

# Math 2060 Class notes Spring 2021

Peter Westerbaan

Last updated: December 31, 2020

Table Of Contents

13.1: Vectors and the Geometry of Space . . . . . 1

13.2: Vectors in Three Dimensions . . . . . 6

13.3: Dot Products . . . . . 12

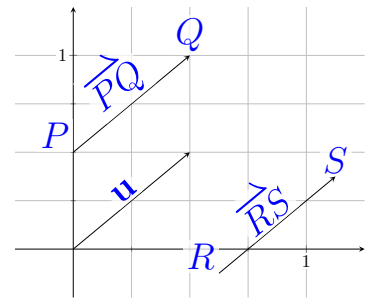
13.4: Cross Products . . . . . 17

## 13.1: Vectors and the Geometry of Space

### Definition.

- **Vectors**

- Have a direction and magnitude,
- vector  $\overrightarrow{PQ}$  has a *tail* at  $P$  and a *head* at  $Q$ ,
- Can be denoted as  $\mathbf{u}$  or  $\vec{u}$ ,
- Equal vectors have the same direction and magnitude (not necessarily the same position)



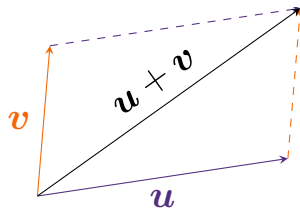
- **Scalars** are quantities with magnitude but no direction (e.g. mass, temperature, price, time, etc.)
- **Zero vector**, denoted  $\mathbf{0}$  or  $\vec{0}$ , has length 0 and no direction

### Scalar-vector multiplication:

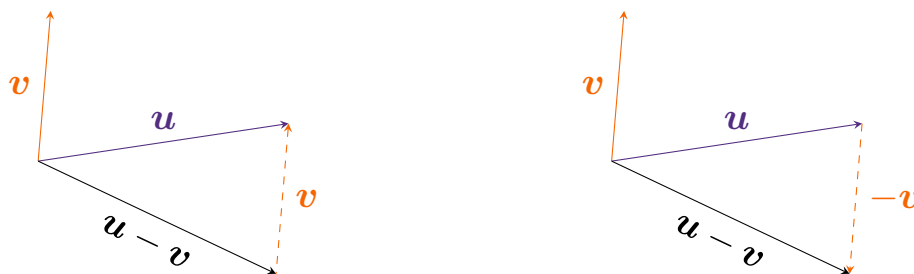
- Denoted  $c\mathbf{v}$  or  $c\vec{v}$ ,
- length of vector multiplied by  $|c|$ ,
- $c\mathbf{v}$  has the same direction as  $\mathbf{v}$  if  $c > 0$ , and has the opposite direction as  $\mathbf{v}$  if  $c < 0$ , (what if  $c = 0$ ?)
- $\mathbf{u}$  and  $\mathbf{v}$  are **parallel** if  $\mathbf{u} = c\mathbf{v}$ . (what vectors are parallel to  $\mathbf{0}$ ?)

### Vector Addition and Subtraction:

Given two vectors  $\mathbf{u}$  and  $\mathbf{v}$ , their sum,  $\mathbf{u} + \mathbf{v}$ , can be represented by the parallelogram (triangle) rule: place the tail of  $\mathbf{v}$  at the head of  $\mathbf{u}$

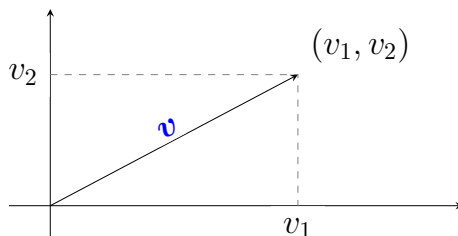


The difference, denoted  $\mathbf{u} - \mathbf{v}$ , is the sum of  $\mathbf{u} + (-\mathbf{v})$ :



### Vector Components:

A vector  $\mathbf{v}$  whose tail is at the origin  $(0, 0)$  and head is at  $(v_1, v_2)$  is a **position vector** (in **standard position**) and is denoted  $\langle v_1, v_2 \rangle$ . The real numbers  $v_1$  and  $v_2$  are the  $x$ - and  $y$ -components of  $\mathbf{v}$ .



