17.6: Surface Integrals

Imagine a sphere with a known temperature distribution. How would we find the average temperature over the sphere?

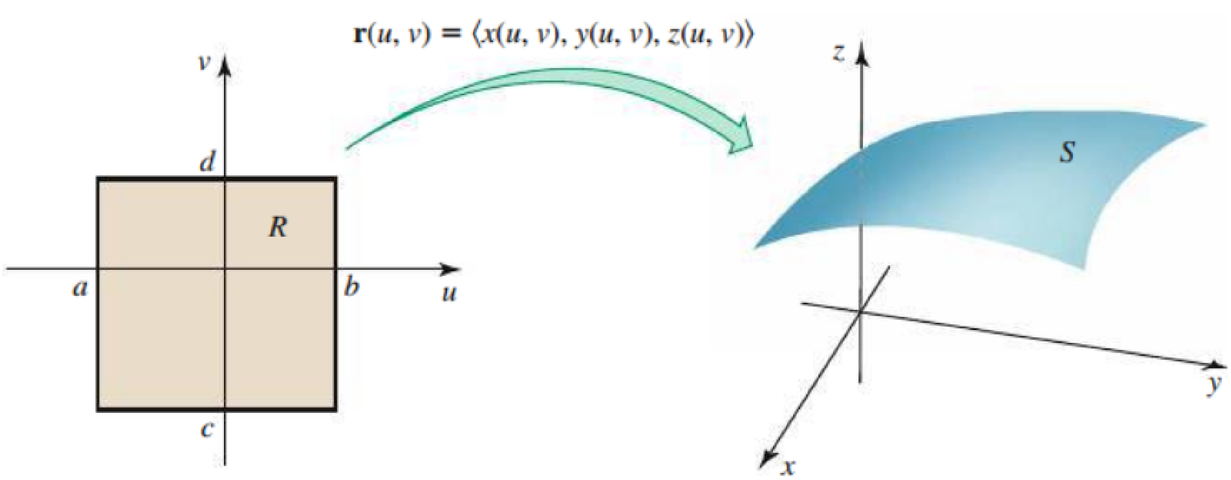
Parallel Concepts	
Curves	**Surfaces**
Arc length	Surface area
Line integrals	Surface integrals
One-parameter description	Two-parameter description

Parameterized Surfaces

Recall that in $\mathbb{R}^2$, we parameterized a curve by $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$ where $a \le t \le b$. In $\mathbb{R}^3$, we parameterize a surface by

$$\mathbf{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle$$

where the parameters are over $R = \{(u,v) : a \le u \le b, c \le v \le d\}$



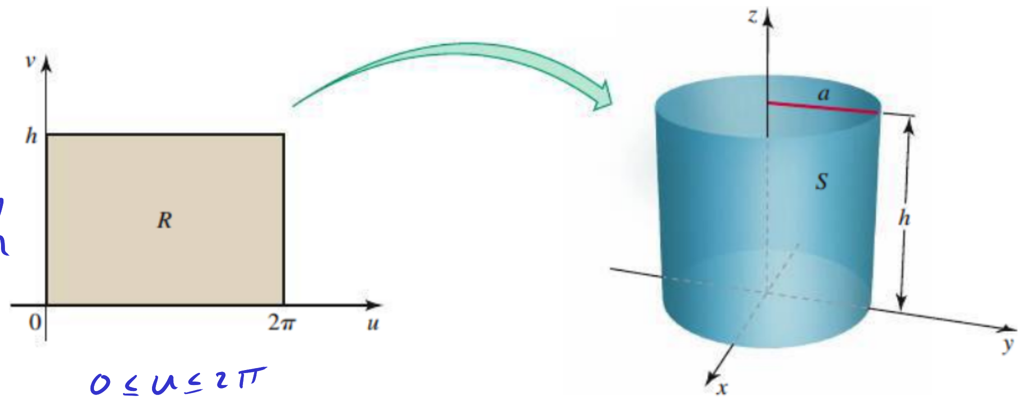
Cylinders:

$$\{(x,\, y,\, z) : x = a\cos(\theta),\ y = a\sin(\theta),\ 0 \le \theta \le 2\pi,\, 0 \le z \le h\}$$

