Math 2060 Class notes
Spring 2021

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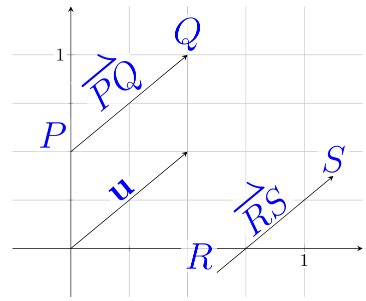
13.1: Vectors and the Geometry of Space

Definition.

- **Vectors**
  - Have a direction and magnitude,
  - vector $\overrightarrow{PQ}$ has a *tail* at P and a *head* at Q,
  - Can be denoted as $\mathbf{u}$ or $\vec{u}$,
  - Equal vectors have the same direction and magnitude (not necessarily the same position)



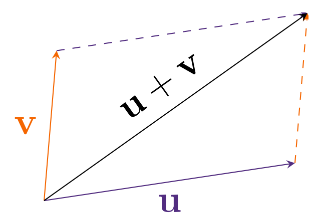
- **Scalars** are quantities with magnitude but no direction (e.g. mass, temperature, price, time, etc.)
- **Zero vector**, denoted $\mathbf{0}$ or $\vec{0}$, has length 0 and no direction

Scalar-vector multiplication:

- Denoted $c\mathbf{v}$ or $c\vec{v}$,
- length of vector multiplied by $|c|$,
- $c\mathbf{v}$ has the same direction as $\mathbf{v}$ if $c > 0$, and has the opposite direction as $\mathbf{v}$ if $c < 0$, (what if $c = 0$?)
- $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if $\mathbf{u} = c\mathbf{v}$. (what vectors are parallel to $\mathbf{0}$?)

Vector Addition and Subtraction:

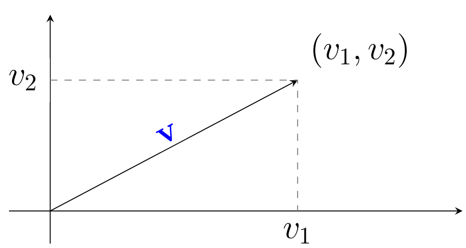
Given two vectors $\mathbf{u}$ and $\mathbf{v}$, their sum, $\mathbf{u}+\mathbf{v}$, can be represented by the parallelogram (triangle) rule: place the tail of $\mathbf{v}$ at the head of $\mathbf{u}$



The difference, denoted $\mathbf{u}-\mathbf{v}$, is the sum of $\mathbf{u}+(-\mathbf{v})$:

Vector Components:

A vector $\mathbf{v}$ whose tail is at the origin $(0,0)$ and head is at (v_1, v_2) is a **position vector** (in **standard position**) and is denoted $\langle v_1, v_2\rangle$. The real numbers v_1 and v_2 are the x- and y-components of $\mathbf{v}$.



Vectors $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ are equal if and only if $u_1 = v_1$ and $u_2 = v_2$.

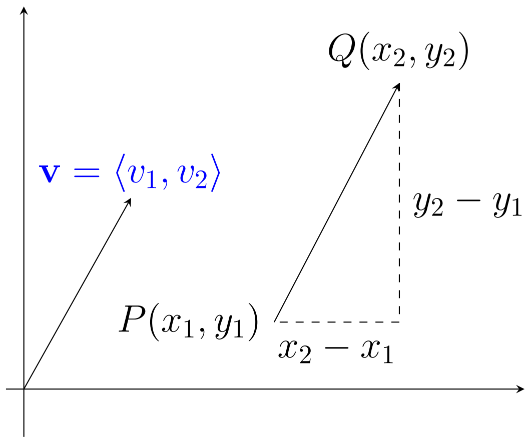
Magnitude:

Given points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the **magnitude**, or **length**, of vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$, denoted $\left|\overrightarrow{PQ}\right|$, is the distance between points P and Q.

$$\left|\overrightarrow{PQ}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The magnitude of position vector $\mathbf{v} = \langle v_1, v_2 \rangle$ is $|\mathbf{v}|$.

(How do $\left|\overrightarrow{PQ}\right|$ and $\left|\overrightarrow{QP}\right|$ relate to each other?)



> <u>Note:</u> The norm, denoted $\|\mathbf{u}\|$ or $\|\mathbf{u}\|_2$, is equivalent to the magnitude of a vector.

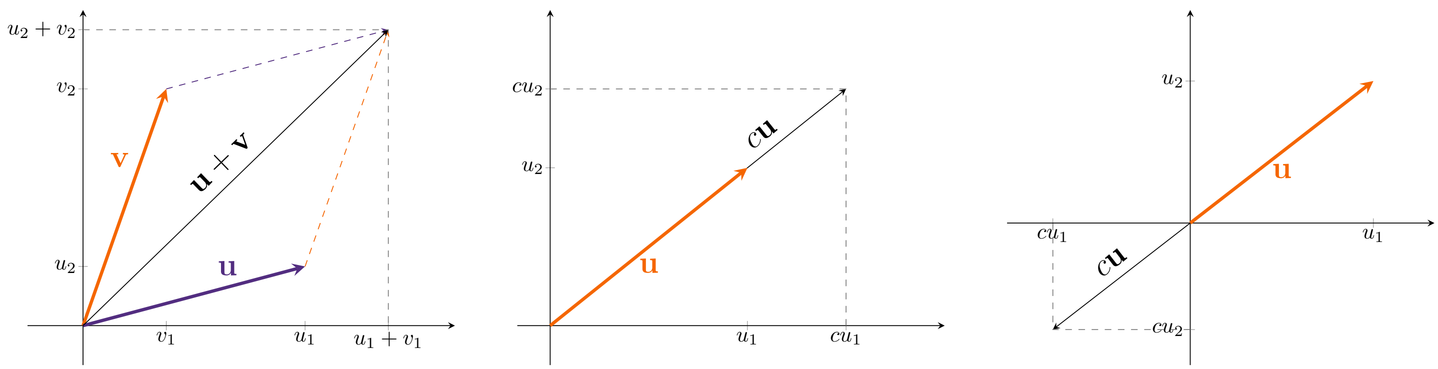
Equation of a Circle:

> **Definition.**
> A **circle** centered at (a, b) with radius r is the set of points satisfying the equation
> $$(x - a)^2 + (y - b)^2 = r^2.$$
> A **disk** centered at (a, b) with radius r is the set of points satisfying the inequality
> $$(x - a)^2 + (y - b)^2 \leq r^2.$$

Vector Operations in Terms of Components

Definition. (Vector Operations in $\mathbb{R}^2$)
Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2\rangle$, and $\mathbf{v} = \langle v_1, v_2\rangle$.

$$\mathbf{u}+\mathbf{v} = \langle u_1+v_1, u_2+v_2\rangle \qquad \text{Vector addition}$$
$$\mathbf{u}-\mathbf{v} = \langle u_1-v_1, u_2-v_2\rangle \qquad \text{Vector subtraction}$$
$$c\mathbf{u} = \langle cu_1, cu_2\rangle \qquad \text{Scalar multiplication}$$



Example. Let $\mathbf{u} = \langle 1, 2\rangle$, $\mathbf{v} = \langle -2, 3\rangle$, $c = 2$, and $d = 3$. Find the following:

$\mathbf{u}+\mathbf{v}$ $\qquad$ $c\mathbf{u}$

$c\mathbf{u}+d\mathbf{v}$ $\qquad$ $\mathbf{u}-c\mathbf{v}$

Definition.
A **unit vector** is any vector with length 1.
In $\mathbb{R}^2$, the **coordinate unit vectors** are $\boldsymbol{i} = \langle 1, 0\rangle$ and $\boldsymbol{j} = \langle 0, 1\rangle$.

Example. Let $\mathbf{u} = \langle -7, 3\rangle$. Find two unit vectors parallel to $\mathbf{u}$. Find another vector parallel to $\mathbf{u}$ with a magnitude of 2.

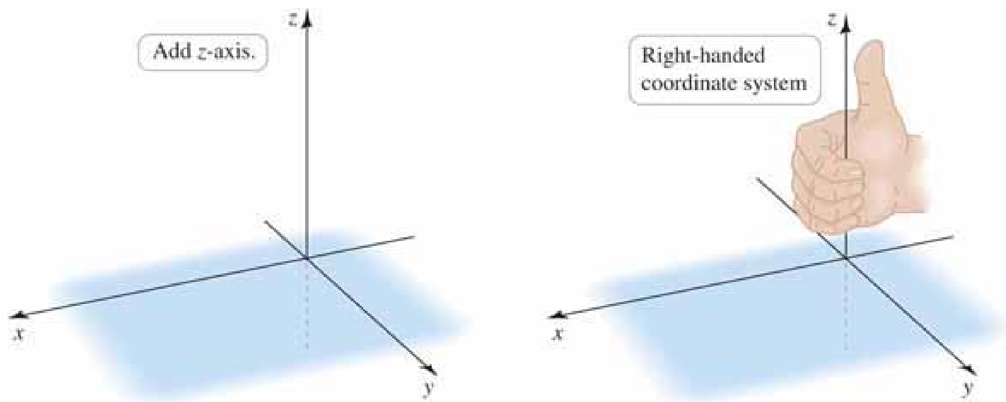
Properties of Vector Operations:
Suppose $\mathbf{u}$, $\mathbf{v}$, and $\boldsymbol{w}$ are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ Commutative property of addition
2. $(\mathbf{u} + \mathbf{v}) + \boldsymbol{w} = \mathbf{u} + (\mathbf{v} + \boldsymbol{w})$ Associative property of addition
3. $\mathbf{v} + \mathbf{0} = \mathbf{v}$ Additive identity
4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ Additive inverse
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ Distributive property 1
6. $(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$ Distributive property 2
7. $0\mathbf{v} = \mathbf{0}$ Multiplication by zero scalar
8. $c\mathbf{0} = \mathbf{0}$ Multiplication by zero vector
9. $1\mathbf{v} = \mathbf{v}$ Multiplicative identity
10. $a(c\mathbf{v}) = (ac)\mathbf{v}$ Associative property of scalar multiplication

13.2: Vectors in Three Dimensions

The xyz- Coordinate System:

The three-dimensional coordinate system is created by adding the z-axis, which is perpendicular to both the x-axis and the y-axis. When looking at the xy-plane, the positive direction of the z-axis protrudes towards the viewer. This can also be shown using the right-hand rule (Figure 13.25 from Briggs):

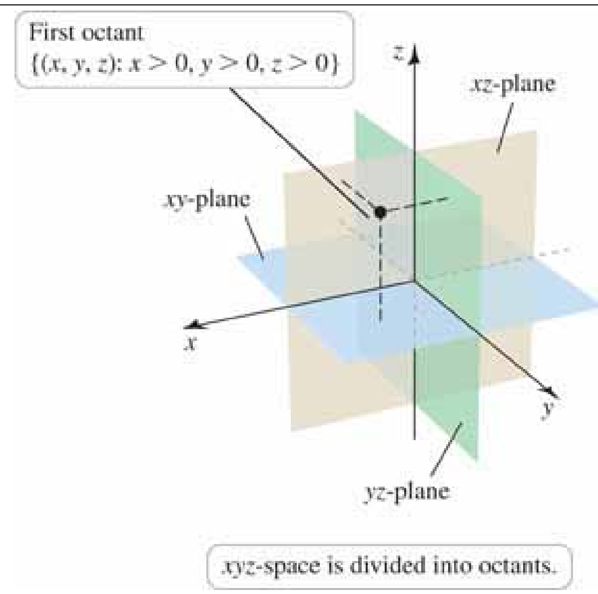


Definition.

This three-dimensional coordinate system is broken up into eight **octants**, which are separated by

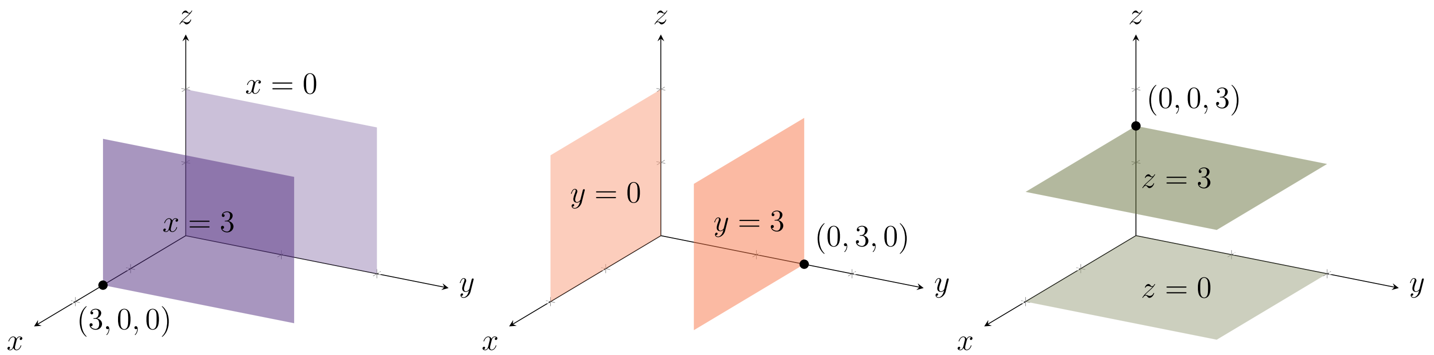
- the xy-**plane** ($z = 0$),
- the xz-**plane** ($y = 0$), and
- the yz-**plane** ($x = 0$).

The **origin** is the location where all three axes intersect.

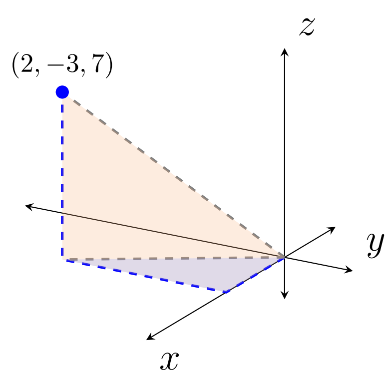


Equations of Simple Planes:

Planes in three-dimensions are analogous to lines in two-dimensions. Below, we see the yz-plane, the xz-plane, and the xy-plane, along with planes that are parallel where x, y, and z are fixed respectively:



Example (Parallel planes)**.** Determine the equation of the plane parallel to the xz-plane passing through the point $(2, -3, 7)$.

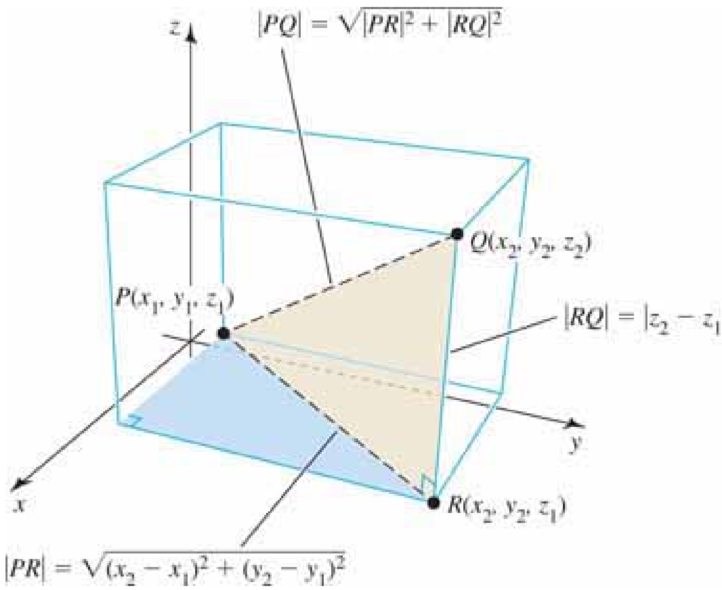


Distances in xyz-Space:

Recall that in $\mathbb{R}^2$, for some vector $\overrightarrow{PR}$, the distance formula is given by

$$|PR| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

where (x_1, y_1) and (x_2, y_2) represent the points P and R respectively. This idea can be further extended into $\mathbb{R}^3$ by considering the two sides of the triangle formed by the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$:



Distance Formula in xyz-Space

The **distance** between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The **midpoint** between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is found by averaging the x-, y-, and z-coordinates:

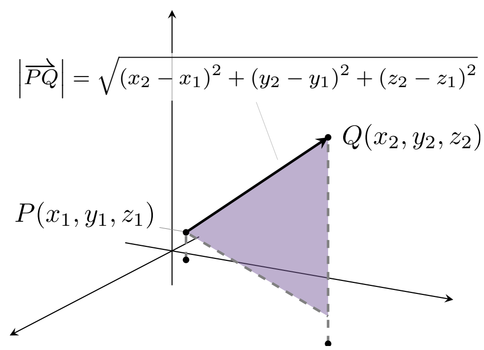
$$\text{Midpoint } = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Magnitude and Unit Vectors:

Definition.
The **magnitude** (or **length**) of the vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ is the distance from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$:

$$\left|\overrightarrow{PQ}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

In $\mathbb{R}^3$, the **coordinate unit vectors** are $\boldsymbol{i} = \langle 1, 0, 0 \rangle$, $\boldsymbol{j} = \langle 0, 1, 0 \rangle$, and $\boldsymbol{k} = \langle 0, 0, 1 \rangle$.



Example. Consider $P(-1, 4, 3)$ and $Q(3, 5, 7)$. Find

- $\left|\overrightarrow{PQ}\right|$

- The midpoint between P and Q

- Two unit vectors parallel to $\overrightarrow{PQ}$

Equation of a Sphere:

Definition.
A **sphere** centered at (a, b, c) with radius r is the set of points satisfying the equation

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

A **ball** centered at (a, b, c) with radius r is the set of points satisfying the inequality

$$(x-a)^2 + (y-b)^2 + (z-c)^2 \leq r^2.$$

Example. Consider $P(-1, 4, 3)$ and $Q(3, 5, 7)$. Find the equation of the sphere centered at the midpoint passing through P and Q

Example. What is the geometry of the intersection between $x^2 + y^2 + z^2 = 50$ and $z = 1$?

Example. Rewrite the following equation into the standard form of a sphere:

$$x^2 + y^2 + z^2 - 2x + 6y - 8z = -1$$

Vector Operations in Terms of Components

Definition. (Vector Operations in $\mathbb{R}^3$)
Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle && \text{Vector addition} \\ \mathbf{u} - \mathbf{v} &= \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle && \text{Vector subtraction} \\ c\mathbf{u} &= \langle cu_1, cu_2, cu_3 \rangle && \text{Scalar multiplication} \end{aligned}$$

Properties of Vector Operations:
Suppose $\mathbf{u}$, $\mathbf{v}$, and $\boldsymbol{w}$ are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

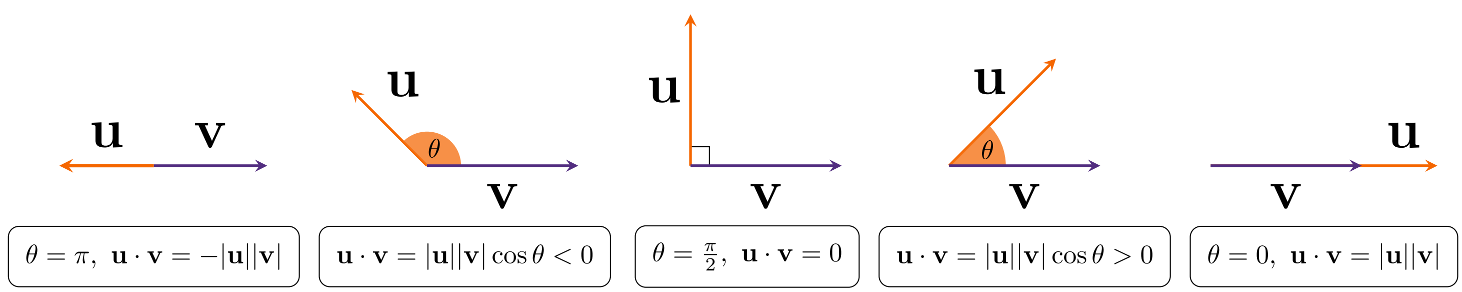
1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ — Commutative property of addition
2. $(\mathbf{u} + \mathbf{v}) + \boldsymbol{w} = \mathbf{u} + (\mathbf{v} + \boldsymbol{w})$ — Associative property of addition
3. $\mathbf{v} + \mathbf{0} = \mathbf{v}$ — Additive identity
4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ — Additive inverse
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ — Distributive property 1
6. $(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$ — Distributive property 2
7. $0\mathbf{v} = \mathbf{0}$ — Multiplication by zero scalar
8. $c\mathbf{0} = \mathbf{0}$ — Multiplication by zero vector
9. $1\mathbf{v} = \mathbf{v}$ — Multiplicative identity
10. $a(c\mathbf{v}) = (ac)\mathbf{v}$ — Associative property of scalar multiplication

13.3: Dot Products

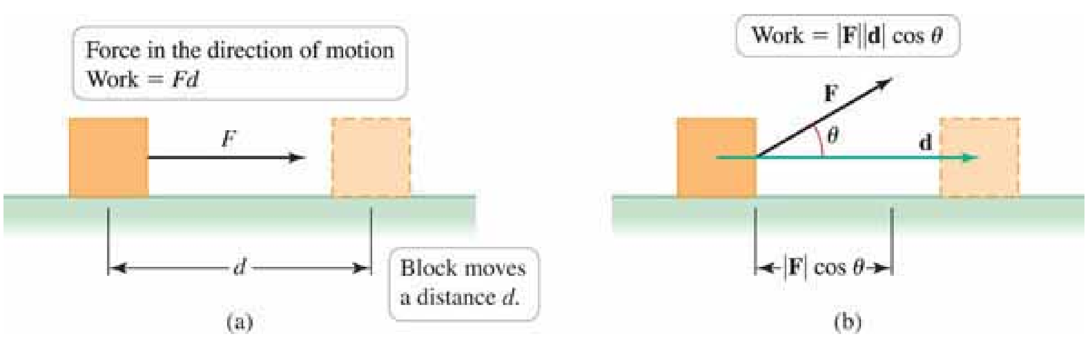
Definition. (Dot Product)
Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in two or three dimensions, their **dot product** is

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ with $0 \le \theta \le \pi$. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then $\mathbf{u}\cdot\mathbf{v} = 0$, and θ is undefined.



A physical example of the dot product is the amount of work done when a force is applied at an angle θ as shown in figure 13.43:



Note: The result of the dot product is a scalar!

Definition. (Orthogonal Vectors)
Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. The zero vector is orthogonal to all vectors. In two or three dimensions, two nonzero orthogonal vectors are perpendicular to each other.

- $\mathbf{u}$ and $\mathbf{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \cdot \mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$.
- $\mathbf{u}$ and $\mathbf{v}$ are perpendicular ($\theta = \frac{\pi}{2}$) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Example. Given $|\mathbf{u}| = 2$ and $|\mathbf{v}| = \sqrt{3}$, compute $\mathbf{u} \cdot \mathbf{v}$ when

- $\theta = \frac{\pi}{4}$
- $\theta = \frac{\pi}{3}$
- $\theta = \frac{5\pi}{6}$

Theorem 31.1: Dot Product
Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$,

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Example. Given vectors $\mathbf{u} = \left\langle \sqrt{3}, 1, 0 \right\rangle$ and $\mathbf{v} = \left\langle 1, \sqrt{3}, 0 \right\rangle$, compute $\mathbf{u} \cdot \mathbf{v}$ and find θ.

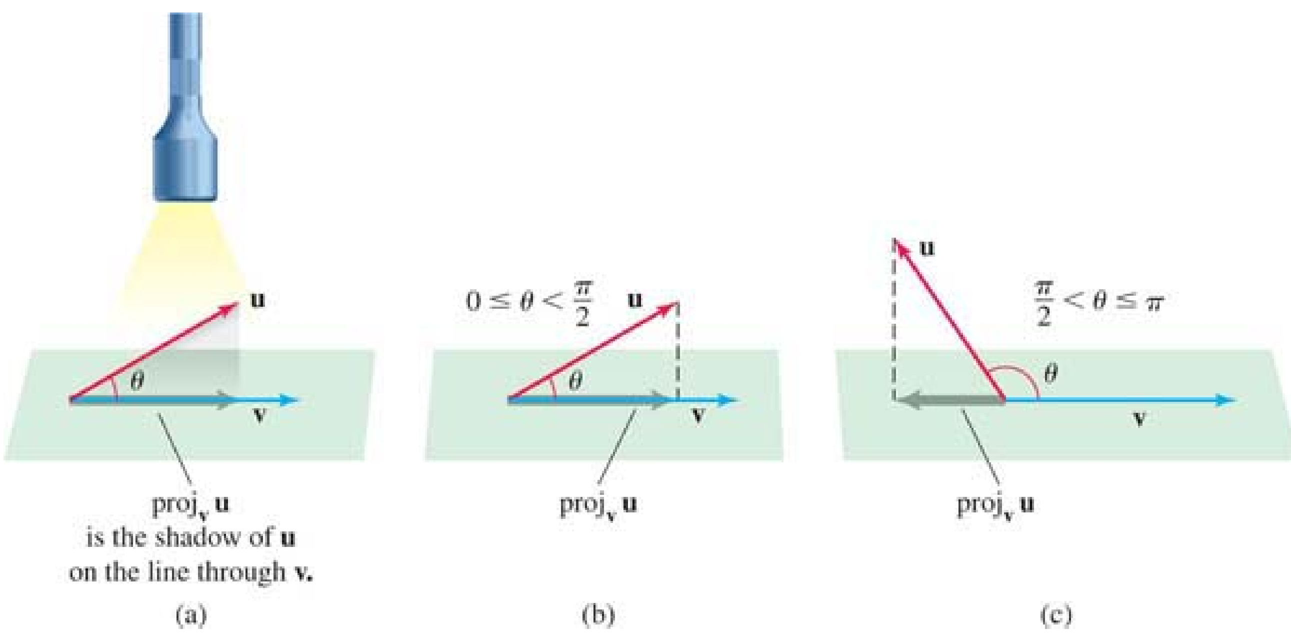
Properties of Dot Products

Theorem 13.2: Properties of the Dot Product
Suppose $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ — Commutative property
2. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$ — Associative property
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ — Distributive property

Orthogonal Projections

Given vectors $\mathbf{u}$ and $\mathbf{v}$, the projection of $\mathbf{u}$ onto $\mathbf{v}$ produces a vector parallel to $\mathbf{v}$ using the "shadow" of $\mathbf{u}$ cast onto $\mathbf{v}$.



Definition. ((Orthogonal) Projection of u onto v)
The **orthogonal projection of u onto v**, denoted $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$, where $\mathbf{v} \neq \mathbf{0}$, is

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \underbrace{|\mathbf{u}| \cos\theta}_{\text{length}} \underbrace{\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)}_{\text{direction}}.$$

The orthogonal projection may also be computed with the formulas

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \operatorname{scal}_{\mathbf{v}} \mathbf{u} \left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v},$$

where the **scalar component of u in the direction of v** is

$$\operatorname{scal}_{\mathbf{v}} \mathbf{u} = |\mathbf{u}| \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}.$$

Example. Find $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$ and $\operatorname{scal}_{\mathbf{v}} \mathbf{u}$ for the following:

- $\mathbf{u} = \langle 1, 1 \rangle$, $\mathbf{v} = \langle -2, 1 \rangle$

- $\mathbf{u} = \langle 7, 1, 7 \rangle$, $\mathbf{v} = \langle 5, 7, 0 \rangle$

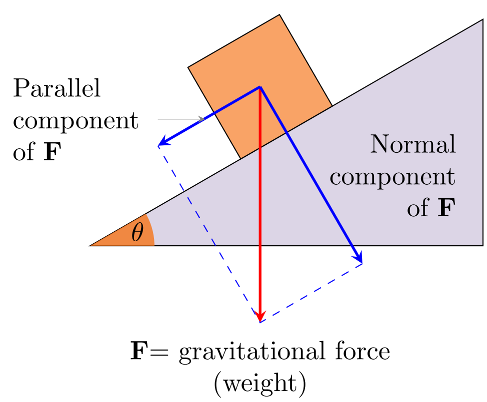
Applications of Dot Products

> **Definition. (Work)**
> Let a constant force $\mathbf{F}$ be applied to an object, producing a displacement $\mathbf{d}$. If the angle between $\mathbf{F}$ and $\mathbf{d}$ is θ, then the **work** done by the force is
> $$W = |\mathbf{F}||\mathbf{d}|\cos\theta = \mathbf{F}\cdot\mathbf{d}$$

Example. A force $\mathbf{F} = \langle 3,3,2\rangle$ (in newtons) moves an object along a line segment from $P(1,1,0)$ to $Q(6,6,0)$ (in meters). What is the work done by the force?

Components of a Force:

Example. A 10-lb block rests on a plane that is inclined at 30° above the horizontal. Find the components of the gravitational force parallel to and normal (perpendicular) to the plane.



13.4: Cross Products

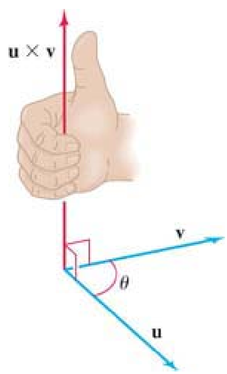
Definition. (Cross Product)
Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the **cross product** $\mathbf{u} \times \mathbf{v}$ is a vector with magnitude

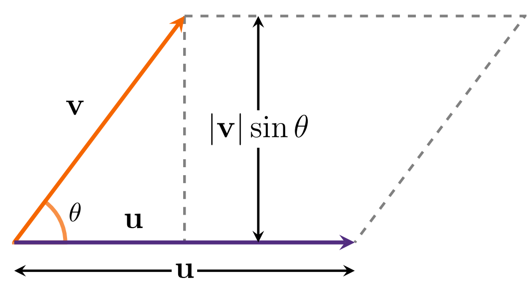
$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta,$$

where $0 \le \theta \le \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
The direction of $\mathbf{u} \times \mathbf{v}$ is given by the **right-hand rule**:

> When you put your the vectors tail to tail and let the fingers of your right hand curl from $\mathbf{u}$ to $\mathbf{v}$, the direction of $\mathbf{u} \times \mathbf{v}$ is the direction of your thumb, orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ (Figure 13.56).

When $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, the direction of $\mathbf{u} \times \mathbf{v}$ is undefined.





Theorem 13.3: Geometry of the Cross Product
Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in $\mathbb{R}^3$.

1. The vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
2. If $\mathbf{u}$ and $\mathbf{v}$ are two sides of a parallelogram, then the area of the parallelogram is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$$

Example. Consider the vectors $\mathbf{u} = \langle 2, 0, 0 \rangle$ and $\mathbf{v} = \left\langle \sqrt{3}, 3, 0 \right\rangle$. The angle between these vectors is $\theta = \frac{\pi}{3}$. Find the area of the parallelogram formed by these vectors.

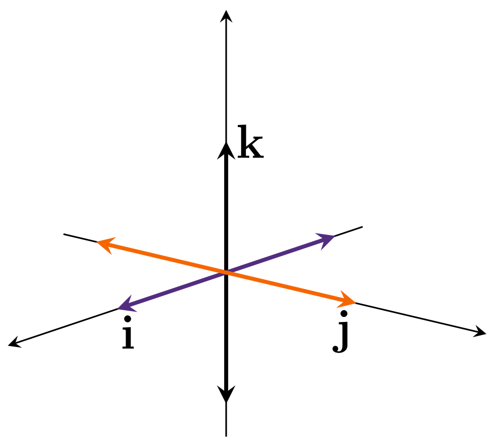
Theorem 13.4: Properties of the Cross Product

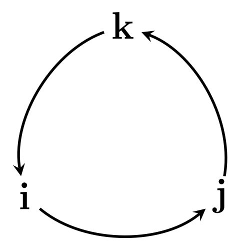
Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^3$, and let a and b be scalars.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ — Anticommutative property
2. $(a\mathbf{u}) \times (b\mathbf{v}) = ab(\mathbf{u} \times \mathbf{v})$ — Associative property
3. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ — Distributive property
4. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$ — Distributive property

Theorem 13.5: Cross Products of Coordinate Unit Vectors

$$\boldsymbol{i} \times \boldsymbol{j} = -(\boldsymbol{j} \times \boldsymbol{i}) = \boldsymbol{k} \qquad \boldsymbol{j} \times \boldsymbol{k} = -(\boldsymbol{k} \times \boldsymbol{j}) = \boldsymbol{i}$$
$$\boldsymbol{k} \times \boldsymbol{i} = -(\boldsymbol{i} \times \boldsymbol{k}) = \boldsymbol{j} \qquad \boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = \boldsymbol{0}$$





$\mathbf{i} \times \mathbf{j} = \mathbf{k}$
$\mathbf{j} \times \mathbf{k} = \mathbf{i}$
$\mathbf{k} \times \mathbf{i} = \mathbf{j}$

Using the unit vectors, we can compute $\mathbf{u} \times \mathbf{v}$:

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}) \times (v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}) \\
&= u_1v_1 \underbrace{(\boldsymbol{i} \times \boldsymbol{i})}_{\boldsymbol{0}} + u_1v_2 \underbrace{(\boldsymbol{i} \times \boldsymbol{j})}_{\boldsymbol{k}} + u_1v_3 \underbrace{(\boldsymbol{i} \times \boldsymbol{k})}_{-\boldsymbol{j}} \\
&+ u_2v_1 \underbrace{(\boldsymbol{j} \times \boldsymbol{i})}_{-\boldsymbol{k}} + u_2v_2 \underbrace{(\boldsymbol{j} \times \boldsymbol{j})}_{\boldsymbol{0}} + u_2v_3 \underbrace{(\boldsymbol{j} \times \boldsymbol{k})}_{\boldsymbol{i}} \\
&+ u_3v_1 \underbrace{(\boldsymbol{k} \times \boldsymbol{i})}_{\boldsymbol{j}} + u_3v_2 \underbrace{(\boldsymbol{k} \times \boldsymbol{j})}_{-\boldsymbol{i}} + u_3v_3 \underbrace{(\boldsymbol{k} \times \boldsymbol{k})}_{\boldsymbol{0}} \\
&= (u_2v_3 - u_3v_2)\boldsymbol{i} - (u_1v_3 - u_3v_1)\boldsymbol{j} + (u_1v_2 - u_2v_1)\boldsymbol{k}
\end{aligned}$$

Theorem 13.6: Evaluating the Cross Product
Let $\mathbf{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$ and $\mathbf{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$. Then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \boldsymbol{k}$$

Note:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\boldsymbol{i} - (u_1v_3 - u_3v_1)\boldsymbol{j} + (u_1v_2 - u_2v_1)\boldsymbol{k}$$

Alternative approach:

$$\left|\begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ u_1 & u_2 & u_3 & u_1 & u_2 \\ v_1 & v_2 & v_3 & v_1 & v_2 \end{array}\right.$$

Example. Compute $\mathbf{u} \times \mathbf{v}$ for $\mathbf{u} = \langle 3, 5, 4 \rangle$ and $\mathbf{v} = \langle 1, -1, 9 \rangle$.

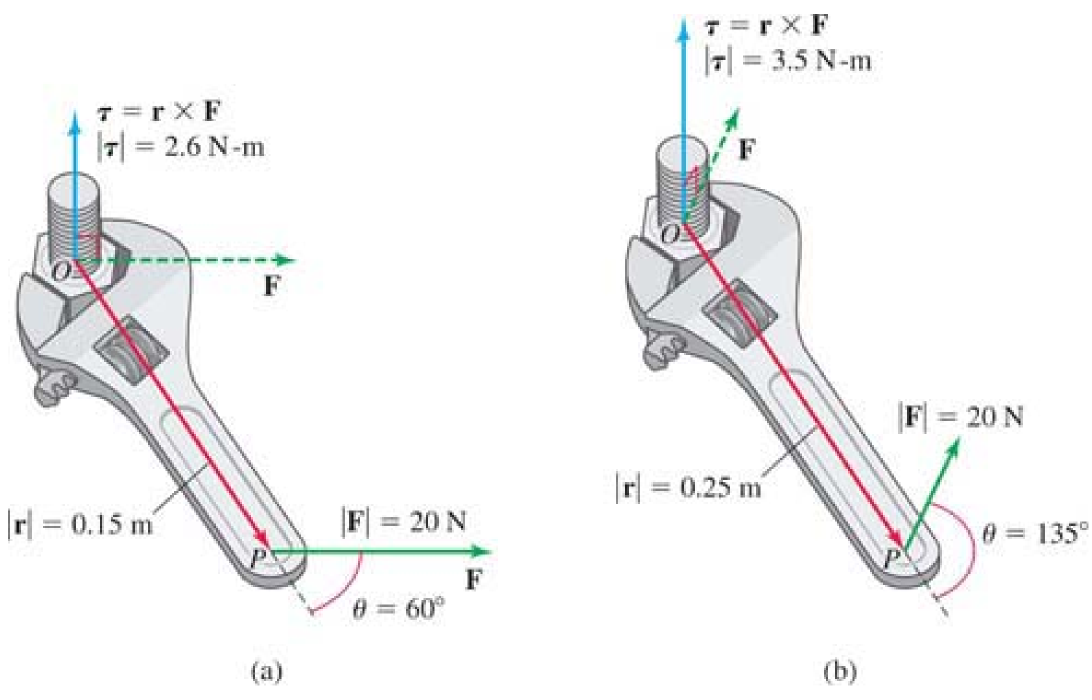
Example. Consider the vectors $\mathbf{u} = \left\langle \sqrt{3}, 1, 0 \right\rangle$ and $\mathbf{v} = \left\langle -\sqrt{3}, 1, 0 \right\rangle$. From the unit circle, we know the angle between these two vectors is $\theta = \frac{2\pi}{3}$. Use the definition of the cross product to show this.

Example. Find the area of the triangle formed by $\mathbf{u} = \langle 1, 2, 3 \rangle$ and $\mathbf{v} = \langle 3, -1, 1 \rangle$.

Example. Given a force $\mathbf{F}$ applied to a point P at the head of the vector $\mathbf{r} = \overrightarrow{OP}$, the **torque** produced at point O is given by $\tau = \mathbf{r} \times \mathbf{F}$ with magnitude

$$|\tau| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin \theta.$$

Now suppose a force of $20N$ is applied to a wrench attached to a bolt in a direction perpendicular to the bolt. Which produces more torque: applying the force at an angle of $60°$ on a wrench that is $0.15m$ long or applying the force at an angle of $135°$ on a wrench that is $0.25m$ long?

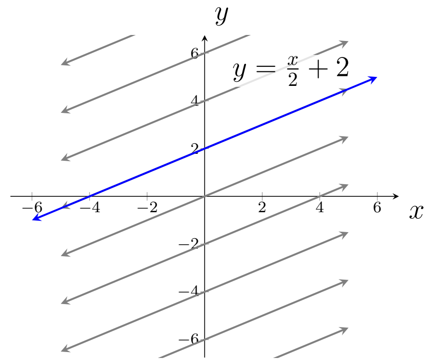


13.5: Lines and Planes in Space

Equation of a Line:

Recall the equation of a line in $\mathbb{R}^2$:

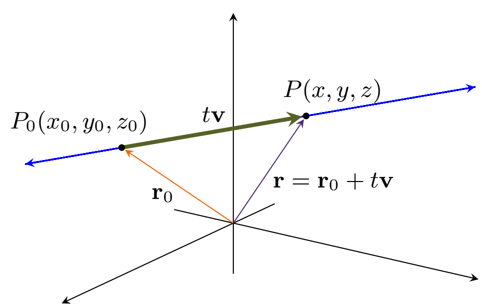
$$y = mx + b$$



where b is the intercept and m is the slope. This idea can be extended into higher dimensions:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

Here, $\mathbf{r}_0$ is a fixed point, and $\mathbf{v}$ is the position vector that is parallel to the line $\mathbf{r}$.



Equation of a Line

A **vector equation of the line** passing through the point $P_0(x_0, y_0, z_0)$ in the direction of the vector $\mathbf{v} = \langle a, b, c\rangle$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, or

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle, \quad \text{for} \quad -\infty < t < \infty$$

Equivalently, the corresponding **parametric equations of the line** are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \text{for} \quad -\infty < t < \infty$$

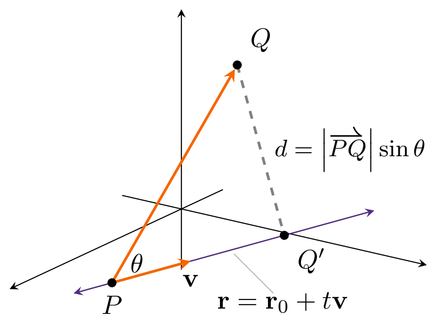
Example. Find the vector equation and parametric equation of the line that

- goes through the points $P(-1,-2,1)$ and $Q(-4,-5,-3)$ where $t=0$ corresponds to P,

- goes through the point $P(1,-3,-3)$ and is parallel to the vector $\mathbf{r} = \langle -4, 1, -1\rangle$,

- goes through the point $P(-2,5,-2)$ and is perpendicular to the lines $x = 3-4t$, $y = 2-3t$, $z = -1-t$, and $x = -2+0t$, $y = 2-t$, $z = 3t$, where $t=0$ corresponds to P.

Distance from a Point to a Line:
Given a point Q and a line ℓ, the shortest distance to the line is the length of $\overrightarrow{QQ'}$.



From the definition of the cross product, we have

$$\left|\mathbf{v} \times \overrightarrow{PQ}\right| = |\mathbf{v}| \underbrace{\left|\overrightarrow{PQ}\right| \sin\theta}_{d} = |\mathbf{v}|d$$

From here, solving for d gives us the following:

Distance Between a Point and a Line
The distance d between the point Q and the $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ is

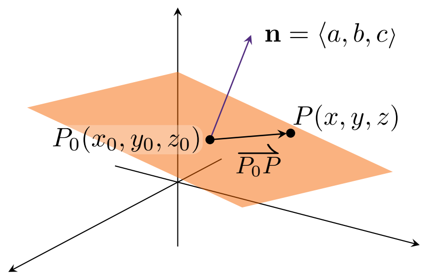
$$d = \frac{\left|\mathbf{v} \times \overrightarrow{PQ}\right|}{|\mathbf{v}|},$$

where P is any point on the line and $\mathbf{v}$ is a vector parallel to the line.

Example. Find the distance from the point $Q(-4, -1, -3)$ and the line $x = -5 - 5t$, $y = -5 + t$, $z = -1 + 4t$. (*Hint:* Let P be the point at $t = 0$)

Equations of Planes:

In $\mathbb{R}^2$, two distinct points determine a line.
In $\mathbb{R}^3$, three noncollinear points determine a unique plane. Alternatively, a plane is uniquely determined by a point and a vector that is orthogonal to the plane.



> **Definition. (Plane in $\mathbb{R}^3$)**
> Given a fixed point P_0 and a nonzero **normal vector n**, the set of points P in $\mathbb{R}^3$ for which $\overrightarrow{P_0P}$ is orthogonal to **n** is called a **plane**.

Consider the normal vector $\mathbf{n} = \langle a, b, c \rangle$ at the point $P_0(x_0, y_0, z_0)$, and any point $P(x, y, z)$ on the plane. Since **n** is orthogonal to the plane, it is also orthogonal to the vector $\overrightarrow{P_0P}$, which is also in the plane. Thus,

$$\begin{aligned}
\mathbf{n} \cdot \overrightarrow{P_0P} &= 0 \\
\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 \\
a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \\
ax + by + cz &= d
\end{aligned}$$

> **General Equation of a Plane in $\mathbb{R}^3$**
> The plane passing through the point $P_0(x_0, y_0, z_0)$ with a nonzero normal vector $\mathbf{n} = \langle a, b, c \rangle$ is described by the equation
>
> $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{or} \quad ax + by + cz = d,$$
>
> where $d = ax_0 + by_0 + cz_0$.

Example. Find the equation of the plane that

- goes through the point $P(-2, 5, 0)$ and is parallel to the plane $x - 5y - 5z = 1$,

- goes through the points $P(5, -2, 1)$, $Q(5, 1, 3)$ and $R(1, -5, -2)$

- that is parallel to the vectors $\langle 4, -2, -3\rangle$ and $\langle 3, 2, 3\rangle$, passing through the point $P(-2, -2, 5)$.

Example. Find the location where the line $\langle -3, 1, 4\rangle + t\langle -1, -4, 2\rangle$ and the plane $2x - 2y - 4z = 5$ intersect.

> **Definition. (Parallel and Orthogonal Planes)**
> Two distinct planes are **parallel** if their respective normal vectors are parallel (that is, the normal vectors are scaling multiples of each other). Two planes are **orthogonal** if their respective normal vectors are orthogonal (that is, the dot product of the normal vectors is *zero*).

Example. Find the line of intersection between the planes $3x - y + 4z = -4$ and $x + 3y - 2z = 0$.

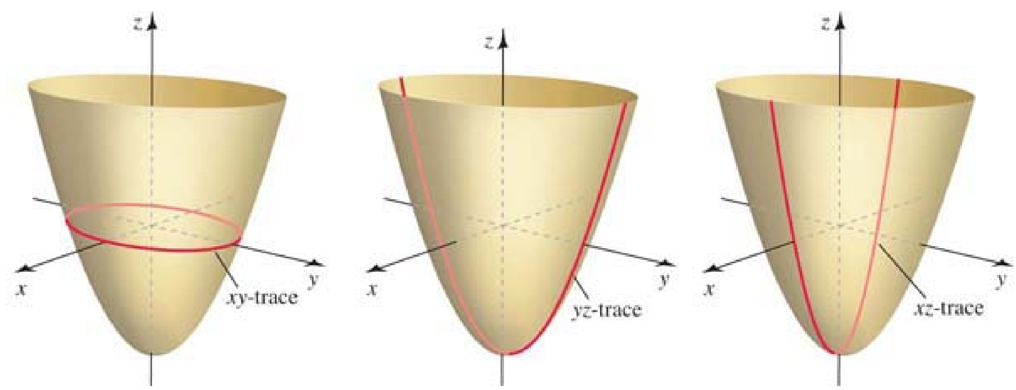
Example. Find the smallest angle between planes $3x - y + 4z = -4$ and $x + 3y - 2z = 0$.

13.6: Cylinders and Quadric Surfaces

Cylinders and Traces:

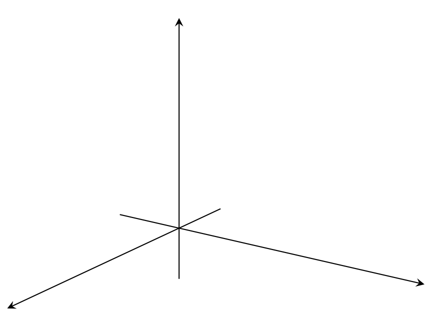
When talking about three-dimensional surfaces, a *cylinder* refers to a surface that is parallel to a line. When considering surfaces that is parallel to one of the coordinate axes, that the associated variable is missing (e.g. $3y^2 + z^2 = 8$ is parallel to the x-axis).

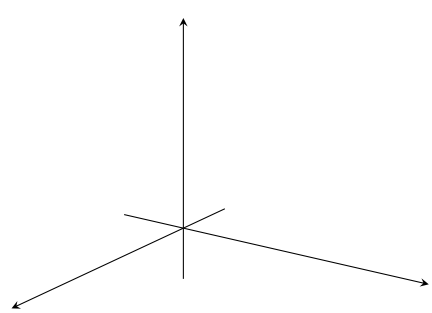
> **Definition. (Trace)**
> A **trace** of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes. The traces in the coordinate planes are called the xy**-trace**, the yz**-trace**, and the xz**-trace** (Figure 13.80).



Example. Roughly sketch the following functions:

1. $x^2 + 4y^2 = 16$
2. $x - \sin(z) = 0$





Quadric Surfaces:

Quadric surfaces are described by the general quadratic (second-degree) equation in three variables,

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0,$$

Where the coefficients $A, \ldots, J$ and not all zero. To sketch quadric surfaces, keep the following ideas in mind:

1. **Intercepts** Determine the points, if any, where the surface intersects the coordinate axes. To find these intercepts, set x, y, and z equal to zero in pairs in the equation of the surface, and solve for the third coordinate.

2. **Traces** Finding traces of the surface helps visualize the surface. Setting x, y, and z equal to zero in pairs gives the planes parallel in that pair's plane.

3. **Completing the figure** Sketch some traces in parallel planes, then draw smooth curves that pass through the traces to fill out the surface.

Example (An ellipsoid)**.** The surface defined by the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. Graph $a = 3$, $b = 4$ and $c = 5$.

Example (An elliptic parabaloid)**.** The surface defined by the equation $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$. Graph the elliptic paraboloid with $a = 4$ and $b = 2$.

Example (A hyperboloid of one sheet).

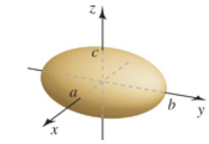
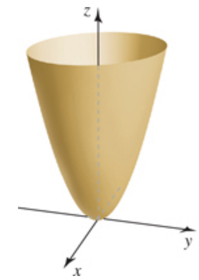
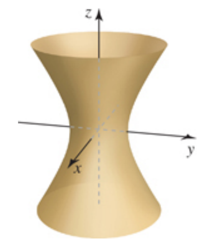
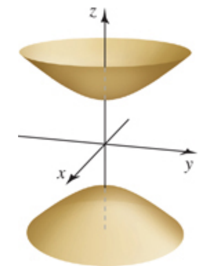
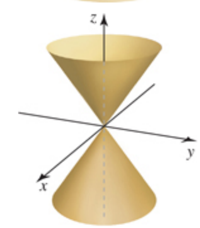
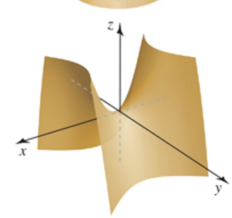
Graph the surface defined by the equation $\dfrac{x^2}{4} + \dfrac{y^2}{9} - z^2 = 1$.

Example (A hyperboloid of two sheets)**.** Graph the surface defined by the equation $-16x^2 - 4y^2 + z^2 + 64x - 80 = 0$.

Example (Elliptic cones)**.** Graph the surface defined by the equation $\dfrac{y^2}{4} + z^2 = 4x^2$.

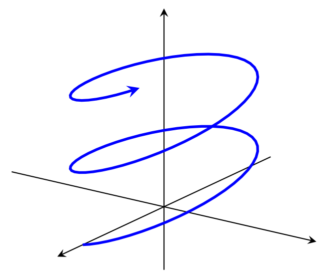
Example (A hyperbolic paraboloid)**.**

Graph the surface defined by the equation $z = x^2 - \dfrac{y^2}{4}$.

Name	Standard Equation	Features	Graph
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	All traces are ellipses.	
Elliptic paraboloid	$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$	Traces with $z=z_0>0$ are ellipses. Traces with $x=x_0$ or $y=y_0$ are parabolas.	
Hyperboloid of one sheet	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	Traces with $z=z_0$ are ellipses for all z_0. Traces with $x=x_0$ or $y=y_0$ are hyperbolas.	
Hyperboloid of two sheets	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	Traces with $z=z_0$ with $\lvert z_0\rvert>\lvert c\rvert$ are ellipses. Traces with $x=x_0$ and $y=y_0$ are hyperbolas.	
Elliptic cone	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$	Traces with $z=z_0\neq 0$ are ellipses. Traces with $x=x_0$ or $y=y_0$ are hyperbolas or intersecting lines.	
Hyperbolic paraboloid	$z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$	Traces with $z=z_0\neq 0$ are hyperbolas. Traces with $x=x_0$ or $y=y_0$ are parabolas.	

14.1: Vector-Valued Functions

Vector-valued functions are functions of the form $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, where $x(t)$, $y(t)$, and $z(t)$ are parametric equations dependent on t.



Curves in Space

Consider

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k},$$

where f, g, and h are defined for $a \leq t \leq b$. The **domain** of $\mathbf{r}$ is the largest set of t for which all of f, g, and h are defined.

Example. What plane does the curve $\mathbf{r}(t) = t\boldsymbol{i} + 6t^3\boldsymbol{k}$ lie?

Example (Lines as vector-valued functions)**.** Find a vector function for the line that passes through the points $P(5, 2, -4)$ and $Q(5, 5, -2)$. What about the line segment that connects P and Q?

Example. Find the domain of

$$\mathbf{r}(t) = \sqrt{16 - t^2}\boldsymbol{i} + \sqrt{t}\boldsymbol{j} + \frac{4}{\sqrt{3+t}}\boldsymbol{k}$$

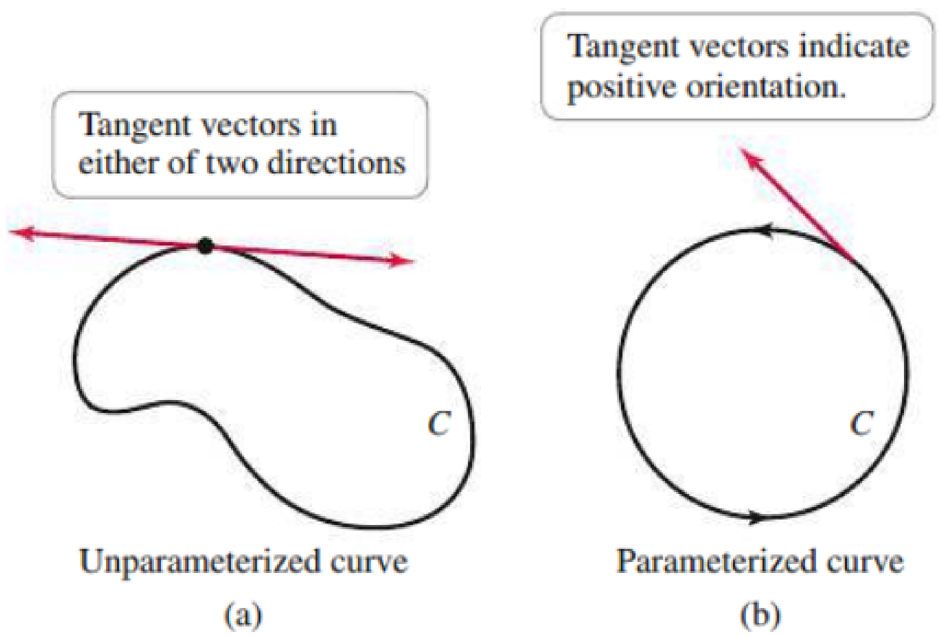
Example. Find the point P on

$$\mathbf{r}(t) = t^2\boldsymbol{i} + 2t\boldsymbol{j} + 2t\boldsymbol{k},$$

closest to $P_0(4, 17, 10)$. What is the distance between P and P_0?

Orientation of Curves

- A **unparameterized curve** is a smooth curve C with no specified direction and the tangent vector can be drawn in two directions.
- A **parameterized curve** is a smooth curve C described by a function $\mathbf{r}(t)$ for $a \leq t \leq b$ and has a direction referred to as its **orientation**.
- The *positive* orientation is the direction of the curve generated when t increases from a to b.
- The tangent vector of a parameterized curve points in the positive orientation of the curve.



Example. Graph the curve described by the equation

$$\mathbf{r}(t) = 4\cos(t)\boldsymbol{i} + \sin(t)\boldsymbol{j} + \frac{t}{2\pi}\boldsymbol{k},$$

where $0 \leq t \leq 2\pi$. Indicate the positive orientation of this curve.

Limits and Continuity for Vector-Valued Functions

The properties of limits extend to vector-valued functions naturally. In particular, for $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, if

$$\lim_{t\to a} f(t) = L_1, \qquad \lim_{t\to a} g(t) = L_2, \qquad \lim_{t\to a} h(t) = L_3$$

then

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} f(t),\ \lim_{t\to a} g(t),\ \lim_{t\to a} h(t) \right\rangle = \langle L_1, L_2, L_3\rangle.$$

> **Definition. (Limit of a Vector-Valued Function)**
> A vector-valued function $\mathbf{r}$ approaches the limit $\mathbf{L}$ as t approaches a, written $\lim\limits_{t\to a} \mathbf{r}(t) = \mathbf{L}$, provided $\lim\limits_{t\to a} |\mathbf{r}(t) - \mathbf{L}| = 0$.
>
> A function $\mathbf{r}(t)$ is **continuous** at $t = a$, provided $\lim\limits_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$.

Example. Evaluate the following limits:

$$\lim_{t\to \pi} \left(\cos(t)\boldsymbol{i} - 7\sin\left(-\frac{t}{2}\right)\boldsymbol{j} + \frac{t}{\pi}\boldsymbol{k} \right)$$

$$\lim_{t\to \infty} \left(\frac{t}{t-3}\boldsymbol{i} + \frac{40}{1+19e^{-t}}\boldsymbol{j} + \frac{1}{2t}\boldsymbol{k} \right)$$

14.2: Calculus of Vector-Valued Functions

Definition. (Derivative and Tangent Vector)
Let $\mathbf{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$, where f,g, and h are differentiable functions on (a, b). Then $\mathbf{r}$ has a **derivative** (or is **differentiable**) on (a, b) and

$$\mathbf{r}'(t) = f'(t)\boldsymbol{i} + g'(t)\boldsymbol{j} + h'(t)\boldsymbol{k}.$$

Provided $\mathbf{r}'(t) \neq \mathbf{0}$, $\mathbf{r}'(t)$ is a **tangent vector** at the point corresponding to $\mathbf{r}(t)$.

Example. For the following functions below, find $\mathbf{r}'(t)$

a) $\mathbf{r}(t) = \left\langle e^{-t^2},\, \log_2(t-4),\, \sin(t) \right\rangle$

b) $\mathbf{r}(t) = 3\boldsymbol{i} - 2\tan(t)\boldsymbol{j} + e^t\boldsymbol{k}$

Example. For $\mathbf{r}(t) = \langle 3t,\, \sec(2t),\, \cos(t) \rangle$ compute $\mathbf{r}'(t)$ at $t = \frac{\pi}{4}$.

Definition. (Unit Tangent Vector)
Let $\mathbf{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$ be a smooth parameterized curve, for $a \le t \le b$. The **unit tangent vector** for a particular value of t is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Example. For $\mathbf{r}(t) = \langle 3\sin(t),\, -2\cos(2t),\, 3\cos(t)\rangle$, find the unit tangent vector.

Example. For $\mathbf{r}(t) = \langle \sin(6t),\, 3t,\, \cos(3t)\rangle$, compute $\mathbf{T}\left(\frac{\pi}{3}\right)$.

Derivative Rules

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector-valued functions, and let f be a differentiable scalar-valued function, all at a point t. Let $\mathbf{c}$ be a constant vector. The following rules apply.

1. $\dfrac{d}{dt}(\mathbf{c}) = \mathbf{0}$ — Constant Rule
2. $\dfrac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$ — Sum Rule
3. $\dfrac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ — Product Rule
4. $\dfrac{d}{dt}(\mathbf{u}(f(t))) = \mathbf{u}'(f(t))f'(t)$ — Chain Rule
5. $\dfrac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ — Dot Product Rule
6. $\dfrac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ — Cross Product Rule

Example. Given $\mathbf{u}(t) = \left\langle 4t^2,\, 1,\, 3t \right\rangle$ and $\mathbf{v}(t) = \left\langle e^{-2t},\, -2e^t,\, e^t \right\rangle$, find $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$.

Definition. (Indefinite Integral of a Vector-Valued Function)
Let $\mathbf{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$ be a vector function, and let
$\mathbf{R}(t) = F(t)\boldsymbol{i} + G(t)\boldsymbol{j} + H(t)\boldsymbol{k}$, where F, G, and H are antiderivatives of f, g, and h, respectively. The **indefinate integral** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\, dt = \mathbf{R}(t) + \mathbf{C},$$

where $\mathbf{C}$ is an arbitrary constant vector. Alternatively, in component form,

$$\int \langle f(t),\, g(t),\, h(t)\rangle\, dt = \langle F(t),\, G(t),\, H(t)\rangle + \langle C_1,\, C_2,\, C_3\rangle.$$

Example. Find $\mathbf{r}(t)$ such that $\mathbf{r}'(t) = \left\langle \frac{t}{t^2+1},\, t^2e^{-t^3},\, \frac{-2t}{\sqrt{t^2+16}} \right\rangle$ and $\mathbf{r}(0) = \left\langle 3,\, \frac{5}{3},\, -5 \right\rangle$.

Definition. (Definite Integral of a Vector-Valued Function)
Let $\mathbf{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$, where f, g, and h are integrable on the interval $[a, b]$. The **definite integral** of $\mathbf{r}$ on $[a, b]$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\boldsymbol{i} + \left(\int_a^b g(t)\,dt\right)\boldsymbol{j} + \left(\int_a^b h(t)\,dt\right)\boldsymbol{k}$$

Example. $\displaystyle\int_{-\pi}^{\pi} \langle \sin(t),\, \cos(t),\, 8t\rangle\,dt$

14.3: Motion in Space

Definition.
Let the **position** of an object moving in three-dimensional space be given by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, for $t \geq 0$. The **velocity** of the object is

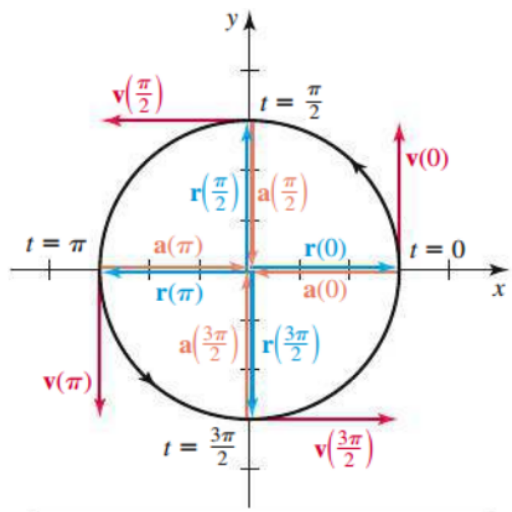
$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle.$$

The **speed** of the object is the scalar function

$$|\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

The **acceleration** of the object is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

Example. Given $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t)\rangle$ for $0 \leq t \leq 2\pi$, find the velocity, speed, and acceleration.



Circular motion: At all times $\mathbf{a}(t) = -\mathbf{r}(t)$ and $\mathbf{v}(t)$ is orthogonal to $\mathbf{r}(t)$ and $\mathbf{a}(t)$.

Theorem 14.2: Motion with constant $|\mathbf{r}|$
Let $\mathbf{r}$ describe a path on which $|\mathbf{r}|$ is constant (motion on a circle or sphere centered at the origin). Then $\mathbf{r} \cdot \mathbf{v} = 0$, which means the position vector and the velocity vector are orthogonal at all times for which the functions are defined.

Example (Path on a sphere)**.** Consider

$$\mathbf{r}(t) = \langle 3\cos(t), 5\sin(t), 4\cos(t)\rangle, \quad \text{for } 0 \le t \le 2\pi.$$

a) Show that an object with this trajectory moves on a sphere and find the radius.

b) Find the velocity and speed of the above trajectory.

c) Show that $\mathbf{r}(t) = \langle 5\cos(t), 5\sin(t), 5\sin(2t)\rangle$ does not lie on a sphere. How could this function be modified so that it does lie on a sphere?

Example. Given $\mathbf{a}(t) = \langle \cos(t), 4\sin(t) \rangle$, with an initial velocity $\langle \mathbf{u}_0, \mathbf{v}_0 \rangle = \langle 0, 4 \rangle$ and an initial position $\langle x_0, y_0 \rangle = \langle 5, 0 \rangle$ where $t \geq 0$, find the velocity and position vector.

Summary: Two-Dimensional Motion in a Gravitational Field
Consider an object moving in a plane with a horizontal x-axis and a vertical y-axis, subject only to the force of gravity. Given the initial velocity $\mathbf{v}(0) = \langle u_0, v_0 \rangle$ and the initial position $\mathbf{r}(0) = \langle x_0, y_0 \rangle$, the velocity of the object, for $t \geq 0$, is

$$\mathbf{v}(t) = \langle x'(t), y'(t) \rangle = \langle u_0, -gt + v_0 \rangle$$

and the position is

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \left\langle u_0 t + x_0, -\frac{1}{2}gt^2 + v_0 t + y_0 \right\rangle.$$

Example. Consider a ball with an initial position of $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$ m and an initial velocity of $\langle u_0, v_0 \rangle = \langle 25, 4 \rangle$ m/s.

a) Find the position and velocity of the ball while it is in the air

Summary: Two-Dimensional Motion
Assume an object traveling over horizontal ground, acted on only by the gravitational force, has an initial position $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$ and initial velocity $\langle u_0, v_0 \rangle = \langle |\mathbf{v}_0| \cos \alpha, |\mathbf{v}_0| \sin \alpha \rangle$. The trajectory, which is a segment of a parabola, has the following properties.

$$\text{time of flight} = T = \frac{2|\mathbf{v}_0| \sin \alpha}{g}$$

$$\text{range} = \frac{|\mathbf{v}_0|^2 \sin(2\alpha)}{g}$$

$$\text{maximum height} = y\left(\frac{T}{2}\right) = \frac{(|\mathbf{v}_0| \sin \alpha)^2}{2g}$$

Example. Consider a ball with an initial position of $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$ m and an initial velocity of $\langle u_0, v_0 \rangle = \langle 25, 4 \rangle$ m/s. Assuming the ground is flat and level:

b) How long is the ball in the air?

c) How far does the ball travel horizontally?

d) What is the maximum height that the ball reaches?

14.4: Length of Curves

Definition. (Arc Length for Vector Functions)
Consider the parameterized curve $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, where f', g', and h' are continuous, and the curve is traversed once for $a \le t \le b$. The **arc length** of the curve between $(f(a), g(a), h(a))$ and $(f(b), g(b), h(b))$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt = \int_a^b |\mathbf{r}'(t)|\, dt.$$

Example (Flight of an eagle)**.** Suppose an eagle rises at a rate of 100 vertical ft/min on a helical path given by

$$\mathbf{r}(t) = \langle 250\cos(t),\ 250\sin(t),\ 100t\rangle$$

where $\mathbf{r}$ is measured in feet and t is measured in minutes. How far does it travel in 10 minutes?

Example. Suppose a particle has a trajectory given by

$$\mathbf{r}(t) = \langle 10\cos(3t),\, 10\sin(3t)\rangle$$

where $0 \leq t \leq \pi$. How far does this particle travel?

Example. Find the length of the curve

$$\mathbf{r}(t) = \left\langle 3t^2 - 5,\, 4t^2 + 5\right\rangle$$

where $0 \leq t \leq 1$.

Example. Find the length of $\mathbf{r}(t) = \left\langle t^2, \dfrac{(4t+1)^{\frac{3}{2}}}{6} \right\rangle$ where $0 \leq t \leq 6$.

Example. Find the length of $\mathbf{r}(t) = \langle 2\sqrt{2}, \sin(t), \cos(t) \rangle$ where $0 \leq t \leq 5$.

Theorem 14.3: Arc Length as a Function of a Parameter
Let $\mathbf{r}(t)$ describe a smooth curve, for $t \geq a$. The arc length is given by

$$s(t) = \int_a^t |\mathbf{v}(u)|\, du,$$

where $|\mathbf{v}| = |\mathbf{r}'|$. Equivalently, $\dfrac{ds}{dt} = |\mathbf{v}(t)|$. If $|\mathbf{v}(t)| = 1$, for all $t \geq a$, then the parameter t corresponds to arc length.

Example. For the following functions, determine if $\mathbf{r}(t)$ uses arc length as a parameter. If not, find a description that uses arc length as a parameter.

a) $\mathbf{r}(t) = \langle -4t+1, 3t-1 \rangle$, $0 \leq t \leq 4$.

b) $\mathbf{r}(t) = \left\langle \frac{1}{\sqrt{10}} \cos(t),\ \frac{3}{\sqrt{10}} \cos(t),\ \sin(t) \right\rangle$, $0 \leq t \leq 2\pi$.

14.5: Curvature and Normal Vectors:

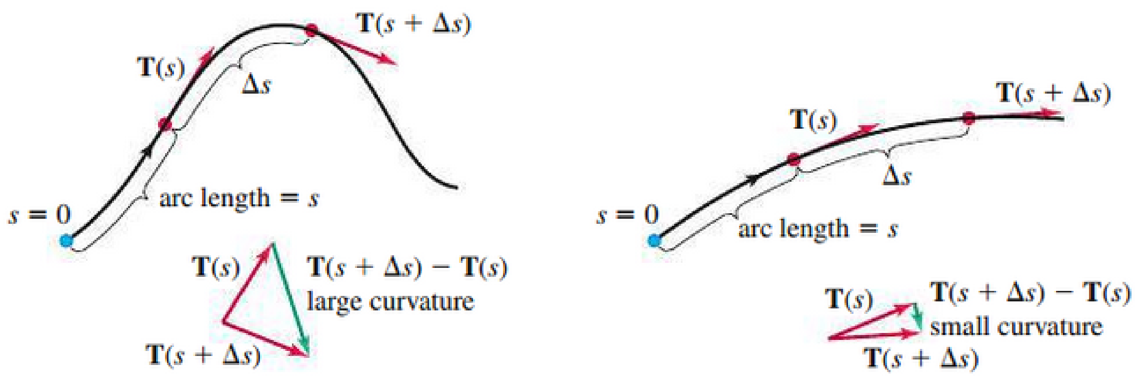
There are two ways to change the velocity, or in other words, to accelerate:

- change in speed
- change in direction

The change in direction is referred to as *curvature*. Recall that if we have a smooth curve $\mathbf{r}(t)$, the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

Specifically, *curvature* of the curve is the magnitude of the rate at which $\mathbf{T}$ changes with respect to arc length.



Definition. (Curvature)
Let $\mathbf{r}$ describe a smooth parameterized curve. If s denotes arc length and $\mathbf{T} = \mathbf{r}'/|\mathbf{r}'|$ is the unit tangent vector, the **curvature** is $\kappa(s) = \left|\frac{d\mathbf{T}}{ds}\right|$.

Theorem 14.4: Curvature Formula
Let $\mathbf{r}(t)$ describe a smooth parameterized curve, where t is any parameter. If $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{T}$ is the unit tangent vector, then the curvature is

$$\kappa(t) = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

- κ is a non-negative scalar-valued function
- Curvature of zero corresponds to a straight line
- A relatively flat curve has a small curvature
- A tight curve has a larger curvature

Example. Consider the line

$$\mathbf{r}(t) = \langle x_0 + at,\, y_0 + bt,\, z_0 + ct\rangle, \text{ for } -\infty < t < \infty.$$

Compute κ.

Example. Consider the circle

$$\mathbf{r}(t) = \langle R\cos(t),\ R\sin(t)\rangle$$

for $0 \leq t \leq 2\pi$, where $R > 0$. Show that $\kappa = 1/R$.

Example. Consider the curve

$$\mathbf{r}(t) = \left\langle 2\cos(t),\ 2\sin(t),\ \sqrt{5}t \right\rangle$$

Compute κ.

An Alternative Curvature Formula:
Consider a smooth function $\mathbf{r}(t)$ with non-zero velocity $\mathbf{v}(t) = \mathbf{r}'(t)$ and non-zero acceleration $\mathbf{a}(t) = \mathbf{v}'(t)$.

