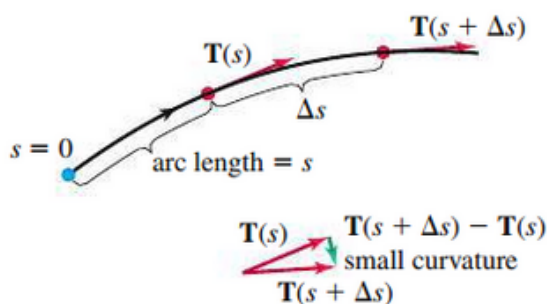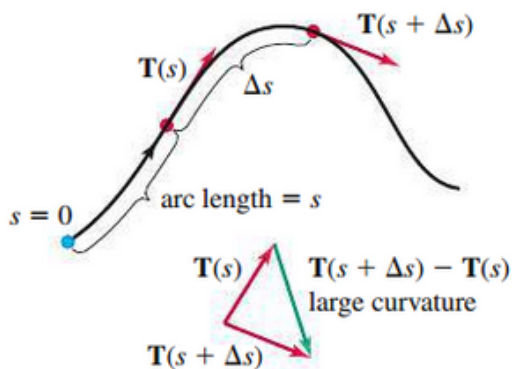# 1   14.5: Curvature and Normal Vectors:

There are two ways to acceleration:

- change in speed

- change in direction

The change in direction is referred to as *curvature*. Recall that if we have a smooth curve $\mathbf{r}(t)$, the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

Specifically, *curvature* of the curve is the magnitude of the rate at which $\mathbf{T}$ changes with respect to arc length.

Definition. (Curvature)

Let $\mathbf{r}$ describe a smooth parameterized curve. If s denotes arc length and $\mathbf{T} = \mathbf{r}'/|\mathbf{r}'|$ is the unit tangent vector, the **curvature** is $\kappa(s) = \left| \dfrac{d\mathbf{T}}{ds} \right|$.

Theorem 14.4: Curvature Formula

Let $\mathbf{r}(t)$ describe a smooth parameterized curve, where t is any parameter. If $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{T}$ is the unit tangent vector, then the curvature is

$$\kappa(t) = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

Theorem 14.5: Alternative Curvature Formula

Let $\mathbf{r}$ be the position of an object moving on a smooth curve. The **curvature** at a point on the curve is

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3},$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{a} = \mathbf{v}'$ is the acceleration.

Definition. (Principal Unit Normal Vector)

Let $\mathbf{r}$ describe a smooth curve parameterized by arc length. The **principal unit normal vector** at a point P on the curve at which $\kappa \neq 0$ is

$$\mathbf{N}(s) = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{1}{\kappa} \frac{d\mathbf{T}}{ds}.$$

For other parameters, we use the equivalent formula

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|},$$

evaluated at the value of t corresponding to P.

Theorem 14.6: Properties of the Principal Unit Normal Vector

Let $\mathbf{r}$ describe a smooth parameterized curve with unit tangent vector $\mathbf{T}$ and principal unit normal vector $\mathbf{N}$.

1. $\mathbf{T}$ and $\mathbf{N}$ are orthogonal at all points of the curve; that is, $\mathbf{T} \cdot \mathbf{N} = 0$ at all points where $\mathbf{N}$ is defined.

2. The principal unit normal vector points to the inside of the curve – in the direction that the curve is turning.

Theorem 14.7: Tangential and Normal Components of the Acceleration

The acceleration vector of an object moving in space along a smooth curve has the following representation in terms of its **tangential component** a_T (in the direction of $\mathbf{T}$) and its **normal component** a_N (in the direction of $\mathbf{N}$):

$$\mathbf{a} = a_N \mathbf{N} + a_T \mathbf{T},$$

where $a_N = \kappa |\mathbf{v}|^2 = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|}$ and $a_T = \dfrac{d^2 s}{dt^2}$.

Definition. (Unit Binormal Vector and Torsion)

Let C be a smooth parameterized curve with unit tangent and principal unit normal vectors $\mathbf{T}$ and $\mathbf{N}$, respectively. Then at each point of the curve at which the curvature is nonzero, the **unit binomial vector** is

$$\mathbf{B} = \mathbf{T} \times \mathbf{N},$$

and the **torsion** is

$$\tau = -\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

Summary: Formula for Curves in Space