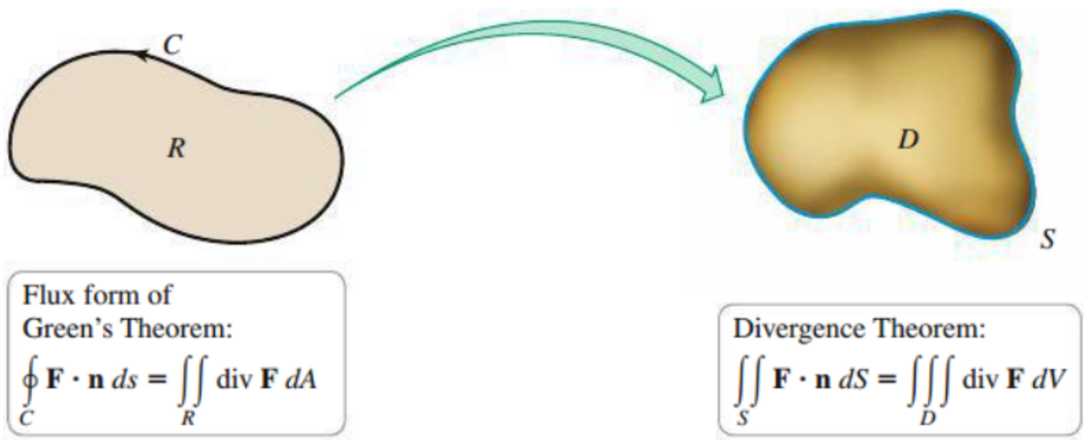
17.8: Divergence Theorem

The Divergence Theorem is the three-dimensional version of the flux form of Green's Theorem. Recall the flux form of Green's Theorem:

$$\underbrace{\oint_C \mathbf{F}\cdot\mathbf{n}\,ds}_{\text{flux across } C} = \iint_R \underbrace{(f_x + g_y)}_{\text{divergence}}\,dA.$$

The above means that the cumulative expansion and contraction throughout R equals the flux across the boundary of R. The Divergence Theorem computes the flux over a surface S in $\mathbb{R}^3$:



Theorem 17.17: Divergence Theorem
Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives in a connected and simply connected region D in $\mathbb{R}^3$ enclosed by an oriented surface S. Then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \nabla\cdot\mathbf{F}\,dV,$$

where $\mathbf{n}$ is the outward unit normal vector on S.

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \nabla\cdot\vec{F}\,dV$$

Example. Verify the Divergence Theorem: Consider the radial field $\mathbf{F} = \langle x, y, z\rangle$ and let S be the sphere $x^2 + y^2 + z^2 = a^2$ that encloses the region D. Assume $\mathbf{n}$ is the outward unit normal vector on the sphere. Evaluate both integrals of the Divergence Theorem.

$$\nabla\cdot\vec{F} = 1+1+1 = 3$$

$$\iiint_D \nabla\cdot\vec{F}\,dV = \iiint_D 3\,dV = 3\underbrace{\iiint_D dV}_{\text{volume of sphere}} \qquad 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3$$

$$\iint_S \vec{F}\cdot\vec{n}\,dS \qquad \vec{r}(u,v) = \langle a\sin(u)\cos(v),\ a\sin(u)\sin(v),\ a\cos(u)\rangle$$

$$0 \le u \le \pi \qquad 0 \le v \le 2\pi$$

$$\vec{t}_u = \langle a\cos(u)\cos(v),\ a\cos(u)\sin(v),\ -a\sin(u)\rangle$$

$$\vec{t}_v = \langle -a\sin(u)\sin(v),\ a\sin(u)\cos(v),\ 0\rangle$$

$$\vec{t}_u\times\vec{t}_v = a^2\sin(u)\langle \sin(u)\cos(v),\ \sin(u)\sin(v),\ \cos(u)\underbrace{(\cos^2(v)+\sin^2(v))}_{1}\rangle$$

$$\vec{F} = \langle a\sin(u)\cos(v),\ a\sin(u)\sin(v),\ a\cos(u)\rangle$$

$$\vec{F}\cdot(\vec{t}_u\times\vec{t}_v) = \ldots = a^3\sin(u)$$

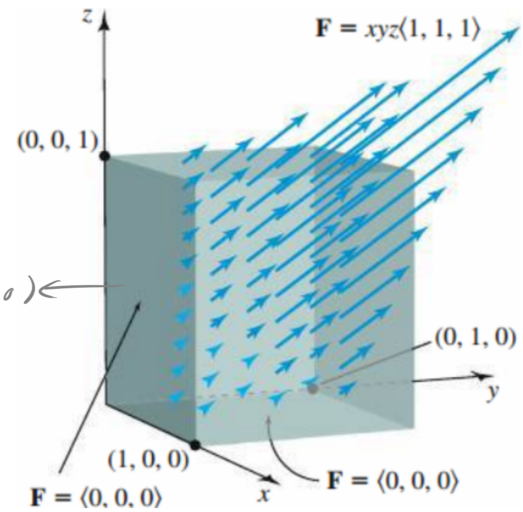
$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_R \vec{F}\cdot(\vec{t}_u\times\vec{t}_v)\,dA = \int_0^{2\pi}\int_0^{\pi} a^3\sin(u)\,du\,dv = \int_0^{2\pi} -a^3\underbrace{\cos(u)\Big|_{u=0}^{u=\pi}}_{(-1-1)}\,dv$$

$$= 2a^3 v\Big|_{v=0}^{v=2\pi} = \boxed{4\pi a^3}$$

Example. Find the net outward flux of the field $\mathbf{F} = xyz\langle 1, 1, 1\rangle$ across the boundaries of the cube $D = \{(x, y, z) : 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$.

$$\text{flux} = \iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_D \nabla\cdot\vec{F}\, dV$$

$\vec{n} = (0, -1, 0)$



- parameterize S
- normal vectors

$$\nabla\cdot\vec{F} = yz + xz + xy$$

$0 \le x \le 1$

$0 \le y \le 1$

$0 \le z \le 1$

$$= \int_0^1\int_0^1\int_0^1 (yz + xz + xy)\, dx\, dy\, dz$$

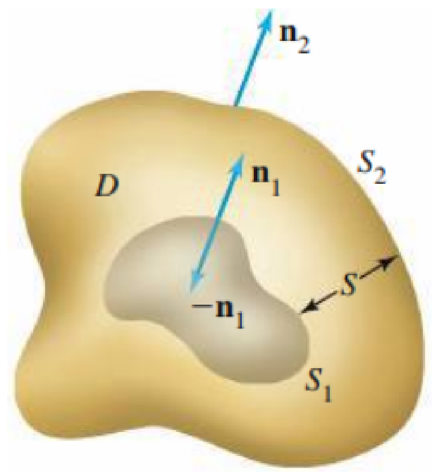
$$= \int_0^1\int_0^1 \underbrace{xyz}_{yz} + \underbrace{\frac{x^2}{2}(z+y)}_{\frac{z}{2}+\frac{y}{2}}\Big|_{x=0}^{x=1} dy\, dz$$

$$= \int_0^1 \underbrace{\frac{y^2}{2}z}_{z/2} + \underbrace{y\frac{z}{2}}_{z/2} + \underbrace{\frac{y^2}{4}}_{1/4}\Big|_{y=0}^{y=1} dz$$

$$= \frac{z^2}{2} + \frac{z}{4}\Big|_{z=0}^{z=1} = \boxed{\frac{3}{4}}$$

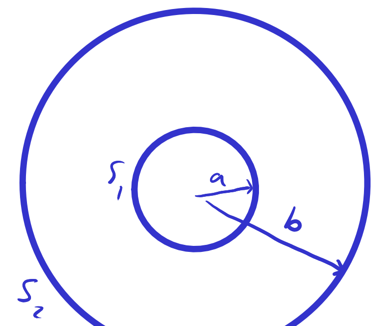
Theorem 17.18: Divergence Theorem for Hollow Regions
Suppose the vector field $\mathbf{F}$ satisfies the conditions of the Divergence Theorem on a region D bounded by two oriented surfaces S_1 and S_2, where S_1 lies within S_2. Let S be the entire boundary of D ($S = S_1 \cup S_2$) and let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the outward unit normal vectors for S_1 and S_2, respectively. Then

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS - \iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1\, dS.$$



$\mathbf{n}_1$ is the outward unit normal to S_1 and points into D. The outward unit normal to S on S_1 is $-\mathbf{n}_1$.

Example. Consider the inverse square vector field



$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{\langle x,\, y,\, z\rangle}{(x^2+y^2+z^2)^{3/2}}$$

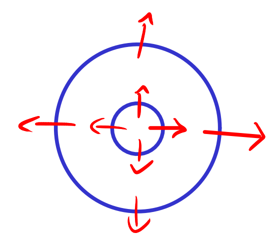
Find the net outward flux of $\mathbf{F}$ across the surface of the region $D = \{(x,y,z) : a^2 \le x^2+y^2+z^2 \le b^2\}$ that lies between concentric spheres with radii a and b.

$$\left.\begin{aligned} S_1 &= \{(x,y,z) : x^2+y^2+z^2 = a^2\} \\ S_2 &= \{(x,y,z) : x^2+y^2+z^2 = b^2\} \end{aligned}\right\} \quad S = S_1 \cup S_2$$

$$\iint_{S_2} \vec{F}\cdot\vec{n}_2\,dS - \iint_{S_1} \vec{F}\cdot\vec{n}_1\,dS = \iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \nabla\cdot\vec{F}\,dV = \iiint_D 0\,dV = 0$$

Theorem 17.10 $\quad \nabla\cdot\dfrac{\vec{r}}{|\vec{r}|^p} = \dfrac{3-p}{|\vec{r}|^p}$

Find the outward flux of $\mathbf{F}$ across any sphere that encloses the origin.



$$\iint_S \vec{F}\cdot\vec{n}\,dS = 0 \Rightarrow \iint_{S_2} \vec{F}\cdot\vec{n}_2\,dS = \iint_{S_1} \vec{F}\cdot\vec{n}_1\,dS$$

sphere of radius a

$\vec{r}\cdot\vec{r} = |\vec{r}|^2$

$$\frac{|\vec{r}|^2}{|\vec{r}|^4} = \frac{1}{|\vec{r}|^2}$$

$$\iint_S \vec{F}\cdot\underbrace{\vec{n}}_{\frac{\vec{r}}{|\vec{r}|}}\,dS = \iint_S \frac{\vec{r}}{|\vec{r}|^3}\cdot\frac{\vec{r}}{|\vec{r}|}\,dS = \iint_S \frac{1}{|\vec{r}|^2}\,dS = \frac{1}{a^2}\underbrace{\iint_S dS}_{SA} \quad \frac{1}{a^2}(4\pi a^2) = 4\pi$$

Example. Use the Divergence Theorem to compute the net outward flux of the field $\mathbf{F} = \langle x^2, y^2, z^2 \rangle$ across the surface S where S is the sphere $\{(x, y, z) : x^2 + y^2 + z^2 = r^2\}$.

