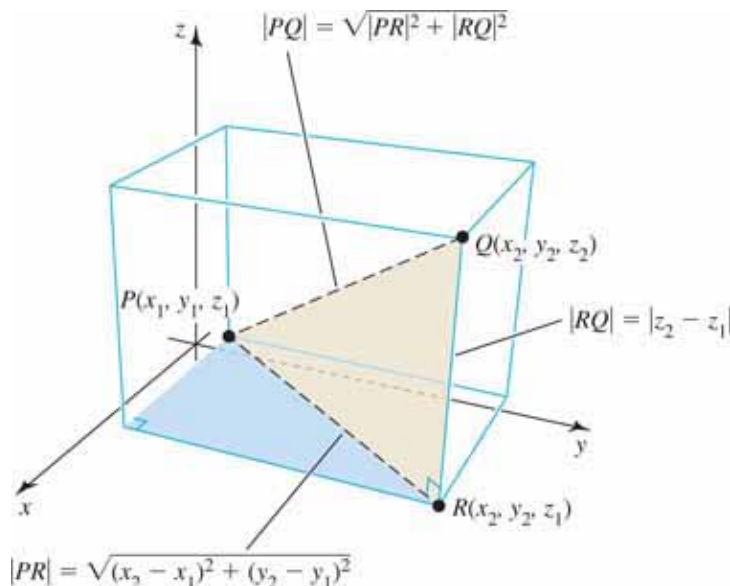


Distances in xyz -Space:

Recall that in \mathbb{R}^2 , for some vector \overrightarrow{PR} , the distance formula is given by

$$|PR| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

where (x_1, y_1) and (x_2, y_2) represent the points P and R respectively. This idea can be further extended into \mathbb{R}^3 by considering the two sides of the triangle formed by the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$:



Distance Formula in xyz -Space

The **distance** between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The **midpoint** between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is found by averaging the x -, y -, and z -coordinates:

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

Example. Consider $P(-1, 4, 3)$ and $Q(3, 5, 7)$. Find

- $\left| \overrightarrow{PQ} \right|$
- The midpoint between P and Q
- Two unit vectors parallel to \overrightarrow{PQ}
- The equation of the sphere centered at the midpoint passing through P and Q

Equation of a Sphere:

Definition.

A **sphere** centered at (a, b, c) with radius r is the set of points satisfying the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

A **ball** centered at (a, b, c) with radius r is the set of points satisfying the inequality

$$(x - a)^2 + (y - b)^2 + (z - c)^2 \leq r^2.$$

Example. Rewrite the following equation into the standard form of a sphere:

$$x^2 + y^2 + z^2 - 2x + 6y - 8z = -1$$

Example. What is the geometry of the intersection between $x^2 + y^2 + z^2 = 50$ and $z = 1$?

Vector Operations in Terms of Components

Definition. (Vector Operations in \mathbb{R}^3)

Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

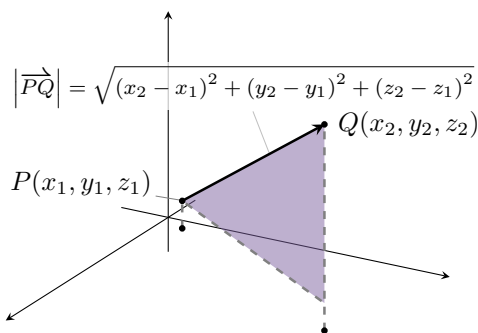
$$\begin{aligned}\mathbf{u} + \mathbf{v} &= \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle && \text{Vector addition} \\ \mathbf{u} - \mathbf{v} &= \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle && \text{Vector subtraction} \\ c\mathbf{u} &= \langle cu_1, cu_2, cu_3 \rangle && \text{Scalar multiplication}\end{aligned}$$

Magnitude and Unit Vectors:

Definition.

The **magnitude** (or **length**) of the vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ is the distance from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$:

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



In \mathbb{R}^3 , the **coordinate unit vectors** are $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$.

Example. Consider $P(-1, 4, 3)$ and $Q(3, 5, 7)$. Find two unit vectors parallel to \overrightarrow{PQ} .

