15.2: Limits and Continuity

Definition. (Limit of a Function of Two Variables)
The function f has the **limit** L as $P(x, y)$ approaches $P_0(a, b)$, written

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{P\to P_0} f(x, y) = L,$$

if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

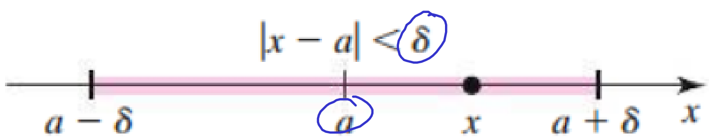
$$|f(x, y) - L| < \varepsilon$$ Interval

whenever (x, y) is in the domain of f and

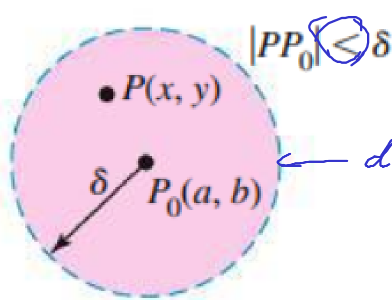
$$0 < |PP_0| = \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$ ball

Note: For functions with 1 independent variable, $|x - a| < \delta$ represents an open interval on a number line. Recall that these limits only exist if the same value is approached from two directions.

For functions with 2 independent variables, $|PP_0| < \delta$ represents an open disk (open ball). Here, the limit only exists if the same value is approached from *all* directions.



(a)



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Theorem 15.1: Limits of Constant and Linear Functions
Let a, b, and c be real numbers.

1. Constant function $f(x,y) = c$: $\displaystyle\lim_{(x,y)\to(a,b)} c = c$
2. Linear function $f(x,y) = x$: $\displaystyle\lim_{(x,y)\to(a,b)} x = a$
3. Linear function $f(x,y) = y$: $\displaystyle\lim_{(x,y)\to(a,b)} y = b$

$3x = 126$

$\lim_{(x,y)\to(4,0)}$

Theorem 15.2: Limit Laws for Functions of Two Variables
Let L and M be real numbers and suppose $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $\displaystyle\lim_{(x,y)\to(a,b)} g(x,y) = M$. Assume c is constant, and $n > 0$ is an integer.

1. **Sum** $\displaystyle\lim_{(x,y)\to(a,b)} (f(x,y) + g(x,y)) = L + M$
2. **Difference** $\displaystyle\lim_{(x,y)\to(a,b)} (f(x,y) - g(x,y)) = L - M$
3. **Constant multiple** $\displaystyle\lim_{(x,y)\to(a,b)} cf(x,y) = cL$
4. **Product** $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)g(x,y) = LM$
5. **Quotient** $\displaystyle\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$, provided $M \neq 0$
6. **Power** $\displaystyle\lim_{(x,y)\to(a,b)} (f(x,y))^n = L^n$
7. **Root** $\displaystyle\lim_{(x,y)\to(a,b)} (f(x,y))^{1/n} = L^{1/n}$, when $L > 0$ if n is even.

Example. Evaluate the following limits:

$$\lim_{(x,y)\to(4,11)} 570 = 570$$

$$\lim_{(x,y)\to(2,8)} \left(3x^2y + \sqrt{xy}\right)$$

$$= 3(4)8 + \sqrt{2\cdot 8}$$

$$= 96 + \sqrt{16}$$

$$= 100$$

$$\lim_{(x,y)\to(0,\pi)} \frac{\sin(xy) + \cos(xy)}{7y}$$

$$= \frac{\sin(0) + \cos(0)}{7(\pi)} = \frac{1}{7\pi}$$

$\left(\frac{0}{0}\right)$

$$\lim_{(x,y)\to(\frac{1}{3},-1)} \frac{9x^2 - y}{3x + y}$$

$$= \lim_{(x,y)\to(\frac{1}{3},-1)} \frac{(3x-y)(3x+y)}{3x+y}$$

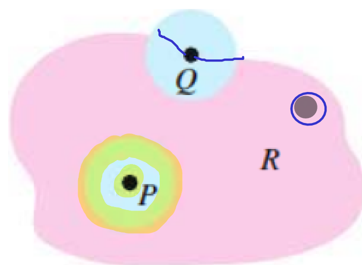
$$= \lim_{(x,y)\to(\frac{1}{3},-1)} 3x-y = 2$$

$0 \le x \le 1$ closed set $x=0$ $x=1$ boundary pts for closed & open

$0 < x < 1$ open set $x = 0.5$ interior pt $x = 0.99999$

Definition. (Interior and Boundary Points)
Let R be a region in $\mathbb{R}^2$. An **interior point** P of R lies entirely within R, which means it is possible to find a disk centered at P that contains only points of R.

A **boundary point** Q of R lies on the edge of R in the sense that every disk centered at Q contains at least one point in R and at least one point not in R.



$(0,1)$

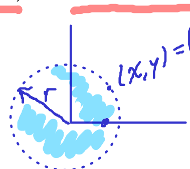
Contains No boundary points

Definition. (Open and Closed Sets)
A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points.

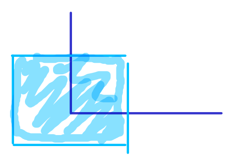
not open and not closed $\Rightarrow$ neither

Example. Identify which regions are open sets and which are closed sets.

