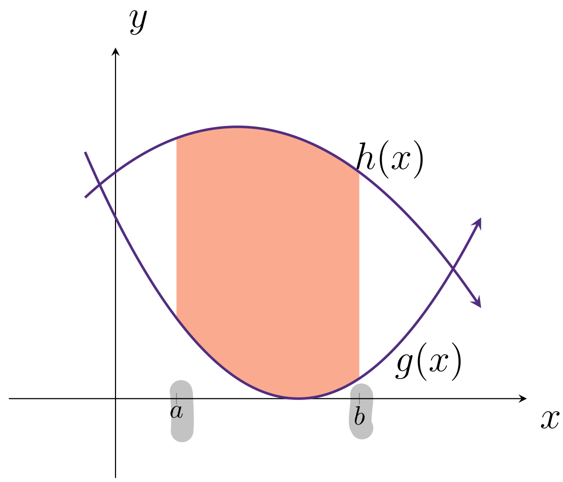
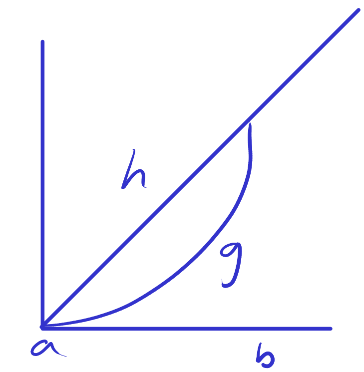
16.2: Double Integrals over General Regions

In this section, we consider double integrals over non-rectangular regions. For instance, my domain for x and y can be constrained where $a \le x \le b$ and $g(x) \le y \le h(x)$:





$a \le x \le b$
$g \le y \le h$

Theorem 16.2: Double Integrals over Nonrectangular Regions
Let R be a region bounded below and above by the graphs of the continuous functions $y = g(x)$ and $y = h(x)$, respectively, and by the lines $x = a$ and $x = b$. If f is continuous on R, then

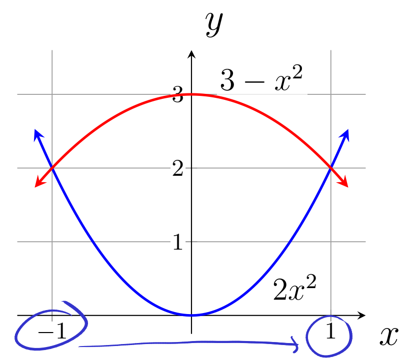
$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx.$$

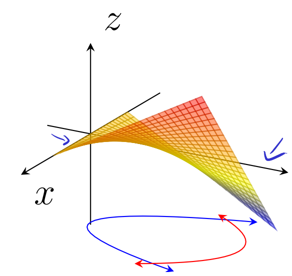
Let R be a region bounded on the left and right by the graphs of the continuous functions $x = g(y)$ and $x = h(y)$, respectively, and the lines $y = c$ and $y = d$. If f is continuous on R, then

$$\iint\limits_R f(x,y)\,dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy.$$

Example. Consider the surface generated by the function $f(x, y) = 3xy$. Find the volume of the solid generated by $f(x, y)$ over the region bounded by $2x^2$ and $3 - x^2$.

$$R = \{(x,y) : 2x^2 \le y \le 3 - x^2,\ -1 \le x \le 1\}$$





$$\int_{-1}^{1}\int_{2x^2}^{3-x^2} 3xy\, dy\, dx$$

$$= \int_{-1}^{1} \frac{3}{2}xy^2 \Big|_{y=2x^2}^{y=3-x^2} dx$$

$$= \int_{-1}^{1} \frac{3}{2}x(3-x^2)^2 - \frac{3}{2}x(2x^2)^2\, dx$$

$$= \frac{3}{2}\int_{-1}^{1} x(x^4-6x^2+9) - 4x^5\, dx$$

$$= \frac{3}{2}\int_{-1}^{1} -3x^5 - 6x^3 + 9x\, dx$$

LC#1

$$= \frac{3}{2}\left(-\frac{1}{2}x^6 - \frac{3}{2}x^4 + \frac{9}{2}x^2 \Big|_{x=-1}^{x=1}\right) = \frac{3}{2}\left[\left(-\frac{1}{2} - \frac{3}{2} + \frac{9}{2}\right) - \left(-\frac{1}{2} - \frac{3}{2} + \frac{9}{2}\right)\right] = 0$$

Area above and below
xy-plane cancel out

Example. Find the area under $f(x, y) = \frac{1}{x} + 1$ over the region formed by the lines $y = 2$, $y = 1 + x$, and $y = 5 - x$.