$u = \theta$
$v = z$

$$\mathbf{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle = \langle a\cos(u),\, a\sin(u),\, v\rangle$$

$0 \le v \le h$



$0 \le u \le 2\pi$

Cones:

$$\{(\underline{r,\, \theta,\, z}) : 0 \underline{\le r \le a},\ 0 \underline{\le \theta \le 2\pi},\ \underline{z} = rh/a\}$$

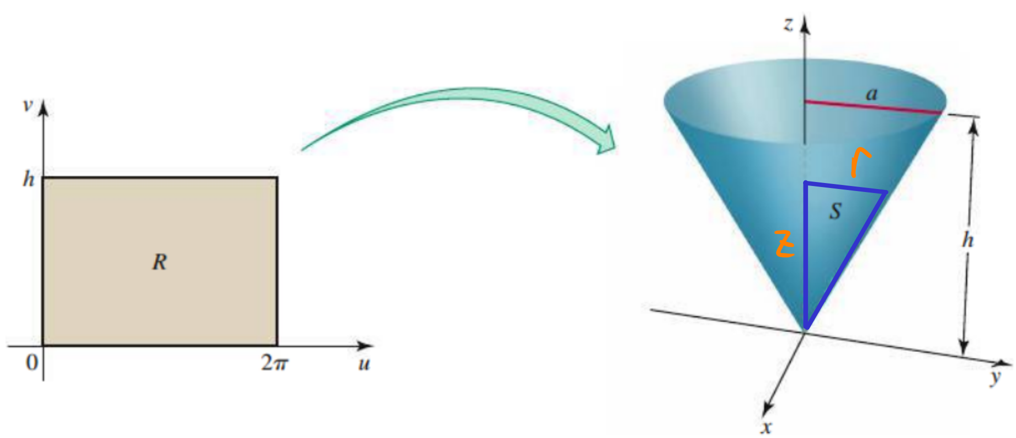
For a fixed value of z, $r = az/h$:

$$x = r\cos(\theta) = \frac{az}{h}\cos(\theta) \text{ and } y = r\sin(\theta) = \frac{az}{h}\sin(\theta)$$

Now, let $u = \theta$ and $v = z$, then

$$\mathbf{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle = \left\langle \frac{av}{h}\cos(u),\, \frac{av}{h}\sin(u),\, v\right\rangle$$

$\frac{z}{r} = \frac{h}{a}$



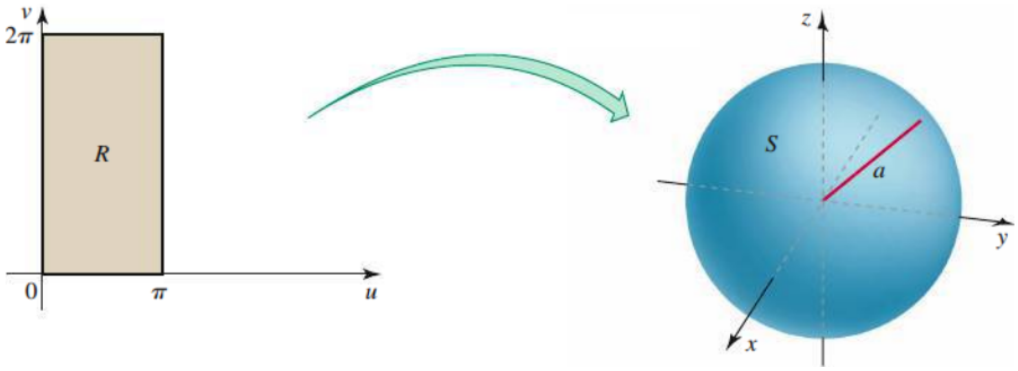
Spheres:

$$\{(\rho,\,\varphi,\,\theta) : \rho = a, 0 \le \varphi \le \pi,\; 0 \le \theta \le 2\pi\}$$

$$x = a\sin(\varphi)\cos(\theta), \qquad y = a\sin(\varphi)\sin(\theta), \qquad z = a\cos(\varphi)$$

Now, let $u = \theta$ and $v = z$, then

$$\mathbf{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle = \langle a\sin(u)\cos(v),\, a\sin(u)\cos(v),\, a\cos(u)\rangle$$



