3.4 The Product and Quotient Rule

Theorem 3.7: Product Rule
If f and g are differentiable at x, then

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

Note: This can also be denoted

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}[f(x)]g(x) + f(x)\frac{d}{dx}[g(x)].$$

Example. For $f(x) = (3x^2)(2x)$, find $f'(x)$ by using the product rule and by

$$f'(x) = \frac{d}{dx}[3x^2](2x) + (3x^2)\frac{d}{dx}[2x]$$
$$= 6x^2(2x) + 3x^2(2x)$$
$$= 12x^2 + 6x^2 = \boxed{18x^2}$$

$f(x) = 6x^3$

$f'(x) = 18x^2$

Example. For $g(x) = \left(x + \dfrac{1}{x}\right)\left(x - \dfrac{1}{x} + 1\right)$, find $g'(x)$.

$$g'(x) = \frac{d}{dx}[x+x^{-1}](x - x^{-1} + 1) + (x+x^{-1})\frac{d}{dx}[x - x^{-1} + 1]$$
$$= (1-x^{-2})(x-x^{-1}+1) + (x+x^{-1})(1+x^{-2})$$
$$= x + 1 - x^{-1} - x^{-1} - x^{-2} + x^{-3} + x + x^{-1} + x^{-1} + x^{-3}$$
$$= 2x + 1 - x^{-2} + 2x^{-3} = \boxed{2x + 1 - \frac{1}{x^2} + \frac{2}{x^3}}$$

$g(x) = x^2 + x + \frac{1}{x} - \frac{1}{x^2}$

$g'(x) = 2x + 1 - \frac{1}{x} + \frac{2}{x^3}$

Example. For $h(x) = (x-1)(x^2+x+1)$, find $h'(x)$.

$$h'(x) = \frac{d}{dx}[x-1](x^2+x+1) + (x-1)\frac{d}{dx}[x^2+x+1]$$
$$= (1)(x^2+x+1) + (x-1)(2x+1)$$
$$= x^2 + x + 1 + 2x^2 - x - 1$$
$$= 3x^2$$

$h(x) = x^3 - 1$

$h'(x) = 3x^2$

$$1-(e^t)^2$$

Example. Use the product rule to find the derivative of $1 - e^{2t}$.

$$\frac{d}{dt}\left[1-e^{2t}\right] = \frac{d}{dt}\left[(1-e^t)(1+e^t)\right] = \frac{d}{dt}\left[1-e^t\right](1+e^t) + (1-e^t)\frac{d}{dt}\left[1+e^t\right]$$

$$= -e^t(1+e^t) + (1-e^t)e^t$$

$$= -e^t - e^{2t} + e^t - e^{2t}$$

$$= -2e^{2t}$$

Theorem 3.8 Quotient Rule
If f and g are differentiable at x and $g(x) \neq 0$, then the derivative of f/g at x exists and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

Note: A common phrase for the quotient rule is
"Lo De Hi minus Hi De Lo over Lo squared"

Example. Find the derivative of $y = \dfrac{t^2+1}{3t^2-2t+1}$.

$$y' = \frac{(3t^2-2t+1)\frac{d}{dt}\left[t^2+1\right] - (t^2+1)\frac{d}{dt}\left[3t^2-2t+1\right]}{(3t^2-2t+1)^2}$$

$$= \frac{2t(3t^2-2t+1) - (t^2+1)(6t-2)}{(3t^2-2t+1)^2}$$

$$= \frac{6t^3-4t^2+2t - 6t^3+2t^2-6t+2}{(3t^2-2t+1)^2} = \frac{-2t^2-4t+2}{(3t^2-2t+1)^2}$$

Example. Find the derivatives of the following functions:

$$f(t) = \frac{2t}{4+t^2}$$

$$f'(t) = \frac{(4+t^2)(2) - 2t(2t)}{(4+t^2)^2}$$

$$= \frac{8+2t^2-4t^2}{(4+t^2)^2}$$

$$= \frac{-2t^2+8}{(4+t^2)^2}$$

$$w = (2x-7)^{-1}(x+5) = \frac{x+5}{2x-7}$$

$$w' = \frac{(2x-7)(1) - (x+5)(2)}{(2x-7)^2}$$

$$= \frac{2x-7-2x-10}{(2x-7)^2}$$

$$= \frac{-17}{(2x-7)^2}$$

$$y = \frac{e^x}{1-e^x}$$

$$y' = \frac{(1-e^x)e^x - e^x(-e^x)}{(1-e^x)^2}$$

$$= \frac{e^x - e^{2x} + e^{2x}}{(1-e^x)^2}$$

$$= \frac{e^x}{(1-e^x)^2}$$

$$h(w) = \frac{w^2-1}{w^2+1}$$

$$h'(w) = \frac{(w^2+1)(2w) - (w^2-1)(2w)}{(w^2+1)^2}$$

$$= \frac{2w^3+2w-2w^3+2w}{(w^2+1)^2}$$

$$= \frac{4w}{(w^2+1)^2}$$

Example. Find the derivative of the following functions. Is using the quotient rule recommended here?

