11.2: Properties of Power Series

From the *geometric series*, we have

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots = \frac{1}{1-x}, \quad \text{provided } |x| < 1.$$

Definition. (Power Series)
A **power series** has the general form

$$\sum_{k=0}^{\infty} c_k (x-a)^k,$$

where a and c_k are real numbers, and x is a variable. The c_k's are the **coefficients** of the power series, and a is the **center** of the power series. The set of values of x for which the series converges is its **interval of convergence**. The **radius of convergence** of the power series, denoted R, is the distance from the center of the series to the boundary of the interval of convergence.

Theorem 11.3: Convergence of Power Series
A power series $\sum_{k=0}^{\infty} c_k (x-a)^k$ centered at a converges in one of three ways:

1. The series converges absolutely for all x. It follows, by Theorem 10.19, that the series converges for all x, in which the interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.
2. There is a real number $R > 0$ such that the series converges absolutely (and therefore converges) for $|x - a| < R$ and diverges for $|x - a| > R$, in which case the radius of converge is R.
3. The series converges only at a, in which case the radius of convergence is $R = 0$.

Summary: Determining the Radius and Interval of Convergence of $\sum c_k(x-a)^k$

1. Use the Ratio Test or the Root Test to find the interval $(a-R, a+R)$ on which the series converges absolutely; the radius of convergence for the series is R.
2. Use the *radius* of convergence to find the *interval* of convergence:
   - If $R=\infty$, the interval of convergence is $(-\infty,\infty)$.
   - If $R=0$, the interval of convergence is the single point $x=a$.
   - If $0<R<\infty$, the interval of convergence consists of the interval $(a-R,a+R)$ and possibly one or both of its endpoints. Determining whether the series $\sum c_k(x-a)^k$ converges at the endpoints $x=a-R$ and $x=a+R$ amounts to analyzing the series $\sum c_k(-R)^k$ and $\sum c_k R^k$.

Example (LC 28.1)**.** Where is the power series $\displaystyle\sum_{k=1}^{\infty} c_k(x-3)^k$ centered?

Could it's interval of convergence be $(-2,8)$?

Example (LC 28.2)**.** Where is the power series $\displaystyle\sum_{k=0}^{\infty} \frac{(4x-1)^k}{k^2+3}$ centered?

Example (LC 28.3)**.** Where is the power series $\displaystyle\sum_{k=1}^{\infty} c_k(x-1)^k$ centered?

Could it's interval of convergence be $(-1,1)$?

Example (LC 28.4-28.5)**.** For the following, determine the radius and interval of convergence.

Power series only converges if $|4x - 8| \leq 40$.

Power series only converges if $|x - 3| < 4$.

Example (LC 28.6-28.9)**.** Consider the power series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-4)^k}{9^k\sqrt{k}}$.

Use the ratio test to compute the radius of convergence.

What is the interval of convergence?

Example (LC 28.10-28.13). Consider the power series $\displaystyle\sum_{k=1}^{\infty} \frac{(x-2)^k}{k^k}$.

Use the root test to compute the radius of convergence.

What is the interval of convergence?

Theorem 11.4: Combining Power Series
Suppose the power series $\sum c_k x^k$ and $\sum d_k x^k$ converge to $f(x)$ and $g(x)$, respectively, on an interval I.

1. **Sum and difference:** The power series $\sum (c_k \pm d_k)x^k$ converges to $f(x) \pm g(x)$ on I
2. **Multiplication by a power:** Suppose m is an integer such that $k + m \geq 0$, for all terms of the power series $x^m \sum c_k x^k = \sum c_k x^{k+m}$. This series converges to $x^m f(x)$, for all $x \neq 0$ in I. When $x = 0$, the series converges to $\lim\limits_{x\to 0} x^m f(x)$.
3. **Composition:** If $h(x) = bx^m$, where m is a positive integer and b is a nonzero real number, the power series $\sum c_k (h(x))^k$ converges to the composite function $f(h(x))$, for all x such that $h(x)$ is in I.

Example (LC 29.1)**.** Using the power series representation of

$$f(x) = \ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k},$$

where $-1 \leq x < 1$, find the power series centered at 0 for $g(x) = x\ln(1-x^3)$.

Example (LC 29.2-29.3)**.** Recall the geometric series:

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots = \frac{1}{1-x}, \quad \text{provided } |x| < 1.$$

Find the function represented by the power series $\displaystyle\sum_{k=0}^{\infty} \left(\sqrt{x} - 2\right)^k$.

What is the interval of convergence?

Example. Find the function represented by the power series $\sum_{k=0}^{\infty} \left(\frac{x^2+3}{7} \right)^k$.

What is the interval of convergence?

Theorem 11.5: Differentiating and Integrating Power Series
Suppose the power series $\sum c_k(x-a)^k$ converges for $|x-a| < R$ and defines a function f on that interval.

1. Then f is differentiable (which implies continuous) for $|x-a| < R$, and f' is found by differentiating the power series for f term by term; that is
$$f'(x) = \sum kc_k(x-a)^{k-1},$$
for $|x-a| < R$.
2. The indefinite integral of f is found by integrating the power series for f term by term; that is
$$\int f(x)\,dx = \sum c_k \frac{(x-a)^{k+1}}{k+1} + C,$$
for $|x-a| < R$, where C is an arbitrary constant.

Note: (LC 29.4) Differentiating or integrating a power series does not change the radius of convergence.

Example (LC 29.5)**.** Evaluate $\displaystyle\int xe^{-x^3}\,dx$ by integrating the power series representation:
$$f(x) = xe^{-x^3} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k+1}}{k!}, \quad \text{for } -\infty < x < \infty.$$

Example (LC 29.6)**.** Compute $f'(x)$ given that

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{2k+1}, \text{ for } |x| \leq 1.$$

Example (LC 29.7)**.** Find the power series representation of $g(x) = \dfrac{2}{(1-2x)^2}$ by using $f(x) = \dfrac{1}{1-2x}$.

Example (LC 29.8-29.10)**.** Find the power series representation of $g(x) = \ln(1-3x)$ by using $f(x) = \dfrac{1}{1-3x}$. What is the interval of convergence of this power series?