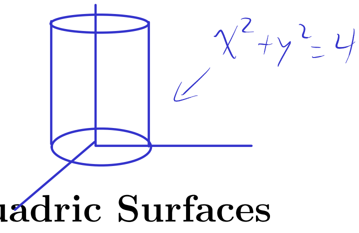
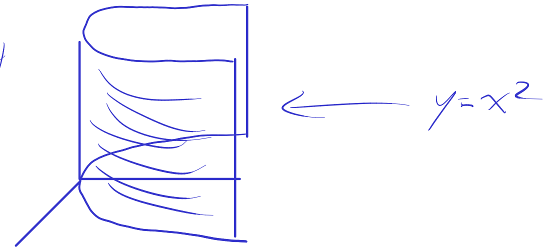


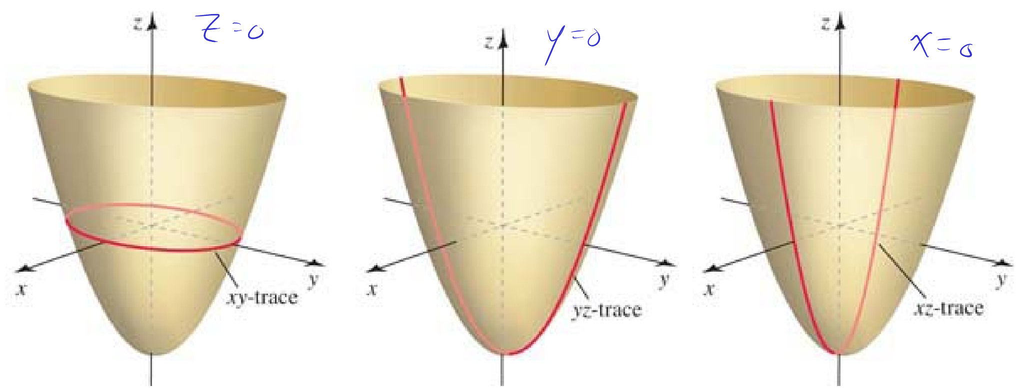




## 13.6: Cylinders and Quadric Surfaces

### Cylinders and Traces:

When talking about three-dimensional surfaces, a *cylinder* refers to a surface that is parallel to a line. When considering surfaces that is parallel to one of the coordinate axes, that the associated variable is missing (e.g. $3y^2 + z^2 = 8$ is parallel to the $x$-axis).

> **Definition. (Trace)**
> A **trace** of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes. The traces in the coordinate planes are called the $xy$**-trace**, the $yz$**-trace**, and the $xz$**-trace** (Figure 13.80).




