7.1 Composition and Inverse Functions

Definition. Given two functions f and g, the composite function $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$. It is evaluated in two steps: $y = f(u)$, where $u = g(x)$. The domain of $f \circ g$ consists of all x in the domain of g such that $u = g(x)$ is in the domain of f.

Example. Given $f(x) = x^2 + 2x + \pi$ and $g(x) = x^2$, find

$f(g(x)) = f(x^2) = x^4 + 2x^2 + \pi$

$g(f(x)) = g(x^2 + 2x + \pi) = (x^2 + 2x + \pi)^2$

Example. Given $f(x) = \dfrac{1}{x+2}$ and $g(x) = x^2 - 1$, find

$f(g(x)) = f(x^2 - 1) = \dfrac{1}{(x^2-1)+2} = \dfrac{1}{x^2+1}$

$g(f(x)) = g\left(\dfrac{1}{x+2}\right) = \left(\dfrac{1}{x+2}\right)^2 - 1 = \dfrac{1}{(x+2)^2} - 1$

Example. Given $f(x) = x^2, g(x) = \sin(x)$ and $h(x) = 2x + 1$, find

$f(g(h(x))) = f(g(2x+1)) = f(\sin(2x+1)) = (\sin(2x+1))^2$

$= \sin^2(2x+1)$

Example. Given $f(x) = x^3, g(x) = \cos(x)$, find

$f(0) = 0^3 = 0$

$f(1) = 1^3 = 1$

$g(0) = \cos(0) = 1$

$(f \circ g)(0) = f(g(0)) = f(1) = 1$

$(f \circ g)(x) = f(g(x)) = f(\cos(x)) = \cos^3(x)$

$(g \circ f)(x) = g(f(x)) = g(x^3) = \cos(x^3)$

$(f \circ f)(x) = f(f(x)) = f(x^3) = (x^3)^3 = x^9$

$(g \circ g)(x) = g(g(x)) = g(\cos(x)) = \cos(\cos(x))$.

Example. Evaluate or explain why the functions value is undefined:

$f(g(2))$ $= f(5) = 4$

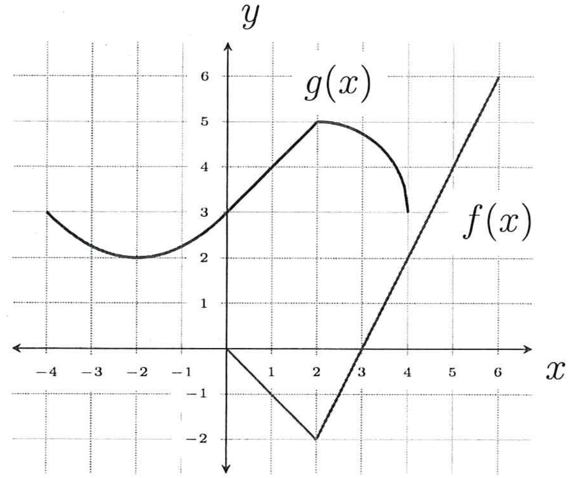
$g(f(2))$ $= g(-2) = 2$

$(f \circ g)(0)$ $= f(g(0)) = f(3) = 0$

$(g \circ f)(6)$ $g(f(6)) = g(6)$ DNE

$(g \circ g)(-2)$ $g(g(-2)) = g(2) = 5$

$(f \circ f)(4)$ $f(f(4)) = f(2) = -2$



Note: $f(g(x))$ is not necessarily the same as $g(f(x))$.

Note: If $f(g(x)) = x$ and $g(f(x)) = x$, then $f(x)$ and $g(x)$ are inverse functions.

7.2: The Ideas of Inverses

Definition (Inverse function). Given a function f, its inverse (if it exists) is a function f^{-1} such that whenever $y = f(x)$, then $f^{-1}(y) = x$.

Note: f and g are inverses if $f(g(x)) = x$ *and* $g(f(x)) = x$.

Note: The domain of $f(x)$ must be the range of $g(x)$.

Note: The domain of $g(x)$ must be the range of $f(x)$.

Note: The inverse, $f^{-1}(x)$, should **not** be confused with $[f(x)]^{-1} = \dfrac{1}{f(x)}$.

