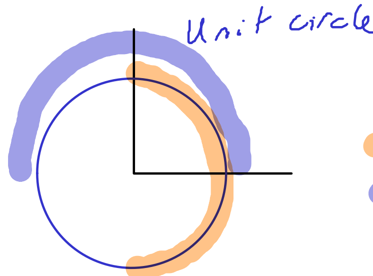
Turns a sum/difference into a product



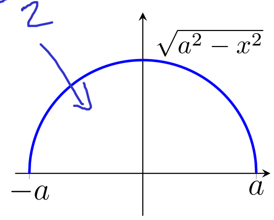
$f(x)$	range
$\sin^{-1}(x)$	$[-\pi/2, \pi/2]$
$\cos^{-1}(x)$	$[0, \pi]$
$\tan^{-1}(x)$	$(-\pi/2, \pi/2)$
$\sec^{-1}(x)$	$[0, \pi/2) \cup (\pi/2, \pi]$

8.4: Trigonometric Substitutions

Example. Verify the formula for the area of a circle with radius a by finding the area under $f(x) = \sqrt{a^2 - x^2}$.

Area of a circle w/ radius r is $A = \pi r^2$

$A = \frac{\pi a^2}{2}$



$$A = 2\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx$$

Let $x = a\sin\theta \longrightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right)$

$dx = a\cos\theta$

$x = -a \to \theta = -\pi/2$

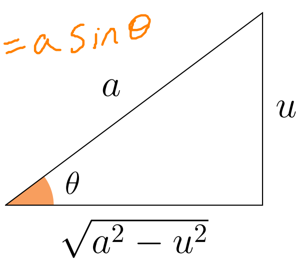
$x = a \to \theta = \pi/2$

$$= 2\int_{-\pi/2}^{\pi/2} \underbrace{\sqrt{a^2 - a^2\sin^2\theta}}_{a^2\cos^2\theta}\; a\cos\theta\, d\theta = 2\int_{-\pi/2}^{\pi/2} a\cos\theta\; a\cos\theta\, d\theta = 2a^2\int_{-\pi/2}^{\pi/2} \cos^2(\theta)\, d\theta$$

$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$

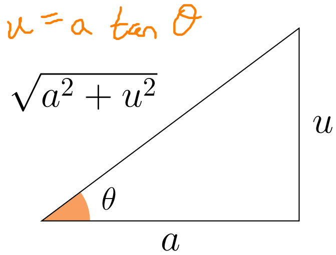
$$= 2a^2\int_{-\pi/2}^{\pi/2} \frac{1}{2} + \frac{\cos(2\theta)}{2}\, d\theta = 2a^2\left(\frac{1}{2}\theta + \frac{\sin(2\theta)}{4}\right)\Bigg|_{-\pi/2}^{\pi/2} = 2a^2\left[\left(\frac{\pi}{4} + 0\right) - \left(-\frac{\pi}{4} + 0\right)\right] = \boxed{\pi a^2}$$

$u = a\sin\theta$



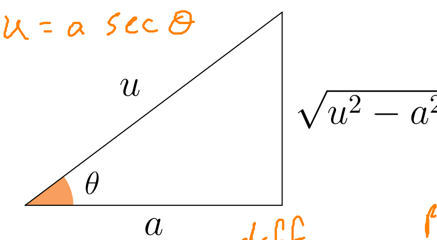
$\tan\theta = \frac{u}{a}$

$u = a\tan\theta$



$\sec\theta = \frac{u}{a}$

$u = a\sec\theta$



diff product

$a^2 - u^2$	$u = a\sin(\theta)$,	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, for $\lvert u\rvert \le a$	$a^2 - a^2\sin^2(\theta) = a^2\cos^2(\theta)$ $1 = \sin^2\theta + \cos^2\theta$
$a^2 + u^2$	$u = a\tan(\theta)$,	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$,	$a^2 + a^2\tan^2(\theta) = a^2\sec^2(\theta)$ $\tan^2\theta + 1 = \sec^2\theta$
$u^2 - a^2$	$u = a\sec(\theta)$,	$\begin{cases} 0 \le \theta < \frac{\pi}{2}, & \text{for } u \ge a \\ \frac{\pi}{2} < \theta \le \pi, & \text{for } u \le -a \end{cases}$	$a^2\sec^2(\theta) - a^2 = a^2\tan^2(\theta)$ $\sec^2\theta - 1 = \tan^2\theta$

$\rightarrow \cos^2 x + \sin^2 x = 1$

$\cos^2(x) = 1 - \sin^2(x)$

$a^2 - u^2$

$\rightarrow 1 + \tan^2 x = \sec^2 x \longrightarrow \tan^2(x) = \sec^2(x) - 1$

$a^2 + u^2$

$u^2 - a^2$

variable

$$a^2 - u^2 \qquad u = a\sin(\theta),$$

$$a^2 + u^2 \qquad u = a\tan(\theta),$$

Example. $\displaystyle\int \frac{\sqrt{x^2-4}}{x^3}\,dx$

$$u^2 - a^2 \qquad u = a\sec(\theta),$$

Let $x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\,d\theta$

$$\sqrt{x^2-2^2} = \sqrt{2^2\sec^2\theta - 2^2} = 2\sqrt{\sec^2\theta - 1} = 2\sqrt{\tan^2\theta} = 2\tan\theta$$

$$\int \frac{\sqrt{x^2-4}}{x^3}\,dx = \int \frac{2\tan\theta}{2^3\sec^3\theta}\,2\sec\theta\tan\theta\,d\theta = \frac{1}{2}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\sec\theta = \frac{1}{\cos\theta}$