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} \;\Rightarrow\; \mathbf{v} = |\mathbf{v}|\,\mathbf{T}.$$

Thus

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}[|\mathbf{v}|\,\mathbf{T}] = \frac{d}{dt}[|\mathbf{v}|]\,\mathbf{T} + |\mathbf{v}|\,\frac{d\mathbf{T}}{dt}.$$

Now we form $\mathbf{v} \times \mathbf{a}$:

$$\begin{aligned}\mathbf{v} \times \mathbf{a} &= |\mathbf{v}|\,\mathbf{T} \times \left(\frac{d}{dt}[|\mathbf{v}|]\,\mathbf{T} + |\mathbf{v}|\,\frac{d\mathbf{T}}{dt}\right) \\ &= \underbrace{|\mathbf{v}|\,\mathbf{T} \times \frac{d}{dt}[|\mathbf{v}|]\,\mathbf{T}}_{\mathbf{0}} + |\mathbf{v}|\,\mathbf{T} \times |\mathbf{v}|\,\frac{d\mathbf{T}}{dt}\end{aligned}$$

Since $\mathbf{T}$ is a unit vector, $\mathbf{T}$ and $d\mathbf{T}/dt$ are orthogonal (Theorem 14.2). Thus

$$|\mathbf{v} \times \mathbf{a}| = \left| |\mathbf{v}|\,\mathbf{T} \times |\mathbf{v}|\,\frac{d\mathbf{T}}{dt}\right| = |\mathbf{v}|\,\underbrace{|\mathbf{T}|}_{1}\,\left||\mathbf{v}|\frac{d\mathbf{T}}{dt}\right|\underbrace{\sin\theta}_{1} = |\mathbf{v}|^2\left|\frac{d\mathbf{T}}{dt}\right|$$

Now, using Theorem 14.4, where $\left|\dfrac{d\mathbf{T}}{dt}\right| = \kappa|\mathbf{v}|$, we have

$$|\mathbf{v} \times \mathbf{a}| = |\mathbf{v}|^2\left|\frac{d\mathbf{T}}{dt}\right| = |\mathbf{v}|^2\,\kappa|\mathbf{v}| = \kappa|\mathbf{v}|^3.$$

Theorem 14.5: Alternative Curvature Formula
Let $\mathbf{r}$ be the position of an object moving on a smooth curve. The **curvature** at a point on the curve is

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3},$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{a} = \mathbf{v}'$ is the acceleration.

Example. Consider the curve

$$\mathbf{r}(t) = \langle -16\cos(t),\, 16\sin(t),\, 0\rangle.$$

Compute the curvature κ using both methods.

Principal Unit Normal Vector

Curvature indicates how quickly a curve turns. The principal unit normal vector determines the *direction* in which a curve turns.

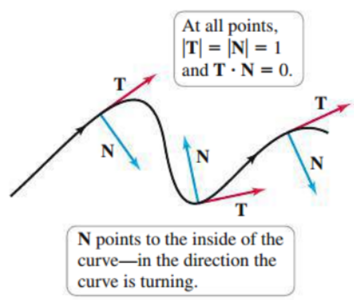
Definition. (Principal Unit Normal Vector)
Let $\mathbf{r}$ describe a smooth curve parameterized by arc length. The **principal unit normal vector** at a point P on the curve at which $\kappa \neq 0$ is

$$\mathbf{N}(s) = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

For other parameters, we use the equivalent formula

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|},$$

evaluated at the value of t corresponding to P.



Theorem 14.6: Properties of the Principal Unit Normal Vector
Let $\mathbf{r}$ describe a smooth parameterized curve with unit tangent vector $\mathbf{T}$ and principal unit normal vector $\mathbf{N}$.

1. $\mathbf{T}$ and $\mathbf{N}$ are orthogonal at all points of the curve; that is, $\mathbf{T} \cdot \mathbf{N} = 0$ at all points where $\mathbf{N}$ is defined.
2. The principal unit normal vector points to the inside of the curve – in the direction that the curve is turning.

Example. For the curve $\mathbf{r}(t) = \langle a\cos(t),\, a\sin(t),\, bt\rangle$, find the unit tangent vector $\mathbf{T}$ and the principal unit normal vector $\mathbf{N}$. Verify $|\mathbf{T}| = |\mathbf{N}| = 1$ and $\mathbf{T}\cdot\mathbf{N} = 0$.

Components of the Acceleration

Recall that the change in velocity, or acceleration, of an object can change in *speed* (in the direction of $\mathbf{T}$) and in *direction* (in the direction of $\mathbf{N}$). $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|} \Longrightarrow \mathbf{v} = \mathbf{T}|\mathbf{v}| = \mathbf{T}\,\dfrac{ds}{dt}$.

$$\begin{aligned}
\mathbf{a} = \frac{d\mathbf{v}}{dt} &= \frac{d}{dt}\left(\mathbf{T}\frac{ds}{dt}\right) \\
&= \frac{d\mathbf{T}}{dt}\,\frac{ds}{dt} + \mathbf{T}\frac{d^2s}{dt^2} \\
&= \underbrace{\frac{d\mathbf{T}}{ds}}_{\kappa\mathbf{N}}\underbrace{\frac{ds}{dt}}_{|\mathbf{v}|}\underbrace{\frac{ds}{dt}}_{|\mathbf{v}|} + \mathbf{T}\frac{d^2s}{dt^2} \\
&= \kappa|\mathbf{v}|^2\,\mathbf{N} + \frac{d^2s}{dt^2}\,\mathbf{T}.
\end{aligned}$$

Theorem 14.7: Tangential and Normal Components of the Acceleration

The acceleration vector of an object moving in space along a smooth curve has the following representation in terms of its **tangential component** a_T (in the direction of $\mathbf{T}$) and its **normal component** a_N (in the direction of $\mathbf{N}$):

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T},$$

where $a_N = \kappa|\mathbf{v}|^2 = \dfrac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|}$ and $a_T = \dfrac{d^2s}{dt^2}$.

Example. Consider the function

$$\mathbf{r}(t) = \langle -2t+2,\ -2t+3,\ -2t+2 \rangle.$$

Find the tangential and normal components of the acceleration.

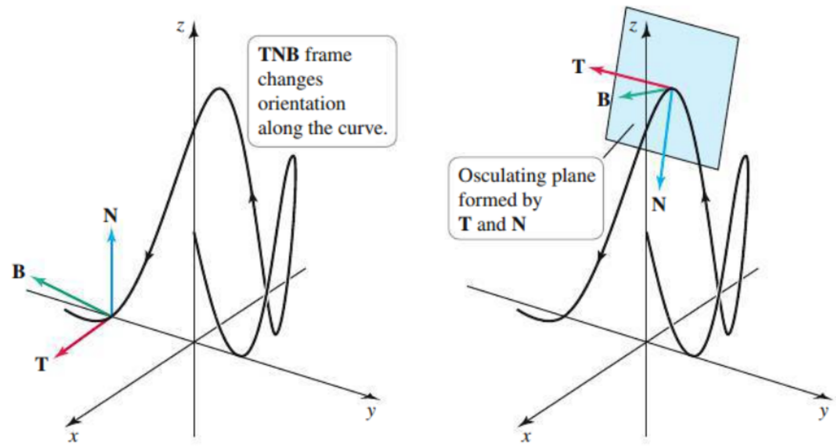
Example. Find the components of the acceleration on the circular trajectory

$$\mathbf{r}(t) = \langle R\cos(\omega t),\ R\sin(\omega t) \rangle.$$

Example. The driver of a car follows the parabolic trajectory $\mathbf{r}(t) = \left\langle t,\, t^2 \right\rangle$, for $-2 \le t \le 2$, through a sharp bend. Find the tangential and normal components of the acceleration of the car.

The Binormal Vector and Torsion

On a smooth parameterized curve C, $\mathbf{T}$ and $\mathbf{N}$ determine a plane called the *osculating plane.*



The coordinate system defined by these vectors is called the **TNB frame**. The rate at which the curve C twists out of the plane is the rate at which $\mathbf{B}$ changes as we move along C, which is $\dfrac{d\mathbf{B}}{ds}$.

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T}\times\mathbf{N}) = \underbrace{\frac{d\mathbf{T}}{ds}\times\mathbf{N}}_{\mathbf{0}} + \mathbf{T}\times\frac{d\mathbf{N}}{ds} = \mathbf{T}\times\frac{d\mathbf{N}}{ds}$$

$\dfrac{d\mathbf{B}}{ds}$ is:

- orthogonal to both $\mathbf{T}$ and $\dfrac{d\mathbf{N}}{ds}$,
- orthogonal to $\mathbf{B}$ (Theorem 14.2),
- parallel with $\mathbf{N}$.

Since $\dfrac{d\mathbf{B}}{ds}$ is parallel to $\mathbf{N}$, we write

$$\frac{d\mathbf{B}}{ds} = -\tau \mathbf{N}$$

where τ is the *torsion* (the negative sign is conventional). We can solve for τ via the dot product:

$$\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\tau \underbrace{\mathbf{N} \cdot \mathbf{N}}_{1} \quad \Longrightarrow \quad \frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\tau$$

Definition. (Unit Binormal Vector and Torsion)
Let C be a smooth parameterized curve with unit tangent and principal unit normal vectors $\mathbf{T}$ and $\mathbf{N}$, respectively. Then at each point of the curve at which the curvature is nonzero, the **unit binomial vector** is

$$\mathbf{B} = \mathbf{T} \times \mathbf{N},$$

and the **torsion** is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

Example. Consider the circle C defined by

$$\mathbf{r}(t) = \langle R\cos(t),\ R\sin(t)\rangle,\ \text{for } 0 \leq t \leq 2\pi,\ \text{with } R > 0.$$

Find the unit binormal vector $\mathbf{B}$ and determine the torsion.

Example. Compute the torsion of the helix

$$\mathbf{r}(t) = \langle a\cos(t),\, a\sin(t),\, bt\rangle,\ \text{for } t \geq 0,\ \text{and } b > 0.$$

Summary: Formula for Curves in Space

Position function: $\mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$

Velocity: $\mathbf{v} = \mathbf{r}'$

Acceleration: $\mathbf{a} = \mathbf{v}'$

Unit tangent vector: $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

Principal unit normal vector:
$\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$ (provided $d\mathbf{T}/dt \neq \mathbf{0}$)

Curvature: $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{1}{|\mathbf{v}|}\left|\dfrac{d\mathbf{T}}{dt}\right| = \dfrac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$

Components of acceleration: $\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}$, where

$$a_N = \kappa|\mathbf{v}|^2 = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|} \text{ and } a_T = \frac{d^2s}{dt^2} = \frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}|}$$

Unit binormal vector: $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \dfrac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|}$

Torsion: $\tau = -\dfrac{d\mathbf{B}}{ds}\cdot\mathbf{N} = \dfrac{(\mathbf{v}\times\mathbf{a})\cdot\mathbf{a}'}{|\mathbf{v}\times\mathbf{a}|^2} = \dfrac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}'\times\mathbf{r}''|^2}$

15.1: Graphs and Level Curves

In the previous chapter, we considered functions of the form

$$\mathbf{r}(t) = \langle f(t),\, g(t),\, h(t)\rangle,$$

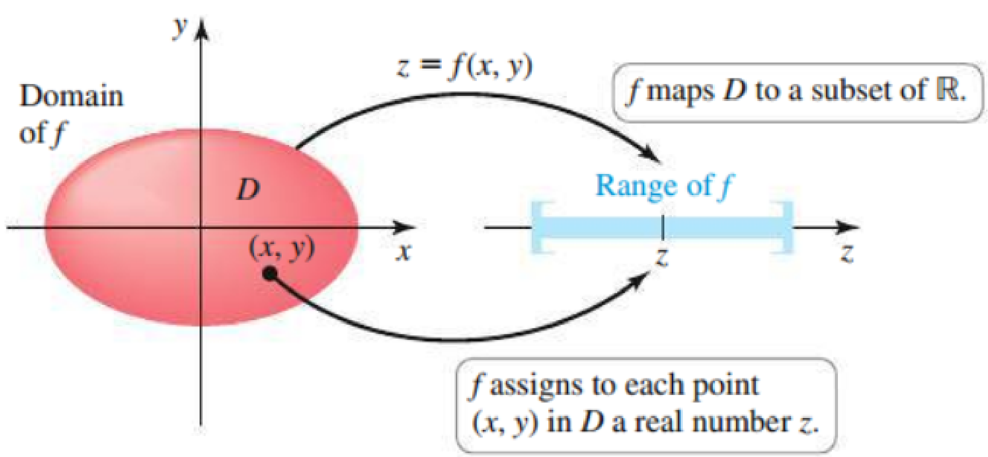
which have one independent variable t and three dependent variables $f(t)$, $g(t)$, and $h(t)$. In this chapter, we consider functions of the form

$$x_{n+1} = f(x_1, x_2, \ldots, x_n),$$

where we have multiple independent variables x_1, $x_2, \ldots,$ x_n and one single dependent variable x_{n+1}. We begin with functions of two variables:

$$z = f(x, y).$$

Definition. (Function, Domain, and Range with 2 Independent Variables) A **function** $z = f(x, y)$ assigns to each point (x, y) in a set D in $\mathbb{R}^2$ a unique real number z in a subset of $\mathbb{R}$. The set D is the **domain** of f. The set D is the **domain** of f. The **range** of f is the set of real numbers z that are assumed as the points (x, y) vary over the domain.



Example. Find the domain of the following functions:

$$f(x,y) = \frac{1}{xy+2} \qquad\qquad g(x,y) = \sqrt{108 - 3x^2 - 3y^2}$$

$$h(x,y) = \log_2\left(x^3 - y^{1/3}\right) \qquad\qquad j(x,y) = \frac{1}{\sqrt{x^2 + y^2 - 16}}$$

Example. Roughly graph the following functions:

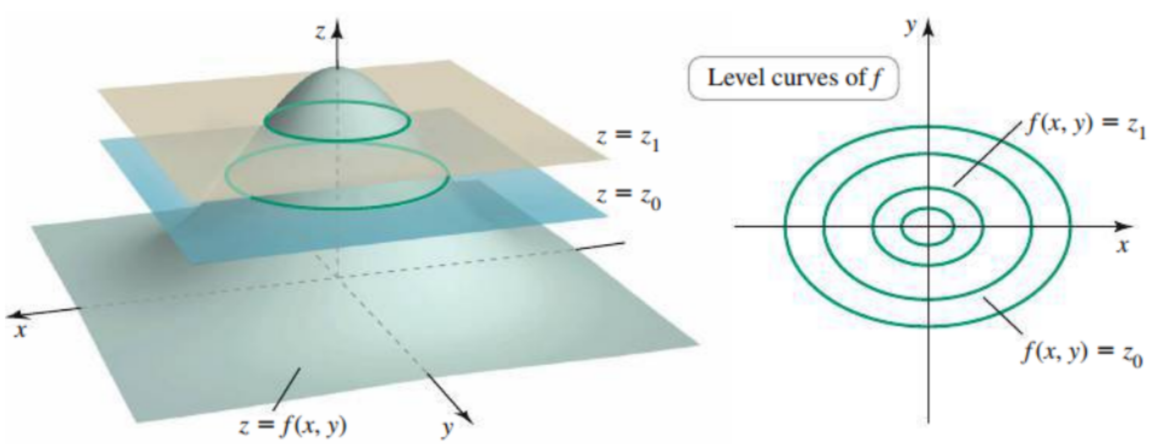
$$f(x, y) = -4x + 3y - 10$$

$$g(x, y) = x^2 + y^2 + 4$$

$$h(x, y) = \sqrt{4 + x^2 + y^2}$$

Level Curves:

A **contour curve** is formed by tracing a three-dimensional surface at a constant height. A **level curve** is formed when a contour curve is projected to the xy-plane.



Example. Find the level curves of the following functions:

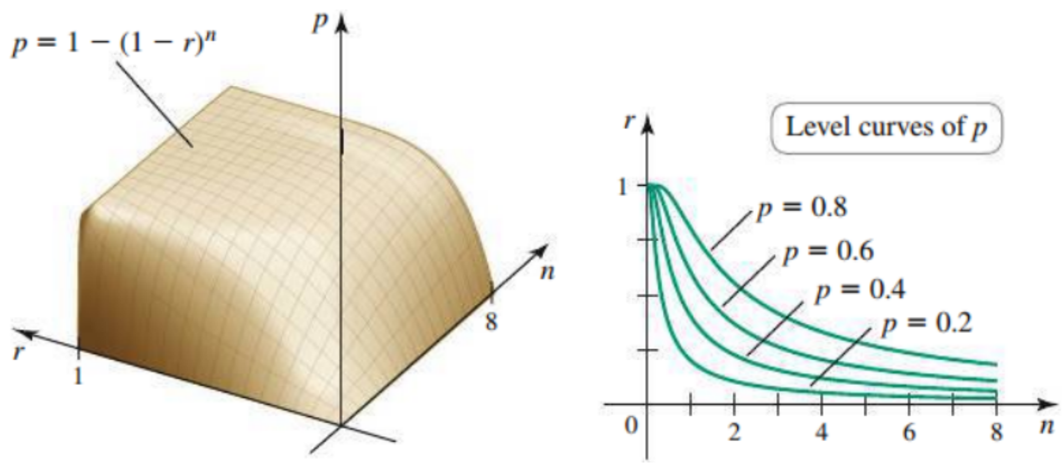
$f(x, y) = y - x^2 - 1$

$g(x, y) = e^{-x^2-y^2}$

$h(x, y) = x^2 + y^2$

Applications of Functions of Two Variables:

Example. A probability function of two variables: Suppose on a particular day, the fraction of students on campus infected with COVID-19 is r, where $0 \leq r \leq 1$. If you have n random (possibly repeated) encounters with students during the day, the probability of meeting *at least* one infected person is $p(n, r) = 1 - (1 - r)^n$.



Functions of More than Two Variables:

Number of Independent Variables	Explicit Form	Implicit Form	Graph Resides In...
1	$y = f(x)$	$F(x, y) = 0$	$\mathbb{R}^2 (xy - \text{plane})$
2	$z = f(x, y)$	$F(x, y, z) = 0$	$\mathbb{R}^3 (xyz - \text{space})$
3	$w = f(x, y, z)$	$F(x, y, z, w) = 0$	$\mathbb{R}^4$
n	$x_{n+1} = f(x_1, x_2, \ldots, x_n)$	$F(x_1, x_2, \ldots, x_n, x_{n+1}) = 0$	$\mathbb{R}^{n+1}$

Definition. (Function, Domain, and Range with n Independent Variables) The **function** $x_{n+1} = f(x_1, x_2, \ldots, x_n)$ assigns a unique real number x_{n+1} to each point $(x_1, x_2, \ldots, x_n)$ in a set D in $\mathbb{R}^4$. The set D is the **domain** of f. The **range** is the set of real numbers x_{n+1} that are assumed as the points $(x_1, x_2, \ldots, x_n)$ vary over the domain.

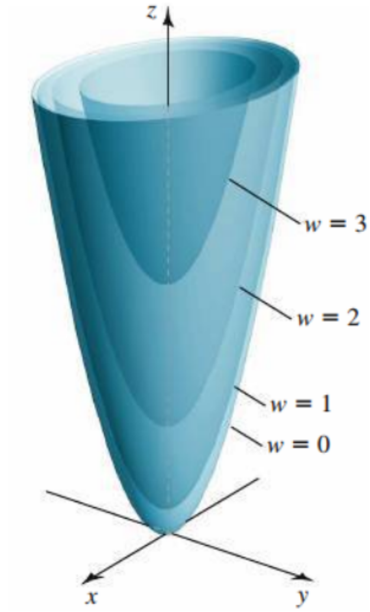
Example. Find the domain of the following functions:

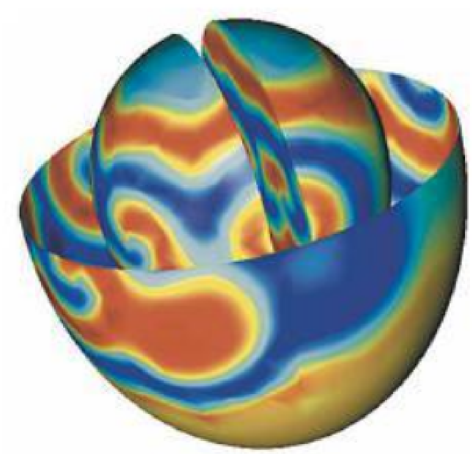
$$f(x, y, z) = 4xyz - 2xz + 5yz$$

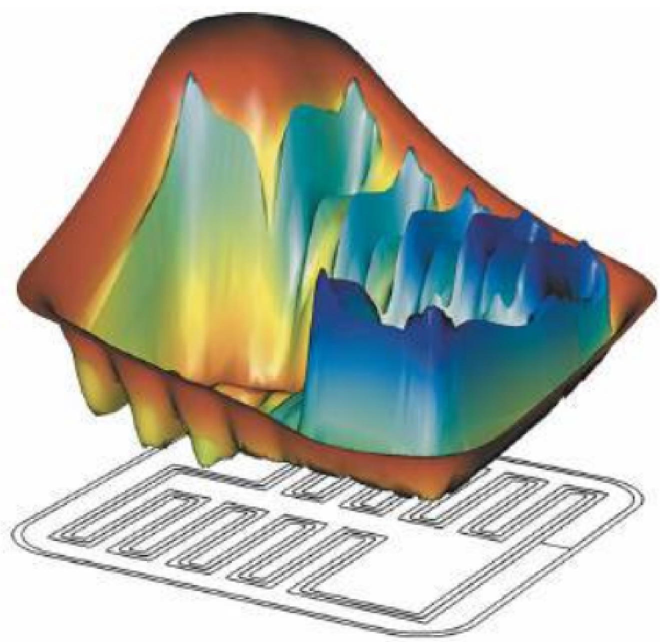
$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 9}$$

Graphs of Functions of More Than Two Variables:

The idea of level curves can be extended to **level surfaces**. Level surfaces can be used to represent functions of three variables:







15.2: Limits and Continuity

Definition. (Limit of a Function of Two Variables)
The function f has the **limit** L as $P(x,y)$ approaches $P_0(a,b)$, written

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{P\to P_0} f(x,y) = L,$$

if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

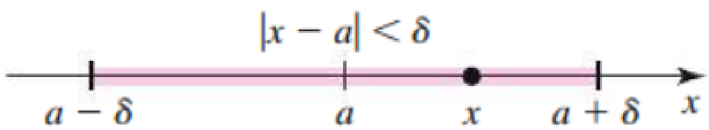
$$|f(x,y) - L| < \varepsilon$$

whenever (x,y) is in the domain of f and

$$0 < |PP_0| = \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

Note: For functions with 1 independent variable, $|x-a| < \delta$ represents an open interval on a number line. Recall that these limits only exist if the same value is approached from two directions.

For functions with 2 independent variables, $|PP_0| < \delta$ represents an open disk (open ball). Here, the limit only exists if the same value is approached from *all* directions.



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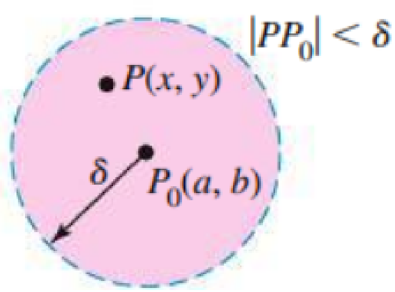


Theorem 15.1: Limits of Constant and Linear Functions
Let a, b, and c be real numbers.

1. Constant function $f(x,y) = c$: $\displaystyle\lim_{(x,y)\to(a,b)} c = c$
2. Linear function $f(x,y) = x$: $\displaystyle\lim_{(x,y)\to(a,b)} x = a$
3. Linear function $f(x,y) = y$: $\displaystyle\lim_{(x,y)\to(a,b)} y = b$

Theorem 15.2: Limit Laws for Functions of Two Variables
Let L and M be real numbers and suppose $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $\displaystyle\lim_{(x,y)\to(a,b)} g(x,y) = M$. Assume c is constant, and $n > 0$ is an integer.

1. **Sum** $\displaystyle\lim_{(x,y)\to(a,b)} (f(x,y) + g(x,y)) = L + M$
2. **Difference** $\displaystyle\lim_{(x,y)\to(a,b)} (f(x,y) - g(x,y)) = L - M$
3. **Constant multiple** $\displaystyle\lim_{(x,y)\to(a,b)} cf(x,y) = cL$
4. **Product** $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)g(x,y) = LM$
5. **Quotient** $\displaystyle\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$, provided $M \neq 0$
6. **Power** $\displaystyle\lim_{(x,y)\to(a,b)} (f(x,y))^n = L^n$
7. **Root** $\displaystyle\lim_{(x,y)\to(a,b)} (f(x,y))^{1/n} = L^{1/n}$, when $L > 0$ if n is even.

Example. Evaluate the following limits:

$$\lim_{(x,y)\to(4,11)} 570$$

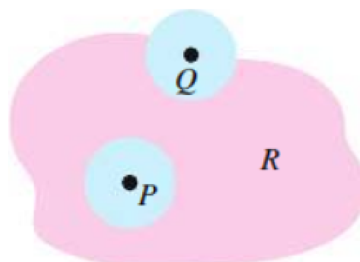
$$\lim_{(x,y)\to(2,8)} \left(3x^2y + \sqrt{xy}\right)$$

$$\lim_{(x,y)\to(0,\pi)} \frac{\sin(xy) + \cos(xy)}{7y}$$

$$\lim_{(x,y)\to(\frac{1}{3},-1)} \frac{9x^2 - y}{3x + y}$$

Definition. (Interior and Boundary Points)
Let R be a region in $\mathbb{R}^2$. An **interior point** P of R lies entirely within R, which means it is possible to find a disk centered at P that contains only points of R.

A **boundary point** Q of R lies on the edge of R in the sense that every disk centered at Q contains at least one point in R and at least one point not in R.



Definition. (Open and Closed Sets)
A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points.

Example. Identify which regions are open sets and which are closed sets.

$\{(x,y) : x^2 + y^2 < 9\}$

$\{(x,y) : |x| \leq 1, |y| \leq 1\}$

$\{(x,y) : x \neq 0, -1 \leq y \leq 3\}$

$\{(x,y) : x + y < 2\}$

A limit at a boundary point $P_0(a,b)$ of a function's domain can exist, provided $f(x,y)$ approaches the same value as (x,y) approaches (a,b) *along all paths that lie in the domain.*

Example. $f(x,y) = \dfrac{x^2-y^2}{x-y}$

Example. Evaluate the following limits

$$\lim_{(x,y)\to(0,\pi)} \frac{\sin(xy)+\cos(xy)}{7y} \qquad \lim_{(x,y)\to(-3,-15)} \frac{y^2-5xy}{y-5x}$$

$$\lim_{(x,y)\to(0,0)} \frac{x+2y}{x-2y} \qquad \lim_{(x,y)\to(1,-1)} \frac{y^5}{(x-1)^{30}+y^5}$$

Procedure: Two-Path Test for Nonexistence of Limits
If $f(x,y)$ approaches two different values as (x,y) approaches (a,b) along two different paths in the domain of f, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Definition. (Continuity)
The function f is continuous at the point (a, b) provided

1. f is defined at (a, b)
2. $\lim_{(x,y)\to(a,b)} f(x, y)$ exists, and
3. $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

Example. Determine if $f(x, y)$ is continuous at $(0, 0)$

$$f(x,y) = \begin{cases} \dfrac{3xy^2}{x^2+y^4}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

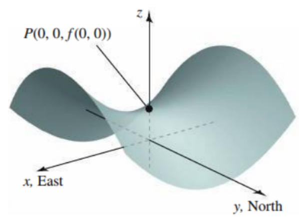
Theorem 15.3: Continuity of Composite Functions
If $u = g(x, y)$ is continuous at (a, b) and $z = f(u)$ is continuous at $g(a, b)$, then the composite function $z = f(g(x, y))$ is continuous at (a, b).

Example. Determine the points at which the following functions are continous:

$f(x, y) = \ln(x^2 + y^2 + 4)$ $\qquad$ $g(x, y) = e^{x/y}$

15.3: Partial Derivatives

Recall that for functions with one independent variable, say $y = f(x)$, the derivative measures the change in y with respect to x. For functions with multiple independent variables, we compute derivatives with respect to each variable.



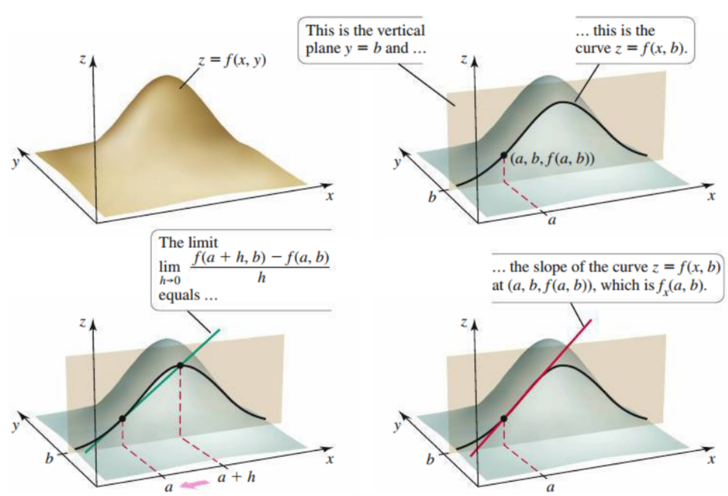
Definition. (Partial Derivatives)
The **partial derivative of f with respect to x at the point (a, b) is**

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

The **partial derivative of f with respect to y at the point (a, b) is**

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h},$$

provided these limits exist.



When evaluating a partial derivative at a point (a, b), we denote this

$$\frac{\partial f}{\partial x}(a,b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a,b) \text{ and } \frac{\partial f}{\partial y}(a,b) = \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a,b)$$

Example. For the following functions, find the first partial derivatives. If a point is provided, evaluate the partial derivatives.

$f(x, y) = x^8 + 3y^9 + 8$

$g(x, y) = 6x^5y^2 + 2x^3y + 5$

$$h(s,t) = \frac{s-t}{4s+t} \text{ at } (s,t) = (2,-3)$$

$$k(x,y) = \tan^{-1}\!\big(3x^2y^2\big) \text{ at } (x,y) = (1,1)$$

$$\ell(w,v) = \int_v^w g(u)\,du$$

Higher-Order Partial Derivatives:

$\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x^2}$	$(f_x)_x = f_{xx}$	"d squared f dx squared or $f - x - x$"
$\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2}$	$(f_y)_y = f_{yy}$	"d squared f dy squared or $f - y - y$"
$\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x \partial y}$	$(f_y)_x = f_{yx}$	"$f - y - x$"
$\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x}$	$(f_x)_y = f_{xy}$	"$f - x - y$"

The order of differentiation is important when finding **mixed partial derivatives** f_{xy} and f_{yx}.

Example. Find the four 2nd-order partial derivatives of the following functions

$z = 4ye^{3x}$

$f(x, y) = \sin^2(x^3 y)$

Theorem 15.4: (Clairaut) Equality of Mixed Partial Derivatives
Assume f is defined on an open set D of $\mathbb{R}^2$, and that f_{xy} and f_{yx} are continuous throughout D. Then $f_{xy} = f_{yx}$ at all points of D.

Note: Clairut's theorem also extends to higher order derivatives of f.

Example. Ideal Gas Law: The pressure P, volume V, and temperature T of an ideal gas are related by the equation $PV = kT$, where $k > 0$ is a constant depending on the amount of gas.

Determine the rate of change of the pressure with respect to the volume

Determine the rate of change of the pressure with respect to the temperature

Definition. (Differentiability)
The function $z = f(x, y)$ is **differentiable at** (a, b) provided $f_x(a, b)$ and $f_y(a, b)$ exist and the change $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$ equals

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y = \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where for fixed a and b, ε_1 and ε_2 are functions that depend only on Δx and δy, with $(\varepsilon_1, \varepsilon_2) \to (0, 0)$ as $(\Delta x, \Delta y) \to (0, 0)$. A function is **differentiable** on an open set R if it is differentiable at every point of R.

Theorem 15.5: Conditions for Differentiability
Suppose the function f has partial derivatives f_x and f_y defined on an open set containing (a, b), with f_x and f_y continuous at (a, b). Then f is differentiable at (a, b).

Theorem 15.6: Differentiable Implies Continuous
If a function f is differentiable at (a, b), then it is continuous at (a, b).

Example. Why is the function

$$f(x, y) = \begin{cases} \frac{3xy}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

not continuous at $(x, y) = (0, 0)$?

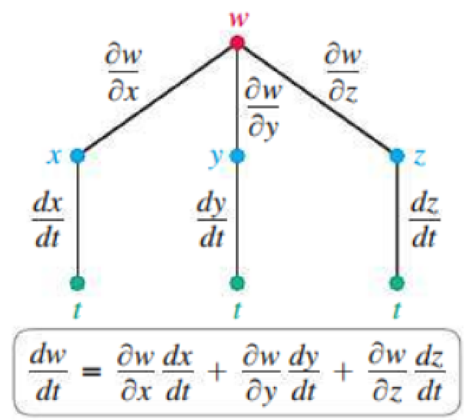
15.4: The Chain Rule

Theorem 15.7: Chain Rule (One Independent Variable)
Let z be a differentiable function of x and y on its domain, where x and y are differentiable functions of t on an interval I. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Note:

- For $z = f(x(t), y(t))$, z is the dependent variable, t is the independent variable, and x and y are **intermediate variables**.
- Since x and y only depend on t, we use the 'ordinary' derivative symbol
- Theorem 15.7 generalizes to functions of n variables



Example. Find the derivative of the following functions using the chain rule where appropriate.

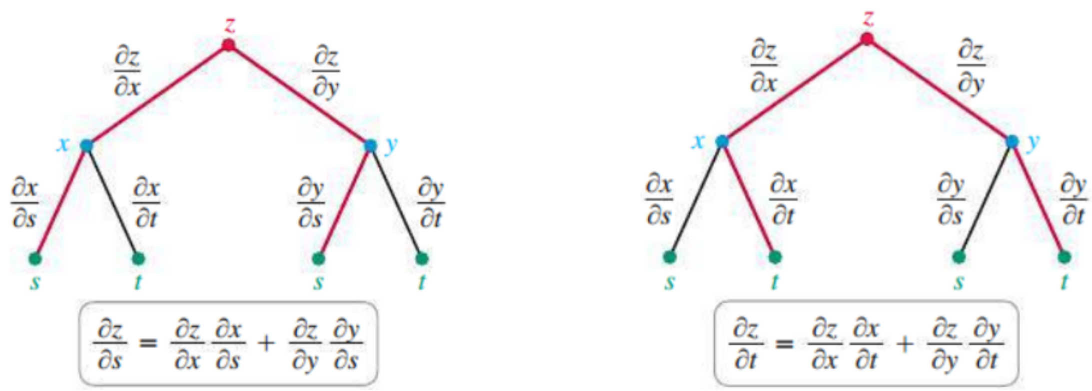
$z = x^2 - 2y^2 + 20$ where $x = 2\cos(t)$ and $y = 2\sin(t)$

$w = \sin(12x)\cos(2y)$ where $x = t/2$ and $y = t^3$

$Q = \sqrt{3x^2 + 3y^2 + 2z^2}$ where $x = \sin(t)$, $y = \cos(t)$, and $z = \cos(t)$.

Theorem 15.8: Chain Rule (Two Independent Variables)
Let z be a differentiable function of x and y, where x and y are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$



Example. For $z = e^{5x+8y}$, where $x = 7st$ and $y = 5s + t$, find z_s and z_t.

Example. For $z = \sin(2x)\cos(3y)$, where $x = s + t$ and $y = s - t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

Example. For $r = \ln(x^2 + xy + y^2)$, where $x = 2st$ and $y = s/t$, find $\partial r/\partial s$ and $\partial r/\partial t$.

Theorem 15.9: Implicit Differentiation
Let F be differentiable on its domain and suppose $F(x,y)=0$ defines y as a differentiable function of x. Provided $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Note: The above derivation comes from using the chain rule on $F(x,y)=0$.

Example. For $4x^3 + 2x^2y - 3y^3 = 0$, find $\frac{dy}{dx}$ implicitly.

