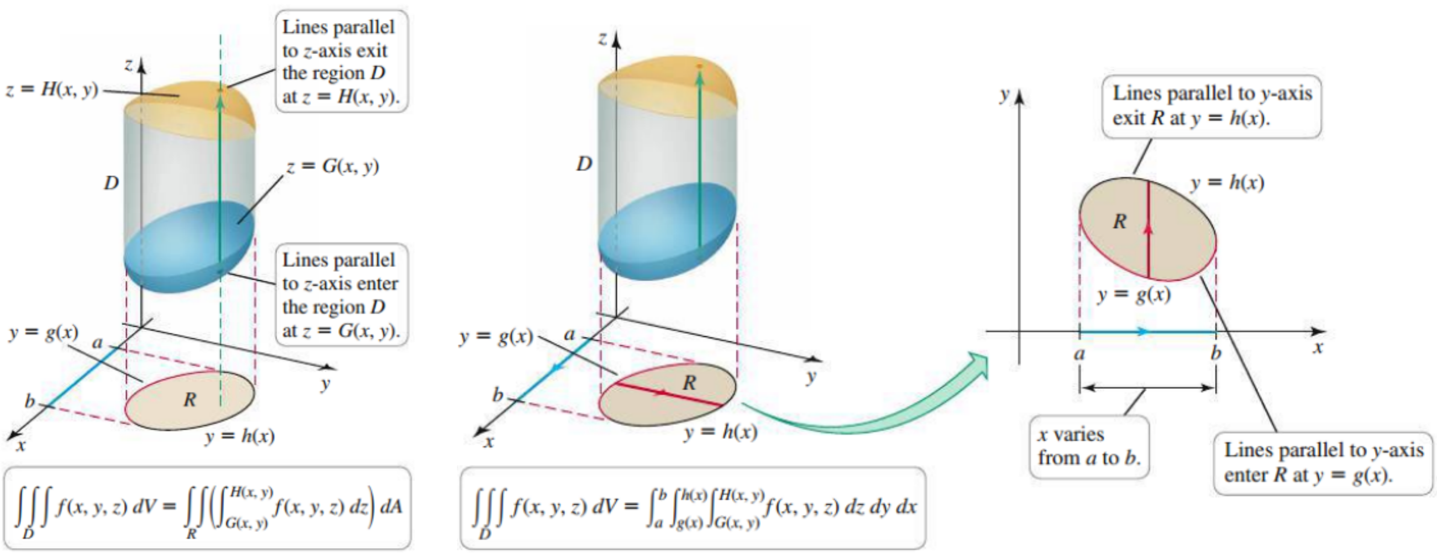
16.4: Triple Integrals

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

> **Theorem 16.5: Triple Integrals**
> Let f be continuous over the region
> $$D = \{(x,y,z) : a \le x \le b,\ g(x) \le y \le h(x),\ G(x,y) \le z \le H(x,y)\},$$
> where g, h, G, and H are continuous functions. Then f is integrable over D and the triple integral is evaluated as the iterated integral
> $$\iiint_D f(x,y,z)\,dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

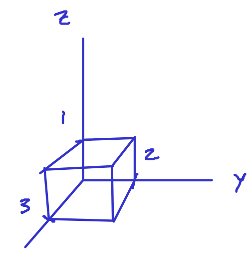


Integral	Variable	Interval
Inner	z	$G(x,y) \le z \le H(x,y)$
Middle	y	$g(x) \le y \le h(x)$
Outer	x	$a \le x \le b$

$0 \le x \le 3$ $\quad$ $0 \le y \le 2$ $\quad$ $0 \le z \le 1$

Example. A solid box D is bounded by the planes $x = 0$, $x = 3$, $y = 0$, $y = 2$, $z = 0$, and $z = 1$. The density of the box decreases linearly in the positive z-direction and is given by $f(x, y, z) = 2 - z$. Find the mass of the box.

$$\int_0^3 \int_0^2 \int_0^1 (2-z)\, dz\, dy\, dx$$



$$= \int_0^3 \int_0^2 2z - \frac{z^2}{2} \Big|_{z=0}^{z=1} dy\, dx = \int_0^3 \int_0^2 \frac{3}{2}\, dy\, dx$$

$$= \int_0^3 \frac{3}{2} y \Big|_{y=0}^{y=2} dx = \int_0^3 3\, dx = 3x \Big|_{x=0}^{x=3} = \boxed{9}$$

$\boxed{LC\#1}$

Example. Find the volume of the prism D in the first octant bounded by the planes $y = 4 - 2x$ and $z = 6$.

