## 17.1: Vector Fields

$F(x,y) = \langle 1, -1 \rangle$
$F(x,y) = \langle 2x, 4y \rangle$
$F(x,y) = \langle x-y, x \rangle$

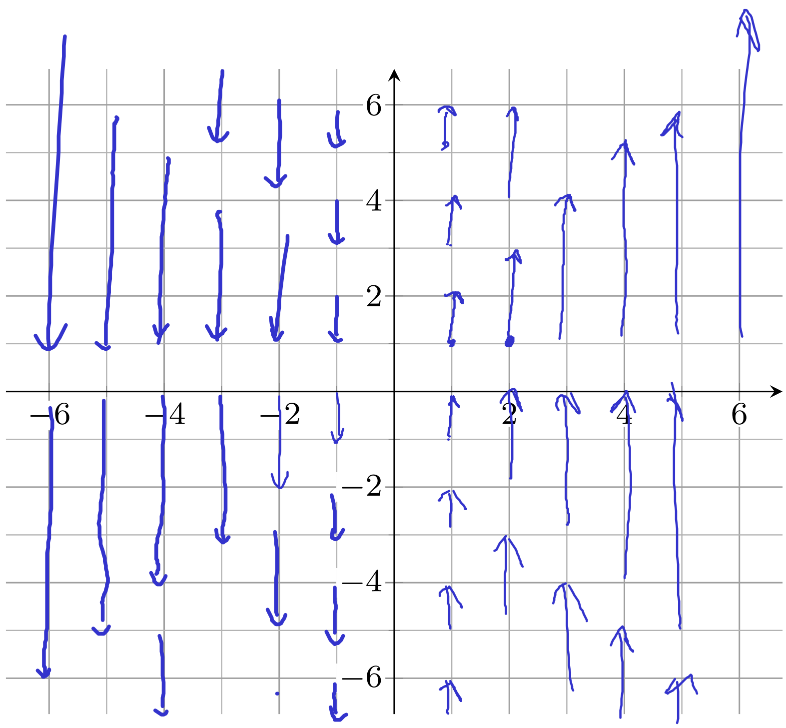
$\vec{F}$

**Definition. (Vector Fields in Two Dimensions)**
Let $f$ and $g$ be defined on a region $R$ of $\mathbb{R}^2$. A **vector field** in $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point in $R$ a vector $\langle f(x, y),\, g(x, y)\rangle$. The vector field is written as

$$\mathbf{F}(x, y) = \langle f(x, y),\, g(x, y)\rangle \quad \text{or}$$
$$\mathbf{F}(x, y) = f(x, y)\boldsymbol{i} + g(x, y)\boldsymbol{j}.$$

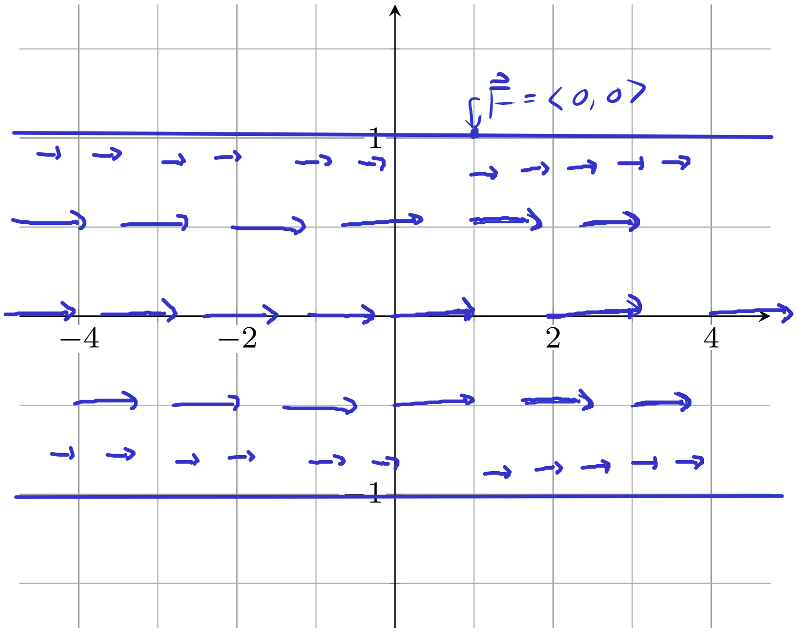
A vector field $\mathbf{F} = \langle f, g\rangle$ is continuous or differentiable on a region $R$ of $\mathbb{R}^2$ if $f$ and $g$ are continuous or differentiable on $R$, respectively.

