10.7: The Ratio and Root Tests

Theorem 10.20: Ratio Test

Let $\sum a_k$ be an infinite series, and let $r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$

1. If $r < 1$, the series converges absolutely, and therefore it converges (by Theorem 10.19)
2. If $r > 1$ (including $r = \infty$), the series diverges.
3. If $r = 1$, the test is inconclusive.

Note: The ratio test is used to determine if a series converges or diverges and indicates nothing about the *value* of the series.

Example. Use the ratio test on the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ and the alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$.

Example. *Note:* $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$

Rewrite $n!n!$ and $\dfrac{(2n)!}{(2n-1)!}$

Example. Consider the series below. Use the ratio test, if appropriate, to show if each of the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{k^2}{2^k}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k k}{k^3+1}$$

$$\sum_{k=1}^{\infty} \frac{5^k k!}{k^k}$$

$$\lim_{k\to\infty} \left(1+\frac{x}{k}\right)^k = e^x$$

$$\sum_{k=1}^{\infty} \frac{(-7)^k}{(2k+1)!}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k \ln(k)}{k}$$

Example. Use the ratio test to determine if the series $\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$ converges or diverges.

Example. Use the ratio test to determine if the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k}{(2k)!}$ converges or diverges.

Example. Use the ratio test to determine if the series $\displaystyle\sum_{k=1}^{\infty} \frac{(2k)!}{(k!)^2}$ converges or diverges.

10.21: Root Test

Let $\sum a_k$ be an infinite series, and let $\rho = \lim_{k\to\infty} \sqrt[k]{|a_k|}$.

1. If $\rho < 1$, the series converges absolutely, and therefore it converges (by Theorem 10.19)
2. If $\rho > 1$ (including $\rho = \infty$), the series diverges.
3. If $\rho = 1$, the test is inconclusive.

Note: The root test is used to determine if a series converges or diverges and indicates nothing about the *value* of the series.

Example. Use the root test to determine if the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}k^k}{3^{k^2}}$ converges.

Example. Consider the series below. Use the root test to show if each of the series converges or diverges.

$$\sum_{k=1}^{\infty} \left(\frac{1}{\ln(k+1)} \right)^k$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{3k^2+1}{k-2k^2} \right)^k$$

$$\sum_{k=1}^{\infty} \left(\frac{k+3}{k+1} \right)^{2k}$$

Example. Use the root test to determine if the series $\sum_{k=1}^{\infty}\left(1-\frac{3}{k}\right)^{k^2}$ converges.

Example. Determine whether each of the series below converges conditionally, converges absolutely, or diverges.

$$\sum_{k=1}^{\infty} (-1)^k k^{-1/3}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\arctan(k)}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k!}$$

Example. Determine if the series $\displaystyle\sum_{k=1}^{\infty}\left(\frac{k}{k+5}\right)^{3k^2}$ converges.

Example. Determine a condition for $x \geq 0$ such that $\displaystyle\sum_{k=1}^{\infty} \frac{4x^k}{5k^2}$ converges.