

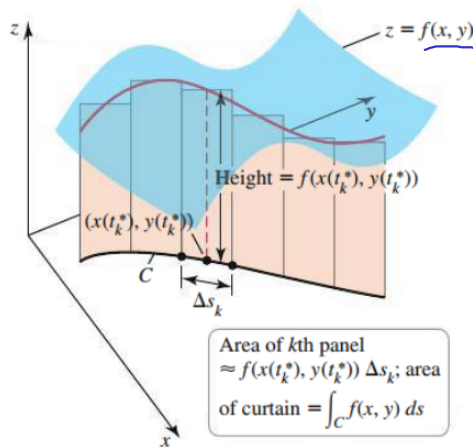
17.2: Line Integrals

Definition. (Scalar Line Integral in the Plane)

Suppose the scalar-valued function f is defined on a region containing the smooth curve C given by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$. The line integral of f over C is

$$\int_C f(x(t), y(t)) ds = \lim_{\Delta \rightarrow 0} \sum_{k=1}^n f(x(t_k^*), y(t_k^*)) \Delta s_k,$$

provided this limit exists over all partitions of $[a, b]$. When the limit exists, f is said to be **integrable** on C .



$$s(t) = \int_a^t \vec{r}'(u) du$$

$$s'(t) = \vec{r}'(t)$$

$$dA = \vec{r}'(t) dt$$

Theorem 17.1: Evaluating Scalar Line Integrals in \mathbb{R}^2

Let f be continuous on a region containing a smooth curve C : $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$. Then

$$\begin{aligned} \int_C f ds &= \int_a^b f(x(t), y(t)) \underbrace{|\mathbf{r}'(t)|}_{ds} dt \\ &= \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} dt. \end{aligned}$$

If $f(x, y) = 1$, $\int_C f ds$ is arc length

Procedure: Evaluating the Line Integral $\int_C f \, ds$

1. Find a parametric description of C in the form $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$.
2. Compute $|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$.
3. Make substitutions for x and y in the integrand and evaluate an ordinary integral:

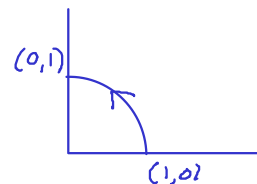
$$\int_C f \, ds = \int_a^b f(\underline{x(t)}, \underline{y(t)}) \underbrace{|\mathbf{r}'(t)|}_{ds} dt.$$

Example. Find the length of the quarter-circle from $(1, 0)$ to $(0, 1)$ with its center at the origin.

$$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle, \quad 0 \leq t \leq \pi/2$$

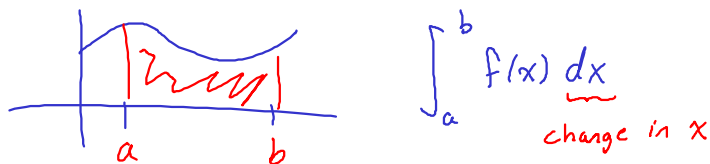
$$|\vec{r}'(t)| = | \langle -\sin(t), \cos(t) \rangle | = 1$$

$$\vec{r}(t) = \langle t, \sqrt{1-t^2} \rangle$$



$$L = \int_C f \, ds = \int_0^{\pi/2} \underbrace{f(x(t), y(t))}_1 \underbrace{|\vec{r}'(t)|}_1 dt = t \Big|_{t=0}^{t=\pi/2} = \boxed{\pi/2}$$

$$\boxed{LC \neq 1, a = 1/2}$$



Example. The temperature of the circular plate $R = \{(x, y) : x^2 + y^2 \leq 1\}$ is $T(x, y) = 100(x^2 + 2y^2)$. Find the average temperature along the edge of the plate.

$$\bar{T} = \frac{1}{L} \int_C T(x, y) ds$$

$$\text{Circle} \Rightarrow L = 2\pi$$

$$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle \quad 0 \leq t \leq 2\pi$$

$$|\vec{r}'(t)| = |\langle -\sin(t), \cos(t) \rangle| = 1$$

$$T(x(t), y(t)) = 100 (\cos^2(t) + 2\sin^2(t))$$

$$\int_C T(x, y) ds = \int_0^{2\pi} T(x(t), y(t)) |\vec{r}'(t)| dt$$

$$= \int_0^{2\pi} 100 (\cos^2(t) + 2\sin^2(t)) (1) dt$$

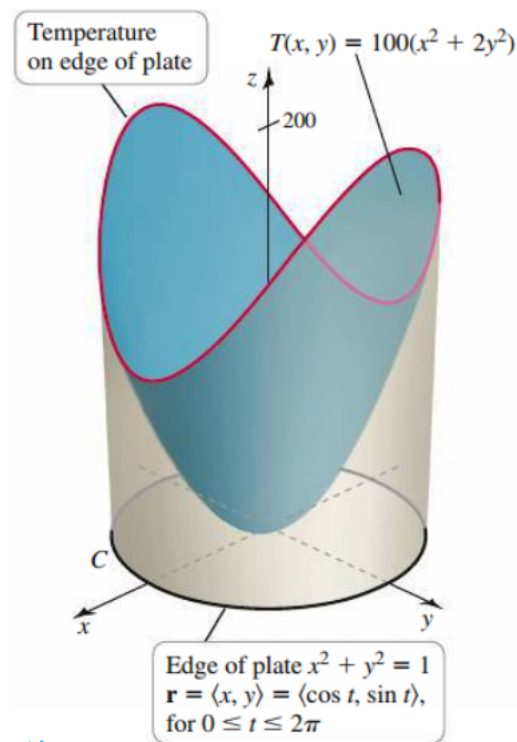
$$1 + \sin^2(t) = 1 + \frac{1 - \cos(2t)}{2}$$

$$= \frac{100}{2} \int_0^{2\pi} 3 - \cos(2t) dt$$

$$= 50 \left[3t - \frac{\sin(2t)}{2} \right]_{t=0}^{t=2\pi}$$

$$= 50 \left[3(2\pi - 0) - \frac{1}{2} (0 - 0) \right] = 300\pi$$

$$\bar{T} = \frac{1}{L} \int_C T(x, y) ds = \frac{300\pi}{2\pi} = \boxed{150}$$



$L \neq 2$

Theorem 17.2: Evaluating Scalar Line Integrals in \mathbb{R}^3

Let f be continuous on a region containing a smooth curve $C : \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $a \leq t \leq b$. Then

$$\begin{aligned} \int_C f \, ds &= \int_a^b \overbrace{f(x(t), y(t), z(t))}^{f(\vec{r}(t))} \overbrace{|\mathbf{r}'(t)| \, dt}^{ds} \\ &= \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt. \end{aligned}$$

Example. Evaluate $\int_C (x - y + 2z) \, ds$, where C is the circle $\mathbf{r}(t) = \langle 1, 3 \cos(t), 3 \sin(t) \rangle$, for $0 \leq t \leq 2\pi$.

$$|\vec{r}'(t)| = |\langle 0, -3 \sin(t), 3 \cos(t) \rangle| = 3$$

$$\int_C (x - y + 2z) \, ds = \int_0^{2\pi} (x(t) - y(t) + 2z(t)) |\vec{r}'(t)| \, dt$$

$$= \int_0^{2\pi} (1 - 3 \cos(t) + 6 \sin(t)) \cdot 3 \, dt$$

$$= 3 \left[t - 3 \sin(t) - 6 \cos(t) \right]_{t=0}^{t=2\pi}$$

$$= 3 \left[(2\pi - 0) - 3(0 - 0) - 6(1 - 1) \right] = \boxed{6\pi}$$

$$\begin{aligned} &LC \# 3 \\ &6\pi + 36 \end{aligned}$$

Example. Evaluate $\int_C x e^{yz} ds$, where C is $\mathbf{r}(t) = \langle \underline{t}, \underline{2t}, \underline{-2t} \rangle$, for $0 \leq t \leq 2$.

