

Math 2060 Class notes Spring 2021

Peter Westerbaan

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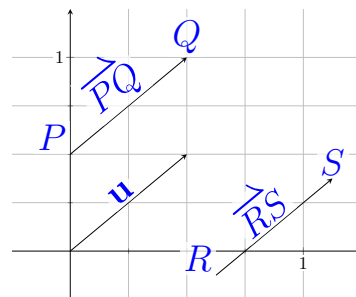
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13.1: Vectors and the Geometry of Space

Definition.

- **Vectors**

- Have a direction and magnitude,
- vector \overrightarrow{PQ} has a *tail* at P and a *head* at Q ,
- Can be denoted as \mathbf{u} or \vec{u} ,
- Equal vectors have the same direction and magnitude (not necessarily the same position)



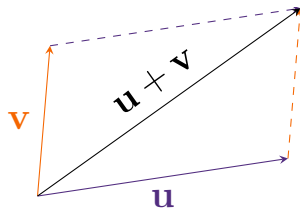
- **Scalars** are quantities with magnitude but no direction (e.g. mass, temperature, price, time, etc.)
- **Zero vector**, denoted $\mathbf{0}$ or $\vec{0}$, has length 0 and no direction

Scalar-vector multiplication:

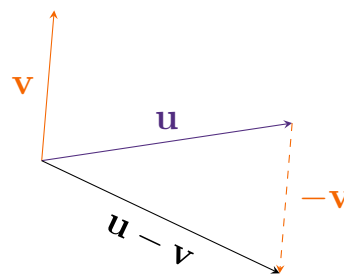
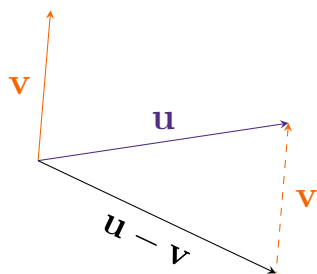
- Denoted $c\mathbf{v}$ or $c\vec{v}$,
- length of vector multiplied by $|c|$,
- $c\mathbf{v}$ has the same direction as \mathbf{v} if $c > 0$, and has the opposite direction as \mathbf{v} if $c < 0$, (what if $c = 0$?)
- \mathbf{u} and \mathbf{v} are **parallel** if $\mathbf{u} = c\mathbf{v}$. (what vectors are parallel to $\mathbf{0}$?)

Vector Addition and Subtraction:

Given two vectors \mathbf{u} and \mathbf{v} , their sum, $\mathbf{u} + \mathbf{v}$, can be represented by the parallelogram (triangle) rule: place the tail of \mathbf{v} at the head of \mathbf{u}

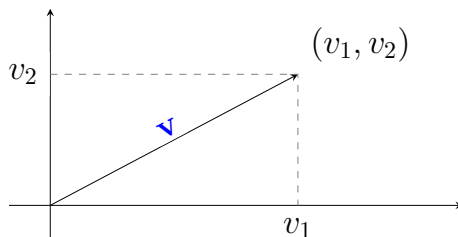


The difference, denoted $\mathbf{u} - \mathbf{v}$, is the sum of $\mathbf{u} + (-\mathbf{v})$:



Vector Components:

A vector \mathbf{v} whose tail is at the origin $(0, 0)$ and head is at (v_1, v_2) is a **position vector** (in **standard position**) and is denoted $\langle v_1, v_2 \rangle$. The real numbers v_1 and v_2 are the x - and y -components of \mathbf{v} .



Vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are equal if and only if $u_1 = v_1$ and $u_2 = v_2$.

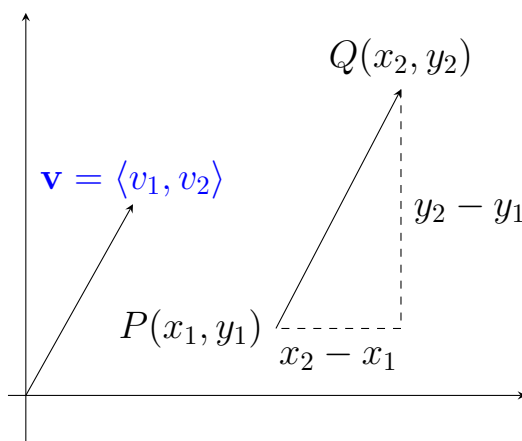
Magnitude:

Given points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the **magnitude**, or **length**, of vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$, denoted $|\overrightarrow{PQ}|$, is the distance between points P and Q .

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The magnitude of position vector $\mathbf{v} = \langle v_1, v_2 \rangle$ is $|\mathbf{v}|$.

(How do $|\overrightarrow{PQ}|$ and $|\overrightarrow{QP}|$ relate to each other?)



Note: The norm, denoted $\|\mathbf{u}\|$ or $\|\mathbf{u}\|_2$, is equivalent to the magnitude of a vector.

Equation of a Circle:

Definition.

A **circle** centered at (a, b) with radius r is the set of points satisfying the equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

A **disk** centered at (a, b) with radius r is the set of points satisfying the inequality

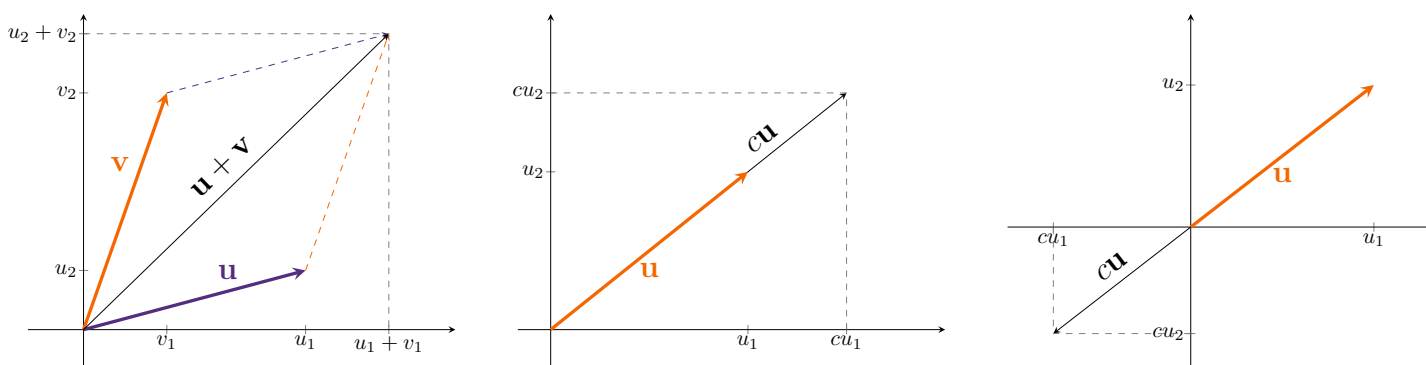
$$(x - a)^2 + (y - b)^2 \leq r^2.$$

Vector Operations in Terms of Components

Definition. (Vector Operations in \mathbb{R}^2)

Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2 \rangle$, and $\mathbf{v} = \langle v_1, v_2 \rangle$.

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= \langle u_1 + v_1, u_2 + v_2 \rangle && \text{Vector addition} \\ \mathbf{u} - \mathbf{v} &= \langle u_1 - v_1, u_2 - v_2 \rangle && \text{Vector subtraction} \\ c\mathbf{u} &= \langle cu_1, cu_2 \rangle && \text{Scalar multiplication}\end{aligned}$$



Example. Let $\mathbf{u} = \langle 1, 2 \rangle$, $\mathbf{v} = \langle -2, 3 \rangle$, $c = 2$, and $d = 3$. Find the following:

$$\mathbf{u} + \mathbf{v}$$

$$c\mathbf{u}$$

$$c\mathbf{u} + d\mathbf{v}$$

$$\mathbf{u} - c\mathbf{v}$$

Definition.

A **unit vector** is any vector with length 1.

In \mathbb{R}^2 , the **coordinate unit vectors** are $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.

Example. Let $\mathbf{u} = \langle -7, 3 \rangle$. Find two unit vectors parallel to \mathbf{u} . Find another vector parallel to \mathbf{u} with a magnitude of 2.

Properties of Vector Operations:

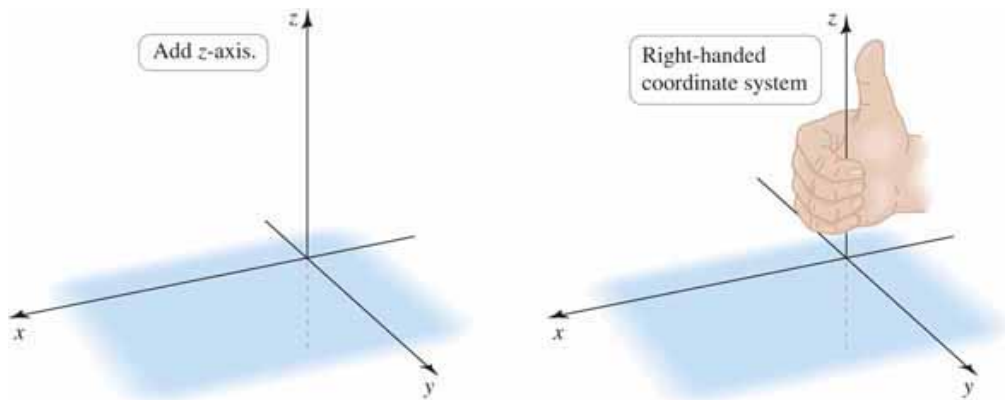
Suppose \mathbf{u} , \mathbf{v} , and \mathbf{w} are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

