$x - 2y + 3z = 7 \longrightarrow$

$F(x, y, z) = x - 2y + 3z - 7 = 0$

15.6: Tangent Planes and Linear Approximation

$\nabla F(x, y, z) = \langle F_x, F_y, F_z \rangle$

Definition. (Equation of the Tangent Plane for $F(x, y, z) = 0$)
Let F be differentiable at the point $P_0(a, b, c)$ with $\nabla F(a, b, c) \neq \mathbf{0}$. The plane tangent to the surface $F(x, y, z) = 0$ at P_0, called the **tangent plane**, is the plane passing through P_0 orthogonal to $\nabla F(a, b, c)$. An equation of the tangent plane is

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

$$\nabla F(a, b, c) \cdot \langle x - a, y - b, z - c \rangle = 0$$

"normal vector"

Example. Consider the ellipsoid

$$F(x, y, z) = \frac{x^2}{9} + \frac{y^2}{25} + z^2 - 1 = 0.$$

a) Find an equation of the plane tangent to the ellipsoid at $(0, 4, \frac{3}{5})$.

$\nabla F(x, y, z) = \langle \frac{2x}{9}, \frac{2y}{25}, 2z \rangle$

$\nabla F(0, 4, 3/5) \cdot \langle x - 0, y - 4, z - 3/5 \rangle = 0$

$\Rightarrow \langle 0, 8/25, 6/5 \rangle \cdot \langle x - 0, y - 4, z - 3/5 \rangle = 0$

$\frac{8}{25}(y - 4) + \frac{6}{5}(z - 3/5) = 0$

$\Rightarrow 8(y - 4) + 30(z - 3/5) = 0 \longrightarrow 8y + 30z = 50$

$4y + 15z = 25$

LC #1 $c_1 = 0$

#2 $c_2 = 4$

#3 $c_3 = 15$

b) At what points on the ellipsoid is the tangent plane horizontal?

$F(x, y, z) = \frac{x^2}{9} + \frac{y^2}{25} + z^2 - 1 = 0$

$\Rightarrow$ normal vector $\langle 0, 0, c \rangle = \nabla F(x, y, z) = \langle \frac{2}{9}x, \frac{2}{25}y, 2z \rangle$

$\Rightarrow x = y = 0$

$\Rightarrow 0 + 0 + z^2 - 1 = 0 \longrightarrow (0, 0, 1)$

$\Rightarrow z = \pm 1 \longrightarrow (0, 0, -1)$

Surfaces of the form $z = f(x, y)$ are a special case of $F(x, y, z) = 0$:
Define $F(x, y, z) = z - f(x, y) = 0$, then

$$\nabla F(a, b, f(a, b)) = \langle -f_x(a, b),\ -f_y(a, b),\ 1\rangle$$

so the tangent plane is $\nabla F(a, b, f(a,b)) \cdot \langle x-a, y-b, z-f(a,b) \rangle = 0$

$$-f_x(a, b)(x - a) - f_y(a, b)(y - b) + 1(z - f(a, b)) = 0$$

Tangent Plane for $z = f(x, y)$
Let f be differentiable at the point (a, b). An equation of the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

tangent line

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b) \qquad y = f(a) + f'(a)(x-a)$$

$z = f(a,b) + \nabla f(a,b) \cdot \langle x-a, y-b \rangle$

Example. Find an equation of the plane tangent to $z = f(x, y) = 4e^{xy^2}$ at $(3, 0, 4)$ and $(0, 2, 4)$.

$\nabla f(x,y) = \langle 4y^2 e^{xy^2}, 8xye^{xy^2} \rangle$

$(3, 0, 4)$: $z = f(3,0) + \nabla f(3,0) \cdot \langle x-3, y-0 \rangle$

$= 4 + \langle 0, 0 \rangle \cdot \langle x-3, y-0 \rangle \longrightarrow z = 4$

LC #4 $c_1 = 0$
#5 $c_2 = 0$
#6 $c_3 = 4$

$(0, 2, 4)$: $z = f(0,2) + \nabla f(0,2) \cdot \langle x-0, y-2 \rangle$

$= 4 + \langle 16, 0 \rangle \cdot \langle x-0, y-2 \rangle$

$\rightarrow z = 4 + 16x$

LC #7 $c_1 = 16$
#8 $c_2 = 0$
#9 $c_3 = 4$

Example. Find an equation of the plane tangent to $f(x,y) = \tan^{-1}(xy)$ at $\left(\sqrt{3},\, 1,\, \frac{\pi}{3}\right)$ and $\left(\frac{\sqrt{3}}{3}, 1, \frac{\pi}{6}\right)$.