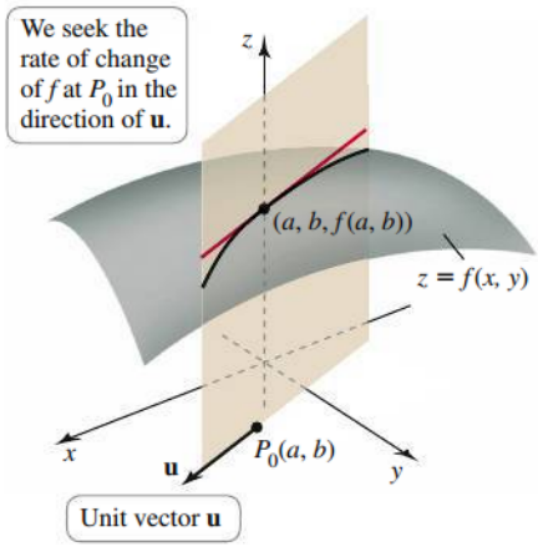
Example. For $xy + xz + 5yz = 42$, find $\partial z/\partial x$ and $\partial z/\partial y$ implicitly.

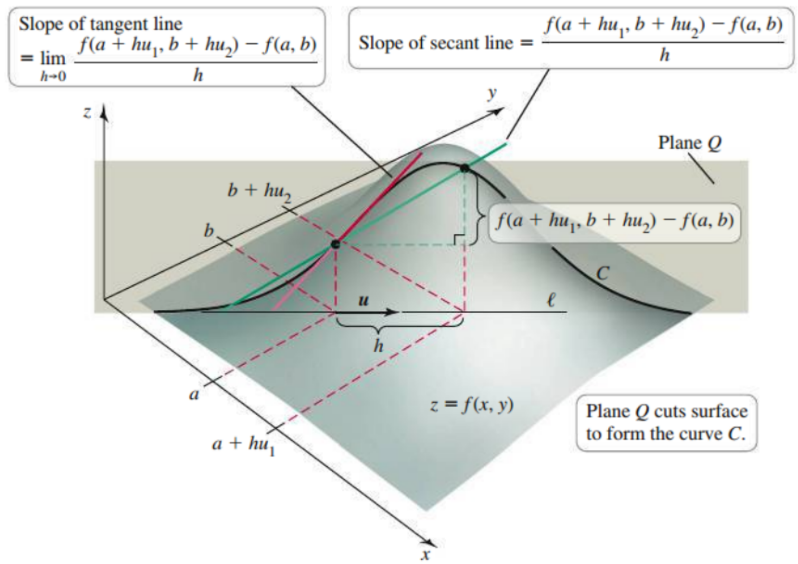
Example. For $xyz + 2yz + 3xz = 4x + 2y - 3z$, find $\partial z/\partial x$ and $\partial z/\partial y$.

Example. Consider the surface $z = f(x, y) = 3x^2 + 9y^2 + 4$ and the curve C given parametrically by $x = \cos(t)$ and $y = \sin(t)$ where $0 \leq t \leq 2\pi$. Find $z'(t)$ and find t such that $z'(t) > 0$.

15.5: Directional Derivatives and the Gradient

Directional derivatives allow us to evaluate the rate of change of a function $f(x, y)$ along any direction (not just parallel with the x-axis and y-axis).





> **Definition. (Directional Derivative)**
> Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The **directional derivative of f at (a, b) in the direction of u** is
> $$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h},$$
> provided the limit exists.

To motivate the formula for the directional derivative, let ℓ be a line going through (a, b) in the direction of the unit vector $\mathbf{u}$. Now, let

$$x = a + su_1, \quad \text{and} \quad y = b + su_2,$$

where $-\infty < s < \infty$ and define

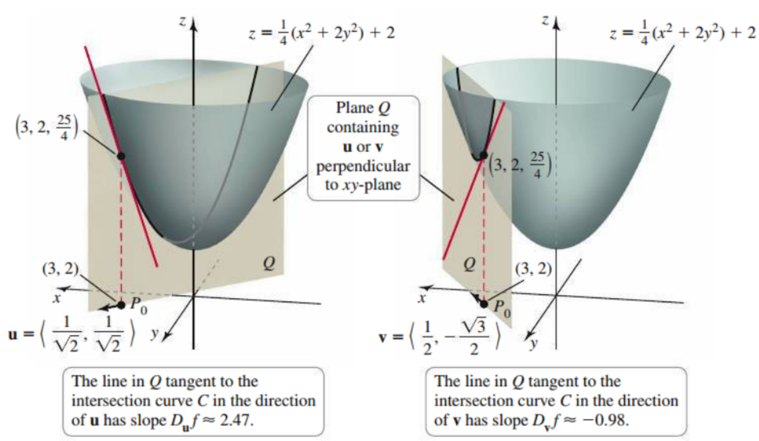
$$g(s) = f(\underbrace{a + su_1}_{x}, \underbrace{b + su_2}_{y}),$$

which evaluates f along ℓ. Thus, $g'(s)$ gives us the derivative along this line, and $g'(0)$ gives us the directional derivative of f at (a, b):

$$\begin{aligned} D_{\mathbf{u}} f(a,b) = g'(0) &= \left(\frac{\partial f}{\partial x} \underbrace{\frac{dx}{ds}}_{u_1} + \frac{\partial f}{\partial y} \underbrace{\frac{dy}{ds}}_{u_2} \right) \Bigg|_{s=0} \\ &= f_x(a,b)u_1 + f_y(a,b)u_2 \\ &= \langle f_x(a,b),\ f_y(a,b) \rangle \cdot \langle u_1, u_2 \rangle. \end{aligned}$$

Theorem 15.10: Directional Derivative
Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The **directional derivative of f at (a, b) in the direction of $\mathbf{u}$** is

$$D_{\mathbf{u}} f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle \cdot \langle u_1, u_2 \rangle.$$



Example. Compute the directional derivatives of the following functions at the given point along the given direction.

$f(x, y) = \sqrt{4 - x^2 - 2y}$; $P(2, -2)$; and $\mathbf{u} = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$,

$g(x, y) = \tan^{-1}(xy)$; $P(\pi, 1/3)$; along $\mathbf{u} = \langle 1, 1 \rangle$,

$h(x, y) = 2x^2 - xy + 3y^2$; $P(1, -3)$; along $\mathbf{u} = \langle 1, -1 \rangle$ and $\mathbf{v} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$.

The Gradient Vector:
The vector of derivatives used in the directional derivative is called the *gradient* of f.

Definition. (Gradient (Two Dimensions))
Let f be differentiable at the point (x, y). The **gradient** of f at (x, y) is the vector-valued function

$$\nabla f(x, y) = \langle f_x(x, y),\ f_y(x, y) \rangle = f_x(x, y)\boldsymbol{i} + f_y(x, y)\boldsymbol{j}.$$

Example. For $f(x, y) = 3 - \dfrac{x^2}{10} + \dfrac{xy^2}{10}$, compute $\nabla f(3, -1)$, then compute $D_{\mathbf{u}} f(3, -1)$, where $\mathbf{u} = \left\langle \dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}} \right\rangle$.

Theorem 15.11: Directions of Change
Let f be differentiable at (a, b) with $\nabla f(a, b) \neq \mathbf{0}$.

1. f has its maximum rate of increase at (a, b) in the direction of the gradient $\nabla f(a, b)$. The rate of change in this direction is $|\nabla f(a, b)|$.
2. f has its maximum rate of decrease at (a, b) in the direction of $-\nabla f(a, b)$. The rate of change in this direction is $-|\nabla f(a, b)|$.
3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$.

Example. For $f = 4 + x^2 + 3y^2$:

What direction is the greatest ascent at $P(2, -\frac{1}{2}, \frac{35}{4})$? What is the rate of change in this direction?

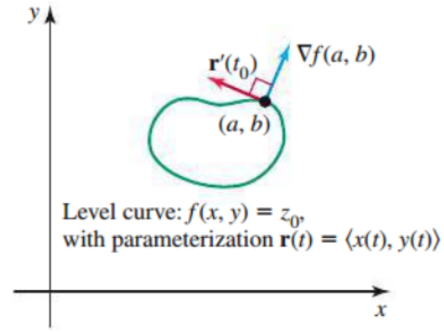
What direction is the greatest descent at $P(\frac{5}{2}, -2, \frac{89}{4})$? What is the rate of change in this direction?

What direction results in no change in function values at $P(3, 1, 16)$?

Theorem 15.12: The Gradient and Level Curves
Given a function f differentiable at (a, b), the line tangent to the level curve of f at (a, b) is orthogonal to the gradient $\nabla f(a, b)$, provided $\nabla f(a, b) \neq \mathbf{0}$.

Note: From Theorem 15.12, we get an equation for the line tangent to the curve $z = f(x, y)$ at (a, b):

$$\nabla f(a, b) \cdot \langle x - a,\ y - b \rangle = 0.$$



Example. Consider the upper sheet $z = f(x, y) = \sqrt{1 + 2x^2 + y^2}$ of a hyperboloid of two sheets.

Verify that the gradient at $(1, 1)$ is orthogonal to the corresponding level curve at that point.

Find an equation of the line tangent to the level curve at $(1, 1)$.

Example. Consider $z = f(x, y) = 15 - \dfrac{x^2}{25} - \dfrac{y^2}{9}$:

Compute the slope of the tangent line at $P(5\sqrt{5}, -6, 6)$.

Verify the gradient is orthogonal to the tangent line.

Definition. (Directional Derivative and Gradient in Three Dimensions) Let f be directional at (a, b, c) and let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ be a unit vector. The **directional derivative of f at (a, b, c) in the direction of $\mathbf{u}$** is

$$D_{\mathbf{u}}(a, b, c) = \lim_{h \to 0} \frac{f(a + hu_1,\, b + hu_2,\, c + hu_3) - f(a, b, c)}{h},$$

provided this limit exists.
The **gradient** of f at this point (x, y, z) is the vector-valued function

$$\begin{aligned} \nabla f(x, y, z) &= \langle f_x(x, y, z),\ f_y(x, y, z),\ f_z(x, y, z) \rangle \\ &= f_x(x, y, z)\boldsymbol{i} + f_y(x, y, z)\boldsymbol{j} + f_z(x, y, z)\boldsymbol{k}. \end{aligned}$$

Theorem 15.13: Directional Derivative and Interpreting the Gradient Let f be differentiable at (a, b, c) and let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ be a unit vector. The directional derivative of f at (a, b, c) in the direction of $\mathbf{u}$ is

$$\begin{aligned} D_{\mathbf{u}} f(a, b, c) &= \nabla f(a, b, c) \cdot \mathbf{u} \\ &= \langle f_x(a, b, c),\ f_y(a, b, c),\ f_z(a, b, c) \rangle \cdot \langle u_1,\ u_2,\ u_3 \rangle. \end{aligned}$$

Assuming $\nabla f(a, b, c) \neq \mathbf{0}$, the gradient in three dimensions has the following properties.

1. f has its maximum rate of increase at (a, b, c) in the direction of the gradient $\nabla f(a, b, c)$ and the rate of change in this direction is $|\nabla f(a, b, c)|$.
2. f has its maximum rate of decrease at (a, b, c) in the direction of $-\nabla f(a, b, c)$ and the rate of change in this direction is $-|\nabla f(a, b, c)|$.
3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b, c)$.

Example. Consider $f(x, y, z) = x^2 + 2y^2 + 4z^2 - 1$ and the level surface $f(x, y, z) = 3$. Find the gradient and the corresponding rate of change at the points $P(2, 0, 0)$, $Q(0, \sqrt{2}, 0)$, $R(0, 0, 1)$, and $S(1, 1, 1/2)$ on the level surface.

15.6: Tangent Planes and Linear Approximation

Definition. (Equation of the Tangent Plane for $F(x, y, z) = 0$)
Let F be differentiable at the point $P_0(a, b, c)$ with $\nabla F(a, b, c) \neq \mathbf{0}$. The plane tangent to the surface $F(x, y, z) = 0$ at P_0, called the **tangent plane**, is the plane passing through P_0 orthogonal to $\nabla F(a, b, c)$. An equation of the tangent plane is

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

Example. Consider the ellipsoid

$$F(x, y, z) = \frac{x^2}{9} + \frac{y^2}{25} + z^2 - 1 = 0.$$

a) Find an equation of the plane tangent to the ellipsoid at $(0, 4, \frac{3}{5})$.

b) At what points on the ellipsoid is the tangent plane horizontal?

Surfaces of the form $z = f(x,y)$ are a special case of $F(x,y,z) = 0$:
Define $F(x,y,z) = z - f(x,y) = 0$, then

$$\nabla F(a,b,f(a,b)) = \langle -f_x(a,b),\ -f_y(a,b),\ 1\rangle$$

so the tangent plane is

$$-f_x(a,b)(x-a) - f_y(a,b)(y-b) + 1(z - f(a,b)) = 0$$

Tangent Plane for $z = f(x,y)$
Let f be differentiable at the point (a,b). An equation of the plane tangent to the surface $z = f(x,y)$ at the point $(a,b,f(a,b))$ is

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

Example. Find an equation of the plane tangent to $f(x,y) = 4e^{xy^2}$ at $(3,0,4)$ and $(0,2,4)$.

Example. Find an equation of the plane tangent to $f(x,y) = \tan^{-1}(xy)$ at $\left(\sqrt{3},\, 1,\, \frac{\pi}{3}\right)$ and $\left(\frac{\sqrt{3}}{3}, 1, \frac{\pi}{6}\right)$.

Definition. (Linear Approximation)
Let f be differentiable at (a,b). The linear approximation to the surface $z = f(x,y)$ at the point $(a,b,f(a,b))$ is the tangent plane at that point, given by the equation

$$L(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b),$$

For a function of three variables, the linear approximation to $w = f(x,y,z)$ at the point $(a,b,c,f(a,b,c))$ is given by

$$L(x,y,z) = f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) + f(a,b,c).$$

Example. Let $f(x,y) = \dfrac{5}{x^2+y^2}$. Find the linear approximation to the function at the point $(-1,2,1)$. Use this to approximate $f(-1.05, 2.1)$.

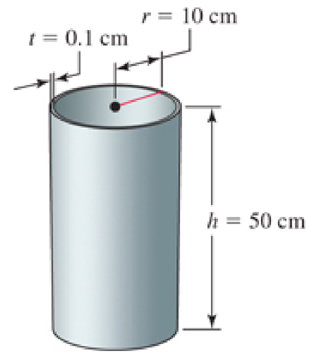
Example. Let $f(x,y) = \sqrt{x^2+y^2}$. Find the linear approximation to the function at the point $(-8,15,17)$. Use this to approximate $f(-7.91, 14.96)$.

Definition. (The differential dz)
Let f be differentiable at the point (x, y). The change in $z = f(x, y)$ as the independent variables change from (x, y) to $(x+dx, y+dy)$ is denoted Δz and is approximated by the differential dz:

$$\Delta z \approx dz = f_x(x, y)\, dx + f_y(x, y)\, dy.$$

Example. Let $z = f(x, y) = \dfrac{5}{x^2 + y^2}$. Approximate the change in z when the variables change from $(-1, 2)$ to $(-0.93, 1.94)$.

Example. A company manufactures cylindrical aluminum tubes to rigid specifications. The tubes are designed to have an outside radius of $r = 10\ cm$, a height of $h = 50\ cm$, and a thickness of $t = 0.1\ cm$. The manufacturing process produces tubes with a maximum error of $\pm 0.05\ cm$ in the radius and height, and a maximum error of $\pm 0.0005\ cm$ in the thickness. The volume of the cylindrical tube is $V(r, h, t) = \pi h t(2r - t)$. Use differentials to estimate the maximum error in the volume of a tube.



15.7: Maximum/Minimum Problems

Example. Consider the function $f(x) = x^3 - 3x + 1$ on the interval $[-1, 2]$. Find the local extrema and absolute extrema of this function.

Definition. (Local Maximum/Minimum Values)
Suppose (a, b) is a point in a region R on which f is defined.

- If $f(x, y) \leq f(a, b)$ for all (x, y) in the domain of f and in some open disk centered at (a, b), then $f(a, b)$ is a **local maximum value** of f.
- If $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f and in some open disk centered at (a, b), then $f(a, b)$ is a **local minimum value** of f.
- Local maximum and local minimum values are also called **local extreme values** or **local extrema**.

Theorem 15.14: Derivatives and Local Maximum/Minimum Values
If f has a local maximum or minimum value at (a, b) and the partial derivatives f_x and f_y exist at (a, b), then $f_x(a, b) = f_y(a, b) = 0$.

Definition. (Critical Point)
An interior point (a, b) in the domain of f is a **critical point** of f if either

1. $f_x(a, b) = f_y(a, b) = 0$, or
2. at least one of the partial derivatives f_x and f_y does not exist at (a, b).

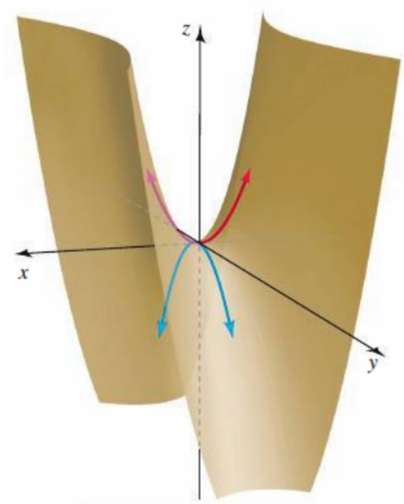
Example. Find the critical points of $f(x, y) = 3(x-1)^2 + 4(2-y)^3$.

Example. Find the critical points of $g(x, y) = x^2 + xy - y^2$.

Example. Find the critical points of $h(x, y) = \dfrac{3}{x} - \dfrac{4}{y}$.

Definition. (Saddle Point)
Consider a function f that is differentiable at a critical point (a, b). Then f has a **saddle point** at (a, b) if, in every open disk centered at (a, b), there are points (x, y) for which $f(x, y) > f(a, b)$ and points for which $f(x, y) < f(a, b)$.

Example. Compute the first and second order partial derivatives of $f(x, y) = x^2 - y^2$.



The hyperbolic paraboloid $z = x^2 - y^2$ has a saddle point at $(0, 0)$.

Theorem 15.15: Second Derivative Test
Suppose the second partial derivatives of f are continuous throughout an open disk centered at the point (a, b), where $f_x(a, b) = f_y(a, b) = 0$. Let

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2.$$

1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum value at (a, b).
2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum value at (a, b).
3. If $D(a, b) < 0$, then f has a saddle point at (a, b).
4. If $D(a, b) = 0$, then the test is inconclusive.

Example. Use the Second Derivative Test to classify the critical points of $f(x, y) = x^2 + 2y^2 - 4x + 4y + 6$.

Example. Use the Second Derivative Test to classify the critical points of $f(x, y) = xy(x - 2)(y + 3)$.

Definition. (Absolute Maximum/Minimum Values)
Let f be defined on a set R in $\mathbb{R}^2$ containing the point (a, b).

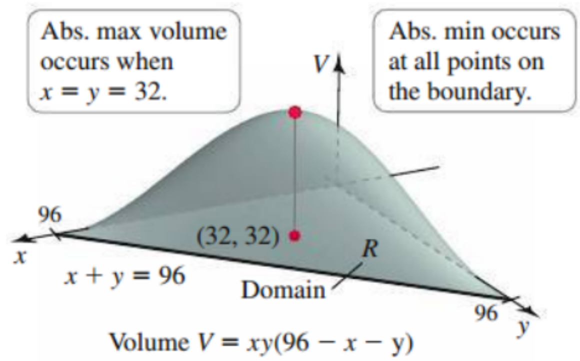
- If $f(a, b) \geq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an **absolute maximum value** of f on R.
- If $f(a, b) \leq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an **absolute minimum value** of f on R.

Procedure:
Finding Absolute Maximum/Minimum Values on Closed Bounded Sets
Let f be continuous on a closed bounded set R in $\mathbb{R}^2$. To find the absolute maximum and minimum values of f on R:

1. Determine the values of f at all critical points in R.
2. Find the maximum and minimum values of f on the boundary of R.
3. The greatest function value found in Steps 1 and 2 is the absolute maximum value of f on R, and the least function value found in Steps 1 and 2 is the absolute minimum value of f on R.

Example. Find the absolute maximum and minimum values of $f(x, y) = xy - 8x - y^2 + 12y + 160$ over the triangular region $R = \{(x, y) : 0 \leq x \leq 15,\ 0 \leq y \leq 15 - x\}$.

Example. A shipping company handles rectangular boxes provided the sum of the length, width, and height of the box does not exceed 96 in. Find the dimensions of the box that meets this condition and has the largest volume.

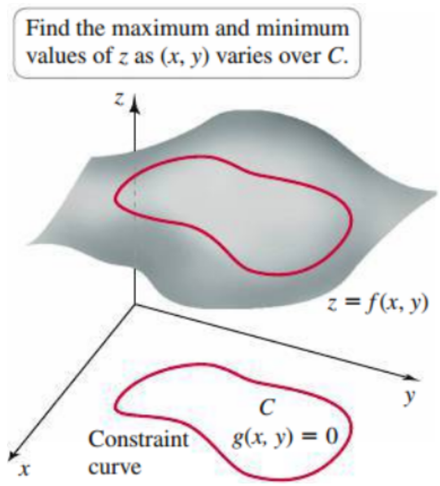


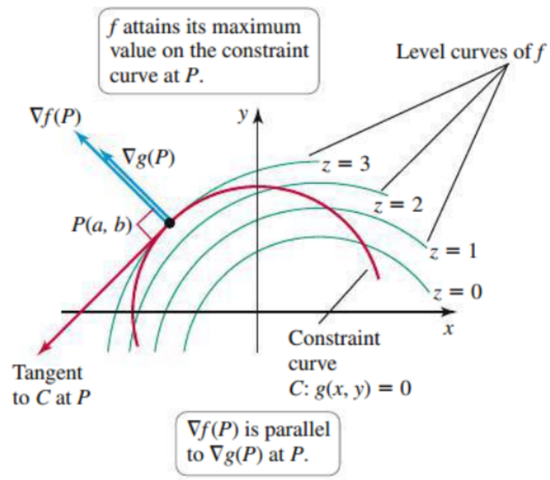
Example. Find the absolute maximum and minimum values of $f(x, y) = 4 - x^2 - y^2$ on the open disk $R = \left\{(x, y) : x^2 + y^2 < 1\right\}$ (if they exist).

Example. Find the point(s) on the plane $x + 2y + z = 2$ closest to the point $P(2, 0, 4)$.

15.8: Lagrange Multipliers

Constrained optimization functions have an **objective function** f with the restriction that the independent variables x and y lie on a **constraint** curve C in the xy-plane given by $g(x, y) = 0$.



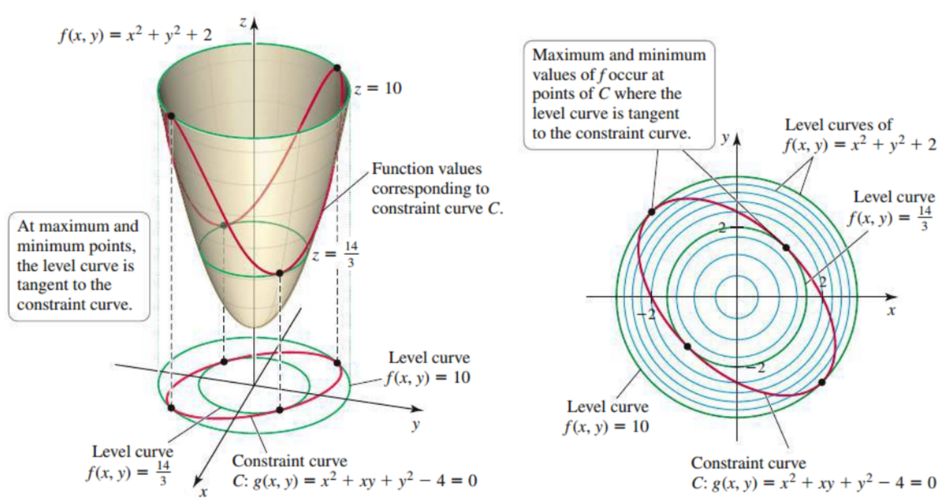


Definition. (Parallel Gradients)
Let f be a differentiable function in a region of $\mathbb{R}^2$ that contains the smooth curve C given by $g(x, y) = 0$. Assume f has a local extreme value on C at a point $P(a, b)$. Then $\nabla f(a, b)$ is orthogonal to the line tangent to C at P. Assuming $\nabla g(a, b) \neq \mathbf{0}$, it follows that there is a real number λ (called a **Lagrange multiplier**) such that $\nabla f(a, b) = \lambda \nabla g(a, b)$.

We consider the three following cases:

- Bounded constraint curves that close on themselves (e.g. circles, ellipses, etc),
- Bounded constraint curves that do not close on themselves, but include endpoints,
- Unbounded constraint curves

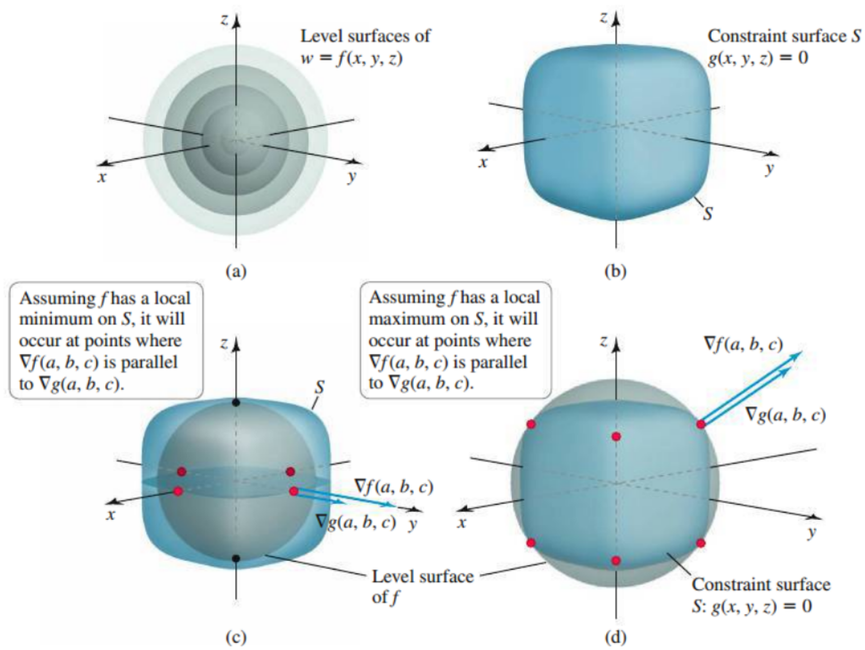
Example. Find the absolute maximum and minimum values of the objective function $f(x,y) = x^2+y^2+2$, where x and y lie on the ellipse C given by $g(x,y) = x^2+xy+y^2-4 = 0$.



Procedure- Lagrange Multipliers: Absolute Extrema on Closed and Bounded Constraint Curves
Let the objective function f and the constraint function g be differentiable on a region $\mathbb{R}^2$ with $\nabla g(x, y) \neq \mathbf{0}$ on the curve $g(x, y) = 0$. To locate the absolute maximum and minimum values of f subject to the constraint $g(x, y) = 0$, carry out the following steps.

1. Find the values of x, y, and λ (if they exist) that satisfy the equations
$$\nabla f(x, y) = \lambda \nabla g(x, y) \text{ and } g(x, y) = 0.$$
2. Evaluate f at the values (x, y) in Step 1 and at the endpoints of the constraint curve (if they exist). Select the largest and smallest corresponding function values. These values are the absolute maximum and minimum values of f subject to the constraint.

Using Lagrange multipliers extends to higher dimensions with three or more independent variables:



Example. Find the least distance between the point $P(3, 4, 0)$ and the surface of the cone $z^2 = x^2 + y^2$.

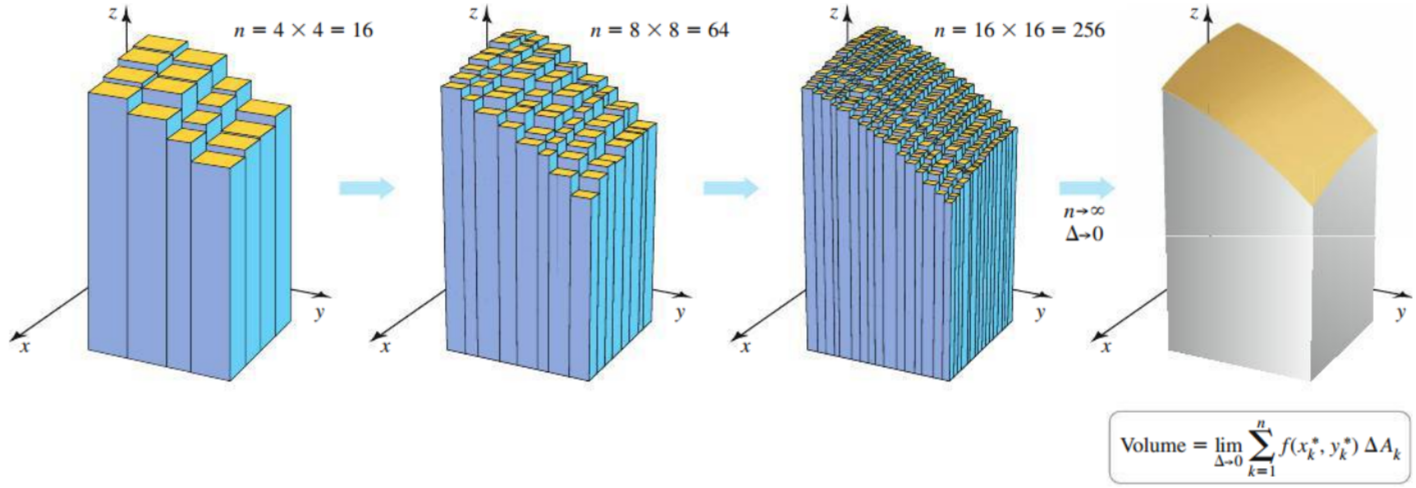
Example. Find the absolute maximum value of the utility function $U = f(\ell, g) = \ell^{1/3}g^{2/3}$, subject to the constraint $G(\ell, g) = 3\ell + 2g - 12 = 0$, where $\ell \geq 0$ and $g \geq 0$.

Example. Find the maximum value of $x_1 + x_2 + x_3 + x_4$ subject to the condition that $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 16$.

Procedure- Lagrange Multipliers: Absolute Extrema on Closed and Bounded Constraint Surfaces
Let f and g be differentiable on a region of $\mathbb{R}^3$ with $\nabla g(x,y,z) \neq \mathbf{0}$ on the surface $g(x,y,z) = 0$. To locate the absolute maximum and minimum values of f subject to the constraint $g(x,y,z) = 0$, carry out the following steps.

1. Find the values of x, y, z, and λ that satisfy the equations
$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) \text{ and } g(x,y,z) = 0.$$
2. Among the points (x,y,z) found in Step 1, select the largest and smallest corresponding function values. These values are the absolute maximum and minimum values of f subject to the constraint.

16.1: Double Integrals over Rectangular Regions



> **Definition. (Double Integrals)**
> A function f defined on a rectangular region R in the xy-plane is **integrable** on R if $\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\,\Delta A_k$ exists for all partitions of R and for all choices of (x_k^*, y_k^*) within those partitions. The limit is the **double integral of f over R**, which we write
>
> $$\iint_R f(x, y)\,dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\,\Delta A_k.$$

Example. Compute the following integral: $\displaystyle\int_0^1 \int_0^2 (6 - 2x - y)\,dy\,dx$

Example. Compute the following integral: $\displaystyle\int_0^2 \int_0^1 (6 - 2x - y)\, dx\, dy$

Theorem 16.1: (Fubini) Double Integrals over Rectangular Regions
Let f be continuous on the rectangular region $R = \{(x, y) : a \leq x \leq b,\ c \leq y \leq d\}$. The double integral of f over R may be evaluated by either of the two iterated integrals:

$$\iint\limits_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

Example. Find the volume of the solid bounded by the surface $f(x, y) = 4 + 9x^2y^2$ over the region $R = \{(x, y) : -1 \leq x \leq 1,\ 0 \leq y \leq 2\}$. Integrate with respect to x first, then with respect to y first.

Example. Evaluate $\displaystyle\iint_R ye^{xy}\,dA$, where $R = \{(x,y) : 0 \le x \le 1, 0 \le y \le \ln(2)\}$.

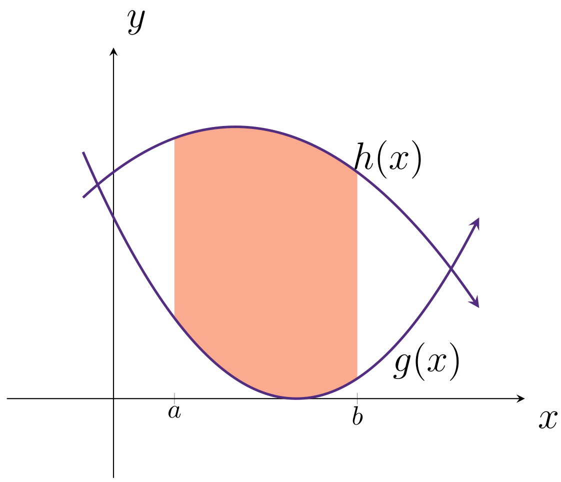
Definition. (Average Value of a Function over a Plane Region)
The **average value** of an integrable function f over a region R is

$$\bar{f} = \frac{1}{\text{area of } R} \iint_R f(x, y)\, dA.$$

Example. Find the average value of $f(x, y) = 2 - x - y$ over the region $R = \{(x, y) : 0 \leq x \leq 2,\ 0 \leq y \leq 2\}$.

16.2: Double Integrals over General Regions

In this section, we consider double integrals over non-rectangular regions. For instance, my domain for x and y can be constrained where $a \leq x \leq b$ and $g(x) \leq y \leq h(x)$:



Theorem 16.2: Double Integrals over Nonrectangular Regions

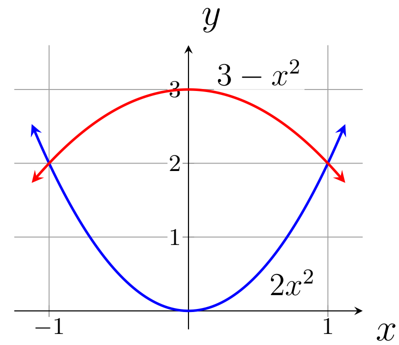
Let R be a region bounded below and above by the graphs of the continuous functions $y = g(x)$ and $y = h(x)$, respectively, and by the lines $x = a$ and $x = b$. If f is continuous on R, then

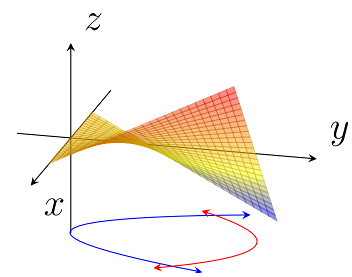
$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx.$$

Let R be a region bounded on the left and right by the graphs of the continuous functions $x = g(y)$ and $x = h(y)$, respectively, and the lines $y = c$ and $y = d$. If f is continuous on R, then

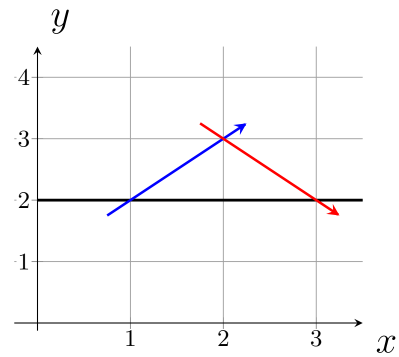
$$\iint\limits_R f(x, y)\, dA = \int_c^d \int_{g(y)}^{h(y)} f(x, y)\, dx\, dy.$$

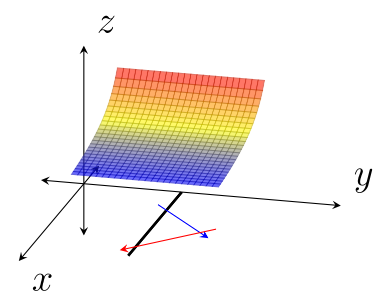
Example. Consider the surface generated by the function $f(x, y) = 3xy$. Find the volume of the solid generated by $f(x, y)$ over the region bounded by $2x^2$ and $3 - x^2$.





