3.2 The Derivative as a Function

Definition (The Derivative Function).
The **derivative** of f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists and x is in the domain of f. If $f'(x)$ exists, we say that f is **differentiable** at x. If f is differentiable at every point on an open interval I, we say that f is differentiable on I.

Note: The derivative of f has several notations:

$$f'(x) \qquad \frac{d}{dx}(f(x)) \qquad D_x(f(x)) \qquad y'(x)$$

Note: The derivative of f evaluated at a has several notations:

$$f'(a) \qquad y'(a) \qquad \left.\frac{df}{dx}\right|_{x=a} \qquad \left.\frac{dy}{dx}\right|_{x=a}$$

Example. Use the limit definition of a derivative to find the derivative function $f'(x)$ for the function $f(x) = 5x^2 - 6x + 1$.

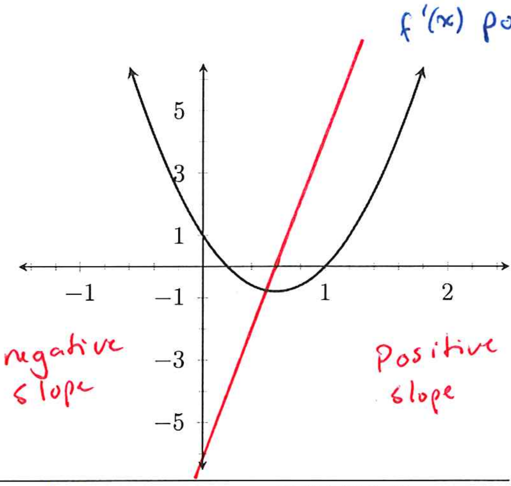
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[5(x+h)^2 - 6(x+h) + 1] - [5x^2 - 6x + 1]}{h}$$

$$= \lim_{h \to 0} \frac{5x^2 + 10xh + 5h^2 - 6x - 6h + 1 - 5x^2 + 6x - 1}{h}$$

$$= \lim_{h \to 0} \frac{10xh + 5h^2 - 6h}{h}$$

$$= \lim_{h \to 0} 10x + 5h - 6 = 10x - 6$$



Example. Find the derivative of the following functions. If a point is specified, find the tangent line at that point.

$f(w) = \sqrt{4w-3},\ w = 3$

$$f'(w) = \lim_{h\to 0} \frac{f(w+h)-f(w)}{h} = \lim_{h\to 0} \frac{\left[\sqrt{4(w+h)-3}\right] - \left[\sqrt{4w-3}\right]}{h}\left(\frac{\sqrt{4(w+h)-3}+\sqrt{4w-3}}{\sqrt{4(w+h)-3}+\sqrt{4w-3}}\right)$$

$$= \lim_{h\to 0} \frac{[4(w+h)-3] - [4w-3]}{h\left(\sqrt{4(w+h)-3}+\sqrt{4w+3}\right)} = \lim_{h\to 0} \frac{4w+4h-3-4w+3}{h\left(\sqrt{4(w+h)-3}+\sqrt{4w+3}\right)}$$

$$= \lim_{h\to 0} \frac{4h}{h\left(\sqrt{4(w+h)-3}+\sqrt{4w+3}\right)} = \lim_{h\to 0} \frac{4}{\sqrt{4(w+h)-3}+\sqrt{4w+3}} = \frac{4}{\sqrt{4w-3}+\sqrt{4w-3}} = \boxed{\frac{2}{\sqrt{4w-3}}}$$

$f(3) = 3$

$f'(3) = 2/3$ $\Rightarrow$ tan lin $y - 3 = \frac{2}{3}(x-3)$

$$\boxed{y = \frac{2}{3}x + 1}$$

$g(v) = \dfrac{v}{v+2}, v = 0$

$$g'(v) = \lim_{h\to 0} \frac{\left(\frac{v+2}{v+2}\right)\left[\frac{v+h}{v+h+2}\right] - \left[\frac{v}{v+2}\right]\left(\frac{v+h+2}{v+h+2}\right)}{h/1} = \lim_{h\to 0} \frac{(v+2)(v+h) - v(v+h+2)}{(v+2)(v+h+2)} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{v^2+2v+hv+2h-v^2-hv-2v}{h(v+2)(v+h+2)} = \lim_{h\to 0} \frac{2h}{h(v+2)(v+h+2)}$$

$$= \lim_{h\to 0} \frac{2}{(v+2)(v+h+2)} = \frac{2}{(v+2)(v+0+2)} = \boxed{\frac{2}{(v+2)^2}}$$