$$|\vec{r}'(t)| = |\langle 1, 2, -2 \rangle| = 3$$

$$\int_C \underline{x} e^{\underline{y}\underline{z}} ds = \int_0^2 \underbrace{t e^{(2t)(-2t)}}_{f(x(t), y(t), z(t))} \underbrace{(3)}_{|\vec{r}'(t)|} dt$$

$$= \int_0^2 \underline{3t} e^{-4t^2} \underline{dt}$$

$$= \int_0^{-16} -\frac{3}{8} e^u du$$

$$= -\frac{3}{8} e^u \Big|_{u=0}^{u=-16}$$

$$= -\frac{3}{8} (e^{-16} - 1) =$$

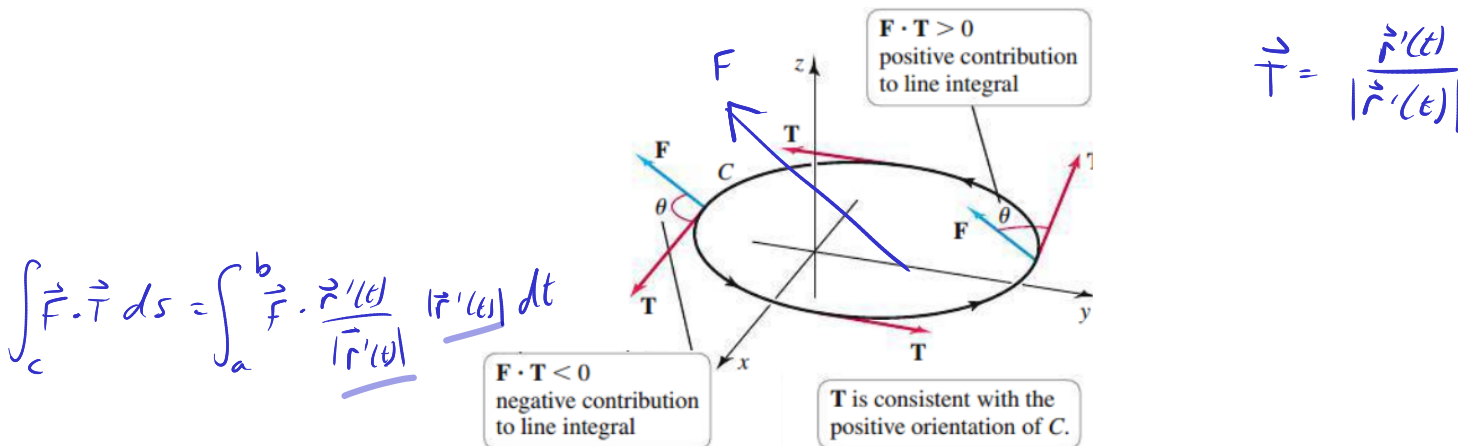
$$\boxed{\frac{3(1 - e^{-16})}{8}}$$

$$\text{LC \# 4}$$

$$a = -16$$

Definition. (Line Integral of a Vector Field)

Let \mathbf{F} be a vector field that is continuous on a region containing a smooth oriented curve C parameterized by arc length. Let \mathbf{T} be the unit tangent vector at each point of C consistent with the orientation. The line integral of \mathbf{F} over C is $\int_C \mathbf{F} \cdot \mathbf{T} ds$.



Different Forms of Line Integrals of Vector Fields

The line integral $\int_C \mathbf{F} \cdot \mathbf{T} ds$ may be expressed in the following forms, where $\mathbf{F} = \langle f, g, h \rangle$ and C has a parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $a \leq t \leq b$:

$$\begin{aligned}
 \int_a^b \mathbf{F} \cdot \mathbf{r}'(t) dt &= \int_a^b (f(t)x'(t) + g(t)y'(t) + h(t)z'(t)) dt \\
 &= \int_C f dx + g dy + h dz \\
 &= \int_C \mathbf{F} \cdot d\mathbf{r}.
 \end{aligned}$$

Handwritten note: $dx = x'(t) dt$

For line integrals in the plane, we let $\mathbf{F} = \langle f, g \rangle$ and assume C is parameterized in the form $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$. Then

$$\int_a^b \mathbf{F} \cdot \mathbf{r}'(t) dt = \int_a^b (f(t)x'(t) + g(t)y'(t)) dt = \int_C f dx + g dy = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

$$\int_C \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F} \cdot \vec{r}'(t) dt$$

t	$\vec{r}(t)$
0	$\langle 0, 1 \rangle \leftarrow P(0, 1)$
$\frac{\pi}{2}$	$\langle 1, 0 \rangle \leftarrow Q(1, 0)$

Example. Evaluate $\int_C \vec{F} \cdot \vec{T} ds$ with $\vec{F} = \langle y - x, x \rangle$ on the following oriented paths in \mathbb{R}^2 .