Example. Find the area under $f(x, y) = \frac{1}{x} + 1$ over the region formed by the lines $x = 2$, $y = 1 + x$, and $y = 5 - x$.





Example. Find the volume of the tetrahedron in the first octant bounded by the plane $z = c - ax - by$ and the coordinate planes ($x = 0$, $y = 0$, and $z = 0$). Assume a, b, and c are positive real numbers.

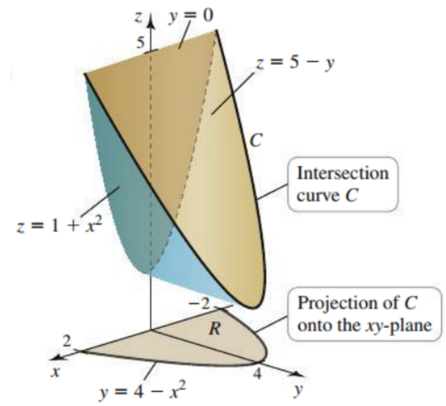
Example. For the following problems, reverse the order of integration

- $\displaystyle\int_0^2 \int_0^{2x} f(x,y)\,dy\,dx$

- $\displaystyle\int_0^1 \int_{x^3}^{\sqrt{x}} f(x,y)\,dy\,dx$

- $\displaystyle\int_{-3}^4 \int_{2x^2}^{2x+24} f(x,y)\,dy\,dx$

Example. Find the volume between $f(x, y) = 5 - y$ and $g(x, y) = 1 + x^2$ over the region $R = \{(x, y) : 0 \leq y \leq 4 - x^2,\ -2 \leq x \leq 2\}$.



Areas of Regions by Double Integrals
Let R be a region in the xy-plane. Then

$$\text{area of } R = \iint\limits_R dA.$$

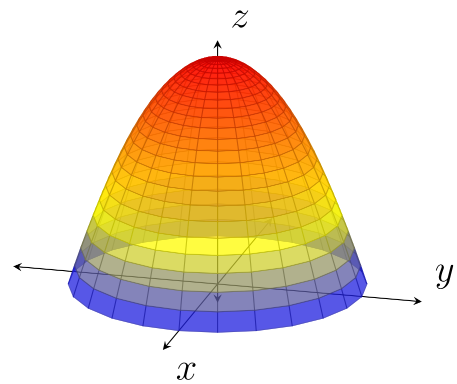
Example. Find the area of the region R bounded by $y = x^2$, $y = 6 - x$, and $y = 6 + 5x$ where $x \geq 0$.

16.3: Double Integrals in Polar Coordinates

Suppose we wish to find the volume bounded by the curve $f(x, y) = 9 - x^2 - y^2$ and the xy-plane. The region of integration would be

$$R = \left\{(x, y) : -3 \leq x \leq 3, -\sqrt{9 - x^2} \leq y \leq \sqrt{9 - x^2}\right\}$$

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 9 - x^2 - y^2 \, dy \, dx$$



Alternatively, we can use polar coordinates where $x = r\cos(\theta)$ and $y = r\sin(\theta)$. The associated region R is called a **polar rectangle**.

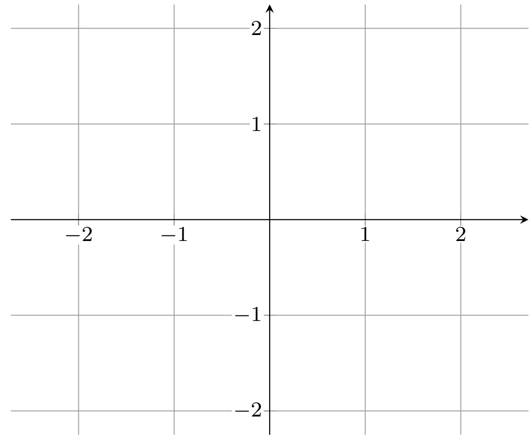
Theorem 16.3: Change of Variables for Double Integrals over Polar Rectangle Regions
Let f be continuous on the region R in the xy-plane expressed in polar coordinates a s$R = \{(r, \theta) : 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\}$, where $\beta - \alpha = 2\pi$. Then f is integrable over R, and the double integral of f over R is

$$\iint_R f(x, y) \, dA = \int_{\alpha}^{\beta} \int_{a}^{b} f(r\cos\theta, \, r\sin\theta) \, r \, dr \, d\theta.$$

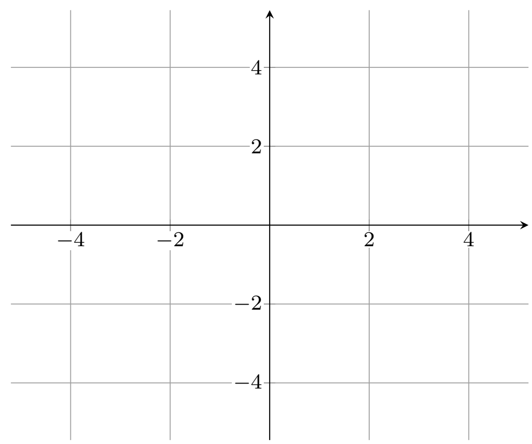
Note: When we convert to polar coordinates, there is an extra factor of r. This is due to the area of the circular segment being $\frac{1}{2}r^2\theta$ (Section 16.7 will elaborate on this).

Example. Graph the following regions:

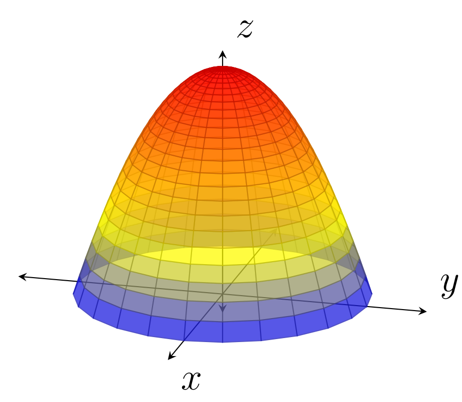
$$R = \left\{ (r, \theta) : 0 \le r \le 1,\ 0 \le \theta \le \frac{5\pi}{4} \right\}$$



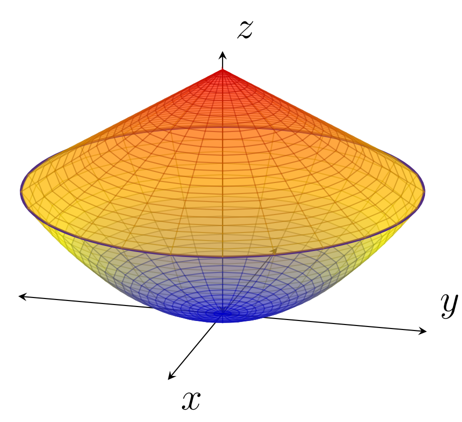
$$R = \left\{ (r, \theta) : 2 \le r \le 4,\ -\frac{\pi}{6} \le \theta \le \frac{7\pi}{6} \right\}$$



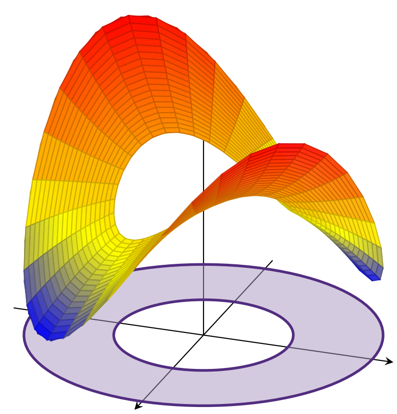
Example. Consider the paraboloid given earlier: Find the volume of the solid bounded above by $z = 9 - x^2 - y^2$ and below by the xy-plane.



Example. Find the area of the solid bounded below by the paraboloid $z = x^2 + y^2$ and bounded above by the cone $z = 2 - \sqrt{x^2 + y^2}$.



Example. Find the volume of the region beneath the surface $z = xy + 10$ and above the annular region $R = \{(r, \theta) : 2 \leq r \leq 4,\ 0 \leq \theta \leq 2\pi\}$.



Theorem 16.4: Change of Variables for Double Integrals over More General Polar Regions
Let f be continuous on the region R in the xy-plane expressed in polar coordinates as

$$R = \{(r, \theta) : 0 \leq g(\theta) \leq r \leq h(\theta),\ \alpha \leq \theta \leq \beta\},$$

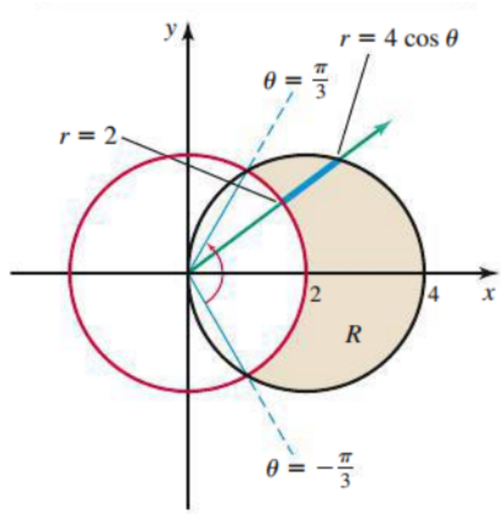
where $0 < \beta - \alpha \leq 2\pi$. Then

$$\iint\limits_R f(x, y)\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

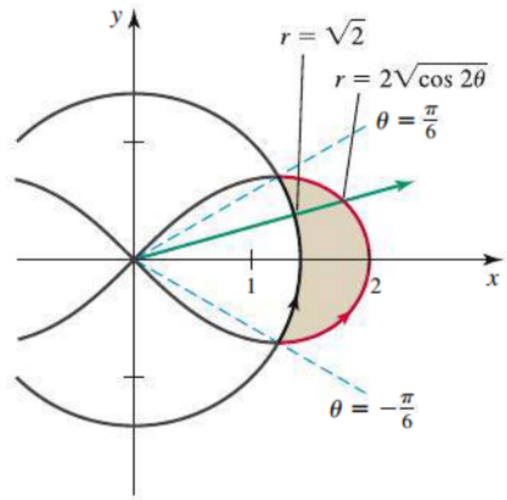
Area of Polar Regions
The area of the polar region $R = \{(r, \theta) : 0 \leq g(\theta) \leq r \leq h(\theta),\ \alpha \leq \theta \leq \beta\}$, where $0 < \beta - \alpha \leq 2\pi$, is

$$A = \iint\limits_R dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} r\, dr\, d\theta.$$

Example. Write an iterated integral in polar coordinates for $\iint_R g(r, \theta)\, dA$ for the region outside the circle $r = 2$ and inside the circle $r = 4\cos(\theta)$.



Example. Compute the area of the region in the first and fourth quadrants outside the circle $r = \sqrt{2}$ and inside the lemniscate $r^2 = 4\cos(2\theta)$.



Example. Find the average value of the y-coordinates of the points in the semicircular disk of radius a given by $R = \{(r, \theta) : 0 \leq r \leq a,\ 0 \leq \theta \leq \pi\}$.

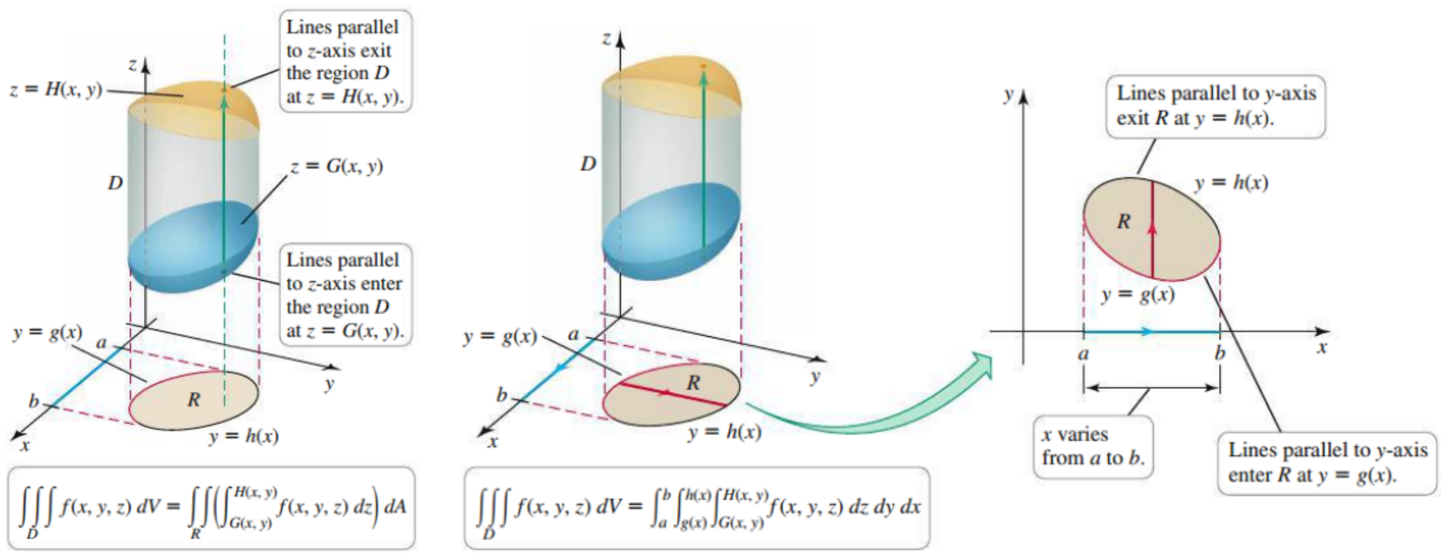
16.4: Triple Integrals

Theorem 16.5: Triple Integrals
Let f be continuous over the region

$$D = \{(x, y, z) : a \leq x \leq b,\ g(x) \leq y \leq h(x),\ G(x, y) \leq z \leq H(x, y)\},$$

where g, h, G, and H are continuous functions. Then f is integrable over D and the triple integral is evaluated as the iterated integral

$$\iiint\limits_D f(x, y, z)\, dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x, y, z)\, dz\, dy\, dx.$$



Integral	**Variable**	**Interval**
Inner	z	$G(x, y) \leq z \leq H(x, y)$
Middle	y	$g(x) \leq y \leq h(x)$
Outer	x	$a \leq x \leq b$

Example. A solid box D is bounded by the planes $x = 0$, $x = 3$, $y = 0$, $y = 2$, $z = 0$, and $z = 1$. The density of the box decreases linearly in the positive z-direction and is given by $f(x, y, z) = 2 - z$. Find the mass of the box.

Example. Find the volume of the prism D in the first octant bounded by the planes $y = 4 - 2x$ and $z = 6$.

Example. Write the triple integral for $\iiint_D f(x,y,z)\,dV$ where D is a sphere of radius r centered at the origin.

Example. Find the volume of the solid D bounded by the paraboloids $y = x^2 + 3z^2 + 1$ and $y = 5 - 3x^2 - z^2$.

The concept of changing the order of integration for double integrals also extends to triple integrals:

Example. Consider the integral

$$\int_0^{\sqrt[4]{\pi}} \int_0^z \int_y^z 12y^2z^3 \sin\left(x^4\right) dx\, dy\, dz.$$

Sketch the region of integration, then evaluate the integral by changing the order of integration.

Definition. (Average Value of a Function of Three Variables)
If f is continuous on a region D of $\mathbb{R}^3$, then the **average value** of f over D is

$$\bar{f} = \frac{1}{\text{volume of } D} \iiint\limits_D f(x, y, z)\, dV.$$

Example. Find the average y-coordinate of the points in the standard simplex
$D = \{(x, y, z) : x + y + z \leq 1,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$.

16.5: Triple Integrals in Cylindrical and Spherical Coordinates

Cylindrical coordinates:

The concept of polar coordinates in $\mathbb{R}^2$ from section 16.3 can be extended to $\mathbb{R}^3$. This coordinate system is called *cylindrical coordinates* where every point P in $\mathbb{R}^3$ has coordinates (r, θ, z), where $0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$, and $-\infty < z < \infty$.

Transformations between Cylindrical and Rectangular Coordinates

Rectangular → Cylindrical	**Cylindrical → Rectangular**
$r^2 = x^2 + y^2$	$x = r\cos\theta$
$\tan\theta = y/x$	$y = r\sin\theta$
$z = z$	$z = z$

Example. Sketch the following sets represented in cylindrical coordinates:

$\{(r, \theta, z) : r = a\}, a > 0$

$\{(r, \theta, z) : 0 < a \leq r \leq b\}$

$\{(r, \theta, z) : z = a\}$

$\{(r, \theta, z) : z = ar\}, a \neq 0$

$\{(r, \theta, z) : \theta = \theta_0\}$

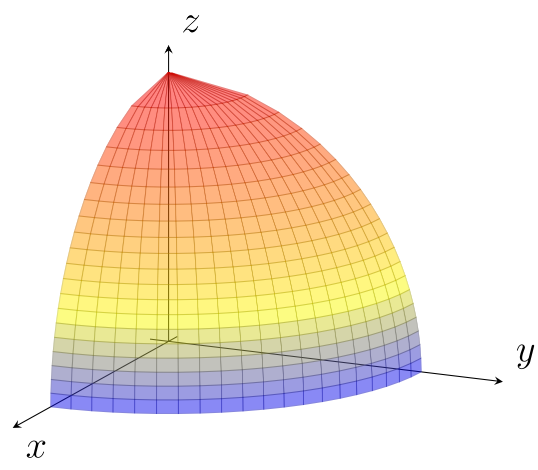
Theorem 16.6: Change of Variables for Triple Integrals in Cylindrical Coordinates
Let f be continuous over the region D, expressed in cylindrical coordinates as

$$D = \{(r, \theta, z) : 0 \leq g(\theta) \leq r \leq h(\theta),\ \alpha \leq \theta \leq \beta,\ G(x, y) \leq z \leq H(x, y)\}$$

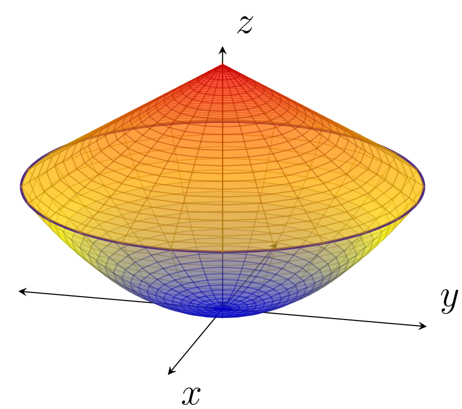
Then f is integrable over D, and the triple integral of f over D is

$$\iiint\limits_D f(x, y, z)\, dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta,\ r\sin\theta)}^{H(r\cos\theta,\ r\sin\theta)} f(r\cos\theta,\ r\sin\theta)\, dz\, r\, dr\, d\theta.$$

Example. Evaluate the following integral using cylindrical coordinates:
$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{x^2+y^2}} \left(x^2+y^2\right)^{-1/2} \, dz \, dy \, dx$$



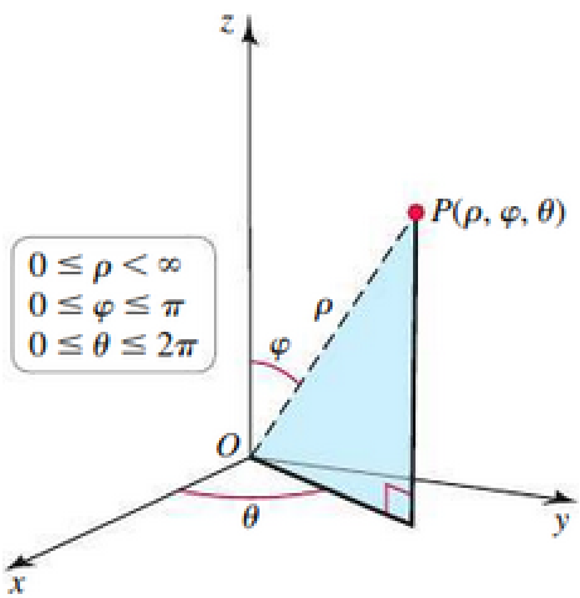
Example. Find the volume of the solid bounded below by the paraboloid $z = x^2 + y^2$ and bounded above by the cone $z = 2 - \sqrt{x^2 + y^2}$.

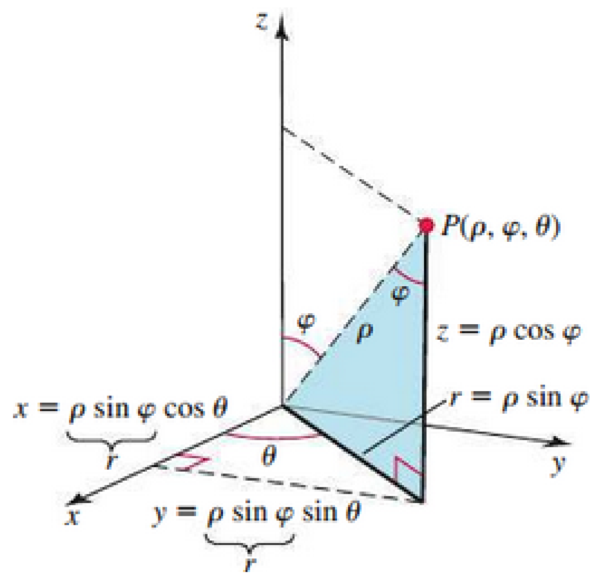


Spherical Coordinates:

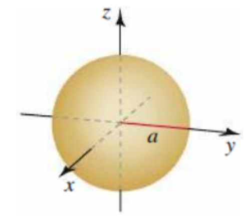
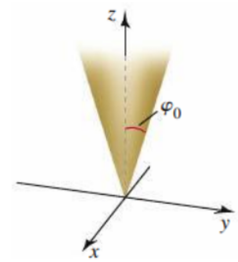
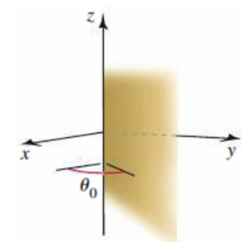
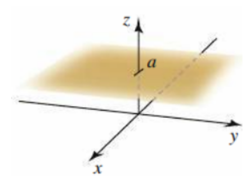
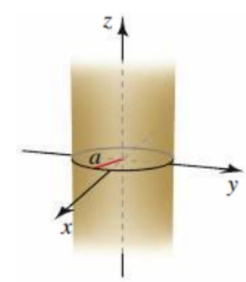
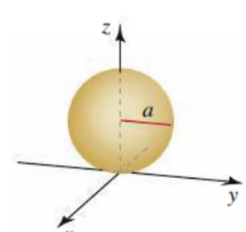
Spherical coordinates can represent a point P in $\mathbb{R}^3$ as (ρ, φ, θ) where

- ρ is the distance from the origin to P,
- φ is the angle between the positive z-axis and the line OP, and
- θ is the same angle as in cylindrical coordinates.





Transformations between Spherical and Rectangular Coordinates	
Rectangular → Spherical	**Spherical → Rectangular**
$\rho^2 = x^2 + y^2 + z^2$	$x = \rho \sin(\varphi)\cos(\theta)$
Use trigonometry to find	$y = \rho \sin(\varphi)\sin(\theta)$
φ and θ.	$z = \rho \cos(\varphi)$

Name	Description	Example
Sphere, radius a, center $(0,0,0)$	$\{(\rho,\varphi,\theta) : \rho = a\}, a > 0$	
Cone	$\{(\rho,\varphi,\theta) : \varphi = \varphi_0\}, \varphi_0 \neq 0, \pi/2, \pi$	
Vertical half-plane	$\{(\rho,\varphi,\theta) : \theta = \theta_0\}$	
Horizontal plane, $z = a$	$a > 0 : \{(\rho,\varphi,\theta) : \rho = a\sec(\varphi),\ 0 \leq \varphi < \pi/2\}$ $a < 0 : \{(\rho,\varphi,\theta) : \rho = a\sec(\varphi),\ \pi/2 < \varphi \leq \pi\}$	
Cylinder, radius $a > 0$	$\{(\rho,\varphi,\theta) : \rho = \alpha\csc(\varphi),\ 0 < \varphi < \pi\}$	
Sphere, radius $a > 0$ center $(0,0,a)$	$\{(\rho,\varphi,\theta) : \rho = 2a\cos(\varphi),\ 0 \leq \varphi \leq \pi/2\}$	

Theorem 16.7: Change of Variables for Triple Integrals in Spherical Coordinates
Let f be continuous over the region D, expressed in spherical coordinates as

$$D = \{(\rho, \varphi, \theta) : 0 \leq g(\varphi, \theta) \leq \rho \leq h(\varphi, \theta),\ a \leq \varphi \leq b,\ \alpha \leq \theta \leq \beta\}.$$

Then f is integrable over D, and the triple integral of f over D is

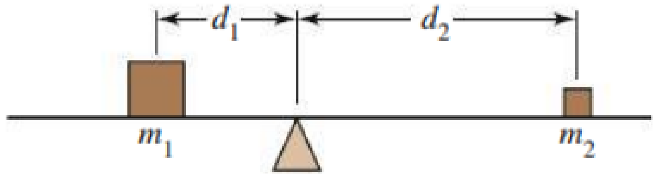
$$\iiint\limits_D f(x, y, z)\, dV$$
$$= \int_\alpha^\beta \int_a^b \int_{g(\varphi,\theta)}^{h(\varphi,\theta)} f(\rho \sin(\varphi) \cos(\theta),\ \rho \sin(\varphi) \sin(\theta),\ \rho \cos(\varphi))\, \rho^2 \sin(\varphi)\ d\rho\, d\varphi\, d\theta.$$

Example. Evaluate $\iiint\limits_D \left(x^2 + y^2 + z^2\right)^{-3/2} dV$, where D is the region in the first octant between two spheres of radius 1 and 2 centered at the origin

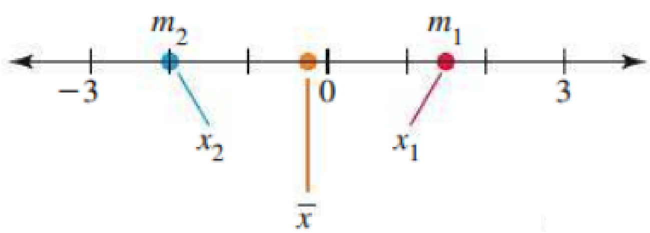
Example. Find the volume of the solid region D that lies inside the cone $\varphi = \pi/6$ and inside the sphere $\rho = 4$.

16.6: Integrals for Mass Calculations

Suppose we have two masses m_1 and m_2 on a beam (with no mass) that are distances d_1 and d_2 away from a pivot point. This beam will be balanced when $m_1 d_1 = m_2 d_2$.



This concept can be used to to find the balance point $\bar{x}$ between 2 objects with masses m_1 and m_2:



$$m_1(x_1 - \bar{x}) = m_2(\bar{x} - x_2) \;\Rightarrow\; m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) = 0.$$

$\Rightarrow \bar{x} =$

Next, we can generalize this to n objects with masses $m_1, \ldots, m_n$:

$$m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) + \cdots + m_n(x_n - \bar{x}) = \sum_{k=1}^{n} m_k(x_k - \bar{x}) = 0.$$

$\Rightarrow \bar{x} =$

Definition. (Center of Mass in One Dimension)
Let ρ be an integrable density function on the interval $[a, b]$ (which represents a thin rod or wire). The **center of mass** is located at the point $\bar{x} = \frac{M}{m}$, where the **total moment** M and mass m are

$$M = \int_a^b x\rho(x)\,dx \quad \text{and} \quad m = \int_a^b \rho(x)\,dx.$$

Example. Find the mass and center of mass of the thin rods with the following density functions:

$\rho(x) = 2 + \cos(x)$, for $0 \leq x \leq \pi$

$$\rho(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1 \\ x(2-x) & \text{if } 1 < x \leq 2 \end{cases}$$

Definition. (Center of Mass in Two Dimensions)
Let ρ be an integrable area density function defined over a closed bounded region R in $\mathbb{R}^2$. The coordinates of the center of mass of the object represented by R are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint\limits_R x\rho(x,y)\,dA \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint\limits_R y\rho(x,y)\,dA,$$

where $m = \iint_R \rho(x,y)\,dA$ is the mass, and M_y and M_x are the moments with respect to the y-axis and x-axis, respectively. If ρ is constant, the center of mass is called the **centroid** and is independent of the density.

Example. Find the center of mass of the following plane regions with variable density:

$R = \{(x,y) : 0 \le x \le 4,\ 0 \le y \le 2\};\ \rho(x,y) = 1 + x/2.$

The quarter disk in the first quadrant bounded by $x^2+y^2=4$ with $\rho(x,y)=1+x^2+y^2$.

Definition. (Center of Mass in Three Dimensions)
Let ρ be an integrable area density function defined over a closed bounded region D in $\mathbb{R}^3$. The coordinates of the center of mass of the region are

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m} \iiint\limits_D x\rho(x,y,z)\,dV$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1}{m} \iiint\limits_D y\rho(x,y,z)\,dV$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m} \iiint\limits_D z\rho(x,y,z)\,dV$$

where $m = \iiint_D \rho(x,y,z)\,dA$ is the mass, and M_{yz}, M_{xz}, and M_{xy} are the moments with respect to the coordinate planes.

16.7: Change of Variables in Multiple Integrals

Recall in Calculus I that the Substitution Rule allow us to perform a *change of variables*. Reversing the order, if we set $x = g(u)$ and $dx = g'(u)du$,

$$\int_{g(c)}^{g(d)} f(x)\,dx \overset{\text{sub}}{=} \int_c^d f(g(u))g'(u)\,du$$

In Sections 16.3 and 16.5, we used a change of variables to rewrite our integrals in terms of polar curves (2D) and cylindrical/spherical (3D). What if we want to change our current coordinate system to *any* coordinate system?

Change of Coordinates in 2D

Definition. (Jacobian)
Suppose $x = g(u, v)$ and $y = h(u, v)$ represent a transformation from the $uv-$coordinate system to $xy-$coordinate system. The **Jacobian** of this transformation is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Example. Compute the Jacobian for the transformation $x = r\cos\theta$ and $y = r\sin\theta$

Theorem 16.8: Change of Variables for Double Integrals
Suppose that the transformation $x = g(u, v)$ and $y = g(u, v)$ has a nonzero Jacobian that maps a closed bounded region S in the $uv-$plane to a region R in the $xy-$plane. Let f be a continuous function on R. Then

$$\iint\limits_{R} f(x, y)\, dA_1 = \iint\limits_{S} f(g(u, v), h(u, v))\, |J(u, v)|\, dA_2$$

where $|J(u, v)|$ denotes the absolute value of the Jacobian.

Example. Perform a change of variables for the integral $\iint_R f(x, y)\, dA$, where

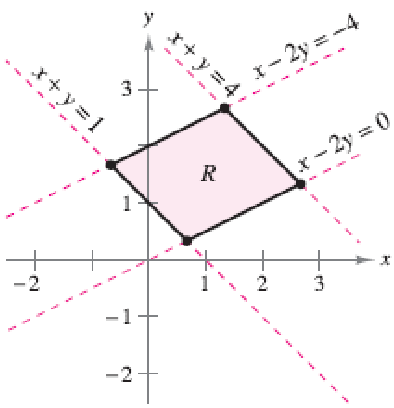
$$R = \left\{(r, \theta) : 0 \le a \le r \le b,\ \alpha \le \theta\beta\right\},\ \beta - \alpha = 2\pi$$

under the transformation $x = r\cos\theta$ and $y = r\sin\theta$.

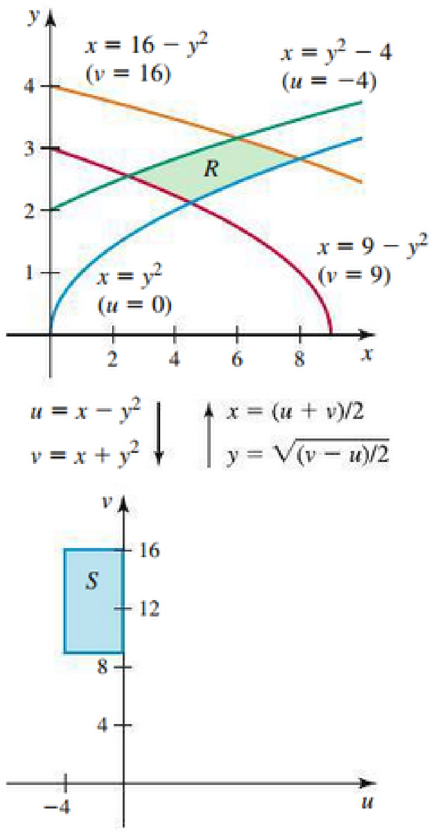
Example. Let R be the region bounded by the lines

$$x - 2y = 0,\ x - 2y = -4,\ x + y = 4,\ \text{and } x + y = 1.$$

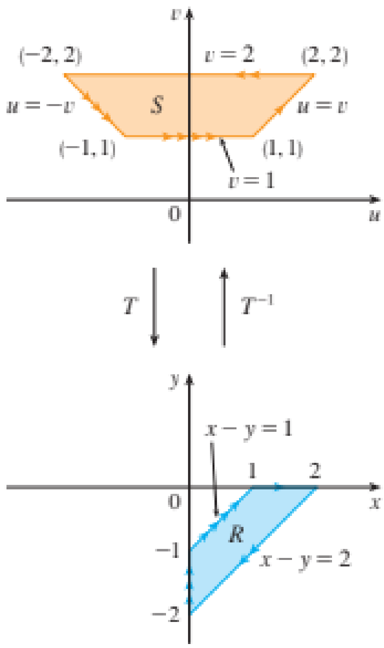
Evaluate the double integral $\displaystyle\iint_R 3xy\, dA$.



Example. Let R be the region in the first quadrant bounded by the parabolas $x = y^2$, $x = y^2 - 4$, $x = 9 - y^2$, and $x = 16 - y^2$. Evaluate $\displaystyle\iint_R y^2 \, dA$



Example. Evaluate the integral $\displaystyle\iint_R e^{(x+y)/(x-y)}\,dA$, where R is the trapezodial region with vertices $(1,0)$, $(2,0)$, $(0,-2)$, and $(0,-1)$.



Change of Coordinates in 3D

We can perform a similar change of variables for triple integrals. For this change in variables, we make the appropriate adjustments from our current work.

Definition. (Jacobian in 3D)
Suppose $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = p(u, v, w)$ represent a transformation from the $uvw-$coordinate system to $xyz-$coordinate system. The **Jacobian** of this transformation is

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Theorem 16.9: Change of Variables for Triple Integrals
Suppose that the transformation $x = g(u, v, w)$ and $y = g(u, v, w)$, and $z = p(u, v, w)$ has a nonzero Jacobian that maps a closed bounded region S in the $uvw-$plane to a region R in the $xyz-$plane. Let f be a continuous function on R. Then

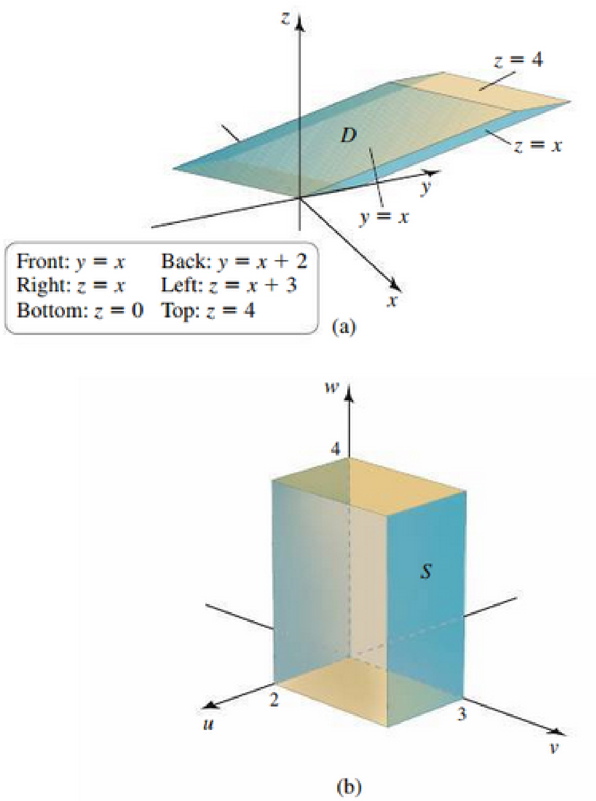
$$\iiint\limits_R f(x, y, z)\, dV_1 = \iiint\limits_S f(g(u, v, w), h(u, v, w), p(u, v, w))\, |J(u, v, w)|\, dV_2$$

Note: In Linear Algebra, the determinant of a matrix represents how much the volume of a region scales under a transformation. Thus, the Jacobian determines how much the volume scales, at a point, under the change of coordinates. This is why we multiply the integrand by the Jacobian, to account for the changes in volume between the two coordinate systems.

