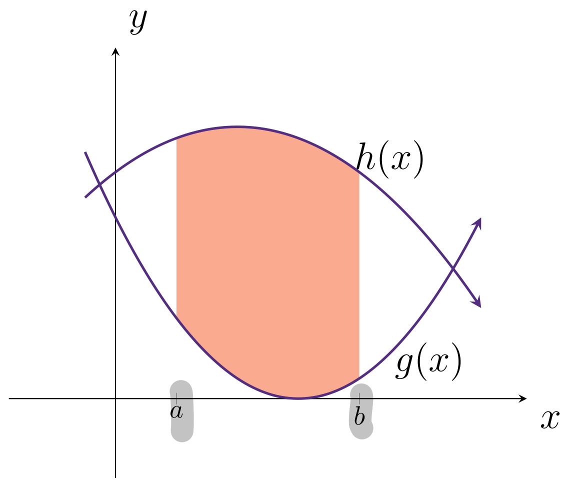
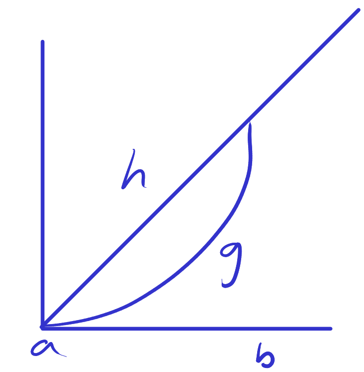
16.2: Double Integrals over General Regions

In this section, we consider double integrals over non-rectangular regions. For instance, my domain for x and y can be constrained where $a \leq x \leq b$ and $g(x) \leq y \leq h(x)$:





$a \leq x \leq b$
$g \leq y \leq h$

Theorem 16.2: Double Integrals over Nonrectangular Regions
Let R be a region bounded below and above by the graphs of the continuous functions $y = g(x)$ and $y = h(x)$, respectively, and by the lines $x = a$ and $x = b$. If f is continuous on R, then

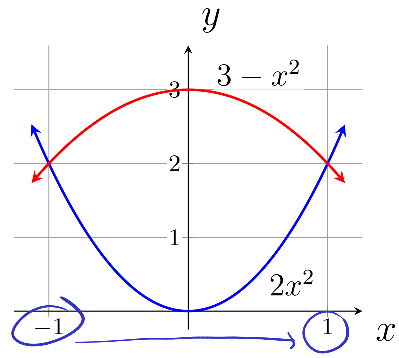
$$\iint_R f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx.$$

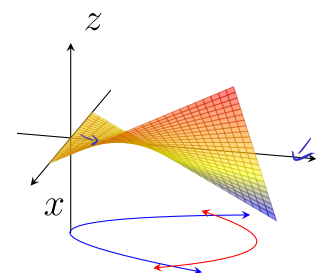
Let R be a region bounded on the left and right by the graphs of the continuous functions $x = g(y)$ and $x = h(y)$, respectively, and the lines $y = c$ and $y = d$. If f is continuous on R, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy.$$

Example. Consider the surface generated by the function $f(x,y) = 3xy$. Find the volume of the solid generated by $f(x,y)$ over the region bounded by $2x^2$ and $3-x^2$.

$$R = \{(x,y) : 2x^2 \le y \le 3-x^2,\ -1 \le x \le 1\}$$





$$\int_{-1}^{1}\int_{2x^2}^{3-x^2} 3xy\, dy\, dx$$

$$= \int_{-1}^{1} \frac{3}{2}xy^2 \Big|_{y=2x^2}^{y=3-x^2} dx$$

$$= \int_{-1}^{1} \frac{3}{2}x(3-x^2)^2 - \frac{3}{2}x(2x^2)^2\, dx$$

$$= \frac{3}{2}\int_{-1}^{1} x(x^4-6x^2+9) - 4x^5\, dx$$

$$= \frac{3}{2}\int_{-1}^{1} -3x^5 - 6x^3 + 9x\, dx$$

LC#1

$$= \frac{3}{2}\left(-\frac{1}{2}x^6 - \frac{3}{2}x^4 + \frac{9}{2}x^2 \Big|_{x=-1}^{x=1}\right) = \frac{3}{2}\left[\left(-\frac{1}{2}-\frac{3}{2}+\frac{9}{2}\right) - \left(-\frac{1}{2}-\frac{3}{2}+\frac{9}{2}\right)\right] = 0$$

