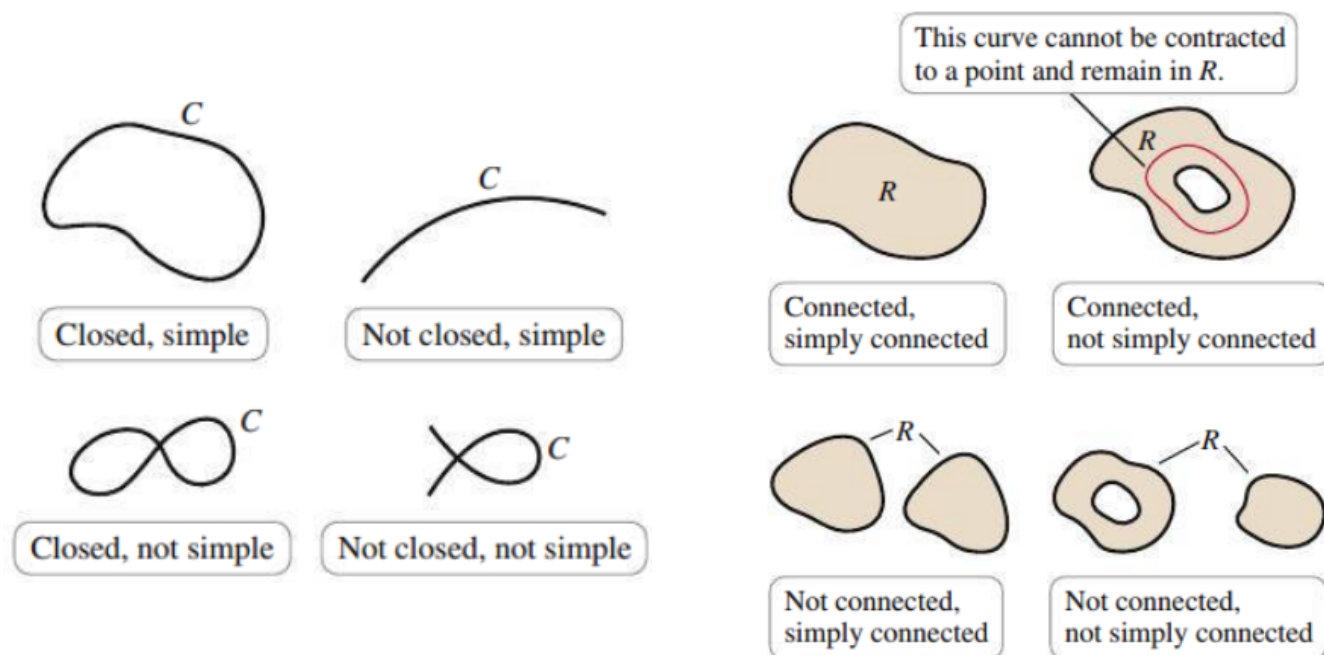


17.3: Conservative Vector Fields

Definition. (Simple and Closed Curves)

Suppose a curve C (in \mathbb{R}^2 or \mathbb{R}^3) is described parametrically by $\mathbf{r}(t)$, where $a \leq t \leq b$. Then C is a **simple curve** if $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ for all t_1 and t_2 , with $a < t_1 < t_2 < b$; that is, C never intersects itself between its endpoints. The curve C is **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$; that is, the initial and terminal points of C are the same.



Definition. (Connected and Simply Connected Regions)

An open region R in \mathbb{R}^2 (or D in \mathbb{R}^3) is **connected** if it is possible to connect any two points of R by a continuous curve lying in R . An open region R is **simply connected** if every closed simple curve in R can be deformed and contracted to a point in R .

Definition. (Conservative Vector Field)

A vector field \mathbf{F} is said to be **conservative** on a region (in \mathbb{R}^2 or \mathbb{R}^3) if there exists a scalar function φ such that $\mathbf{F} = \nabla\varphi$ on that region.

Assume that $\mathbf{F} = \langle f, g, h \rangle$ is a conservative vector field. Then, there exists φ such that

$$\langle f, g, h \rangle = \nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z \rangle$$

Now, we consider the second partial derivatives:

$$\varphi_{xy} = \varphi_{yx} \Rightarrow$$

$$\varphi_{xz} = \varphi_{zx} \Rightarrow$$

$$\varphi_{yz} = \varphi_{zy} \Rightarrow$$

Theorem 17.3: Test for Conservative Vector Fields

Let $\mathbf{F} = \langle f, g, h \rangle$ be a vector field defined on a connected and simply connected region D of \mathbb{R}^3 , where f , g , and h have continuous first partial derivatives on D . Then \mathbf{F} is a conservative vector field on D (there is a potential function φ such that $\mathbf{F} = \nabla\varphi$) if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \text{and} \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$$

For vector fields in \mathbb{R}^2 , we have the single condition $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

Example. Determine if the following vector fields are conservative:

$$\mathbf{F} = \langle e^x \cos(y), -e^x \sin(y) \rangle$$

$$\mathbf{F} = \langle 2xy - z^2, x^2 + 2z, 2y - 2xz \rangle$$

Procedure: Finding Potential Functions in \mathbb{R}^3

Suppose $\mathbf{F} = \langle f, g, h \rangle$ is a conservative vector field. To find φ such that $\mathbf{F} = \nabla\varphi$, use the following steps:

1. Integrate $\varphi_x = f$ with respect to x to obtain φ , which includes an arbitrary function $c(y, z)$.
2. Compute φ_y and equate it to g to obtain an expression for $c_y(y, z)$.
3. Integrate $c_y(y, z)$ with respect to y to obtain $c(y, z)$, including an arbitrary function $d(z)$.
4. Compute φ_z and equate it to h to get $d(z)$.

