

8.9: Improper Integrals

Definition. (Improper Integrals over Infinite Intervals)

1. If f is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx.$$

2. If f is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx.$$

3. If f is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{a \rightarrow -\infty} \int_a^c f(x) dx + \lim_{b \rightarrow \infty} \int_c^b f(x) dx.$$

where c is any real number. It can be shown that the choice of c does not affect the value or convergence of the original integral.

If the limits in cases 1.– 3. exist, then the improper integrals **converge**; otherwise they **diverge**.

Example. Evaluate $\int_1^{\infty} \frac{\ln(x)}{x} dx$ and determine if the integral converges or diverges.

$$= \lim_{b \rightarrow \infty} \int_1^b \frac{\ln(x)}{x} dx$$

$$u = \ln(x)$$

$$du = \frac{1}{x} dx$$

$$x = 1, u = 0$$

$$x = b, u = \ln(b)$$

$$= \lim_{b \rightarrow \infty} \int_0^{\ln(b)} u du = \lim_{b \rightarrow \infty} \left. \frac{u^2}{2} \right|_0^{\ln(b)} = \lim_{b \rightarrow \infty} \frac{(\ln(b))^2}{2} = \infty$$

Diverges

Example. Evaluate $\int_{-\infty}^{\infty} \frac{e^{3x}}{1+e^{6x}} dx$. arbitrary (e.g. $c=0$)

$$\lim_{a \rightarrow -\infty} \int_a^c \frac{e^{3x}}{1+e^{6x}} dx + \lim_{b \rightarrow \infty} \int_c^b \frac{e^{3x}}{1+e^{6x}} dx$$

$$\begin{aligned} u &= e^{3x} \\ du &= 3e^{3x} dx \\ \frac{du}{3} &= e^{3x} dx \end{aligned} \quad \begin{aligned} x=a, u &= e^{3a} \\ x=c, u &= e^{3c} \\ x=b, u &= e^{3b} \end{aligned} \quad c=0 \rightarrow u=1$$

$$= \lim_{a \rightarrow -\infty} \frac{1}{3} \int_{e^{3a}}^1 \frac{du}{1+u^2} + \lim_{b \rightarrow \infty} \frac{1}{3} \int_1^{e^{3b}} \frac{du}{1+u^2}$$

$$= \lim_{a \rightarrow -\infty} \frac{1}{3} \tan^{-1}(u) \Big|_{e^{3a}}^1 + \lim_{b \rightarrow \infty} \frac{1}{3} \tan^{-1}(u) \Big|_1^{e^{3b}}$$

$$= \lim_{a \rightarrow -\infty} \frac{1}{3} \left(\tan^{-1}(1) - \underbrace{\tan^{-1}(e^{3a})}_0 \right) + \lim_{b \rightarrow \infty} \frac{1}{3} \left(\underbrace{\tan^{-1}(e^{3b})}_{\pi/2} - \tan^{-1}(1) \right)$$

$$= \boxed{\frac{\pi}{6}} \rightarrow \boxed{\text{converges}}$$

Example. For what values of p does $\int_1^{\infty} \frac{1}{x^p} dx$ converge?

$$\lim_{b \rightarrow \infty} \int_1^b x^{-p} dx \stackrel{p \neq 1}{=} \lim_{b \rightarrow \infty} \left. \frac{x^{1-p}}{1-p} \right|_1^b = \lim_{b \rightarrow \infty} \frac{b^{1-p} - 1}{1-p}$$

$$p < 1, \quad 1-p > 0$$

$$\lim_{b \rightarrow \infty} \frac{1}{1-p} \left(\underset{\uparrow}{b^{1-p}} - 1 \right) = \infty$$

$$p > 1, \quad 1-p < 0$$

$$\lim_{b \rightarrow \infty} \frac{1}{1-p} \left(\underset{\uparrow}{b^{1-p}} - 1 \right) = \frac{-1}{1-p} = \frac{1}{p-1}$$

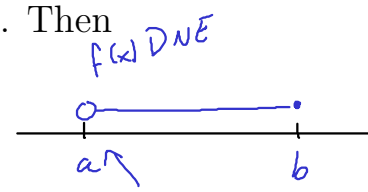
$$p = 1$$

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x} dx = \lim_{b \rightarrow \infty} \left. \ln|x| \right|_1^b \\ &= \lim_{b \rightarrow \infty} \ln(b) = \infty \end{aligned}$$

