Math 2060 Class notes
Spring 2021

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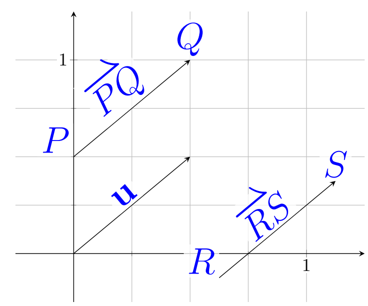
13.1: Vectors and the Geometry of Space

Definition.

- **Vectors**
  - Have a direction and magnitude,
  - vector $\overrightarrow{PQ}$ has a *tail* at P and a *head* at Q,
  - Can be denoted as $\mathbf{u}$ or $\vec{u}$,
  - Equal vectors have the same direction and magnitude (not necessarily the same position)



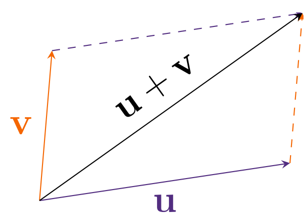
- **Scalars** are quantities with magnitude but no direction (e.g. mass, temperature, price, time, etc.)
- **Zero vector**, denoted $\mathbf{0}$ or $\vec{0}$, has length 0 and no direction

Scalar-vector multiplication:

- Denoted $c\mathbf{v}$ or $c\vec{v}$,
- length of vector multiplied by $|c|$,
- $c\mathbf{v}$ has the same direction as $\mathbf{v}$ if $c > 0$, and has the opposite direction as $\mathbf{v}$ if $c < 0$, (what if $c = 0$?)
- $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if $\mathbf{u} = c\mathbf{v}$. (what vectors are parallel to $\mathbf{0}$?)

Vector Addition and Subtraction:

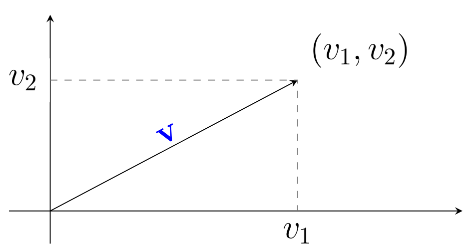
Given two vectors $\mathbf{u}$ and $\mathbf{v}$, their sum, $\mathbf{u}+\mathbf{v}$, can be represented by the parallelogram (triangle) rule: place the tail of $\mathbf{v}$ at the head of $\mathbf{u}$



The difference, denoted $\mathbf{u}-\mathbf{v}$, is the sum of $\mathbf{u}+(-\mathbf{v})$:

Vector Components:

A vector $\mathbf{v}$ whose tail is at the origin $(0,0)$ and head is at (v_1, v_2) is a **position vector** (in **standard position**) and is denoted $\langle v_1, v_2\rangle$. The real numbers v_1 and v_2 are the x- and y-components of $\mathbf{v}$.



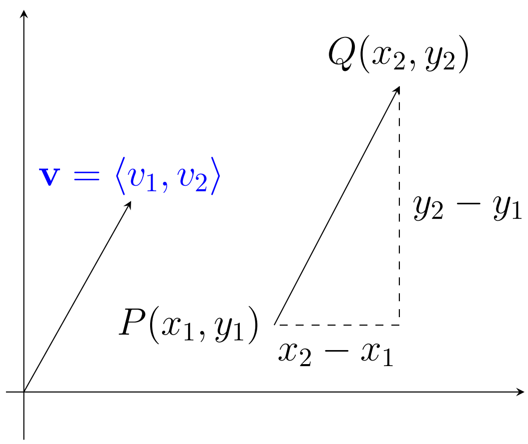
Vectors $\mathbf{u}=\langle u_1, u_2\rangle$ and $\mathbf{v}=\langle v_1, v_2\rangle$ are equal if and only if $u_1=v_1$ and $u_2=v_2$.

Magnitude:

Given points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the **magnitude**, or **length**, of vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$, denoted $\left|\overrightarrow{PQ}\right|$, is the distance between points P and Q.

$$\left|\overrightarrow{PQ}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The magnitude of position vector $\mathbf{v} = \langle v_1, v_2 \rangle$ is $|\mathbf{v}|$.
(How do $\left|\overrightarrow{PQ}\right|$ and $\left|\overrightarrow{QP}\right|$ relate to each other?)



Note: The norm, denoted $\|\mathbf{u}\|$ or $\|\mathbf{u}\|_2$, is equivalent to the magnitude of a vector.

Equation of a Circle:

Definition.
A **circle** centered at (a, b) with radius r is the set of points satisfying the equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

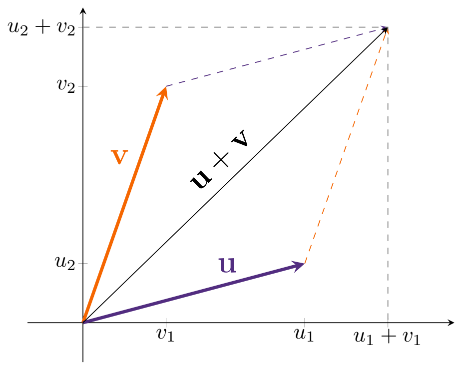
A **disk** centered at (a, b) with radius r is the set of points satisfying the inequality

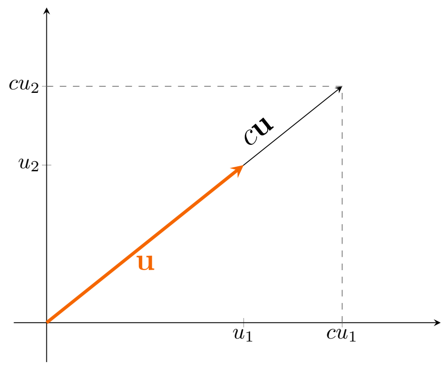
$$(x - a)^2 + (y - b)^2 \le r^2.$$

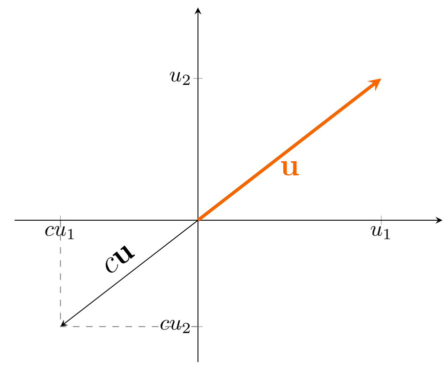
Vector Operations in Terms of Components

Definition. (Vector Operations in $\mathbb{R}^2$)
Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2 \rangle$, and $\mathbf{v} = \langle v_1, v_2 \rangle$.