$z = f(a,b) + \nabla f(a,b) \cdot \langle x-a, y-b \rangle$

$\nabla f(x,y) = \left\langle \frac{y}{1+x^2y^2}, \frac{x}{1+x^2y^2} \right\rangle$

$(\sqrt{3}, 1, \pi/3)$: $z = f(\sqrt{3},1) + \nabla f(\sqrt{3},1) \cdot \langle x-\sqrt{3}, y-1 \rangle$

$= \pi/3 + \left\langle \frac{1}{4}, \frac{\sqrt{3}}{4} \right\rangle \cdot \langle x-\sqrt{3}, y-1 \rangle$

$= \frac{1}{4}x + \frac{\sqrt{3}}{4}y - \frac{\sqrt{3}}{2} + \frac{\pi}{3}$

$\frac{1}{1+1/3} = \frac{3}{4}$

$\frac{\sqrt{3}/3}{1+1/3} = \frac{\sqrt{3}}{4}$

$(\sqrt{3}/3, 1, \pi/6)$: $z = f\left(\frac{\sqrt{3}}{3}, 1\right) + \nabla f\left(\frac{\sqrt{3}}{3}, 1\right) \cdot \left\langle x - \sqrt{3}/3, y-1 \right\rangle$

$= \pi/6 + \left\langle \frac{1}{1+1/3}, \frac{\sqrt{3}/3}{1+1/3} \right\rangle \cdot \left\langle x - \frac{\sqrt{3}}{3}, y-1 \right\rangle$

$= \frac{3}{4}x - \frac{\sqrt{3}}{4}y - \frac{\sqrt{3}}{2} + \pi/6$

Definition. (Linear Approximation)
Let f be differentiable at (a,b). The linear approximation to the surface $z = f(x,y)$ at the point $(a,b,f(a,b))$ is the tangent plane at that point, given by the equation

$$L(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b),$$

For a function of three variables, the linear approximation to $w = f(x,y,z)$ at the point $(a,b,c,f(a,b,c))$ is given by

$$L(x,y,z) = f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) + f(a,b,c).$$

$= f(a,b,c) + \nabla f(a,b,c) \cdot \langle x-a, y-b, z-c \rangle$

$5(x^2+y^2)^{-1}$

Example. Let $f(x,y) = \dfrac{5}{x^2+y^2}$. Find the linear approximation to the function at the point $(-1,2,1)$. Use this to approximate $f(-1.05,2.1)$.

$$\nabla f(x,y) = \left\langle \frac{-10x}{(x^2+y^2)^2}, \frac{-10y}{(x^2+y^2)^2} \right\rangle$$

$$L(x,y) = f(-1,2) + \nabla f(-1,2) \cdot \langle x+1, y-2 \rangle$$

$$= 1 + \left\langle \frac{10}{25}, \frac{-20}{25} \right\rangle \cdot \langle x+1, y-2 \rangle$$

$$= \frac{2}{5}(x+1) - \frac{4}{5}(y-2) + 1$$

$$f(-1.05,2.1) \approx L(-1.05,2.1) = \frac{2}{5}(\underset{-5/100}{-0.05}) - \frac{4}{5}(\underset{1/10}{0.1}) + 1 = -\frac{1}{50} - \frac{4}{50} + 1 = 1 - 1/10 = 9/10$$

Example. Let $f(x,y) = \sqrt{x^2+y^2}$. Find the linear approximation to the function at the point $(-8,15,17)$. Use this to approximate $f(-7.91,14.96)$.

