8.5: Partial Fractions

Example. Simplify $f(x) = \overset{\left(\frac{x+4}{x+4}\right)}{\frac{1}{x-2}} + \overset{\left(\frac{x-2}{x-2}\right)}{\frac{2}{x+4}}$ by finding a common denominator.

$$f(x) = \frac{1(x+4) + 2(x-2)}{(x-2)(x+4)} = \frac{3x}{x^2+2x-8}$$

Procedure: Partial Fractions with Simple Linear Factors
Suppose $f(x) = p(x)/q(x)$, where p and q are polynomials with no common factors and with the degree of P less than the degree of q. Assume q is the product of simple linear factors. The partial fraction decomposition is obtained as follows.

Step 1: Factor the denominator q in the form $(x - r_1)(x - r_2) \ldots (x - r_n)$

Step 2: Partial fraction decomposition

$$\frac{p(x)}{q(x)} = \frac{A_1}{(x - r_1)} + \frac{A_2}{(x - r_2)} + \cdots + \frac{A_n}{(x - r_n)}.$$

Step 3: Clear denominators Multiply both sides of the equation in Step 2 by $q(x) = (x - r_1)(x - r_2) \ldots (x - r_n)$

Step 4: Solve for coefficients Equate like powers of x in Step 3 to solve for the undetermined coefficients $A_1, \ldots, A_n$.

Example. Perform partial fraction decomposition on $f(x) = \dfrac{3x}{x^2 + 2x - 8}$.

$$x^2+2x-8 = (x+4)(x-2)$$

$$\Rightarrow (x^2+2x-8)\frac{3x}{x^2+2x-8} = \left(\frac{A}{x+4} + \frac{B}{x-2}\right)(x^2+2x-8)$$

$$\Rightarrow 3x+0 = A(x-2) + B(x+4) = Ax - 2A + Bx + 4B = \underbrace{(A+B)}_{3}x + \underbrace{(-2A+4B)}_{0}$$

$$\boxed{A+B = 3} \quad (1)$$

$$-2A+4B = 0 \quad (2)$$

$\rightarrow 2(1)+(2)$

$$\begin{array}{r} 2A+2B=6 \\ + \ -2A+4B=0 \\ \hline 6B=6 \end{array} \longrightarrow B=1 \quad (1) \Rightarrow A+1=3, \ A=2$$

$$\boxed{\frac{3x}{x^2+2x-8} = \frac{2}{x+4} + \frac{1}{x-2}}$$

Example. $\displaystyle\int \frac{28x^3 - 56x^2 + 9}{x^2 - 2x}\, dx$

$$\begin{array}{r} 28x \qquad\qquad\quad \\ x^2-2x \;\big)\overline{28x^3-56x^2+0x \;+9} \\ -(28x^3-56x^2) \qquad\quad \\ \hline 0 \qquad\qquad\quad \\ \nearrow 9 \end{array}$$

Remainder

$$\int \frac{28x^3-56x^2+9}{x^2-2x}\,dx = \int 28x + \underbrace{\frac{9}{x^2-2x}}_{\text{PFD}}\,dx$$

$$(x^2-2x)\left(\frac{9}{x^2-2x}\right) = \left(\frac{A}{x} + \frac{B}{x-2}\right)(x^2-2x)$$

$$0x + 9 = A(x-2) + Bx = Ax - 2A + Bx$$

$$= \underbrace{(A+B)}_{0}x + \underbrace{(-2A)}_{9} \longrightarrow \begin{array}{l} -2A = 9 \\ A = -9/2 \end{array} \longrightarrow \begin{array}{r} -9/2 + B = 0 \\ B = 9/2 \end{array}$$

$$\int 28x + \frac{9}{x^2-2x}\,dx = \int 28x - \frac{9}{2x} + \frac{9}{2(x-2)}\,dx$$

$$= 14x^2 - \frac{9}{2}\ln|x| + \frac{9}{2}\ln|x-2| + C$$

$$\frac{}{x^4(x+1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x+1} + \frac{F}{(x+1)^2} + \frac{G}{(x+1)^3}$$

Procedure: Partial Fractions for Repeated Linear Factors
Suppose the repeated linear factor $(x-r)^m$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition has a partial fraction for each power of $(x-r)$ up to and including the mth power; that is, the partial fraction decomposition contains the sum

$$\frac{A_1}{(x-r)} + \frac{A_3}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}$$

where $A_1, \ldots, A_m$ are constants to be determined.

