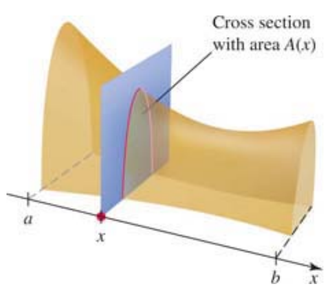
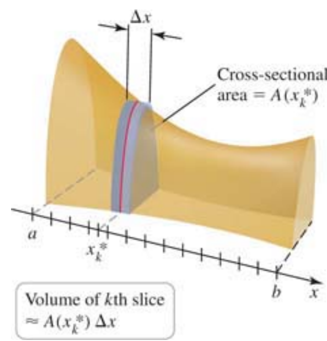
6.3: Volume by Slicing

General Slicing Method
Suppose a solid object extends from $x = a$ to $y = b$, and the cross section of the solid perpendicular to the x-axis has an area given by a function A that is integrable on $[a, b]$. The volume of the solid is

$$V = \int_a^b A(x)\,dx.$$





Example. Use the general slicing method to find the volume of the solid whose base is the region bounded by the semicircle $y = \sqrt{1 - x^2}$ and the x-axis, and whose cross sections through the solid perpendicular to the x-axis are squares.

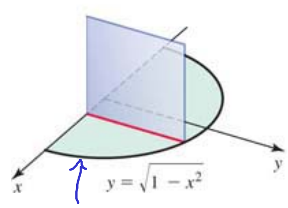
$A(x) = \left(\sqrt{1-x^2}\right)^2 = 1 - x^2$

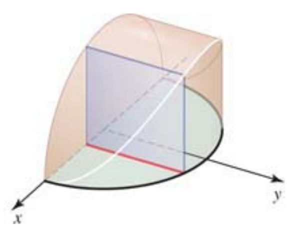
solve where $y=0$ (x-axis)

$\sqrt{1-x^2} = 0$

$1 = x^2$

$\pm 1 = x$





$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} 1 - x^2\,dx$$

semicircle

$\frac{\pi}{2} r^2$

$\frac{\pi}{2}\left(\frac{\sqrt{1-x^2}}{2}\right)^2$

$$= x - \frac{x^3}{3}\Big|_{-1}^{1} = \left(1 - \tfrac{1}{3}\right) - \left(-1 - \tfrac{-1}{3}\right)$$

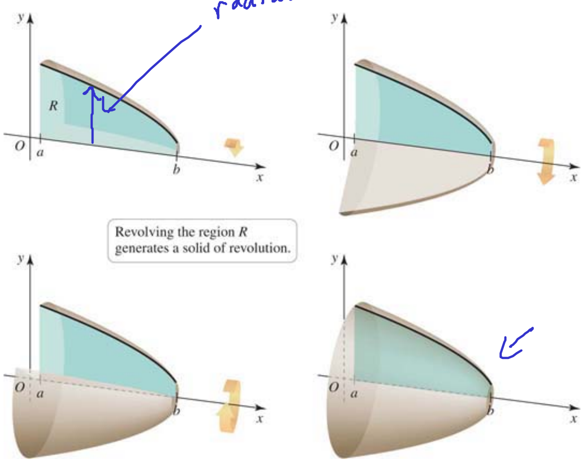
$$= 2 - \tfrac{2}{3} = 4/3$$

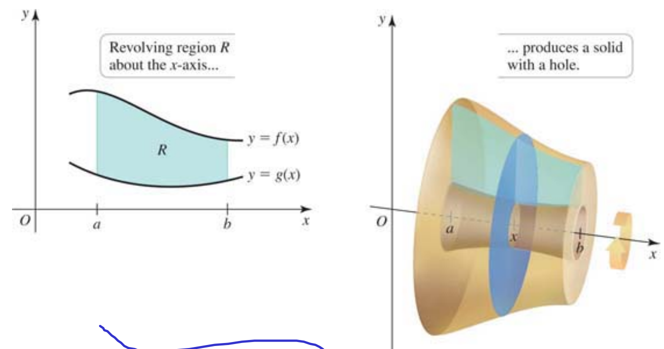
Disk Method about the x-Axis
Let f be continuous with $f(x) \geq 0$ on the interval $[a, b]$. If the region R bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi \underbrace{f(x)}_{\substack{\text{disk}\\\text{radius}}}{}^2 dx.$$

area of a circle πr^2

radius = f(x)





