

## 14.5: Curvature and Normal Vectors:

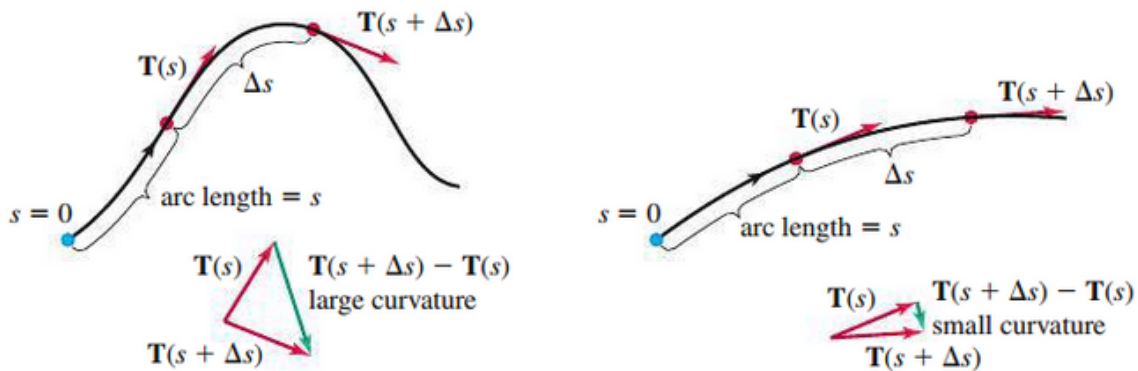
There are two ways to change the velocity, or in other words, to accelerate:

- change in speed
- change in direction

The change in direction is referred to as *curvature*. Recall that if we have a smooth curve  $\mathbf{r}(t)$ , the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} \Rightarrow \left| \frac{d\mathbf{T}}{ds} \right| = 1$$

Specifically, *curvature* of the curve is the magnitude of the rate at which  $\mathbf{T}$  changes with respect to arc length.



### Definition. (Curvature)

Let  $\mathbf{r}$  describe a smooth parameterized curve. If  $s$  denotes arc length and  $\mathbf{T} = \mathbf{r}'/|\mathbf{r}'|$  is the unit tangent vector, the **curvature** is  $\kappa(s) = \left| \frac{d\mathbf{T}}{ds} \right|$ .   
*Kappa*  $\geq 0$  change unit tangent vector w.r.t. arc length

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| \quad \left| \frac{d\vec{T}}{dt} \right| = \left| \frac{d\vec{T}}{ds} \cdot \frac{ds}{dt} \right| \Rightarrow \left| \frac{d\vec{T}}{ds} \right| = \frac{|d\vec{T}/dt|}{|ds/dt|}$$

$\vec{v}(t)$   
↓

### Theorem 14.4: Curvature Formula

Let  $\mathbf{r}(t)$  describe a smooth parameterized curve, where  $t$  is any parameter. If  $\mathbf{v} = \mathbf{r}'$  is the velocity and  $\mathbf{T}$  is the unit tangent vector, then the curvature is

$$\kappa(t) = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \quad \leftarrow \quad \left| \frac{\vec{T}'(t)}{\vec{r}'(t)} \right|$$

$\vec{v}$  speed

- $\kappa$  is a non-negative scalar-valued function
- Curvature of zero corresponds to a straight line
- A relatively flat curve has a small curvature
- A tight curve has a larger curvature

~~$$\sqrt{\frac{0}{0} + \frac{0}{0} + \frac{0}{0}} = \sqrt{0 + 0 + 0} = 0$$~~

**Example.** Consider the line

$$\mathbf{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle, \text{ for } -\infty < t < \infty.$$

Compute  $\kappa$ .

$$\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\langle 0, 0, 0 \rangle|}{\sqrt{a^2 + b^2 + c^2}} = 0$$

