17.6: Surface Integrals

Imagine a sphere with a known temperature distribution. How would we find the average temperature over the sphere?

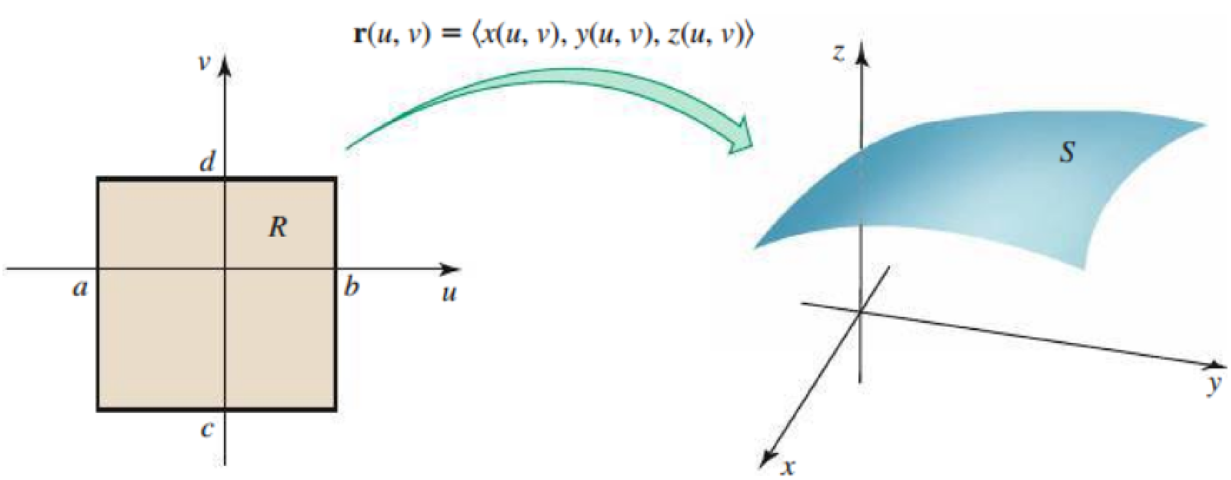
Parallel Concepts	
Curves	**Surfaces**
Arc length	Surface area
Line integrals	Surface integrals
One-parameter description	Two-parameter description

Parameterized Surfaces

Recall that in $\mathbb{R}^2$, we parameterized a curve by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ where $a \leq t \leq b$. In $\mathbb{R}^3$, we parameterize a surface by

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$

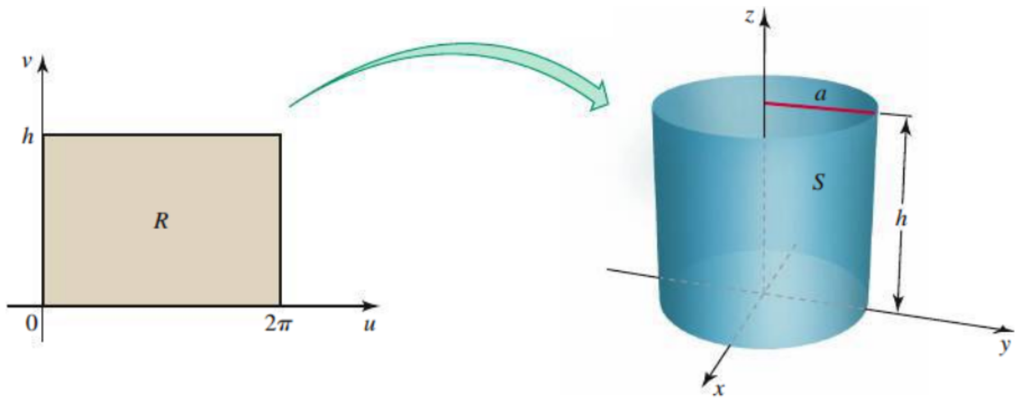
where the parameters are over $R = \{(u, v) : a \leq u \leq b, c \leq v \leq d\}$