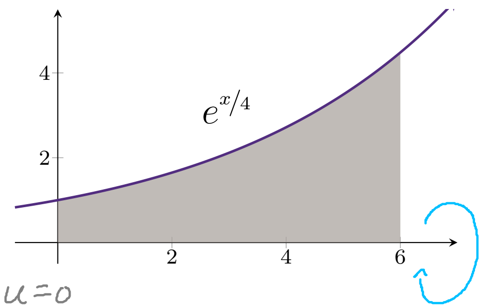
Washer Method about the x-Axis
Let f and g be continuous functions with $f(x) \geq g(x) \geq 0$ on $[a, b]$. Let R be the region bounded by $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$. When R is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi\left(\underbrace{f(x)}_{\substack{\text{outer}\\\text{radius}}}{}^2 - \underbrace{g(x)}_{\substack{\text{inner}\\\text{radius}}}{}^2\right) dx.$$

Disk method: $g(x) = 0$

$\underbrace{\pi R^2}_{\text{outer circle}} - \underbrace{\pi r^2}_{\text{inner circle}}$

Example. Consider the region bounded by $y = e^{x/4}$, $y = 0$, $x = 0$, and $x = 6$. Find the volume of the solid generated by rotating the region about the x-axis.



$$V = \int_0^6 \pi \left(e^{x/4}\right)^2 dx$$

$$e^{x/4} \cdot e^{x/4}$$

$$= \int_0^6 \pi e^{x/2} dx$$

$\frac{1}{2}x$

$u = x/2$

$du = \frac{1}{2} dx$

$2du = dx$

$x = 0, \quad u = 0$

$x = 6, \quad u = 3$

$$= \frac{\pi}{1/2} e^{x/2} \Big|_0^6$$

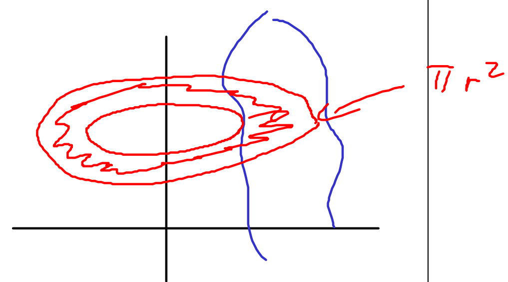
$$2\pi \int_0^3 e^u \, du = 2\pi e^u \Big|_0^3 = 2\pi\left(e^3 - 1\right)$$

$$= 2\pi\left(e^{6/2} - e^0\right) = \boxed{2\pi\left(e^3 - 1\right)}$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + c$$

Disk and Washer Methods about the y-Axis
Let p and q be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let R be the region bounded by $x = p(y)$, $x = q(y)$, and the lines $y = c$ and $y = d$. When R is revolved around the y-axis, the volume of the resulting solid of revolution is given by

