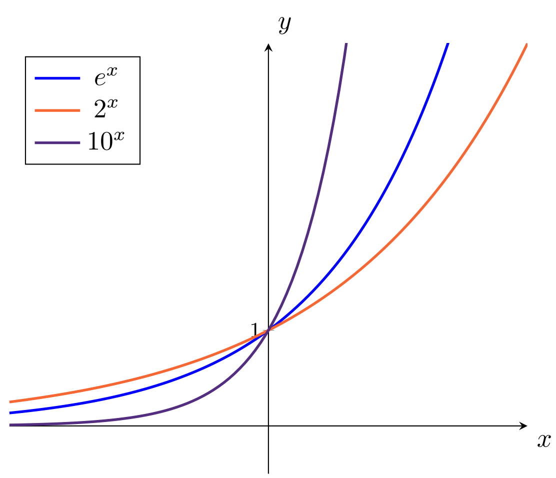
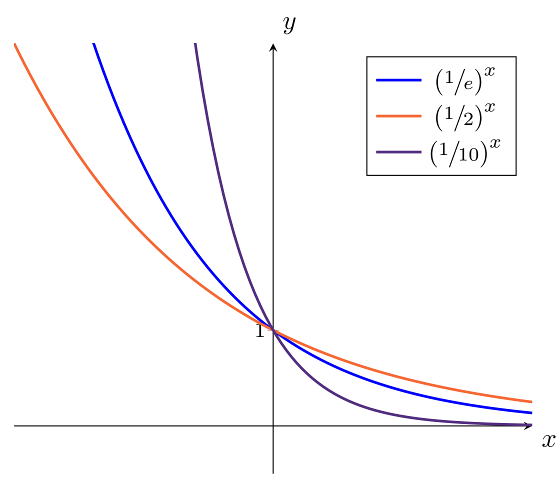
1 1.3: Inverse, Exponential and Logarithmic Functions

Definition.
An **exponential function** has the form

$$f(x) = b^x$$

where $b \neq 1$ is a positive real number. Exponential functions have a horizontal asymptote of $y =$ ____ and y-intercept of $(0, \_\_)$. When b is such that $0 < b < 1$, then $f(x)$ is ________________________________ and when $b > 1$, then $f(x)$ is ________________________________. Exponential functions have domain _______ and range _______.





Definition.
The **natural exponential function** is

$$f(x) = e^x.$$

where e is the irrational constant $e \approx 2.718281828459045\ldots$.

Laws of Exponents:
For $a > 0$, we have the following laws:

a) $a^{x+y} = a^x a^y$

b) $a^{x-y} = \frac{a^x}{a^y}$

c) $(a^x)^y = a^{xy}$

d) $(ab)^x = a^x b^x$

Example. For the following expressions, use the Laws of Exponents to simplify:

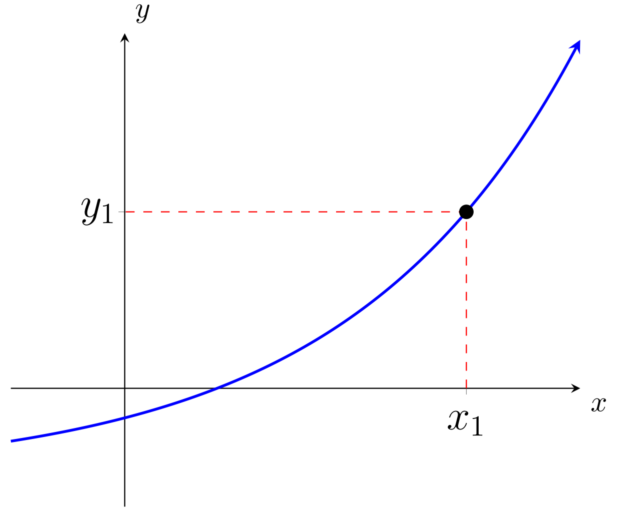
a) $\left(x^2 y^3\right)^5$

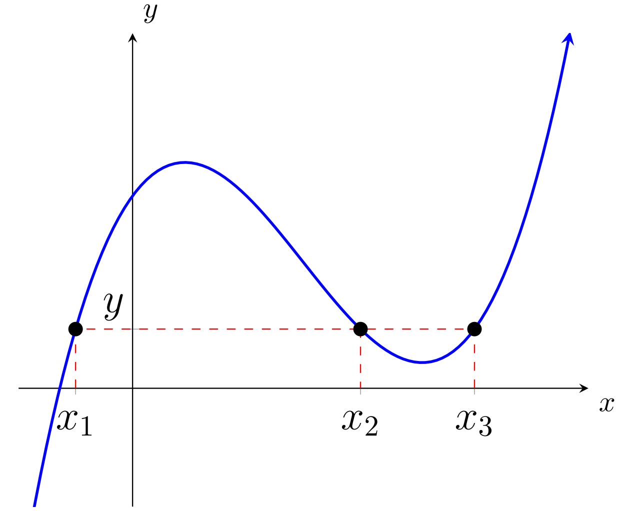
b) $\left(\sqrt{3}\right)^{1/2} \cdot \left(\sqrt{12}\right)^{1/2}$

