2.5 Limits at Infinity

Definition. Limits at Infinity and Horizontal Asymptotes
If $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently large and positive x, then we write

$$\lim_{x\to\infty} f(x) = L$$

We say the limit of $f(x)$ as x approaches infinity is L. In this case, the line $y = L$ is a **horizontal asymptote** of f. The limit at negative infinity,

$$\lim_{x\to-\infty} f(x) = M$$

is defined analogously. When this limit exists, $y = M$ is a horizontal asymptote.

Note: The function *can* cross it's horizontal asymptote (consider $\frac{\sin x}{x}$).

Note: A function can have 0, 1 or 2 horizontal asymptotes.

Example. For each of the following functions, find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

a) $f(x) = \dfrac{1}{x^2}$

$\lim\limits_{x\to-\infty} \dfrac{1}{x^2} = 0$

$\lim\limits_{x\to\infty} \dfrac{1}{x^2} = 0$

b) $f(x) = \dfrac{1}{x^3}$

$\lim\limits_{x\to\pm\infty} \dfrac{1}{x^3} = 0$

c) $f(x) = 2 + \dfrac{10}{x^2}$

$\lim\limits_{x\to\pm\infty} 2 + \underbrace{\dfrac{10}{x^2}}$

$= 2 + 0 = 2$

d) $f(x) = 5 + \dfrac{\sin x}{\sqrt{x}}$

$\lim\limits_{x\to\pm\infty} 5 + \underbrace{\dfrac{\sin x}{\sqrt{x}}}_{0}$

$= 5$

e) $f(x) = \left(5 + \dfrac{1}{x} + \dfrac{10}{x^2}\right)$

$\lim\limits_{x\to\pm\infty}\left(5 + \dfrac{1}{x} + \dfrac{10}{x^2}\right) = 5 + 0 + 0 = 5$

f) $f(x) = \left(3x^{12} - 9x^7\right)$

$\lim\limits_{x\to\pm\infty} 3x^{12} - 9x^7 = \infty$

g) $f(x) = \sin(x)$

$\lim\limits_{x\to\pm\infty} \sin(x)$

h) $f(x) = \dfrac{\sin x}{x}$

$\lim\limits_{x\to\pm\infty} \dfrac{\sin x}{x} > 0$

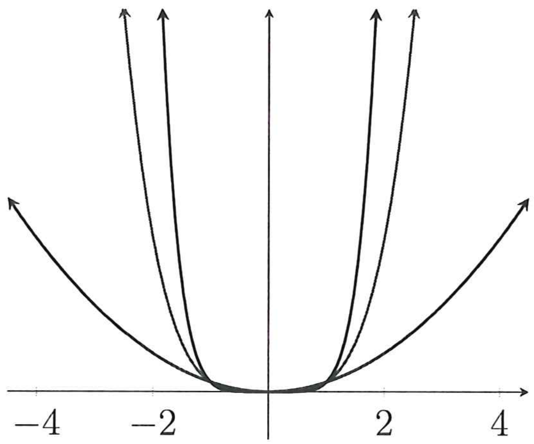
Definition. Infinite Limits at Infinity

If $f(x)$ becomes arbitrarily large as x becomes arbitrarily large, then we write

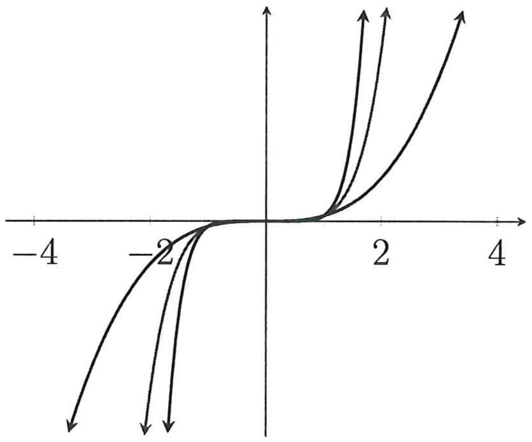
$$\lim_{x\to\infty} f(x) = \infty$$

The limits $\lim\limits_{x\to\infty} = -\infty$, $\lim\limits_{x\to-\infty} = \infty$ and $\lim\limits_{x\to-\infty} = -\infty$ are defined similarly.

