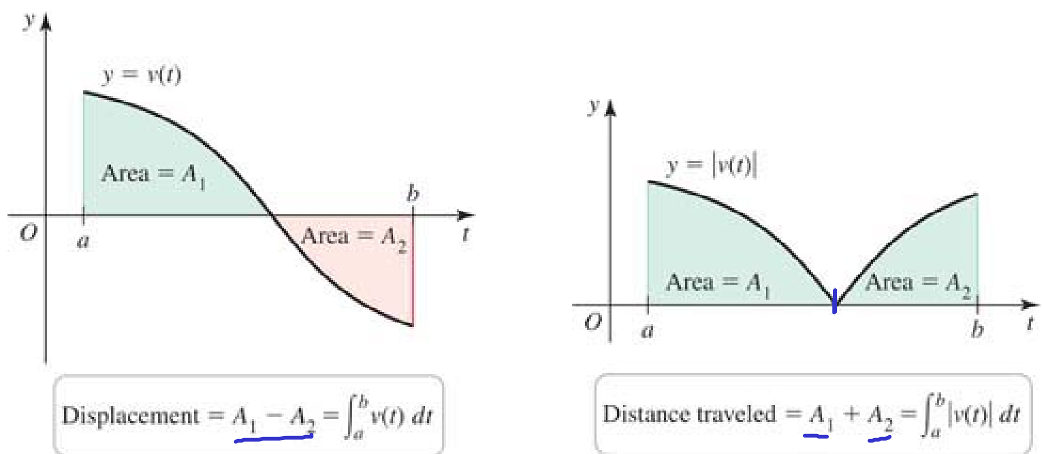
6.1: Velocity and Net Change

Definition. (Position, Velocity, Displacement, and Distance)

1. The **position** of an object moving along a line at time t, denoted $s(t)$, is the location of the object relative to the origin.
2. The **velocity** of an object at time t is $v(t) = s'(t)$.
3. The **displacement** of the object between $t = a$ and $t = b > a$ is
$$s(b) - s(a) = \int_a^b v(t)\,dt.$$
4. The **distance traveled** by the object between $t = a$ and $t = b > a$ is
$$\int_a^b |v(t)|\,dt$$
where $|v(t)|$ is the **speed** of the object at time t.



(a) (b)

Example. Suppose an object moves along a line with velocity (in ft/s) $v(t) = 6 - 2t$, for $0 \le t \le 5$, where t is measured in seconds.

- Find the displacement of the object on the interval $0 \le t \le 5$.

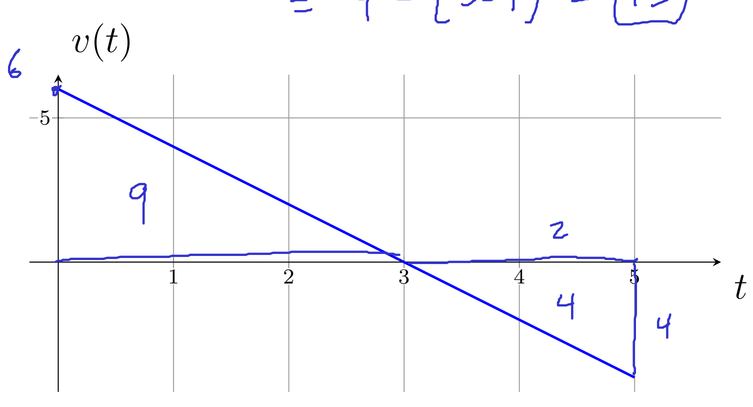
$$\text{displacement} = \int_0^5 6-2t\,dt = 6t - t^2\Big|_0^5$$
$$= (30-25)-(0-0)$$
$$= 5$$
$$= 9-4 = 5$$

- Find the distance traveled by the object on the interval $0 \le t \le 5$.

$$\text{distance} = \int_0^5 |6-2t|\,dt = \int_0^3 (6-2t)\,dt - \int_3^5 (6-2t)\,dt$$

$$|6-2t| = \begin{cases} 6-2t & t \le 3 \\ -(6-2t) & t > 3 \end{cases}$$

$$= 6t - t^2\Big|_0^3 - \left(6t-t^2\right)\Big|_3^5$$
$$= \left[(18-9)-(0)\right] - \left[(30-25)-(18-9)\right]$$
$$= 9 - [5-9] = \boxed{13}$$



Example. A cyclist rides down a long straight road at a velocity (in m/min) given by $v(t) = 400 - 20t$, for $0 \leq t \leq 10$.

$$|400-20t| = \begin{cases} 400-20t, & t \leq 20 \\ -(400-20t), & t > 20 \end{cases}$$

- How far does the cyclists travel in the first 5 minutes?

$$\int_0^5 |v(t)|\,dt = \int_0^5 400-20t\,dt$$

$$= 400t - 10t^2 \Big|_0^5$$

$$= 2000 - 250 = 1750 \text{ m}$$

- How far does the cyclists travel in the first 10 minutes?

$$\int_0^{10} |v(t)|\,dt = \int_0^{10} 400-20t\,dt = 400t - 10t^2 \Big|_0^{10}$$

$$= 4000 - 1000 = \boxed{3000 \text{ m}}$$

- How far has the cyclist traveled when her velocity is 250 m/min?

