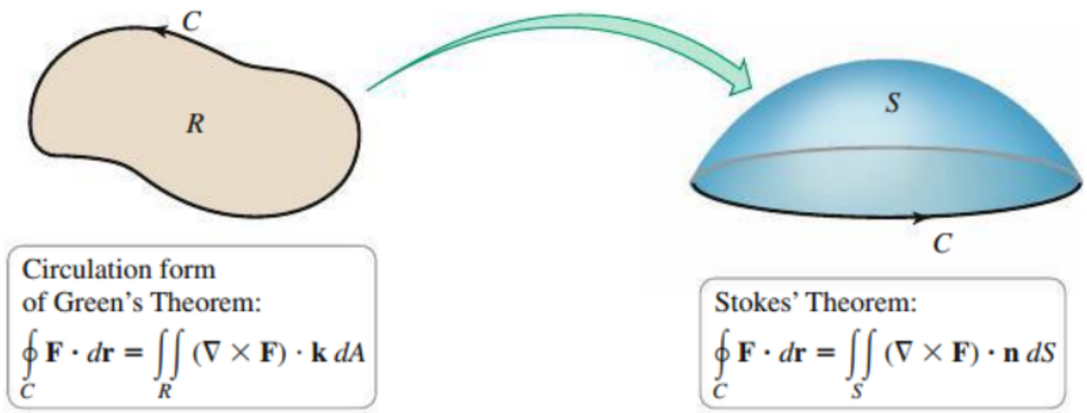
17.7: Stokes' Theorem

Stokes' Theorem is the three-dimensional version of the circulation form of Green's Theorem. Recall the circulation form of Green's Theorem:

$$\underbrace{\oint_C \mathbf{F}\cdot d\mathbf{r}}_{\text{circulation}} = \iint_R \underbrace{(g_x - f_y)}_{\text{curl or rotation}} \; dA.$$

The above means that the cumulative rotation within R equals the circulation along the boundary of R. Stokes' Theorem computes the circulation over a surface S in $\mathbb{R}^3$:



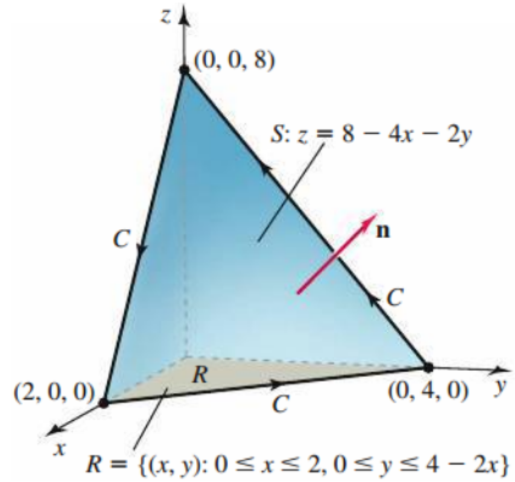
Theorem 17.15: Stokes' Theorem
Let S be an oriented surface in $\mathbb{R}^3$ with a piecewise-smooth closed boundary C whose orientation is consistent with that of S. Assume $\mathbf{F} = \langle f, g, h\rangle$ is a vector field whose components have continuous first partial derivatives on S. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS,$$

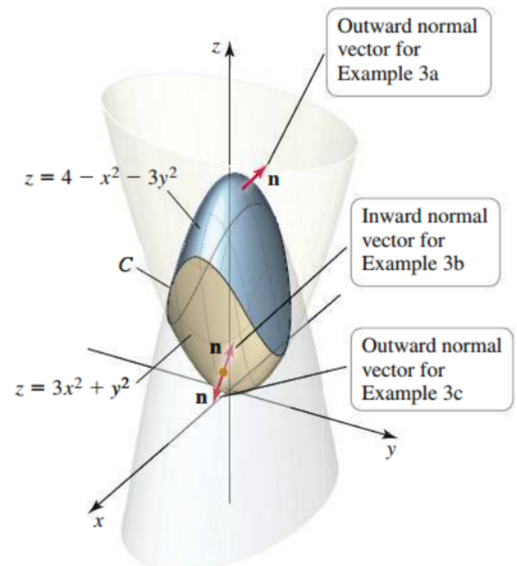
where $\mathbf{n}$ is the unit vector normal to S determined by the orientation of S.

Example. Verify Stokes' Theorem: Confirm that Stokes' Theorem holds for the vector field $\mathbf{F} = \langle z - y,\, x,\, -x \rangle$, where S is the hemisphere $x^2 + y^2 + z^2 = 4$, for $z \geq 0$, and C is the circle $x^2 + y^2 = 4$, oriented counterclockwise.

