

Math 1080 Class notes

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5.5: Substitution Rule

Theorem 5.6: Substitution Rule for Indefinite Integrals

Let $u = g(x)$, where g is differentiable on an interval, and let f be continuous on the corresponding range of g . On that interval,

$$\int f(g(x))g'(x) dx = \int f(u) du$$

Example. We know

$$\frac{d}{dx} \left[\frac{(2x+1)^4}{4} \right] = 2(2x+1)^3$$

Thus, if $f(x) = x^3$ and $g(x) = 2x + 1$ then $g'(x) = 2$, so we let $u = 2x + 1$, then

$$\begin{aligned} \int 2(2x+1)^3 dx &= \int f(g(x))g'(x) dx \\ &= \int u^3 du \\ &= \frac{u^4}{4} + C \\ &= \frac{(2x+1)^4}{4} + C \end{aligned}$$

Procedure: Substitution Rule (Change of Variables)

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x) dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u .
4. Write the result in terms of x using $u = g(x)$.

Example. Evaluate the following integrals:

a) $\int 2x(x^2 + 3)^4 dx$

b) $\int (2x + 1)^3 dx$

c) $\int x^2 \sqrt{x^3 + 1} dx$

d) $\int \theta \sqrt[4]{1 - \theta^2} d\theta$

e) $\int \sqrt{4 - t} dt$

f) $\int (2 - x)^6 dx$

Example. Evaluate the following integrals:

a) $\int \sec(2\theta) \tan(2\theta) d\theta$

b) $\int \csc^2\left(\frac{t}{3}\right) dt$

c) $\int \frac{\sin(x)}{1 + \cos^2(x)} dx$

d) $\int \frac{\tan^{-1}(x)}{1 + x^2} dx$

The acceleration of a particle moving back and forth on a line is $a(t) = \frac{d^2s}{dt^2} = \pi^2 \cos(\pi t) \text{ m/s}^2$ for all t . If $s = 0$ and $v = 8 \text{ m/s}$ when $t = 0$, find the value of s when $t = 1$ sec.

Example. Evaluate the following integrals:

a) $\int (6x^2 + 2) \sin(x^3 + x + 1) dx$

b) $\int \frac{\sin(\theta)}{\cos^5(\theta)} d\theta$

c) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

d) $\int \frac{2^t}{2^t + 3} dt$

e) $\int 6x^2 4^{x^3} dx$

f) $\int \frac{dx}{\sqrt{36 - 4x^2}}$

g) $\int \sin(t) \sec^2(\cos(t)) dt$

h) $\int \frac{1}{\sqrt{x}(1 + \sqrt{x})^2} dx$

i) $\int \frac{\sin(\sqrt{x})}{\sqrt{x}} dx$

j) $\int 5 \cos(7x + 5) dx$

k) $\int \frac{3}{\sqrt{1 - 25x^2}} dx$

l) $\int \frac{dx}{\sqrt{1 - 9x^2}}$

Example. Evaluate the following integrals using the recommended substitution:

a) $\int \sec^2(x) \tan(x) \, dx$
where $u = \tan(x)$.

b) $\int \sec^2(x) \tan(x) \, dx$
where $u = \sec(x)$.

Example. Solve the initial value problem: $\frac{dy}{dx} = 4x(x^2 + 8)^{-1/3}, y(0) = 0$.

Example. Evaluate the following integrals:

a) $\int x e^{-x^2} dx$

b) $\int \frac{e^{1/x}}{x^2} dx$

c) $\int \frac{dt}{8-3t}$

d) $\int 5^t \sin(5^t) dt$

e) $\int \frac{e^w}{36 + e^{2w}} dw$

Theorem 5.7: Substitution Rule for Definite Integrals

Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g . Then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

Example. Evaluate the integrals:

a) $\int_0^1 \frac{x^3}{\sqrt{x^4 + 9}} dx$

b) $\int_1^3 \frac{dt}{(t - 4)^2}$

c) $\int_0^3 \frac{v^2 + 1}{\sqrt{v^3 + 3v + 4}} dv$

d) $\int_0^1 2x(4 - x^2) dx$

e) $\int_2^3 \frac{x}{\sqrt[3]{x^2-1}} dx$

f) $\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{1+\cos(x)} dx$

g) $\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos^2(x)} dx$

h) $\int_{-\frac{\pi}{12}}^{\frac{\pi}{8}} \sec^2(2y) dy$

i) $\int_0^1 (1 - 2x^9) dx$

j) $\int_0^1 (1 - 2x)^9 dx$

k) $\int_0^{\frac{1}{2}} \frac{1}{1 + 4x^2} dx$

l) $\int_0^4 \frac{x}{x^2 + 1} dx$

m) $\int_0^\pi 3 \cos^2(x) \sin(x) \, dx$

n) $\int_0^{\frac{\pi}{8}} \sec(2\theta) \tan(2\theta) \, d\theta$

o) $\int_0^1 (3t - 1)^{50} \, dt$

p) $\int_0^3 \frac{1}{5x + 1} \, dx$

q) $\int_0^1 x e^{-x^2} dx$

r) $\int_e^{e^4} \frac{1}{x \sqrt{\ln(x)}} dx$

s) $\int_0^{\frac{1}{2}} \frac{\sin^{-1}(x)}{\sqrt{1-x^2}} dx$

t) $\int_0^1 \frac{e^z + 1}{e^z + z} dz$

$$\text{u)} \int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2}$$

$$\text{v)} \int_{\ln(\frac{\pi}{4})}^{\ln(\frac{\pi}{2})} e^w \cos(e^w) dw$$

$$\text{w)} \int_0^{\frac{1}{8}} \frac{x}{\sqrt{1-16x^2}} dx$$

$$\text{x)} \int_1^{e^2} \frac{\ln(p)}{p} dp$$

$$\text{y) } \int_0^{\frac{\pi}{4}} e^{\sin^2(x)} \sin(2x) \, dx$$

$$\text{z) } \int_{-\pi}^{\pi} x^2 \sin(7x^3) \, dx$$

Example. Average velocity: An object moves in one dimension with a velocity in m/s given by $v(t) = 8 \sin(\pi t) + 2t$. Find its average velocity over the time interval from $t = 0$ to $t = 10$, where t is measured in seconds.

Example. Prove $\int \tan(x) \, dx = \ln |\sec(x)| + C$.

Example. Evaluate the integrals:

a) $\int \frac{x}{(x-2)^3} \, dx$

b) $\int x\sqrt{x-1} \, dx$

c) $\int x^3(1+x^2)^{\frac{3}{2}} dx$

d) $\int \frac{y^2}{(y+1)^4} dy$

e) $\int (z+1)\sqrt{3z+2} dz$

f) $\int_0^1 \frac{x}{(x+2)^3} dx$

Half-Angle Formulas

$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$$

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$$

Example. Evaluate the integrals:

a) $\int \cos^2(x) \, dx$

b) $\int_0^{\frac{\pi}{2}} \cos^2(x) \, dx$

c) $\int \frac{1}{x^2} \cos^2\left(\frac{1}{x}\right) dx$

d) $\int x \sin^2(x^2) dx$

e) $\int \sin^2\left(\theta + \frac{\pi}{6}\right) d\theta$

f) $\int_0^{\frac{\pi}{4}} \cos^2(8\theta) d\theta$

Example. If f is continuous and $\int_0^4 f(x) dx = 10$, find $\int_0^2 f(2x) dx$.

Example. If f is continuous and $\int_0^9 f(x) dx = 4$, find $\int_0^3 xf(x^2) dx$.

Example. Suppose f is an even function with $\int_0^8 f(x) dx = 9$. Evaluate the following:

a) $\int_{-1}^1 xf(x^2) dx$.

b) $\int_{-2}^2 x^2 f(x^3) dx$.

Example. Evaluate the integrals:

a) $\int \sec^2(10x) \, dx$

b) $\int \tan^{10}(4x) \sec^2(4x) \, dx$

c) $\int \left(x^{\frac{3}{2}} + 8\right)^5 \sqrt{x} \, dx$

d) $\int \frac{2x}{\sqrt{3x+2}} \, dx$

e) $\int \frac{7x^2 + 2x}{x} dx$

f) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

g) $\int_0^{\sqrt{3}} \frac{3}{9 + x^2} dx$

h) $\int_0^{\frac{\pi}{6}} \frac{\sin(2y)}{\sin^2(y) + 2} dy$

i) $\int \frac{\sec(z) \tan(z)}{\sqrt{\sec(z)}} dz$

j) $\int \frac{1}{\sin^{-1}(x) \sqrt{1-x^2}} dx$

k) $\int \frac{x}{\sqrt{4-9x^2}} dx$

l) $\int \frac{x}{1+x^4} dx$

$$\text{m) } \int \frac{\cos \sqrt{\theta}}{\sqrt{\theta} \sin^2 \sqrt{\theta}} d\theta$$