Vectors  $\mathbf{u} = \langle u_1, u_2 \rangle$  and  $\mathbf{v} = \langle v_1, v_2 \rangle$  are equal if and only if  $u_1 = v_1$  and  $u_2 = v_2$ .

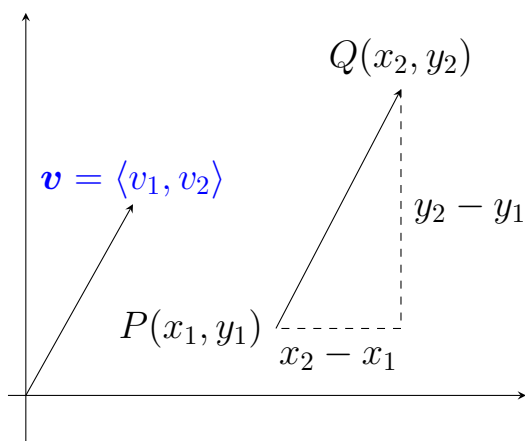
### Magnitude:

Given points  $P(x_1, y_1)$  and  $Q(x_2, y_2)$ , the **magnitude**, or **length**, of vector  $\vec{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$ , denoted  $|\vec{PQ}|$ , is the distance between points  $P$  and  $Q$ .

$$|\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The magnitude of position vector  $\mathbf{v} = \langle v_1, v_2 \rangle$  is  $|\mathbf{v}|$ .

(How do  $|\vec{PQ}|$  and  $|\vec{QP}|$  relate to each other?)



Note: The norm, denoted  $\|\mathbf{u}\|$  or  $\|\mathbf{u}\|_2$ , is equivalent to the magnitude of a vector.

### Equation of a Circle:

#### Definition.

A **circle** centered at  $(a, b)$  with radius  $r$  is the set of points satisfying the equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

A **disk** centered at  $(a, b)$  with radius  $r$  is the set of points satisfying the inequality

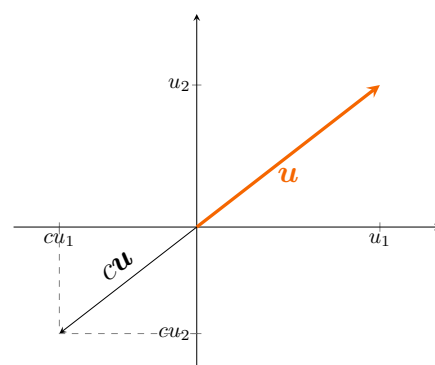
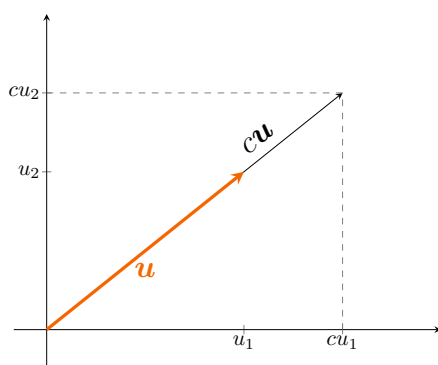
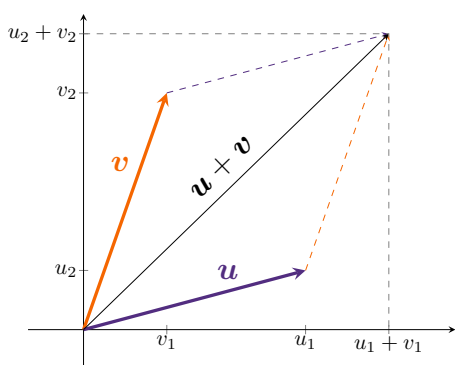
$$(x - a)^2 + (y - b)^2 \leq r^2.$$

## Vector Operations in Terms of Components

### Definition. (Vector Operations in $\mathbb{R}^2$ )

Suppose  $c$  is a scalar,  $\mathbf{u} = \langle u_1, u_2 \rangle$ , and  $\mathbf{v} = \langle v_1, v_2 \rangle$ .