- | | |
|--|---|
| 1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | Commutative property of addition |
| 2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | Associative property of addition |
| 3. $\mathbf{v} + \mathbf{0} = \mathbf{v}$ | Additive identity |
| 4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ | Additive inverse |
| 5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ | Distributive property 1 |
| 6. $(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$ | Distributive property 2 |
| 7. $0\mathbf{v} = \mathbf{0}$ | Multiplication by zero scalar |
| 8. $c\mathbf{0} = \mathbf{0}$ | Multiplication by zero vector |
| 9. $1\mathbf{v} = \mathbf{v}$ | Multiplicative identity |
| 10. $a(c\mathbf{v}) = (ac)\mathbf{v}$ | Associative property of scalar multiplication |

13.2: Vectors in Three Dimensions

The xyz - Coordinate System:

The three-dimensional coordinate system is created by adding the z -axis, which is perpendicular to both the x -axis and the y -axis. When looking at the xy -plane, the positive direction of the z -axis protrudes towards the viewer. This can also be shown using the right-hand rule (Figure 13.25 from Briggs):

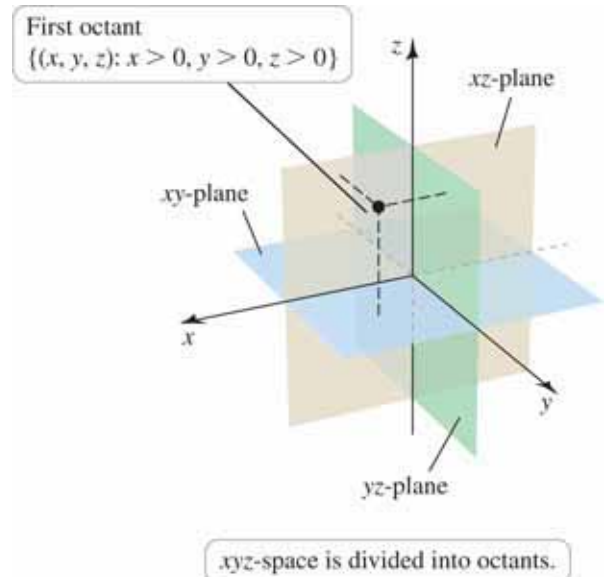


Definition.

This three-dimensional coordinate system is broken up into eight **octants**, which are separated by

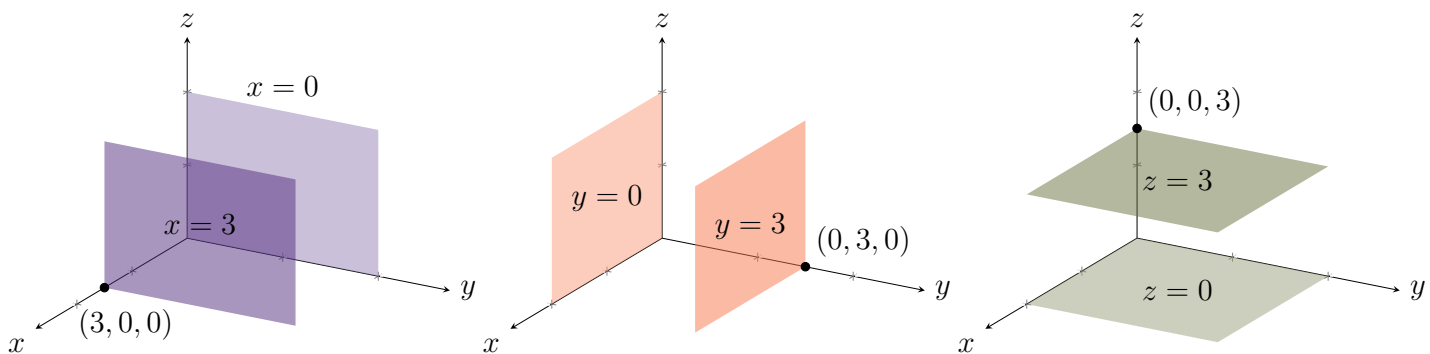
- the xy -**plane** ($z = 0$),
- the xz -**plane** ($y = 0$), and
- the yz -**plane** ($x = 0$).

The **origin** is the location where all three axes intersect.

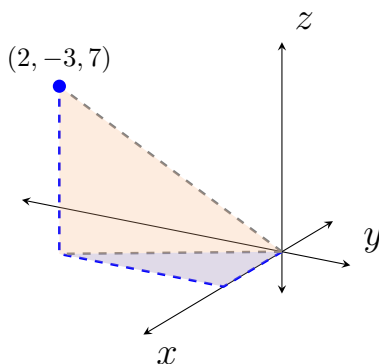


Equations of Simple Planes:

Planes in three-dimensions are analogous to lines in two-dimensions. Below, we see the yz -plane, the xz -plane, and the xy -plane, along with planes that are parallel where x , y , and z are fixed respectively:



Example (Parallel planes). Determine the equation of the plane parallel to the xz -plane passing through the point $(2, -3, 7)$.



Distances in xyz -Space:

Recall that in \mathbb{R}^2 , for some vector \overrightarrow{PR} , the distance formula is given by

$$|PR| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

where (x_1, y_1) and (x_2, y_2) represent the points P and R respectively. This idea can be further extended into \mathbb{R}^3 by considering the two sides of the triangle formed by the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$:



Distance Formula in xyz -Space

The **distance** between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The **midpoint** between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is found by averaging the x -, y -, and z -coordinates:

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

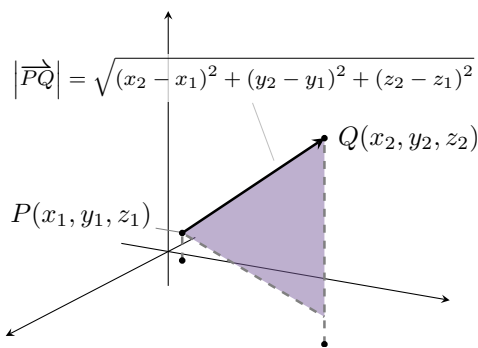
Magnitude and Unit Vectors:

Definition.

The **magnitude** (or **length**) of the vector $\vec{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ is the distance from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$:

$$|\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

In \mathbb{R}^3 , the **coordinate unit vectors** are $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$.



Example. Consider $P(-1, 4, 3)$ and $Q(3, 5, 7)$. Find

- $|\vec{PQ}|$
- The midpoint between P and Q
- Two unit vectors parallel to \vec{PQ}

Equation of a Sphere:

Definition.

A **sphere** centered at (a, b, c) with radius r is the set of points satisfying the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

A **ball** centered at (a, b, c) with radius r is the set of points satisfying the inequality

$$(x - a)^2 + (y - b)^2 + (z - c)^2 \leq r^2.$$

Example. Consider $P(-1, 4, 3)$ and $Q(3, 5, 7)$. Find the equation of the sphere centered at the midpoint passing through P and Q

Example. What is the geometry of the intersection between $x^2 + y^2 + z^2 = 50$ and $z = 1$?

Example. Rewrite the following equation into the standard form of a sphere:

$$x^2 + y^2 + z^2 - 2x + 6y - 8z = -1$$

Vector Operations in Terms of Components

Definition. (Vector Operations in \mathbb{R}^3)

Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$	Vector addition
$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$	Vector subtraction
$c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle$	Scalar multiplication

Properties of Vector Operations:

Suppose \mathbf{u} , \mathbf{v} , and \mathbf{w} are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

- | | |
|--|---|
| 1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | Commutative property of addition |
| 2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | Associative property of addition |
| 3. $\mathbf{v} + \mathbf{0} = \mathbf{v}$ | Additive identity |
| 4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ | Additive inverse |
| 5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ | Distributive property 1 |
| 6. $(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$ | Distributive property 2 |
| 7. $0\mathbf{v} = \mathbf{0}$ | Multiplication by zero scalar |
| 8. $c\mathbf{0} = \mathbf{0}$ | Multiplication by zero vector |
| 9. $1\mathbf{v} = \mathbf{v}$ | Multiplicative identity |
| 10. $a(c\mathbf{v}) = (ac)\mathbf{v}$ | Associative property of scalar multiplication |

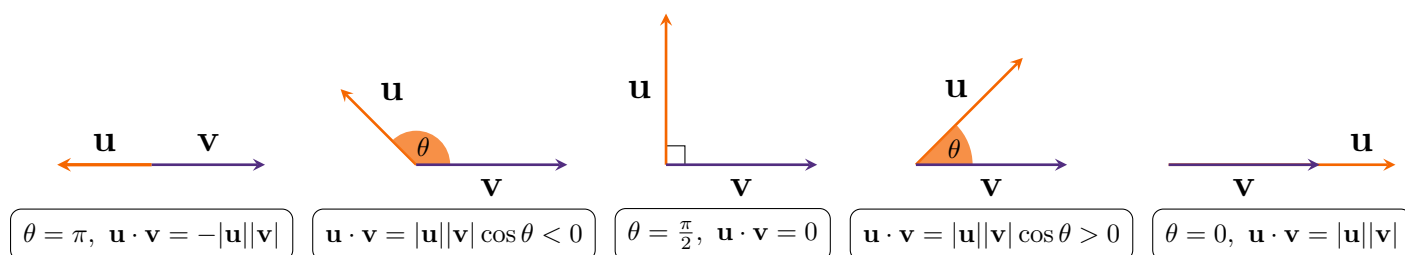
13.3: Dot Products

Definition. (Dot Product)

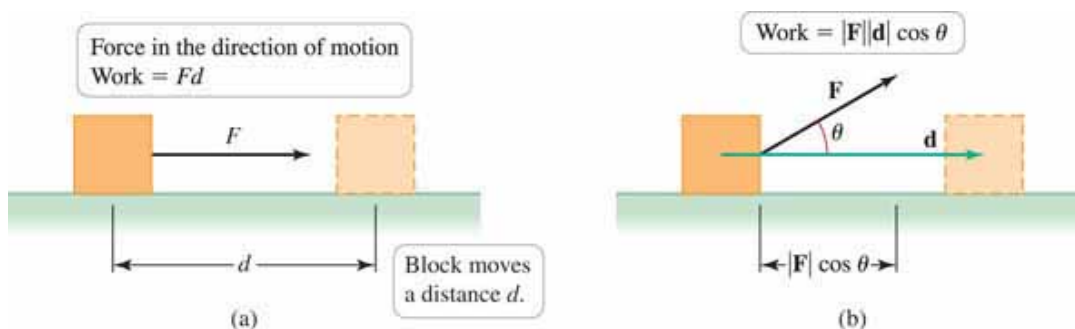
Given two nonzero vectors \mathbf{u} and \mathbf{v} in two or three dimensions, their **dot product** is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta,$$

where θ is the angle between \mathbf{u} and \mathbf{v} with $0 \leq \theta \leq \pi$. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$, and θ is undefined.