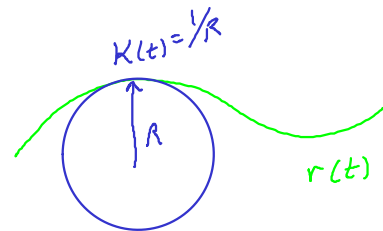
$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\langle a, b, c \rangle}{\sqrt{a^2 + b^2 + c^2}} \longrightarrow \vec{T}'(t) = \langle 0, 0, 0 \rangle$$

$$R \rightarrow \infty \Rightarrow \kappa = 0$$

**Example.** Consider the circle

$$\mathbf{r}(t) = \langle R \cos(t), R \sin(t) \rangle$$

for  $0 \leq t \leq 2\pi$ , where  $R > 0$ . Show that  $\kappa = 1/R$ .



$$\kappa = \frac{|d\vec{T}/dt|}{|ds/dt|} = \frac{|\langle -\cos(t), -\sin(t) \rangle|}{R} = \frac{1}{R} \sqrt{\underbrace{\cos^2(t) + \sin^2(t)}_1} = \frac{1}{R}$$

$$\begin{aligned} \vec{T}(t) &= \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\langle -R \sin(t), R \cos(t) \rangle}{\sqrt{(-R \sin(t))^2 + (R \cos(t))^2}} = \frac{\langle -R \sin(t), R \cos(t) \rangle}{\sqrt{R^2(\sin^2(t) + \cos^2(t))}} = \frac{\langle -R \sin(t), R \cos(t) \rangle}{R} \\ &= \langle -\sin(t), \cos(t) \rangle \end{aligned}$$

**Example.** Consider the curve

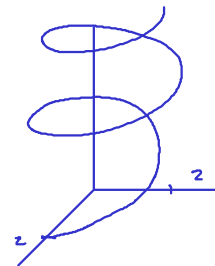
$$\mathbf{r}(t) = \left\langle 2 \cos(t), 2 \sin(t), \sqrt{5}t \right\rangle$$

Compute  $\kappa$ .

$$\kappa = \frac{1}{2}$$

$$\kappa = \frac{|d\vec{T}/dt|}{|ds/dt|} = \frac{\left| \frac{1}{3} \langle -2 \cos(t), -2 \sin(t), 0 \rangle \right|}{3} = \frac{1}{9} \sqrt{4(\cos^2(t) + \sin^2(t))} = \left( \frac{2}{9} \right)$$

$$\vec{T}(t) = \frac{\langle -2 \sin(t), 2 \cos(t), \sqrt{5} \rangle}{\sqrt{2^2(\sin^2(t) + \cos^2(t)) + 5}} = \frac{1}{3} \langle -2 \sin(t), 2 \cos(t), \sqrt{5} \rangle$$



### An Alternative Curvature Formula:

Consider a smooth function  $\mathbf{r}(t)$  with non-zero velocity  $\mathbf{v}(t) = \mathbf{r}'(t)$  and non-zero acceleration  $\mathbf{a}(t) = \mathbf{v}'(t)$ .

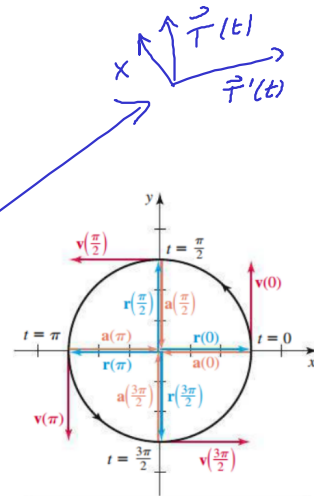
$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} \Rightarrow \mathbf{v} = |\mathbf{v}| \mathbf{T}.$$

Thus

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}[|\mathbf{v}| \mathbf{T}] = \frac{d}{dt}[|\mathbf{v}|] \mathbf{T} + |\mathbf{v}| \frac{d\mathbf{T}}{dt}.$$

