

## 17.6: Surface Integrals

Imagine a sphere with a known temperature distribution. How would we find the average temperature over the sphere?

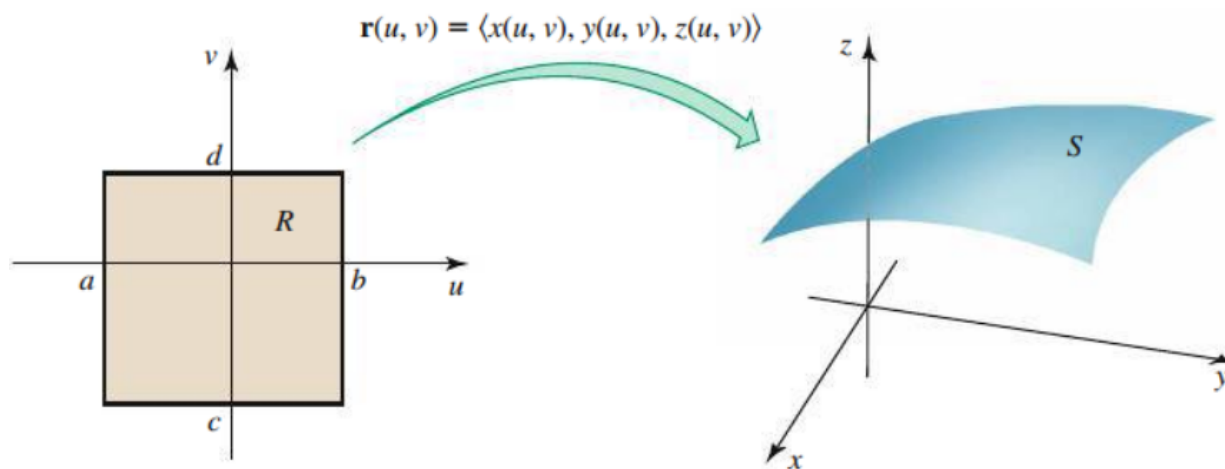
Parallel Concepts	
Curves	Surfaces
Arc length	Surface area
Line integrals	Surface integrals
One-parameter description	Two-parameter description

### Parameterized Surfaces

Recall that in  $\mathbb{R}^2$ , we parameterized a curve by  $\mathbf{r}(t) = \langle x(t), y(t) \rangle$  where  $a \leq t \leq b$ . In  $\mathbb{R}^3$ , we parameterize a surface by

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$

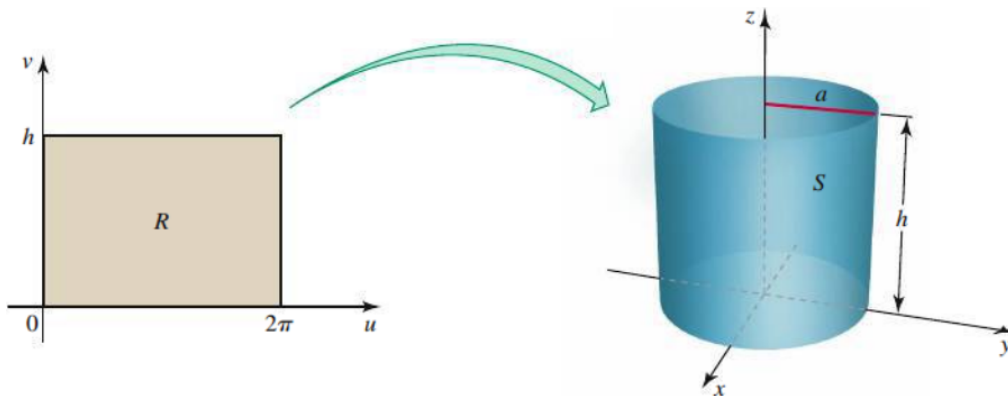
where the parameters are over  $R = \{(u, v) : a \leq u \leq b, c \leq v \leq d\}$



## Cylinders:

$$\{(x, y, z) : x = a \cos(\theta), y = a \sin(\theta), 0 \leq \theta \leq 2\pi, 0 \leq z \leq h\}$$

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle = \langle a \cos(u), a \sin(u), v \rangle$$



