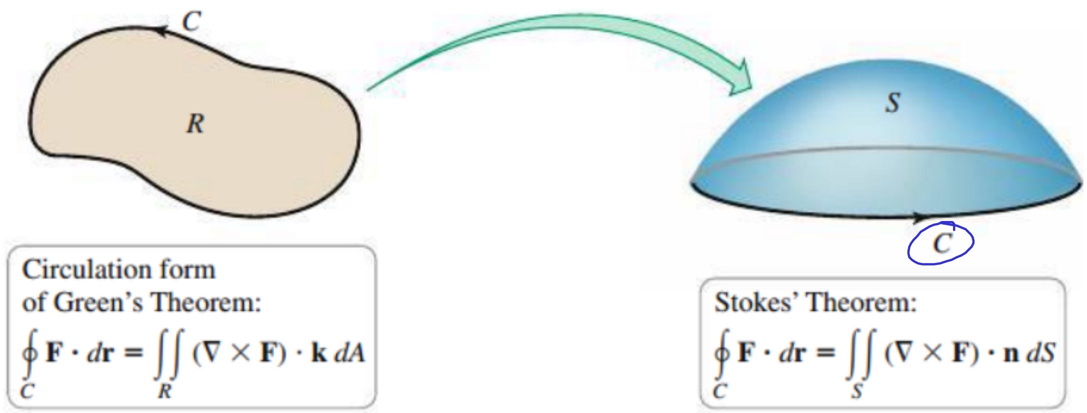
17.7: Stokes' Theorem

Stokes' Theorem is the three-dimensional version of the circulation form of Green's Theorem. Recall the circulation form of Green's Theorem:

$$\underbrace{\oint_C \mathbf{F}\cdot d\mathbf{r}}_{\text{circulation}} = \iint_R \underbrace{(g_x - f_y)}_{\text{curl or rotation}}\; dA.$$

C is boundary of R

The above means that the cumulative rotation within R equals the circulation along the boundary of R. Stokes' Theorem computes the circulation over a surface S in $\mathbb{R}^3$:



Theorem 17.15: Stokes' Theorem
Let S be an oriented surface in $\mathbb{R}^3$ with a piecewise-smooth closed boundary C whose orientation is consistent with that of S. Assume $\mathbf{F} = \langle f,\, g,\, h\rangle$ is a vector field whose components have continuous first partial derivatives on S. Then

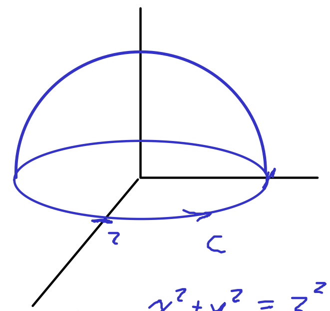
$$\overset{\text{Boundary}}{\oint_C \mathbf{F}\cdot d\mathbf{r}} = \iint_S \overset{\text{Surface}}{(\nabla\times\mathbf{F})}\cdot\mathbf{n}\, dS,$$

where $\mathbf{n}$ is the unit vector normal to S determined by the orientation of S.

Example. Verify Stokes' Theorem: Confirm that Stokes' Theorem holds for the vector field $\mathbf{F} = \langle z - y,\, x,\, -x\rangle$, where S is the hemisphere $x^2 + y^2 + z^2 = 4$, for $z \geq 0$, and C is the circle $x^2 + y^2 = 4$, oriented counterclockwise.

$\vec{r}(t) = \langle 2\cos(t), 2\sin(t), 0\rangle \qquad 0 \leq t \leq 2\pi$

$\vec{r}\,'(t) = \langle -2\sin(t), 2\cos(t), 0\rangle$



$F = \langle 0 - 2\sin(t), 2\cos(t), -2\cos(t)\rangle$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle -2\sin(t), 2\cos(t), -2\cos(t)\rangle \cdot \langle -2\sin(t), 2\cos(t), 0\rangle\, dt$$

$$= \int_0^{2\pi} 4(\underbrace{\sin^2(t) + \cos^2(t)}_{1})\, dt = 4\int_0^{2\pi} dt = \boxed{8\pi}$$

$F = \langle z-y, x, -x\rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z-y & x & -x \end{vmatrix} = \langle 0, 2, 2\rangle$$

$\vec{n} = \langle -z_x, -z_y, 1\rangle = \left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle$

