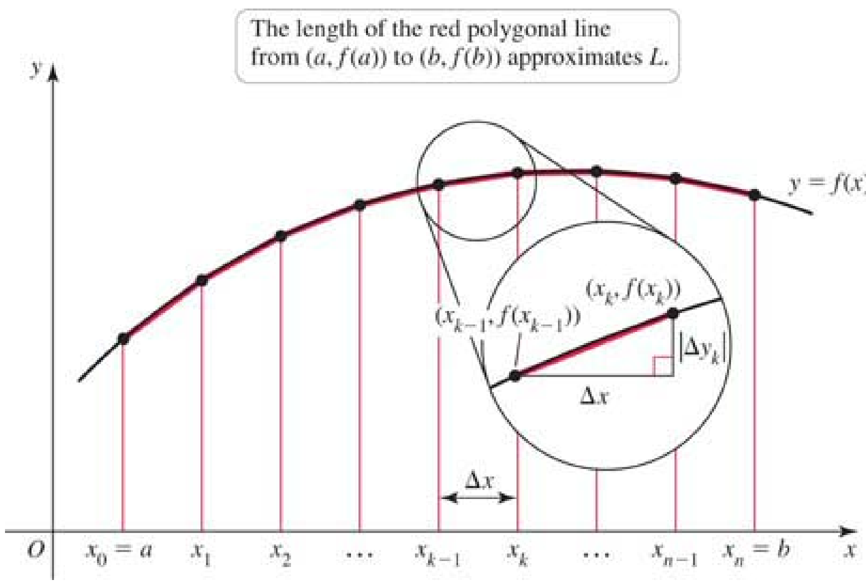
6.5: Length of Curves

Definition. (Arc Length for $y = f(x)$)
Let f have a continuous first derivative on the interval $[a, b]$. The length of the curve from $(a, f(a))$ to $(b, f(b))$ is

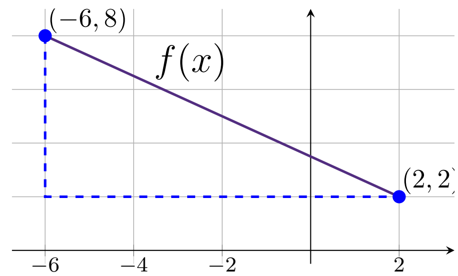
$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$



Definition. (Arc Length for $x = g(y)$)
Let g have a continuous first derivative on the interval $[c, d]$. The length of the curve from $(g(c), c)$ to $(g(d), d)$ is

$$L = \int_c^d \sqrt{1 + g'(y)^2}\, dy.$$

Example. Using a geometric argument, we can see that the length of $f(x) = -\frac{3}{4}x + \frac{7}{2}$ on the interval $[-6, 2]$ is $L = 10$. Compute this using the arc-length formula.



Example. Find the arc length of the curve $y = \dfrac{1}{4}x^2 - \dfrac{1}{2}\ln(x)$, for $1 \leq x \leq 2$.

Example. Find the arc length of the curve $y = \dfrac{1}{3}x^{3/2}$ on $[0, 12]$.

Example. Find a curve that passes through $(1, 2)$ on $[2, 6]$ whose arc length is computed using

$$\int_2^6 \sqrt{1 + 16x^{-2}}\, dx.$$

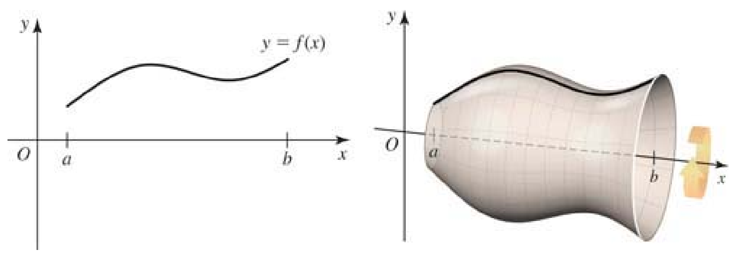
Example. Suppose f has length L on $[a, b]$. Evaluate

$$\int_{a/c}^{b/c} \sqrt{1 + f'(cx)^2}\, dx.$$

6.6: Surface Area

Definition. (Area of a Surface of Revolution)
Let f be a nonnegative function with a continuous first derivative on the interval $[a, b]$. The area of the surface generated when the graph of f on the interval $[a, b]$ is revolved around the x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx.$$



Example. Find the exact area of the surface obtained by rotating the curve $y = x^3$, $0 \le x \le 2$ about the x-axis.