$1 + x = 5 - x$

$2x = 4$

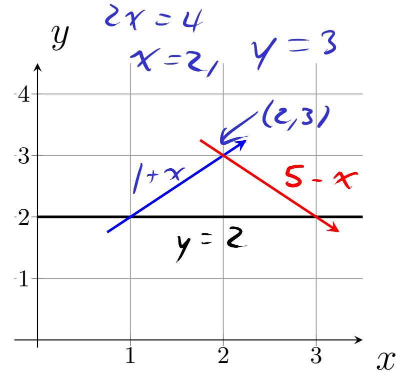
$x = 2, \quad y = 3$

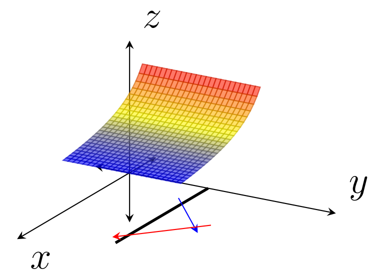
$1 + x = y$

$x = y - 1$

$5 - x = y$

$x = 5 - y$





$$R = \{(x,y) : y - 1 \le x \le 5 - y,\ 2 \le y \le 3\}$$

$$\int_2^3 \int_{y-1}^{5-y} \frac{1}{x} + 1 \, dx \, dy$$

$$= \int_2^3 \ln(x) + x \Big|_{x=y-1}^{x=5-y} dy$$

$$= \int_2^3 \big(\ln(5-y) + 5 - y\big) - \big(\ln(y-1) + y - 1\big) \, dy$$

$$= \int_2^3 \ln(5-y) - \ln(y-1) - 2y + 6 \, dy$$

IBP

$$= (5 + y)\ln(5-y) + 5 - y + (y-1)\ln(y-1) - (y-1) - y^2 + 6y \Big|_{y=2}^{y=3}$$

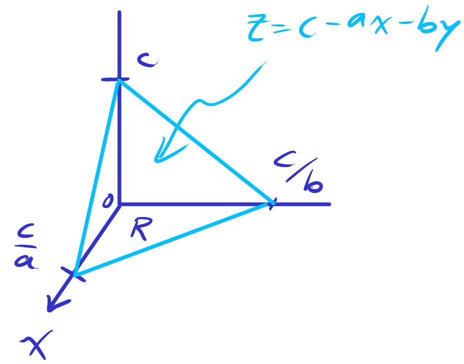
$$= \big(10\ln(2) + 9\big) - \big(7\ln(3) + 10\big)$$

ew...

LC #2

Put 0

Example. Find the volume of the tetrahedron in the first octant bounded by the plane $f(x,y) = z = c - ax - by$ and the coordinate planes ($x = 0$, $y = 0$, and $z = 0$). Assume a, b, and c are positive real numbers.



$$R = \left\{ (x,y) : 0 \le y \le \frac{c}{b} - \frac{a}{b}x,\ 0 \le x \le \frac{c}{a} \right\}$$

$$\int_0^{c/a} \int_0^{\frac{c}{b} - \frac{a}{b}x} c - ax - by \; dy \, dx$$

$$= \int_0^{c/a} (c - ax)y - \frac{b}{2}y^2 \Big|_{y=0}^{y = \frac{c}{b} - \frac{a}{b}x} dx$$

$$m = \frac{0 - c/b}{\frac{c}{a} - 0} = -\frac{a}{b}$$

$$y = -\frac{a}{b}x + \frac{c}{b}$$

$$= \int_0^{c/a} \frac{(c-ax)^2}{b} - \frac{b}{2}\left(\frac{c-ax}{b}\right)^2 dx$$

$$= \int_0^{c/a} \frac{(c-ax)^2}{2b} \, dx$$

$$u = c - ax \qquad x = 0,\ u = c$$

$$du = -a \, dx \qquad x = \frac{c}{a},\ u = 0$$

$$-\frac{1}{a} du = dx$$

$$= -\int_c^0 \frac{u^2}{2ab} du = \int_0^c \frac{u^2}{2ab} du$$

$$= \frac{u^3}{6ab}\Big|_{u=0}^{u=c} = \boxed{\frac{c^3}{6ab}} = \frac{1}{3} \underbrace{\frac{c^2}{2ab}}_{\text{area base}} \cdot \underset{\text{height}}{c}$$

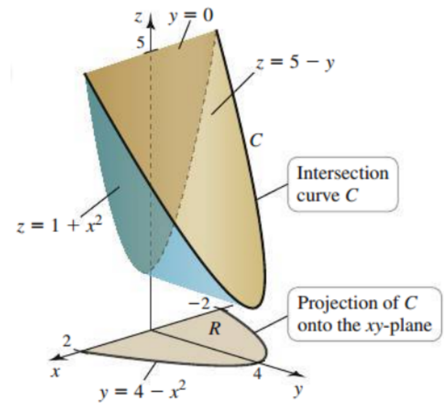
Example. For the following problems, reverse the order of integration

- $\displaystyle\int_0^2 \int_0^{2x} f(x,y)\,dy\,dx$

- $\displaystyle\int_0^1 \int_{x^3}^{\sqrt{x}} f(x,y)\,dy\,dx$

- $\displaystyle\int_{-3}^4 \int_{2x^2}^{2x+24} f(x,y)\,dy\,dx$

Example. Find the volume between $f(x, y) = 5 - y$ and $g(x, y) = 1 + x^2$ over the region $R = \{(x, y) : 0 \le y \le 4 - x^2,\ -2 \le x \le 2\}$.



Areas of Regions by Double Integrals
Let R be a region in the xy-plane. Then

$$\text{area of } R = \iint_R dA.$$

Example. Find the area of the region R bounded by $y = x^2$, $y = 6 - x$, and $y = 6 + 5x$.