$$\text{n) } \int x^2 \sqrt{2+x} dx$$

$$\text{o) } \int (\sin^5(x) + 3 \sin^3(x) - \sin(x)) \cos(x) dx$$

p) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (x^3 + x^4 \tan(x)) \, dx$

q) $\int_0^{\frac{\pi}{2}} \cos(x) \sin(\sin(x)) \, dx$

r) $\int \frac{1+x}{1+x^2} \, dx$

Example. Evaluate these more challenging integrals:

a) $\int \frac{dx}{\sqrt{1 + \sqrt{1 + x}}}$

b) $\int x \sin^4(x^2) \cos(x^2) dx$

6.1: Velocity and Net Change

Definition. (Position, Velocity, Displacement, and Distance)

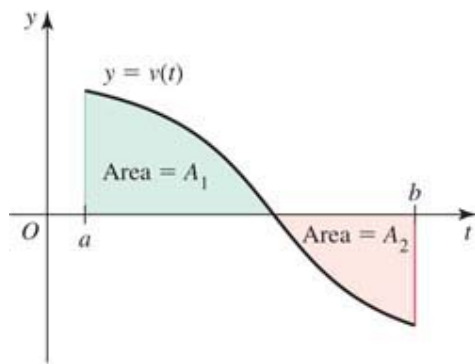
1. The **position** of an object moving along a line at time t , denoted $s(t)$, is the location of the object relative to the origin.
2. The **velocity** of an object at time t is $v(t) = s'(t)$.
3. The **displacement** of the object between $t = a$ and $t = b > a$ is

$$s(b) - s(a) = \int_a^b v(t) dt.$$

4. The **distance traveled** by the object between $t = a$ and $t = b > a$ is

$$\int_a^b |v(t)| dt$$

where $|v(t)|$ is the **speed** of the object at time t .



$$\text{Displacement} = A_1 - A_2 = \int_a^b v(t) dt$$

(a)



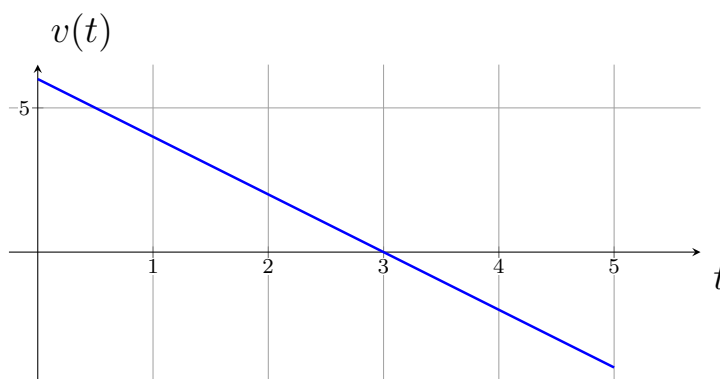
$$\text{Distance traveled} = A_1 + A_2 = \int_a^b |v(t)| dt$$

(b)

Example. Suppose an object moves along a line with velocity (in ft/s) $v(t) = 6 - 2t$, for $0 \leq t \leq 5$, where t is measured in seconds.

- Find the displacement of the object on the interval $0 \leq t \leq 5$.

- Find the distance traveled by the object on the interval $0 \leq t \leq 5$.



Example. A cyclist rides down a long straight road at a velocity (in m/min) given by $v(t) = 400 - 20t$, for $0 \leq t \leq 10$.

- How far does the cyclists travel in the first 5 minutes?
- How far does the cyclists travel in the first 10 minutes?
- How far has the cyclist traveled when her velocity is 250 m/min?

Example. The population of a community of foxes is observed to fluctuate on a 10-year cycle due to variations in the availability of prey. When population measurements began ($t = 0$), the population was 35 foxes. The growth rate in units of foxes/year was observed to be:

$$P'(t) = 5 + 10 \sin\left(\frac{\pi t}{5}\right)$$

- Find $P(t)$.
- Find the population of foxes after the first 5 years, rounded to the nearest whole number of foxes.