Example. Find parametric descriptions for the following surfaces

The plane $3x - 2y + z = 2$

$z = 2 - 3x + 2y$

Let $x = u$, $y = v$

$z = 2 - 3u + 2v \longrightarrow \vec{r}(u,v) = \langle u, v, 2 - 3u + 2v\rangle$

$-\infty \le u \le \infty, \quad -\infty \le v \le \infty$

The paraboloid $z = \underbrace{x^2 + y^2}$, for $0 \le z \le 9$ $\longrightarrow$ $0 \le r \le 3$

$x = r\cos\theta$, $y = r\sin\theta$, $z = r^2$

Let $u = \theta$, $v = r^2 \longrightarrow \vec{r}(u,v) = \langle \sqrt{v}\cos(u), \sqrt{v}\sin(u), v\rangle$

$0 \le u \le 2\pi$

$0 \le v \le 9$

Let $u = \theta$, $v = r$

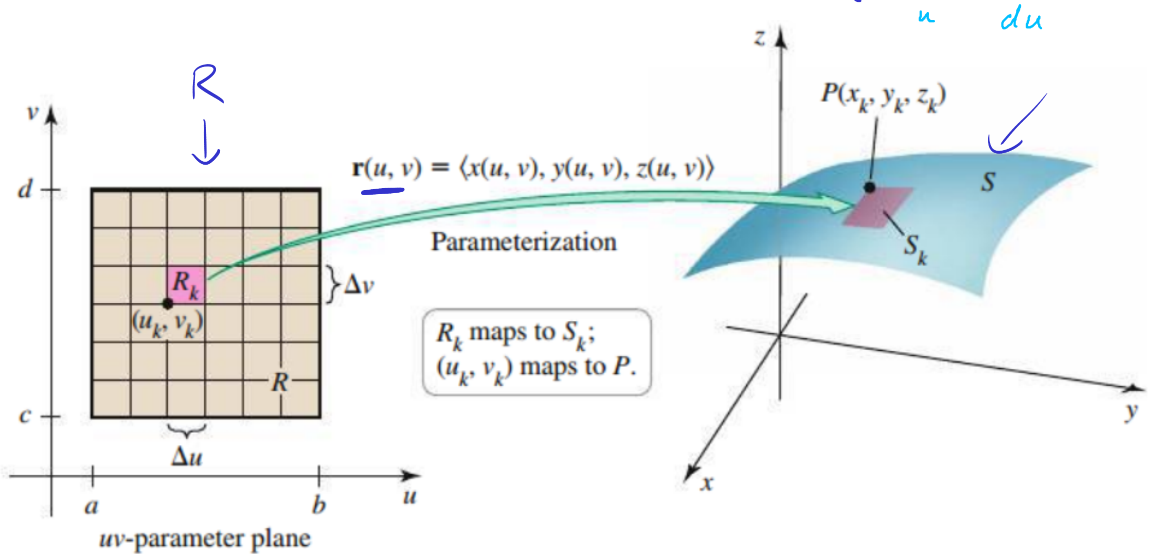
$\vec{r}(u,v) = \langle v\cos(u), v\sin(u), v^2\rangle$

$0 \le u \le 2\pi$

$0 \le v \le 3$

Surface Integrals of Scalar-Valued Functions

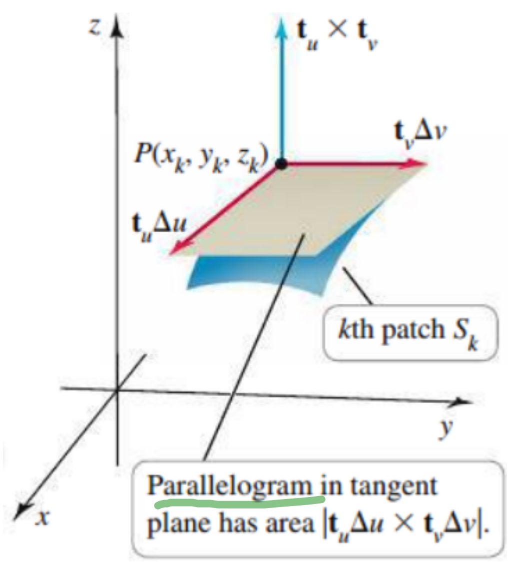
$\int f(\underbrace{g(x)}_{u})\, \underbrace{g'(x)\, dx}_{du} = \int f(u)\, du$



Using the parameterization

$$\mathbf{r}(u, v) = \langle x(u, v),\, y(u, v),\, z(u, v)\rangle$$

over the region $R = \{(u, v) : a \le u \le b,\, c \le v \le d\}$, it is important that we know ΔS_k, which is the area of S_k.



