$\vec{u}, \vec{v}$

$\vec{u} \pm \vec{v} = \langle u_1 \pm v_1, u_2 \pm v_2, u_3 \pm v_3 \rangle$

13.3: Dot Products

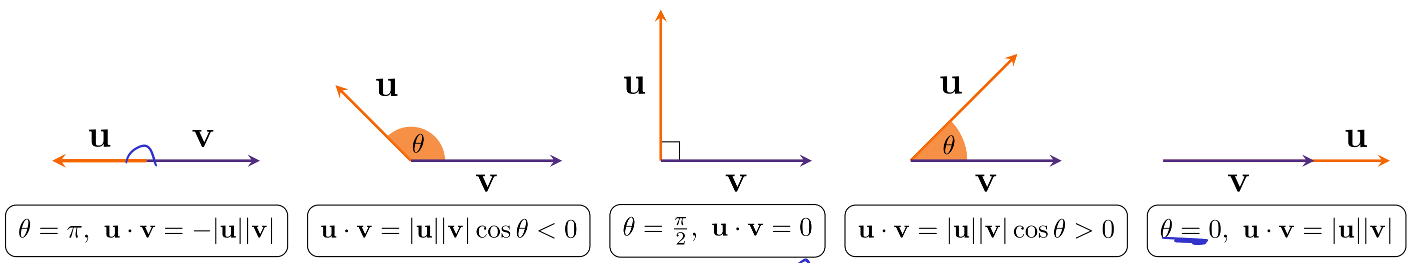
$a\vec{u} = \langle au_1, au_2, au_3 \rangle$

$\cancel{\vec{u}\vec{v}}$?

Definition. (Dot Product)
Given two nonzero vectors **u** and **v** in two or three dimensions, their **dot product** is

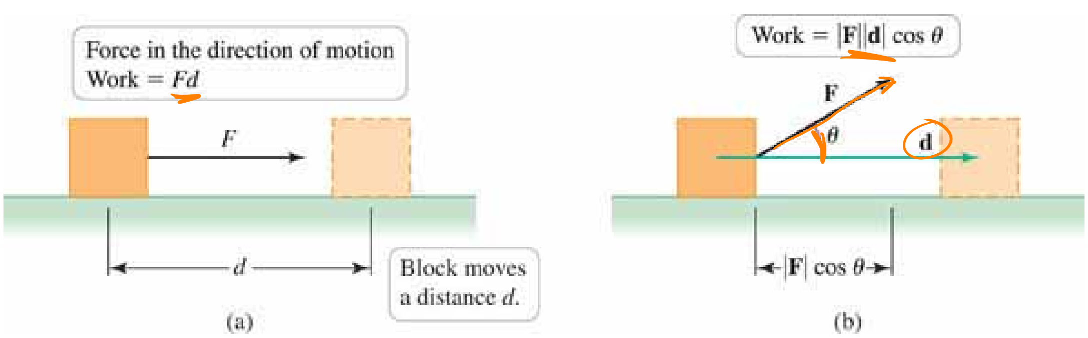
$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta, \quad = \vec{v} \cdot \vec{u}$$

where θ is the angle between **u** and **v** with $0 \leq \theta \leq \pi$. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$, and θ is undefined.



Represents how much they have in common

A physical example of the dot product is the amount of work done when a force is applied at an angle θ as shown in figure 13.43:

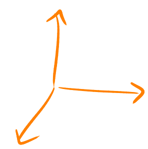


Note: The result of the dot product is a scalar!

$\vec{u}\cdot\vec{0} = |\vec{u}||\vec{0}|\cos\theta = 0$

Definition. (Orthogonal Vectors)
Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if and only if $\mathbf{u}\cdot\mathbf{v} = 0$. The zero vector is orthogonal to all vectors. In two or three dimensions, two nonzero orthogonal vectors are perpendicular to each other.

