$x = f(\theta)\cos\theta$

$y = f(\theta)\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{\underbrace{f'(\theta)\cos\theta - f(\theta)\sin\theta}_{\text{chain rule}}}$$

12.3: Calculus in Polar Coordinates

Theorem 12.2: Slope of a Tangent Line
Let f be a differentiable function at θ_0. The slope of the line tangent to the curve $r = f(\theta)$ at the point $(f(\theta_0), \theta_0)$ is

$$\left.\frac{dy}{dx}\right|_{\theta=\theta_0} = \frac{f'(\theta_0)\sin(\theta_0) + f(\theta_0)\cos(\theta_0)}{f'(\theta_0)\cos(\theta_0) - f(\theta_0)\sin(\theta_0)},$$

provided the denominator is nonzero at the point. At angles θ_0 for which $f(\theta_0) = 0$, $f'(\theta_0) \neq 0$, and $\cos(\theta_0) \neq 0$, the tangent line is $\theta = \theta_0$ with slope $\tan(\theta_0)$.

Example. Compute the slope of the line tangent to the polar curve $\overset{f(\theta)}{r = e^{-\theta}}$ at $\theta = \pi$.

$f'(\theta) = -e^{-\theta}$

$$\frac{dy}{dx} = \frac{-e^{-\theta}\sin\theta + e^{-\theta}\cos\theta}{-e^{-\theta}\cos\theta - e^{-\theta}\sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{-e^{-\pi}(0) + e^{-\pi}(-1)}{-e^{-\pi}(-1) - e^{-\pi}(0)} = \boxed{-1}$$

Example (LC 34.1-34.3)**.** Consider the polar curve $r = 2\sin(\theta)$.

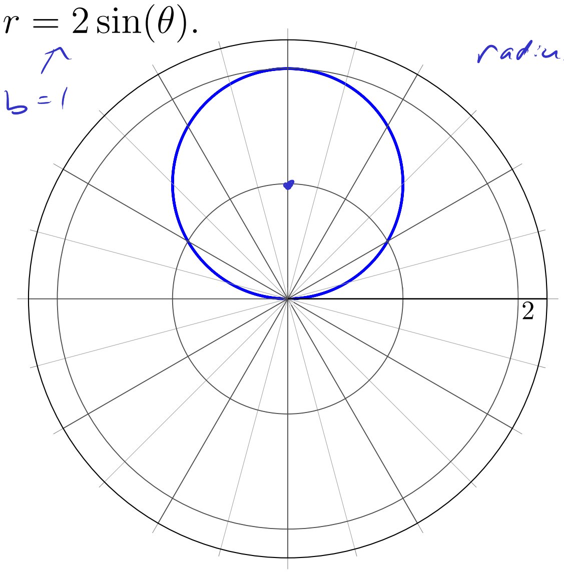
$r = 2a\cos\theta + 2b\sin\theta$

radius $= \sqrt{a^2+b^2}$

(a, b)

$b = 1$

Express this polar curve in Cartesian coordinates



$r = 2\sin\theta$

$x = r\cos\theta = 2\sin\theta\cos\theta$

$y = r\sin\theta = 2\sin^2\theta$

$r = 2\sin\theta$

$x = r\cos\theta = 2\sin\theta\cos\theta$

$y = r\sin\theta = 2\sin^2\theta$

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} \qquad r := f(\theta) = 2\sin\theta \qquad f'(\theta) = 2\cos\theta$$

Locate all points (r, θ), where this curve has a horizontal tangent line

$$\frac{dy}{dx} = \frac{2\cos\theta(\sin\theta) + 2\sin\theta(\cos\theta)}{2\cos\theta(\cos\theta) - 2\sin\theta(\sin\theta)} = \frac{4\sin\theta\cos\theta}{2(\cos^2\theta - \sin^2\theta)} = \frac{2\sin\theta\cos\theta}{\cos(2\theta)}$$