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= \langle u_1 + v_1, u_2 + v_2 \rangle && \text{Vector addition} \\ \mathbf{u} - \mathbf{v} &= \langle u_1 - v_1, u_2 - v_2 \rangle && \text{Vector subtraction} \\ c\mathbf{u} &= \langle cu_1, cu_2 \rangle && \text{Scalar multiplication}\end{aligned}$$



### Definition.

A **unit vector** is any vector with length 1.

In  $\mathbb{R}^2$ , the **coordinate unit vectors** are  $\mathbf{i} = \langle 1, 0 \rangle$  and  $\mathbf{j} = \langle 0, 1 \rangle$ .

**Properties of Vector Operations:**

Suppose  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  are vectors and  $a$  and  $c$  are scalars. Then the following properties hold (for vectors in any number of dimensions).

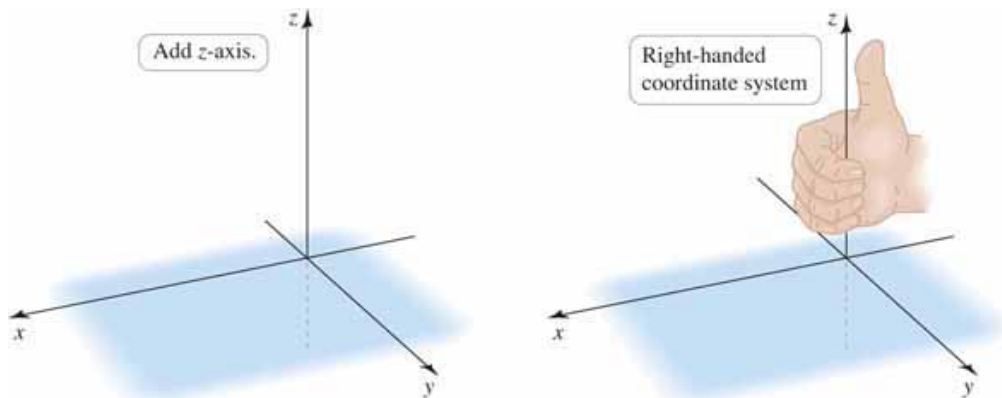
- |  |   |
|--|---|
| 1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$                               | Commutative property of addition              |
| 2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | Associative property of addition              |
| 3. $\mathbf{v} + \mathbf{0} = \mathbf{v}$  | Additive identity                             |
| 4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$   | Additive inverse                              |
| 5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$                          | Distributive property 1                       |
| 6. $(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$                                   | Distributive property 2                       |
| 7. $0\mathbf{v} = \mathbf{0}$  | Multiplication by zero scalar                 |
| 8. $c\mathbf{0} = \mathbf{0}$  | Multiplication by zero vector                 |
| 9. $1\mathbf{v} = \mathbf{v}$  | Multiplicative identity                       |
| 10. $a(c\mathbf{v}) = (ac)\mathbf{v}$  | Associative property of scalar multiplication |

**Applications of Vectors:**

## 13.2: Vectors in Three Dimensions

### The $xyz$ - Coordinate System:

The three-dimensional coordinate system is created by adding the  $z$ -axis, which is perpendicular to both the  $x$ -axis and the  $y$ -axis. When looking at the  $xy$ -plane, the positive direction of the  $z$ -axis protrudes towards the viewer. This can also be shown using the right-hand rule (Figure 13.25 from Briggs):

