17.2: Line Integrals

Definition. (Scalar Line Integral in the Plane)
Suppose the scalar-valued function f is defined on a region containing the smooth curve C given by $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$. The **line integral of f over** C is

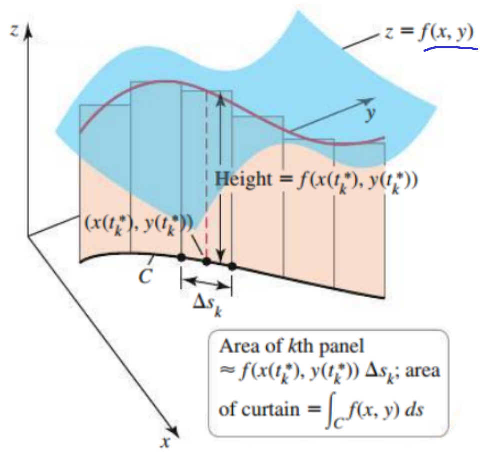
$$\int_C f(x(t), y(t))\, ds = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x(t_k^*),\, y(t_k^*))\Delta s_k,$$

provided this limit exists over all partitions of $[a, b]$. When the limit exists, f is said to be **integrable** on C.

$s(t) = \int_a^t \vec{r}(u)\, du$

$s'(t) = \vec{r}(t)$

$ds = \vec{r}'(t)\, dt$



Theorem 17.1: Evaluating Scalar Line Integrals in $\mathbb{R}^2$
Let f be continuous on a region containing a smooth curve C: $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$. Then

ds

$$\int_C f\, ds = \int_a^b f(x(t),\, y(t))|\mathbf{r}'(t)|\, dt$$
$$= \int_a^b f(x(t),\, y(t))\sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

If $f(x,y) = 1$, $\int_C f\, ds$ is arc length

Procedure: Evaluating the Line Integral $\int_C f\, ds$

1. Find a parametric description of C in the form $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, for $a \le t \le b$.
2. Compute $|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$.
3. Make substitutions for x and y in the integrand and evaluate an ordinary integral:

$$\int_C f\, ds = \int_a^b f(x(t), y(t)) \underbrace{|\mathbf{r}'(t)|\, dt}_{ds}.$$

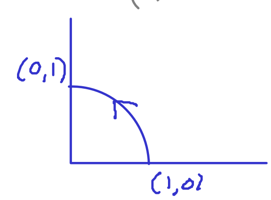
$\boxed{f(x,y) = 1}$

Example. Find the length of the quarter-circle from $(1, 0)$ to $(0, 1)$ with its center at the origin.

$\vec{r}(t) = \langle t, \sqrt{1-t^2}\rangle$

$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle, \quad 0 \le t \le \pi/2$

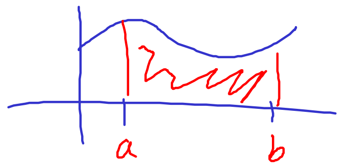
$|\vec{r}\,'(t)| = |\langle -\sin(t), \cos(t)\rangle| = 1$



$$L = \int_C f\, ds = \int_0^{\pi/2} \underbrace{f(x(t), y(t))}_{1} \underbrace{|\vec{r}\,'(t)|}_{1}\, dt = t\Big|_{t=0}^{t=\pi/2} = \boxed{\pi/2}$$