A physical example of the dot product is the amount of work done when a force is applied at an angle θ as shown in figure 13.43:



Note: The result of the dot product is a scalar!

Definition. (Orthogonal Vectors)

Two vectors \mathbf{u} and \mathbf{v} are **orthogonal** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. The zero vector is orthogonal to all vectors. In two or three dimensions, two nonzero orthogonal vectors are perpendicular to each other.

- \mathbf{u} and \mathbf{v} are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \cdot \mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$.
- \mathbf{u} and \mathbf{v} are perpendicular ($\theta = \frac{\pi}{2}$) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Example. Given $|\mathbf{u}| = 2$ and $|\mathbf{v}| = \sqrt{3}$, compute $\mathbf{u} \cdot \mathbf{v}$ when

- $\theta = \frac{\pi}{4}$
- $\theta = \frac{\pi}{3}$
- $\theta = \frac{5\pi}{6}$

Theorem 31.1: Dot Product

Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$,

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

Example. Given vectors $\mathbf{u} = \langle \sqrt{3}, 1, 0 \rangle$ and $\mathbf{v} = \langle 1, \sqrt{3}, 0 \rangle$, compute $\mathbf{u} \cdot \mathbf{v}$ and find θ .

Properties of Dot Products

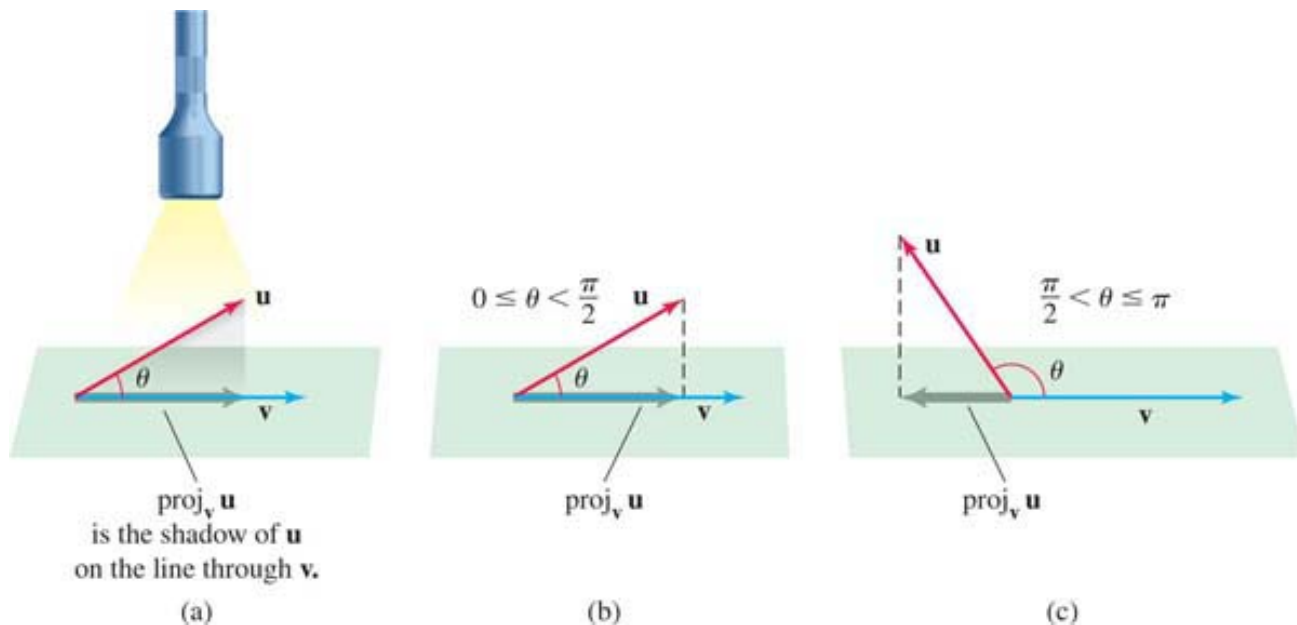
Theorem 13.2: Properties of the Dot Product

Suppose \mathbf{u} , \mathbf{v} and \mathbf{w} are vectors and let c be a scalar.

- | | |
|---|-----------------------|
| 1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ | Commutative property |
| 2. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$ | Associative property |
| 3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ | Distributive property |

Orthogonal Projections

Given vectors \mathbf{u} and \mathbf{v} , the projection of \mathbf{u} onto \mathbf{v} produces a vector parallel to \mathbf{v} using the “shadow” of \mathbf{u} cast onto \mathbf{v} .



Definition. ((Orthogonal) Projection of \mathbf{u} onto \mathbf{v})

The **orthogonal projection of \mathbf{u} onto \mathbf{v}** , denoted $\text{proj}_{\mathbf{v}} \mathbf{u}$, where $\mathbf{v} \neq \mathbf{0}$, is

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \underbrace{|\mathbf{u}| \cos \theta}_{\text{length}} \underbrace{\left(\frac{\mathbf{v}}{|\mathbf{v}|} \right)}_{\text{direction}}.$$

The orthogonal projection may also be computed with the formulas

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \text{scal}_{\mathbf{v}} \mathbf{u} \left(\frac{\mathbf{v}}{|\mathbf{v}|} \right) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v},$$

where the **scalar component of \mathbf{u} in the direction of \mathbf{v}** is

$$\text{scal}_{\mathbf{v}} \mathbf{u} = |\mathbf{u}| \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}.$$

Example. Find $\text{proj}_{\mathbf{v}} \mathbf{u}$ and $\text{scal}_{\mathbf{v}} \mathbf{u}$ for the following:

- $\mathbf{u} = \langle 1, 1 \rangle$, $\mathbf{v} = \langle -2, 1 \rangle$

- $\mathbf{u} = \langle 7, 1, 7 \rangle$, $\mathbf{v} = \langle 5, 7, 0 \rangle$

Applications of Dot Products

Definition. (Work)

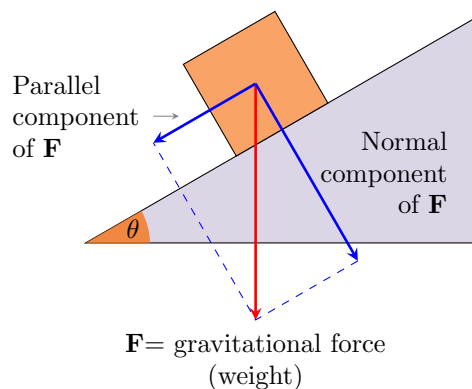
Let a constant force \mathbf{F} be applied to an object, producing a displacement \mathbf{d} . If the angle between \mathbf{F} and \mathbf{d} is θ , then the **work** done by the force is

$$W = |\mathbf{F}||\mathbf{d}| \cos \theta = \mathbf{F} \cdot \mathbf{d}$$

Example. A force $\mathbf{F} = \langle 3, 3, 2 \rangle$ (in newtons) moves an object along a line segment from $P(1, 1, 0)$ to $Q(6, 6, 0)$ (in meters). What is the work done by the force?

Components of a Force:

Example. A 10-lb block rests on a plane that is inclined at 30° above the horizontal. Find the components of the gravitational force parallel to and normal (perpendicular) to the plane.



13.4: Cross Products

Definition. (Cross Product)

Given two nonzero vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^3 , the **cross product** $\mathbf{u} \times \mathbf{v}$ is a vector with magnitude

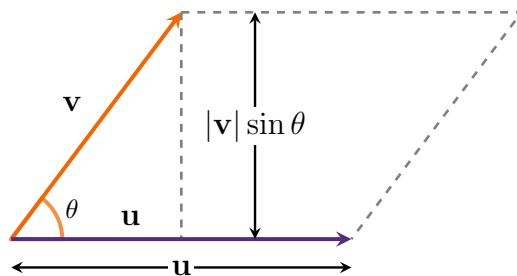
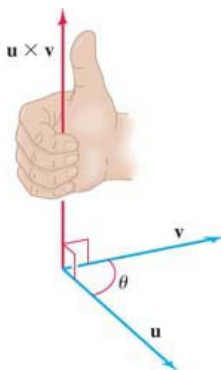
$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta,$$

where $0 \leq \theta \leq \pi$ is the angle between \mathbf{u} and \mathbf{v} .

The direction of $\mathbf{u} \times \mathbf{v}$ is given by the **right-hand rule**:

When you put your the vectors tail to tail and let the fingers of your right hand curl from \mathbf{u} to \mathbf{v} , the direction of $\mathbf{u} \times \mathbf{v}$ is the direction of your thumb, orthogonal to both \mathbf{u} and \mathbf{v} (Figure 13.56).

When $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, the direction of $\mathbf{u} \times \mathbf{v}$ is undefined.



Theorem 13.3: Geometry of the Cross Product

Let \mathbf{u} and \mathbf{v} be two nonzero vectors in \mathbb{R}^3 .

1. The vectors \mathbf{u} and \mathbf{v} are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
2. If \mathbf{u} and \mathbf{v} are two sides of a parallelogram, then the area of the parallelogram is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta$$

Example. Consider the vectors $\mathbf{u} = \langle 2, 0, 0 \rangle$ and $\mathbf{v} = \langle \sqrt{3}, 3, 0 \rangle$. The angle between these vectors is $\theta = \frac{\pi}{3}$. Find the area of the parallelogram formed by these vectors.