Example. Derive the formula for the triple integration in spherical coordinates.

Example. Evaluate $\displaystyle\iiint_D xz\,dV$, where D is a parallelpiped bounded by the planes

$$y = x,\ y = x + 2,\ z = x,\ z = x + 3,\ z = 0,\ \text{and } z = 4$$



Strategies for Choosing New Variables

As when you learned the Substitution Rule for the first time, learning how to choose new variables for multiple integrals will be a bumpy road. Below is a guideline to aid you when you change variables. The guideline is made with respect to double integrals, but you can translate this to triple integrals as well:

1. **Aim for simple regions of integration in the uv−plane:** One goal of changing the variables is to make the region you're integrating over as simple as possible. Hence, perform the transformation where you make the new region straightforward to calculate. It would be ideal if you can transform the new region to a rectangular region.

2. **Determine the easier coordinate system:** You will encounter problems where it's easier to write (x, y) as functions of (u, v); in other problems, the opposite is true.

   - If you already know (x, y) in terms of (u, v), it's straightforward to get the Jacobian and sketching the region R. However, you must invert the transformation to determine the region S.
   - If you already know (u, v) in terms of (x, y), knowing the region S is straightforward. However, the transformation must be inverted to compute the Jacobian.

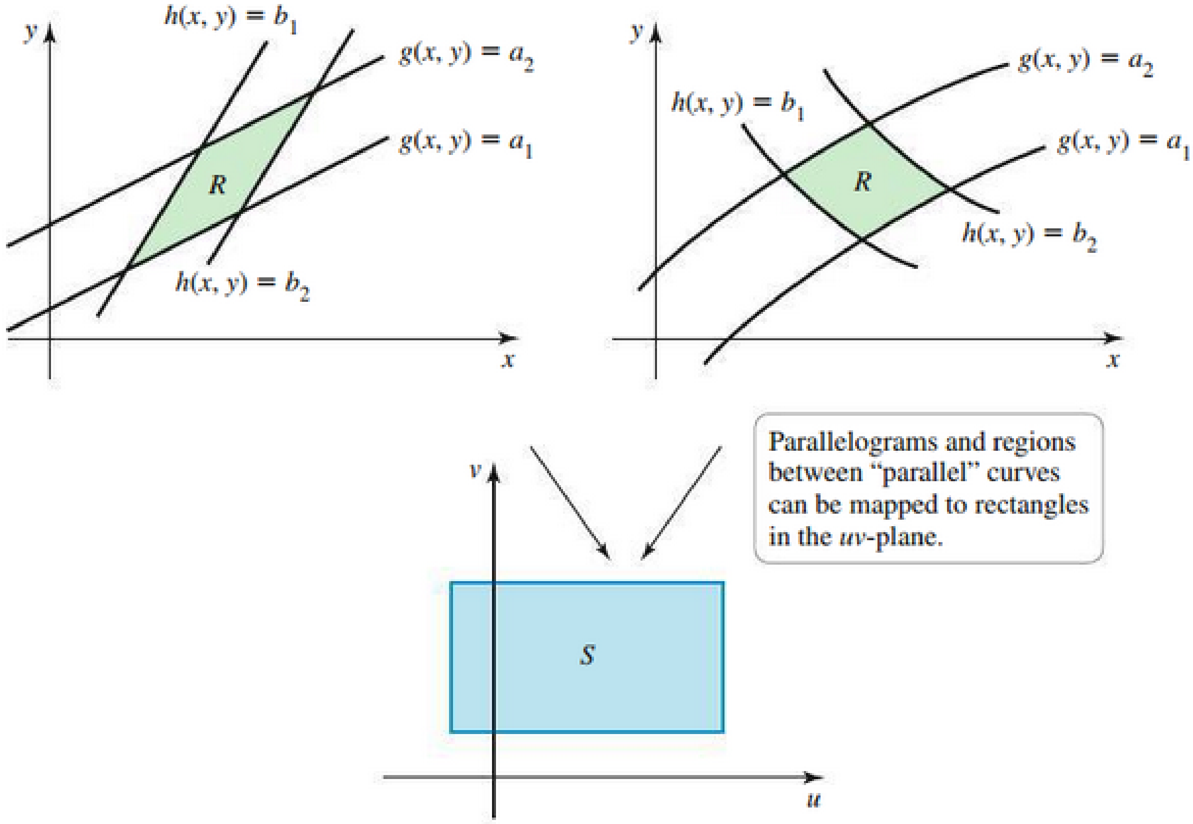
3. **Take a hint from the integrand:** Sometimes the new variables are often chosen based on the integrand. Look for a composition appearing in the integrand and choose the inside functions. For example, if

   $$\sqrt{\frac{x-y}{x+y}}$$

   was the integrand of a double integral, choose the new variables $u = x - y$ and $v = x + y$.

   If a combination of variables appears, let one new variable be the entire combination and set the other new variable as one of the original variables. For example, if the integrand of the double integral is $(x + 4y)^{3/2}$ choose the new variables $u = x + 4y$ and $v = y$.

4. **Take a hint from the given region:** If you have a region that's bounded by two pairs of "parallel" curves, you can transform the given region into a rectangular region. An illustration is provided below. In both regions, we can transform them into the rectangle
$$S = \{(u, v) : a_1 \leq u \leq a_2,\ b_1 \leq v \leq b_2\}$$
with $u = g(x, y)$ and $v = h(x, y)$.



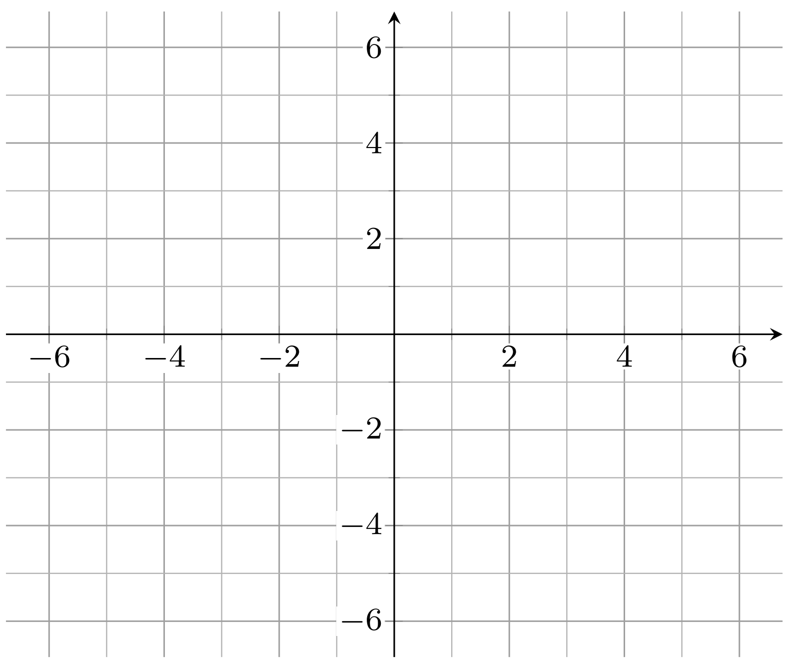
17.1: Vector Fields

Definition. (Vector Fields in Two Dimensions)
Let f and g be defined on a region R of $\mathbb{R}^2$. A **vector field** in $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point in R a vector $\langle f(x,y),\, g(x,y)\rangle$. The vector field is written as

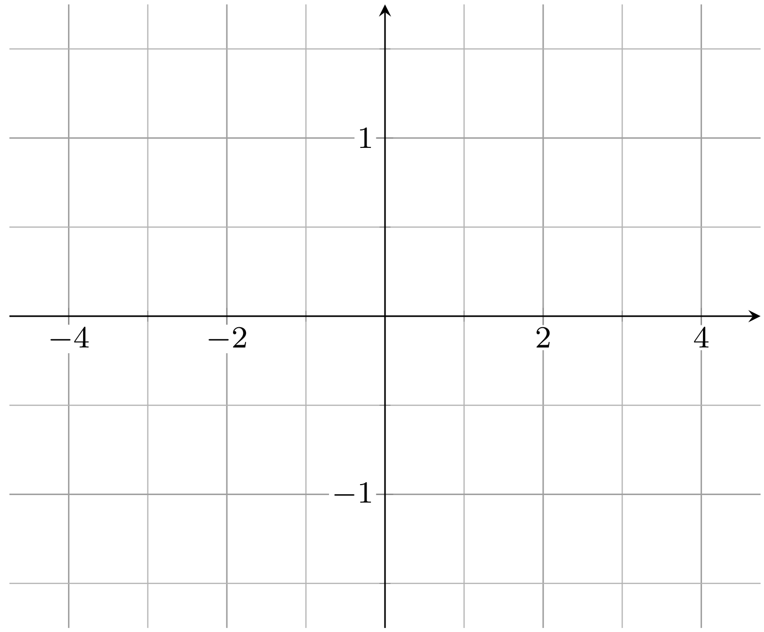
$$\mathbf{F}(x,y) = \langle f(x,y),\, g(x,y)\rangle \quad \text{or}$$
$$\mathbf{F}(x,y) = f(x,y)\boldsymbol{i} + g(x,y)\boldsymbol{j}.$$

A vector field $\mathbf{F} = \langle f, g\rangle$ is continuous or differentiable on a region R of $\mathbb{R}^2$ if f and g are continuous or differentiable on R, respectively.

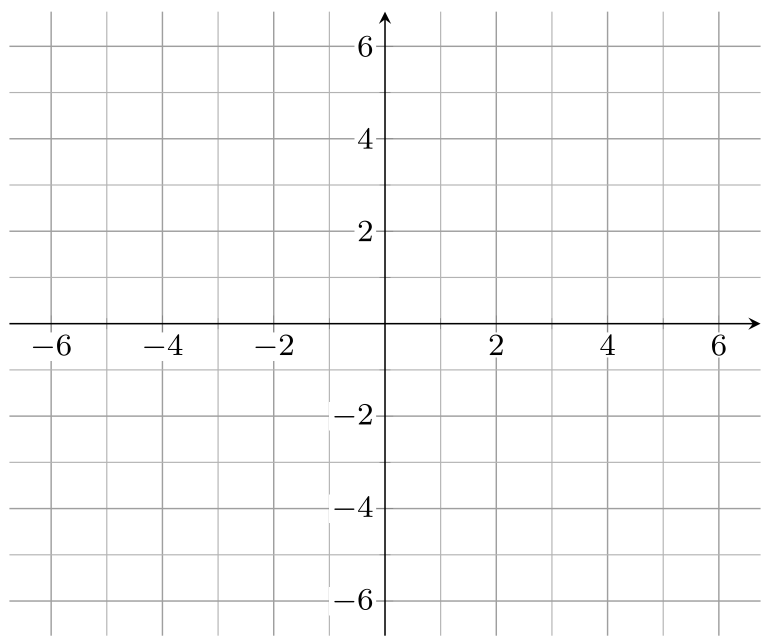
Example. Sketch the vector field $\mathbf{F} = \langle 0, x\rangle$.



Example. Sketch the vector field $\mathbf{F} = \left\langle 1 - y^2, 0 \right\rangle$ for $|y| \leq 1$.



Example. Sketch the vector field $\mathbf{F} = \langle -y, x \rangle$.



Definition. (Radial Vector Fields in $\mathbb{R}^2$)
Let $\mathbf{r} = \langle x, y \rangle$. A vector field of the form $\mathbf{F} = f(x, y)\mathbf{r}$, where f is a scalar valued function, is a **radial vector field**. Of specific interest are the radial vector fields

$$\mathbf{F}(x, y) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y \rangle}{|\mathbf{r}|^p} = \frac{\mathbf{r}}{|\mathbf{r}|} \frac{1}{|\mathbf{r}|^{p-1}},$$

where p is a real number. At every point (expect the origin), the vectors of this field are directed outward from the origin with a magnitude of $|\mathbf{F}| = \dfrac{1}{|\mathbf{r}|^{p-1}}$.

Example. Let C be the circle $x^2 + y^2 = a^2$, where $a > 0$.

a) Show that at each point of C, the radial vector field $\mathbf{F}(x, y) = \dfrac{\mathbf{r}}{|\mathbf{r}|} = \dfrac{\langle x, y \rangle}{\sqrt{x^2 + y^2}}$ is orthogonal to the line tangent to C at that point.

b) Show that at each point of C, the rotation vector field $\mathbf{G}(x, y) = \dfrac{\langle -y, x \rangle}{\sqrt{x^2 + y^2}}$ is parallel to the line tangent to C at that point.

Definition. (Vector Fields and Radial Vector Fields in $\mathbb{R}^3$)
Let f, g, and h be defined on a region D of $\mathbb{R}^3$. A **vector field** in $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point in D a vector $\langle f(x,y,z),\, g(x,y,z),\, h(x,y,z)\rangle$. The vector field is written as

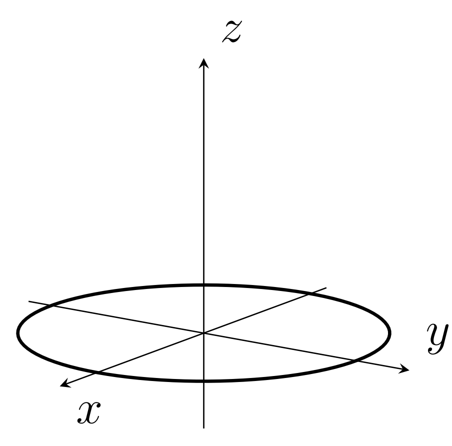
$$\mathbf{F}(x,y,z) = \langle f(x,y,z),\, g(x,y,z),\, h(x,y,z)\rangle \quad \text{or}$$
$$\mathbf{F}(x,y,z) = f(x,y,z)\boldsymbol{i} + g(x,y,z)\boldsymbol{j} + h(x,y,z)\boldsymbol{k}.$$

A vector field $\mathbf{F} = \langle f, g, h\rangle$ is continuous or differentiable on a region D of $\mathbb{R}^3$ if f, g, and h are continuous or differentiable on D, respectively. Of particular importance are the **radial vector fields**

$$\mathbf{F}(x,y,z) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x,y,z\rangle}{|\mathbf{r}|^p},$$

where p is a real number.

Example. Sketch the vector field $\mathbf{F}(x,y,z) = \left\langle 0, 0, 1 - x^2 - y^2 \right\rangle$, for $x^2 + y^2 \leq 1$.



Definition. (Gradient Fields and Potential Functions)
Let φ be differentiable on a region of $\mathbb{R}^2$ or $\mathbb{R}^3$. The vector field $=\nabla\varphi$ is a **gradient field** and the function φ is a **potential function** for $\mathbf{F}$.

Example. Sketch and interpret the gradient field associated with the temperature function $T = 200 - x^2 - y^2$ on the cirular plane $R = \left\{(x, y) : x^2 + y^2 \leq 25\right\}$.

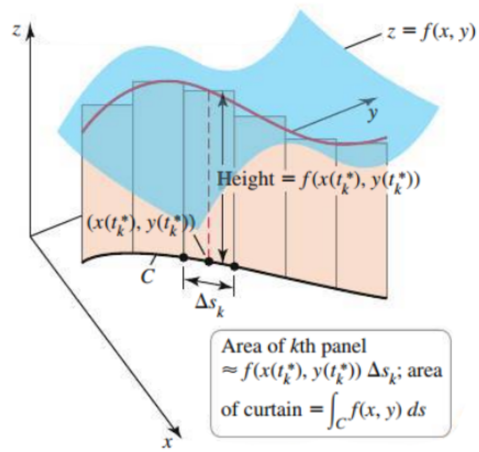
Example. Sketch and interpret the gradient field associated with the velocity potential $\varphi = \tan^{-1}(xy)$.

17.2: Line Integrals

Definition. (Scalar Line Integral in the Plane)
Suppose the scalar-valued function f is defined on a region containing the smooth curve C given by $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$. The **line integral of f over C** is

$$\int_C f(x(t), y(t))\, ds = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x(t_k^*),\, y(t_k^*))\Delta s_k,$$

provided this limit exists over all partitions of $[a, b]$. When the limit exists, f is said to be **integrable** on C.



Theorem 17.1: Evaluating Scalar Line Integrals in $\mathbb{R}^2$
Let f be continuous on a region containing a smooth curve C: $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$. Then

$$\begin{aligned}\int_C f\, ds &= \int_a^b f(x(t),\, y(t))|\mathbf{r}'(t)|\, dt \\ &= \int_a^b f(x(t),\, y(t))\sqrt{x'(t)^2 + y'(t)^2}\, dt.\end{aligned}$$

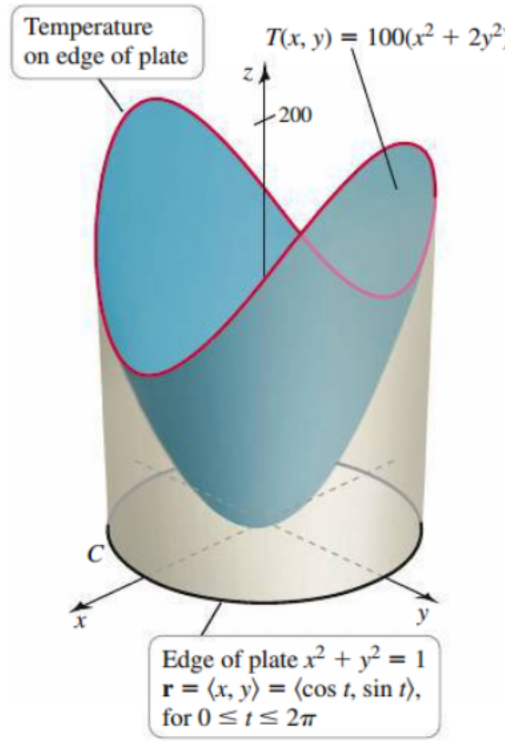
Procedure: Evaluating the Line Integral $\int_C f\, ds$

1. Find a parametric description of C in the form $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$.
2. Compute $|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$.
3. Make substitutions for x and y in the integrand and evaluate an ordinary integral:

$$\int_C f\, ds = \int_a^b f(x(t),\, y(t))|\mathbf{r}'(t)|\, dt.$$

Example. Find the length of the quarter-circle from $(1, 0)$ to $(0, 1)$ with its center at the origin.

Example. The temperature of the circular plate $R = \{(x, y) : x^2 + y^2 \leq 1\}$ is $T(x, y) = 100(x^2 + 2y^2)$. Find the average temperature along the edge of the plate.



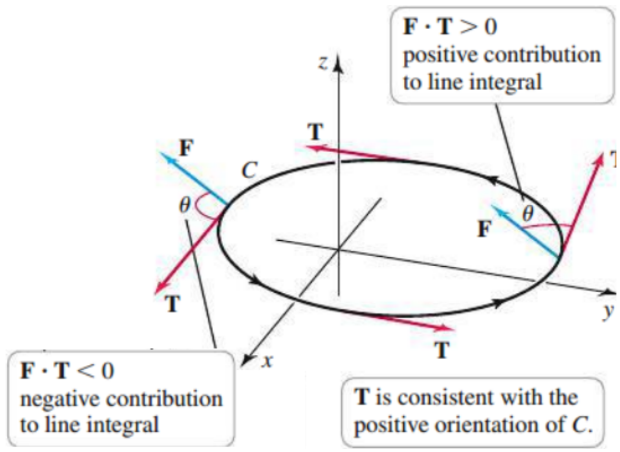
Theorem 17.2: Evaluating Scalar Line Integrals in $\mathbb{R}^3$
Let f be continuous on a region containing a smooth curve $C : \mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$, for $a \le t \le b$. Then

$$\begin{aligned}\int_C f\,ds &= \int_a^b f(x(t),\, y(t),\, z(t))|\mathbf{r}'(t)|\,dt \\ &= \int_a^b f(x(t),\, y(t),\, z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt.\end{aligned}$$

Example. Evaluate $\displaystyle\int_C (x - y + 2z)\,ds$, where C is the circle $\mathbf{r}(t) = \langle 1,\, 3\cos(t),\, 3\sin(t)\rangle$, for $0 \le t \le 2\pi$.

Example. Evaluate $\int_C xe^{yz}\,ds$, where C is $\mathbf{r}(t) = \langle t, 2t, -2t\rangle$, for $0 \le t \le 2$.

Definition. (Line Integral of a Vector Field)
Let $\mathbf{F}$ be a vector field that is continuous on a region containing a smooth oriented curve C parameterized by arc length. Let $\mathbf{T}$ be the unit tangent vector at each point of C consistent with the orientation. The line integral of $\mathbf{F}$ over C is $\displaystyle\int_C \mathbf{F} \cdot \mathbf{T}\, ds$.



Different Forms of Line Integrals of Vector Fields
The line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ may be expressed in the following forms, where $\mathbf{F} = \langle f, g, h\rangle$ and C has a parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, for $a \le t \le b$:

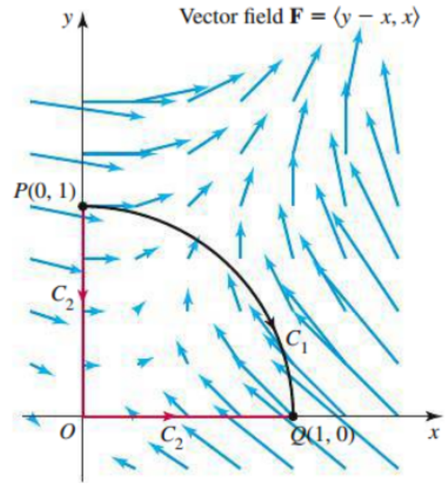
$$\begin{aligned}\int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt &= \int_a^b (f(t)x'(t) + g(t)y'(t) + h(t)z'(t))\, dt \\ &= \int_C f\, dx + g\, dy + h\, dz \\ &= \int_C \mathbf{F} \cdot d\mathbf{r}.\end{aligned}$$

For line integrals in the plane, we let $\mathbf{F} = \langle f, g\rangle$ and assume C is parameterized in the form $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, for $a \le t \le b$. Then

$$\int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt = \int_a^b (f(t)x'(t) + g(t)y'(t))\, dt = \int_C f\, dx + g\, dy = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

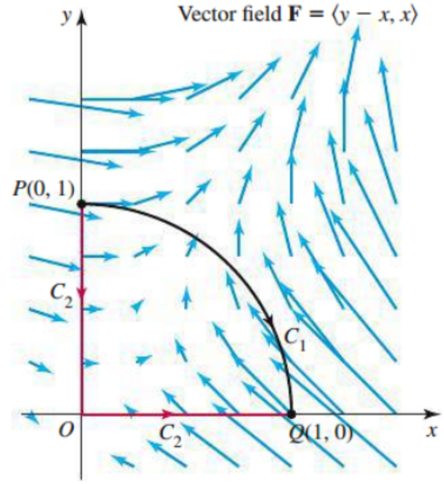
Example. Evaluate $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ with $\mathbf{F} = \langle y - x,\, x\rangle$ on the following oriented paths in $\mathbb{R}^2$.

a) The quarter-circle C_1 from $P(0,1)$ to $Q(1,0)$



b) The quarter-circle $-C_1$ from $Q(1,0)$ to $P(0,1)$

c) the path C_2 from $P(0,1)$ to $Q(1,0)$ via two line segments through $O(0,0)$.



Definition. (Work Done in a Force Field)
Let $\mathbf{F}$ be a continuous force field in a region D of $\mathbb{R}^3$. Let

$$C : \mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle, \text{ for } a \le t \le b,$$

be a smooth curve in D with a unit tangent vector $\mathbf{T}$ consistent with the orientation. The work done in moving an object along C in the positive direction is

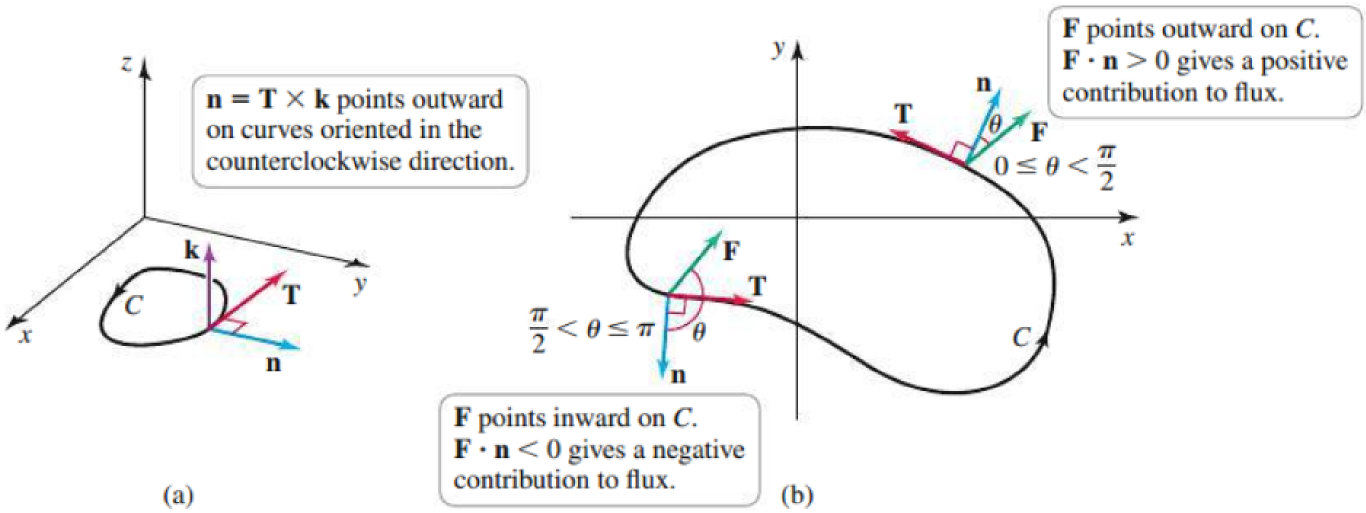
$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt.$$

Example. For the force field $\mathbf{F} = \dfrac{\langle x, y, z\rangle}{(x^2 + y^2 + z^2)^{3/2}}$, calculate the work required to move an object from $(1, 1, 1)$ to $(10, 10, 10)$.

Definition. (Circulation)
Let $\mathbf{F}$ be a continuous vector field on a region D of $\mathbb{R}^3$, and let C be a closed smooth oriented curve in D. The **circulation** of $\mathbf{F}$ on C is $\displaystyle\int_C \mathbf{F}\cdot\mathbf{T}\,ds$, where $\mathbf{T}$ is the unit vector tangent to C consistent with the orientation.

Example. Compute the circulation in the vector field $\mathbf{F} = \dfrac{\langle y, -2x\rangle}{\sqrt{4x^2+y^2}}$ along the curve C given by $\mathbf{r}(t) = \langle 2\cos(t),\, 4\sin(t)\rangle$, for $0 \le t \le 2\pi$.

Flux of the vector field is the total forces orthogonal to each point on the curve C. Let $\mathbf{F} = \langle f, g \rangle$ be a continuous vector field in a region R of $\mathbb{R}^2$. Using $\mathbf{n}$ to represent a unit vector normal to C, the component of $\mathbf{F}$ that is normal to C is $\mathbf{F} \cdot \mathbf{n}$.



Since C is in the xy-plane, the unit tangent vector $\mathbf{T} = \langle T_x, T_y, 0 \rangle$ is also in the xy-plane. We let $\mathbf{n}$ be in the xy-plane as well, but using the cross product of $\mathbf{T}$ and $\boldsymbol{k}$:

$$\mathbf{n} = \mathbf{T} \times \boldsymbol{k} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ T_x & T_y & 0 \\ 0 & 0 & 1 \end{vmatrix} = T_y \boldsymbol{i} - T_x \boldsymbol{j}.$$

Since $\mathbf{T} = \dfrac{\mathbf{r}(t)}{|\mathbf{r}(t)|}$, we have

$$\mathbf{n} = T_y \boldsymbol{i} - T_x \boldsymbol{j} = \frac{y'(t)}{|\mathbf{r}'(t)|}\boldsymbol{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\boldsymbol{j} = \frac{\langle y'(t), -x'(t) \rangle}{|\mathbf{r}'(t)|}.$$

Thus, we have the flux integral

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b \mathbf{F} \cdot \frac{\langle y'(t), -x'(t) \rangle}{|\mathbf{r}'(t)|} |\mathbf{r}'(t)|\, dt = \int_a^b (f(t) y'(t) - g(t) x'(t))\, dt = \int_C f\, dy - g\, dx.$$

Definition. (Flux)
Let $\mathbf{F} = \langle f, g\rangle$ be a continuous vector field on a region R of $\mathbb{R}^2$. Let $C : \mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, $a \le t \le b$, be a smooth orientated curve in R that does not intersect itself. The **flux** of the vector field $\mathbf{F}$ across C is

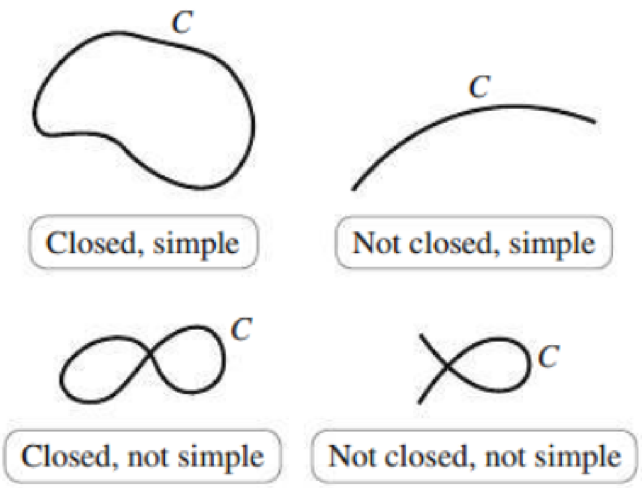
$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_a^b \left(f(t)y'(t) - g(t)x'(t)\right)dt,$$

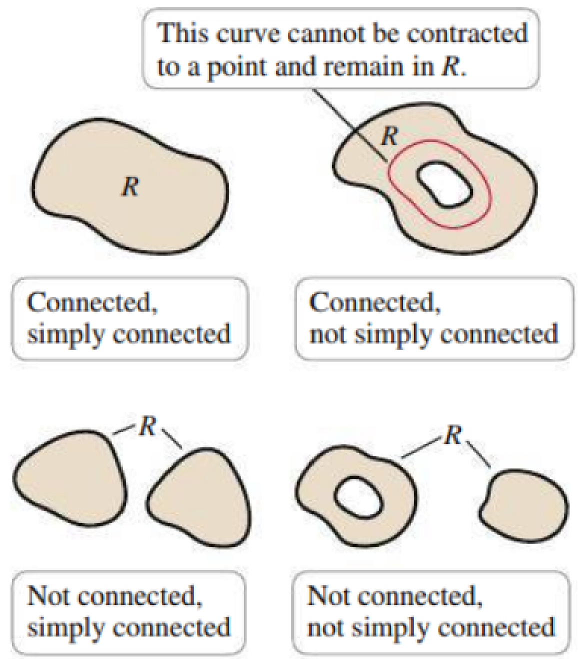
where $\mathbf{n} = \mathbf{T}\times\boldsymbol{k}$ is the unit normal vector and $\mathbf{T}$ is the unit tangent vector consistent with the orientation. If C is a closed curve with counterclockwise orientation, $\mathbf{n}$ is the outward normal vector, and the flux integral gives the **outward flux** across C.

Example. Compute the flux in the vector field $\mathbf{F} = \dfrac{\langle y, -2x\rangle}{\sqrt{4x^2+y^2}}$ along the curve C given by $\mathbf{r}(t) = \langle 2\cos(t),\, 4\sin(t)\rangle$, for $0 \le t \le 2\pi$.

17.3: Conservative Vector Fields

Definition. (Simple and Closed Curves)
Suppose a curve C (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is described parametrically by $\mathbf{r}(t)$, where $a \le t \le b$. Then C is a **simple curve** if $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ for all t_1 and t_2, with $a < t_1 < t_2 < b$; that is, C never intersects itself between its endpoints. The curve C is **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$; that is, the initial and terminal points of C are the same.





Definition. (Connected and Simply Connected Regions)
An open region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) is **connected** if it is possible to connect any two points of R by a continuous curve lying in R. An open region R is **simply connected** if every closed simple curve in R can be deformed and contracted to a point in R.

Definition. (Conservative Vector Field)
A vector field $\mathbf{F}$ is said to be **conservative** on a region (in $\mathbb{R}^2$ or $\mathbb{R}^3$) if there exists a scalar function φ such that $\mathbf{F} = \nabla\varphi$ on that region.

Assume that $\mathbf{F} = \langle f, g, h\rangle$ is a conservative vector field. Then, there exists φ such that

$$\langle f, g, h\rangle = \nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z\rangle$$

Now, we consider the second partial derivatives:

$\varphi_{xy} = \varphi_{yx} \Rightarrow$

$\varphi_{xz} = \varphi_{zx} \Rightarrow$

$\varphi_{yz} = \varphi_{zy} \Rightarrow$

Theorem 17.3: Test for Conservative Vector Fields
Let $\mathbf{F} = \langle f, g, h\rangle$ be a vector field defined on a connected and simply connected region D of $\mathbb{R}^3$, where f, g, and h have continuous first partial derivatives on D. Then $\mathbf{F}$ is a conservative vector field on D (there is a potential function φ such that $\mathbf{F} = \nabla\varphi$) if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \qquad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \qquad \text{and} \qquad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$$

For vector fields in $\mathbb{R}^2$, we have the single condition $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$.

Example. Determine if the following vector fields are conservative:

$$\mathbf{F} = \langle e^x \cos(y), -e^x \sin(y) \rangle$$

$$\mathbf{F} = \left\langle 2xy - z^2, x^2 + 2z, 2y - 2xz \right\rangle$$

Procedure: Finding Potential Functions in $\mathbb{R}^3$

Suppose $\mathbf{F} = \langle f, g, h \rangle$ is a conservative vector field. To find φ such that $\mathbf{F} = \nabla\varphi$, use the following steps:

1. Integrate $\varphi_x = f$ with respect to x to obtain φ, which includes an arbitrary function $c(y, z)$.
2. Compute φ_y and equate it to g to obtain an expression for $c_y(y, z)$.
3. Integrate $c_y(y, z)$ with respect to y to obtain $c(y, z)$, including an arbitrary function $d(z)$.
4. Compute φ_z and equate it to h to get $d(z)$.

A similar procedure beginning with $\varphi_y = g$ or $\varphi_z = h$ may be easier in some cases.