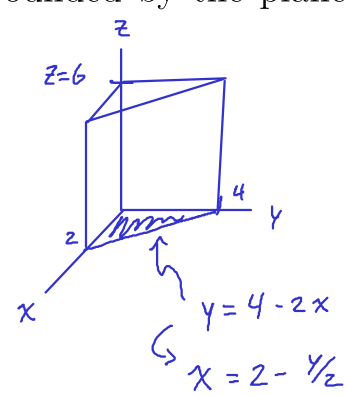
$0 \le z \le 6$

$0 \le y \le 4$

$0 \le x \le 2 - y/2$

$0 \le x \le 2$

$0 \le y \le 4 - 2x$



$$\int_0^4 \int_0^{2-y/2} \int_0^6 1\, dz\, dx\, dy$$

$$= \int_0^4 \int_0^{2-y/2} z \Big|_{z=0}^{z=6} dx\, dy = 6 \int_0^4 x \Big|_{x=0}^{x=2-y/2} dy = 6 \int_0^4 (2 - y/2)\, dy$$

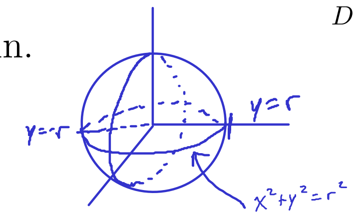
$$= 6 \left(2y - \frac{y^2}{4}\right) \Big|_{y=0}^{y=4}$$

$$= 6(8-4) = 24$$

$\boxed{LC\#2}$

Example. Write the triple integral for $\iiint_D f(x,y,z)\,dV$ where D is a sphere of radius r centered at the origin.

$0 \le x^2+y^2+z^2 \le r^2$



$-r \le y \le r$

$-\sqrt{r^2-y^2} \le x \le \sqrt{r^2-y^2}$

$-\sqrt{r^2-x^2-y^2} \le z \le \sqrt{r^2-x^2-y^2}$

$z = \pm\sqrt{r^2-x^2-y^2}$

$$\int_{-r}^{r}\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}}\int_{-\sqrt{r^2-x^2-y^2}}^{\sqrt{r^2-x^2-y^2}} f(x,y,z)\,dz\,dx\,dy$$

Example. Find the volume of the solid D bounded by the paraboloids $y = x^2+3z^2+1$ and $y = 5-3x^2-z^2$.

$x^2+3z^2+1 \le y \le 5-3x^2-z^2$

$-\sqrt{1-x^2} \le z \le \sqrt{1-x^2}$

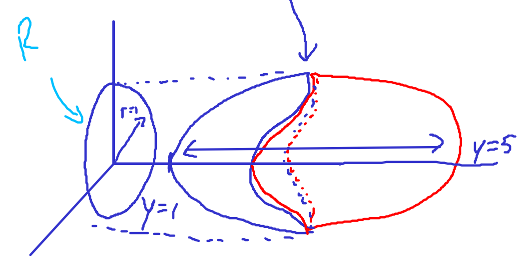
$-1 \le x \le 1$

$x^2+3z^2+1 = 5-3x^2-z^2$

$4x^2+4z^2 = 4$

$x^2+z^2 = 1$

$\hookrightarrow \; z = \pm\sqrt{1-x^2}$



$$\iiint_D dV = \iint_R \left(\int_{x^2+3z^2+1}^{5-3x^2-z^2} dy\right) dA$$

$$= \iint_R y\Big|_{y=x^2+3z^2+1}^{y=5-3x^2-z^2} dA = 4\iint_R 1 - x^2+z^2\,dA$$

LC #3

2π

$\Rightarrow c=2$

$x = r\cos(\theta)$

$z = r\sin(\theta)$

$x^2+z^2 = r^2$

$$= 4\int_0^{2\pi}\int_0^1 (1-r^2)r\,dr\,d\theta$$

$$= 4\int_0^{2\pi}\int_0^1 r - r^3\,dr\,d\theta = 4\int_0^{2\pi} \underbrace{\frac{r^2}{2}-\frac{r^4}{4}\Big|_{r=0}^{r=1}}_{1/4} d\theta$$

$$= \int_0^{2\pi} d\theta$$

$$= \theta\Big|_{\theta=0}^{\theta=2\pi} = 2\pi$$

The concept of changing the order of integration for double integrals also extends to triple integrals:

Example. Consider the integral

$$\int_0^{\sqrt[4]{\pi}} \int_0^{z} \int_y^{z} 12y^2z^3\sin\left(x^4\right)\,dx\,dy\,dz.$$

$x = z$

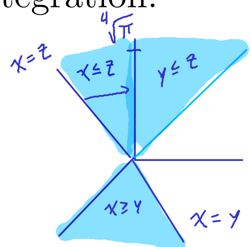
$x = y$

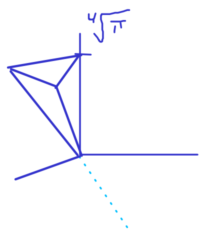
$0 \le z \le \sqrt[4]{\pi}$

$0 \le y \le z$

$y \le x \le z$

Sketch the region of integration, then evaluate the integral by changing the order of integration.

