10.6: Alternating Series

Theorem 10.16: Alternating Series Test
The alternating series $\sum(-1)^{k+1}a_k$ converges provided

1. the terms of the series are nonincreasing in magnitude ($0 < \underline{a_{k+1} \leq a_k}$, for k greater than some index N) and
2. $\lim_{k\to\infty} a_k = 0$.

show $f(k) = a_k$
& $f'(x) < 0$

Example. Which of the following are considered alternating series?

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k+2} \quad ✓$$

$$\sum_{k=0}^{\infty} (-1)^{k+1} a_k$$

$$a_k = \frac{1}{k+2}$$

$$\sum_{k=4}^{\infty} \left(\frac{-3}{2}\right)^k \quad ✓$$

$$\sum_{k=4}^{\infty} (-1)^k \underbrace{\left(\frac{3}{2}\right)^k}_{a_k = \left(\frac{3}{2}\right)^k}$$

$$\sum_{k=0}^{\infty} (-1)\left(\frac{1}{2}\right)^k \quad ✗$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \underbrace{\left(\frac{1}{2}\right)^k}_{a_k} \quad ✓$$

Diverges
$\lim_{k\to\infty} a_k = \infty$

$$(*) \quad \sum_{k=-3}^{\infty} \frac{\cos(k\pi)}{(k+4)^2} \quad ✓$$

k	$\cos(k\pi)$
-3	-1
-2	1
-1	-1
0	1
1	-1

$$(*) = \sum_{k=-3}^{\infty} \frac{(-1)^k}{(k+4)^2}$$

$$\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2} \quad ✗$$

Can't rewrite $\sin(k)$ as $(-1)^k$

$$\sum_{k=0}^{\infty} (-1)^{k+1} \left(\frac{1}{-2}\right)^k = \sum_{k=0}^{\infty} \underbrace{(-1)^{k+1}(-1)^k}_{(-1)^{2k+1} = -1} \left(\frac{1}{2}\right)^k \quad ✗$$

Example. Consider the series $\sum_{k=1}^{\infty}(-1)^k \underbrace{\boxed{\frac{\sqrt{k}}{2k+3}}}_{a_k}$. Let a_k represent that magnitude of the terms of the given series.

- What is $\lim_{k\to\infty} a_k$?

$$\lim_{k\to\infty} \frac{\sqrt{k}}{2k+3}\left(\frac{1/k}{1/k}\right) = \lim_{k\to\infty} \frac{1/\sqrt{k}}{2+3/k} = 0$$

LC#4

$$a_{k+1} = \frac{\sqrt{k+1}}{2(k+1)+3} = \frac{\sqrt{k+1}}{2k+5}$$

LC#3

- Compute $f'(x)$ where $f(k) = a_k$. $f(x) = \frac{\sqrt{x}}{2x+3}$

Since f'(x) is negative for x>3/2, the end behavior of the function is decreasing. We can have finitely many terms that are increasing.

$$f'(x) = \frac{(2x+3)\frac{1}{2\sqrt{x}} - 2\sqrt{x}}{(2x+3)^2} = \frac{2x+3-4x}{2\sqrt{x}(2x+3)^2} = \frac{-2x+3}{2\sqrt{x}(2x+3)^2}$$

Want $f'(x) < 0$

$$\frac{-2x+3}{2\sqrt{x}(2x+3)^2} < 0$$

$\Rightarrow -2x+3 < 0$

$x > \frac{3}{2} \Rightarrow$ a_k decreasing for $k \geq 2$

LC#5

- Use the Alternating Series Test to determine if the given series converges.

① a_k are non-increasing (decreasing)

② $\lim_{k\to\infty} a_k = 0$

$\Longrightarrow$ By the AST, $\sum_{k=1}^{\infty} (-1)^k \frac{\sqrt{k}}{2k+1}$ converges

Example. Does the series $\sum_{k=0}^{\infty}(-1)^{k+1}\left(\frac{4}{3}\right)^k$ converge?

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty}\left(\frac{4}{3}\right)^k = \infty \longrightarrow \text{Diverges by the divergence test}$$

Example. Does the series $\sum_{k=1}^{\infty}\cos(\pi k)e^{-k}$ converge?

