16.4: Triple Integrals

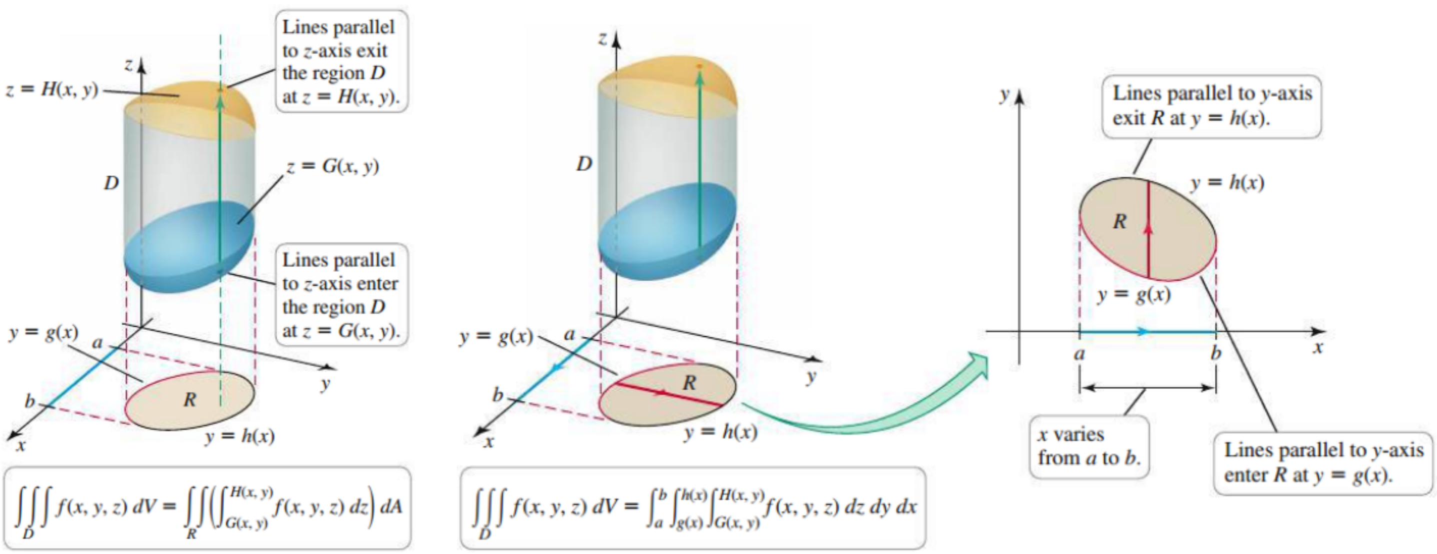
Theorem 16.5: Triple Integrals

Let f be continuous over the region

$$D = \{(x, y, z) : a \leq x \leq b,\ g(x) \leq y \leq h(x),\ G(x, y) \leq z \leq H(x, y)\},$$

where g, h, G, and H are continuous functions. Then f is integrable over D and the triple integral is evaluated as the iterated integral

$$\iiint_D f(x, y, z)\, dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x, y, z)\, dz\, dy\, dx.$$



Integral	Variable	Interval
Inner	z	$G(x, y) \leq z \leq H(x, y)$
Middle	y	$g(x) \leq y \leq h(x)$
Outer	x	$a \leq x \leq b$

Example. A solid box D is bounded by the planes $x = 0$, $x = 3$, $y = 0$, $y = 2$, $z = 0$, and $z = 1$. The density of the box decreases linearly in the positive z-direction and is given by $f(x, y, z) = 2 - z$. Find the mass of the box.

Example. Find the volume of the prism D in the first octant bounded by the planes $y = 4 - 2x$ and $z = 6$.

Example. Write the triple integral for $\iiint\limits_D f(x, y, z)\, dV$ where D is a sphere of radius r centered at the origin.

Example. Find the volume of the solid D bounded by the paraboloids $y = x^2 + 3z^2 + 1$ and $y = 5 - 3x^2 - z^2$.

The concept of changing the order of integration for double integrals also extends to triple integrals:

Example. Consider the integral

$$\int_0^{\sqrt[4]{\pi}} \int_0^z \int_y^z 12y^2z^3 \sin\left(x^4\right) dx\, dy\, dz.$$

Sketch the region of integration, then evaluate the integral by changing the order of integration.

Definition. (Average Value of a Function of Three Variables)
If f is continuous on a region D of $\mathbb{R}^3$, then the **average value** of f over D is

$$\bar{f} = \frac{1}{\text{volume of } D} \iiint\limits_{D} f(x, y, z)\, dV.$$

Example. Find the average y-coordinate of the points in the standard simplex
$D = \{(x, y, z) : x + y + z \leq 1,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$.