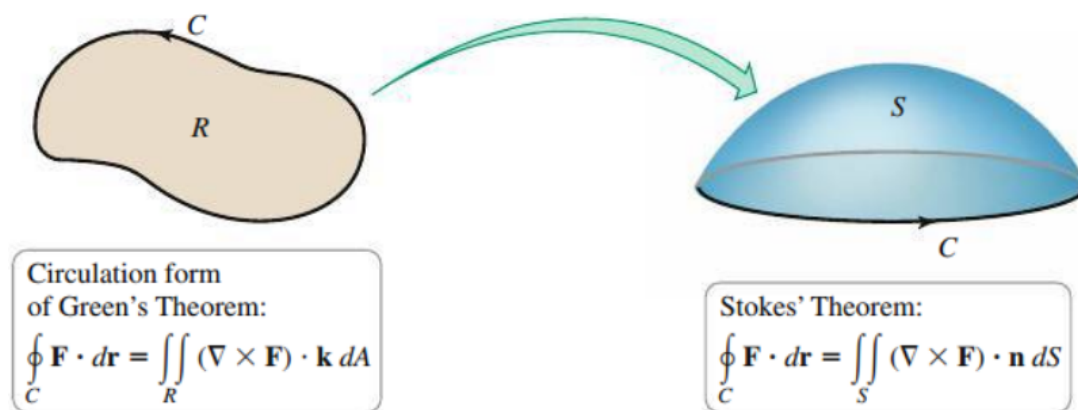


## 17.7: Stokes' Theorem

Stokes' Theorem is the three-dimensional version of the circulation form of Green's Theorem. Recall the circulation form of Green's Theorem:

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{circulation}} = \iint_R \underbrace{(g_x - f_y)}_{\text{curl or rotation}} dA.$$

The above means that the cumulative rotation within  $R$  equals the circulation along the boundary of  $R$ . Stokes' Theorem computes the circulation over a surface  $S$  in  $\mathbb{R}^3$ :



### Theorem 17.15: Stokes' Theorem

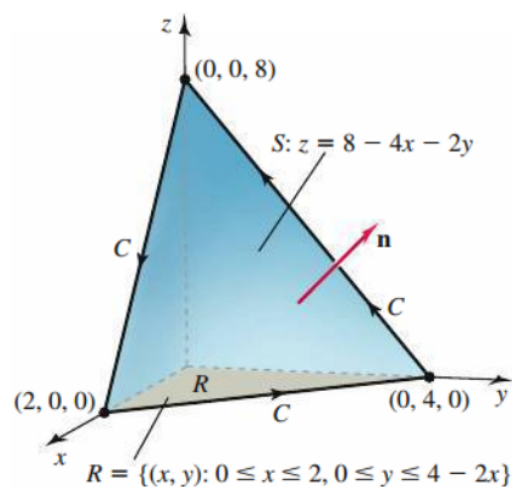
Let  $S$  be an oriented surface in  $\mathbb{R}^3$  with a piecewise-smooth closed boundary  $C$  whose orientation is consistent with that of  $S$ . Assume  $\mathbf{F} = \langle f, g, h \rangle$  is a vector field whose components have continuous first partial derivatives on  $S$ . Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS,$$

where  $\mathbf{n}$  is the unit vector normal to  $S$  determined by the orientation of  $S$ .

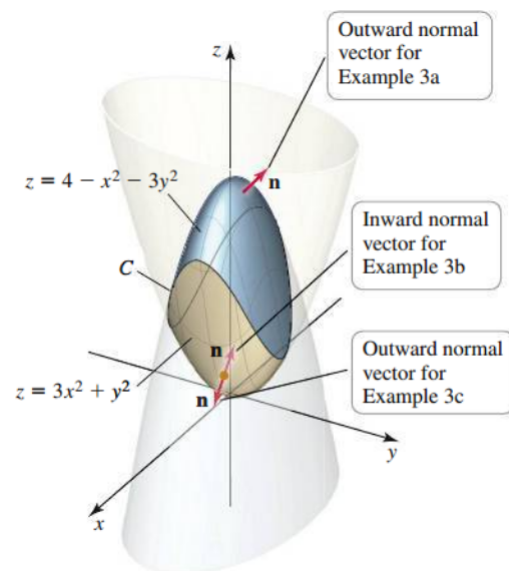
**Example. Verify Stokes' Theorem:** Confirm that Stokes' Theorem holds for the vector field  $\mathbf{F} = \langle z - y, x, -x \rangle$ , where  $S$  is the hemisphere  $x^2 + y^2 + z^2 = 4$ , for  $z \geq 0$ , and  $C$  is the circle  $x^2 + y^2 = 4$ , oriented counterclockwise.