Now we form  $\mathbf{v} \times \mathbf{a}$ :

$$\begin{aligned} \mathbf{v} \times \mathbf{a} &= |\mathbf{v}| \mathbf{T} \times \left( \frac{d}{dt}[|\mathbf{v}|] \mathbf{T} + |\mathbf{v}| \frac{d\mathbf{T}}{dt} \right) \\ &= \underbrace{|\mathbf{v}| \mathbf{T} \times \frac{d}{dt}[|\mathbf{v}|] \mathbf{T}}_0 + \underbrace{|\mathbf{v}| \mathbf{T} \times |\mathbf{v}| \frac{d\mathbf{T}}{dt}} \end{aligned}$$



Since  $\mathbf{T}$  is a unit vector,  $\mathbf{T}$  and  $d\mathbf{T}/dt$  are orthogonal (Theorem 14.2). Thus

$$|\mathbf{v} \times \mathbf{a}| = \left| |\mathbf{v}| \mathbf{T} \times |\mathbf{v}| \frac{d\mathbf{T}}{dt} \right| = |\mathbf{v}| \underbrace{|\mathbf{T}|}_1 \left| |\mathbf{v}| \frac{d\mathbf{T}}{dt} \right| \underbrace{\sin \theta}_1 = |\mathbf{v}|^2 \left| \frac{d\mathbf{T}}{dt} \right|$$

Now, using Theorem 14.4, where  $\left| \frac{d\mathbf{T}}{dt} \right| = \kappa |\mathbf{v}|$ , we have

$$K = \frac{|d\hat{T}/dt|}{|ds/dt|} = \frac{1}{|\dot{\mathbf{r}}|} \left| \frac{d\hat{T}}{dt} \right|$$

$$|\mathbf{v} \times \mathbf{a}| = |\mathbf{v}|^2 \left| \frac{d\mathbf{T}}{dt} \right| = |\mathbf{v}|^2 \kappa |\mathbf{v}| = \kappa |\mathbf{v}|^3.$$

### Theorem 14.5: Alternative Curvature Formula

Let  $\mathbf{r}$  be the position of an object moving on a smooth curve. The **curvature** at a point on the curve is

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3},$$

where  $\mathbf{v} = \mathbf{r}'$  is the velocity and  $\mathbf{a} = \mathbf{v}'$  is the acceleration.

$$K = \frac{1}{|\vec{r}|} \left| \frac{d\vec{T}}{dt} \right| = \frac{|\vec{r} \times \vec{a}|}{|\vec{r}|^3}$$

$$K = \frac{1}{16}$$

**Example.** Consider the curve

$$\mathbf{r}(t) = \langle -16 \cos(t), 16 \sin(t), 0 \rangle.$$

$$K = \frac{1}{a} \quad (a=16)$$

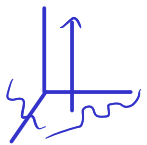
Compute the curvature  $\kappa$  using both methods.

$$\vec{r}(t) = \langle 16 \sin(t), 16 \cos(t), 0 \rangle \quad |\vec{r}(t)| = \sqrt{16^2 \sin^2(t) + 16^2 \cos^2(t)} = 16$$

$$\vec{a}(t) = \langle 16 \cos(t), -16 \sin(t), 0 \rangle$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\langle 16 \sin(t), 16 \cos(t), 0 \rangle}{16} = \langle \sin(t), \cos(t), 0 \rangle$$

$$K = \frac{1}{|\vec{r}|} \left| \frac{d\vec{T}}{dt} \right| = \frac{1}{16} |\langle \cos(t), -\sin(t), 0 \rangle| = \frac{1}{16} \sqrt{\cos^2(t) + \sin^2(t)} = \frac{1}{16}$$



$$K = \frac{|\vec{r} \times \vec{a}|}{|\vec{r}|^3} = \frac{|\langle 0, -0, -16^2 \sin^2(t) - 16^2 \cos^2(t) \rangle|}{16^3} = \frac{16^2}{16^3} = \frac{1}{16}$$

## Principal Unit Normal Vector

Curvature indicates how quickly a curve turns. The principal unit normal vector determines the *direction* in which a curve turns.