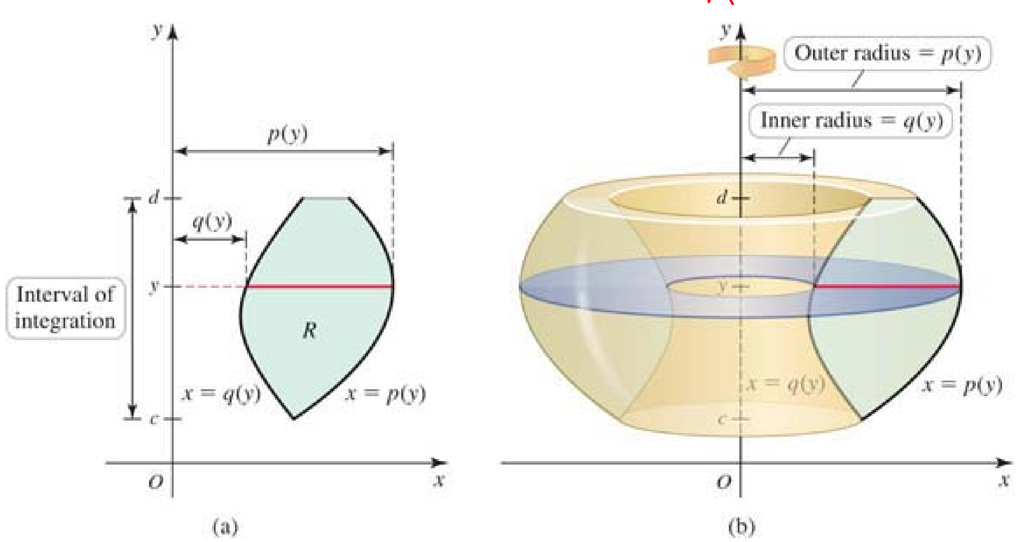
$$V = \int_c^d \pi(\underbrace{p(y)^2}_{\substack{\text{outer}\\\text{radius}}} - \underbrace{q(y)^2}_{\substack{\text{inner}\\\text{radius}}})\, dy.$$



If $q(y) = 0$, the disk method results:

$$V = \int_c^d \pi \underbrace{p(y)^2}_{\substack{\text{disk}\\\text{radius}}} dy.$$

$x=0$

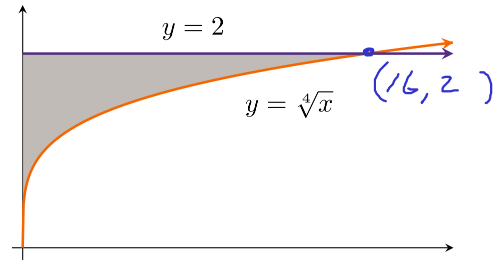


(a) (b)

Example. Consider the region bounded between $y = \sqrt[4]{x}$, $y = 2$, and $x = 0$.

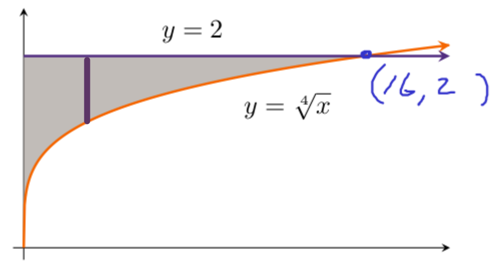
$\sqrt[4]{x} = 2$

$x = 16$



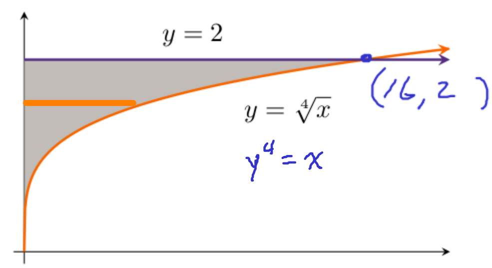
Setup the integral with respect to x that gives the area of the region.

$$A = \int_0^{16} 2 - \sqrt[4]{x} \, dx$$



Setup the integral with respect to y that gives the area of the region.

$$A = \int_0^{2} y^4 - 0 \, dy$$

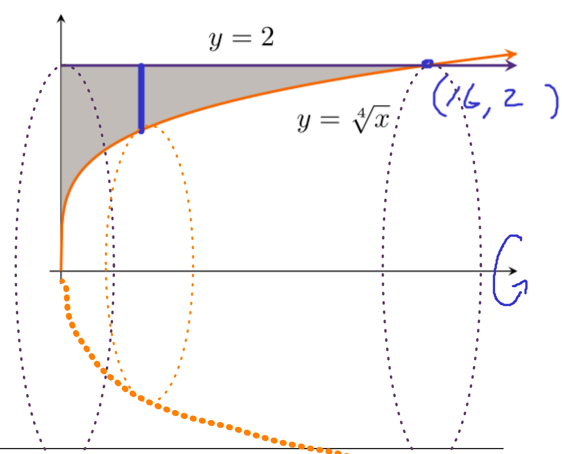


Use the disk/washer method to setup the that represents the volume of the solid generated by rotating the region about the x-axis.

