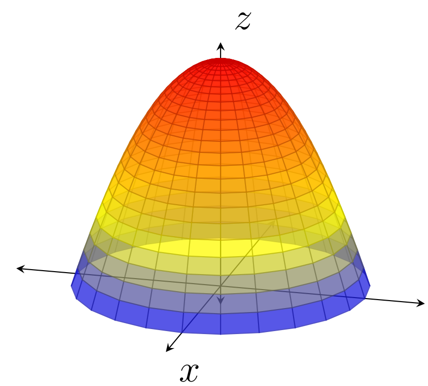
16.3: Double Integrals in Polar Coordinates

Suppose we wish to find the volume bounded by the curve $f(x, y) = 9 - x^2 - y^2$ and the xy-plane. The region of integration would be

$$R = \left\{ (x, y) : -3 \le x \le 3, -\sqrt{9 - x^2} \le y \le \sqrt{9 - x^2} \right\}$$

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 9 - x^2 - y^2 \, dy \, dx$$



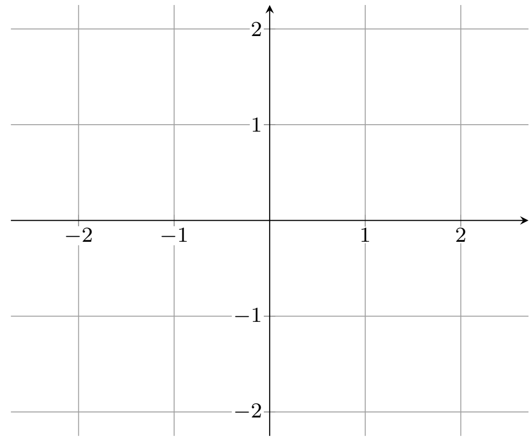
Alternatively, we can use polar coordinates where $x = r\cos(\theta)$ and $y = r\sin(\theta)$. The associated region R is called a **polar rectangle**.

> **Theorem 16.3: Change of Variables for Double Integrals over Polar Rectangle Regions**
> Let f be continuous on the region R in the xy-plane expressed in polar coordinates a s$R = \{(r, \theta) : 0 \le a \le r \le b, \alpha \le \theta \le \beta\}$, where $\beta - \alpha = 2\pi$. Then f is integrable over R, and the double integral of f over R is
>
> $$\iint\limits_R f(x, y) \, dA = \int_{\alpha}^{\beta} \int_{a}^{b} f(r\cos\theta, \, r\sin\theta) \, r \, dr \, d\theta.$$

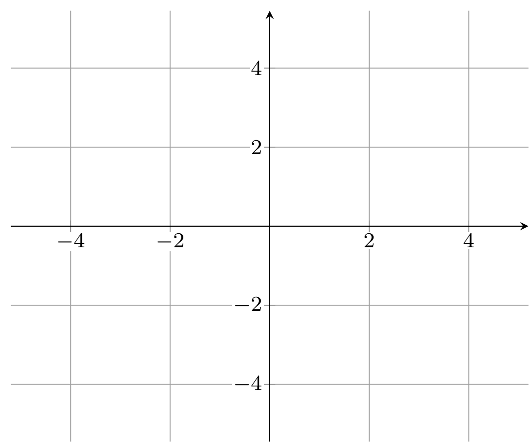
Note: When we convert to polar coordinates, there is an extra factor of r. This is due to the area of the circular segment being $\frac{1}{2}r^2\theta$ (Section 16.7 will elaborate on this).

Example. Graph the following regions:

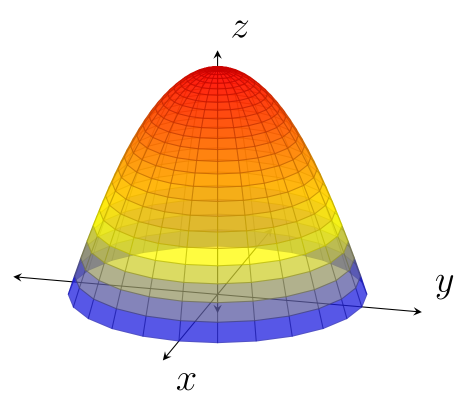
$$R = \left\{(r,\theta) : 0 \leq r \leq 1,\ 0 \leq \theta \leq \frac{5\pi}{4}\right\}$$