c) $\left(\dfrac{x^{-2}}{x^8}\right)^{-2}$

d) $\left(\dfrac{-1}{27}\right)^{4/3}$

Definition. (One-to-One Functions and the Horizontal Line Test)
A function f is **one-to-one** on a domain D if each value of $f(x)$ corresponds to exactly one value of x in D. More precisely, f is one-to-one on d if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, for x_1 and x_2 in D.
The **horizontal line test** says that every horizontal line intersects the graph of a one-to-one function at most once.





Finding an Inverse Function
Suppose f is one-to-one on an interval I. To find f^{-1}, use the following steps:

1. Solve $y = f(x)$ for x. If necessary, choose the function that corresponds to I.
2. Interchange x and y and write $y = f^{-1}(x)$.

Example. Find $f^{-1}(x)$:

$f(x) = x^2 - 2x + 1,\ x \geq 1$

$g(x) = \frac{x}{2} - \frac{7}{2}$

$h(x) = \sqrt[3]{5x+1}$

$j(x) = \frac{2x}{1-x}$

$k(x) = e^x$

Existence of Inverse Functions

Let f be a one-to-one function on a domain D with a range R. Then f has a unique inverse f^{-1} with domain R and range D such that

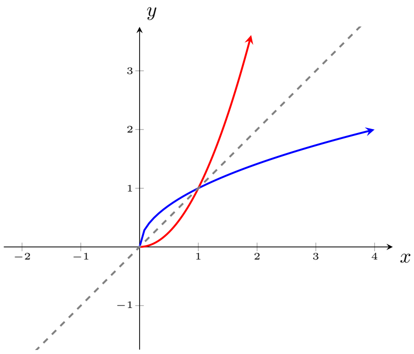
$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f\big(f^{-1}(y)\big) = y$$

where x is in D and y is in R.

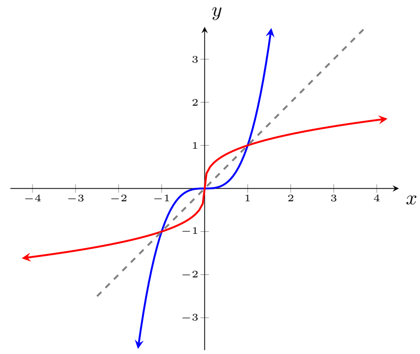
Example. For $f(x) = \sqrt[3]{4x-1} + 2$, show that $f^{-1}(f(x)) = f\big(f^{-1}(x)\big) = x$

Note: A function is symmetric with it's inverse with respect to $y = x$.

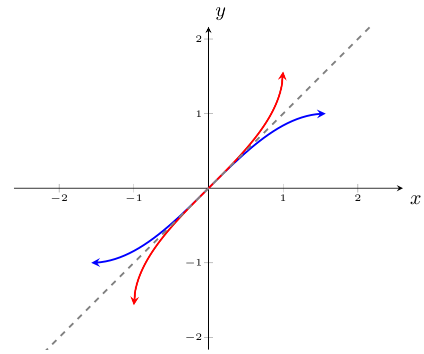
$f(x) = \sqrt{x}$
$f^{-1}(x) = x^2,\ x > 0$



