1 10.1: An Overview of Sequences and Infinite Series

Definition. (Sequence)
A **sequence** $\{a_n\}$ is an ordered list of numbers of the form

$$\{a_1, a_2, a_3, \ldots, a_n, \ldots\}.$$

A sequence may be generated by a **recurrence relation** of the form $a_{n+1} = f(a_n)$, for $n = 1, 2, 3, \ldots$, where a_1 is given. A sequence may also be defined with an **explicit formula** of the form $a_n = f(n)$, for $n = 1, 2, 3, \ldots$.

Example. Consider the sequence $a_n = \frac{2^{n+1}}{2^n+1}$; Compute a_1, a_2, a_3, and a_4.

Definition. (Limit of a Sequence)
If the terms of a sequence $\{a_n\}$ approach a unique number L as n increases— that is, if a_n can be made arbitrarily close to L by taking n sufficiently large— then we say $\lim_{n\to\infty} a_n = L$ exists, and the sequence **converges** to L. If the terms of the sequence do not approach a single number as n increases, the sequence has no limit, and the sequence **diverges**.

Example. Determine if the sequence given by

$$a_n = \frac{3+5n^2}{n+n^2}$$

converges or diverges. If it converges, find the value that the sequence converges to.

Example. Determine if the sequence given by

$$a_n = (-1)^n \frac{3+5n^2}{n+n^2}$$

converges or diverges. If it converges, find the value that the sequence converges to.

Example. A ball is thrown upward to a height of 10 meters. After each bounce, the ball rebounds to $2/3$ of its previous height. Let h_n be the height after the nth bounce. Find an explicit formula for the nth term of the sequence $\{h_n\}$.

Definition. (Infinite series)
Given a sequence $\{a_1, a_2, a_3, \ldots\}$, the sum of its terms

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$$

is called an **infinite series**. The **sequence of partial sums** $\{S_n\}$ associated with this series has the terms

$$\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
S_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k, \qquad \text{for } n = 1, 2, 3, \ldots.
\end{aligned}$$

If the sequence of partial sums $\{S_n\}$ has a limit L, the infinite series **converges** to that limit, and we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \underbrace{\sum_{k=1}^{\infty} a_k}_{S_n} = \lim_{n\to\infty} S_n = L.$$

If the sequence of partial sums diverges, the infinite series also **diverges**.

Example. Consider the infinite series $4 + 0.9 + 0.09 + 0.009 + \ldots$. Compute S_1, S_2, S_3, and S_4. What is the value of this series?

Example. A sequence $\{a_n\}$ has partial sums given by the formula $S_n = 5 - \frac{1}{\sqrt{n}}$.

What is the value of the series $\sum_{n=1}^{\infty} a_n$?

What is the formula for a_n?

What is the limit $\lim_{n\to\infty} a_n$?