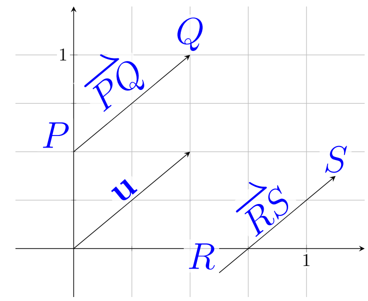
13.1: Vectors and the Geometry of Space

Definition.

- **Vectors**
  - Have a direction and magnitude,
  - vector $\overrightarrow{PQ}$ has a *tail* at P and a *head* at Q,
  - Can be denoted as $\mathbf{u}$ or $\vec{u}$,
  - Equal vectors have the same direction and magnitude (not necessarily the same position)



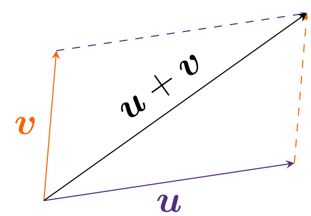
- **Scalars** are quantities with magnitude but no direction (e.g. mass, temperature, price, time, etc.)
- **Zero vector**, denoted $\mathbf{0}$ or $\vec{0}$, has length 0 and no direction

Scalar-vector multiplication:

- Denoted $c\mathbf{v}$ or $c\vec{v}$,
- length of vector multiplied by $|c|$,
- $c\mathbf{v}$ has the same direction as $\mathbf{v}$ if $c > 0$, and has the opposite direction as $\mathbf{v}$ if $c < 0$, (what if $c = 0$?)
- $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if $\mathbf{u} = c\mathbf{v}$. (what vectors are parallel to $\mathbf{0}$?)

Vector Addition and Subtraction:

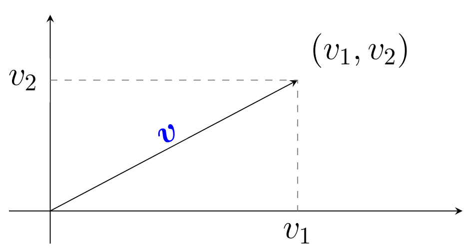
Given two vectors $\mathbf{u}$ and $\mathbf{v}$, their sum, $\mathbf{u}+\mathbf{v}$, can be represented by the parallelogram (triangle) rule: place the tail of $\mathbf{v}$ at the head of $\mathbf{u}$



The difference, denoted $\mathbf{u}-\mathbf{v}$, is the sum of $\mathbf{u}+(-\mathbf{v})$:

Vector Components:

A vector $\mathbf{v}$ whose tail is at the origin $(0,0)$ and head is at (v_1, v_2) is a **position vector** (in **standard position**) and is denoted $\langle v_1, v_2\rangle$. The real numbers v_1 and v_2 are the x- and y-components of $\mathbf{v}$.



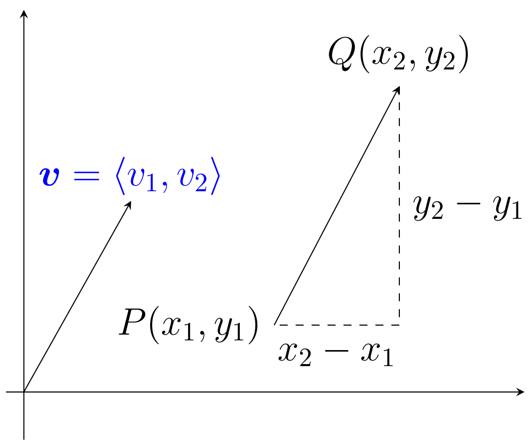
Vectors $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ are equal if and only if $u_1 = v_1$ and $u_2 = v_2$.

Magnitude:

Given points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the **magnitude**, or **length**, of vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$, denoted $\left|\overrightarrow{PQ}\right|$, is the distance between points P and Q.

$$\left|\overrightarrow{PQ}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The magnitude of position vector $\mathbf{v} = \langle v_1, v_2 \rangle$ is $|\mathbf{v}|$.
(How do $\left|\overrightarrow{PQ}\right|$ and $\left|\overrightarrow{QP}\right|$ relate to each other?)



Note: The norm, denoted $\|\mathbf{u}\|$ or $\|\mathbf{u}\|_2$, is equivalent to the magnitude of a vector.

