1. Follow the steps below to show $9.\bar{9} = 10$. $= 0.\bar{9} = 1$

Geometric Sequence converges

$-1 < r \leq 1$

$\{r, r^2, r^3, \ldots\}$

(a) (_/1 pts.) Write $9.\bar{9} = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots$ as a geometric series.

n

$9 + \frac{9}{10} + \frac{9}{10^2} + \ldots$

$\frac{1}{10}$ $\frac{1}{10}$

$a = 9$

$r = \frac{1}{10}$

$$9.\bar{9} = \sum_{k=0}^{\infty} 9\left(\frac{1}{10}\right)^k$$

Geometric Series

Converge

$|r| < 1$

$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^n$

$$9.\bar{9} = \sum_{k=0}^{\infty} 9\left(\frac{1}{10}\right)^k = \lim_{n \to \infty} \sum_{k=0}^{n} 9\left(\frac{1}{10}\right)^k = \lim_{n \to \infty} S_n$$

2. For the following infinite series,

- Find a formula for the partial sum S_n
- Evaluate the infinite series

$$\sum_{k=0}^{n} ar^n = S_n = a\,\frac{1-r^n}{1-r}$$

(a) (_/3 pts.) $\frac{1}{16} + \frac{3}{64} + \frac{9}{256} + \ldots = \sum_{k=0}^{\infty} \frac{1}{16}\left(\frac{3}{4}\right)^k$

Geometric

4

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}, \quad |r| < 1$$

$a = \frac{1}{16}$ $\quad r = \frac{3}{4}$

(b) (_/3 pts.) $\sum_{k=2}^{\infty} \frac{6}{k^2 + 2k}$ (*Hint:* Use PFD)

Telescoping

$$S_n = \sum_{k=2}^{n} \frac{3}{k} - \frac{3}{k+2} = \underbrace{\left[\boxed{\frac{3}{2}} - \cancel{\frac{3}{4}}\right]}_{k=2} + \underbrace{\left[\boxed{\frac{3}{3}} - \cancel{\frac{3}{5}}\right]}_{k=3} + \underbrace{\left[\cancel{\frac{3}{4}} - \cancel{\frac{3}{6}}\right]}_{k=4} + \underbrace{\left[\cancel{\frac{3}{5}} - \frac{3}{7}\right]}_{k=5} + \ldots$$

$$+ \left[\cancel{\frac{3}{n-1}} - \frac{3}{n+1}\right] + \left[\cancel{\frac{3}{n}} - \frac{3}{n+2}\right] = \frac{5}{2} - \frac{3}{n+1} - \frac{3}{n+2}$$

(c) (_/3 pts.) $\sum_{k=1}^{\infty} \left(\frac{-4}{3}\right)^{-k} = \sum_{k=1}^{\infty} \left(-\frac{3}{4}\right)^k$

$a = -\frac{3}{4}$

$r = -\frac{3}{4}$

Geometric

$$\sum_{k=1}^{n} \left(\frac{4}{\sqrt{k+5}} - \frac{4}{\sqrt{k+7}}\right) = \underbrace{\left(\boxed{\frac{4}{\sqrt{6}}} - \cancel{\frac{4}{\sqrt{8}}}\right)}_{k=1} + \underbrace{\left(\boxed{\frac{4}{\sqrt{7}}} - \cancel{\frac{4}{\sqrt{9}}}\right)}_{k=2} + \underbrace{\left(\cancel{\frac{4}{\sqrt{8}}} - \cancel{\frac{4}{\sqrt{10}}}\right)}_{k=3} + \ldots$$

$$\sum_{k=1}^{\infty} \left(4 - \frac{4}{3}\right)\frac{1}{3^k}$$

$$+ \left(\cancel{\frac{4}{\sqrt{n+4}}} - \frac{4}{\sqrt{n+6}}\right) + \left(\cancel{\frac{4}{\sqrt{n+5}}} - \frac{4}{\sqrt{n+7}}\right)$$

$$= \frac{4}{\sqrt{6}} + \frac{4}{\sqrt{7}} - \frac{4}{\sqrt{n+6}} - \frac{4}{\sqrt{n+7}}$$

$$\sum_{k=1}^{\infty} \left(\frac{4}{3^k} - \frac{4}{3^{k+1}}\right)$$

Geometric

Telescoping

$$\sum_{k=1}^{\infty} \frac{4}{3^k} - \sum_{k=1}^{\infty} \frac{4}{3^{k+1}}$$

$$\sum_{k=1}^{\infty} \frac{4}{3^k} = \frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \dots = \frac{4/3}{1-1/3} = \frac{4}{3}\frac{1}{2/3} = 2$$

$a = 4/3$

$r = 1/3$

$$\sum_{k=1}^{\infty} \frac{4}{3^k} = \sum_{K=0}^{\infty} \frac{4}{3^{K+1}} = \sum_{K=0}^{\infty} \frac{4}{3}\left(\frac{1}{3}\right)^K$$

a r

10.4: The Divergence and Integral Tests

Harmonic Series $\sum_{K=1}^{\infty} \frac{1}{K}$ Diverges

$4\sum_{K=1}^{\infty} \frac{1}{3^K}$

Theorem 10.9: Divergence Test
If $\sum a_k$ converges, then $\lim_{k\to\infty} a_k = 0$. Equivalently, if $\lim_{k\to\infty} a_k \neq 0$, then the series diverges.