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \langle u_1 + v_1, u_2 + v_2 \rangle && \text{Vector addition} \\ \mathbf{u} - \mathbf{v} &= \langle u_1 - v_1, u_2 - v_2 \rangle && \text{Vector subtraction} \\ c\mathbf{u} &= \langle cu_1, cu_2 \rangle && \text{Scalar multiplication} \end{aligned}$$







Example. Let $\mathbf{u} = \langle 1, 2 \rangle$, $\mathbf{v} = \langle -2, 3 \rangle$, $c = 2$, and $d = 3$. Find the following:

$\mathbf{u} + \mathbf{v}$ $\qquad\qquad$ $c\mathbf{u}$

$c\mathbf{u} + d\mathbf{v}$ $\qquad\qquad$ $\mathbf{u} - c\mathbf{v}$

Definition.
A **unit vector** is any vector with length 1.
In $\mathbb{R}^2$, the **coordinate unit vectors** are $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.

Example. Let $\mathbf{u} = \langle -7, 3 \rangle$. Find two unit vectors parallel to $\mathbf{u}$. Find another vector parallel to $\mathbf{u}$ with a magnitude of 2.

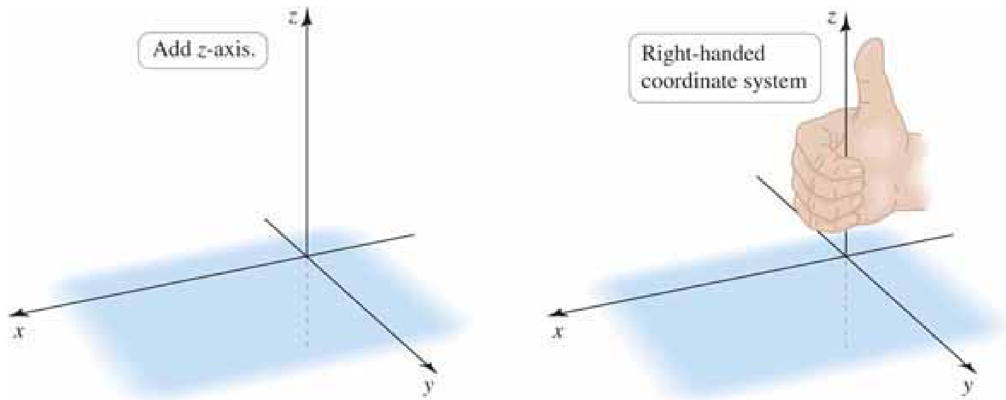
Properties of Vector Operations:
Suppose $\mathbf{u}$, $\mathbf{v}$, and $\boldsymbol{w}$ are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$	Commutative property of addition
2. $(\mathbf{u}+\mathbf{v})+\boldsymbol{w}=\mathbf{u}+(\mathbf{v}+\boldsymbol{w})$	Associative property of addition
3. $\mathbf{v}+\mathbf{0}=\mathbf{v}$	Additive identity
4. $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$	Additive inverse
5. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$	Distributive property 1
6. $(a+c)\mathbf{v}=a\mathbf{v}+c\mathbf{v}$	Distributive property 2
7. $0\mathbf{v}=\mathbf{0}$	Multiplication by zero scalar
8. $c\mathbf{0}=\mathbf{0}$	Multiplication by zero vector
9. $1\mathbf{v}=\mathbf{v}$	Multiplicative identity
10. $a(c\mathbf{v})=(ac)\mathbf{v}$	Associative property of scalar multiplication

13.2: Vectors in Three Dimensions

The xyz- Coordinate System:

The three-dimensional coordinate system is created by adding the z-axis, which is perpendicular to both the x-axis and the y-axis. When looking at the xy-plane, the positive direction of the z-axis protrudes towards the viewer. This can also be shown using the right-hand rule (Figure 13.25 from Briggs):

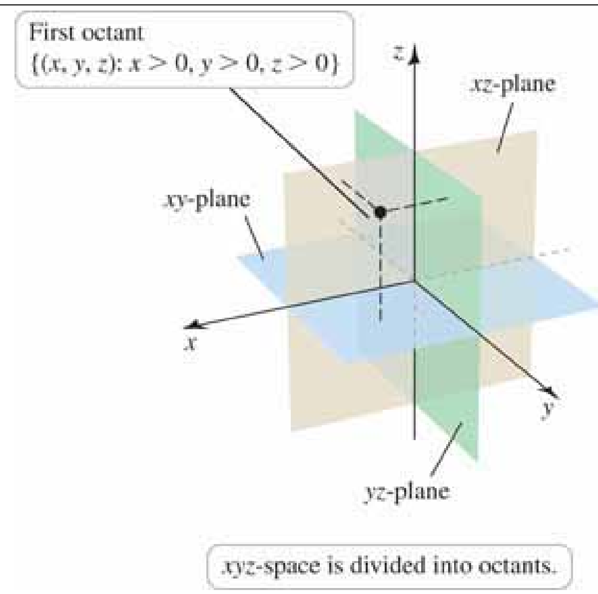


Definition.

This three-dimensional coordinate system is broken up into eight **octants**, which are separated by

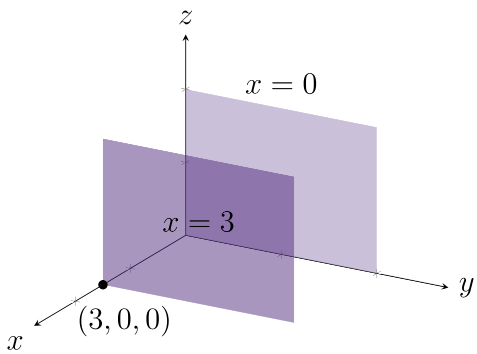
- the xy**-plane** ($z = 0$),
- the xz**-plane** ($y = 0$), and
- the yz**-plane** ($x = 0$).

