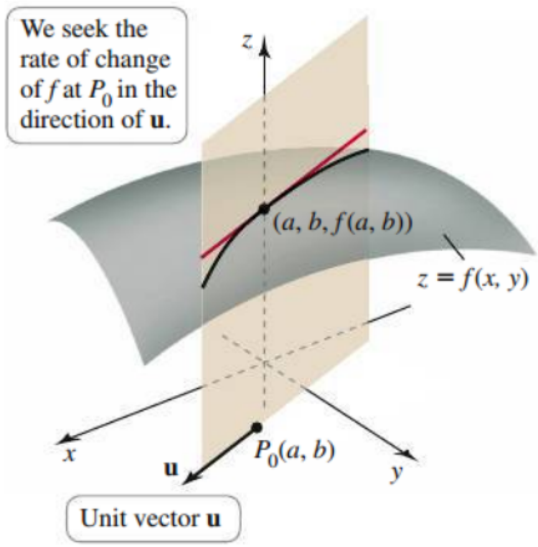
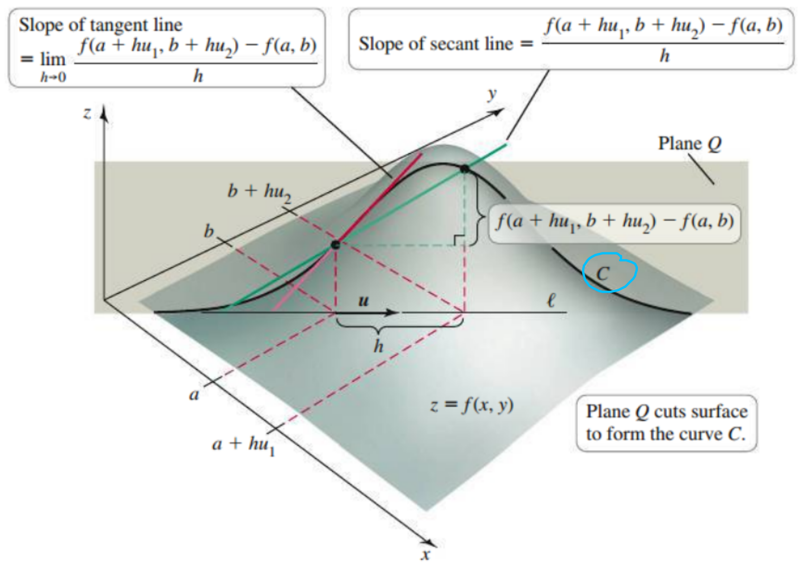
15.5: Directional Derivatives and the Gradient

Directional derivatives allow us to evaluate the rate of change of a function $f(x, y)$ along any direction (not just parallel with the x-axis and y-axis).



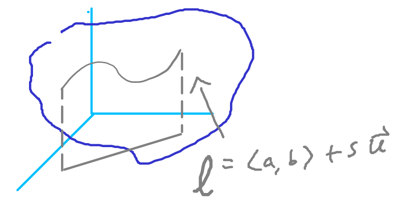


Definition. (Directional Derivative)
Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2\rangle$ be a unit vector in the xy-plane. The **directional derivative of** f **at** (a, b) **in the direction of** u is

$$D_{\mathbf{u}} f(a, b) = \lim_{h\to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h},$$

provided the limit exists.

To motivate the formula for the directional derivative, let ℓ be a line going through (a, b) in the direction of the unit vector $\mathbf{u}$. Now, let



$$x = a + su_1, \quad \text{and} \quad y = b + su_2,$$

where $-\infty < s < \infty$ and define

$$g(s) = f(\underbrace{a + su_1}_{x}, \underbrace{b + su_2}_{y}),$$

which evaluates f along ℓ. Thus, $g'(s)$ gives us the derivative along this line, and $g'(0)$ gives us the directional derivative of f at (a, b):

