8.5: Partial Fractions

Example. Simplify $f(x) = \dfrac{1}{x-2} + \dfrac{2}{x+4}$ by finding a common denominator.

Procedure: Partial Fractions with Simple Linear Factors
Suppose $f(x) = p(x)/q(x)$, where p and q are polynomials with no common factors and with the degree of P less than the degree of q. Assume q is the product of simple linear factors. The partial fraction decomposition is obtained as follows.

Step 1: Factor the denominator q in the form $(x - r_1)(x - r_2) \dots (x - r_n)$

Step 2: Partial fraction decomposition

$$\frac{p(x)}{q(x)} = \frac{A_1}{(x-r_1)} + \frac{A_2}{(x-r_2)} + \cdots + \frac{A_n}{(x-r_n)}.$$

Step 3: Clear denominators Multiply both sides of the equation in Step 2 by $q(x) = (x - r_1)(x - r_2) \dots (x - r_n)$

Step 4: Solve for coefficients Equate like powers of x in Step 3 to solve for the undetermined coefficients $A_1, \dots, A_n$.

Example. Perform partial fraction decomposition on $f(x) = \dfrac{3x}{x^2 + 2x - 8}$.

Example. $\displaystyle\int \frac{28x^3 - 56x^2 + 9}{x^2 - 2x}$

Procedure: Partial Fractions for Repeated Linear Factors
Suppose the repeated linear factor $(x-r)^m$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition has a partial fraction for each power of $(x-r)$ up to and including the mth power; that is, the partial fraction decomposition contains the sum

$$\frac{A_1}{(x-r)} + \frac{A_3}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}$$

where $A_1, \ldots, A_m$ are constants to be determined.

Example. Setup the partial fraction decomposition for $f(x) = \dfrac{x^3 - 8x + 19}{x^4 + 3x^3}$.

Example. Setup the partial fraction decomposition for $g(x) = \dfrac{2}{x^5 - 6x^4 + 9x^3}$.

Example. Evaluate $\displaystyle\int \frac{x^2+1}{(2x-3)(x-2)^2}\,dx$.

Example. Evaluate $\displaystyle\int \frac{8}{3x^3 + 7x^2 + 4x}\,dx$.

Procedure: Partial Fractions with Simple Irreducible Quadratic Factors
Suppose a simple irreducible factor ax^2+bx+c appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition contains a term of the form

$$\frac{Ax+B}{ax^2+bx+c},$$

where A and B are unknown coefficients to be determined.

Example. Perform partial fraction decomposition on the following fractions or identify them as irreducible.

$$\frac{1}{x^2-13x+43}$$

$$\frac{x^2}{(x-4)(x+5)}$$

Example. Perform partial fraction decomposition on the following fractions or identify them as irreducible.

$$\frac{7}{(x^2+1)^2}$$

$$\frac{1}{x^2+11x+28}$$

Example. Evaluate $\displaystyle\int \frac{4x}{(x+1)(x^2+1)}\,dx$

Example. Evaluate $\displaystyle\int \frac{3x^2 + 2x + 12}{(x^2 + 4)^2}\, dx$

Example. Evaluate $\displaystyle\int \frac{1}{x\sqrt{1+2x}}\,dx$ using the substitution $u = \sqrt{1+2x}$.

Summary: Partial Fraction Decomposition
Let $f(x) = p(x)/q(x)$ be a proper rational function in reduced form. Assume the denominator q has been factored completely over the real numbers and m is a positive integer.

1. **Simple linear factor:** A factor $x - r$ in the denominator requires the partial fraction $\dfrac{A}{x-r}$.
2. **Repeated linear factor:** A factor $(x-r)^m$ with $m > 1$ in the denominator requires the partial fractions
$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}.$$
3. **Simple irreducible quadratic factor:** An irreducible factor $ax^2 + bx + c$ in the denominator requires the partial fraction
$$\frac{Ax+B}{ax^2+bx+c}.$$
4. **Repeated irreducible quadratic factor:** An irreducible factor $(ax^2+bx+c)^m$ with $m > 1$ in the denominator requires the partial fractions
$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}.$$