Example. Setup the partial fraction decomposition for $f(x) = \dfrac{x^3 - 8x + 19}{x^4 + 3x^3}$.

$$x^4 + 3x^3 = x^3(x+3)$$

$$\frac{x^3 - 8x + 19}{x^4 + 3x^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+3}$$

Example. Setup the partial fraction decomposition for $g(x) = \dfrac{2}{x^5 - 6x^4 + 9x^3}$.

$$x^5 - 6x^4 + 9x^3 = x^3(x^2 - 6x + 9) = x^3(x-3)^2$$

$$\frac{2}{x^5 - 6x^4 + 9x^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-3} + \frac{E}{(x-3)^2}$$

Example. Evaluate $\int \frac{\overbrace{x^2+1}^{\text{quadratic}}}{\underbrace{(2x-3)(x-2)^2}_{\text{cubic}}}\,dx.$

$$(2x-3)(x-2)^2\left(\frac{x^2+1}{(2x-3)(x-2)^2}\right) = \left(\frac{A}{2x-3}+\frac{B}{x-2}+\frac{C}{(x-2)^2}\right)(2x-3)(x-2)^2$$

$$\underbrace{x^2+0x+1}_{x=2} = A(x-2)^2+B(2x-3)(x-2)+C(2x-3)$$

$$= A(x^2-4x+4)+B(2x^2-7x+6)+C(2x-3)$$

$$= \underbrace{(A+2B)}_{1}x^2+(\underbrace{-4A-7B+2C}_{0})x+(\underbrace{4A+6B-3C}_{1})$$

$$\begin{aligned} A+2B &= 1\\ -4A-7B+2C &= 0\\ 4A+6B-3C &= 1 \end{aligned}$$

Alternatively

$x=2$, $x=3/2$ are roots

When $x=2$

$x^2+1=5$ $\quad A\cancel{(x-2)^2}+B\cancel{(2x-3)(x-2)}+C(2x-3) = C(4-3)=C$

$\Rightarrow C=5$

when $x=3/2$

$x^2+1=\left(\frac{3}{2}\right)^2+1=\frac{13}{4}$ $\quad A(x-2)^2+B\cancel{(2x-3)}(x-2)+C\cancel{(2x-3)} = A\left(\frac{3}{2}-2\right)^2$

$= \frac{A}{4}$

$\Rightarrow A=13$

$$\int \frac{x^2+1}{(2x-3)(x-2)^2}dx = \frac{13}{2}\ln|2x-3|-6\ln|x-2|-\frac{5}{x-2}+C$$

$$\left.\begin{aligned} A+2B&=1\\ A&=13 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} 13+2B&=1\\ 2B&=-12\\ &\boxed{B=-6} \end{aligned}$$

$$\frac{A}{2x-3}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$$

$$= \frac{13}{2x-3}-\frac{6}{x-2}+\frac{5}{(x-2)^2}$$

Example. Evaluate $\displaystyle\int \frac{8}{3x^3+7x^2+4x}\,dx$.

Procedure: Partial Fractions with Simple Irreducible Quadratic Factors
Suppose a simple irreducible factor ax^2+bx+c appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition contains a term of the form

$$\frac{\boxed{Ax+B}}{ax^2+bx+c},$$

$u = ax^2+bx+c$

$du = 2ax+b \quad dx$

where A and B are unknown coefficients to be determined.

Example. Perform partial fraction decomposition on the following fractions or identify them as irreducible.

$$\frac{1}{x^2-13x+43}$$

a b c

roots:

$$x = \frac{13 \pm \sqrt{13^2-4(1)(43)}}{2} = \frac{13 \pm \boxed{\sqrt{169-172}}}{2}$$

neg

⇒ irreducible

b^2-4ac : determinant

If $b^2-4ac<0$, then the eqn is irreducible

$$\frac{x^2}{(x-4)(x+5)} = \frac{x^2}{x^2+x-20}$$

① Factorable denom

② do poly long div first

$$\begin{array}{r|l} & \quad 1 \\ \hline x^2 & x^2+x-20 \\ & -(x^2) \\ \hline & \quad x-20 \end{array}$$

$$\frac{x^2}{(x-4)(x+5)} = 1 + \frac{x-20}{(x-4)(x+5)} = 1 + \frac{A}{x-4} + \frac{B}{x+5}$$

Example. Perform partial fraction decomposition on the following fractions or identify them as irreducible.

$$\frac{7}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$$\frac{1}{x^2+11x+28}$$

Determinant:

$$b^2-4ac = 11^2-4(1)(28)$$

$$= \frac{A}{(x+4} + \frac{B}{(x+7)}$$

$$= 121 - 112 > 0 \rightarrow \underline{\underline{\text{is}}} \text{ reducible}$$

Example. Evaluate $\displaystyle\int \frac{4x}{(x+1)(x^2+1)}\,dx$

Example. Evaluate $\displaystyle\int \frac{3x^2+2x+12}{(x^2+4)^2}\,dx$

Example. Evaluate $\displaystyle\int \frac{1}{x\sqrt{1+2x}}\,dx$ using the substitution $u = \sqrt{1+2x}$.

Summary: Partial Fraction Decomposition
Let $f(x) = p(x)/q(x)$ be a proper rational function in reduced form. Assume the denominator q has been factored completely over the real numbers and m is a positive integer.

1. **Simple linear factor:** A factor $x - r$ in the denominator requires the partial fraction $\dfrac{A}{x-r}$.
2. **Repeated linear factor:** A factor $(x-r)^m$ with $m > 1$ in the denominator requires the partial fractions
$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}.$$
3. **Simple irreducible quadratic factor:** An irreducible factor $ax^2 + bx + c$ in the denominator requires the partial fraction
$$\frac{Ax+B}{ax^2+bx+c}.$$
4. **Repeated irreducible quadratic factor:** An irreducible factor $(ax^2+bx+c)^m$ with $m > 1$ in the denominator requires the partial fractions
$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}.$$