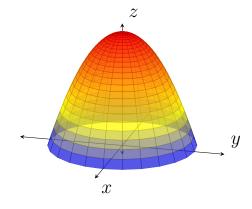
## 16.3: Double Integrals in Polar Coordinates

Suppose we wish to find the volume bounded by the curve  $f(x,y) = 9 - x^2 - y^2$  and the xy-plane. The region of integration would be

$$R = \left\{ (x, y) : -3 \le x \le 3, -\sqrt{9 - x^2} \le y \le \sqrt{9 - x^2} \right\}$$

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 9 - x^2 - y^2 \, dy \, dx$$



Alternatively, we can use polar coordinates where  $x = r\cos(\theta)$  and  $y = r\sin(\theta)$ . The associated region R is called a **polar rectangle**.

## Theorem 16.3: Change of Variables for Double Integrals over Polar Rectangle Regions

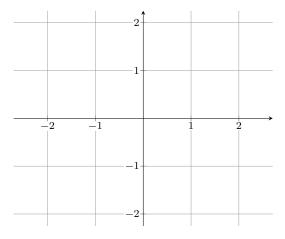
Let f be continuous on the region R in the xy-plane expressed in polar coordinates a  $sR = \{(r, \theta) : 0 \le a \le r \le b, \alpha \le \theta \le \beta\}$ , where  $\beta - \alpha = 2\pi$ . Then f is integrable over R, and the double integral of f over R is

$$\iint\limits_{R} f(x,y) dA = \int_{\alpha}^{\beta} \int_{a}^{b} f(r\cos\theta, r\sin\theta) r dr d\theta.$$

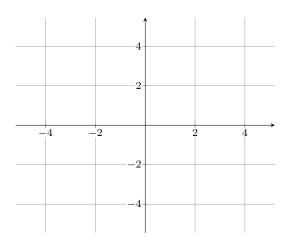
Note: When we convert to polar coordinates, there is an extra factor of r. This is due to the area of the circular segment being  $\frac{1}{2}r^2\theta$  (Section 16.7 will elaborate on this).

**Example.** Graph the following regions:

$$R = \left\{ (r, \theta) : 0 \le r \le 1, \ 0 \le \theta \le \frac{5\pi}{4} \right\}$$

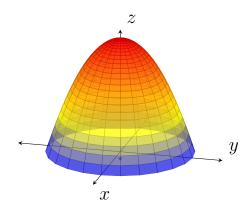


$$R = \left\{ (r, \theta) : 2 \le r \le 4, -\frac{\pi}{6} \le \theta \le \frac{7\pi}{6} \right\}$$

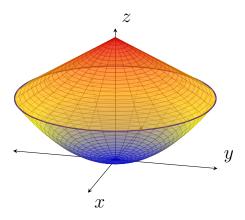


Spring 2021

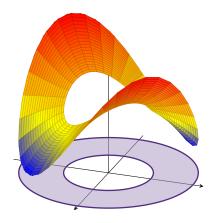
**Example.** Consider the paraboloid given earlier: Find the volume of the solid bounded above by  $z = 9 - x^2 - y^2$  and below by the xy-plane.



**Example.** Find the area of the solid bounded below by the paraboloid  $z = x^2 + y^2$  and bounded above by the cone  $z = 2 - \sqrt{x^2 + y^2}$ .



**Example.** Find the volume of the region beneath the surface z = xy + 10 and above the annular region  $R = \{(r, \theta) : 2 \le r \le 4, \ 0 \le \theta \le 2\pi\}.$ 



## Theorem 16.4: Change of Variables for Double Integrals over More General Polar Regions

Let f be continuous on the region R in the xy-plane expressed in polar coordinates as

$$R = \{(r, \theta) : 0 \le g(\theta) \le r \le h(\theta), \ \alpha \le \theta \le \beta\},\$$

where  $0 < \beta - \alpha \le 2\pi$ . Then

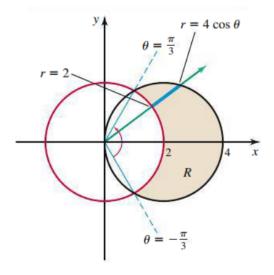
$$\iint\limits_{R} f(x,y) dA = \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta) r dr d\theta$$

## Area of Polar Regions

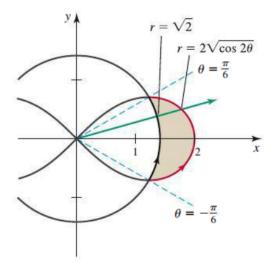
The area of the polar region  $R = \{(r, \theta) : 0 \le g(\theta) \le r \le h(\theta), \ \alpha \le \theta \le \beta\}$ , where  $0 < \beta - \alpha \le 2\pi$ , is

$$A = \iint\limits_{R} dA = \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} r \, dr \, d\theta.$$

**Example.** Write an iterated integral in polar coordinates for  $\iint_R g(r,\theta) dA$  for the region outside the circle r=2 and inside the circle  $r=4\cos(\theta)$ .



**Example.** Compute the area of the region in the first and fourth quadrants outside the circle  $r = \sqrt{2}$  and inside the lemniscate  $r^2 = 4\cos(2\theta)$ .



<b>Example.</b> Find the average value of disk of radius $a$ given by $R = \{(r, \theta)\}$		s in the semicircular	
16.3: Double Integrals in Polar Coordinates	150	Math 2060 Class notes Spring 2021	