14.3: Motion in Space

Definition.
Let the **position** of an object moving in three-dimensional space be given by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, for $t \geq 0$. The **velocity** of the object is

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle.$$

The **speed** of the object is the scalar function

$$|\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$ Scalar

The **acceleration** of the object is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

Example. Given $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t)\rangle$ for $0 \leq t \leq 2\pi$, find the velocity, speed, and acceleration.

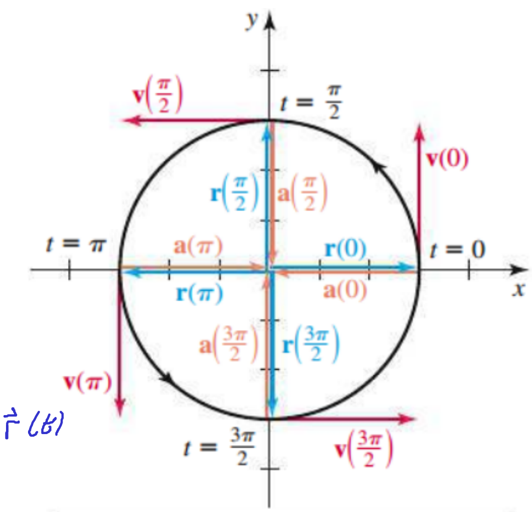
$$\vec{v}(t) = \vec{r}\,'(t) = \langle -3\sin(t), 3\cos(t)\rangle$$

$$|\vec{v}(t)| = \sqrt{(-3\sin(t))^2 + (3\cos(t))^2}$$

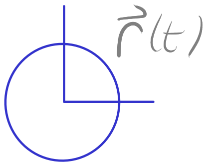
$$= \sqrt{9(\underbrace{\sin^2(t) + \cos^2(t)})}$$

$$= 3$$

$$\vec{a}(t) = \vec{r}\,''(t) = \langle -3\cos(t), -3\sin(t)\rangle = -\vec{r}(t)$$



Circular motion: At all times $\mathbf{a}(t) = -\mathbf{r}(t)$ and $\mathbf{v}(t)$ is orthogonal to $\mathbf{r}(t)$ and $\mathbf{a}(t)$.



$\vec{r}(t)$ constant

Theorem 14.2: Motion with constant $|\mathbf{r}|$
Let $\mathbf{r}$ describe a path on which $|\mathbf{r}|$ is constant (motion on a circle or sphere centered at the origin). Then $\mathbf{r} \cdot \mathbf{v} = 0$, which means the position vector and the velocity vector are orthogonal at all times for which the functions are defined.

Example (Path on a sphere)**.** Consider

$$\mathbf{r}(t) = \langle 3\cos(t),\, 5\sin(t),\, 4\cos(t)\rangle, \quad \text{for } 0 \le t \le 2\pi.$$

a) Show that an object with this trajectory moves on a sphere and find the radius.

$$|\vec{r}(t)| = \sqrt{(3\cos t)^2 + (5\sin t)^2 + (4\cos t)^2}$$
$$= \sqrt{25\cos^2(t) + 25\sin^2(t)}$$
$$= 5\sqrt{\cos^2(t) + \sin^2(t)} = 5$$

b) Find the velocity and speed of the above trajectory.

$$\vec{v}(t) = \langle -3\sin(t),\, 5\cos(t),\, -4\sin(t)\rangle$$

$$|\vec{v}(t)| = \sqrt{9\sin^2(t) + 25\cos^2(t) + 16\sin^2(t)} = \sqrt{25(\underbrace{\sin^2(t) + \cos^2(t)}_{1})} = 5$$

c) Show that $\mathbf{r}(t) = \langle 5\cos(t),\, 5\sin(t),\, 5\sin(2t)\rangle$ does not lie on a sphere. How could this function be modified so that it does lie on a sphere?

$$|\vec{r}(t)| = \sqrt{25\cos^2(t) + 25\sin^2(t) + 25\sin^2(2t)}$$
$$= 5\sqrt{\underbrace{\cos^2(t) + \sin^2(t)}_{1} + \sin^2(2t)} = 5\sqrt{1 + \sin^2(2t)}$$

Not constant

$$\vec{r}_m(t) = \frac{1}{5\sqrt{1+\sin^2(2t)}}\langle 5\cos(t), 5\sin(t), 5\sin(2t)\rangle \longrightarrow |\vec{r}_m(t)| = 1$$

– or –

$$\vec{r}_m(t) = \frac{1}{\sqrt{1+\sin^2(2t)}}\langle 5\cos(t), 5\sin(t), 5\sin(2t)\rangle \longrightarrow |\vec{r}_m(t)| = 5$$

Example. Given $\mathbf{a}(t) = \langle \cos(t), 4\sin(t) \rangle$, with an initial velocity $\langle \mathbf{u}_0, \mathbf{v}_0 \rangle = \langle 0, 4 \rangle$ and an initial position $\langle x_0, y_0 \rangle = \langle 5, 0 \rangle$ where $t \geq 0$, find the velocity and position vector.

