15.4: The Chain Rule $\frac{dy}{dx}$ derivative of y wrt to the single var x $\quad \frac{\partial z}{\partial x}$ partial deriv of z wrt to one of the vars e.g. (x,y)

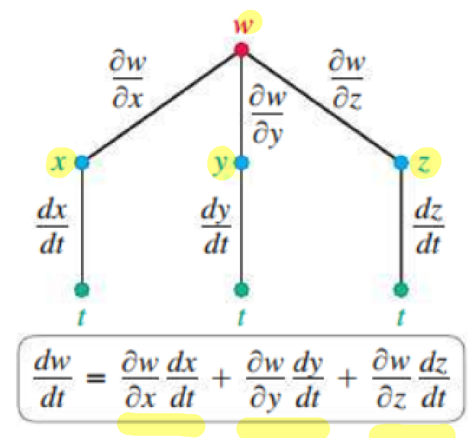
Theorem 15.7: Chain Rule (One Independent Variable)
Let z be a differentiable function of x and y on its domain, where x and y are differentiable functions of t on an interval I. Then

$z=f(x,y)$
$x=g(t)$, $y=h(t)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Note:

- For $z = f(x(t), y(t))$, z is the dependent variable, t is the independent variable, and x and y are **intermediate variables**.
- Since x and y only depend on t, we use the 'ordinary' derivative symbol
- Theorem 15.7 generalizes to functions of n variables



Example. Find the derivative of the following functions using the chain rule where appropriate.

$z = x^2 - 2y^2 + 20$ where $x = 2\cos(t)$ and $y = 2\sin(t)$

LC 1

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$= (2x)(-2\sin(t)) + (-4y)(2\cos(t))$

$= -4x\sin(t) - 8y\cos(t)$

$= -8\cos(t)\sin(t) - 16\sin(t)\cos(t)$

$= -24\sin(t)\cos(t) \longleftarrow$ ok.

$= -12\sin(2t)$

$w = \sin(12x)\cos(2y)$ where $x = t/2$ and $y = t^3$

LC2

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= \left(\cos(12x)\cos(2y)\,12\right)\left(\tfrac{1}{2}\right) + \left(\sin(12x)(-\sin(2y)\,2)\right)(3t^2)$$

$$= 6\cos(12x)\cos(2y) - 6t^2\sin(12x)\sin(2y)$$

$$= 6\left(\cos(6t)\cos(2t^3) - t^2\sin(6t)\cos(2t^3)\right)$$

$\cancel{\frac{\partial x}{dt}}$

$\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}$ $\quad z(m) \leftarrow \frac{dz}{dm}$

$m(x, y)$

$x(s,t) \quad y(s,t)$

$Q = \sqrt{3x^2 + 3y^2 + 2z^2}$ where $x = \sin(t)$, $y = \cos(t)$, and $z = \cos(t)$.

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial Q}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial Q}{\partial z}\cdot\frac{dz}{dt}$$

$$= \left(\frac{\overset{3}{6}x}{2\sqrt{3x^2+3y^2+2z^2}}\right)\left(\cos(t)\right) + \left(\frac{\overset{3}{6}y}{2\sqrt{3x^2+3y^2+2z^2}}\right)\left(-\sin(t)\right) + \left(\frac{\overset{2}{4}z}{2\sqrt{3x^2+3y^2+2z^2}}\right)\left(-\sin(t)\right)$$

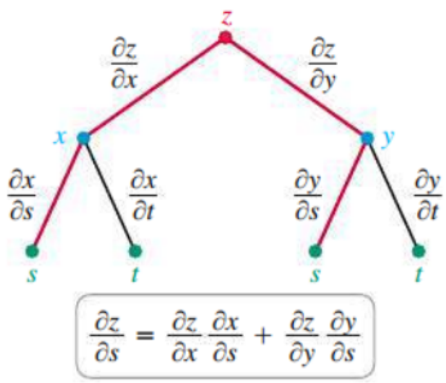
$$= \frac{3x\cos(t) - (3y+2z)\sin(t)}{\sqrt{3x^2+3y^2+2z^2}} = \frac{3\sin(t)\cos(t) - 5\cos(t)\sin(t)}{\sqrt{3\sin^2(t) + 5\cos^2(t)}}$$

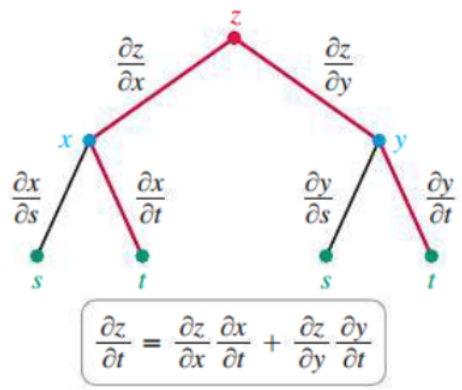
$\underbrace{3\sin^2(t) + 3\cos^2(t)}_{3} + 2\cos^2(t)$

$$= \frac{-2\cos(t)\sin(t)}{\sqrt{3 + 2\cos^2(t)}}$$

Theorem 15.8: Chain Rule (Two Independent Variables)
Let z be a differentiable function of x and y, where x and y are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$





Example. For $z = e^{5x+8y}$, where $x = 7st$ and $y = 5s + t$, find z_s and z_t.

