10.8: Choosing a Convergence Test

Example. Consider the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$. Is this series conditionally convergent, absolutely convergent, or divergent? Which test do you use?

abs convergence : $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges b/c p-series w/ $p = 1 \leq 1$

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges by AST

Conditionally convergent

Example. Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$. Is this series conditionally convergent, absolutely convergent, or divergent? Which test do you use?

Example. Which of the following series can be rewritten as a p-series?

$$\sum_{k=1}^{\infty} \frac{(-1)^{2k}}{k\sqrt{k}}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k}}{k^5}$$

$$\sum_{k=1}^{\infty} \frac{k^2+k+1}{k^4+2}$$

$$\sum_{k=1}^{\infty} \frac{3^k}{k^4}$$

$$\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2}$$

Example. Which test *cannot* be used to determine the convergence of $\displaystyle\sum_{k=1}^{\infty} \frac{k^2\, 2^{k-1}}{(-5)^k}$?

Example. For the following series, which test should be used to determine if the series converges or diverges? Use your selected test to show convergence or divergence.

$$\sum_{k=1}^{\infty} (-1)^k \frac{k}{k+2}$$

$$\sum_{k=1}^{\infty}(-1)^k\frac{k}{k+2}$$

$$\sum_{k=1}^{\infty}\frac{k!}{2^k(k+2)!}$$

$$\sum_{k=1}^{\infty} \frac{|\sin(2k)|}{1+2^k}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\sqrt{k}-1}$$

$$\sum_{k=2}^{\infty} \frac{1}{k\sqrt{\ln(k)}}$$

$$\sum_{k=1}^{\infty} \left(2^{1/k} - 1\right)^k$$

$$\sum_{k=3}^{\infty} \frac{1}{k^{2/5} \ln(k)}$$

$$\sum_{k=1}^{\infty} \frac{8(3k)!}{(k!)^3}$$

$$\sum_{k=1}^{\infty} \sin\left(\frac{9}{k^{12}}\right)$$

Series or Test	**Form of Series**	**Condition for Convergence**	**Condition for Divergence**	**Comments**
Geometric series	$\sum_{k=0}^{\infty} ar^k$, $a \neq 0$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	If $\lvert r\rvert < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$.
Divergence Test	$\sum_{k=1}^{\infty} a_k$	Does not apply	$\lim_{k\to\infty} a_k \neq 0$	Cannot be used to prove convergence.
Integral Test	$\sum_{k=1}^{\infty} a_k$, where $a_k = f(k)$ and f is continuous, positive, and decreasing.	$\int_1^{\infty} f(x)\,dx$ converges.	$\int_1^{\infty} f(x)\,dx$ diverges.	The value of the integral is not the value of the series.
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	Useful for comparison tests.
Ratio Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert < 1$	$\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert > 1$	Inconclusive if $\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert = 1$
Root Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} < 1$	$\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} > 1$	Inconclusive if $\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} = 1$
Comparison Test (DCT)	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$a \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges.	$b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Limit Comparison Test (LCT)	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$, $b_k > 0$	$0 \leq \lim_{k\to\infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges.	$\lim_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Alternating Series Test (AST)	$\sum_{k=1}^{\infty} (-1)^k a_k$, where $a_k > 0$	$\lim_{k\to\infty} a_k$ and $0 < a_{k+1} \leq a_k$	$\lim_{k\to\infty} a_k \neq 0$	Remainder R_n satisfies $\lvert R_n\rvert \leq a_{n+1}$
Absolute Convergence	$\sum_{k=1}^{\infty} a_k$, a_k arbitrary	$\sum_{k=1}^{\infty} \lvert a_k\rvert$ converges.		Applies to arbitrary series