14.4: Length of Curves

Definition. (Arc Length for Vector Functions)
Consider the parameterized curve $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, where f', g', and h' are continuous, and the curve is traversed once for $a \leq t \leq b$. The **arc length** of the curve between $(f(a), g(a), h(a))$ and $(f(b), g(b), h(b))$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt = \int_a^b |\mathbf{r}'(t)|\, dt.$$

Example (Flight of an eagle)**.** Suppose an eagle rises at a rate of 100 vertical ft/min on a helical path given by

$$\mathbf{r}(t) = \langle 250\cos(t),\ 250\sin(t),\ 100t\rangle$$

where $\mathbf{r}$ is measured in feet and t is measured in minutes. How far does it travel in 10 minutes?

Example. Suppose a particle has a trajectory given by

$$\mathbf{r}(t) = \langle 10\cos(3t),\, 10\sin(3t)\rangle$$

where $0 \leq t \leq \pi$. How far does this particle travel?

Example. Find the length of the curve

$$\mathbf{r}(t) = \left\langle 3t^2 - 5,\, 4t^2 + 5\right\rangle$$

where $0 \leq t \leq 1$.

Example. Find the length of $\mathbf{r}(t) = \left\langle t^2, \dfrac{(4t+1)^{\frac{3}{2}}}{6} \right\rangle$ where $0 \leq t \leq 6$.

Example. Find the length of $\mathbf{r}(t) = \left\langle 2\sqrt{2}, \sin(t), \cos(t) \right\rangle$ where $0 \leq t \leq 5$.

Theorem 14.3: Arc Length as a Function of a Parameter
Let $\mathbf{r}(t)$ describe a smooth curve, for $t \geq a$. The arc length is given by

$$s(t) = \int_a^t |\mathbf{v}(u)|\, du,$$

where $|\mathbf{v}| = |\mathbf{r}'|$. Equivalently, $\dfrac{ds}{dt} = |\mathbf{v}(t)|$. If $|\mathbf{v}(t)| = 1$, for all $t \geq a$, then the parameter t corresponds to arc length.

Example. For the following functions, determine if $\mathbf{r}(t)$ uses arc length as a parameter. If not, find a description that uses arc length as a parameter.

a) $\mathbf{r}(t) = \langle -4t + 1, 3t - 1 \rangle$, $0 \leq t \leq 4$.

b) $\mathbf{r}(t) = \left\langle \frac{1}{\sqrt{10}} \cos(t),\ \frac{3}{\sqrt{10}} \cos(t),\ \sin(t) \right\rangle$, $0 \leq t \leq 2\pi$.