8.1: Basic Approaches (to Integration)

Example. Derive the integral formula $\displaystyle\int \sec(ax)\,dx = \frac{1}{a}\ln|\sec(ax)+\tan(ax)| + C.$

$\frac{d}{dx}[\sec(ax)] = a\sec(ax)\tan(ax)$

$$\int \sec(ax)\underbrace{\frac{\sec(ax)+\tan(ax)}{\sec(ax)+\tan(ax)}}_{1}\,dx = \int \frac{\sec^2(ax)+\sec(ax)\tan(ax)}{\sec(ax)+\tan(ax)}\,dx$$

$u = \sec(ax)+\tan(ax)$

$du = a\sec(ax)\tan(ax) + a\sec^2(ax)\,dx$

$$= \frac{1}{a}\int \frac{1}{u}\,du$$

$$= \frac{1}{a}\ln|u| + c = \frac{1}{a}\ln|\sec(ax)+\tan(ax)| + c$$

$$\boxed{\int \csc(ax)\,dx = \frac{1}{a}\ln|\csc(ax) - \cot(ax)| + c}$$

Example. Evaluate $\displaystyle\int \frac{dx}{e^{3x}+e^{-3x}}\cdot\left(\frac{e^{3x}}{e^{3x}}\right)$

$u = e^{3x}$

$du = 3e^{3x}\,dx$

$\frac{du}{3} = e^{3x}\,dx$

$$= \int \frac{e^{3x}}{(e^{3x})^2+1}\,dx$$

$$= \frac{1}{3}\int \frac{du}{u^2+1} = \frac{1}{3}\tan^{-1}(u) + c = \frac{1}{3}\tan^{-1}(e^{3x}) + c$$

$u = \csc(x)$

$du = -\csc(x)\cot(x)\,dx$

$\csc(x) = \frac{1}{\sin(x)}$

Example. Evaluate $\displaystyle\int \frac{\sin(x) + \cos^4(x)}{\csc(x)}\,dx.$

Note: $\begin{cases} \cos^2(x) = \dfrac{1+\cos(2x)}{2} \\ \sin^2(x) = \dfrac{1-\cos(2x)}{2} \end{cases}$

$$= \int \frac{\sin(x)}{\csc(x)} + \frac{\cos^4(x)}{\csc(x)}\,dx$$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$$= \int \sin^2(x) + \sin(x)\left(\cos(x)\right)^4 dx$$

$$= \int \frac{1-\cos(2x)}{2}\,dx - \int u^4\,du$$

$$= \frac{x}{2} - \frac{\sin(2x)}{4} - \frac{u^5}{5} + C = \frac{x}{2} - \frac{\sin(2x)}{4} - \frac{\cos^5(x)}{5} + C$$

Example. Evaluate $\displaystyle\int \frac{2x^2 + 3x - 4}{x-2}\,dx.$

$$\begin{array}{r|l} & \quad 2x + 7 \\ \hline x-2 & 2x^2 + 3x - 4 \\ & -(2x^2 - 4x) \\ \hline & \qquad 7x - 4 \\ & \quad -(7x - 14) \\ \hline & \qquad\quad 10 \end{array}$$

$$= \int 2x + 7 + \frac{10}{x-2}\,dx$$

$$= x^2 + 7x + 10\ln|x-2| + C$$

$$\frac{1}{\sqrt{1-u^2}}$$

Example. Evaluate $\displaystyle\int \frac{dx}{\sqrt{7-6x-x^2}}$.

$2b = 6$
$b = 3$

$u = 7 - 6x - x^2$
$du = -6 - 2x \; dx$

Completing the square

$\left(\frac{6}{2}\right)^2 = 3^2 = 9$

$-x^2 - 6x + 7 = -(x^2 + 6x \; \underline{+9} \; \underline{-9}) + 7$

factor this out

$= -(x^2+6x+9) - (-9) + 7$

$(\; + b)^2 = x^2 + \underline{2b}x + b^2$ $\qquad \left(\frac{2b}{2}\right)^2$

$= -(x+3)^2 + 16$

$-x^2 - 6x + 7$
$-(x^2 + 6x - 7)$
$-(x - 1)(x + 7)$

$$\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}}$$

$$\int \frac{dx}{\sqrt{7-6x-x^2}} = \int \frac{dx}{\sqrt{16-(x+3)^2}}$$

Let $u = \frac{x+3}{4}$

$du = \frac{1}{4}\,dx$

$$= \int \frac{dx}{4\sqrt{1-\left(\frac{x+3}{4}\right)^2}}$$

$$= \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(u) + C = \boxed{\sin^{-1}\left(\frac{x+3}{4}\right) + C}$$