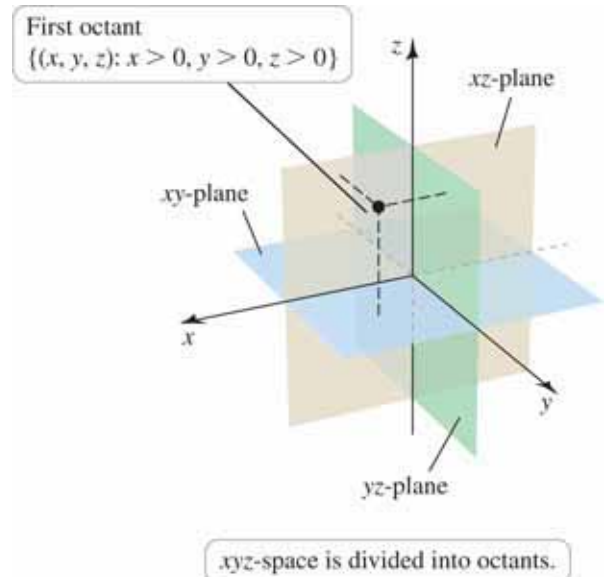


### Definition.

This three-dimensional coordinate system is broken up into eight **octants**, which are separated by

- the  $xy$ -**plane** ( $z = 0$ ),
- the  $xz$ -**plane** ( $y = 0$ ), and
- the  $yz$ -**plane** ( $x = 0$ ).

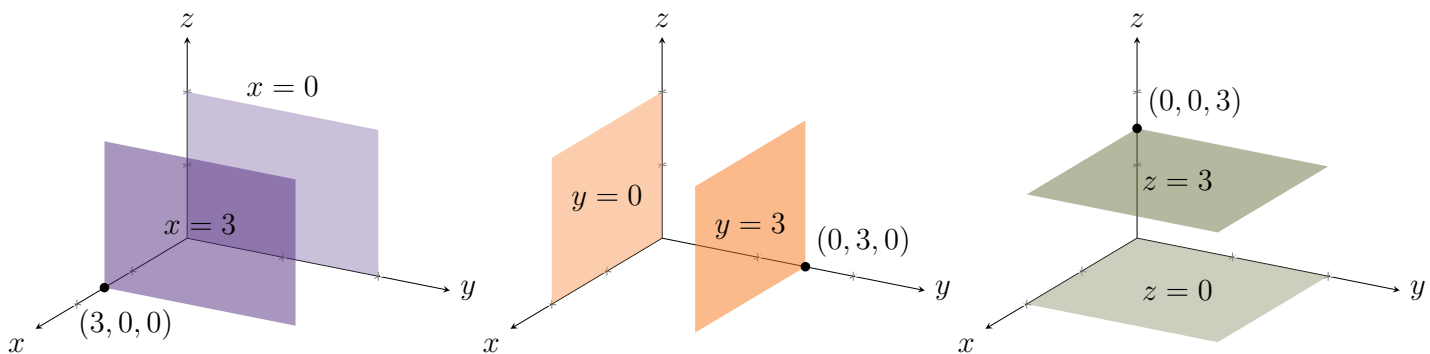
The **origin** is the location where all three axes intersect.



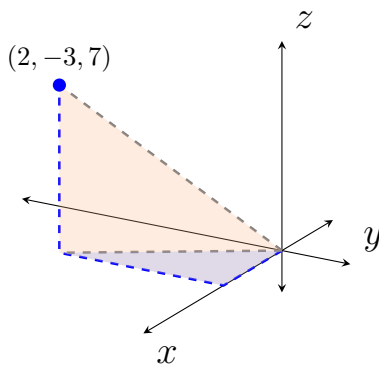


### Equations of Simple Planes:

Planes in three-dimensions are analogous to lines in two-dimensions. Below, we see the  $yz$ -plane, the  $xz$ -plane, and the  $xy$ -plane, along with planes that are parallel where  $x$ ,  $y$ , and  $z$  are fixed respectively:



**Example** (Parallel planes). Determine the equation of the plane parallel to the  $xz$ -plane passing through the point  $(2, -3, 7)$ .



