

13.4: Cross Products

Definition. (Cross Product)

Given two nonzero vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^3 , the **cross product** $\mathbf{u} \times \mathbf{v}$ is a vector with magnitude

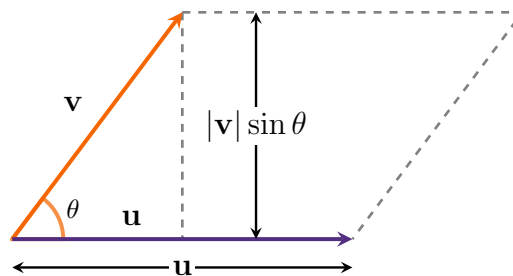
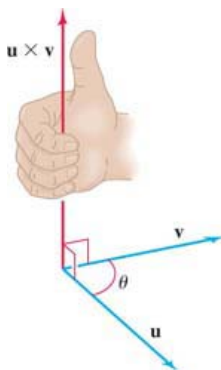
$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta,$$

where $0 \leq \theta \leq \pi$ is the angle between \mathbf{u} and \mathbf{v} .

The direction of $\mathbf{u} \times \mathbf{v}$ is given by the **right-hand rule**:

When you put your the vectors tail to tail and let the fingers of your right hand curl from \mathbf{u} to \mathbf{v} , the direction of $\mathbf{u} \times \mathbf{v}$ is the direction of your thumb, orthogonal to both \mathbf{u} and \mathbf{v} (Figure 13.56).

When $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, the direction of $\mathbf{u} \times \mathbf{v}$ is undefined.



Theorem 13.3: Geometry of the Cross Product

Let \mathbf{u} and \mathbf{v} be two nonzero vectors in \mathbb{R}^3 .

1. The vectors \mathbf{u} and \mathbf{v} are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
2. If \mathbf{u} and \mathbf{v} are two sides of a parallelogram, then the area of the parallelogram is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta$$

Example. Consider the vectors $\mathbf{u} = \langle 2, 0, 0 \rangle$ and $\mathbf{v} = \langle \sqrt{3}, 3, 0 \rangle$. The angle between these vectors is $\theta = \frac{\pi}{3}$. Find the area of the parallelogram formed by these vectors.

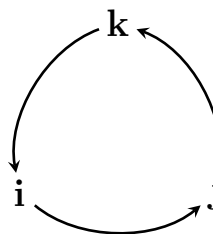
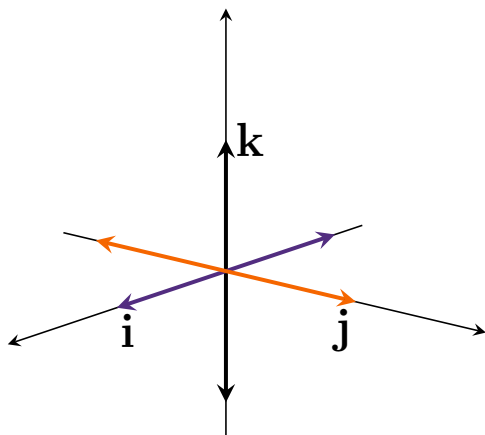
Theorem 13.4: Properties of the Cross Product Let \mathbf{u} , \mathbf{v} , and \mathbf{w} be nonzero vectors in \mathbb{R}^3 , and let a and b be scalars.

- | | |
|--|--------------------------|
| 1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ | Anticommutative property |
| 2. $(a\mathbf{u}) \times (b\mathbf{v}) = ab(\mathbf{u} \times \mathbf{v})$ | Associative property |
| 3. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ | Distributive property |
| 4. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$ | Distributive property |

Theorem 13.5: Cross Products of Coordinate Unit Vectors

$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= -(\mathbf{j} \times \mathbf{i}) = \mathbf{k} \\ \mathbf{k} \times \mathbf{i} &= -(\mathbf{i} \times \mathbf{k}) = \mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{j} \times \mathbf{k} &= -(\mathbf{k} \times \mathbf{j}) = \mathbf{i} \\ \mathbf{i} \times \mathbf{i} &= \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}\end{aligned}$$



$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j}\end{aligned}$$

Using the unit vectors, we can compute $\mathbf{u} \times \mathbf{v}$:

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1 \underbrace{(\mathbf{i} \times \mathbf{i})}_{\mathbf{0}} + u_1v_2 \underbrace{(\mathbf{i} \times \mathbf{j})}_{\mathbf{k}} + u_1v_3 \underbrace{(\mathbf{i} \times \mathbf{k})}_{-\mathbf{j}} \\ &\quad + u_2v_1 \underbrace{(\mathbf{j} \times \mathbf{i})}_{-\mathbf{k}} + u_2v_2 \underbrace{(\mathbf{j} \times \mathbf{j})}_{\mathbf{0}} + u_2v_3 \underbrace{(\mathbf{j} \times \mathbf{k})}_{\mathbf{i}} \\ &\quad + u_3v_1 \underbrace{(\mathbf{k} \times \mathbf{i})}_{\mathbf{j}} + u_3v_2 \underbrace{(\mathbf{k} \times \mathbf{j})}_{-\mathbf{i}} + u_3v_3 \underbrace{(\mathbf{k} \times \mathbf{k})}_{\mathbf{0}} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}\end{aligned}$$

Theorem 13.6: Evaluating the Cross Product

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

Note:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Alternative approach:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \begin{vmatrix} \mathbf{i} & \mathbf{j} \\ u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

Example. Compute $\mathbf{u} \times \mathbf{v}$ for $\mathbf{u} = \langle 3, 5, 4 \rangle$ and $\mathbf{v} = \langle 1, -1, 9 \rangle$.

Example. Consider the vectors $\mathbf{u} = \langle \sqrt{3}, 1, 0 \rangle$ and $\mathbf{v} = \langle -\sqrt{3}, 1, 0 \rangle$. From the unit circle, we know the angle between these two vectors is $\theta = \frac{2\pi}{3}$. Use the definition of the cross product to show this.

Example. Find the area of the triangle formed by $\mathbf{u} = \langle 1, 2, 3 \rangle$ and $\mathbf{v} = \langle 3, -1, 1 \rangle$.

Example. Given a force \mathbf{F} applied to a point P at the head of the vector $\mathbf{r} = \overrightarrow{OP}$, the **torque** produced at point O is given by $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ with magnitude

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin \theta.$$

Now suppose a force of 20N is applied to a wrench attached to a bolt in a direction perpendicular to the bolt. Which produces more torque: applying the force at an angle of 60° on a wrench that is 0.15m long or applying the force at an angle of 135° on a wrench that is 0.25m long?

