8.9: Improper Integrals

Definition. (Improper Integrals over Infinite Intervals)

1. If f is continuous on $[a, \infty)$, then
$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$
2. If f is continuous on $(-\infty, b]$, then
$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$
3. If f is continuous on $(-\infty, \infty)$, then
$$\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to-\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx.$$
where c is any real number. It can be shown that the choice of c does not affect the value or convergence of the original integral.

If the limits in cases 1.– 3. exist, then the improper integrals **converge**; otherwise they **diverge**.

Example. Evaluate $\displaystyle\int_1^\infty \frac{\ln(x)}{x}\,dx$ and determine if the integral converges or diverges.

Example. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{e^{3x}}{1+e^{6x}}\,dx$.

Example. For what values of p does $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converge?

Example (Gabriel's Horn)**.** Let R be the region bounded by the graph of $y = 1/x$ and the x-axis for $x \geq 1$.

What is the volume of the solid generated when R is revolved around the x-axis?

What is the surface area of the solid generated when R is revolved about the x-axis?

Definition. (Improper Integrals with Unbounded Integrand)

1. Suppose f is continuous on $(a, b]$ with $\lim_{x \to a^+} f(x) = \pm\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2. Suppose f is continuous on $[a, b)$ with $\lim_{x \to b^-} f(x) = \pm\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. Suppose f is continuous on $[a, b]$ except at the interior point p where f is unbounded. Then

$$\int_a^b f(x)\,dx = \lim_{c \to p^-} \int_a^c f(x)\,dx + \lim_{d \to p^+} \int_d^b f(x)\,dx.$$

If the limits in cases 1.– 3. exist, then the improper integrals **converge**; otherwise, they **diverge**.

Example. Determine which of the following integrals are improper integrals

$$\int_0^1 \sec(x)\,dx$$

$$\int_{\pi/2}^{3\pi/4} \tan(x)\,dx$$

$$\int_1^e \ln(x)\,dx$$

$$\int_0^1 \arctan(x)\,dx$$

$$\int_0^{0.5} \ln(x)\,dx$$

$$\int_{-10}^{-1} \frac{1}{x^{1/3}}\,dx$$

Example. Evaluate $\displaystyle\int_1^9 \frac{dx}{(x-1)^{2/3}}$. Does this integral converge or diverge?

Example. Evaluate $\int_{-1}^{1} \frac{e^{2/x}}{x^2}\, dx$. Does this integral converge or diverge?

Theorem 8.2: Comparison Test for Improper Integrals
Suppose the functions f and g are continuous on the interval $[a, \infty)$, with $f(x) \geq g(x) \geq 0$, for $x \geq a$.

1. If $\displaystyle\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.
2. If $\displaystyle\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

Example. Determine if the integral $\displaystyle\int_2^\infty \frac{x^3}{x^4 - x^3 - 1}\,dx$ converges or diverges.