Theorem 6.1: Position from Velocity

Given the velocity $v(t)$ of an object moving along a line and its initial position $s(0)$, the position function of the object for future times $t \geq 0$ is

$$\underbrace{s(t)}_{\substack{\text{position} \\ \text{at } t}} = \underbrace{s(0)}_{\substack{\text{initial} \\ \text{position}}} + \underbrace{\int_0^t v(x) \, dx}_{\substack{\text{displacement} \\ \text{over } [0, t]}}.$$

Theorem 6.2: Velocity from Acceleration

Given the acceleration $a(t)$ of an object moving along a line and its initial velocity $v(0)$, the velocity of the object for future times $t \geq 0$ is

$$v(t) = v(0) + \int_0^t a(x) \, dx.$$

Example. At $t = 0$, a train approaching a station begins decelerating from a speed of 80 miles/hour according to the acceleration function $a(t) = -1280(1 + 8t)^{-3}$, where $t \geq 0$ is measured in hours. The units of acceleration are mi/hr².

- Find the velocity of the train at $t = 0.25$.
- How far does the train travel in the first 15 minutes (1/4 hour)?
- How long does it take the train to travel 9 miles?

Theorem 6.3: Net Change and Future Value

Suppose a quantity Q changes over time at a known rate Q' . Then the **net change** in Q between $t = a$ and $t = b > a$ is

$$\underbrace{Q(b) - Q(a)}_{\text{net change in } Q} = \int_a^b Q'(t) dt.$$

Given the initial value $Q(0)$, the **future value** of Q at time $t \geq 0$ is

$$Q(t) = Q(0) + \int_0^t Q'(x) dx.$$

Velocity-Displacement Problems

Position $s(t)$

Velocity: $s'(t) = v(t)$

Displacement: $s(b) - s(a) = \int_a^b v(t) dt$

Future position: $s(t) = s(0) + \int_0^t v(x) dx$

General Problems

Quantity $Q(t)$ (such as volume or population)

Rate of change: $Q'(t)$

Net change: $Q(b) - Q(a) = \int_a^b Q'(t) dt$

Future value of Q : $Q(t) = Q(0) + \int_0^t Q'(x) dx$

6.2: Regions Between Curves

Definition. (Area of a Region Between Two Curves)

Suppose f and g are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$. The area of the region bounded by the graphs of f and g on $[a, b]$ is

$$A = \int_a^b (f(x) - g(x)) \, dx.$$



Example. Consider the region bounded by the curves $y = \cos(x)$ and $y = 1 - \cos(x)$, $0 \leq x \leq \pi$. Set up the integral(s) representing the area of this region.



Example. Find the area of the region by integrating with respect to x .



Example. Find the volume of the solid whose base is bounded by the graphs of $y = x + 1$ and $y = x^2 - 1$, with the cross sections in the shape of rectangles of height 2 taken perpendicular to the x -axis.



Definition. (Area of a Region Between Two Curves with Respect to y)

Suppose f and g are continuous functions with $f(y) \geq g(y)$ on the interval $[c, d]$. The area of the region bounded by the graphs $x = f(y)$ and $x = g(y)$ on $[c, d]$ is

$$A = \int_c^d (f(y) - g(y)) dy.$$

Example. Find the area of the region bounded by $x = 3y$, and $x = y^2 - 10$

by integrating with respect to x

by integrating with respect to y

Example. Find the area of the region bounded by $y = x^3$, and $y = \sqrt{x}$
by integrating with respect to x

by integrating with respect to y

Example. Find the area of the region bounded by $y = 4\sqrt{2x}$, $y = 2x^2$, and $y = -4x + 6$