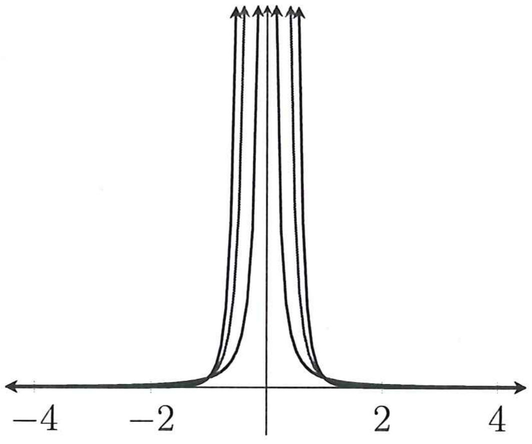
Even functions



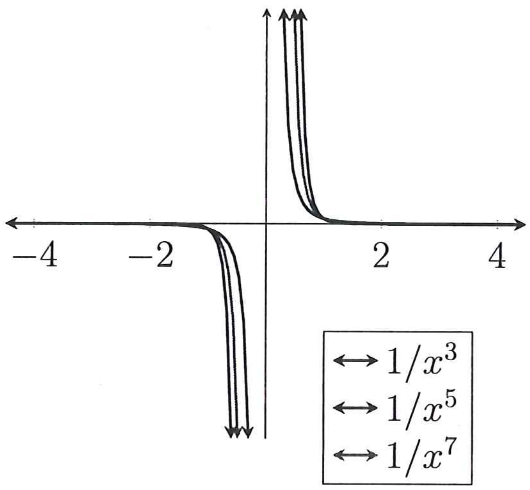
Odd functions



$1/x^n, n$ Even



$1/x^n, n$ Odd



Theorem. Limits at Infinity of Powers and Polynomials
Let n be a positive integer and let p be the polynomial

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0, \text{ where } a_n \neq 0.$$

1. $\lim_{x\to\pm\infty} x^n = \infty$ when n is even.
2. $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to-\infty} x^n = -\infty$ when n is odd.
3. $\lim_{x\to\pm\infty} \frac{1}{x^n} = \lim_{x\to\pm\infty} x^{-n} = 0.$
4. $\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} a_n x^n = \pm\infty$, depending on the degree of the polynomial and the sign of the leading coefficient a_n.

Note: All previous limit laws still apply (e.g. constant multiplier rule)

Note: This theorem *ONLY* applies for $x \to \pm\infty$. When $x \to a$, $|a| < \infty$, we compute the left and right limits and use sm+/sm- (as done in section 2.4).

Example. For the following, find the limits as $x \to -\infty$ and $x \to \infty$:

1. $f(x) = 2x^{-8}$

$\lim_{x\to-\infty} 2x^{-8} = \lim_{x\to-\infty} \frac{2}{x^8} = 0$

$\lim_{x\to\infty} 2x^{-8} = \lim_{x\to\infty} \frac{2}{x^8} = 0$

2. $g(x) = -12x^{-5}$

$\lim_{x\to-\infty} \frac{-12}{x^5} = 0$ $\quad \lim_{x\to\infty} \frac{-12}{x^5} = 0$

3. $h(x) = 3x^{12} - 9x^7$

$\lim_{x\to\pm\infty} 3x^{12} - 9x^7 = \lim_{x\to\pm\infty} 3x^{12} = \infty$

4. $\ell(x) = 2x^{-8} + 4x^3$

$\lim_{x\to-\infty} \frac{2}{x^8} + 4x^3 = \lim_{x\to-\infty} 4x^3 = -\infty$

$\lim_{x\to\infty} \frac{2}{x^8} + 4x^3 = \lim_{x\to\infty} 4x^3 = \infty$

When finding the limit as $x \to \pm\infty$ of a rational function, $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions, we multiply the function by $\frac{1/x^n}{1/x^n}$, where n is the highest degree in the denominator $q(x)$.

Note: To receive full credit for questions of this type, you must show all the fractions in your intermediate steps.

Example.

a) $\lim\limits_{x\to\infty} \frac{1-x}{2x}\left(\frac{1/x}{1/x}\right) = \lim\limits_{x\to\infty} \frac{1/x - 1}{2} = \frac{0-1}{2} = -1/2$

b) $\lim\limits_{x\to\infty} \frac{1-x}{x^2}\left(\frac{1/x^2}{1/x^2}\right) = \lim\limits_{x\to\infty} \frac{1/x^2 - 1/x}{1} = \frac{0-0}{1} = 0$

c) $\lim\limits_{x\to\infty} \frac{1-x^2}{2x}\left(\frac{1/x}{1/x}\right) = \lim\limits_{x\to\infty} \frac{1/x - x}{2} = \frac{0-\infty}{2} = -\infty.$

Theorem. End Behavior and Asymptotes of Rational Functions

Suppose $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function, where

$$\begin{aligned} p(x) &= a_m x^m + a_{m-1}x^{m-1} + \cdots + a_2x^2 + a_1x + a_0 \\ q(x) &= b_n x^n + b_{n-1}x^{n-1} + \cdots + b_2x^2 + b_1x + b_0 \end{aligned}$$

with $a_m \neq 0$ and $b_n \neq 0$.