## Cones:

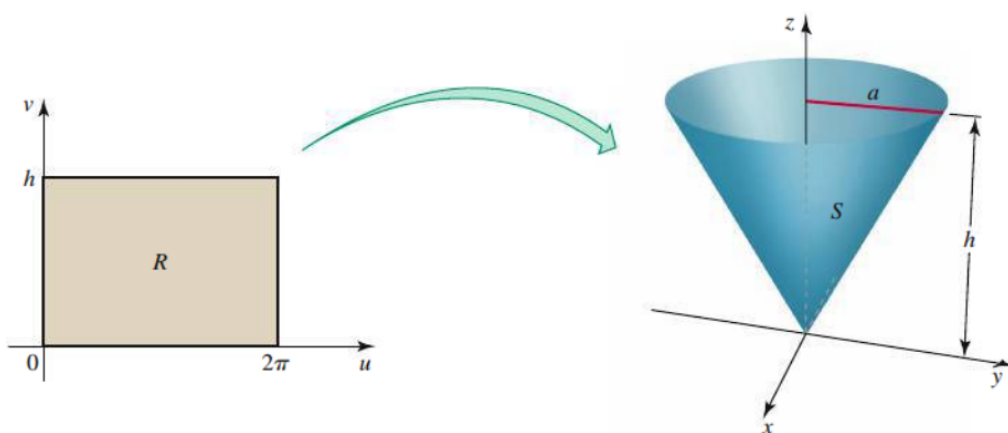
$$\{(r, \theta, z) : 0 \leq r \leq a, 0 \leq \theta \leq 2\pi, z = rh/a\}$$

For a fixed value of  $z$ ,  $r = az/h$ :

$$x = r \cos(\theta) = \frac{az}{h} \cos(\theta) \text{ and } y = r \sin(\theta) = \frac{az}{h} \sin(\theta)$$

Now, let  $u = \theta$  and  $v = z$ , then

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle = \left\langle \frac{av}{h} \cos(u), \frac{av}{h} \sin(u), v \right\rangle$$



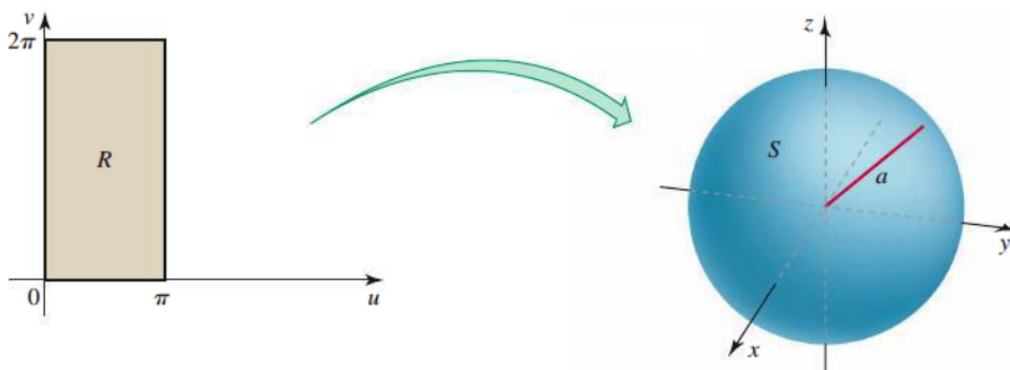
### Spheres:

$$\{(\rho, \varphi, \theta) : \rho = a, 0 \leq \varphi \leq \pi, 0 \leq \theta \leq 2\pi\}$$

$$x = a \sin(\varphi) \cos(\theta), \quad y = a \sin(\varphi) \sin(\theta), \quad z = a \cos(\varphi)$$

Now, let  $u = \theta$  and  $v = \varphi$ , then

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle = \langle a \sin(u) \cos(v), a \sin(u) \sin(v), a \cos(u) \rangle$$

