1 16.5: Triple Integrals in Cylindrical and Spherical Coordinates

Transformations between Cylindrical and Rectangular Coordinates

Rectangular → Cylindrical	**Cylindrical → Rectangular**
$r^2 = x^2 + y^2$	$x = r\cos\theta$
$\tan\theta = y/x$	$y = r\sin\theta$
$z = z$	$z = z$

Theorem 16.6: Change of Variables for Triple Integrals in Cylindrical Coordinates

Let f be continuous over the region D, expressed in cylindrical coordinates as

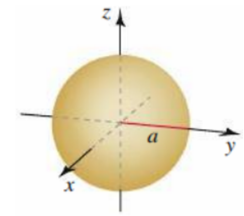
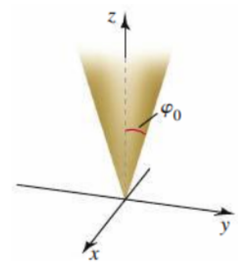
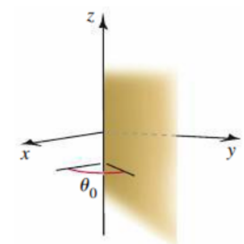
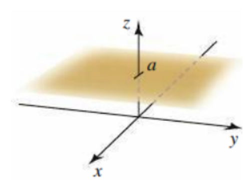
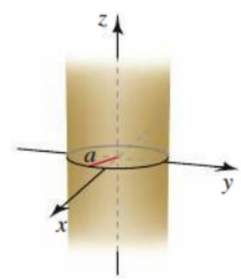
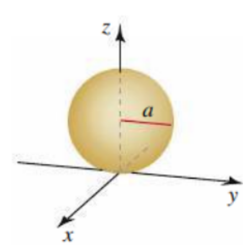
$$D = \{(r,\theta,z) : 0 \le g(\theta) \le r \le h(\theta),\ \alpha \le \theta \le \beta,\ G(x,y) \le z \le H(x,y)\}$$

Then f is integrable over D, and the triple integral of f over D is

$$\iiint\limits_D f(x,y,z)\,dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta,\ r\sin(\theta))}^{H(r\cos\theta,\ r\sin(\theta))} f(r\cos(\theta), r\sin(\theta))\,dz\,r\,dr\,d\theta.$$

Transformations between Spherical and Rectangular Coordinates

Rectangular → Spherical	**Spherical → Rectangular**
$\rho^2 = x^2 + y^2 + z^2$	$x = \rho\sin(\varphi)\cos(\theta)$
Use trigonometry to find	$y = \rho\sin(\varphi)\sin(\theta)$
φ and θ.	$z = \rho\cos(\varphi)$

Name	**Description**	**Example**
Sphere, radius a, center $(0,0,0)$	$\{(\rho,\varphi,\theta) : \rho = a\}, a > 0$	
Cone	$\{(\rho,\varphi,\theta) : \varphi = \varphi_0\}, \varphi_0 \neq 0, \pi/2, \pi$	
Vertical half-plane	$\{(\rho,\varphi,\theta) : \theta = \theta_0\}$	
Horizontal plane, $z = a$	$a > 0 : \{(\rho,\varphi,\theta) : \rho = a\sec(\varphi),\ 0 \leq \varphi < \pi/2\}$ $a < 0 : \{(\rho,\varphi,\theta) : \rho = a\sec(\varphi),\ \pi/2 < \varphi \leq \pi\}$	
Cylinder, radius $a > 0$	$\{(\rho,\varphi,\theta) : \rho = \alpha\csc(\varphi),\ 0 < \varphi < \pi\}$	
Sphere, radius $a > 0$ center $(0,0,a)$	$\{(\rho,\varphi,\theta) : \rho = 2a\cos(\varphi),\ 0 \leq \varphi \leq \pi/2\}$	

Theorem 16.7: Change of Variables for Triple Integrals in Spherical Coordinates
Let f be continuous over the region D, expressed in spherical coordinates as

$$D = \{(\rho, \varphi, \theta) : 0 \leq g(\varphi, \theta) \leq \rho \leq h(\varphi, \theta),\ a \leq \varphi \leq b,\ \alpha \leq \theta \leq \beta\}.$$

Then f is integrable over D, and the triple integral of f over D is

$$\iiint\limits_D f(x, y, z)\, dV$$
$$= \int_\alpha^\beta \int_a^b \int_{g(\varphi,\theta)}^{h(\varphi,\theta)} f(\rho\sin(\varphi)\cos(\theta),\ \rho\sin(\varphi)\sin(\theta),\ \rho\cos(\varphi))\rho^2\sin(\varphi)\ d\rho\, d\varphi\, d\theta.$$