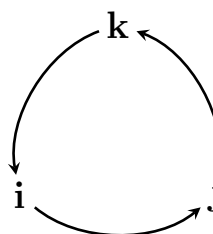
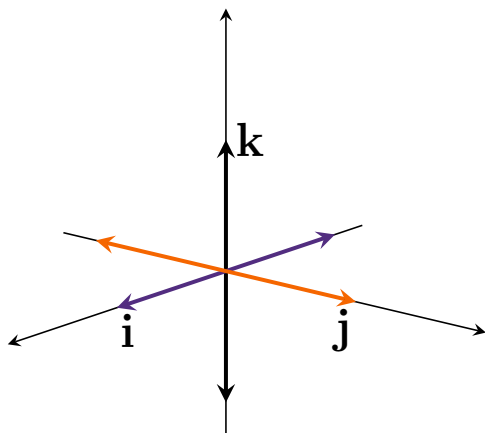
Theorem 13.4: Properties of the Cross Product Let \mathbf{u} , \mathbf{v} , and \mathbf{w} be nonzero vectors in \mathbb{R}^3 , and let a and b be scalars.

- | | |
|--|--------------------------|
| 1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ | Anticommutative property |
| 2. $(a\mathbf{u}) \times (b\mathbf{v}) = ab(\mathbf{u} \times \mathbf{v})$ | Associative property |
| 3. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ | Distributive property |
| 4. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$ | Distributive property |

Theorem 13.5: Cross Products of Coordinate Unit Vectors

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= -(\mathbf{j} \times \mathbf{i}) = \mathbf{k} \\ \mathbf{k} \times \mathbf{i} &= -(\mathbf{i} \times \mathbf{k}) = \mathbf{j} \end{aligned}$$

$$\begin{aligned} \mathbf{j} \times \mathbf{k} &= -(\mathbf{k} \times \mathbf{j}) = \mathbf{i} \\ \mathbf{i} \times \mathbf{i} &= \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0} \end{aligned}$$



$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j} \end{aligned}$$

Using the unit vectors, we can compute $\mathbf{u} \times \mathbf{v}$:

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1 \underbrace{(\mathbf{i} \times \mathbf{i})}_{\mathbf{0}} + u_1v_2 \underbrace{(\mathbf{i} \times \mathbf{j})}_{\mathbf{k}} + u_1v_3 \underbrace{(\mathbf{i} \times \mathbf{k})}_{-\mathbf{j}} \\ &\quad + u_2v_1 \underbrace{(\mathbf{j} \times \mathbf{i})}_{-\mathbf{k}} + u_2v_2 \underbrace{(\mathbf{j} \times \mathbf{j})}_{\mathbf{0}} + u_2v_3 \underbrace{(\mathbf{j} \times \mathbf{k})}_{\mathbf{i}} \\ &\quad + u_3v_1 \underbrace{(\mathbf{k} \times \mathbf{i})}_{\mathbf{j}} + u_3v_2 \underbrace{(\mathbf{k} \times \mathbf{j})}_{-\mathbf{i}} + u_3v_3 \underbrace{(\mathbf{k} \times \mathbf{k})}_{\mathbf{0}} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} \end{aligned}$$

Theorem 13.6: Evaluating the Cross Product

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

Note:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Alternative approach:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \begin{vmatrix} \mathbf{i} & \mathbf{j} \\ u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

Example. Compute $\mathbf{u} \times \mathbf{v}$ for $\mathbf{u} = \langle 3, 5, 4 \rangle$ and $\mathbf{v} = \langle 1, -1, 9 \rangle$.

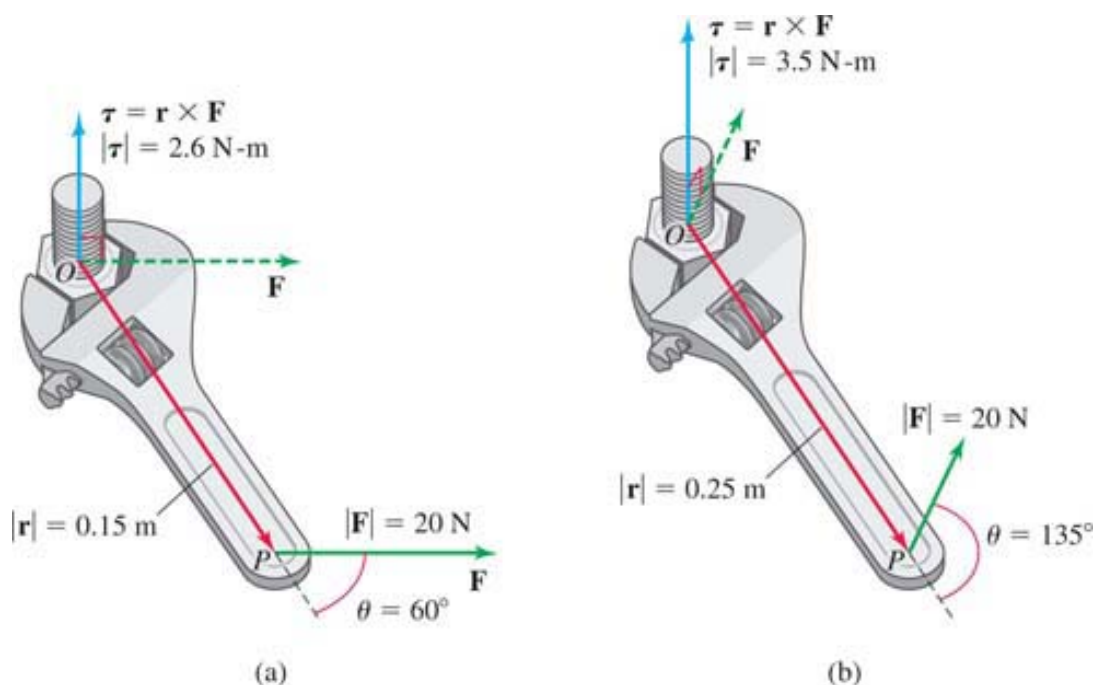
Example. Consider the vectors $\mathbf{u} = \langle \sqrt{3}, 1, 0 \rangle$ and $\mathbf{v} = \langle -\sqrt{3}, 1, 0 \rangle$. From the unit circle, we know the angle between these two vectors is $\theta = \frac{2\pi}{3}$. Use the definition of the cross product to show this.

Example. Find the area of the triangle formed by $\mathbf{u} = \langle 1, 2, 3 \rangle$ and $\mathbf{v} = \langle 3, -1, 1 \rangle$.

Example. Given a force \mathbf{F} applied to a point P at the head of the vector $\mathbf{r} = \overrightarrow{OP}$, the **torque** produced at point O is given by $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ with magnitude

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin \theta.$$

Now suppose a force of 20N is applied to a wrench attached to a bolt in a direction perpendicular to the bolt. Which produces more torque: applying the force at an angle of 60° on a wrench that is 0.15m long or applying the force at an angle of 135° on a wrench that is 0.25m long?

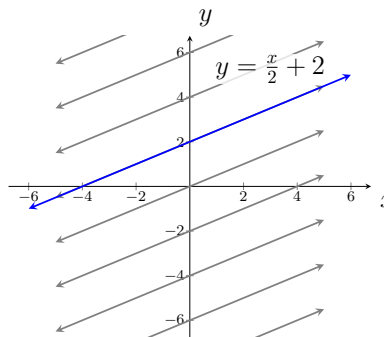


13.5: Lines and Planes in Space

Equation of a Line:

Recall the equation of a line in \mathbb{R}^2 :

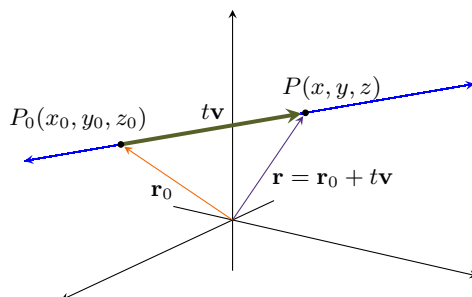
$$y = mx + b$$



where b is the intercept and m is the slope. This idea can be extended into higher dimensions:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

Here, \mathbf{r}_0 is a fixed point, and \mathbf{v} is the position vector that is parallel to the line \mathbf{r} .



Equation of a Line

A **vector equation of the line** passing through the point $P_0(x_0, y_0, z_0)$ in the direction of the vector $\mathbf{v} = \langle a, b, c \rangle$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, or

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle, \quad \text{for } -\infty < t < \infty$$

Equivalently, the corresponding **parametric equations of the line** are

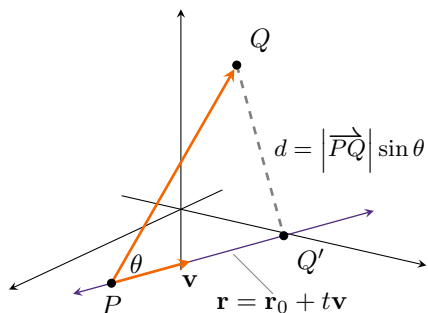
$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \text{for } -\infty < t < \infty$$

Example. Find the vector equation and parametric equation of the line that

- goes through the points $P(-1, -2, 1)$ and $Q(-4, -5, -3)$ where $t = 0$ corresponds to P ,
- goes through the point $P(1, -3, -3)$ and is parallel to the vector $\mathbf{r} = \langle -4, 1, -1 \rangle$,
- goes through the point $P(-2, 5, -2)$ and is perpendicular to the lines $x = 3 - 4t$, $y = 2 - 3t$, $z = -1 - t$, and $x = -2 + 0t$, $y = 2 - t$, $z = 3t$, where $t = 0$ corresponds to P .

