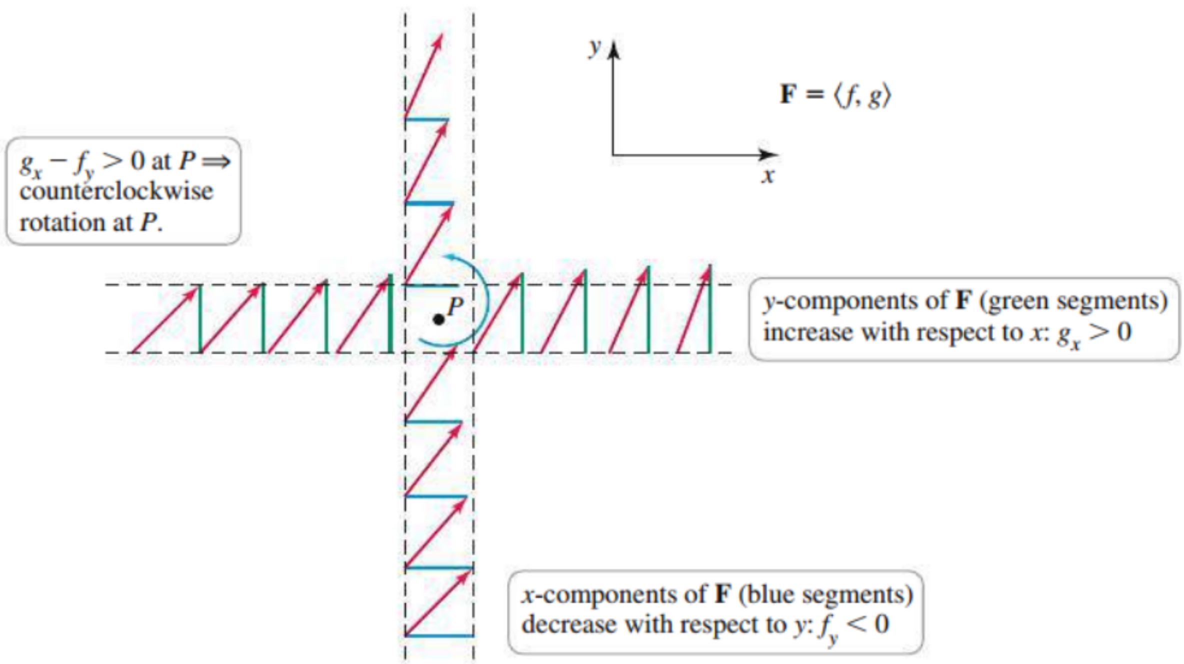
17.4: Green's Theorem

Green's Theorem — Circulation Form

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

line integral

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{circulation}} = \underbrace{\oint_C f\,dx + g\,dy}_{\text{circulation}} = \underbrace{\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA}_{\text{over region}}.$$



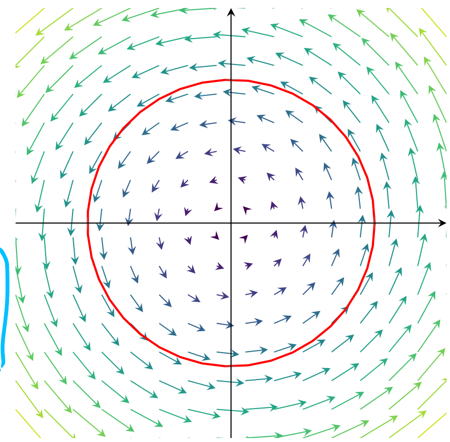
Definition. (Two-Dimensional Curl)

The **two-dimensional curl** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}$. If the curl is zero throughout a region, the vector field is **irrotational** on the region.