1. **Degree of numerator less than degree of denominator**

   If $m < n$, then $\lim_{x\to\pm\infty} f(x) = 0$ and $y = 0$ is a horizontal asymptote of f.

2. **Degree of numerator equals degree of denominator**

   If $m = n$, then $\lim_{x\to\pm\infty} f(x) = a_m/b_n$ and $y = a_m/b_n$ is a horizontal asymptote of f.

3. **Degree of numerator greater than degree of denominator**

   If $m > n$, then $\lim_{x\to\pm\infty} f(x) = \infty$ or $-\infty$ and f has no horizontal asymptote.

4. **Slant Asymptote**

   If $m = n+1$, then $\lim_{x\to\pm\infty} f(x) = \infty$ or $-\infty$, and f has no horizontal asymptote, but f has a slant asymptote.

5. **Vertical asymptotes**

   Assuming f is in reduced form (p and q share no common factors), vertical asymptotes occur at the zeros of q.

Example. Evaluate the limits of the following as $x \to -\infty$ and $x \to \infty$. State the equation of the horizontal asymptote.

1. $f(x) = \dfrac{2x^3+7}{x^3-x^2+x+7} \left(\dfrac{1/x^3}{1/x^3}\right)$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2 + \cancel{\frac{7}{x^3}}^{\to 0}}{1 - \cancel{\frac{1}{x}}^{\to 0} + \cancel{\frac{1}{x^2}}^{\to 0} + \cancel{\frac{7}{x^3}}^{\to 0}} = \frac{2}{1} = \boxed{2}$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2 + \cancel{\frac{7}{x^3}}^{\to 0}}{1 - \cancel{\frac{1}{x}}^{\to 0} + \cancel{\frac{1}{x^2}}^{\to 0} + \cancel{\frac{7}{x^3}}^{\to 0}} = \frac{2}{1} = \boxed{2}$$

$\boxed{\text{H.A. } y = 2}$

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2. $g(x) = \dfrac{1}{x^3-4x+1} \left(\dfrac{1/x^3}{1/x^3}\right)$

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{\cancel{\frac{1}{x^3}}^{\to 0}}{1 - \cancel{\frac{4}{x^2}}^{\to 0} + \cancel{\frac{1}{x^3}}^{\to 0}} = \frac{0}{1} = \boxed{0}$$

$$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} \frac{\cancel{\frac{1}{x^3}}^{\to 0}}{1 - \cancel{\frac{4}{x^2}}^{\to 0} + \cancel{\frac{1}{x^3}}^{\to 0}} = \frac{0}{1} = \boxed{0}$$

$\boxed{\text{H.A. } y = 0}$

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3. $h(x) = \dfrac{3x^5+2x^2-2}{4x^4-3x} \left(\dfrac{1/x^4}{1/x^4}\right)$

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{3x + \cancel{\frac{2}{x^2}}^{\to 0} - \cancel{\frac{2}{x^4}}^{\to 0}}{4 - \cancel{\frac{3}{x^3}}^{\to 0}} = \infty \quad \text{b/c } \frac{3x}{4} \to \infty$$

$$\lim_{x\to-\infty} h(x) = \lim_{x\to-\infty} \frac{3x + \cancel{\frac{2}{x^2}}^{\to 0} - \cancel{\frac{2}{x^4}}^{\to 0}}{4 - \cancel{\frac{3}{x^3}}^{\to 0}} = -\infty \quad \text{b/c } \frac{3x}{4} \to -\infty$$

$\boxed{\text{No H.A.}}$

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4. $j(x) = \dfrac{4x^2-2x+3}{7x^2-1} \left(\dfrac{1/x^2}{1/x^2}\right)$

$$\lim_{x\to\infty} j(x) = \lim_{x\to\infty} \frac{4 - \cancel{\frac{2}{x}}^{\to 0} + \cancel{\frac{3}{x^2}}^{\to 0}}{7 - \cancel{\frac{1}{x^2}}^{\to 0}} = \boxed{\frac{4}{7}}$$

$$\lim_{x\to-\infty} j(x) = \lim_{x\to-\infty} \frac{4 - \cancel{\frac{2}{x}}^{\to 0} + \cancel{\frac{3}{x^2}}^{\to 0}}{7 - \cancel{\frac{1}{x^2}}^{\to 0}} = \boxed{\frac{4}{7}}$$

$\boxed{\text{H.A. } y = 4/7}$

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5. $\ell(x) = \dfrac{1-x^2}{3+2x-x^3} \left(\dfrac{1/x^3}{1/x^3}\right)$

$$\lim_{x\to\infty} \ell(x) = \lim_{x\to\infty} \frac{\cancel{\frac{1}{x^3}}^{\to 0} - \cancel{\frac{1}{x}}^{\to 0}}{\cancel{\frac{3}{x^3}}^{\to 0} + \cancel{\frac{2}{x^2}}^{\to 0} - 1} = \frac{0}{-1} = \boxed{0}$$