Equation of a Circle:

Definition.
A **circle** centered at (a, b) with radius r is the set of points satisfying the equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

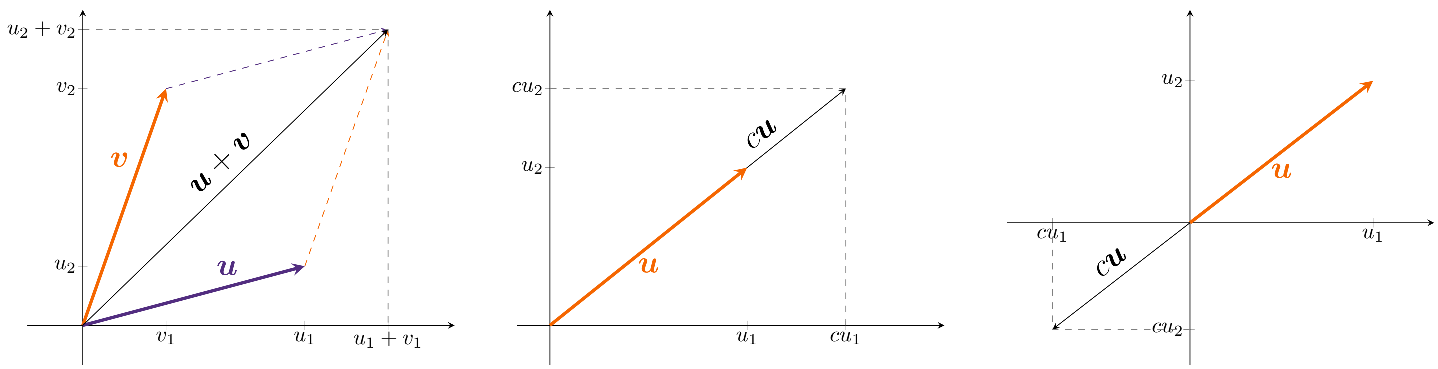
A **disk** centered at (a, b) with radius r is the set of points satisfying the inequality

$$(x - a)^2 + (y - b)^2 \leq r^2.$$

Vector Operations in Terms of Components

Definition. (Vector Operations in $\mathbb{R}^2$)
Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2 \rangle$, and $\mathbf{v} = \langle v_1, v_2 \rangle$.

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle \qquad \text{Vector addition}$$
$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2 \rangle \qquad \text{Vector subtraction}$$
$$c\mathbf{u} = \langle cu_1, cu_2 \rangle \qquad \text{Scalar multiplication}$$



Example. Let $\mathbf{u} = \langle 1, 2 \rangle$, $\mathbf{v} = \langle -2, 3 \rangle$, $c = 2$, and $d = 3$. Find the following:

$\mathbf{u} + \mathbf{v}$

$c\mathbf{u}$

$c\mathbf{u} + d\mathbf{v}$

$\mathbf{u} - c\mathbf{v}$

Definition.
A **unit vector** is any vector with length 1.
In $\mathbb{R}^2$, the **coordinate unit vectors** are $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.

Example. Let $\mathbf{u} = \langle -7, 3 \rangle$. Find two unit vectors parallel to $\mathbf{u}$. Find another vector parallel to $\mathbf{u}$ with a magnitude of 2.

Properties of Vector Operations:
Suppose $\mathbf{u}$, $\mathbf{v}$, and $\boldsymbol{w}$ are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ — Commutative property of addition
2. $(\mathbf{u}+\mathbf{v})+\boldsymbol{w}=\mathbf{u}+(\mathbf{v}+\boldsymbol{w})$ — Associative property of addition
3. $\mathbf{v}+\mathbf{0}=\mathbf{v}$ — Additive identity
4. $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$ — Additive inverse
5. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$ — Distributive property 1
6. $(a+c)\mathbf{v}=a\mathbf{v}+c\mathbf{v}$ — Distributive property 2
7. $0\mathbf{v}=\mathbf{0}$ — Multiplication by zero scalar
8. $c\mathbf{0}=\mathbf{0}$ — Multiplication by zero vector
9. $1\mathbf{v}=\mathbf{v}$ — Multiplicative identity
10. $a(c\mathbf{v})=(ac)\mathbf{v}$ — Associative property of scalar multiplication