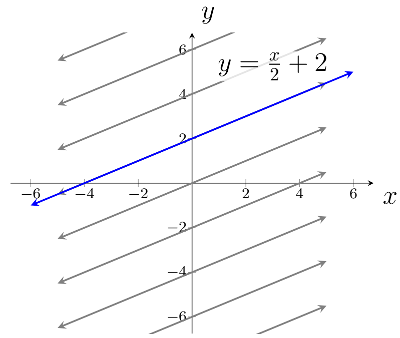
13.5: Lines and Planes in Space

Equation of a Line:

Recall the equation of a line in $\mathbb{R}^2$:

Slope-intercept $y = mx + b \longrightarrow -mx + y = b$

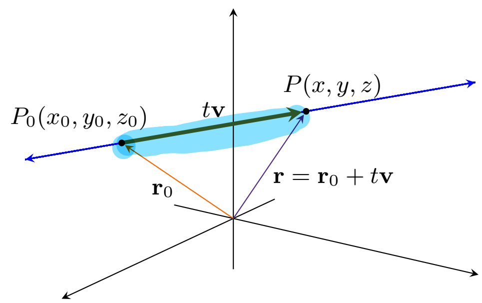
standard form $Ax + By = C$



where b is the intercept and m is the slope. This idea can be extended into higher dimensions:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

Here, $\mathbf{r}_0$ is a fixed point, and $\mathbf{v}$ is the position vector that is parallel to the line $\mathbf{r}$.



Equation of a Line

A **vector equation of the line** passing through the point $P_0(x_0, y_0, z_0)$ in the direction of the vector $\mathbf{v} = \langle a, b, c\rangle$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, or

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle, \quad \text{for} \quad -\infty < t < \infty$$

Equivalently, the corresponding **parametric equations of the line** are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \text{for} \quad -\infty < t < \infty$$

$\langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle = \langle x_0 + at, y_0 + bt, z_0 + ct\rangle$

$$\vec r = \vec r_0 + t\vec v$$

Example. Find the vector equation and parametric equation of the line that

- goes through the points $P\underbrace{(-1,-2,1)}_{\vec r_0}$ and $Q(-4,-5,-3)$ where $t = 0$ corresponds to P,

$\vec r_0 = \langle -1,-2,1\rangle$

$\vec v = \cancel{QR} = \langle -4-(-1), -5-(-2), -3-1\rangle = \langle -3,-3,-4\rangle$

$\vec{PQ} = \langle 3,3,4\rangle$ $\qquad \vec r = \langle -1,\underset{y_0}{-2},\underset{z_0}{1}\rangle + t\langle \underset{a}{3},3,\underset{c}{4}\rangle$

$\vec r = \langle -1,-2,1\rangle + t\langle -3,-3,-4\rangle$

same line
diff directions

$x = -1+3t,\quad y = -2+3t,\quad z = 1+4t$

- goes through the point $P(1,-3,-3)$ and is parallel to the vector $\mathbf{r} = \langle -4,1,-1\rangle$,

$\vec r = \vec r_0 + t\vec v$

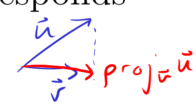
$\vec \ell = \langle 1,-3,-3\rangle + t\langle -4,1,-1\rangle$

$x = 1-4t,\quad y = -3+t,\quad z = -3-t$

- goes through the point $P(-2,5,-2)$ and is perpendicular to the lines $x = 3-4t$, $y = 2-3t$, $z = -1-t$, and $x = -2+0t$, $y = 2-t$, $z = 3t$, where $t = 0$ corresponds to P.

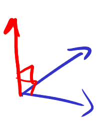
$\vec r = \vec r_0 + t\vec v$

$$\begin{array}{ccc|cc} \hat i & \hat j & \hat k & \hat i & \hat j \\ -4 & -3 & -1 & -4 & -3 \\ 0 & -1 & 3 & 0 & -1 \end{array}$$



$\vec r_0 = \langle -2,5,-2\rangle$

$\vec v = \underbrace{\langle -4,-3,-1\rangle}_{\vec u} \times \underbrace{\langle 0,-1,3\rangle}_{\vec v} = \langle -10,12,4\rangle$



$\vec r = \langle -2,5,-2\rangle + t\langle -10,12,4\rangle$

$x = -2-10t$
$y = 5+12t$
$z = -2+4t$

LC

$z = -2 + ct$

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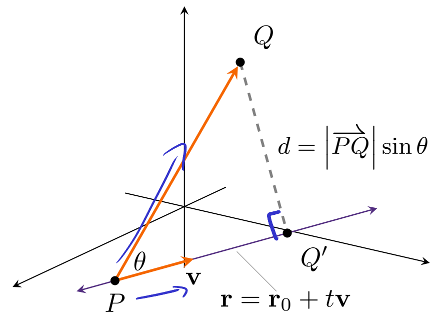
Distance from a Point to a Line:

Given a point Q and a line ℓ, the shortest distance to the line is the length of $\overrightarrow{QQ'}$.



