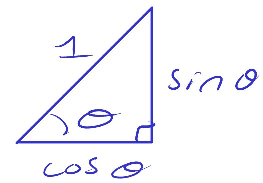
8.3: Trigonometric Integrals

Important trigonometric identities



Pythagorean Identities	$\frac{1}{\sin^2(\theta)}$ $\quad$ $\sin^2(\theta) + \cos^2(\theta) = 1$ $\tan^2(\theta) + 1 = \sec^2\theta$ $1 + \cot^2(\theta) = \csc^2\theta$
Angle sum formulas	$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$ $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
Double angle formulas	$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ * Pythagorean Identity $2\cos^2\theta - 1$ $1 - 2\sin^2\theta$
Half angle formulas	$\sin^2(\theta) = \dfrac{1 - \cos(2\theta)}{2}$ $\cos^2(\theta) = \dfrac{1 + \cos(2\theta)}{2}$

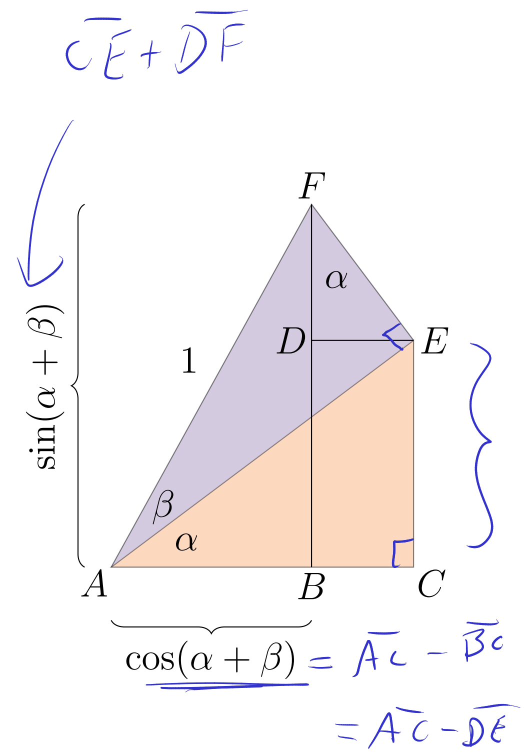
Derivation of angle sum formulas

$$\sin(\alpha) = \frac{\overline{DE}}{\overline{EF}} = \frac{\overline{DE}}{\sin(\beta)} \quad \Rightarrow \overline{DE} = \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha) = \frac{\overline{DF}}{\overline{EF}} = \frac{\overline{DF}}{\sin(\beta)} \quad \Rightarrow \overline{DF} = \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha) = \frac{\overline{CE}}{\overline{AE}} = \frac{\overline{CE}}{\cos(\beta)} \quad \Rightarrow \overline{CE} = \sin(\alpha)\cos(\beta)$$

$$\cos(\alpha) = \frac{\overline{AC}}{\overline{AE}} = \frac{\overline{AC}}{\cos(\beta)} \quad \Rightarrow \overline{AC} = \cos(\alpha)\cos(\beta)$$



Derivation of the double angle formulas

$$\sin(2\theta) = \sin(\theta+\theta) = \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos(\theta+\theta) = \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta) = \cos^2(\theta) - \sin^2(\theta)$$

Derivation of the half angle formulas

Start with the cosine double angle formula:

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \boxed{2\cos^2(\theta) - 1} = \boxed{1 - 2\sin^2(\theta)}$$

Solve for either $\sin^2(\theta)$ or $\cos^2(\theta)$:

$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2} \qquad\qquad \cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$

$\sin^2(x) + \cos^2(x) = 1$

$\cos^2(x) = 1 - \sin^2(x)$

$\left(\cos(x)\right)^5$

$\sin^m(x)\cos^n(x)$

$m = 0$ even

$n = 5$ odd

Example. Evaluate the integral $\int \cos^5(x)\,dx$.

Split off $\cos(x)$, rewrite the resulting even power of $\cos(x)$ in terms of $\sin(x)$, and then use $u = \sin(x)$.