Example. Consider the following vector fields $\mathbf{F}$ over the region $R = \{(x, y) : x^2 + y^2 \leq 1\}$. Compute the circulation using Green's Theorem.

$$\mathbf{F} = \langle -y, x\rangle = \langle f, g\rangle$$

$$\text{curl} = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 1 - (-1) = 2$$



$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

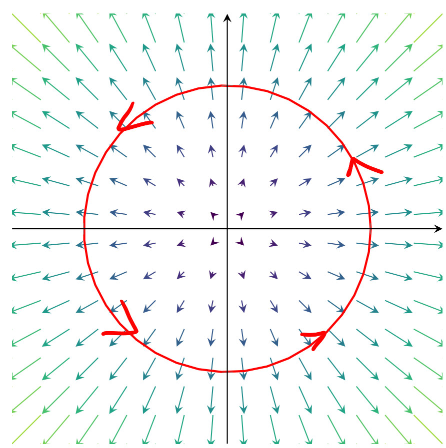
$$= \iint_R 2\,dA = 2\underbrace{\iint_R dA}_{\text{area of } R} = \boxed{2\pi}$$

LC #1

a=2

$$\mathbf{F} = \langle x, y\rangle = \langle f, g\rangle$$

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0 - 0 = 0$$



circulation $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \underbrace{\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)}_{0} dA$

$$= 0$$

LC #2

a=0

$\langle f, g \rangle$

Example. Compute the curl of $\mathbf{F} = \langle x^2, 2y^2 \rangle$ where C is the upper half of the unit circle and the line segment $-1 \leq x \leq 1$.

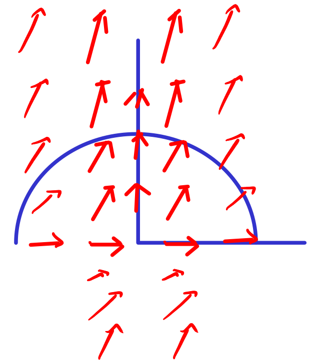
$$\text{curl} = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0 - 0 = 0$$

$\uparrow$

LC #3

0

$\Rightarrow \vec{F}$ irrotational



Area of a Plane Region by Line Integrals

Under the conditions of Green's Theorem, the area of a region R enclosed by a curve C is

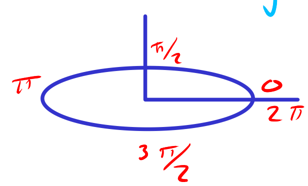
$$\oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C \underbrace{(x\,dy - y\,dx)}_{\langle x,y\rangle \cdot \langle dy, -dx\rangle}.$$

$\vec{F} = \langle 0, x \rangle$

$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 1$

$\oint \vec{F} \cdot d\vec{r} = \iint_R dA$

Example. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



$\vec{r}(t) = \langle a\cos(t), b\sin(t) \rangle \quad 0 \leq t \leq 2\pi$

$\vec{r}\,'(t) = \langle -a\sin(t), b\cos(t) \rangle$

$\langle dy, -dx \rangle$

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} \langle a\cos(t), b\sin(t)\rangle \cdot \langle b\cos(t), a\sin(t)\rangle\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} ab\big(\underbrace{\cos^2(t) + \sin^2(t)}_{1}\big)\,dt$$

