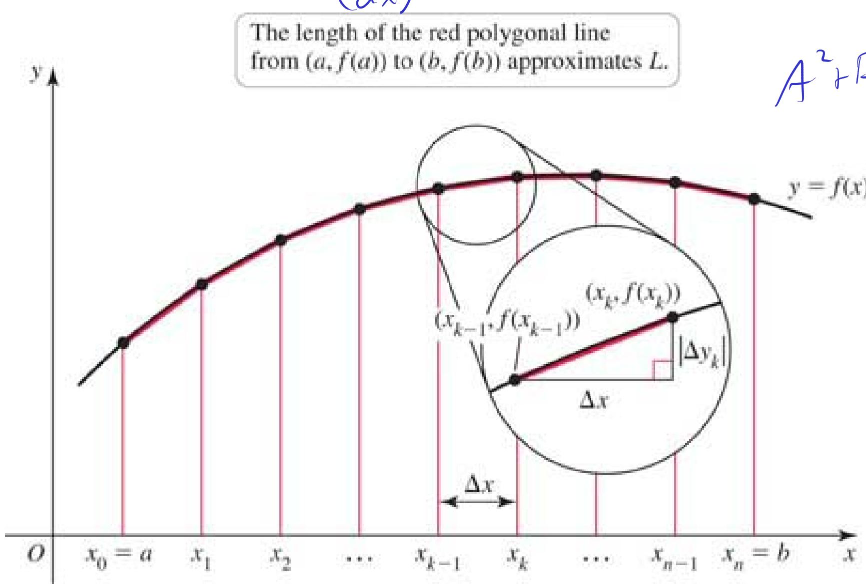
6.5: Length of Curves

Definition. (Arc Length for $y = f(x)$)
Let f have a continuous first derivative on the interval $[a, b]$. The length of the curve from $(a, f(a))$ to $(b, f(b))$ is

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

$\left(\frac{dx}{dx}\right)^2$ $\left(\frac{dy}{dx}\right)^2$

$A^2 + B^2 = c^2$

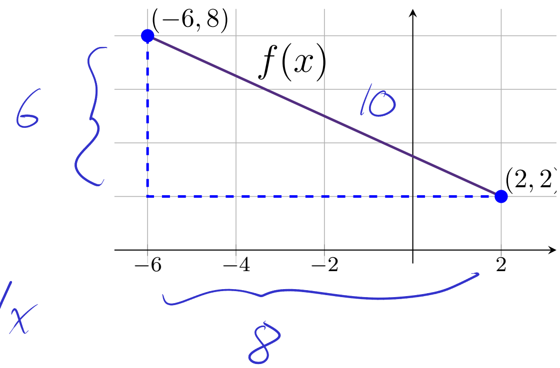


Definition. (Arc Length for $x = g(y)$)
Let g have a continuous first derivative on the interval $[c, d]$. The length of the curve from $(g(c), c)$ to $(g(d), d)$ is

$$L = \int_c^d \sqrt{1 + g'(y)^2}\, dy.$$

$\left(\frac{dy}{dy}\right)^2$ $\left(\frac{dx}{dy}\right)^2$

Example. Using a geometric argument, we can see that the length of $f(x) = -\dfrac{3}{4}x + \dfrac{7}{2}$ on the interval $[-6, 2]$ is $L = 10$. Compute this using the arc-length formula.



$f'(x) = -\frac{3}{4}$

$$\boxed{L = \int_{-6}^{2} \sqrt{1 + \left(-3/4\right)^2}\, dx} = \int_{-6}^{2} \sqrt{25/16}\, dx$$

$$= \frac{5}{4} x \Big|_{-6}^{2} = \frac{5}{4}\left(2 - (-6)\right) = \frac{5}{4}(8) = 10$$

Example. Find the arc length of the curve $y = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$, for $1 \leq x \leq 2$.

$$y' = \frac{1}{2}x - \frac{1}{2x}$$

$$L = \int_1^2 \sqrt{1 + \left(\frac{1}{2}x - \frac{1}{2x}\right)^2}\, dx = \int_1^2 \sqrt{1 + \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2}}\, dx$$

$$= \int_1^2 \sqrt{\frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2}}\, dx \xrightarrow{\text{factor}} = \int_1^2 \sqrt{\left(\frac{1}{2}x + \frac{1}{2x}\right)^2}\, dx = \int_1^2 \frac{1}{2}x + \frac{1}{2x}\, dx$$

$$= \frac{1}{4}x^2 + \frac{1}{2}\ln(x)\Big|_1^2 = \left(1 + \frac{1}{2}\ln(2)\right) - \left(\frac{1}{4} + \frac{1}{2}\ln(1)\right) = \boxed{\frac{3}{4} + \frac{1}{2}\ln(2)}$$

Example. Find the arc length of the curve $y = \frac{1}{3}x^{3/2}$ on $[0, 12]$.