$u = \sin(x)$

$du = \cos(x)\,dx$

$= \int \cancel{\boxed{\cos^2(x)}\cos^3(x)\,dx}$

$= \int \cancel{\left(1 - \sin^2(x)\right)\cos^3(x)\,dx}$

u^3

$$= \int \cos^4(x)\,\cos(x)\,dx$$

$\left(\cos^2(x)\right)^2$

$$= \int \left(1 - \underbrace{\sin^2(x)}_{u^2}\right)^2 \underbrace{\cos(x)\,dx}_{du}$$

$\cancel{t = 1 - \sin^2(x)}$

$\cancel{dt = -2\sin(x)\cos(x)\,dx}$

$$= \int \left(1 - u^2\right)^2 du$$

$$= \int 1 - 2u^2 + u^4\,du$$

$$= u - \frac{2}{3}u^3 + \frac{u^5}{5} + C$$

$$= \boxed{\sin(x) - \frac{2}{3}\sin^3(x) + \frac{1}{5}\sin^5(x) + C}$$

Split off $\sin(x)$, rewrite the resulting even power of $\sin(x)$ in terms of $\cos(x)$, and then use $u = \cos(x)$.

Example. Evaluate the integral $\int \underset{\text{odd}}{\sin^3}(x) \underset{\text{real}}{\cos^{3/2}}(x)\, dx$.

$$= \int \sin^2(x) \cos^{3/2}(x) \sin(x)\, dx$$

$$= \int (1-\cos^2(x)) \cos^{3/2}(x) \sin(x)\, dx$$

$u = \cos(x)$

$du = -\sin(x)\, dx$

$$= -\int (1-u^2) u^{3/2}\, du$$

$$= -\int u^{3/2} - u^{7/2}\, du$$

$$= -\left(\frac{2}{5} u^{5/2} - \frac{2}{9} u^{9/2}\right) + C$$

$u = \cos(x)$

$$= \boxed{-\frac{2}{5} \cos^{5/2}(x) + \frac{2}{9} \cos^{9/2}(x) + C}$$

Use half-angle formulas to transform the integrand into a polynomial in $\cos(2x)$, and apply the preceding strategies once again to powers of $\cos(2x)$ greater than 1.

$\sin^6(x) = \left(\sin^2(x)\right)^3$

Example. Evaluate the integral $\int 20\sin^2(x)\cos^2(x)\,dx$

$\sin^2(x) = \dfrac{1-\cos(2x)}{2}$

$\cos^2(x) = \dfrac{1+\cos(2x)}{2}$

$$= 20\int \left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right) dx$$

$$= 5\int 1 - \cos^2(2x)\ dx$$

$\cos^2(2x) = \dfrac{1+\cos(4x)}{2}$

$$= 5\int 1 - \left(\frac{1+\cos(4x)}{2}\right) dx$$

Alternatively...

$1 - \cos^2(2x) = \sin^2(2x)$

$$= \frac{5}{2}\int 1 - \cos(4x)\ dx$$

$$= \frac{5}{2}\left(x - \frac{\sin(4x)}{4}\right) + C$$

Split off $\sec^2(x)$, rewrite the remaining even power of $\sec(x)$ in terms of $\tan(x)$, and use $u = \tan(x)$.

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

Example. Evaluate the integral $\int \sec^6(x)\tan^4(x)\,dx$.

$$= \int \underbrace{\sec^4(x)}_{(\sec^2(x))^2} \underbrace{\tan^4(x)}_{u^4} \underline{\sec^2(x)\,dx}$$

$u = \tan(x)$

$du = \sec^2(x)\,dx$

$$= \int (\tan^2(x)+1)^2 \tan^4(x) \sec^2(x)\,dx$$

$$= \int (u^2+1)^2 u^4\,du$$

$$= \int (u^4+2u^2+1)u^4\,du$$

$$= \int u^8 + 2u^6 + u^4\,du$$

$$= \frac{1}{9}u^9 + \frac{2}{7}u^7 + \frac{1}{5}u^5 + C$$

$$= \boxed{\frac{1}{9}\tan^9(x) + \frac{2}{7}\tan^7(x) + \frac{1}{5}\tan^5(x) + C}$$

$$\left(\sin^2\theta + \cos^2\theta = 1\right) / \cos^2(\theta)$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cancel{\tan^2\theta = \sec^2\theta - 1}$$