$\sin\theta = \dfrac{d}{|\overrightarrow{PQ}|}$

From the definition of the cross product, we have

$$\left|\mathbf{v}\times\overrightarrow{PQ}\right| = |\mathbf{v}|\underbrace{\left|\overrightarrow{PQ}\right|\sin\theta}_{d} = |\mathbf{v}|d$$

From here, solving for d gives us the following:

Distance Between a Point and a Line

The distance d between the point Q and the $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ is

$$d = \frac{\left|\mathbf{v}\times\overrightarrow{PQ}\right|}{|\mathbf{v}|},$$

where P is any point on the line and $\mathbf{v}$ is a vector parallel to the line.

Example. Find the distance from the point $Q(-4,-1,-3)$ and the line $x = -5-5t$, $y = -5+t$, $z = -1+4t$. (*Hint:* Let P be the point at $t = 0$)

$P = (-5,-5,-1)$ $\quad \overrightarrow{PQ} = \langle -4-(-5), -1-(-5), -3-(-1)\rangle = \langle 1, 4, -2\rangle$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & 1 & 4 \\ 1 & 4 & -2 \end{vmatrix}$$

$\vec{v}\times\overrightarrow{PQ} = \langle -2-16, -(10-4), -20-1\rangle = \langle -18, -6, -21\rangle = 3\langle -6,-2,-7\rangle$

$|\vec{v}\times\overrightarrow{PQ}| = 3\sqrt{36+4+49} = 3\sqrt{89}$

$|\vec{v}| = \sqrt{42}$

$d = \dfrac{3\sqrt{89}}{\sqrt{42}}$

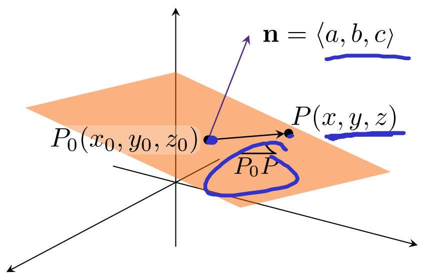
Equations of Planes:

In $\mathbb{R}^2$, two distinct points determine a line.

In $\mathbb{R}^3$, three noncollinear points determine a unique plane. Alternatively, a plane is uniquely determined by a point and a vector that is orthogonal to the plane.

$\vec{w} = \vec{u} \times \vec{v}$

$\vec{u} \cdot \vec{w}$



> **Definition. (Plane in $\mathbb{R}^3$)**
> Given a fixed point P_0 and a nonzero **normal vector n**, the set of points P in $\mathbb{R}^3$ for which $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$ is called a **plane**.

Consider the normal vector $\mathbf{n} = \langle a, b, c \rangle$ at the point $P_0(x_0, y_0, z_0)$, and any point $P(x, y, z)$ on the plane. Since $\mathbf{n}$ is orthogonal to the plane, it is also orthogonal to the vector $\overrightarrow{P_0P}$, which is also in the plane. Thus,

$$\begin{aligned}
\mathbf{n} \cdot \overrightarrow{P_0P} &= 0 \\
\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 \\
a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \\
ax + by + cz &= d
\end{aligned}$$

> **General Equation of a Plane in $\mathbb{R}^3$**
> The plane passing through the point $P_0(x_0, y_0, z_0)$ with a nonzero normal vector $\mathbf{n} = \langle a, b, c \rangle$ is described by the equation
>
> $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{or} \quad ax + by + cz = d,$$
>
> where $d = ax_0 + by_0 + cz_0$.

Example. Find the equation of the plane that

- goes through the point $P(-2,5,0)$ and is parallel to the plane $\underbrace{x-5y-5z}=1$,

$\vec{n}=\langle 1,-5,-5\rangle$

$\overrightarrow{P_0P}=\langle x-(-2), y-5, z-0\rangle$

$\vec{n}\cdot\overrightarrow{P_0P}=\langle 1,-5,-5\rangle\langle x+2, y-5, z\rangle=0$

$(x+2)-5(y-5)-5z=0$

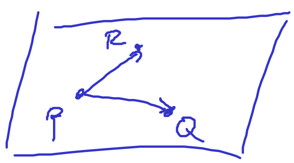
$x+2-5y+25-5z=0$

$x-5y-5z=-27$

- goes through the points $\underline{P}(5,-2,1)$, $Q(5,1,3)$ and $R(1,-5,-2)$

$\overrightarrow{PQ}\times\overrightarrow{PR}=\langle 0,3,2\rangle\times\langle -4,-3,-3\rangle=\langle -9-(-6), -(-(-8)), -(-12)\rangle$

$=\langle -3,-8,12\rangle=\vec{n}$



$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & 2 \\ -4 & -3 & -3 \end{vmatrix}$$

