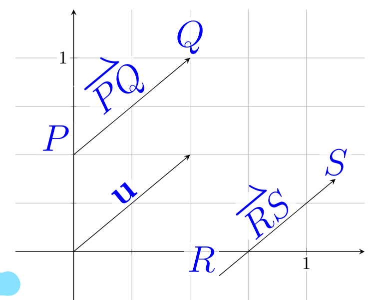
13.1: Vectors and the Geometry of Space

Definition.

- **Vectors**
  - Have a direction and magnitude,
  - vector $\overrightarrow{PQ}$ has a *tail* at P and a *head* at Q,
  - Can be denoted as $\mathbf{u}$ or $\vec{u}$, $\vec{u}$
  - Equal vectors have the same direction and magnitude (not necessarily the same position)

$\overrightarrow{PQ} = \overrightarrow{RS} = \vec{u}$



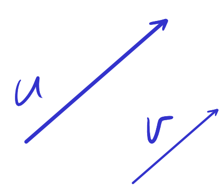
- **Scalars** are quantities with magnitude but no direction (e.g. mass, temperature, price, time, etc.)
- **Zero vector**, denoted $\mathbf{0}$ or $\vec{0}$, has length 0 and no direction

scalar $\downarrow$ $2\vec{u}$

$0 \quad \vec{0}$

Scalar-vector multiplication:

- Denoted $c\mathbf{v}$ or $c\vec{v}$, $c\vec{v}$ ← another vector
- length of vector multiplied by $|c|$, length is always positive
- $c\mathbf{v}$ has the same direction as $\mathbf{v}$ if $c > 0$, and has the opposite direction as $\mathbf{v}$ if $c < 0$, (what if $c = 0$?)
- $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if $\mathbf{u} = c\mathbf{v}$. (what vectors are parallel to $\mathbf{0}$?)



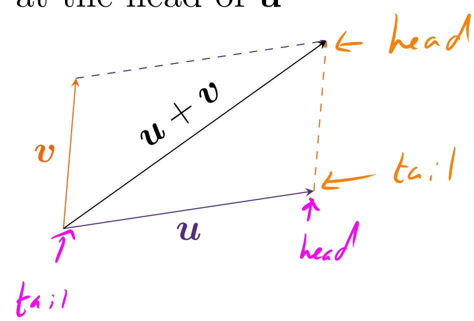
$\vec{0} = 0\vec{u}$

all vectors

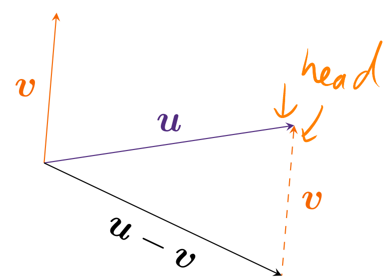
$v = \frac{1}{2}u$

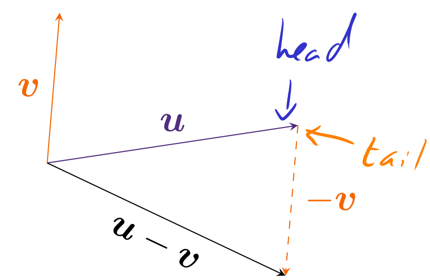
Vector Addition and Subtraction:

Given two vectors $\mathbf{u}$ and $\mathbf{v}$, their sum, $\mathbf{u}+\mathbf{v}$, can be represented by the parallelogram (triangle) rule: place the tail of $\mathbf{v}$ at the head of $\mathbf{u}$



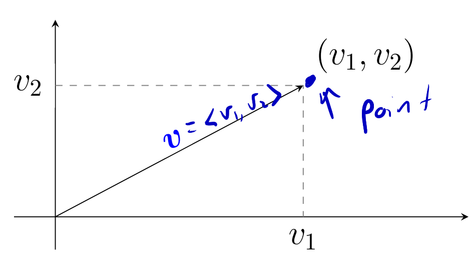
The difference, denoted $\mathbf{u}-\mathbf{v}$, is the sum of $\mathbf{u}+(-\mathbf{v})$:





Vector Components:

A vector $\mathbf{v}$ whose tail is at the origin $(0,0)$ and head is at (v_1, v_2) is a **position vector** (in **standard position**) and is denoted $\boxed{\langle v_1, v_2\rangle.}$ The real numbers v_1 and v_2 are the x- and y-components of $\mathbf{v}$.



Vectors $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ are equal if and only if $u_1 = v_1$ and $u_2 = v_2$.