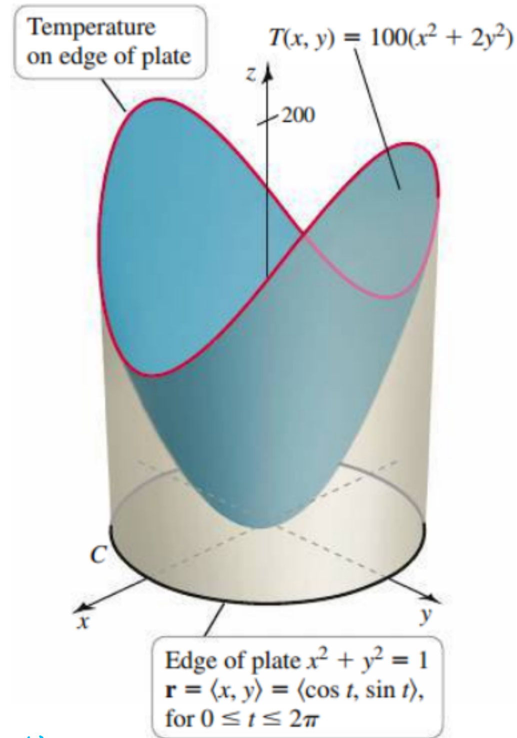
LC #1

$a = 1/2$



$$\int_a^b f(x)\,\underset{\text{change in } x}{dx}$$

Example. The temperature of the circular plate $R = \{(x, y) : x^2 + y^2 \leq 1\}$ is $T(x, y) = 100(x^2 + 2y^2)$. Find the average temperature along the edge of the plate.



$$\bar{T} = \frac{1}{L}\int_C T(x,y)\,ds \qquad \text{Circle} \Rightarrow L = 2\pi$$

$$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle \qquad 0 \leq t \leq 2\pi$$

$$|\vec{r}\,'(t)| = |\langle -\sin(t), \cos(t)\rangle| = 1$$

$$T(x(t), y(t)) = 100\left(\cos^2(t) + 2\sin^2(t)\right)$$

$$\int_C T(x,y)\,ds = \int_0^{2\pi} T(x(t), y(t))\,|\vec{r}\,'(t)|\,dt$$

$$= \int_0^{2\pi} 100\left(\underbrace{\cos^2(t) + 2\sin^2(t)}_{1 + \sin^2(t) = 1 + \frac{1-\cos(2t)}{2}}\right)(1)\,dt$$

$$= \frac{100}{2}\int_0^{2\pi} 3 - \cos(2t)\ dt$$

$$= 50\left[3t - \frac{\sin(2t)}{2}\right]_{t=0}^{t=2\pi}$$

$$= 50\left[3(2\pi - 0) - \tfrac{1}{2}(0 - 0)\right] = 300\pi$$

LC #2

$$\bar{T} = \frac{1}{L}\int_C T(x,y)\,ds = \frac{300\pi}{2\pi} = \boxed{150}$$

Theorem 17.2: Evaluating Scalar Line Integrals in $\mathbb{R}^3$
Let f be continuous on a region containing a smooth curve $C : \mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$, for $a \le t \le b$. Then

$$\int_C f\,ds = \int_a^b \underbrace{f(x(t),\, y(t),\, z(t))}_{f}\; \overbrace{|\mathbf{r}'(t)|\,dt}^{ds}$$

$$= \int_a^b f(x(t),\, y(t),\, z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt.$$

f $|\vec{r}'|$

Example. Evaluate $\displaystyle\int_C (x - y + 2z)\,ds$, where C is the circle $\mathbf{r}(t) = \langle 1,\, 3\cos(t),\, 3\sin(t)\rangle$, for $0 \le t \le 2\pi$.

$|\vec{r}'(t)| = |\langle 0, -3\sin(t), 3\cos(t)\rangle| = 3$

$$\int_C (x - y + 2z)\,ds = \int_0^{2\pi} \big(x(t) - y(t) + 2z(t)\big)\,|\vec{r}'(t)|\,dt$$

$$= \int_0^{2\pi} \big(1 - 3\cos(t) + 6\sin(t)\big)\,3\,dt$$

$$= 3\Big[t - 3\sin(t) - 6\cos(t)\Big]_{t=0}^{t=2\pi}$$

$$= 3\big[(2\pi - 0) - 3(0 - 0) - 6(1 - 1)\big] = \boxed{6\pi}$$

LC # 3
$6\pi + 36$

Example. Evaluate $\int_C xe^{yz}\,ds$, where C is $\mathbf{r}(t) = \langle t, 2t, -2t\rangle$, for $0 \le t \le 2$.