### Definition. (Principal Unit Normal Vector)

Let  $\mathbf{r}$  describe a smooth curve parameterized by arc length. The **principal unit normal vector** at a point  $P$  on the curve at which  $\kappa \neq 0$  is

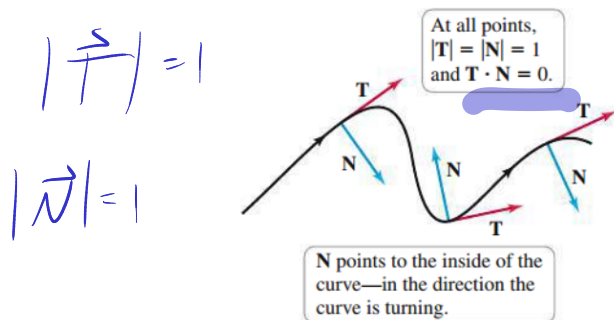
$$\mathbf{N}(s) = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{1}{\kappa} \frac{d\mathbf{T}}{ds}.$$

unit change in unit tangent vector wrt arc length

For other parameters, we use the equivalent formula

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|},$$

evaluated at the value of  $t$  corresponding to  $P$ .



### Theorem 14.6: Properties of the Principal Unit Normal Vector

Let  $\mathbf{r}$  describe a smooth parameterized curve with unit tangent vector  $\mathbf{T}$  and principal unit normal vector  $\mathbf{N}$ .

1.  $\mathbf{T}$  and  $\mathbf{N}$  are orthogonal at all points of the curve; that is,  $\mathbf{T} \cdot \mathbf{N} = 0$  at all points where  $\mathbf{N}$  is defined.
2. The principal unit normal vector points to the inside of the curve – in the direction that the curve is turning.

**Example.** For the curve  $\mathbf{r}(t) = \langle a \cos(t), a \sin(t), bt \rangle$ , find the unit tangent vector  $\mathbf{T}$  and the principal unit normal vector  $\mathbf{N}$ . Verify  $|\mathbf{T}| = |\mathbf{N}| = 1$  and  $\mathbf{T} \cdot \mathbf{N} = 0$ .

$$\vec{v}(t) = \langle -a \sin(t), a \cos(t), b \rangle$$

$$|\vec{v}(t)| = \sqrt{\underbrace{(-a \sin(t))^2 + (a \cos(t))^2}_{a^2} + b^2} = \sqrt{a^2 + b^2} \quad \text{Constant}$$

$$\vec{T} = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\langle -a \sin(t), a \cos(t), b \rangle}{\sqrt{a^2 + b^2}}$$

$$|\vec{T}| = \left| \frac{\vec{v}(t)}{|\vec{v}(t)|} \right| = \frac{1}{|\vec{v}(t)|} |\vec{v}(t)| = 1$$

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} = \frac{\frac{1}{\sqrt{a^2+b^2}} \langle -a \cos(t), -a \sin(t), 0 \rangle}{\frac{\sqrt{(-a \cos(t))^2 + (-a \sin(t))^2}}{\sqrt{a^2+b^2}}} = \frac{1}{a} \langle -a \cos(t), -a \sin(t), 0 \rangle = \langle -\cos(t), -\sin(t), 0 \rangle$$

$$|\vec{N}| = 1$$

$$\begin{aligned} \vec{T} \cdot \vec{N} &= \frac{1}{\sqrt{a^2+b^2}} \langle -a \sin(t), a \cos(t), b \rangle \cdot \langle -\cos(t), -\sin(t), 0 \rangle \\ &= \frac{1}{\sqrt{a^2+b^2}} (a \sin(t) \cos(t) - a \sin(t) \cos(t) + 0) = 0 \end{aligned}$$