$\boxed{\text{H.A. } y = 0.}$

$$\lim_{x\to-\infty} \ell(x) = \lim_{x\to-\infty} \frac{\cancel{\frac{1}{x^3}}^{\to 0} - \cancel{\frac{1}{x}}^{\to 0}}{\cancel{\frac{3}{x^3}}^{\to 0} + \cancel{\frac{2}{x^2}}^{\to 0} - 1} = \frac{0}{-1} = \boxed{0}$$

Definition. When the degree of the numerator, m is greater than the degree of the denominator, n, the function has an oblique asymptote:

$$f(x) = \frac{p(x)}{q(x)} = a(x) + \frac{r(x)}{q(x)}$$

where $a(x)$ is the resulting polynomial that we get from polynomial long division and $r(x)$ is the remainder. We are interested in the special case where $m = n + 1$, and $f(x)$ has a **slant asymptote**.

Example. For the following functions, find the vertical asymptotes and the slant asymptotes:

1. $y = \dfrac{2x^3 + x^2 + x + 3}{x^2 + 2x}$

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2. $f(x) = \dfrac{x^2 - 1}{x + 2}$

3. $g(t) = \dfrac{t^2 - 1}{2t + 4}$

4. $h(u) = \dfrac{u^2}{u - 1}$

① $y = \frac{2x^3+x^2+x+3}{x^2+2x}$

V.A. - Vertical asymptotes occur when $y \to \frac{c}{0},\ c \neq 0.$

$$x^2+2x=0$$

$$\Rightarrow x(x+2)=0 \quad \Rightarrow x=0,\ x=-2$$

If $x=0$, $y \to \frac{3}{0} \Rightarrow$ Vertical asymptote

If $x=-2$, $y \to \frac{-16+4-2+3}{0} \to \frac{-11}{0} \Rightarrow$ Vertical asymptote

$$\lim_{x\to 0^-} y = \lim_{x\to 0^-} \frac{2x^3+x^2+x+3}{x(x+2)} = \frac{3}{(\text{sm}-)(2)} = -\infty$$

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} \frac{2x^3+x^2+x+3}{x(x+2)} = \frac{3}{(\text{sm}+)(2)} = \infty$$

$$\lim_{x\to -2^-} y = \lim_{x\to -2^-} \frac{2x^3+x^2+x+3}{x(x+2)} = \frac{-11}{(-2)(\text{sm}-)} = -\infty$$

$$\lim_{x\to -2^+} y = \lim_{x\to -2^+} \frac{2x^3+x^2+x+3}{x(x+2)} = \frac{-11}{(-2)(\text{sm}+)} = \infty$$

$$\boxed{\text{V.A. } x=0,\ x=-2}$$

Slant asymptote

Recall: $\frac{p(x)}{q(x)} \to \begin{array}{r} a(x) \\ q(x)\,\overline{)\,p(x)} \\ \vdots \\ r(x) \end{array} \to a(x) + \frac{r(x)}{q(x)}$

$$\begin{array}{r} 2x-3 \\ x^2+2x\,\overline{)\,2x^3+x^2+x+3} \\ -(2x^3+4x^2) \\ \hline -3x^2+x \\ -(-3x^2-6x) \\ \hline 7x+3 \end{array} \Rightarrow y = \underbrace{2x-3} + \frac{7x+3}{x^2+2x}$$

$$\boxed{\begin{array}{c}\text{Slant asymptote is} \\ 2x-3\end{array}}$$

Remember: This is the same function! Try finding a common denominator.

② $f(x)=\dfrac{x^2-1}{x+2} \Rightarrow$ V.A.: $x \neq -2 \Rightarrow f(-2) \longrightarrow \dfrac{3}{0}$

$$\begin{cases} \lim\limits_{x\to -2^-} \dfrac{x^2-1}{x+2} = \dfrac{3}{sm-} = -\infty \\ \lim\limits_{x\to -2^+} \dfrac{x^2-1}{x+2} = \dfrac{3}{sm+} = \infty \end{cases}$$

$\boxed{\text{V.A. } x = -2}$

Slant asymptote:

$$\begin{array}{r|l} & \quad\quad\; x-2 \\ \hline x+2 & x^2+0x-1 \\ & -(x^2+2x)\downarrow \\ \hline & \quad -2x-1 \\ & -(-2x-4) \\ \hline & \quad\quad 3 \end{array}$$