$\mathbf{u} = c\mathbf{v}.$

$\vec{u} = \langle 1, 1\rangle$

$\langle 2, 2\rangle \leftarrow 2\cdot\vec{u}$

$\langle -1, -1\rangle \leftarrow -1\cdot\vec{u}$

$\langle 0, 0\rangle \leftarrow 0\cdot\vec{u}$

Not $\left\{ \begin{array}{l} \langle -1, 1\rangle \\ \langle 1, -1\rangle \end{array} \right.$

Magnitude:

Given points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the **magnitude**, or **length**, of vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$, denoted $\left|\overrightarrow{PQ}\right|$, is the distance between points P and Q.

$$\longrightarrow \left|\overrightarrow{PQ}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$P = (4, 1)$

$Q = (7, 5)$

$\langle 7-4, 5-1 \rangle$

The magnitude of position vector $\mathbf{v} = \langle v_1, v_2 \rangle$ is $|\mathbf{v}|$.

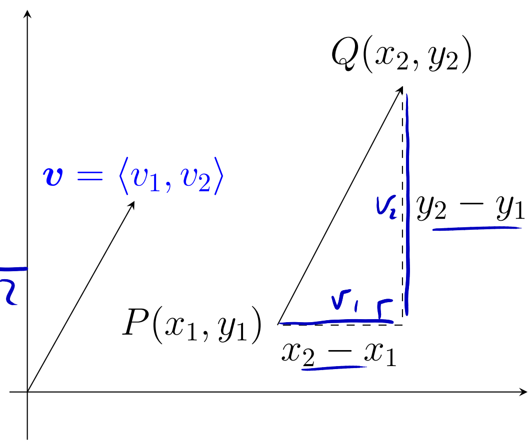
(How do $\left|\overrightarrow{PQ}\right|$ and $\left|\overrightarrow{QP}\right|$ relate to each other?)

$\overrightarrow{PQ} = -\overrightarrow{QP}$

equal

$|\overrightarrow{PQ}| = |\overrightarrow{QP}|$

$|\overrightarrow{QP}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$



$\vec{v} = \overrightarrow{PQ}$

$u = \langle 3, -4 \rangle$

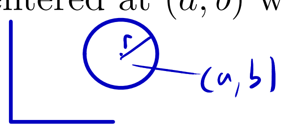
$|u| = \sqrt{(3)^2 + (-4)^2}$

$= 5$

Note: The norm, denoted $\|\mathbf{u}\|$ or $\|\mathbf{u}\|_2$, is equivalent to the magnitude of a vector.

Equation of a Circle:

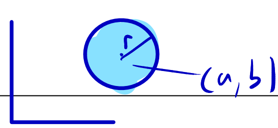
Definition.
A **circle** centered at (a, b) with radius r is the set of points satisfying the equation



$$(x - a)^2 + (y - b)^2 = r^2.$$

$r = \sqrt{(x-a)^2 + (y-b)^2}$

A **disk** centered at (a, b) with radius r is the set of points satisfying the inequality



$$(x - a)^2 + (y - b)^2 \leq r^2.$$

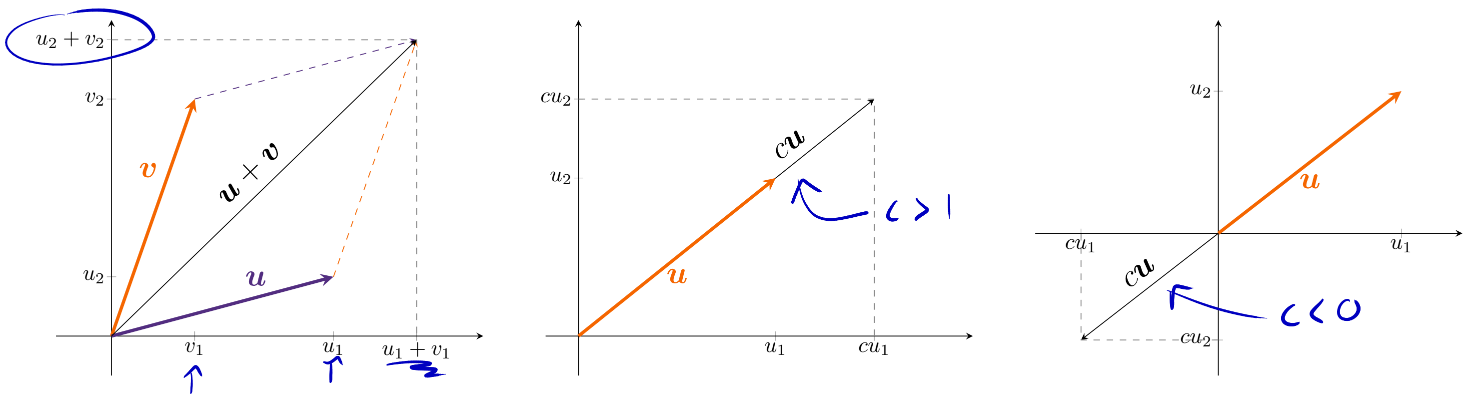
Vector Operations in Terms of Components