Cylinders:

$$\{(x,\, y,\, z) : x = a\cos(\theta),\; y = a\sin(\theta),\; 0 \le \theta \le 2\pi,\, 0 \le z \le h\}$$

$$\mathbf{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle = \langle a\cos(u),\, a\sin(u),\, v\rangle$$



Cones:

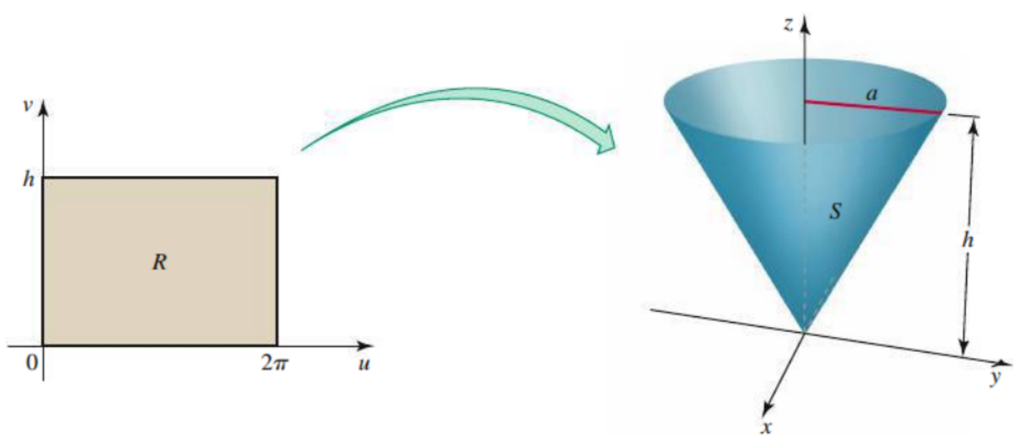
$$\{(r,\, \theta,\, z) : 0 \le r \le a,\, 0 \le \theta \le 2\pi,\, z = rh/a\}$$

For a fixed value of z, $r = az/h$:

$$x = r\cos(\theta) = \frac{az}{h}\cos(\theta) \text{ and } y = r\sin(\theta) = \frac{az}{h}\sin(\theta)$$

Now, let $u = \theta$ and $v = z$, then

$$\mathbf{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle = \left\langle \frac{av}{h}\cos(u),\, \frac{av}{h}\sin(u),\, v\right\rangle$$



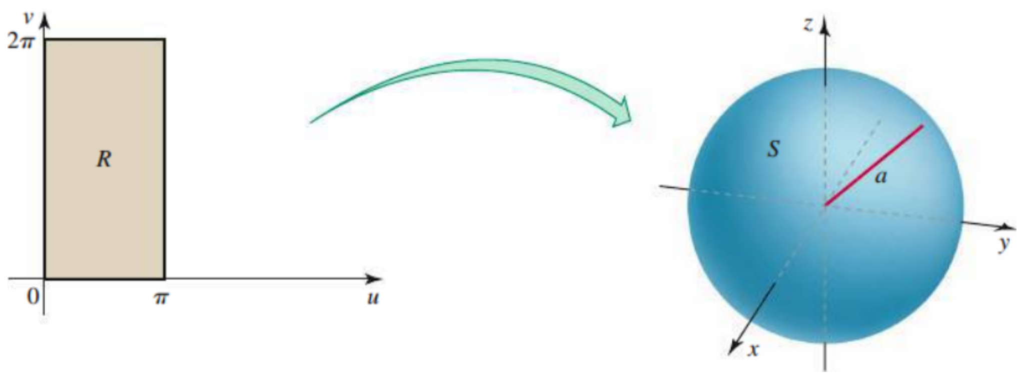
Spheres:

$$\{(\rho,\, \varphi,\, \theta) : \rho = a, 0 \le \varphi \le \pi,\, 0 \le \theta \le 2\pi\}$$

$$x = a\sin(\varphi)\cos(\theta), \qquad y = a\sin(\varphi)\sin(\theta), \qquad z = a\cos(\varphi)$$