## Components of the Acceleration

Recall that the change in velocity, or acceleration, of an object can change in *speed* (in the direction of  $\mathbf{T}$ ) and in *direction* (in the direction of  $\mathbf{N}$ ).  $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} \Rightarrow \mathbf{v} = \mathbf{T}|\mathbf{v}| = \mathbf{T} \frac{ds}{dt}$ .

$$\frac{d}{dt}[\vec{v}] = \vec{a}$$

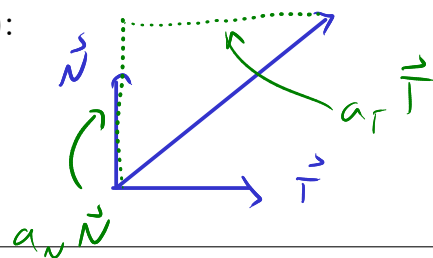
$$\begin{aligned} \rightarrow \mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d}{dt} \left( \mathbf{T} \frac{ds}{dt} \right) && \text{product rule} \\ &= \frac{d\mathbf{T}}{dt} \frac{ds}{dt} + \mathbf{T} \frac{d^2s}{dt^2} \\ &= \underbrace{\frac{d\mathbf{T}}{ds}}_{\kappa \mathbf{N}} \underbrace{\frac{ds}{dt}}_{|\mathbf{v}|} \underbrace{\frac{ds}{dt}}_{|\mathbf{v}|} + \mathbf{T} \frac{d^2s}{dt^2} \\ &= \kappa |\mathbf{v}|^2 \mathbf{N} + \frac{d^2s}{dt^2} \mathbf{T}. \end{aligned}$$

### Theorem 14.7: Tangential and Normal Components of the Acceleration

The acceleration vector of an object moving in space along a smooth curve has the following representation in terms of its **tangential component**  $a_T$  (in the direction of  $\mathbf{T}$ ) and its **normal component**  $a_N$  (in the direction of  $\mathbf{N}$ ):

$$\mathbf{a} = a_N \mathbf{N} + a_T \mathbf{T},$$

where  $a_N = \kappa |\mathbf{v}|^2 = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|}$  and  $a_T = \frac{d^2s}{dt^2}$ .



$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$$



**Example.** Consider the function

$$\mathbf{r}(t) = \langle -2t + 2, -2t + 3, -2t + 2 \rangle. \quad \rightarrow \vec{a}(t) = \langle 0, 0, 0 \rangle$$

(Note: In the original image, arrows point from the terms  $-2t$  in each component to the word "linear" written above them.)

Find the tangential and normal components of the acceleration.

$$a_N = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|} \quad a_T = \frac{d^2 s}{dt^2} = \frac{d}{dt} [|\vec{v}(t)|] = 0 \quad \text{LC 3}$$

$$\vec{v}(t) = \langle -2, -2, -2 \rangle$$

$$|\vec{v}(t)| = 2\sqrt{3}$$

$$\vec{a}(t) = \langle 0, 0, 0 \rangle$$

$$a_N = \frac{|\langle 0, 0, 0 \rangle|}{2\sqrt{3}} = 0$$

LC 2

constant

**Example.** Find the components of the acceleration on the circular trajectory

$$\mathbf{r}(t) = \langle R \cos(\omega t), R \sin(\omega t), 0 \rangle \quad K = \frac{1}{R}$$

~~$$\vec{v}(t) = \langle -R\omega \sin(\omega t), R\omega \cos(\omega t), 0 \rangle$$~~

$$|\vec{v}(t)| = \sqrt{(-R\omega \sin(\omega t))^2 + (R\omega \cos(\omega t))^2}$$

