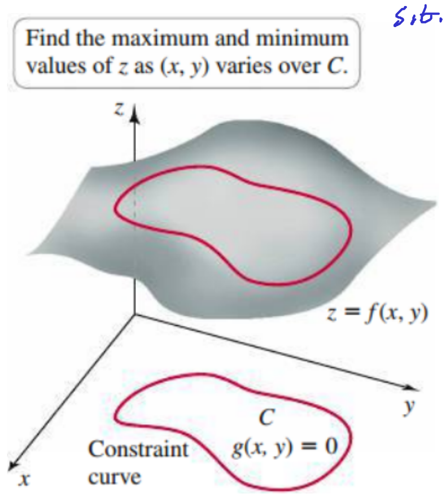
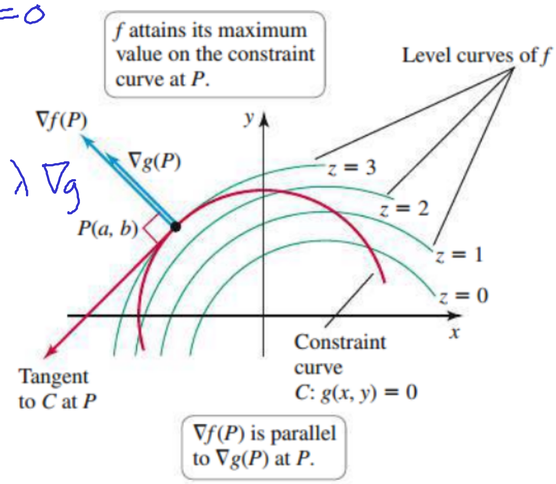
15.8: Lagrange Multipliers

Constrained optimization functions have an **objective function** f with the restriction that the independent variables x and y lie on a **constraint** curve C in the xy-plane given by $g(x, y) = 0$.

min $f(x,y)$
s.t. $g(x,y)=0$



$\nabla f = \lambda \nabla g$



Definition. (Parallel Gradients)
Let f be a differentiable function in a region of $\mathbb{R}^2$ that contains the smooth curve C given by $g(x, y) = 0$. Assume f has a local extreme value on C at a point $P(a, b)$. Then $\nabla f(a, b)$ is orthogonal to the line tangent to C at P. Assuming $\nabla g(a, b) \neq \mathbf{0}$, it follows that there is a real number λ (called a **Lagrange multiplier**) such that $\boxed{\nabla f(a, b) = \lambda \nabla g(a, b).}$

lambda

We consider the three following cases:

- Bounded constraint curves that close on themselves (e.g. circles, ellipses, etc),
- Bounded constraint curves that do not close on themselves, but include endpoints,
- Unbounded constraint curves

Example. Find the absolute maximum and minimum values of the objective function $f(x, y) = x^2+y^2+2$, where x and y lie on the ellipse C given by $g(x, y) = x^2+xy+y^2-4=0$.

$\nabla f(x,y) = \lambda \nabla g(x,y) \longleftarrow$

$\nabla f(x,y) = \langle 2x, 2y \rangle$

$\nabla g(x,y) = \langle 2x+y, 2y+x \rangle$

$$\Rightarrow \begin{cases} 2x = \lambda(2x+y) & (1) \\ 2y = \lambda(2y+x) & (2) \\ 0 = x^2+xy+y^2-4 & (3) \end{cases}$$