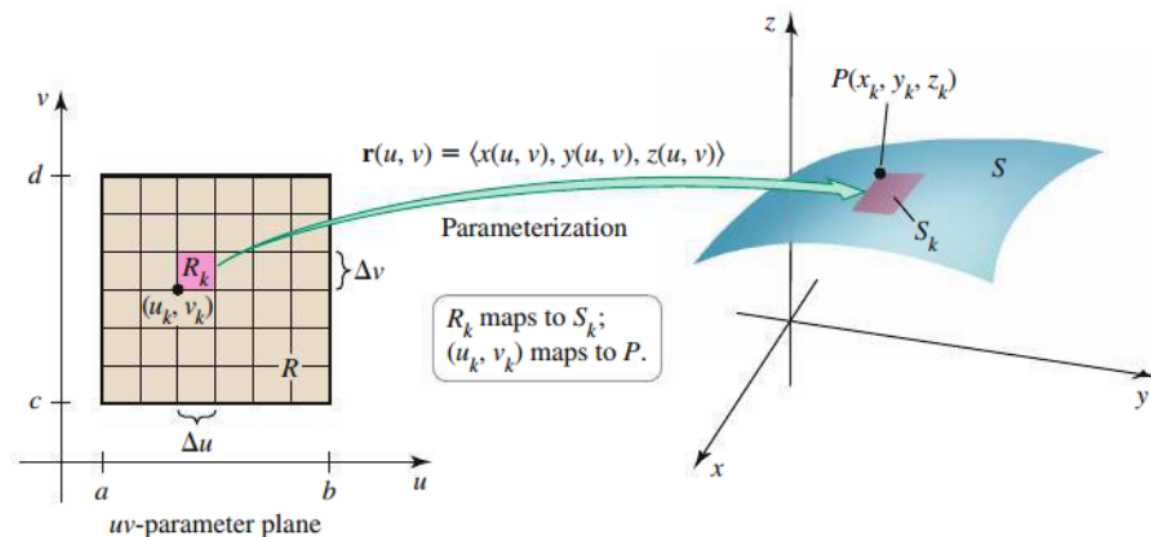


**Example.** Find parametric descriptions for the following surfaces

The plane  $3x - 2y + z = 2$

The paraboloid  $z = x^2 + y^2$ , for  $0 \leq z \leq 9$

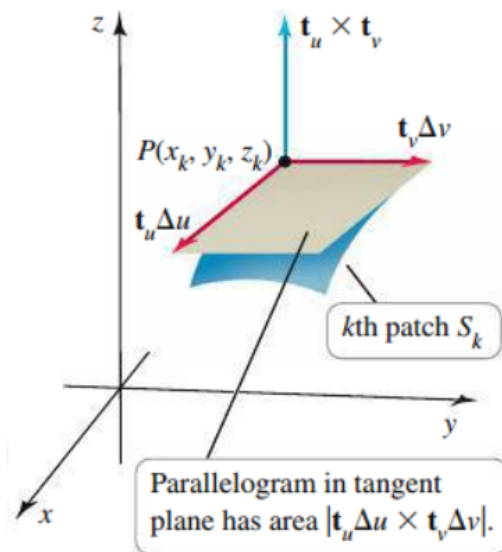
## Surface Integrals of Scalar-Valued Functions



Using the parameterization

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$

over the region  $R = \{(u, v) : a \leq u \leq b, c \leq v \leq d\}$ , it is important that we know  $\Delta S_k$ , which is the area of  $S_k$ .



**Definition. (Surface Integral of Scalar-Valued Functions on Parameterized Surfaces)**

Let  $f$  be a continuous scalar-valued function on a smooth surface  $S$  given parametrically by  $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ , where  $u$  and  $v$  vary over

$R = \{(u, v) : a \leq u \leq b, c \leq v \leq d\}$ . Assume also that the tangent vectors

$$\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \text{ and } \mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

are continuous on  $R$  and the normal vector  $\mathbf{t}_u \times \mathbf{t}_v$  is nonzero on  $R$ . Then the **surface integral of  $f$  over  $S$**  is

$$\iint_S f(x, y, z) dS = \iint_R f(x(u, v), y(u, v), z(u, v)) |\mathbf{t}_u \times \mathbf{t}_v| dA$$

If  $f(x, y, z) = 1$ , this integral equals the surface area of  $S$ .

**Example.** Find the surface area of the following surfaces

A cylinder with radius  $a > 0$  and height  $h$  (open ends)

A sphere of radius  $a$

**Example.** The temperature on the surface of a sphere of radius  $a$  varies with latitude according to the function  $T(\varphi, \theta) = 10 + 50 \sin(\varphi)$ , for  $0 \leq \varphi \leq \pi$  and  $0 \leq \theta \leq 2\pi$ . Find the average temperature over the sphere.

## Surface Integrals on Explicitly Defined Surfaces

Suppose a smooth surface  $S$  is defined explicitly as  $z = g(x, y)$ . Here, we let  $u = x$  and  $v = y$ . This gives us

$$\mathbf{t}_u = \mathbf{t}_x = \langle 1, 0, z_x \rangle, \quad \mathbf{t}_v = \mathbf{t}_y = \langle 0, 1, z_y \rangle$$

