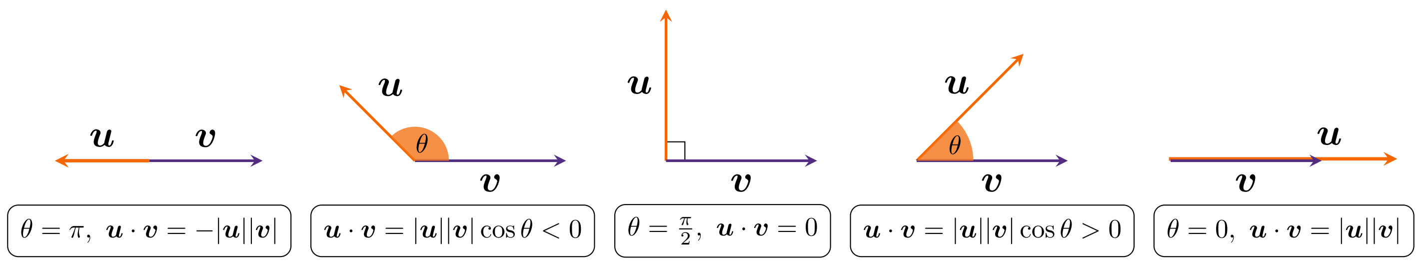
1 13.3: Dot Products

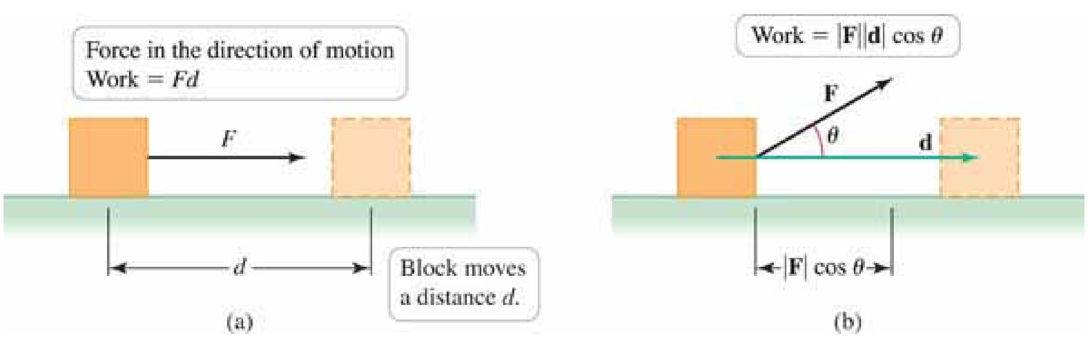
Definition. (Dot Product)
Given two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in two or three dimensions, their **dot product** is

$$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}| \cos\theta,$$

where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ with $0 \le \theta \le \pi$. If $\boldsymbol{u} = \mathbf{0}$ or $\boldsymbol{v} = \mathbf{0}$, then $\boldsymbol{u} \cdot \boldsymbol{v} = 0$, and θ is undefined.



A physical example of the dot product is the amount of work done when a force is applied at an angle θ as shown in figure 13.43:



Note: The result of the dot product is a scalar!

Definition. (Orthogonal Vectors)
Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **orthogonal** if and only if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$. The zero vector is orthogonal to all vectors. In two or three dimensions, two nonzero orthogonal vectors are perpendicular to each other.

- $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\boldsymbol{u} \cdot \boldsymbol{v} = \pm|\boldsymbol{u}||\boldsymbol{v}|$.
- $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular ($\theta = \frac{\pi}{2}$) if and only if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

Theorem 31.1: Dot Product
Given two vectors $\boldsymbol{u} = \langle u_1, u_2, u_3 \rangle$ and $\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle$,

