1 15.1: Graphs and Level Curves

In the previous chapter, we considered functions of the form

$$\mathbf{r}(t) = \langle f(t),\, g(t),\, h(t)\rangle,$$

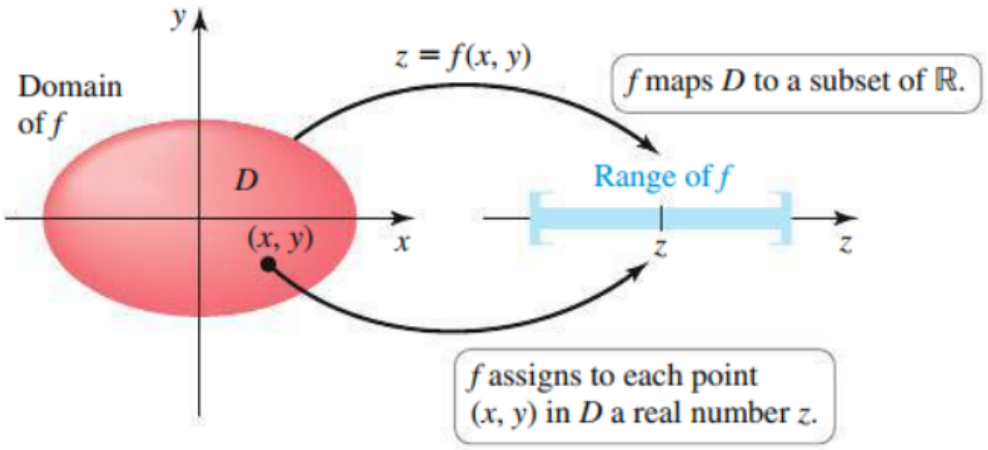
which have one independent variable t and three dependent variables $f(t)$, $g(t)$, and $h(t)$. In this chapter, we consider functions of the form

$$x_{n+1} = f(x_1, x_2, \ldots, x_n),$$

where we have multiple independent variables x_1, $x_2, \ldots,$ x_n and one single dependent variable x_{n+1}. We begin with functions of two variables:

$$z = f(x, y).$$

Definition. (Function, Domain, and Range with 2 Independent Variables) A **function** $z = f(x, y)$ assigns to each point (x, y) in a set D in $\mathbb{R}^2$ a unique real number z in a subset of $\mathbb{R}$. The set D is the **domain** of f. The **range** of f is the set of real numbers z that are assumed as the points (x, y) vary over the domain.



Example. Find the domain of the following functions:

$$f(x,y) = \frac{1}{xy+2} \qquad g(x,y) = \sqrt{108 - 3x^2 - 3y^2}$$

$$h(x,y) = \log_2\left(x^3 - y^{1/3}\right) \qquad j(x,y) = \frac{1}{\sqrt{x^2+y^2-16}}$$

Example. Roughly graph the following functions:

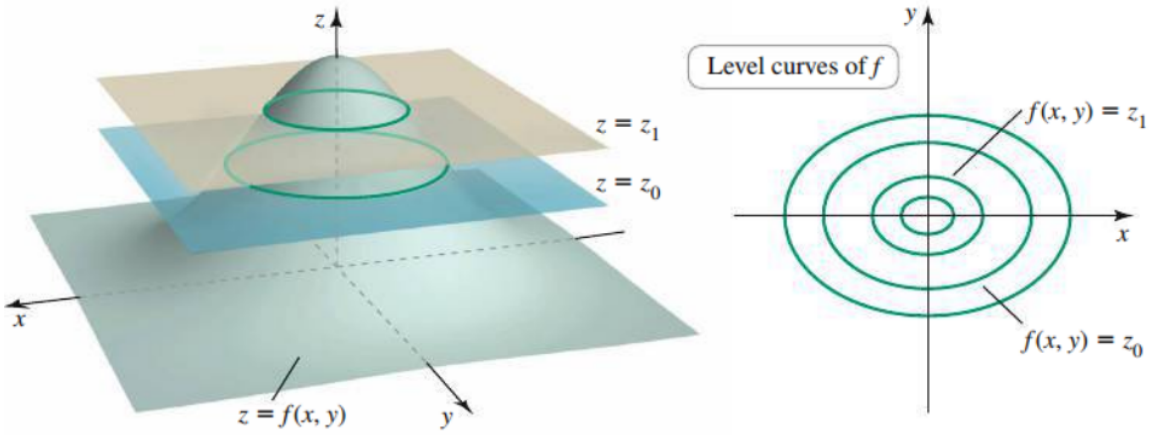
$f(x, y) = -4x + 3y - 10$

$g(x, y) = x^2 + y^2 + 4$

$h(x, y) = \sqrt{4 + x^2 + y^2}$

Level Curves:

A **contour curve** is formed by tracing a three-dimensional surface at a constant height. A **level curve** is formed when a contour curve is projected to the xy-plane.



Example. Find the level curves of the following functions:

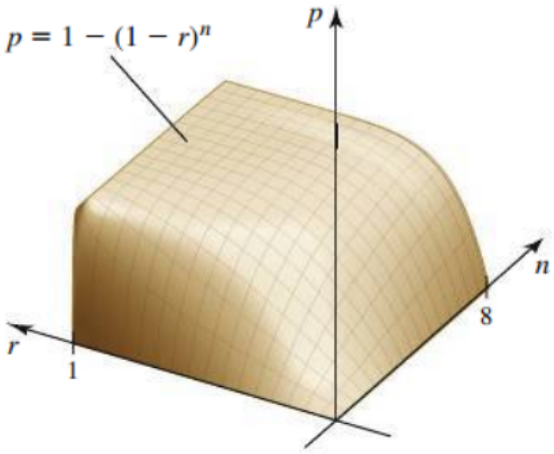
$f(x, y) = y - x^2 - 1$

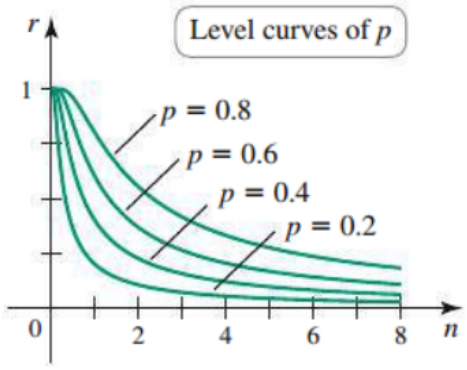
$g(x, y) = e^{-x^2-y^2}$

$h(x, y) = x^2 + y^2$

Applications of Functions of Two Variables:

Example. A probability function of two variables: Suppose on a particular day, the fraction of students on campus infected with COVID-19 is r, where $0 \leq r \leq 1$. If you have n random (possibly repeated) encounters with students during the day, the probability of meeting *at least* one infected person is $p(n, r) = 1 - (1 - r)^n$.





Functions of More than Two Variables:

Number of Independent Variables	Explicit Form	Implicit Form	Graph Resides In...
1	$y{=}f(x)$	$F(x, y){=}0$	$\mathbb{R}^2(xy-\text{plane})$
2	$z{=}f(x, y)$	$F(x, y, z){=}0$	$\mathbb{R}^3(xyz-\text{space})$
3	$w{=}f(x, y, z)$	$F(x, y, z, w){=}0$	$\mathbb{R}^4$
n	$x_{n+1}{=}f(x_1, x_2, \ldots, x_n)$	$F(x_1, x_2, \ldots, x_n, x_{n+1}){=}0$	$\mathbb{R}^{n+1}$

Definition. (Function, Domain, and Range with n Independent Variables) The **function** $x_{n+1} = f(x_1, x_2, \ldots, x_n)$ assigns a unique real number x_{n+1} to each point $(x_1, x_2, \ldots, x_n)$ in a set D in $\mathbb{R}^4$. The set D is the **domain** of f. The **range** is the set of real numbers x_{n+1} that are assumed as the points $(x_1, x_2, \ldots, x_n)$ vary over the domain.

Example. Find the domain of the following functions:

$$f(x, y, z) = 4xyz - 2xz + 5yz$$

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 9}$$

Graphs of Functions of More Than Two Variables:

The idea of level curves can be extended to **level surfaces**. Level surfaces can be used to represent functions of three variables: