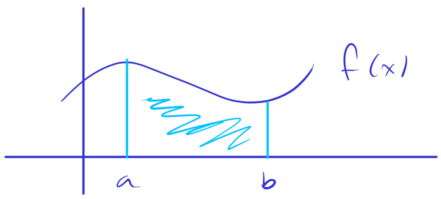
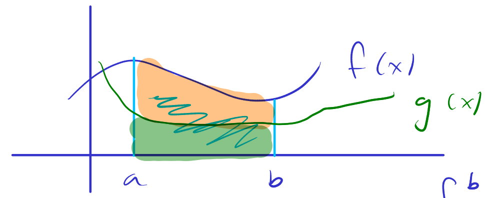


$\int_a^b f(x)\,dx$

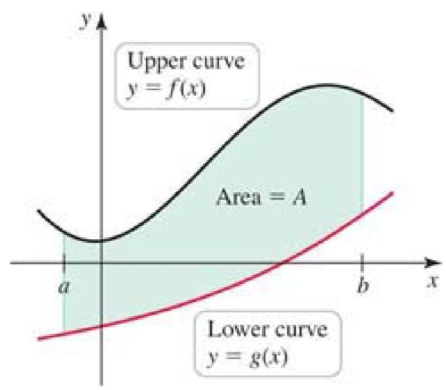


$\int_a^b f(x)\,dx$

6.2: Regions Between Curves

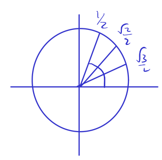
Definition. (Area of a Region Between Two Curves)
Suppose f and g are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$. The area of the region bounded by the graphs of f and g on $[a, b]$ is

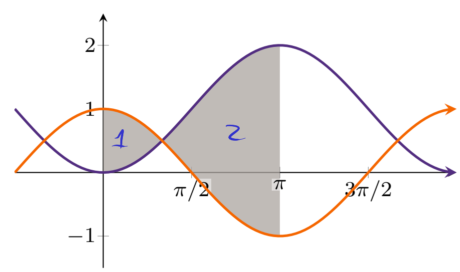
$$A = \int_a^b (f(x) - g(x))\,dx.$$



Example. Consider the region bounded by the curves $y = \cos(x)$ and $y = 1 - \cos(x)$, $0 \leq x \leq \pi$. Set up the integral(s) representing the area of this region.

$\cos(x) \geq 1 - \cos(x)$ $\quad 0 \leq x \leq \pi/3$





$\cos(x) = 1 - \cos(x)$

$\cos(x) = \frac{1}{2}$

$x = \pi/3$

$\cos(x) \leq 1 - \cos(x)$ $\quad \frac{\pi}{3} \leq x \leq \pi$

$$A = \underbrace{\int_0^{\pi/3} \cos(x) - (1 - \cos(x))\,dx}_{1} + \underbrace{\int_{\pi/3}^{\pi} (1 - \cos(x)) - \cos(x)\,dx}_{2}$$

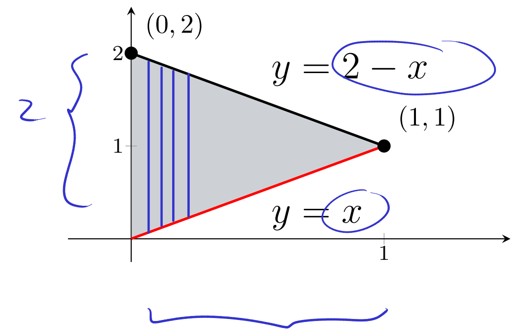
Solve $2-x=x \rightarrow 2=2x$
$x=1$

Example. Find the area of the region by integrating with respect to x.

$$A=\int_0^1 \underbrace{(2-x)}_{\text{upper}} - \underbrace{x}_{\text{lower}}\, dx = \int_0^1 2-2x\, dx$$

$$= 2x-x^2\Big|_0^1$$

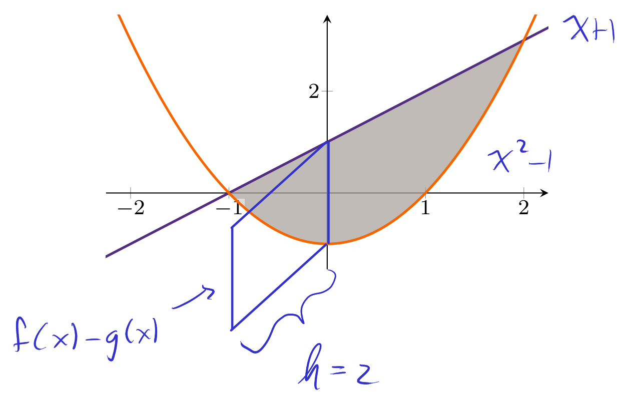
$$=(2-1)-(0-0) = 1$$



$A=\frac{1}{2}(2)(1)=1$

Example. Find the volume of the solid whose base is bounded by the graphs of $y=x+1$ and $y=x^2-1$, with the cross sections in the shape of rectangles of height 2 taken perpendicular to the x-axis.