Converges when $p > 1$

Definition. (Improper Integrals with Unbounded Integrand)

1. Suppose f is continuous on $(a, b]$ with $\lim_{x \rightarrow a^+} f(x) = \pm\infty$. Then

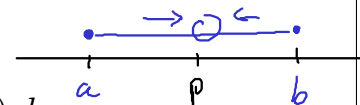
$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx.$$


2. Suppose f is continuous on $[a, b)$ with $\lim_{x \rightarrow b^-} f(x) = \pm\infty$. Then

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx.$$

3. Suppose f is continuous on $[a, b]$ except at the interior point p where f is unbounded. Then

$$\int_a^b f(x) dx = \lim_{c \rightarrow p^-} \int_a^c f(x) dx + \lim_{d \rightarrow p^+} \int_d^b f(x) dx.$$



If the limits in cases 1.– 3. exist, then the improper integrals **converge**; otherwise, they **diverge**.

Example. Determine which of the following integrals are improper integrals

$$\int_0^1 \sec(x) dx$$

Domain $\sec(x)$

Not Improper

$$\sec(x) = \frac{1}{\cos(x)}$$

$\sec(x)$ DNE at $x = \frac{\pi}{2} + n\pi$

$$\pi \approx 3.14$$

$$\frac{\pi}{2} \approx 1.57 \dots$$



$$\int_1^e \ln(x) dx$$

$x \in (0, \infty)$

Not improper

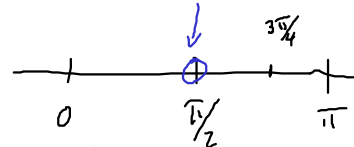


$$\int_{\pi/2}^{3\pi/4} \tan(x) dx$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

Domain $x \neq \frac{\pi}{2} + n\pi$

$\tan(x)$ DNE

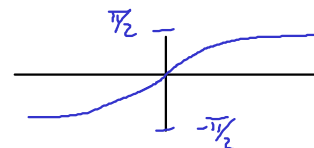


$$\lim_{a \rightarrow \pi/2^+} \int_a^{3\pi/4} \tan(x) dx$$

$$\int_0^1 \arctan(x) dx$$

Domain $(-\infty, \infty)$

Not improper



$$\int_0^{0.5} \ln(x) dx$$

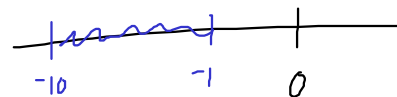
$x \in (0, \infty)$

Improper

$$\lim_{a \rightarrow 0^+} \int_a^{0.5} \ln(x) dx$$

$$\int_{-10}^{-1} \frac{1}{x^{1/3}} dx$$

Domain: $x \neq 0$

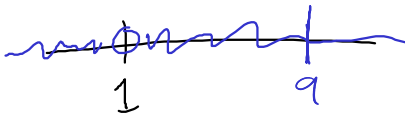


Not Improper

Example. Evaluate $\int_1^9 \frac{dx}{(x-1)^{2/3}}$. Does this integral converge or diverge?

$$= \lim_{a \rightarrow 1^+} \int_a^9 \frac{dx}{(x-1)^{2/3}}$$

$x \neq 1$



$$= \lim_{a \rightarrow 1^+} \int_{a-1}^8 u^{-2/3} du$$

$u = x-1$
 $du = dx$

$x = a, u = a-1$
 $x = 9, u = 8$

$$= \lim_{a \rightarrow 1^+} 3 u^{1/3} \Big|_{a-1}^8 = \lim_{a \rightarrow 1^+} 3 \left(8^{1/3} - \underbrace{(a-1)^{1/3}}_0 \right) = 3(2-0) = \boxed{6}$$

Converges

Example. Evaluate $\int_{-1}^1 \frac{e^{2/x}}{x^2} dx$. Does this integral converge or diverge?

