11.4: Working with Taylor Series

Limits by Taylor Series

Example (LC 31.1-31.2). Evaluate the following limit using its Taylor series:

$$\lim_{x\to 0} \frac{12x - 8x^3 - 6\sin(2x)}{x^5}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \ldots \qquad = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!},$$

$$L = \lim_{x\to 0} \frac{12x - 8x^3 - 6\left((2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \frac{(2x)^9}{9!} \cdots\right)}{x^5}$$

$$= \lim_{x\to 0} \frac{\boxed{12x}\boxed{-8x^3}\boxed{-12x}\boxed{+\frac{48}{6}x^3} - 6\left(\frac{32x^5}{5!} + \frac{(2x)^7}{7!} - \frac{(2x)^9}{9!} \cdots\right)}{x^5}$$

$$= \lim_{x\to 0} -\frac{6\cdot 2^5}{5!} + 6\left(\underbrace{\frac{128x^2}{7!}}_{0} - \underbrace{\frac{512x^4}{9!}}_{0} + \underbrace{\ldots}_{0}\right) = -\frac{6\cdot 2^5}{5!}$$

$$= -\frac{\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdot 8}{5\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}$$

$$= \boxed{\frac{-8}{5}}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \ldots \qquad = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

Example. Evaluate the following limit using its Taylor series:

$$\lim_{x\to\infty} 2x^2\left(e^{-2/x^2} - 1\right)$$

$\infty \cdot 0$

$$\lim_{x\to\infty} 2x^2\left(-1 + e^{-2/x^2}\right) = \lim_{x\to\infty} 2x^2\Big(\cancel{-1} + \cancel{1} + \underbrace{\left(\frac{-2}{x^2}\right) + \frac{\left(-2/x^2\right)^2}{2!} + \frac{\left(-2/x^2\right)^3}{3!} + \ldots}_{e^{-2/x^2}}\Big)$$

$$= \lim_{x\to\infty} -4 + \underbrace{\frac{2x^2\left(-2/x^2\right)^2}{2!}}_{} + \frac{2x^2\left(-2/x^2\right)^3}{3!} + \ldots$$

$\dfrac{2x^2(-8)}{2\,(x^2)^2}$

$$= \lim_{x\to\infty} -4 \underbrace{- \frac{8}{x^2} - \frac{16}{6x^2} + \ldots}_{\to 0}$$

$$= -4$$

Differentiating Power Series

Example (LC 31.3-31.4). The differential equation

$$y'(t) + 4y = 8; \quad y(0) = 0$$

is satisfied by the function

$$y(t) = \sum_{k=1}^{\infty} \frac{8(-4)^{k-1}t^k}{k!}$$

Find $y'(t)$ as a power series.

$$y'(t) = \sum_{k=1}^{\infty} \frac{8(-4)^{k-1}k t^{k-1}}{\underbrace{k!}_{k\cdot(k-1)!}} = \sum_{k=1}^{\infty} \frac{8(-4)^{k-1} t^{k-1}}{(k-1)!}$$

Identify the function $y(t)$ represented by this power series.

$$e^x = \underbrace{1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}}$$

$$e^x - 1$$

$$\sum_{k=1}^{\infty} \frac{8(-4)^{k-1}t^k}{k!} = \sum_{k=1}^{\infty} \frac{8(-4t)^k}{-4k!} = -2\underbrace{\sum_{k=1}^{\infty} \frac{(-4t)^k}{k!}}_{e^{-4t}-1}$$

$$= \boxed{2 - 2e^{-4t}}$$

Integrating Power Series

Example (LC 31.5-31.6)**.** Given that

$$x\cos\left(x^3\right) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{6k+1}}{(2k)!}, \text{ for } |x| < \infty$$

Evalute $\displaystyle\int_0^1 x\cos\left(x^3\right)dx$ as an infinite series

Using the Alternating Series Estimation Theorem, what is the bound on $|R_3|$?

Representing Real Numbers

Example (LC 31.7)**.** Given that $\tan^{-1}(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$, for $|x| \leq 1$, can we approximate $\dfrac{\pi}{3}$ using $x = \sqrt{3}$?

Example (LC 31.8)**.** Evaluate $\displaystyle\sum_{k=0}^{\infty} \frac{(\ln(2))^k}{k!}$.

Example. Let $f(x) = \begin{cases} \frac{e^x - 1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$. Using $f(x)$ and $f'(x)$, evaluate

$$\sum_{k=1}^{\infty} \frac{k\, 2^{k-1}}{(k+1)!}$$

Representing Functions as Power Series

Example (LC 31.9-31.10)**.** Consider the following Taylor series:

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{3^k}{k\,5^k}$$

What function is being represented by this power series?

What does the sum of the series equal?

Example. Identify the function represented by

$$\sum_{k=0}^{\infty}(-1)^k\frac{x^{5k}}{3^k}$$