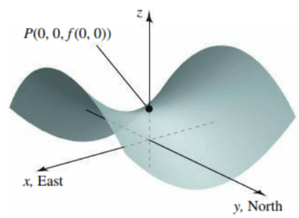
15.3: Partial Derivatives

Recall that for functions with one independent variable, say $y = f(x)$, the derivative measures the change in y with respect to x. For functions with multiple independent variables, we compute derivatives with respect to each variable.



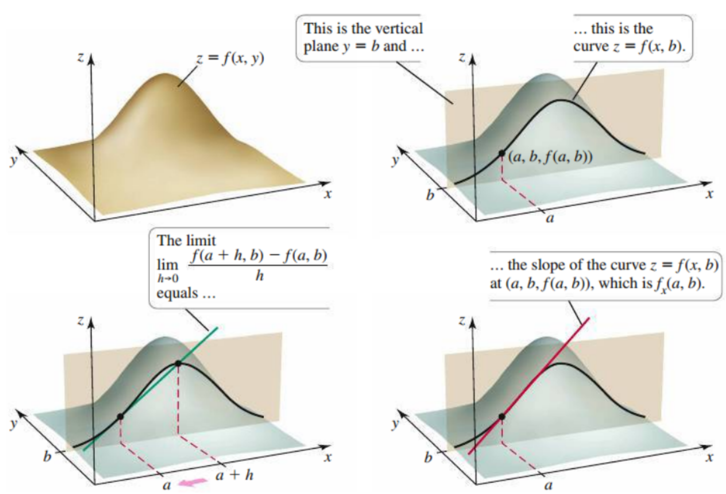
Definition. (Partial Derivatives)
The **partial derivative of f with respect to x at the point** (a, b) **is**

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

The **partial derivative of f with respect to y at the point** (a, b) **is**

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h},$$

provided these limits exist.



When evaluating a partial derivative at a point (a, b), we denote this

$$\frac{\partial f}{\partial x}(a,b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a,b) \text{ and } \frac{\partial f}{\partial y}(a,b) = \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a,b)$$

Example. For the following functions, find the first partial derivatives. If a point is provided, evaluate the partial derivatives.

$f(x, y) = x^8 + 3y^9 + 8$

$g(x, y) = 6x^5y^2 + 2x^3y + 5$

$$h(s,t) = \frac{s-t}{4s+t} \text{ at } (s,t) = (2,-3)$$

$$k(x,y) = \tan^{-1}\big(3x^2y^2\big) \text{ at } (x,y) = (1,1)$$

$$\ell(w,v) = \int_v^w g(u)\,du$$

Higher-Order Partial Derivatives:

$\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x^2}$	$(f_x)_x = f_{xx}$	"d squared f dx squared or $f - x - x$"
$\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2}$	$(f_y)_y = f_{yy}$	"d squared f dy squared or $f - y - y$"
$\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x \partial y}$	$(f_y)_x = f_{yx}$	"$f - y - x$"
$\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x}$	$(f_x)_y = f_{xy}$	"$f - x - y$"

The order of differentiation is important when finding **mixed partial derivatives** f_{xy} and f_{yx}.

Example. Find the four 2nd-order partial derivatives of the following functions

$$z = 4ye^{3x}$$

$$f(x, y) = \sin^2(x^3 y)$$

Theorem 15.4: (Clairut) Equality of Mixed Partial Derivatives Assume f is defined on an open set D of $\mathbb{R}^2$, and that f_{xy} and f_{yx} are continuous throughout D. Then $f_{xy} = f_{yx}$ at all points of D.

Note: Clairut's theorem also extends to higher order derivatives of f.

Example. Ideal Gas Law: The pressure P, volume V, and temperature T of an ideal gas are related by the equation $PV = kT$, where $k > 0$ is a constant depending on the amount of gas.

Determine the rate of change of the pressure with respect to the volume

Determine the rate of change of the pressure with respect to the temperature

Definition. (Differentiability)
The function $z = f(x, y)$ is **differentiable at** (a, b) provided $f_x(a, b)$ and $f_y(a, b)$ exist and the change $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$ equals

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y = \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where for fixed a and b, ε_1 and ε_2 are functions that depend only on Δx and δy, with $(\varepsilon_1, \varepsilon_2) \to (0, 0)$ as $(\Delta x, \Delta y) \to (0, 0)$. A function is **differentiable** on an open set R if it is differentiable at every point of R.

Theorem 15.5: Conditions for Differentiability
Suppose the function f has partial derivatives f_x and f_y defined on an open set containing (a, b), with f_x and f_y continuous at (a, b). Then f is differentiable at (a, b).

Theorem 15.6: Differentiable Implies Continuous
If a function f is differentiable at (a, b), then it is continuous at (a, b).

Example. Why is the function

$$f(x, y) = \begin{cases} \frac{3xy}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

not continuous at $(x, y) = (0, 0)$?