Now, let $u = \theta$ and $v = z$, then

$$\mathbf{r}(u, v) = \langle x(u, v),\, y(u, v),\, z(u, v)\rangle = \langle a\sin(u)\cos(v),\, a\sin(u)\cos(v),\, a\cos(u)\rangle$$

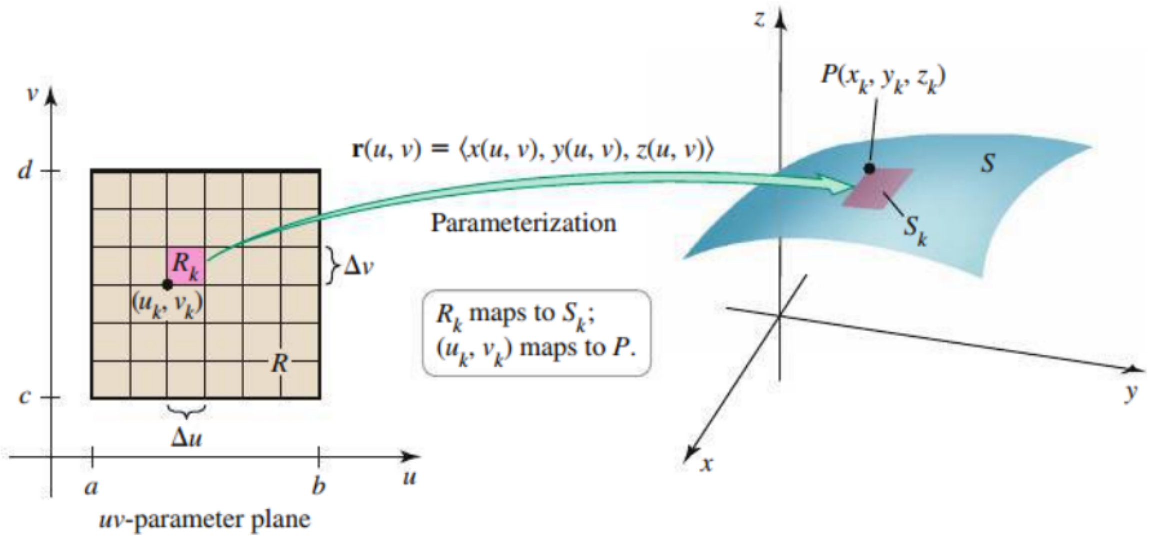


Example. Find parametric descriptions for the following surfaces

The plane $3x - 2y + z = 2$

The paraboloid $z = x^2 + y^2$, for $0 \le z \le 9$

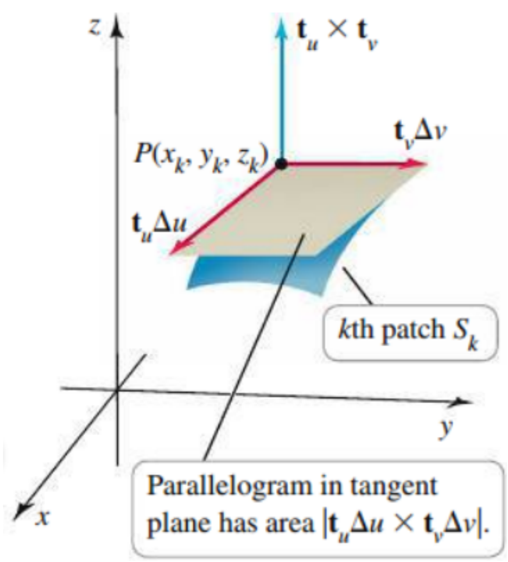
Surface Integrals of Scalar-Valued Functions



Using the parameterization

$$\mathbf{r}(u, v) = \langle x(u, v),\, y(u, v),\, z(u, v)\rangle$$

over the region $R = \{(u, v) : a \le u \le b,\ c \le v \le d\}$, it is important that we know ΔS_k, which is the area of S_k.