Solve for lambda $\lambda = \frac{2x}{2x+y}$

$(1)-(2)$: $2x - 2y = \lambda\big((2x+y)-(2y+x)\big)$

$= \lambda(x-y)$

$\rightarrow 2(x-y) - \lambda(x-y) = 0$

$(2-\lambda)(x-y) = 0$

$2 = \lambda \longrightarrow x = -y$

$x = y$

$\lambda = 2$: $(1) \Rightarrow 2x = 2(2x+y)$

$\rightarrow -2x = 2y$

$\rightarrow x = -y$

$(2) \Rightarrow 2y = 2(2y+x)$

$\rightarrow x = -y$

$x = y$: $(3) \longrightarrow$

$x^2 + x^2 + x^2 - 4 = 0$

$3x^2 = 4$

$x = \pm\frac{2}{\sqrt{3}}$

$\rightarrow y = \pm\frac{2}{\sqrt{3}}$

$(x,y) = \left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$

$(x,y) = \left(-\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$

$x = -y$: $(3) \longrightarrow x^2 - x^2 + x^2 - 4 = 0$

$x^2 = 4$

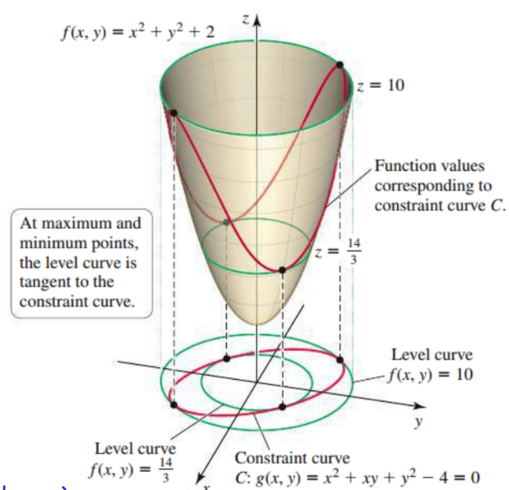
$x = \pm 2$

$y = \mp 2$

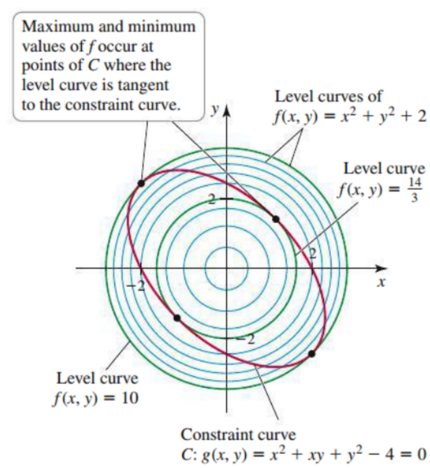
$(x,y) = (2,-2)$

$(x,y) = (-2,2)$

(x,y)	$f(x,y) = x^2+y^2+2$	
$\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$	$14/3$	abs min
$\left(-2/\sqrt{3}, -2/\sqrt{3}\right)$	$14/3 = 4.\bar{6}$	
$(2,-2)$	10	abs max
$(-2,2)$	10	



$g(x,y)$

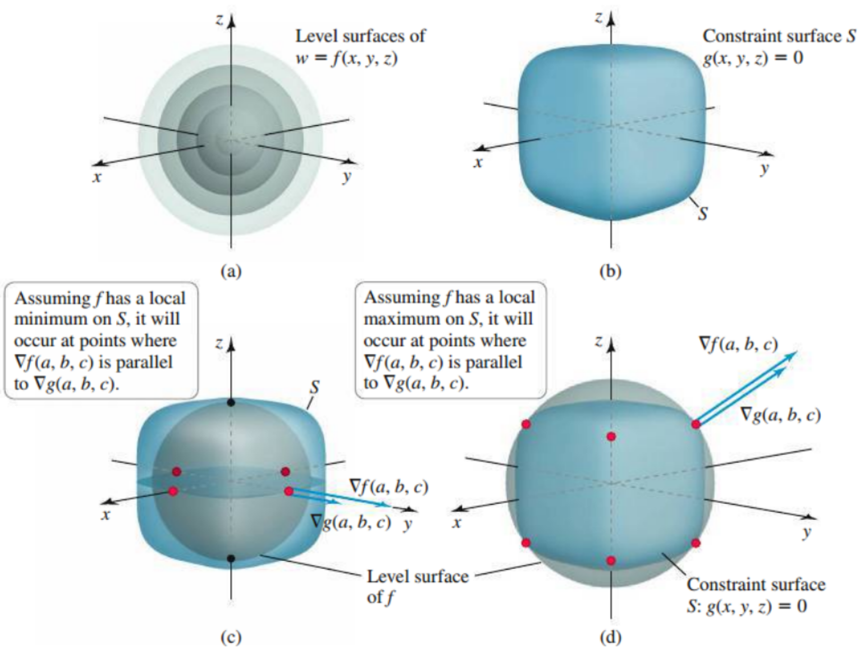


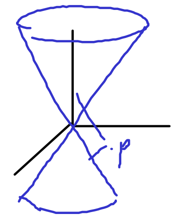
Procedure- Lagrange Multipliers: Absolute Extrema on Closed and Bounded Constraint Curves

Let the objective function f and the constraint function g be differentiable on a region $\mathbb{R}^2$ with $\nabla g(x, y) \neq \mathbf{0}$ on the curve $g(x, y) = 0$. To locate the absolute maximum and minimum values of f subject to the constraint $g(x, y) = 0$, carry out the following steps.