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Properties of Dot Products

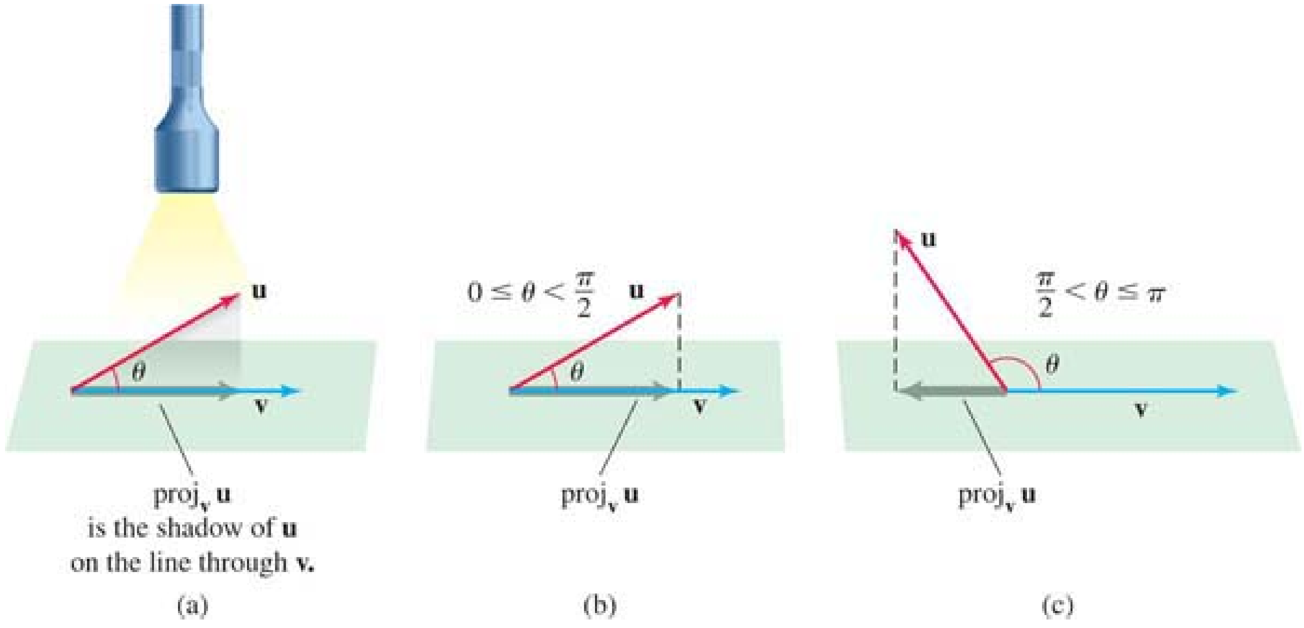
Theorem 13.2: Properties of the Dot Product

Suppose $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ are vectors and let c be a scalar.

1. $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$ Commutative property
2. $c(\boldsymbol{u} \cdot \boldsymbol{v}) = (c\boldsymbol{u}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot (c\boldsymbol{v})$ Associative property
3. $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$ Distributive property

Orthogonal Projections

Given vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, the projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$ produces a vector parallel to $\boldsymbol{v}$ using the "shadow" of $\boldsymbol{u}$ cast onto $\boldsymbol{v}$.



Definition. ((Orthogonal) Projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$)
The **orthogonal projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$**, denoted $\text{proj}_{\boldsymbol{v}}\,\boldsymbol{u}$, where $\boldsymbol{v} \neq \mathbf{0}$, is

$$\text{proj}_{\boldsymbol{v}}\,\boldsymbol{u} = \underbrace{|\boldsymbol{u}|\cos\theta}_{\text{length}}\underbrace{\left(\frac{\boldsymbol{v}}{|\boldsymbol{v}|}\right)}_{\text{direction}}.$$

The orthogonal projection may also be computed with the formulas

$$\text{proj}_{\boldsymbol{v}}\,\boldsymbol{u} = \text{scal}_{\boldsymbol{v}}\,\boldsymbol{u}\left(\frac{\boldsymbol{v}}{|\boldsymbol{v}|}\right) = \left(\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\boldsymbol{v}\cdot\boldsymbol{v}}\right)\boldsymbol{v},$$

where the **scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$** is

$$\text{scal}_{\boldsymbol{v}}\,\boldsymbol{u} = |\boldsymbol{u}|\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{v}|}.$$

Applications of Dot Products

Definition. (Work)
Let a constant force $\mathbf{F}$ be applied to an object, producing a displacement $\mathbf{d}$. If the angle between $\mathbf{F}$ and $\mathbf{d}$ is θ, then the **work** done by the force is

$$W = |\mathbf{F}||\mathbf{d}|\cos\theta = \mathbf{F}\cdot\mathbf{d}$$

Example.

<u>Parallel and Normal Forces:</u>
Example.