**Example.** Sketch the vector field $\mathbf{F} = \langle 0, x\rangle$.




$F(1,1) = \langle 0, 1 \rangle$
$F(2,1) = \langle 0, 2 \rangle$

**Example.** Sketch the vector field $\mathbf{F} = \left\langle 1 - y^2, 0 \right\rangle$ for $|y| \le 1$.




$F(1,1) = \langle 0,0 \rangle$

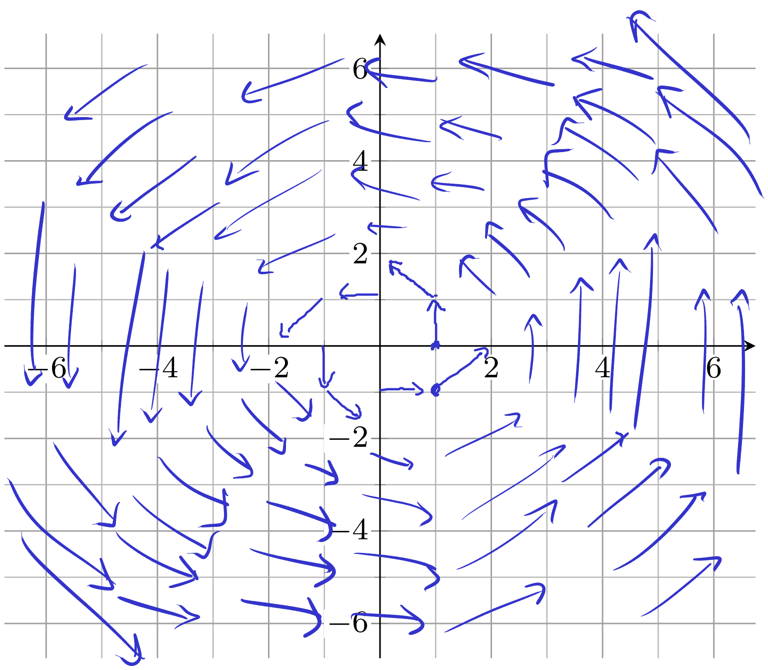
$F(1,\tfrac{1}{2}) = \langle \tfrac{3}{4}, 0 \rangle$

$F(3,\tfrac{1}{2}) = \langle \tfrac{3}{4}, 0 \rangle$

$F(0,2) = \langle 1,0 \rangle$

**Example.** Sketch the vector field $\mathbf{F} = \langle -y, x \rangle$.

$\vec{F} \cdot \langle x,y \rangle = 0$




$\vec{F}(1,1) = \langle -1,1 \rangle$

$\vec{F}(1,0) = \langle 0,1 \rangle$

$\vec{F}(1,-1) = \langle 1,1 \rangle$

**Definition. (Radial Vector Fields in $\mathbb{R}^2$)**
Let $\mathbf{r} = \langle x, y \rangle$. A vector field of the form $\mathbf{F} = f(x, y)\mathbf{r}$, where $f$ is a scalar valued function, is a **radial vector field**. Of specific interest are the radial vector fields

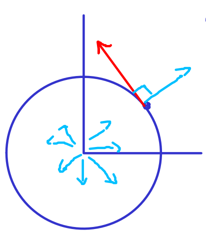
$$\mathbf{F}(x, y) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y \rangle}{|\mathbf{r}|^p} = \boxed{\frac{\mathbf{r}}{|\mathbf{r}|}} \underbrace{\frac{1}{|\mathbf{r}|^{p-1}}}_{\text{magnitude}},$$

where $p$ is a real number. At every point (expect the origin), the vectors of this field are directed outward from the origin with a magnitude of $|\mathbf{F}| = \dfrac{1}{|\mathbf{r}|^{p-1}}$.

**Example.** Let $C$ be the circle $x^2 + y^2 = a^2$, where $a > 0$.

a) Show that at each point of $C$, the radial vector field $\mathbf{F}(x, y) = \dfrac{\mathbf{r}}{|\mathbf{r}|} = \dfrac{\langle x, y \rangle}{\sqrt{x^2 + y^2}}$ is orthogonal to the line tangent to $C$ at that point.

Let $g(x,y) = x^2 + y^2$



$\nabla g(x,y) = \langle 2x, 2y \rangle$ orthogonal to tangent vector

↳ parallel to radial vector field

Theorem 15.12
Section 15.5
p 103 of notes

b) Show that at each point of $C$, the rotation vector field $\mathbf{G}(x, y) = \dfrac{\langle -y, x \rangle}{\sqrt{x^2 + y^2}}$ is parallel to the line tangent to $C$ at that point.

$\nabla g(x,y) = \langle 2x, 2y \rangle$

$g(a,b) + \nabla g(a,b) \cdot \langle x-a, y-b \rangle$

$\langle 2x, 2y \rangle \cdot \vec{G}(x,y) = 2\langle x, y \rangle \cdot \dfrac{\langle -y, x \rangle}{\sqrt{x^2+y^2}} = \dfrac{2}{\sqrt{x^2+y^2}}(-xy+xy) = 0$

**Definition. (Vector Fields and Radial Vector Fields in $\mathbb{R}^3$)**
Let $f$, $g$, and $h$ be defined on a region $D$ of $\mathbb{R}^3$. A **vector field** in $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point in $D$ a vector $\langle f(x,y,z),\, g(x,y,z),\, h(x,y,z)\rangle$. The vector field is written as

$$\mathbf{F}(x,y,z) = \langle f(x,y,z),\, g(x,y,z),\, h(x,y,z)\rangle \quad \text{or}$$
$$\mathbf{F}(x,y,z) = f(x,y,z)\boldsymbol{i} + g(x,y,z)\boldsymbol{j} + h(x,y,z)\boldsymbol{k}.$$