**Example.** Roughly sketch the following functions:

1. $x^2 + 4y^2 = 16$
2. $x - \sin(z) = 0$

$4y^2 = 16$

$y^2 = 4$

$y = \pm 2$

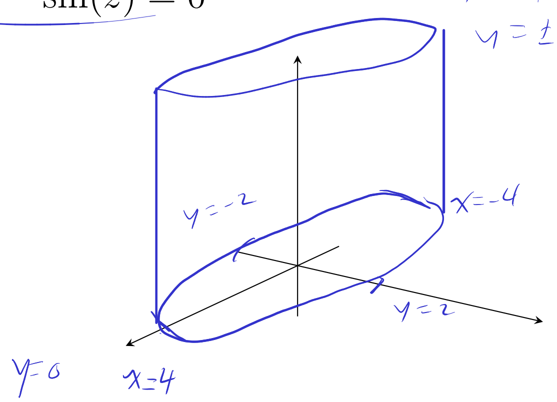
Let $y = 0$

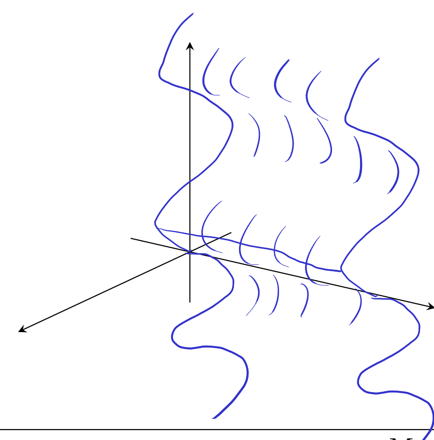
$x = 0, \quad -\sin(z) = 0$

$z = 0, \pi, 2\pi, \ldots$

$= n\pi, \; n \in \mathbb{Z}$

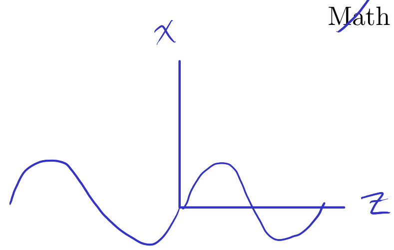
$x = \sin(z)$





Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2$



## Quadric Surfaces:

**Quadric surfaces** are described by the general quadratic (second-degree) equation in three variables,

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0,$$

Where the coefficients $A, \ldots, J$ and not all zero. To sketch quadric surfaces, keep the following ideas in mind:

x-intercept → set $y = z = 0$

1. **Intercepts** Determine the points, if any, where the surface intersects the coordinate axes. To find these intercepts, set $x$, $y$, and $z$ equal to zero in pairs in the equation of the surface, and solve for the third coordinate.

2. **Traces** Finding traces of the surface helps visualize the surface. Setting $x$, $y$, and $z$ equal to zero in pairs gives the planes parallel in that pair's plane.

3. **Completing the figure** Sketch some traces in parallel planes, then draw smooth curves that pass through the traces to fill out the surface.

**Example** (An ellipsoid)**.** The surface defined by the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. Graph $a = 3$, $b = 4$ and $c = 5$.

Intercepts

$$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{25} = 1$$

x-int ($y=0$, $z=0$)

$$\frac{x^2}{9} = 1 \rightarrow x = \pm 3$$

y-int

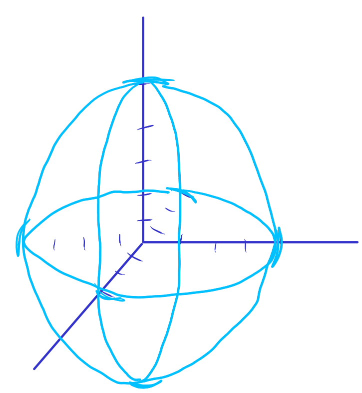
$$\frac{y^2}{16} = 1 \rightarrow y = \pm 4$$

$$z = \pm 5$$

Traces

$$x = 0 \rightarrow \frac{y^2}{16} + \frac{z^2}{25} = 1$$

Sketch



**Example** (An elliptic parabaloid)**.** The surface defined by the equation $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$. Graph the elliptic paraboloid with $a = 4$ and $b = 2$.

$$z = \frac{x^2}{16} + \frac{y^2}{4}$$

$$x = \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

Int

$x = 0$ $(y=0, z=0)$

$y = 0$

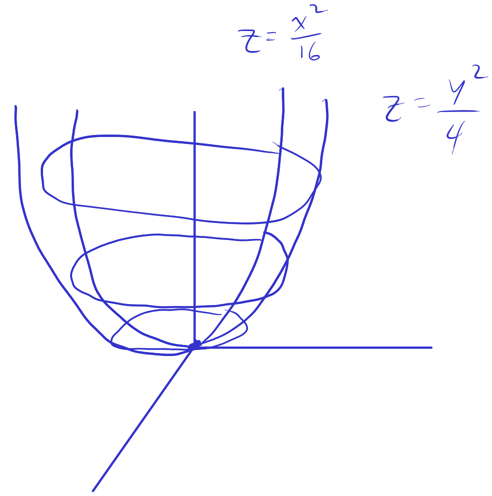
$z = 0$

Traces

$x=0 \rightarrow z = \frac{y^2}{4}$

$y=0 \rightarrow z = \frac{x^2}{16}$

$z=0 \rightarrow 0 = \frac{x^2}{16} + \frac{y^2}{4}$




$z=1 = \frac{x^2}{16} + \frac{y^2}{4}$

$z=4 = \frac{x^2}{16} + \frac{y^2}{4}$

$z=9 = \frac{x^2}{16} + \frac{y^2}{4}$

Shape?
ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

**Example** (A hyperboloid of one sheet).

Graph the surface defined by the equation $\dfrac{x^2}{4} + \dfrac{y^2}{9} - \underbrace{z^2}_{c^2} = 1$.