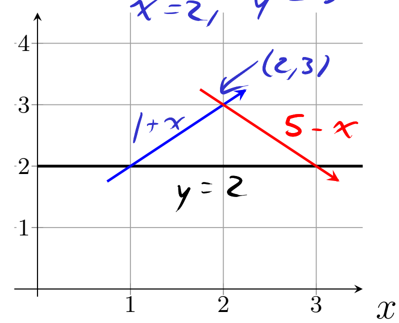
Area above and below
xy-plane cancel out

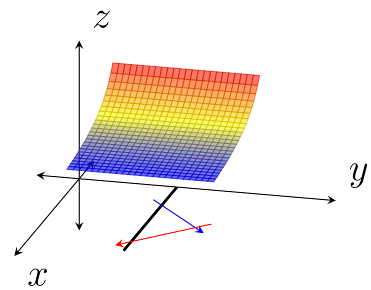
Example. Find the area under $f(x, y) = \frac{1}{x} + 1$ over the region formed by the lines $y = 2$, $y = 1 + x$, and $y = 5 - x$.

$1 + x = 5 - x$
$2x = 4$
$x = 2, \quad y = 3$

$1 + x = y$
$x = y - 1$

$5 - x = y$
$x = 5 - y$





$$R = \{(x, y) : y - 1 \le x \le 5 - y,\ 2 \le y \le 3\}$$

$$\int_2^3 \int_{y-1}^{5-y} \frac{1}{x} + 1 \, dx \, dy$$

$$= \int_2^3 \ln(x) + x \Big|_{x=y-1}^{x=5-y} dy$$

$$= \int_2^3 \left(\ln(5-y) + 5 - y\right) - \left(\ln(y-1) + y - 1\right) dy$$

$$= \int_2^3 \ln(5-y) - \ln(y-1) - 2y + 6 \, dy$$

IBP

$$= (5+y)\ln(5-y) + 5 - y + (y-1)\ln(y-1) - (y-1) - y^2 + 6y \Big|_{y=2}^{y=3}$$

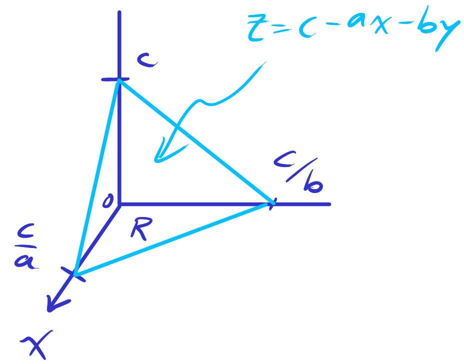
$$= (10\ln(2) + 9) - (7\ln(3) + 10)$$ ew...

LC #2

Put 0

Example. Find the volume of the tetrahedron in the first octant bounded by the plane $f(x,y) = z = c - ax - by$ and the coordinate planes ($x = 0$, $y = 0$, and $z = 0$). Assume a, b, and c are positive real numbers.

$$R = \left\{(x,y) : 0 \le y \le \frac{c}{b} - \frac{a}{b}x,\ 0 \le x \le \frac{c}{a}\right\}$$



$$\int_0^{c/a} \int_0^{\frac{c}{b} - \frac{a}{b}x} c - ax - by \, dy \, dx$$

$$= \int_0^{c/a} (c - ax)y - \frac{b}{2}y^2 \Big|_{y=0}^{y = \frac{c}{b} - \frac{a}{b}x} dx$$

$$m = \frac{0 - c/b}{\frac{c}{a} - 0} = -\frac{a}{b}$$

$$y = -\frac{a}{b}x + \frac{c}{b}$$

$$= \int_0^{c/a} \frac{(c-ax)^2}{b} - \frac{b}{2}\left(\frac{c-ax}{b}\right)^2 dx$$