Definition. (Surface Integral of Scalar-Valued Functions on Parameterized Surfaces)
Let f be a continuous scalar-valued function on a smooth surface S given parametrically by $\mathbf{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle$, where u and v vary over $R = \{(u,v) : a \le u \le b,\, c \le v \le d\}$. Assume also that the tangent vectors

$$\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \text{ and } \mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

are continuous on R and the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R. Then the **surface integral of f over S** is

Recall line integrals $\int_C f\, ds = \int_C f \cdot |\vec{r}\,'|\, dt$

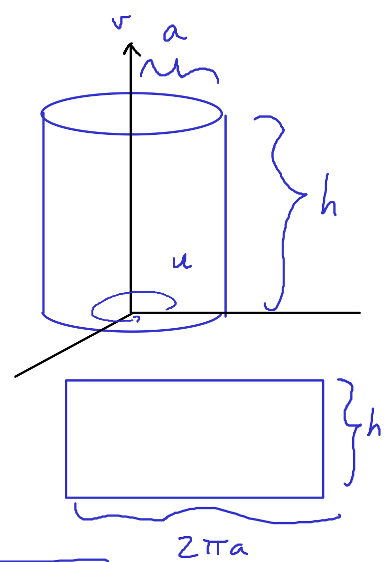
$$\iint_S f(x,y,z)\, dS = \iint_R \underbrace{f(x(u,v),\, y(u,v),\, z(u,v))}_{\text{scalar}} \underbrace{|\mathbf{t}_u \times \mathbf{t}_v|}_{\text{scalar}}\, dA$$

If $f(x,y,z) = 1$, this integral equals the surface area of S. ← arc length, SA (1080)

Example. Find the surface area of the following surfaces

A cylinder with radius $a > 0$ and height h (open ends)

$\vec{r}(u,v) = \langle a\cos(u), a\sin(u), v\rangle$ $\quad 0 \le u \le 2\pi$, $\; 0 \le v \le h$



$\vec{t}_u = \langle -a\sin(u),\; a\cos(u),\; 0 \rangle$

$\vec{t}_v = \langle 0,\; 0,\; 1 \rangle$

$|\vec{t}_u \times \vec{t}_v| = |\langle a\cos(u),\, a\sin(u),\, 0\rangle| = \sqrt{a^2\cos^2(u) + a^2\sin^2(v)} = a$

$SA = \iint_S 1\, dS = \iint_R 1 \cdot \underbrace{|\vec{t}_u \times \vec{t}_v|}_{a}\, dA = \int_0^{2\pi}\int_0^h a\, dv\, du = \boxed{2\pi a h}$

LC # 1

Polar $\quad x = r\cos\theta$, $y = r\sin\theta$, $z = z$ → Fixed r

$$V = \frac{4}{3}\pi r^3 \longrightarrow \underset{r=a}{SA = 4\pi r^2} \longrightarrow 4\pi a^2$$

Spherical

$x = \rho \sin\varphi \cos\theta$

$y = \rho \sin\varphi \sin\theta$

$z = \rho \cos\varphi$

Fixed $\rho = a$

A sphere of radius a

$$\vec{r}(u,v) = \langle a\sin(u)\cos(v),\ a\sin(u)\sin(v),\ a\cos(u)\rangle \qquad 0 \le u \le \pi,\ 0 \le v \le 2\pi$$

$$R = \{(u,v) : 0 \le u \le \pi,\ 0 \le v \le 2\pi\}$$

$$\vec{t}_u = \langle a\cos(u)\cos(v),\ a\cos(u)\sin(v),\ -a\sin(u)\rangle$$

$$\vec{t}_v = \langle -a\sin(u)\sin(v),\ a\sin(u)\cos(v),\ 0\rangle$$

$$|\vec{t}_u \times \vec{t}_v| = |\langle a^2\sin^2(u)\cos(v),\ a^2\sin^2(u)\sin(v),\ a^2\cos(u)\sin(u)\cos^2(v) + a^2\cos(u)\sin(u)\sin^2(v)\rangle|$$