$$L(x,y) = f(-8,15) + \nabla f(-8,15) \cdot \langle x+8, y-15 \rangle$$

$$\nabla f(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

$$L(x,y) = 17 + \left\langle \frac{-8}{17}, \frac{15}{17} \right\rangle \cdot \langle x+8, y-15 \rangle$$

$$= -\frac{8}{17}(x+8) + \frac{15}{17}(y-15) + 17$$

$$f(-7.91,14.96) \approx L(-7.91,14.96) = \frac{-8}{17}(0.09) + \frac{15}{17}(-0.04) + 17$$

$$= \frac{-0.72 - 0.6}{17} + 17 = 17 - \frac{1.32}{17}$$

From (x_1, y_1) to (x_2, y_2) $\longrightarrow$ $dx = x_2 - x_1$

$dy = y_2 - y_1$

$$\Delta z = \underbrace{f(a+dx, y+dz)}_{L(a+dx,\, y+dz)} - f(a,b)$$

cancels

$$L(x,y) = f(a,b) + \nabla f(a,b) \cdot \langle x-a, y-b \rangle$$

Definition. (The differential dz)
Let f be differentiable at the point (x, y). The change in $z = f(x, y)$ as the independent variables change from (x, y) to $(x+dx, y+dy)$ is denoted Δz and is approximated by the differential dz:

$$\Delta z \approx dz = f_x(x, y)\, dx + f_y(x, y)\, dy. = \nabla f(x,y) \cdot \langle dx, dy \rangle$$

Example. Let $z = f(x, y) = \dfrac{5}{x^2 + y^2}$. Approximate the change in z when the variables change from $(-1, 2)$ to $(-0.93, 1.94)$. $\longrightarrow$ $dx = -0.93 + 1 = 0.07$

$dy = 1.94 - 2 = -0.06$

$$\nabla f(x,y) = \left\langle \frac{-10x}{(x^2+y^2)^2}, \frac{-10y}{(x^2+y^2)^2} \right\rangle$$

$$\Delta z = f(-0.93, 1.94) - f(-1,2)$$

$$\Delta z \simeq dz = \nabla f(-1,2) \cdot \langle 0.07, -0.06 \rangle$$

$$= \left\langle \frac{10}{25}, \frac{-20}{25} \right\rangle \cdot \left\langle \frac{7}{100}, \frac{-6}{100} \right\rangle$$

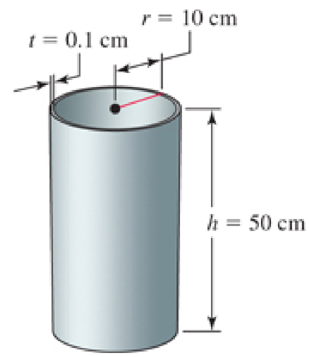
$$= \frac{\overset{1}{\cancel{10}}}{25} \cdot \frac{7}{\underset{10}{\cancel{100}}} + \frac{\overset{2}{\cancel{20}}}{25} \cdot \frac{6}{\underset{10}{\cancel{100}}}$$

$$= 19/250$$

Example. A company manufactures cylindrical aluminum tubes to rigid specifications. The tubes are designed to have an outside radius of $r = 10\ cm$, a height of $h = 50\ cm$, and a thickness of $t = 0.1\ cm$. The manufacturing process produces tubes with a maximum error of $\pm 0.05\ cm$ in the radius and height, and a maximum error of $\pm 0.0005\ cm$ in the thickness. The volume of the cylindrical tube is $V(r, h, t) = \pi h t(2r - t)$. Use differentials to estimate the maximum error in the volume of a tube.

dr dh

dt



$$\nabla V(r,h,t) = \langle 2\pi h t,\ \pi t(2r-t),\ \pi h(2r-t) + \pi h t(-1)\rangle$$

$$= \langle 2\pi h t,\ \pi t(2r-t),\ 2\pi h(r-t)\rangle$$

$1/10$

$$\Delta V \approx dV = \nabla V(10, 50, 0.1) \cdot \langle 0.05, 0.05, 0.0005\rangle$$

$2\pi(50)(1/10)$ $\pi(\frac{1}{10})(20 - 1/10) \to \frac{19.9\pi}{10}$ $2\pi\, 50(10 - 1/10) \to 100\pi(9.9)$

$$= \langle 10\pi,\ 1.99\pi,\ 990\pi\rangle \cdot \langle 5/100,\ 5/100,\ 5/10000\rangle$$

$$= \frac{\pi}{2} + \frac{9.95\pi}{100} + \frac{5 \cdot 990\pi}{10{,}000} \approx 3.44$$