$$V=\int_{-1}^{2} 2\left((x+1)-(x^2-1)\right)dx$$



Definition. (Area of a Region Between Two Curves with Respect to y) Suppose f and g are continuous functions with $f(y) \geq g(y)$ on the interval $[c, d]$. The area of the region bounded by the graphs $x = f(y)$ and $x = g(y)$ on $[c, d]$ is

$$A = \int_c^d (f(y) - g(y))\, dy.$$

Example. Find the area of the region bounded by $x = 3y$, and $x = y^2 - 10$

$\frac{x}{3} = y$ $y = \pm\sqrt{x+10}$

$\sqrt{x+10} = x/3$ by integrating with respect to x

$x+10 = x^2/9$

$0 = x^2 - 9x - 90$

$= x^2 - 15x + 6x - 90$

$= x(x-15) + 6(x-15)$

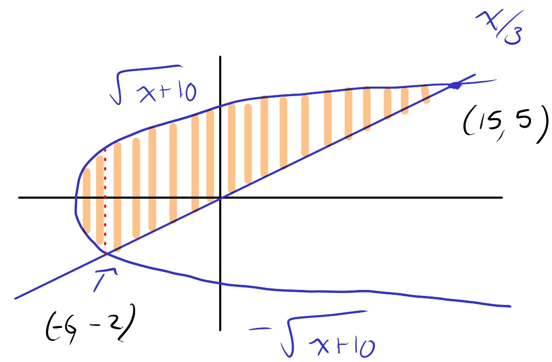
$= (x+6)(x-15) \to x = -6,\ x = 15$

1·90
2·45
3·30
5·18
6·15

$\sqrt{x+10} = -\sqrt{x+10}$

$2\sqrt{x+10} = 0$

$x = -10$



$$A = \int_{-10}^{-6} \sqrt{x+10} - \left(-\sqrt{x+10}\right) dx + \int_{-6}^{15} \sqrt{x+10} - x/3 \; dx$$

by integrating with respect to y

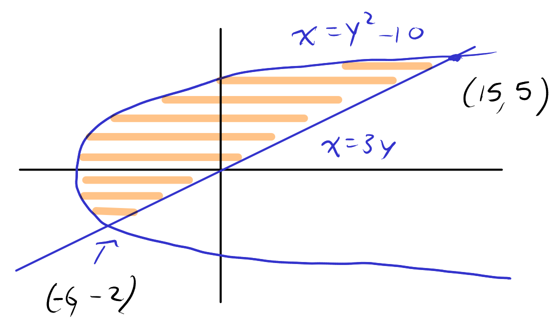
$y^2 - 10 = 3y$

$y^2 - 3y - 10 = 0$

$(y-5)(y+2) = 0$

$y = 5,\ y = -2$

$$\int_{-2}^{5} (3y) - (y^2 - 10)\, dy$$

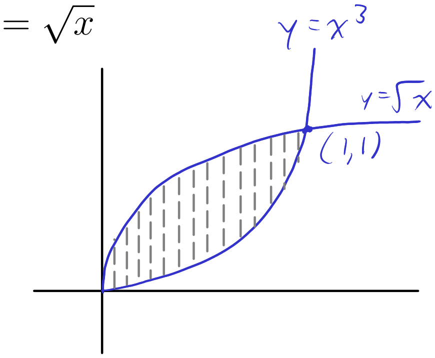


$$= \frac{3}{2}y^2 - \frac{y^3}{3} + 10y \Big|_{-2}^{5}$$

y	$3y$	$y^2 - 10$
0	0	-10

$$= \left(\frac{3}{2}(25) - \frac{125}{3} + 50\right) - \left(6 + \frac{8}{3} - 20\right) = \frac{343}{6}$$