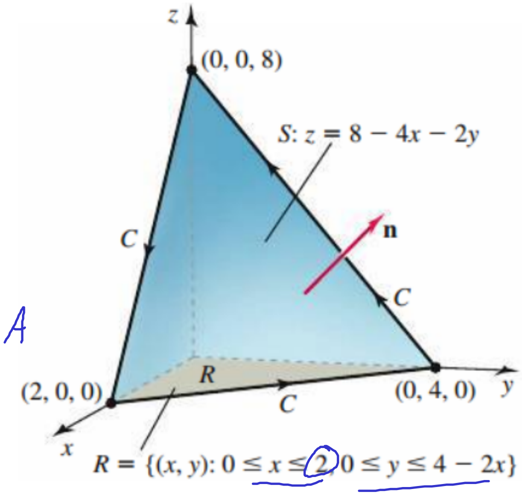
$R = \{(x,y) : x^2+y^2 \leq 4\}$

$= \{(r,\theta) : 0 \leq r \leq 2\}$

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\, dS = \iint_R \langle 0,2,2\rangle \cdot \left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle dA$$

Example. Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle z, -z, x^2 - y^2 \rangle$ and C consists of the three line segments that bound the plane $z = 8 - 4x - 2y$ in the first octant.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & -z & x^2-y^2 \end{vmatrix} = \langle 1-2y, 1-2x, 0 \rangle$$



$$\vec{t}_x \times \vec{t}_y = \langle -z_x, -z_y, 1 \rangle = \langle 4, 2, 1 \rangle$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS = \iint_R \langle 1-2y, 1-2x, 0 \rangle \cdot \langle 4, 2, 1 \rangle \, dA$$

$$= \int_0^2 \int_0^{4-2x} 6 - 4x - 8y \; dy \, dx$$

$$= \int_0^2 (6-4x)y - 4y^2 \Big|_{y=0}^{y=4-2x} dx$$

$$= \int_0^2 -8x^2 + 36x - 40 \; dx$$

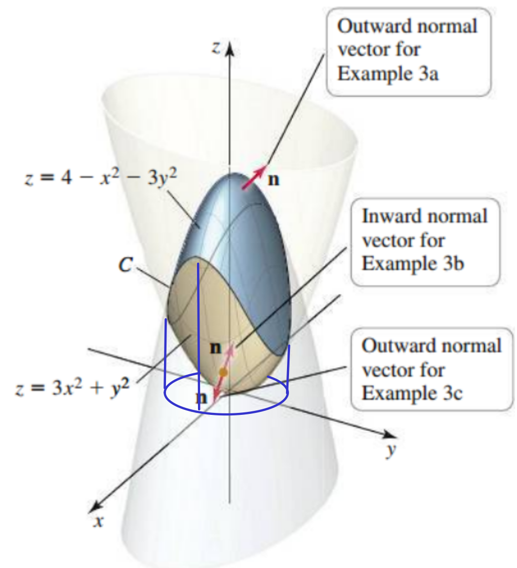
$$= -\frac{8}{3}x^3 + 18x^2 - 40x \Big|_{x=0}^{x=2}$$

$$= \frac{-64}{3} + \underbrace{72 - 80}_{-8 = \frac{-24}{3}}$$

$$= \boxed{-\frac{88}{3}}$$

Example. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$, where $\mathbf{F} = \langle -y,\, x,\, z \rangle$, where:

- S is the part of the paraboloid $z = 4 - x^2 - 3y^2$ contained within $z = 3x^2 + y^2$, with $\mathbf{n}$ pointing upwards.



$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & z \end{vmatrix} = \langle 0, 0, 2 \rangle$$

$$\vec{t}_x \times \vec{t}_y = \langle 2x, 6y, 1 \rangle$$

What is S? $\vec{r}(u,v) = ?$

Boundary is intersection of paraboloids

$$4 - x^2 - 3y^2 = 3x^2 + y^2$$
$$4 = 4x^2 + 4y^2$$
$$1 = x^2 + y^2$$

$\vec{r}(t) = \langle \cos(t), \sin(t), 4 - \cos^2(t) - 3\sin^2(t) \rangle$ $\quad 0 \le t \le 2\pi$

$\vec{r}\,'(t) = \langle -\sin(t), \cos(t), -4\cos(t)\sin(t) \rangle$

$z = 4 - x^2 - 3y^2$ or $z = 3x^2 + y^2$

$$\vec{F} = \langle -y, x, z \rangle = \langle -\sin(t), \cos(t), 4 - \cos^2(t) - 3\sin^2(t) \rangle$$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS = \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \underbrace{\sin^2(t) + \cos^2(t)}_{1} \underbrace{- 16\sin(t)\cos(t)}_{-8\sin(2t)} \overbrace{+ 4\cos^3(t)\sin(t) - 12\cos(t)\sin^3(t)}^{\to 0}\, dt$$

