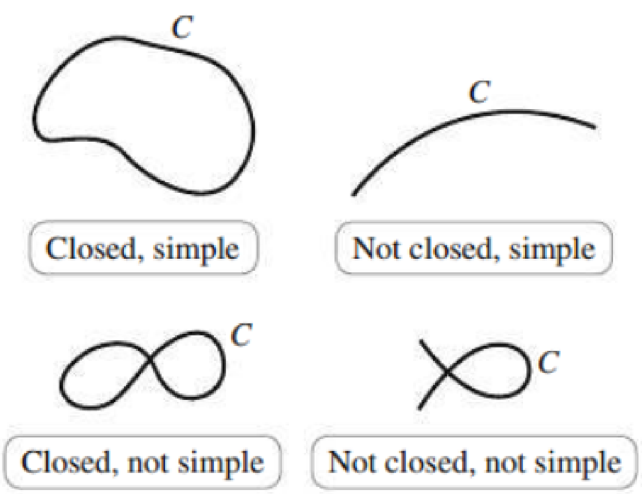
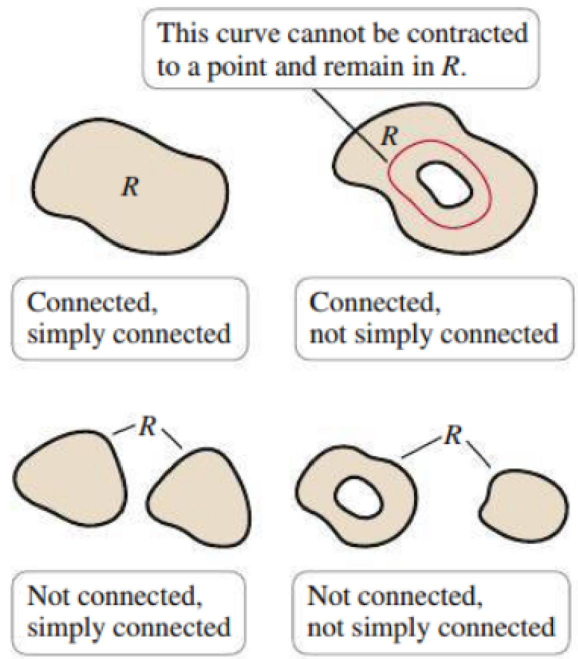
17.3: Conservative Vector Fields

Definition. (Simple and Closed Curves)
Suppose a curve C (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is described parametrically by $\mathbf{r}(t)$, where $a \le t \le b$. Then C is a **simple curve** if $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ for all t_1 and t_2, with $a < t_1 < t_2 < b$; that is, C never intersects itself between its endpoints. The curve C is **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$; that is, the initial and terminal points of C are the same.





Definition. (Connected and Simply Connected Regions)
An open region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) is **connected** if it is possible to connect any two points of R by a continuous curve lying in R. An open region R is **simply connected** if every closed simple curve in R can be deformed and contracted to a point in R.

Definition. (Conservative Vector Field)
A vector field $\mathbf{F}$ is said to be **conservative** on a region (in $\mathbb{R}^2$ or $\mathbb{R}^3$) if there exists a scalar function φ such that $\mathbf{F} = \nabla\varphi$ on that region.

Assume that $\mathbf{F} = \langle f, g, h\rangle$ is a conservative vector field. Then, there exists φ such that

$$\langle f, g, h\rangle = \nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z\rangle$$

Now, we consider the second partial derivatives:

$\varphi_{xy} = \varphi_{yx} \Rightarrow$

$\varphi_{xz} = \varphi_{zx} \Rightarrow$

$\varphi_{yz} = \varphi_{zy} \Rightarrow$

Theorem 17.3: Test for Conservative Vector Fields
Let $\mathbf{F} = \langle f, g, h\rangle$ be a vector field defined on a connected and simply connected region D of $\mathbb{R}^3$, where f, g, and h have continuous first partial derivatives on D. Then $\mathbf{F}$ is a conservative vector field on D (there is a potential function φ such that $\mathbf{F} = \nabla\varphi$) if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \qquad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \qquad \text{and} \qquad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$$

For vector fields in $\mathbb{R}^2$, we have the single condition $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$.

Example. Determine if the following vector fields are conservative:

$$\mathbf{F} = \langle e^x \cos(y), -e^x \sin(y) \rangle$$

$$\mathbf{F} = \left\langle 2xy - z^2, x^2 + 2z, 2y - 2xz \right\rangle$$

Procedure: Finding Potential Functions in $\mathbb{R}^3$
Suppose $\mathbf{F} = \langle f, g, h \rangle$ is a conservative vector field. To find φ such that $\mathbf{F} = \nabla\varphi$, use the following steps:

1. Integrate $\varphi_x = f$ with respect to x to obtain φ, which includes an arbitrary function $c(y, z)$.
2. Compute φ_y and equate it to g to obtain an expression for $c_y(y, z)$.
3. Integrate $c_y(y, z)$ with respect to y to obtain $c(y, z)$, including an arbitrary function $d(z)$.
4. Compute φ_z and equate it to h to get $d(z)$.