1. Find the values of x, y, and λ (if they exist) that satisfy the equations
$$\nabla f(x, y) = \lambda \nabla g(x, y) \text{ and } g(x, y) = 0.$$
2. Evaluate f at the values (x, y) in Step 1 and at the endpoints of the constraint curve (if they exist). Select the largest and smallest corresponding function values. These values are the absolute maximum and minimum values of f subject to the constraint.

Using Lagrange multipliers extends to higher dimensions with three or more independent variables:





objective

Example. Find the least distance between the point $P(3, 4, 0)$ and the surface of the cone $z^2 = x^2 + y^2$.

constraint

$$d(x,y,z) = \sqrt{(x-3)^2 + (y-4)^2 + z^2}$$

$$f(x,y,z) = (d(x,y,z))^2 = (x-3)^2 + (y-4)^2 + z^2$$

$$g(x,y,z) = z^2 - x^2 - y^2 = 0$$

$$\nabla f(x,y,z) = \langle 2(x-3), 2(y-4), 2z \rangle$$

$$\nabla g(x,y,z) = \langle -2x, -2y, 2z \rangle$$

(1) $2(x-3) = -2\lambda x \longrightarrow 2x - 6 + 2\lambda x = 0 \longrightarrow x(1+\lambda) = 3$

(2) $2(y-4) = -2\lambda y \longrightarrow y + \lambda y = 4 \longrightarrow y(1+\lambda) = 4$

(3) $2z = 2\lambda z \longrightarrow z - \lambda z = 0 \longrightarrow z(1-\lambda) = 0 \longrightarrow z = 0$, $\lambda = 1$

(4) $0 = z^2 - x^2 - y^2$

$z = 0$: (4) $\xrightarrow{0 = 0 - x^2 - y^2}$ $x = 0, y = 0 \longrightarrow$ violates (1) & (2) X

$\lambda = 1$: (1) $\longrightarrow 2x = 3$, $x = 3/2$ (2) $\longrightarrow 2y = 4$, $y = 2$ (4) $0 = z^2 - \left(\frac{3}{2}\right)^2 - (2)^2 = z^2 - \frac{25}{4} \longrightarrow z = \pm\frac{5}{2}$

$$f(3/2, 2, 5/2) = \frac{9}{4} + 4 + \frac{25}{4} = \frac{50}{4} = \boxed{\frac{25}{2}}$$

$$f(3/2, 2, -5/2) = \frac{9}{4} + 4 + \frac{25}{4} = \frac{50}{4} = \boxed{\frac{25}{2}}$$

LC #2 $3/2$
LC #3 2
LC #4 $5/2$

$$f(x,y,z) = (d(x,y,z))^2 = (x-3)^2 + (y-4)^2 + z^2$$

Example. Find the absolute maximum value of the utility function $U = f(\ell, g) = \ell^{1/3}g^{2/3}$, subject to the constraint $G(\ell, g) = 3\ell + 2g - 12 = 0$, where $\ell \geq 0$ and $g \geq 0$.

max $f(\ell,g) = \ell^{1/3}g^{2/3}$

s.t. $G(\ell,g) = 3\ell + 2g - 12 = 0$

$\nabla f(\ell,g) = \langle \frac{1}{3}\ell^{-2/3}g^{2/3}, \frac{2}{3}\ell^{1/3}g^{-1/3} \rangle = \langle \frac{1}{3}\left(\frac{g}{\ell}\right)^{2/3}, \frac{2}{3}\left(\frac{\ell}{g}\right)^{1/3} \rangle$

$\nabla G(\ell,g) = \langle 3, 2 \rangle$

$\nabla f = \lambda \nabla G$: (1) $\frac{1}{3}\left(\frac{g}{\ell}\right)^{2/3} = 3\lambda$ $\left(\frac{1}{3}\right)$

$\qquad$ (2) $\frac{2}{3}\left(\frac{\ell}{g}\right)^{1/3} = 2\lambda$ $\left(\frac{1}{2}\right)$

$\qquad$ (3) $3\ell + 2g - 12 = 0 \longrightarrow g = 6 - \frac{3}{2}\ell$ $\qquad$?