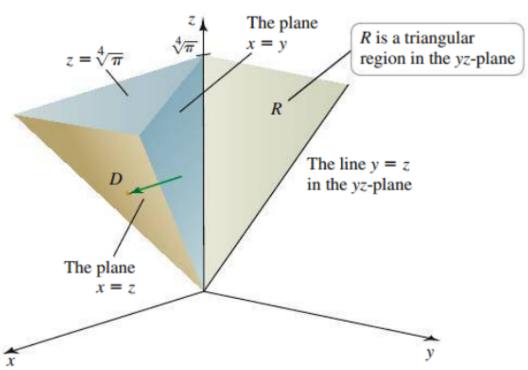




$0 \le z \le \sqrt[4]{\pi}$

$0 \le x \le z$

$0 \le y \le x$



$$\int_0^{\sqrt[4]{\pi}} \int_0^{z} \int_0^{x} 12y^2z^3\sin(x^4)\,dy\,dx\,dz$$

$$= \int_0^{\sqrt[4]{\pi}} \int_0^{z} 4y^3z^3\sin(x^4)\Big|_{y=0}^{y=x}\,dx\,dz$$

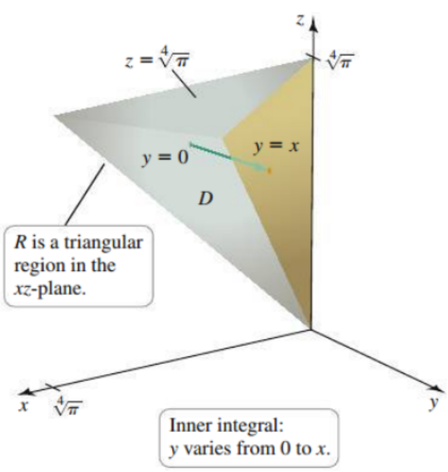
$$= \int_0^{\sqrt[4]{\pi}} \int_0^{z} 4x^3z^3\sin(x^4)\,dx\,dz$$

$u = x^4$

$du = 4x^3\,dx$

$x=0,\ u=0$

$x=z,\ u=z^4$



$$= \int_0^{\sqrt[4]{\pi}} \int_0^{z^4} z^3\sin(u)\,du\,dz$$

$$= \int_0^{\sqrt[4]{\pi}} -z^3\cos(u)\Big|_{u=0}^{u=z^4}\,dz$$

$$= \int_0^{\sqrt[4]{\pi}} -z^3\cos(z^4)\,dz + \int_0^{\sqrt[4]{\pi}} z^3\,dz$$

$u = z^4$

$du = 4z^3\,dz$

$\frac{1}{4}du = z^3\,dz$

$z=0,\ u=0$

$z=\sqrt[4]{\pi},\ u=\pi$

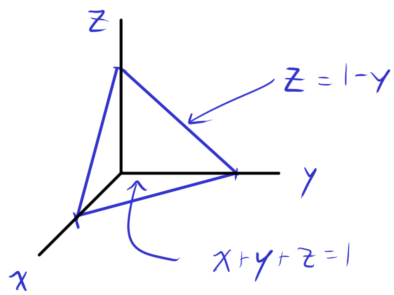
$$= \int_0^{\pi} -\frac{1}{4}\cos(u)\,du + \frac{z^4}{4}\Big|_{z=0}^{z=\sqrt[4]{\pi}}$$

$$= \underbrace{-\frac{1}{4}\sin(u)\Big|_0^{\pi}}_{0} + \left(\frac{\pi}{4} - 0\right) = \frac{\pi}{4}$$

Definition. (Average Value of a Function of Three Variables)
If f is continuous on a region D of $\mathbb{R}^3$, then the **average value** of f over D is

$$\bar{f} = \frac{1}{\text{volume of } D} \iiint\limits_D f(x,y,z)\, dV.$$

Example. Find the average y-coordinate of the points in the standard simplex $D = \{(x,y,z) : x+y+z \le 1,\ x \ge 0,\ y \ge 0,\ z \ge 0\}$.



$0 \le z \le 1$

$0 \le y \le 1-z$

$0 \le x \le 1-y-z$

$V = 1/6$

This volume can be found using the triple integral of f(x,y,z)=1

f(x,y,z)=y since we are trying to find the average y-value

$$\bar{f} = \frac{1}{1/6} \int_0^1 \int_0^{1-z} \int_0^{1-y-z} y \, dx\, dy\, dz$$

$$= 6 \int_0^1 \int_0^{1-z} xy \Big|_{x=0}^{x=1-y-z} dy\, dz = 6 \int_0^1 \int_0^{1-z} y(1-z) - y^2 \, dy\, dz$$

$$= 6 \int_0^1 \frac{y^2}{2}(1-z) - \frac{y^3}{3} \Big|_0^{1-z} dz = 6 \int_0^1 \frac{(1-z)^3}{6}\, dz$$

$u = 1-z$

$du = -dz$

$z=0, u=1$

$z=1, u=0$

$$= -\int_1^0 u^3\, du = -\frac{u^4}{4}\Big|_{u=1}^{u=0} = \frac{1}{4}$$