1 17.4: Green's Theorem

Green's Theorem — Circulation Form

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected regin R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{circulation}} = \underbrace{\oint_C f\,dx + g\,dy}_{\text{circulation}} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA.$$

Area of a Plane Region by Line Integrals

Under the conditions of Green's Theorem, the area of a region R enclosed by a curve C is

$$\oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2} \oint_C (x\,dy - y\,dx).$$

Green's Theorem — Flux Form

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n} ds}_{\text{outward flux}} = \underbrace{\oint_C f\,dy - g\,dx}_{\text{outward flux}} = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA,$$

where $\mathbf{n}$ is the outward unit normal vector on the curve.

Definition. (Two-Dimensional Divergence)
The **two-dimensional divergence** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y}$. If the divergence is zero throughout a region, the vector field is **source free** on that region.

Conservative Fields $\mathbf{F} = \langle f, g \rangle$

$$\text{curl} = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0$$

Potential function φ with

$$\mathbf{F} = \nabla\varphi \qquad \text{or} \qquad f = \frac{\partial \varphi}{\partial x}, \qquad g = \frac{\partial \varphi}{\partial y}$$

Circulation $= \displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all closed curves C.

Source-Free Fields $\mathbf{F} = \langle f, g \rangle$

$$\text{divergence} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0$$

Stream function ψ with

$$f = \frac{\partial \psi}{\partial y}, \qquad g = -\frac{\partial \psi}{\partial x}$$

Flux $= \displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = 0$ on all closed curves C.