$g(0) = 0$

$g'(0) = \frac{2}{(2)^2} = \frac{1}{2}$ $\Rightarrow$ $y - 0 = \frac{1}{2}(x-0) \longrightarrow \boxed{y = \frac{1}{2}x}$

$h(m) = 1 + \sqrt{m},\ m = 1/4,\ m = 1$

$$h'(m) = \lim_{h\to 0} \frac{[1+\sqrt{m+h}]-[1+\sqrt{m}]}{h} = \lim_{h\to 0} \frac{\sqrt{m+h}-\sqrt{m}}{h}\left(\frac{\sqrt{m+h}+\sqrt{m}}{\sqrt{m+h}+\sqrt{m}}\right)$$

$$= \lim_{h\to 0} \frac{(m+h)-(m)}{h(\sqrt{m+h}+\sqrt{m})} = \lim_{h\to 0} \frac{h}{h(\sqrt{m+h}+\sqrt{m})} = \lim_{h\to 0} \frac{1}{\sqrt{m+h}+\sqrt{m}} = \boxed{\frac{1}{2\sqrt{m}}}$$

$h(1/4) = 1 + \sqrt{1/4} = 1 + 1/2 = 3/2$

$h'(1/4) = \dfrac{1}{2\sqrt{1/4}} = 1$

$\Rightarrow\ y - 3/2 = 1(x - 1/4)$

$\boxed{y = x + \frac{5}{4}}$

$h(1) = 2$

$h'(1) = \frac{1}{2}$

$\Longrightarrow\ y - 2 = \frac{1}{2}(x-1)$

$\boxed{y = \frac{1}{2}x + \frac{3}{2}}$

$\dfrac{d}{dx}(\sqrt{ax+b})$. Then find $\dfrac{d}{dx}(f(x))$ where $f(x) = \sqrt{5x+9}$ and find $f'(-1)$.

$$\frac{d}{dx}\left(\sqrt{ax+b}\right) = \lim_{h\to 0} \frac{\sqrt{a(x+h)+b}-\sqrt{ax+b}}{h}\left(\frac{\sqrt{a(x+h)+b}+\sqrt{ax+b}}{\sqrt{a(x+h)+b}+\sqrt{ax+b}}\right)$$

$$= \lim_{h\to 0} \frac{[a(x+h)+b]-[ax+b]}{h(\sqrt{a(x+h)+b}+\sqrt{ax+b})} = \lim_{h\to 0} \frac{ah}{h(\sqrt{a(x+h)+b}+\sqrt{ax+b})}$$

$$= \lim_{h\to 0} \frac{a}{\sqrt{a(x+h)+b}+\sqrt{ax+b}} = \frac{a}{2\sqrt{ax+b}}$$

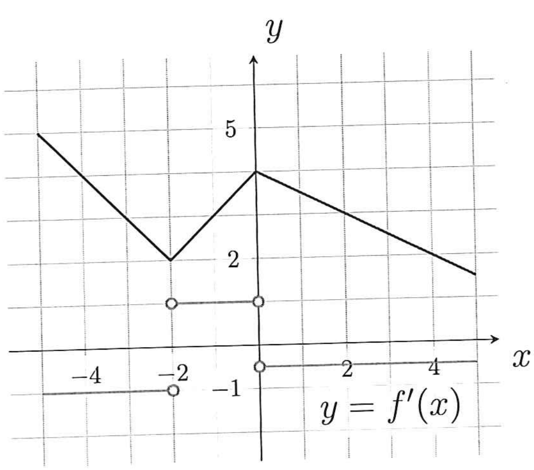
$$\Rightarrow \frac{d}{dx}\left(\sqrt{5x+9}\right) = \frac{5}{2\sqrt{5x+9}} = f'(x) \qquad f'(-1) = \frac{5}{2\sqrt{5(-1)+9}} = \frac{5}{2\sqrt{4}} = \boxed{\frac{5}{4}}$$

