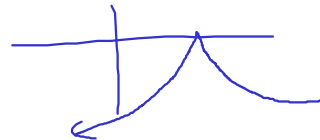## 14.2: Calculus of Vector-Valued Functions

Definition. (Derivative and Tangent Vector)
Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions on (a, b).
Then $\mathbf{r}$ has a **derivative** (or is **differentiable**) on (a, b) and

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}. \quad = \langle f(t), g(t), h(t) \rangle$$

Provided $\mathbf{r}'(t) \neq \mathbf{0}$, $\mathbf{r}'(t)$ is a **tangent vector** at the point corresponding to $\mathbf{r}(t)$.

Example. For the following functions below, find $\mathbf{r}'(t)$ $\qquad \log_2(t) = \frac{\ln(t)}{\ln(2)}$

a) $\mathbf{r}(t) = \left\langle e^{-t^2}, \log_2(t - 4), \sin(t) \right\rangle$

$$\vec{r}'(t) = \left\langle -2t\, e^{-t^2}, \frac{1}{\ln(2)(t-4)}, \cos(t) \right\rangle$$

b) $\mathbf{r}(t) = 3\mathbf{i} - 2\tan(t)\mathbf{j} + e^t\mathbf{k}$

$$\vec{r}'(t) = 0\,\hat{\imath} - 2\sec^2(t)\,\hat{\jmath} + e^t\,\hat{k}$$

Example. For $\mathbf{r}(t) = \langle 3t, \sec(2t), \cos(t) \rangle$ compute $\mathbf{r}'(t)$ at $t = \frac{\pi}{4}$.

$$\vec{r}'(t) = \langle 3, 2\sec(2t)\tan(2t), -\sin(t) \rangle$$

$$\vec{r}'\left(\tfrac{\pi}{4}\right) = \left\langle 3, \boxed{2(1)\quad 4}, -\tfrac{\sqrt{2}}{2} \right\rangle$$

$$\tan\left(\tfrac{\pi}{2}\right) \text{ DNE}$$

$$\angle\angle\ 2\sqrt{2}$$

Definition. (Unit Tangent Vector)

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a smooth parameterized curve, for $a \leq t \leq b$. The **unit tangent vector** for a particular value of t is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Example. For $\mathbf{r}(t) = \langle 3\sin(t), -2\cos(2t), 3\cos(t) \rangle$, find the unit tangent vector.

$$\vec{r}\,'(t) = \langle 3\cos(t), 4\sin(2t), -3\sin(t) \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{9\cos^2(t) + 16\sin^2(2t) + 9\sin^2(t)} = \sqrt{9 + 16\sin^2(2t)}$$

$$\vec{T}(t) = \frac{\langle 3\cos(t), 4\sin(2t), -3\sin(t) \rangle}{\sqrt{9 + 16\sin^2(2t)}}$$

Example. For $\mathbf{r}(t) = \langle \sin(6t), 3t, \cos(3t) \rangle$, compute $\mathbf{T}\left(\frac{\pi}{3}\right)$.

$$T(t) = \frac{\langle 6\cos(6t), 3, -3\sin(3t) \rangle}{\sqrt{36\cos^2(6t) + 9 + 9\sin^2(3t)}}$$

$$T\left(\frac{\pi}{3}\right) = \frac{\langle 6(1), 3, -3(0) \rangle}{3\sqrt{4(1) + 1 + 0}} = \frac{\langle 6, 3, 0 \rangle}{3\sqrt{5}}$$

$$= \frac{\langle 2, 1, 0 \rangle}{\sqrt{5}}$$

Derivative Rules

Let **u** and **v** be differentiable vector-valued functions, and let f be a differentiable scalar-valued function, all at a point t. Let **c** be a constant vector. The following rules apply.

1. $\dfrac{d}{dt}(\mathbf{c}) = \mathbf{0}$ Constant Rule

2. $\dfrac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$ Sum Rule

3. $\dfrac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ Product Rule

4. $\dfrac{d}{dt}(\mathbf{u}(f(t))) = \mathbf{u}'(f(t))f'(t)$ Chain Rule

5. $\dfrac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ Dot Product Rule

6. $\dfrac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ Cross Product Rule

Example. Given $\mathbf{u}(t) = \langle 4t^2, 1, 3t \rangle$ and $\mathbf{v}(t) = \langle e^{-2t}, -2e^t, e^t \rangle$, find $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$.