Example. Find the exact area of the surface obtained by rotating the curve $y = \sqrt{8x - x^2}$, $1 \leq x \leq 7$ about the x-axis.

Example. Find the exact area of the surface obtained by rotating the curve $y = \frac{1}{2}(e^x + e^{-x})$, $-\ln(2) \leq x \leq \ln(2)$ about the x-axis.

6.7: Physical Applications

Definition. (Mass of a One-Dimensional Object)
Suppose a thin bar or wire is represented by the interval $a \le x \le b$ with a density function ρ (with units of mass per length). The **mass** of the object is

$$m = \int_a^b \rho(x)\, dx.$$

Example. A thin bar, represented by the interval $0 \le x \le 4$, has density in units of kg/m given by $\rho(x) = 5e^{-2x}$. What is the mass of the bar?

Definition. (Work)
The work done by a variable force F moving an object along a line from $x = a$ to $x = b$ in the direction of the force is

$$W = \int_a^b F(x)\,dx.$$

Example. According to **Hooke's Law**, the force required to keep a spring in a compressed or stretched position x units from the equilibrium position is $F(x) = kx$, where the positive spring constant k measures the stiffness of the spring.

Suppose a force of 40 N is required to stretch a spring 0.1 m from its equilibrium position. Assuming the spring obeys Hooke's Law, how much work is required to stretch the spring 0.4 m beyond is equilibrium position?

Example (Work from force)**.** How much work is required to move an object from $x = 1$ to $x = 3$ (measured in meters) in the presence of a force (in N) given by $F(x) = \dfrac{2}{x^2}$ acting along the x-axis?

Example. Imagine a chain of length L meters with constant density ρ kg/m is hanging vertically. Using g to represent the acceleration due to gravity, the work required to lift the chain is

$$W = \int_0^L \rho g(L - y)\, dy$$

A 50 meter long chain hangs vertically from a cylinder attached to a winch. Assume there is no friction in the system and the chain has a density of 3 kg/m. How much work is required to wind the entire chain onto the cylinder if a 60-kg load is attached to the end of the chain? Use g for the acceleration due to gravity.

Example. A 30-meter long rope hangs freely from a ledge. The rope has a density of 5 kg/m. How much work is done if the top 1/3 of the rope is pulled up to the ledge? Use g for the acceleration due to gravity.

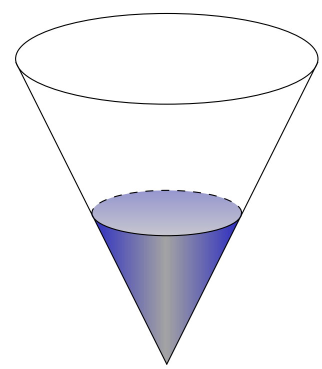
Procedure: Solving Pumping Problems

1. Draw a y-axis in the vertical direction (parallel to gravity) and choose a convenient origin. Assume the interval $[a, b]$ corresponds to the vertical extent of the fluid.
2. For $a \leq y \leq b$, find the cross-sectional area $A(y)$ of the horizontal slices and the distance $D(y)$ the slices must be lifted.
3. The work required to lift the water is

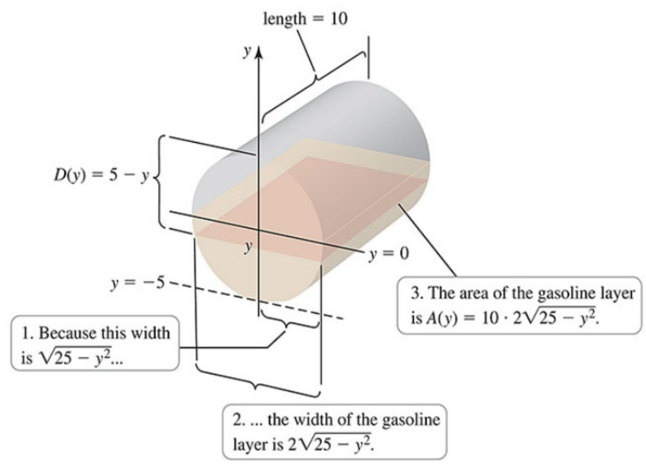
$$W = \int_a^b \rho g A(y) D(y)\, dy.$$

Note: Lifting problems are a special case of pumping problems where $A(y) = 1$.

Example. A water tank is shaped like an inverted cone with height 6 meters and base radius 1.5 meters. If the tank is full, how much work is required to pump the water to the level of the top of the tank and out of the tank? Use g for the acceleration due to gravity and note that the density of water is 1000 kg/m^3.



