12.1 Decomposition of Functions

Example. Decompose the following functions:

1. A function under a power

a) $y(x) = \left(x^3 - 1\right)^2$

$f(x) = x^2$

$g(x) = x^3 - 1$

b) $y(x) = \left(\sqrt[5]{x} - 1\right)^{\frac{2}{3}}$

$f(x) = x^{2/3}$

$g(x) = \sqrt[5]{x} - 1$

c) $y(x) = \tan^2(x)$

$f(x) = x^2$

$g(x) = \tan(x)$

2. The argument of a trig function

a) $y(x) = \cos\left(x^5\right)$

$f(x) = \cos(x)$

$g(x) = x^5$

b) $y(x) = \sin\sqrt{x}$

$f(x) = \sin(x)$

$g(x) = \sqrt{x}$

c) $y(x) = \sin\left(3^x\right)$

$f(x) = \sin(x)$

$g(x) = 3^x$

3. The functional power of an exponent

a) $y(x) = e^{3x+1}$

$f(x) = e^x$

$g(x) = 3x + 1$

4. Various combinations

a) $y(x) = \tan^3(2x)$

$f(x) = x^3$

$g(x) = \tan(x)$

$h(x) = 2x$

b) $y(x) = 2^{\sqrt{\sin(x)}}$

$f(x) = 2^x$

$g(x) = \sqrt{x}$

$h(x) = \sin(x)$

c) $y(x) = \cos\left(x^3 - 2\right)^{2/7}$

$f(x) = x^{2/7}$

$g(x) = \cos(x)$

$h(x) = x^3 - 2$

3.7 The Chain Rule

Theorem 3.13 The Chain Rule
Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at x. The composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed in two equivalent ways.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{1}$$

$$\frac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x) \tag{2}$$

Example. Take the derivatives of the following functions

a) $y = \left(3x^3 + 1\right)^2$

$y' = 2(3x^3+1)(9x^2)$
$= 54x^5 + 18x^2$

$\boxed{\begin{aligned} y &= 9x^6+6x^3+1 \\ y' &= 54x^5+18x^2 \end{aligned}}$

b) $y = \left(3x^3 + 1\right)^7$

$y' = 7(3x^3+1)^6(9x^2)$
$= 63x^2(3x^3+1)^6$

c) $y = 6\cos^2(x)$

$y' = 12\cos(x) \cdot \sin(x)$

d) $y = \sin\left(x + \cot(x)\right)$

$y = \cos(x + \cot(x))\left[1 - \csc^2(x)\right]$

To use the chain rule,

- Identify the inner and outer function
- Take the derivative of the outside, leaving the original inner function
- Multiply by the derivative of the inner function

e) $y(x) = e^{-4x}$

$y' = e^{-4x}(-4)$

f) $y(x) = \sin(x + \cot(x))$

$y' = \cos(x + \cot(x))\left[1 - \csc^2(x)\right]$

g) $y(x) = \sqrt{\sec(x)} = [\sec(x)]^{1/2}$

$y' = \frac{1}{2}[\sec(x)]^{-1/2}\sec(x)\tan(x)$

$= \frac{1}{2}\sqrt{\sec(x)}\ \tan(x)$

h) $y(x) = 2(8x-1)^3$

$y' = 6(8x-1)^2(8)$

$= 48(8x-1)^2$

i) $y(x) = \left(\frac{x}{2} - 1\right)^{-10}$

$y' = -10\left(\frac{x}{2} - 1\right)^{-11}\left(\frac{1}{2}\right)$

$= -5\left(\frac{x}{2} - 1\right)^{-11}$

j) $y(x) = e^{\sin(t)} + \sin(e^t)$

$y' = e^{\sin(t)}\cos(t) + \cos(e^t)e^t$

k) $y(x) = x^2 e^{x^2}$

$$y'(x) = \frac{d}{dx}\left[x^2\right]e^{x^2} + x^2\frac{d}{dx}\left[e^{x^2}\right]$$

$$= 2xe^{x^2} + x^2e^{x^2}\frac{d}{dx}\left[x^2\right]$$

$$= 2xe^{x^2} + x^2e^{x^2}(2x)$$

$$= \boxed{2xe^{x^2}(1+x^2)}$$

l) $\dfrac{f(x)}{g(x)} = f(x)\cdot[g(x)]^{-1}$

$$\frac{d}{dx}\left[f(x)[g(x)]^{-1}\right] = f'(x)[g(x)]^{-1} + f(x)(-1)[g(x)]^{-2}g'(x)$$

$$= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2}$$

$$= \boxed{\frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}}$$

m) $y(x) = f\big(g(h(x))\big)$

$$\boxed{y'(x) = f'(g(h(x)))\cdot g'(h(x))\cdot h'(x)}$$

n) $y(x) = -12e^{3x^7}$

$$y'(x) = -12e^{3x^7}(21x^6)$$

$$= \boxed{-252x^6e^{3x^7}}$$

o) $y(x) = \dfrac{\cos^2(x)}{e^x(x^2+4)}$

$$y'(x) = \frac{e^x(x^2+4)\frac{d}{dx}\left[\cos^2(x)\right] - \cos^2(x)\frac{d}{dx}\left[e^x(x^2+4)\right]}{\left[e^x(x^2+4)\right]^2}$$

$$= \frac{e^x(x^2+4)\left[2\cos(x)(-\sin(x))\right] - \cos^2(x)\overbrace{\left[e^x(x^2+4) + e^x(2x)\right]}^{\text{product rule}}}{e^{2x}(x^2+4)^2}$$

$$= \frac{-2(x^2+4)\sin(x)\cos(x) - \cos^2(x)\left[x^2+2x+4\right]}{e^x(x^2+4)^2}$$