$$\frac{d}{dx}(ax^2+bx+c) = \lim_{h\to 0} \frac{[a(x+h)^2+b(x+h)+c]-[ax^2+bx+c]}{h}$$

$$= \lim_{h\to 0} \frac{ax^2+2ahx+ah^2+bx+bh+c-ax^2-bx-c}{h}$$

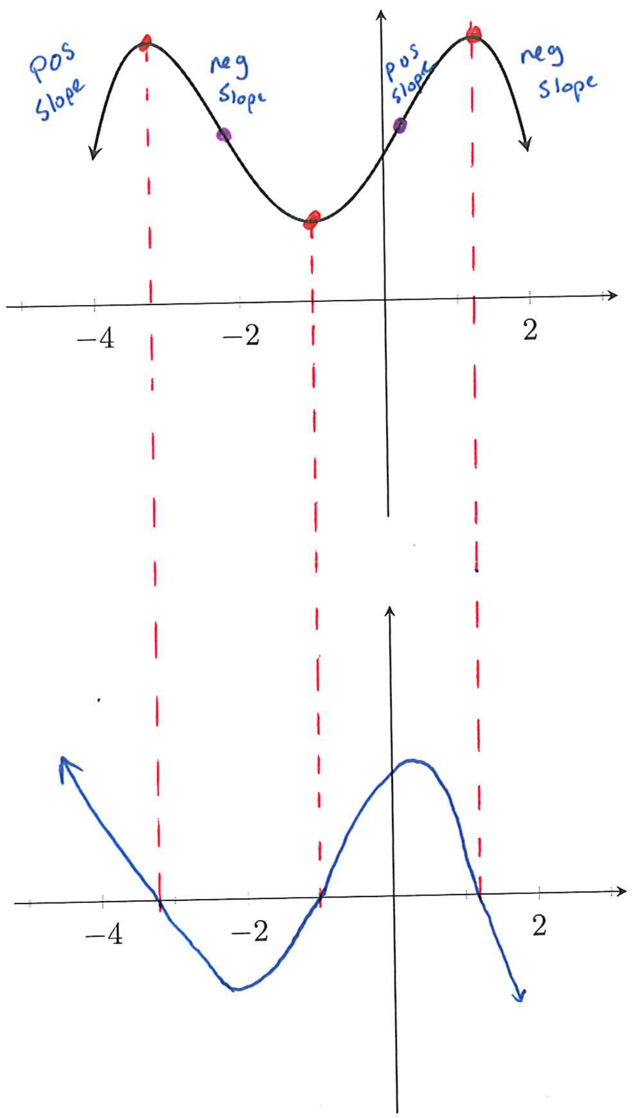
$$= \lim_{h\to 0} \frac{2ahx+ah^2+bh}{h}$$

$$= \lim_{h\to 0} 2ax+ah+b = \boxed{2ax+b}$$



Function	Derivative
Increasing	Positive
Decreasing	Negative
Smooth Min/Max	Zero (at a point)
Constant	Zero (on an interval)
Linear	Constant
Quadratic	linear