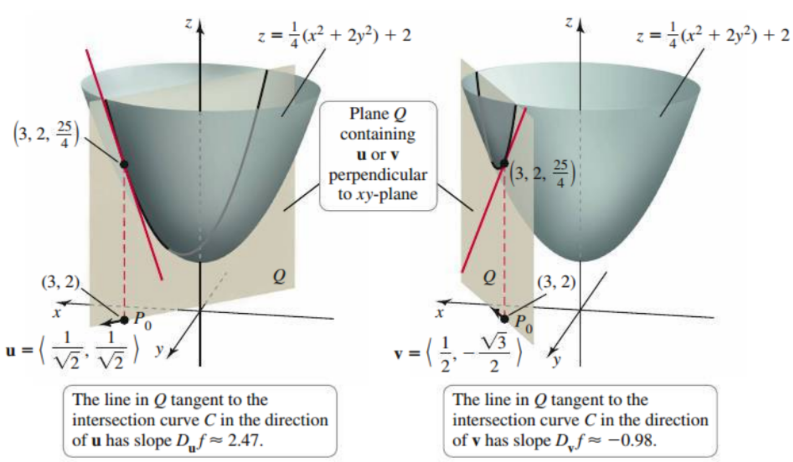
$$\begin{aligned} D_{\mathbf{u}}f(a,b) = g'(0) &= \left(\frac{\partial f}{\partial x}\underbrace{\frac{dx}{ds}}_{u_1} + \frac{\partial f}{\partial y}\underbrace{\frac{dy}{ds}}_{u_2}\right)\Bigg|_{s=0} \\ &= f_x(a,b)u_1 + f_y(a,b)u_2 \\ &= \langle f_x(a,b),\ f_y(a,b)\rangle \cdot \langle u_1, u_2\rangle. \end{aligned}$$

Unit vector!

Theorem 15.10: Directional Derivative

Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2\rangle$ be a unit vector in the xy-plane. The **directional derivative of f at (a, b) in the direction of $\mathbf{u}$** is

$$D_{\mathbf{u}}f(a,b) = \langle f_x(a,b), f_y(a,b)\rangle \cdot \langle u_1, u_2\rangle.$$



Example. Compute the directional derivatives of the following functions at the given point along the given direction.

$$|\vec{u}| = \sqrt{\left(\tfrac{1}{\sqrt{10}}\right)^2 + \left(\tfrac{3}{\sqrt{10}}\right)^2} = \sqrt{\tfrac{1}{10} + \tfrac{9}{10}} = 1$$

$$f(x, y) = \sqrt{4 - x^2 - 2y};\ P(2, -2);\ \text{and } \mathbf{u} = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle,$$

$$D_{\vec{u}} f(x,y) = \left\langle \frac{-x}{\sqrt{4-x^2-2y}}, \frac{-1}{\sqrt{4-x^2-2y}} \right\rangle \cdot \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$$

$$= \frac{-x-3}{\sqrt{10}\sqrt{4-x^2-2y}}$$

$$D_{\vec{u}} f(2,-2) = \frac{-5}{\sqrt{10}\sqrt{4-4+4}} = \frac{-5}{2\sqrt{10}} = \frac{-\sqrt{10}}{4}$$

$$D_{\vec{u}} f = \langle f_x, f_y \rangle \cdot \frac{\vec{u}}{|\vec{u}|}$$

$$g(x, y) = \tan^{-1}(xy);\ P(\pi, 1/3);\ \text{along } \mathbf{u} = \langle 1, 1\rangle, \longrightarrow |\vec{u}| = \sqrt{2}$$

$$D_{\vec{u}} g(x,y) = \left\langle \frac{1}{1+(xy)^2} y, \frac{1}{1+(xy)^2} x \right\rangle \cdot \underbrace{\frac{\langle 1, 1\rangle}{\sqrt{2}}}_{\text{unit vector}}$$

$$= \frac{y+x}{\sqrt{2}(1+(xy)^2)}$$

$$D_{\vec{u}} g(\pi, 1/3) = \frac{1/3 + \pi}{\sqrt{2}\left(1 + \frac{\pi^2}{9}\right)}$$

$$h(x, y) = 2x^2 - xy + 3y^2;\ P(1, -3);\ \text{along } \mathbf{u} = \langle 1, -1\rangle \text{ and } \mathbf{v} = \left\langle \tfrac{3}{5}, \tfrac{4}{5} \right\rangle.$$

$$D_{\vec{u}} h(x,y) = \langle 4x - y, -x + 6y \rangle \cdot \frac{\langle 1, -1\rangle}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{2}}(4x - y + x - 6y) = \frac{1}{\sqrt{2}}(5x - 7y)$$

$$D_{\vec{u}} h(1,-3) = \frac{1}{\sqrt{2}}(5 + 21) = \frac{26}{\sqrt{2}} = 13\sqrt{2}$$

$$D_{\vec{v}} h(x,y) = \langle 4x - y, -x + 6y \rangle \cdot \frac{\langle 3/5, 4/5 \rangle}{\sqrt{\frac{9}{25} + \frac{16}{25}}} = \frac{12}{5}x - \frac{3}{5}y - \frac{4}{5}x + \frac{24}{5}y = \frac{8}{5}x + \frac{21}{5}y$$

$$D_{\vec{v}} h(x,y)\Big|_{(x,y)=(1,-3)} = \frac{8}{5} - \frac{63}{5} = -11$$

$D_{\vec{u}} f(3,-1) = \langle -\frac{1}{2}, -\frac{3}{5} \rangle \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle = -\frac{1}{2\sqrt{2}} + \frac{3}{5\sqrt{2}} = \left(\frac{5}{5}\right) -\frac{\sqrt{2}}{4} + \frac{3\sqrt{2}}{10}\left(\frac{2}{2}\right) = \frac{\sqrt{2}}{20}$

$$D_u f(x,y) = \nabla f(x,y) \cdot \frac{\vec{u}}{|\vec{u}|}$$

"nabla"