**Example.** Evaluate the line integral  $\oint_C \mathbf{F} \cdot d\mathbf{r}$ , where  $\mathbf{F} = \langle z, -z, x^2 - y^2 \rangle$  and  $C$  consists of the three line segments that bound the plane  $z = 8 - 4x - 2y$  in the first octant.

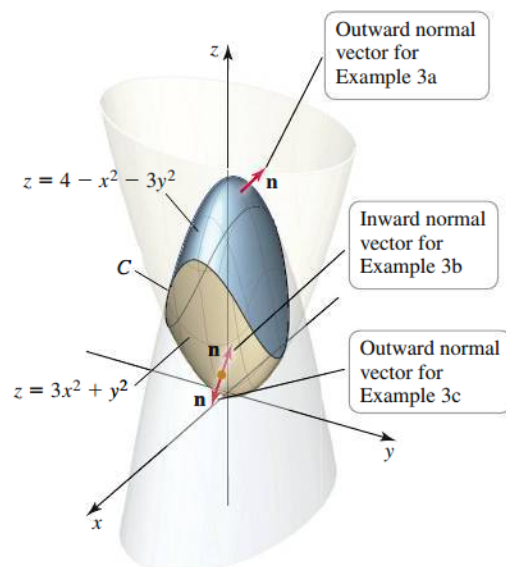


**Example.** Evaluate  $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS$ , where  $\mathbf{F} = \langle -y, x, z \rangle$ , where:

- $S$  is the part of the paraboloid  $z = 4 - x^2 - 3y^2$  contained within  $z = 3x^2 + y^2$ , with  $\mathbf{n}$  pointing upwards.



- $S$  is the part of the paraboloid  $z = 3x^2 + y^2$  contained within  $z = 4 - x^2 - 3y^2$  with  $\mathbf{n}$  pointing upwards.



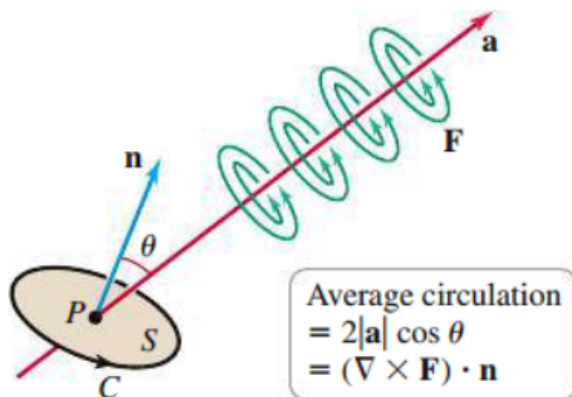
- $S$  is the part of the paraboloid  $z = 3x^2 + y^2$  contained within  $z = 4 - x^2 - 3y^2$  with  $\mathbf{n}$  pointing downwards.

### Interpreting the Curl:

The **average circulation** is

$$\frac{1}{\text{area of } S} \oint_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{\text{area of } S} \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS.$$

Consider a general rotation field  $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ , where  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{r} = \langle x, y, z \rangle$ . Now, let  $S$  be a small circular disk centered at a point  $P$ , whose normal vector  $\mathbf{n}$  makes an angle  $\theta$  with the axis  $\mathbf{a}$ :



The average circulation of this vector field on  $S$  is

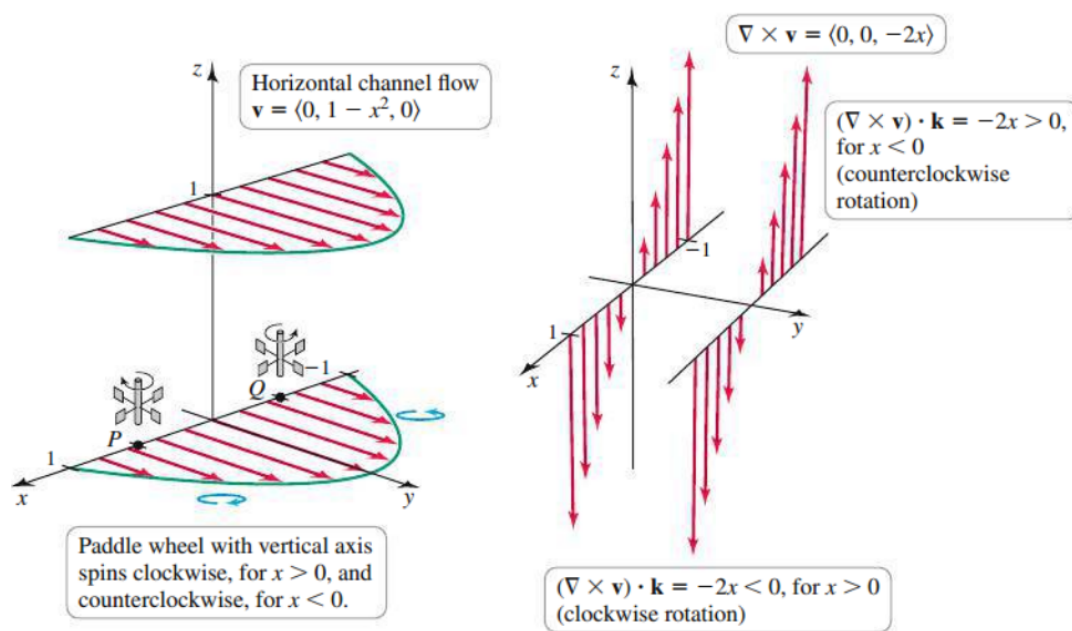
$$\begin{aligned} \frac{1}{\text{area of } S} \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS &= \frac{(\nabla \times \mathbf{F}) \cdot \mathbf{n}}{\text{area of } S} (\text{area of } S) \\ &= 2\mathbf{a} \cdot \mathbf{n} \\ &= 2|\mathbf{a}| \cos(\theta) \end{aligned}$$

