8.5: Partial Fractions

Example. Simplify $f(x) = \frac{1}{x-2}\left(\frac{x+4}{x+4}\right) + \frac{2}{x+4}\left(\frac{x-2}{x-2}\right)$ by finding a common denominator.

$$f(x) = \frac{1(x+4)+2(x-2)}{(x-2)(x+4)} = \frac{3x}{x^2+2x-8}$$

Procedure: Partial Fractions with Simple Linear Factors
Suppose $f(x) = p(x)/q(x)$, where p and q are polynomials with no common factors and with the degree of P less than the degree of q. Assume q is the product of simple linear factors. The partial fraction decomposition is obtained as follows.

Step 1: Factor the denominator q in the form $(x-r_1)(x-r_2)\dots(x-r_n)$

Step 2: Partial fraction decomposition

$$\frac{p(x)}{q(x)} = \frac{A_1}{(x-r_1)} + \frac{A_2}{(x-r_2)} + \dots + \frac{A_n}{(x-r_n)}.$$

Step 3: Clear denominators Multiply both sides of the equation in Step 2 by $q(x) = (x-r_1)(x-r_2)\dots(x-r_n)$

Step 4: Solve for coefficients Equate like powers of x in Step 3 to solve for the undetermined coefficients $A_1, \dots, A_n$.

Example. Perform partial fraction decomposition on $f(x) = \frac{3x}{x^2+2x-8}$.

$$x^2+2x-8 = (x+4)(x-2)$$

$$\Rightarrow (x^2+2x-8)\frac{3x}{x^2+2x-8} = \left(\frac{A}{x+4} + \frac{B}{x-2}\right)(x^2+2x-8)$$

$$\Rightarrow 3x+0 = A(x-2)+B(x+4) = Ax-2A+Bx+4B = \underbrace{(A+B)}_{3}x + \underbrace{(-2A+4B)}_{0}$$

$$\boxed{A+B = 3} \quad (1)$$

$$-2A+4B = 0 \quad (2)$$

$\rightarrow 2(1)+(2)$

$$\begin{array}{r} 2A+2B=6 \\ +\ -2A+4B=0 \\ \hline 6B=6 \end{array} \longrightarrow B=1 \quad (1)\Rightarrow A+1=3, \quad A=2$$

$$\boxed{\frac{3x}{x^2+2x-8} = \frac{2}{x+4} + \frac{1}{x-2}}$$

Example. $\displaystyle\int \frac{28x^3 - 56x^2 + 9}{x^2 - 2x}\,dx$

$$x^2-2x \,\overline{\big)\, 28x^3 - 56x^2 + 0x + 9}^{\;28x}$$
$$\underline{-(28x^3 - 56x^2)}$$
$$0 \quad 9 \leftarrow \text{Remainder}$$

$$\int \frac{28x^3-56x^2+9}{x^2-2x}\,dx = \int 28x + \underbrace{\frac{9}{x^2-2x}}_{\text{PFD}}\,dx$$

$$(x^2-2x)\left(\frac{9}{x^2-2x}\right) = \left(\frac{A}{x} + \frac{B}{x-2}\right)(x^2-2x)$$

$$0x + 9 = A(x-2) + Bx = Ax - 2A + Bx$$

$$= \underbrace{(A+B)}_{0}x + \underbrace{(-2A)}_{9} \longrightarrow \begin{array}{l} -2A = 9 \\ A = -9/2 \end{array} \longrightarrow \begin{array}{l} -9/2 + B = 0 \\ B = 9/2 \end{array}$$

$$\int 28x + \frac{9}{x^2-2x}\,dx = \int 28x - \frac{9}{2x} + \frac{9}{2(x-2)}\,dx$$

$$= 14x^2 - \frac{9}{2}\ln|x| + \frac{9}{2}\ln|x-2| + C$$

$$\frac{\quad}{x^4(x+1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x+1} + \frac{F}{(x+1)^2} + \frac{G}{(x+1)^3}$$

Procedure: Partial Fractions for Repeated Linear Factors
Suppose the repeated linear factor $(x-r)^m$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition has a partial fraction for each power of $(x-r)$ up to and including the mth power; that is, the partial fraction decomposition contains the sum

$$\frac{A_1}{(x-r)} + \frac{A_3}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}$$

where $A_1, \ldots, A_m$ are constants to be determined.