Solve $4\sin\theta\cos\theta = 0 \rightarrow \theta = 0, \pi/2, \pi, 3\pi/2$

$$\left.\frac{dy}{dx}\right|_{\theta=0} = \frac{0}{\cos(0)} = \frac{0}{1} = 0 \qquad \boxed{(0,0)}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{0}{\cos(2\pi)} = \frac{0}{1} = 0 \qquad \boxed{(0, 2\pi)}$$

$r = 2\sin(\frac{\pi}{2}) = 2$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \frac{0}{\cos(\pi)} = \frac{0}{-1} = 0 \qquad \boxed{(2, \pi/2)}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{3\pi}{2}} = \frac{0}{\cos(3\pi)} = \frac{0}{-1} = 0 \qquad \boxed{(0, 3\pi)}$$

Locate all points (r, θ), where this curve has a vertical tangent line

$$\frac{dy}{dx} = \frac{4\sin\theta\cos\theta}{2(\cos^2\theta - \sin^2\theta)} = \frac{2\sin\theta\cos\theta}{\cos(2\theta)}$$

Solve either $\cos^2\theta - \sin^2\theta = 0$ or $\cos(2\theta) = 0$

$\cos^2\theta = \sin^2\theta$

$\tan^2\theta = 1$

$\rightarrow \tan\theta = \pm 1$

$\rightarrow \theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$

$\rightarrow 2\theta = \frac{\pi}{2} + n\pi$

$\theta = \frac{\pi}{4} + 2n\frac{\pi}{4}$

$r = 2\sin(\pi/4) = \sqrt{2}$

$\boxed{(\sqrt{2}, \pi/4)}$

$r = 2\sin(\frac{3\pi}{4}) = \sqrt{2}$

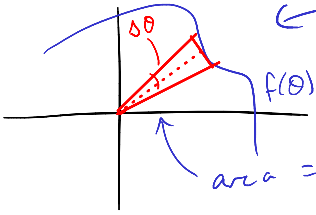
$\boxed{(\sqrt{2}, 3\pi/4)}$

$r = 2\sin(\frac{5\pi}{4}) = -\sqrt{2}$

$(-\sqrt{2}, \frac{5\pi}{4})$

$r = 2\sin(7\pi/4) = -\sqrt{2}$

$(-\sqrt{2}, \frac{7\pi}{4})$



← Curve in polar coordinates

area $= \frac{1}{2} f(\theta)^2 \Delta\theta$ ↖ Area of segment of a circle

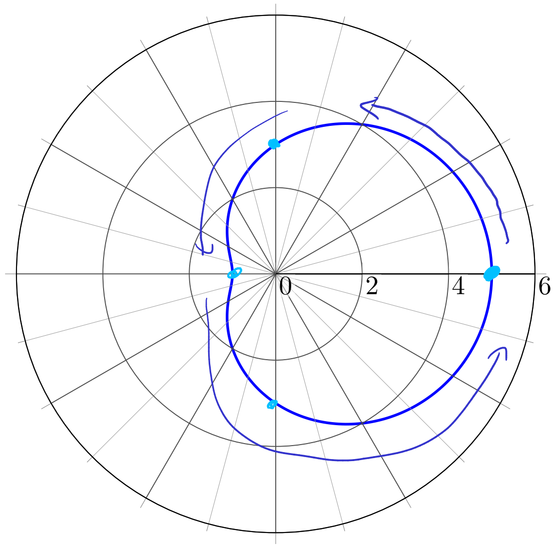
Definition. (Area of Regions in Polar Coordinates)
Let R be the region bounded by the graphs of $r = f(\theta)$ and $r = g(\theta)$, between $\theta = \alpha$ and $\theta = \beta$, where f and g are continuous and $f(\theta) \geq g(\theta) \geq 0$ on $[\alpha, \beta]$. The area of R is

$$\int_\alpha^\beta \frac{1}{2}\left(f(\theta)^2 - g(\theta)^2\right) d\theta.$$

Example (LC 34.5-34.6)**.** Find the area enclosed by the polar curve $\overset{f(\theta)}{r} = 3 + 2\cos(\theta)$.