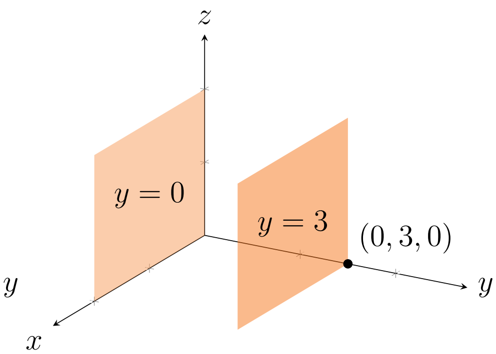
The **origin** is the location where all three axes intersect.

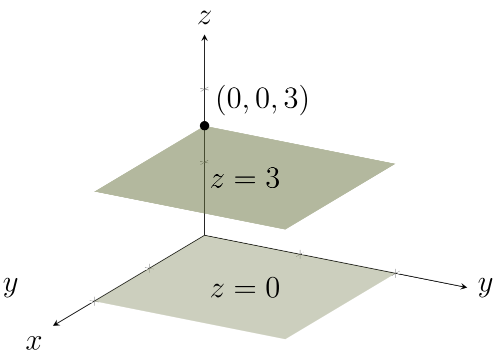


Equations of Simple Planes:

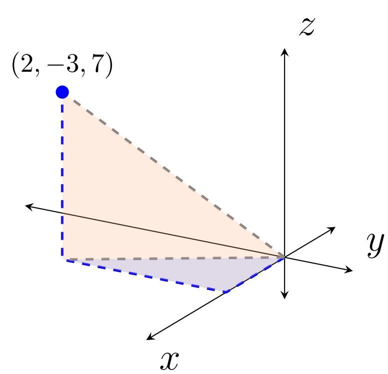
Planes in three-dimensions are analogous to lines in two-dimensions. Below, we see the yz-plane, the xz-plane, and the xy-plane, along with planes that are parallel where x, y, and z are fixed respectively:







Example (Parallel planes)**.** Determine the equation of the plane parallel to the xz-plane passing through the point $(2, -3, 7)$.

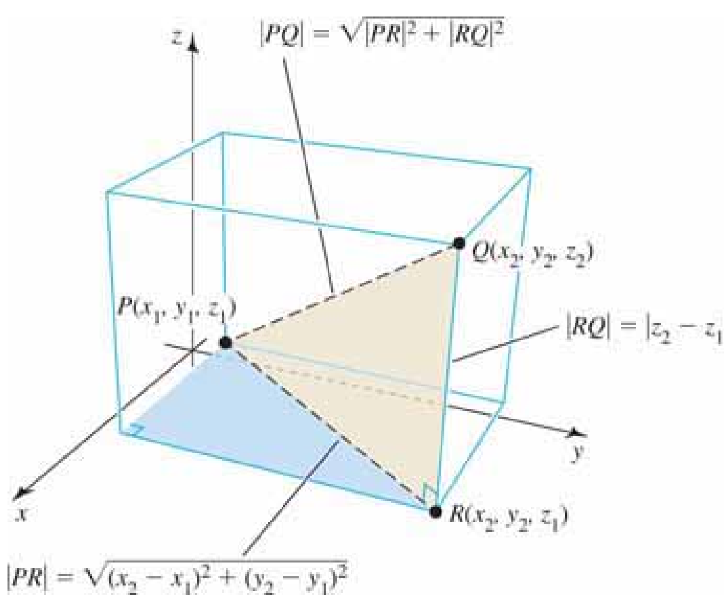


Distances in xyz-Space:

Recall that in $\mathbb{R}^2$, for some vector $\overrightarrow{PR}$, the distance formula is given by

$$|PR| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

where (x_1, y_1) and (x_2, y_2) represent the points P and R respectively. This idea can be further extended into $\mathbb{R}^3$ by considering the two sides of the triangle formed by the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$:



Distance Formula in xyz-Space

The **distance** between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The **midpoint** between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is found by averaging the x-, y-, and z-coordinates:

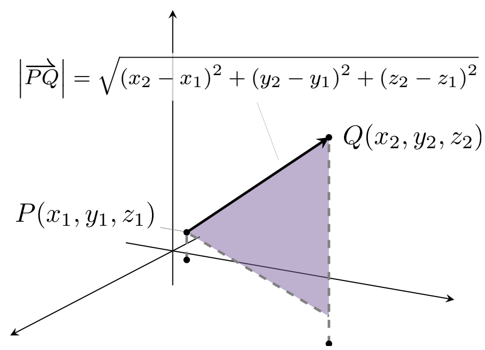
$$\text{Midpoint } = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Magnitude and Unit Vectors:

Definition.
The **magnitude** (or **length**) of the vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ is the distance from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$:

$$\left|\overrightarrow{PQ}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

In $\mathbb{R}^3$, the **coordinate unit vectors** are $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$.



Example. Consider $P(-1, 4, 3)$ and $Q(3, 5, 7)$. Find

- $\left|\overrightarrow{PQ}\right|$

- The midpoint between P and Q

- Two unit vectors parallel to $\overrightarrow{PQ}$

Equation of a Sphere:

Definition.
A **sphere** centered at (a, b, c) with radius r is the set of points satisfying the equation

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

A **ball** centered at (a, b, c) with radius r is the set of points satisfying the inequality

$$(x-a)^2 + (y-b)^2 + (z-c)^2 \leq r^2.$$

Example. Consider $P(-1, 4, 3)$ and $Q(3, 5, 7)$. Find the equation of the sphere centered at the midpoint passing through P and Q

Example. What is the geometry of the intersection between $x^2+y^2+z^2 = 50$ and $z = 1$?