$$= \frac{ab}{2}\int_0^{2\pi} dt = \frac{ab}{2}t\Big|_0^{2\pi} = ab\pi$$

LC #4

42

Circle $a = b = r$

$\Rightarrow A = \pi r^2$

Circulation

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

Green's Theorem — Flux Form

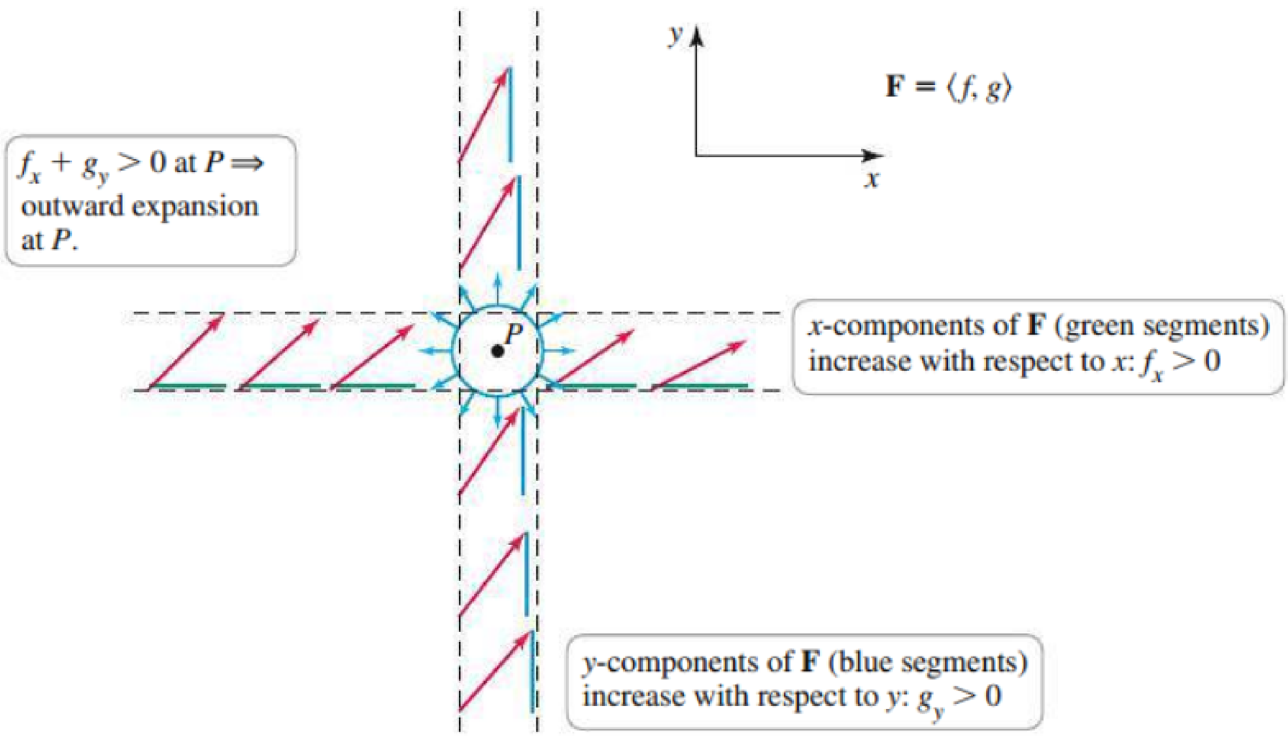
Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g\rangle$, where f and g have continuous first partial derivatives in R. Then

$\langle f,g\rangle\cdot\langle dy,-dx\rangle$

$$\underbrace{\oint_C \mathbf{F}\cdot\mathbf{n}\,ds}_{\text{outward flux}} = \underbrace{\oint_C f\,dy - g\,dx}_{\text{outward flux}} = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA,$$

divergence

where $\mathbf{n}$ is the outward unit normal vector on the curve.

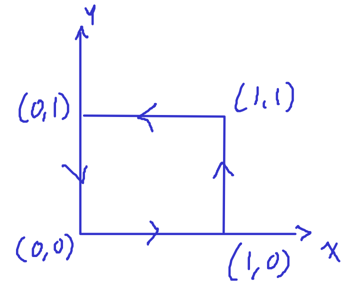


Definition. (Two-Dimensional Divergence)

The **two-dimensional divergence** of the vector field $\mathbf{F} = \langle f, g\rangle$ is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y}$. If the divergence is zero throughout a region, the vector field is **source free** on that region.

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C f\,dy - g\,dx = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA$$

Example. Integrate $\oint_C \Big(\underbrace{2x + e^{y^2}}_{f}\Big)\,dy - \Big(\underbrace{4y^2 + e^{x^2}}_{g}\Big)\,dx$, where C is the boundary of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$.



$$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA$$

$$= \int_0^1\int_0^1 2 + 8y \;\; dy\,dx$$

$$= \int_0^1 2y + 4y^2\Big|_{y=0}^{y=1} dx$$

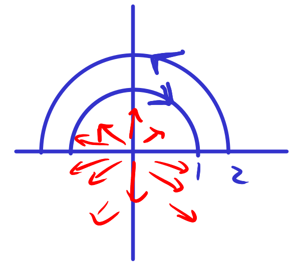
$$= \int_0^1 6\,dx = 6x\Big|_{x=0}^{x=1} = \boxed{6}$$