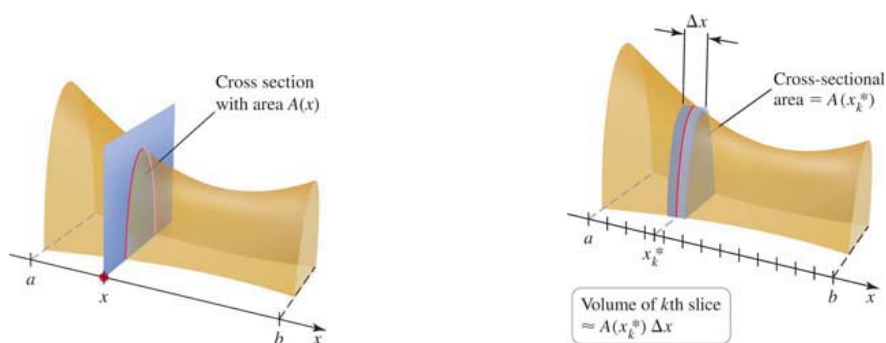


6.3: Volume by Slicing

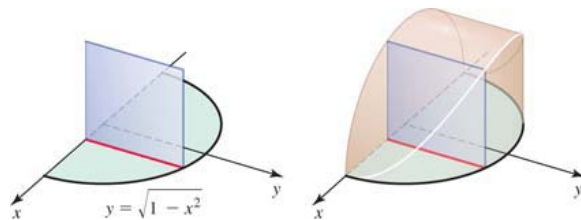
General Slicing Method

Suppose a solid object extends from $x = a$ to $x = b$, and the cross section of the solid perpendicular to the x -axis has an area given by a function A that is integrable on $[a, b]$. The volume of the solid is

$$V = \int_a^b A(x) dx.$$



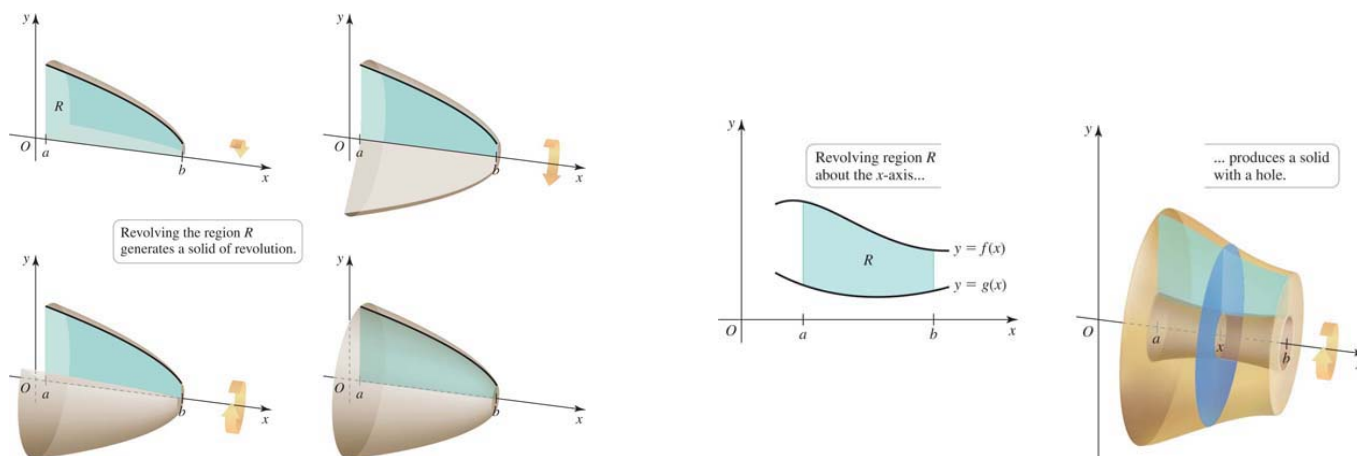
Example. Use the general slicing method to find the volume of the solid whose base is the region bounded by the semicircle $y = \sqrt{1 - x^2}$ and the x -axis, and whose cross sections through the solid perpendicular to the x -axis are squares.



Disk Method about the x -Axis

Let f be continuous with $f(x) \geq 0$ on the interval $[a, b]$. If the region R bounded by the graph of f , the x -axis, and the lines $x = a$ and $x = b$ is revolved about the x -axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \underbrace{\pi f(x)^2}_{\text{disk radius}} dx.$$

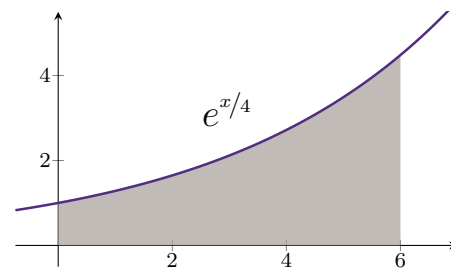


Washer Method about the x -Axis

Let f and g be continuous functions with $f(x) \geq g(x) \geq 0$ on $[a, b]$. Let R be the region bounded by $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$. When R is revolved about the x -axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi \left(\underbrace{f(x)^2}_{\text{outer radius}} - \underbrace{g(x)^2}_{\text{inner radius}} \right) dx.$$

Example. Consider the region bounded by $y = e^{x/4}$, $y = 0$, $x = 0$, and $x = 6$. Find the volume of the solid generated by rotating the region about the x -axis.