Example. Rewrite the following equation into the standard form of a sphere:

$$x^2 + y^2 + z^2 - 2x + 6y - 8z = -1$$

Vector Operations in Terms of Components

Definition. (Vector Operations in $\mathbb{R}^3$)
Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$	Vector addition
$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$	Vector subtraction
$c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle$	Scalar multiplication

Properties of Vector Operations:
Suppose $\mathbf{u}$, $\mathbf{v}$, and $\boldsymbol{w}$ are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

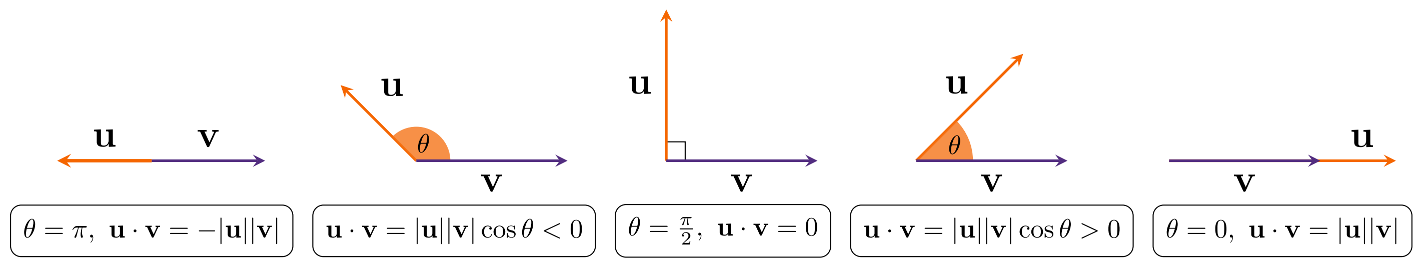
1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ — Commutative property of addition
2. $(\mathbf{u} + \mathbf{v}) + \boldsymbol{w} = \mathbf{u} + (\mathbf{v} + \boldsymbol{w})$ — Associative property of addition
3. $\mathbf{v} + \mathbf{0} = \mathbf{v}$ — Additive identity
4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ — Additive inverse
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ — Distributive property 1
6. $(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$ — Distributive property 2
7. $0\mathbf{v} = \mathbf{0}$ — Multiplication by zero scalar
8. $c\mathbf{0} = \mathbf{0}$ — Multiplication by zero vector
9. $1\mathbf{v} = \mathbf{v}$ — Multiplicative identity
10. $a(c\mathbf{v}) = (ac)\mathbf{v}$ — Associative property of scalar multiplication

13.3: Dot Products

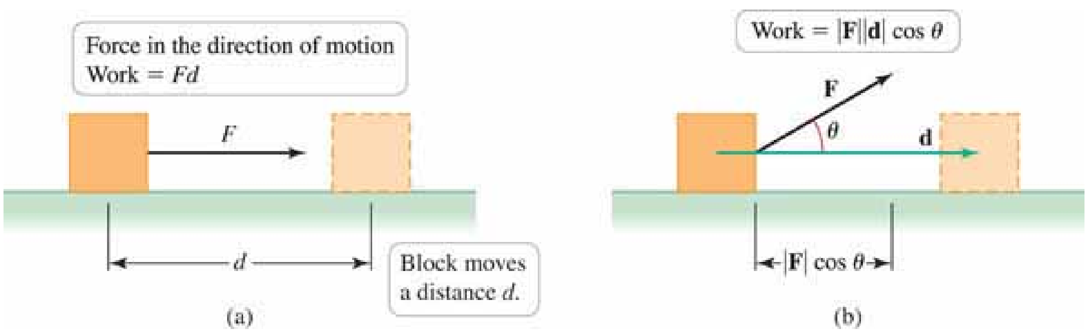
Definition. (Dot Product)
Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in two or three dimensions, their **dot product** is

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ with $0 \le \theta \le \pi$. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then $\mathbf{u}\cdot\mathbf{v} = 0$, and θ is undefined.



A physical example of the dot product is the amount of work done when a force is applied at an angle θ as shown in figure 13.43:



Note: The result of the dot product is a scalar!

Definition. (Orthogonal Vectors)
Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. The zero vector is orthogonal to all vectors. In two or three dimensions, two nonzero orthogonal vectors are perpendicular to each other.

- $\mathbf{u}$ and $\mathbf{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \cdot \mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$.
- $\mathbf{u}$ and $\mathbf{v}$ are perpendicular ($\theta = \frac{\pi}{2}$) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Example. Given $|\mathbf{u}| = 2$ and $|\mathbf{v}| = \sqrt{3}$, compute $\mathbf{u} \cdot \mathbf{v}$ when

- $\theta = \frac{\pi}{4}$
- $\theta = \frac{\pi}{3}$
- $\theta = \frac{5\pi}{6}$

Theorem 31.1: Dot Product
Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$,

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Example. Given vectors $\mathbf{u} = \left\langle \sqrt{3}, 1, 0 \right\rangle$ and $\mathbf{v} = \left\langle 1, \sqrt{3}, 0 \right\rangle$, compute $\mathbf{u} \cdot \mathbf{v}$ and find θ.

Properties of Dot Products

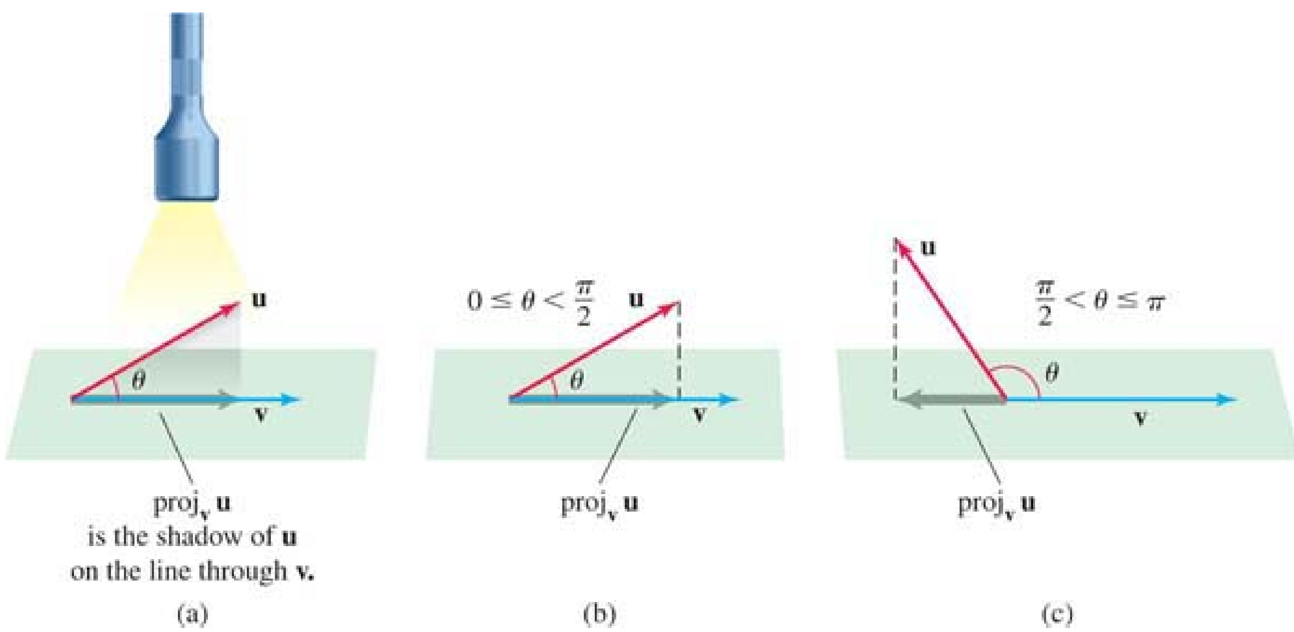
Theorem 13.2: Properties of the Dot Product