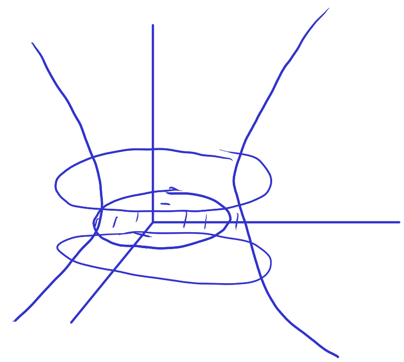
$c = 1$

Int

$y = z = 0 \Rightarrow x = \pm 2$

$x = z = 0 \Rightarrow y = \pm 3$

$x = y = 0 \Rightarrow -z^2 = 1$ No z-intercept



Traces

$z = 0 \rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1$

$z = \pm 1 \rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 2$

$y = 0 \rightarrow \frac{x^2}{4} - z = 1 \rightarrow$ hyperbola

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

**Example** (A hyperboloid of two sheets)**.** Graph the surface defined by the equation $-16x^2 - 4y^2 + z^2 + 64x - 80 = 0$.

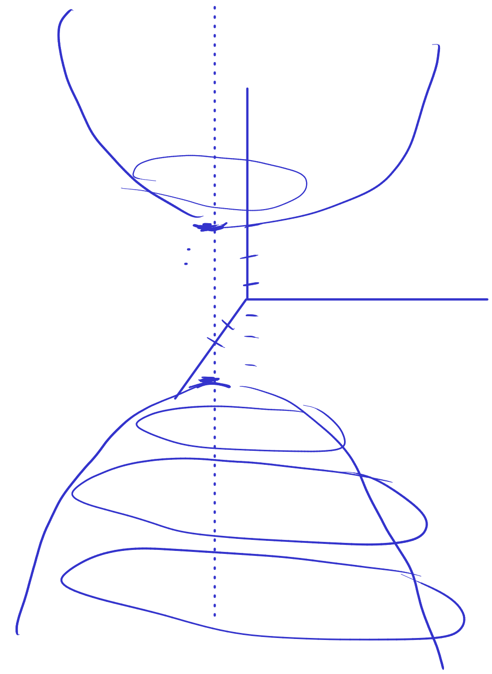
$$-16x^2 + 64x - 4y^2 + z^2 = 80$$

$$-16(x^2 - 4x + 4 - 4) - 4y^2 + z^2 = 80$$

$$-16(x-2)^2 + 64 - 4y^2 + z^2 = 80$$

$$-16(x-2)^2 - 4y^2 + z^2 = 16$$

$$-(x-2)^2 - \frac{y^2}{4} + \frac{z^2}{16} = 1$$



Int

$y = z = 0 \to -(x-2)^2 = 1$ DNE

$x = z = 0 \to$ DNE

$x = y = 0 \to z = \pm 4$

Traces

$x = 0 \to -\frac{y^2}{4} + \frac{z^2}{16} = 1 \to$ hyperbola

$y = 0 \longrightarrow$ hyperbola

$z = 0 \to \underbrace{-(x-2)^2}_{\geq 0} - \underbrace{\frac{y^2}{4}}_{\geq 0} = 1$



$z = 8 \to -(x-2)^2 - \frac{y^2}{4} + \frac{64}{16} = 1$

$-(x-2)^2 - \frac{y^2}{4} = -3$

$(x-2)^2 + \frac{y^2}{4} = 3 \longrightarrow$ ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

**Example** (Elliptic cones)**.** Graph the surface defined by the equation $\dfrac{y^2}{4} + z^2 = 4x^2$.

Intercepts

$x = y = 0 \Longrightarrow z = 0$

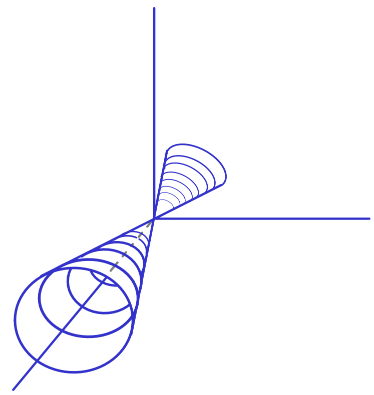
$x = z = 0 \Longrightarrow y = 0$

$y = z = 0 \Longrightarrow x = 0$

Trace

let $x = 0 \longrightarrow \frac{y^2}{4} + z^2 = 0 \longrightarrow$ point @ $(0,0,0)$

