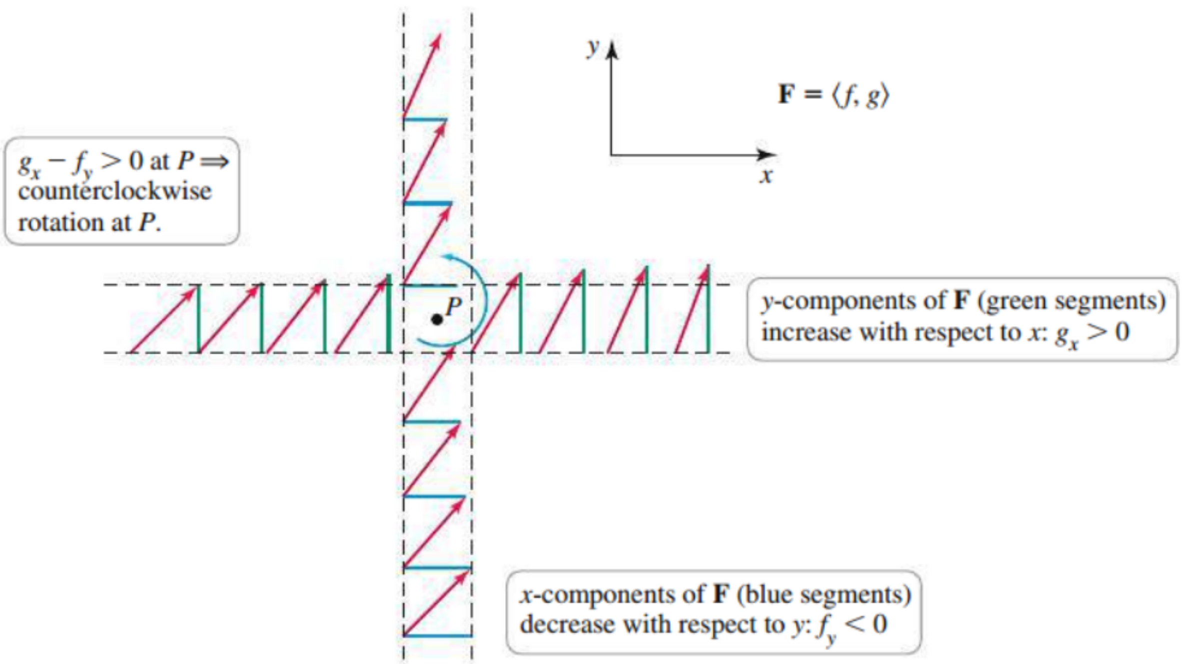
1 17.4: Green's Theorem

Green's Theorem — Circulation Form

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected regin R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{circulation}} = \underbrace{\oint_C f\,dx + g\,dy}_{\text{circulation}} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA.$$

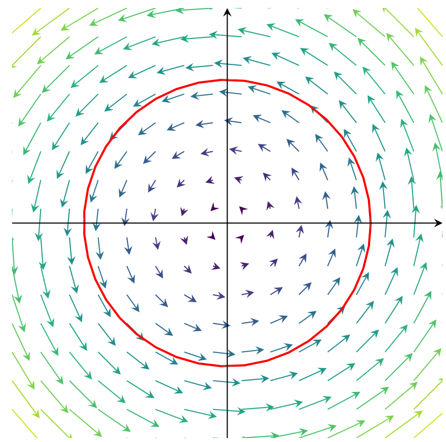


Definition. (Two-Dimensional Curl)

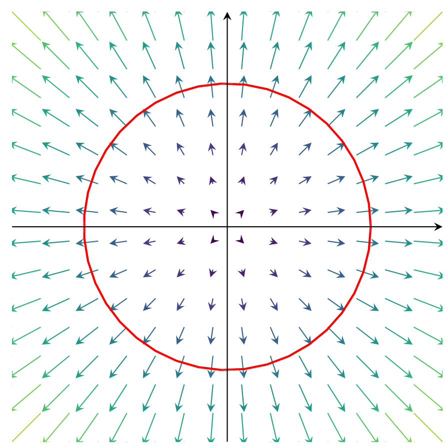
The **two-dimensional curl** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}$. If the curl is zero throughout a region, the vector field is **irrotational** on the region.

Example. Consider the following vector fields $\mathbf{F}$ over the region $R = \left\{(x, y) : x^2 + y^2 \leq 1\right\}$. Compute the circulation using Green's Theorem.

$\mathbf{F} = \langle -y, x \rangle$



$\mathbf{F} = \langle x, y \rangle$



Area of a Plane Region by Line Integrals

Under the conditions of Green's Theorem, the area of a region R enclosed by a curve C is

$$\oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C (x\,dy - y\,dx).$$

Example. Find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Green's Theorem — Flux Form

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n} ds}_{\text{outward flux}} = \underbrace{\oint_C f\, dy - g\, dx}_{\text{outward flux}} = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA,$$

where $\mathbf{n}$ is the outward unit normal vector on the curve.

Definition. (Two-Dimensional Divergence)

The **two-dimensional divergence** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y}$. If the divergence is zero throughout a region, the vector field is **source free** on that region.

Conservative Fields $\mathbf{F} = \langle f, g \rangle$	**Source-Free Fields $\mathbf{F} = \langle f, g \rangle$**
curl $= \dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 0$	divergence $= \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 0$
Potential function φ with $\mathbf{F} = \nabla\varphi \quad$ or $\quad f = \dfrac{\partial \varphi}{\partial x}, \quad g = \dfrac{\partial \varphi}{\partial y}$	Stream function ψ with $f = \dfrac{\partial \psi}{\partial y}, \qquad g = -\dfrac{\partial \psi}{\partial x}$
Circulation $= \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all closed curves C.	Flux $= \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = 0$ on all closed curves C.
Evaluation of the line integral $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$	Evaluation of the line integral $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \psi(B) - \psi(A)$

Circulation/work integrals: $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C f\,dx + g\,dy$		
	C closed	**C not closed**
F conservative ($\mathbf{F} = \nabla\varphi$)	$\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$	$\int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(B) - \varphi(A)$
F not conservative	Green's Theorem $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R (g_x - f_y)\,dA$	Direct evaluation $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b (fx' + gy')\,dt$

Flux integrals: $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_C f\,dy - g\,dx$		
	C closed	**C not closed**
F source free ($f = \psi_y, g = -\psi_x$)	$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = 0$	$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \psi(B) - \psi(A)$
F not source free	Green's Theorem $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R (f_x + g_y)\,dA$	Direct evaluation $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_a^b (fy' - gx')\,dt$