10.6: Alternating Series

Theorem 10.16: Alternating Series Test
The alternating series $\sum(-1)^{k+1}a_k$ converges provided

1. the terms of the series are nonincreasing in magnitude ($0 < a_{k+1} \leq a_k$, for k greater than some index N) and
2. $\lim_{k\to\infty} a_k = 0$.

Example. Which of the following are considered alternating series?

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k+2} \qquad \sum_{k=4}^{\infty} \left(\frac{-3}{2}\right)^k \qquad \sum_{k=0}^{\infty} (-1)\left(\frac{1}{2}\right)^k \qquad \sum_{k=1}^{\infty} (-1)^{k+1}\left(\frac{1}{2}\right)^k$$

$$\sum_{k=-3}^{\infty} \frac{\cos(k\pi)}{(k+4)^2} \qquad \sum_{k=1}^{\infty} \frac{\sin(k)}{k^2} \qquad \sum_{k=0}^{\infty} (-1)^{k+1}\left(\frac{1}{-2}\right)^k$$

Example. Consider the series $\sum_{k=1}^{\infty}(-1)^k \frac{\sqrt{k}}{2k+3}$. Let a_k represent that magnitude of the terms of the given series.

- What is $\lim_{k\to\infty} a_k$?

- Compute $f'(x)$ where $f(k) = a_k$.

- Use the Alternating Series Test to determine if the given series converges.

Example. Does the series $\displaystyle\sum_{k=0}^{\infty}(-1)^{k+1}\left(\frac{4}{3}\right)^{k}$ converge?

Example. Does the series $\displaystyle\sum_{k=1}^{\infty}\cos(\pi k)e^{-k}$ converge?

Theorem 10.17: Alternating Harmonic Series
The alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges (even though the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges).

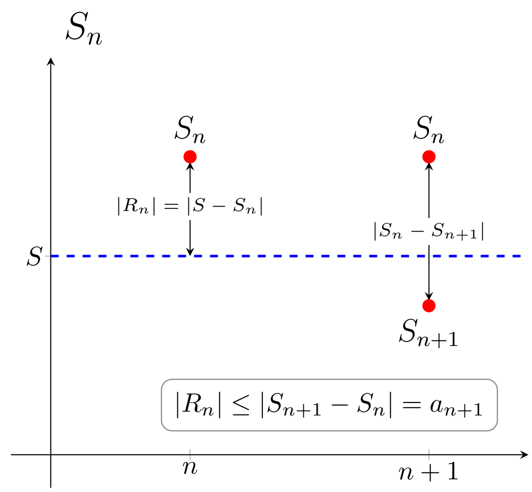
Example. Use the Alternating Series Test to show that the alternating harmonic series converges.

Theorem 10.18: Remainder in Alternating Series

Let $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ be a convergent alternating series with terms that are nonincreasing in magnitude. Let $R_n = S - S_n$ be the remainder in approximating the value of that series by the sum of its first n terms. Then $|R_n| \le a_{n+1}$. In other words, the magnitude of the remainder is less than or equal to the magnitude of the first neglected term.

Example. Find the minimum value of n such that $|R_n| < 10^{-4}$ for the following series:

$$\ln(2) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$



Definition. (Absolute and Conditional Convergence)
If $\sum |a_k|$ converges, then $\sum a_k$ **converges absolutely**.
If $\sum |a_k|$ diverges and $\sum a_k$ converges, then $\sum a_k$ **converges conditionally**.

Example. Can a series of strictly positive terms converge conditionally?

Example. Consider the series $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{4+k}{k^2}$. Determine if this series converges absolute, converges conditionally, or diverges.

Example. Determine if the following series converge absolute, converge conditionally, or diverge.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2\sqrt{k}-1}$$

$$\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^{k}$$

Theorem 10.19: Absolute Convergence Implies Convergence
If $\sum |a_k|$ converges, then $\sum a_k$ converges (absolute convergence implies convergence). Equivalently, if $\sum a_k$ diverges, then $\sum |a_k|$ diverges.