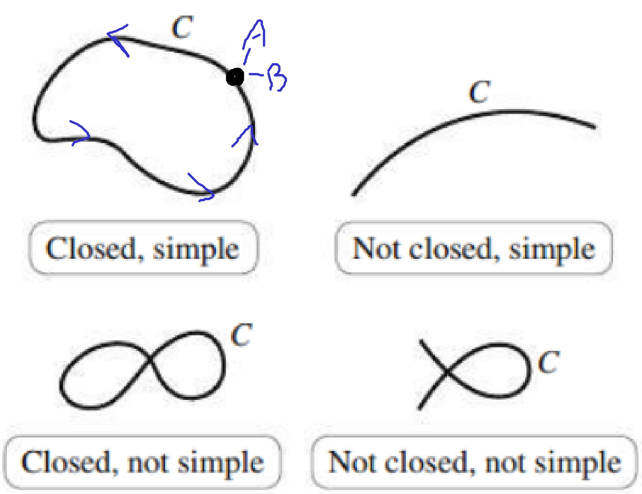
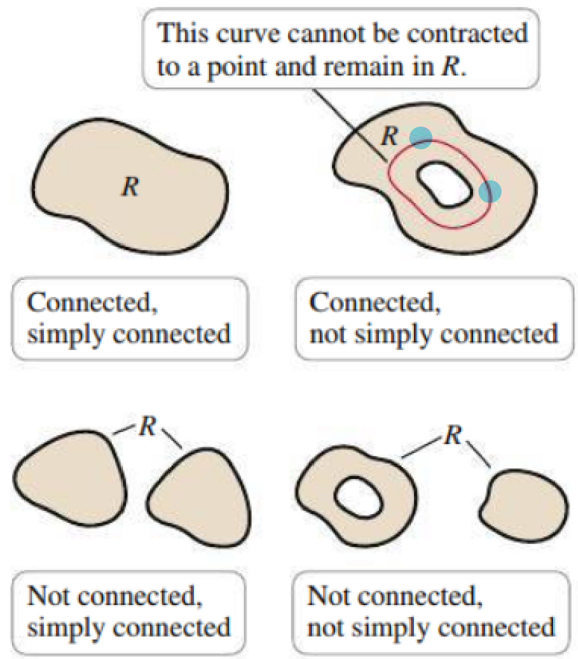
17.3: Conservative Vector Fields

Definition. (Simple and Closed Curves)
Suppose a curve C (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is described parametrically by $\mathbf{r}(t)$, where $a \le t \le b$. Then C is a **simple curve** if $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ for all t_1 and t_2, with $a < t_1 < t_2 < b$; that is, C never intersects itself between its endpoints. The curve C is **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$; that is, the initial and terminal points of C are the same.





Definition. (Connected and Simply Connected Regions)
An open region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) is **connected** if it is possible to connect any two points of R by a continuous curve lying in R. An open region R is **simply connected** if every closed simple curve in R can be deformed and contracted to a point in R.

Definition. (Conservative Vector Field)
A vector field $\mathbf{F}$ is said to be **conservative** on a region (in $\mathbb{R}^2$ or $\mathbb{R}^3$) if there exists a scalar function φ such that $\mathbf{F} = \nabla\varphi$ on that region.

Assume that $\mathbf{F} = \langle f, g, h\rangle$ is a conservative vector field. Then, there exists φ such that

$$\langle f, g, h\rangle = \nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z\rangle$$

Now, we consider the second partial derivatives:

$\varphi_{xy} = \varphi_{yx} \Rightarrow$ $(\varphi_x)_y = (\varphi_y)_x \Rightarrow f_y = g_x$

$\varphi_{xz} = \varphi_{zx} \Rightarrow$ $(\varphi_x)_z = (\varphi_z)_x \Rightarrow f_z = h_x$

$\varphi_{yz} = \varphi_{zy} \Rightarrow$ $(\varphi_y)_z = (\varphi_z)_y \Rightarrow g_z = h_y$

Theorem 17.3: Test for Conservative Vector Fields
Let $\mathbf{F} = \langle f, g, h\rangle$ be a vector field defined on a connected and simply connected region D of $\mathbb{R}^3$, where f, g, and h have continuous first partial derivatives on D. Then $\mathbf{F}$ is a conservative vector field on D (there is a potential function φ such that $\mathbf{F} = \nabla\varphi$) if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \qquad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \qquad \text{and} \qquad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$$

For vector fields in $\mathbb{R}^2$, we have the single condition $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$.