$$= \int_0^{c/a} \frac{(c-ax)^2}{2b} \, dx$$

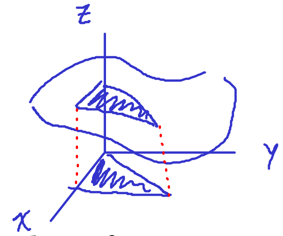
$$u = c - ax \qquad x = 0,\ u = c$$

$$du = -a \, dx \qquad x = \frac{c}{a},\ u = 0$$

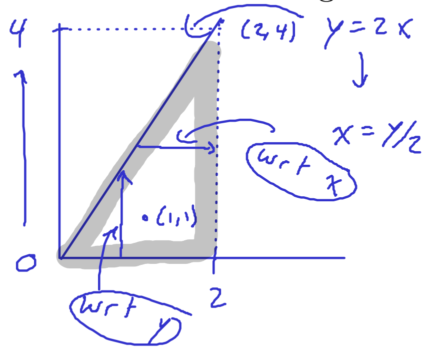
$$-\frac{1}{a} du = dx$$

$$= -\int_c^0 \frac{u^2}{2ab} du = \int_0^c \frac{u^2}{2ab} du$$

$$= \frac{u^3}{6ab}\Big|_{u=0}^{u=c} = \boxed{\frac{c^3}{6ab}} = \frac{1}{3} \underbrace{\frac{c^2}{2ab}}_{\text{area base}} \cdot \underbrace{c}_{\text{height}}$$



Example. For the following problems, reverse the order of integration



(1,1)