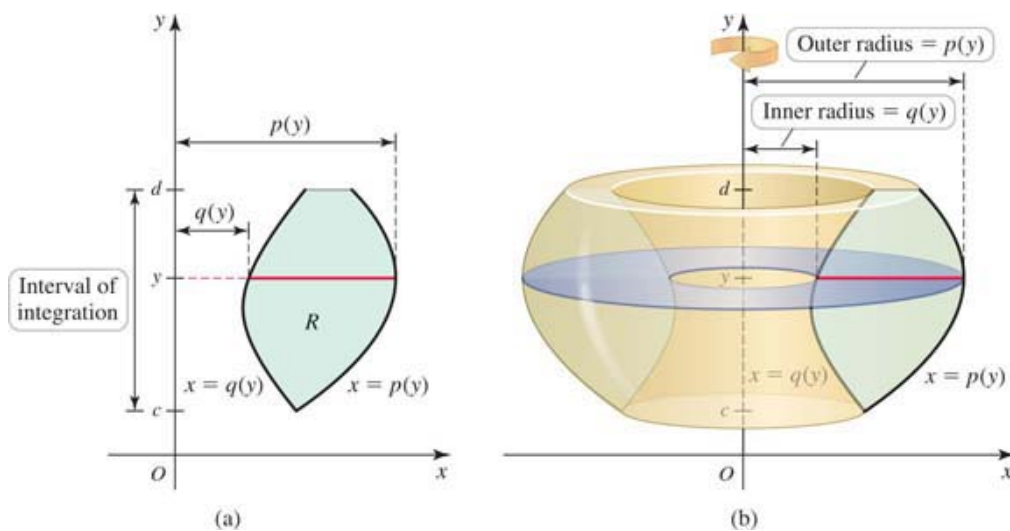
Disk and Washer Methods about the y -Axis

Let p and q be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let R be the region bounded by $x = p(y)$, $x = q(y)$, and the lines $y = c$ and $y = d$. When R is revolved around the y -axis, the volume of the resulting solid of revolution is given by

$$V = \int_c^d \pi \underbrace{(p(y)^2)}_{\text{outer radius}} - \underbrace{(q(y)^2)}_{\text{inner radius}} dy.$$

If $q(y) = 0$, the disk method results:

$$V = \int_c^d \pi \underbrace{p(y)^2}_{\text{disk radius}} dy.$$



Example. Consider the region bounded between $y = \sqrt[4]{x}$, $y = 2$, and $x = 0$.

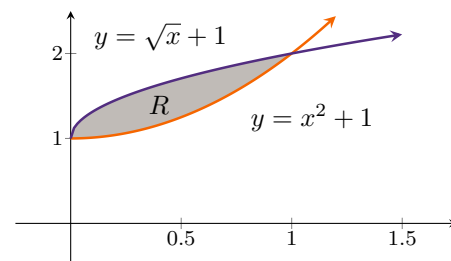


Setup the integral with respect to x that gives the area of the region.

Setup the integral with respect to y that gives the area of the region.

Use the disk/washer method to setup the that represents the volume of the solid generated by rotating the region about the x -axis.

Example. Consider the region R between $y = \sqrt{x} + 1$ and $y = x^2 + 1$. Setup the integrals which find the volume of the solid obtained by rotating the region R as indicated below.