Example. Find a potential function for the following conservative vector fields:

$$\mathbf{F} = \langle e^x \cos(y), -e^x \sin(y) \rangle$$

$$\mathbf{F} = \left\langle 2xy - z^2, x^2 + 2z, 2y - 2xz \right\rangle$$

Fundamental Theorem for Line Integrals and Path Independence:
Suppose that $\mathbf{F}$ is a conservative vector field in $\mathbb{R}^3$ with potential function φ.

$$\begin{aligned}\frac{d\varphi}{dt} &= \frac{\partial\varphi}{\partial x}\frac{dx}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dt} + \frac{\partial\varphi}{\partial z}\frac{dz}{dt} \\ &= \left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \\ &= \nabla\varphi \cdot \mathbf{r}'(t) \\ &= \mathbf{F} \cdot \mathbf{r}'(t),\end{aligned}$$

where $\mathbf{r}(t)$ defines a curve C for $a \le t \le b$. Now, we integrate $\mathbf{F}$ over the curve C:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt = \int_a^b \frac{d\varphi}{dt}\, dt = \varphi(B) - \varphi(A)$$

where A and B are points corresponding to $\mathbf{r}(a)$ and $\mathbf{r}(b)$ respectively.

Theorem 17.4: Fundamental Theorem for Line Integrals
Let R be a region in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let φ be a differentiable potential function defined on R. If $\mathbf{F} = \nabla\varphi$ (which means that $\mathbf{F}$ is conservative), then

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A),$$

for all points A and B in R and all piecewise-smooth oriented curves C in R from A to B.

Definition. (Independence of Path)
Let $\mathbf{F}$ be a continuous vector field with domain R. If $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for all piecewise-smooth curves C_1 and C_2 in R with the same initial and terminal points, then the line integral is **independent of path**.

Theorem 17.5
Let $\mathbf{F}$ be a continuous vector field on an open connected region R in $\mathbb{R}^2$. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, then $\mathbf{F}$ is conservative; that is, there exists a potential function φ such that $\mathbf{F} = \nabla\varphi$ on R.

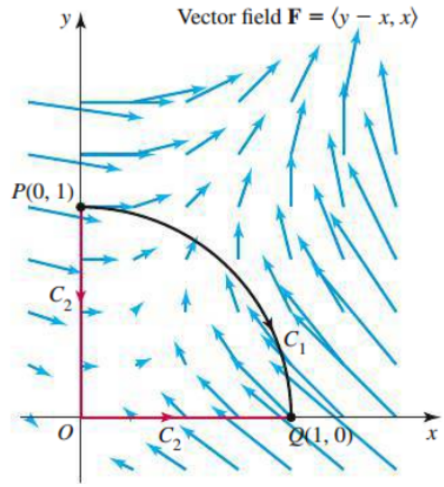
Example. Consider the potential function $\varphi(x, y) = \left(x^2 - y^2\right)/2$ with gradient field $\mathbf{F} = \langle x, -y\rangle$.

- Let C_1 be the quarter-circle $\mathbf{r}(t) = \langle \cos(t), \sin(t)\rangle$, for $0 \le t \le \pi/2$, from $A(1, 0)$ to $B(0, 1)$,
- let C_2 be the line $\mathbf{r}(t) = \langle 1 - t, t\rangle$, for $0 \le t \le 1$, also from A to B.

Evaluate the line integrals of $\mathbf{F}$ on C_1 and C_2, and show that both are equal to $\varphi(B) - \varphi(A)$.

Example. With $\mathbf{F} = \langle y - x,\, x\rangle$ on the following oriented paths in $\mathbb{R}^2$.

a) Find the potential function $\varphi(x, y)$



b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along

the quarter-circle C_1 from $P(0, 1)$ to $Q(1, 0)$,

the path C_2 from $P(0, 1)$ to $Q(1, 0)$ via two line segments through $O(0, 0)$.

Example. Evaluate
$$\int_C \left\langle 2xy - z^2,\, x^2 + 2z,\, 2y - 2xz \right\rangle d\mathbf{r}$$
where C is the curve from $A(-3, -2, 1)$ to $B(1, 2, 3)$.

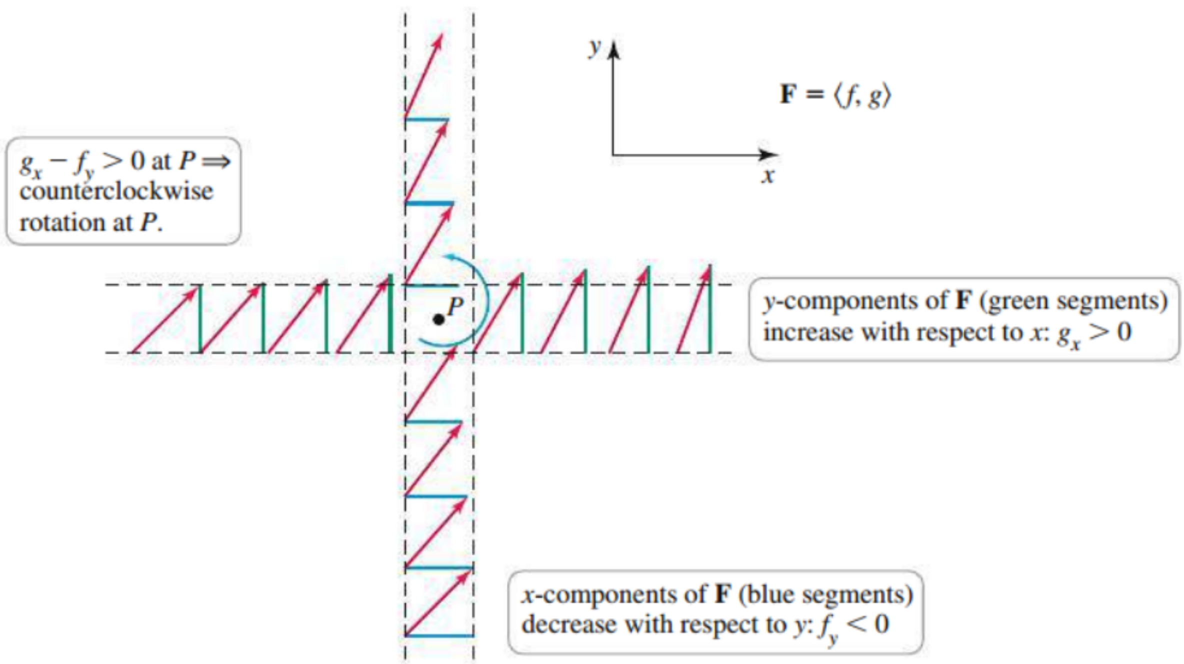
Theorem 17.6: Line Integrals on Closed Curves
Let R be on open connected region in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\mathbf{F}$ is a conservative vector field on R if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple closed piecewise-smooth oriented curves C in R.

Example. Evaluate $\displaystyle\int_C \left\langle 2xy + z^2,\, x^2,\, 2xz \right\rangle \cdot d\mathbf{r}$ where C is the circle $\mathbf{r}(t) = \langle 3\cos(t),\, 4\cos(t),\, 5\sin(t)\rangle$, for $0 \leq t \leq 2\pi$.

17.4: Green's Theorem

Green's Theorem — Circulation Form
Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

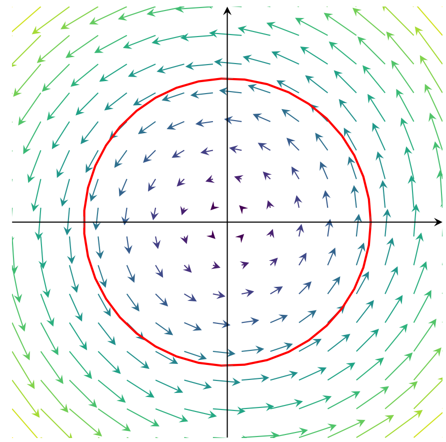
$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{circulation}} = \underbrace{\oint_C f\,dx + g\,dy}_{\text{circulation}} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA.$$



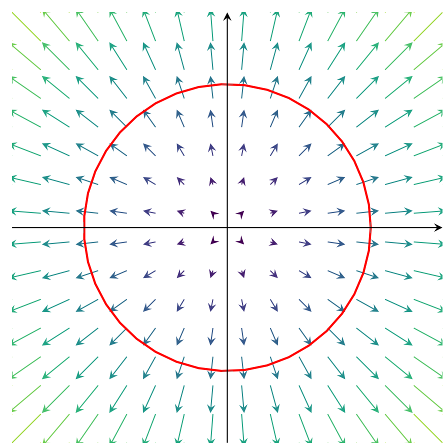
Definition. (Two-Dimensional Curl)
The **two-dimensional curl** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}$. If the curl is zero throughout a region, the vector field is **irrotational** on the region.

Example. Consider the following vector fields $\mathbf{F}$ over the region $R = \{(x, y) : x^2 + y^2 \leq 1\}$. Compute the circulation using Green's Theorem.

$\mathbf{F} = \langle -y, x \rangle$



$\mathbf{F} = \langle x, y \rangle$



Example. Compute the curl of $\mathbf{F} = \left\langle x^2, 2y^2 \right\rangle$ where C is the upper half of the unit circle and the line segment $-1 \leq x \leq 1$.

Area of a Plane Region by Line Integrals

Under the conditions of Green's Theorem, the area of a region R enclosed by a curve C is

$$\oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C (x\,dy - y\,dx).$$

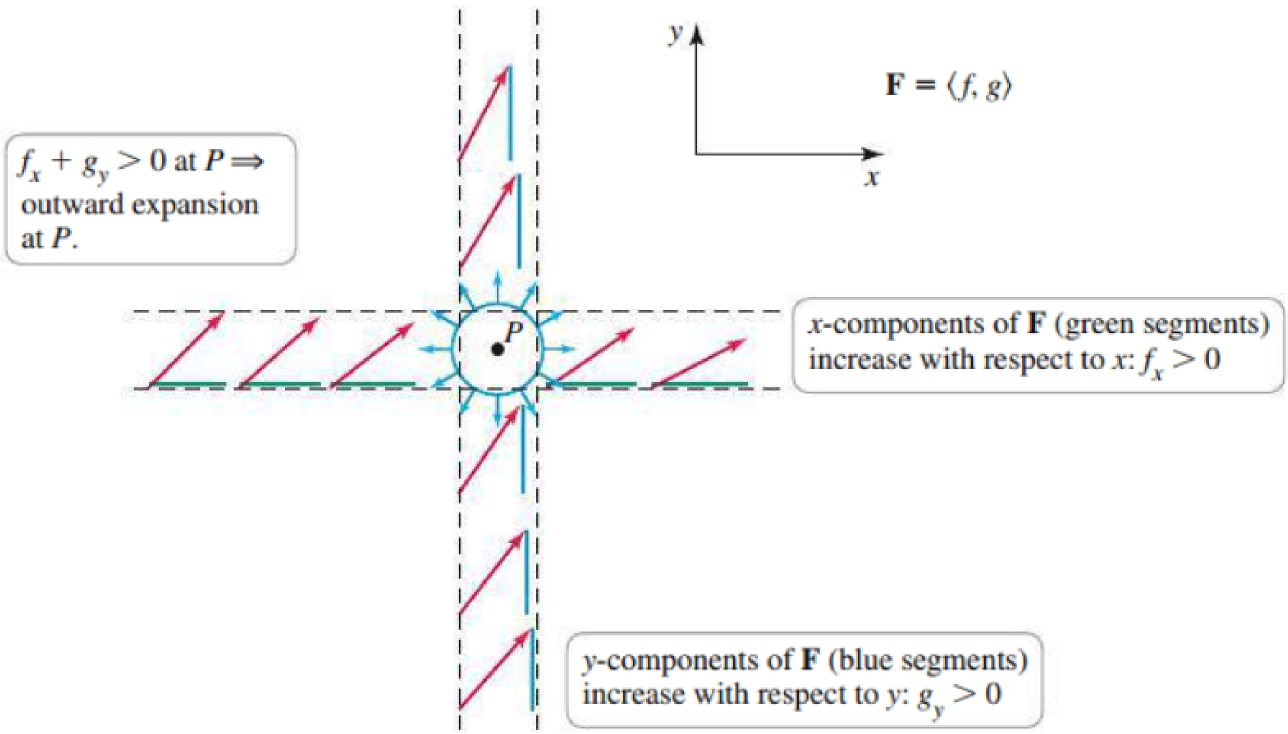
Example. Find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Green's Theorem — Flux Form

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n}\, ds}_{\text{outward flux}} = \underbrace{\oint_C f\, dy - g\, dx}_{\text{outward flux}} = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA,$$

where $\mathbf{n}$ is the outward unit normal vector on the curve.



Definition. (Two-Dimensional Divergence)

The **two-dimensional divergence** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y}$. If the divergence is zero throughout a region, the vector field is **source free** on that region.

Example. Integrate $\oint_C \left(2x + e^{y^2}\right) dy - \left(4y^2 + e^{x^2}\right) dx$, where C is the boundary of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$.

Example. Compute the circulation and outward flux across the boundary of the given regions:

$\mathbf{F} = \langle x, y \rangle$; R is the half-annulus $\{(r, \theta) : 1 \le r \le 2, 0 \le \theta \le \pi\}$,

$\mathbf{F} = \langle -y, x \rangle$; R is the annulus $\{(r, \theta) : 1 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$.

Stream functions:

In the same way that a vector field is conservative if there exists a potential function φ, a vector field is source free if a **stream function** ψ exists such that

$$\frac{\partial \psi}{\partial y} = f, \qquad \frac{\partial \psi}{\partial x} = -g.$$

If such a function exists, then the divergence is zero:

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)}_{\psi_{yx} = \psi_{xy}} = 0$$

If a vector field is both conservative and source-free, then it has both a potential function and a stream function. Furthermore, the level curves of the potential and stream functions form orthogonal families. These vector fields have zero divergence

$$0 = f_x + g_y = \varphi_{xx} + \varphi_{yy},$$

and zero curl

$$0 = g_x - f_y = -\psi_{xx} - \psi_{yy}.$$

Thus, conservative, source-free vector fields satisfy **Laplace's equation**:

$$\varphi_{xx} + \varphi_{yy} = 0 \qquad \text{and} \qquad \psi_{xx} + \psi_{yy} = 0.$$

Example. For $\mathbf{F} = \langle -e^{-x}\sin(y),\, e^{-x}\cos(y)\rangle$

Show $\mathbf{F}$ is conservative and source-free field

Find the potential function φ and the stream function ψ

Conservative Fields $\mathbf{F} = \langle f, g \rangle$	**Source-Free Fields $\mathbf{F} = \langle f, g \rangle$**
curl $= \dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 0$	divergence $= \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 0$
Potential function φ with $\mathbf{F} = \nabla \varphi$ or $f = \dfrac{\partial \varphi}{\partial x}, \quad g = \dfrac{\partial \varphi}{\partial y}$	Stream function ψ with $f = \dfrac{\partial \psi}{\partial y}, \quad g = -\dfrac{\partial \psi}{\partial x}$
Circulation $= \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all closed curves C.	Flux $= \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = 0$ on all closed curves C.
Evaluation of the line integral $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$	Evaluation of the line integral $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \psi(B) - \psi(A)$

Circulation/work integrals: $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C f\,dx + g\,dy$		
	C **closed**	C **not closed**
F conservative ($\mathbf{F} = \nabla\varphi$)	$\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$	$\int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(B) - \varphi(A)$
F not conservative	Green's Theorem $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R (g_x - f_y)\,dA$	Direct evaluation $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b (fx' + gy')\,dt$

Flux integrals: $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_C f\,dy - g\,dx$		
	C **closed**	C **not closed**
F source free ($f = \psi_y, g = -\psi_x$)	$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = 0$	$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \psi(B) - \psi(A)$
F not source free	Green's Theorem $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R (f_x + g_y)\,dA$	Direct evaluation $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_a^b (fy' - gx')\,dt$

Example. Suppose C is a circle centered at the origin, oriented counterclockwise, that encloses disk R in the plane. For $\mathbf{F} = \left\langle 4x^3y, xy^2 + x^4 \right\rangle$

a) Calculate the two-dimensional curl of $\mathbf{F}$

b) Calculate the two-dimensional divergence of $\mathbf{F}$

c) Is $\mathbf{F}$ irrotational on R?

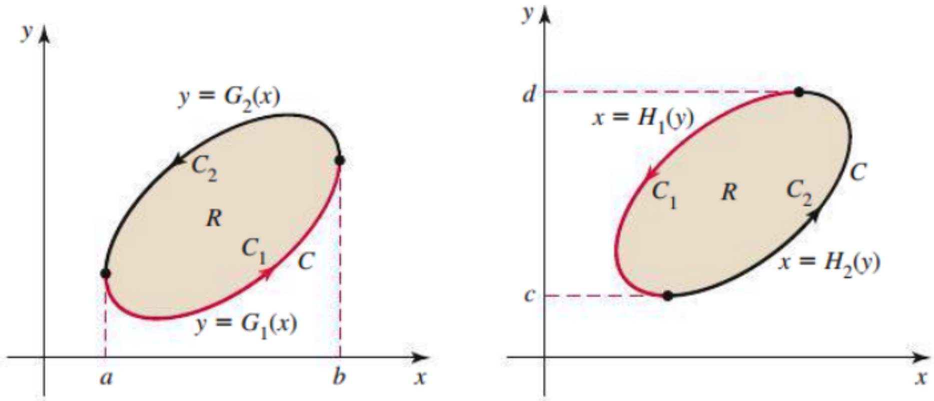
d) Is $\mathbf{F}$ source free on R?

Proof. Consider the regions R enclosed by a simple closed smooth curve C oriented in a counterclockwise direction, given by

$$R = \{(x, y) : a \leq x \leq b, G_1(x) \leq y \leq G_2(x)\}$$

or

$$R = \{(x, y) : H_1(y) \leq x \leq H_2(y), c \leq y \leq d\}.$$



To prove the circulation form of Green's Theorem, we have

$$\begin{aligned}
&\iint_R \frac{\partial f}{\partial y}\, dA \\
&= \int_a^b \int_{G_1(x)}^{G_2(x)} \frac{\partial f}{\partial y}\, dy\, dx \\
&= \int_a^b \Big(\underbrace{f(x, G_2(x))}_{\text{on } C_2} - \underbrace{f(x, G_1(x))}_{\text{on } C_1} \Big)\, dx \\
&= \int_{-C_2} f\, dx - \int_{C_1} f\, dx \\
&= -\int_{C_2} f\, dx - \int_{C_1} f\, dx \\
&= -\oint_C f\, dx
\end{aligned}$$

$$\begin{aligned}
&\iint_R \frac{\partial g}{\partial x}\, dA \\
&= \int_c^d \int_{H_1(y)}^{H_2(y)} \frac{\partial g}{\partial x}\, dx\, dy \\
&= \int_c^d \Big(\underbrace{g(H_2(y), y)}_{C_2} - \underbrace{f(H_1(y), y)}_{-C_1} \Big)\, dy \\
&= \int_{C_2} g\, dy - \int_{-C_1} g\, dy \\
&= \int_{C_2} g\, dy + \int_{C_1} g\, dy \\
&= \oint_C g\, dy
\end{aligned}$$

$\square$

17.5: Divergence and Curl

The idea behind Green's Theorem can be extended from $\mathbb{R}^2$ to $\mathbb{R}^3$. The following tools are needed to accomplish this:

- three-dimensional divergence and curl (17.5)
- *surface integrals* (17.6)
- *Stokes' Theorem* (17.7): relates line integrals over a simple closed oriented curve in $\mathbb{R}^3$ to a double integral over a surface whose boundary is that curve
- *Divergence Theorem* (17.8): relates integrals over a closed oriented surface in $\mathbb{R}^3$ to triple integrals over the corresponding region

Divergence:

Recall the *del operator* ∇:

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$$

When f is a scalar valued function, we obtain the gradient:

$$\nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k} = \langle f_x, f_y, f_z \rangle$$

The dot product of ∇ and a vector field $\mathbf{F}$, produces the three dimensional divergence:

$$\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f,\, g,\, h \rangle = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

Definition. (Divergence of a Vector Field)
The **divergence** of a vector field $\mathbf{F} = \langle f,\, g,\, h \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

If $\nabla \cdot \mathbf{F} = 0$, the vector field is **source free**.

Example. Compute the divergence of the following vector fields

$$\mathbf{F} = \langle x, -2y, 3z \rangle$$

$$\mathbf{F} = \langle -y, x - z, y \rangle$$

$$\mathbf{F} = \langle 4yz\cos(x),\ 3xz\tan(y),\ -5xy\csc(z) \rangle$$

Example. Compute the divergence of the radial vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\langle x,\, y,\, z\rangle}{\sqrt{x^2 + y^2 + z^2}}.$$

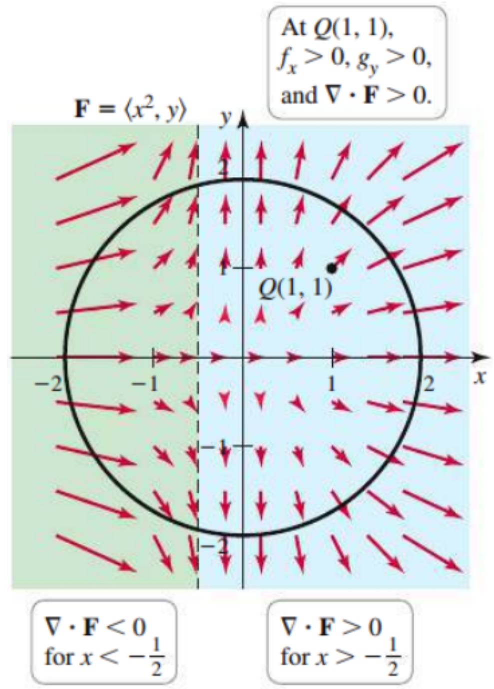
Theorem 17.10: Divergence of Radial Vector Fields

For a real number p, the divergence of the radial vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x,\, y,\, z\rangle}{(x^2 + y^2 + z^2)^{p/2}} \quad \text{is} \quad \nabla \cdot \mathbf{F} = \frac{3 - p}{|\mathbf{r}|^p}.$$

Example. Consider the two-dimensional vector field $\mathbf{F} = \langle x^2, y \rangle$ and a circle C of radius 2 centered at the origin.

Compute the two-dimensional divergence at Q.



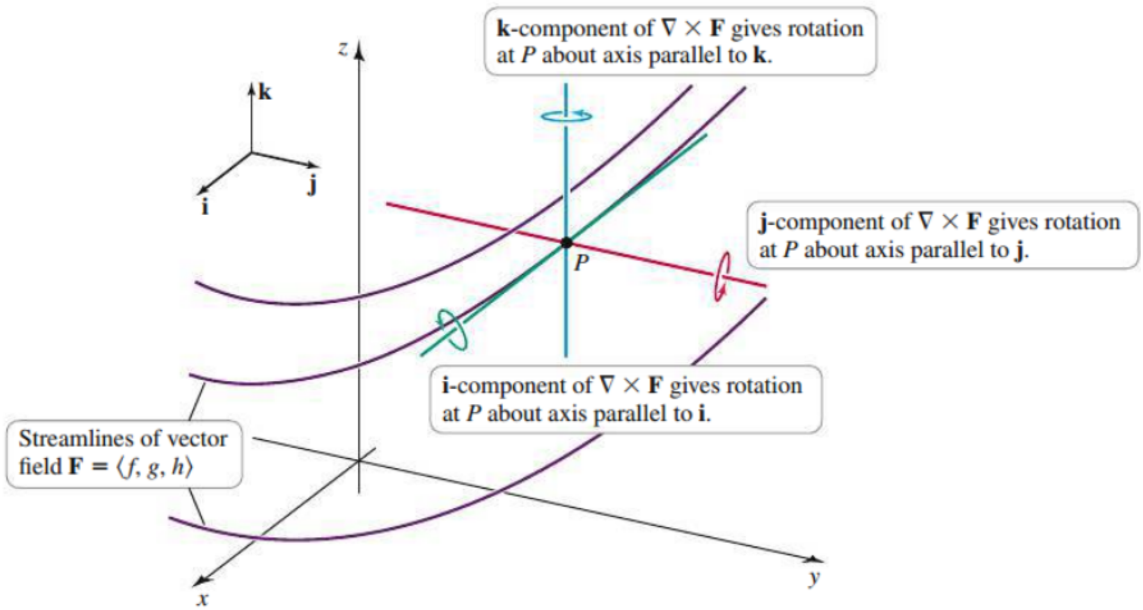
Where is the divergence positive? Negative?

Where on C is the flux outward? Inward?

Is the net flux across C positive or negative?

Curl:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle$$



Definition. (Curl of a Vector Field)
The **curl** of a vector field $\mathbf{F} = \langle f, g, h \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

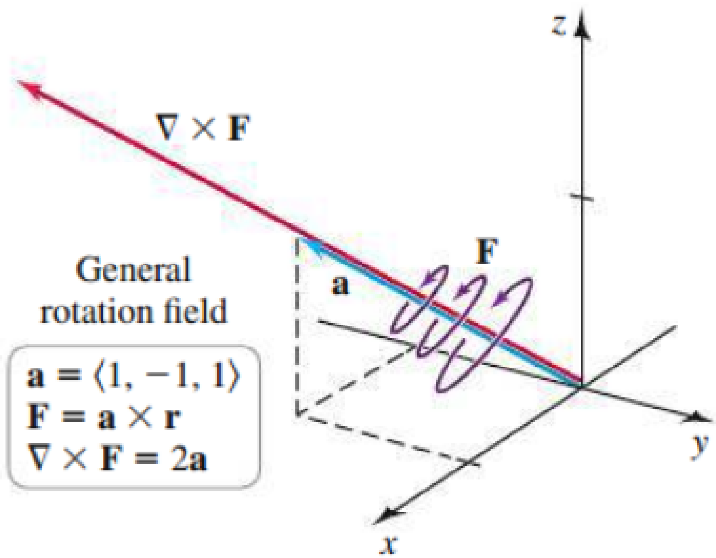
$$\begin{aligned} \nabla \times \mathbf{F} &= \operatorname{curl} \mathbf{F} \\ &= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\boldsymbol{k} \end{aligned}$$

If $\nabla \times \mathbf{F} = \mathbf{0}$, the vector field is **irrotational**.

Curl of a General Rotation Vector Field

Let $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$. Then

$$\nabla \cdot \mathbf{F} = 0 \qquad\qquad \nabla \times \mathbf{F} = 2\mathbf{a}$$



General Rotation Vector Field

The **general rotation vector field** is $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, when the nonzero constant vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is the axis of rotation and $\mathbf{r} = \langle x, y, z \rangle$. For all nonzero choices of a, $|\nabla \times \mathbf{F}| = 2|\mathbf{a}|$ and $\nabla \cdot \mathbf{F} = 0$. If $\mathbf{F}$ is a vector field, then the constant angular speed of the field is

$$\omega = |\mathbf{a}| = \frac{1}{2}|\nabla \times \mathbf{F}|.$$

Example. Compute the curl of the rotational field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a} = \langle -3, 2, 1 \rangle$ and $\mathbf{r} = \langle x, y, z \rangle$. What are the direction and magnitude of the curl?

Properties of Divergence and Curl:

Divergence Properties	Curl Properties
$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$	$\nabla \times (\mathbf{F} + \mathbf{G}) = (\nabla \times \mathbf{F}) + (\nabla \times \mathbf{G})$
$\nabla \cdot (c\mathbf{F}) = c(\nabla \cdot \mathbf{F})$	$\nabla \times (c\mathbf{F}) = c(\nabla \times \mathbf{F})$

Theorem 17.11: Curl of a Conservative Vector Field
Suppose $\mathbf{F}$ is a conservative vector field on an open region D of $\mathbb{R}^3$. Let $\mathbf{F} = \nabla\varphi$, where φ is a potential function with continuous second partial derivatives on D. Then $\nabla \times \mathbf{F} = \nabla \times \nabla_\varphi = \mathbf{0}$: The curl of the gradient is the zero vector and $\mathbf{F}$ is irrotational.

Proof.

$$\nabla \times \mathbf{F} = \nabla \times \nabla\varphi = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \varphi_x & \varphi_y & \varphi_z \end{vmatrix} = \langle \varphi_{zy} - \varphi_{yz}, \varphi_{xz} - \varphi_{zx}, \varphi_{yx} - \varphi_{xy} \rangle = \mathbf{0}$$

$\square$

Theorem 17.12: Divergence of the Curl
Suppose $\mathbf{F} = \langle f, g, h \rangle$, where f, g, and h have continuous second partial derivatives. Then $\nabla \cdot (\nabla \times \mathbf{F}) = 0$: The divergence of the curl is zero.

Proof.

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{F}) &= \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \\ &= (h_{yx} - h_{xy}) + (g_{xz} - g_{zx}) + (f_{zy} - f_{yz}) = 0 \end{aligned}$$

$\square$

The **Laplacian**, denoted $\nabla^2 u$ or Δu, arises from $\nabla \cdot \nabla u$:

$$\nabla \cdot \nabla u = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial}{\partial z}\frac{\partial u}{\partial z} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Theorem 17.13: Product Rule for the Divergence
Let u be a scalar-valued function that is differentiable on a region D and let $\mathbf{F}$ be a vector field that is differentiable on D. Then

$$\nabla \cdot (u\mathbf{F}) = \nabla u \cdot \mathbf{F} + u(\nabla \cdot \mathbf{F}).$$

Example. Let $\mathbf{r} = \langle x, y, z \rangle$ and let $\varphi = \frac{1}{|\mathbf{r}|}$ be a potential function.

Find the associated gradient field $\mathbf{F} = \nabla\left(\frac{1}{|\mathbf{r}|}\right)$

Compute $\nabla \cdot \mathbf{F}$

Properties of a Conservative Vector Field
Let $\mathbf{F}$ be a conservative vector field whose components have continuous second partial derivatives on an open connected region D in $\mathbb{R}^3$. Then $\mathbf{F}$ has the following equivalent properties.

1. There exists a potential function φ such that $\mathbf{F} = \nabla_\varphi$ (definition).
2. $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$ for all points A and B in D and all piecewise smooth oriented curves C in D from A to B.
3. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple piecewise-smooth closed oriented curves C in D.
4. $\nabla \times \mathbf{F} = \mathbf{0}$ at all points of D.

17.6: Surface Integrals

Imagine a sphere with a known temperature distribution. How would we find the average temperature over the sphere?

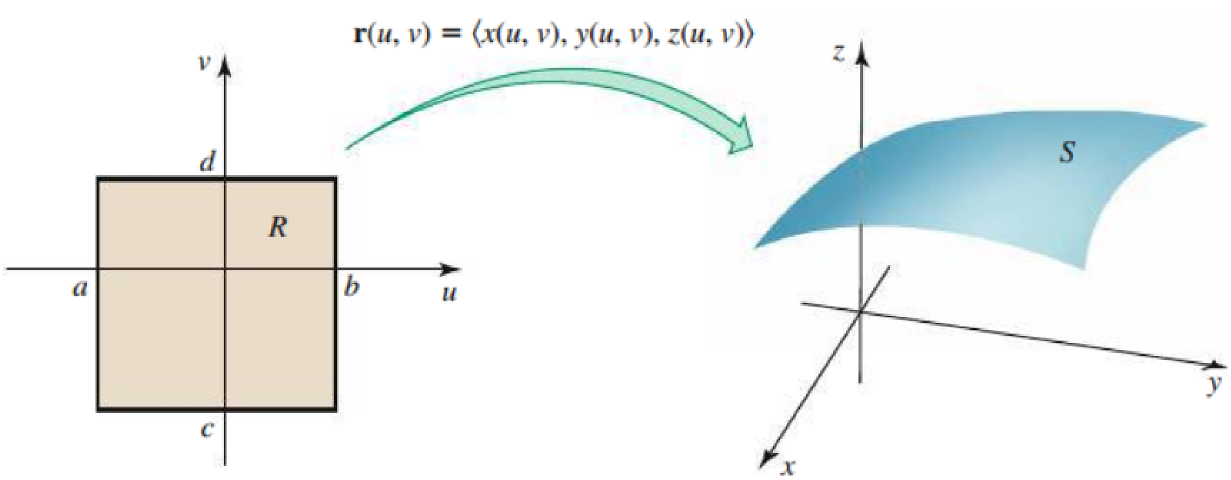
Parallel Concepts	
Curves	**Surfaces**
Arc length	Surface area
Line integrals	Surface integrals
One-parameter description	Two-parameter description

Parameterized Surfaces

Recall that in $\mathbb{R}^2$, we parameterized a curve by $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$ where $a \le t \le b$. In $\mathbb{R}^3$, we parameterize a surface by

$$\mathbf{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle$$

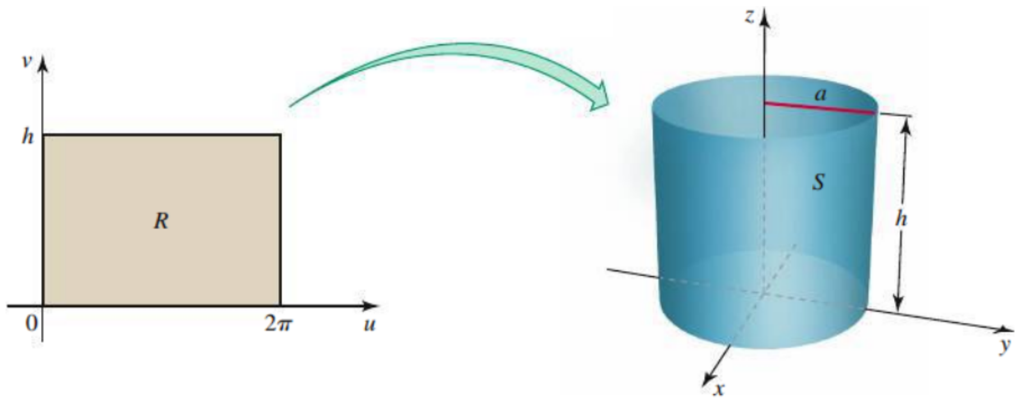
where the parameters are over $R = \{(u,v) : a \le u \le b, c \le v \le d\}$



Cylinders:

$$\{(x,\, y,\, z) : x = a\cos(\theta),\, y = a\sin(\theta),\; 0 \le \theta \le 2\pi,\, 0 \le z \le h\}$$

$$\mathbf{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle = \langle a\cos(u),\, a\sin(u),\, v\rangle$$



Cones:

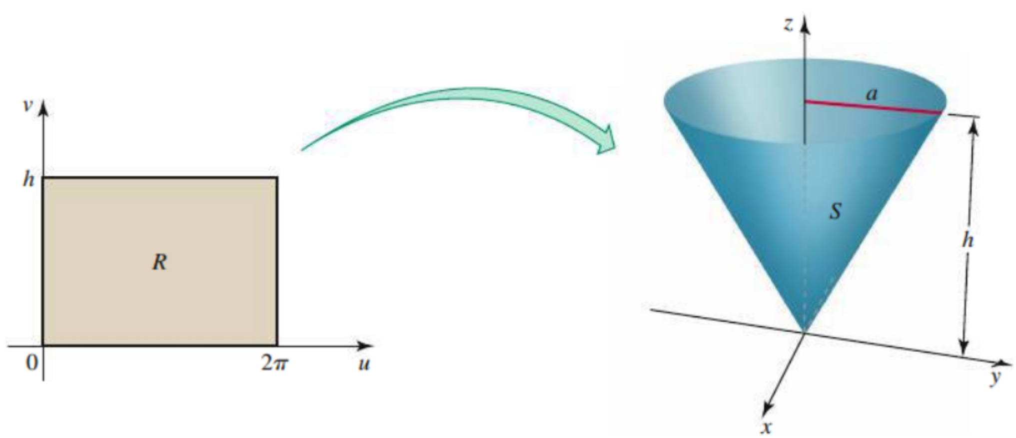
$$\{(r,\, \theta,\, z) : 0 \le r \le a,\, 0 \le \theta \le 2\pi,\, z = rh/a\}$$

For a fixed value of z, $r = az/h$:

$$x = r\cos(\theta) = \frac{az}{h}\cos(\theta) \text{ and } y = r\sin(\theta) = \frac{az}{h}\sin(\theta)$$

Now, let $u = \theta$ and $v = z$, then

$$\mathbf{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle = \left\langle \frac{av}{h}\cos(u),\, \frac{av}{h}\sin(u),\, v\right\rangle$$



