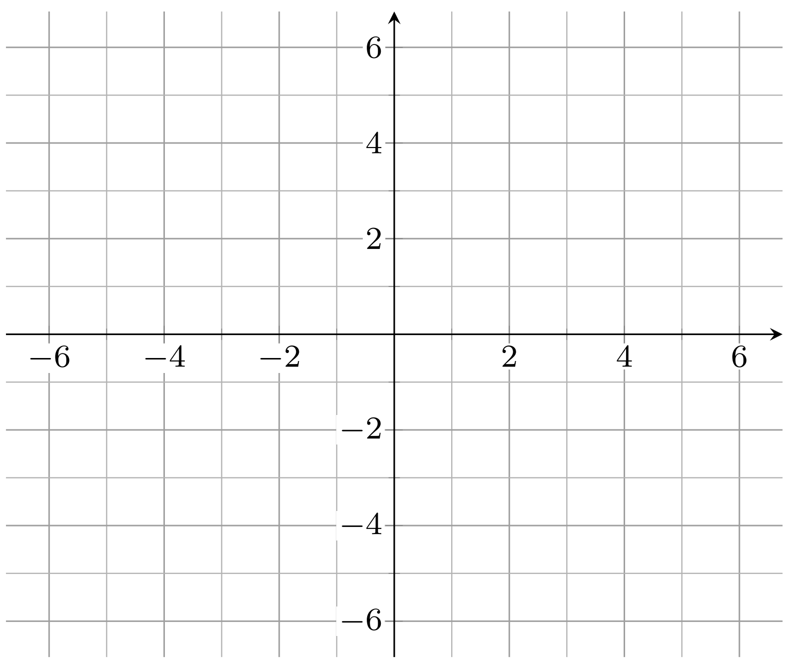
1 17.1: Vector Fields

Definition. (Vector Fields in Two Dimensions)
Let f and g be defined on a region R of $\mathbb{R}^2$. A **vector field** in $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point in R a vector $\langle f(x,y),\, g(x,y)\rangle$. The vector field is written as

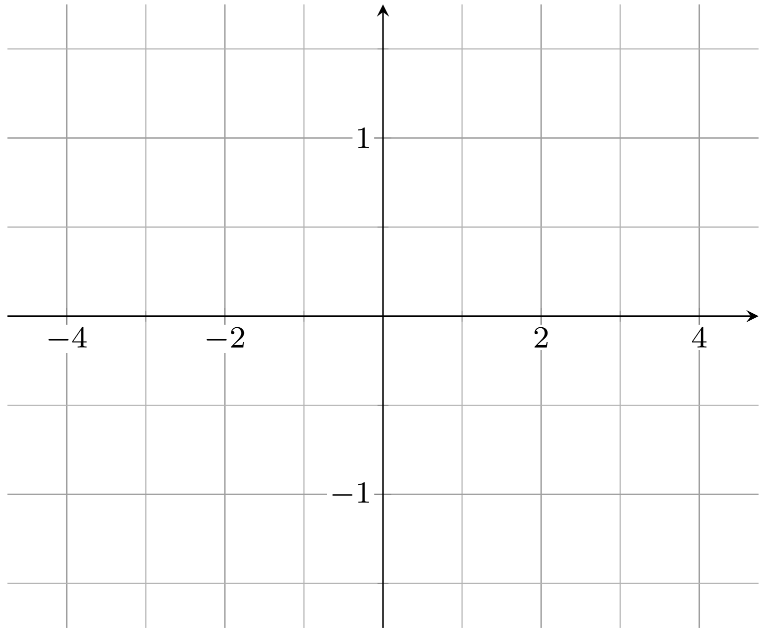
$$\mathbf{F}(x,y) = \langle f(x,y),\, g(x,y)\rangle \quad \text{or}$$
$$\mathbf{F}(x,y) = f(x,y)\boldsymbol{i} + g(x,y)\boldsymbol{j}.$$

A vector field $\mathbf{F} = \langle f, g\rangle$ is continuous or differentiable on a region R of $\mathbb{R}^2$ if f and g are continuous or differentiable on R, respectively.

