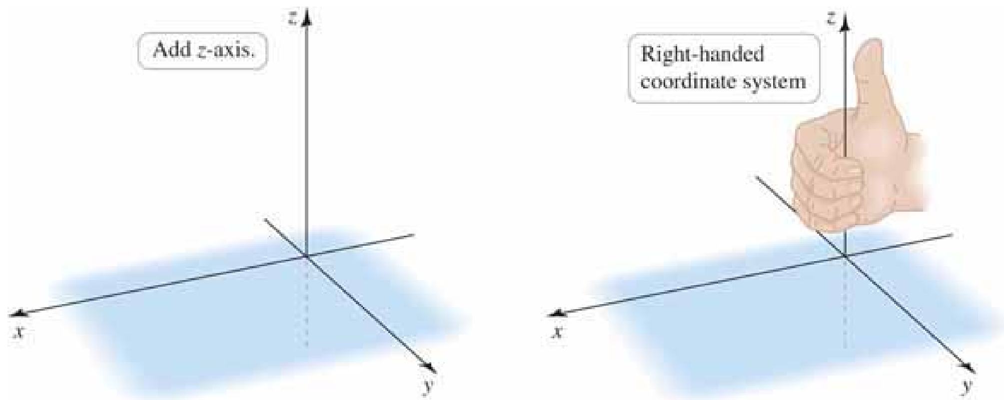
1 13.2: Vectors in Three Dimensions

The xyz- Coordinate System:
The three-dimensional coordinate system is created by adding the z-axis, which is perpendicular to both the x-axis and the y-axis. When looking at the xy-plane, the positive direction of the z-axis protrudes towards the viewer. This can also be shown using the right-hand rule (Figure 13.25 from Briggs):

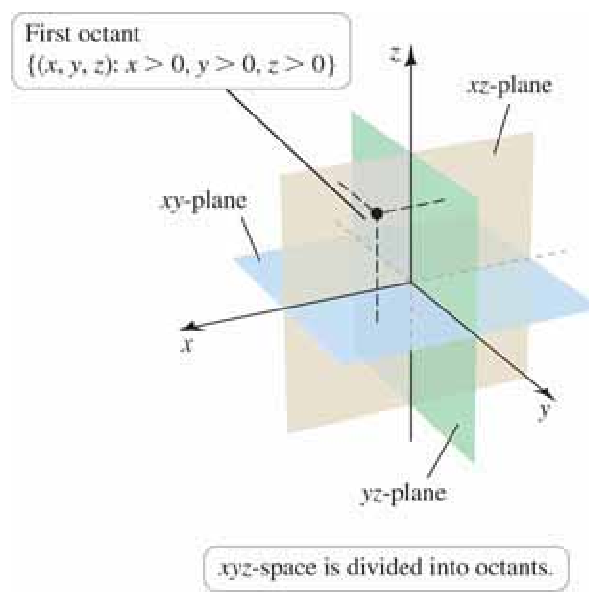


Definition.

This three-dimensional coordinate system is broken up into eight **octants**, which are separated by

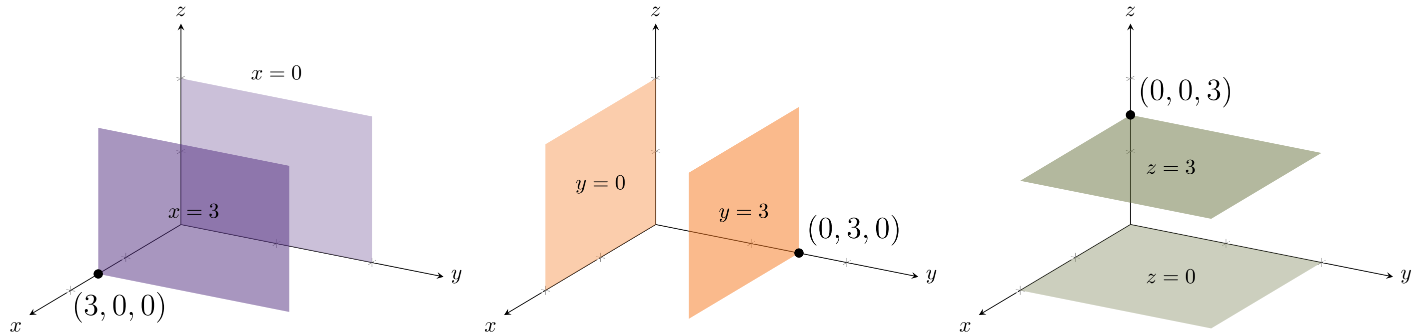
- the xy-**plane** ($z = 0$),
- the xz-**plane** ($y = 0$), and
- the yz-**plane** ($x = 0$).

The **origin** is the location where all three axes intersect.

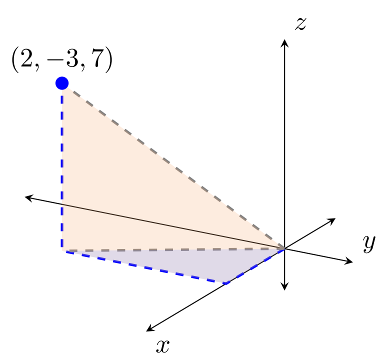


Equations of Simple Planes:

Planes in three-dimensions are analogous to lines in two-dimensions. Below, we see the yz-plane, the xz-plane, and the xy-plane, along with planes that are parallel where x, y, and z are fixed respectively:



Example (Parallel planes)**.** Determine the equation of the plane parallel to the xz-plane passing through the point $(2,-3,7)$.