LC #5

6

Example. Compute the circulation and outward flux across the boundary of the given regions:

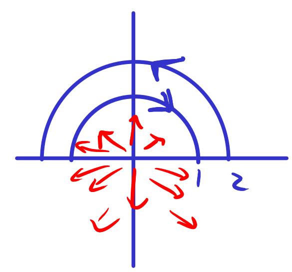
$\mathbf{F} = \langle x, y\rangle$; R is the half-annulus $\{(r,\theta) : 1 \le r \le 2, 0 \le \theta \le \pi\}$,



$$\text{circulation} = \oint_C \vec{F}\cdot d\vec{r} = \oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

$$= \iint_R 0 - 0 \, dA = 0$$

$\boxed{\begin{array}{l} LC\#6 \\ a=0 \end{array}}$



$$\text{flux} = \oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C f\,dy - g\,dx = \iint_R \Big(\underbrace{\frac{\partial f}{\partial x}}_{1} + \underbrace{\frac{\partial g}{\partial y}}_{1}\Big) dA$$

$$= \int_0^{\pi}\int_1^2 2r\,dr\,d\theta$$

$$= \int_0^{\pi} r^2\Big|_{r=1}^{r=2} d\theta$$

$$= 3\int_0^{\pi} d\theta = 3\theta\Big|_{\theta=0}^{\theta=\pi} = \boxed{3\pi}$$

$\boxed{\begin{array}{l} LC\#7 \\ a=3 \end{array}}$

circulation

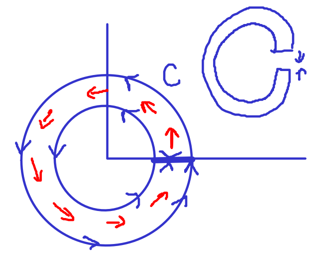
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

Flux

$$\oint \vec{F}\circ\vec{n}\, ds = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA$$

$\mathbf{F} = \langle -y, x\rangle$; R is the annulus $\{(r,\theta) : 1 \le r \le 3, 0 \le \theta \le 2\pi\}$.

$$\frac{\partial g}{\partial x} = 1 \qquad \frac{\partial f}{\partial y} = -1$$



circulation $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \int_0^{2\pi}\int_1^3 \underbrace{(1-(-1))}_{2}\, r\, dr\, d\theta$

$\vec{F} = \langle -y, x\rangle$

$$= \int_0^{2\pi} r^2 \Big|_{r=1}^{r=3} d\theta = \int_0^{2\pi} 8\, d\theta = 8\theta \Big|_{\theta=0}^{\theta=2\pi} = \boxed{16\pi}$$

LC #8
16

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flux $\oint \vec{F}\circ\vec{n}\, ds = \iint_R \underbrace{\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)}_{\text{divergence}} dA \quad \iint_R 0 + 0\, dA = 0$

LC #9
0

$\int_C \vec{F} \cdot d\vec{r} = \varphi(B) - \varphi(A)$ $\qquad$ $\vec{F} = \nabla \varphi$

Stream functions:

In the same way that a vector field is conservative if there exists a potential function φ, a vector field is source free if a **stream function** ψ exists such that

phi

$F = \langle \psi_y, -\psi_x \rangle$

$$\frac{\partial \psi}{\partial y} = f, \qquad \frac{\partial \psi}{\partial x} = -g.$$

$\int_C \vec{F} \cdot \vec{n}\, ds = \psi(B) - \psi(A)$

If such a function exists, then the divergence is zero:

$F = \langle f, g \rangle$

$\psi = \int f\, dy$

$-\int g\, dx$

$$\underbrace{\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}}_{\text{divergence}} = \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)}_{\psi_{yx} = \psi_{xy}} = 0$$