Example. Determine if the following vector fields are conservative:

$$\mathbf{F} = \langle e^x \cos(y), -e^x \sin(y) \rangle = \langle f, g \rangle \qquad f_y = g_x$$

$$\left.\begin{aligned} f_y &= \frac{\partial}{\partial y}\left(e^x \cos(y)\right) = -e^x \sin(y) \\ g_x &= \frac{\partial}{\partial x}\left(-e^x \sin(y)\right) = -e^x \sin(y) \end{aligned}\right\} \Rightarrow \text{conservative.}$$

$$\vec{F} = \langle x, x \rangle \qquad \left.\begin{aligned} f_y &= 0 \\ g_x &= 1 \end{aligned}\right\} \text{Not conservative}$$

$$\mathbf{F} = \left\langle 2xy - z^2, x^2 + 2z, 2y - 2xz \right\rangle$$

$$f_y = 2x \qquad f_z = -2z$$

$$g_x = 2x \qquad g_z = 2$$

$$h_x = -2z \qquad h_y = 2$$

Procedure: Finding Potential Functions in $\mathbb{R}^3$
Suppose $\mathbf{F} = \langle f, g, h \rangle$ is a conservative vector field. To find φ such that $\mathbf{F} = \nabla\varphi$, use the following steps: $\to \langle \varphi_x, \varphi_y, \varphi_z \rangle$

1. Integrate $\varphi_x = f$ with respect to x to obtain φ, which includes an arbitrary function $c(y, z)$.
2. Compute φ_y and equate it to g to obtain an expression for $c_y(y, z)$.
3. Integrate $c_y(y, z)$ with respect to y to obtain $c(y, z)$, including an arbitrary function $d(z)$.
4. Compute φ_z and equate it to h to get $d(z)$.

A similar procedure beginning with $\varphi_y = g$ or $\varphi_z = h$ may be easier in some cases.

Example. Find a potential function for the following conservative vector fields:

$$\mathbf{F} = \langle e^x \cos(y), -e^x \sin(y) \rangle = \langle \varphi_x, \varphi_y \rangle$$

$$\varphi(x,y) = \int \varphi_x \, dx = \int e^x \cos(y) \, dx = e^x \cos(y) + c(y)$$

$$\varphi_y = \frac{\partial}{\partial y} \varphi(x,y) = -e^x \sin(y) + c_y(y) = -e^x \sin(y)$$

$$\implies c_y(y) = 0$$

$$\implies c(y) = \int c_y(y) \, dy = 0$$

$$\implies \varphi(x,y) = e^x \cos(y)$$

$$\varphi(x,y) = \int \varphi_x \, dx = e^x \cos(y)$$

$$\varphi(x,y) = \int \varphi_y \, dy = e^x \cos(y)$$

$\mathbf{F} = \langle 2xy - z^2, x^2 + 2z, 2y - 2xz \rangle = \langle \varphi_x, \varphi_y, \varphi_z \rangle$

$\varphi(x,y,z) = \int \varphi_x \, dx = \int 2xy - z^2 \, dx = x^2 y - x z^2 + c(y,z)$

$\varphi_y = \frac{\partial}{\partial y} \varphi(x,y,z) = x^2 - 0 + c_y(y,z) = x^2 + 2z$

$\Rightarrow c_y(y,z) = 2z$

$\Rightarrow c(y,z) = \int c_y(y,z)\, dy = 2yz + d(z)$

$\varphi_z = \frac{\partial}{\partial z} \varphi(x,y,z) = \frac{\partial}{\partial z}\left(x^2 y - x z^2 + 2yz + d(z) \right) = 0 - 2xz + 2y + d_z(z)$

$= 2y - 2xz$

$\Rightarrow \varphi(x,y,z) = x^2 y - x z^2 + 2yz \qquad \Rightarrow d_z(z) = 0 \qquad d(z) = \int d_z(z)\, dz = 0$

$\int \varphi_x \, dx = x^2 y - x z^2$

$\int \varphi_y \, dy = x^2 y + 2yz$

$\int \varphi_z \, dz = 2yz - x z^2$

$\varphi(x,y,z) = x^2 y + 2yz - x z^2$

Fundamental Theorem for Line Integrals and Path Independence:
Suppose that $\mathbf{F}$ is a conservative vector field in $\mathbb{R}^3$ with potential function φ.

