3.7 The Chain Rule

Theorem 3.13 The Chain Rule
Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at x. The composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed in two equivalent ways.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{1}$$

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x) \tag{2}$$

Example. Take the derivatives of the following functions

a) $y = \left(3x^3 + 1\right)^2$

$y' = 2(3x^3+1)^1[9x^2]$

b) $y = \left(3x^3 + 1\right)^7$

$y' = 7(3x^3+1)^6[9x^2]$

c) $y = 6\cos^2(x)$

$y' = 12\cos(x) \cdot [-\sin(x)]$

d) $y = \sin\left(x + \cot(x)\right)$

$y' = \cos(x+\cot(x))[1-\csc^2(x)]$

To use the chain rule,

- Identify the inner and outer function
- Take the derivative of the outside, leaving the original inner function
- Multiply by the derivative of the inner function

e) $y(x) = e^{-4x}$

$y'(x) = e^{-4x}[-4]$

f) $y(x) = \left(\dfrac{x-2}{2x+1}\right)^9$

$y'(x) = 9\left(\dfrac{x-2}{2x+1}\right)^8 \left[\dfrac{(2x+1)1 - (x-2)2}{(2x+1)^2}\right]$

g) $y(x) = \sqrt{\sec(x)}$ $= (\sec(x))^{1/2}$

$y'(x) = \frac{1}{2}(\sec(x))^{-1/2}[\sec(x)\tan(x)]$

$= \frac{1}{2}\sqrt{\sec(x)}\tan(x)$

h) $y(x) = 2(8x-1)^3$

$y'(x) = 6(8x-1)^2[8]$

i) $y(x) = \left(\frac{x}{2} - 1\right)^{-10}$

$y'(x) = -10\left(\frac{x}{2} - 1\right)^{-11}\left[\frac{1}{2}\right]$

j) $y(t) = e^{\sin(t)} + \sin(e^t)$

$y'(t) = e^{\sin(t)}[\cos(t)] + \cos(e^t)[e^t]$

k) $y(x) = x^2 e^{x^2}$

$$y'(x) = \frac{d}{dx}[x^2]e^{x^2} + x^2\frac{d}{dx}[e^{x^2}]$$

$$= 2x e^{x^2} + x^2 e^{x^2}[2x]$$

$$= 2xe^{x^2}(1+x^2)$$

l) $\dfrac{f(x)}{g(x)} = f(x)\cdot[g(x)]^{-1}$

$$\frac{d}{dx}\left[f(x)[g(x)]^{-1}\right] = f'(x)[g(x)]^{-1} + f(x)(-1)[g(x)]^{-2}g'(x)$$

$$= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2}$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

m) $y(x) = f(g(h(x)))$

$$y'(x) = f'(g(h(x)))\cdot g'(h(x))\cdot h'(x)$$

n) $y(x) = -12e^{3x^7}$

$$y'(x) = -12e^{3x^7}[21x^6]$$

o) $y(x) = \dfrac{\cos^2(x)}{e^x(x^2+4)}$

$$y'(x) = \frac{e^x(x^2+4)\frac{d}{dx}[\cos^2(x)] - \cos^2(x)\frac{d}{dx}[e^x(x^2+4)]}{(e^x(x^2+4))^2}$$

$$= \frac{e^x(x^2+4)[2\cos(x)(-\sin(x))] - \cos^2(x)[e^x(x^2+4)+e^x(2x)]}{e^{2x}(x^2+4)^2}$$