$$= a^2\sin(u)\,|\langle \sin(u)\underbrace{\cos(v)}_{\cos^2(v)},\ \sin(u)\underbrace{\sin(v)}_{\sin^2(v)},\ \cos(u)\underbrace{(\cos^2(v)+\sin^2(v))}_{1}\rangle|$$

$$= a^2\sin(u)$$

$$SA = \iint_S 1\,dS = \iint_R |\vec{t}_u \times \vec{t}_v|\,dA = \int_0^{2\pi}\int_0^{\pi} a^2\sin(u)\,du\,dv$$

$$= \int_0^{2\pi} -a^2\cos(u)\Big|_{u=0}^{u=\pi} dv$$

$$= 2a^2\int_0^{2\pi} dv = 4\pi a^2$$

LC #2

Example. The temperature on the surface of a sphere of radius a varies with latitude according to the function $T(\varphi, \theta) = 10 + 50\sin(\varphi)$, for $0 \le \varphi \le \pi$ and $0 \le \theta \le 2\pi$. Find the average temperature over the sphere.

$$\vec{r}(u,v) = \langle a\sin(u)\cos(v),\ a\sin(u)\sin(v),\ a\cos(\overset{\overset{u}{\uparrow}}{u})\rangle \qquad R = \{(u,v) : 0 \le u \le \pi,\ 0 \le v \le 2\pi\}$$

$$\iint_S T(\varphi, \theta)\, dS = \int_0^{2\pi}\int_0^{\pi} (10 + 50\sin(u))\, \underbrace{a^2\sin(u)}_{|\vec{t}_u \times \vec{t}_v|}\, du\, dv$$

$$= 10\int_0^{2\pi}\int_0^{\pi} a^2\sin(u) + 5a^2\underbrace{\sin^2(u)}_{\frac{1-\cos(2u)}{2}}\, du\, dv$$

$$= 10\int_0^{2\pi} -a^2\cos(u) + \frac{5}{2}a^2\left(u - \frac{\sin(2u)}{2}\right)\Big|_{u=0}^{u=\pi} dv$$

$$= 10\int_0^{2\pi} 2a^2 + \frac{5}{2}a^2\pi\, dv$$

$$= 10a^2\left(2 + \frac{5\pi}{2}\right)v\Big|_{v=0}^{v=2\pi} = 10a^2(4 + 5\pi)$$

LC #3
$a = 26$

LC #4
$a = 256$

Surface Integrals on Explicitly Defined Surfaces

Suppose a smooth surface S is defined explicitly as $z = g(x, y)$. Here, we let $u = x$ and $v = y$. This gives us

$$\mathbf{t}_u = \mathbf{t}_x = \langle 1,\, 0,\, z_x \rangle, \quad \mathbf{t}_v = \mathbf{t}_y = \langle 0,\, 1,\, z_y \rangle$$

thus

$$\mathbf{t}_x \times \mathbf{t}_y = \langle -z_x,\, -z_y,\, 1 \rangle$$

and

$$|\mathbf{t}_x \times \mathbf{t}_y| = \sqrt{z_x^2 + z_y^2 + 1}$$

Theorem 17.14: Evaluation of Surface Integrals of Scalar-Valued Functions on Explicitly Defined Surfaces

Let f be a continuous function on a smooth surface S given by $z = g(x, y)$, for (x, y) in a region R. The surface integral of f over S is

$$\iint_S f(x, y, z)\, dS = \iint_S f(x, y, g(x, y)) \underbrace{\sqrt{z_x^2 + z_y^2 + 1}}_{|\vec{t}_u \times \vec{t}_v|}\, dA.$$

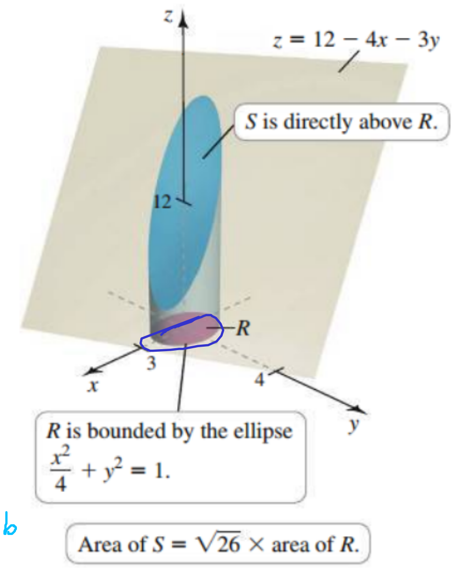
If $f(x, y, z) = 1$, the surface integral equals the area of the surface.