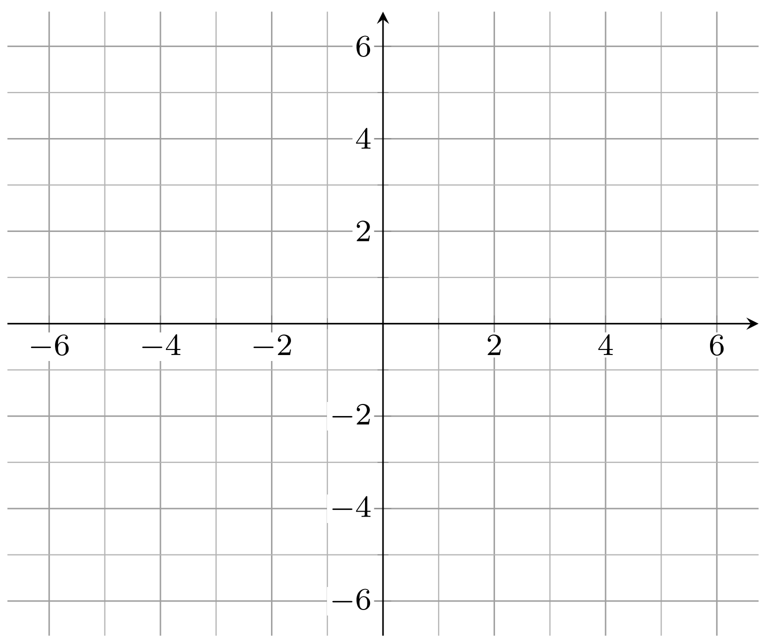
Example. Sketch the vector field $\mathbf{F} = \langle 0, x\rangle$.



Example. Sketch the vector field $\mathbf{F} = \left\langle 1 - y^2, 0 \right\rangle$ for $|y| \leq 1$.



Example. Sketch the vector field $\mathbf{F} = \langle -y, x \rangle$.



Definition. (Radial Vector Fields in $\mathbb{R}^2$)
Let $\mathbf{r} = \langle x, y \rangle$. A vector field of the form $\mathbf{F} = f(x,y)\mathbf{r}$, where f is a scalar valued function, is a **radial vector field**. Of specific interest are the radial vector fields

$$\mathbf{F}(x,y) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y \rangle}{|\mathbf{r}|^p} = \frac{\mathbf{r}}{|\mathbf{r}|}\,\frac{1}{|\mathbf{r}|^{p-1}},$$

where p is a real number. At every point (expect the origin), the vectors of this field are directed outward from the origin with a magnitude of $|\mathbf{F}| = \dfrac{1}{|\mathbf{r}|^{p-1}}$.

Example. Let C be the circle $x^2 + y^2 = a^2$, where $a > 0$.

a) Show that at each point of C, the radial vector field $\mathbf{F}(x,y) = \dfrac{\mathbf{r}}{|\mathbf{r}|} = \dfrac{\langle x, y \rangle}{\sqrt{x^2 + y^2}}$ is orthogonal to the line tangent to C at that point.

b) Show that at each point of C, the rotation vector field $\mathbf{G}(x,y) = \dfrac{\langle -y, x \rangle}{\sqrt{x^2 + y^2}}$ is parallel to the line tangent to C at that point.

Definition. (Vector Fields and Radial Vector Fields in $\mathbb{R}^3$)
Let f, g, and h be defined on a region D of $\mathbb{R}^3$. A **vector field** in $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point in D a vector $\langle f(x,y,z),\, g(x,y,z),\, h(x,y,z)\rangle$. The vector field is written as

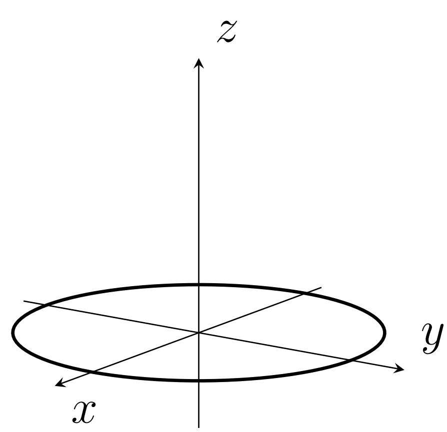
$$\mathbf{F}(x,y,z) = \langle f(x,y,z),\, g(x,y,z),\, h(x,y,z)\rangle \quad \text{or}$$
$$\mathbf{F}(x,y,z) = f(x,y,z)\boldsymbol{i} + g(x,y,z)\boldsymbol{j} + h(x,y,z)\boldsymbol{k}.$$

A vector field $\mathbf{F} = \langle f, g, h\rangle$ is continuous or differentiable on a region D of $\mathbb{R}^3$ if f, g, and h are continuous or differentiable on D, respectively. Of particular importance are the **radial vector fields**

$$\mathbf{F}(x,y,z) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x,y,z\rangle}{|\mathbf{r}|^p},$$

where p is a real number.

Example. Sketch the vector field $\mathbf{F}(x,y,z) = \left\langle 0, 0, 1 - x^2 - y^2\right\rangle$, for $x^2 + y^2 \leq 1$.



Definition. (Gradient Fields and Potential Functions)
Let φ be differentiable on a region of $\mathbb{R}^2$ or $\mathbb{R}^3$. The vector field $=\nabla\varphi$ is a **gradient field** and the function φ is a **potential function** for $\mathbf{F}$.

Example. Sketch and interpret the gradient field associated with the temperature function $T = 200 - x^2 - y^2$ on the cirular plane $R = \{(x, y) : x^2 + y^2 \leq 25\}$.

Example. Sketch and interpret the gradient field associated with the velocity potential $\varphi = \tan^{-1}(xy)$.