Distance from a Point to a Line:

Given a point Q and a line ℓ , the shortest distance to the line is the length of $\overrightarrow{QQ'}$.



From the definition of the cross product, we have

$$|\mathbf{v} \times \overrightarrow{PQ}| = |\mathbf{v}| \underbrace{|\overrightarrow{PQ}| \sin \theta}_d = |\mathbf{v}| d$$

From here, solving for d gives us the following:

Distance Between a Point and a Line

The distance d between the point Q and the $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ is

$$d = \frac{|\mathbf{v} \times \overrightarrow{PQ}|}{|\mathbf{v}|},$$

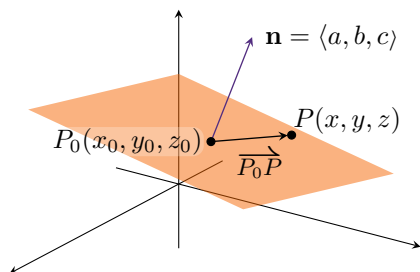
where P is any point on the line and \mathbf{v} is a vector parallel to the line.

Example. Find the distance from the point $Q(-4, -1, -3)$ and the line $x = -5 - 5t$, $y = -5 + t$, $z = -1 + 4t$. (*Hint:* Let P be the point at $t = 0$)

Equations of Planes:

In \mathbb{R}^2 , two distinct points determine a line.

In \mathbb{R}^3 , three noncollinear points determine a unique plane. Alternatively, a plane is uniquely determined by a point and a vector that is orthogonal to the plane.



Definition. (Plane in \mathbb{R}^3)

Given a fixed point P_0 and a nonzero **normal vector** \mathbf{n} , the set of points P in \mathbb{R}^3 for which $\overrightarrow{P_0P}$ is orthogonal to \mathbf{n} is called a **plane**.

Consider the normal vector $\mathbf{n} = \langle a, b, c \rangle$ at the point $P_0(x_0, y_0, z_0)$, and any point $P(x, y, z)$ on the plane. Since \mathbf{n} is orthogonal to the plane, it is also orthogonal to the vector $\overrightarrow{P_0P}$, which is also in the plane. Thus,

$$\begin{aligned}\mathbf{n} \cdot \overrightarrow{P_0P} &= 0 \\ \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \\ ax + by + cz &= d\end{aligned}$$

General Equation of a Plane in \mathbb{R}^3

The plane passing through the point $P_0(x_0, y_0, z_0)$ with a nonzero normal vector $\mathbf{n} = \langle a, b, c \rangle$ is described by the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{or} \quad ax + by + cz = d,$$

where $d = ax_0 + by_0 + cz_0$.

Example. Find the equation of the plane that

- goes through the point $P(-2, 5, 0)$ and is parallel to the plane $x - 5y - 5z = 1$,
- goes through the points $P(5, -2, 1)$, $Q(5, 1, 3)$ and $R(1, -5, -2)$
- that is parallel to the vectors $\langle 4, -2, -3 \rangle$ and $\langle 3, 2, 3 \rangle$, passing through the point $P(-2, -2, 5)$.

Example. Find the location where the line $\langle -3, 1, 4 \rangle + t\langle -1, -4, 2 \rangle$ and the plane $2x - 2y - 4z = 5$ intersect.

Definition. (Parallel and Orthogonal Planes)

Two distinct planes are **parallel** if their respective normal vectors are parallel (that is, the normal vectors are scaling multiples of each other). Two planes are **orthogonal** if their respective normal vectors are orthogonal (that is, the dot product of the normal vectors is *zero*).

Example. Find the line of intersection between the planes $3x - y + 4z = -4$ and $x + 3y - 2z = 0$.

Example. Find the smallest angle between planes $3x - y + 4z = -4$ and $x + 3y - 2z = 0$.

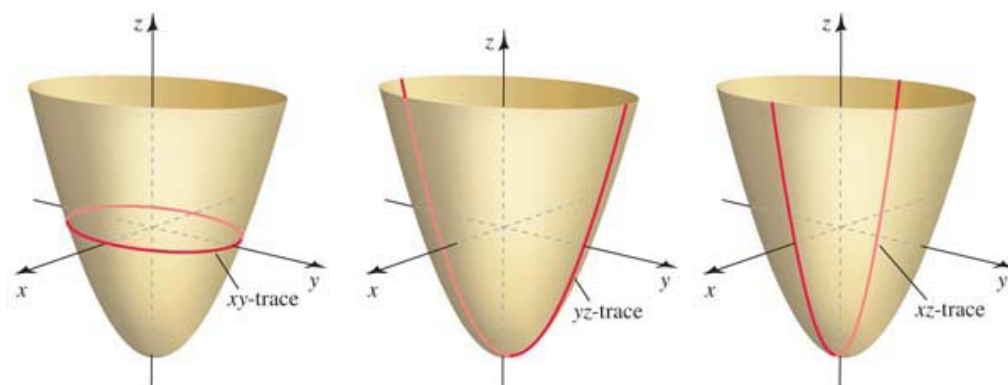
13.6: Cylinders and Quadric Surfaces

Cylinders and Traces:

When talking about three-dimensional surfaces, a *cylinder* refers to a surface that is parallel to a line. When considering surfaces that is parallel to one of the coordinate axes, that the associated variable is missing (e.g. $3y^2 + z^2 = 8$ is parallel to the x -axis).

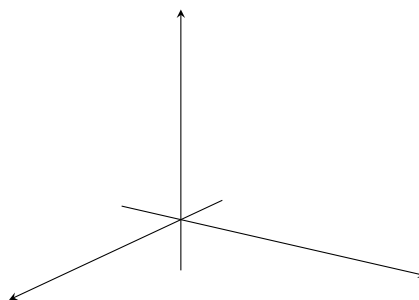
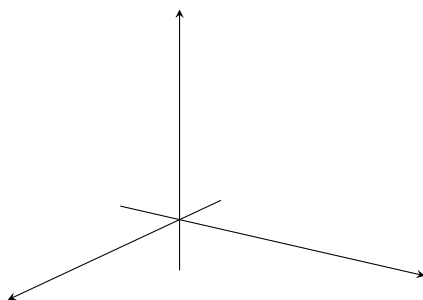
Definition. (Trace)

A **trace** of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes. The traces in the coordinate planes are called the **xy -trace**, the **yz -trace**, and the **xz -trace** (Figure 13.80).



Example. Roughly sketch the following functions:

1. $x^2 + 4y^2 = 16$
2. $x - \sin(z) = 0$



Quadric Surfaces:

Quadric surfaces are described by the general quadratic (second-degree) equation in three variables,

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0,$$

Where the coefficients A, \dots, J and not all zero. To sketch quadric surfaces, keep the following ideas in mind:

1. **Intercepts** Determine the points, if any, where the surface intersects the coordinate axes. To find these intercepts, set x , y , and z equal to zero in pairs in the equation of the surface, and solve for the third coordinate.
2. **Traces** Finding traces of the surface helps visualize the surface. Setting x , y , and z equal to zero in pairs gives the planes parallel in that pair's plane.
3. **Completing the figure** Sketch some traces in parallel planes, then draw smooth curves that pass through the traces to fill out the surface.

Example (An ellipsoid). The surface defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Graph $a = 3$, $b = 4$ and $c = 5$.

Example (An elliptic paraboloid). The surface defined by the equation $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. Graph the elliptic paraboloid with $a = 4$ and $b = 2$.

Example (A hyperboloid of one sheet).

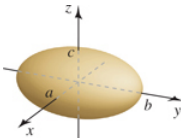
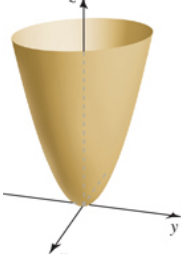
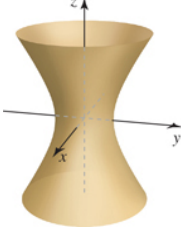
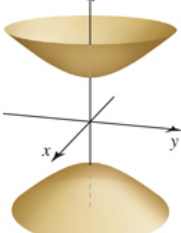
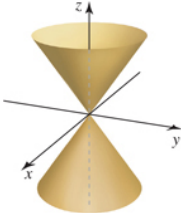
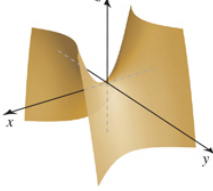
Graph the surface defined by the equation $\frac{x^2}{4} + \frac{y^2}{9} - z^2 = 1$.

Example (A hyperboloid of two sheets). Graph the surface defined by the equation $-16x^2 - 4y^2 + z^2 + 64x - 80 = 0$.

Example (Elliptic cones). Graph the surface defined by the equation $\frac{y^2}{4} + z^2 = 4x^2$.

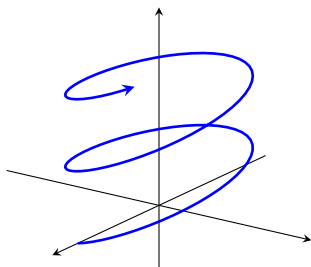
Example (A hyperbolic paraboloid).

Graph the surface defined by the equation $z = x^2 - \frac{y^2}{4}$.

Name	Standard Equation	Features	Graph
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	All traces are ellipses.	
Elliptic paraboloid	$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	Traces with $z = z_0 > 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are parabolas.	
Hyperboloid of one sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	Traces with $z = z_0$ are ellipses for all z_0 . Traces with $x = x_0$ or $y = y_0$ are hyperbolas.	
Hyperboloid of two sheets	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Traces with $z = z_0$ with $ z_0 > c $ are ellipses. Traces with $x = x_0$ and $y = y_0$ are hyperbolas.	
Elliptic cone	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$	Traces with $z = z_0 \neq 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are hyperbolas or intersecting lines.	
Hyperbolic paraboloid	$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$	Traces with $z = z_0 \neq 0$ are hyperbolas. Traces with $x = x_0$ or $y = y_0$ are parabolas.	