Suppose $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ — Commutative property
2. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$ — Associative property
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ — Distributive property

Orthogonal Projections

Given vectors $\mathbf{u}$ and $\mathbf{v}$, the projection of $\mathbf{u}$ onto $\mathbf{v}$ produces a vector parallel to $\mathbf{v}$ using the "shadow" of $\mathbf{u}$ cast onto $\mathbf{v}$.



Definition. ((Orthogonal) Projection of u onto v)
The **orthogonal projection of u onto v**, denoted $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$, where $\mathbf{v} \neq \mathbf{0}$, is

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \underbrace{|\mathbf{u}| \cos\theta}_{\text{length}} \underbrace{\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)}_{\text{direction}}.$$

The orthogonal projection may also be computed with the formulas

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \operatorname{scal}_{\mathbf{v}} \mathbf{u} \left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v},$$

where the **scalar component of u in the direction of v** is

$$\operatorname{scal}_{\mathbf{v}} \mathbf{u} = |\mathbf{u}| \cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}.$$

Example. Find $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$ and $\operatorname{scal}_{\mathbf{v}} \mathbf{u}$ for the following:

- $\mathbf{u} = \langle 1, 1\rangle$, $\mathbf{v} = \langle -2, 1\rangle$

- $\mathbf{u} = \langle 7, 1, 7\rangle$, $\mathbf{v} = \langle 5, 7, 0\rangle$

Applications of Dot Products

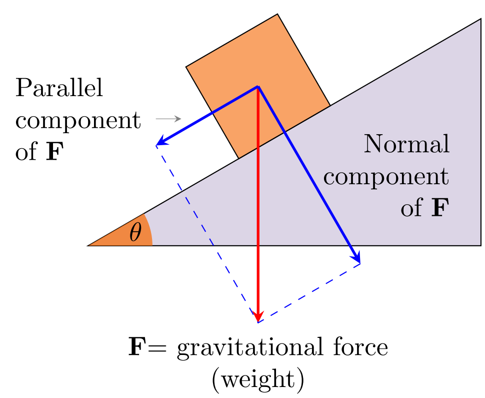
Definition. (Work)
Let a constant force $\mathbf{F}$ be applied to an object, producing a displacement $\mathbf{d}$. If the angle between $\mathbf{F}$ and $\mathbf{d}$ is θ, then the **work** done by the force is

$$W = |\mathbf{F}||\mathbf{d}|\cos\theta = \mathbf{F}\cdot\mathbf{d}$$

Example. A force $\mathbf{F} = \langle 3, 3, 2\rangle$ (in newtons) moves an object along a line segment from $P(1, 1, 0)$ to $Q(6, 6, 0)$ (in meters). What is the work done by the force?

<u>Parallel and Normal Forces:</u>

Example. A 10-lb block rests on a plane that is inclined at 30° above the horizontal. Find the components of the gravitational force parallel to and normal (perpendicular) to the plane.



13.4: Cross Products

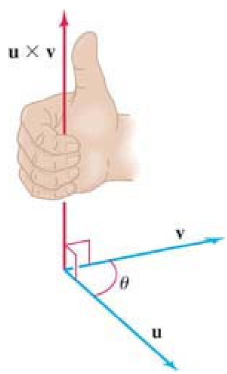
Definition. (Cross Product)
Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the **cross product** $\mathbf{u} \times \mathbf{v}$ is a vector with magnitude

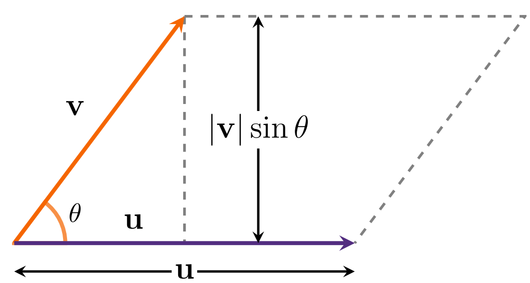
$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta,$$

where $0 \le \theta \le \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
The direction of $\mathbf{u} \times \mathbf{v}$ is given by the **right-hand rule**:

> When you put your the vectors tail to tail and let the fingers of your right hand curl from $\mathbf{u}$ to $\mathbf{v}$, the direction of $\mathbf{u} \times \mathbf{v}$ is the direction of your thumb, orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ (Figure 13.56).

When $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, the direction of $\mathbf{u} \times \mathbf{v}$ is undefined.





Theorem 13.3: Geometry of the Cross Product
Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in $\mathbb{R}^3$.

1. The vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
2. If $\mathbf{u}$ and $\mathbf{v}$ are two sides of a parallelogram, then the area of the parallelogram is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$$

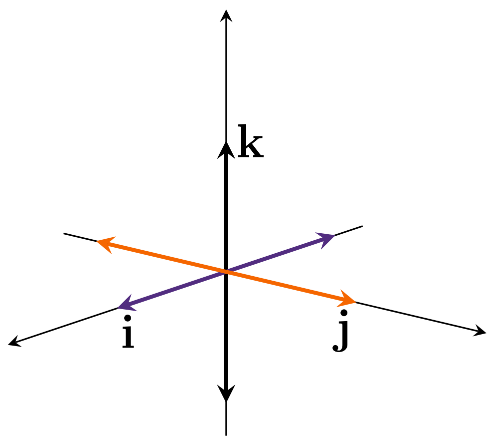
Example. Consider the vectors $\mathbf{u} = \langle 2, 0, 0 \rangle$ and $\mathbf{v} = \left\langle \sqrt{3}, 3, 0 \right\rangle$. The angle between these vectors is $\theta = \frac{\pi}{3}$. Find the area of the parallelogram formed by these vectors.