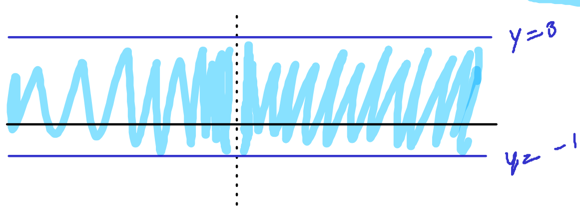
$\{(x, y) : x^2 + y^2 < 9\}$ $x = 2.5,\ y=0$ $(x,y) = \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ $r=3$ open



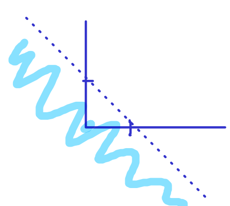
$\{(x, y) : |x| \le 1, |y| \le 1\}$ closed



$\{(x, y) : x \ne 0, -1 \le y \le 3\}$ neither



$\{(x, y) : x + y < 2\}$ open



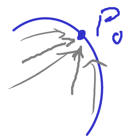
$\frac{1}{x}$ Domain $x \neq 0$

$\lim_{x \to -1} \frac{x^2-1}{x+1} = \lim_{x \to -1} x-1 = -2$

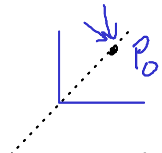
A limit at a boundary point $P_0(a, b)$ of a function's domain can exist, provided $f(x, y)$ approaches the same value as (x, y) approaches (a, b) *along all paths that lie in the domain.*

Example. $f(x, y) = \dfrac{x^2 - y^2}{x - y}$

$= \dfrac{(x-y)(x+y)}{x-y}$



Domain
$x - y \neq 0$
$x \neq y$



$\lim_{(x,y) \to P_0} f(x,y) = \lim_{(x,y) \to (a,b)} x + y = a + b$

Example. Evaluate the following limits

$$\lim_{(x,y)\to(0,\pi)} \frac{\sin(xy) + \cos(xy)}{7y} = \frac{0+1}{7\pi}$$

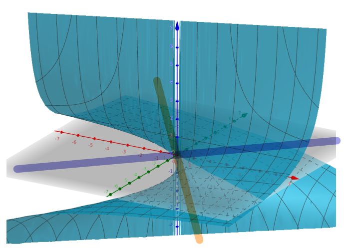
$$\lim_{(x,y)\to(-3,-15)} \frac{y^2 - 5xy}{y - 5x}$$

$= \lim_{(x,y) \to (-3,-15)} \dfrac{y\cancel{(y-5x)}}{\cancel{y-5x}}$

$= \lim_{(x,y) \to (-3,-15)} y = -15$

$\left(\frac{0}{0}\right)$ Indet form

$$\lim_{(x,y)\to(0,0)} \frac{x + 2y}{x - 2y}$$



$$\lim_{(x,y)\to(1,-1)} \frac{y^5}{(x - 1)^{30} + y^5} = \frac{-1}{0-1} = 1$$

$y = -x$ $\lim_{(x,-x) \to (0,0)} \dfrac{x-2x}{x+2x} = \lim_{(x,-x) \to (0,0)} \dfrac{-x}{3x} = \lim_{(x,-x) \to (0,0)} -\dfrac{1}{3} = -\dfrac{1}{3}$

$y = x$ $\lim_{(x,-x) \to (0,0)} \dfrac{x+2x}{x-2x} = \lim_{(x,-x) \to (0,0)} \dfrac{3x}{-x} = \lim_{(x,-x) \to (0,0)} -3 = -3$

Procedure: Two-Path Test for Nonexistence of Limits
If $f(x, y)$ approaches two different values as (x, y) approaches (a, b) along two different paths in the domain of f, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

$y = mx$: $\lim_{(x,y) \to (0,0)} \dfrac{x+2y}{x-2y} = \lim_{(x,mx) \to (0,0)} \dfrac{x+2mx}{x-2mx} = \lim_{(x,mx) \to (0,0)} \dfrac{1+2m}{1-2m} = \dfrac{1+2m}{1-2m}$ ← not constant

$m \neq \frac{1}{2}$

Definition. (Continuity)
The function f is continuous at the point (a, b) provided

1. f is defined at (a, b)
2. $\lim_{(x,y)\to(a,b)} f(x, y)$ exists, and
3. $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

Example. Determine if $f(x, y)$ is continuous at $(0, 0)$

$y = mx$

$$f(x,y) = \begin{cases} \dfrac{3xy^2}{x^2+y^4}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

① $f(0,0) = 0$

② $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \dfrac{3xy^2}{x^2+y^4} = \lim_{(x,mx)\to(0,0)} \dfrac{3x(mx)^2}{x^2+(mx)^4} = \lim_{(x,mx)\to(0,0)} \dfrac{3m^2x^3}{x^2+m^4x^4}$

$= \lim_{(x,mx)\to(0,0)} \dfrac{3m^2x}{1+m^4x^2} = \dfrac{0}{1+0} = 0$

③ $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) \Rightarrow$ continuous @ $(0,0)$

Theorem 15.3: Continuity of Composite Functions
If $u = g(x, y)$ is continuous at (a, b) and $z = f(u)$ is continuous at $g(a, b)$, then the composite function $z = f(g(x, y))$ is continuous at (a, b).

Example. Determine the points at which the following functions are continous:

$f(x, y) = \ln(\underline{x^2 + y^2 + 4})$

$x^2 + y^2 + 4 > 0$

$x^2 + y^2 > -4$

$\{(x,y) : -\infty < x < \infty, -\infty < y < \infty\}$

$g(x, y) = e^{x/\underline{y}}$

$-\infty < x/y < \infty$

$y \neq 0$

$\{(x,y) : y \neq 0\}$