Definition. (Vector Operations in $\mathbb{R}^2$)
Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2 \rangle$, and $\mathbf{v} = \langle v_1, v_2 \rangle$.

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle \quad \text{Vector addition}$$
$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2 \rangle \quad \text{Vector subtraction}$$
$$c\mathbf{u} = \langle cu_1, cu_2 \rangle \quad \text{Scalar multiplication}$$

$u + (-v) = \langle u_1 + (-v_1), u_2 + (-v_2) \rangle$

$2u = \langle 2u_1, 2u_2 \rangle$



Example. Let $\mathbf{u} = \langle 1, 2 \rangle$, $\mathbf{v} = \langle -2, 3 \rangle$, $c = 2$, and $d = 3$. Find the following:

$\mathbf{u} + \mathbf{v} = \langle 1 + (-2), 2 + 3 \rangle = \langle -1, 5 \rangle$

$c\mathbf{u} = 2\langle 1, 2 \rangle = \langle 2, 4 \rangle$

$c\mathbf{u} + d\mathbf{v} = \langle 2(1) + 3(-2), 2(2) + 3(3) \rangle = \langle -4, 13 \rangle$

$\mathbf{u} - c\mathbf{v} = \langle 1 - 2(-2), 2 - 2(3) \rangle = \langle 5, -4 \rangle$

Definition.
A **unit vector** is any vector with length 1.
In $\mathbb{R}^2$, the **coordinate unit vectors** are $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.

$\hat{i}$ $\hat{j}$

Example. Let $\mathbf{u} = \langle -7, 3 \rangle$. Find two unit vectors parallel to $\mathbf{u}$. Find another vector parallel to $\mathbf{u}$ with a magnitude of 2.

$\vec{u}_1 = c \cdot \vec{u}$

$|\vec{u}| = \sqrt{(-7)^2 + (3)^2} = \sqrt{58} \longrightarrow \vec{u}_1 = \frac{1}{\sqrt{58}} \vec{u} = \left\langle \frac{-7}{\sqrt{58}}, \frac{3}{\sqrt{58}} \right\rangle = \left\langle \frac{-7\sqrt{58}}{58}, \frac{3\sqrt{58}}{58} \right\rangle$

To find a parallel unit vector, divide by the magnitude

$|\vec{u}_1| = \sqrt{\left(\frac{-7}{\sqrt{58}}\right)^2 + \left(\frac{3}{\sqrt{58}}\right)^2}$

$\vec{u}_2 = -\frac{1}{\sqrt{58}} \vec{u} = \left\langle \frac{7}{\sqrt{58}}, \frac{-3}{\sqrt{58}} \right\rangle \rightarrow |\vec{u}_1| = |u_2| = 1$

$\vec{u}_3 = \frac{2}{\sqrt{58}} \vec{u} = \left\langle \frac{-7\sqrt{58}}{29}, \frac{3\sqrt{58}}{29} \right\rangle \rightarrow |\vec{u}_3| = 2$

$\rightarrow \quad |c \cdot \vec{u}| = |c| \cdot |\vec{u}|$

Properties of Vector Operations:

Suppose $\mathbf{u}$, $\mathbf{v}$, and $\boldsymbol{w}$ are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property of addition
2. $(\mathbf{u} + \mathbf{v}) + \boldsymbol{w} = \mathbf{u} + (\mathbf{v} + \boldsymbol{w})$	Associative property of addition
3. $\mathbf{v} + \mathbf{0} = \mathbf{v}$	Additive identity
4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$	Additive inverse
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property 1
6. $(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$	Distributive property 2
7. $0\mathbf{v} = \mathbf{0}$	Multiplication by zero scalar
8. $c\mathbf{0} = \mathbf{0}$	Multiplication by zero vector
9. $1\mathbf{v} = \mathbf{v}$	Multiplicative identity
10. $a(c\mathbf{v}) = (ac)\mathbf{v}$	Associative property of scalar multiplication

$\cancel{\vec{0}\,\vec{v}}$