Definition. (Surface Integral of Scalar-Valued Functions on Parameterized Surfaces)
Let f be a continuous scalar-valued function on a smooth surface S given parametrically by $\mathbf{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle$, where u and v vary over $R = \{(u,v) : a \le u \le b,\, c \le v \le d\}$. Assume also that the tangent vectors

$$\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \text{ and } \mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

are continuous on R and the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R. Then the **surface integral of f over S** is

$$\iint\limits_S f(x,y,z)\, dS = \iint\limits_R f(x(u,v),\, y(u,v),\, z(u,v))|\mathbf{t}_u \times \mathbf{t}_v|\, dA$$

If $f(x,y,z) = 1$, this integral equals the surface area of S.

Example. Find the surface area of the following surfaces

A cylinder with radius $a > 0$ and height h (open ends)

A sphere of radius a

Example. The temperature on the surface of a sphere of radius a varies with latitude according to the function $T(\varphi, \theta) = 10 + 50\sin(\varphi)$, for $0 \leq \varphi \leq \pi$ and $0 \leq \theta \leq 2\pi$. Find the average temperature over the sphere.

Surface Integrals on Explicitly Defined Surfaces

Suppose a smooth surface S is defined explicitly as $z = g(x, y)$. Here, we let $u = x$ and $v = y$. This gives us

$$\mathbf{t}_u = \mathbf{t}_x = \langle 1,\, 0,\, z_x \rangle, \quad \mathbf{t}_v = \mathbf{t}_y = \langle 0,\, 1,\, z_y \rangle$$

thus

$$\mathbf{t}_x \times \mathbf{t}_y = \langle -z_x,\, -z_y,\, 1 \rangle$$

and

$$|\mathbf{t}_x \times \mathbf{t}_y| = \sqrt{z_x^2 + z_y^2 + 1}$$

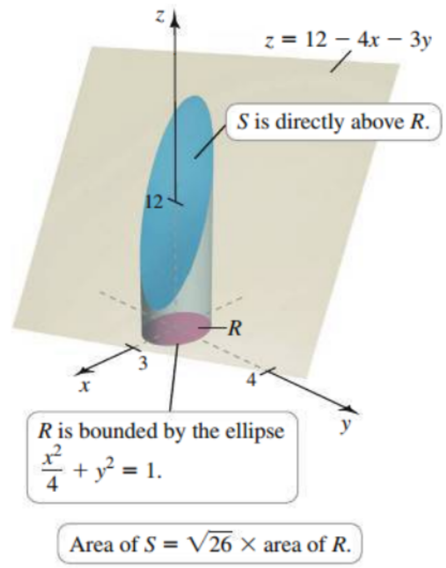
Theorem 17.14: Evaluation of Surface Integrals of Scalar-Valued Functions on Explicitly Defined Surfaces

Let f be a continuous function on a smooth surface S given by $z = g(x, y)$, for (x, y) in a region R. The surface integral of f over S is

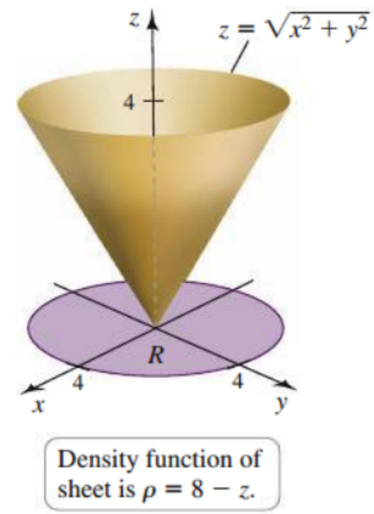
$$\iint_S f(x, y, z)\, dS = \iint_S f(x, y, g(x, y))\sqrt{z_x^2 + z_y^2 + 1}\, dA.$$

If $f(x, y, z) = 1$, the surface integral equals the area of the surface.