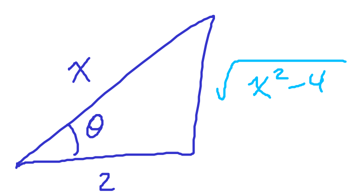
$$= \frac{1}{2}\int \frac{\sin^2\theta}{\cos^2\theta}\cdot\frac{\cos^2\theta}{1}\,d\theta = \frac{1}{2}\int \underbrace{\sin^2\theta}_{\frac{1-\cos(2\theta)}{2}}\,d\theta = \frac{1}{2}\int \frac{1}{2} - \frac{\cos(2\theta)}{2}\,d\theta$$

$$= \frac{\theta}{4} - \frac{\sin(2\theta)}{8} + C$$

double angle formula

$x = 2\sec\theta \rightarrow \frac{2}{x} = \cos\theta$

$\sec^{-1}\left(\frac{x}{2}\right) = \theta$



$$= \frac{1}{4}\sec^{-1}\left(\frac{x}{2}\right) - \frac{2}{8}\sin(\theta)\cos(\theta) + C$$

$$= \frac{1}{4}\sec^{-1}\left(\frac{x}{2}\right) - \frac{1}{4}\left(\frac{\sqrt{x^2-4}}{x}\right)\left(\frac{2}{x}\right) + C$$

$$= \boxed{\frac{1}{4}\sec^{-1}\left(\frac{x}{2}\right) - \frac{\sqrt{x^2-4}}{2x^2} + C}$$

A. $\displaystyle\frac{1}{4}\left[\sec^{-1}\left(\frac{x}{2}\right) - \left(\frac{2}{x}\right)\left(\frac{\sqrt{x^2-4}}{x}\right)\right] + C$

$a^2 - u^2$	$u = a\sin(\theta)$,
$a^2 + u^2$	$u = a\tan(\theta)$,
$u^2 - a^2$	$u = a\sec(\theta)$,

Example. $\displaystyle\int \frac{\sqrt{16-x^2}}{x}\,dx$

Let $x = 4\sin\theta$

$dx = 4\cos\theta\, d\theta$

$$\sqrt{16-x^2} = \sqrt{4^2 - 4^2\sin^2\theta} = 4\sqrt{1-\sin^2\theta} = 4\sqrt{\cos^2\theta} = 4\cos\theta$$

$$\sqrt{4^2(1-\sin^2\theta)}$$

$\csc(\theta)\cot(\theta) = \frac{1}{\sin\theta}\,\frac{\cos(\theta)}{\sin(\theta)}$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$$\int \frac{\sqrt{16-x^2}}{x}\,dx = \int \frac{4\cos\theta}{4\sin\theta}\,4\cos\theta\, d\theta = 4\int \frac{\cos^2\theta}{\sin\theta}\,d\theta$$

$$= 4\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta = 4\int \csc\theta - \sin\theta\, d\theta$$

$$= 4\int \underbrace{\csc\theta\,\frac{\csc(\theta)+\cot(\theta)}{\underbrace{\csc(\theta)+\cot(\theta)}_{u}}\,d\theta}_{-du} + 4\cos\theta + C$$

$u = \csc\theta + \cot\theta$

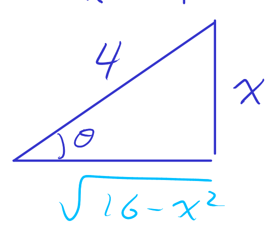
$du = -\csc\theta\cot\theta - \csc^2\theta\, d\theta$

$$= -4\int \frac{1}{u}\,du + 4\cos\theta + C$$

$$= -4\ln\left|\csc\theta + \cot\theta\right| + 4\cos\theta + C$$

$$= \boxed{-4\ln\left|\frac{4}{x} + \frac{\sqrt{16-x^2}}{x}\right| + \cancel{4}\,\frac{\sqrt{16-x^2}}{\cancel{4}} + C}$$