about the y -axis

about the x -axis

about the line $x = 1$

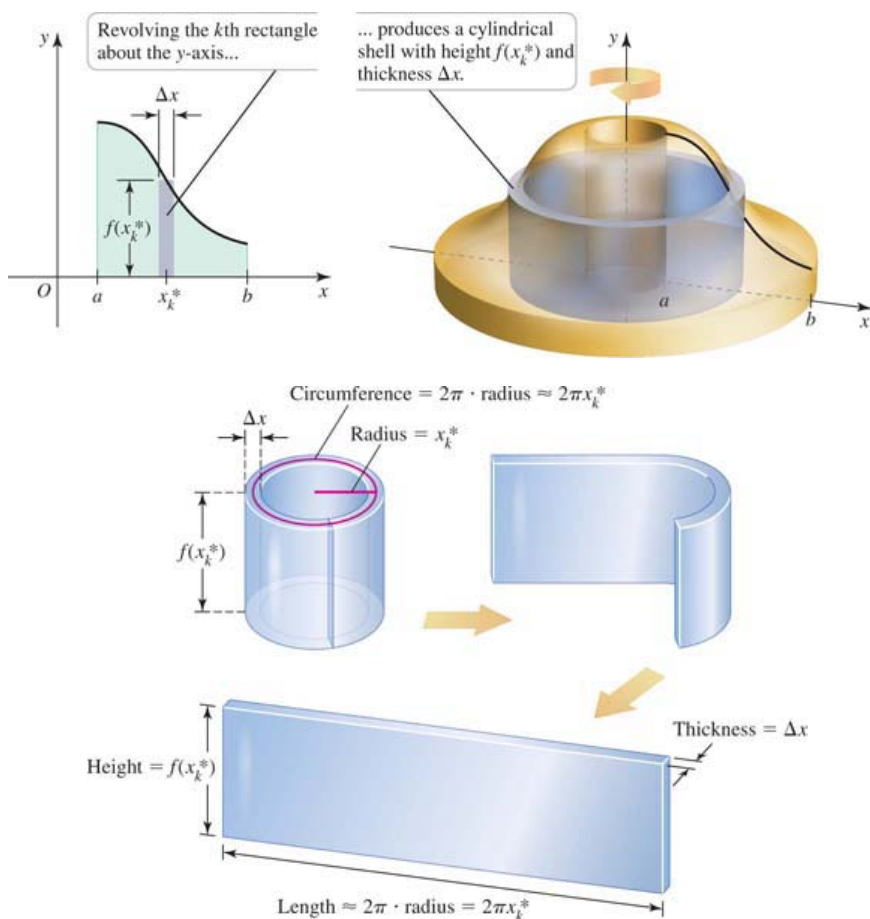
about the line $y = -1$

6.4: Volume by Shells

Volume by the Shell Method

Let f and g be continuous functions with $f(x) \geq g(x)$ on $[a, b]$. If R is the region bounded by the curves $y = f(x)$ and $y = g(x)$ between the lines $x = a$ and $x = b$, the volume of the solid generated when R is revolved about the y -axis is

$$V = \int_a^b \underbrace{2\pi x}_{\text{shell circumference}} \underbrace{(f(x) - g(x))}_{\text{shell height}} dx.$$



Example. Consider a general region R revolved around the y -axis.

When using the **disk/washer** method, we integrate with respect to _____

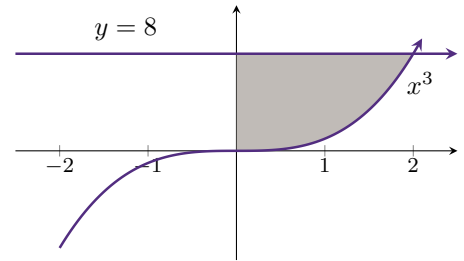
When using the **shell** method, we integrate with respect to _____

Example. Consider a general region R revolved around the x -axis.

When using the **disk/washer** method, we integrate with respect to _____

When using the **shell** method, we integrate with respect to _____

Example. Consider the region bounded between $y = x^3$, $y = 8$ and $x = 0$.



Use the disk/washer method to setup the integral that represents the volume of the solid generated by rotating the region about the x -axis.

about the y -axis.

Use the disk/washer method to setup the integral that represents the volume of the solid generated by rotating the region about the line $x = -1$.

about the line $y = 8$.

Example. Consider the region bounded by $y = \frac{1}{x+1}$ and $y = 1 - \frac{x}{3}$. Use both the disk/washer method and shell method to find the volume of the solid generated when R is rotated about the x -axis.

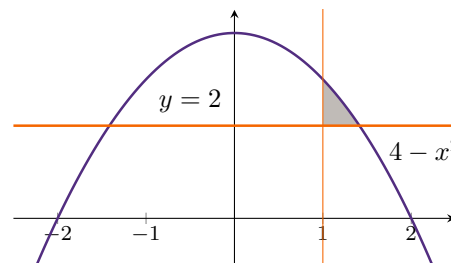
Example. Determine if the following statements are true.

When using the shell method, the axis of the cylindrical shells is parallel to the axis of revolution.

If a region is revolved about the y -axis, then the shell method must be used.

If a region is revolved about the x -axis, it is possible to use the disk/washer method and integrate with respect to x .

Example. Consider the region R bounded by $y = 4 - x^2$, $y = 2$, and $x = 1$. Use the shell method to setup the integral that represents the volume of the solid generated by rotating the region R about the indicated axis of rotation.



about x -axis,

about y -axis,

about the line $x = -2$,

about the line $y = 2$.