$$w(z) = \frac{4}{z^{3}} = 4z^{-3}$$

$$w'(z) = \frac{z^3(0) - 4(3z^2)}{(z^3)^2} = \frac{-12z^2}{z^6} = -\frac{12}{z^4}$$

$$w'(z) = 4(-3)z^{-4} = \frac{-12}{z^4}$$

$$f(x) = \frac{x^2 - 2ax + a^2}{x - a} = \frac{(x-a)^2}{x-a}$$

$$f'(x) = \frac{(x-a)(2x-2a) - (x^2-2ax+a^2)(1)}{(x-a)^2}$$

$$= \frac{2x^2 - 4ax + 2a^2 - x^2 + 2ax - a^2}{(x-a)^2}$$

$$= \frac{x^2 - 2ax + a^2}{(x-a)^2} = 1$$

$$f'(x) = \frac{d}{dx}[x-a] = 1$$

Example. Find the second derivative of the following functions.

$$f(x) = x^{\frac{5}{2}}e^x$$

$$f'(x) = (x^{5/2})[e^x] + [\tfrac{5}{2}x^{3/2}](e^x)$$

$$= (x^{5/2} + \tfrac{5}{2}x^{3/2})e^x$$

$$f''(x) = (x^{5/2} + 5/2\, x^{3/2})[e^x] + [\tfrac{5}{2}x^{3/2} + \tfrac{15}{4}x^{1/2}]e^x$$

$$= [x^{5/2} + 5x^{3/2} + \tfrac{15}{4}x^{1/2}]e^x$$

$$y(t) = \frac{t}{t+2}$$

$$y'(t) = \frac{(t+2)(1) - t(1)}{(t+2)^2} = \frac{2}{t^2+4t+4}$$

$$y''(t) = \frac{(t^2+4t+4)[0] - (2)[2t+4]}{(t^2+4t+4)}$$

$$= \frac{-4(t+2)}{(t+2)^4} = \boxed{\frac{-4}{(t+2)^3}}$$

Example. Use the table below to evaluate the following

x	1	2	3	4	5
$f(x)$	5	4	3	2	1
$f'(x)$	3	5	2	1	4
$g(x)$	4	2	5	3	1
$g'(x)$	2	4	3	1	5

$$\frac{d}{dx}[f(x)\cdot g(x)]\bigg|_{x=5}$$

$= f(5)\,g'(5) + f'(5)\cdot g(5)$

$= (1)(5) + (4)(1)$

$= \boxed{9}$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]\bigg|_{x=3} = \frac{g(3)\,f'(3) - f(3)\,g'(3)}{[g(3)]^2}$$

$= \dfrac{(5)(2) - (3)(3)}{[5]^2}$

$= \boxed{\dfrac{1}{25}}$

$$\frac{d}{dx}[x\cdot g(x)]\bigg|_{x=2}$$

$= x\cdot g'(x) + 1\cdot g(x)\Big|_{x=2}$

$= 2\cdot g'(2) + g(2)$

$= 2(4) + 2$

$= \boxed{10}$

$$\frac{d}{dx}\left[\frac{x\cdot f(x)}{g(x)}\right]\bigg|_{x=4}$$

$= \dfrac{g(x)\frac{d}{dx}[x\,f(x)] - x\,f(x)\,g'(x)}{[g(x)]^2}\Bigg|_{x=4}$

$= \dfrac{g(4)[4\cdot f'(4) + (1)\,f(4)] - 4\,f(4)\,g'(4)}{[g(4)]^2}$

$= \dfrac{3[4\cdot(1) + (1)2] - 4(2)(1)}{[3]^2} = \boxed{\dfrac{10}{9}}$

$h(x) = (-2x^3)\cdot f(x)$, find $h'(4)$.

$h'(x) = -2x^3 f'(x) - 6x^2 f(x)$

$h'(4) = -128(1) - 96(2)$

$= -224$

$r(x) = \dfrac{2g(x)}{-3\sqrt[4]{x}}$, find $r'(1)$.

$\sqrt[4]{x} = x^{1/4}$

$r'(x) = \dfrac{2}{-3}\left[\dfrac{x^{1/4}g'(x) - g(x)\frac{1}{4}x^{-3/4}}{(x^{1/4})^2}\right]$

$= -\dfrac{2}{3}\left[\dfrac{x^{1/4}g'(x) - 1/4\,x^{-3/4}g(x)}{x^{1/2}}\right]$

$r'(1) = -\dfrac{2}{3}\left[\dfrac{1\,g'(1) - \frac{1}{4}(1)\,g(1)}{1}\right]$

$= -\dfrac{2}{3}\left(2 - \frac{1}{4}(4)\right) = \boxed{-\dfrac{2}{3}}$