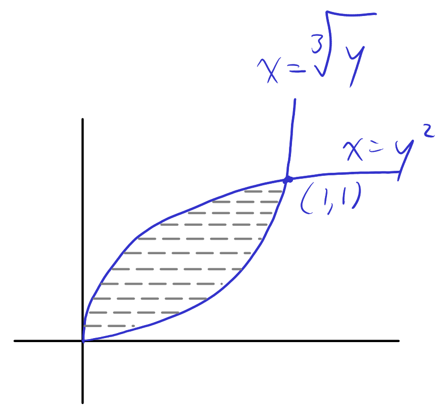
Example. Find the area of the region bounded by $y = x^3$, and $y = \sqrt{x}$



by integrating with respect to x

$$A = \int_0^1 \sqrt{x} - x^3 \, dx = \frac{2}{3}x^{3/2} - \frac{x^4}{4}\Bigg|_0^1$$

$$= \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$



by integrating with respect to y

$$A = \int_0^1 \sqrt[3]{y} - y^2 \, dy = \frac{3}{4}y^{4/3} - \frac{y^3}{3}\Bigg|_0^1$$

$$= \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$$

Example. Find the area of the region bounded by $y = 4\sqrt{2x}$, $y = 2x^2$, and $y = -4x + 6$

$4\sqrt{2x} = 2x^2$

$\sqrt{2x} = \frac{x^2}{2}$

$2x = \frac{x^4}{4}$

$0 = x^4 - 8x$

$= x(x^3 - 8)$

$x = 0 \quad x = 2$

$4\sqrt{2x} = -4x + 6$

$32x = 16x^2 + 36 - 48x$

$0 = 16x^2 - 80x + 36$

$0 = 4(4x^2 - 20x + 9)$

$= 4(4x^2 - 2x - 18x + 9)$

$= 4(2x - 9)(2x - 1) \quad x = 1/2$

$\rightarrow x = 9/2$

This point corresponds to y=-12

1.36

2.18

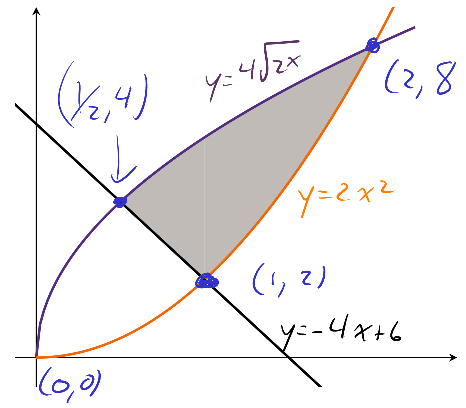
$2x^2 = -4x + 6$

$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$x = -3, \; x = 1$

negative



If we integrate with respect to x:

The intervals are $[1/2, 1]$ and $[1, 2]$

$$A = \int_{1/2}^{1} 4\sqrt{2x} - (-4x+6)\, dx + \int_{1}^{2} 4\sqrt{2x} - 2x^2\, dx = \left[\frac{8\sqrt{2}}{3} x^{3/2} + 2x^2 - 6x\right]_{1/2}^{1} - \left[\frac{8\sqrt{2}}{3} x^{3/2} - \frac{2}{3}x^3\right]_{1}^{2} = \frac{19}{6}$$

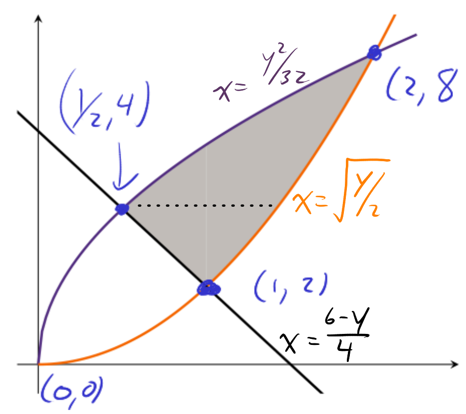
If we integrate with respect to y:

The intervals are $[2, 4]$ and $[4, 8]$

$$A = \int_{2}^{4} \left(\sqrt{y/2}\right) - \left(\frac{6-y}{4}\right) dy + \int_{4}^{8} \left(\sqrt{y/2}\right) - \left(y^2/32\right) dy$$

$$= \left[\frac{\sqrt{2}}{3} y^{3/2} - \frac{3}{2}y + \frac{y^2}{8}\right]_{2}^{4} + \left[\frac{\sqrt{2}}{3} y^{3/2} - \frac{y^3}{96}\right]_{4}^{8}$$

$$= \frac{19}{6}$$



Tip: Take the square root first when evaluating 4^(3/2)
Tip: Factor 96 into 8*12 when evaluating 8^3/96