Use table for correct index

$$= \sum_{k=1}^{\infty} (-1)^k e^{-k}$$

k	$\cos(\pi k)$
1	-1
2	1
3	-1

$f(x) = e^{-x}$

(1) $f'(x) = -e^{-x} < 0$, for all x $\quad -\infty < x < \infty \to$ decreasing a_k for all k

(2) $\lim_{k\to\infty} a_k = \lim_{k\to\infty} e^{-k} = \lim_{k\to\infty} \frac{1}{e^k} = 0$

$\Longrightarrow$ Converges by the AST

Theorem 10.17: Alternating Harmonic Series

The alternating harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges (even though the harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges).

$\displaystyle\sum_{k=1}^{\infty} |a_k|$ where $a_k = \dfrac{(-1)^{k+1}}{k}$

Example. Use the Alternating Series Test to show that the alternating harmonic series converges.

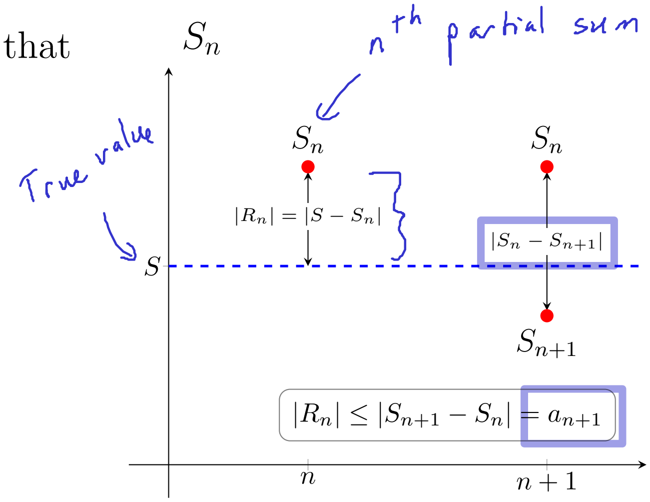
$$\left.\begin{array}{l} (1) \quad a_{k+1} = \dfrac{1}{k+1} < \dfrac{1}{k} = a_k \\ (2) \quad \displaystyle\lim_{k \to \infty} \frac{1}{k} = 0 \end{array}\right\} \text{By AST, } \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \text{ converges}$$

Theorem 10.18: Remainder in Alternating Series

Let $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ be a convergent alternating series with terms that are nonincreasing in magnitude. Let $R_n = S - S_n$ be the remainder in approximating the value of that series by the sum of its first n terms. Then $|R_n| \le a_{n+1}$. In other words, the magnitude of the remainder is less than or equal to the magnitude of the first neglected term.

Example. Find the minimum value of n such that $|R_n| < 10^{-4}$ for the following series:

$$\ln(2) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$



$$|R_n| \le a_{n+1} < 10^{-4}$$

$$\frac{1}{n+1} < 10^{-4}$$

$$10^4 < n+1$$

$$9{,}999 < n$$

$$\boxed{10{,}000 \le n}$$

$$S_{n+1} = a_1 + a_2 + \cdots + a_{n-1} + a_n + a_{n+1}$$

$$-\ S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$$

$$S_{n+1} - S_n = a_{n+1}$$

Quiz 8

$$\sum \frac{(-1)^{k+1}}{k^2} \to \sum |a_k| \to \sum \frac{1}{k^2} \leftarrow \text{Still converges, converges abs.}$$

converge

$$\sum \frac{(-1)^{k+1}}{k} \to \sum |a_k| \to \sum \frac{1}{k} \leftarrow \text{diverges, converges cond.}$$

Definition. (Absolute and Conditional Convergence)
If $\sum |a_k|$ converges, then $\sum a_k$ **converges absolutely**.
If $\sum |a_k|$ diverges and $\sum a_k$ converges, then $\sum a_k$ **converges conditionally**.

Example. Can a series of strictly positive terms converge conditionally?