$$\begin{aligned}\frac{d\varphi}{dt} &= \frac{\partial\varphi}{\partial x}\frac{dx}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dt} + \frac{\partial\varphi}{\partial z}\frac{dz}{dt} \\ &= \left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \\ &= \nabla\varphi \cdot \mathbf{r}'(t) \\ &= \mathbf{F} \cdot \mathbf{r}'(t),\end{aligned}$$

where $\mathbf{r}(t)$ defines a curve C for $a \le t \le b$. Now, we integrate $\mathbf{F}$ over the curve C:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\,dt = \int_a^b \frac{d\varphi}{dt}\,dt = \varphi(B) - \varphi(A)$$

where A and B are points corresponding to $\mathbf{r}(a)$ and $\mathbf{r}(b)$ respectively.

Theorem 17.4: Fundamental Theorem for Line Integrals
Let R be a region in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let φ be a differentiable potential function defined on R. If $\mathbf{F} = \nabla\varphi$ (which means that $\mathbf{F}$ is conservative), then

$$\int_C \mathbf{F} \cdot \mathbf{T}\,ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A),$$

for all points A and B in R and all piecewise-smooth oriented curves C in R from A to B.

Definition. (Independence of Path)
Let $\mathbf{F}$ be a continuous vector field with domain R. If $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for all piecewise-smooth curves C_1 and C_2 in R with the same initial and terminal points, then the line integral is **independent of path**.

Theorem 17.5
Let $\mathbf{F}$ be a continuous vector field on an open connected region R in $\mathbb{R}^2$. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, then $\mathbf{F}$ is conservative; that is, there exists a potential function φ such that $\mathbf{F} = \nabla\varphi$ on R.

Example. Consider the potential function $\varphi(x, y) = \left(x^2 - y^2\right)/2$ with gradient field $\mathbf{F} = \langle x, -y \rangle$.

- Let C_1 be the quarter-circle $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$, for $0 \leq t \leq \pi/2$, from $A(1, 0)$ to $B(0, 1)$,
- let C_2 be the line $\mathbf{r}(t) = \langle 1 - t, t \rangle$, for $0 \leq t \leq 1$, also from A to B.

Evaluate the line integrals of $\mathbf{F}$ on C_1 and C_2, and show that both are equal to $\varphi(B) - \varphi(A)$.

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \langle \cos(t), -\sin(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle \, dt = \int_0^{\pi/2} -2\sin(t)\cos(t) \, dt$$

$$= -\int_0^{\pi/2} \sin(2t) \, dt = \left. \frac{\cos(2t)}{2} \right|_{t=0}^{t=\pi/2} = \frac{1}{2}\left(-1 - 1\right) = \boxed{-1}$$

LC #1
-1

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 \langle 1-t, -t \rangle \cdot \langle -1, 1 \rangle \, dt = \int_0^1 -1 \, dt = \left. -t \right|_{t=0}^{t=1} = \boxed{-1}$$

LC #2
-1

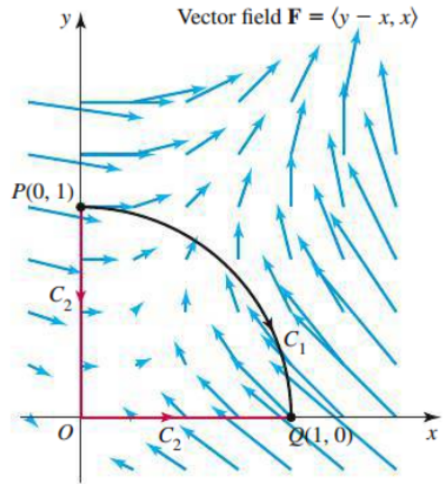
$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} = \varphi(B) - \varphi(A) = \varphi(0,1) - \varphi(1,0) = \frac{-1}{2} - \frac{1}{2} = \boxed{-1}$$

$B(0,1)$ $A(1,0)$

$$\varphi(x, y) = \frac{x^2 - y^2}{2}$$

Example. With $\mathbf{F} = \langle y - x,\, x\rangle$ on the following oriented paths in $\mathbb{R}^2$.