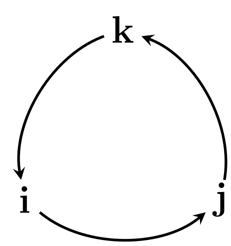
Theorem 13.4: Properties of the Cross Product Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^3$, and let a and b be scalars.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ — Anticommutative property
2. $(a\mathbf{u}) \times (b\mathbf{v}) = ab(\mathbf{u} \times \mathbf{v})$ — Associative property
3. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ — Distributive property
4. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$ — Distributive property

Theorem 13.5: Cross Products of Coordinate Unit Vectors

$$\mathbf{i}\times\mathbf{j} = -(\mathbf{j}\times\mathbf{i}) = \mathbf{k} \qquad \mathbf{j}\times\mathbf{k} = -(\mathbf{k}\times\mathbf{j}) = \mathbf{i}$$
$$\mathbf{k}\times\mathbf{i} = -(\mathbf{i}\times\mathbf{k}) = \mathbf{j} \qquad \mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$$





$$\mathbf{i}\times\mathbf{j}=\mathbf{k}$$
$$\mathbf{j}\times\mathbf{k}=\mathbf{i}$$
$$\mathbf{k}\times\mathbf{i}=\mathbf{j}$$

Using the unit vectors, we can compute $\mathbf{u}\times\mathbf{v}$:

$$\begin{aligned}
\mathbf{u}\times\mathbf{v} &= (u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k})\times(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k})\\
&= u_1v_1\underbrace{(\mathbf{i}\times\mathbf{i})}_{\mathbf{0}} + u_1v_2\underbrace{(\mathbf{i}\times\mathbf{j})}_{\mathbf{k}} + u_1v_3\underbrace{(\mathbf{i}\times\mathbf{k})}_{-\mathbf{j}}\\
&+ u_2v_1\underbrace{(\mathbf{j}\times\mathbf{i})}_{-\mathbf{k}} + u_2v_2\underbrace{(\mathbf{j}\times\mathbf{j})}_{\mathbf{0}} + u_2v_3\underbrace{(\mathbf{j}\times\mathbf{k})}_{\mathbf{i}}\\
&+ u_3v_1\underbrace{(\mathbf{k}\times\mathbf{i})}_{\mathbf{j}} + u_3v_2\underbrace{(\mathbf{k}\times\mathbf{j})}_{-\mathbf{i}} + u_3v_3\underbrace{(\mathbf{k}\times\mathbf{k})}_{\mathbf{0}}\\
&= (u_2v_3-u_3v_2)\mathbf{i} - (u_1v_3-u_3v_1)\mathbf{j} + (u_1v_2-u_2v_1)\mathbf{k}
\end{aligned}$$

Theorem 13.6: Evaluating the Cross Product
Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

Note:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Alternative approach:

$$\left|\begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ u_1 & u_2 & u_3 & u_1 & u_2 \\ v_1 & v_2 & v_3 & v_1 & v_2 \end{array}\right.$$

Example. Compute $\mathbf{u} \times \mathbf{v}$ for $\mathbf{u} = \langle 3, 5, 4 \rangle$ and $\mathbf{v} = \langle 1, -1, 9 \rangle$.

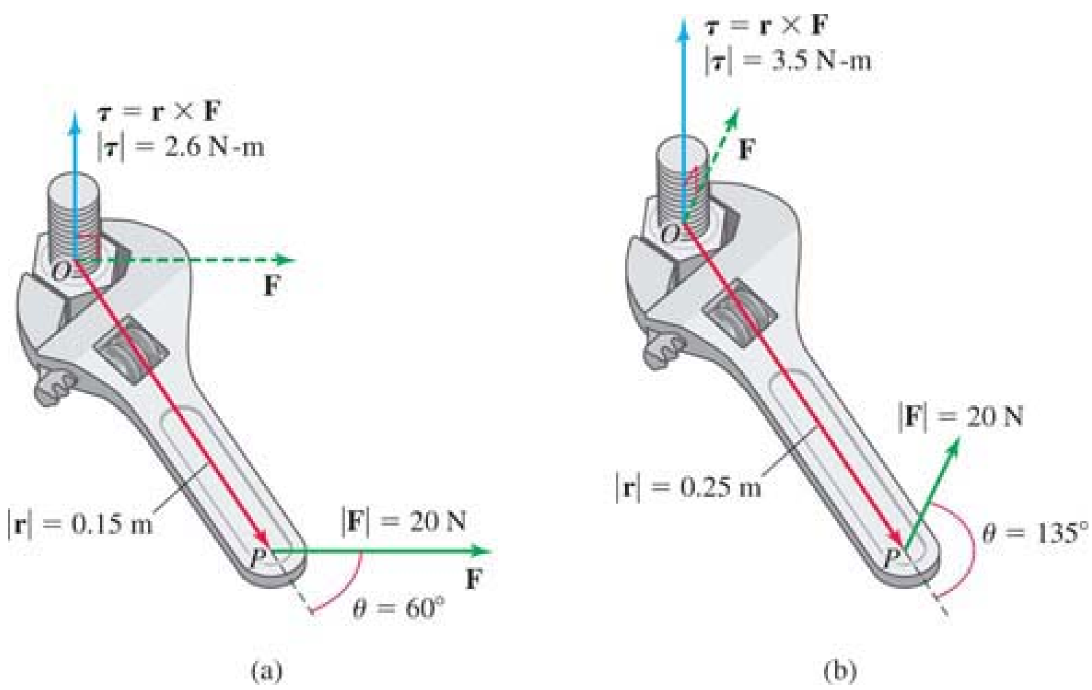
Example. Consider the vectors $\mathbf{u} = \left\langle \sqrt{3}, 1, 0 \right\rangle$ and $\mathbf{v} = \left\langle -\sqrt{3}, 1, 0 \right\rangle$. From the unit circle, we know the angle between these two vectors is $\theta = \frac{2\pi}{3}$. Use the definition of the cross product to show this.

Example. Find the area of the triangle formed by $\mathbf{u} = \langle 1, 2, 3 \rangle$ and $\mathbf{v} = \langle 3, -1, 1 \rangle$.