$\cos(t)$ $\sin(t)$

a) The quarter-circle C_1 from $P(0, 1)$ to $Q(1, 0)$

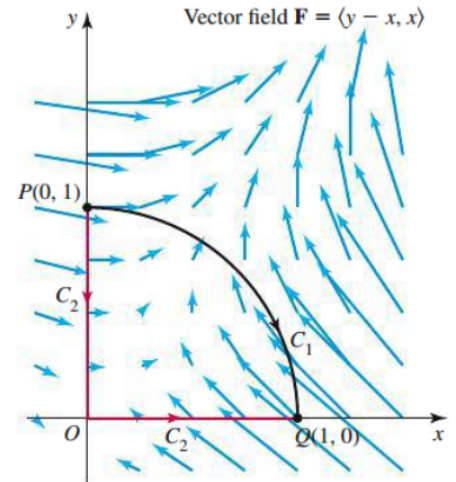
$$\vec{r}(t) = \langle \sin(t), \cos(t) \rangle, \quad 0 \leq t < \pi/2$$

$$\vec{r}'(t) = \langle \cos(t), -\sin(t) \rangle$$

$$\vec{F} \cdot \vec{r}'(t) = \langle \cos(t) - \sin(t), \sin(t) \rangle \cdot \langle \cos(t), -\sin(t) \rangle$$

$$= \cos^2(t) - \sin(t)\cos(t) - \sin^2(t)$$

$$= \cos(2t) - \frac{1}{2} \sin(2t)$$



$$\bullet \cos(2t) = \cos^2(t) - \sin^2(t)$$

$$\bullet \sin(2t) = 2 \sin(t) \cos(t)$$

$$\int_C \vec{F} \cdot \vec{T} ds = \int_0^{\pi/2} \vec{F} \cdot \vec{r}'(t) dt = \int_0^{\pi/2} \cos(2t) - \frac{1}{2} \sin(2t) dt = \left[\frac{\sin(2t)}{2} + \frac{1}{4} \cos(2t) \right]_{t=0}^{t=\pi/2}$$

denotes opposite direction

$$= (0 - 0) + \frac{1}{4}(-1 - 1) = \boxed{-\frac{1}{2}}$$

b) The quarter-circle $-C_1$ from $Q(1, 0)$ to $P(0, 1)$

$$\textcircled{1} \vec{r}(t) = \langle \cos t, \sin t \rangle, \quad 0 \leq t \leq \pi/2, \dots$$

$$\textcircled{2} \int_{-C} \vec{F} \cdot \vec{T} ds = - \int_C \vec{F} \cdot \vec{T} ds \Rightarrow \int_{-C} \vec{F} \cdot \vec{T} ds = \boxed{\frac{1}{2}}$$

Lc #6

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

c) the path C_2 from $P(0, 1)$ to $Q(1, 0)$ via two line segments through $O(0, 0)$.

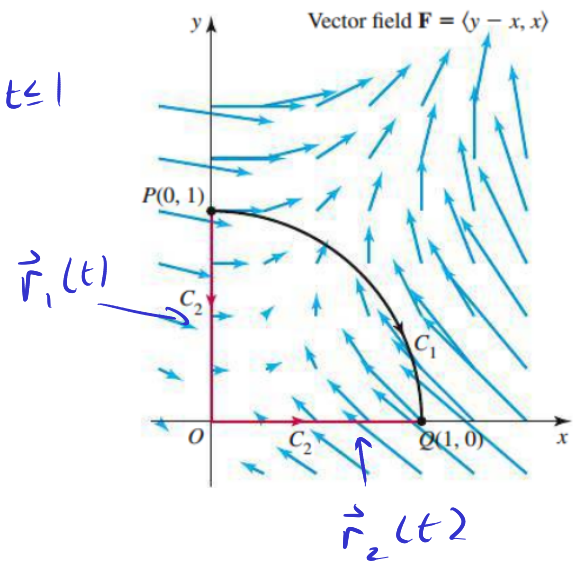
$$(0, 1) \rightarrow (0, 0)$$

$$\vec{r}_1(t) = \langle 0, 1 \rangle + t \langle 0, -1 \rangle = \langle 0, 1-t \rangle, \quad 0 \leq t \leq 1$$