The Gradient Vector:
The vector of derivatives used in the directional derivative is called the *gradient* of f.

> **Definition. (Gradient (Two Dimensions))**
> Let f be differentiable at the point (x, y). The **gradient** of f at (x, y) is the vector-valued function
>
> $$\nabla f(x, y) = \langle f_x(x, y),\ f_y(x, y)\rangle = f_x(x, y)\boldsymbol{i} + f_y(x, y)\boldsymbol{j}.$$

Example. For $f(x, y) = 3 - \dfrac{x^2}{10} + \dfrac{xy^2}{10}$, compute $\nabla f(3, -1)$, then compute $D_{\mathbf{u}} f(3, -1)$, where $\mathbf{u} = \left\langle \dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}} \right\rangle$. ← unit vector

$\nabla f(x,y) = \langle -\frac{x}{5} + \frac{y^2}{10}, \frac{xy}{5} \rangle$ $\qquad \nabla f(3,-1) = \nabla f(x,y)\Big|_{(3,-1)} = \langle -\frac{3}{5} + \frac{1}{10}, -\frac{3}{5} \rangle$

$= \langle -\frac{1}{2}, -\frac{3}{5} \rangle$

$D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \frac{\vec{u}}{|\vec{u}|} = \langle -\frac{x}{5} + \frac{y^2}{10}, \frac{xy}{5} \rangle \cdot \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle = \frac{-x}{5\sqrt{2}} + \frac{y^2}{10\sqrt{2}} - \frac{xy}{5\sqrt{2}}$

$D_{\vec{u}} f(3,-1) = \frac{-3}{5\sqrt{2}} + \frac{1}{10\sqrt{2}} + \frac{3}{5\sqrt{2}} = \frac{1}{10\sqrt{2}} = \frac{\sqrt{2}}{20}$

> **Theorem 15.11: Directions of Change**
> Let f be differentiable at (a, b) with $\nabla f(a, b) \neq \mathbf{0}$.
>
> 1. f has its maximum rate of increase at (a, b) in the direction of the gradient $\nabla f(a, b)$. The rate of change in this direction is $|\nabla f(a, b)|$.
> 2. f has its maximum rate of decrease at (a, b) in the direction of $-\nabla f(a, b)$. The rate of change in this direction is $-|\nabla f(a, b)|$.
> 3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$.

Example. For $f = 4 + x^2 + 3y^2$:

$f(2,-\frac{1}{2})$

What direction is the greatest ascent at $P(2, -\frac{1}{2}, \frac{35}{4})$? What is the rate of change in this direction?

$\nabla f(x,y)$

$\nabla f(x,y) = \langle 2x, 6y \rangle$ direction of greatest increase

$\nabla f(x,y) = \langle 4, -3 \rangle$

$|\nabla f(2,-\frac{1}{2})| = |\langle 4, -3\rangle| = \sqrt{4^2+(-3)^2} = \sqrt{16+9} = \sqrt{25} = 5$ rate of greatest increase

What direction is the greatest descent at $P(\frac{5}{2}, -2, \frac{89}{4})$? What is the rate of change in this direction?

$-\nabla f(x,y) = \langle -2x, -6y \rangle$ $\quad -\nabla f(\frac{5}{2}, -2) = \langle -5, 12 \rangle \Leftarrow$ direction

$-|\nabla f(\frac{5}{2}, -2)| = -|\langle -5, 12\rangle| = -\sqrt{25+144} = -\sqrt{169} = -13$

rate.

What direction results in no change in function values at $P(3, 1, 16)$?