thus

$$\mathbf{t}_x \times \mathbf{t}_y = \langle -z_x, -z_y, 1 \rangle$$

and

$$|\mathbf{t}_x \times \mathbf{t}_y| = \sqrt{z_x^2 + z_y^2 + 1}$$

### Theorem 17.14: Evaluation of Surface Integrals of Scalar-Valued Functions on Explicitly Defined Surfaces

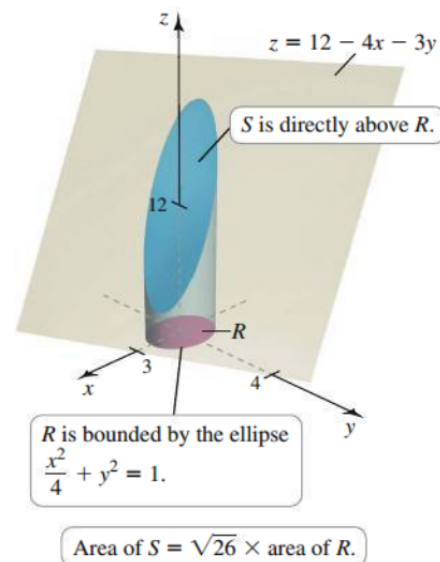
Let  $f$  be a continuous function on a smooth surface  $S$  given by  $z = g(x, y)$ , for  $(x, y)$  in a region  $R$ . The surface integral of  $f$  over  $S$  is

$$\iint_S f(x, y, z) \, dS = \iint_S f(x, y, g(x, y)) \sqrt{z_x^2 + z_y^2 + 1} \, dA.$$

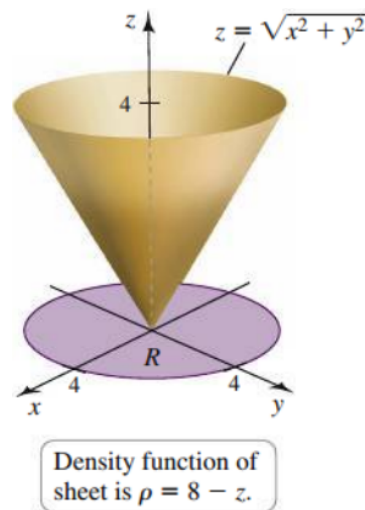
If  $f(x, y, z) = 1$ , the surface integral equals the area of the surface.



**Example.** Find the area of the surface  $S$  that lies in the plane  $z = 12 - 4x - 3y$  directly above the region  $R$  bounded by the ellipse  $x^2/4 + y^2 = 1$



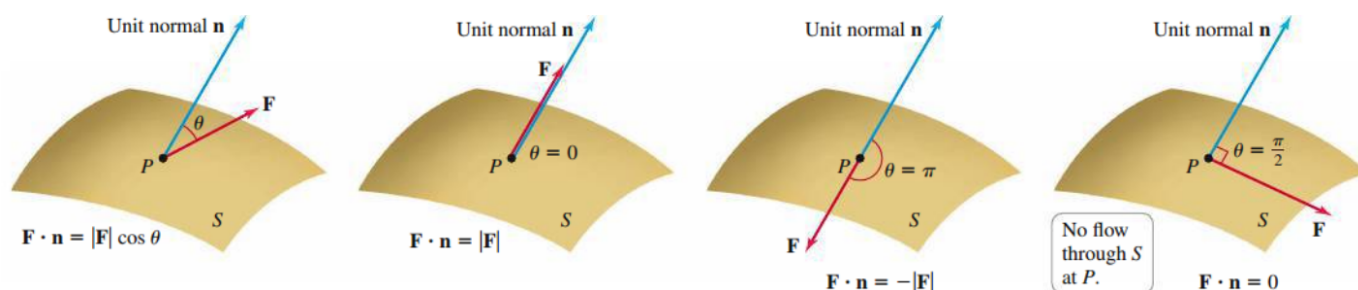
**Example.** A thin conical sheet is described by the surface  $z = (x^2 + y^2)^{\frac{1}{2}}$ , for  $0 \leq z \leq 4$ . The density of the sheet in  $\text{g/cm}^2$  is  $\rho = f(x, y, z) = (8 - z)$ . What is the mass of the cone?



