15.7: Maximum/Minimum Problems

Example. Consider the function $f(x) = x^3 - 3x + 1$ on the interval $[-1, 2]$. Find the local extrema and absolute extrema of this function.

c.p. not on boundary

$f'(x) = 3x^2 - 3 \implies$ solve $f'(x) = 0$

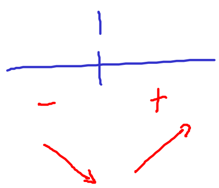
$3x^2 - 3 = 0$

$x^2 - 1 = 0 \implies x = \pm 1$

$\implies$ crit. pt. $x = 1$

1st deriv. test

$f'(x) = 3(x^2 - 1)$



$\implies$ local min

$f''(x) = 6x$

$f''(1) = 6 > 0 \implies$ concave up $\implies$ local min

$f''(cp) < 0 \implies$ concave down $\implies$ local max

$f''(cp) = 0 \implies$ Inconclusive $\leftarrow$ $y = x^3$

global extrema:

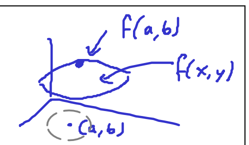
$f(x) = x^3 - 3x + 1$

x	$f(x)$
−1	3
1	−1
2	3

$\implies$ global min is -1 @ $x = 1$

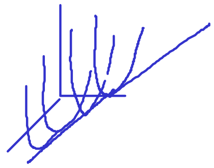
$\implies$ global max is 3 @ $x = -1$ and @ $x = 2$

Definition. (Local Maximum/Minimum Values)



Suppose (a, b) is a point in a region R on which f is defined.

- If $f(x, y) \le f(a, b)$ for all (x, y) in the domain of f and in some open disk centered at (a, b), then $f(a, b)$ is a **local maximum value** of f.
- If $f(x, y) \ge f(a, b)$ for all (x, y) in the domain of f and in some open disk centered at (a, b), then $f(a, b)$ is a **local minimum value** of f.
- Local maximum and local minimum values are also called **local extreme values** or **local extrema**.



Theorem 15.14: Derivatives and Local Maximum/Minimum Values
If f has a local maximum or minimum value at (a, b) and the partial derivatives f_x and f_y exist at (a, b), then $f_x(a, b) = f_y(a, b) = 0$.

Definition. (Critical Point)
An interior point (a, b) in the domain of f is a **critical point** of f if either

1. $f_x(a, b) = f_y(a, b) = 0$, or
2. at least one of the partial derivatives f_x and f_y does not exist at (a, b).

Example. Find the critical points of $f(x, y) = 3(x-1)^2 + 4(2-y)^3$.

$f_x(x,y) = 6(x-1) \Longrightarrow$ solve $f_x(x,y)=0 \longrightarrow 6(x-1)=0 \Rightarrow x=1$

$f_y(x,y) = -12(2-y)^2 \Longrightarrow$ solve $f_y(x,y)=0 \longrightarrow -12(2-y)^2 \Rightarrow y=2$

crit. point @ $(x,y)=(1,2)$

Example. Find the critical points of $g(x, y) = x^2 + xy - y^2$.

$g_x(x,y) = 2x+y \Rightarrow$ solve $g_x(x,y)=0 \longrightarrow 2x+y=0 \Rightarrow y=-2x \longrightarrow y=0$

$g_y(x,y) = x-2y$ solve $g_y(x,y)=0 \longrightarrow x-2y=0$

$x-2(-2x)=0$

$5x=0 \Rightarrow x=0$

crit. pt. $(x,y)=(0,0)$

Example. Find the critical points of $h(x, y) = \dfrac{3}{x} - \dfrac{4}{y}$.

$h_x(x,y) = -\frac{3}{x^2} \longrightarrow$ solve $h_x(x,y)=0 \longrightarrow \cancel{-\frac{3}{x^2} \stackrel{?}{=} 0}$

$h_y(x,y) = \frac{4}{y^2}$

$h_x(x,y)$ DNE w/ $x=0$
$h_y(x,y)$ DNE w/ $y=0$ $\Big\}$ crit. pt. $(x,y)=(0,0)$

Definition. (Saddle Point)
Consider a function f that is differentiable at a critical point (a, b). Then f has a **saddle point** at (a, b) if, in every open disk centered at (a, b), there are points (x, y) for which $f(x, y) > f(a, b)$ and points for which $f(x, y) < f(a, b)$.