14.1: Vector-Valued Functions

Vector-valued functions are functions of the form $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, where $x(t)$, $y(t)$, and $z(t)$ are parametric equations dependent on t .



Curves in Space

Consider

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k},$$

where f , g , and h are defined for $a \leq t \leq b$. The **domain** of \mathbf{r} is the largest set of t for which all of f , g , and h are defined.

Example. What plane does the curve $\mathbf{r}(t) = t\mathbf{i} + 6t^3\mathbf{k}$ lie?

Example (Lines as vector-valued functions). Find a vector function for the line that passes through the points $P(5, 2, -4)$ and $Q(5, 5, -2)$. What about the line segment that connects P and Q ?

Example. Find the domain of

$$\mathbf{r}(t) = \sqrt{16 - t^2}\mathbf{i} + \sqrt{t}\mathbf{j} + \frac{4}{\sqrt{3 + t}}\mathbf{k}$$

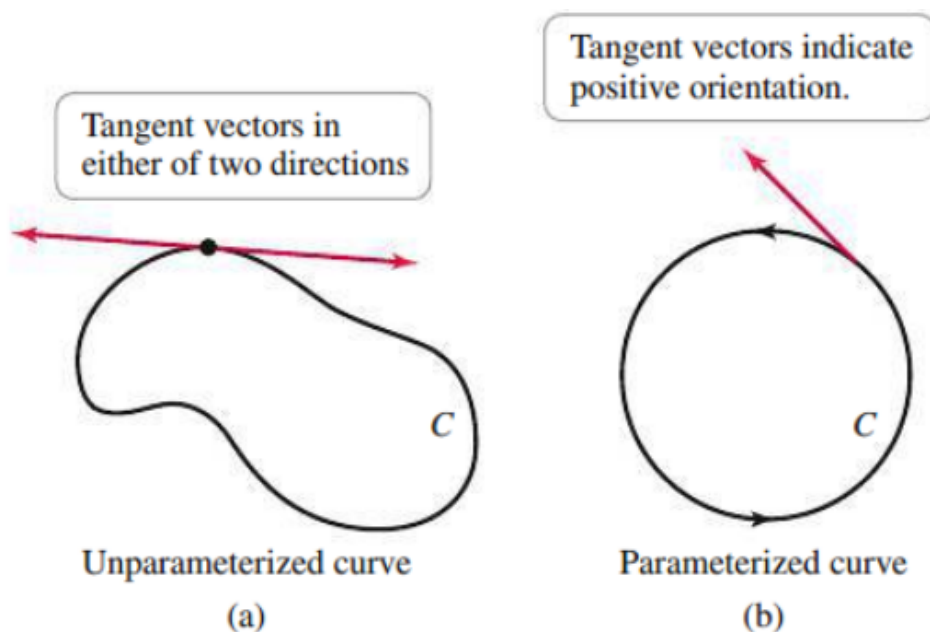
Example. Find the point P on

$$\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j} + 2t\mathbf{k},$$

closest to $P_0(4, 17, 10)$. What is the distance between P and P_0 ?

Orientation of Curves

- A **unparameterized curve** is a smooth curve C with no specified direction and the tangent vector can be drawn in two directions.
- A **parameterized curve** is a smooth curve C described by a function $\mathbf{r}(t)$ for $a \leq t \leq b$ and has a direction referred to as its **orientation**.
- The *positive* orientation is the direction of the curve generated when t increases from a to b .
- The tangent vector of a parameterized curve points in the positive orientation of the curve.



Example. Graph the curve described by the equation

$$\mathbf{r}(t) = 4 \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + \frac{t}{2\pi}\mathbf{k},$$

where $0 \leq t \leq 2\pi$. Indicate the positive orientation of this curve.

Limits and Continuity for Vector-Valued Functions

The properties of limits extend to vector-valued functions naturally. In particular, for $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, if

$$\lim_{t \rightarrow a} f(t) = L_1, \quad \lim_{t \rightarrow a} g(t) = L_2, \quad \lim_{t \rightarrow a} h(t) = L_3$$

then

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \left\langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \right\rangle = \langle L_1, L_2, L_3 \rangle.$$

Definition. (Limit of a Vector-Valued Function)

A vector-valued function \mathbf{r} approaches the limit \mathbf{L} as t approaches a , written $\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{L}$, provided $\lim_{t \rightarrow a} |\mathbf{r}(t) - \mathbf{L}| = 0$.

A function $\mathbf{r}(t)$ is **continuous** at $t = a$, provided $\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{r}(a)$.

Example. Evaluate the following limits:

$$\lim_{t \rightarrow \pi} \left(\cos(t) \mathbf{i} - 7 \sin \left(-\frac{t}{2} \right) \mathbf{j} + \frac{t}{\pi} \mathbf{k} \right)$$

$$\lim_{t \rightarrow \infty} \left(\frac{t}{t-3} \mathbf{i} + \frac{40}{1+19e^{-t}} \mathbf{j} + \frac{1}{2t} \mathbf{k} \right)$$

14.2: Calculus of Vector-Valued Functions

Definition. (Derivative and Tangent Vector)

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g , and h are differentiable functions on (a, b) . Then \mathbf{r} has a **derivative** (or is **differentiable**) on (a, b) and

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

Provided $\mathbf{r}'(t) \neq \mathbf{0}$, $\mathbf{r}'(t)$ is a **tangent vector** at the point corresponding to $\mathbf{r}(t)$.

Example. For the following functions below, find $\mathbf{r}'(t)$

a) $\mathbf{r}(t) = \langle e^{-t^2}, \log_2(t - 4), \sin(t) \rangle$

b) $\mathbf{r}(t) = 3\mathbf{i} - 2\tan(t)\mathbf{j} + e^t\mathbf{k}$

Example. For $\mathbf{r}(t) = \langle 3t, \sec(2t), \cos(t) \rangle$ compute $\mathbf{r}'(t)$ at $t = \frac{\pi}{4}$.

Definition. (Unit Tangent Vector)

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a smooth parameterized curve, for $a \leq t \leq b$. The **unit tangent vector** for a particular value of t is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Example. For $\mathbf{r}(t) = \langle 3 \sin(t), -2 \cos(2t), 3 \cos(t) \rangle$, find the unit tangent vector.

Example. For $\mathbf{r}(t) = \langle \sin(6t), 3t, \cos(3t) \rangle$, compute $\mathbf{T}\left(\frac{\pi}{3}\right)$.

Derivative Rules

Let \mathbf{u} and \mathbf{v} be differentiable vector-valued functions, and let f be a differentiable scalar-valued function, all at a point t . Let \mathbf{c} be a constant vector. The following rules apply.

- | | |
|---|--------------------|
| 1. $\frac{d}{dt}(\mathbf{c}) = \mathbf{0}$ | Constant Rule |
| 2. $\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$ | Sum Rule |
| 3. $\frac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ | Product Rule |
| 4. $\frac{d}{dt}(\mathbf{u}(f(t))) = \mathbf{u}'(f(t))f'(t)$ | Chain Rule |
| 5. $\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ | Dot Product Rule |
| 6. $\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ | Cross Product Rule |

Example. Given $\mathbf{u}(t) = \langle 4t^2, 1, 3t \rangle$ and $\mathbf{v}(t) = \langle e^{-2t}, -2e^t, e^t \rangle$, find $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$.

Definition. (Indefinite Integral of a Vector-Valued Function)

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function, and let

$\mathbf{R}(t) = F(t)\mathbf{i} + G(t)\mathbf{j} + H(t)\mathbf{k}$, where F , G , and H are antiderivatives of f , g , and h , respectively. The **indefinite integral** of \mathbf{r} is

$$\int \mathbf{r}(t) dt = \mathbf{R}(t) + \mathbf{C},$$

where \mathbf{C} is an arbitrary constant vector. Alternatively, in component form,

$$\int \langle f(t), g(t), h(t) \rangle dt = \langle F(t), G(t), H(t) \rangle + \langle C_1, C_2, C_3 \rangle.$$

Example. Find $\mathbf{r}(t)$ such that $\mathbf{r}'(t) = \left\langle \frac{t}{t^2+1}, t^2e^{-t^3}, \frac{-2t}{\sqrt{t^2+16}} \right\rangle$ and $\mathbf{r}(0) = \left\langle 3, \frac{5}{3}, -5 \right\rangle$.

Definition. (Definite Integral of a Vector-Valued Function)

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f , g , and h are integrable on the interval $[a, b]$. The **definite integral** of \mathbf{r} on $[a, b]$ is

$$\int_a^b \mathbf{r}(t) dt = \left(\int_a^b f(t) dt \right) \mathbf{i} + \left(\int_a^b g(t) dt \right) \mathbf{j} + \left(\int_a^b h(t) dt \right) \mathbf{k}$$

Example. $\int_{-\pi}^{\pi} \langle \sin(t), \cos(t), 8t \rangle dt$

14.3: Motion in Space

Definition.

Let the **position** of an object moving in three-dimensional space be given by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $t \geq 0$. The **velocity** of the object is

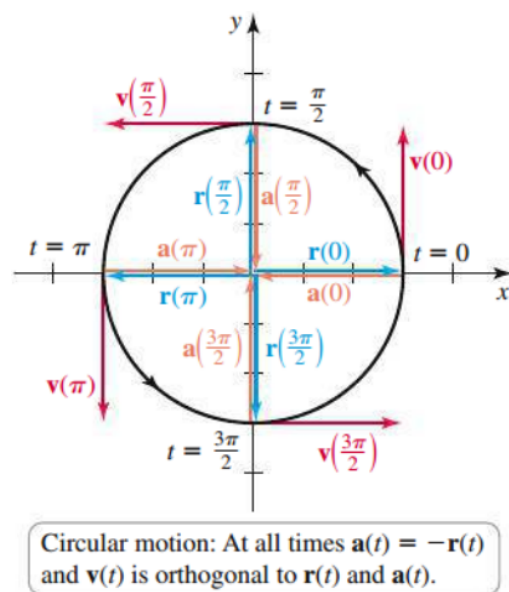
$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.$$

The **speed** of the object is the scalar function

$$|\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

The **acceleration** of the object is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

Example. Given $\mathbf{r}(t) = \langle 3 \cos(t), 3 \sin(t) \rangle$ for $0 \leq t \leq 2\pi$, find the velocity, speed, and acceleration.