$x = r\cos\theta$
$y = r\sin\theta$

θ	r	(x, y)
0	5	$(5, 0)$
$\pi/2$	3	$(0, 3)$
π	1	$(-1, 0)$
$\frac{3\pi}{2}$	3	$(0, -3)$
2π	5	$(5, 0)$



$$A = \int_0^{2\pi} \frac{1}{2}\Big[\overbrace{(3+2\cos\theta)}^{f^2}{}^2 - \overbrace{(0)}^{g^2}{}^2\Big]\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 9 + 12\cos\theta + 4\underbrace{\cos^2\theta}_{\frac{1+\cos(2\theta)}{2}}\, d\theta = \frac{1}{2}\left[11\theta + 12\sin\theta + \frac{\sin(2\theta)}{4}\right]_0^{2\pi} = \boxed{11\pi}$$

Example (LC 34.7-34.9). Find the area of the region inside the polar curve $r = 2$ and outside of the polar curve $r = 3 + 2\sin(\theta)$.

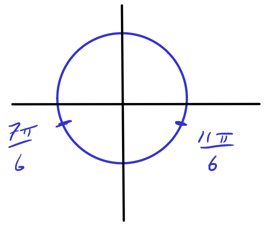
Intersection

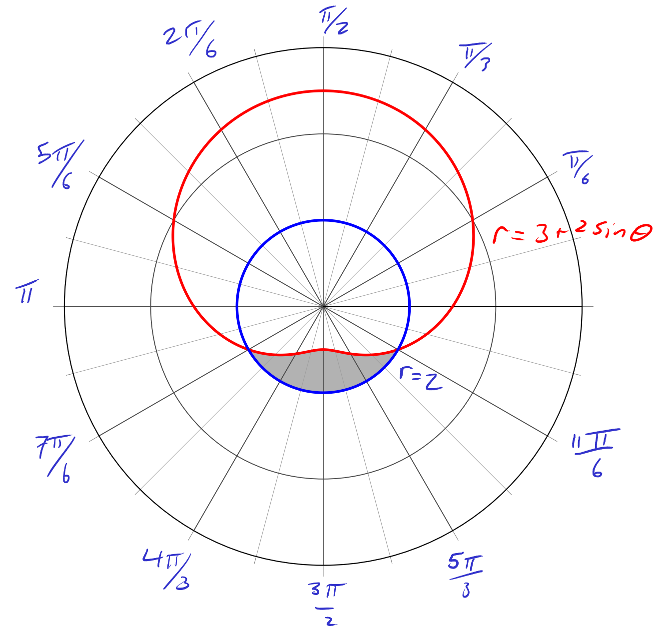
$$r = 2 = 3 + 2\sin\theta$$

$$-\tfrac{1}{2} = \sin\theta$$

$$\theta = \frac{7\pi}{6},\ \frac{11\pi}{6}$$

$$\left(2, \frac{7\pi}{6}\right),\ \left(2, \frac{11\pi}{6}\right)$$





Test point:

$$\theta = \frac{3\pi}{2} \longrightarrow r = 3 + 2\sin\left(\frac{3\pi}{2}\right) = 1 < \underset{\substack{\uparrow \\ \text{other curve}}}{2}$$

$$\Longrightarrow r = 2 \quad \text{outer curve}$$

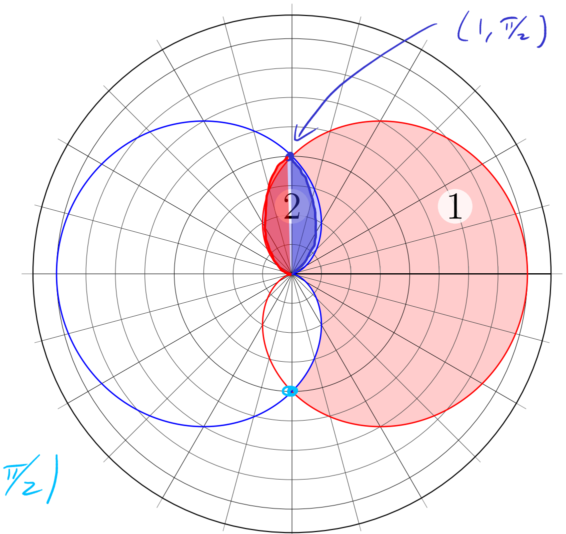
$$\int_{7\pi/6}^{11\pi/6} \frac{1}{2}\left((2)^2 - (3 + 2\sin(\theta))^2\right) d\theta = \frac{1}{2}\int_{7\pi/6}^{11\pi/6} -5 - 12\sin(\theta) - 4\underbrace{\sin^2\theta}_{\frac{1-\cos(2\theta)}{2}}\, d\theta$$