Example. Graph the slope graph of the following function

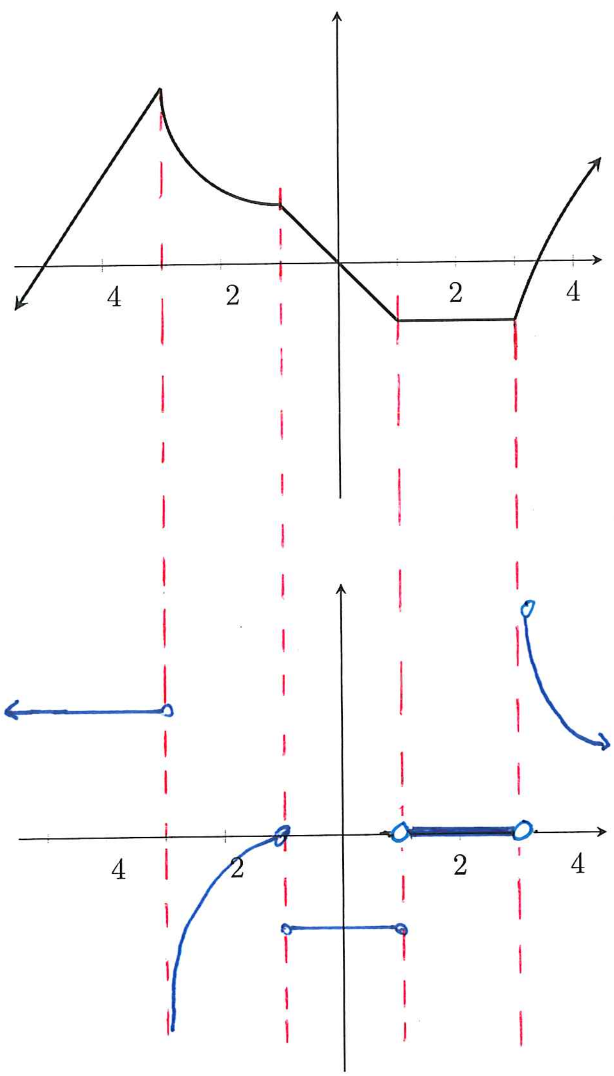


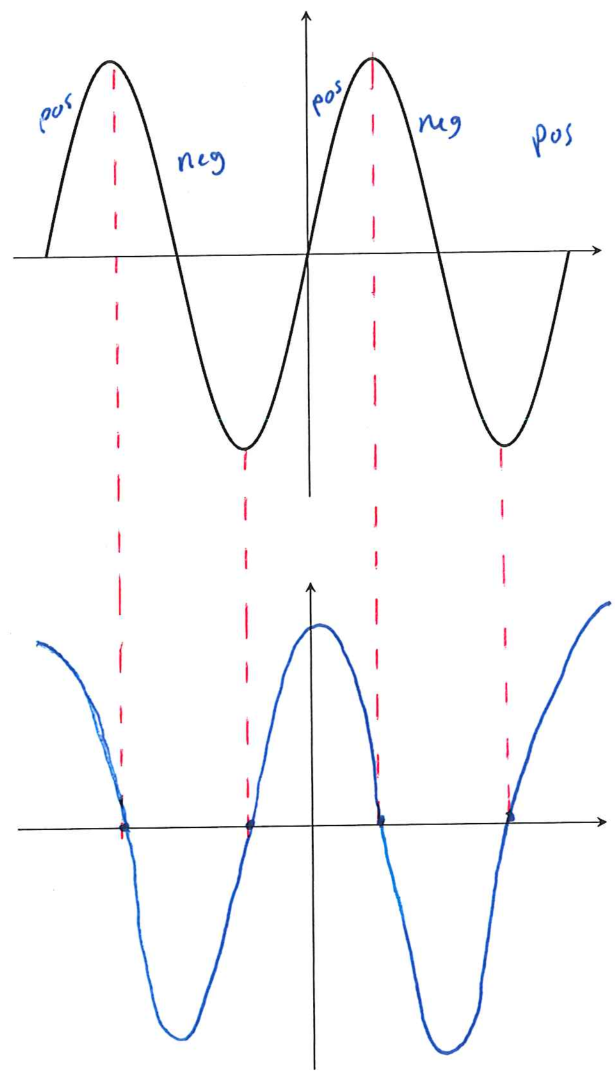
* ① Find where slope is zero (horizontal tan lines)

② Note the intervals where f(x) is increasing or decreasing

* ③ The max/min of f'(x) occurs where f(x) is steepest.

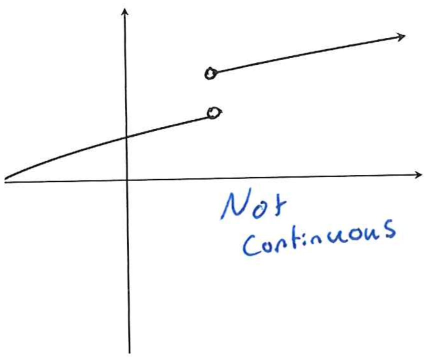
Example. Graph the slope graph of the following functions