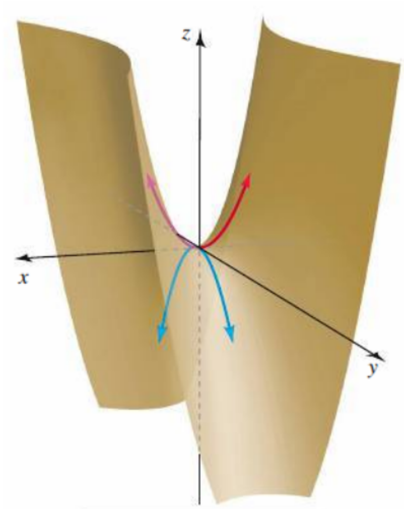
Example. Compute the first and second order partial derivatives of $f(x, y) = x^2 - y^2$.

$f_x(x, y) = 2x$ $\quad$ $f_y(x, y) = -2y$ $\Rightarrow$ crit. pt $(0,0)$

$f_{xx}(x, y) = 2$ $\quad$ $f_{xy}(x, y) = 0$

$f_{yx}(x, y) = 0$ $\quad$ $f_{yy}(x, y) = -2$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$



The hyperbolic paraboloid $z = x^2 - y^2$ has a saddle point at $(0, 0)$.

Theorem 15.15: Second Derivative Test
Suppose the second partial derivatives of f are continuous throughout an open disk centered at the point (a, b), where $f_x(a, b) = f_y(a, b) = 0$. Let

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2.$$

1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum value at (a, b).
2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum value at (a, b).
3. If $D(a, b) < 0$, then f has a saddle point at (a, b).
4. If $D(a, b) = 0$, then the test is inconclusive.

Example. Use the Second Derivative Test to classify the critical points of $f(x, y) = x^2 + 2y^2 - 4x + 4y + 6$.

$f_x(x,y) = 2x - 4$ $\qquad$ $f_y(x,y) = 4y + 4$

$\to$ solve $\quad 2x - 4 = 0 \to x = 2$, $\quad 4y + 4 = 0 \to y = -1$ $\quad \Big\}$ $(x,y) = (2,-1)$

$f_{xx}(x,y) = 2 \qquad f_{xy}(x,y) = 0$

$f_{yx}(x,y) = 0 \qquad f_{yy}(x,y) = 4$

$D(2,-1) = f_{xx}(2,-1)\, f_{yy}(2,-1) - (f_{xy}(2,-1))^2 = 8$

$f_{xx}(2,-1) = 2 > 0 \to$ concave up $\to$ local minimum at $(2,-1)$

Example. Use the Second Derivative Test to classify the critical points of $f(x, y) = xy(x - 2)(y + 3)$.

$f_x(x,y) = y(x-2)(y+3) + xy(y+3) = (y+3)\big(y(x-2) + xy\big) = (y+3)(2xy - 2y)$

$f_y(x,y) = x(x-2)(y+3) + xy(x-2) = (x-2)(2xy + 3x)$

Solve $\quad (y+3)(x-1)2y = 0 \longrightarrow y = -3,\ y = 0,\ x = 1$

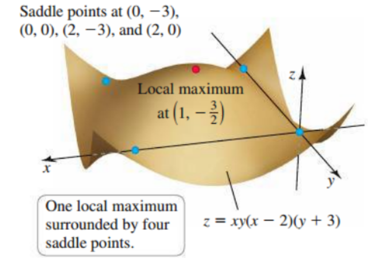
$\qquad (x-2)x(2y+3) = 0 \longrightarrow x = 2,\ x = 0,\ y = -3/2$

$(x-2)x(-3) = 0$

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$$= \begin{vmatrix} (y+3)2y & 2(x-1)(2y+3) \\ 2(x-1)(2y+3) & 2(x-2)x \end{vmatrix}$$

(x,y)	$D(x,y)$	Conclusion
$(2,-3)$	$0 - (2\cdot 1\cdot(-3))^2 = -36 < 0$	S.P.
$(0,-3)$	$0 - (6)^2 = -36 < 0$	S.P.
$(2,0)$	$0 - (6)^2 = -36 < 0$	S.P.
$(0,0)$	$0\cdot 0 - (-6)^2 = -36 < 0$	S.P.
$(1,-3/2)$	$9 > 0$	$f_{xx}(1,-3/2) = -9/2 < 0 \to$ local max.



One local maximum surrounded by four saddle points.

$(2x-2)(2y+3)$ $\qquad$ $(x-1)(2y + 2y + 6)$

$2(x-1)(2y+3)$ $\qquad$ $2(x-1)(2y+3)$

Definition. (Absolute Maximum/Minimum Values)
Let f be defined on a set R in $\mathbb{R}^2$ containing the point (a, b).