Let $x = 1 \longrightarrow \frac{y^2}{4} + z^2 = 4 \longrightarrow$ ellipse

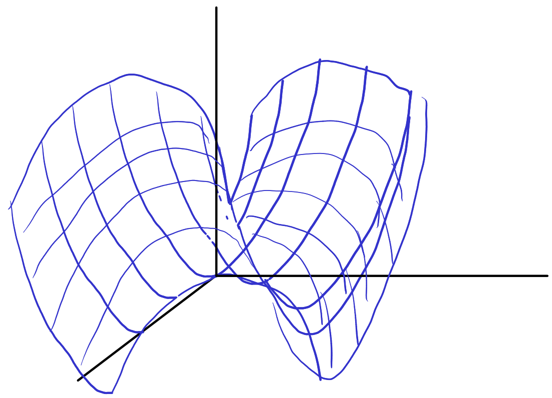


$z = \frac{x^2}{a^2} - \frac{y^2}{c^2}$

**Example** (A hyperbolic paraboloid)**.**

Graph the surface defined by the equation $z = x^2 - \dfrac{y^2}{4}$.

"pringles chip"



Int

$x = y = 0 \Rightarrow z = 0$

$x = z = 0 \longrightarrow y = 0$

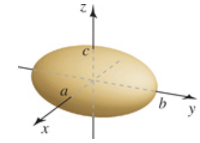
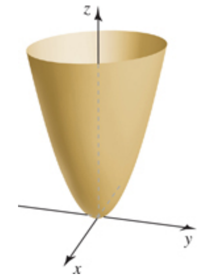
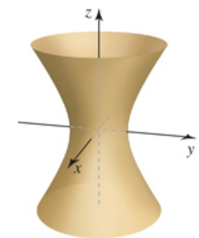
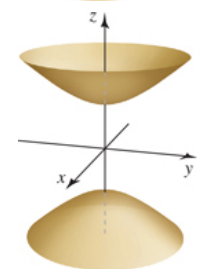
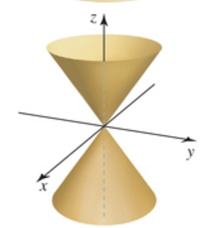
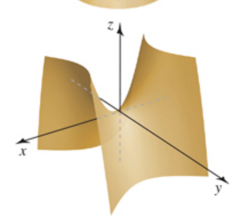
$y = z = 0 \longrightarrow x = 0$

Trace

$x = 0 \longrightarrow z = -\frac{y^2}{4}$ $\cap$

$y = 0 \longrightarrow z = x^2$ $\cup$

$z = 0 \longrightarrow x^2 = \frac{y^2}{4} \longrightarrow$ hyperbola $)\ ($

| **Name** | **Standard Equation** | **Features** | **Graph** |
|---|---|---|---|
| Ellipsoid | $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ | All traces are ellipses. |  |
| Elliptic paraboloid | $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ | Traces with $z=z_0>0$ are ellipses. Traces with $x=x_0$ or $y=y_0$ are parabolas. |  |
| Hyperboloid of one sheet | $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ | Traces with $z=z_0$ are ellipses for all $z_0$. Traces with $x=x_0$ or $y=y_0$ are hyperbolas. |  |
| Hyperboloid of two sheets | $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ | Traces with $z=z_0$ with $\|z_0\|>\|c\|$ are ellipses. Traces with $x=x_0$ and $y=y_0$ are hyperbolas. |  |
| Elliptic cone | $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$ | Traces with $z=z_0\neq 0$ are ellipses. Traces with $x=x_0$ or $y=y_0$ are hyperbolas or intersecting lines. |  |
| Hyperbolic paraboloid | $z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ | Traces with $z=z_0\neq 0$ are hyperbolas. Traces with $x=x_0$ or $y=y_0$ are parabolas. |  |