10.3: Infinite Series

A **Geometric sum** with n terms has the form

initial term

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k$$

ratio

Derivation of partial sum formula:

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$-rS_n = -(ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n)$$

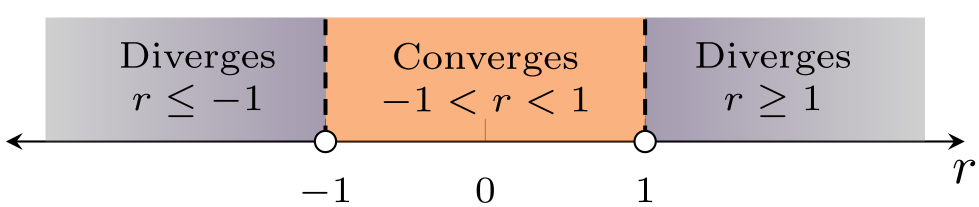
$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r} \leftarrow \text{Partial sum}$$

Theorem 10.7: Geometric Series

Let $a \neq 0$ and r be real numbers. If $|r| < 1$, then $\boxed{\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}}$. If $|r| \geq 1$, then the series diverges.



Example. Evaluate the following geometric series or state that the series diverges

$$\sum_{k=0}^{\infty} 1.1^k = 1 + 1.1 + 1.1^2 + 1.1^3 + \dots$$

a

$a = 1$

$r = 1.1$

$S_n = \dfrac{1(1-1.1^n)}{1-1.1}$

$|r| = |1.1| > 1$ diverges

$$\sum_{k=0}^{\infty} e^{-k} = 1 + e^{-k} + e^{-2k} + e^{-3k} + \dots$$

$a = 1$

$r = e^{-1} = \frac{1}{e}$

$S_n = \dfrac{1\left(1 - \frac{1}{e^n}\right)}{1 - \frac{1}{e}}$

$|r| < 1 \longrightarrow$ converges

$\sum a r^k$

$$\frac{27}{16} \qquad \frac{27}{16}\left(-\frac{3}{4}\right)^1$$

$$\sum_{k=2}^{\infty} 3(\underbrace{-0.75}_{r})^k = 3\left(-\frac{3}{4}\right)^2 + 3\left(-\frac{3}{4}\right)^3 + \dots$$

$$= \frac{27}{16} - \frac{81}{64} + \dots$$

$a = \frac{27}{16}$

$r = \frac{-3}{4}$

$S_n = \dfrac{27}{16} \dfrac{\left(1 - \left(-\frac{3}{4}\right)^n\right)}{\underbrace{1 - \left(-\frac{3}{4}\right)}}$

$= \frac{27}{28}\left(1 - \left(-\frac{3}{4}\right)^n\right)$

$\lim_{n \to \infty} S_n = \boxed{\frac{27}{28}}$

$$\sum_{k=1}^{\infty} \frac{7}{10^k} = \sum_{k=0}^{\infty} \frac{7}{10^{k+1}} = \sum_{k=0}^{\infty} \underbrace{\frac{7}{10}}_{a}\Bigg(\underbrace{\frac{1}{10}}_{r}\Bigg)^k$$

$$= \frac{7}{10} + \frac{7}{10^2} + \frac{7}{10^3} + \frac{7}{10^4} + \dots$$

a

$S_n = \dfrac{a(1-r^n)}{1-r} = \dfrac{7}{10}\dfrac{\left(1 - \left(\frac{1}{10}\right)^n\right)}{\underbrace{1 - \frac{1}{10}}_{9/10}}$

$= \frac{7}{9}\left(1 - \left(\frac{1}{10}\right)^n\right)$

$$\sum_{k=2}^{\infty} 3(\underbrace{-0.75}_{r})^k = \sum_{k=0}^{\infty} 3(-0.75)^{k+2} = \sum_{k=0}^{\infty} \underbrace{3(-0.75)^2}_{a = \frac{27}{16}}(\underbrace{-0.75}_{r=-0.75})^k$$

Telescoping Series:

Example. Evaluate the following series

$$\sum_{k=1}^{\infty} \cos\left(\frac{1}{k^2}\right) - \cos\left(\frac{1}{(k+1)^2}\right)$$

