

## 8.9: Improper Integrals

### Definition. (Improper Integrals over Infinite Intervals)

1. If  $f$  is continuous on  $[a, \infty)$ , then

$$\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx.$$

2. If  $f$  is continuous on  $(-\infty, b]$ , then

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx.$$

3. If  $f$  is continuous on  $(-\infty, \infty)$ , then

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{a \rightarrow -\infty} \int_a^c f(x) dx + \lim_{b \rightarrow \infty} \int_c^b f(x) dx.$$

where  $c$  is any real number. It can be shown that the choice of  $c$  does not affect the value or convergence of the original integral.

If the limits in cases 1.– 3. exist, then the improper integrals **converge**; otherwise they **diverge**.

**Example.** Evaluate  $\int_1^{\infty} \frac{\ln(x)}{x} dx$  and determine if the integral converges or diverges.

**Example.** Evaluate  $\int_{-\infty}^{\infty} \frac{e^{3x}}{1 + e^{6x}} dx$ .

**Example.** For what values of  $p$  does  $\int_1^\infty \frac{1}{x^p} dx$  converge?

**Definition. (Improper Integrals with Unbounded Integrand)**

1. Suppose  $f$  is continuous on  $(a, b]$  with  $\lim_{x \rightarrow a^+} f(x) = \pm\infty$ . Then

$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx.$$

2. Suppose  $f$  is continuous on  $[a, b)$  with  $\lim_{x \rightarrow b^-} f(x) = \pm\infty$ . Then

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx.$$

3. Suppose  $f$  is continuous on  $[a, b]$  except at the interior point  $p$  where  $f$  is unbounded. Then

$$\int_a^b f(x) dx = \lim_{c \rightarrow p^-} \int_a^c f(x) dx + \lim_{d \rightarrow p^+} \int_d^b f(x) dx.$$

If the limits in cases 1.– 3. exist, then the improper integrals **converge**; otherwise, they **diverge**.

**Example.** Determine which of the following integrals are improper integrals

$$\int_0^1 \sec(x) \, dx$$

$$\int_{\pi/2}^{3\pi/4} \tan(x) \, dx$$

$$\int_1^e \ln(x) \, dx$$

$$\int_0^1 \arctan(x) \, dx$$

$$\int_0^{0.5} \ln(x) \, dx$$

$$\int_{-10}^{-1} \frac{1}{x^{1/3}} \, dx$$

**Example.** Evaluate  $\int_1^9 \frac{dx}{(x-1)^{2/3}}$ . Does this integral converge or diverge?

**Example.** Evaluate  $\int_{-1}^1 \frac{e^{2/x}}{x^2} dx$ . Does this integral converge or diverge?

**Theorem 8.2: Comparison Test for Improper Integrals**

Suppose the functions  $f$  and  $g$  are continuous on the interval  $[a, \infty)$ , with  $f(x) \geq g(x) \geq 0$ , for  $x \geq a$ .

1. If  $\int_a^\infty f(x) dx$  converges, then  $\int_a^\infty g(x) dx$  converges.
2. If  $\int_a^\infty g(x) dx$  diverges, then  $\int_a^\infty f(x) dx$  diverges.

**Example.** Determine if the integral  $\int_2^\infty \frac{x^3}{x^4 - x^3 - 1} dx$  converges or diverges.



**Example** (Gabriel's Horn). Let  $R$  be the region bounded by the graph of  $y = 1/x$  and the  $x$ -axis for  $x \geq 1$ .

What is the volume of the solid generated when  $R$  is revolved around the  $x$ -axis?

What is the surface area of the solid generated when  $R$  is revolved about the  $x$ -axis?