Explicit Description $z = g(x, y)$				Parametric Description		
Surface	Equation	Normal vector $\pm \langle -z_x, -z_y, 1 \rangle$	magnitude $ \langle -z_x, -z_y, 1 \rangle $	Equation	Normal vector $\mathbf{t}_u \times \mathbf{t}_v$	magnitude $ \mathbf{t}_u \times \mathbf{t}_v $
<b>Cylinder</b>	$x^2 + y^2 = a^2,$ $0 \leq z \leq h$	$\langle x, y, 0 \rangle$	$a$	$\mathbf{r} = \langle a \cos(u), a \sin(u), v \rangle,$ $0 \leq u \leq 2\pi, 0 \leq v \leq h$	$\langle a \cos(u), a \sin(u), 0 \rangle$	$a$
<b>Cone</b>	$z^2 = x^2 + y^2,$ $0 \leq z \leq h$	$\langle x/z, y/z, -1 \rangle$	$\sqrt{2}$	$\mathbf{r} = \langle v \cos(u), v \sin(u), v \rangle,$ $0 \leq u \leq 2\pi, 0 \leq v \leq h$	$\langle v \cos(u), v \sin(u), -v \rangle$	$\sqrt{2}v$
<b>Sphere</b>	$x^2 + y^2 + z^2 = a^2$	$\langle x/z, y/z, 1 \rangle;$	$a/z$	$\mathbf{r} = \langle a \sin(u) \cos(v),$ $a \sin(u) \sin(v),$ $a \cos(u) \rangle$ $0 \leq u \leq \pi, 0 \leq v \leq 2\pi$	$\langle a^2 \sin^2(u) \cos(v),$ $a^2 \sin^2(u) \sin(v),$ $a^2 \sin(u) \cos(u) \rangle$	$a^2 \sin(u)$
<b>Paraboloid</b>	$z = x^2 + y^2,$ $0 \leq z \leq h$	$\langle 2x, 2y, -1 \rangle$	$\sqrt{1 + 4(x^2 + y^2)}$	$\mathbf{r} = \langle v \cos(u), v \sin(u), v^2 \rangle,$ $0 \leq u \leq 2\pi, 0 \leq v \leq \sqrt{h}$	$\langle 2v^2 \cos(u), 2v^2 \sin(u), -v \rangle$	$v\sqrt{1 + 4v^2}$

## Flux Integrals:

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$



The unit normal vector we use is

$$\mathbf{n} = \frac{\mathbf{t}_u \times \mathbf{t}_v}{|\mathbf{t}_u \times \mathbf{t}_v|}$$

giving us

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \iint_R \mathbf{F} \cdot \mathbf{n} |\mathbf{t}_u \times \mathbf{t}_v| \, dA \\ &= \iint_R \mathbf{F} \cdot \frac{\mathbf{t}_u \times \mathbf{t}_v}{|\mathbf{t}_u \times \mathbf{t}_v|} |\mathbf{t}_u \times \mathbf{t}_v| \, dA \\ &= \iint_R \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v) \, dA \end{aligned}$$

When the surface  $S$  is explicitly given as  $z = s(x, y)$ , then

$$\mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v) = -fz_x - gz_y + h$$

**Definition. (Surface Integral of a Vector Field)**

Suppose  $\mathbf{F} = \langle f, g, h \rangle$  is a continuous vector field on a region of  $\mathbb{R}^3$  containing a smooth oriented surface  $S$ . If  $S$  is defined parametrically as  $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ , for  $(u, v)$  in a region  $R$ , then

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v) \, dA,$$

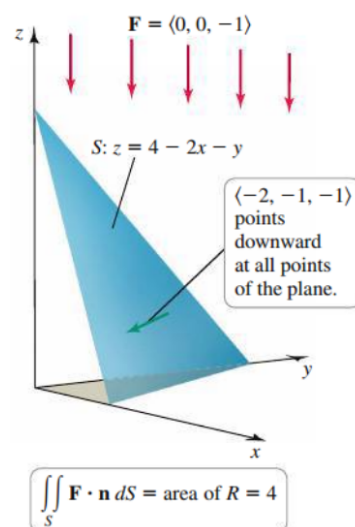
where

$$\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \text{ and } \mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

are continuous on  $R$ , the normal vector  $\mathbf{t}_u \times \mathbf{t}_v$  is nonzero on  $R$ , and the direction of the normal vector is consistent with the orientation of  $S$ . If  $S$  is defined in the form  $z = s(x, y)$ , for  $(x, y)$  in a region  $R$ , then

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_R (-f z_x - g z_y + h) \, dA.$$

**Example.** Consider the vertical field  $\mathbf{F} = \langle 0, 0, -1 \rangle$ . Find the flux in the downward direction across the surface  $S$ , which is the plane  $z = 4 - 2x - y$  in the first octant.



**Example.** Consider the radial vector field  $\mathbf{F} = \langle f, g, h \rangle = \langle x, y, z \rangle$ . Compute the upward flux of the field across the:

hemisphere  $x^2 + y^2 + z^2 = 1$ , for  $z \geq 0$ ,

paraboloid  $z = 1 - x^2 - y^2$ , for  $z \geq 0$ .