$\Rightarrow f(x) = \underbrace{x-2}_{} + \dfrac{3}{x+2}$

Slant asymptote $\boxed{y = x-2}$

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③ $g(t) = \dfrac{t^2-1}{2t+4} = \dfrac{(t+1)(t-1)}{2(t+2)} \Rightarrow t \neq -2$

V.A.: $\lim\limits_{t\to -2^-} \dfrac{t^2-1}{2t+4} = \dfrac{3}{sm-} = -\infty$

$\boxed{\text{V.A. } x = -2}$

$\lim\limits_{t\to -2^+} \dfrac{t^2-1}{2t+4} = \dfrac{3}{sm+} = \infty$

Slant asymptote:

$$\begin{array}{r|l} & \quad\quad\; \tfrac{1}{2}t-1 \\ \hline 2t+4 & t^2+0t-1 \\ & -(t^2+2t) \\ \hline & \quad -2t-1 \\ & -(-2t-4) \\ \hline & \quad\quad 3 \end{array}$$

$\Rightarrow g(t) = \underbrace{\tfrac{1}{2}t-1}_{} + \dfrac{3}{2t+4}$

Slant asymptote

$\boxed{y = \frac{1}{2}t-1}$

④ $h(u) = \dfrac{u^2}{u-1}$

V.A.: $u \neq 1$

$$\lim_{u \to 1^-} \frac{u^2}{u-1} = \frac{1}{sm-} = -\infty$$

$$\boxed{V.A.\ x = 1}$$

$$\lim_{u \to 1^+} \frac{u^2}{u-1} = \frac{1}{sm+} = \infty$$

Slant Asymptote:

$$\begin{array}{r|l} & \quad\quad\quad u+1 \\ \hline u-1 & u^2 + 0u + 0 \downarrow \\ & -(u^2 - u) \\ \hline & \quad\quad u + 0 \\ & \quad -(u-1) \\ \hline & \quad\quad\quad 1 \end{array}$$

$$\Rightarrow h(u) = \underbrace{u+1} + \frac{1}{u-1}$$

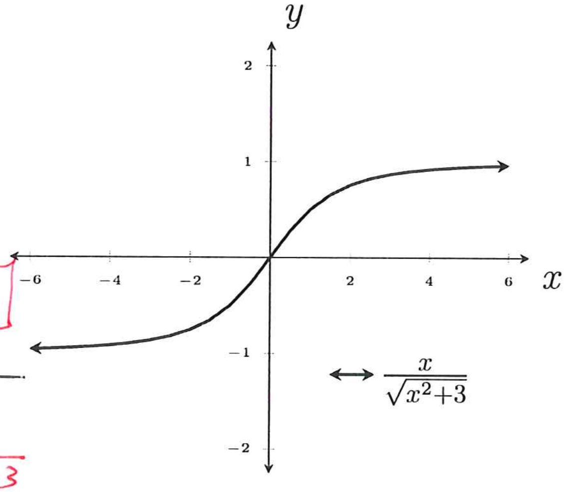
slant asymptote

$$\boxed{y = u+1}$$

For n odd $\quad \frac{1}{x^n} = \begin{cases} \frac{1}{\sqrt{x^{2n}}} \\ -\frac{1}{\sqrt{x^{2n}}} \end{cases}$ $\qquad$ For n even, $\frac{1}{x^n} = \frac{1}{\sqrt{x^{2n}}}$

If the denominator has a square root, we need to change our work depending on if $x \to -\infty$ or $x \to \infty$:

Example. For the following, find the equation of the horizontal asymptotes:



a) $\lim\limits_{x\to\infty} \frac{x}{\sqrt{x^2+3}}\left(\frac{1/x}{1/x}\right) = \lim\limits_{x\to\infty} \frac{1}{\frac{1}{\sqrt{x^2}}\sqrt{x^2+3}}$

$\frac{1}{x} = \frac{1}{\sqrt{x^2}}$ $\qquad = \lim\limits_{x\to\infty} \frac{1}{\sqrt{1+\cancel{3/x^2}^{\,0}}} = \boxed{1}$

$\boxed{\text{H.A. } y=1}$

b) $\lim\limits_{x\to-\infty} \frac{x}{\sqrt{x^2+3}}\left(\frac{1/x}{1/x}\right) = \lim\limits_{x\to-\infty} \frac{1}{-\frac{1}{\sqrt{x^2}}\sqrt{x^2+3}}$

$\frac{1}{x} = -\frac{1}{\sqrt{x^2}}$ $\qquad = \lim\limits_{x\to-\infty} \frac{1}{-\sqrt{1+\cancel{3/x^2}^{\,0}}} = \frac{1}{-\sqrt{1}} = \boxed{-1}$