### Distances in $xyz$ -Space:

Recall that in  $\mathbb{R}^2$ , for some vector  $\overrightarrow{PR}$ , the distance formula is given by

$$|PR| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

where  $(x_1, y_1)$  and  $(x_2, y_2)$  represent the points  $P$  and  $R$  respectively. This idea can be further extended into  $\mathbb{R}^3$  by considering the two sides of the triangle formed by the points  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$ :



### Distance Formula in $xyz$ -Space

The **distance** between points  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$  is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The **midpoint** between points  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$  is found by averaging the  $x$ -,  $y$ -, and  $z$ -coordinates:

$$\text{Midpoint} = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

## Equation of a Sphere:

### Definition.

A **sphere** centered at  $(a, b, c)$  with radius  $r$  is the set of points satisfying the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

A **ball** centered at  $(a, b, c)$  with radius  $r$  is the set of points satisfying the inequality

$$(x - a)^2 + (y - b)^2 + (z - c)^2 \leq r^2.$$

**Example.** Rewrite the following equation into the standard form of a sphere:

$$x^2 + y^2 + z^2 - 2x + 6y - 8z = -1$$

## Vector Operations in Terms of Components

### Definition. (Vector Operations in $\mathbb{R}^3$ )

Suppose  $c$  is a scalar,  $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ , and  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ .

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle \quad \text{Vector addition}$$

$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle \quad \text{Vector subtraction}$$

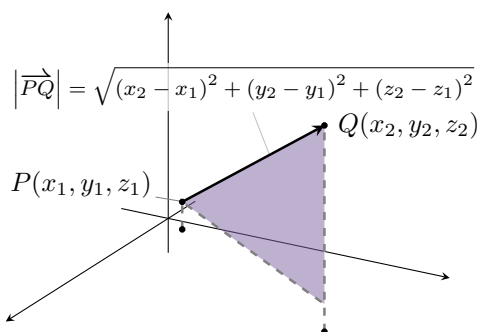
$$c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle \quad \text{Scalar multiplication}$$

## Magnitude and Unit Vectors:

### Definition.

The **magnitude** (or **length**) of the vector  $\vec{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$  is the distance from  $P(x_1, y_1, z_1)$  to  $Q(x_2, y_2, z_2)$ :

$$|\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



In  $\mathbb{R}^3$ , the **coordinate unit vectors** are  $\mathbf{i} = \langle 1, 0, 0 \rangle$ ,  $\mathbf{j} = \langle 0, 1, 0 \rangle$ , and  $\mathbf{k} = \langle 0, 0, 1 \rangle$ .

### Properties of Vector Operations:

Suppose  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  are vectors and  $a$  and  $c$  are scalars. Then the following properties hold (for vectors in any number of dimensions).

- |  |   |
|--|---|
| 1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$                               | Commutative property of addition              |
| 2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | Associative property of addition              |
| 3. $\mathbf{v} + \mathbf{0} = \mathbf{v}$  | Additive identity                             |
| 4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$   | Additive inverse                              |
| 5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$                          | Distributive property 1                       |
| 6. $(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$                                   | Distributive property 2                       |
| 7. $0\mathbf{v} = \mathbf{0}$  | Multiplication by zero scalar                 |
| 8. $c\mathbf{0} = \mathbf{0}$  | Multiplication by zero vector                 |
| 9. $1\mathbf{v} = \mathbf{v}$  | Multiplicative identity                       |
| 10. $a(c\mathbf{v}) = (ac)\mathbf{v}$  | Associative property of scalar multiplication |