Example. Find the area of the surface S that lies in the plane $z = 12 - 4x - 3y$ directly above the region R bounded by the ellipse $x^2/4 + y^2 = 1$



Example. A thin conical sheet is described by the surface $z=\left(x^2+y^2\right)^{\frac{1}{2}}$, for $0 \leq z \leq 4$. The density of the sheet in g/ cm^2 is $\rho=f(x, y, z)=(8-z)$. What is the mass of the cone?



	Explicit Description $z = g(x,y)$			**Parametric Description**		
Surface	**Equation**	**Normal vector** $\pm\langle -z_x,\, -z_y,\, 1\rangle$	**magnitude** $\lvert\langle -z_x,\, -z_y,\, 1\rangle\rvert$	**Equation**	**Normal vector** $\mathbf{t}_u \times \mathbf{t}_v$	**magnitude** $\lvert\mathbf{t}_u \times \mathbf{t}_v\rvert$
Cylinder	$x^2+y^2=a^2$, $0 \le z \le h$	$\langle x, y, 0\rangle$	a	$\mathbf{r} = \langle a\cos(u),\, a\sin(u),\, v\rangle$, $0 \le u \le 2\pi$, $0 \le v \le h$	$\langle a\cos(u),\, a\sin(u),\, 0\rangle$	a
Cone	$z^2 = x^2+y^2$, $0 \le z \le h$	$\langle x/z,\, y/z,\, -1\rangle$	$\sqrt{2}$	$\mathbf{r} = \langle v\cos(u),\, v\sin(u),\, v\rangle$, $0 \le u \le 2\pi$, $0 \le v \le h$	$\langle v\cos(u),\, v\sin(u),\, -v\rangle$	$\sqrt{2}v$
Sphere	$x^2+y^2+z^2=a^2$	$\langle x/z,\, y/z,\, 1\rangle$;	a/z	$\mathbf{r} = \langle a\sin(u)\cos(v),\, a\sin(u)\sin(v),\, a\cos(u)\rangle$ $0 \le u \le \pi$, $0 \le v \le 2\pi$	$\langle a^2\sin^2(u)\cos(v),\, a^2\sin^2(u)\sin(v),\, a^2\sin(u)\cos(u)\rangle$	$a^2\sin(u)$
Paraboloid	$z = x^2+y^2$, $0 \le z \le h$	$\langle 2x,\, 2y,\, -1\rangle$	$\sqrt{1+4(x^2+y^2)}$	$\mathbf{r} = \left\langle v\cos(u),\, v\sin(u),\, v^2\right\rangle$, $0 \le u \le 2\pi$, $0 \le v \le \sqrt{h}$	$\left\langle 2v^2\cos(u), 2v^2\sin(u), -v\right\rangle$	$v\sqrt{1+4v^2}$

Surface Integrals of Vector Fields:

The surfaces we consider must be

- **two-sided** or **orientable**
- **oriented**

Definition. (Surface Integral of a Vector Field)
Suppose $\mathbf{F} = \langle f,\, g,\, h\rangle$ is a continuous vector field on a region of $\mathbb{R}^3$ containing a smooth oriented surface S. If S is defined parametrically as $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, for (u, v) in a region R, then

$$\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,ds = \iint\limits_R \mathbf{F}\cdot(\mathbf{t}_u \times \mathbf{t}_v)\,dA,$$

where

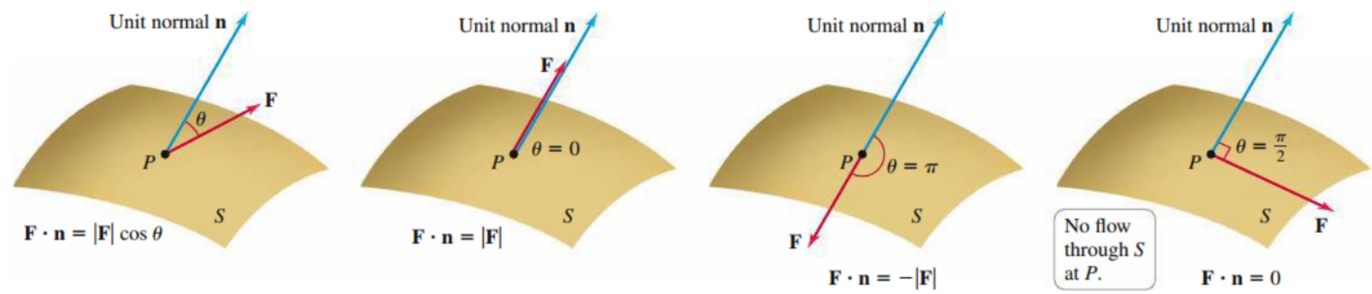
$$\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right\rangle \text{ and } \mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right\rangle$$