Example. Find the area of the surface S that lies in the plane $z = 12 - 4x - 3y$ directly above the region R bounded by the ellipse $x^2/4 + y^2 = 1$

$\vec{t}_u \times \vec{t}_v = \vec{t}_x \times \vec{t}_y = \langle -z_x, -z_y, 1 \rangle$

$= \langle 4, 3, 1 \rangle$

$|\vec{t}_x \times \vec{t}_y| = \sqrt{26}$



$$\iint_S 1\, dS = \iint_R \underbrace{|\vec{t}_x \times \vec{t}_y|}_{\sqrt{26}}\, dA = \sqrt{26} \underbrace{\iint_R dA}_{\text{area of ellipse} = \pi ab}$$

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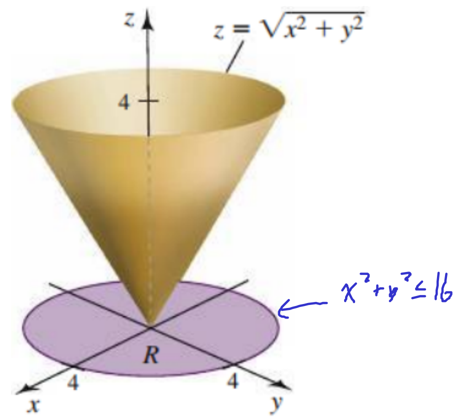
$a = 2$
$b = 1$

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

$= 2\pi\sqrt{26}$

$z^2 = x^2 + y^2 \overset{\text{Polar}}{=} r^2$

Example. A thin conical sheet is described by the surface $z = \left(x^2 + y^2\right)^{\frac{1}{2}}$, for $0 \le z \le 4$. The density of the sheet in g/ cm^2 is $\rho = f(x, y, z) = (8 - z)$. What is the mass of the cone?



Density function of sheet is $\rho = 8 - z$.

$$z_x = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{z} \qquad z_y = \frac{y}{z}$$

$$|\vec{t}_x \times \vec{t}_y| = \sqrt{z_x^2 + z_y^2 + 1} = \sqrt{\underbrace{\frac{x^2+y^2}{z^2}}_{1} + 1} = \sqrt{2}$$

$$m = \iint_S f(x,y,z)\, dS = \iint_R (8-z)\sqrt{2}\, dA$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^4 (8-r)\, r\, dr\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi} 4r^2 - \frac{r^3}{3}\Big|_{r=0}^{r=4} d\theta$$

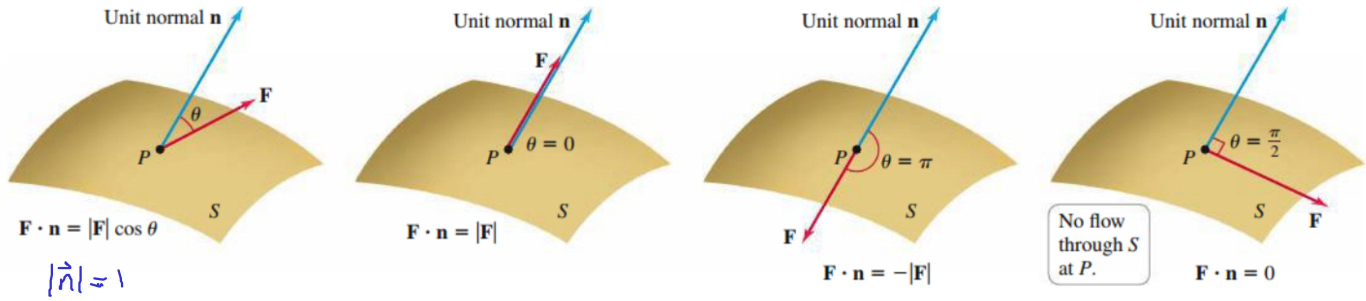
$$= \sqrt{2}\ \frac{128}{3}\int_0^{2\pi} d\theta$$