If a vector field is both conservative and source-free, then it has both a potential function and a stream function. Furthermore, the level curves of the potential and stream functions form orthogonal families. These vector fields have zero divergence

$$0 = f_x + g_y = \varphi_{xx} + \varphi_{yy},$$

and zero curl

$$0 = g_x - f_y = -\psi_{xx} - \psi_{yy}.$$

Thus, conservative, source-free vector fields satisfy **Laplace's equation**:

$\Delta \psi = 0$

$$\varphi_{xx} + \varphi_{yy} = 0 \qquad \text{and} \qquad \psi_{xx} + \psi_{yy} = 0.$$

$\nabla^2 \psi = 0$

Example. For $\mathbf{F} = \langle -\underline{e^{-x}\sin(y)},\ \underline{e^{-x}}\cos(y)\rangle$

Show $\mathbf{F}$ is conservative and source-free field

conservative

$\Rightarrow \vec{F} = \nabla\varphi \quad \Rightarrow (\varphi_x)_y = (\varphi_y)_x$

$\langle f, g\rangle = \langle \varphi_x, \varphi_y\rangle \qquad f_y = g_x$

Source free

$\Rightarrow \vec{F} = \langle \psi_y, -\psi_x\rangle \qquad (\psi_x)_y = (\psi_y)_x$

$\quad //$

$\langle f, g\rangle \qquad\qquad -g_y = f_x$

$$\left.\begin{aligned} f_y &= -e^{-x}\cos(y) \\ g_x &= -e^{-x}\cos(y) \end{aligned}\right\} \text{conservative}$$

$$\left.\begin{aligned} f_x &= e^{-x}\sin(y) \\ -g_y &= -\left(-e^{-x}\sin(y)\right) \end{aligned}\right\} \text{source free}$$

Find the potential function φ and the stream function ψ

$$\varphi(x,y) = \int \varphi_x\,dx = \underline{e^{-x}\sin(y) + C(y)}$$

$$\Rightarrow \varphi_y = \frac{\partial}{\partial y}\left(\underline{e^{-x}\sin(y) + C(y)}\right) = e^{-x}\cos(x) + C_y(y) = \overbrace{e^{-x}\cos(x)}^{g}$$

$$\Rightarrow C_y(y) = 0 \Rightarrow \int C_y(y)\,dy = 0$$

$$\Rightarrow \boxed{\varphi(x,y) = e^{-x}\sin(y)}$$

$$\psi(x,y) = \int \psi_x\,dx = \int -g\,dx = \underline{e^{-x}\cos(y) + c(y)}$$

$$\Rightarrow \psi_y = \frac{\partial}{\partial y}\left(\underline{e^{-x}\cos(y) + c(y)}\right) = -e^{-x}\sin(y) + c_y(y) = \overbrace{-e^{-x}\sin(y)}^{f}$$

$$\Rightarrow c_y(y) = 0 \quad \Rightarrow c(y) = \int c_y(y)\,dy = 0$$

$$\Rightarrow \boxed{\psi(x,y) = e^{-x}\cos(y)}$$

Conservative Fields $\mathbf{F} = \langle f, g \rangle$	**Source-Free Fields $\mathbf{F} = \langle f, g \rangle$**
$\text{curl} = \dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 0$	$\text{divergence} = \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 0$
Potential function φ with $\mathbf{F} = \nabla\varphi \quad \text{or} \quad f = \dfrac{\partial \varphi}{\partial x}, \quad g = \dfrac{\partial \varphi}{\partial y}$	Stream function ψ with $f = \dfrac{\partial \psi}{\partial y}, \qquad g = -\dfrac{\partial \psi}{\partial x}$
Circulation $= \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all closed curves C.	Flux $= \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = 0$ on all closed curves C.
Evaluation of the line integral $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$	Evaluation of the line integral $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \psi(B) - \psi(A)$

Circulation/work integrals: $\displaystyle\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C f\,dx + g\,dy$

	C **closed**	C **not closed**
F conservative ($\mathbf{F} = \nabla\varphi$)	$\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$	$\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(B) - \varphi(A)$
F not conservative	Green's Theorem $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R (g_x - f_y)\,dA$	Direct evaluation $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b (fx' + gy')\,dt$