Theorem 14.2: Motion with constant $|\mathbf{r}|$

Let \mathbf{r} describe a path on which $|\mathbf{r}|$ is constant (motion on a circle or sphere centered at the origin). Then $\mathbf{r} \cdot \mathbf{v} = 0$, which means the position vector and the velocity vector are orthogonal at all times for which the functions are defined.

Example (Path on a sphere). Consider

$$\mathbf{r}(t) = \langle 3 \cos(t), 5 \sin(t), 4 \cos(t) \rangle, \quad \text{for } 0 \leq t \leq 2\pi.$$

a) Show that an object with this trajectory moves on a sphere and find the radius.

b) Find the velocity and speed of the above trajectory.

c) Show that $\mathbf{r}(t) = \langle 5 \cos(t), 5 \sin(t), 5 \sin(2t) \rangle$ does not lie on a sphere. How could this function be modified so that it does lie on a sphere?

Example. Given $\mathbf{a}(t) = \langle \cos(t), 4 \sin(t) \rangle$, with an initial velocity $\langle \mathbf{u}_0, \mathbf{v}_0 \rangle = \langle 0, 4 \rangle$ and an initial position $\langle x_0, y_0 \rangle = \langle 5, 0 \rangle$ where $t \geq 0$, find the velocity and position vector.

Summary: Two-Dimensional Motion in a Gravitational Field

Consider an object moving in a plane with a horizontal x -axis and a vertical y -axis, subject only to the force of gravity. Given the initial velocity $\mathbf{v}(0) = \langle u_0, v_0 \rangle$ and the initial position $\mathbf{r}(0) = \langle x_0, y_0 \rangle$, the velocity of the object, for $t \geq 0$, is

$$\mathbf{v}(t) = \langle x'(t), y'(t) \rangle = \langle u_0, -gt + v_0 \rangle$$

and the position is

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \left\langle u_0 t + x_0, -\frac{1}{2}gt^2 + v_0 t + y_0 \right\rangle.$$

Example. Consider a ball with an initial position of $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$ m and an initial velocity of $\langle u_0, v_0 \rangle = \langle 25, 4 \rangle$ m/s.

a) Find the position and velocity of the ball while it is in the air

Summary: Two-Dimensional Motion

Assume an object traveling over horizontal ground, acted on only by the gravitational force, has an initial position $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$ and initial velocity $\langle u_0, v_0 \rangle = \langle |\mathbf{v}_0| \cos \alpha, |\mathbf{v}_0| \sin \alpha \rangle$. The trajectory, which is a segment of a parabola, has the following properties.

$$\text{time of flight} = T = \frac{2|\mathbf{v}_0| \sin \alpha}{g}$$

$$\text{range} = \frac{|\mathbf{v}_0|^2 \sin(2\alpha)}{g}$$

$$\text{maximum height} = y\left(\frac{T}{2}\right) = \frac{(|\mathbf{v}_0| \sin \alpha)^2}{2g}$$

Example. Consider a ball with an initial position of $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$ m and an initial velocity of $\langle u_0, v_0 \rangle = \langle 25, 4 \rangle$ m/s. Assuming the ground is flat and level:

b) How long is the ball in the air?

c) How far does the ball travel horizontally?

d) What is the maximum height that the ball reaches?

14.4: Length of Curves

Definition. (Arc Length for Vector Functions)

Consider the parameterized curve $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where f' , g' , and h' are continuous, and the curve is traversed once for $a \leq t \leq b$. The **arc length** of the curve between $(f(a), g(a), h(a))$ and $(f(b), g(b), h(b))$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} dt = \int_a^b |\mathbf{r}'(t)| dt.$$

Example (Flight of an eagle). Suppose an eagle rises at a rate of 100 vertical ft/min on a helical path given by

$$\mathbf{r}(t) = \langle 250 \cos(t), 250 \sin(t), 100t \rangle$$

where \mathbf{r} is measured in feet and t is measured in minutes. How far does it travel in 10 minutes?

Example. Suppose a particle has a trajectory given by

$$\mathbf{r}(t) = \langle 10 \cos(3t), 10 \sin(3t) \rangle$$

where $0 \leq t \leq \pi$. How far does this particle travel?

Example. Find the length of the curve

$$\mathbf{r}(t) = \langle 3t^2 - 5, 4t^2 + 5 \rangle$$

where $0 \leq t \leq 1$.

Example. Find the length of $\mathbf{r}(t) = \left\langle t^2, \frac{(4t+1)^{\frac{3}{2}}}{6} \right\rangle$ where $0 \leq t \leq 6$.

Example. Find the length of $\mathbf{r}(t) = \langle 2\sqrt{2}, \sin(t), \cos(t) \rangle$ where $0 \leq t \leq 5$.

Theorem 14.3: Arc Length as a Function of a Parameter

Let $\mathbf{r}(t)$ describe a smooth curve, for $t \geq a$. The arc length is given by

$$s(t) = \int_a^t |\mathbf{v}(u)| \, du,$$

where $|\mathbf{v}| = |\mathbf{r}'|$. Equivalently, $\frac{ds}{dt} = |\mathbf{v}(t)|$. If $|\mathbf{v}(t)| = 1$, for all $t \geq a$, then the parameter t corresponds to arc length.

Example. For the following functions, determine if $\mathbf{r}(t)$ uses arc length as a parameter. If not, find a description that uses arc length as a parameter.

a) $\mathbf{r}(t) = \langle -4t + 1, 3t - 1 \rangle, 0 \leq t \leq 4$.

b) $\mathbf{r}(t) = \left\langle \frac{1}{\sqrt{10}} \cos(t), \frac{3}{\sqrt{10}} \cos(t), \sin(t) \right\rangle, 0 \leq t \leq 2\pi$.

14.5: Curvature and Normal Vectors:

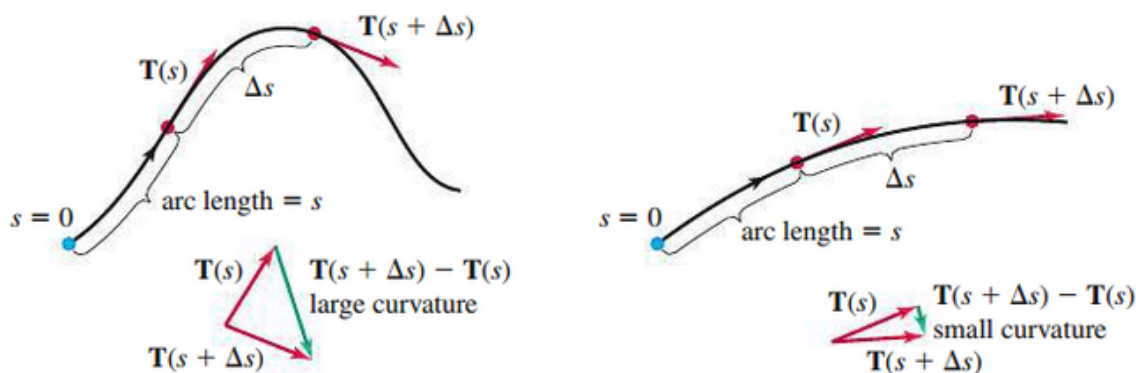
There are two ways to change the velocity, or in other words, to accelerate:

- change in speed
- change in direction

The change in direction is referred to as *curvature*. Recall that if we have a smooth curve $\mathbf{r}(t)$, the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

Specifically, *curvature* of the curve is the magnitude of the rate at which \mathbf{T} changes with respect to arc length.



Definition. (Curvature)

Let \mathbf{r} describe a smooth parameterized curve. If s denotes arc length and $\mathbf{T} = \mathbf{r}'/|\mathbf{r}'|$ is the unit tangent vector, the **curvature** is $\kappa(s) = \left| \frac{d\mathbf{T}}{ds} \right|$.

Theorem 14.4: Curvature Formula

Let $\mathbf{r}(t)$ describe a smooth parameterized curve, where t is any parameter. If $\mathbf{v} = \mathbf{r}'$ is the velocity and \mathbf{T} is the unit tangent vector, then the curvature is

$$\kappa(t) = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

- κ is a non-negative scalar-valued function
- Curvature of zero corresponds to a straight line
- A relatively flat curve has a small curvature
- A tight curve has a larger curvature

Example. Consider the line

$$\mathbf{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle, \text{ for } -\infty < t < \infty.$$

Compute κ .

Example. Consider the circle

$$\mathbf{r}(t) = \langle R \cos(t), R \sin(t) \rangle$$

for $0 \leq t \leq 2\pi$, where $R > 0$. Show that $\kappa = 1/R$.

Example. Consider the curve

$$\mathbf{r}(t) = \langle 2 \cos(t), 2 \sin(t), \sqrt{5}t \rangle$$

Compute κ .

An Alternative Curvature Formula:

Consider a smooth function $\mathbf{r}(t)$ with non-zero velocity $\mathbf{v}(t) = \mathbf{r}'(t)$ and non-zero acceleration $\mathbf{a}(t) = \mathbf{v}'(t)$.