$\boxed{\text{H.A. } y=-1}$

c) $\lim\limits_{x\to\infty} \frac{x}{\sqrt{x^2+x}}\left(\frac{1/x}{1/x}\right) = \lim\limits_{x\to\infty} \frac{1}{\frac{1}{\sqrt{x^2}}\sqrt{x^2+x}} = \lim\limits_{x\to\infty} \frac{1}{\sqrt{1+\cancel{1/x}^{\,0}}} = \boxed{1}$

$\boxed{\text{H.A. } y=1}$

d) $\lim\limits_{x\to-\infty} \frac{x}{\sqrt{x^2+x}}\left(\frac{1/x}{1/x}\right) = \lim\limits_{x\to-\infty} \frac{1}{-\frac{1}{\sqrt{x^2}}\sqrt{x^2+x}} = \lim\limits_{x\to-\infty} \frac{1}{-\sqrt{1+1/x}} = \frac{1}{-\sqrt{1}} = \boxed{-1}$

$\boxed{\text{H.A. } y=-1}$

e) $\dfrac{7x^3-2}{-x^3+\sqrt{25x^6+4}}\left(\dfrac{1/x^3}{1/x^3}\right)$

$$\lim_{x\to\infty}\frac{7-2/x^3}{-1+\frac{1}{\sqrt{x^6}}\sqrt{25x^6+4}}=\lim_{x\to\infty}\frac{7-\cancel{2/x^3}^{\,0}}{-1+\sqrt{25+\cancel{4/x^6}^{\,0}}}$$

$$=\frac{7}{-1+\sqrt{25}}=\boxed{\frac{7}{4}}$$

$$\lim_{x\to-\infty}\frac{7-2/x^3}{-1-\frac{1}{\sqrt{x^6}}\sqrt{25x^6+4}}=\lim_{x\to-\infty}\frac{7-\cancel{2/x^3}^{\,0}}{-1-\sqrt{25+\cancel{4/x^6}^{\,0}}}$$

$$=\frac{7}{-1-\sqrt{25}}=\boxed{\frac{7}{-6}}$$

$\boxed{\text{H.A.: } y=7/4 \text{ and } y=-7/6}$

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f) $\dfrac{\sqrt[3]{x^6+8}}{4x^2+\sqrt{3x^4+1}}\left(\dfrac{1/x^2}{1/x^2}\right)$

$$\lim_{x\to\infty}\frac{\frac{1}{\sqrt[3]{x^6}}\sqrt[3]{x^6+8}}{4+\frac{1}{\sqrt{x^4}}\sqrt{3x^4+1}}$$

$$=\lim_{x\to\infty}\frac{\sqrt[3]{1+\cancel{8/x^6}^{\,0}}}{4+\sqrt{3+\cancel{1/x^4}^{\,0}}}=\frac{\sqrt[3]{1}}{4+\sqrt{3}}=\boxed{\frac{1}{4+\sqrt3}}$$

$$\lim_{x\to-\infty}\frac{\frac{1}{\sqrt[3]{x^6}}\sqrt[3]{x^6+8}}{4+\frac{1}{\sqrt{x^4}}\sqrt{3x^4+1}}=\lim_{x\to-\infty}\frac{\sqrt[3]{1+\cancel{8/x^6}^{\,0}}}{4+\sqrt{3+\cancel{1/x^4}^{\,0}}}$$

$$=\frac{\sqrt[3]{1}}{4+\sqrt{3}}=\boxed{\frac{1}{4+\sqrt3}}$$

$\boxed{\text{H.A. } y=\frac{1}{4+\sqrt3}}$

g) $\dfrac{2x}{\sqrt{x^2-x-2}}\left(\dfrac{1/x}{1/x}\right)$

$$\lim_{x\to\infty}\frac{2}{\frac{1}{\sqrt{x^2}}\sqrt{x^2-x-2}}=\lim_{x\to\infty}\frac{2}{\sqrt{1-\cancel{1/x}^{\,0}-\cancel{2/x^2}^{\,0}}}=\frac{2}{\sqrt1}=\boxed{2}$$

$$\lim_{x\to-\infty}\frac{2}{-\frac{1}{\sqrt{x^2}}\sqrt{x^2-x-2}}=\lim_{x\to-\infty}\frac{2}{-\sqrt{1-\cancel{1/x}^{\,0}-\cancel{2/x^2}^{\,0}}}=\frac{2}{-\sqrt1}=\boxed{-2}$$

$\boxed{\text{H.A. } y=2 \text{ and } y=-2}$

Answers may vary!