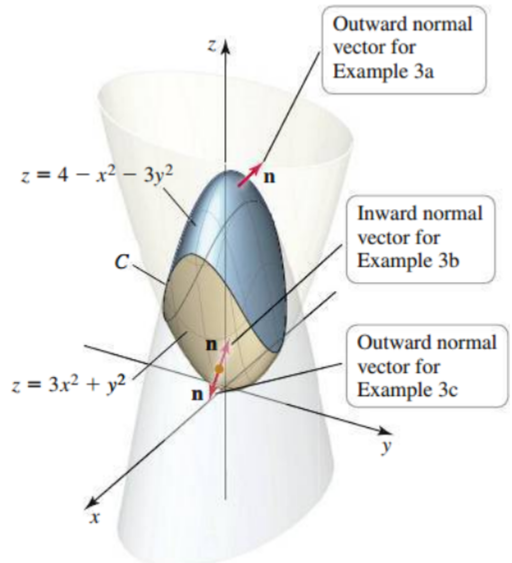
$$\int \underbrace{\cos(t)\sin^3(t)\, dt}_{0}$$

$$= \int_0^{2\pi} 1 - 8\sin(2t)\, dt = t + 4\cos(2t) \Big|_{t=0}^{t=2\pi}$$

$$= 2\pi + 4(1 - 1) = \boxed{2\pi}$$

- S is the part of the paraboloid $z = 3x^2 + y^2$ contained within $z = 4 - x^2 - 3y^2$ with $\mathbf{n}$ pointing upwards.

Same boundary
same vector field
same normal
$\Big\} \to$ Same line integral



$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS = 2\pi$$

- S is the part of the paraboloid $z = 3x^2 + y^2$ contained within $z = 4 - x^2 - 3y^2$ with $\mathbf{n}$ pointing downwards.

$\hookrightarrow$ C is clockwise

$\vec{r} = \langle \sin t, \cos t, 3\cos^2 t + \sin^2 t \rangle \Rightarrow \vec{F} \cdot \vec{r}' = -1 - 12\cos^3(t)\sin(t) - 4\sin^3(t)\cos(t)$

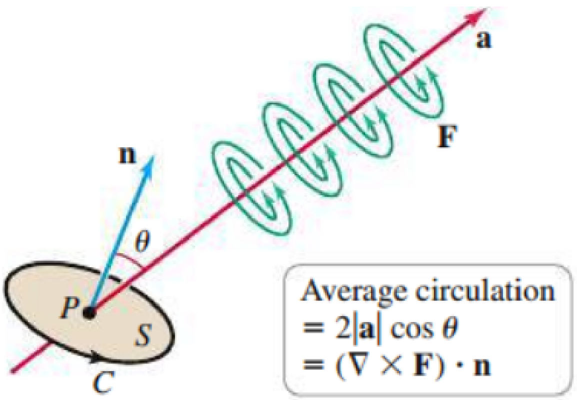
$\Longrightarrow -2\pi$

Interpreting the Curl:

The **average circulation** is

$$\frac{1}{\text{area of } S}\oint_C \mathbf{F}\cdot d\mathbf{r} = \frac{1}{\text{area of } S}\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS.$$

Consider a general rotation field $\mathbf{F} = \mathbf{a}\times\mathbf{r}$, where $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{r} = \langle x, y, z\rangle$. Now, let S be a small circular disk centered at a point P, whose normal vector $\mathbf{n}$ makes an angle θ with the axis $\mathbf{a}$:



The average circulation of this vector field on S is

$$\frac{1}{\text{area of } S}\iint_S \underbrace{(\nabla\times\mathbf{F})\cdot\mathbf{n}}_{\text{indep of } S}\,dS = \frac{(\nabla\times\mathbf{F})\cdot\mathbf{n}}{\cancel{\text{area of } S}}(\cancel{\text{area of } S})$$

