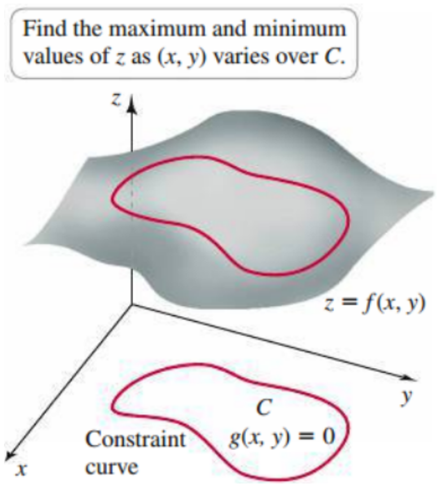
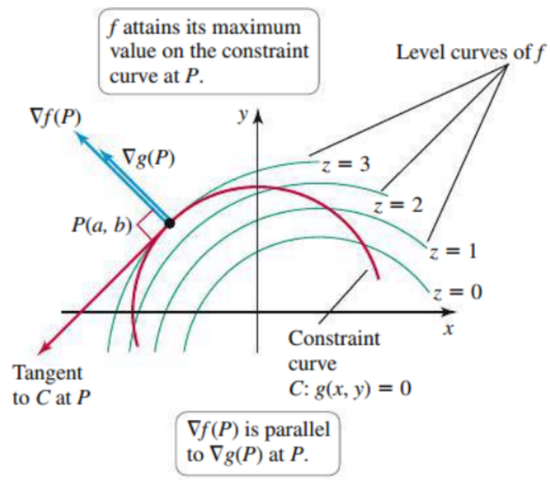
15.8: Lagrange Multipliers

Constrained optimization functions have an **objective function** f with the restriction that the independent variables x and y lie on a **constraint** curve C in the xy-plane given by $g(x,y)=0$.



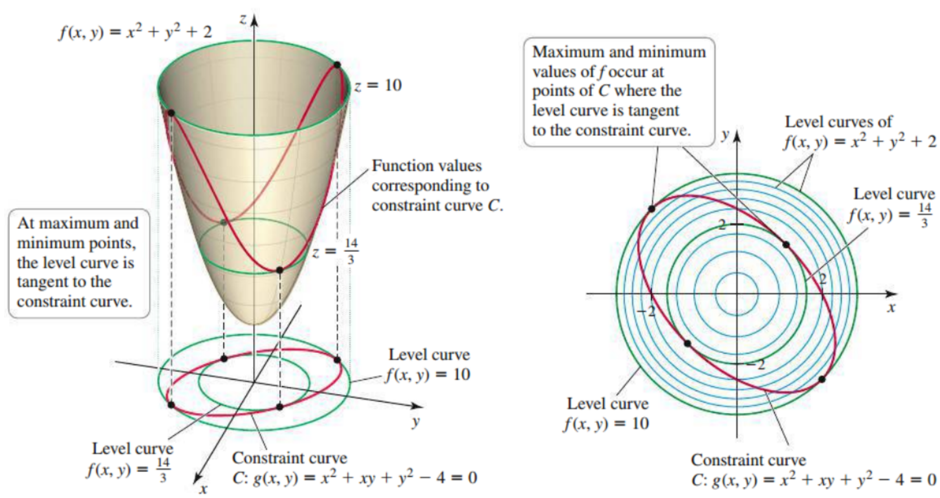


Definition. (Parallel Gradients)
Let f be a differentiable function in a region of $\mathbb{R}^2$ that contains the smooth curve C given by $g(x,y)=0$. Assume f has a local extreme value on C at a point $P(a,b)$. Then $\nabla f(a,b)$ is orthogonal to the line tangent to C at P. Assuming $\nabla g(a,b)\neq \mathbf{0}$, it follows that there is a real number λ (called a **Lagrange multiplier**) such that $\nabla f(a,b)=\lambda\nabla g(a,b)$.

