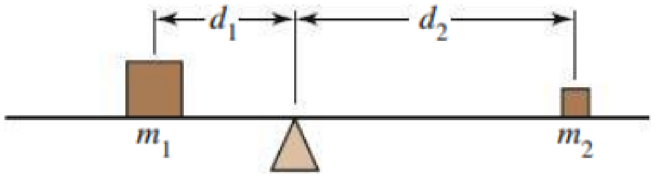
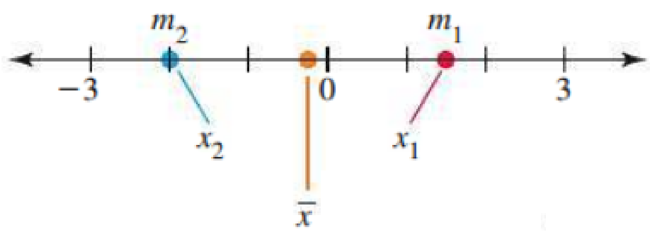
1 16.6: Integrals for Mass Calculations

Suppose we have two masses m_1 and m_2 on a beam (with no mass) that are distances d_1 and d_2 away from a pivot point. This beam will be balanced when $m_1 d_1 = m_2 d_2$.



This concept can be used to to find the balance point $\bar{x}$ between 2 objects with masses m_1 and m_2:



$$m_1(x_1 - \bar{x}) = m_2(\bar{x} - x_2) \;\Rightarrow\; m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) = 0.$$

$$\Rightarrow \bar{x} =$$

Next, we can generalize this to n objects with masses $m_1, \ldots, m_n$:

$$m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) + \cdots + m_n(x_n - \bar{x}) = \sum_{k=1}^{n} m_k(x_k - \bar{x}) = 0.$$

$$\Rightarrow \bar{x} =$$

Definition. (Center of Mass in One Dimension)
Let ρ be an integrable density function on the interval $[a, b]$ (which represents a thin rod or wire). The **center of mass** is located at the point $\bar{x} = \frac{M}{m}$, where the **total moment** M and mass m are

$$M = \int_a^b x\rho(x)\,dx \quad \text{and} \quad m = \int_a^b \rho(x)\,dx.$$

Example. Find the mass and center of mass of the thin rods with the following density functions:

$\rho(x) = 2 + \cos(x)$, for $0 \leq x \leq \pi$

$$\rho(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1 \\ x(2-x) & \text{if } 1 < x \leq 2 \end{cases}$$

Definition. (Center of Mass in Two Dimensions)
Let ρ be an integrable area density function defined over a closed bounded region R in $\mathbb{R}^2$. The coordinates of the center of mass of the object represented by R are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint\limits_R x\rho(x,y)\,dA \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint\limits_R y\rho(x,y)\,dA,$$

where $m = \iint_R \rho(x,y)\,dA$ is the mass, and M_y and M_x are the moments with respect to the y-axis and x-axis, respectively. If ρ is constant, the center of mass is called the **centroid** and is independent of the density.

Example. Find the center of mass of the following plane regions with variable density:

$R = \{(x,y) : 0 \leq x \leq 4,\ 0 \leq y \leq 2\};\ \rho(x,y) = 1 + x/2.$

The quarter disk in the first quadrant bounded by $x^2+y^2 = 4$ with $\rho(x, y) = 1+x^2+y^2$.

Definition. (Center of Mass in Two Dimensions)
Let ρ be an integrable area density function defined over a closed bounded region D in $\mathbb{R}^3$. The coordinates of the center of mass of the region are

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m} \iiint\limits_D x\rho(x,y,z)\,dV$$
$$\bar{y} = \frac{M_{xz}}{m} = \frac{1}{m} \iiint\limits_D y\rho(x,y,z)\,dV$$
$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m} \iiint\limits_D z\rho(x,y,z)\,dV$$

where $m = \iiint_D \rho(x,y,z)\,dA$ is the mass, and M_{yz}, M_{xz}, and M_{xy} are the moments with respect to the coordinate planes.