Solve $\nabla f(x,y) \cdot \vec{u} = 0$

$\nabla f(x,y) = \langle 2x, 6y \rangle$

$\langle 2x, 6y \rangle \cdot \langle u_1, u_2 \rangle = 0$

$\langle 6, 6 \rangle \cdot \langle u_1, u_2 \rangle = 0$

$6u_1 + 6u_2 = 0$

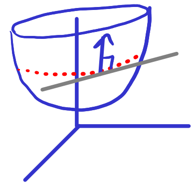
Let $u_1 = 1$

$\Longrightarrow 6 + 6u_2 = 0 \Longrightarrow u_2 = -1$

$\vec{u} = \langle 1, -1 \rangle$

What if $w = f(x,y,z)$

$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$



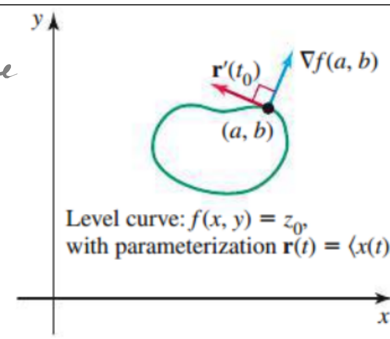
Theorem 15.12: The Gradient and Level Curves
Given a function f differentiable at (a, b), the line tangent to the level curve of f at (a, b) is orthogonal to the gradient $\nabla f(a, b)$, provided $\nabla f(a, b) \neq \mathbf{0}$.

similar to finding a plane

Note: From Theorem 15.12, we get an equation for the line tangent to the curve $z = f(x, y)$ at (a, b):

$$\nabla f(a, b) \cdot \langle x - a, y - b \rangle = 0.$$

$$\nabla f(a,b) \cdot \langle x, y \rangle = \nabla f(a,b) \cdot \langle a, b \rangle$$



Example. Consider the upper sheet $z = f(x, y) = \sqrt{1 + 2x^2 + y^2}$ of a hyperboloid of two sheets.

$f(1,1) = 2 \rightarrow 2 = \sqrt{1+2x^2+y^2}$

Verify that the gradient at $(1, 1)$ is orthogonal to the corresponding level curve at that point.

$1 + 2x^2 + y^2 = 4$

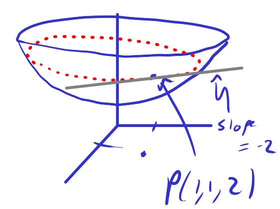
$\underbrace{2x^2 + y^2}_{F(x,y)} = 3 \leftarrow$ ellipse

$$\nabla f(x,y) = \left\langle \frac{2x}{\sqrt{1+2x^2+y^2}}, \frac{y}{\sqrt{1+2x^2+y^2}} \right\rangle$$

$$\nabla f(1,1) = \langle 1, 1/2 \rangle$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{4x}{2y} = -\frac{2x}{y}$$

Verify $\nabla f(1,1) \cdot \langle 1, -2 \rangle = 0$



$$\left.\frac{dy}{dx}\right|_{(1,1)} = -2 \Rightarrow \langle 1, -2 \rangle$$

$\rightarrow \langle 1, 1/2 \rangle \cdot \langle 1, -2 \rangle = 1 + (-1) = 0$

Find an equation of the line tangent to the level curve at $(1, 1)$.

$$\nabla f(1,1) \cdot \langle x-1, y-1 \rangle = 0$$

$$\langle 1, 1/2 \rangle \cdot \langle x-1, y-1 \rangle = 0$$

$$x - 1 + y/2 - 1/2 = 0$$

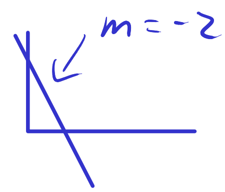
$$y = -2x + 3$$

$m = -2$ LC2

$b = 3$ LC3

$\vec{r}(t)$ be tangent line

$$\vec{r}(t) = \langle 1, 1 \rangle + t \langle \underset{dx/dx}{1}, \underset{dy/dx}{-2} \rangle$$



Example. Consider $z = f(x,y) = 15 - \frac{x^2}{25} - \frac{y^2}{9}$:

Compute the slope of the tangent line at $P(5\sqrt{5}, -6, 6)$.

$$\nabla f(5\sqrt{5}, -6) \cdot \langle x - 5\sqrt{5}, y+6 \rangle = 0$$

$$\nabla f(x,y) = \left\langle -\frac{2x}{25}, -\frac{2y}{9} \right\rangle$$

$$\nabla f(5\sqrt{5}, -6) = \left\langle -\frac{2\sqrt{5}}{5}, \frac{4}{3} \right\rangle$$

$$\nabla f(5\sqrt{5}, -6) \cdot \langle x - 5\sqrt{5}, y+6 \rangle = \left\langle -\frac{2\sqrt{5}}{5}, 4/3 \right\rangle \cdot \langle x - 5\sqrt{5}, y+6 \rangle$$

$$= -\frac{2\sqrt{5}}{5} x + 10 + 4/3\, y + 8 = 0$$

$$\Rightarrow 4/3\, y = \frac{2\sqrt{5}}{5} x - 18 \Rightarrow y = \frac{3\sqrt{5}}{10} x - \frac{27}{2}$$

LC 4 $\quad m = \frac{5\sqrt{5}}{6}$ $\quad b = -\frac{89}{6}$

Verify the gradient is orthogonal to the tangent line.

LC 5 $\quad m = -2$

$$y = \frac{3\sqrt{5}}{10} x - \frac{27}{2} \longrightarrow \left\langle 1, \frac{3\sqrt{5}}{10} \right\rangle$$

$$\nabla f(5\sqrt{5}, -6) = \left\langle -\frac{2\sqrt{5}}{5}, 4/3 \right\rangle$$

$$\left\langle -\frac{2\sqrt{5}}{5}, 4/3 \right\rangle \cdot \left\langle 1, \frac{3\sqrt{5}}{10} \right\rangle = -\frac{2\sqrt{5}}{5} + \frac{\overset{2}{\cancel{12}}\sqrt{5}}{\underset{5}{\cancel{30}}} = 0.$$

Definition. (Directional Derivative and Gradient in Three Dimensions) Let f be directional at (a, b, c) and let $\mathbf{u} = \langle u_1,\, u_2,\, u_3\rangle$ be a unit vector. The **directional derivative of f at (a, b, c) in the direction of u** is

$$D_{\mathbf{u}}(a, b, c) = \lim_{h\to 0} \frac{f(a + hu_1,\, b + hu_2,\, c + hu_3) - f(a, b, c)}{h},$$

$\nabla f(a,b,c) \cdot \vec{u}$

provided this limit exists.

The **gradient** of f at this point (x, y, z) is the vector-valued function

$$\begin{aligned}\nabla f(x, y, z) &= \langle f_x(x, y, z),\; f_y(x, y, z),\; f_z(x, y, z)\rangle \\ &= f_x(x, y, z)\boldsymbol{i} + f_y(x, y, z)\boldsymbol{j} + f_z(x, y, z)\boldsymbol{k}.\end{aligned}$$

Theorem 15.13: Directional Derivative and Interpreting the Gradient

Let f be differentiable at (a, b, c) and let $\mathbf{u} = \langle u_1,\, u_2,\, u_3\rangle$ be a unit vector. The directional derivative of f at (a, b, c) in the direction of $\mathbf{u}$ is

$$\begin{aligned}D_{\mathbf{u}}f(a, b, c) &= \nabla f(a, b, c) \cdot \mathbf{u} \\ &= \langle f_x(a, b, c),\; f_y(a, b, c),\; f_z(a, b, c)\rangle \cdot \langle u_1,\, u_2,\, u_3\rangle.\end{aligned}$$

Assuming $\nabla f(a, b, c) \neq \mathbf{0}$, the gradient in three dimensions has the following properties.

1. f has its maximum rate of increase at (a, b, c) in the direction of the gradient $\nabla f(a, b, c)$ and the rate of change in this direction is $|\nabla f(a, b, c)|$.
2. f has its maximum rate of decrease at (a, b, c) in the direction of $-\nabla f(a, b, c)$ and the rate of change in this direction is $-|\nabla f(a, b, c)|$.
3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b, c)$.

Example. Consider $f(x, y, z) = x^2+2y^2+4z^2-1$ and the level surface $f(x, y, z) = 3$. Find the gradient and the corresponding rate of change at the points $P(2, 0, 0)$, $Q(0, \sqrt{2}, 0)$, $R(0, 0, 1)$, and $S(1, 1, 1/2)$ on the level surface.

$$\nabla f(x,y,z) = \langle 2x, 4y, 8z \rangle$$

P $\quad \nabla f(2,0,0) = \langle 4, 0, 0 \rangle \qquad |\nabla f(2,0,0)| = 4$

Q $\quad \nabla f(0,\sqrt{2},0) = \langle 0, 4\sqrt{2}, 0 \rangle \qquad |\nabla f(0,4\sqrt{2},0)| = 4\sqrt{2}$

R $\quad \nabla f(0,0,1) = \langle 0, 0, 8 \rangle \qquad |\nabla f(0,0,1)| = 8$

S $\quad \nabla f(1,1,\tfrac{1}{2}) = \langle 2, 4, 4 \rangle \qquad |\nabla f(1,1,\tfrac{1}{2})| = \sqrt{2^2+4^2+4^2}$

$$= \sqrt{36} = 6$$