$$= \frac{256\pi\sqrt{2}}{3}$$

	Explicit Description $z = g(x,y)$			Parametric Description		
Surface	**Equation**	**Normal vector** $\pm\langle -z_x,\, -z_y,\, 1\rangle$	**magnitude** $\lvert\langle -z_x,\, -z_y,\, 1\rangle\rvert$	**Equation**	**Normal vector** $\mathbf{t}_u \times \mathbf{t}_v$	**magnitude** $\lvert\mathbf{t}_u \times \mathbf{t}_v\rvert$
Cylinder	$x^2 + y^2 = a^2$, $0 \le z \le h$	$\langle x, y, 0\rangle$	a	$\mathbf{r} = \langle a\cos(u),\, a\sin(u),\, v\rangle$, $0 \le u \le 2\pi$, $0 \le v \le h$	$\langle a\cos(u),\, a\sin(u),\, 0\rangle$	a
Cone	$z^2 = x^2 + y^2$, $0 \le z \le h$	$\langle x/z,\, y/z,\, -1\rangle$	$\sqrt{2}$	$\mathbf{r} = \langle v\cos(u),\, v\sin(u),\, v\rangle$, $0 \le u \le 2\pi$, $0 \le v \le h$	$\langle v\cos(u),\, v\sin(u),\, -v\rangle$	$\sqrt{2}v$
Sphere	$x^2 + y^2 + z^2 = a^2$	$\langle x/z,\, y/z,\, 1\rangle$;	a/z	$\mathbf{r} = \langle a\sin(u)\cos(v),\, a\sin(u)\sin(v),\, a\cos(u)\rangle$ $0 \le u \le \pi$, $0 \le v \le 2\pi$	$\langle a^2\sin^2(u)\cos(v),\, a^2\sin^2(u)\sin(v),\, a^2\sin(u)\cos(u)\rangle$	$a^2\sin(u)$
Paraboloid	$z = x^2 + y^2$, $0 \le z \le h$	$\langle 2x,\, 2y,\, -1\rangle$	$\sqrt{1 + 4(x^2 + y^2)}$	$\mathbf{r} = \left\langle v\cos(u),\, v\sin(u),\, v^2\right\rangle$, $0 \le u \le 2\pi$, $0 \le v \le \sqrt{h}$	$\left\langle 2v^2\cos(u), 2v^2\sin(u), -v\right\rangle$	$v\sqrt{1 + 4v^2}$

Flux Integrals:

$$\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS$$



$|\vec{n}| = 1$

The unit normal vector we use is

$$\mathbf{n} = \frac{\mathbf{t}_u\times\mathbf{t}_v}{|\mathbf{t}_u\times\mathbf{t}_v|}$$

giving us

$$\begin{aligned}
\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint\limits_R \mathbf{F}\cdot\mathbf{n}\overbrace{|\mathbf{t}_u\times\mathbf{t}_v|\,dA}^{dS}\\
&= \iint\limits_R \mathbf{F}\cdot\frac{\mathbf{t}_u\times\mathbf{t}_v}{\cancel{|\mathbf{t}_u\times\mathbf{t}_v|}}\cancel{|\mathbf{t}_u\times\mathbf{t}_v|}\,dA\\
&= \iint\limits_R \mathbf{F}\cdot(\mathbf{t}_u\times\mathbf{t}_v)\,dA
\end{aligned}$$

When the surface S is explicitly given as $z = s(x,y)$, then

$$\mathbf{F}\cdot\underbrace{(\mathbf{t}_u\times\mathbf{t}_v)}_{\langle -z_x,\, -z_y,\, 1\rangle} = -fz_x - gz_y + h$$

$F \cdot (\vec{t}_x \times \vec{t}_v)\, dA$

Definition. (Surface Integral of a Vector Field)
Suppose $\mathbf{F} = \langle f, g, h \rangle$ is a continuous vector field on a region of $\mathbb{R}^3$ containing a smooth oriented surface S. If S is defined parametrically as $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, for (u, v) in a region R, then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v)\, dA,$$