Example. (Pumping gasoline) A cylindrical tank with a length of 10 m and a radius of 5 m is on its side and half full of gasoline. How much work is required to empty the tank through an outlet pipe at the top of the tank? The density of gasoline is $\rho = 737\,\mathrm{kg/m^3}$

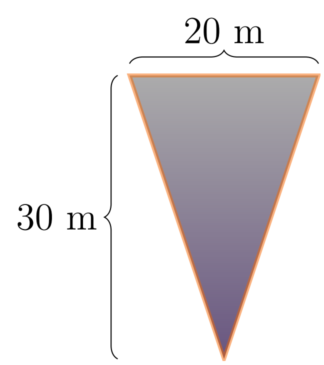


Procedure: Solving Force-on-Dam Problems

1. Draw a y-axis on the face of the dam in the vertical direction and choose a convenient origin (often taken to be the base of the dam).
2. Find the width function $w(y)$ for each value of y on the face of the dam.
3. If the base of the dam is at $y = 0$ and the top of the dam is at $y = a$, then the total force on the dam is

$$F = \int_0^a \rho g \underbrace{(a-y)}_{\text{depth}} \underbrace{w(y)}_{\text{width}} \, dy.$$

Example. The figure to the right shows the shape and dimensions of a small dam. Assuming the water level is at the top of the dam, find the total force on the face of the dam. Use ρ for the density of the water and g for the acceleration due to gravity.



Example. Force on a building A large building shaped like a box is 50 m high with a face that is 80 m wide. A strong wind blows directly at the face of the building, exerting a pressure of 150 $\mathrm{N/m^2}$ at the ground and increasing with height according to $P(y) = 150 + 2y$, where y is the height above the ground. Calculate the total force on the building, which is a measure of the resistance that must be included in the design of the building.

8.1: Basic Approaches (to Integration)

Example. Derive the integral formula $\int \sec(ax)\,dx = \dfrac{1}{a}\ln|\sec(ax)+\tan(ax)| + C$.

Example. Evaluate $\displaystyle\int \frac{dx}{e^{3x}+e^{-3x}}$.

Example. Evaluate $\displaystyle\int \frac{\sin(x)+\cos^4(x)}{\csc(x)}\,dx.$

Note: $\begin{cases} \cos^2(x) = \dfrac{1+\cos(2x)}{2} \\ \sin^2(x) = \dfrac{1-\cos(2x)}{2} \end{cases}$

Example. Evaluate $\displaystyle\int \frac{2x^2+3x-4}{x-2}\,dx.$

Example. Evaluate $\displaystyle\int \frac{dx}{\sqrt{7-6x-x^2}}$.

8.2: Integration by Parts

Integraton by Parts
Suppose u and v are differentiable functions. Then
$$\int u\,dv = uv - \int v\,du.$$

A good mnemonic is ILATE.

Example. Evaluate $\int xe^{-\frac{x}{2}}\,dx$.

Integration by Parts for Definite Integrals
Let u and v be differentiable. Then

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx$$

Example. Find the area of the region between the x-axis and $f(x) = \dfrac{\ln(x)}{x^2}$ on $[1, e]$.

Example. Evaluate $\int x^2 \cos(2x)\, dx$.

Example. Evaluate $\int e^{-x} \sin(3x)\, dx$.

Example. Evaluate $\int e^{4x} \cos(3x)\, dx$.

Example. Derive the integral formula

$$\int \ln(x)\,dx = x\ln(x) - x + C$$

Example. Evaluate $\displaystyle\int 10\cos(\sqrt{x})\,dx$

Example. Evaluate $\int_1^e \ln(2x)\,dx$.

8.3: Trigonometric Integrals

Important trigonometric identities

Pythagorean Identities	$\sin^2(\theta) + \cos^2(\theta) = 1$
Angle sum formulas	$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$ $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
Double angle formulas	$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
Half angle formulas	$\sin^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$ $\cos^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$

