2.1 The Idea of Limits

Definition.
The **average velocity** is the distance traveled over some time period.
The **instantaneous velocity** is the limit of the average velocities as the length of the time period goes to zero.

Example. An unladen swallow is flying from Camelot to the Castle Anthrax and back. It's current position, in miles, is given by

$$s(t) = -16t^2 + 96t$$

where t is given in hours. Find the average velocity between:

a) $t = 1$ and $t = 3$,

$$\frac{s(3) - s(1)}{3-1} = \frac{144-80}{2} = \boxed{32 \text{ mph}}$$

b) $t = 1$ and $t = 2$,

$$\frac{s(2) - s(1)}{2-1} = \frac{128-80}{1} = \boxed{48 \text{ mph}}$$

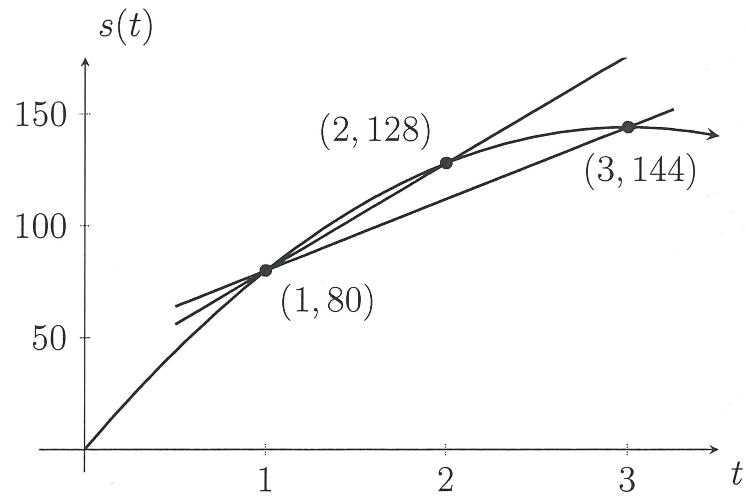
Example. Find the instantaneous velocity using $s(t)$ by computing the average velocity between $t = 1$ and $t = h$:

$$\frac{s(h) - s(1)}{h-1} = \frac{-16h^2 + 96h - 80}{h-1}$$

$$= \frac{-16(h^2 - 6h + 5)}{h-1}$$

$$= \frac{-16(h-5)(h-1)}{h-1}$$

$$= -16(h-5)$$



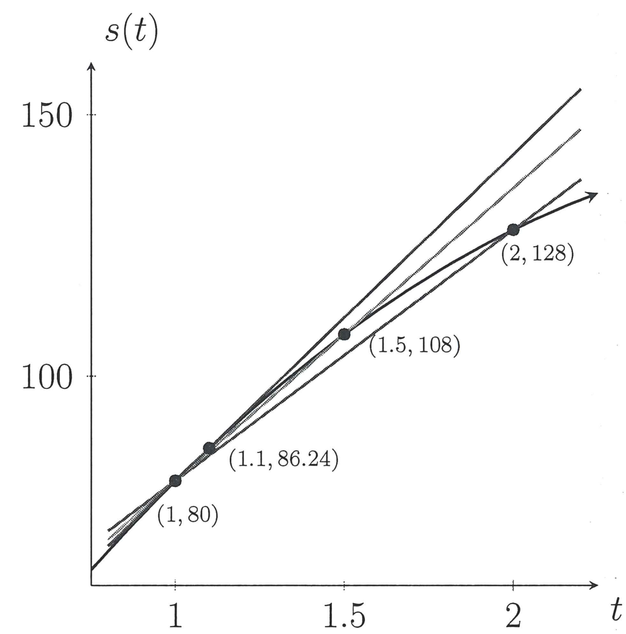
> **Definition.**
> The **secant line** is the line that intersects the function in two places.
> The **tangent line** is the line that intersects the function in exactly one place (locally).

Note– The average velocity is the slope of the secant line. The instantaneous velocity is the slope of the tangent line.

Example. Using the average velocity between $t = 1$ and $t = h$:

$$v_{avg} = -16(h - 5)$$

compute the instantaneous velocity at $h = 1$.



$$V_{inst} = -16(-4) = \boxed{64 \text{ mph}}$$

Example. Find the instantaneous velocity of

$$f(x) = 2x^2 - 4x + 1$$

for any value of x.

$$\lim_{x \to a} \frac{f(a) - f(x)}{a - x} = \lim_{x \to a} \frac{[2a^2 - 4a + 1] - [2x^2 - 4x + 1]}{a - x} = \lim_{x \to a} \frac{2(a^2 - x^2) - 4(a - x)}{(a - x)}$$

$$= \lim_{x \to a} \frac{2(a + x)(a - x) - 4(a - x)}{(a - x)} = \lim_{x \to a} 2(a + x) - 4 = \boxed{4a - 4}$$