$$\vec{v}(t) = \int \vec{a}(t)\, dt = \langle \sin(t), -4\cos(t) \rangle + \langle c_1, c_2 \rangle$$
$$= \langle \sin(t), 8 - 4\cos(t) \rangle$$

$$\vec{v}(0) = \langle 0, -4 \rangle + \langle c_1, c_2 \rangle = \underbrace{\langle 0, 4 \rangle}_{\text{I.C.}} \Longrightarrow \begin{array}{l} c_1 = 0 \\ c_2 = 8 \end{array}$$

$$\vec{r}(t) = \int \vec{v}(t)\, dt = \langle -\cos(t), 8t - 4\sin(t) \rangle + \langle c_3, c_4 \rangle$$
$$= \langle 6 - \cos(t), 8t - 4\sin(t) \rangle$$

$$\langle 5, 0 \rangle = \vec{r}(0) = \langle -1, 0 \rangle + \langle c_3, c_4 \rangle \longrightarrow \begin{array}{l} c_3 = 6 \\ c_4 = 0 \end{array}$$

Summary: Two-Dimensional Motion in a Gravitational Field
Consider an object moving in a plane with a horizontal x-axis and a vertical y-axis, subject only to the force of gravity. Given the initial velocity $\mathbf{v}(0) = \langle u_0, v_0 \rangle$ and the initial position $\mathbf{r}(0) = \langle x_0, y_0 \rangle$, the velocity of the object, for $t \geq 0$, is

$$\mathbf{v}(t) = \langle x'(t), y'(t) \rangle = \langle u_0, -gt + v_0 \rangle$$

and the position is

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \left\langle u_0 t + x_0, -\frac{1}{2}gt^2 + v_0 t + y_0 \right\rangle.$$

Example. Consider a ball with an initial position of $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$ m and an initial velocity of $\langle u_0, v_0 \rangle = \langle 25, 4 \rangle$ m/s.

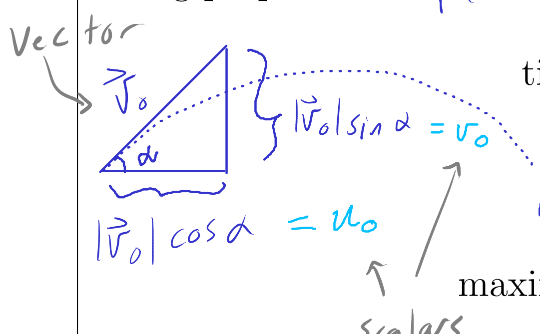
a) Find the position and velocity of the ball while it is in the air

$g = 9.81 \ m/s^2$ or $g = 32 \ ft/s^2$

$\vec{v}(t) = \langle 25, -gt + 4 \rangle$ m/s

$\vec{r}(t) = \langle 25t + 0, -\frac{1}{2}gt^2 + 4t + 3 \rangle$ m

Summary: Two-Dimensional Motion
Assume an object traveling over horizontal ground, acted on only by the gravitational force, has an initial position $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$ and initial velocity $\langle u_0, v_0 \rangle = \langle |\mathbf{v}_0| \cos\alpha, |\mathbf{v}_0| \sin\alpha \rangle$. The trajectory, which is a segment of a parabola, has the following properties. $\vec{r}(t) = \langle u_0 t, -\frac{1}{2}gt^2 + v_0 t \rangle$ $\quad \vec{v}(t) = \langle u_0, -gt + v_0 \rangle$



$$\text{time of flight} = T = \frac{2|\mathbf{v}_0| \sin\alpha}{g} = \boxed{\frac{2v_0}{g}}$$

$$u_0 T \quad \text{range} = \frac{|\mathbf{v}_0|^2 \sin(2\alpha)}{g} = \frac{2|\vec{v}_0|\sin(\alpha)|\vec{v}_0|\cos\alpha}{g} = \boxed{\frac{2v_0 u_0}{g}}$$

$$\text{maximum height} = y\left(\frac{T}{2}\right) = \frac{(|\mathbf{v}_0| \sin\alpha)^2}{2g} = \boxed{\frac{v_0^2}{2g}}$$

$-\frac{1}{2}gt^2 + v_0 t = 0 \qquad T = 0$

$t\left(-\frac{1}{2}gt + v_0\right) = 0$

$v_0 = \frac{1}{2}gt \longrightarrow \boxed{T = \frac{2v_0}{g}} = \frac{2|\vec{v}_0|\sin\alpha}{g}$

$$-\frac{1}{2}g\left(\frac{v_0}{g}\right)^2 + v_0\left(\frac{v_0}{g}\right) = \frac{-v_0^2}{2g} + \frac{v_0^2}{g} = \frac{v_0^2}{2g} = \frac{(|\vec{v}_0|\sin\alpha)^2}{2g}$$

Example. Consider a ball with an initial position of $\langle x_0, y_0\rangle = \langle 0, 0\rangle$ m and an initial velocity of $\langle u_0, v_0\rangle = \langle 25, 4\rangle$ m/s. Assuming the ground is flat and level:

b) How long is the ball in the air?

$$t = \frac{2v_0}{g} = \frac{2(4)}{g} = \frac{8}{g} \text{ seconds}$$

c) How far does the ball travel horizontally?

$$\text{range} = \frac{2(25)(4)}{g} = \frac{200}{g}\ m \qquad \frac{2v_0u_0}{g}$$

d) What is the maximum height that the ball reaches?

$$\frac{v_0^2}{2g} \qquad \frac{(4)^2}{2g} = \frac{8}{g}\ m$$