A similar procedure beginning with $\varphi_y = g$ or $\varphi_z = h$ may be easier in some cases.

Example. Find a potential function for the following conservative vector fields:

$\mathbf{F} = \langle e^x \cos(y), -e^x \sin(y) \rangle$

$$\mathbf{F} = \left\langle 2xy - z^2, x^2 + 2z, 2y - 2xz \right\rangle$$

Fundamental Theorem for Line Integrals and Path Independence:
Suppose that $\mathbf{F}$ is a conservative vector field in $\mathbb{R}^3$ with potential function φ.

$$\begin{aligned}\frac{d\varphi}{dt} &= \frac{\partial\varphi}{\partial x}\frac{dx}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dt} + \frac{\partial\varphi}{\partial z}\frac{dz}{dt} \\ &= \left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \\ &= \nabla\varphi \cdot \mathbf{r}'(t) \\ &= \mathbf{F} \cdot \mathbf{r}'(t),\end{aligned}$$

where $\mathbf{r}(t)$ defines a curve C for $a \le t \le b$. Now, we integrate $\mathbf{F}$ over the curve C:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt = \int_a^b \frac{d\varphi}{dt}\, dt = \varphi(B) - \varphi(A)$$

where A and B are points corresponding to $\mathbf{r}(a)$ and $\mathbf{r}(b)$ respectively.

Theorem 17.4: Fundamental Theorem for Line Integrals
Let R be a region in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let φ be a differentiable potential function defined on R. If $\mathbf{F} = \nabla\varphi$ (which means that $\mathbf{F}$ is conservative), then

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A),$$

for all points A and B in R and all piecewise-smooth oriented curves C in R from A to B.

Definition. (Independence of Path)
Let $\mathbf{F}$ be a continuous vector field with domain R. If $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for all piecewise-smooth curves C_1 and C_2 in R with the same initial and terminal points, then the line integral is **independent of path**.

Theorem 17.5
Let $\mathbf{F}$ be a continuous vector field on an open connected region R in $\mathbb{R}^2$. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, then $\mathbf{F}$ is conservative; that is, there exists a potential function φ such that $\mathbf{F} = \nabla\varphi$ on R.

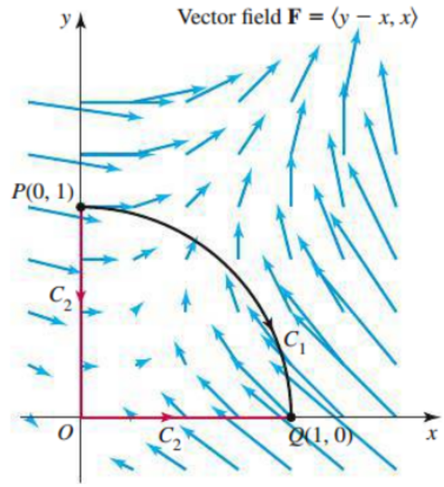
Example. Consider the potential function $\varphi(x,y) = \big(x^2 - y^2\big)/2$ with gradient field $\mathbf{F} = \langle x, -y\rangle$.

- Let C_1 be the quarter-circle $\mathbf{r}(t) = \langle \cos(t), \sin(t)\rangle$, for $0 \le t \le \pi/2$, from $A(1,0)$ to $B(0,1)$,
- let C_2 be the line $\mathbf{r}(t) = \langle 1-t, t\rangle$, for $0 \le t \le 1$, also from A to B.

Evaluate the line integrals of $\mathbf{F}$ on C_1 and C_2, and show that both are equal to $\varphi(B) - \varphi(A)$.

Example. With $\mathbf{F} = \langle y - x,\, x \rangle$ on the following oriented paths in $\mathbb{R}^2$.

a) Find the potential function $\varphi(x, y)$



b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along

the quarter-circle C_1 from $P(0, 1)$ to $Q(1, 0)$,

the path C_2 from $P(0, 1)$ to $Q(1, 0)$ via two line segments through $O(0, 0)$.

Example. Evaluate

$$\int_C \left\langle 2xy - z^2,\, x^2 + 2z,\, 2y - 2xz \right\rangle d\mathbf{r}$$

where C is the curve from $A(-3, -2, 1)$ to $B(1, 2, 3)$.

Theorem 17.6: Line Integrals on Closed Curves
Let R be on open connected region in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\mathbf{F}$ is a conservative vector field on R if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple closed piecewise-smooth oriented curves C in R.

Example. Evaluate $\displaystyle\int_C \left\langle 2xy + z^2,\, x^2,\, 2xz \right\rangle \cdot d\mathbf{r}$ where C is the circle $\mathbf{r}(t) = \langle 3\cos(t),\, 4\cos(t),\, 5\sin(t)\rangle$, for $0 \leq t \leq 2\pi$.