$$= 2\mathbf{a}\cdot\mathbf{n} \quad \text{17.5 (p214)}$$

$$= 2|\mathbf{a}|\cos(\theta)$$

$|n|=1$

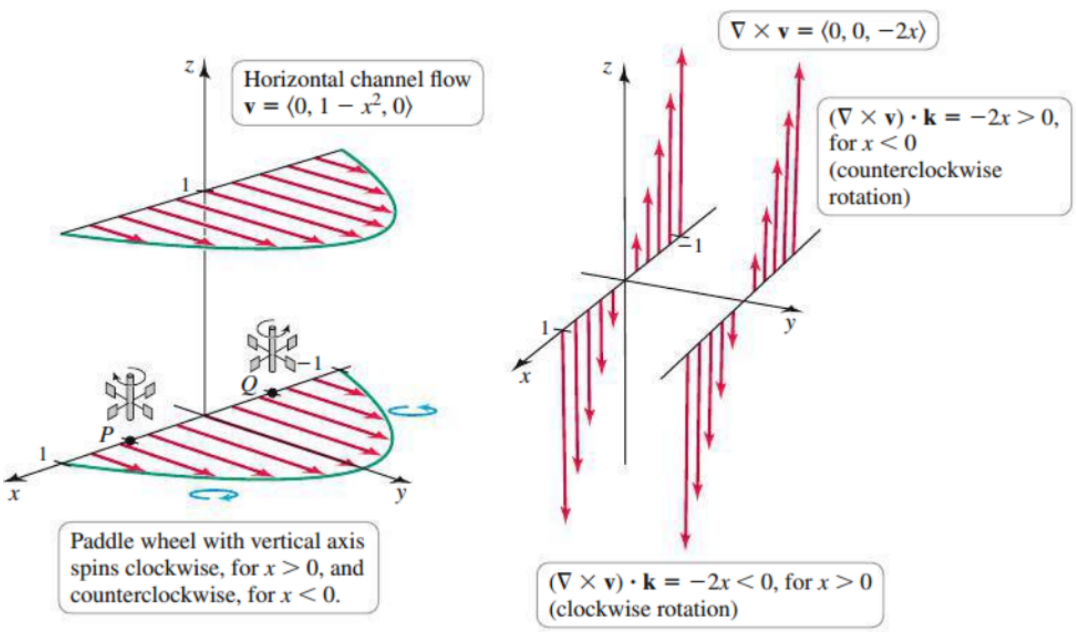
From this, we see

- The scalar component of $\nabla\times\mathbf{F}$ at P in the direction of $\mathbf{n}$ is the average circulation of $\mathbf{F}$ on S.
- The direction of $\nabla\times\mathbf{F}$ at P is the direction that maximizes the average circulation of $\mathbf{F}$ on S.

A similar argument for the curl can be applied to more general vector fields.

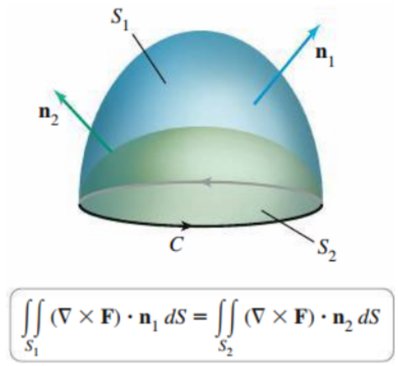
Example. Consider the vector field $\mathbf{v} = \langle 0,\, 1 - x^2,\, 0 \rangle$ for $|x| \le 1$ and $|z| \le 1$. Compute the curl of $\mathbf{v}$.

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 1-x^2 & 0 \end{vmatrix} = \langle 0, 0, -2x \rangle$$



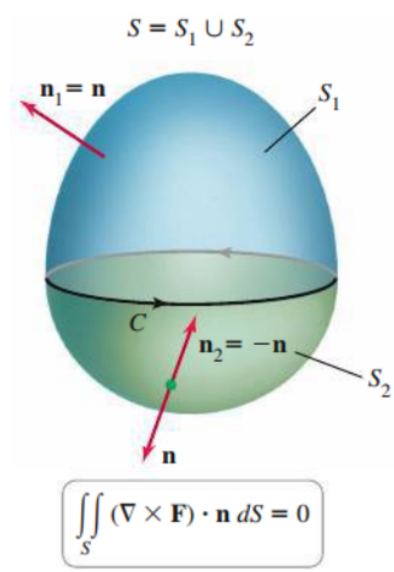
Since, using Stokes' Theorem, we evaluate the surface integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ using only the boundary C, then for any two smooth oriented surfaces S_1 and S_2 both with a consistent orientation with that of C,

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_1\, dS = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2\, dS$$



Furthermore, if S is a closed surface consisting of S_1 and S_2, with $\mathbf{n} = \mathbf{n}_1$ and $\mathbf{n} = -n_2$, then

$$\iint_{S} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS + \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = 0$$



Theorem 17.11 (Section 17.5) states that if $\mathbf{F}$ is conservative, then $\nabla \times \mathbf{F} = \mathbf{0}$. Now, the converse follows using Stokes' Theorem:

Theorem 17.16: Curl $\mathbf{F} = \mathbf{0}$ implies $\mathbf{F}$ Is Conservative
Suppose $\nabla \times \mathbf{F} = \mathbf{0}$ throughout an open simply connected region D of $\mathbb{R}^3$. Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all closed simple smooth curves C in D, and $\mathbf{F}$ is a conservative vector field on D.

Proof. Given a closed simple smooth curve C, it can be shown that C is the boundary of at least one smooth oriented surface S in D. By Stokes' Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \underbrace{(\nabla \times \mathbf{F})}_{0} \cdot \mathbf{n}\, dS = 0$$

Since the line integral equals zero over all such curves in D, the vector field is conservative on D. □