$|\vec{r}\,'(t)| = |\langle 1, 2, -2\rangle| = 3$

$$\int_C xe^{yz}\,ds = \int_0^2 \underbrace{t\,e^{(2t)(-2t)}}_{f(x(t),\,y(t),\,z(t))}\,\underbrace{(3)}_{|\vec{r}\,'(t)|}\,dt$$

$$= \int_0^2 3\,t\,e^{-4t^2}\,dt$$

$u = -4t^2$ $\quad t=0,\ u=0$

$du = -8t\,dt$ $\quad t=2,\ u=-16$

$-\frac{du}{8} = t\,dt$

$$= \int_0^{-16} -\frac{3}{8}\,e^{u}\,du$$

$$= -\frac{3}{8}e^{u}\Big|_{u=0}^{u=-16} = -\frac{3}{8}\left(e^{-16}-1\right) = \boxed{\frac{3(1-e^{-16})}{8}}$$

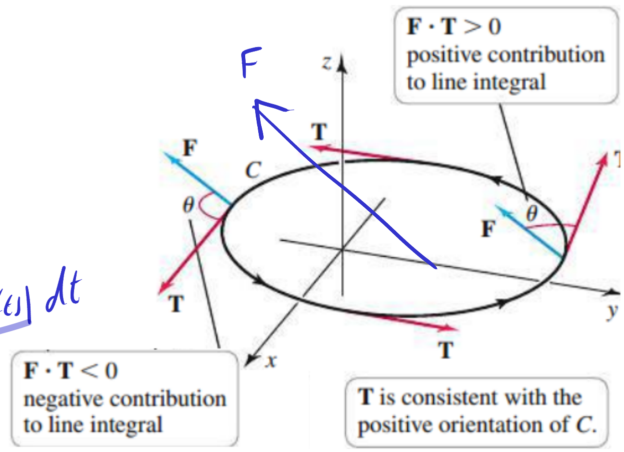
LC # 4

$a = -16$

Definition. (Line Integral of a Vector Field)
Let $\mathbf{F}$ be a vector field that is continuous on a region containing a smooth oriented curve C parameterized by arc length. Let $\mathbf{T}$ be the unit tangent vector at each point of C consistent with the orientation. The line integral of $\mathbf{F}$ over C is $\displaystyle\int_C \mathbf{F}\cdot\mathbf{T}\,ds$.

$$\vec{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}\cdot\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\,|\vec{r}\,'(t)|\,dt$$



Different Forms of Line Integrals of Vector Fields
The line integral $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ may be expressed in the following forms, where $\mathbf{F} = \langle f,\, g,\, h\rangle$ and C has a parameterization $\mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$, for $a \le t \le b$:

$$\begin{aligned}\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt &= \int_a^b \left(f(t)x'(t) + g(t)y'(t) + h(t)z'(t)\right)dt\\ &= \int_C f\,dx + g\,dy + h\,dz\\ &= \int_C \mathbf{F}\cdot d\mathbf{r}.\end{aligned}$$

$$dx = x'(t)\,dt$$

For line integrals in the plane, we let $\mathbf{F} = \langle f,\, g\rangle$ and assume C is parameterized in the form $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$. Then

$$\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_a^b \left(f(t)x'(t) + g(t)y'(t)\right)dt = \int_C f\,dx + g\,dy = \int_C \mathbf{F}\cdot d\mathbf{r}.$$

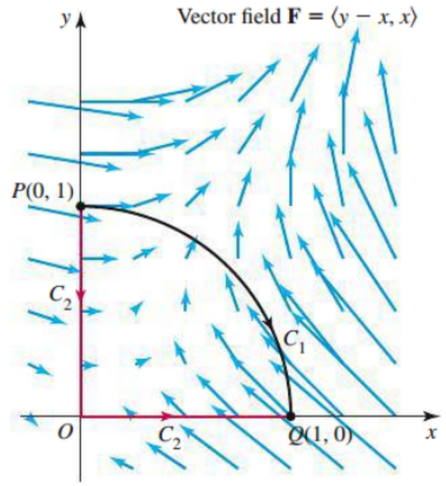
$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}\cdot\vec{r}\,'(t)\,dt$$

t	$\vec{r}(t)$
0	$\langle 0,1\rangle \leftarrow P(0,1)$
$\frac{\pi}{2}$	$\langle 1,0\rangle \leftarrow Q(1,0)$

Example. Evaluate $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ with $\mathbf{F} = \langle y - x, x\rangle$ on the following oriented paths in $\mathbb{R}^2$.