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_D \nabla \cdot \vec{F}\, dV \longleftarrow \text{spherical coords}$$

$x = \rho \sin(\varphi)\cos(\theta)$

$y = \rho \sin(\varphi)\sin(\theta)$

$z = \rho \cos(\varphi)$

$0 \le \rho \le r$

$0 \le \varphi \le \pi$

$0 \le \theta \le 2\pi$

$$\nabla \cdot \vec{F} = 2x + 2y + 2z$$

$$\int_0^r \int_0^\pi \int_0^{2\pi} [2\rho \sin(\varphi) \cos\theta + 2\rho \sin(\varphi) \sin(\theta) + 2\rho \cos(\varphi)]\rho^2 \sin(\varphi)\, d\theta\, d\varphi\, d\rho$$

$$= \int_0^r \int_0^\pi 2\rho^3 \sin^2\varphi\,(\underbrace{\sin\theta}_{0-0} - \underbrace{\cos\theta}_{1-1}) + 2\theta\rho^3 \underbrace{\sin\varphi\cos\varphi}_{\frac{\sin(2\varphi)}{2}} \Big|_{\theta=0}^{\theta=2\pi} d\varphi\, d\rho$$

$$= \int_0^r \int_0^\pi 2\pi \rho^3 \sin(2\varphi)\, d\varphi\, d\rho$$

$$= \int_0^r -\pi\rho^3 \underbrace{\cos(2\varphi)}_{1-1} \Big|_{\varphi=0}^{\varphi=\pi} = \boxed{0}$$

Example. Use the Divergence Theorem to compute the net outward flux of the field $\mathbf{F} = \langle x^2, y^2, z^2 \rangle$ across the surface S where S is the sphere $\{(x, y, z) : x^2 + y^2 + z^2 = r^2\}$.

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iiint_D \nabla \cdot \vec{F} \, dV$$

Sphere w/ radius r: $\vec{r}(u,v) = \langle r\sin(u)\cos(v), r\sin(u)\sin(v), r\cos(u) \rangle$

$0 \le u \le \pi$

$0 \le v \le 2\pi$

$\vec{F} = \langle x^2, y^2, z^2 \rangle = \langle r^2\sin^2(u)\cos^2(v), r^2\sin^2(u)\sin^2(v), r^2\cos^2(u) \rangle$

$\vec{t}_u \times \vec{t}_v = r^2\sin(u)\langle \sin(u)\cos(v), \sin(u)\sin(v), \cos(u) \rangle$

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iint_R \vec{F} \cdot (\vec{t}_u \times \vec{t}_v) \, dA$$

$$= r^4 \int_0^{\pi} \int_0^{2\pi} \underbrace{\sin^4(u)\cos^3(v)}_{0} + \underbrace{\sin^4(u)\sin^3(v)}_{0} + \cos^3(u)\sin(u) \, dv \, du$$

$$= r^4 \int_0^{\pi} 2\pi \cos^3(u)\sin(u) \, du$$

$w = \cos(u)$ $\quad u=0, w=1$

$dw = -\sin(u)\,du$ $\quad u=\pi, w=-1$

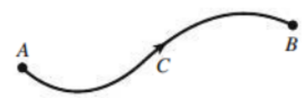
$$= -\frac{\pi r^4}{2} w^4 \Big|_{-1}^{1} = \boxed{0}$$

Fundamental Theorem of Calculus

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Fundamental Theorem for Line Integrals

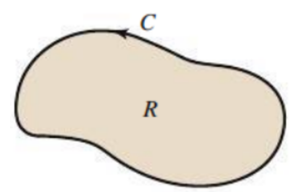
$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$$



Green's Theorem (Circulation Form)

$$\iint_R (g_x - f_y)\,dA = \oint_C f\,dx + g\,dy$$

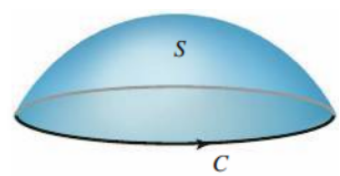
$R \subseteq \mathbb{R}^2$



Stokes' Theorem

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

$S \subseteq \mathbb{R}^3$



Divergence Theorem

$$\iiint_D \nabla \cdot \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot \mathbf{n}\,dS$$