$$= \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} 2\cos(2\theta) - 12\sin\theta - 7\, d\theta = \frac{1}{2}\Big[\sin(2\theta) + 12\cos\theta - 7\theta\Big]_{7\pi/6}^{11\pi/6}$$

$$= \frac{1}{2}\left[\left(-\frac{\sqrt{3}}{2} + 12\left(\frac{\sqrt{3}}{2}\right) - \frac{49\pi}{6}\right) - \left(\frac{\sqrt{3}}{2} + 12\left(-\frac{\sqrt{3}}{2}\right) - \frac{77\pi}{6}\right)\right] = \boxed{\frac{11\sqrt{3}}{2} - \frac{7\pi}{3}}$$

Example. Consider the polar curves $r = 1 + \cos(\theta)$ and $r = 1 - \cos(\theta)$.

Setup the integral(s) that finds the area of area 1.



Solve $1+\cos(\theta) = 1-\cos\theta$

$2\cos(\theta) = 0$

$\cos(\theta) = 0 \longrightarrow \theta = -\pi/2,\ \pi/2$

$(1, -\pi/2)$, $(1, \pi/2)$

Test point

$\theta = 0$

$1+\cos(0) = 2 \rightarrow 1+\cos(\theta) \geq 1-\cos(\theta)$ on $(-\frac{\pi}{2}, \pi/2)$

$1-\cos(0) = 0$

$$\int_{-\pi/2}^{\pi/2} \frac{1}{2}\left((1+\cos(\theta))^2 - (1-\cos(\theta))^2\right) d\theta$$

Setup and solve the integral(s) that finds the area of area 2.

$$\text{Area} = \int_0^{\pi/2} \frac{1}{2}(1-\cos(\theta))^2\, d\theta + \int_{\pi/2}^{\pi} \frac{1}{2}(1+\cos(\theta))^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 1 - 2\cos(\theta) + \underbrace{\cos^2\theta}_{\frac{1+\cos(2\theta)}{2}}\, d\theta + \frac{1}{2}\int_{\pi/2}^{\pi} 1 + 2\cos(\theta) + \underbrace{\cos^2(\theta)}_{\frac{1+\cos(2\theta)}{2}}\, d\theta$$

$$= \frac{1}{2}\left[\frac{3}{2}\theta - 2\sin\theta + \frac{\sin(2\theta)}{4}\right]_0^{\pi/2} + \frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin(2\theta)}{4}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{2}\left(\left[\frac{3}{2}\theta + \frac{\sin(2\theta)}{4}\right]_0^{\pi} - 2\sin\theta\Big|_0^{\pi/2} + 2\sin\theta\Big|_{\pi/2}^{\pi}\right) = \boxed{\frac{3\pi}{4} - 2}$$

Arc Length of a Polar Curve
Let f have a continuous derivative on the interval $[\alpha, \beta]$. The **arc length** of the polar curve $r = f(\theta)$ on $[\alpha, \beta]$ is

$$L = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta.$$

Example (LC 34.13)**.** Find the length of the polar curve $r = e^{-a\theta}$ for $\theta \geq 0$ and $a > 0$ (a is constant).

$$L = \int_0^{\infty} \sqrt{(e^{-a\theta})^2 + (-a e^{-a\theta})^2}\; d\theta$$

$$= \lim_{b \to \infty} \int_0^b e^{-a\theta} \underbrace{\sqrt{1+a^2}}_{\text{constant}}\; d\theta$$

$$= \lim_{b \to \infty} \frac{e^{-a\theta}}{-a} \sqrt{1+a^2} \Bigg|_{\theta=0}^{\theta=b} = \frac{\sqrt{1+a^2}}{a}$$