Flux integrals: $\displaystyle\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_C f\,dy - g\,dx$

	C **closed**	C **not closed**
F source free ($f = \psi_y, g = -\psi_x$)	$\displaystyle\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = 0$	$\displaystyle\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \psi(B) - \psi(A)$
F not source free	Green's Theorem $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R (f_x + g_y)\,dA$	Direct evaluation $\displaystyle\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_a^b (fy' - gx')\,dt$

Example. Suppose C is a circle centered at the origin, oriented counterclockwise, that encloses disk R in the plane. For $\mathbf{F} = \left\langle 4x^3y, xy^2 + x^4 \right\rangle$

a) Calculate the two-dimensional curl of $\mathbf{F}$

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = (y^2 + 4x^3) - (4x^3) = y^2$$

b) Calculate the two-dimensional divergence of $\mathbf{F}$

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = (12x^2y) + (2xy) = 2xy(6x+1)$$

c) Is $\mathbf{F}$ irrotational on R?

No, since curl is non-zero.

d) Is $\mathbf{F}$ source free on R?

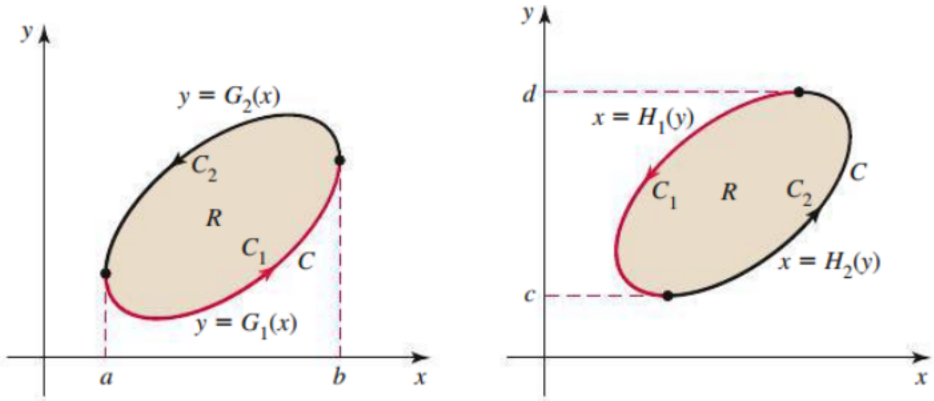
No, since divergence is non-zero.

Proof. Consider the regions R enclosed by a simple closed smooth curve C oriented in a counterclockwise direction, given by

$$R = \{(x, y) : a \leq x \leq b, G_1(x) \leq y \leq G_2(x)\}$$

or

$$R = \{(x, y) : H_1(y) \leq x \leq H_2(y), c \leq y \leq d\}.$$



To prove the circulation form of Green's Theorem, we have

$$\begin{aligned}
&\iint\limits_R \frac{\partial f}{\partial y}\, dA \\
&= \int_a^b \int_{G_1(x)}^{G_2(x)} \frac{\partial f}{\partial y}\, dy\, dx \\
&= \int_a^b \big(\underbrace{f(x, G_2(x))}_{\text{on } C_2} - \underbrace{f(x, G_1(x))}_{\text{on } C_1} \big)\, dx \\
&= \int_{-C_2} f\, dx - \int_{C_1} f\, dx \\
&= -\int_{C_2} f\, dx - \int_{C_1} f\, dx \\
&= -\oint_C f\, dx
\end{aligned}$$

$$\begin{aligned}
&\iint\limits_R \frac{\partial g}{\partial x}\, dA \\
&= \int_c^d \int_{H_1(y)}^{H_2(y)} \frac{\partial g}{\partial x}\, dx\, dy \\
&= \int_c^d \big(\underbrace{g(H_2(y), y)}_{C_2} - \underbrace{f(H_1(y), y)}_{-C_1} \big)\, dy \\
&= \int_{C_2} g\, dy - \int_{-C_1} g\, dy \\
&= \int_{C_2} g\, dy + \int_{C_1} g\, dy \\
&= \oint_C g\, dy
\end{aligned}$$

$\square$