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} \Rightarrow \mathbf{v} = |\mathbf{v}| \mathbf{T}.$$

Thus

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}[|\mathbf{v}| \mathbf{T}] = \frac{d}{dt}[|\mathbf{v}|] \mathbf{T} + |\mathbf{v}| \frac{d\mathbf{T}}{dt}.$$

Now we form $\mathbf{v} \times \mathbf{a}$:

$$\begin{aligned} \mathbf{v} \times \mathbf{a} &= |\mathbf{v}| \mathbf{T} \times \left(\frac{d}{dt}[|\mathbf{v}|] \mathbf{T} + |\mathbf{v}| \frac{d\mathbf{T}}{dt} \right) \\ &= \underbrace{|\mathbf{v}| \mathbf{T} \times \frac{d}{dt}[|\mathbf{v}|] \mathbf{T}}_0 + |\mathbf{v}| \mathbf{T} \times |\mathbf{v}| \frac{d\mathbf{T}}{dt} \end{aligned}$$

Since \mathbf{T} is a unit vector, \mathbf{T} and $d\mathbf{T}/dt$ are orthogonal (Theorem 14.2). Thus

$$|\mathbf{v} \times \mathbf{a}| = \left| |\mathbf{v}| \mathbf{T} \times |\mathbf{v}| \frac{d\mathbf{T}}{dt} \right| = |\mathbf{v}| \underbrace{|\mathbf{T}|}_1 \left| |\mathbf{v}| \frac{d\mathbf{T}}{dt} \right| \underbrace{\sin \theta}_1 = |\mathbf{v}|^2 \left| \frac{d\mathbf{T}}{dt} \right|$$

Now, using Theorem 14.4, where $\left| \frac{d\mathbf{T}}{dt} \right| = \kappa |\mathbf{v}|$, we have

$$|\mathbf{v} \times \mathbf{a}| = |\mathbf{v}|^2 \left| \frac{d\mathbf{T}}{dt} \right| = |\mathbf{v}|^2 \kappa |\mathbf{v}| = \kappa |\mathbf{v}|^3.$$

Theorem 14.5: Alternative Curvature Formula

Let \mathbf{r} be the position of an object moving on a smooth curve. The **curvature** at a point on the curve is

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3},$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{a} = \mathbf{v}'$ is the acceleration.

Example. Consider the curve

$$\mathbf{r}(t) = \langle -16 \cos(t), 16 \sin(t), 0 \rangle.$$

Compute the curvature κ using both methods.

Principal Unit Normal Vector

Curvature indicates how quickly a curve turns. The principal unit normal vector determines the *direction* in which a curve turns.

Definition. (Principal Unit Normal Vector)

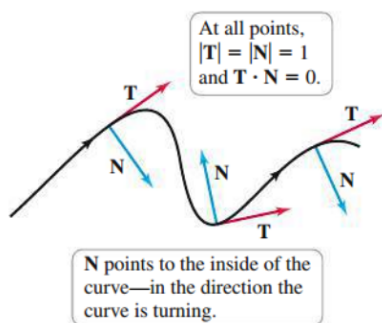
Let \mathbf{r} describe a smooth curve parameterized by arc length. The **principal unit normal vector** at a point P on the curve at which $\kappa \neq 0$ is

$$\mathbf{N}(s) = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{1}{\kappa} \frac{d\mathbf{T}}{ds}.$$

For other parameters, we use the equivalent formula

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|},$$

evaluated at the value of t corresponding to P .



Theorem 14.6: Properties of the Principal Unit Normal Vector

Let \mathbf{r} describe a smooth parameterized curve with unit tangent vector \mathbf{T} and principal unit normal vector \mathbf{N} .

1. \mathbf{T} and \mathbf{N} are orthogonal at all points of the curve; that is, $\mathbf{T} \cdot \mathbf{N} = 0$ at all points where \mathbf{N} is defined.
2. The principal unit normal vector points to the inside of the curve – in the direction that the curve is turning.

Example. For the curve $\mathbf{r}(t) = \langle a \cos(t), a \cos(t), bt \rangle$, find the unit tangent vector \mathbf{T} and the principal unit normal vector \mathbf{N} . Verify $|\mathbf{T}| = |\mathbf{N}| = 1$ and $\mathbf{T} \cdot \mathbf{N} = 0$.

Components of the Acceleration

Recall that the change in velocity, or acceleration, of an object can change in *speed* (in the direction of \mathbf{T}) and in *direction* (in the direction of \mathbf{N}). $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} \implies \mathbf{v} = \mathbf{T}|\mathbf{v}| = \mathbf{T} \frac{ds}{dt}$.

$$\begin{aligned} \mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d}{dt} \left(\mathbf{T} \frac{ds}{dt} \right) \\ &= \frac{d\mathbf{T}}{dt} \frac{ds}{dt} + \mathbf{T} \frac{d^2s}{dt^2} \\ &= \underbrace{\frac{d\mathbf{T}}{ds}}_{\kappa \mathbf{N}} \underbrace{\frac{ds}{dt}}_{|\mathbf{v}|} \underbrace{\frac{ds}{dt}}_{|\mathbf{v}|} + \mathbf{T} \frac{d^2s}{dt^2} \\ &= \kappa |\mathbf{v}|^2 \mathbf{N} + \frac{d^2s}{dt^2} \mathbf{T}. \end{aligned}$$

Theorem 14.7: Tangential and Normal Components of the Acceleration

The acceleration vector of an object moving in space along a smooth curve has the following representation in terms of its **tangential component** a_T (in the direction of \mathbf{T}) and its **normal component** a_N (in the direction of \mathbf{N}):

$$\mathbf{a} = a_N \mathbf{N} + a_T \mathbf{T},$$

where $a_N = \kappa |\mathbf{v}|^2 = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|}$ and $a_T = \frac{d^2s}{dt^2}$.

Example. Consider the function

$$\mathbf{r}(t) = \langle -2t + 2, -2t + 3, -2t + 2 \rangle.$$

Find the tangential and normal components of the acceleration.

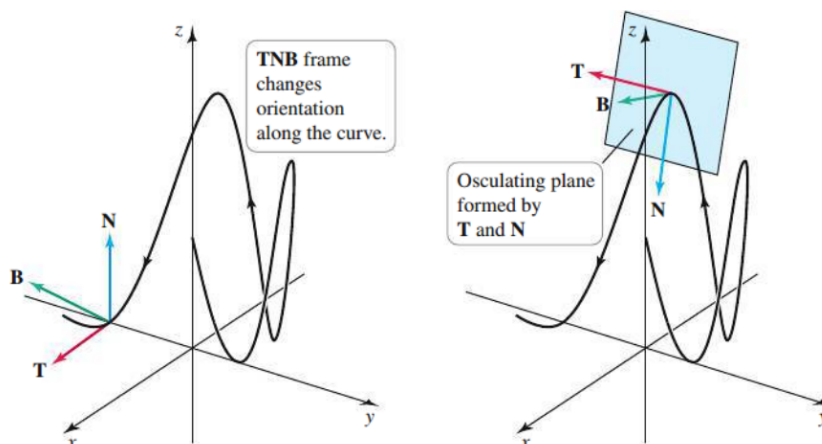
Example. Find the components of the acceleration on the circular trajectory

$$\mathbf{r}(t) = \langle R \cos(\omega t), R \sin(\omega t) \rangle.$$

Example. The driver of a car follows the parabolic trajectory $\mathbf{r}(t) = \langle t, t^2 \rangle$, for $-2 \leq t \leq 2$, through a sharp bend. Find the tangential and normal components of the acceleration of the car.

The Binormal Vector and Torsion

On a smooth parameterized curve C , \mathbf{T} and \mathbf{N} determine a plane called the *osculating plane*.



The coordinate system defined by these vectors is called the **TNB frame**. The rate at which the curve C twists out of the plane is the rate at which \mathbf{B} changes as we move along C , which is $\frac{d\mathbf{B}}{ds}$.

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T} \times \mathbf{N}) = \underbrace{\frac{d\mathbf{T}}{ds} \times \mathbf{N}}_0 + \mathbf{T} \times \frac{d\mathbf{N}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds}$$

$\frac{d\mathbf{B}}{ds}$ is:

- orthogonal to both \mathbf{T} and $\frac{d\mathbf{N}}{ds}$,
- orthogonal to \mathbf{B} (Theorem 14.2),
- parallel with \mathbf{N} .

Since $\frac{d\mathbf{B}}{ds}$ is parallel to \mathbf{N} , we write

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

where τ is the *torsion* (the negative sign is conventional). We can solve for τ via the dot product:

$$\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\tau \underbrace{\mathbf{N} \cdot \mathbf{N}}_1 \implies \frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\tau$$

Definition. (Unit Binormal Vector and Torsion)

Let C be a smooth parameterized curve with unit tangent and principal unit normal vectors \mathbf{T} and \mathbf{N} , respectively. Then at each point of the curve at which the curvature is nonzero, the **unit binomial vector** is

$$\mathbf{B} = \mathbf{T} \times \mathbf{N},$$

and the **torsion** is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

Example. Consider the circle C defined by

$$\mathbf{r}(t) = \langle R \cos(t), R \sin(t) \rangle, \text{ for } 0 \leq t \leq 2\pi, \text{ with } R > 0.$$

Find the unit binormal vector \mathbf{B} and determine the torsion.

Example. Compute the torsion of the helix

$$\mathbf{r}(t) = \langle a \cos(t), a \sin(t), bt \rangle, \text{ for } t \geq 0, \text{ and } b > 0.$$

Summary: Formula for Curves in Space

Position function: $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$

Velocity: $\mathbf{v} = \mathbf{r}'$

Acceleration: $\mathbf{a} = \mathbf{v}'$

Unit tangent vector: $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$

Principal unit normal vector: $\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$ (provided $d\mathbf{T}/dt \neq \mathbf{0}$)

Curvature: $\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$

Components of acceleration: $\mathbf{a} = a_N \mathbf{N} + a_T \mathbf{T}$, where

$$a_N = \kappa |\mathbf{v}|^2 = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|} \text{ and } a_T = \frac{d^2 s}{dt^2} = \frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}|}$$

Unit binormal vector: $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v} \times \mathbf{a}|}$

Torsion: $\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}$