$x^{2/4}$ $y=0$

$$V = \int_0^{16} \pi\left((2)^2 - (\sqrt[4]{x})^2\right) dx$$

$$= \int_0^{16} \pi(4 - \sqrt{x}) \, dx$$



$$V = \int_a^b \pi \left(f^2 - g^2\right) dx$$

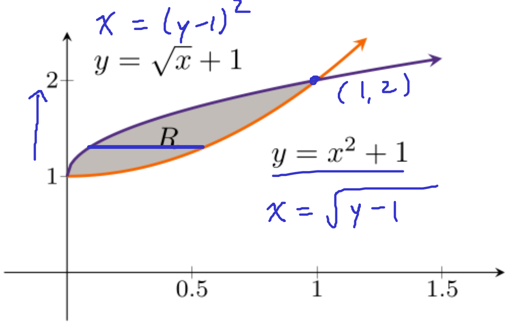
around x-axis: y=0

$$V = \int_a^b \pi \left((f-c)^2 - (g-c)^2\right) dx$$

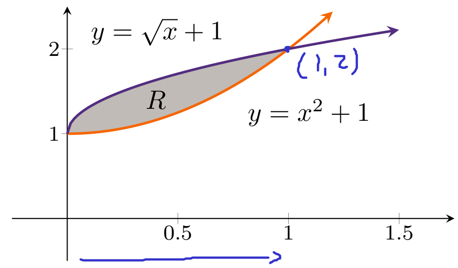
around y=c

Note: Revolving around a different axis may change your upper and lower functions!!

Example. Consider the region R between $y = \sqrt{x}+1$ and $y = x^2+1$. Setup the integrals which find the volume of the solid obtained by rotating the region R as indicated below.



about the y-axis $(x=0)$



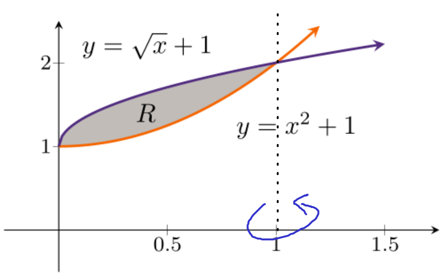
about the x-axis $(y=0)$

horizontal slices
need a func of y

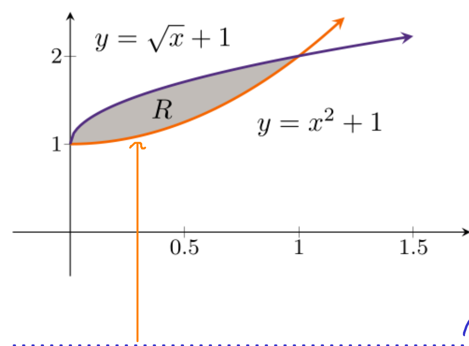
$$V = \int_1^2 \pi \left(\left(\sqrt{y-1}\right)^2 - \left((y-1)^2\right)^2 \right) dy$$

$$V = \int_0^1 \pi \left(\left(\sqrt{x}+1\right)^2 - \left(x^2+1\right)^2 \right) dx$$

about the line $x = 1$



about the line $y = -1$



$y=-1$

$$V = \int_1^2 \pi \left(\left(\underbrace{(y-1)^2 - 1}_{\leq 0}\right)^2 - \left(\underbrace{\sqrt{y-1}-1}_{\leq 0}\right)^2 \right) dy$$

$$V = \int_0^1 \pi \left(\left(\underbrace{\sqrt{x}+1-(-1)}_{\geq 0}\right)^2 - \left(\underbrace{x^2+1-(-1)}_{\geq 0}\right)^2 \right) dx$$

$$= \int_1^2 \pi \left(\left(\underbrace{1-(y-1)^2}_{\geq 0}\right)^2 - \left(\underbrace{1-\sqrt{y-1}}_{\geq 0}\right)^2 \right) dy$$