Example. For the following, verify that $f(x)$ and $g(x)$ are inverses:

$f(x) = x^2, \ x > 0$
$g(x) = \sqrt{x}$

$f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x, \ x > 0$
$g(f(x)) = g(x^2) = \sqrt{x^2} = x, \ x > 0$

$f(x) = {}^1/_x$
$g(x) = {}^1/_x$

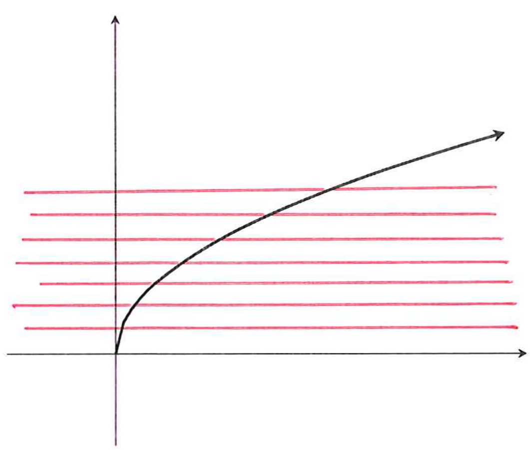
$f(g(x)) = f\left(\frac{1}{x}\right) = \frac{1}{(1/x)} = x$
$g(f(x)) = g\left(\frac{1}{x}\right) = \frac{1}{(1/x)} = x$

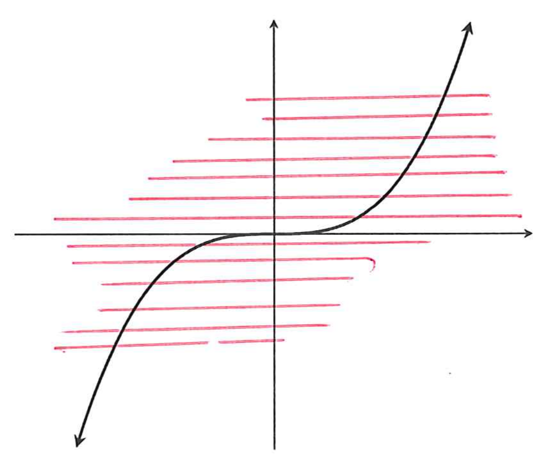
$f(x) = 3x + 2$
$g(x) = \dfrac{1}{3}(x - 2)$

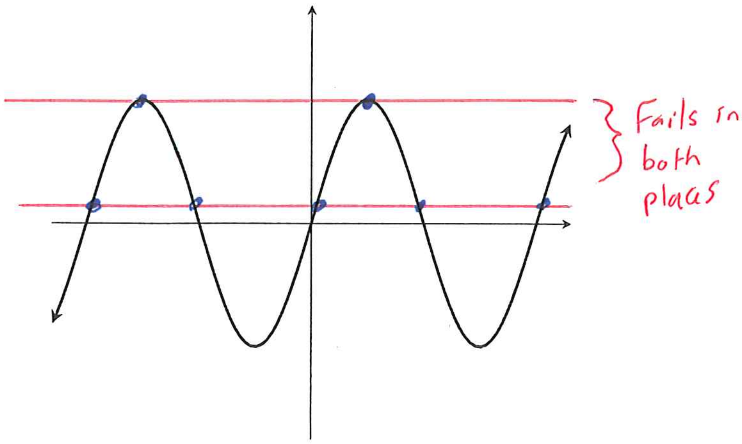
$f(g(x)) = f\left(\frac{1}{3}(x-2)\right) = 3\left[\frac{1}{3}(x-2)\right]+2$

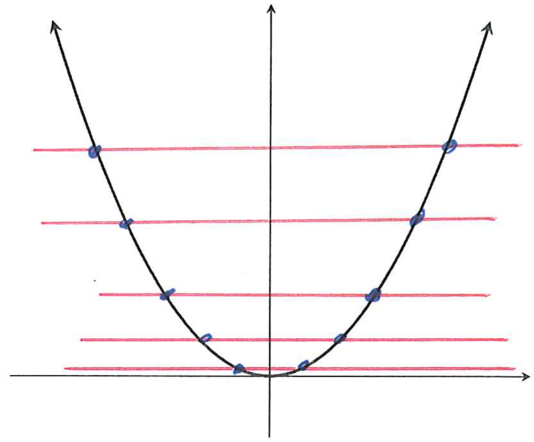
Definition (One-to-One Functions and the Horizontal Line Test)**.** A function f is **one-to-one** on a domain D if each value of $f(x)$ corresponds to exactly one value of x in D. More precisely, f is one-to-one on D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, for x_1 and x_2 in D.

The **horizontal line test** says that every horizontal line intercepts the graph of a one-to-one function at most once.









Existence of Inverse Functions
Let f be a one-to-one function on a domain D with a range R. Then f has a unique inverse f^{-1} with domain R and range D such that

$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f\big(f^{-1}(y)\big) = y$$

where x is in D and y is in R.

