10.7: The Ratio and Root Tests

Theorem 10.20: Ratio Test

Let $\sum a_k$ be an infinite series, and let $\boxed{r = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|}$

1. If $r < 1$, the series converges absolutely, and therefore it converges (by Theorem 10.19)
2. If $r > 1$ (including $r = \infty$), the series diverges.
3. If $r = 1$, the test is inconclusive.

Note: The ratio test is used to determine if a series converges or diverges and indicates nothing about the *value* of the series.

Example. Use the ratio test on the harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ and the alternating harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$.

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$

$$r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{1/(k+1)}{1/k}\right| = \lim_{k\to\infty} \left|\frac{1}{k+1}\cdot\frac{k}{1}\right| = \lim_{k\to\infty} \left|\frac{k}{k+1}\right| = 1$$

Inconclusive

Diverges

Harmonic series

& p-series w/ $p = 1 \le 1$

$\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$

$$r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{1/(k+1)}{1/k}\right| = \lim_{k\to\infty} \left|\frac{1}{k+1}\cdot\frac{k}{1}\right| = \lim_{k\to\infty} \left|\frac{k}{k+1}\right| = 1$$

Inconclusive

Converges

by AST

$0! = 1$

$n=3 \rightarrow \frac{(2n)!}{(2n-1)!} = \frac{6!}{5!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1} = 6$

Example. *Note:* $n! = n\cdot(n-1)\cdots3\cdot2\cdot1$

$(2n)! = 2n\cdot\underbrace{(2n-1)\cdot\ldots\cdot1}_{(2n-1)!}$

Rewrite $n!n! = (n!)^2 \neq (2n)!$ and $\frac{(2n)!}{(2n-1)!} = \frac{2n\cdot(2n-1)!}{(2n-1)!} = 2n$

$n\cdot(n-1)\cdot(n-2)\cdots1$
$\times n\cdot(n-1)\cdots1$

$2n\cdot(2n-1)\cdot(2n-2)\cdots1$

Example. Consider the series below. Use the ratio test, if appropriate, to show if each of the series converges or diverges.

$$\sum_{k=1}^{\infty}\frac{k^2}{2^k} \qquad r = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)^2}{\underbrace{2^{k+1}}_{2^k\cdot2}}\cdot\frac{2^k}{k^2}\right| = \lim_{k\to\infty}\left|\frac{\overbrace{(k+1)^2}^{k^2}}{2k^2}\right| = \frac{1}{2}$$

Since $r=\frac{1}{2}<1$, the series $\underbrace{\text{converges absolutely}}_{\text{implies convergence}}$ by the Ratio Test

Ratio test helpful

$$\sum_{k=1}^{\infty}\frac{(-1)^k k}{k^3+1} \qquad r = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\cancel{(-1)^{k+1}}(k+1)}{(k+1)^3+1}\cdot\frac{k^3+1}{\cancel{(-1)^k}k}\right| = \lim_{k\to\infty}\left|\frac{\overbrace{(k+1)(k^3+1)}^{k^4}}{\underbrace{[(k+1)^3+1]k}_{k^4}}\right| = 1$$

The $\underbrace{\text{ratio test is inconclusive}}_{\text{not helpful}}$ since $r=1$

$a_k \xrightarrow{r} a_{k+1}$ Geometric Behavior

Divergence test is inconclusive $\lim_{k\to\infty}\frac{(-1)^k k}{k^3+1} = 0$

$\lim_{k\to\infty}\frac{k}{k^3+1} = 0$

AST

① $\lim_{k\to\infty} a_k = 0$
② $0 < a_{k+1} \le a_k$
} Converges by AST

ew

~~$(k^3+1)(k+1) < k((k+1)^3+1)$~~

~~$\frac{k+1}{(k+1)^3+1} < \frac{k}{k^3+1}$~~ ← want

$f(x) = \frac{x}{x^3+1}$

$f'(x) = \frac{(x^3+1) - x(3x^2)}{(x^3+1)^2} = \frac{1-2x^3}{(x^3+1)^2} < 0$ when $1-2x^3<0$
$\sqrt[3]{\frac{1}{2}} < x$ $k\ge1$, this works

positive a_k w/ $\lim_{k\to\infty} a_k = 0$ Does not imply non increasing

$a_k = \dfrac{1-\sin(k)}{k} \geq 0$, oscillating sequence

$\dfrac{k+x}{k} \xrightarrow{x=1} \dfrac{k+1}{k}$

$$\sum_{k=1}^{\infty} \frac{5^k k!}{k^k}$$

$\lim_{k\to\infty}\left|\dfrac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\dfrac{\overbrace{5^{k+1}}^{5^k 5}\overbrace{(k+1)!}^{(k+1)k!}}{\underbrace{(k+1)^{k+1}}_{(k+1)^k(k+1)}}\cdot\dfrac{k^k}{5^k k!}\right|$

$$\lim_{k\to\infty}\overbrace{\left(1+\frac{x}{k}\right)}^{}{}^{k} = e^x$$

$= \lim_{k\to\infty}\left|\dfrac{5(k+1)\;k^k}{(k+1)^k(k+1)}\right| = \lim_{k\to\infty}\left|5\left(\dfrac{k}{k+1}\right)^k\right|$

L'Hopital's Rule

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

$= 5\lim_{k\to\infty}\left|\left(\dfrac{k+1}{k}\right)^{-k}\right| = 5\lim_{k\to\infty}\left|\left(1+\frac{1}{k}\right)^{-k}\right| = 5\cdot e^{-1} > 1$

Diverges by ratio test

Ratio Test helpful

$$\sum_{k=1}^{\infty} \frac{(-7)^k}{(2k+1)!}$$

$r = \lim_{k\to\infty}\left|\dfrac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\dfrac{\overbrace{7^{k+1}}^{7^k\cdot 7}}{\underbrace{(2(k+1)+1)!}_{(2k+3)!}}\cdot\dfrac{(2k+1)!}{7^k}\right|$

$(2k+3)! = (2k+3)(2k+2)(2k+1)!$

$= \lim_{k\to\infty}\left|\dfrac{7}{(2k+3)(2k+2)}\right| = 0 < 1$ $\to$ Converges by Ratio Test

Ratio Test helpful

$$\sum_{k=1}^{\infty} \frac{(-1)^k \ln(k)}{k}$$

$r = \lim_{k\to\infty}\left|\dfrac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\dfrac{\ln(k+1)}{k+1}\cdot\dfrac{k}{\ln(k)}\right| = \lim_{k\to\infty}\left|\dfrac{k}{k+1}\;\dfrac{\ln(k+1)}{\ln(k)}\right|$

$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

Ratio Test not helpful

Example. Use the ratio test to determine if the series $\displaystyle\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^{k}$ converges or diverges.

Example. Use the ratio test to determine if the series $\sum_{k=1}^{\infty} \frac{(-1)^k k}{(2k)!}$ converges or diverges.

Example. Use the ratio test to determine if the series $\sum_{k=1}^{\infty} \frac{(2k)!}{(k!)^2}$ converges or diverges.

10.21: Root Test
Let $\sum a_k$ be an infinite series, and let $\rho = \lim_{k\to\infty} \sqrt[k]{|a_k|}$.

1. If $\rho < 1$, the series converges absolutely, and therefore it converges (by Theorem 10.19)
2. If $\rho > 1$ (including $\rho = \infty$), the series diverges.
3. If $\rho = 1$, the test is inconclusive.

Note: The root test is used to determine if a series converges or diverges and indicates nothing about the *value* of the series.

Example. Use the root test to determine if the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}k^k}{3^{k^2}}$ converges.

Example. Consider the series below. Use the root test to show if each of the series converges or diverges.

$$\sum_{k=1}^{\infty} \left(\frac{1}{\ln(k+1)} \right)^k$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{3k^2+1}{k-2k^2} \right)^k$$

$$\sum_{k=1}^{\infty} \left(\frac{k+3}{k+1} \right)^{2k}$$

Example. Use the root test to determine if the series $\displaystyle\sum_{k=1}^{\infty}\left(1-\frac{3}{k}\right)^{k^2}$ converges.

Example. Determine whether each of the series below converges conditionally, converges absolutely, or diverges.

$$\sum_{k=1}^{\infty} (-1)^k k^{-1/3}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\arctan(k)}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k!}$$

Example. Determine if the series $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k}{k+5}\right)^{3k^2}$ converges.

Example. Determine a condition for $x \geq 0$ such that $\displaystyle\sum_{k=1}^{\infty} \frac{4x^k}{5k^2}$ converges.