$\overrightarrow{P_0P}=\langle x-5, y+2, z-1\rangle$

$\vec{n}\cdot\overrightarrow{P_0P}=0$

$\langle -3,-8,12\rangle\langle x-5, y+2, z-1\rangle=0$

$-3x+15-8y-16+12z-12=0$

$-3x-8y+12z=13$

- that is parallel to the vectors $\langle 4,-2,-3\rangle$ and $\langle 3,2,3\rangle$, passing through the point

Point $P(-2,-2,5)$. $\longrightarrow$ $\overrightarrow{P_0P}=\langle x+2, y+2, z-5\rangle$

normal vector $\vec{n}=\langle 4,-2,-3\rangle\times\langle 3,2,3\rangle=\langle 0,-21,14\rangle$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & -3 \\ 3 & 2 & 3 \end{vmatrix}$$

$\Longrightarrow$ $\vec{n}\cdot\overrightarrow{P_0P}=0$

$\langle 0,-21,14\rangle\langle x+2, y+2, z-5\rangle=0$

$-21y-42+14z-70=0$

$-21y+14z=112$

Example. Find the location where the line $\langle -3, 1, 4\rangle + t\langle -1, -4, 2\rangle$ and the plane $2x - 2y - 4z = 5$ intersect.

$$\left.\begin{aligned} x &= -3 - t \\ y &= 1 - 4t \\ z &= 4 + 2t \end{aligned}\right\} \quad \begin{aligned} x &= -3 - \left(-\tfrac{29}{2}\right) = 23/2 \\ y &= 1 - 4\left(\tfrac{-29}{2}\right) = 59 \\ z &= 4 + 2\left(\tfrac{-29}{2}\right) = -25 \end{aligned}$$

$$2(-3-t) - 2(1-4t) - 4(4+2t) = 5$$

$$-6 - 2t - 2 + 8t - 16 - 8t = 5$$

$$-2t = 29$$

$$t = -\frac{29}{2}$$

Definition. (Parallel and Orthogonal Planes)
Two distinct planes are **parallel** if their respective normal vectors are parallel (that is, the normal vectors are scaling multiples of each other). Two plans are **orthogonal** if their respective normal vectors are orthogonal (that is, the dot product of the normal vectors is *zero*).

Example. Find the line of intersection between the planes $3x - y + 4z = -4$ and $x + 3y - 2z = 0$.

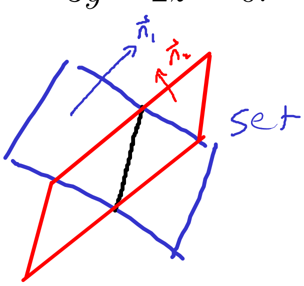
Point: $P = (-6/5, 2/5, 0)$

$\vec{n}$: orthog to $\vec{n}_1$ and $\vec{n}_2$

$\vec{n}_1 = \langle 3, -1, 4\rangle$ $\quad \vec{n}_2 = \langle 1, 3, -2\rangle$

$\vec{n}_1 \neq f\,\vec{n}_2 \Rightarrow$ not parallel

$\Rightarrow$ do intersect



$$z = 0 \rightarrow \begin{pmatrix} 3x - y = -4 \\ x + 3y = 0 \end{pmatrix} \; -3$$

$$0x - 10y = -4$$

$$y = 2/5$$

$$\longrightarrow 3x - \tfrac{2}{5} = -4$$

$$x = -6/5$$

$$\vec{n} = \vec{n}_1 \times \vec{n}_2 = \langle -10, 10, 10\rangle$$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -1 & 4 \\ 1 & 3 & -2 \end{vmatrix}$$

$$\vec{\ell} = \langle -6/5, 2/5, 0\rangle + t\langle -10, 10, 10\rangle$$

$$= (-6/5 - 10t)\,\hat{\imath} + (2/5 + 10t)\,\hat{\jmath} + (0 + 10t)\,\hat{k}$$

Example. Find the smallest angle between the planes $\underbrace{3x-y+4z}_{\vec{n}_1=\langle 3,-1,4\rangle} = -4$ and $\underbrace{x+3y-2z}_{\vec{n}_2=\langle 1,3,-2\rangle} = 0$.

$$\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1||\vec{n}_2| \cos\theta \longleftarrow \theta \in [0,\pi] \ :)$$

$$|\vec{n}_1 \times \vec{n}_2| = |\vec{n}_1||\vec{n}_2| \sin\theta \longleftarrow \theta \in [-\pi/2, \pi/2] \ :($$

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{\langle 3,-1,4\rangle \cdot \langle 1,3,-2\rangle}{\sqrt{9+1+16}\ \sqrt{1+9+4}} = \frac{3-3-8}{\sqrt{26}\sqrt{14}} = \frac{-8}{2\sqrt{91}} = \frac{-4}{\sqrt{91}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-4}{\sqrt{91}}\right) \approx 2 \text{ radians}$$