$$\frac{d}{dt}\left[\vec{u}(t) \cdot \vec{r}(t)\right] = \vec{u}'(t) \cdot \vec{r}(t) + \vec{u}(t) \cdot \vec{r}'(t)$$

$$= \langle 8t, 0, 3 \rangle \cdot \langle e^{-2t}, -2e^t, e^t \rangle + \langle 4t^2, 1, 3t \rangle \cdot \langle -2e^{-2t}, -2e^t, e^t \rangle$$

$$= 8t e^{-2t} + 3e^t - 8t^2 e^{-2t} - 2e^t + 3t e^t$$

$$= 8t e^{-2t}(1-t) + e^t(1+3t)$$

Definition. (Indefinite Integral of a Vector-Valued Function)
Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function, and let
$\mathbf{R}(t) = F(t)\mathbf{i} + G(t)\mathbf{j} + H(t)\mathbf{k}$, where F, G, and H are antiderivatives of f, g, and h, respectively. The **indefinate integral** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C},$$

where $\mathbf{C}$ is an arbitrary constant vector. Alternatively, in component form,

$$\int \langle f(t),\, g(t),\, h(t) \rangle\,dt = \langle F(t),\, G(t),\, H(t) \rangle + \langle C_1,\, C_2,\, C_3 \rangle.$$

Example. Find $\mathbf{r}(t)$ such that $\mathbf{r}'(t) = \left\langle \frac{t}{t^2+1},\ t^2 e^{-t^3},\ \frac{-2t}{\sqrt{t^2+16}} \right\rangle$ and $\mathbf{r}(0) = \langle 3, \frac{5}{3}, -5 \rangle$.

$$\int \frac{t}{t^2+1}\,dt$$

$u = t^2+1$
$du = 2t\,dt$
$\frac{1}{2}du = t\,dt$

$$= \frac{1}{2}\int u^{-1}\,du$$

$$= \frac{1}{2}\ln|u| + C_1$$

$$= \frac{1}{2}\ln|t^2+1| + C_1$$

$$\int t^2 e^{-t^3}\,dt$$

$u = -t^3$
$du = -3t^2\,dt$
$-\frac{1}{3}du = t^2\,dt$

$$-\frac{1}{3}\int e^u\,du = -\frac{1}{3}e^{-t^3} + C_2$$

$$\int \frac{-2t}{\sqrt{t^2+16}}\,dt$$

$u = t^2+16$
$du = 2t\,dt$

$$-\int u^{-1/2}\,du$$

$$= -2u^{1/2} + C_3$$

$$= -2(t^2+16)^{1/2} + C_3$$

$$\vec{r}(t) = \left\langle \frac{1}{2}\ln(t^2+1),\ -\frac{1}{3}e^{-t^3},\ -2\sqrt{t^2+16} \right\rangle + \langle C_1, C_2, C_3 \rangle$$

$$\langle 3, \tfrac{5}{3}, -5 \rangle = \vec{r}\langle 0 \rangle = \langle 0, -\tfrac{1}{3}, -8 \rangle + \langle C_1, C_2, C_3 \rangle \longrightarrow \begin{array}{l} C_1 = 3 \\ C_2 = 2 \\ C_3 = 3 \end{array}$$

Definition. (Definite Integral of a Vector-Valued Function)
Let $\mathbf{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$, where f, g, and h are integrable on the interval $[a, b]$. The **definite integral** of $\mathbf{r}$ on $[a, b]$ is

$$\int_a^b \mathbf{r}(t)\, dt = \left(\int_a^b f(t)\, dt \right)\boldsymbol{i} + \left(\int_a^b g(t)\, dt \right)\boldsymbol{j} + \left(\int_a^b h(t)\, dt \right)\boldsymbol{k}$$

Example. $\displaystyle\int_{-\pi}^{\pi} \langle \sin(t),\ \cos(t),\ 8t \rangle\, dt$

$$= \left\langle \int_{-\pi}^{\pi} \sin(t)\, dt,\ \int_{-\pi}^{\pi} \cos(t)\, dt,\ \int_{-\pi}^{\pi} 8t\, dt \right\rangle$$

$$= \left\langle -\cos(t) \Big|_{-\pi}^{\pi},\ \sin(t) \Big|_{-\pi}^{\pi},\ 4t^2 \Big|_{-\pi}^{\pi} \right\rangle$$

$$= \left\langle 1 - 1,\ 0 - 0,\ 4\pi^2 - 4\pi^2 \right\rangle$$

$$= \langle 0, 0, 0 \rangle$$

$$= \vec{0}$$