Example. Setup the partial fraction decomposition for $f(x) = \dfrac{x^3 - 8x + 19}{x^4 + 3x^3}$.

$$x^4 + 3x^3 = x^3(x+3)$$

$$\frac{x^3 - 8x + 19}{x^4 + 3x^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+3}$$

Example. Setup the partial fraction decomposition for $g(x) = \dfrac{2}{x^5 - 6x^4 + 9x^3}$.

$$x^5 - 6x^4 + 9x^3 = x^3(x^2 - 6x + 9) = x^3(x-3)^2$$

$$\frac{2}{x^5 - 6x^4 + 9x^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-3} + \frac{E}{(x-3)^2}$$

Example. Evaluate $\displaystyle\int \frac{\overbrace{x^2+1}^{\text{quadratic}}}{\underbrace{(2x-3)(x-2)^2}_{\text{cubic}}}\,dx.$

$$(2x-3)(x-2)^2\left(\frac{x^2+1}{(2x-3)(x-2)^2}\right) = \left(\frac{A}{2x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}\right)(2x-3)(x-2)^2$$

$$\underbrace{x^2+0x+1}_{x=2} = A(x-2)^2 + B(2x-3)(x-2) + C(2x-3)$$

$$= A(x^2-4x+4) + B(2x^2-7x+6) + C(2x-3)$$

$$= (\underbrace{A+2B}_{1})x^2 + (\underbrace{-4A-7B+2C}_{0})x + (\underbrace{4A+6B-3C}_{1})$$

$$\begin{aligned} A + 2B \quad\quad &= 1 \\ -4A - 7B + 2C &= 0 \\ 4A + 6B - 3C &= 1 \end{aligned}$$

Alternatively

$x=2$, $x=3/2$ are roots

When $x=2$

$x^2+1=5$ $\quad A\cancel{(x-2)^2} + B\cancel{(2x-3)(x-2)} + C(2x-3) = C(4-3) = C$

$\Rightarrow C=5$

when $x=3/2$

$x^2+1 = \left(\frac{3}{2}\right)^2+1 = \frac{13}{4}$ $\quad A(x-2)^2 + B\cancel{(2x-3)}(x-2) + C\cancel{(2x-3)} = A\left(\frac{3}{2}-2\right)^2 = \frac{A}{4}$

$\Rightarrow A=13$

$$\int \frac{x^2+1}{(2x-3)(x-2)^2}\,dx = \frac{13}{2}\ln|2x-3| - 6\ln|x-2| - \frac{5}{x-2} + C$$

$\left.\begin{aligned} A + 2B &= 1 \\ A &= 13 \end{aligned}\right\} \Rightarrow$ $\quad 13 + 2B = 1$

$2B = -12$

$\boxed{B=-6}$

$$\frac{A}{2x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$= \frac{13}{2x-3} - \frac{6}{x-2} + \frac{5}{(x-2)^2}$$

Example. Evaluate $\displaystyle\int \frac{8}{3x^3+7x^2+4x}\,dx$.

$$\frac{8}{3x^3+7x^2+4x} = \frac{A}{x} + \frac{B}{3x+4} + \frac{C}{x+1}$$

$$\Rightarrow 0x^2+0x+8 = A(3x+4)(x+1) + Bx(x+1) + Cx(3x+4)$$

$$= A(3x^2+7x+4) + B(x^2+x) + C(3x^2+4x)$$

$$= \underbrace{(3A+B+3C)}_{0}x^2 + \underbrace{(7A+B+4C)}_{0}x + \underbrace{(4A)}_{8}$$

$$3A + B + 3C = 0$$
$$7A + B + 4C = 0$$
$$4A = 8 \rightarrow A = 2$$

When $x=-1$,

$$\left.\begin{array}{l} 8 = 8 \\ A(3x+4)(x+1) + Bx(x+1) + Cx(3x+4) = -C \end{array}\right\} \Rightarrow C = -8$$

