

14.2: Calculus of Vector-Valued Functions

Definition. (Derivative and Tangent Vector)

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g , and h are differentiable functions on (a, b) . Then \mathbf{r} has a **derivative** (or is **differentiable**) on (a, b) and

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

Provided $\mathbf{r}'(t) \neq \mathbf{0}$, $\mathbf{r}'(t)$ is a **tangent vector** at the point corresponding to $\mathbf{r}(t)$.

Example. For the following functions below, find $\mathbf{r}'(t)$

a) $\mathbf{r}(t) = \langle e^{-t^2}, \log_2(t - 4), \sin(t) \rangle$

b) $\mathbf{r}(t) = 3\mathbf{i} - 2\tan(t)\mathbf{j} + e^t\mathbf{k}$

Example. For $\mathbf{r}(t) = \langle 3t, \sec(2t), \cos(t) \rangle$ compute $\mathbf{r}'(t)$ at $t = \frac{\pi}{4}$.

Definition. (Unit Tangent Vector)

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a smooth parameterized curve, for $a \leq t \leq b$. The **unit tangent vector** for a particular value of t is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Example. For $\mathbf{r}(t) = \langle 3 \sin(t), -2 \cos(2t), 3 \cos(t) \rangle$, find the unit tangent vector.

Example. For $\mathbf{r}(t) = \langle \sin(6t), 3t, \cos(3t) \rangle$, compute $\mathbf{T}\left(\frac{\pi}{3}\right)$.

Derivative Rules

Let \mathbf{u} and \mathbf{v} be differentiable vector-valued functions, and let f be a differentiable scalar-valued function, all at a point t . Let \mathbf{c} be a constant vector. The following rules apply.

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| 1. $\frac{d}{dt}(\mathbf{c}) = \mathbf{0}$ | Constant Rule |
| 2. $\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$ | Sum Rule |
| 3. $\frac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ | Product Rule |
| 4. $\frac{d}{dt}(\mathbf{u}(f(t))) = \mathbf{u}'(f(t))f'(t)$ | Chain Rule |
| 5. $\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ | Dot Product Rule |
| 6. $\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ | Cross Product Rule |

Example. Given $\mathbf{u}(t) = \langle 4t^2, 1, 3t \rangle$ and $\mathbf{v}(t) = \langle e^{-2t}, -2e^t, e^t \rangle$, find $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$.

Definition. (Indefinite Integral of a Vector-Valued Function)

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function, and let

$\mathbf{R}(t) = F(t)\mathbf{i} + G(t)\mathbf{j} + H(t)\mathbf{k}$, where F , G , and H are antiderivatives of f , g , and h , respectively. The **indefinite integral** of \mathbf{r} is

$$\int \mathbf{r}(t) dt = \mathbf{R}(t) + \mathbf{C},$$

where \mathbf{C} is an arbitrary constant vector. Alternatively, in component form,

$$\int \langle f(t), g(t), h(t) \rangle dt = \langle F(t), G(t), H(t) \rangle + \langle C_1, C_2, C_3 \rangle.$$

Example. Find $\mathbf{r}(t)$ such that $\mathbf{r}'(t) = \left\langle \frac{t}{t^2+1}, t^2e^{-t^3}, \frac{-2t}{\sqrt{t^2+16}} \right\rangle$ and $\mathbf{r}(0) = \left\langle 3, \frac{5}{3}, -5 \right\rangle$.

Definition. (Definite Integral of a Vector-Valued Function)

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f , g , and h are integrable on the interval $[a, b]$. The **definite integral** of \mathbf{r} on $[a, b]$ is

$$\int_a^b \mathbf{r}(t) dt = \left(\int_a^b f(t) dt \right) \mathbf{i} + \left(\int_a^b g(t) dt \right) \mathbf{j} + \left(\int_a^b h(t) dt \right) \mathbf{k}$$

Example. $\int_{-\pi}^{\pi} \langle \sin(t), \cos(t), 8t \rangle dt$