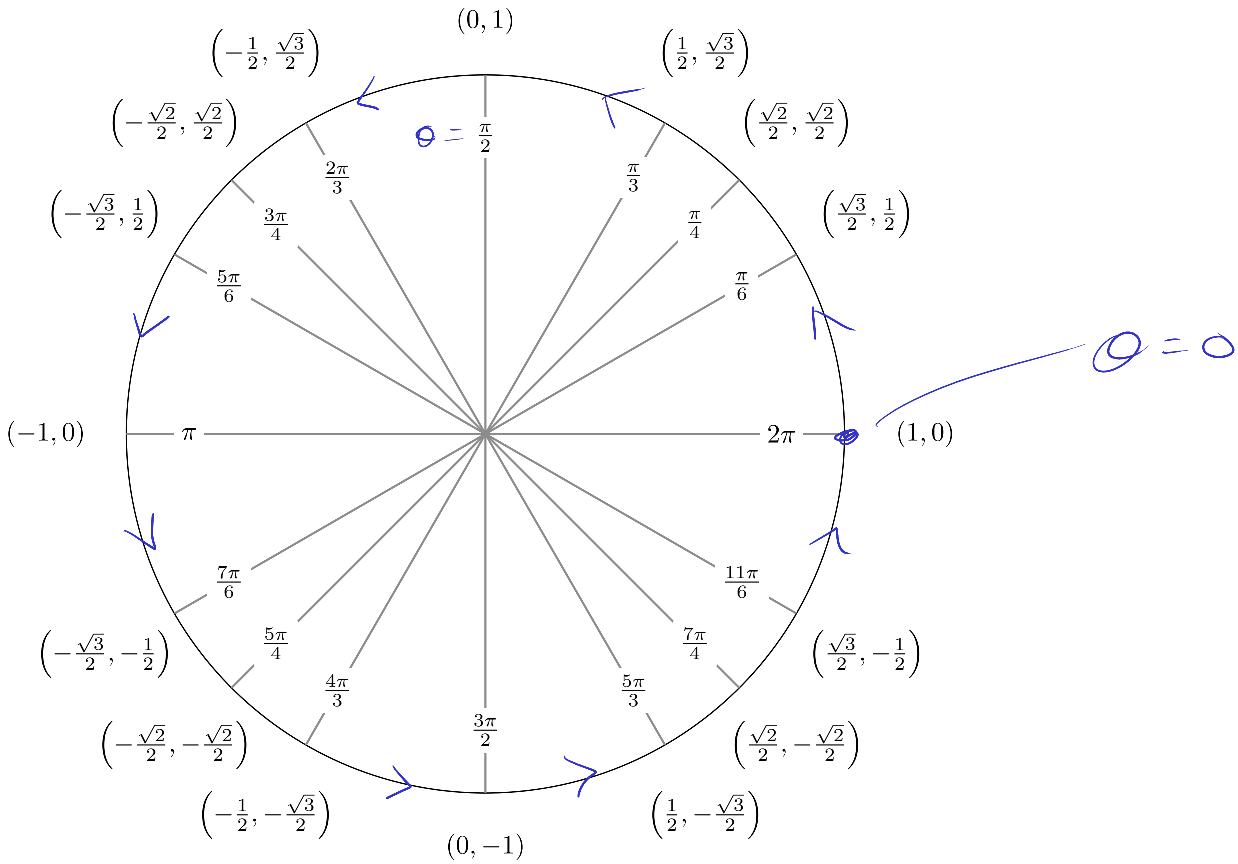
12.1: Parametric Equations

We've already seen a parametric equation represented by the unit circle. Here, we have

$$x(\theta) = \cos(\theta) \text{ and } y(\theta) = \sin(\theta), \text{ where } 0 \leq \theta \leq 2\pi$$



Definition. (Positive Orientation)
The direction in which a parametric curve is generated as the parameter increases is called the **positive orientation** of the curve (and is indicated by arrows on the curve).