Example. Evaluate the integral $\int 35 \tan^5(x) \sec^4(x)\, dx$.

$$u = \tan(x)$$

$$du = \sec^2(x)\, dx$$

$$\underbrace{\tan^5(x)}_{u^5} \sec^4(x) \quad du$$

e.g. $\sec^8(x) = \left(\sec^2(x)\right)^4$

$= \left(\tan^2(x) + 1\right)^4$

$$= \int 35 \underbrace{\tan^5(x)}_{u^5} \underbrace{\sec^2(x)}_{\substack{(\tan^2 x + 1) \\ (u^2+1)}} \underbrace{\sec^2(x)\, dx}_{du}$$

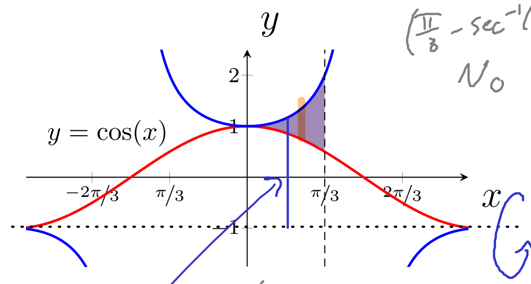
$$= \int 35 u^5 (u^2+1)\, du = 35 \int u^7 + u^5\, du$$

$$= 35\left(\frac{u^8}{8} + \frac{u^6}{6}\right) + C = \boxed{35\left(\frac{\tan^8(x)}{8} + \frac{\tan^6(x)}{6}\right) + C}$$

Example. Consider the region bounded by $y = \sec(x)$ and $y = \cos(x)$ for $0 \leq x \leq \pi/3$. Find the volume of the solid generated when rotating this region about the line $y = -1$.

$\pi(R^2 - r^2)$

Use disk/washer so we one integral



$(\frac{\pi}{3} - \sec^{-1}(x))$ No

$(\pi/3 - \cos^{-1}(x))$

$\sec(x) - (-1)$

$$V = \int_0^{\pi/3} \pi\left(\underbrace{(\sec(x)+1)^2}_{\sec^2(x)+2\sec(x)+1} - (\cos(x)+1)^2\right)dx$$

$\cos^2(x) + 2\cos(x) + 1$

$$= \pi\int_0^{\pi/3} \sec^2(x) + 2\sec(x) - \cos^2(x) - 2\cos(x)\,dx$$

$-2\int \cos(x)\,dx = -2\sin(x) + C$

$\int \sec^2(x)\,dx = \tan(x) + C$

$$2\sec(x)\left(\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\right)$$

$$\frac{2\sec^2(x) + 2\sec(x)\tan(x)}{\sec(x)+\tan(x)}$$

$u = \sec(x) + \tan(x)$

$du = \sec(x)\tan(x) + \sec^2(x)\,dx$

$$\boxed{\cos^2(x) = \frac{1+\cos(2x)}{2}}$$

$\cos(2x) = \cos^2(x) - \sin^2(x)$

$= 2\cos^2(x) - 1$

$= 1 - 2\sin^2(x)$

$\int \frac{2}{u}\,du$

$$= \pi\int_0^{\pi/3} \sec^2(x) + 2\,\frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x)+\tan(x)} - \frac{1+\cos(2x)}{2} - 2\cos(x)\,dx$$

$$= \tan(x) + 2\ln\left|\sec(x)+\tan(x)\right| - \frac{x}{2} - \frac{\sin(2x)}{4} - 2\sin(x)\Big|_0^{\pi/3}$$

Example. Find the length of the curve $y = \ln(2\sec(x))$ on the interval $[0, \pi/6]$.