Example. If $\lim_{k\to\infty} a_k = 1$, what can we conclude about $\sum_{k=1}^{\infty} a_k$?

$\sum_{K=1}^{\infty} a_K$ Diverges

Example. If $\sum_{k=1}^{\infty} a_k = 42$, what can we conclude about $\lim_{k\to\infty} a_k$?

$\lim_{K\to\infty} a_K = 0$

Example. If $\lim_{k\to\infty} a_k = 0$, what can we conclude about $\sum_{k=1}^{\infty} a_k$?

Nothing

Example. Determine which of the following series diverge by the divergence test.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}}$$

$\lim_{k\to\infty} \frac{1}{\sqrt{k+1}} = 0$ $\boxed{\text{Don't know}}$

Diverges

$$\sum_{k=1}^{\infty} \frac{k^3+100}{3k^3+k+1}$$

$\lim_{k\to\infty} \frac{k^3+100}{3k^3+k+1}\left(\frac{1/3k^3}{1/3k^3}\right) = \frac{1}{3} \neq 0$

$\Rightarrow$ series diverges

$$\sum_{k=1}^{\infty} \frac{e^k}{k^2}$$

$\ln(x) \ll x^n \ll b^m \ll n! \ll n^n$

grows faster than k^2

$\lim_{k\to\infty} \frac{e^k}{k^2} = \infty$

diverges $\Longrightarrow$ series diverges

$\lim_{n\to\infty} a_n = 4$

$\lim_{n\to\infty} a_n = 0$

$\lim_{n\to\infty} a_n = \infty$

Table 1 Series Convergence

Scenario	Sequence of Terms $\{a_1, a_2, a_3, \ldots\}$	Sequence of Partial Sums $\{s_1, s_2, s_3, \ldots\}$	Series $\sum_{n=1}^{\infty} a_n$	Possible or Impossible?
A	* Converges	Diverges	Diverges	Possible
B	Converges	Diverges	Converges	Impossible
C	Converges	Converges	Diverges	Impossible
* D	Converges	Converges	Converges	Possible
E	Diverges	Converges	Diverges	Impossible
F	Diverges	Converges	Converges	Impossible
G	Diverges	Diverges	Diverges	Possible
H	Diverges	Diverges	Converges	Impossible

Theorem 10.10: Harmonic Series

The harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$ diverges—even though the terms of the series approach zero.

$\lim_{k\to\infty} \frac{1}{k} = 0$ But $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

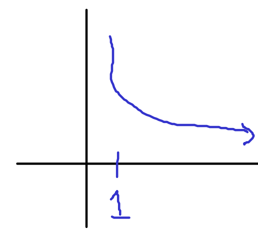
Theorem 10.11: Integral Test
Suppose f is a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Then

$$\sum_{k=1}^{\infty} a_k \text{ and } \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge. In the case of convergence, the value of the integral is *not* equal to the value of the series.

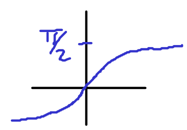
Example. Which of the following series below satisfy all the conditions to use the Integral Test?

Continuous
Positive
decreasing



$$\sum_{k=1}^{\infty} \arctan(k)$$

Cont. on $[1, \infty)$
Pos. on $(0, \infty)$
dec ?



$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

cont?

$\dfrac{(-1)^{3/2}}{(3/2)^2}$ DNE

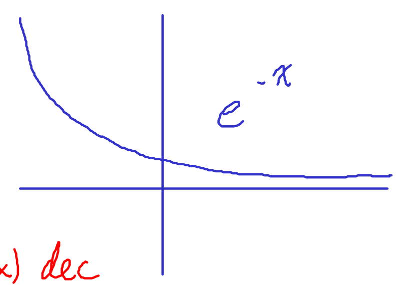
$$\sum_{k=1}^{\infty} \frac{1}{e^k}$$

Cont on $[1, \infty)$

Pos on $[1, \infty)$ $\quad f(x) = e^{-x} > 0$

e^{-k}

dec $\longrightarrow$ $\quad f'(x) = -e^{-x} < 0$ $\longrightarrow$ $f(x)$ dec



Example. Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

Use the integral test to show that the Harmonic Series diverges. For what values of p does this series converge?

cont, pos, dec $[1,\infty)$

$f(x) = \frac{1}{x^p}$ $\quad x \neq 0 \quad 1 \le x < \infty$

$\frac{1}{x^p} > 0, \; x > 0 \quad 1 \le x < \infty$

$f'(x) = -p\,x^{-p-1} = \frac{-p}{x^{p+1}} < 0$ when $x > 0$ and $p > 0$ $\quad \frac{1}{k}, \frac{1}{k^2}, \frac{1}{k^3} \cdots$