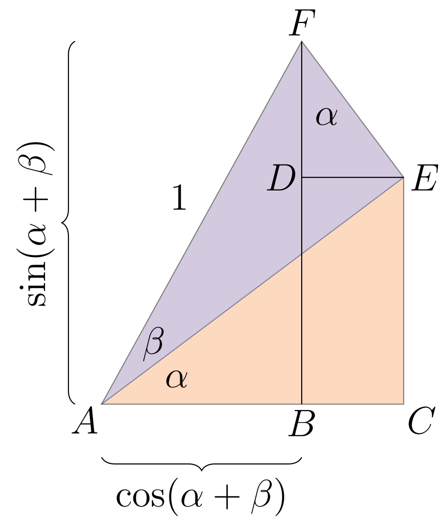
Derivation of angle sum formulas

$$\sin(\alpha) = \frac{\overline{DE}}{\overline{EF}} = \frac{\overline{DE}}{\sin(\beta)} \quad \Rightarrow \overline{DE} = \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha) = \frac{\overline{DF}}{\overline{EF}} = \frac{\overline{DF}}{\sin(\beta)} \quad \Rightarrow \overline{DF} = \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha) = \frac{\overline{CE}}{\overline{AE}} = \frac{\overline{CE}}{\cos(\beta)} \quad \Rightarrow \overline{CE} = \sin(\alpha)\cos(\beta)$$

$$\cos(\alpha) = \frac{\overline{AC}}{\overline{AE}} = \frac{\overline{AC}}{\cos(\beta)} \quad \Rightarrow \overline{AC} = \cos(\alpha)\cos(\beta)$$



Derivation of the double angle formulas

$$\sin(2\theta) = \sin(\theta+\theta) = \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos(\theta+\theta) = \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta) = \cos^2(\theta) - \sin^2(\theta)$$

Derivation of the half angle formulas

Start with the cosine double angle formula:

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \boxed{2\cos^2(\theta) - 1} = \boxed{1 - 2\sin^2(\theta)}$$

Solve for either $\sin^2(\theta)$ or $\cos^2(\theta)$:

$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2} \qquad\qquad \cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$

Example. Evaluate the integral $\int \cos^5(x)\,dx$.

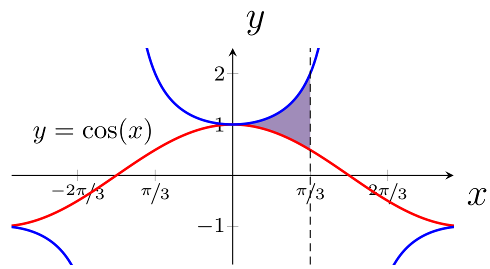
Example. Evaluate the integral $\displaystyle\int \sin^3(x)\cos^{3/2}(x)\,dx$.

Example. Evaluate the integral $\displaystyle\int 20\sin^2(x)\cos^2(x)\,dx$

Example. Evaluate the integral $\int \sec^6(x)\tan^4(x)\,dx$.

Example. Evaluate the integral $\int 35 \tan^5(x) \sec^4(x)\, dx$.

Example. Consider the region bounded by $y = \sec(x)$ and $y = \cos(x)$ for $0 \leq x \leq {}^{\pi}/_{3}$. Find the volume of the solid generated when rotating this region about the line $y = -1$.



Example. Find the length of the curve $y = \ln(2\sec(x))$ on the interval $[0, \pi/6]$.

$\int \sin^m(x)\cos^n(x)\,dx$	**Strategy**
m odd and positive, n real	Split off $\sin(x)$, rewrite the resulting even power of $\sin(x)$ in terms of $\cos(x)$, and then use $u = \cos(x)$.
n odd and positive, m real	Split off $\cos(x)$, rewrite the resulting even power of $\cos(x)$ in terms of $\sin(x)$, and then use $u = \sin(x)$.
m and n both even, nonnegative integers	Use half-angle formulas to transform the integrand into a polynomial in $\cos(2x)$, and apply the preceding strategies once again to powers of $\cos(2x)$ greater than 1.
$\int \tan^m(x)\sec^n(x)\,dx$	
n even and positive, m real	Split off $\sec^2(x)$, rewrite the remaining even power of $\sec(x)$ in terms of $\tan(x)$, and use $u = \tan(x)$.
m odd and positive, n real	Split off $\sec(x)\tan(x)$, rewrite the remaining even power of $\tan(x)$ in terms of $\sec(x)$, and use $u = \sec(x)$.
m even and positive, n odd and positive	Rewrite $\tan^m(x)$ in terms of $\sec(x)$
$\int \sec^n(x)\,dx$	
n odd	Use integration by parts with $u = \sec^{n-2}(x)$ and $dv = \sec^2(x)\,dx$
n even	Split off $\sec^2(x)$, rewrite the remaining powers of $\sec(x)$ in terms of $\tan(x)$, and use $u = \tan(x)$.
$\int \tan^m(x)\,dx$	Split off $\tan^2(x)$ and rewrite in terms of $\sec(x)$. Expand into difference of integrals substituting $u = \tan(x)$. Repeat the process as needed for remaining powers of $\tan(x)$.