When $x = -4/3$

$$\left.\begin{array}{l} 8 = 8 \\ A(3x+4)(x+1) + Bx(x+1) + Cx(3x+4) = \frac{4}{9}B \end{array}\right\} \Rightarrow B = 18$$

$$\int \frac{8}{3x^3+7x^2+4x}\,dx = \int \frac{2}{x} + \frac{18}{3x+4} - \frac{8}{x+1}\,dx = \boxed{2\ln|x| + 6\ln|3x+4| - 8\ln|x+1| + C}$$

Procedure: Partial Fractions with Simple Irreducible Quadratic Factors
Suppose a simple irreducible factor ax^2+bx+c appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition contains a term of the form

$$\frac{Ax+B}{ax^2+bx+c},$$

$u = ax^2+bx+c$

$du = 2ax+b \quad dx$

where A and B are unknown coefficients to be determined.

Example. Perform partial fraction decomposition on the following fractions or identify them as irreducible.

$$\frac{1}{x^2-13x+43}$$

$a \quad b \quad c$

roots: $x = \frac{13 \pm \sqrt{13^2 - 4(1)(43)}}{2} = \frac{13 \pm \sqrt{169-172}}{2}$ neg

$\Rightarrow$ irreducible

b^2-4ac : determinant

If $b^2-4ac<0$, then the eqn is irreducible

$$\frac{x^2}{(x-4)(x+5)} = \frac{x^2}{x^2+x-20}$$

① Factorable denom

② do poly long div first

$$\begin{array}{r} 1 \\ x^2 \enclose{longdiv}{x^2+x-20} \\ -(x^2) \\ \hline x-20 \end{array}$$

$$\frac{x^2}{(x-4)(x+5)} = 1 + \frac{x-20}{(x-4)(x+5)} = 1 + \frac{A}{x-4} + \frac{B}{x+5}$$

Example. Perform partial fraction decomposition on the following fractions or identify them as irreducible.

$$\frac{7}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$$\frac{1}{x^2+11x+28}$$

Determinant:

$$b^2-4ac = 11^2-4(1)(28)$$

$$= 121 - 112 > 0 \rightarrow \underline{\underline{\text{is}}} \text{ reducible}$$

$$= \frac{A}{(x+4} + \frac{B}{(x+7)}$$

Example. Evaluate $\displaystyle\int \frac{4x}{(x+1)(x^2+1)}\,dx$

$$\frac{4x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

irreducible quadratic factor

$$0x^2 + 4x + 0 = A(x^2+1) + (Bx+C)(x+1)$$
$$= Ax^2 + A + Bx^2 + Bx + Cx + C$$
$$= \underbrace{(A+B)}_{0}x^2 + \underbrace{(B+C)}_{4}x + \underbrace{(A+C)}_{0}$$

$\cancel{(a+b)^2 = a^2 + b^2}$

$\cancel{\sqrt{a^2 - x^2} = a - x}$

$$A + B = 0 \longrightarrow B = 2$$
$$B + C = 4 \longrightarrow C = 2$$
$$A + C = 0$$

$$A + B = A + C \Rightarrow B + C \Rightarrow \begin{array}{l} B+C=4 \\ 2B=4 \\ B=2 \end{array}$$

when $x = -1$

$$\left.\begin{array}{r} 4x = -4 \\ A(x^2+1) + (Bx+C)(x+1) = 2A \end{array}\right\} A = -2$$

$$\frac{4x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{-2}{x+1} + \frac{2x+2}{x^2+1}$$

$$\int \frac{4x}{(x+1)(x^2+1)}\,dx = \int \frac{-2}{x+1}\,dx + \int \frac{2x+2}{x^2+1}\,dx \longrightarrow \int \frac{2x}{x^2+1}\,dx + \int \frac{2}{x^2+1}\,dx$$