a) Find the potential function $\varphi(x, y)$



$F = \langle y - x,\, x\rangle = \langle \varphi_x, \varphi_y\rangle$

$\varphi(x,y) = \int \varphi_y \, dy = \int x \, dy = xy + c(x)$

$\varphi_x = \frac{\partial}{\partial x}\left(xy + c(x)\right) = y + c_x(x) = y - x$

$\Rightarrow c_x(x) = -x$

$c(x) = \int c_x(x)\, dx = -\frac{x^2}{2}$

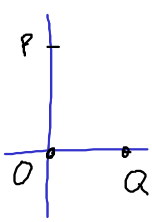
$\varphi(x,y) = xy - \frac{x^2}{2}$

b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along

the quarter-circle C_1 from $P(0, 1)$ to $Q(1, 0)$,

$\int_{C_1} \vec{F} \cdot d\vec{r} = \varphi(Q) - \varphi(P) = \varphi(1,0) - \varphi(0,1)$

$= -\frac{1}{2} - 0 = \boxed{-\frac{1}{2}}$



the path C_2 from $P(0, 1)$ to $Q(1, 0)$ via two line segments through $O(0, 0)$.

$\int_{C_2} \vec{F} \cdot d\vec{r} = \varphi(Q) - \varphi(P) = \boxed{-\frac{1}{2}}$

$= \left(\varphi(0) - \varphi(P)\right) + \left(\varphi(Q) - \varphi(0)\right)$

Example. Evaluate

$$\int_C \langle 2xy - z^2,\ x^2 + 2z,\ 2y - 2xz\rangle\, d\mathbf{r}$$

where C is the curve from $A(-3,-2,1)$ to $B(1,2,3)$.

Old Line integral method

Find $\vec{r}$

Find $\vec{r}\,'(t)$

Rewrite $\vec{F}$

Find $\varphi(x,y,z)$

$$\varphi(x,y,z) = \int \varphi_y \, dy = \int x^2 + 2z \, dy = x^2y + 2yz + C(x,z)$$

$$\varphi_x = \frac{\partial}{\partial x}\left(x^2y + 2yz + C(x,z)\right) = 2xy + C_x(x,z) = 2xy - z^2$$

$$\Rightarrow C_x(x,z) = -z^2$$

$$C(x,z) = \int C_x(x,z)\,dx = -xz^2 + d(z)$$

$$\varphi(x,y,z) = x^2y + 2yz - xz^2 + d(z)$$

$$\varphi_z = \frac{\partial}{\partial z}\left(x^2y + 2yz - xz^2 + d(z)\right) = 2y - 2xz + d_z(z) = 2y - 2xz$$

$$d_z(z) = 0$$

$$d(z) = \int d_z(z)\,dz = 0$$

$$\Longrightarrow \varphi(x,y,z) = x^2y + 2yz - xz^2$$

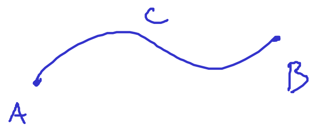
$$\int_C \langle 2xy - z^2,\ x^2 + 2z,\ 2y - xz\rangle \cdot d\vec{r} = \varphi(B) - \varphi(A)$$

$A(-3,-2,1)$

$B(1,2,3)$

$$= [2 + 12 - 9] - [-18 - 4 + 3]$$

$$= 23 + 19 = \boxed{42}$$



$\int_C \vec{F} \cdot d\vec{r} = \varphi(B) - \varphi(A)$

Line Integrals on Closed Curves
Let R be on open connected region in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\mathbf{F}$ is a conservative vector field on R if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple closed piecewise-smooth oriented curves C in R.

Example. Evaluate $\int_C \langle 2xy + z^2, x^2, 2xz \rangle \cdot d\mathbf{r}$ where C is the circle $\mathbf{r}(t) = \langle 3\cos(t), 4\cos(t), 5\sin(t) \rangle$, for $0 \le t \le 2\pi$.

Show conservative

$f_y = 2x \qquad f_z = 2z$

$g_x = 2x \qquad g_z = 0$

$h_x = 2z \qquad h_y = 0$

$\left.\begin{array}{l} \vec{r}(0) = \langle 3, 4, 0 \rangle \\ \vec{r}(2\pi) = \langle 3, 4, 0 \rangle \end{array}\right\} \to C$ closed curve $\Rightarrow \oint_C \langle 2xy + z^2, x^2, 2xz \rangle \cdot d\vec{r} = 0$