

## 17.5: Divergence and Curl

The idea behind Green's Theorem can be extended from  $\mathbb{R}^2$  to  $\mathbb{R}^3$ . The following tools are needed to accomplish this:

- three-dimensional divergence and curl (17.5)
- *surface integrals* (17.6)
- *Stokes' Theorem* (17.7): relates line integrals over a simple closed oriented curve in  $\mathbb{R}^3$  to a double integral over a surface whose boundary is that curve
- *Divergence Theorem* (17.8): relates integrals over a closed oriented surface in  $\mathbb{R}^3$  to triple integrals over the corresponding region

### Divergence:

Recall the *del operator*  $\nabla$ :

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$$

When  $f$  is a scalar valued function, we obtain the gradient:

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} = \langle f_x, f_y, f_z \rangle$$

The dot product of  $\nabla$  and a vector field  $\mathbf{F}$ , produces the three dimensional divergence:

$$\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, h \rangle = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

#### Definition. (Divergence of a Vector Field)

The **divergence** of a vector field  $\mathbf{F} = \langle f, g, h \rangle$  that is differentiable on a region of  $\mathbb{R}^3$  is

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

If  $\nabla \cdot \mathbf{F} = 0$ , the vector field is **source free**.

**Example.** Compute the divergence of the following vector fields

$$\mathbf{F} = \langle x, -2y, 3z \rangle$$

$$\mathbf{F} = \langle -y, x - z, y \rangle$$

$$\mathbf{F} = \langle 4yz \cos(x), 3xz \tan(y), -5xy \csc(z) \rangle$$

**Example.** Compute the divergence of the radial vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}}.$$

**Theorem 17.10: Divergence of Radial Vector Fields**

For a real number  $p$ , the divergence of the radial vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{p/2}} \quad \text{is} \quad \nabla \cdot \mathbf{F} = \frac{3 - p}{|\mathbf{r}|^p}.$$

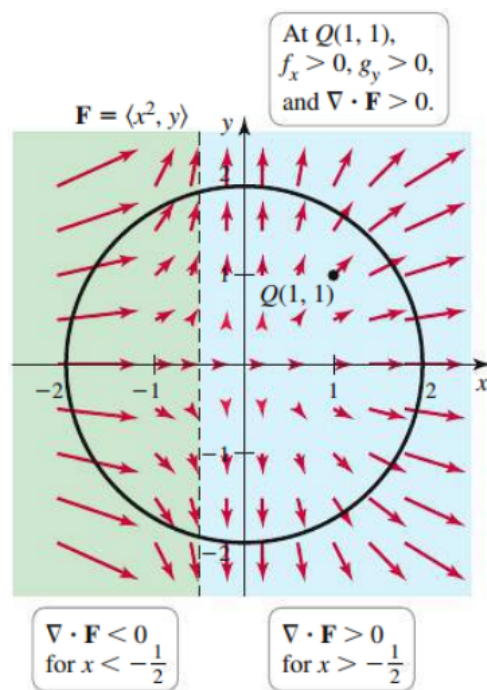
**Example.** Consider the two-dimensional vector field  $\mathbf{F} = \langle x^2, y \rangle$  and a circle  $C$  of radius 2 centered at the origin.

Compute the two-dimensional divergence at  $Q$ .

Where is the divergence positive? Negative?