$p=1$ $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln|x|\Big|_1^b = \lim_{b\to\infty} \ln(b) - \underbrace{\ln(1)}_{0} = \infty$

Diverges

By the integral test $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ also diverges

$p>1$ $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx = \lim_{b\to\infty} \frac{x^{-p+1}}{-p+1}\Bigg|_1^b = \lim_{b\to\infty} \underbrace{\frac{1}{(1-p)\,b^{p-1}}}_{0} - \underbrace{\frac{1}{(1-p)\,1^{p-1}}}_{\frac{1}{1-p}}$

$\to p-1>0$

$\to$ By the integral test, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ $= -\frac{1}{1-p} = \frac{1}{p-1}$

Converges when $p>1$

$\int \frac{1}{x^p}\,dx = \int x^{-p}\,dx = \frac{x^{-p+1}}{-p+1} + C \quad p \neq 1$

$p>1 \quad \lim_{b\to\infty} \frac{1}{b^{p-1}} = 0$

$p<1 \quad b\to\infty \quad \frac{1}{b^{p-1}} = \infty$

Theorem 10.12: Convergence of the p-series

The **p-series** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p \le 1$.

Geometric $\sum a r^k$

p-series $\sum \frac{1}{k^p}$

Example. Determine if the following p-series converge or diverge.

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \qquad p = 2$$

$\Rightarrow$ Converges

$$\sum_{k=1}^{\infty} k^{-1/3} = \sum_{k=1}^{\infty} \frac{1}{k^{1/3}} \qquad p = 1/3 < 1$$

diverges

$$\sum_{k=1}^{\infty} \frac{k^2}{k^{\pi}} = \sum_{k=1}^{\infty} \frac{1}{k^{\pi-2}} \qquad p = \pi - 2 > 1$$

Converges

$$\sum_{k=1}^{\infty} \frac{2}{k} = 2\sum_{k=1}^{\infty} \frac{1}{k^1} \qquad p = 1$$

diverges

$$\sum_{k=1}^{\infty} \frac{-3}{\sqrt[3]{k^4}} = -3\sum_{k=1}^{\infty} \frac{1}{k^{4/3}} \qquad p = 4/3 > 1$$

Converges

$$\sum_{k=1}^{\infty} \frac{k^3+1}{k^5} = \sum_{k=1}^{\infty} \frac{k^3}{k^5} + \sum_{k=1}^{\infty} \frac{1}{k^5}$$

$p = 2$ $\qquad$ $p = 5$

Converges

Example. Apply the Integral Test to determine if the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}}$ converges or diverges.

cont

pos

dec

$f(x) = \frac{1}{\sqrt{x+1}}$ $\quad x \neq -1 \quad 1 \leq x < \infty$

$(x+1)^{-1/2} = \frac{1}{\sqrt{x+1}} > 0, \quad x > -1 \quad 1 \leq x < \infty$

$f'(x) = -\frac{1}{2}(x+1)^{-3/2} = \frac{-1}{2\sqrt{(x+1)^3}} < 0 \rightarrow f(x)$ dec

$x > -1$

$$\lim_{b\to\infty} \int_1^b \frac{1}{\sqrt{x+1}}\,dx = \lim_{b\to\infty} \int_2^{b+1} u^{-1/2}\,du = \lim_{b\to\infty} 2u^{1/2}\Big|_2^{b+1}$$

$u = x+1$

$du = dx$

$x=1, u=2$

$x=b, u=b+1$

$$= \lim_{b\to\infty} 2\left(\underbrace{\sqrt{b+1}}_{\infty} - \sqrt{2}\right)$$

$$= \infty$$

By the integral test, $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}}$ diverges

exam 2

Theorem 10.13: Estimating Series with Positive Terms
Let f be a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Let $S = \sum_{k=1}^{\infty} a_k$ be a convergent series and let $S_n = \sum_{k=1}^{n} a_k$ be the sum of the first n terms of the series. The remainder $R_n = S - S_n$ satisfies

$$\boxed{R_n < \int_n^{\infty} f(x)\,dx.}$$

Furthermore, the exact value of the series is bounded as follows:

$$L_n = S_n + \int_{n+1}^{\infty} f(x)\,dx < \sum_{k=1}^{\infty} a_k < S_n + \int_n^{\infty} f(x)\,dx = U_n.$$

S

Example. How many terms of the convergent p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ must be summed to obtain an approximation that is within 10^{-3} of the exact value of the series?

$$R_n < \lim_{b\to\infty} \int_n^b \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} -x^{-1}\Big|_n^b$$

$$= \lim_{b\to\infty} -\frac{1}{b} - \frac{-1}{n} = \frac{1}{n} = 10^{-3}$$

$$\boxed{1000 = 10^3 = n}$$