We consider the three following cases:

- Bounded constraint curves that close on themselves (e.g. circles, ellipses, etc),
- Bounded constraint curves that do not close on themselves, but include endpoints,
- Unbounded constraint curves

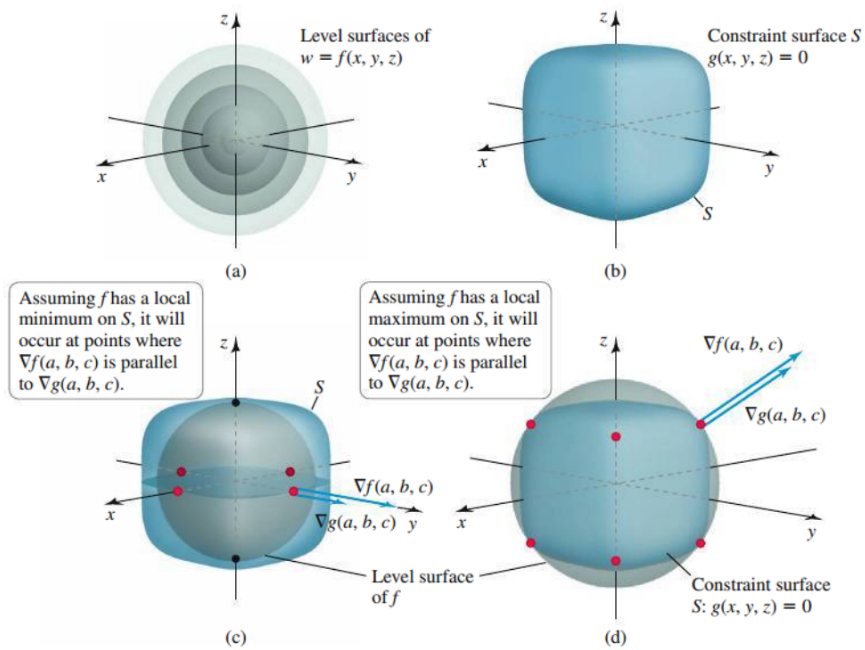
Example. Find the absolute maximum and minimum values of the objective function $f(x, y) = x^2+y^2+2$, where x and y lie on the ellipse C given by $g(x, y) = x^2+xy+y^2-4 = 0$.



Procedure- Lagrange Multipliers: Absolute Extrema on Closed and Bounded Constraint Curves
Let the objective function f and the constraint function g be differentiable on a region $\mathbb{R}^2$ with $\nabla g(x, y) \neq \mathbf{0}$ on the curve $g(x, y) = 0$. To locate the absolute maximum and minimum values of f subject to the constraint $g(x, y) = 0$, carry out the following steps.

1. Find the values of x, y, and λ (if they exist) that satisfy the equations
$$\nabla f(x, y) = \lambda \nabla g(x, y) \text{ and } g(x, y) = 0.$$
2. Evaluate f at the values (x, y) in Step 1 and at the endpoints of the constraint curve (if they exist). Select the largest and smallest corresponding function values. These values are the absolute maximum and minimum values of f subject to the constraint.

Using Lagrange multipliers extends to higher dimensions with three or more independent variables:



Example. Find the least distance between the point $P(3, 4, 0)$ and the surface of the cone $z^2 = x^2 + y^2$.

Example. Find the absolute maximum value of the utility function $U = f(\ell, g) = \ell^{1/3}g^{2/3}$, subject to the constraint $G(\ell, g) = 3\ell + 2g - 12 = 0$, where $\ell \geq 0$ and $g \geq 0$.

Example. Find the maximum value of $x_1 + x_2 + x_3 + x_4$ subject to the condition that $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 16$.

Procedure- Lagrange Multipliers: Absolute Extrema on Closed and Bounded Constraint Surfaces

Let f and g be differentiable on a region of $\mathbb{R}^3$ with $\nabla g(x, y, z) \neq \mathbf{0}$ on the surface $g(x, y, z) = 0$. To locate the absolute maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$, carry out the following steps.

1. Find the values of x, y, z, and λ that satisfy the equations

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \text{ and } g(x, y, z) = 0.$$

2. Among the points (x, y, z) found in Step 1, select the largest and smallest corresponding function values. These values are the absolute maximum and minimum values of f subject to the constraint.