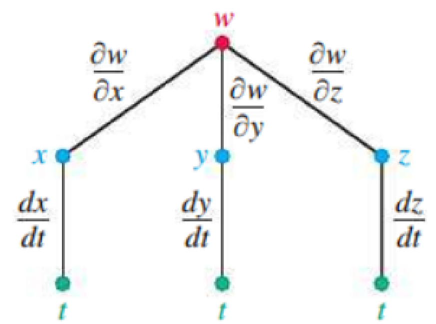
1 15.4: The Chain Rule

Theorem 15.7: Chain Rule (One Independent Variable)
Let z be a differentiable function of x and y on its domain, where x and y are differentiable functions of t on an interval I. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Note:

- For $z = f(x(t), y(t))$, z is the dependent variable, t is the independent variable, and x and y are **intermediate variables**.
- Since x and y only depend on t, we use the 'ordinary' derivative symbol
- Theorem 15.7 generalizes to functions of n variables



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Example. Find the derivative of the following functions using the chain rule where appropriate.

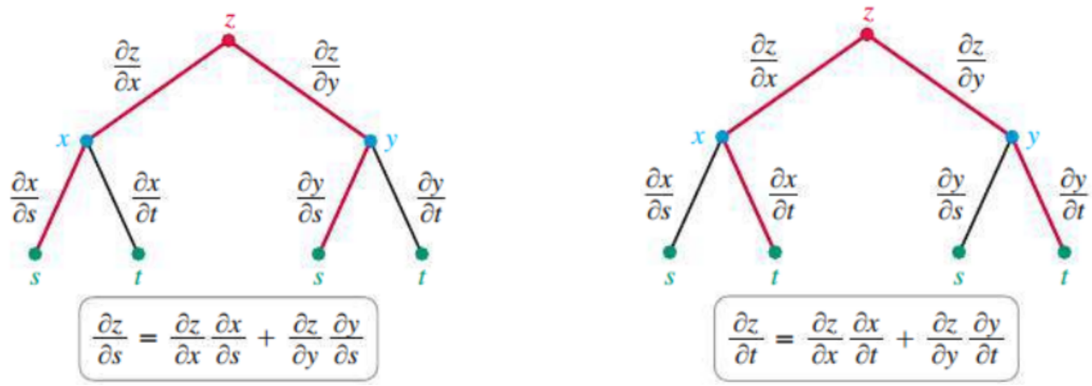
$$z = x^2 - 2y^2 + 20 \text{ where } x = 2\cos(t) \text{ and } y = 2\sin(t)$$

$w = \sin(12x)\cos(2y)$ where $x = t/2$ and $y = t^3$

$Q = \sqrt{3x^2 + 3y^2 + 2z^2}$ where $x = \sin(t)$, $y = \cos(t)$, and $z = \cos(t)$.

Theorem 15.8: Chain Rule (Two Independent Variables)
Let z be a differentiable function of x and y, where x and y are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$



Example. For $z = e^{5x+8y}$, where $x = 7st$ and $y = 5s + t$, find z_s and z_t.

Example. For $z = \sin(2x)\cos(3y)$, where $x = s + t$ and $y = s - t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

Example. For $r = \ln(x^2 + xy + y^2)$, where $x = 2st$ and $y = s/t$, find $\partial r/\partial s$ and $\partial r/\partial t$.

Theorem 15.9: Implicit Differentiation
Let F be differentiable on its domain and suppose $F(x, y) = 0$ defines y as a differentiable function of x. Provided $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Note: The above derivation comes from using the chain rule on $F(x, y) = 0$.

Example. For $4x^3 + 2x^2y - 3y^3 = 0$, find $\frac{dy}{dx}$ implicitly.

Example. For $xy + xz + 5yz = 42$, find $\partial z/\partial x$ and $\partial z/\partial y$ implicitly.

Example. For $xyz + 2yz + 3xz = 4x + 2y - 3z$, find $\partial z/\partial x$ and $\partial z/\partial y$.

Example. Consider the surface $z = f(x, y) = 3x^2 + 9y^2 + 4$ and the curve C given parametrically by $x = \cos(t)$ and $y = \sin(t)$ where $0 \leq t \leq 2\pi$. Find $z'(t)$ and find t such that $z'(t) > 0$.