$\Rightarrow \sum a_k$ converges, $\sum |a_k|$ diverges → impossible

Converges absolutely

$a_k > 0 \to |a_k| = a_k$

Example. Consider the series $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{4+k}{k^2}$. Determine if this series converges absolutely, converges conditionally, or diverges.

$$\sum_{k=1}^{\infty}\left|(-1)^{k+1}\frac{4+k}{k^2}\right| = \sum_{k=1}^{\infty}\underbrace{\frac{4+k}{k^2}}_{a_k} \to \sum_{k=1}^{\infty}\frac{4}{k^2} + \boxed{\sum_{k=1}^{\infty}\frac{1}{k}}$$

diverges by p-series

Direct Comparison Test

$$\sum_{k=1}^{\infty}\underbrace{\frac{1}{k}}_{b_k} = \sum \frac{k}{k^2}$$

divergent

$4k + k^2 > k^2$

$\dfrac{4+k}{k^2} > \dfrac{1}{k}$

Since $b_k \le a_k$, $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges
$\sum_{k=1}^{\infty}\frac{4+k}{k^2}$ also diverges by DCT

$\boxed{\text{Does not converge abs.}}$

Cond conv: $\sum |a_k|$ diverges
$\sum a_k$ converges

① $0 < a_{k+1} \le a_k$ $\qquad \dfrac{4+k}{k^2} = \underbrace{\dfrac{4}{k^2} + \dfrac{1}{k}}$ both decreasing

② $\displaystyle\lim_{k\to\infty} a_k = \lim_{k\to\infty}\frac{4+k}{k^2} = 0$

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{4+k}{k^2}$$

$\Rightarrow$ converges by AST

$\Rightarrow$ converges conditionally.

Example. Determine if the following series converge absolutely, converge conditionally, or diverge.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2\sqrt{k}-1}$$

absolute convergence

LCT w/ $\frac{1}{\sqrt{k}}$

Divergent p-series w/ $p = 1/2$

$\sum_{k=1}^{\infty} \underbrace{\frac{1}{2\sqrt{k}-1}}_{a_k}$ $\quad \sum_{k=1}^{\infty} \underbrace{\frac{1}{\sqrt{k}}}_{b_k}$

$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\sqrt{k}}{2\sqrt{k}-1} = 1/2 \Rightarrow$ Series diverges by LCT

conditional convergence

AST

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2\sqrt{k}-1}$

1. $f(x) = \frac{1}{2\sqrt{x}-1}$

   $f'(x) = \frac{-1}{\sqrt{x}(2\sqrt{x}-1)^2} < 0$, when $x \neq \frac{1}{4}$

   $\Rightarrow k \geq 1$

2. $\lim_{k\to\infty} \frac{1}{2\sqrt{k}-1} = 0$

Converges by AST

$\longrightarrow$ conditionally convergent

$$\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k$$

absolute convergence

$\underbrace{\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k}_{\text{Same as original series}} \longleftarrow$ convergent geometric series w/ $r = 3/4$

$\Longrightarrow$ absolutely convergent

$\sum |a_k|$ $\quad$ $\sum a_k$

abs convergence $\Longrightarrow$ convergence

no abs convergence $\Longleftarrow$ diverges

$\boxed{\text{check } \sum |a_k| \text{ convergence}} \xrightarrow{\text{yes}}$ absolute convergence

$\downarrow$ no

$\boxed{\text{check } \sum a_k \text{ convergence}} \xrightarrow{\text{yes}}$ cond convergence

no $\downarrow$ diverges

Theorem 10.19: Absolute Convergence Implies Convergence
If $\sum |a_k|$ converges, then $\sum a_k$ converges (absolute convergence implies convergence). Equivalently, if $\sum a_k$ diverges, then $\sum |a_k|$ diverges.