$$x \neq 0 \rightarrow \lim_{c \rightarrow 0^-} \int_{-1}^c \frac{e^{2/x}}{x^2} dx + \lim_{d \rightarrow 0^+} \int_d^1 \frac{e^{2/x}}{x^2} dx$$

$$u = 2/x$$

$$du = -2/x^2 dx$$

$$x = -1, u = -2$$

$$x = c, u = 2/c$$

$$x = d, u = 2/d$$

$$x = 1, u = 2$$

$$= -\lim_{c \rightarrow 0^-} \frac{1}{2} \int_{-2}^{2/c} e^u du - \lim_{d \rightarrow 0^+} \frac{1}{2} \int_{2/d}^2 e^u du$$

$$= -\lim_{c \rightarrow 0^-} \left. \frac{1}{2} e^u \right|_{-2}^{2/c} - \lim_{d \rightarrow 0^+} \left. \frac{1}{2} e^u \right|_{2/d}^2$$

$$= -\frac{1}{2} \left[\lim_{c \rightarrow 0^-} \underbrace{e^{2/c}}_0 - e^{-2} \right] - \frac{1}{2} \left[\lim_{d \rightarrow 0^+} \underbrace{e^2}_\infty - e^{2/d} \right]$$

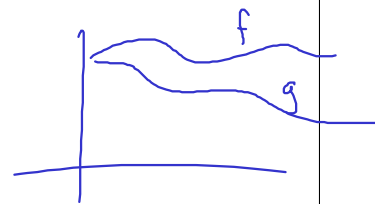
$$= \infty \rightarrow \text{diverges}$$

Theorem 8.2: Comparison Test for Improper Integrals

Suppose the functions f and g are continuous on the interval $[a, \infty)$, with $f(x) \geq g(x) \geq 0$, for $x \geq a$.

1. If $\int_a^\infty f(x) dx$ converges, then $\int_a^\infty g(x) dx$ converges.

2. If $\int_a^\infty g(x) dx$ diverges, then $\int_a^\infty f(x) dx$ diverges.



Example. Determine if the integral $\int_2^\infty \frac{x^3}{x^4 - x^3 - 1} dx$ converges or diverges.

$$f(x) = \frac{x^3}{x^4 - x^3 - 1} \geq 0 \quad g(x) = \frac{x^3}{x^4} = \frac{1}{x} \geq 0$$

which is larger

smaller denominator \rightarrow bigger fraction

$$\frac{1}{3} \text{ vs. } \frac{1}{2}$$

$$\frac{x^3}{x^4 - x^3 - 1} > \frac{1}{x}$$

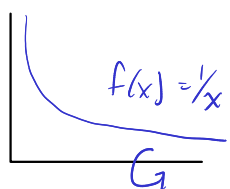
$$x^4 > x^4 - x^3 - 1$$

$$\int_2^\infty g(x) dx = \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x} dx = \lim_{b \rightarrow \infty} \ln|x| \Big|_2^b = \infty$$

$$\int_2^\infty g(x) dx \text{ diverges} \Rightarrow \int_2^\infty f(x) dx \text{ diverges}$$

Example (Gabriel's Horn). Let R be the region bounded by the graph of $y = 1/x$ and the x -axis for $x \geq 1$.

What is the volume of the solid generated when R is revolved around the x -axis?



$$\begin{aligned}
 V &= \int_1^{\infty} \pi \left(\frac{1}{x}\right)^2 dx = \lim_{b \rightarrow \infty} \pi \int_1^b x^{-2} dx \\
 &= \lim_{b \rightarrow \infty} \left. \frac{-\pi}{x} \right|_1^b \\
 &= \lim_{b \rightarrow \infty} \underbrace{\frac{-\pi}{b}}_0 - \frac{-\pi}{1} = \boxed{\pi}
 \end{aligned}$$

What is the surface area of the solid generated when R is revolved about the x -axis?

$$\begin{aligned}
 f(x) &= \frac{1}{x} \\
 f'(x) &= -\frac{1}{x^2} \\
 SA &= \int_1^{\infty} 2\pi \frac{1}{x} \sqrt{1 + \left(\frac{-1}{x^2}\right)^2} dx = \lim_{b \rightarrow \infty} 2\pi \int_1^b \frac{1}{x} \sqrt{\frac{x^2 + 1}{(x^2)^2}} dx \\
 &= \lim_{b \rightarrow \infty} 2\pi \int_1^b \frac{\sqrt{x^4 + 1}}{x^3} dx
 \end{aligned}$$

Note: $\sqrt{x^4 + 1} > \sqrt{x^4} = x^2$

$$\begin{aligned}
 &\geq \lim_{b \rightarrow \infty} 2\pi \int_1^b \frac{x^2}{x^3} dx = \lim_{b \rightarrow \infty} 2\pi \int_1^b \frac{1}{x} dx \\
 &= \lim_{b \rightarrow \infty} 2\pi \left| \ln(x) \right|_1^b = \infty
 \end{aligned}$$

Since the smaller function diverges then the surface area of this "horn" also diverges