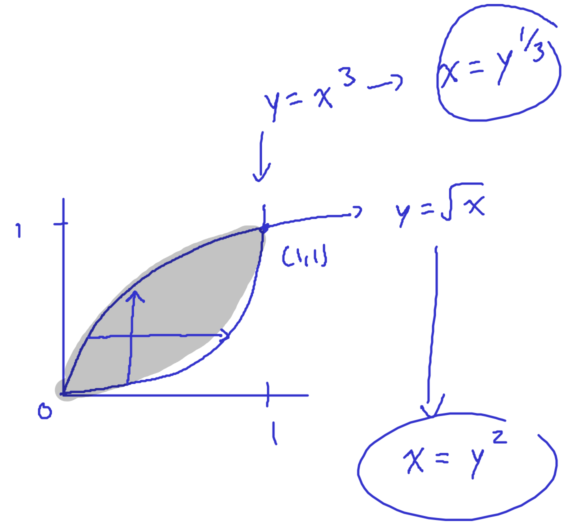
$0 \le 1 \le 2$

- $\displaystyle\int_0^2 \int_0^{2x} f(x,y)\, dy\, dx$

$R = \{(x,y) : 0 \le x \le 2,\ 0 \le y \le 2x\}$

$= \{(x,y) : \frac{y}{2} \le x \le 2,\ 0 \le y \le 4\}$

$$\int_0^4 \int_{y/2}^{2} f(x,y)\, dx\, dy$$

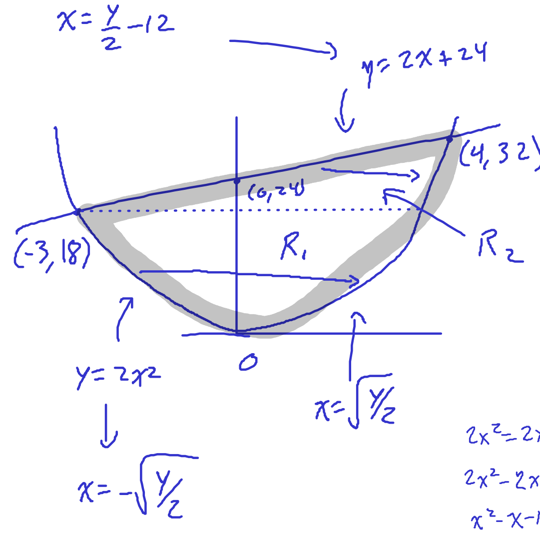


- $\displaystyle\int_0^1 \int_{x^3}^{\sqrt{x}} f(x,y)\, dy\, dx$

$R = \{(x,y) : 0 \le x \le 1,\ x^3 \le y \le \sqrt{x}\}$

$= \{(x,y) : y^2 \le x \le y^{1/3},\ 0 \le y \le 1\}$

$$\int_0^1 \int_{y^2}^{y^{1/3}} f(x,y)\, dx\, dy$$



- $\displaystyle\int_{-3}^4 \int_{2x^2}^{2x+24} f(x,y)\, dy\, dx$

$R = \{(x,y) : -3 \le x \le 4,\ 2x^2 \le y \le 2x+24\}$

$R_1 = \{(x,y) : -\sqrt{y/2} \le x \le \sqrt{y/2},\ 0 \le y \le 18\}$

$R_2 = \{(x,y) : \frac{y}{2} - 12 \le x \le \sqrt{y/2},\ 18 \le y \le 32\}$

$$\int_0^{18} \int_{-\sqrt{y/2}}^{\sqrt{y/2}} f(x,y)\, dx\, dy + \int_{18}^{32} \int_{\frac{y}{2}-12}^{\sqrt{y/2}} f(x,y)\, dx\, dy$$

$2x^2 = 2x + 24$

$2x^2 - 2x - 24 = 0$

$x^2 - x - 12 = 0$

$(x-4)(x+3) = 0$

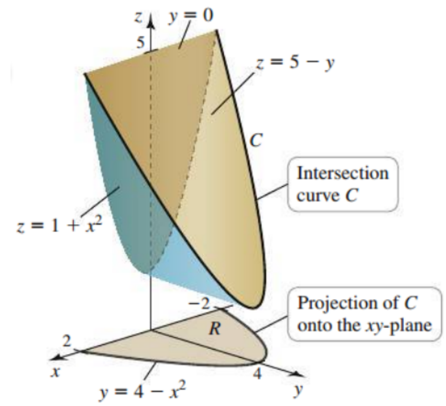
$x = 4$

$x = -3$

$$\int_0 \left(\int_{-\sqrt{y/2}}^{\sqrt{y/2}} f(x,y)\, dx + \int_{\frac{y}{2}-12}^{\sqrt{y/2}} f(x,y)\, dx \right) dy$$

$$\int f-g \; dx$$

Example. Find the volume between $f(x, y) = 5 - y$ and $g(x, y) = 1 + x^2$ over the region $R = \{(x, y) : 0 \le y \le 4 - x^2,\ -2 \le x \le 2\}$.



$$\int_{-2}^{2}\int_{0}^{4-x^2} f(x,y) - g(x,y)\, dy \; dx$$

$$= \int_{-2}^{2}\int_{0}^{4-x^2} (5-y) - (1+x^2)\, dy \; dx$$

$$= \int_{-2}^{2}\int_{0}^{4-x^2} 4 - y - x^2 \; dy \; dx$$

$$= \int_{-2}^{2} \underbrace{y(4-x^2)}_{(4-x^2)^2} - \underset{\frac{(4-x^2)^2}{2}}{\frac{y^2}{2}} \Bigg|_{y=0}^{y=4-x^2} dx$$

$$= \int_{-2}^{2} \frac{(4-x^2)^2}{2} \, dx$$

$$= \frac{1}{2}\int_{-2}^{2} 16 - 8x^2 + x^4 \, dx$$

$$= \frac{1}{2}\left(16x - \frac{8}{3}x^3 + \frac{x^5}{5}\right)\Bigg|_{x=-2}^{x=2}$$

$$= \frac{1}{2}\left[\left(32 - \frac{64}{3} + \frac{32}{5}\right) - \left(-32 + \frac{64}{3} - \frac{32}{5}\right)\right]$$

$$= 16\left(\frac{12}{5} - \frac{4}{3}\right) = 64\left(\frac{3}{5} - \frac{1}{3}\right) = 64\left(\frac{4}{15}\right) = \frac{256}{15}$$