A vector field $\mathbf{F} = \langle f, g, h\rangle$ is continuous or differentiable on a region $D$ of $\mathbb{R}^3$ if $f$, $g$, and $h$ are continuous or differentiable on $D$, respectively. Of particular importance are the **radial vector fields**

$$\mathbf{F}(x,y,z) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x,y,z\rangle}{|\mathbf{r}|^p}, \qquad \frac{1}{|\vec{r}|^{p-1}} \quad \text{magnitude}$$

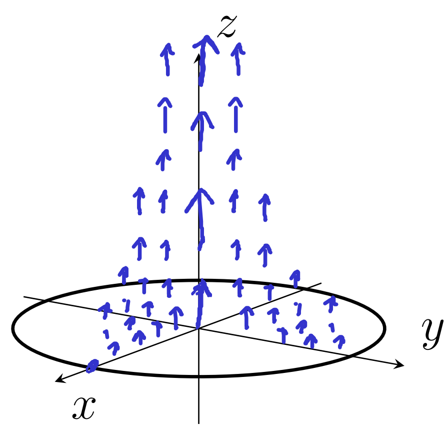
where $p$ is a real number.

**Example.** Sketch the vector field $\mathbf{F}(x,y,z) = \langle 0, 0, 1 - x^2 - y^2\rangle$, for $x^2 + y^2 \le 1$.

$\vec{F}(0,0,0) = \langle 0,0,1\rangle$

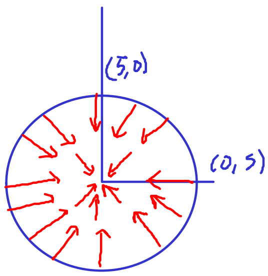
$\vec{F}(1,0,0) = \langle 0,0,1-1-0\rangle = \langle 0,0,0\rangle$

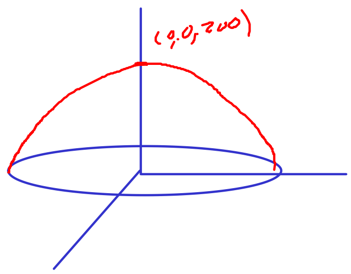
If $x^2+y^2=1$, then $F(x,y,z) = \langle 0,0,0\rangle$



> **Definition. (Gradient Fields and Potential Functions)**
> Let $\varphi$ be differentiable on a region of $\mathbb{R}^2$ or $\mathbb{R}^3$. The vector field $=\nabla\varphi$ is a **gradient field** and the function $\varphi$ is a **potential function** for **F**.

**Example.** Sketch and interpret the gradient field associated with the temperature function $T = 200 - x^2 - y^2$ on the cirular plane $R = \{(x, y) : x^2 + y^2 \leq 25\}$.

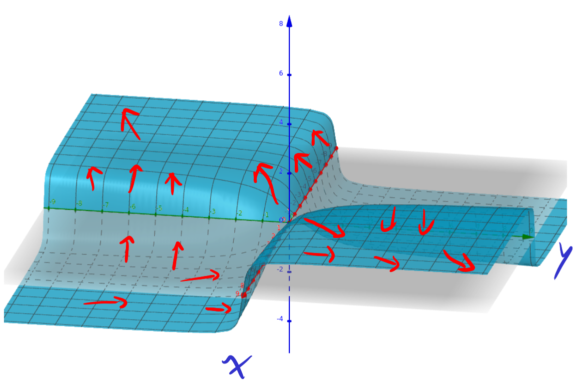


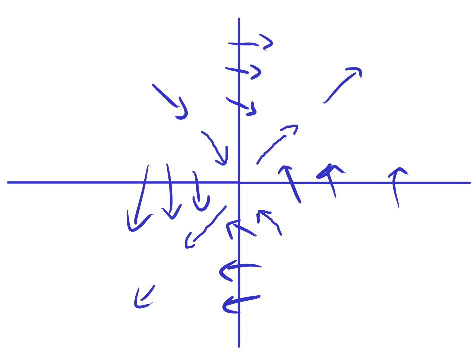




$\nabla T = \langle -2x, -2y \rangle = -2\langle x, y \rangle$

**Example.** Sketch and interpret the gradient field associated with the velocity potential $\varphi = \tan^{-1}(xy)$.

$$\nabla \varphi = \left\langle \frac{y}{1+(xy)^2}, \frac{x}{1+(xy)^2} \right\rangle$$



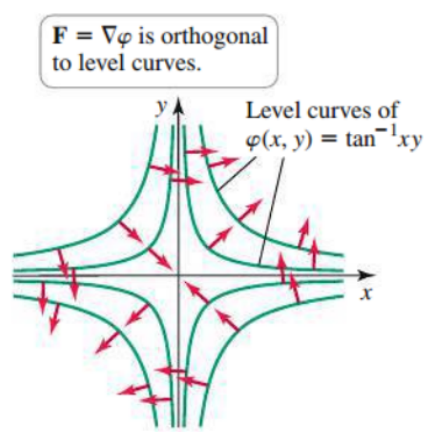



Figure 17.13