$$(0, 0) \rightarrow (1, 0)$$

$$\vec{r}_2(t) = \langle 0, 0 \rangle + t \langle 1, 0 \rangle = \langle t, 0 \rangle, \quad 0 \leq t \leq 1$$

$$\vec{F} = \langle y-x, x \rangle$$



$$\vec{F} \cdot \vec{r}_1'(t) = \langle (1-t)-0, 0 \rangle \cdot \langle 0, -1 \rangle = 0$$

$$\langle 1-t, 0 \rangle$$

$$\vec{F} \cdot \vec{r}_2'(t) = \langle 0-t, t \rangle \cdot \langle 1, 0 \rangle = -t$$

$$\int_{C_2} \vec{F} \cdot d\vec{s} = \int_0^1 \underbrace{\vec{F} \cdot \vec{r}_1'(t)}_0 dt + \int_0^1 \vec{F} \cdot \vec{r}_2'(t) dt$$

$$= \int_0^1 -t dt$$

$$= -\frac{t^2}{2} \Big|_{t=0}^{t=1} = \boxed{-\frac{1}{2}}$$

LC #7

Definition. (Work Done in a Force Field)

Let \mathbf{F} be a continuous force field in a region D of \mathbb{R}^3 . Let

$$C : \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, \text{ for } a \leq t \leq b,$$

be a smooth curve in D with a unit tangent vector \mathbf{T} consistent with the orientation. The work done in moving an object along C in the positive direction is

$$\underline{W} = \int_C \underline{\mathbf{F} \cdot \mathbf{T}} ds = \int_a^b \underline{\mathbf{F} \cdot \mathbf{r}'(t)} dt.$$

Example. For the force field $\mathbf{F} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}$, calculate the work required to move an object from $(1, 1, 1)$ to $(10, 10, 10)$.

$$\vec{r}(t) = \langle 1, 1, 1 \rangle + t \langle 9, 9, 9 \rangle = \langle 1+9t, 1+9t, 1+9t \rangle, \quad 0 \leq t \leq 1$$

$$\vec{r}'(t) = \langle 9, 9, 9 \rangle = 9 \langle 1, 1, 1 \rangle \quad \leftarrow \langle \frac{10-1}{1}, \frac{10-1}{1}, \frac{10-1}{1} \rangle$$

$$\vec{r}(t) = \langle 1+t, 1+t, 1+t \rangle, \quad 0 \leq t \leq 9$$

$$\leftarrow \langle \frac{10-1}{9}, \frac{10-1}{9}, \frac{10-1}{9} \rangle$$

$$\vec{F} = \frac{\langle 1+9t, 1+9t, 1+9t \rangle}{(3(1+9t)^2)^{3/2}} = \frac{(1+9t) \langle 1, 1, 1 \rangle}{3^{3/2} (1+9t)^3} = \frac{\langle 1, 1, 1 \rangle}{3^{3/2} (1+9t)^2}$$

$$W = \int_C \vec{F} \cdot \vec{T} ds = \int_0^1 \vec{F} \cdot \vec{r}'(t) dt = \int_0^1 \frac{\langle 1, 1, 1 \rangle}{3^{3/2} (1+9t)^2} \cdot 9 \langle 1, 1, 1 \rangle dt$$

$$= \int_0^1 \frac{3(9)}{3^{3/2} (1+9t)^2} dt = \int_1^{10} \frac{u^{-2}}{3^{1/2}} du = \frac{-1}{3^{1/2} u} \bigg|_{u=1}^{u=10} = \frac{-1}{\sqrt{3} \cdot 10} + \frac{1}{\sqrt{3}} = \frac{3\sqrt{3}}{10}$$

