1 15.2: Limits and Continuity

Definition. (Limit of a Function of Two Variables)
The function f has the **limit** L as $P(x,y)$ approaches $P_0(a,b)$, written

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{P\to P_0} f(x,y) = L,$$

if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

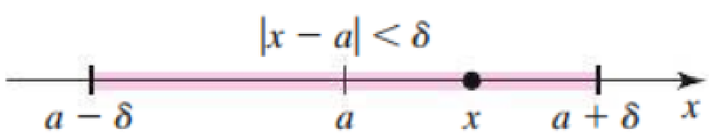
$$|f(x,y) - L| < \varepsilon$$

whenever (x,y) is in the domain of f and

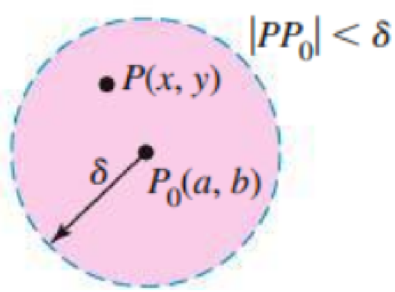
$$0 < |PP_0| = \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

Note: For functions with 1 independent variable, $|x-a| < \delta$ represents an open interval on a number line. Recall that these limits only exist if the same value is approached from two directions.

For functions with 2 independent variables, $|PP_0| < \delta$ represents an open disk (open ball). Here, the limit only exists if the same value is approached from *all* directions.



(a)



Theorem 15.1: Limits of Constant and Linear Functions
Let a, b, and c be real numbers.

1. Constant function $f(x,y) = c$: $\lim_{(x,y)\to(a,b)} c = c$
2. Linear function $f(x,y) = x$: $\lim_{(x,y)\to(a,b)} x = a$
3. Linear function $f(x,y) = y$: $\lim_{(x,y)\to(a,b)} y = b$

Theorem 15.2: Limit Laws for Functions of Two Variables
Let L and M be real numbers and suppose $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim_{(x,y)\to(a,b)} g(x,y) = M$. Assume c is constant, and $n > 0$ is an integer.

1. **Sum** $\lim_{(x,y)\to(a,b)} (f(x,y) + g(x,y)) = L + M$
2. **Difference** $\lim_{(x,y)\to(a,b)} (f(x,y) - g(x,y)) = L - M$
3. **Constant multiple** $\lim_{(x,y)\to(a,b)} cf(x,y) = cL$
4. **Product** $\lim_{(x,y)\to(a,b)} f(x,y)g(x,y) = LM$
5. **Quotient** $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$, provided $M \neq 0$
6. **Power** $\lim_{(x,y)\to(a,b)} (f(x,y))^n = L^n$
7. **Root** $\lim_{(x,y)\to(a,b)} (f(x,y))^{1/n} = L^{1/n}$, when $L > 0$ if n is even.

Example. Evaluate the following limits:

$$\lim_{(x,y)\to(4,11)} 570$$

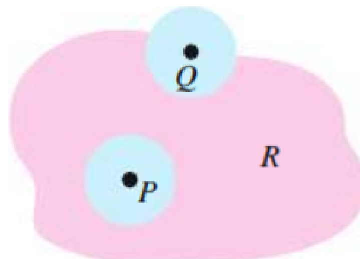
$$\lim_{(x,y)\to(2,8)} \left(3x^2y + \sqrt{xy}\right)$$

$$\lim_{(x,y)\to(0,\pi)} \frac{\sin(xy) + \cos(xy)}{7y}$$

$$\lim_{(x,y)\to(\frac{1}{3},-1)} \frac{9x^2 - y}{3x + y}$$

Definition. (Interior and Boundary Points)
Let R be a region in $\mathbb{R}^2$. An **interior point** P of R lies entirely within R, which means it is possible to find a disk centered at P that contains only points of R.

A **boundary point** Q of R lies on the edge of R in the sense that every disk centered at Q contains at least one point in R and at least one point not in R.



Definition. (Open and Closed Sets)
A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points.

Example. Identify which regions are open sets and which are closed sets.

$$\left\{(x, y) : x^2 + y^2 < 9\right\} \qquad \{(x, y) : |x| \leq 1, |y| \leq 1\}$$

$$\{(x, y) : x \neq 0, -1 \leq y \leq 3\} \qquad \{(x, y) : x + y < 2\}$$

A limit at a boundary point $P_0(a, b)$ of a function's domain can exist, provided $f(x, y)$ approaches the same value as (x, y) approaches (a, b) *along all paths that lie in the domain.*

Example. $f(x, y) = \dfrac{x^2 - y^2}{x - y}$

Example. Evaluate the following limits

$$\lim_{(x,y)\to(0,\pi)} \frac{\sin(xy) + \cos(xy)}{7y}$$

$$\lim_{(x,y)\to(-3,-15)} \frac{y^2 - 5xy}{y - 5x}$$

$$\lim_{(x,y)\to(0,0)} \frac{x + 2y}{x - 2y}$$

$$\lim_{(x,y)\to(1,-1)} \frac{y^5}{(x-1)^{30} + y^5}$$

Procedure: Two-Path Test for Nonexistence of Limits
If $f(x, y)$ approaches two different values as (x, y) approaches (a, b) along two different paths in the domain of f, then $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Definition. (Continuity)
The function f is continuous at the point (a, b) provided

1. f is defined at (a, b)
2. $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y)$ exists, and
3. $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

Example. Determine if $f(x, y)$ is continuous at $(0, 0)$

$$f(x,y) = \begin{cases} \dfrac{3xy^2}{x^2+y^4}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

Theorem 15.3: Continuity of Composite Functions
If $u = g(x, y)$ is continuous at (a, b) and $z = f(u)$ is continuous at $g(a, b)$, then the composite function $z = f(g(x, y))$ is continuous at (a, b).

Example. Determine the points at which the following functions are continous:

$f(x, y) = \ln(x^2 + y^2 + 4)$ $\qquad$ $g(x, y) = e^{x/y}$