

## 1 14.4: Length of Curves

### **Definition. (Arc Length for Vector Functions)**

Consider the parameterized curve  $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ , where  $f'$ ,  $g'$ , and  $h'$  are continuous, and the curve is traversed once for  $a \leq t \leq b$ . The **arc length** of the curve between  $(f(a), g(a), h(a))$  and  $(f(b), g(b), h(b))$  is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} dt = \int_a^b |\mathbf{r}'(t)| dt.$$

**Example** (Flight of an eagle). Suppose an eagle rises at a rate of 100 vertical ft/min on a helical path given by

$$\mathbf{r}(t) = \langle 250 \cos(t), 250 \sin(t), 100t \rangle$$

where  $\mathbf{r}$  is measured in feet and  $t$  is measured in minutes. How far does it travel in 10 minutes?

**Example.** Suppose a particle has a trajectory given by

$$\mathbf{r}(t) = \langle 10 \cos(3t), 10 \sin(3t) \rangle$$

where  $0 \leq t \leq \pi$ . How far does this particle travel?

**Example.** Find the length of the curve

$$\mathbf{r}(t) = \langle 3t^2 - 5, 4t^2 + 5 \rangle$$

where  $0 \leq t \leq 1$ .

**Example.** Find the length of  $\mathbf{r}(t) = \left\langle t^2, \frac{(4t+1)^{\frac{3}{2}}}{6} \right\rangle$  where  $0 \leq t \leq 6$ .

**Example.** Find the length of  $\mathbf{r}(t) = \langle 2\sqrt{2}, \sin(t), \cos(t) \rangle$  where  $0 \leq t \leq 5$ .

**Theorem 14.3: Arc Length as a Function of a Parameter**

Let  $\mathbf{r}(t)$  describe a smooth curve, for  $t \geq a$ . The arc length is given by

$$s(t) = \int_a^t |\mathbf{v}(u)| \, du,$$

where  $|\mathbf{v}| = |\mathbf{r}'|$ . Equivalently,  $\frac{ds}{dt} = |\mathbf{v}(t)|$ . If  $|\mathbf{v}(t)| = 1$ , for all  $t \geq a$ , then the parameter  $t$  corresponds to arc length.

**Example.** For the following functions, determine if  $\mathbf{r}(t)$  uses arc length as a parameter. If not, find a description that uses arc length as a parameter.

a)  $\mathbf{r}(t) = \langle -4t + 1, 3t - 1 \rangle, 0 \leq 4$ .

b)  $\mathbf{r}(t) = \left\langle \frac{1}{\sqrt{10}} \cos(t), \frac{3}{\sqrt{10}} \sin(t) \right\rangle, 0 \leq t \leq 2\pi$ .