$x = 4\sin\theta$



Example. $\int \frac{x^2}{(25-4x^2)^{3/2}}\,dx$

a^2-u^2	$u = a\sin(\theta)$,
a^2+u^2	$u = a\tan(\theta)$,
u^2-a^2	$u = a\sec(\theta)$,

$$\left(4\left(\frac{25}{4}-x^2\right)\right)^{3/2}$$

$$\to x = \frac{5}{2}\sin\theta$$

$$2x = 5\sin\theta$$

$$x = \frac{5}{2}\sin\theta$$

$$dx = \frac{5}{2}\cos\theta\, d\theta$$

$$\int \frac{x^2}{(25-4x^2)^{3/2}}\,dx = \int \frac{\frac{25}{4}\sin^2\theta}{\left(25-4\left(\frac{25}{4}\sin^2\theta\right)\right)^{3/2}}\,\frac{5}{2}\cos\theta\, d\theta$$

$$= \frac{125}{8}\int \frac{\sin^2\theta\cos\theta}{\left(25(1-\sin^2\theta)\right)^{3/2}}\,d\theta = \frac{1}{8}\int \frac{\sin^2\theta\cos\theta}{(\cos^2\theta)^{3/2}}\,d\theta$$

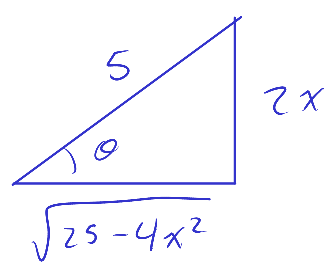
$$25^{3/2} = 5^3 = 125$$

$$= \frac{1}{8}\int \frac{\sin^2\theta\cos\theta}{\cos^3\theta}\,d\theta = \frac{1}{8}\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \frac{1}{8}\int \tan^2\theta\, d\theta$$

$$2x = 5\sin\theta$$

$$= \frac{1}{8}\int \sec^2\theta - 1\, d\theta = \frac{1}{8}\tan\theta - \frac{\theta}{8} + C$$

$$= \boxed{\frac{1}{8}\,\frac{2x}{\sqrt{25-4x^2}} - \frac{\sin^{-1}\left(\frac{2x}{5}\right)}{8} + C}$$



$$a^2 - u^2 \qquad u = a\sin(\theta),$$

$$\boxed{a^2 + u^2} \qquad u = a\tan(\theta),$$

$$u^2 - a^2 \qquad u = a\sec(\theta),$$

Example. $\displaystyle\int_0^{1/3} \frac{dx}{(\underbrace{9x^2}_{u^2}+1)^{3/2}}$

$3x = \tan\theta \longrightarrow x = \dfrac{\tan\theta}{3}$

$\downarrow \qquad dx = \dfrac{\sec^2\theta}{3}\, d\theta$

$9x^2+1 = \tan^2\theta + 1 = \sec^2\theta$

$\theta = \tan^{-1}(3x)$

$x = 0 \longrightarrow \theta = 0$

$x = 1/3 \longrightarrow \theta = \tan^{-1}(3(1/3)) = \tan^{-1}(1) = \pi/4$

$$\int_0^{1/3} \frac{dx}{(9x^2+1)^{3/2}} = \int_0^{\pi/4} \frac{1}{\underbrace{(\sec^2\theta)^{3/2}}_{\sec^3\theta}} \frac{\sec^2\theta}{3}\, d\theta = \frac{1}{3}\int_0^{\pi/4} \sec\theta\, d\theta$$

$$= \frac{1}{3} \ln|\sec\theta + \tan\theta| \Big|_0^{\pi/4} = \frac{1}{3}\left(\ln|\sqrt{2}+1| - \underbrace{\ln|1+0|}_{0}\right)$$

$$= \boxed{\frac{\ln(1+\sqrt{2})}{3}}$$

Example. $\displaystyle\int \frac{x}{\sqrt{\underbrace{x^2-2x+10}_{\text{complete the square}}}}$

a^2-u^2	$u = a\sin(\theta)$,
a^2+u^2	$u = a\tan(\theta)$,
u^2-a^2	$u = a\sec(\theta)$,

$\left(\frac{1}{2}(2)\right)^2$

$$x^2-2x+\underline{1}-\underline{1}+10$$

$$=(x^2-2x+1)+9$$

$$=(x-1)^2+9 \Rightarrow x-1=3\tan\theta \to x=3\tan\theta+1$$

$$dx = 3\sec^2\theta\, d\theta$$

$$\int \frac{x}{\sqrt{x^2-2x+10}}\,dx = \int \frac{3\tan\theta+1}{\sqrt{3^2\tan^2\theta+3^2}}\,3\sec^2\theta\, d\theta = \frac{3}{3}\int \frac{(3\tan\theta+1)\sec^2\theta}{\sqrt{\underbrace{\tan^2\theta+1}_{\sqrt{\sec^2\theta}}}}\,d\theta$$

$$= \int (3\tan\theta+1)\sec\theta\, d\theta = 3\int \sec\theta\tan\theta\, d\theta + \int \sec\theta\, d\theta$$

$$\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}$$

$$= 3\sec\theta + \ln\left|\sec\theta+\tan\theta\right| + C$$

$$= \boxed{\cancel{3}\frac{\sqrt{x^2-2x+10}}{\cancel{3}} + \ln\left|\frac{\sqrt{x^2-2x+10}}{3} + \frac{x-1}{3}\right| + C}$$

$$x-1 = 3\tan\theta$$

$$\frac{x-1}{3} = \tan\theta$$