Example. Given a force $\mathbf{F}$ applied to a point P at the head of the vector $\mathbf{r} = \overrightarrow{OP}$, the **torque** produced at point O is given by $\tau = \mathbf{r} \times \mathbf{F}$ with magnitude

$$|\tau| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin \theta.$$

Now suppose a force of $20N$ is applied to a wrench attached to a bolt in a direction perpendicular to the bolt. Which produces more torque: applying the force at an angle of $60°$ on a wrench that is $0.15m$ long or applying the force at an angle of $135°$ on a wrench that is $0.25m$ long?

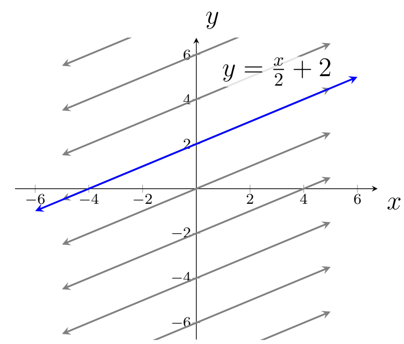


13.5: Lines and Planes in Space

Equation of a Line:

Recall the equation of a line in $\mathbb{R}^2$:

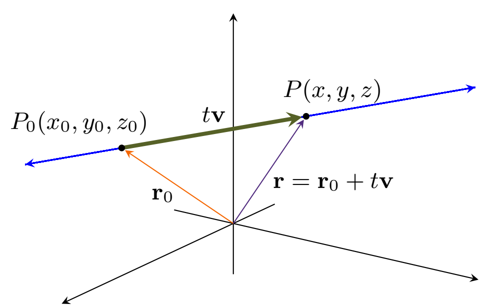
$$y = mx + b$$



where b is the intercept and m is the slope. This idea can be extended into higher dimensions:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

Here, $\mathbf{r}_0$ is a fixed point, and $\mathbf{v}$ is the position vector that is parallel to the line $\mathbf{r}$.



Equation of a Line

A **vector equation of the line** passing through the point $P_0(x_0, y_0, z_0)$ in the direction of the vector $\mathbf{v} = \langle a, b, c\rangle$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, or

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle, \quad \text{for} \quad -\infty < t < \infty$$

Equivalently, the corresponding **parametric equations of the line** are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \text{for} \quad -\infty < t < \infty$$

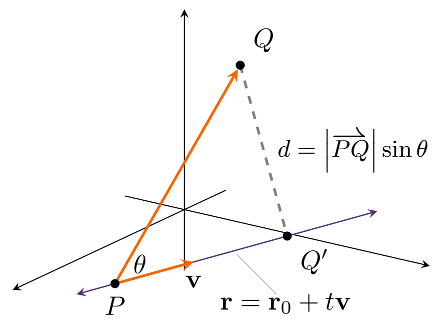
Example. Find the vector equation and parametric equation of the line that

- goes through the points $P(-1,-2,1)$ and $Q(-4,-5,-3)$ where $t=0$ corresponds to P,

- goes through the point $P(1,-3,-3)$ and is parallel to the vector $\mathbf{r} = \langle -4, 1, -1\rangle$,

- goes through the point $P(-2,5,-2)$ and is perpendicular to the lines $x = 3-4t$, $y = 2-3t$, $z = -1-t$, and $x = -2+0t$, $y = 2-t$, $z = 3t$, where $t = 0$ corresponds to P.

Distance from a Point to a Line:
Given a point Q and a line ℓ, the shortest distance to the line is the length of $\overrightarrow{QQ'}$.



From the definition of the cross product, we have

$$\left|\mathbf{v} \times \overrightarrow{PQ}\right| = |\mathbf{v}| \underbrace{\left|\overrightarrow{PQ}\right| \sin\theta}_{d} = |\mathbf{v}| d$$

From here, solving for d gives us the following:

Distance Between a Point and a Line
The distance d between the point Q and the $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ is

$$d = \frac{\left|\mathbf{v} \times \overrightarrow{PQ}\right|}{|\mathbf{v}|},$$

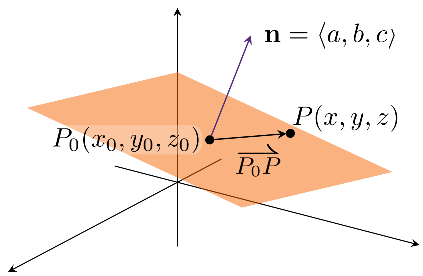
where P is any point on the line and $\mathbf{v}$ is a vector parallel to the line.

Example. Find the distance from the point $Q(-4, -1, -3)$ and the line $x = -5 - 5t$, $y = -5 + t$, $z = -1 + 4t$. (*Hint:* Let P be the point at $t = 0$)

Equations of Planes:

In $\mathbb{R}^2$, two distinct points determine a line.

In $\mathbb{R}^3$, three noncollinear points determine a unique plane. Alternatively, a plane is uniquely determined by a point and a vector that is orthogonal to the plane.



Definition. (Plane in $\mathbb{R}^3$)
Given a fixed point P_0 and a nonzero **normal vector n**, the set of points P in $\mathbb{R}^3$ for which $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$ is called a **plane**.

Consider the normal vector $\mathbf{n} = \langle a, b, c\rangle$ at the point $P_0(x_0, y_0, z_0)$, and any point $P(x, y, z)$ on the plane. Since $\mathbf{n}$ is orthogonal to the plane, it is also orthogonal to the vector $\overrightarrow{P_0P}$, which is also in the plane. Thus,

$$\begin{aligned} \mathbf{n} \cdot \overrightarrow{P_0P} &= 0 \\ \langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \\ ax + by + cz &= d \end{aligned}$$

General Equation of a Plane in $\mathbb{R}^3$
The plane passing through the point $P_0(x_0, y_0, z_0)$ with a nonzero normal vector $\mathbf{n} = \langle a, b, c\rangle$ is described by the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{or} \quad ax + by + cz = d,$$

where $d = ax_0 + by_0 + cz_0$.

Example. Find the equation of the plane that

- goes through the point $P(-2, 5, 0)$ and is parallel to the plane $x - 5y - 5z = 1$,

- goes through the points $P(5, -2, 1)$, $Q(5, 1, 3)$ and $R(1, -5, -2)$

- that is parallel to the vectors $\langle 4, -2, -3\rangle$ and $\langle 3, 2, 3\rangle$, passing through the point $P(-2, -2, 5)$.

Example. Find the location where the line $\langle -3, 1, 4\rangle + t\langle -1, -4, 2\rangle$ and the plane $2x - 2y - 4z = 5$ intersect.