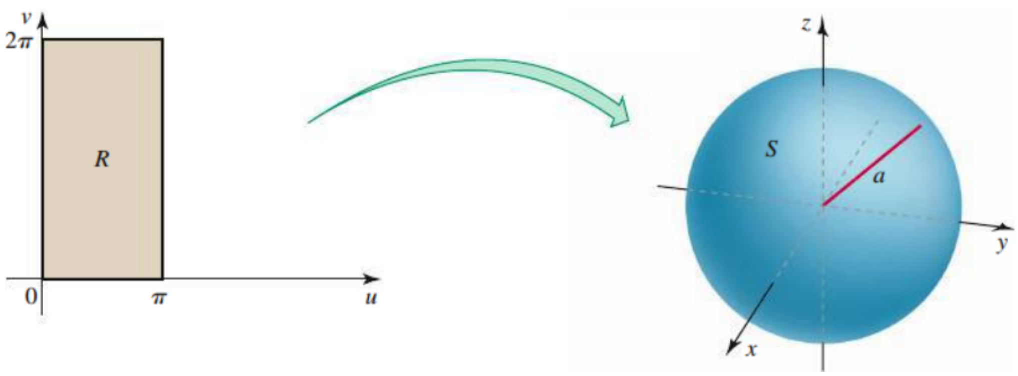
Spheres:

$$\{(\rho,\,\varphi,\,\theta) : \rho = a, 0 \le \varphi \le \pi,\; 0 \le \theta \le 2\pi\}$$

$$x = a\sin(\varphi)\cos(\theta), \qquad y = a\sin(\varphi)\sin(\theta), \qquad z = a\cos(\varphi)$$

Now, let $u = \theta$ and $v = z$, then

$$\mathbf{r}(u, v) = \langle x(u, v),\, y(u, v),\, z(u, v)\rangle = \langle a\sin(u)\cos(v),\, a\sin(u)\cos(v),\, a\cos(u)\rangle$$

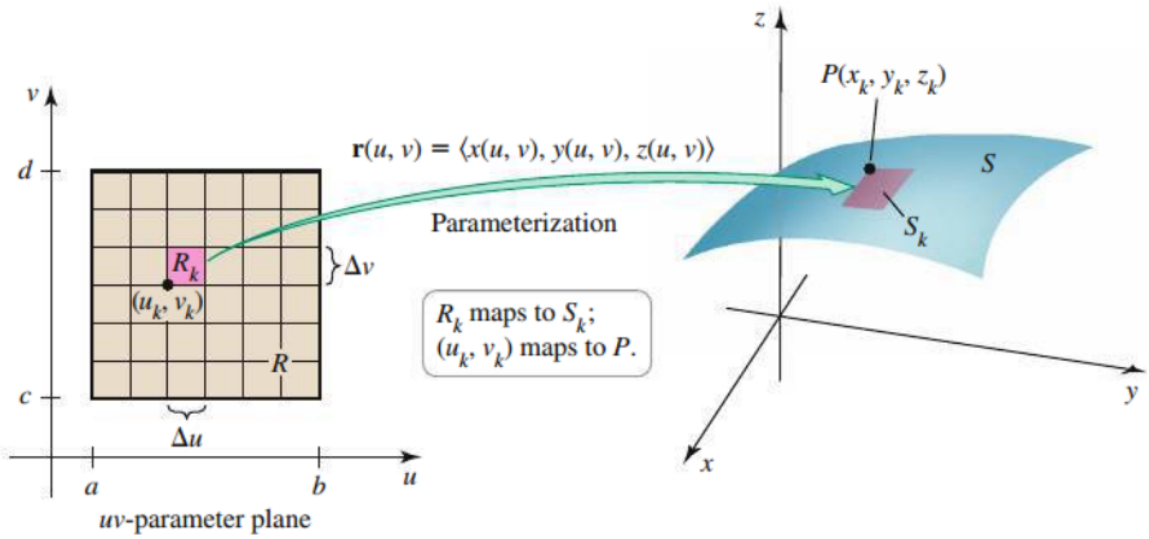


Example. Find parametric descriptions for the following surfaces

The plane $3x - 2y + z = 2$

The paraboloid $z = x^2 + y^2$, for $0 \le z \le 9$

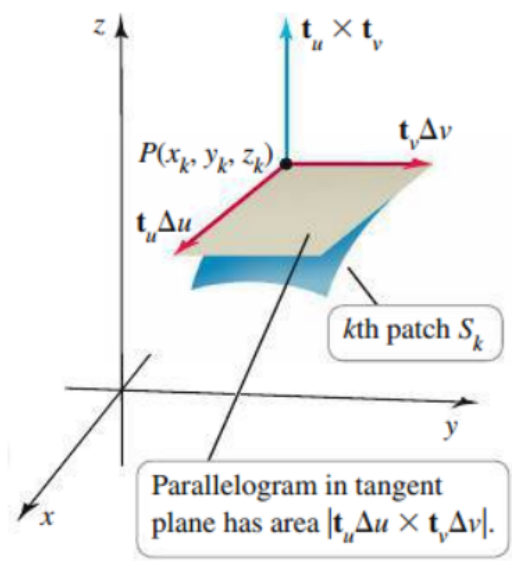
Surface Integrals of Scalar-Valued Functions



Using the parameterization

$$\mathbf{r}(u, v) = \langle x(u, v),\, y(u, v),\, z(u, v)\rangle$$

over the region $R = \{(u, v) : a \le u \le b,\, c \le v \le d\}$, it is important that we know ΔS_k, which is the area of S_k.



Definition. (Surface Integral of Scalar-Valued Functions on Parameterized Surfaces)
Let f be a continuous scalar-valued function on a smooth surface S given parametrically by $\mathbf{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle$, where u and v vary over $R = \{(u,v) : a \le u \le b,\, c \le v \le d\}$. Assume also that the tangent vectors

$$\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \text{ and } \mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

are continuous on R and the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R. Then the **surface integral of f over S** is

$$\iint\limits_S f(x,y,z)\, dS = \iint\limits_R f(x(u,v),\, y(u,v),\, z(u,v)) |\mathbf{t}_u \times \mathbf{t}_v|\, dA$$

If $f(x,y,z) = 1$, this integral equals the surface area of S.

Example. Find the surface area of the following surfaces

A cylinder with radius $a > 0$ and height h (open ends)

A sphere of radius a

Example. The temperature on the surface of a sphere of radius a varies with latitude according to the function $T(\varphi, \theta) = 10 + 50\sin(\varphi)$, for $0 \leq \varphi \leq \pi$ and $0 \leq \theta \leq 2\pi$. Find the average temperature over the sphere.

Surface Integrals on Explicitly Defined Surfaces

Suppose a smooth surface S is defined explicitly as $z = g(x, y)$. Here, we let $u = x$ and $v = y$. This gives us

$$\mathbf{t}_u = \mathbf{t}_x = \langle 1,\, 0,\, z_x \rangle, \quad \mathbf{t}_v = \mathbf{t}_y = \langle 0,\, 1,\, z_y \rangle$$

thus

$$\mathbf{t}_x \times \mathbf{t}_y = \langle -z_x,\, -z_y,\, 1 \rangle$$

and

$$|\mathbf{t}_x \times \mathbf{t}_y| = \sqrt{z_x^2 + z_y^2 + 1}$$

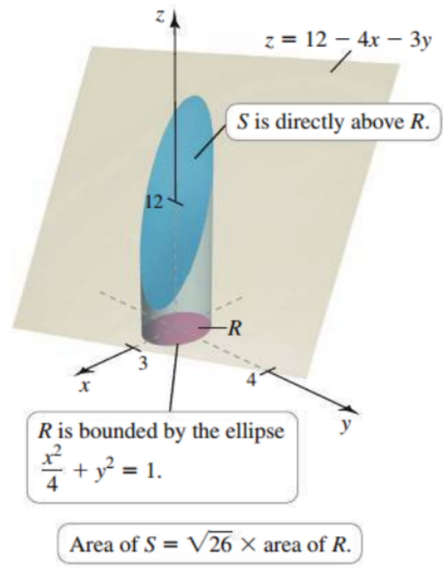
Theorem 17.14: Evaluation of Surface Integrals of Scalar-Valued Functions on Explicitly Defined Surfaces

Let f be a continuous function on a smooth surface S given by $z = g(x, y)$, for (x, y) in a region R. The surface integral of f over S is

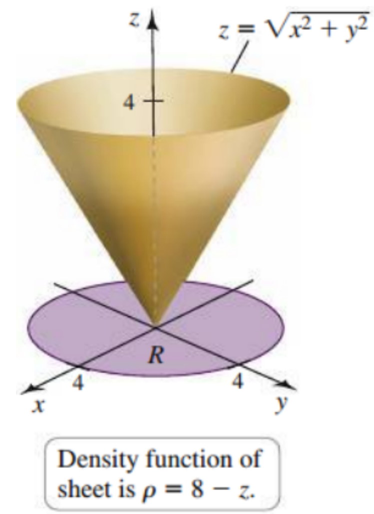
$$\iint\limits_S f(x, y, z)\, dS = \iint\limits_S f(x, y, g(x, y))\sqrt{z_x^2 + z_y^2 + 1}\, dA.$$

If $f(x, y, z) = 1$, the surface integral equals the area of the surface.

Example. Find the area of the surface S that lies in the plane $z = 12 - 4x - 3y$ directly above the region R bounded by the ellipse $x^2/4 + y^2 = 1$



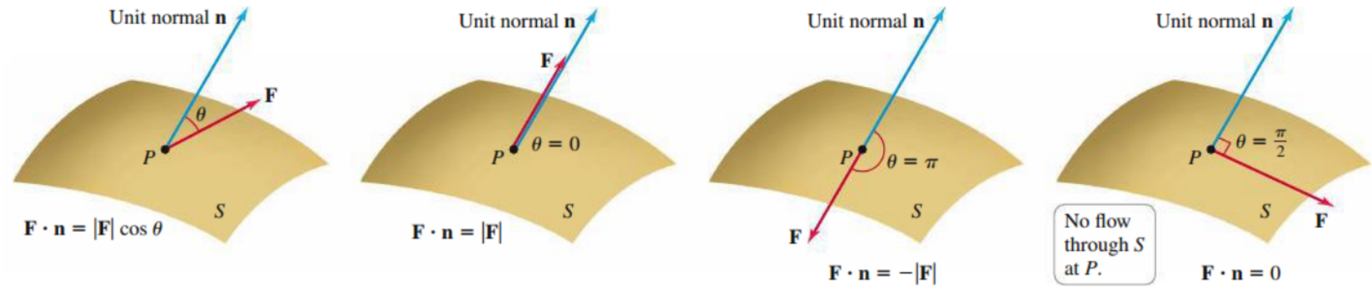
Example. A thin conical sheet is described by the surface $z = \left(x^2 + y^2\right)^{\frac{1}{2}}$, for $0 \le z \le 4$. The density of the sheet in $\mathrm{g}/\ \mathrm{cm}^2$ is $\rho = f(x, y, z) = (8 - z)$. What is the mass of the cone?



	Explicit Description $z = g(x, y)$			**Parametric Description**		
Surface	**Equation**	**Normal vector** $\pm\langle -z_x,\, -z_y,\, 1\rangle$	**magnitude** $\lvert\langle -z_x,\, -z_y,\, 1\rangle\rvert$	**Equation**	**Normal vector** $\mathbf{t}_u \times \mathbf{t}_v$	**magnitude** $\lvert\mathbf{t}_u \times \mathbf{t}_v\rvert$
Cylinder	$x^2 + y^2 = a^2$, $0 \le z \le h$	$\langle x, y, 0\rangle$	a	$\mathbf{r} = \langle a\cos(u),\, a\sin(u),\, v\rangle$, $0 \le u \le 2\pi$, $0 \le v \le h$	$\langle a\cos(u),\, a\sin(u),\, 0\rangle$	a
Cone	$z^2 = x^2 + y^2$, $0 \le z \le h$	$\langle x/z,\, y/z,\, -1\rangle$	$\sqrt{2}$	$\mathbf{r} = \langle v\cos(u),\, v\sin(u),\, v\rangle$, $0 \le u \le 2\pi$, $0 \le v \le h$	$\langle v\cos(u),\, v\sin(u),\, -v\rangle$	$\sqrt{2}v$
Sphere	$x^2 + y^2 + z^2 = a^2$	$\langle x/z,\, y/z,\, 1\rangle$;	a/z	$\mathbf{r} = \langle a\sin(u)\cos(v),\, a\sin(u)\sin(v),\, a\cos(u)\rangle$ $0 \le u \le \pi$, $0 \le v \le 2\pi$	$\langle a^2\sin^2(u)\cos(v),\, a^2\sin^2(u)\sin(v),\, a^2\sin(u)\cos(u)\rangle$	$a^2\sin(u)$
Paraboloid	$z = x^2 + y^2$, $0 \le z \le h$	$\langle 2x,\, 2y,\, -1\rangle$	$\sqrt{1 + 4(x^2 + y^2)}$	$\mathbf{r} = \left\langle v\cos(u),\, v\sin(u),\, v^2\right\rangle$, $0 \le u \le 2\pi$, $0 \le v \le \sqrt{h}$	$\left\langle 2v^2\cos(u), 2v^2\sin(u), -v\right\rangle$	$v\sqrt{1 + 4v^2}$

Flux Integrals:

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$



The unit normal vector we use is

$$\mathbf{n} = \frac{\mathbf{t}_u \times \mathbf{t}_v}{|\mathbf{t}_u \times \mathbf{t}_v|}$$

giving us

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \iint_R \mathbf{F} \cdot \mathbf{n} |\mathbf{t}_u \times \mathbf{t}_v| \, dA \\ &= \iint_R \mathbf{F} \cdot \frac{\mathbf{t}_u \times \mathbf{t}_v}{|\mathbf{t}_u \times \mathbf{t}_v|} |\mathbf{t}_u \times \mathbf{t}_v| \, dA \\ &= \iint_R \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v) \, dA \end{aligned}$$

When the surface S is explicitly given as $z = s(x, y)$, then

$$\mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v) = -f z_x - g z_y + h$$

Definition. (Surface Integral of a Vector Field)
Suppose $\mathbf{F} = \langle f, g, h\rangle$ is a continuous vector field on a region of $\mathbb{R}^3$ containing a smooth oriented surface S. If S is defined parametrically as $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, for (u,v) in a region R, then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \mathbf{F}\cdot(\mathbf{t}_u \times \mathbf{t}_v)\,dA,$$

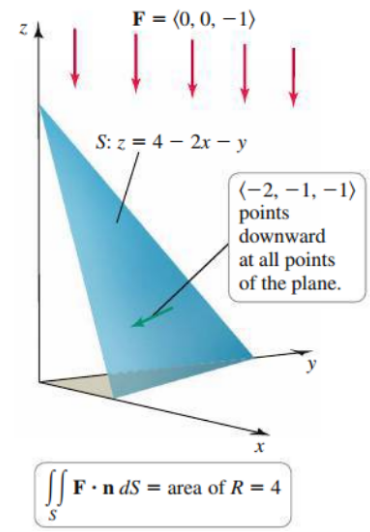
where

$$\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right\rangle \text{ and } \mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right\rangle$$

are continuous on R, the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R, and the direction of the normal vector is consistent with the orientation of S. If S is defined in the form $z = s(x,y)$, for (x,y) in a region R, then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S (-f z_x - g z_y + h)\,dA.$$

Example. Consider the vertical field $\mathbf{F} = \langle 0, 0, -1 \rangle$. Find the flux in the downward direction across the surface S, which is the plane $z = 4 - 2x - y$ in the first octant.



Example. Consider the radial vector field $\mathbf{F} = \langle f, g, h \rangle = \langle x, y, z \rangle$. Compute the upward flux of the field across the:

hemisphere $x^2 + y^2 + z^2 = 1$, for $z \geq 0$,

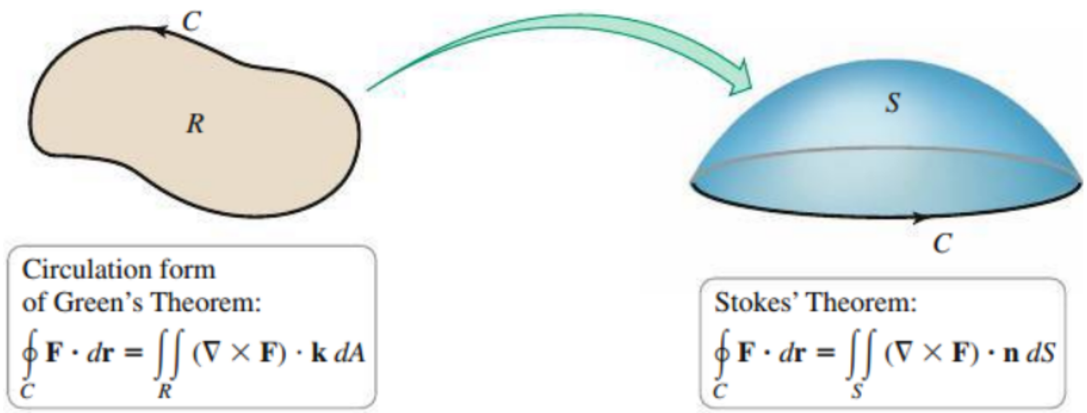
paraboloid $z = 1 - x^2 - y^2$, for $z \geq 0$.

17.7: Stokes' Theorem

Stokes' Theorem is the three-dimensional version of the circulation form of Green's Theorem. Recall the circulation form of Green's Theorem:

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{circulation}} = \iint_R \underbrace{(g_x - f_y)}_{\text{curl or rotation}} \; dA.$$

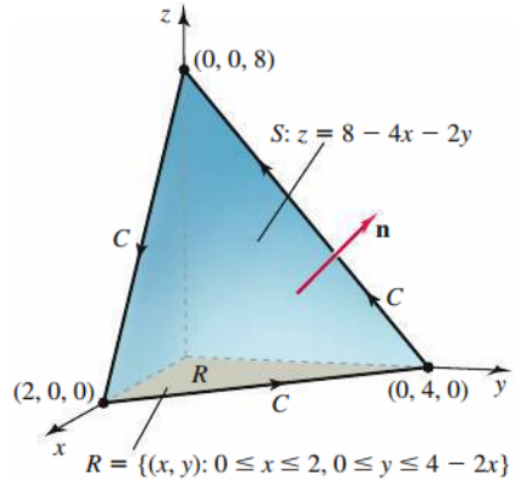
The above means that the cumulative rotation within R equals the circulation along the boundary of R. Stokes' Theorem computes the circulation over a surface S in $\mathbb{R}^3$:



> **Theorem 17.15: Stokes' Theorem**
> Let S be an oriented surface in $\mathbb{R}^3$ with a piecewise-smooth closed boundary C whose orientation is consistent with that of S. Assume $\mathbf{F} = \langle f,\, g,\, h\rangle$ is a vector field whose components have continuous first partial derivatives on S. Then
>
> $$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS,$$
>
> where $\mathbf{n}$ is the unit vector normal to S determined by the orientation of S.

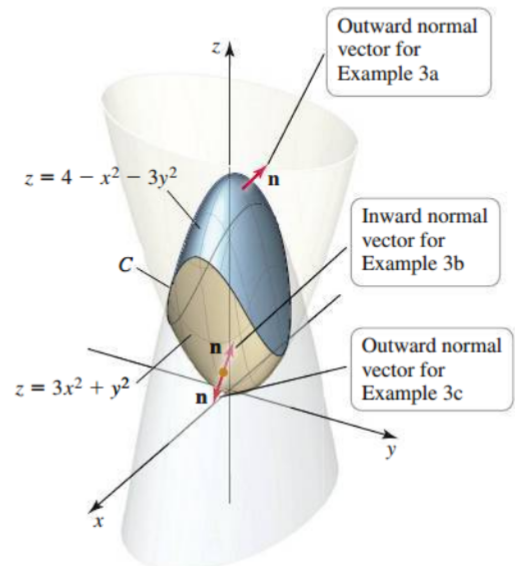
Example. Verify Stokes' Theorem: Confirm that Stokes' Theorem holds for the vector field $\mathbf{F} = \langle z - y, x, -x \rangle$, where S is the hemisphere $x^2 + y^2 + z^2 = 4$, for $z \geq 0$, and C is the circle $x^2 + y^2 = 4$, oriented counterclockwise.

Example. Evaluate the line integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \left\langle z,\, -z,\, x^2 - y^2 \right\rangle$ and C consists of the three line segments that bound the plane $z = 8 - 4x - 2y$ in the first octant.

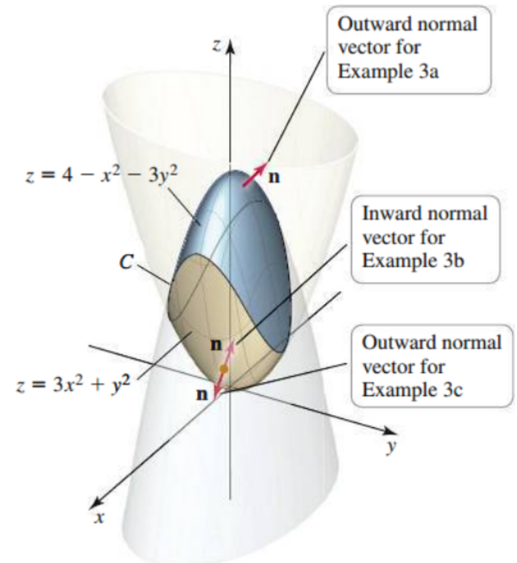


Example. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$, where $\mathbf{F} = \langle -y,\, x,\, z\rangle$, where:

- S is the part of the paraboloid $z = 4 - x^2 - 3y^2$ contained within $z = 3x^2 + y^2$, with $\mathbf{n}$ pointing upwards.



- S is the part of the paraboloid $z = 3x^2 + y^2$ contained within $z = 4 - x^2 - 3y^2$ with $\mathbf{n}$ pointing upwards.



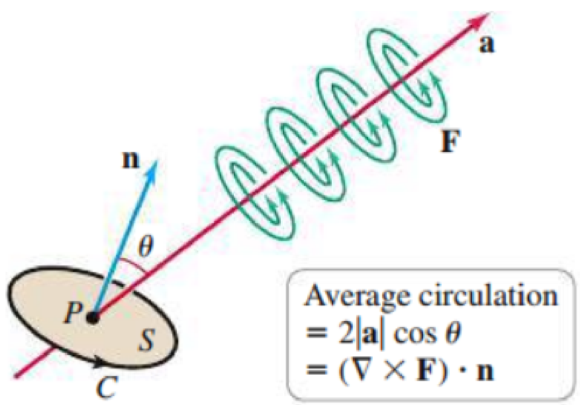
- S is the part of the paraboloid $z = 3x^2 + y^2$ contained within $z = 4 - x^2 - 3y^2$ with $\mathbf{n}$ pointing downwards.

Interpreting the Curl:

The **average circulation** is

$$\frac{1}{\text{area of } S}\oint_C \mathbf{F}\cdot d\mathbf{r} = \frac{1}{\text{area of } S}\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS.$$

Consider a general rotation field $\mathbf{F} = \mathbf{a}\times\mathbf{r}$, where $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{r} = \langle x, y, z\rangle$. Now, let S be a small circular disk centered at a point P, whose normal vector $\mathbf{n}$ makes an angle θ with the axis $\mathbf{a}$:



The average circulation of this vector field on S is

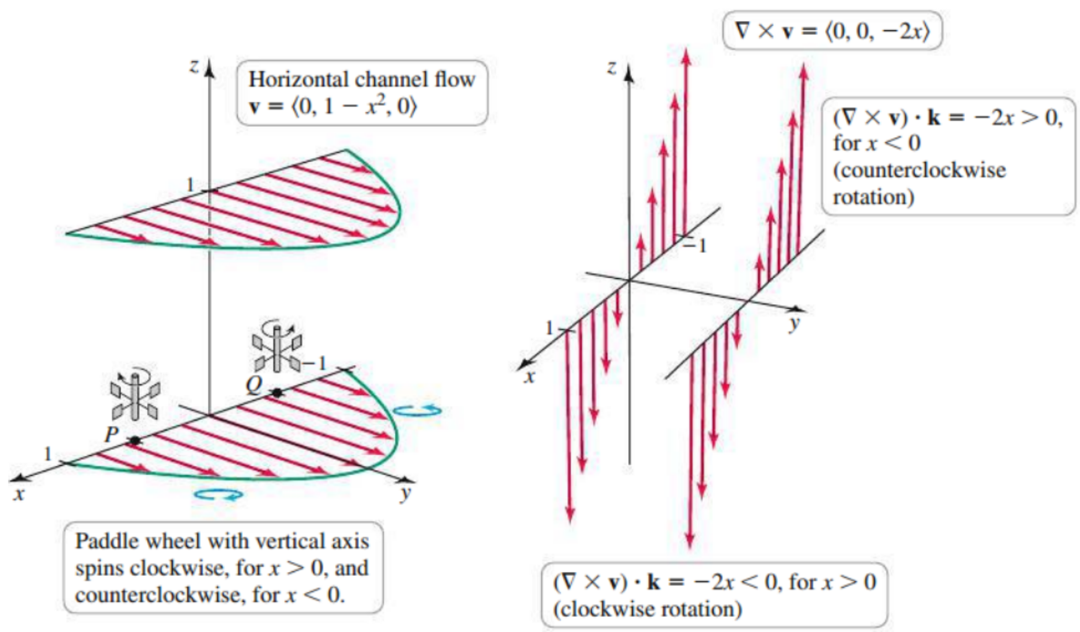
$$\begin{aligned}\frac{1}{\text{area of } S}\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS &= \frac{(\nabla\times\mathbf{F})\cdot\mathbf{n}}{\text{area of } S}(\text{area of } S)\\ &= 2\mathbf{a}\cdot\mathbf{n}\\ &= 2|\mathbf{a}|\cos(\theta)\end{aligned}$$

From this, we see

- The scalar component of $\nabla\times\mathbf{F}$ at P in the direction of $\mathbf{n}$ is the average circulation of $\mathbf{F}$ on S.
- The direction of $\nabla\times\mathbf{F}$ at P is the direction that maximizes the average circulation of $\mathbf{F}$ on S.

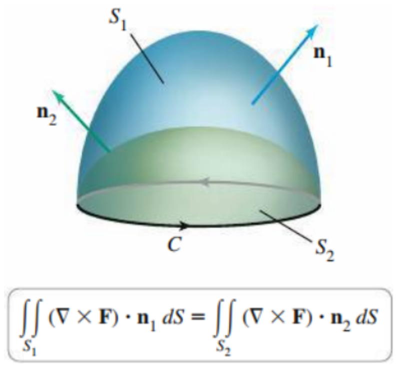
A similar argument for the curl can be applied to more general vector fields.

Example. Consider the vector field $\mathbf{v} = \left\langle 0,\, 1 - x^2,\, 0 \right\rangle$ for $|x| \leq 1$ and $|z| \leq 1$. Compute the curl of $\mathbf{v}$.



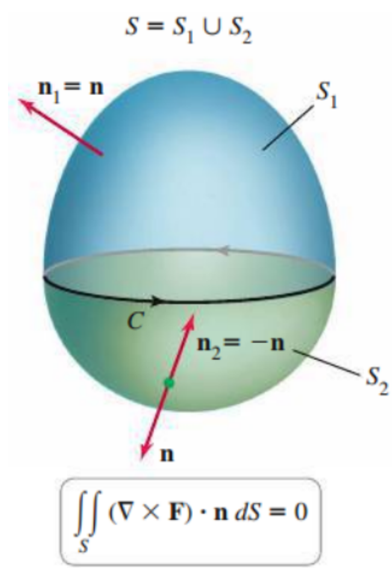
Since, using Stokes' Theorem, we evaluate the surface integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ using only the boundary C, then for any two smooth oriented surfaces S_1 and S_2 both with a consistent orientation with that of C,

$$\iint\limits_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_1\, dS = \iint\limits_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2\, dS$$



Furthermore, if S is a closed surface consisting of S_1 and S_2, with $\mathbf{n} = \mathbf{n}_1$ and $\mathbf{n} = -n_2$, then

$$\iint\limits_{S} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint\limits_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS + \iint\limits_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = 0$$



Theorem 17.11 (Section 17.5) states that if $\mathbf{F}$ is conservative, then $\nabla \times \mathbf{F} = \mathbf{0}$. Now, the converse follows using Stokes' Theorem:

Theorem 17.16: Curl $\mathbf{F} = \mathbf{0}$ implies $\mathbf{F}$ Is Conservative
Suppose $\nabla \times \mathbf{F} = \mathbf{0}$ throughout an open simply connected region D of $\mathbb{R}^3$. Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all closed simple smooth curves C in D, and $\mathbf{F}$ is a conservative vector field on D.

Proof. Given a closed simple smooth curve C, it can be shown that C is the boundary of at least one smooth oriented surface S in D. By Stokes' Theorem,

$$\oint\limits_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_S \underbrace{(\nabla \times \mathbf{F})}_{0} \cdot \mathbf{n}\, dS = 0$$

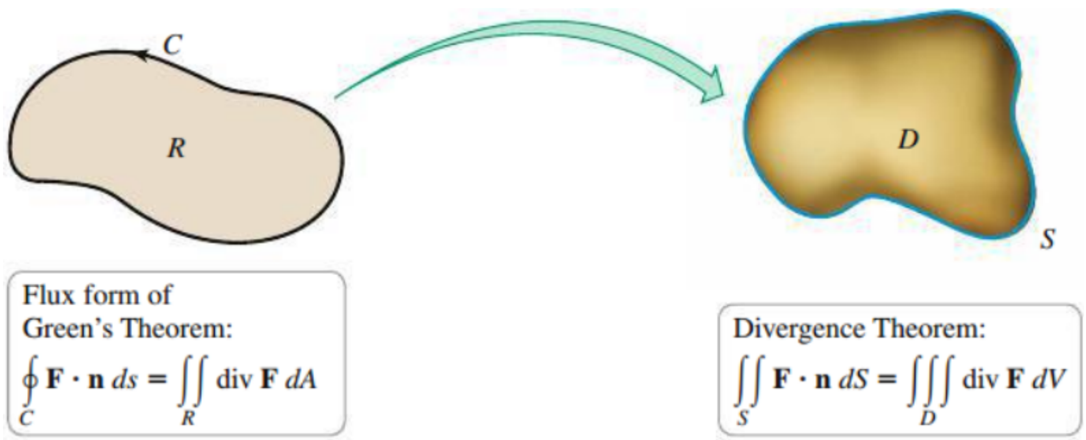
Since the line integral equals zero over all such curves in D, the vector field is conservative on D. $\square$

17.8: Divergence Theorem

The Divergence Theorem is the three-dimensional version of the flux form of Green's Theorem. Recall the flux form of Green's Theorem:

$$\underbrace{\oint_C \mathbf{F}\cdot\mathbf{n}\,ds}_{\text{flux across } C} = \iint_R \underbrace{(f_x + g_y)}_{\text{divergence}}\,dA.$$

The above means that the cumulative expansion and contraction throughout R equals the flux across the boundary of R. The Divergence Theorem computes the flux over a surface S in $\mathbb{R}^3$:



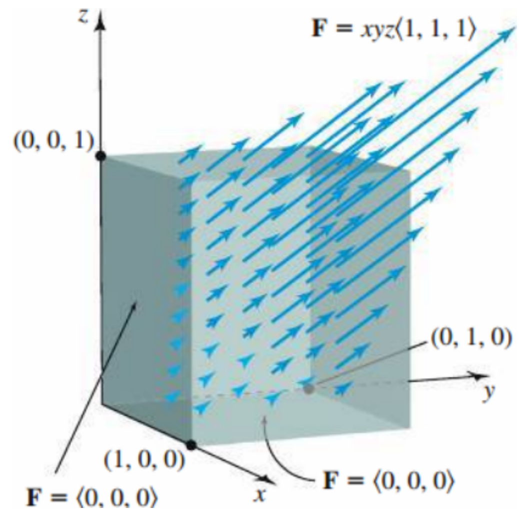
Theorem 17.17: Divergence Theorem
Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives in a connected and simply connected region D in $\mathbb{R}^3$ enclosed by an oriented surface S. Then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \nabla\cdot\mathbf{F}\,dV,$$

where $\mathbf{n}$ is the outward unit normal vector on S.

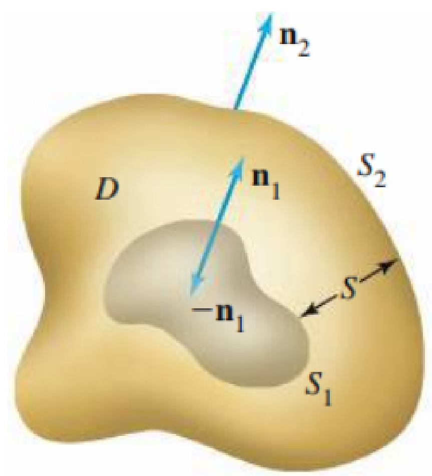
Example. Verify the Divergence Theorem: Consider the radial field $\mathbf{F} = \langle x, y, z \rangle$ and let S be the sphere $x^2 + y^2 + z^2 = a^2$ that encloses the region D. Assume $\mathbf{n}$ is the outward unit normal vector on the sphere. Evaluate both integrals of the Divergence Theorem.

Example. Find the net outward flux of the field $\mathbf{F} = xyz\langle 1, 1, 1\rangle$ across the boundaries of the cube $D = \{(x, y, z) : 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$.



Theorem 17.18: Divergence Theorem for Hollow Regions
Suppose the vector field $\mathbf{F}$ satisfies the conditions of the Divergence Theorem on a region D bounded by two oriented surfaces S_1 and S_2, where S_1 lies within S_2. Let S be the entire boundary of D ($S = S_1 \cup S_2$) and let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the outward unit normal vectors for S_1 and S_2, respectively. Then

$$\iiint\limits_D \nabla \cdot \mathbf{F}\, dV = \iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint\limits_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS - \iint\limits_{S_1} \mathbf{F} \cdot \mathbf{n}_2\, dS.$$



$\mathbf{n}_1$ is the outward unit normal to S_1 and points into D. The outward unit normal to S on S_1 is $-\mathbf{n}_1$.

Example. Consider the inverse square vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{\langle x,\, y,\, z\rangle}{\left(x^2+y^2+z^2\right)^{3/2}}$$

Find the net outward flux of $\mathbf{F}$ across the surface of the region $D = \left\{(x,y,z) : a^2 \le x^2+y^2+z^2 \le b^2\right\}$ that lies between concentric spheres with radii a and b.

Find the outward flux of $\mathbf{F}$ across any sphere that encloses the origin.

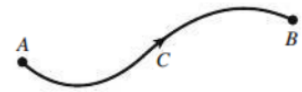
Example. Use the Divergence Theorem to compute the net outward flux of the field $\mathbf{F} = \left\langle x^2,\, y^2,\, z^2 \right\rangle$ across the surface S where S is the sphere $\left\{ (x, y, z) : x^2 + y^2 + z^2 = r^2 \right\}$.

Fundamental Theorem of Calculus

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

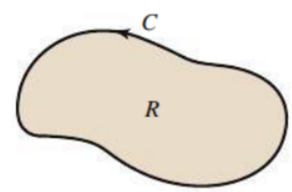
Fundamental Theorem for Line Integrals

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$$



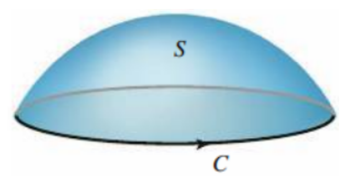
Green's Theorem (Circulation Form)

$$\iint_R (g_x - f_y)\,dA = \oint_C f\,dx + g\,dy$$



Stokes' Theorem

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$$



Divergence Theorem

$$\iiint_D \nabla \cdot \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot \mathbf{n}\,dS$$