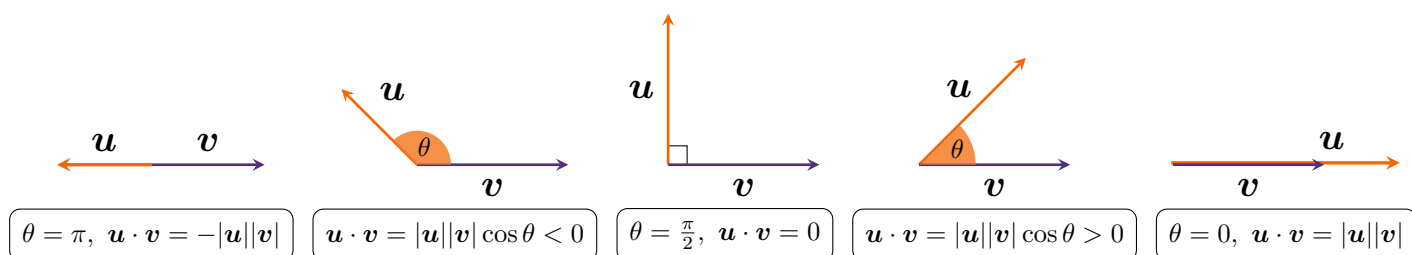
### 13.3: Dot Products

#### Definition. (Dot Product)

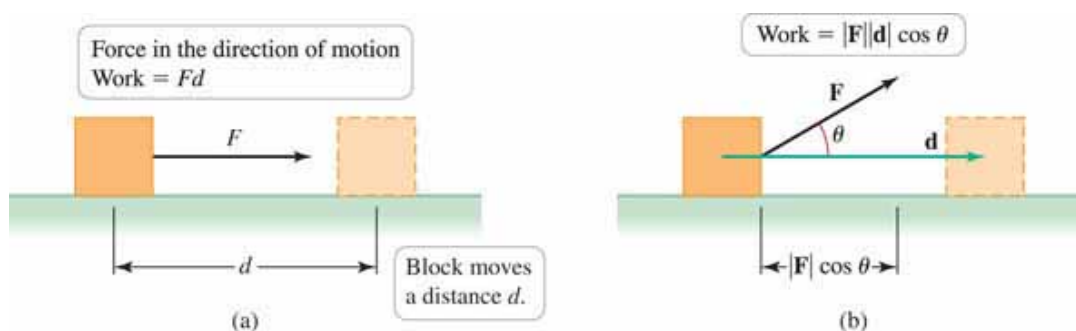
Given two nonzero vectors  $\mathbf{u}$  and  $\mathbf{v}$  in two or three dimensions, their **dot product** is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta,$$

where  $\theta$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$  with  $0 \leq \theta \leq \pi$ . If  $\mathbf{u} = \mathbf{0}$  or  $\mathbf{v} = \mathbf{0}$ , then  $\mathbf{u} \cdot \mathbf{v} = 0$ , and  $\theta$  is undefined.



A physical example of the dot product is the amount of work done when a force is applied at an angle  $\theta$  as shown in figure 13.43:



*Note:* The result of the dot product is a scalar!

**Definition. (Orthogonal Vectors)**

Two vectors  $\mathbf{u}$  and  $\mathbf{v}$  are **orthogonal** if and only if  $\mathbf{u} \cdot \mathbf{v} = 0$ . The zero vector is orthogonal to all vectors. In two or three dimensions, two nonzero orthogonal vectors are perpendicular to each other.

- $\mathbf{u}$  and  $\mathbf{v}$  are parallel ( $\theta = 0$  or  $\theta = \pi$ ) if and only if  $\mathbf{u} \cdot \mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$ .
- $\mathbf{u}$  and  $\mathbf{v}$  are perpendicular ( $\theta = \frac{\pi}{2}$ ) if and only if  $\mathbf{u} \cdot \mathbf{v} = 0$ .

**Theorem 31.1: Dot Product**

Given two vectors  $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$  and  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ ,

