1 15.2: Limits and Continuity

Definition. (Limit of a Function of Two Variables)
The function f has the **limit** L as $P(x, y)$ approaches $P_0(a, b)$, written

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{P\to P_0} f(x, y) = L,$$

if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon$$

whenever (x, y) is in the domain of f and

$$0 < |PP_0| = \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

Theorem 15.1: Limits of Constant and Linear Functions
Let a, b, and c be real numbers.

1. Constant function $f(x, y) = c$: $\lim_{(x,y)\to(a,b)} c = c$
2. Linear function $f(x, y) = x$: $\lim_{(x,y)\to(a,b)} x = a$
3. Linear function $f(x, y) = y$: $\lim_{(x,y)\to(a,b)} y = b$

Theorem 15.2: Limit Laws for Functions of Two Variables
Let L and M be real numbers and suppose $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim\limits_{(x,y)\to(a,b)} g(x,y) = M$. Assume c is constant, and $n > 0$ is an integer.

1. **Sum** $\lim\limits_{(x,y)\to(a,b)} (f(x,y) + g(x,y)) = L + M$
2. **Difference** $\lim\limits_{(x,y)\to(a,b)} (f(x,y) - g(x,y)) = L - M$
3. **Constant multiple** $\lim\limits_{(x,y)\to(a,b)} cf(x,y) = cL$
4. **Product** $\lim\limits_{(x,y)\to(a,b)} f(x,y)g(x,y) = LM$
5. **Quotient** $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$, provided $M \neq 0$
6. **Power** $\lim\limits_{(x,y)\to(a,b)} (f(x,y))^n = L^n$
7. **Root** $\lim\limits_{(x,y)\to(a,b)} (f(x,y))^{1/n} = L^{1/n}$, when $L > 0$ if n is even.

Definition. (Interior and Boundary Points)
Let R be a region in $\mathbb{R}^2$. An **interior point** P of R lies entirely within R, which means it is possible to find a disk centered at P that contains only points of R.

A **boundary point** Q of R lies on the edge of R in the sense that every disk centered at Q contains at least one point in R and at least one point not in R.

Definition. (Open and Closed Sets)

A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points.

Procedure: Two-Path Test for Nonexistence of Limits
If $f(x, y)$ approaches two different values as (x, y) approaches (a, b) along two different paths in the domain of f, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Definition. (Continuity)
The function f is continuous at the point (a, b) provided

1. f is defined at (a, b)
2. $\lim_{(x,y)\to(a,b)} f(x, y)$ exists, and
3. $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

Theorem 15.3: Continuity of Composite Functions
If $u = g(x, y)$ is continuous at (a, b) and $z = f(u)$ is continuous at $g(a, b)$, then the composite function $z = f(g(x, y))$ is continuous at (a, b).