- $\mathbf{u}$ and $\mathbf{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u}\cdot\mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$.
- $\mathbf{u}$ and $\mathbf{v}$ are perpendicular ($\theta = \frac{\pi}{2}$) if and only if $\mathbf{u}\cdot\mathbf{v} = 0$.

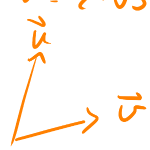


Example. Given $|\mathbf{u}| = 2$ and $|\mathbf{v}| = \sqrt{3}$, compute $\mathbf{u}\cdot\mathbf{v}$ when

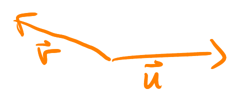
- $\theta = \frac{\pi}{4}$
- $\theta = \frac{\pi}{3}$
- $\theta = \frac{5\pi}{6}$

$u\cdot v = |u||v|\cos\left(\frac{\pi}{4}\right) = 2\cdot\sqrt{3}\left(\frac{\sqrt{2}}{2}\right) = \boxed{\sqrt{6}}$

$u\cdot v = 2\cdot\sqrt{3}\left(\frac{1}{2}\right) = \boxed{\sqrt{3}}$



$u\cdot v = 2\cdot\sqrt{3}\left(-\frac{\sqrt{3}}{2}\right) = \boxed{-3}$



Theorem 31.1: Dot Product
Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$,

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3. = |\vec{u}||\vec{v}|\cos\theta$$

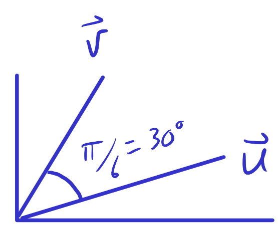
Example. Given vectors $\mathbf{u} = \langle\sqrt{3}, 1, 0\rangle$ and $\mathbf{v} = \langle 1, \sqrt{3}, 0\rangle$, compute $\mathbf{u}\cdot\mathbf{v}$ and find θ.

$|u| = \sqrt{(\sqrt{3})^2 + 1^2 + 0^2} = \sqrt{4} = 2$

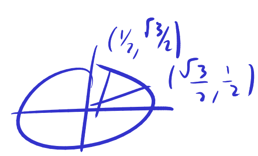
$u\cdot v = \sqrt{3}\cdot 1 + 1\cdot\sqrt{3} + 0\cdot 0 = 2\sqrt{3}$

$u\cdot v = |u||v|\cos\theta \rightarrow \cos\theta = \frac{u\cdot v}{|u||v|} = \frac{\cancel{2}\sqrt{3}}{\cancel{2}\cdot 2} = \frac{\sqrt{3}}{2}$

$\cos\theta = \frac{\sqrt{3}}{2} \rightarrow \boxed{\theta = \frac{\pi}{6}}$



$|u| = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{u_1\cdot u_1 + u_2\cdot u_2 + u_3\cdot u_3}$