$$L = \int_0^{\pi/6} \sqrt{1 + (y')^2}\, dx \qquad y' = \frac{2\sec(x)\tan(x)}{2\sec(x)} = \tan(x)$$

$$= \int_0^{\pi/6} \sqrt{1+\tan^2(x)}\, dx = \int_0^{\pi/6} \sqrt{\sec^2(x)}\, dx$$

$$= \int_0^{\pi/6} \sec(x)\, \frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\, dx = \int_0^{\pi/6} \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x)+\tan(x)}\, dx$$

let $t = \sec(x)+\tan(x)$

$dt = \sec(x)\tan(x) + \sec^2(x)\, dx$

$x = 0,\ t = 1$

$x = \pi/6,\ t = \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$

$$= \int \frac{1}{t}\, dt = \ln|t| \Big|_1^{\sqrt{3}} = \boxed{\frac{\ln(3)}{2}}$$

$\int \sin^m(x)\cos^n(x)\,dx$	**Strategy**
m odd and positive, n real	Split off $\sin(x)$, rewrite the resulting even power of $\sin(x)$ in terms of $\cos(x)$, and then use $u = \cos(x)$.
n odd and positive, m real	Split off $\cos(x)$, rewrite the resulting even power of $\cos(x)$ in terms of $\sin(x)$, and then use $u = \sin(x)$.
m and n both even, nonnegative integers	Use half-angle formulas to transform the integrand into a polynomial in $\cos(2x)$, and apply the preceding strategies once again to powers of $\cos(2x)$ greater than 1.
$\int \tan^m(x)\sec^n(x)\,dx$	
n even and positive, m real	Split off $\sec^2(x)$, rewrite the remaining even power of $\sec(x)$ in terms of $\tan(x)$, and use $u = \tan(x)$.
m odd and positive, n real	Split off $\sec(x)\tan(x)$, rewrite the remaining even power of $\tan(x)$ in terms of $\sec(x)$, and use $u = \sec(x)$.
m even and positive, n odd and positive	Rewrite $\tan^m(x)$ in terms of $\sec(x)$
$\int \sec^n(x)\,dx$	
n odd	Use integration by parts with $u = \sec^{n-2}(x)$ and $dv = \sec^2(x)\,dx$
n even	Split off $\sec^2(x)$, rewrite the remaining powers of $\sec(x)$ in terms of $\tan(x)$, and use $u = \tan(x)$.
$\int \tan^m(x)\,dx$	Split off $\tan^2(x)$ and rewrite in terms of $\sec(x)$. Expand into difference of integrals substituting $u = \tan(x)$. Repeat the process as needed for remaining powers of $\tan(x)$.

$$\int \sec(x)\,\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx$$

$$\int \sec^n(x)\,dx$$

n odd — Use integration by parts with $u = \sec^{n-2}(x)$ and $dv = \sec^2(x)\,dx$

$$\int \sec^3(x)\,dx = \int \sec(x)\,\sec^2(x)\,dx$$

$u = \sec(x)$ $\quad v = \tan(x)$

$du = \sec(x)\tan(x)\,dx$ $\quad dv = \sec^2(x)\,dx$

$$\int u\,dv = uv - \int v\,du$$

$$= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx$$

$\sec(x)$ odd power

$\tan(x)$ even power

$$= \sec(x)\tan(x) - \int \sec(x)\left(\sec^2(x) - 1\right)dx$$

$$\Rightarrow \int \sec^3(x)\,dx = \sec(x)\tan(x) - \int \sec^3(x)\,dx + \int \sec(x)\,dx$$

$$2\int \sec^3(x)\,dx = \sec(x)\tan(x) + \int \sec(x)\,\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx$$

$$= \sec(x)\tan(x) + \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx$$

let $t = \sec(x) + \tan(x)$

$dt = \sec(x)\tan(x) + \sec^2(x)\,dx$

$$\Rightarrow 2\int \sec^3(x)\,dx = \sec(x)\tan(x) + \int \frac{1}{t}\,dt$$

$$\Rightarrow \boxed{\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{\ln|\sec(x)+\tan(x)|}{2} + C}$$