> **Definition. (Parallel and Orthogonal Planes)**
> Two distinct planes are **parallel** if their respective normal vectors are parallel (that is, the normal vectors are scaling multiples of each other). Two plans are **orthogonal** if their respective normal vectors are orthogonal (that is, the dot product of the normal vectors is *zero*).

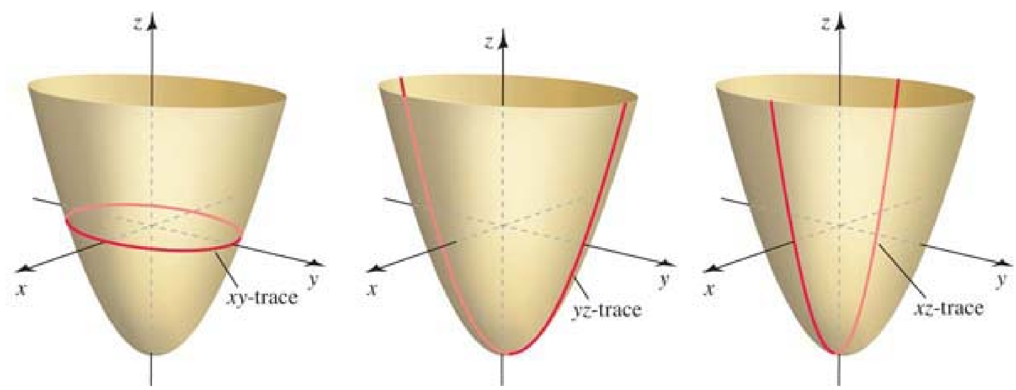
Example. Find the line of intersection between the planes $3x - y + 4z = -4$ and $x + 3y - 2z = 0$.

Example. Find the smallest angle between the planes $3x - y + 4z = -4$ and $x + 3y - 2z = 0$.

13.6: Cylinders and Quadric Surfaces

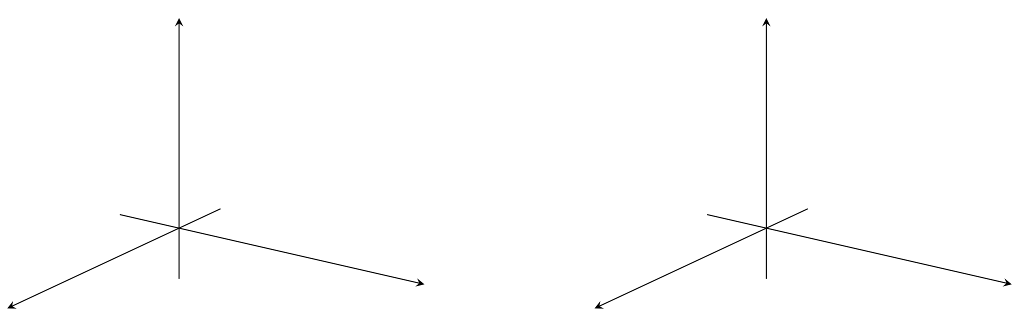
Cylinders and Traces:
When talking about three-dimensional surfaces, a *cylinder* refers to a surface that is parallel to a line. When considering surfaces that is parallel to one of the coordinate axes, that the associated variable is missing (e.g. $3y^2 + z^2 = 8$ is parallel to the x-axis).

Definition. (Trace)
A **trace** of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes. The traces in the coordinate planes are called the xy**-trace**, the yz**-trace**, and the xz**-trace** (Figure 13.80).



Example. Roughly sketch the following functions:

1. $x^2 + 4y^2 = 16$
2. $x - \sin(z) = 0$



Quadric Surfaces:

Quadric surfaces are described by the general quadratic (second-degree) equation in three variables,

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0,$$

Where the coefficients $A, \ldots, J$ and not all zero.

Example (An ellipsoid)**.** The surface defined by the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. Graph $a = 3$, $b = 4$ and $c = 5$.

Example (An elliptic parabaloid)**.** The surface defined by the equation $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$. Graph the elliptic paraboloid with $a = 4$ and $b = 2$.

Example (A hyperboloid of one sheet)**.**

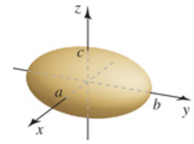
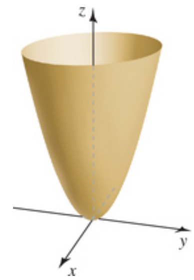
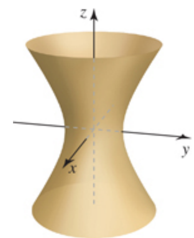
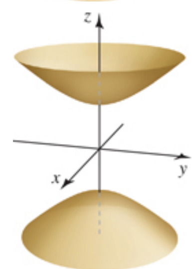
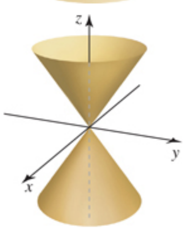
Graph the surface defined by the equation $\dfrac{x^2}{4} + \dfrac{y^2}{9} - z^2 = 1$.

Example (A hyperboloid of two sheets)**.** Graph the surface defined by the equation $-16x^2 - 4y^2 + z^2 + 64x - 80 = 0$.

Example (Elliptic cones)**.** Graph the surface defined by the equation $\dfrac{y^2}{4} + z^2 = 4x^2$.

Example (A hyperbolic paraboloid)**.**

Graph the surface defined by the equation $z = x^2 - \dfrac{y^2}{4}$.

Name	Standard Equation	Features	Graph
Ellipsoid	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$	All traces are ellipses.	
Elliptic paraboloid	$z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$	Traces with $z = z_0 > 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are parabolas.	
Hyperboloid of one sheet	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$	Traces with $z = z_0$ are ellipses for all z_0. Traces with $x = x_0$ or $y = y_0$ are hyperbolas.	
Hyperboloid of two sheets	$-\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$	Traces with $z = z_0$ with $\lvert z_0 \rvert > \lvert c \rvert$ are ellipses. Traces with $x = x_0$ and $y = y_0$ are hyperbolas.	
Elliptic cone	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}$	Traces with $z = z_0 \neq 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are hyperbolas or intersecting lines.	
Hyperbolic paraboloid	$z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$	Traces with $z = z_0 \neq 0$ are hyperbolas. Traces with $x = x_0$ or $y = y_0$ are parabolas.