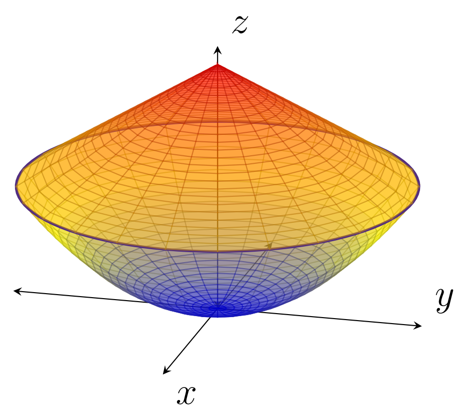
$$R = \left\{(r,\theta) : 2 \leq r \leq 4,\ -\frac{\pi}{6} \leq \theta \leq \frac{7\pi}{6}\right\}$$



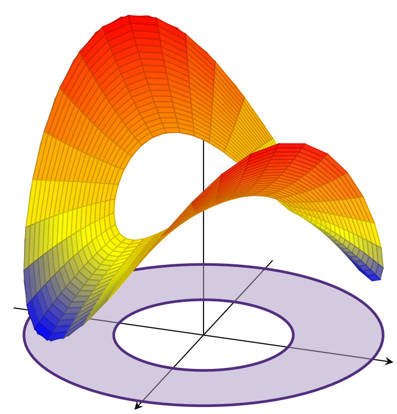
Example. Consider the paraboloid given earlier: Find the volume of the solid bounded above by $z = 9 - x^2 - y^2$ and below by the xy-plane.



Example. Find the area of the solid bounded below by the paraboloid $z = x^2 + y^2$ and bounded above by the cone $z = 2 - \sqrt{x^2 + y^2}$.



Example. Find the volume of the region beneath the surface $z = xy + 10$ and above the annular region $R = \{(r, \theta) : 2 \leq r \leq 4,\ 0 \leq \theta \leq 2\pi\}$.



Theorem 16.4: Change of Variables for Double Integrals over More General Polar Regions
Let f be continuous on the region R in the xy-plane expressed in polar coordinates as

$$R = \{(r, \theta) : 0 \leq g(\theta) \leq r \leq h(\theta),\ \alpha \leq \theta \leq \beta\},$$

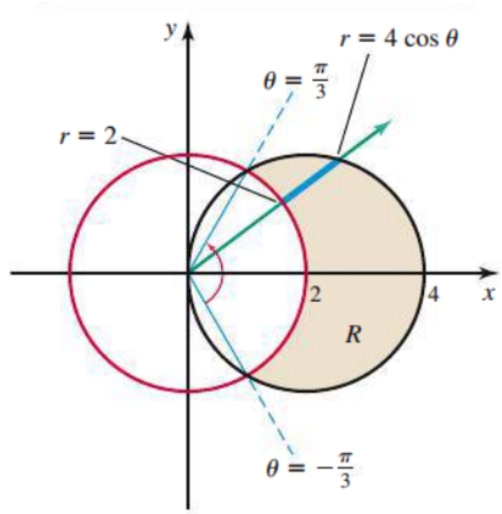
where $0 < \beta - \alpha \leq 2\pi$. Then

$$\iint\limits_R f(x, y)\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

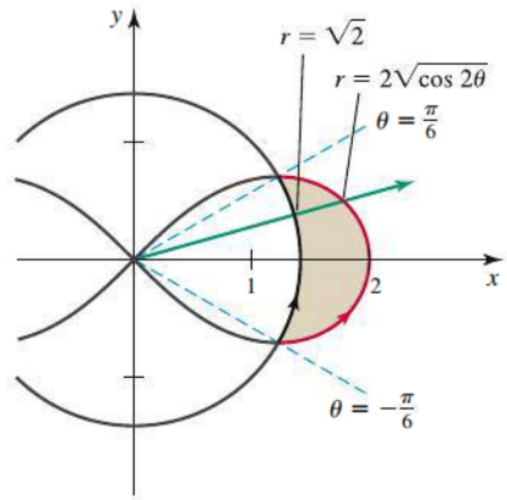
Area of Polar Regions
The area of the polar region $R = \{(r, \theta) : 0 \leq g(\theta) \leq r \leq h(\theta),\ \alpha \leq \theta \leq \beta\}$, where $0 < \beta - \alpha \leq 2\pi$, is

$$A = \iint\limits_R dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} r\, dr\, d\theta.$$

Example. Write an iterated integral in polar coordinates for $\iint_R g(r, \theta)\, dA$ for the region outside the circle $r = 2$ and inside the circle $r = 4\cos(\theta)$.



Example. Compute the area of the region in the first and fourth quadrants outside the circle $r = \sqrt{2}$ and inside the lemniscate $r^2 = 4\cos(2\theta)$.



Example. Find the average value of the y-coordinates of the points in the semicircular disk of radius a given by $R = \{(r, \theta) : 0 \leq r \leq a,\, 0 \leq \theta \leq \pi\}$.