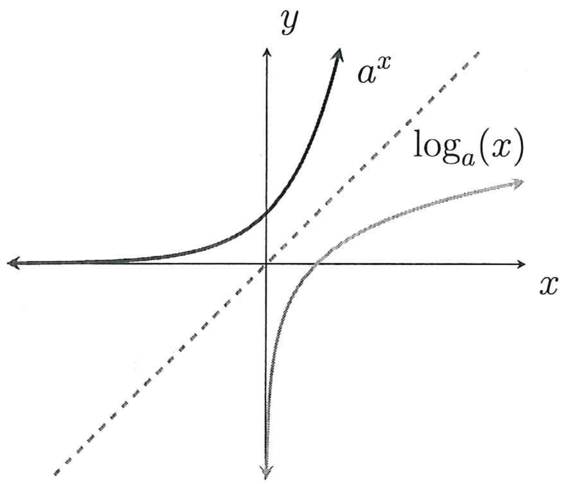
8.1 Definitions of Logarithms

Definition (Logarithmic Function Base b)**.** For any base $b > 0$, with $b \neq 1$, the **logarithmic function base** b, denoted $y = \log_b(x)$, is the inverse of the exponential function $y = b^x$. The inverse of the natural exponential function with base $b = e$ is the **natural logarithm function**, denoted $y = \ln(x)$.



Note:

	a^x	$\log_a(x)$
Domain:	$(-\infty, \infty)$	$(0, \infty)$
Range:	$(0, \infty)$	$(-\infty, \infty)$

$$\boxed{\begin{array}{c} y = b^x \quad \Longleftrightarrow \quad \log_b(y) = x \\ \text{Think "the base stays the base"} \end{array}}$$

Example. Evaluate:

$\log_9(81) = 2 \quad \because \; 9^2 = 81$

$\log_3(\sqrt{3}) = \log_3 (3^{1/2}) = 1/2$

$\log_{\frac{1}{2}}(8) = \log_{\frac{1}{2}} \left(\frac{1}{8}\right)^{-1} = \log_{\frac{1}{2}} \left(\frac{1}{2}\right)^{-3} = -3$

$(\log_5(5^{-3}))^2 = (-3)^2 = 9$

Note: In this course, the **common logarithm** is $\log_{10}(x)$ and is denoted by $\log(x)$.
– Sometimes other disciplines use $\log(x)$ to represent other bases.

Example. Evaluate:

$\log 100000 = \log 10^5 = 5$

$\log \frac{1}{1000} = \log (1000)^{-1}$
$= \log (10)^{-3} = -3$

8.2 Logs as Inverses of Exponential Functions

Recall that for a function f and its inverse g:

- $f(g(x)) = x$
- $g(f(x)) = x$
- Domain of f=Range of g
- Domain of g=Range of f

Inverse Relations for Exponential and Logarithmic Functions
For any base $b > 0$, with $b \neq 1$, the following inverse relations hold:

$$b^{\log_b x} = x \qquad \log_b(b^x) = x, \text{ for all real values of } x$$

Example. Evaluate:

$2^{\log_2 8} = 8$ $\qquad \log_b b^\pi = \pi$ $\qquad \log 10^3 = 3$

8.3 Laws of Logarithms

Example. Write each expression in terms of one logarithm:

$$\log_2 6 - \log_2 15 + \log_2 20 = \log_2\left(\frac{6\cdot 20}{15}\right)$$
$$= \log_2(2\cdot 4)$$
$$= \log_2(8) = \boxed{3}$$

$$\log_3 100 - (\log_3 18 + \log_3 50)$$
$$= \log_3\left(\frac{100}{18\cdot 50}\right) = \log_3\left(\frac{1}{9}\right) = \boxed{-2}$$

Laws of Logarithms
For $x, y > 0$:

1. $\log_a(xy) = \log_a(x) + \log_a(y)$
2. $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(x)$
3. $\log_a(x^r) = r\log_a(x)$
4. $\log_a(1) = 0$
5. $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$

Example. Solve each equation checking for extraneous solutions:

$\log_{64} x^2 = \frac{1}{3}$ $\implies 64^{1/3} = x^2 \implies 4 = x^2 \implies \pm 2 = x$

Check: $\log_{64}(-2)^2 = \log_{64} 4 = \frac{1}{3}$ ✓

$\log_{64}(2)^2 = \log_{64} 4 = 1/3$ ✓

$\log(3x + 2) + \log(x - 1) = 2$

$\log((3x+2)(x-1)) = 2$

$\log(3x^2 - x - 2) = 2$

$3x^2 - x - 2 = 10^2$

$\to 3x^2 - x - 102 = 0$

$(3x+17)(x-6) = 0$

$\Rightarrow \cancel{x = -\frac{17}{3}}$

$x = 6$

Check:

- $\log(3(\frac{-17}{3}) + 2)$ DNE
- $\log(3(6)+2) + \log(6-1)$
  $= \log(20) + \log(5)$
  $= \log(100) = 2$ ✓