A similar procedure beginning with $\varphi_y = g$ or $\varphi_z = h$ may be easier in some cases.

Example. Find a potential function for the following conservative vector Fields:

$$\mathbf{F} = \langle e^x \cos(y), -e^x \sin(y) \rangle$$

$$\mathbf{F} = \langle 2xy - z^2, x^2 + 2z, 2y - 2xz \rangle$$

Fundamental Theorem for Line Integrals and Path Independence:

Suppose that \mathbf{F} is a conservative vector field in \mathbb{R}^3 with potential function φ .

$$\begin{aligned}\frac{d\varphi}{dt} &= \frac{\partial\varphi}{\partial x} \frac{dx}{dt} + \frac{\partial\varphi}{\partial y} \frac{dy}{dt} + \frac{\partial\varphi}{\partial z} \frac{dz}{dt} \\ &= \left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \\ &= \nabla\varphi \cdot \mathbf{r}'(t) \\ &= \mathbf{F} \cdot \mathbf{r}'(t),\end{aligned}$$

where $\mathbf{r}(t)$ defines a curve C for $a \leq t \leq b$. Now, we integrate \mathbf{F} over the curve C :

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t) dt = \int_a^b \frac{d\varphi}{dt} dt = \varphi(B) - \varphi(A)$$

where A and B are points corresponding to $\mathbf{r}(a)$ and $\mathbf{r}(b)$ respectively.

Theorem 17.4: Fundamental Theorem for Line Integrals

Let R be a region in \mathbb{R}^2 or \mathbb{R}^3 and let φ be a differentiable potential function defined on R . If $\mathbf{F} = \nabla\varphi$ (which means that \mathbf{F} is conservative), then

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A),$$

for all points A and B in R and all piecewise-smooth oriented curves C in R from A to B .

Definition. (Independence of Path)

Let \mathbf{F} be a continuous vector field with domain R . If $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for all piecewise-smooth curves C_1 and C_2 in R with the same initial and terminal points, then the line integral is **independent of path**.

Theorem 17.5

Let \mathbf{F} be a continuous vector field on an open connected region R in \mathbb{R}^2 . If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, then \mathbf{F} is conservative; that is, there exists a potential function φ such that $\mathbf{F} = \nabla\varphi$ on R .

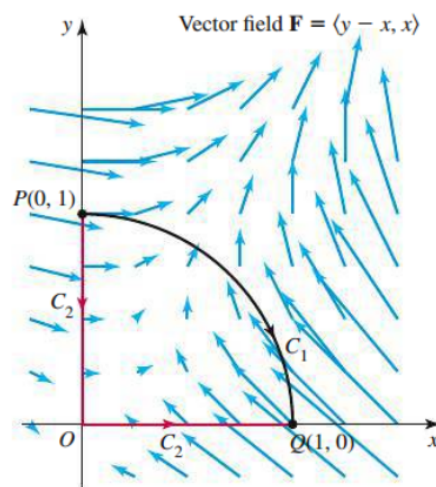
Example. Consider the potential function $\varphi(x, y) = (x^2 - y^2)/2$ with gradient field $\mathbf{F} = \langle x, -y \rangle$.

- Let C_1 be the quarter-circle $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$, for $0 \leq t \leq \pi/2$, from $A(1, 0)$ to $B(0, 1)$,
- let C_2 be the line $\mathbf{r}(t) = \langle 1 - t, t \rangle$, for $0 \leq t \leq 1$, also from A to B .

Evaluate the line integrals of \mathbf{F} on C_1 and C_2 , and show that both are equal to $\varphi(B) - \varphi(A)$.

Example. with $\mathbf{F} = \langle y - x, x \rangle$ on the following oriented paths in \mathbb{R}^2 .

a) Find the potential function $\varphi(x, y)$



b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along

the quarter-circle C_1 from $P(0, 1)$ to $Q(1, 0)$,

the path C_2 from $P(0, 1)$ to $Q(1, 0)$ via two line segments through $O(0, 0)$.

Example. Evaluate

$$\int_C \langle 2xy - z^2, x^2 + 2z, 2y - 2xz \rangle d\mathbf{r}$$

where C is the curve from $A(-3, -2, 1)$ to $B(1, 2, 3)$.

Line Integrals on Closed Curves

Let R be an open connected region in \mathbb{R}^2 or \mathbb{R}^3 . Then \mathbf{F} is a conservative vector field on R if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple closed piecewise-smooth oriented curves C in R .

Example. Evaluate $\int_C \langle 2xy + z^2, x^2, 2xz \rangle \cdot d\mathbf{r}$ where C is the circle $\mathbf{r}(t) = \langle 3 \cos(t), 4 \cos(t), 5 \sin(t) \rangle$, for $0 \leq t \leq 2\pi$.