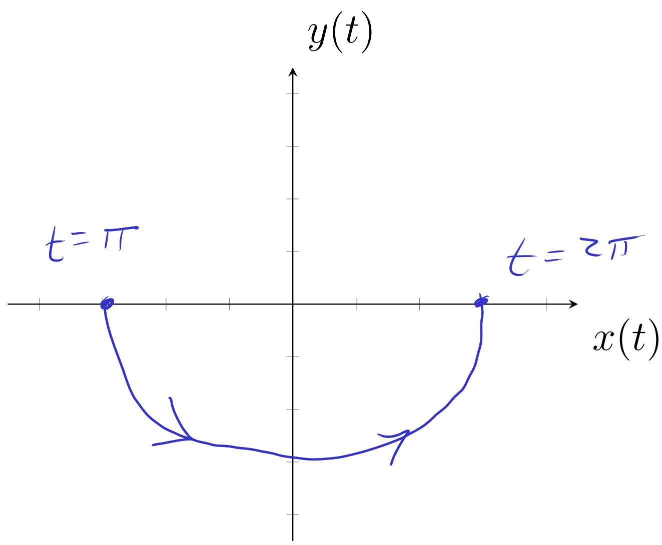
Example (LC 32.1-32.2)**.** Consider the parametric equations

$$x = 3\cos(t),\;\; y = 3\sin(t); \pi \leq t \leq 2\pi$$

Eliminate the parameter t and rewrite as a function of x and y.

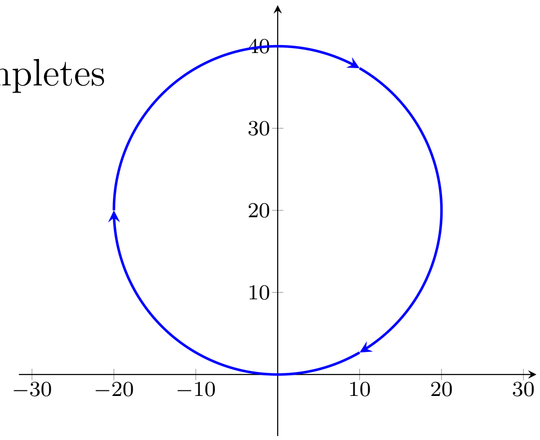
$$\left.\begin{aligned} & x^2 + y^2 \\ &= (3\cos(t))^2 + (3\sin(t))^2 \\ &= 9\cos^2(t) + 9\sin^2(t) \\ &= 9(\cos^2(t) + \sin^2(t)) = 9 \end{aligned}\right\} \quad x^2 + y^2 = 9 \;\; ; \;\; -3 \leq y \leq 0$$

Graph the equation found above indicating the positive orientation.



Example (LC 32.3-32.4). A Ferris wheel has a radius of 20 m and completes a revolution in the **clockwise** direction at constant speed in 3 minutes. Assume x and y measure the horizontal and vertical positions of a seat on the Ferris wheel relative to a coordinate system whose origin is at the low point of the wheel. Assume the seat begins moving at the origin.

What is the domain of t such that the Ferris wheel completes one revolution?



$x(t)$ and $y(t)$ will be parameterized using $\sin(bt)$ and $\cos(bt)$. What is b?

What parametric equations describe the path of the seat on the Ferris wheel?

Summary: Parametric Equations of a Line
The equations

$$x = x_0 + at,\ y = y_0 + bt, \quad \text{for } -\infty < t < \infty,$$

where x_0, y_0, a, and b are constants with $a \neq 0$, describe a line with slope $\frac{b}{a}$ passing through the point (x_0, y_0). If $a = 0$ and $b \neq 0$, the line is vertical.

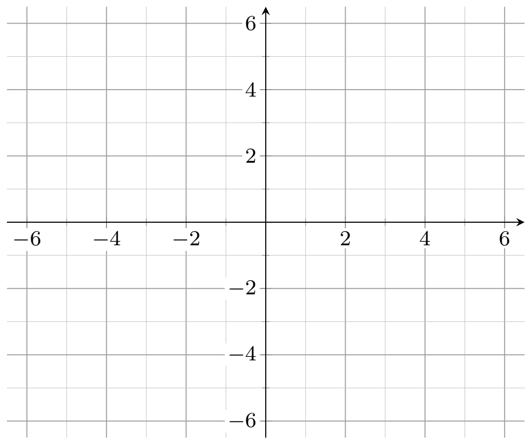
Example. Find 2 parameterized equations of the line that goes through the points $(3, -4)$ and $(-2, 3)$.

Example. Find a parameterized equation for the line segment that connects the points $(3, 0)$ and $(-1, 3)$.