Example. Using the table below, solve the following:

$(f \circ f)(-1)$ $= f(f(-1)) = f(-2) = -8$

$f^{-1}(2)$ $= 1$ because $f(1) = 2$

$f^{-1}(6)$ $= 2$ because $f(2) = 6$

$f\big(f^{-1}(6)\big)$ $= f(2) = 6$

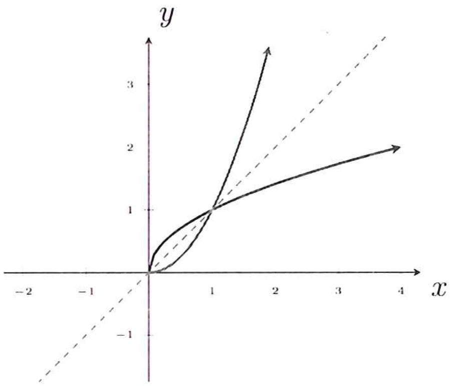
$f^{-1}\big(f^{-1}(6)\big)$ $= f^{-1}(2) = 1$

x	$f(x)$
-2	-8
-1	-2
0	0
1	2
2	6

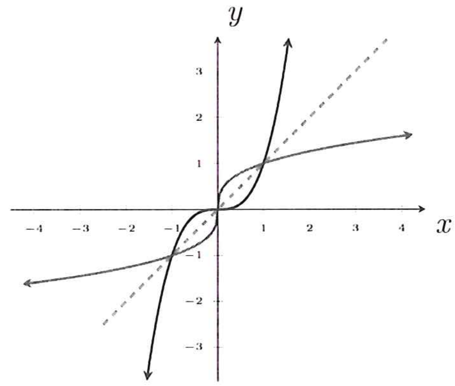
7.3 Finding the Inverse of f Given a Graph

Note: A function is symmetric with it's inverse with respect to $y = x$.

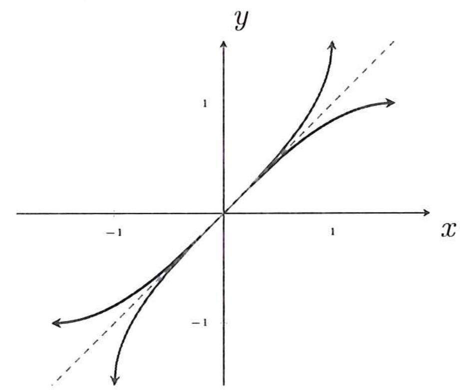
$$f(x) = \sqrt{x}$$
$$f^{-1}(x) = x^2, \ x > 0$$