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

## Properties of Dot Products

### Theorem 13.2: Properties of the Dot Product

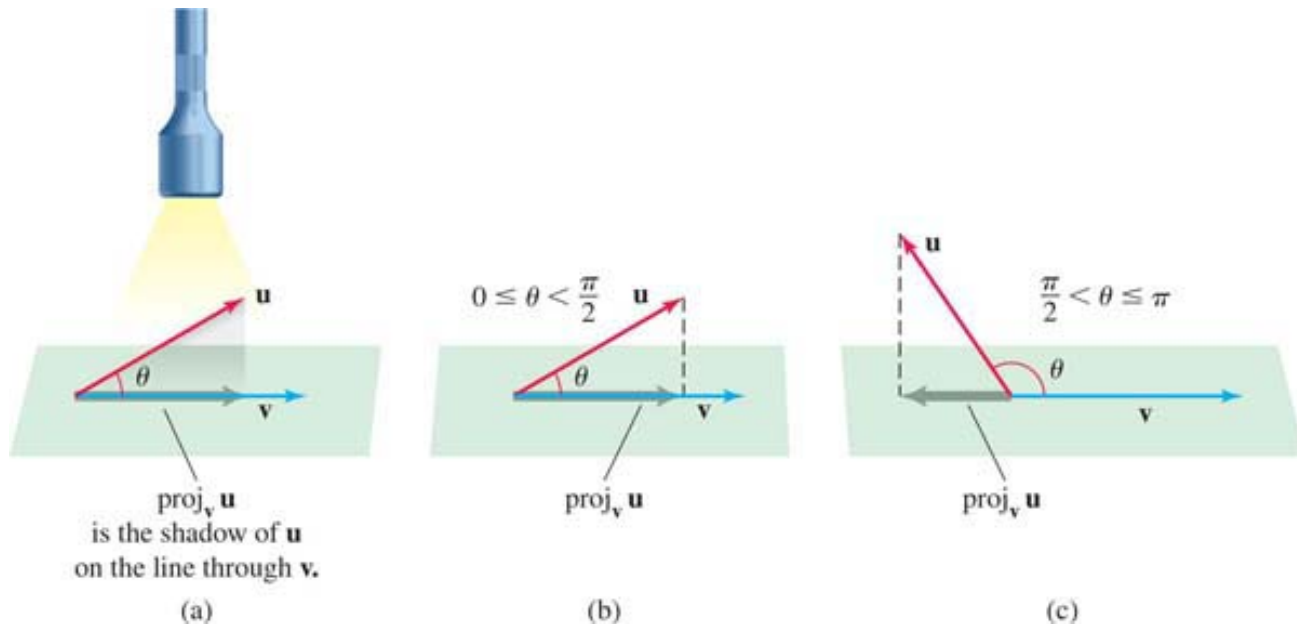
Suppose  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  are vectors and let  $c$  be a scalar.

- |   |                       |
|---|-----------------------|
| 1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$  | Commutative property  |
| 2. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$       | Associative property  |
| 3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ | Distributive property |



## Orthogonal Projections

Given vectors  $\mathbf{u}$  and  $\mathbf{v}$ , the projection of  $\mathbf{u}$  onto  $\mathbf{v}$  produces a vector parallel to  $\mathbf{v}$  using the “shadow” of  $\mathbf{u}$  cast onto  $\mathbf{v}$ .



### Definition. ((Orthogonal) Projection of $\mathbf{u}$ onto $\mathbf{v}$ )

The **orthogonal projection** of  $\mathbf{u}$  onto  $\mathbf{v}$ , denoted  $\text{proj}_{\mathbf{v}} \mathbf{u}$ , where  $\mathbf{v} \neq \mathbf{0}$ , is

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \underbrace{|\mathbf{u}| \cos \theta}_{\text{length}} \underbrace{\left( \frac{\mathbf{v}}{|\mathbf{v}|} \right)}_{\text{direction}}.$$

The orthogonal projection may also be computed with the formulas

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \text{scal}_{\mathbf{v}} \mathbf{u} \left( \frac{\mathbf{v}}{|\mathbf{v}|} \right) = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v},$$

where the **scalar component** of  $\mathbf{u}$  in the direction of  $\mathbf{v}$  is

$$\text{scal}_{\mathbf{v}} \mathbf{u} = |\mathbf{u}| \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}.$$

## Applications of Dot Products

### Definition. (Work)

Let a constant force  $\mathbf{F}$  be applied to an object, producing a displacement  $\mathbf{d}$ . If the angle between  $\mathbf{F}$  and  $\mathbf{d}$  is  $\theta$ , then the **work** done by the force is

$$W = |\mathbf{F}||\mathbf{d}| \cos \theta = \mathbf{F} \cdot \mathbf{d}$$

### Example.

Parallel and Normal Forces:

### Example.