Properties of Vector Operations:

Suppose \mathbf{u} , \mathbf{v} , and \mathbf{w} are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

- | | |
|--|---|
| 1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | Commutative property of addition |
| 2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | Associative property of addition |
| 3. $\mathbf{v} + \mathbf{0} = \mathbf{v}$ | Additive identity |
| 4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ | Additive inverse |
| 5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ | Distributive property 1 |
| 6. $(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$ | Distributive property 2 |
| 7. $0\mathbf{v} = \mathbf{0}$ | Multiplication by zero scalar |
| 8. $c\mathbf{0} = \mathbf{0}$ | Multiplication by zero vector |
| 9. $1\mathbf{v} = \mathbf{v}$ | Multiplicative identity |
| 10. $a(c\mathbf{v}) = (ac)\mathbf{v}$ | Associative property of scalar multiplication |

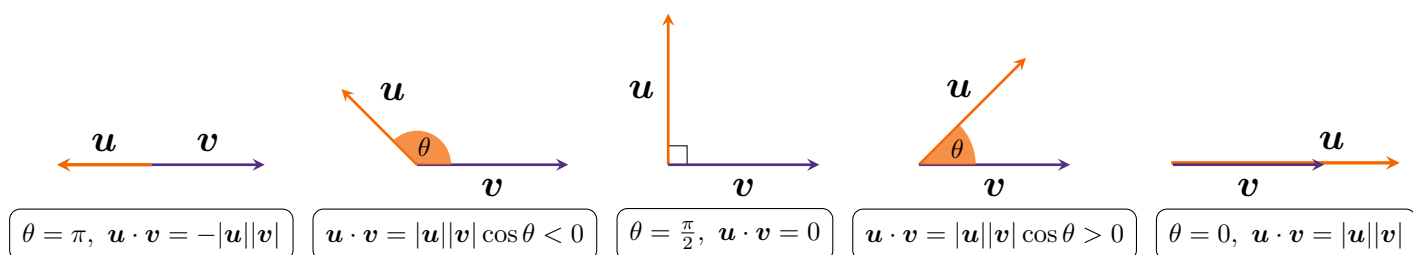
13.3: Dot Products

Definition. (Dot Product)

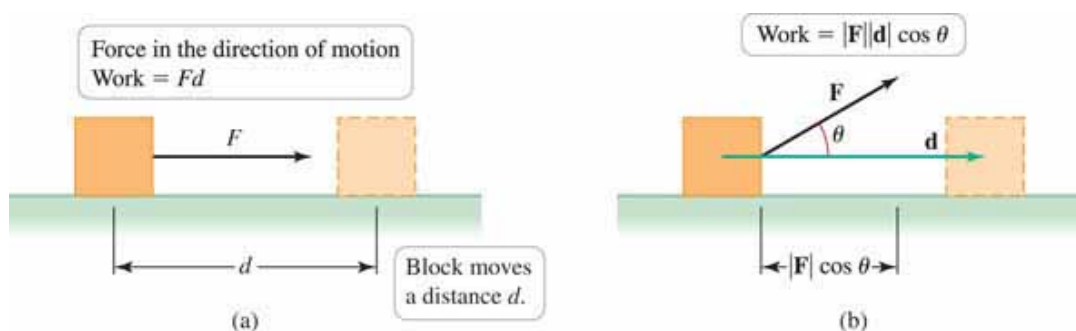
Given two nonzero vectors \mathbf{u} and \mathbf{v} in two or three dimensions, their **dot product** is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta,$$

where θ is the angle between \mathbf{u} and \mathbf{v} with $0 \leq \theta \leq \pi$. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$, and θ is undefined.



A physical example of the dot product is the amount of work done when a force is applied at an angle θ as shown in figure 13.43:



Note: The result of the dot product is a scalar!

Definition. (Orthogonal Vectors)

Two vectors \mathbf{u} and \mathbf{v} are **orthogonal** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. The zero vector is orthogonal to all vectors. In two or three dimensions, two nonzero orthogonal vectors are perpendicular to each other.

- \mathbf{u} and \mathbf{v} are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \cdot \mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$.
- \mathbf{u} and \mathbf{v} are perpendicular ($\theta = \frac{\pi}{2}$) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Theorem 31.1: Dot Product

Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$,

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

Properties of Dot Products

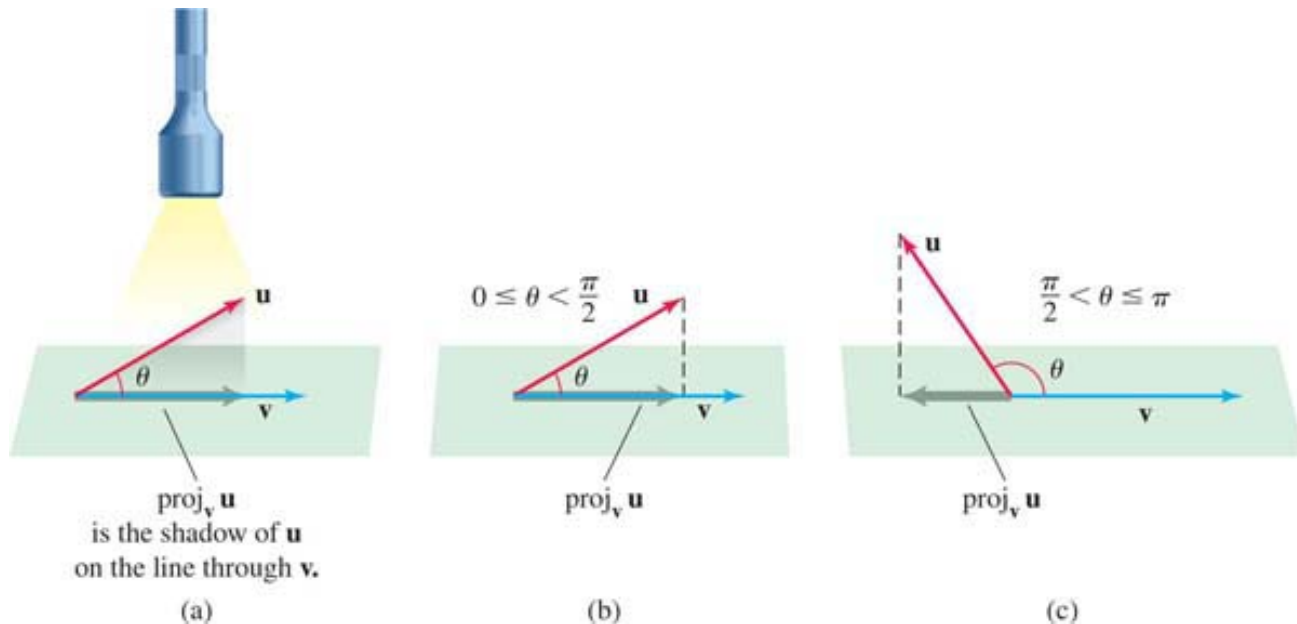
Theorem 13.2: Properties of the Dot Product

Suppose \mathbf{u} , \mathbf{v} and \mathbf{w} are vectors and let c be a scalar.

- | | |
|---|-----------------------|
| 1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ | Commutative property |
| 2. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$ | Associative property |
| 3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ | Distributive property |

Orthogonal Projections

Given vectors \mathbf{u} and \mathbf{v} , the projection of \mathbf{u} onto \mathbf{v} produces a vector parallel to \mathbf{v} using the “shadow” of \mathbf{u} cast onto \mathbf{v} .



Definition. ((Orthogonal) Projection of \mathbf{u} onto \mathbf{v})

The **orthogonal projection** of \mathbf{u} onto \mathbf{v} , denoted $\text{proj}_{\mathbf{v}} \mathbf{u}$, where $\mathbf{v} \neq \mathbf{0}$, is

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \underbrace{|\mathbf{u}| \cos \theta}_{\text{length}} \underbrace{\left(\frac{\mathbf{v}}{|\mathbf{v}|} \right)}_{\text{direction}}.$$

The orthogonal projection may also be computed with the formulas

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \text{scal}_{\mathbf{v}} \mathbf{u} \left(\frac{\mathbf{v}}{|\mathbf{v}|} \right) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v},$$

where the **scalar component** of \mathbf{u} in the direction of \mathbf{v} is

$$\text{scal}_{\mathbf{v}} \mathbf{u} = |\mathbf{u}| \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}.$$

Applications of Dot Products

Definition. (Work)

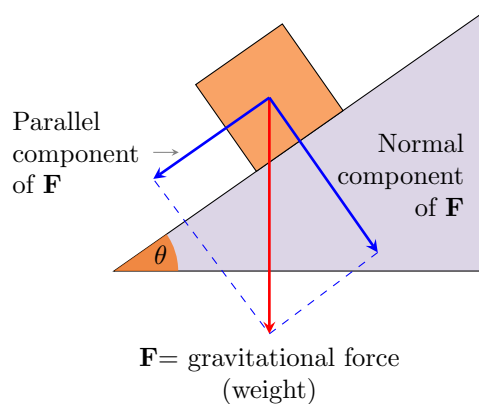
Let a constant force \mathbf{F} be applied to an object, producing a displacement \mathbf{d} . If the angle between \mathbf{F} and \mathbf{d} is θ , then the **work** done by the force is

$$W = |\mathbf{F}||\mathbf{d}| \cos \theta = \mathbf{F} \cdot \mathbf{d}$$

Example.

Parallel and Normal Forces:

Example.