Example. Consider the parametric equations

$$x(t) = 6 - 2t \text{ and } y(t) = -2 + t,$$

Graph the curve indicating the positive orientation



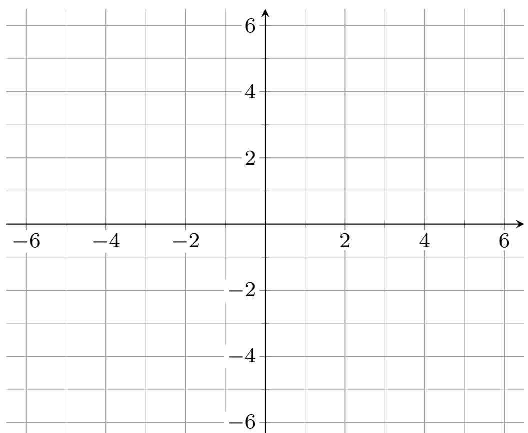
Eliminate the parameter to find an equation in x and y.

Example (LC 32.5-32.7). Consider the parametric equations

$$x = 1 + e^{2t} \text{ and } y = e^t,$$

Eliminate the parameter to find an equation in x and y

Graph the curve indicating the positive orientation



Which of the following parametric equations are equivalent?

$$x = 2t^2, \qquad y = 4 + t; \qquad -4 \le t \le 4$$

$$x = 2t^4, \qquad y = 4 + t^2; \qquad -2 \le t \le 2$$

$$x = 2t^{2/3}, \qquad y = 4 + t^{1/3}; \qquad -64 \le t \le 64$$

Theorem 12.1: Derivative for Parametric Curves
Let $x = f(t)$ and $y = g(t)$, where f and g are differentiable on an interval $[a, b]$. Then the slope of the line tangent to the curve at the point corresponding to t is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)},$$

provided $f'(t) \neq 0$.

Example (LC 32.8-32.9)**.** Consider the parametric equations

$$x = \sqrt{t}, \qquad y = 2t,$$

Find $\dfrac{dy}{dt}$.

Find the equation of the line tangent to the curve at $t = 4$.

Definition. (Arc Length for Curves Defined by Parametric Equations) Consider the curve described by the parametric equations $x = f(t)$, $y = g(t)$, where f' and g' are continuous, and the curve is traversed once for $a \leq t \leq b$. The **arc length** of the curve between $(f(a), g(a))$ and $(f(b), g(b))$ is

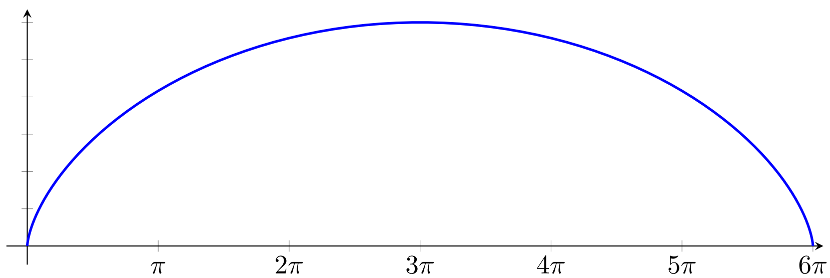
$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

Example (LC 33.1-33.2)**.** Find the arc length of the curve given by $x = 6t^2$, $y = 2t^3$, for $0 \leq t \leq 4$.

Example (Arc length)**.** Suppose the function $y = h(x)$ is nonnegative and continuous on $[\alpha, \beta]$, which implies that the area bounded by the graph of h and the x-axis on $[\alpha, \beta]$ equals $\int_\alpha^\beta h(x)\,dx$ or $\int_\alpha^\beta y\,dx$. If the graph of $y = h(x)$ on $[\alpha, \beta]$ is traced exactly once by the parametric equations $x = f(t)$, $y = g(t)$, for $a \le t \le b$, then it follows by substitution that the area bounded by h is

$$\int_\alpha^\beta h(x)\,dx = \int_a^b g(t)f'(t)\,dt \text{ if } \alpha = f(a) \text{ and } \beta = f(b)$$

Find the area under one arch of the cycloid $x = 3(t - \sin(t))$, $y = 3(1 - \cos(t))$.



Example (33.3 Surface area)**.** Let C be the curve $x = f(t)$, $y = g(t)$, for $a \leq t \leq b$, where f' and g' are continuous on $[a, b]$ and C does not intersect itself, except possibly at its endpoints. If g is nonnegative on $[a, b]$, then the area of the surface obtained by revolving C about the x-axis is

$$S = \int_a^b 2\pi g(t)\sqrt{f'(t)^2 + g'(t)^2}\,dt.$$

Find the area of the surface obtained by revolving the curve $x = t\sin(t)$, $y = t\cos(t)$, for $0 \leq t \leq \pi/2$, about the x-axis.