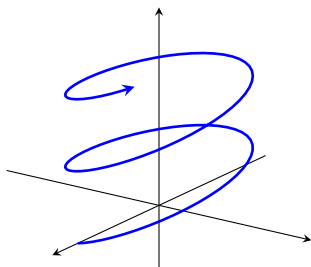


14.1: Vector-Valued Functions

Vector-valued functions are functions of the form $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, where $x(t)$, $y(t)$, and $z(t)$ are parametric equations dependent on t .



Curves in Space

Consider

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k},$$

where f , g , and h are defined for $a \leq t \leq b$. The **domain** of \mathbf{r} is the largest set of t for which all of f , g , and h are defined.

Example. What plane does the curve $\mathbf{r}(t) = t\mathbf{i} + 6t^3\mathbf{k}$ lie?

Example (Lines as vector-valued functions). Find a vector function for the line that passes through the points $P(5, 2, -4)$ and $Q(5, 5, -2)$. What about the line segment that connects P and Q ?

Example. Find the domain of

$$\mathbf{r}(t) = \sqrt{16 - t^2}\mathbf{i} + \sqrt{t}\mathbf{j} + \frac{4}{\sqrt{3 + t}}\mathbf{k}$$

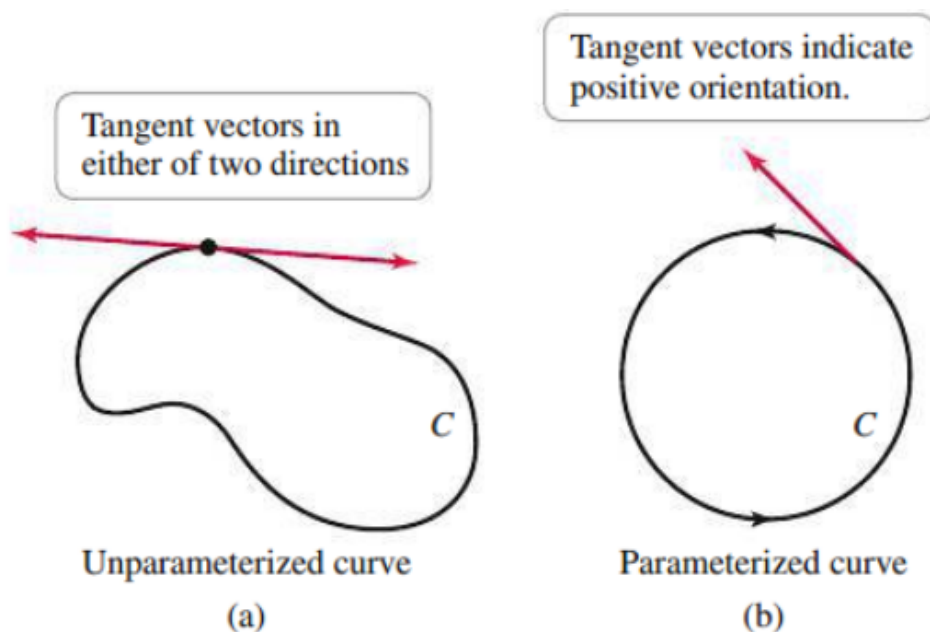
Example. Find the point P on

$$\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j} + 2t\mathbf{k},$$

closest to $P_0(4, 17, 10)$. What is the distance between P and P_0 ?

Orientation of Curves

- A **unparameterized curve** is a smooth curve C with no specified direction and the tangent vector can be drawn in two directions.
- A **parameterized curve** is a smooth curve C described by a function $\mathbf{r}(t)$ for $a \leq t \leq b$ and has a direction referred to as its **orientation**.
- The *positive* orientation is the direction of the curve generated when t increases from a to b .
- The tangent vector of a parameterized curve points in the positive orientation of the curve.



Example. Graph the curve described by the equation

$$\mathbf{r}(t) = 4 \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + \frac{t}{2\pi}\mathbf{k},$$

where $0 \leq t \leq 2\pi$. Indicate the positive orientation of this curve.

Limits and Continuity for Vector-Valued Functions

The properties of limits extend to vector-valued functions naturally. In particular, for $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, if

$$\lim_{t \rightarrow a} f(t) = L_1, \quad \lim_{t \rightarrow a} g(t) = L_2, \quad \lim_{t \rightarrow a} h(t) = L_3$$

then

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \left\langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \right\rangle = \langle L_1, L_2, L_3 \rangle.$$

Definition. (Limit of a Vector-Valued Function)

A vector-valued function \mathbf{r} approaches the limit \mathbf{L} as t approaches a , written $\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{L}$, provided $\lim_{t \rightarrow a} |\mathbf{r}(t) - \mathbf{L}| = 0$.

A function $\mathbf{r}(t)$ is **continuous** at $t = a$, provided $\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{r}(a)$.

Example. Evaluate the following limits:

$$\lim_{t \rightarrow \pi} \left(\cos(t) \mathbf{i} - 7 \sin \left(-\frac{t}{2} \right) \mathbf{j} + \frac{t}{\pi} \mathbf{k} \right)$$

$$\lim_{t \rightarrow \infty} \left(\frac{t}{t-3} \mathbf{i} + \frac{40}{1+19e^{-t}} \mathbf{j} + \frac{1}{2t} \mathbf{k} \right)$$