Example. Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \left\langle z,\, -z,\, x^2 - y^2 \right\rangle$ and C consists of the three line segments that bound the plane $z = 8 - 4x - 2y$ in the first octant.



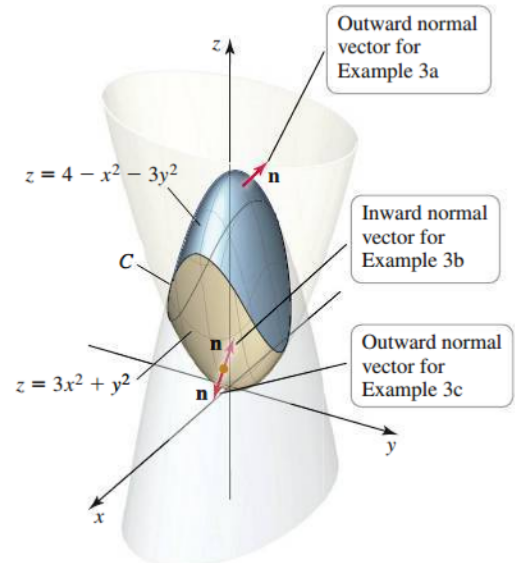
Example. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$, where $\mathbf{F} = \langle -y,\, x,\, z\rangle$, where:



- S is the part of the paraboloid $z = 4 - x^2 - 3y^2$ contained within $z = 3x^2 + y^2$, with $\mathbf{n}$ pointing upwards.

- S is the part of the paraboloid $z = 3x^2 + y^2$ contained within $z = 4 - x^2 - 3y^2$ with $\mathbf{n}$ pointing upwards.

- S is the part of the paraboloid $z = 3x^2 + y^2$ contained within $z = 4 - x^2 - 3y^2$ with $\mathbf{n}$ pointing downwards.

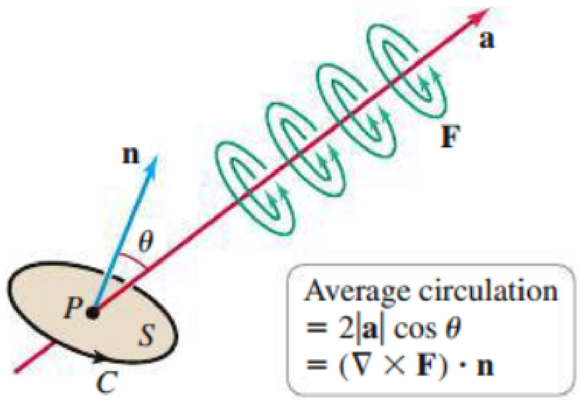


Interpreting the Curl:

The **average circulation** is

$$\frac{1}{\text{area of } S}\oint_C \mathbf{F}\cdot d\mathbf{r} = \frac{1}{\text{area of } S}\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS.$$

Consider a general rotation field $\mathbf{F} = \mathbf{a}\times\mathbf{r}$, where $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{r} = \langle x, y, z\rangle$. Now, let S be a small circular disk centered at a point P, whose normal vector $\mathbf{n}$ makes an angle θ with the axis $\mathbf{a}$:



The average circulation of this vector field on S is

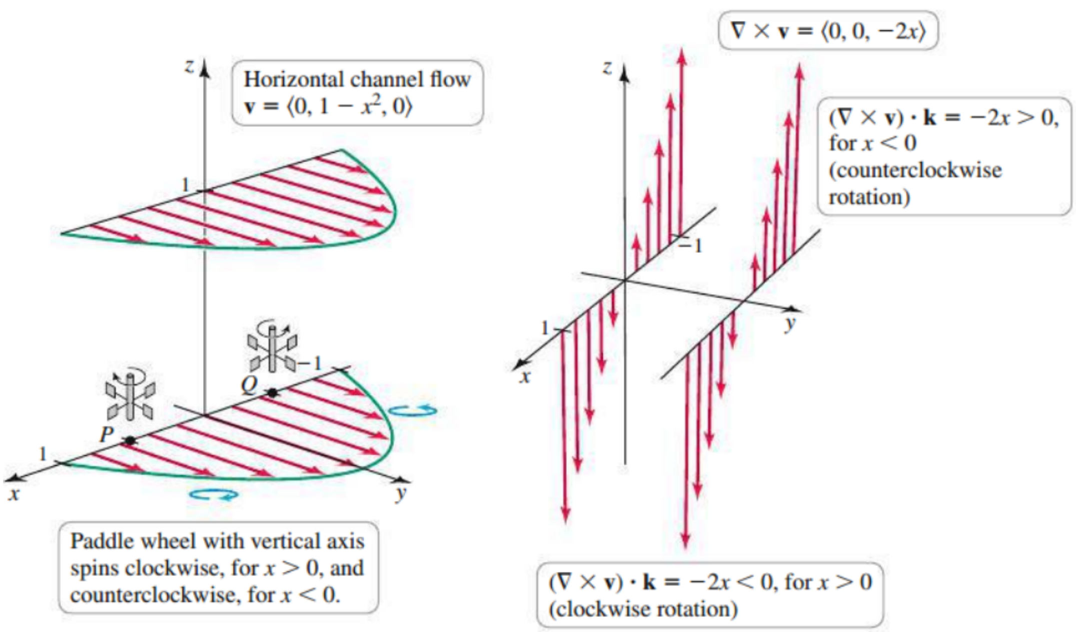
$$\begin{aligned}\frac{1}{\text{area of } S}\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS &= \frac{(\nabla\times\mathbf{F})\cdot\mathbf{n}}{\text{area of } S}(\text{area of } S)\\ &= 2\mathbf{a}\cdot\mathbf{n}\\ &= 2|\mathbf{a}|\cos(\theta)\end{aligned}$$

From this, we see

- The scalar component of $\nabla\times\mathbf{F}$ at P in the direction of $\mathbf{n}$ is the average circulation of $\mathbf{F}$ on S.
- The direction of $\nabla\times\mathbf{F}$ at P is the direction that maximizes the average circulation of $\mathbf{F}$ on S.

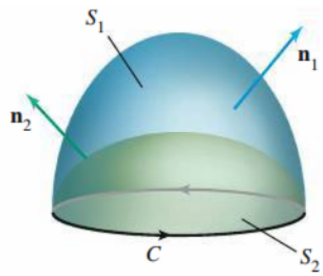
A similar argument for the curl can be applied to more general vector fields.

Example. Consider the vector field $\mathbf{v} = \left\langle 0,\, 1 - x^2,\, 0 \right\rangle$ for $|x| \le 1$ and $|z| \le 1$. Compute the curl of $\mathbf{v}$.



Since, using Stokes' Theorem, we evaluate the surface integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ using only the boundary C, then for any two smooth oriented surfaces S_1 and S_2 both with a consistent orientation with that of C,

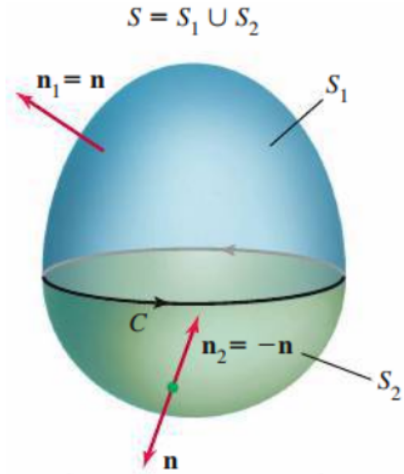
$$\iint\limits_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_1\, dS = \iint\limits_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2\, dS$$



$$\iint\limits_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_1\, dS = \iint\limits_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2\, dS$$

Furthermore, if S is a closed surface consisting of S_1 and S_2, with $\mathbf{n} = \mathbf{n}_1$ and $\mathbf{n} = -n_2$, then

$$\iint\limits_{S} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint\limits_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS + \iint\limits_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = 0$$



$$\boxed{\iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = 0}$$

Theorem 17.11 (Section 17.5) states that if $\mathbf{F}$ is conservative, then $\nabla \times \mathbf{F} = \mathbf{0}$. Now, the converse follows using Stokes' Theorem:

Theorem 17.16: Curl $\mathbf{F} = \mathbf{0}$ implies $\mathbf{F}$ Is Conservative
Suppose $\nabla \times \mathbf{F} = \mathbf{0}$ throughout an open simply connected region D of $\mathbb{R}^3$. Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all closed simple smooth curves C in D, and $\mathbf{F}$ is a conservative vector field on D.

Proof. Given a closed simple smooth curve C, it can be shown that C is the boundary of at least one smooth oriented surface S in D. By Stokes' Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \underbrace{(\nabla \times \mathbf{F})}_{0} \cdot \mathbf{n}\, dS = 0$$

Since the line integral equals zero over all such curves in D, the vector field is conservative on D. $\square$