and continuous on R, the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R, and the direction of the normal vector is consistent with the orientation of S. If S is defined in the form $z = s(x, y)$, for (x, y) in a region R, then

$$\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint\limits_S (-fz_x - gz_y + h)\,dA.$$

Flux Integrals:

$$\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS$$



The unit normal vector we use is

$$\mathbf{n} = \frac{\mathbf{t}_u \times \mathbf{t}_v}{|\mathbf{t}_u \times \mathbf{t}_v|}$$

giving us

$$\begin{aligned}\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint\limits_R \mathbf{F}\cdot\mathbf{n}|\mathbf{t}_u \times \mathbf{t}_v|\,dA\\ &= \iint\limits_R \mathbf{F}\cdot\frac{\mathbf{t}_u \times \mathbf{t}_v}{|\mathbf{t}_u \times \mathbf{t}_v|}|\mathbf{t}_u \times \mathbf{t}_v|\,dA\\ &= \iint\limits_R \mathbf{F}\cdot(\mathbf{t}_u \times \mathbf{t}_v)\,dA\end{aligned}$$

When the surface S is explicitly given as $z = s(x, y)$, then

$$\mathbf{F}\cdot(\mathbf{t}_u \times \mathbf{t}_v) = -fz_x - gz_y + h$$

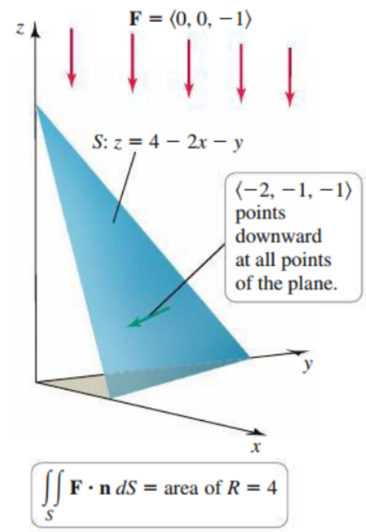
Definition. (Surface Integral of a Vector Field)
Suppose $\mathbf{F} = \langle f, g, h\rangle$ is a continuous vector field on a region of $\mathbb{R}^3$ containing a smooth oriented surface S. If S is defined parametrically as $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, for (u,v) in a region R, then

$$\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint\limits_R \mathbf{F}\cdot(\mathbf{t}_u \times \mathbf{t}_v)\,dA,$$

where $\mathbf{t}_u = \dfrac{\partial \mathbf{r}}{\partial u} = \left\langle \dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u}, \dfrac{\partial z}{\partial u}\right\rangle$ and $\mathbf{t}_v = \dfrac{\partial \mathbf{r}}{\partial u} = \left\langle \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial v}, \dfrac{\partial z}{\partial v}\right\rangle$ are continuous on R, the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R, and the direction of the normal vector is consistent with the orientation of S. If S is defined in the form $z = s(x,y)$, for (x,y) in a region R, then

$$\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint\limits_R (-fz_x - gz_y + h)\,dA.$$

Example. Consider the vertical field $\mathbf{F} = \langle 0, 0, -1 \rangle$. Find the flux in the downward direction across the surface S, which is the plane $z = 4 - 2x - y$ in the first octant.



Example. Consider the radial vector field $\mathbf{F} = \langle f,\, g,\, h \rangle = \langle x,\, y,\, z \rangle$.