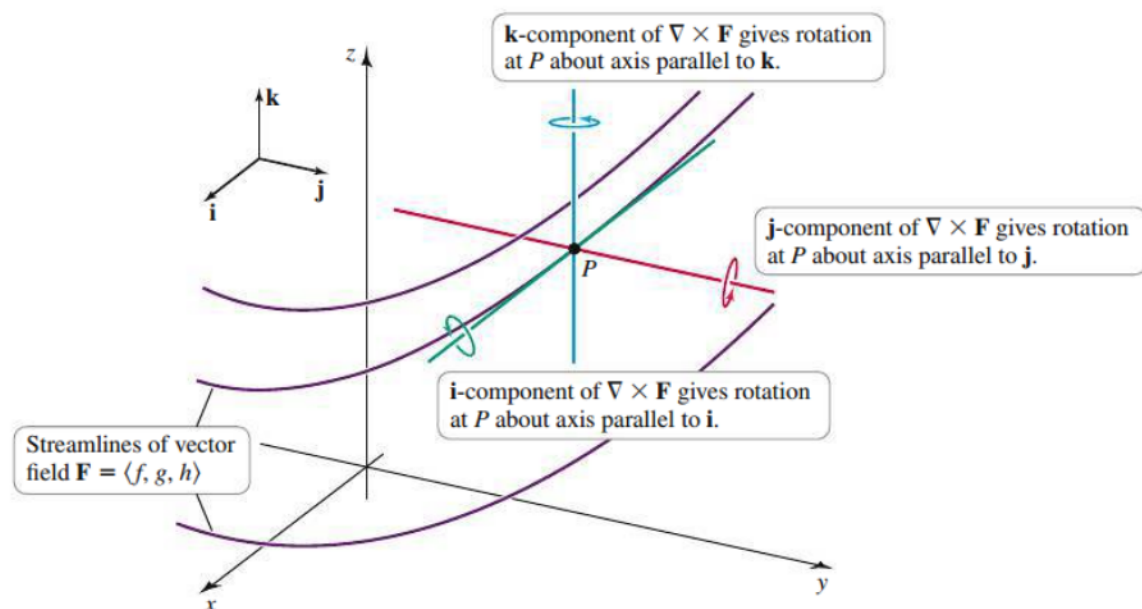
Where on  $C$  is the flux outward? Inward?

Is the net flux across  $C$  positive or negative?



## Curl:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle$$



### Definition. (Curl of a Vector Field)

The **curl** of a vector field  $\mathbf{F} = \langle f, g, h \rangle$  that is differentiable on a region of  $\mathbb{R}^3$  is

$$\begin{aligned} \nabla \times \mathbf{F} &= \text{curl } \mathbf{F} \\ &= \left( \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \mathbf{i} + \left( \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \mathbf{j} + \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathbf{k} \end{aligned}$$

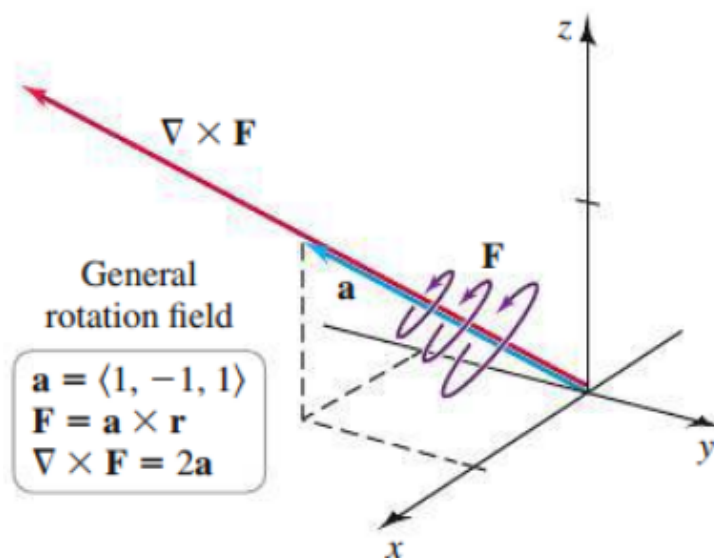
If  $\nabla \times \mathbf{F} = \mathbf{0}$ , the vector field is **irrotational**.

## Curl of a General Rotation Vector Field

Let  $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ , where  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ . Then

$$\nabla \cdot \mathbf{F} = 0$$

$$\nabla \times \mathbf{F} = 2\mathbf{a}$$



### General Rotation Vector Field

The **general rotation vector field** is  $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ , when the nonzero constant vector  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  is the axis of rotation and  $\mathbf{r} = \langle x, y, z \rangle$ . For all nonzero choices of  $\mathbf{a}$ ,  $|\nabla \times \mathbf{F}| = 2|\mathbf{a}|$  and  $\nabla \cdot \mathbf{F} = 0$ . If  $\mathbf{F}$  is a vector field, then the constant angular speed of the field is

$$\omega = |\mathbf{a}| = \frac{1}{2}|\nabla \times \mathbf{F}|.$$

**Example.** Compute the curl of the rotational field  $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ , where  $\mathbf{a} = \langle -3, 2, 1 \rangle$  and  $\mathbf{r} = \langle x, y, z \rangle$ . What are the direction and magnitude of the curl?