~~$$\vec{a}(t) = \langle -R\omega^2 \cos(\omega t), -R\omega^2 \sin(\omega t), 0 \rangle$$~~

$$= R\omega$$

$$a_N = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|} = \frac{|\langle 0, 0, R^2 \omega^3 \sin^2(\omega t) + R^2 \omega^3 \cos^2(\omega t) \rangle|}{R\omega^2} = \frac{R^2 \omega^3}{R\omega}$$

$$R=2, \omega=3$$

$$\text{LC 5} \rightarrow 2 \cdot 3^2 = 18$$

$$= R\omega^2$$

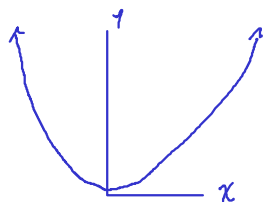
$$a_T = \frac{d}{dt} [|\vec{v}(t)|] = 0$$

$$\text{LC 4} \rightarrow 0$$

LC

1. 16
2. 0
3. 0
4. 0
5. 18

**Example.** The driver of a car follows the parabolic trajectory  $\mathbf{r}(t) = \langle t, t^2 \rangle$ , for  $-2 \leq t \leq 2$ , through a sharp bend. Find the tangential and normal components of the acceleration of the car.



$$a_N = k |\dot{\mathbf{r}}|^2 = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|} \quad a_T = \frac{d^2 s}{dt^2}$$

$$\dot{\mathbf{r}}(t) = \langle 1, 2t, 0 \rangle \quad |\dot{\mathbf{r}}(t)| = \sqrt{1+4t^2}$$

$$\ddot{\mathbf{r}}(t) = \langle 0, 2, 0 \rangle$$

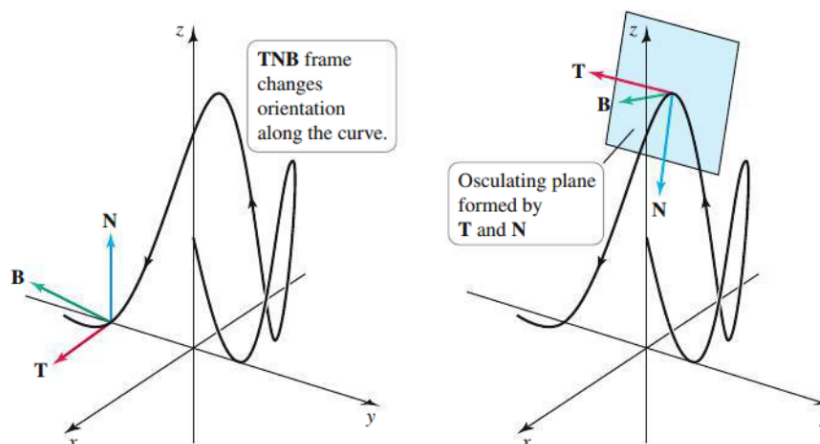
$$a_N = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|} = \frac{|\langle 0, 0, 2 \rangle|}{\sqrt{1+4t^2}} = \frac{2}{\sqrt{1+4t^2}}$$

$$a_T = \frac{d^2 s}{dt^2} = \frac{d}{dt} [|\dot{\mathbf{r}}(t)|] = \frac{8t}{2\sqrt{1+4t^2}} = \frac{4t}{\sqrt{1+4t^2}}$$

$$\begin{matrix} \vec{T} \\ \vec{N} \end{matrix} \longrightarrow \vec{a} = a_N \vec{N} + a_T \vec{T}$$

## The Binormal Vector and Torsion

On a smooth parameterized curve  $C$ ,  $\mathbf{T}$  and  $\mathbf{N}$  determine a plane called the *osculating plane*.



The coordinate system defined by these vectors is called the **TNB frame**. The rate at which the curve  $C$  twists out of the plane is the rate at which  $\mathbf{B}$  changes as we move along  $C$ , which is  $\frac{d\mathbf{B}}{ds}$ .