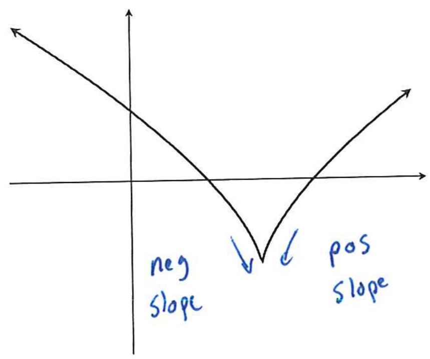


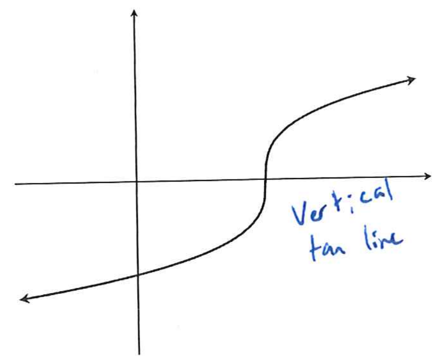


When is a Function Not Differentiable at a Point?
A function f is *not* differentiable at a if at least one of the following conditions holds:

1. f is not continuous at a
2. f has a corner at a
3. f has a vertical tangent at a

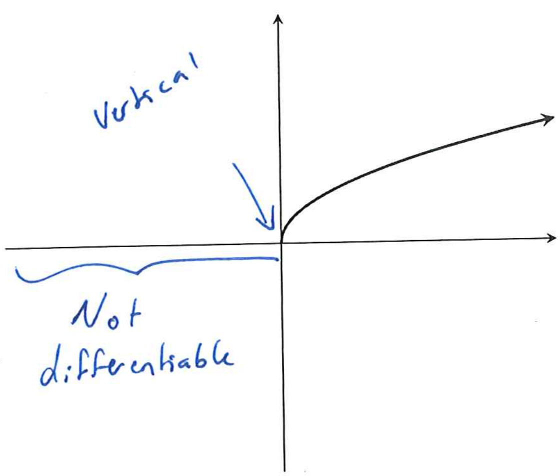


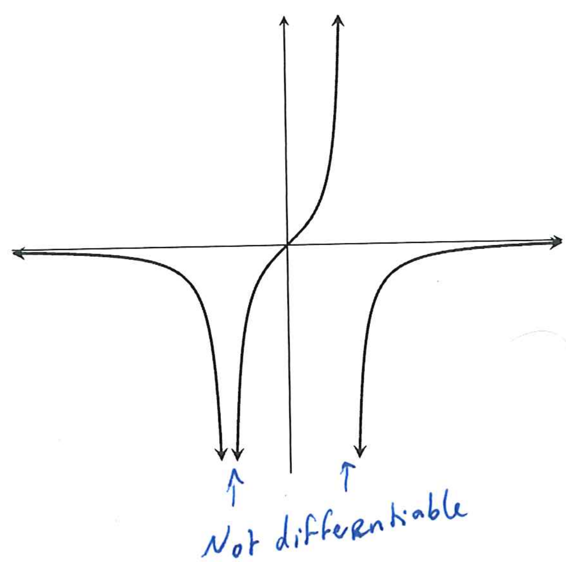




Theorem: Differentiable Implies Continuous
If f is differentiable at a, then f is continuous at a.

Theorem: Not Continuous Implies Not Differentiable
If f is not continuous at a, then f is not differentiable at a.





Definition. The **normal** line at $(a, f(a))$ is the line perpendicular to the tangent line that crosses the point $(a, f(a))$.

Example. Find the derivative of $g(x) = \sqrt{x-2}$. Use your result to find the tangent line and the normal line at $x = 11$.

$$g(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h-2} - \sqrt{x-2}}{h}\left(\frac{\sqrt{x+h-2}+\sqrt{x-2}}{\sqrt{x+h-2}+\sqrt{x-2}}\right)$$

$$= \lim_{h\to 0} \frac{[x+h-2]-[x-2]}{h\left(\sqrt{x+h-2}+\sqrt{x-2}\right)}$$

$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h-2}+\sqrt{x-2}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h-2}+\sqrt{x-2}} = \frac{1}{\sqrt{x-2}+\sqrt{x-2}} = \boxed{\frac{1}{2\sqrt{x-2}}}$$

$$\Rightarrow \boxed{g'(x) = \frac{1}{2\sqrt{x-2}}} \qquad \boxed{g'(11) = \frac{1}{2\sqrt{11-2}} = \frac{1}{2\sqrt{9}} = \frac{1}{2\cdot 3} = \frac{1}{6}}$$

$$\boxed{g(11) = \sqrt{11-2} = \sqrt{9} = 3}$$

<u>tangent line:</u>

$$y - g(11) = g'(11)(x-11)$$
$$y - 3 = \frac{1}{6}(x-11)$$
$$y = \frac{1}{6}x - \frac{11}{6} + 3$$
$$\boxed{y = \frac{1}{6}x + \frac{7}{6}}$$