Example. Determine whether each of the following series converges absolutely, converges conditionally or diverges.

$$\sum_{k=1}^{\infty}(-1)^k e^{1/k}$$

abs. conv. $\quad \sum_{k=1}^{\infty} |(-1)^k e^{1/k}| = \sum_{k=1}^{\infty} e^{1/k}$ $\quad$ Divergence Test $\lim_{k\to\infty} e^{1/k} = e^0 = 1$

$\Rightarrow$ diverges by div test

$\downarrow$

cond. conv. $\quad \sum_{k=1}^{\infty} (-1)^k e^{1/k}$ $\quad$ ① $0 < a_{k+1} \le a_k$ $\quad$ $f(x) = e^{1/x}$

$f'(x) = -\dfrac{e^{1/x}}{x^2} < 0, \ x > 0$

② $\underbrace{\lim_{k\to\infty} e^{1/k} = e^0 = 1}$

$\Rightarrow$ diverges by divergence test

$\downarrow$

diverges

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^6}$$

abs conv $\longrightarrow \sum_{k=1}^{\infty} \frac{1}{k^6}$ converge? $\longrightarrow$ convergent p-series w/ $p = 6 > 1$

$\longrightarrow$ Converges abs. $\Longrightarrow$ converges

$\sum |a_k| \qquad \sum a_k$

$\sum |a_k| \longrightarrow$ p-series $\longrightarrow \sum a_k$ converges

convergence of $\sum \frac{(-1)^{k+1}}{k^6}$

$\longrightarrow$ ast ① $a_{k+1} < a_k$ ② $\lim_{k\to\infty} a_k = 0$ $\Big\} \Rightarrow$ converges

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$$ abs conv. $\sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k$ Geometric w/ $r = 1/3 < 1$

Converges absolutely

$\Rightarrow$ converges

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abs conv $\sum 1/k^{1/2}$ divergent p-series $p = 1/2 \le 1$

$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ cond conv: (1) $a_{k+1} \le a_k$ $\because$ $\frac{1}{\sqrt{k+1}} < \frac{1}{\sqrt{k}}$

(2) $\lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$

converges by AST

$$\sum_{k=1}^{\infty} \frac{(-5)^k}{3^k}$$

$\longrightarrow$ converges conditionally

Divergence test $\lim_{k\to\infty} \left(\frac{-5}{3}\right)^k$ diverges $\boxed{\text{oscillates b/w } \pm\infty}$

$$\sum_{k=1}^{\infty} \frac{(-2)^{k-1}}{3^k}$$ Diverges $\rightarrow$ not abs convergent

$\sum a_k$ $\sum |a_k|$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{3^k}$$ abs conv $\sum_{k=1}^{\infty} \frac{2^{k-1}}{3^k} = \sum_{k=1}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^k$ convergent geometric series w/ $r = 2/3$

$\Rightarrow$ abs conv $\Rightarrow$ $\sum a_k$ converges

Example. Does the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2\sqrt{k}-1}$ converge conditionally, converge absolutely, or diverge?

LCT w/ $\frac{1}{\sqrt{k}}$

Divergent p-series w/ $p = 1/2$

absolute convergence

$$\sum_{k=1}^{\infty} \underbrace{\frac{1}{2\sqrt{k}-1}}_{a_k} \qquad \sum_{k=1}^{\infty} \underbrace{\frac{1}{\sqrt{k}}}_{b_k} \qquad \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\sqrt{k}}{2\sqrt{k}-1} = \frac{1}{2} \Rightarrow$$

Series diverges by LCT

AST

conditional convergence

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2\sqrt{k}-1}$$

① $f(x) = \frac{1}{2\sqrt{x}-1}$

$f'(x) = \frac{-1}{\sqrt{x}(2\sqrt{x}-1)^2} < 0$, when $x \neq \frac{1}{4}$

$\Rightarrow k \geq 1$

② $\lim_{k\to\infty} \frac{1}{2\sqrt{k}-1} = 0$

Converges by AST

$\longrightarrow$ conditionally convergent

Remainder

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2\sqrt{k}-1}$$

Given n, find remainder

$n = 8$

$$|R_8| \leq a_9 = \frac{1}{2\sqrt{9}-1} = \frac{1}{5}$$

Given "remainder", find min n

Want $|R_n| < 10^{-2}$

$$a_{n+1} < 10^{-2}$$

$$\frac{1}{2\sqrt{n+1}-1} < \frac{1}{100}$$

$$100 < 2\sqrt{n+1} - 1$$

$$\frac{101}{2} < \sqrt{n+1}$$

$$\left(\frac{101}{2}\right)^2 - 1 < n$$

$$2549.25 < n$$

$\longrightarrow n = 2550$

$2550 < n$

$\longrightarrow n = 2551$