$\log_2 x^2 - \log_2(3x - 8) = 2$

$\log_2\left(\frac{x^2}{3x-8}\right) = 2$

$\frac{x^2}{3x-8} = 2^2$

$\to x^2 = 4(3x-8)$

$x^2 - 12x + 32 = 0$

$(x-8)(x-4) = 0$

$x = 8, x = 4$

Check:

- $\log_2(8)^2 - \log_2\underbrace{(3(8)-8)}_{16}$
  $= 6 - 4 = 2$ ✓
- $\log_2(4)^2 - \log_2\underbrace{(3(4)-8)}_{4}$
  $= 4 - 2 = 2$ ✓

$\log_4 x - \log_4(x - 1) = \frac{1}{2}$

$\log_4\left(\frac{x}{x-1}\right) = \frac{1}{2}$

$\frac{x}{x-1} = 4^{1/2}$

$\to x = 2(x-1)$

$2 = x$

Check:

- $\log_4(2) - \log_4(2-1) = \frac{1}{2} - 0 = 1/2$ ✓

$\log_3(x + 6) - \log_3(x - 6) = 2$

$\log_3\left(\frac{x+6}{x-6}\right) = 2 \to$

$x + 6 = 3^2(x-6)$

$60 = 8x$

$\frac{15}{2} = x$

Check:

$\log_3(\frac{15}{2} + 6) - \log_3(\frac{15}{2} - 6)$

$\log_3\left(\frac{27/2}{3/2}\right) = \log_3 9 = 2$ ✓

$\log_3(x^2 - 5) = 2$

$x^2 - 5 = 3^2$

$\to x^2 = 9 + 5 = 14$

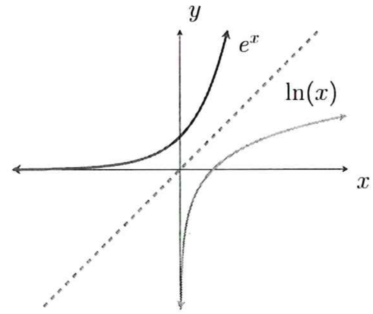
$x = \pm\sqrt{14}$

$\to$ Check: $\log_3((\sqrt{14})^2 - 5)$

$= \log_3(14-5) = \log_3(9) = 2$ ✓

8.4 The Natural Logarithm

Definition. The **Natural Logarithmic Function** uses base e and is denoted $\ln(x) = \log_e x$.



Note: The natural log is the inverse of e^x:

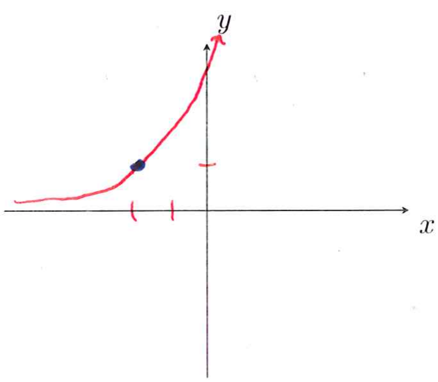
$$\ln(x) = y \iff e^y = x$$

Inverse Properties for a^x and $\log_a x$

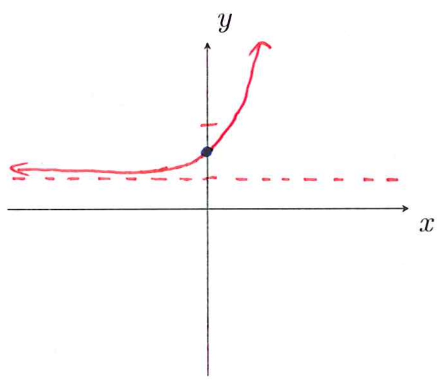
Base a:	$a^{\log_a x} = x,$	$\log_a a^x = x,$	$a > 0, a \neq 1, x > 0$
Base e:	$e^{\ln x} = x,$	$\ln e^x = x,$	$x > 0$

Example. Graph the following functions:

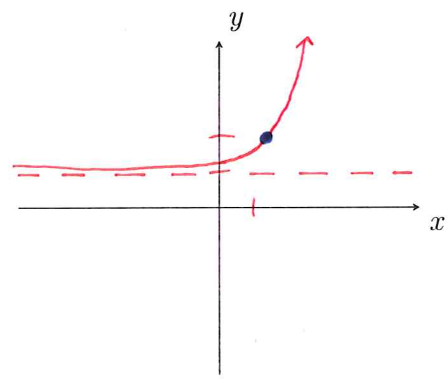
e^{x+2}



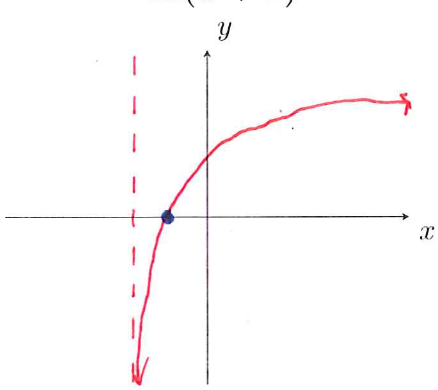
$e^x + 2$



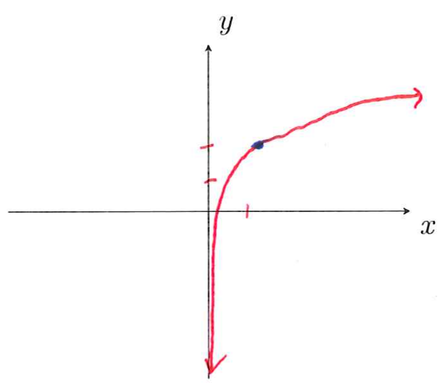
$e^{x-1} + 2$



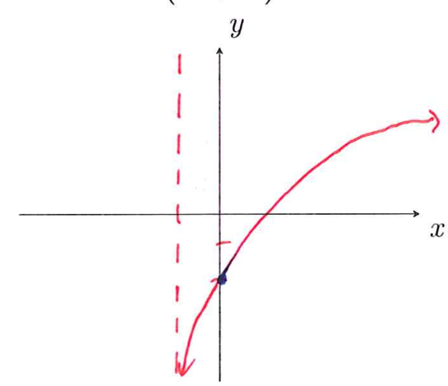
$\ln(x+2)$



$\ln x + 2$



$\ln(x+1) - 2$



Example. Simplify

$e^{-\ln 0.3} = e^{\ln\left(\frac{3}{10}\right)^{-1}} = \left(\frac{3}{10}\right)^{-1} = \boxed{\frac{10}{3}}$

$e^{\ln \pi x - \ln 2} = e^{\ln\left(\frac{\pi x}{2}\right)} = \boxed{\frac{\pi x}{2}}$

$\ln\left(\frac{1}{e}\right) = \ln(e^{-1}) = \boxed{-1}$

or $= \ln(e^{-1}) = -1\ln(e) = -1(1) = \boxed{-1}$

$e^{4\ln x} = e^{\ln(x^4)} = \boxed{x^4}$

Example. Write each expression in terms of one logarithm:

$\ln(a+b) + \ln(a-b) - 2\ln c$

$\ln\left(\frac{(a+b)(a-b)}{c^2}\right) = \ln\left(\frac{a^2-b^2}{c^2}\right)$

Laws of the Natural Logarithm
For $x, y > 0$:

1. $\ln(xy) = \ln(x) + \ln(y)$
2. $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(x)$
3. $\ln(x^r) = r\ln(x)$
4. $\ln(1) = 0$
5. $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$

$\frac{1}{3}\ln(x+2)^3 + \frac{1}{2}\left[\ln x - \ln(x^2+3x+2)^2\right]$

$\ln(x+2) + \ln x^{1/2} - \ln(x^2+3x+2)$

$\ln\left(\frac{(x+2)\,x^{1/2}}{x^2+3x+2}\right) = \ln\left(\frac{(x+2)\sqrt{x}}{(x+2)(x+1)}\right)$

$= \ln\left(\frac{\sqrt{x}}{x+1}\right)$

Note: Many common mistakes come from using the logarithm rules incorrectly:

$$\ln A - \ln B \neq \frac{\ln A}{\ln B} \qquad \ln(A+B) \neq \ln(A)\ln(B)$$

Example. Solve:

$2^x = 55$ $\longrightarrow$ $\log_2 55 = x$

$5^{3x} = 29$ $\longrightarrow$ $\log_5 5^{3x} = \log_5 29$

$3x = \log_5 29$

$\boxed{x = \frac{1}{3}\log_5 29}$

$e^{2x} - 5e^x - 14 = 0$

$(e^x - 7)(e^x + 2) = 0$

$e^x = 7 \quad e^x = -2$

$\boxed{x = \ln(7)}$

$\cancel{x = \ln(-2)}$

$4e^{2x} - 7e^x = 15$

$4e^{2x} - 7e^x - 15 = 0$

$(4e^x + 5)(e^x - 3) = 0$

$\Rightarrow e^x = -\frac{5}{4}, \; e^x = 3$

$\cancel{x = \ln(-5/4)}, \; \boxed{x = \ln(3)}$

Example. Solve for y in terms of x:

$\ln(y - 40) = 5x$

$y - 40 = e^{5x}$

$y = e^{5x} + 40$

$\ln(y^2 - 1) - \ln(y + 1) = \ln(\sin x)$

$\ln\left(\frac{y^2-1}{y+1}\right) = \ln(\sin x)$

$\frac{(y-1)(y+1)}{y+1} = \sin(x)$

$y - 1 = \sin(x)$

$\boxed{y = \sin(x) + 1}$

Example. Solve:

$\ln(t) + \ln(t^2) = 6$

$\ln(t^3) = 6$

$t^3 = e^6 \longrightarrow \boxed{t = e^2}$

$e^{x^2+2x-3} = 1 = e^0$

$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$\boxed{x = -3, \; x = 1}$

$\ln x = -1$

$\boxed{x = e^{-1}}$

$0.3 = \frac{3}{10}$

$e^{-0.3t} = 27$

$e^{-\frac{t}{10}} = 3$

$-\frac{t}{10} = \ln(3)$

$\boxed{t = -10\ln(3)}$