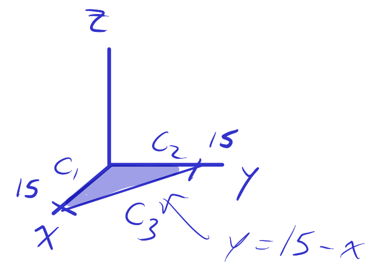
- If $f(a, b) \geq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an **absolute maximum value** of f on R.
- If $f(a, b) \leq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an **absolute minimum value** of f on R.

Procedure:
Finding Absolute Maximum/Minimum Values on Closed Bounded Sets
Let f be continuous on a closed bounded set R in $\mathbb{R}^2$. To find the absolute maximum and minimum values of f on R:

1. Determine the values of f at all critical points in R.
2. Find the maximum and minimum values of f on the boundary of R.
3. The greatest function value found in Steps 1 and 2 is the absolute maximum value of f on R, and the least function value found in Steps 1 and 2 is the absolute minimum value of f on R.

LC #1 & #2

Example. Find the absolute maximum and minimum values of $f(x, y) = xy - 8x - y^2 + 12y + 160$ over the triangular region $R = \{(x, y) : 0 \le x \le 15,\ 0 \le y \le 15 - x\}$.



① $f_x(x,y) = y - 8$ $\longrightarrow$ $0x + y = 8$

$f_y(x,y) = x - 2y + 12$ $\quad x - 2y = -12$

Solve $f_x(x,y) = 0 \longrightarrow y = 8$

$f_y(x,y) = 0 \longrightarrow x - 16 + 12 = 0$

$x = 4$

} crit. pt. $(4,8)$ SP

$f(4,8) = -64 + 96 + 160 = 192$

②

$C_1: \{(x,y): y = 0,\ 0 \le x \le 15\}$ $\quad g_1(x) = f(x,0) = -8x + 160$

$g_1'(x) = -8 \longrightarrow$ No crit. pts.

Check boundaries $\quad g_1(0) = 160 \longrightarrow (0,0)$

$g_1(15) = 40 \longrightarrow (15,0)$

36

$-36 \cancel{+72} + 160$

$C_2: \{(x,y): x = 0,\ 0 \le y \le 15\}$ $\quad g_2(y) = f(0,y) = -y^2 + 12y + 160$

$-15^2 + 12(15) + 160$

$15(\underbrace{12-15}_{-3}) + 160$

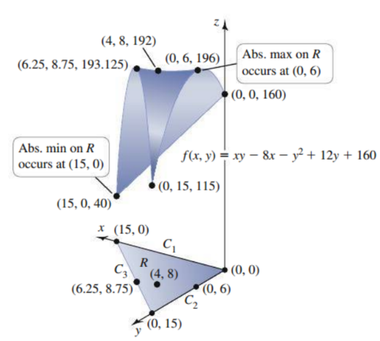
$-45 + 160 = 115$

$g_2'(y) = -2y + 12 \overset{\text{solve}}{=} 0 \rightarrow y = 6$

endpoints $\quad g_2(0) = 160 \longrightarrow (0,0)$

$g_2(6) = 196 \longrightarrow (0,6)$

$g_2(15) = 115 \longrightarrow (0,15)$



$C_3: \{(x,y): y = 15 - x,\ 0 \le x \le 15\}$

$g_3(x) = f(x, 15-x) = x(15-x) - 8x - (15-x)^2 + 12(15-x) + 160$

$= -2x^2 + 25x + 115$

$g_3'(x) = -4x + 25 \overset{\text{solve}}{=} 0 \longrightarrow x = \frac{25}{4} = 6.25 \longrightarrow y = 15 - x$

$y = 8.75$

$g_3(0) = 115 \longrightarrow (0, 15)$

$g_3\left(\frac{25}{4}\right) = \frac{25^2}{8} + 115 = 193.125 \longrightarrow (6.25, 8.75)$

$g_3(15) = 40 \longrightarrow (15, 0)$

3

(x, y)	$f(x,y)$	
$(4, 8)$	192	
$(0, 0)$	160	
abs. min $\longrightarrow$ $(15, 0)$	40	LC #2
abs. max $\longrightarrow$ $(0, 6)$	196	LC #1
$(0, 15)$	115	
$(6.25, 8.75)$	193.125	

LC #3, 4, 5

Example. A shipping company handles rectangular boxes provided the sum of the length, width, and height of the box does not exceed 96 in. Find the dimensions of the box that meets this condition and has the largest volume.