Solve $v(t_0) = 250$ m/min

$$400 - 20t_0 = 250$$

$$150 = 20t_0$$

$$7.5 = t_0$$

$$\begin{array}{r} {}^{2} \\ 7.5 \\ \times\ 7.5 \\ \hline 56.25 \end{array}$$

$$\text{distance} = \int_0^{7.5} |v(t)|\,dt$$

$$= 400t - 10t^2 \Big|_0^{7.5}$$

$$= (3000 - 562.5) - (0)$$

$$= 2437.5 \text{ m}$$

Example. The population of a community of foxes is observed to fluctuate on a 10-year cycle due to variations in the availability of prey. When population measurements began ($t = 0$), the population was 35 foxes. The growth rate in units of foxes/year was observed to be:

$$P'(t) = 5 + 10\sin\left(\frac{\pi t}{5}\right) \qquad P(b) - P(a) = \int_a^b P'(t)\,dt$$

- Find $P(t)$.

$$P(t) = P(0) + \int_0^t P'(x)\,dx$$

$$= 35 + \int_0^t 5 + 10\sin\left(\frac{\pi}{5}x\right)dx$$

$u = \frac{\pi}{5}x$
$du = \frac{\pi}{5}dx \longrightarrow \frac{5}{\pi}du = dx$

$$= 35 + \left[5x - \frac{50}{\pi}\cos\left(\frac{\pi}{5}x\right)\right]_0^t = 35 + 5t - \frac{50}{\pi}\cos\left(\frac{\pi}{5}t\right) - \left[0 - \frac{50}{\pi}\right]$$

$$= 35 + \frac{50}{\pi} + 5t - \frac{50}{\pi}\cos\left(\frac{\pi}{5}t\right)$$

- Find the population of foxes after the first 5 years, rounded to the nearest whole number of foxes.

$$P(5) = P(0) + \int_0^5 P'(t)\,dt \;\ldots$$

$$= 35 + \frac{50}{\pi} + 25 - \frac{50}{\pi}\underbrace{\cos\left(\frac{\pi}{5}5\right)}_{(-1)} \approx 92 \text{ foxes}$$

Theorem 6.1: Position from Velocity
Given the velocity $v(t)$ of an object moving along a line and its initial position $s(0)$, the position function of the object for future times $t \geq 0$ is

$$\underbrace{s(t)}_{\substack{\text{position}\\ \text{at } t}} = \underbrace{s(0)}_{\substack{\text{initial}\\ \text{position}}} + \underbrace{\int_0^t v(x)\,dx}_{\substack{\text{displacement}\\ \text{over } [0,t]}}.$$

Theorem 6.2: Velocity from Acceleration
Given the acceleration $a(t)$ of an object moving along a line and its initial velocity $v(0)$, the velocity of the object for future times $t \geq 0$ is

$$v(t) = v(0) + \int_0^t a(x)\,dx.$$

Example. At $t = 0$, a train approaching a station begins decelerating from a speed of 80 miles/hour according to the acceleration function $a(t) = -1280(1 + 8t)^{-3}$, where $t \geq 0$ is measured in hours. The units of acceleration are mi/hr^2.

- Find the velocity of the train at $t = 0.25$.

$u = 1+8x \quad x=0, \; u = 1$

$du = 8\,dx \quad x = t, \; u = 1+8t$

$\frac{du}{8} = dx$

$$v(t) = v(0) + \int_0^t -1280(1+8x)^{-3}\,dx$$

$$= 80 - \int_1^{1+8t} 160u^{-3}\,du$$

$$= 80 + 80u^{-2}\Big|_1^{1+8t} = 80 + \left[\frac{80}{(1+8t)^2} - \frac{80}{1}\right] = \frac{80}{(1+8t)^2}$$

$$v(1/4) = \frac{80}{(1+2)^2} = \frac{80}{9} \text{ mi/hr}$$

- How far does the train travel in the first 15 minutes (1/4 hour)?

$u = 1+8x \quad x=0, u=1$

$du = 8\,dx \quad x=t, u=1+8t$

$\frac{du}{8} = dx$

$$s(t) = s(0) + \int_0^t v(x)\,dx = 0 + \int_0^t 80(1+8x)^{-2}\,dx$$

$$= \int_1^{1+8t} 10u^{-2}\,du = -\frac{10}{u}\Big|_1^{1+8t} = 10 - \frac{10}{1+8t}$$

$$s(1/4) = 10 - \frac{10}{3} = \frac{20}{3} \text{ miles}$$

- How long does it take the train to travel 9 miles?

Solve $s(t) = 9$

$$10 - \frac{10}{1+8t} = 9$$

$$1 + 8t = 10$$

$$t = 9/8 \text{ hr.}$$

Theorem 6.3: Net Change and Future Value
Suppose a quantity Q changes over time at a known rate Q'. Then the **net change** in Q between $t = a$ and $t = b > a$ is

$$\underbrace{Q(b) - Q(a)}_{\text{net change in } Q} = \int_a^b Q'(t)\,dt.$$

Given the initial value $Q(0)$, the **future value** of Q at time $t \geq 0$ is

$$Q(t) = Q(0) + \int_0^t Q'(x)\,dx.$$

Velocity-Displacement Problems	**General Problems**
Position $s(t)$	Quantity $Q(t)$ (such as volume or population)
Velocity: $s'(t) = v(t)$	Rate of change: $Q'(t)$
Displacement: $s(b) - s(a) = \int_a^b v(t)\,dt$	Net change: $Q(b) - Q(a) = \int_a^b Q'(t)\,dt$
Future position: $s(t) = s(0) + \int_0^t v(x)\,dx$	Future value of Q: $Q(t) = Q(0) + \int_0^t Q'(x)\,dx$