8.3: Trigonometric Integrals

Important trigonometric identities

Pythagorean Identities	$\sin^2(\theta) + \cos^2(\theta) = 1$
Angle sum formulas	$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\alpha)$ $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
Double angle formulas	$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
Half angle formulas	$\sin^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$ $\cos^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$

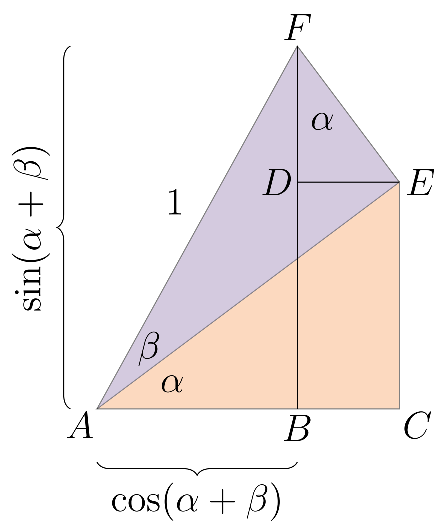
Derivation of angle sum formulas

$$\sin(\alpha) = \frac{\overline{DE}}{\overline{EF}} = \frac{\overline{DE}}{\sin(\beta)} \quad \Rightarrow \overline{DE} = \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha) = \frac{\overline{DF}}{\overline{EF}} = \frac{\overline{DF}}{\sin(\beta)} \quad \Rightarrow \overline{DF} = \cos(\alpha)\sin(\beta)$$

$$\sin(\beta) = \frac{\overline{CE}}{\overline{AE}} = \frac{\overline{CE}}{\cos(\beta)} \quad \Rightarrow \overline{CE} = \sin(\alpha)\cos(\beta)$$

$$\cos(\beta) = \frac{\overline{AC}}{\overline{AE}} = \frac{\overline{AC}}{\cos(\beta)} \quad \Rightarrow \overline{AC} = \cos(\alpha)\cos(\beta)$$



Derivation of the double angle formulas

$$\sin(2\theta) = \sin(\theta+\theta) = \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos(\theta+\theta) = \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta) = \cos^2(\theta) - \sin^2(\theta)$$

Derivation of the half angle formulas

Start with the cosine double angle formula:

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \boxed{2\cos^2(\theta) - 1} = \boxed{1 - 2\sin^2(\theta)}$$

Solve for either $\sin^2(\theta)$ or $\cos^2(\theta)$:

$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2} \qquad\qquad \cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$

Example. Evaluate the integral $\displaystyle\int \cos^5(x)\,dx$.

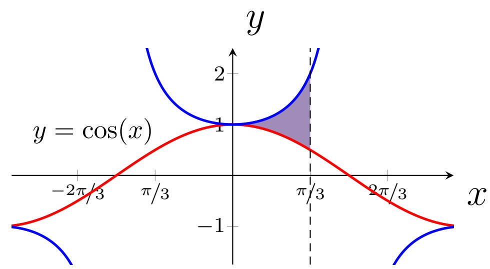
Example. Evaluate the integral $\int \sin^3(x) \cos^{3/2}(x)\, dx$.

Example. Evaluate the integral $\int 20\sin^2(x)\cos^2(x)\,dx$

Example. Evaluate the integral $\int \sec^6(x) \tan^4(x) \, dx$.

Example. Evaluate the integral $\displaystyle\int 35\tan^5(x)\sec^4(x)\,dx$.

Example. Consider the region bounded by $y = \sec(x)$ and $y = \cos(x)$ for $0 \leq x \leq {}^{\pi}/_{3}$. Find the volume of the solid generated when rotating this region about the line $y = -1$.



Example. Find the length of the curve $y = \ln(2\sec(x))$ on the interval $[0, \pi/6]$.

$\int \sin^m(x)\cos^n(x)\,dx$	**Strategy**
m odd and positive, n real	Split off $\sin(x)$, rewrite the resulting even power of $\sin(x)$ in terms of $\cos(x)$, and then use $u = \cos(x)$.
n odd and positive, m real	Split off $\cos(x)$, rewrite the resulting even power of $\cos(x)$ in terms of $\sin(x)$, and then use $u = \sin(x)$.
m and n both even, nonnegative integers	Use half-angle formulas to transform the integrand into a polynomial in $\cos(2x)$, and apply the preceding strategies once again to powers of $\cos(2x)$ greater than 1.
$\int \tan^m(x)\sec^n(x)\,dx$	
n even and positive, m real	Split off $\sec^2(x)$, rewrite the remaining even power of $\sec(x)$ in terms of $\tan(x)$, and use $u = \tan(x)$.
m odd and positive, n real	Split off $\sec(x)\tan(x)$, rewrite the remaining even power of $\tan(x)$ in terms of $\sec(x)$, and use $u = \sec(x)$.
n even and positive, n odd and positive	Rewrite $\tan^m(x)$ in terms of $\sec(x)$
$\int \sec^n(x)\,dx$	
n odd	Use integration by parts with $u = \sec^{n-2}(x)$ and $dv = \sec^2(x)\,dx$
m even	Split off $\sec^2(x)$, rewrite the remaining powers of $\sec(x)$ in terms of $\tan(x)$, and use $u = \tan(x)$.
$\int \tan^m(x)\,dx$	Split off $\tan^2(x)$ and rewrite in terms of $\sec(x)$. Expand into difference of integrals substituting $u = \tan(x)$. Repeat the process as needed for remaining powers of $\tan(x)$.