x: length

y: width

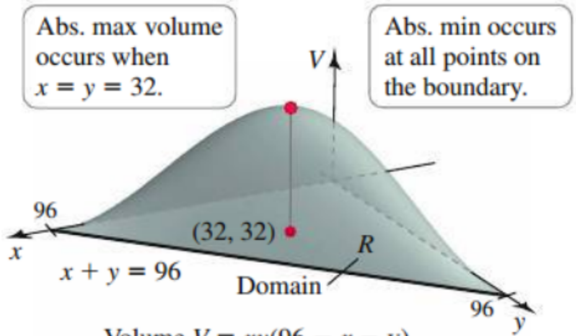
z: height

max $V = xyz$

$x + y + z \le 96$

$\to z = 96 - x - y$

$\to f(x,y) = V = xy(96-x-y)$



Volume $V = xy(96 - x - y)$

① $f_x(x,y) = y(96-x-y) + xy(-1) = 96y - 2xy - y^2 = y(96-2x-y) \overset{\text{solve}}{=} 0$

$f_y(x,y) = x(96-x-y) - xy = 96x - 2xy - x^2 = x(96-2y-x) \overset{\text{solve}}{=} 0$

Let $y=0 \to x=0$; $\to 96-x=0 \to x=96$

Let $x=0 \to y=0$; $\to 96-y=0 \to y=96$

$(0,0)$, $(96,0)$, $(0,96)$ $\to V=0$

$2x+y=96$

$-2(x+2y=96)$

$-3y=-96 \to y=32 \to x=32$

$(x,y)=(32,32)$

$z = 96-x-y$

$z = 32$

$V = 32^3$

② Boundaries

all zero

$\to$ Nothing to check

LC #3 length = 32

#4 width = 32

#5 height = 32

Example. Find the absolute maximum and minimum values of $f(x, y) = 4 - x^2 - y^2$ on the open disk $R = \{(x, y) : x^2 + y^2 < 1\}$ (if they exist).

① $f_x(x,y) = -2x$ $\longrightarrow$ critical point $(0,0)$ $\quad f(0,0) = 4$

$f_y(x,y) = -2y$

② Boundary

Since the boundary is open, there will not be any global max or min there. However, since the MLM has a question like this with a closed circular boundary, I'm going to assume it is closed just to demonstrate.

$x^2 + y^2 = 1$ $\longleftarrow$ Solve for one variable

$y^2 = 1 - x^2$ $\longleftarrow$ If I solve for y, then I have 2 equations to consider

$y = \pm\sqrt{1-x^2}$

$\longrightarrow g_1(x) = 4 - x^2 - (1 - x^2) = 3$

$g_1'(x) = 0 \longrightarrow$ constant

Let y=0, then we have the domain for x: $-1 \le x \le 1$

and the following domain for y based on x: $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

$g_1(-1) = 3$

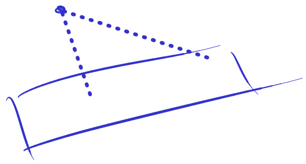
$g_1(1) = 3$ $\Rightarrow$ global min on the boundary

Let's verify that $(0,0)$ is a local max

$f_{xx}(x,y) = -2 \qquad f_{xy}(x,y) = 0$

$f_{yx}(x,y) = 0 \qquad f_{yy}(x,y) = -2$

$\left.\begin{array}{l} D(0,0) = 4 > 0 \\ f_{xx}(x,y) = -2 < 0 \end{array}\right\}$ local max



$\nearrow z = 2 - x - 2y$

Example. Find the point(s) on the plane $x + 2y + z = 2$ closest to the point $P(2, 0, 4)$.

min distance

$$d(x,y,z) = \sqrt{(x-2)^2 + (y-0)^2 + (z-4)^2}$$

$$f(x,y) = \left(d(x,y,z)\right)^2 = (x-2)^2 + y^2 + (\underbrace{-x-2y}_{\geq 0}-2)^2$$

$x \geq 0$

$y \geq 0$

① $f_x(x,y) = 2(x-2) - 2(-x-2y-2) = 4x + 4y \overset{\text{solve}}{=} 0 \rightarrow x = -y$

$f_y(x,y) = 2y - 4(-x-2y-2) = 4x + 10y + 8 \overset{\text{solve}}{=} 0$

$\rightarrow -4y + 10y + 8 = 0$

$y = -4/3$

② No boundary

$\Rightarrow x = 4/3$

Verify min

$f_{xx}(x,y) = 4$ $\quad f_{xy}(x,y) = 4$

$f_{yx}(x,y) = 4$ $\quad f_{yy}(x,y) = 10$

$$D(4/3, -4/3) = 4(10) - (4)^2 = 36 > 0$$

$f_{xx}(x,y) = 4 > 0 \longrightarrow$ local min.