13.4: Cross Products

Definition. (Cross Product)

Given two nonzero vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^3 , the **cross product** $\mathbf{u} \times \mathbf{v}$ is a vector with magnitude

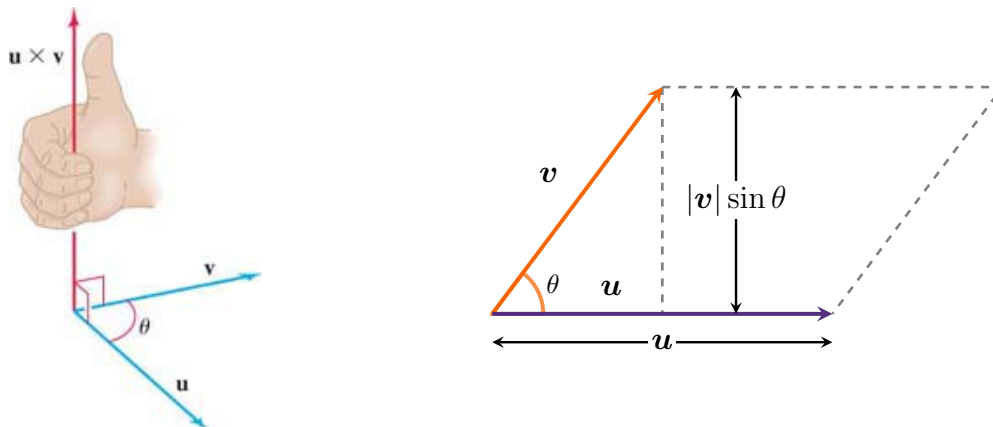
$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta,$$

where $0 \leq \theta \leq \pi$ is the angle between \mathbf{u} and \mathbf{v} .

The direction of $\mathbf{u} \times \mathbf{v}$ is given by the **right-hand rule**:

When you put your the vectors tail to tail and let the fingers of your right hand curl from \mathbf{u} to \mathbf{v} , the direction of $\mathbf{u} \times \mathbf{v}$ is the direction of your thumb, orthogonal to both \mathbf{u} and \mathbf{v} (Figure 13.56).

When $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, the direction of $\mathbf{u} \times \mathbf{v}$ is undefined.



Theorem 13.3: Geometry of the Cross Product

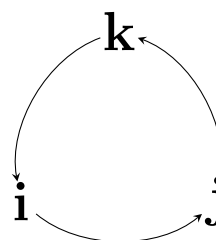
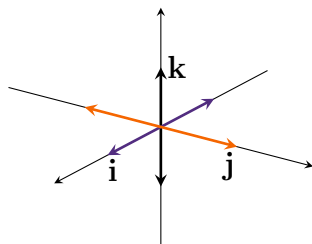
Let \mathbf{u} and \mathbf{v} be two nonzero vectors in \mathbb{R}^3 .

1. The vectors \mathbf{u} and \mathbf{v} are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
2. If \mathbf{u} and \mathbf{v} are two sides of a parallelogram, then the area of the parallelogram is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta$$

Theorem 13.4: Properties of the Cross Product Let \mathbf{u} , \mathbf{v} , and \mathbf{w} be nonzero vectors in \mathbb{R}^3 , and let a and b be scalars.

- | | |
|--|--------------------------|
| 1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ | Anticommutative property |
| 2. $(a\mathbf{u}) \times (b\mathbf{v}) = ab(\mathbf{u} \times \mathbf{v})$ | Associative property |
| 3. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ | Distributive property |
| 4. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$ | Distributive property |



$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j} \end{aligned}$$

Theorem 13.5: Cross Products of Coordinate Unit Vectors

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= -(\mathbf{j} \times \mathbf{i}) = \mathbf{k} & \mathbf{j} \times \mathbf{k} &= -(\mathbf{k} \times \mathbf{j}) = \mathbf{i} \\ \mathbf{k} \times \mathbf{i} &= -(\mathbf{i} \times \mathbf{k}) = \mathbf{j} & \mathbf{i} \times \mathbf{i} &= \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0} \end{aligned}$$

Using the unit vectors, we can compute $\mathbf{u} \times \mathbf{v}$:

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1 \underbrace{(\mathbf{i} \times \mathbf{i})}_0 + u_1v_2 \underbrace{(\mathbf{i} \times \mathbf{j})}_\mathbf{k} + u_1v_3 \underbrace{(\mathbf{i} \times \mathbf{k})}_{-\mathbf{j}} \\ &\quad + u_2v_1 \underbrace{(\mathbf{j} \times \mathbf{i})}_{-\mathbf{k}} + u_2v_2 \underbrace{(\mathbf{j} \times \mathbf{j})}_0 + u_2v_3 \underbrace{(\mathbf{j} \times \mathbf{k})}_\mathbf{i} \\ &\quad + u_3v_1 \underbrace{(\mathbf{k} \times \mathbf{i})}_\mathbf{j} + u_3v_2 \underbrace{(\mathbf{k} \times \mathbf{j})}_{-\mathbf{i}} + u_3v_3 \underbrace{(\mathbf{k} \times \mathbf{k})}_0 \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} \end{aligned}$$

Theorem 13.6: Evaluating the Cross Product

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

Note:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Alternative approach:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \begin{matrix} \mathbf{i} & \mathbf{j} \\ u_1 & u_2 \\ v_1 & v_2 \end{matrix}$$