$u\cdot u = |u|^2$

$u\cdot u = |u||u|\cos(0) = |u|^2$

Properties of Dot Products

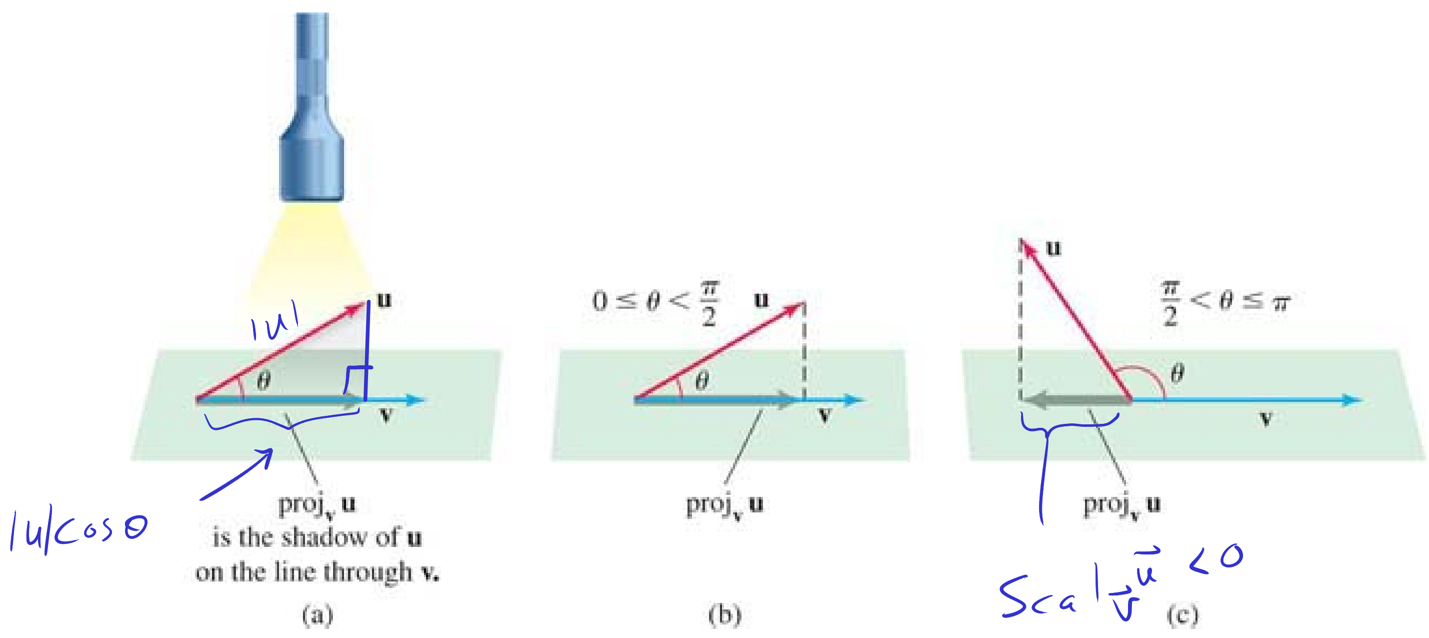
Theorem 13.2: Properties of the Dot Product
Suppose $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ — Commutative property
2. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$ — Associative property
3. $\mathbf{u} \cdot \underbrace{(\mathbf{v} + \mathbf{w})} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ — Distributive property

$\langle u_1, u_2, u_3 \rangle \cdot \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$

Orthogonal Projections

Given vectors $\mathbf{u}$ and $\mathbf{v}$, the projection of $\mathbf{u}$ onto $\mathbf{v}$ produces a vector parallel to $\mathbf{v}$ using the "shadow" of $\mathbf{u}$ cast onto $\mathbf{v}$.



$\vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta$

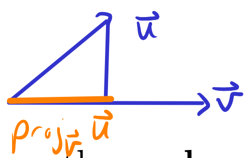
scalar   vector

$$\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\left(\frac{\vec{v}}{|\vec{v}|}\right) = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$$

Definition. ((Orthogonal) Projection of u onto v)
The **orthogonal projection of u onto v**, denoted $\text{proj}_{\mathbf{v}}\,\mathbf{u}$, where $\mathbf{v} \neq \mathbf{0}$, is

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \underbrace{|\mathbf{u}|\cos\theta}_{\text{length}}\underbrace{\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)}_{\text{direction}}.$$

The orthogonal projection may also be computed with the formulas



$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \text{scal}_{\mathbf{v}}\,\mathbf{u}\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v},$$

vector

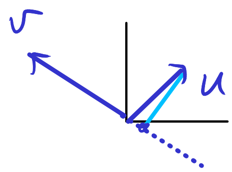
where the **scalar component of u in the direction of v** is

$$\text{scal}_{\mathbf{v}}\,\mathbf{u} = |\mathbf{u}|\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}.$$

length of $\text{proj}_{\vec{v}}\vec{u}$

~~$|\text{proj}_{\vec{v}}\vec{u}| \geq 0$~~

Example. Find $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ and $\text{scal}_{\mathbf{v}}\,\mathbf{u}$ for the following:



- $\mathbf{u} = \langle 1, 1\rangle$, $\mathbf{v} = \langle -2, 1\rangle$

$$\text{proj}_{\vec{v}}\vec{u} = \underbrace{\left(\frac{\langle 1,1\rangle\cdot\langle -2,1\rangle}{\langle -2,1\rangle\cdot\langle -2,1\rangle}\right)}_{\text{scalar}}\langle -2,1\rangle = \frac{-2+1}{4+1}\langle -2,1\rangle = \frac{-1}{5}\langle -2,1\rangle = \left\langle \frac{2}{5}, -\frac{1}{5}\right\rangle$$

$$\text{scal}_{\vec{v}}\vec{u} = \frac{-1}{\sqrt{5}}$$

- $\mathbf{u} = \langle 7, 1, 7\rangle$, $\mathbf{v} = \langle 5, 7, 0\rangle$

$$\text{proj}_{\vec{v}}\vec{u} = \frac{35+7}{25+49}\langle 5,7,0\rangle = \frac{42}{74}\langle 5,7,0\rangle = \frac{21}{37}\langle 5,7,0\rangle$$

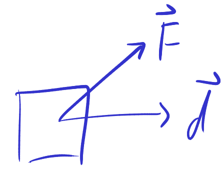
$$\text{scal}_{\vec{v}}\vec{u} = \frac{35+7}{\sqrt{25+49}} = \frac{42}{\sqrt{74}}$$

$|-6| = \sqrt{(-6)^2}$

Applications of Dot Products

Definition. (Work)
Let a constant force **F** be applied to an object, producing a displacement **d**. If the angle between **F** and **d** is θ, then the **work** done by the force is

$$W = |\mathbf{F}||\mathbf{d}|\cos\theta = \mathbf{F}\cdot\mathbf{d}$$



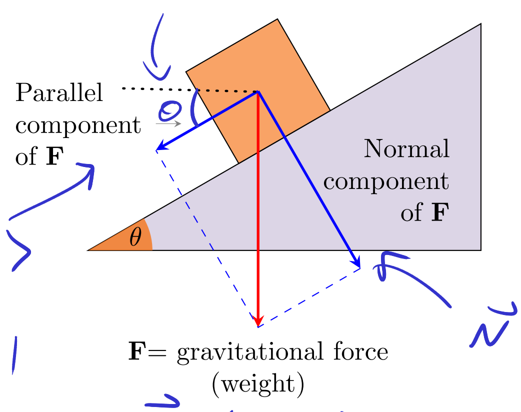
Example. A force $\mathbf{F} = \langle 3, 3, 2\rangle$ (in newtons) moves an object along a line segment from $P(1, 1, 0)$ to $Q(6, 6, 0)$ (in meters). What is the work done by the force?

$\vec{PQ} = \vec{d} = \langle 6-1, 6-1, 0-0\rangle = \langle 5, 5, 0\rangle$

$W = \vec{F}\cdot\vec{d} = \langle 3,3,2\rangle\cdot\langle 5,5,0\rangle$
$= 15+15+0 = 30\ Nm$

Parallel and Normal Forces:

Example. A 10-lb block rests on a plane that is inclined at 30° above the horizontal. Find the components of the gravitational force parallel to and normal (perpendicular) to the plane.



$\vec{v} = \langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}\rangle$

$|\vec{v}| = 1$

$\vec{F} = \langle 0, -10\rangle$

$\text{proj}_{\vec{v}}\vec{F} = \left(\frac{\vec{F}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} = 5\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}\rangle$

$\vec{F} = \text{proj}_{\vec{v}}\vec{F} + \vec{N} \rightarrow \vec{N} = \vec{F} - \underbrace{\text{proj}_{\vec{v}}\vec{F}} = \langle 0,-10\rangle - 5\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}\rangle = \left\langle \frac{5\sqrt{3}}{2}, -\frac{15}{2}\right\rangle$