$u = x^2+1$
$du = 2x\,dx$

$$= -2\ln|x+1| + \int \frac{du}{u} + \int \frac{2}{x^2+1}\,dx$$

$$= -2\ln|x+1| + \ln|u| + 2\tan^{-1}(x^2+1) + C$$

$$= \boxed{-2\ln|x+1| + \ln|x^2+1| + 2\tan^{-1}(x) + C}$$

Example. Evaluate $\int \frac{3x^2+2x+12}{(x^2+4)^2}\,dx$

Repeated Irreducible Quadratic Factor

$$\frac{3x^2+2x+12}{(x^2+4)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$$

$$0x^3 + 3x^2+2x+12 = (Ax+B)(x^2+4) + (Cx+D)$$

$$= Ax^3 + 4Ax + Bx^2 + 4B + Cx + D$$

$$= \underset{\substack{\uparrow\\0}}{(A)}x^3 + \underset{\substack{\uparrow\\3}}{(B)}x^2 + \underset{\substack{\uparrow\\2}}{(4A+C)}x + \underset{\substack{\uparrow\\12}}{(4B+D)}$$

$$\left.\begin{aligned} &\boxed{A \qquad\qquad = 0}\\ &\quad\boxed{B \qquad = 3}\\ &4A \qquad +C \quad = 2\\ &\quad 4B \qquad +D = 12 \end{aligned}\right\}$$

$C = 2$

$12 + D = 12 \to D = 0$

$$\frac{3x^2+2x+12}{(x^2+4)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2} = \frac{3}{x^2+4} + \frac{2x}{(x^2+4)^2}$$

$$\int \frac{3x^2+2x+12}{(x^2+4)^2}\,dx = 3\int \frac{1}{x^2+4}\,dx + \int \frac{2x}{(x^2+4)^2}\,dx$$

$u = x^2+4$

$du = 2x\,dx$

$$= \frac{3}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx + \int u^{-2}\,du$$

$$= \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) - u^{-1} + C$$

$$= \boxed{\frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{x^2+4} + C}$$

Example. Evaluate $\int \frac{1}{x\sqrt{1+2x}}\,dx$ using the substitution $u = \sqrt{1+2x}$. $= (1+2x)^{1/2}$

$$du = \frac{1}{2}(1+2x)^{-1/2}(2)\,dx$$

$$= \frac{1}{\sqrt{1+2x}}\,dx$$

$$\int \frac{1}{x\sqrt{1+2x}}\,dx = \int \frac{2}{u^2-1}\,du$$

$$u = \sqrt{1+2x}$$

$$u^2 = 1+2x$$

$$x = \frac{u^2-1}{2}$$

Alternatively:

let $u = \sec\theta$

$du = \sec\theta\tan\theta\,d\theta$

$$\frac{2}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1}$$

$$2 = A(u-1) + B(u+1)$$

$$= \underbrace{(A+B)}_{0}u + \underbrace{(-A+B)}_{2}$$

$$\begin{array}{rl} & A+B=0 \\ + & -A+B=2 \\ \hline & 2B=2 \to B=1 \\ & A=-1 \end{array}$$

$$\int \frac{2}{u^2-1}\,dx = \int \frac{-1}{u+1} + \frac{1}{u-1}\,du = -\ln|u+1| + \ln|u-1| + C$$

$$= -\ln\left|1+\sqrt{2x+1}\right| + \ln\left|-1+\sqrt{2x+1}\right| + C$$

$$\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$$

$$= \boxed{\ln\left|\frac{-1+\sqrt{2x+1}}{1+\sqrt{2x+1}}\right| + C}$$

Summary: Partial Fraction Decomposition
Let $f(x) = p(x)/q(x)$ be a proper rational function in reduced form. Assume the denominator q has been factored completely over the real numbers and m is a positive integer.

1. **Simple linear factor:** A factor $x - r$ in the denominator requires the partial fraction $\dfrac{A}{x-r}$.
2. **Repeated linear factor:** A factor $(x-r)^m$ with $m > 1$ in the denominator requires the partial fractions
$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}.$$
3. **Simple irreducible quadratic factor:** An irreducible factor $ax^2 + bx + c$ in the denominator requires the partial fraction
$$\frac{Ax+B}{ax^2+bx+c}.$$
4. **Repeated irreducible quadratic factor:** An irreducible factor $(ax^2+bx+c)^m$ with $m > 1$ in the denominator requires the partial fractions
$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}.$$