Example. For the following, sketch a graph with the following properties:

1. $\lim_{x\to 0} f(x) = -\infty$

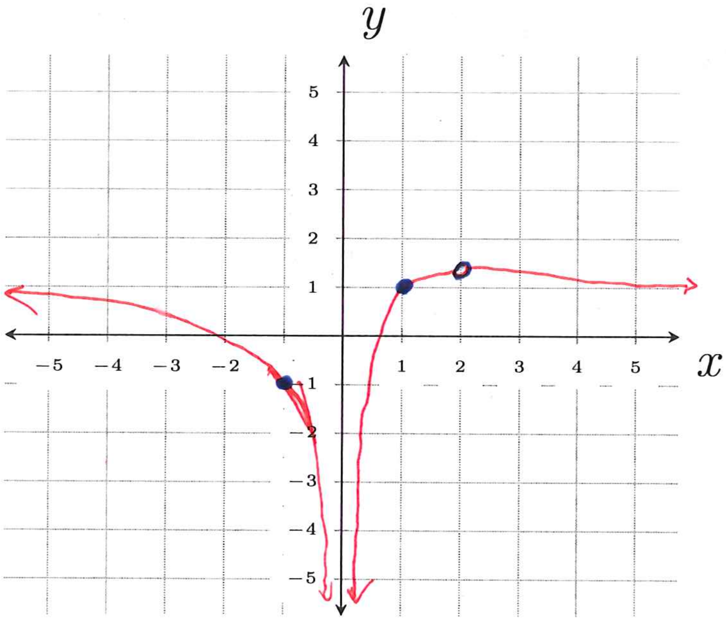
   $\lim_{x\to 2} f(x) = \frac{5}{4}$

   $\lim_{x\to \pm\infty} f(x) = 1$

   $f(2)$ DNE

   $f(1) = 1$

   $f(-1) = -1$



2. $\lim_{x\to -1^-} f(x) = \infty$

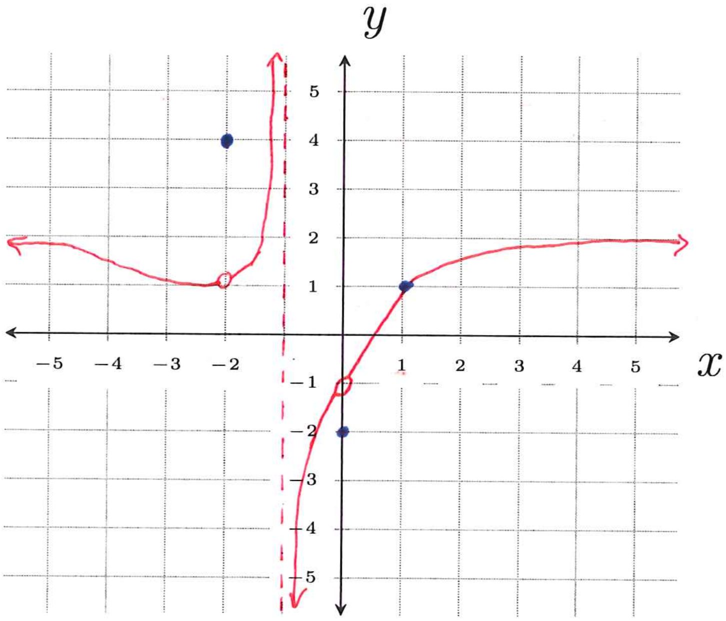
   $\lim_{x\to -1^+} f(x) = -\infty$

   $\lim_{x\to \pm\infty} f(x) = 2$

   $f(0) = -2$

   $f(1) = 1$

   $f(-2) = 4$



Example. Find *all* asymptotes (vertical, horizontal, slant)

1. $\dfrac{x^3 - 10x^2 + 16x}{x^2 - 8x}$

V.A. $x \neq 0, x \neq 8$

$\rightarrow \dfrac{\cancel{x}(\cancel{x-8})(x-2)}{\cancel{x}(\cancel{x-8})} = x-2;\ x \neq 0,\ x \neq 8$

No V.A. since cancellation

↑

degree in num = 3

degree in denom = 2

→ Slant asymptote

$$\begin{array}{r|l} & x - 2 \\ x^2 - 8x & x^3 - 10x^2 + 16x + 0 \\ & -(x^3 - 8x^2) \quad \downarrow \\ & \quad -2x^2 + 16x \\ & \quad -(-2x^2 + 16x) \\ & \qquad 0 \end{array}$$

$\boxed{\text{Slant asymptote } y = x - 2}$

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2. $\dfrac{\cos x + 2\sqrt{x}}{\sqrt{x}}$

V.A. $\sqrt{x} \Rightarrow x \geq 0$

$\dfrac{1}{\sqrt{x}} \Rightarrow x > 0$ ← only limit as $x \longrightarrow 0^+$

$$\lim_{x \to 0^+} \frac{\cos(x) + \sqrt{x}}{\sqrt{x}} = \frac{1 + 0}{sm+} = \infty \qquad \boxed{V.A.\ x = 0}$$

Note: $\cos(x)$ is not a polynomial function ⇒ doesn't affect the degree of numerator.