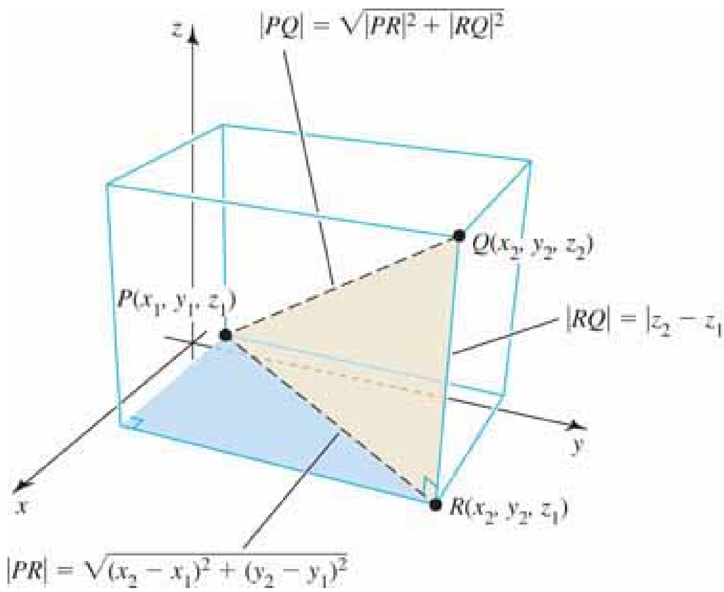


Distances in xyz-Space:

Recall that in $\mathbb{R}^2$, for some vector $\overrightarrow{PR}$, the distance formula is given by

$$|PR| = \sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2}$$

where (x_1, y_1) and (x_2, y_2) represent the points P and R respectively. This idea can be further extended into $\mathbb{R}^3$ by considering the two sides of the triangle formed by the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$:



Distance Formula in xyz-Space

The **distance** between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The **midpoint** between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is found by averaging the x-, y-, and z-coordinates:

$$\text{Midpoint } = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Example. Consider $P(-1, 4, 3)$ and $Q(3, 5, 7)$. Find

- $\left|\overrightarrow{PQ}\right|$

- The midpoint between P and Q

- Two unit vectors parallel to $\overrightarrow{PQ}$

- The equation of the sphere centered at the midpoint passing through P and Q

Equation of a Sphere:

Definition.
A **sphere** centered at (a, b, c) with radius r is the set of points satisfying the equation

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

A **ball** centered at (a, b, c) with radius r is the set of points satisfying the inequality

$$(x-a)^2 + (y-b)^2 + (z-c)^2 \leq r^2.$$

Example. Rewrite the following equation into the standard form of a sphere:

$$x^2 + y^2 + z^2 - 2x + 6y - 8z = -1$$

Example. What is the geometry of the intersection between $x^2 + y^2 + z^2 = 50$ and $z = 1$?

Vector Operations in Terms of Components

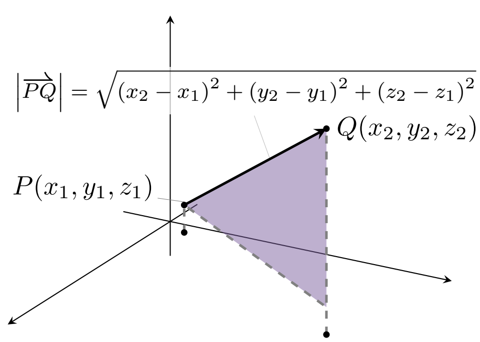
Definition. (Vector Operations in $\mathbb{R}^3$)
Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle \qquad \text{Vector addition}$$
$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle \qquad \text{Vector subtraction}$$
$$c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle \qquad \text{Scalar multiplication}$$

Magnitude and Unit Vectors:

Definition.
The **magnitude** (or **length**) of the vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ is the distance from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$:

$$\left|\overrightarrow{PQ}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



In $\mathbb{R}^3$, the **coordinate unit vectors** are $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$.

Example. Consider $P(-1,4,3)$ and $Q(3,5,7)$. Find two unit vectors parallel to $\overrightarrow{PQ}$.

Properties of Vector Operations:
Suppose $\mathbf{u}$, $\mathbf{v}$, and $\boldsymbol{w}$ are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$	Commutative property of addition
2. $(\mathbf{u}+\mathbf{v})+\boldsymbol{w}=\mathbf{u}+(\mathbf{v}+\boldsymbol{w})$	Associative property of addition
3. $\mathbf{v}+\mathbf{0}=\mathbf{v}$	Additive identity
4. $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$	Additive inverse
5. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$	Distributive property 1
6. $(a+c)\mathbf{v}=a\mathbf{v}+c\mathbf{v}$	Distributive property 2
7. $0\mathbf{v}=\mathbf{0}$	Multiplication by zero scalar
8. $c\mathbf{0}=\mathbf{0}$	Multiplication by zero vector
9. $1\mathbf{v}=\mathbf{v}$	Multiplicative identity
10. $a(c\mathbf{v})=(ac)\mathbf{v}$	Associative property of scalar multiplication