8.1: Basic Approaches (to Integration)

Example. Derive the integral formula $\int \sec(ax)\,dx = \frac{1}{a}\ln|\sec(ax)+\tan(ax)| + C$.

$\frac{d}{dx}[\sec(ax)] = a\sec(ax)\tan(ax)$

$$\int \sec(ax)\,\underbrace{\frac{\sec(ax)+\tan(ax)}{\sec(ax)+\tan(ax)}}_{1}\,dx = \int \frac{\sec^2(ax)+\sec(ax)\tan(ax)}{\sec(ax)+\tan(ax)}\,dx$$

$u = \sec(ax) + \tan(ax)$

$du = a\sec(ax)\tan(ax) + a\sec^2(ax)\,dx$

$$= \frac{1}{a}\int \frac{1}{u}\,du$$

$$= \frac{1}{a}\ln|u| + c = \frac{1}{a}\ln|\sec(ax)+\tan(ax)| + c$$

$$\boxed{\int \csc(ax)\,dx = \frac{1}{a}\ln|\csc(ax) - \cot(ax)| + c}$$

Example. Evaluate $\int \frac{dx}{e^{3x}+e^{-3x}} \cdot \left(\frac{e^{3x}}{e^{3x}}\right)$

$u = e^{3x}$

$du = 3e^{3x}\,dx$

$\frac{du}{3} = e^{3x}\,dx$

$$= \int \frac{e^{3x}}{(e^{3x})^2 + 1}\,dx$$

$$= \frac{1}{3}\int \frac{du}{u^2+1} = \frac{1}{3}\tan^{-1}(u) + c = \frac{1}{3}\tan^{-1}(e^{3x}) + c$$

$u = \csc(x)$
$du = -\csc(x)\cot(x)\,dx$

$\csc(x) = \dfrac{1}{\sin(x)}$

Example. Evaluate $\displaystyle\int \frac{\sin(x) + \cos^4(x)}{\csc(x)}\,dx.$

Note: $\begin{cases} \cos^2(x) = \dfrac{1+\cos(2x)}{2} \\ \sin^2(x) = \dfrac{1-\cos(2x)}{2} \end{cases}$

$$= \int \frac{\sin(x)}{\csc(x)} + \frac{\cos^4(x)}{\csc(x)}\,dx$$

$u = \cos(x)$
$du = -\sin(x)\,dx$

$$= \int \sin^2(x) + \sin(x)(\cos(x))^4\,dx$$

$$= \int \frac{1-\cos(2x)}{2}\,dx - \int u^4\,du$$

$$= \frac{x}{2} - \frac{\sin(2x)}{4} - \frac{u^5}{5} + C = \frac{x}{2} - \frac{\sin(2x)}{4} - \frac{\cos^5(x)}{5} + C$$

Example. Evaluate $\displaystyle\int \frac{2x^2+3x-4}{x-2}\,dx.$

$$\begin{array}{r} 2x+7 \\ x-2\,\overline{)\,2x^2+3x-4} \\ -(2x^2-4x) \\ \hline 7x-4 \\ -(7x-14) \\ \hline 10 \end{array}$$

$$= \int 2x+7 + \frac{10}{x-2}\,dx$$

$$= x^2 + 7x + 10\ln|x-2| + C$$

Example. Evaluate $\displaystyle\int \frac{dx}{\sqrt{7-6x-x^2}}$.