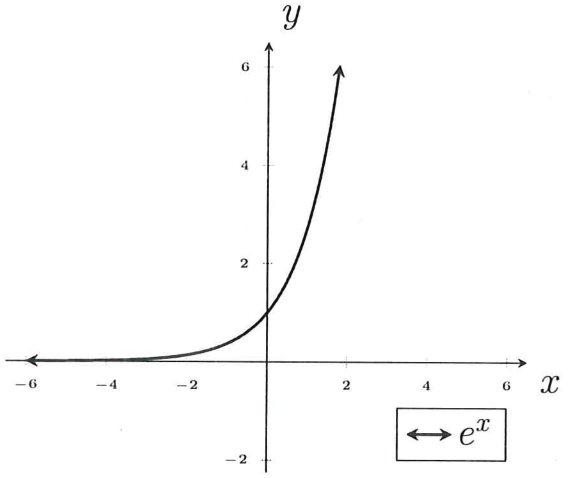
deg of num = $1/2$

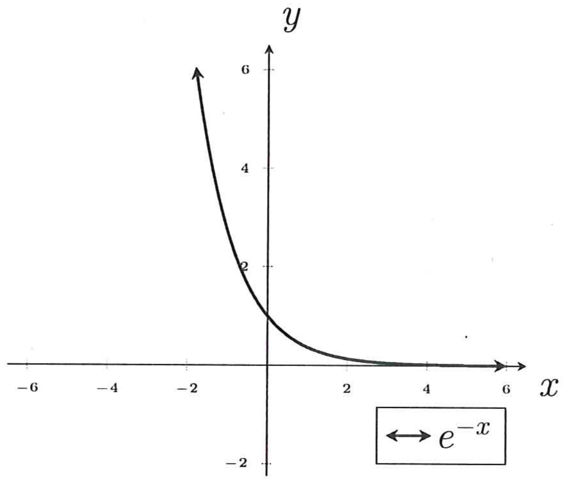
deg of denom = $1/2$ }

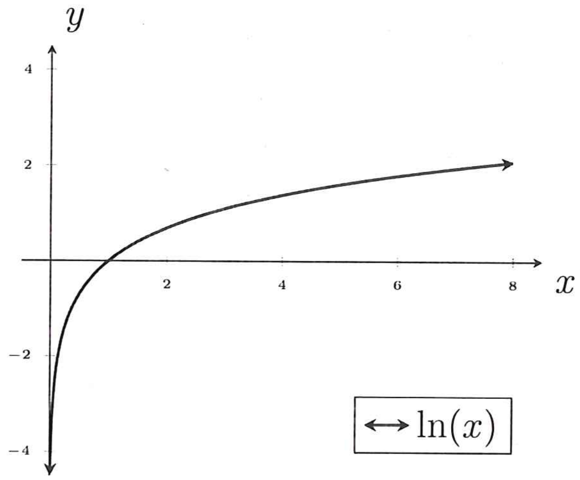
$$\frac{\cos x + 2\sqrt{x}}{\sqrt{x}}\left(\frac{1/\sqrt{x}}{1/\sqrt{x}}\right) \rightarrow \lim_{x \to \infty} \frac{\frac{\cos(x)}{\sqrt{x}} + 2}{1} = \frac{0 + 2}{1} = \boxed{2}$$

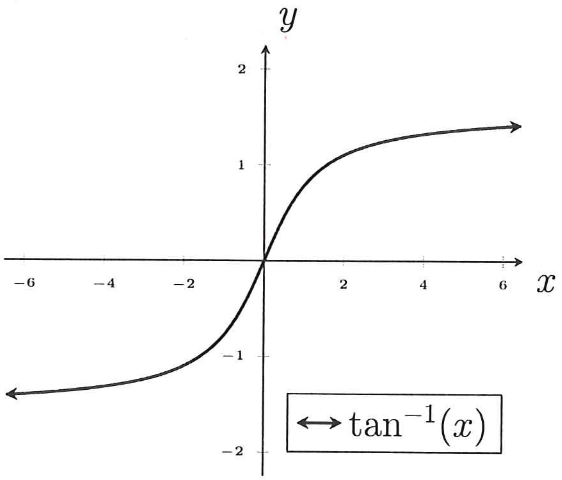
$\boxed{H.A.\ y = 2}$

Other function end behavior to consider include $e^x, e^{-x}, \ln(x)$ and $\tan^{-1}(x)$:









a) $\lim\limits_{x\to-\infty} \sin x$ DNE

b) $\lim\limits_{x\to\infty} \sin x$ DNE

c) $\lim\limits_{x\to-\infty} \cos x$ DNE

d) $\lim\limits_{x\to\infty} \cos x$ DNE