$\left[\cos(1/9) - \cos(1/16)\right]$

$$S_n = \underbrace{\left[\cos\left(\frac{1}{1}\right) - \cos\left(\frac{1}{4}\right)\right]}_{k=1} + \underbrace{\left[\cos\left(\frac{1}{4}\right) - \cos\left(\frac{1}{9}\right)\right]}_{k=2} + \dots + \left[\cos\left(\frac{1}{n^2}\right) - \cos\left(\frac{1}{(n+1)^2}\right)\right]$$

$$= \cos(1) - \cos\left(\frac{1}{(n+1)^2}\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \cos(1) - \cos\Big(\underbrace{\frac{1}{(n+1)^2}}_{\to 0}\Big) = \boxed{\cos(1) - 1}$$

Converges

$$\sum_{k=3}^{\infty} \frac{1}{(k-2)(k-1)}$$

$$(k-2)(k-1)\frac{1}{(k-2)(k-1)} = \left(\frac{A}{k-2} + \frac{B}{k-1}\right)(k-2)(k-1)$$

$$0k + 1 = A(k-1) + B(k-2)$$
$$= Ak - A + Bk - 2B$$
$$= \underbrace{(A+B)}_{0}k + \underbrace{(-A-2B)}_{1}$$

$A + B = 0$ (1)

$-A - 2B = 1$ (2)

$\to A = -B$

$\to B - 2B = 1$

$B = -1$

$(1) + (2) \to -B = 1 \xrightarrow{(1)} A = 1$

$\left[\frac{1}{n-3} - \frac{1}{n-2}\right]$

$$S_n = \sum_{k=3}^{n} \frac{1}{(k-2)(k-1)} = \sum_{k=3}^{n} \frac{1}{k-2} - \frac{1}{k-1} = \underbrace{\left[\frac{1}{1} - \cancel{\frac{1}{2}}\right]}_{k=3} + \underbrace{\left[\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right]}_{k=4} + \dots + \left[\cancel{\frac{1}{n-2}} - \frac{1}{n-1}\right]$$

$$\sum_{k=3}^{\infty} \frac{1}{(k-2)(k-1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n-1} = \boxed{1}$$

Theorem 10.8: Properties of Convergent Series

1. Suppose $\sum a_k$ converges to A and c is a real number. The series $\sum ca_k$ converges, and $\sum ca_k = c\sum a_k = cA$.
2. Suppose $\sum a_k$ diverges. Then $\sum ca_k$ also diverges, for any real number $c \neq 0$.
3. Suppose $\sum a_k$ converges to A and $\sum b_k$ converges to B. The series $\sum(a_k \pm b_k)$ converges and $\sum(a_k \pm b_k) = \sum a_k \pm \sum b_k = A \pm B$.
4. Suppose $\sum a_k$ diverges and $\sum b_k$ converges. Then $\sum(a_k \pm b_k)$ diverges.
   $\cancel{\infty \pm A}$ ← Bad notation
5. If M is a positive integer, then $\displaystyle\sum_{k=1}^{\infty} a_k$ and $\displaystyle\sum_{k=M}^{\infty} a_k$ either both converge or both diverge. In general, *whether* a series converges does not depend on a finite number of terms added to or removed from the series. However, the *value* of a convergent series does change if nonzero terms are added or removed.

Example. Evaluate

$$\sum_{k=1}^{\infty}\left[\frac{1}{2}\left(\frac{2}{5}\right)^k + \frac{2}{3}\left(\frac{1}{6}\right)^k\right] \overset{(3)}{=} \sum_{k=1}^{\infty}\frac{1}{2}\left(\frac{2}{5}\right)^k + \sum_{k=1}^{\infty}\frac{2}{3}\left(\frac{1}{6}\right)^k$$

$$\overset{(1)}{=} \frac{1}{2}\underbrace{\sum_{k=1}^{\infty}\left(\frac{2}{5}\right)^k}_{\substack{\text{Geometric} \\ a=\frac{2}{5},\ r=\frac{2}{5}}} + \frac{2}{3}\underbrace{\sum_{k=1}^{\infty}\left(\frac{1}{6}\right)^k}_{\substack{\text{Geometric} \\ a=1/6,\ r=1/6}}$$

$$= \frac{1}{2}\cdot\frac{2/5}{1-2/5} + \frac{2}{3}\cdot\frac{1/6}{1-1/6}$$

$$= \frac{1}{2}\left(\frac{2}{3}\right) + \frac{2}{3}\left(\frac{1}{5}\right) = \boxed{\frac{7}{15}}$$