1 14.3: Motion in Space

Definition.
Let the **position** of an object moving in three-dimensional space be given by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, for $t \geq 0$. The **velocity** of the object is

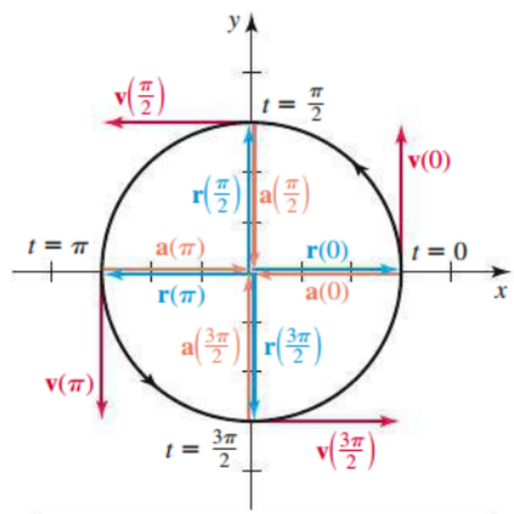
$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle.$$

The **speed** of the object is the scalar function

$$\left|\mathbf{v}(t) = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\right|$$

The **acceleration** of the object is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

Example. Given $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t)\rangle$ for $0 \leq t \leq 2\pi$, find the velocity, speed, and acceleration.



Circular motion: At all times $\mathbf{a}(t) = -\mathbf{r}(t)$ and $\mathbf{v}(t)$ is orthogonal to $\mathbf{r}(t)$ and $\mathbf{a}(t)$.

Theorem 14.2: Motion with constant $|\mathbf{r}|$
Let $\mathbf{r}$ describe a path on which $|\mathbf{r}|$ is constant (motion on a circle or sphere centered at the origin). Then $\mathbf{r} \cdot \mathbf{v} = 0$, which means the position vector and the velocity vector are orthogonal at all times for which the functions are defined.

Example (Path on a sphere)**.** Consider

$$\mathbf{r}(t) = \langle 3\cos(t),\, 5\sin(t),\, 4\cos(t)\rangle, \quad \text{for } 0 \le t \le 2\pi.$$

a) Show that an object with this trajectory moves on a sphere and find the radius.

b) Find the velocity and speed of the above trajectory.

c) Show that $\mathbf{r}(t) = \langle 5\cos(t),\, 5\sin(t),\, 5\sin(2t)\rangle$ does not lie on a sphere. How could this function be modified so that it does lie on a sphere?

Example. Given $\mathbf{a}(t) = \langle \cos(t), 4\sin(t) \rangle$, with an initial velocity $\langle \mathbf{u}_0, \mathbf{v}_0 \rangle = \langle 0, 4 \rangle$ and an initial position $\langle x_0, y_0 \rangle = \langle 5, 0 \rangle$ where $t \geq 0$, find the velocity and position vector.x‘

Summary: Two-Dimensional Motion in a Gravitational Field
Consider an object moving in a plane with a horizontal x-axis and a vertical y-axis, subject only to the force of gravity. Given the initial velocity $\mathbf{v}(0) = \langle u_0, v_0 \rangle$ and the initial position $\mathbf{r}(0) = \langle x_0, y_0 \rangle$, the velocity of the object, for $t \geq 0$, is

$$\mathbf{v}(t) = \langle x'(t), y'(t) \rangle = \langle u_0, -gt + v_0 \rangle$$

and the position is

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \left\langle u_0 t + x_0, -\frac{1}{2}gt^2 + v_0 t + y_0 \right\rangle.$$

Example. Consider a ball with an initial position of $\langle x_0, y_0 \rangle = \langle 0, 3 \rangle$ m and an initial velocity of $\langle u_0, v_0 \rangle = \langle 25, 4 \rangle$ m/s.

a) Find the position and velocity of the ball while it is in the air

Summary: Two-Dimensional Motion
Assume an object traveling over horizontal ground, acted on only by the gravitational force, has an initial position $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$ and initial velocity $\langle u_0, v_0 \rangle = \langle |\mathbf{v}_0| \cos\alpha, |\mathbf{v}_0| \sin\alpha \rangle$. The trajectory, which is a segment of a parabola, has the following properties.

$$\text{time of flight} = T = \frac{2|\mathbf{v}_0| \sin\alpha}{g}$$
$$\text{range} = \frac{|\mathbf{v}_0|^2 \sin(2\alpha)}{g}$$
$$\text{maximum height} = y\left(\frac{T}{2}\right) = \frac{(|\mathbf{v}_0| \sin\alpha)^2}{2g}$$

Example. Consider a ball with an initial position of $\langle x_0, y_0 \rangle = \langle 0, 3 \rangle$ m and an initial velocity of $\langle u_0, v_0 \rangle = \langle 25, 4 \rangle$ m/s.

b) Assuming the ground is flat and level, how far does the ball travel horizontally?

c) What is the maximum height that the ball reaches?