$f(x) = x^3$
$f^{-1}(x) = \sqrt[3]{x} = x^{1/3}$

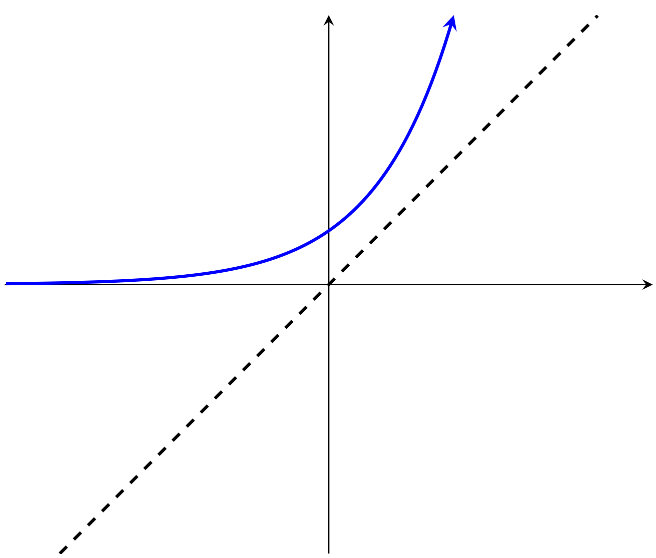


$f(x) = \sin x$ on $[-\pi/2, \pi/2]$
$f^{-1}(x) = \sin^{-1} x$

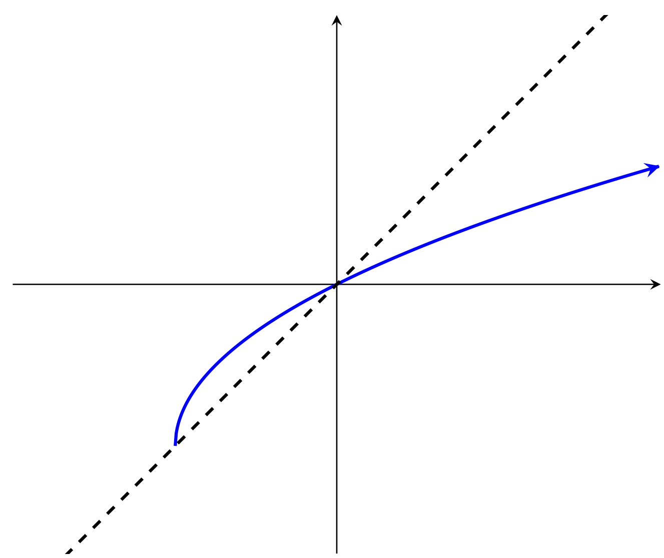


Example. Draw the function inverses:

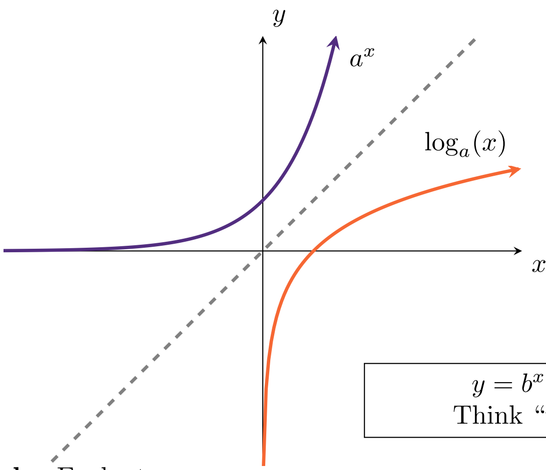
$f(x) = 2^x$



$f(x) = \sqrt{x+1} - 2$



Definition. (Logarithmic Function Base b)
For any base $b > 0$, with $b \neq 1$, the **logarithmic function base** b, denoted $y = \log_b(x)$, is the inverse of the exponential function $y = b^x$. The inverse of the natural exponential function with base $b = e$ is the **natural logarithm function**, denoted $y = \ln(x)$.



Note:

	a^x	$\log_a(x)$
Domain:	$(-\infty, \infty)$	$(0, \infty)$
Range:	$(0, \infty)$	$(-\infty, \infty)$

$$y = b^x \iff \log_b(y) = x$$
Think "the base stays the base"

Example. Evaluate:

$\log_9(81)$

$\log_3(\sqrt{3})$

$\log_{\frac{1}{2}}(8)$

$\left(\log_5(5^{-3})\right)^2$

Note: In this course, the **common logarithm** is $\log_{10}(x)$ and is denoted by $\log(x)$.
– Sometimes other disciplines use $\log(x)$ to represent other bases.

Example. Evaluate:

$\log 100000$

$\log \frac{1}{1000}$

Recall that for a function f and its inverse g:

- $f(g(x)) = x$
- $g(f(x)) = x$
- Domain of f=Range of g
- Domain of g=Range of f

Inverse Relations for Exponential and Logarithmic Functions
For any base $b > 0$, with $b \neq 1$, the following inverse relations hold:

$$b^{\log_b x} = x \qquad \log_b(b^x) = x, \text{ for all real values of } x$$

Example. Evaluate:

$2^{\log_2 8}$ $\qquad$ $\log_b b^\pi$ $\qquad$ $\log 10^3$

Example. Write each expression in terms of one logarithm:

$\log_2 6 - \log_2 15 + \log_2 20$

$\log_3 100 - (\log_3 18 + \log_3 50)$

Laws of Logarithms
For $x, y > 0$:

1. $\log_a(xy) = \log_a(x) + \log_a(y)$
2. $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(x)$
3. $\log_a(x^r) = r\log_a(x)$
4. $\log_a(1) = 0$
5. $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$

Example. Solve each equation checking for extraneous solutions:

$$\log_{64} x^2 = \frac{1}{3}$$

$$\log(3x+2) + \log(x-1) = 2$$

$$\log_2 x^2 - \log_2(3x-8) = 2$$

$$\log_4 x - \log_4(x-1) = \frac{1}{2}$$

$$\log_3(x+6) - \log_3(x-6) = 2$$

$$\log_3(x^2-5) = 2$$