<u>Normal line:</u>

$$y - g(11) = -\frac{1}{g'(11)}(x-11)$$
$$y - 3 = -6(x-11)$$
$$y = -6x + 66 + 3$$
$$\boxed{y = -6x + 69}$$

Example. Find the tangent line and normal line of $h(x) = \dfrac{2}{\sqrt{x^2+x-2}}$ at $x = 4$.

$$h'(x) = \lim_{h\to 0} \frac{h(x+h) - h(x)}{h} = \lim_{h\to 0} \frac{\dfrac{2}{\sqrt{(x+h)^2+(x+h)-2}} - \dfrac{2}{\sqrt{x^2+x-2}}}{h}$$

$$= \lim_{h\to 0} \frac{2\sqrt{x^2+x-2} - 2\sqrt{(x+h)^2+(x+h)-2}}{\sqrt{x^2+x-2}\,\sqrt{(x+h)^2+(x+h)-2}} \cdot \frac{1}{h}\left(\frac{\sqrt{x^2+x-2} + \sqrt{(x+h)^2+(x+h)-2}}{\sqrt{x^2+x-2} + \sqrt{(x+h)^2+(x+h)-2}}\right)$$

$$= \lim_{h\to 0} \frac{2(x^2+x-2) - 2[(x+h)^2+(x+h)-2]}{h\sqrt{x^2+x-2}\,\sqrt{(x+h)^2+(x+h)-2}\left(\sqrt{x^2+x-2} + \sqrt{(x+h)^2+(x+h)-2}\right)}$$

$$= \lim_{h\to 0} \frac{2(x^2+x-2) - 2[x^2+2xh+h^2+x+h-2]}{h\sqrt{x^2+x-2}\,\sqrt{(x+h)^2+(x+h)-2}\left(\sqrt{x^2+x-2} + \sqrt{(x+h)^2+(x+h)-2}\right)}$$

$$= \lim_{h\to 0} \frac{-4xh - 2h^2 - 2h}{h\sqrt{x^2+x-2}\,\sqrt{(x+h)^2+(x+h)-2}\left(\sqrt{x^2+x-2} + \sqrt{(x+h)^2+(x+h)-2}\right)}$$

$$= \lim_{h\to 0} \frac{-4x - 2h - 2}{\sqrt{x^2+x-2}\,\sqrt{(x+h)^2+(x+h)-2}\left(\sqrt{x^2+x-2} + \sqrt{(x+h)^2+(x+h)-2}\right)}$$

$$= \frac{-4x-2}{(x^2+x-2)\,2\sqrt{x^2+x-2}} = \frac{-2x-1}{(x^2+x-2)^{3/2}} = h'(x)$$

$$\boxed{h(4) = \frac{2}{\sqrt{18}} = \frac{2}{3\sqrt{2}} \qquad h'(4) = \frac{-9}{(3\sqrt{2})^3} = \frac{-1}{6\sqrt{2}}}$$

Tangent line:

$$y - \frac{2}{3\sqrt{2}} = -\frac{1}{6\sqrt{2}}(x-4)$$

$$y = -\frac{1}{6\sqrt{2}}x + \frac{2}{3\sqrt{2}} + \frac{2}{3\sqrt{2}}$$

$$\boxed{y = -\frac{1}{6\sqrt{2}}x + \frac{4}{3\sqrt{2}}}$$

Normal line:

$$y - \frac{2}{3\sqrt{2}} = 6\sqrt{2}(x-4)$$

$$\boxed{y = 6\sqrt{2}x - 24\sqrt{2} + \frac{2}{3\sqrt{2}}}$$