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T} \times \mathbf{N}) = \underbrace{\frac{d\mathbf{T}}{ds} \times \mathbf{N}}_0 + \mathbf{T} \times \frac{d\mathbf{N}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds}$$

Parallel

$\frac{d\mathbf{B}}{ds}$  is:

- orthogonal to both  $\mathbf{T}$  and  $\frac{d\mathbf{N}}{ds}$ ,
- orthogonal to  $\mathbf{B}$  (Theorem 14.2),
- parallel with  $\mathbf{N}$ .  $\leftarrow$  b/c orthog to  $\mathbf{T}$  & orthog to  $\mathbf{B}$

Since  $\frac{d\mathbf{B}}{ds}$  is parallel to  $\mathbf{N}$ , we write

$$\vec{\mathcal{N}} \cdot \frac{d\mathbf{B}}{ds} = -\tau \mathbf{N} \cdot \vec{\mathcal{N}}$$

where  $\tau$  is the *torsion* (the negative sign is conventional). We can solve for  $\tau$  via the dot product:

$$\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\tau \underbrace{\mathbf{N} \cdot \mathbf{N}}_1 \implies \frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\tau$$

$$\vec{\mathcal{N}} \cdot \vec{\mathcal{N}} = |\vec{\mathcal{N}}|^2 = 1$$

### Definition. (Unit Binormal Vector and Torsion)

Let  $C$  be a smooth parameterized curve with unit tangent and principal unit normal vectors  $\mathbf{T}$  and  $\mathbf{N}$ , respectively. Then at each point of the curve at which the curvature is nonzero, the **unit binomial vector** is

$$\mathbf{B} = \mathbf{T} \times \mathbf{N},$$

and the **torsion** is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

**Example.** Consider the circle  $C$  defined by

$$\mathbf{r}(t) = \langle R \cos(t), R \sin(t) \rangle, \text{ for } 0 \leq t \leq 2\pi, \text{ with } R > 0.$$

Find the unit binormal vector  $\mathbf{B}$  and determine the torsion.

$$\vec{B} = \vec{T} \times \vec{N} \qquad \vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \qquad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\vec{r} = \langle -R \sin(t), R \cos(t), 0 \rangle \qquad |\vec{v}(t)| = R$$

$$\vec{a} = \langle -R \cos(t), -R \sin(t), 0 \rangle$$

$$\vec{T}(t) = \frac{\langle -R \sin(t), R \cos(t), 0 \rangle}{R} = \langle -\sin(t), \cos(t), 0 \rangle$$

$$\vec{N}(t) = \frac{\langle -\cos(t), -\sin(t), 0 \rangle}{\sqrt{\cos^2(t) + \sin^2(t)}} = \langle -\cos(t), -\sin(t), 0 \rangle$$

$$\vec{B} = \langle 0, 0, \sin^2(t) + \cos^2(t) \rangle = \langle 0, 0, 1 \rangle$$

**Example.** Compute the torsion of the helix

$$\mathbf{r}(t) = \langle a \cos(t), a \sin(t), bt \rangle, \text{ for } t \geq 0, \text{ and } b > 0.$$

### Summary: Formula for Curves in Space

Position function:  $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$

Velocity:  $\mathbf{v} = \mathbf{r}'$

Acceleration:  $\mathbf{a} = \mathbf{v}'$

Unit tangent vector:  $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$

Principal unit normal vector:  $\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$  (provided  $d\mathbf{T}/dt \neq \mathbf{0}$ )

Curvature:  $\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$

Components of acceleration:  $\mathbf{a} = a_N \mathbf{N} + a_T \mathbf{T}$ , where

$$a_N = \kappa |\mathbf{v}|^2 = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|} \text{ and } a_T = \frac{d^2s}{dt^2} = \frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}|}$$

Unit binormal vector:  $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v} \times \mathbf{a}|}$

Torsion:  $\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}$