$\frac{\partial z}{\partial s}$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = \left(5e^{5x+8y}\right)(7t) + \left(8e^{5x+8y}\right)(5)$$

$$= 5e^{5x+8y}(7t+8)$$

$$= 5e^{35st+40s+8t}(7t+8)$$

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}(5s+t)$$
$$0 + 1$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = \left(5e^{5x+8y}\right)(7s) + \left(8e^{5x+8y}\right)(1)$$

$$= e^{5x+8y}(35s+8)$$

$$= e^{35st+40s+8t}(35s+8)$$

Example. For $z = \sin(2x)\cos(3y)$, where $x = s + t$ and $y = s - t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = \big(2\cos(2x)\cos(3y)\big)(1) + \big(-3\sin(2x)\sin(3y)\big)(1)$$

$(1) \qquad (-1)$

$$= 2\cos(2s+2t)\cos(3s-3t) - 3\sin(2s+2t)\sin(3s-3t)$$

partial: ∂

"orig" derivative d

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = 2\cos(2s+2t)\cos(3s-3t) + 3\sin(2s+2t)\sin(3s-3t)$$

Example. For $r = \ln(x^2 + xy + y^2)$, where $x = 2st$ and $y = s/t$, find $\partial r/\partial s$ and $\partial r/\partial t$.

$$\frac{\partial r}{\partial s} = \frac{\partial r}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial r}{\partial y}\cdot\frac{\partial y}{\partial s} = \left(\frac{2x+y}{x^2+xy+y^2}\right)(2t) + \left(\frac{x+2y}{x^2+xy+y^2}\right)\left(\frac{1}{t}\right)$$

$$= \frac{2t\left(4st + \frac{s}{t}\right) + \frac{1}{t}\left(2st + \frac{2s}{t}\right)}{4s^2t^2 + 2s^2 + \frac{s^2}{t^2}}$$

$$\frac{\partial r}{\partial t} = \frac{\partial r}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial r}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$z = \log(m) \qquad m = 2x + 3y, \quad x = st$

$y = s + t$

func of s & t

$$\frac{\partial z}{\partial s} = \frac{dz}{dm}\cdot\frac{\partial m}{\partial x}\frac{\partial x}{\partial s} + \frac{dz}{dm}\frac{\partial m}{\partial y}\frac{\partial y}{\partial s}$$

$$= \frac{dz}{dm}\left(\frac{\partial m}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial m}{\partial y}\frac{\partial y}{\partial s}\right)$$

$xy + \cos(xy) = 2$

$(y + x\frac{dy}{dx}) - \sin(xy)(y + x\frac{dy}{dx}) = 0 \xrightarrow{\text{solve}} \frac{dy}{dx} = \underline{\qquad}$

Theorem 15.9: Implicit Differentiation
Let F be differentiable on its domain and suppose $F(x, y) = 0$ defines y as a differentiable function of x. Provided $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Note: The above derivation comes from using the chain rule on $F(x, y) = 0$.

$\frac{\partial}{\partial x}\left(F(x,y)=0\right) \longrightarrow F_x \cdot \underbrace{\frac{dx}{dx}}_{1} + F_y \frac{dy}{dx} = 0 \longrightarrow F_y \frac{dy}{dx} = -F_x$ ↑ constant

$\frac{dy}{dx} = -\frac{F_x}{F_y}$

Example. For $4x^3 + 2x^2y - 3y^3 = 0$, find $\frac{dy}{dx}$ implicitly.

$$\frac{dy}{dx} = -\frac{12x^2 + 4xy}{2x^2 - 9y^2}$$

$\underbrace{F(x,y,z)}$ $= 42$

Example. For $xy + xz + 5yz = 42$, find $\partial z/\partial x$ and $\partial z/\partial y$ implicitly.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y+z}{x+5y}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x+5z}{x+5y}$$

Example. For $xyz + 2yz + 3xz = 4x + 2y - 3z$, find $\partial z/\partial x$ and $\partial z/\partial y$.

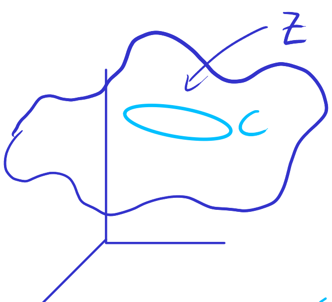
$$\underbrace{xyz + 2yz + 3xz - 4x - 2y + 3z}_{F} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz + 3z - 4}{xy + 2y + 3x + 3}$$

$$z(t) = 3\underbrace{(\cos^2(t) + \sin^2(t))}_{1} + 6\sin^2(t) + 4$$

$$= 6\sin^2(t) + 7 \Rightarrow \boxed{z'(t) = 12\sin(t)\cos(t)}$$

Example. Consider the surface $z = f(x, y) = 3x^2 + 9y^2 + 4$ and the curve C given parametrically by $x = \cos(t)$ and $y = \sin(t)$ where $0 \le t \le 2\pi$. Find $z'(t)$ and find t such that $z'(t) > 0$.



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= -6x\sin(t) + 18x\cos(t)$$

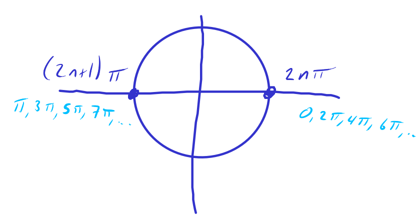
$$= -6\cos(t)\sin(t) + 18\sin(t)\cos(t)$$

$$= 12\sin(t)\cos(t)$$

$$= 6\sin(2t)$$

$2\sin(\alpha)\cos(\alpha) = \sin(2\alpha)$

$n \in \mathbb{Z}$



Solve $6\sin(2t) > 0$

$$\sin(2t) > 0$$

$$2n\pi < 2t < (2n+1)\pi$$

$$n\pi < t < \frac{(2n+1)}{2}\pi$$

LC $2n\pi < t < (2n+1)\pi$