## Properties of Divergence and Curl:

### Divergence Properties

$$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$$

$$\nabla \cdot (c\mathbf{F}) = c(\nabla \cdot \mathbf{F})$$

### Curl Properties

$$\nabla \times (\mathbf{F} + \mathbf{G}) = (\nabla \times \mathbf{F}) + (\nabla \times \mathbf{G})$$

$$\nabla \times (c\mathbf{F}) = c(\nabla \times \mathbf{F})$$

### Theorem 17.11: Curl of a Conservative Vector Field

Suppose  $\mathbf{F}$  is a conservative vector field on an open region  $D$  of  $\mathbb{R}^3$ . Let  $\mathbf{F} = \nabla\varphi$ , where  $\varphi$  is a potential function with continuous second partial derivatives on  $D$ . Then  $\nabla \times \mathbf{F} = \nabla \times \nabla\varphi = \mathbf{0}$ : The curl of the gradient is the zero vector and  $\mathbf{F}$  is irrotational.

*Proof.*

$$\nabla \times \mathbf{F} = \nabla \times \nabla\varphi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi_x & \varphi_y & \varphi_z \end{vmatrix} = \langle \varphi_{zy} - \varphi_{yz}, \varphi_{xz} - \varphi_{zx}, \varphi_{yx} - \varphi_{xy} \rangle = \mathbf{0}$$

□

### Theorem 17.12: Divergence of the Curl

Suppose  $\mathbf{F} = \langle f, g, h \rangle$ , where  $f$ ,  $g$ , and  $h$  have continuous second partial derivatives. Then  $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ : The divergence of the curl is zero.

*Proof.*

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{F}) &= \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) + \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial z} \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \\ &= (h_{yx} - h_{xy}) + (g_{xz} - g_{zx}) + (f_{zy} - f_{yz}) = 0 \end{aligned}$$

□



The **Laplacian**, denoted  $\nabla^2 u$  or  $\Delta u$ , arises from  $\nabla \cdot \nabla u$ :

$$\nabla \cdot \nabla u = \frac{\partial}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial}{\partial y} \frac{\partial u}{\partial y} + \frac{\partial}{\partial z} \frac{\partial u}{\partial z} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

**Theorem 17.13: Product Rule for the Divergence**

Let  $u$  be a scalar-valued function that is differentiable on a region  $D$  and let  $\mathbf{F}$  be a vector field that is differentiable on  $D$ . Then

$$\nabla \cdot (u\mathbf{F}) = \nabla u \cdot \mathbf{F} + u(\nabla \cdot \mathbf{F}).$$

**Example.** Let  $\mathbf{r} = \langle x, y, z \rangle$  and let  $\varphi = \frac{1}{|\mathbf{r}|}$  be a potential function.

Find the associated gradient field  $\mathbf{F} = \nabla \left( \frac{1}{|\mathbf{r}|} \right)$

Compute  $\nabla \cdot \mathbf{F}$

### Properties of a Conservative Vector Field

Let  $\mathbf{F}$  be a conservative vector field whose components have continuous second partial derivatives on an open connected region  $D$  in  $\mathbb{R}^3$ . Then  $\mathbf{F}$  has the following equivalent properties.

1. There exists a potential function  $\varphi$  such that  $\mathbf{F} = \nabla \varphi$  (definition).
2.  $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$  for all points  $A$  and  $B$  in  $D$  and all piecewise smooth oriented curves  $C$  in  $D$  from  $A$  to  $B$ .
3.  $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$  on all simple piecewise-smooth closed oriented curves  $C$  in  $D$ .
4.  $\nabla \times \mathbf{F} = \mathbf{0}$  at all points of  $D$ .