1 13.4: Cross Products

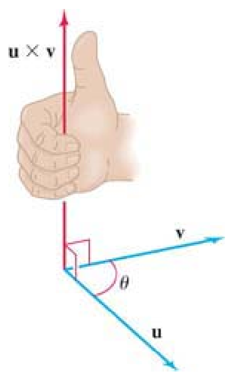
Definition. (Cross Product)
Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the **cross product** $\mathbf{u} \times \mathbf{v}$ is a vector with magnitude

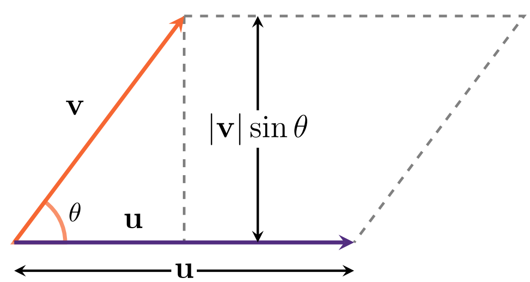
$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta,$$

where $0 \leq \theta \leq \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
The direction of $\mathbf{u} \times \mathbf{v}$ is given by the **right-hand rule**:

> When you put your the vectors tail to tail and let the fingers of your right hand curl from $\mathbf{u}$ to $\mathbf{v}$, the direction of $\mathbf{u} \times \mathbf{v}$ is the direction of your thumb, orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ (Figure 13.56).

When $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, the direction of $\mathbf{u} \times \mathbf{v}$ is undefined.





Theorem 13.3: Geometry of the Cross Product
Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in $\mathbb{R}^3$.

1. The vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
2. If $\mathbf{u}$ and $\mathbf{v}$ are two sides of a parallelogram, then the area of the parallelogram is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta$$

Example. Consider the vectors $\mathbf{u} = \langle 2, 0, 0 \rangle$ and $\mathbf{v} = \left\langle \sqrt{3}, 3, 0 \right\rangle$. The angle between these vectors is $\theta = \frac{\pi}{3}$. Find the area of the parallelogram formed by these vectors.

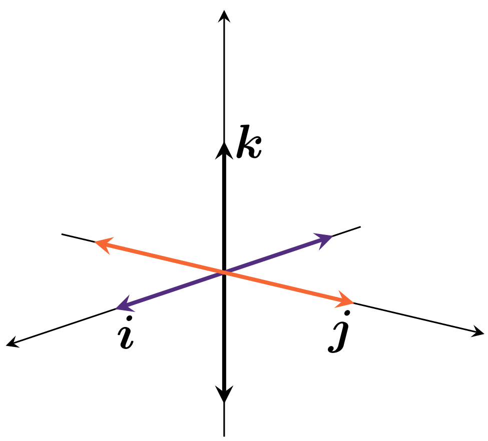
Theorem 13.4: Properties of the Cross Product
Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^3$, and let a and b be scalars.

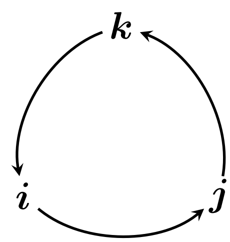
1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ — Anticommutative property
2. $(a\mathbf{u}) \times (b\mathbf{v}) = ab(\mathbf{u} \times \mathbf{v})$ — Associative property
3. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ — Distributive property
4. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$ — Distributive property

Theorem 13.5: Cross Products of Coordinate Unit Vectors

$$\boldsymbol{i} \times \boldsymbol{j} = -(\boldsymbol{j} \times \boldsymbol{i}) = \boldsymbol{k} \qquad \boldsymbol{j} \times \boldsymbol{k} = -(\boldsymbol{k} \times \boldsymbol{j}) = \boldsymbol{i}$$

$$\boldsymbol{k} \times \boldsymbol{i} = -(\boldsymbol{i} \times \boldsymbol{k}) = \boldsymbol{j} \qquad \boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = \boldsymbol{0}$$





$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$$
$$\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$$
$$\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$$

Using the unit vectors, we can compute $\mathbf{u} \times \mathbf{v}$:

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}) \times (v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}) \\
&= u_1v_1 \underbrace{(\boldsymbol{i} \times \boldsymbol{i})}_{\boldsymbol{0}} + u_1v_2 \underbrace{(\boldsymbol{i} \times \boldsymbol{j})}_{\boldsymbol{k}} + u_1v_3 \underbrace{(\boldsymbol{i} \times \boldsymbol{k})}_{-\boldsymbol{j}} \\
&+ u_2v_1 \underbrace{(\boldsymbol{j} \times \boldsymbol{i})}_{-\boldsymbol{k}} + u_2v_2 \underbrace{(\boldsymbol{j} \times \boldsymbol{j})}_{\boldsymbol{0}} + u_2v_3 \underbrace{(\boldsymbol{j} \times \boldsymbol{k})}_{\boldsymbol{i}} \\
&+ u_3v_1 \underbrace{(\boldsymbol{k} \times \boldsymbol{i})}_{\boldsymbol{j}} + u_3v_2 \underbrace{(\boldsymbol{k} \times \boldsymbol{j})}_{-\boldsymbol{i}} + u_3v_3 \underbrace{(\boldsymbol{k} \times \boldsymbol{k})}_{\boldsymbol{0}} \\
&= (u_2v_3 - u_3v_2)\boldsymbol{i} - (u_1v_3 - u_3v_1)\boldsymbol{j} + (u_1v_2 - u_2v_1)\boldsymbol{k}
\end{aligned}$$

Theorem 13.6: Evaluating the Cross Product
Let $\mathbf{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$ and $\mathbf{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$. Then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \boldsymbol{k}$$

Note:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\boldsymbol{i} - (u_1v_3 - u_3v_1)\boldsymbol{j} + (u_1v_2 - u_2v_1)\boldsymbol{k}$$

Alternative approach:

$$\left|\begin{array}{ccc|cc} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{i} & \boldsymbol{j} \\ u_1 & u_2 & u_3 & u_1 & u_2 \\ v_1 & v_2 & v_3 & v_1 & v_2 \end{array}\right.$$

Example. Compute $\mathbf{u} \times \mathbf{v}$ for $\mathbf{u} = \langle 3, 5, 4 \rangle$ and $\mathbf{v} = \langle 1, -1, 9 \rangle$.

Example. Consider the vectors $\mathbf{u} = \left\langle \sqrt{3}, 1, 0 \right\rangle$ and $\mathbf{v} = \left\langle -\sqrt{3}, 1, 0 \right\rangle$. From the unit circle, we know the angle between these two vectors is $\theta = \frac{2\pi}{3}$. Use the definition of the cross product to show this.

Example. Find the area of the triangle formed by $\mathbf{u} = \langle 1, 2, 3 \rangle$ and $\mathbf{v} = \langle 3, -1, 1 \rangle$.

Example. Given a force $\mathbf{F}$ applied to a point P at the head of the vector $\mathbf{r} = \overrightarrow{OP}$, the **torque** produced at point O is given by $\tau = \mathbf{r} \times \mathbf{F}$ with magnitude

$$|\tau| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin \theta.$$

Now suppose a force of $20N$ is applied to a wrench attached to a bolt in a direction perpendicular to the bolt. Which produces more torque: applying the force at an angle of $60°$ on a wrench that is $0.15m$ long or applying the force at an angle of $135°$ on a wrench that is $0.25m$ long?