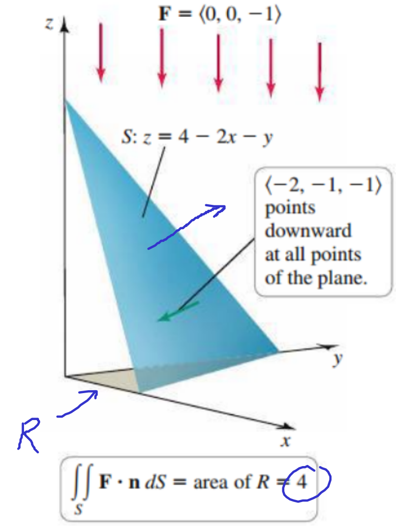
where

$$\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \text{ and } \mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

are continuous on R, the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R, and the direction of the normal vector is consistent with the orientation of S. If S is defined in the form $z = s(x, y)$, for (x, y) in a region R, then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S (-f z_x - g z_y + h)\, dA.$$

Example. Consider the vertical field $\mathbf{F} = \langle 0, 0, -1\rangle$. Find the flux in the downward direction across the surface S, which is the plane $z = 4 - 2x - y$ in the first octant.



$\langle -z_x, -z_y, 1\rangle = \langle 2, 1, 1\rangle$ upward

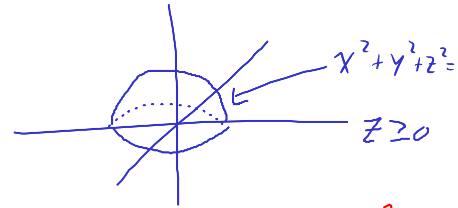
$\Rightarrow \vec{n} = \langle -2, -1, -1\rangle$

$$\text{flux} = \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_R \underbrace{\langle 0,0,-1\rangle \cdot \langle -2,-1,-1\rangle}_{1}\, dA$$

$$= \iint_R dA = 4$$

Example. Consider the radial vector field $\mathbf{F} = \langle f, g, h\rangle = \langle x, y, z\rangle$. Compute the upward flux of the field across the:

hemisphere $x^2 + y^2 + z^2 = 1$, for $z \geq 0$,



$\vec{n} = \langle -z_x, -z_y, 1\rangle = \langle \frac{x}{z}, \frac{y}{z}, 1\rangle$ $\qquad z = \sqrt{1-x^2-y^2} = \sqrt{1-r^2}$

$R = \{(r,\theta) : 0 \le r \le 1, 0 \le \theta \le 2\pi\}$

$$\text{flux} = \iint_S \vec{F}\cdot n \, dS = \iint_R \langle x, y, z\rangle \cdot \langle \tfrac{x}{z}, \tfrac{y}{z}, 1\rangle \, dA$$

$u = 1-r^2$

$du = -2r\,dr$

$$= \iint_R \frac{x^2+y^2}{z} + z \, dA = \iint_R \frac{\overbrace{x^2+y^2+z^2}^{1}}{z} \, dA = \int_0^{2\pi}\int_0^1 \frac{1}{\sqrt{1-r^2}}\, r\,dr\,d\theta = -\frac{1}{2}\int_0^{2\pi}\int_1^0 u^{-1/2}\,du\,d\theta$$

$$= \int_0^{2\pi} u^{1/2}\Big|_{u=0}^{u=1} d\theta = \boxed{2\pi}$$

paraboloid $z = 1 - x^2 - y^2$, for $z \geq 0$.

$\vec{n} = \langle -z_x, -z_y, 1\rangle = \langle 2x, 2y, 1\rangle$ $\qquad z = 1 - r^2$

$$\text{flux} = \iint_S \vec{F}\cdot n \, dS = \iint_R \langle x, y, z\rangle \cdot \langle 2x, 2y, 1\rangle \, dA = \iint_R \underbrace{2x^2+2y^2+1-x^2-y^2}_{x^2+y^2+1} \, dA$$

$$= \int_0^{2\pi}\int_0^1 (r^2+1)\, r\,dr\,d\theta = \int_0^{2\pi} \frac{r^4}{4} + \frac{r^2}{2}\Big|_{r=0}^{r=1} d\theta$$

$$= \frac{3}{4}\int_0^{2\pi} d\theta = \boxed{\frac{3\pi}{2}}$$