$$f(x) = x^3$$
$$f^{-1}(x) = \sqrt[3]{x} = x^{1/3}$$

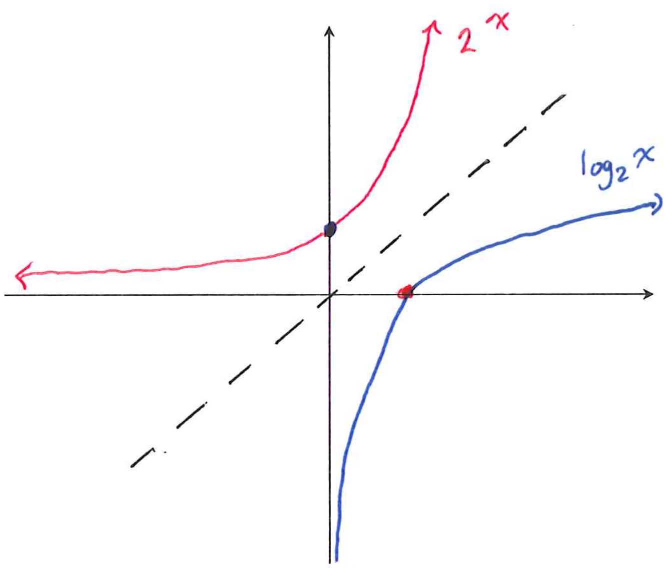


$$f(x) = \sin x \text{ on } [-\pi/2, \pi/2]$$
$$f^{-1}(x) = \sin^{-1} x$$

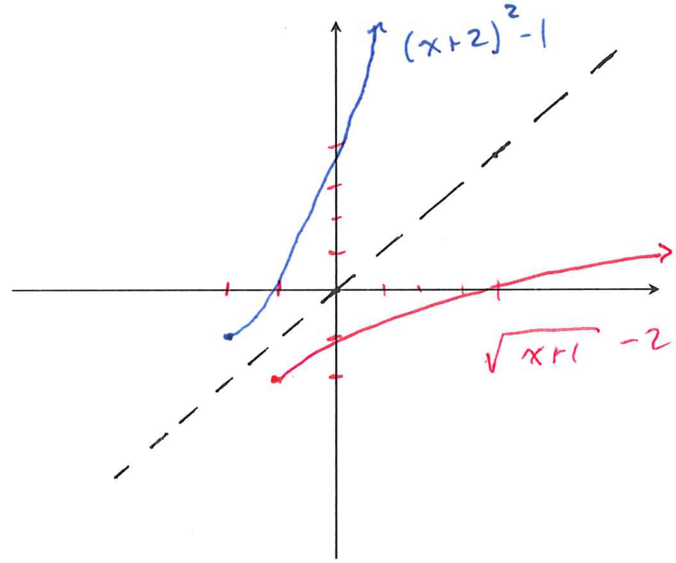


Example. Draw the function's inverse:

$$f(x) = 2^x$$



$$f(x) = \sqrt{x+1} - 2$$



7.4 Finding the Inverse of f Given by an Expression

Finding an Inverse Function

Suppose f is one-to-one on an interval I. To find f^{-1}, use the following steps:

1. Solve $y = f(x)$ for x. If necessary, choose the function that corresponds to I.
2. Interchange x and y and write $y = f^{-1}(x)$.

Example. Find $f^{-1}(x)$:

$f(x) = x^2 - 2x + 1, \ x \geq 1$

$y = x^2 - 2x + 1 = (x-1)^2$

$\sqrt{y} = x - 1$

$\sqrt{y} + 1 = x \iff \boxed{f^{-1}(x) = \sqrt{x} + 1}$

$g(x) = \dfrac{x}{2} - \dfrac{7}{2}$

$y = \frac{x}{2} - \frac{7}{2}$

$y + \frac{7}{2} = \frac{x}{2}$

$2y + 7 = x \iff \boxed{f^{-1}(x) = 2x + 7}$

$h(x) = \sqrt[3]{5x+1}$

$y = \sqrt[3]{5x+1} \Rightarrow y^3 = 5x + 1$

$\Rightarrow y^3 - 1 = 5x$

$\Rightarrow \frac{y^3 - 1}{5} = x \iff \boxed{f^{-1}(x) = \frac{x^3 - 1}{5}}$

$j(x) = \dfrac{2x}{1-x}$

$y = \frac{2x}{1-x} \Rightarrow y(1-x) = 2x$

$y - xy = 2x$

$y = 2x + xy$

$y = (2+y)x$

$\frac{y}{y+2} = x \iff \boxed{f^{-1}(x) = \frac{x}{x+2}}$

$k(x) = e^x$

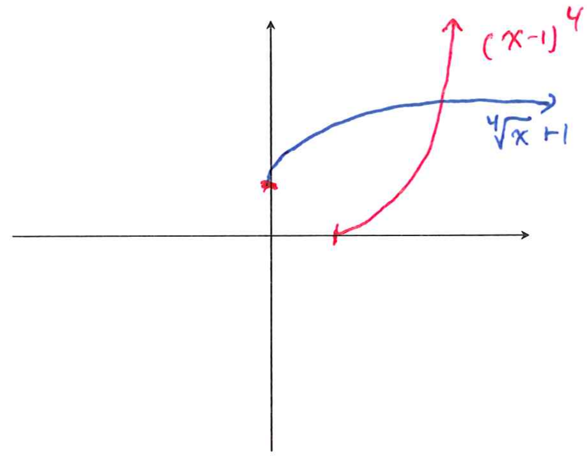
$y = e^x \iff \ln(y) = x$

Example. Find the inverse of $f(x) = (x-1)^4$ (on a restricted domain) and graph $f(x)$ and $f^{-1}(x)$.

$y = (x-1)^4$, $x \geq 1$

$\sqrt[4]{y} = x - 1$

$\sqrt[4]{y} + 1 = x \iff \boxed{f^{-1}(x) = \sqrt[4]{x} + 1}$



If $x \leq 1$, the domain of $f(x)$ is $(-\infty, 1]$ and range is $[0, \infty)$. This means $f^{-1}(x)$ has domain $[0, \infty)$ and range $(-\infty, 1]$

$y = (x-1)^4$, $x \leq 1$

$-\sqrt[4]{y} = x - 1$ because $x - 1 \leq 0$

$-\sqrt[4]{y} + 1 = x \iff \boxed{f^{-1}(x) = -\sqrt[4]{x} + 1}$