$y' = \frac{1}{2}x^{1/2}$

$L = \int_0^{12} \sqrt{1 + \left(\frac{1}{2}x^{1/2}\right)^2}\, dx = \int_0^{12} \sqrt{1 + \frac{1}{4}x}\, dx$

$u = 1 + \frac{1}{4}x \qquad x = 0, u = 1$

$du = \frac{1}{4}dx \qquad x = 12, u = 4$

$4du = dx$

$= 4\int_1^4 u^{1/2}\, du$

$= 4\left(\frac{2}{3}u^{3/2}\right)\Big|_1^4$

$= \frac{8}{3}\left(4^{3/2} - 1^{3/2}\right)$

$= \frac{8}{3}(8-1) = \frac{56}{3}$

$$f(x) = \frac{1}{4}\left(e^{2x} + e^{-2x}\right) \qquad [a, b]$$

$$f'(x) = \frac{1}{4}\left(2e^{2x} - 2e^{-2x}\right) = \frac{1}{2}\left(e^{2x} - e^{-2x}\right)$$

$$L = \int_a^b \sqrt{1 + \left(\frac{1}{2}\left(e^{2x} - e^{-2x}\right)\right)^2}\, dx$$

$$\left(e^{2x}\right)^2 - 2\underbrace{e^{2x}e^{-2x}}_{1} + \left(e^{-2x}\right)^2$$

$$= \int_a^b \sqrt{1 + \frac{1}{4}\left(e^{4x} - 2 + e^{-4x}\right)}\, dx$$

$$= \int_a^b \sqrt{\underline{1} + \frac{e^{4x}}{4} - \underline{\frac{1}{2}} + \frac{e^{-4x}}{4}}\, dx$$

$$= \int_a^b \sqrt{\frac{e^{4x}}{4} + \frac{1}{2} + \frac{e^{-4x}}{4}}\, dx \qquad \frac{1}{4}\left(\left(e^{2x}\right)^2 + 2 + \left(e^{-2x}\right)^2\right)$$

$$= \int_a^b \sqrt{\left(\frac{e^{2x}}{2} + \frac{e^{-2x}}{2}\right)^2}\, dx$$

$$\frac{d}{dx}\left[-\frac{e^{-2x}}{4}\right] = +\frac{e^{-2x}}{2}$$

$$= \int_a^b \frac{e^{2x}}{2} + \frac{e^{-2x}}{2}\, dx = \frac{e^{2x}}{4} - \frac{e^{-2x}}{4}\Bigg|_a^b$$

$$\frac{1}{4}\left(e^{4x} + 2 + e^{-4x}\right) = \frac{e^{-4x}}{4}\left(\underbrace{e^{8x}} + 2e^{4x} + 1\right)$$

$$w = e^{4x}$$

$$= \frac{w^{-1}}{4}\left(w^2 + 2w + 1\right)$$

$$= \frac{w^{-1}}{4}\left(w+1\right)^2$$

$$= \frac{1}{4}\left(w^{-1/2}(w+1)\right)^2$$

$$w = e^{4x} \qquad = \frac{1}{4}\left(w^{1/2} + w^{-1/2}\right)^2$$

$$= \frac{1}{4}\left(e^{2x} + e^{-2x}\right)^2$$

$$= \left(\frac{e^{2x}}{2} + \frac{e^{-2x}}{2}\right)^2$$

Example. Find a curve that passes through $(1, 2)$ on $[2, 6]$ whose arc length is computed using

$$\int_2^6 \sqrt{1 + 16x^{-2}}\, dx.$$

$$f'(x)^2 = \frac{16}{x^2} \Rightarrow f'(x) = \frac{4}{x} \Rightarrow f(x) = 4\ln(x) + C$$

$$f(2) = 4\ln(2) + C = 6$$

$$\Rightarrow C = 6 - 4\ln(2)$$

$$\Rightarrow f(x) = 4\ln(x) - 4\ln(2) + 6$$

$$= 4\ln\left(\frac{x}{2}\right) + 6$$

Example. Suppose f has length L on $[a, b]$. Evaluate

$$\int_{a/c}^{b/c} \sqrt{1 + f'(cx)^2}\, dx. = \frac{1}{c}\underbrace{\int_a^b \sqrt{1 + f'(u)^2}\, du}_{L}$$

$$= \frac{L}{c}$$

Let $u = cx$ $\quad x = a/c,\ u = a$

$du = c\,dx$ $\quad x = b/c,\ u = b$

$\frac{1}{c} du = dx$