From this, we see

- The scalar component of  $\nabla \times \mathbf{F}$  at  $P$  in the direction of  $\mathbf{n}$  is the average circulation of  $\mathbf{F}$  on  $S$ .
- The direction of  $\nabla \times \mathbf{F}$  at  $P$  is the direction that maximizes the average circulation of  $\mathbf{F}$  on  $S$ .

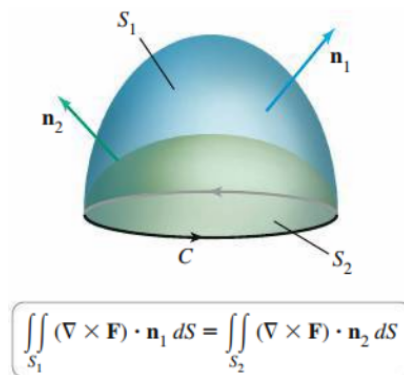
A similar argument for the curl can be applied to more general vector fields.

**Example.** Consider the vector field  $\mathbf{v} = \langle 0, 1 - x^2, 0 \rangle$  for  $|x| \leq 1$  and  $|z| \leq 1$ . Compute the curl of  $\mathbf{v}$ .



Since, using Stokes' Theorem, we evaluate the surface integral  $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$  using only the boundary  $C$ , then for any two smooth oriented surfaces  $S_1$  and  $S_2$  both with a consistent orientation with that of  $C$ ,

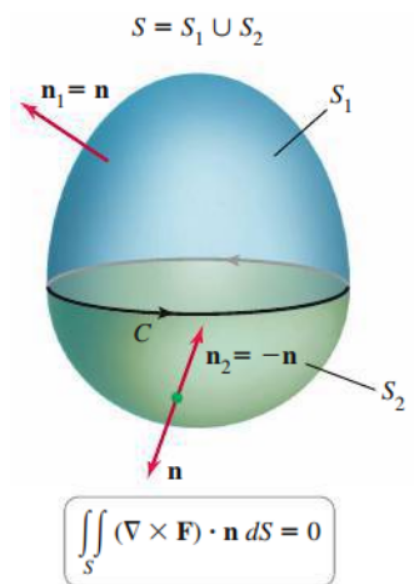
$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_1 dS = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 dS$$



Furthermore, if  $S$  is a closed surface consisting of  $S_1$  and  $S_2$ , with  $\mathbf{n} = \mathbf{n}_1$  and  $\mathbf{n} = -\mathbf{n}_2$ , then

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_1 dS + \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 dS = 0$$





Theorem 17.11 (Section 17.5) states that if  $\mathbf{F}$  is conservative, then  $\nabla \times \mathbf{F} = \mathbf{0}$ . Now, the converse follows using Stokes' Theorem:

**Theorem 17.16: Curl  $\mathbf{F} = \mathbf{0}$  implies  $\mathbf{F}$  Is Conservative**

Suppose  $\nabla \times \mathbf{F} = \mathbf{0}$  throughout an open simply connected region  $D$  of  $\mathbb{R}^3$ . Then  $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$  on all closed simple smooth curves  $C$  in  $D$ , and  $\mathbf{F}$  is a conservative vector field on  $D$ .

*Proof.* Given a closed simple smooth curve  $C$ , it can be shown that  $C$  is the boundary of at least one smooth oriented surface  $S$  in  $D$ . By Stokes' Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \underbrace{(\nabla \times \mathbf{F}) \cdot \mathbf{n}}_0 dS = 0$$

Since the line integral equals zero over all such curves in  $D$ , the vector field is conservative on  $D$ .  $\square$