$y \leftarrow \cos(t)$, $x \leftarrow \sin(t)$

a) The quarter-circle C_1 from $P(0,1)$ to $Q(1,0)$



$\vec{r}(t) = \langle \sin(t), \cos(t)\rangle, \quad 0 \le t \le \pi/2$

$\vec{r}\,'(t) = \langle \cos(t), -\sin(t)\rangle$

$\vec{F}\cdot\vec{r}\,'(t) = \langle \cos(t) - \sin(t), \sin(t)\rangle \cdot \langle \cos(t), -\sin(t)\rangle$

$= \cos^2(t) - \sin(t)\cos(t) - \sin^2(t)$

$= \cos(2t) - \frac{1}{2}\sin(2t)$

$\cos(2t) = \cos^2(t) - \sin^2(t)$

$\sin(2t) = 2\sin(t)\cos(t)$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_0^{\pi/2} \vec{F}\cdot\vec{r}\,'(t)\,dt = \int_0^{\pi/2} \cos(2t) - \frac{1}{2}\sin(2t)\,dt = \frac{\sin(2t)}{2} + \frac{1}{4}\cos(2t)\Big|_{t=0}^{t=\frac{\pi}{2}}$$

$$= (0-0) + \frac{1}{4}(-1-1) = \boxed{-\frac{1}{2}}$$

LC #5

denotes opposite direction

b) The quarter-circle $-C_1$ from $Q(1,0)$ to $P(0,1)$

① $\vec{r}(t) = \langle \cos t, \sin t\rangle, \quad 0 \le t \le \pi/2, \ \ldots$

② $\int_{-C} \vec{F}\cdot\vec{T}\,ds = -\int_C \vec{F}\cdot\vec{T}\,ds \Rightarrow \int_{-C_1} \vec{F}\cdot\vec{T}\,ds = \boxed{\frac{1}{2}}$

$\boxed{\text{LC \#6}}$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

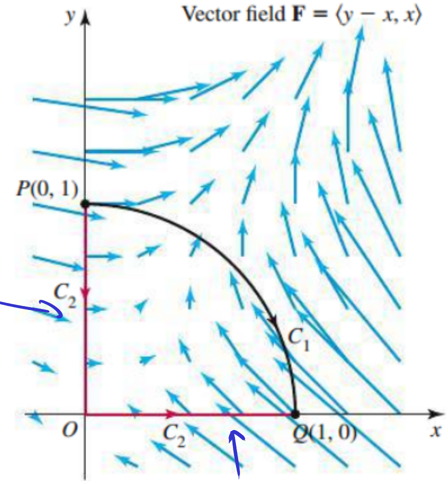
c) the path C_2 from $P(0,1)$ to $Q(1,0)$ via two line segments through $O(0,0)$.

$(0,1) \longrightarrow (0,0)$

$\vec{r}_1(t) = \langle 0,1\rangle + t\langle 0,-1\rangle = \langle 0, 1-t\rangle, \quad 0 \le t \le 1$

$(0,0) \rightarrow (1,0)$

$\vec{r}_2(t) = \langle 0,0\rangle + t\langle 1,0\rangle = \langle t, 0\rangle, \quad 0 \le t \le 1$



$F = \langle y-x, x\rangle$

$\vec{F}\cdot\vec{r}_1{}'(t) = \langle (1-t)-0, 0\rangle \cdot \langle 0,-1\rangle = 0$

$\langle 1-t, 0\rangle$

$\vec{F}\cdot\vec{r}_2{}'(t) = \langle 0-t, t\rangle \cdot \langle 1,0\rangle = -t$

$$\int_{C_2} \vec{F}\cdot\vec{T}\,ds = \int_0^1 \underbrace{\vec{F}\cdot\vec{r}_1{}'(t)}_{0}\,dt + \int_0^1 \vec{F}\cdot\vec{r}_2{}'(t)\,dt$$

$$= \int_0^1 -t\,dt$$

$\boxed{LC \#7}$

$$= -\frac{t^2}{2}\Big|_{t=0}^{t=1} = \boxed{-1/2}$$

Definition. (Work Done in a Force Field)
Let $\mathbf{F}$ be a continuous force field in a region D of $\mathbb{R}^3$. Let

$$C : \mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle,\ \text{for } a \le t \le b,$$

be a smooth curve in D with a unit tangent vector $\mathbf{T}$ consistent with the orientation. The work done in moving an object along C in the positive direction is

$$W = \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt.$$

Example. For the force field $\mathbf{F} = \dfrac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}}$, calculate the work required to move an object from $(1,1,1)$ to $(10,10,10)$.

$$\vec{r}(t) = \langle 1,1,1\rangle + t\langle 9,9,9\rangle = \langle 1+9t, 1+9t, 1+9t\rangle, \quad 0 \le t \le 1$$

$$\left\langle \frac{10-1}{1}, \frac{10-1}{1}, \frac{10-1}{1}\right\rangle$$

$$\vec{r}(t) = \langle 1+t, 1+t, 1+t\rangle, \quad 0 \le t \le 9$$

$$\left\langle \frac{10-1}{9}, \frac{10-1}{9}, \frac{10-1}{9}\right\rangle$$

$$\vec{r}\,'(t) = \langle 9,9,9\rangle = 9\langle 1,1,1\rangle$$

$$\vec{F} = \frac{\langle 1+9t, 1+9t, 1+9t\rangle}{(3(1+9t)^2)^{3/2}} = \frac{(1+9t)\langle 1,1,1\rangle}{3^{3/2}(1+9t)^3} = \frac{\langle 1,1,1\rangle}{3^{3/2}(1+9t)^2}$$

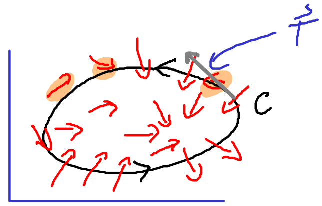
$$W = \int_C \vec{F}\cdot\vec{T}\,ds = \int_0^1 \vec{F}\cdot\vec{r}\,'(t)\,dt = \int_0^1 \frac{\langle 1,1,1\rangle}{3^{3/2}(1+9t)^2}\cdot 9\langle 1,1,1\rangle\,dt$$

$$= \int_0^1 \frac{3(9)}{3^{3/2}(1+9t)^2}\,dt = \int_1^{10} \frac{u^{-2}}{3^{1/2}}\,du = \left.\frac{-1}{3^{1/2}u}\right|_{u=1}^{u=10} = \frac{-1}{\sqrt{3}\cdot 10} + \frac{1}{\sqrt{3}}$$

$$= \frac{3\sqrt{3}}{10}$$

$$u = 1+9t \qquad t=0,\ u=1$$
$$du = 9\,dt \qquad t=1,\ u=10$$

$$\boxed{\begin{array}{l} LC\ \#8 \\ a = 3/10 \end{array}}$$



		against	with
circulation	0	pos	neg

Definition. (Circulation)
Let $\mathbf{F}$ be a continuous vector field on a region D of $\mathbb{R}^3$, and let C be a closed smooth oriented curve in D. The **circulation** of $\mathbf{F}$ on C is $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$, where $\mathbf{T}$ is the unit vector tangent to C consistent with the orientation.

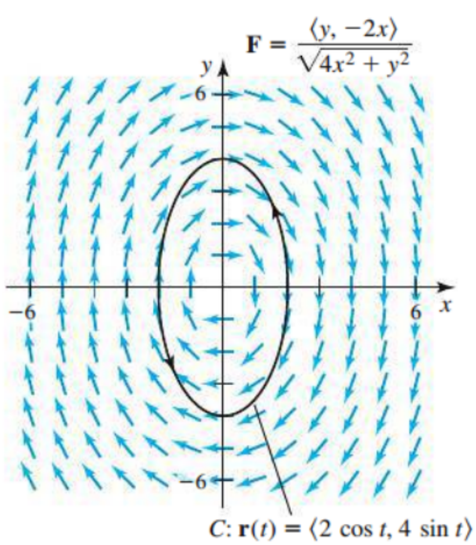
$$\int_C \vec{F}\cdot\vec{r}\,'(t)\,dt = \int_C \vec{F}\cdot d\vec{r}$$

Example. Compute the circulation in the vector field $\mathbf{F} = \dfrac{\langle y, -2x\rangle}{\sqrt{4x^2+y^2}}$ along the curve C given by $\mathbf{r}(t) = \langle 2\cos(t), 4\sin(t)\rangle$, for $0 \le t \le 2\pi$.

$$\vec{r}\,'(t) = \langle -2\sin(t), 4\cos(t)\rangle$$

$$\vec{F} = \frac{\langle 4\sin(t), -2(2\cos(t))\rangle}{\sqrt{4\underbrace{(2\cos(t))^2}_{4\cos^2(t)} + (4\sin(t))^2}} = \frac{\langle 4\sin(t), -4\cos(t)\rangle}{4\sqrt{\cos^2(t)+\sin^2(t)}} = \langle \sin(t), -\cos(t)\rangle$$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_0^{2\pi} \vec{F}\cdot\vec{r}\,'(t)\,dt = \int_0^{2\pi} \langle \sin(t), -\cos(t)\rangle \cdot \langle -2\sin(t), 4\cos(t)\rangle\,dt$$



$$= \int_0^{2\pi} -2\sin^2(t) \underbrace{-4\cos^2(t)}_{-2\cos^2(t)\,-2\cos^2(t)}\,dt$$

$$= \int_0^{2\pi} -2 - \underbrace{2\cos^2(t)}_{\cos(2t)+1}\,dt$$

$\cos(2t) = \cos^2(t) - \sin^2(t)$
$= 2\cos^2(t) - 1$
$= 1 - 2\sin^2(t)$

$$= \int_0^{2\pi} -3 - \cos(2t)\,dt$$

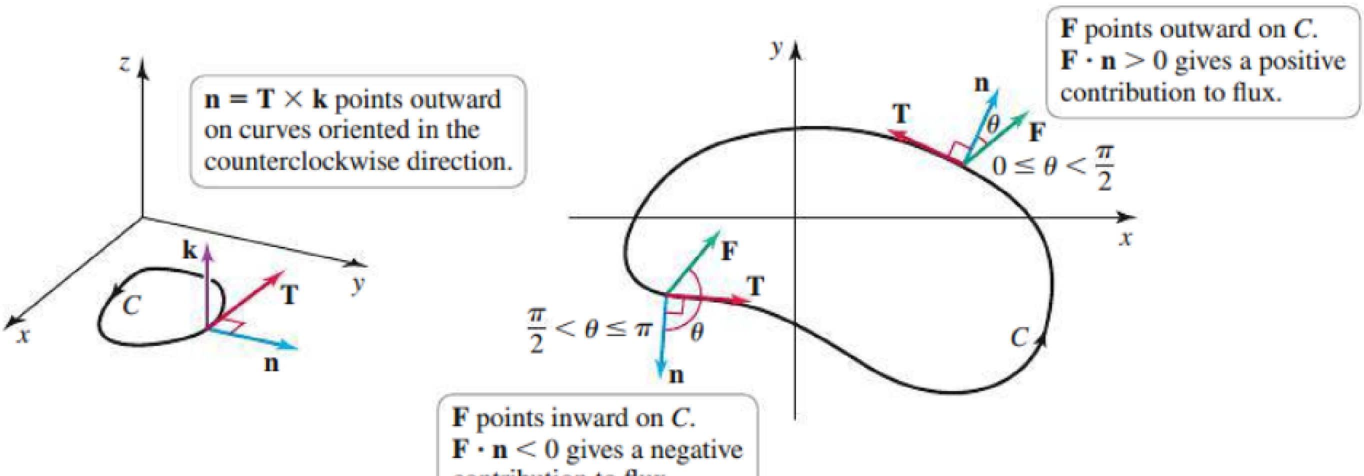
$$= -3t\Big|_0^{2\pi} - \underbrace{\frac{\sin(2t)}{2}\Big|_0^{2\pi}}_{0}$$

$$= -6\pi$$

LC # 9

-2π

Flux of the vector field is the total forces orthogonal to each point on the curve C. Let $\mathbf{F} = \langle f, g\rangle$ be a continuous vector field in a region R of $\mathbb{R}^2$. Using $\mathbf{n}$ to represent a unit vector normal to C, the component of $\mathbf{F}$ that is normal to C is $\mathbf{F}\cdot\mathbf{n}$.