$$u = 1+9t$$

$$du = 9 dt$$

$$t=0, u=1$$

$$t=1, u=10$$

$$\text{LC \#8}$$

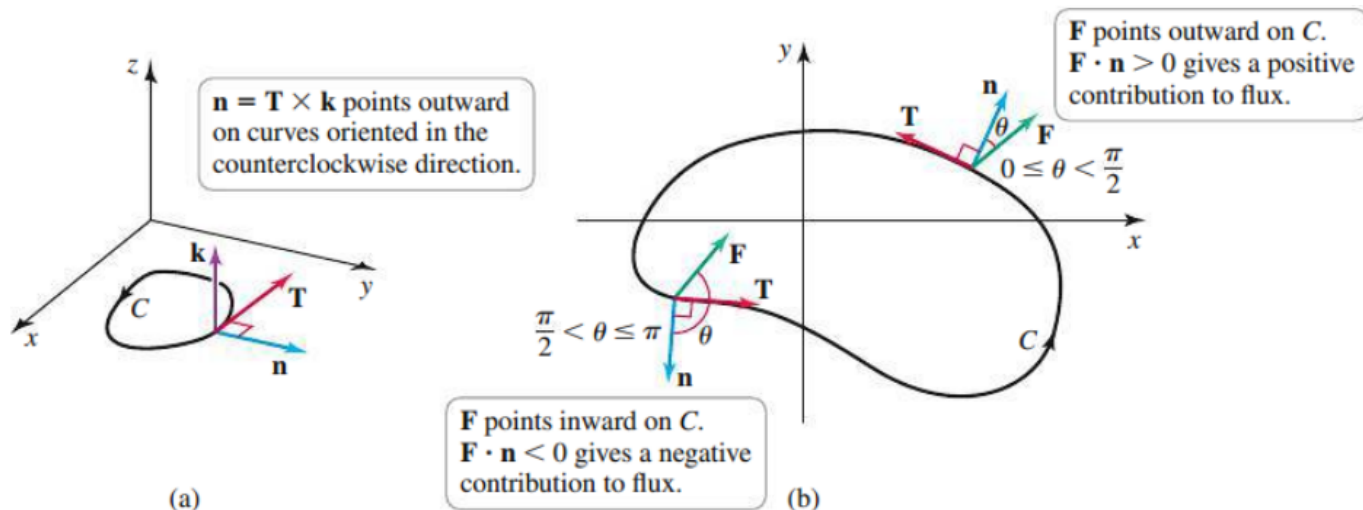
$$a = 3/10$$

Definition. (Circulation)

Let \mathbf{F} be a continuous vector field on a region D of \mathbb{R}^3 , and let C be a closed smooth oriented curve in D . The **circulation** of \mathbf{F} on C is $\int_C \mathbf{F} \cdot \mathbf{T} \, ds$, where \mathbf{T} is the unit vector tangent to C consistent with the orientation.

Example. Compute the circulation in the vector field $\mathbf{F} = \frac{\langle y, -2x \rangle}{\sqrt{4x^2 + y^2}}$ along the curve C given by $\mathbf{r}(t) = \langle 2 \cos(t), 4 \sin(t) \rangle$, for $0 \leq t \leq 2\pi$.

Flux of the vector field is the total forces orthogonal to each point on the curve C . Let $\mathbf{F} = \langle f, g \rangle$ be a continuous vector field in a region R of \mathbb{R}^2 . Using \mathbf{n} to represent a unit vector normal to C , the component of \mathbf{F} that is normal to C is $\mathbf{F} \cdot \mathbf{n}$.



Since C is in the xy -plane, the unit tangent vector $\mathbf{T} = \langle T_x, T_y, 0 \rangle$ is also in the xy -plane. We let \mathbf{n} be in the xy -plane as well, but using the cross product of \mathbf{T} and \mathbf{k} :

$$\mathbf{n} = \mathbf{T} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ T_x & T_y & 0 \\ 0 & 0 & 1 \end{vmatrix} = T_y \mathbf{i} - T_x \mathbf{j}.$$

Since $\mathbf{T} = \frac{\mathbf{r}(t)}{|\mathbf{r}(t)|}$, we have

$$\mathbf{n} = T_y \mathbf{i} - T_x \mathbf{j} = \frac{y'(t)}{|\mathbf{r}'(t)|} \mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|} \mathbf{j} = \frac{\langle y'(t), -x'(t) \rangle}{|\mathbf{r}'(t)|}.$$

Thus, we have the flux integral

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_a^b \mathbf{F} \cdot \frac{\langle y'(t), -x'(t) \rangle}{|\mathbf{r}'(t)|} |\mathbf{r}'(t)| \, dt = \int_a^b (f(t)y'(t) - g(t)x'(t)) \, dt = \int_C f \, dy - g \, dx.$$