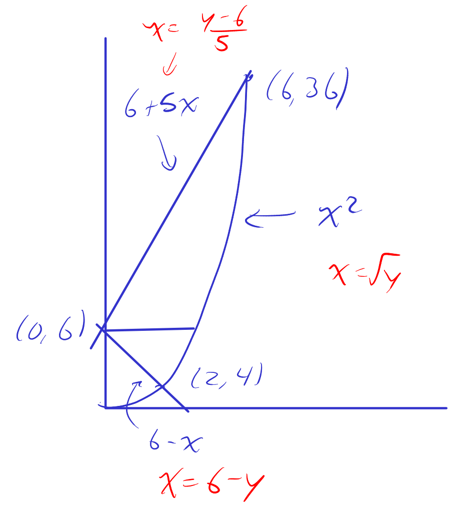
Areas of Regions by Double Integrals
Let R be a region in the xy-plane. Then

$$\text{area of } R = \iint_R dA.$$

Example. Find the area of the region R bounded by $y = x^2$, $y = 6 - x$, and $y = 6 + 5x$ where $x \geq 0$.

$x^2 = 6 - x \rightarrow x^2 + x - 6 = 0$
$(x+3)(x-2) = 0$
$x = -3$
$x = 2$

$x^2 = 6 + 5x \rightarrow x^2 - 5x - 6 = 0$
$(x-6)(x+1) = 0$
$x = 6$
$x = -1$



$R_1 = \{(x,y) : 0 \leq x \leq 2, \ 6 - x \leq y \leq 6 + 5x\}$

$R_2 = \{(x,y) : 2 \leq x \leq 6, \ x^2 \leq y \leq 6 + 5x\}$

$$\text{area} = \int_0^2 \int_{6-x}^{6+5x} dy\, dx + \int_2^6 \int_{x^2}^{6+5x} dy\, dx$$

$$= \int_0^2 y \Big|_{y=6-x}^{y=6+5x} dx + \int_2^6 y \Big|_{y=x^2}^{y=6+5x} dx$$

$$= \int_0^2 6x\, dx + \int_2^6 6 + 5x - x^2\, dx$$

$$= 3x^2 \Big|_{x=0}^{x=2} + \left(6x + \tfrac{5}{2}x^2 - \tfrac{1}{3}x^3\right)\Big|_{x=2}^{x=6}$$

$$= 12 + \left(36 + 5 \cdot 18 - \tfrac{1}{3} 6^3\right) - \left(12 + 10 - \tfrac{8}{3}\right)$$

CC 4

typo $\rightarrow$ 36

$$= 36 + 90 - 72 - 10 + 8/3$$

$$= 44 + 8/3 = \frac{140}{3} = \boxed{46.\overline{6}}$$

IBP

$$\int_a^b x e^x \, dx = x e^x \Big|_a^b - \int_a^b e^x \, dx$$

$u = x \qquad v = e^x$

$du = dx \qquad dv = e^x dx$

$$\int u \, dv = uv - \int v \, du$$

Max/min $f(x,y)$ → $\nabla f(x,y) = \vec{0}$ (x_0, y_0)

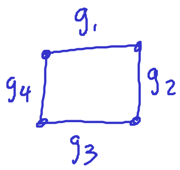
2nd deriv test

$D(x_0,y_0) < 0$ S.P.

$D(x_0,y_0) > 0$ → $f_{xx}(x_0,y_0) > 0$ min

→ $f_{xx}(x_0,y_0) < 0$ max

Over Region →



Take deriv of g_1, g_2, g_3, g_4 → Find crit pts

Also check end points

	(x,y)	$f(x,y)$
gradient c.p. →		
boundary c.p. →		
region end points →		

Lagrange multiplies (Teitloff exam, hw 8 #1)

Tangent planes $F(x,y,z) = 0$ vs. $z = f(x,y)$