$(2) \longrightarrow \frac{2}{3}\left(\frac{\ell}{6-\frac{3}{2}\ell}\right)^{1/3} = 2\lambda \longrightarrow \frac{1}{3}\left(\frac{\ell}{6-\frac{3}{2}\ell}\right)^{1/3} = \lambda$

$(1) \longrightarrow \frac{1}{3}\left(\frac{g}{\ell}\right)^{2/3} = \cancel{3}\cdot\frac{1}{\cancel{3}}\left(\frac{\ell}{6-\frac{3}{2}\ell}\right)^{1/3}$

$\frac{1}{27}\frac{g^2}{\ell^2} = \frac{\ell}{6-3/2\,\ell}$

$\frac{1}{3}(1) = \frac{1}{2}(2)$

$\left(\frac{g}{\ell}\right)^{1/3}\frac{1}{9}\left(\frac{g}{\ell}\right)^{2/3} = \frac{1}{3}\left(\frac{\ell}{g}\right)^{1/3}\left(\frac{g}{\ell}\right)^{1/3}$

$\frac{1}{9}\frac{g}{\ell} = \frac{1}{3}$

$g = 3\ell \xrightarrow{(3)} 3\ell + 6\ell - 12 = 0$

$9\ell = 12$

$\ell = 4/3 \longrightarrow g = 4$

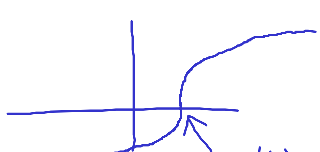
∇f DNE $\qquad$ f defined

$\begin{cases} \ell = 0 \xrightarrow{(3)} g = 6 \\ g = 0 \xrightarrow{(3)} \ell = 4 \end{cases}$

(ℓ,g)	$U = f(\ell,g) = \ell^{1/3}g^{2/3}$
$(4/3, 4)$	$\left(\frac{4}{3}\right)^{1/3}(4)^{2/3} = \frac{4}{\sqrt[3]{3}}$ $\longleftarrow$ max
$(0, 6)$	0
$(4, 0)$	0

LC #5 $\ell = 4/3$

LC #6 $g = 4$



$(x-1)^{1/3}$

$f'(1)$ DNE

$\vec{x} = \langle x_1, x_2, x_3, x_4 \rangle$

Example. Find the maximum value of $\underbrace{x_1 + x_2 + x_3 + x_4}_{f(x_1, x_2, x_3, x_4)}$ subject to the condition that $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 16$.

$$g(\vec{x}) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - 16 = 0$$
$$= \langle x, x \rangle - 16 = 0$$

$$\nabla f(\vec{x}) = \langle 1, 1, 1, 1 \rangle$$
$$\nabla g(\vec{x}) = \langle 2x_1, 2x_2, 2x_3, 2x_4 \rangle$$

(1) $1 = 2\lambda x_1 \longrightarrow$ since $1 \neq 0$, $\lambda \neq 0$ $\quad x_1 = \frac{1}{2\lambda}$

(2) $1 = 2\lambda x_2 \longrightarrow \quad x_i = \frac{1}{2\lambda}, \ i = 1, 2, 3, 4$

(3) $1 = 2\lambda x_3$

(4) $1 = 2\lambda x_4$

(5) $0 = x_1^2 + x_2^2 + x_3^2 + x_4^2 - 16 \quad$ (5) $\longrightarrow 0 = 4\left(\frac{1}{2\lambda}\right)^2 - 16$

$$= \frac{1}{\lambda^2} - 16 \rightarrow \lambda = \pm\frac{1}{4}$$

$\lambda \neq 0 \Rightarrow x_1 = x_2 = x_3 = x_4$

$\Rightarrow 0 = 4x_1^2 - 16 \rightarrow x_1 = \pm 2$

$$x_1 = \frac{1}{2\lambda} = \pm 2$$

$$x_1 = x_2 = x_3 = x_4 = 2$$
$$f(\vec{x}) = 8$$

LC #7 $\boxed{x_1 = 2}$

Procedure- Lagrange Multipliers: Absolute Extrema on Closed and Bounded Constraint Surfaces

Let f and g be differentiable on a region of $\mathbb{R}^3$ with $\nabla g(x, y, z) \neq \mathbf{0}$ on the surface $g(x, y, z) = 0$. To locate the absolute maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$, carry out the following steps.

1. Find the values of x, y, z, and λ that satisfy the equations

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \text{ and } g(x, y, z) = 0.$$

2. Among the points (x, y, z) found in Step 1, select the largest and smallest corresponding function values. These values are the absolute maximum and minimum values of f subject to the constraint.