1 14.2: Calculus of Vector-Valued Functions

Definition. (Derivative and Tangent Vector)
Let $\mathbf{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$, where f,g, and h are differentiable functions on (a, b). Then $\mathbf{r}$ has a **derivative** (or is **differentiable**) on (a, b) and

$$\mathbf{r}'(t) = f'(t)\boldsymbol{i} + g'(t)\boldsymbol{j} + h'(t)\boldsymbol{k}.$$

Provided $\mathbf{r}'(t) \neq \mathbf{0}$, $\mathbf{r}'(t)$ is a **tangent vector** at the point corresponding to $\mathbf{r}(t)$.

Example. For the following functions below, find $\mathbf{r}'(t)$

a) $\mathbf{r}(t) = \left\langle e^{-t^2}, \log_2(t-4), \sin(t) \right\rangle$

b) $\mathbf{r}(t) = 3\boldsymbol{i} - 2\tan(t)\boldsymbol{j} + e^t\boldsymbol{k}$

Example. For $\mathbf{r}(t) = \langle 3t, \sec(2t), \cos(t)\rangle$ compute $\mathbf{r}'(t)$ at $t = \frac{\pi}{4}$.

Definition. (Unit Tangent Vector)
Let $\mathbf{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$ be a smooth parameterized curve, for $a \leq t \leq b$. The **unit tangent vector** for a particular value of t is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Example. For $\mathbf{r}(t) = \langle 3\sin(t),\, -2\cos(2t),\, 3\cos(t)\rangle$, find the unit tangent vector.

Example. For $\mathbf{r}(t) = \langle \sin(6t),\, 3t,\, \cos(3t)\rangle$, compute $\mathbf{T}\left(\frac{\pi}{3}\right)$.

Derivative Rules

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector-valued functions, and let f be a differentiable scalar-valued function, all at a point t. Let $\mathbf{c}$ be a constant vector. The following rules apply.

1. $\dfrac{d}{dt}(\mathbf{c}) = \mathbf{0}$ — Constant Rule
2. $\dfrac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$ — Sum Rule
3. $\dfrac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ — Product Rule
4. $\dfrac{d}{dt}(\mathbf{u}(f(t))) = \mathbf{u}'(f(t))f'(t)$ — Chain Rule
5. $\dfrac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ — Dot Product Rule
6. $\dfrac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ — Cross Product Rule

Example. Given $\mathbf{u}(t) = \left\langle 4t^2,\, 1,\, 3t \right\rangle$ and $\mathbf{v}(t) = \left\langle e^{-2t},\, -2e^t,\, e^t \right\rangle$, find $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$.

Definition. (Indefinite Integral of a Vector-Valued Function)
Let $\mathbf{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$ be a vector function, and let
$\mathbf{R}(t) = F(t)\boldsymbol{i} + G(t)\boldsymbol{j} + H(t)\boldsymbol{k}$, where F, G, and H are antiderivatives of f, g, and h, respectively. The **indefinate integral** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C},$$

where $\mathbf{C}$ is an arbitrary constant vector. Alternatively, in component form,

$$\int \langle f(t),\, g(t),\, h(t)\rangle\,dt = \langle F(t),\, G(t),\, H(t)\rangle + \langle C_1,\, C_2,\, C_3\rangle.$$

Example. Find $\mathbf{r}(t)$ such that $\mathbf{r}'(t) = \left\langle \frac{t}{t^2+1},\, t^2e^{-t^3},\, \frac{-2t}{\sqrt{t^2+16}} \right\rangle$ and $\mathbf{r}(0) = \left\langle 3,\, \frac{5}{3},\, -5 \right\rangle$.

Definition. (Definite Integral of a Vector-Valued Function)
Let $\mathbf{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$, where f, g, and h are integrable on the interval $[a, b]$. The **definite integral** of $\mathbf{r}$ on $[a, b]$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\boldsymbol{i} + \left(\int_a^b g(t)\,dt\right)\boldsymbol{j} + \left(\int_a^b h(t)\,dt\right)\boldsymbol{k}$$

Example. $\displaystyle\int_{-\pi}^{\pi} \langle \sin(t),\, \cos(t),\, 8t \rangle\,dt$