(a) (b)

Since C is in the xy-plane, the unit tangent vector $\mathbf{T} = \langle T_x, T_y, 0\rangle$ is also in the xy-plane. We let $\mathbf{n}$ be in the xy-plane as well, but using the cross product of $\mathbf{T}$ and $\boldsymbol{k}$:

$$\mathbf{n} = \mathbf{T}\times\boldsymbol{k} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ T_x & T_y & 0 \\ 0 & 0 & 1 \end{vmatrix} = T_y\boldsymbol{i} - T_x\boldsymbol{j}.$$

Since $\mathbf{T} = \dfrac{\mathbf{r}(t)}{|\mathbf{r}(t)|}$, we have

$$\mathbf{n} = T_y\boldsymbol{i} - T_x\boldsymbol{j} = \frac{y'(t)}{|\mathbf{r}'(t)|}\boldsymbol{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\boldsymbol{j} = \frac{\langle y'(t), -x'(t)\rangle}{|\mathbf{r}'(t)|}.$$

Thus, we have the flux integral

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_a^b \mathbf{F}\cdot\frac{\langle y'(t), -x'(t)\rangle}{|\mathbf{r}'(t)|}|\mathbf{r}'(t)|\,dt = \int_a^b (f(t)y'(t) - g(t)x'(t))\,dt = \int_C \underbrace{f\,dy - g\,dx}_{\langle f,g\rangle\cdot\langle dy,-dx\rangle}.$$

Definition. (Flux)
Let $\mathbf{F} = \langle f, g \rangle$ be a continuous vector field on a region R of $\mathbb{R}^2$. Let $C : \mathbf{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$, be a smooth orientated curve in R that does not intersect itself. The **flux** of the vector field $\mathbf{F}$ across C is

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b \underbrace{(f(t)y'(t) - g(t)x'(t))}_{\langle f, g\rangle \cdot \langle y', -x'\rangle}\, dt, = \int_C f\,dy - g\,dx$$

where $\mathbf{n} = \mathbf{T} \times \boldsymbol{k}$ is the unit normal vector and $\mathbf{T}$ is the unit tangent vector consistent with the orientation. If C is a closed curve with counterclockwise orientation, $\mathbf{n}$ is the outward normal vector, and the flux integral gives the **outward flux** across C.

Example. Compute the flux in the vector field $\mathbf{F} = \dfrac{\langle y, -2x \rangle}{\sqrt{4x^2 + y^2}}$ along the curve C given by $\mathbf{r}(t) = \langle 2\cos(t), 4\sin(t) \rangle$, for $0 \le t \le 2\pi$.

$y'(t), \quad -x'(t)$

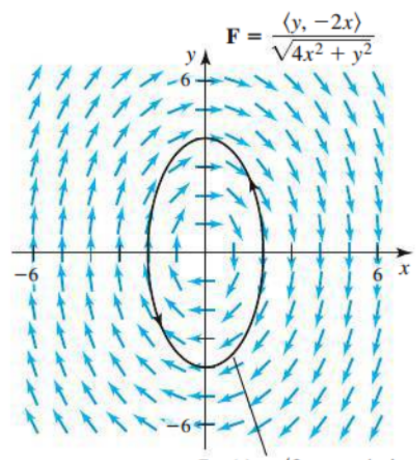
$$\int_C \vec{F} \cdot \vec{n}\, ds = \int_0^{2\pi} \langle \sin(t), -\cos(t) \rangle \cdot \langle 4\cos(t), 2\sin(t) \rangle\, dt$$

$$= \int_0^{2\pi} 2\sin(t)\cos(t)\, dt$$

$$= \int_0^{2\pi} \sin(2t)\, dt$$

$$= -\frac{\cos(2t)}{2} \Big|_{t=0}^{t=2\pi}$$

$$= -\frac{1}{2}(1 - 1) = 0$$



LC #10
$a = 0$