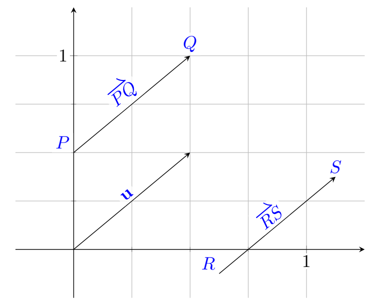
1 13.1: Vectors and the Geometry of Space

Definition.

- **Vectors**
  - Have a direction and magnitude,
  - vector $\overrightarrow{PQ}$ has a *tail* at P and a *head* at Q,
  - Can be denoted as $\mathbf{u}$ or $\vec{u}$,
  - Equal vectors have the same direction and magnitude (not necessarily the same position)

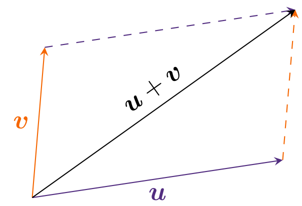


- **Scalars** are quantities with magnitude but no direction (e.g. mass, temperature, price, time, etc.)
- **Zero vector**, denoted $\mathbf{0}$ or $\vec{0}$, has length 0 and no direction

Scalar-vector multiplication:

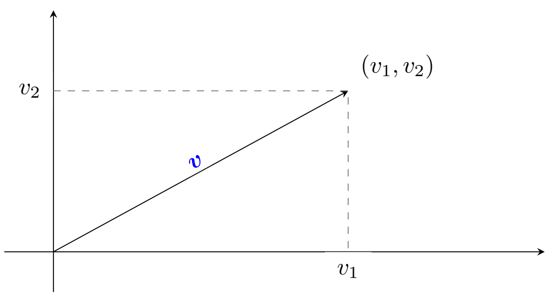
- Denoted $c\mathbf{v}$ or $c\vec{v}$,
- length of vector multiplied by $|c|$,
- $c\mathbf{v}$ has the same direction as $\mathbf{v}$ if $c > 0$, and has the opposite direction as $\mathbf{v}$ if $c < 0$, (what if $c = 0$?)
- $\boldsymbol{u}$ and $\boldsymbol{v}$ are **parallel** if $\boldsymbol{u} = c\boldsymbol{v}$. (what vectors are parallel to $\mathbf{0}$?)

Vector Addition and Subtraction:
Given two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, their sum, $\boldsymbol{u}+\boldsymbol{v}$, can be represented by the parallelogram (triangle) rule: place the tail of $\boldsymbol{v}$ at the head of $\boldsymbol{u}$



The difference, denoted $\boldsymbol{u}-\boldsymbol{v}$, is the sum of $\boldsymbol{u}+(-\boldsymbol{v})$:

Vector Components:
A vector $\boldsymbol{v}$ whose tail is at the origin $(0,0)$ and head is at (v_1, v_2) is a **position vector** (in **standard position**) and is denoted $\langle v_1, v_2\rangle$. The real numbers v_1 and v_2 are the x- and y-components of $\boldsymbol{v}$.



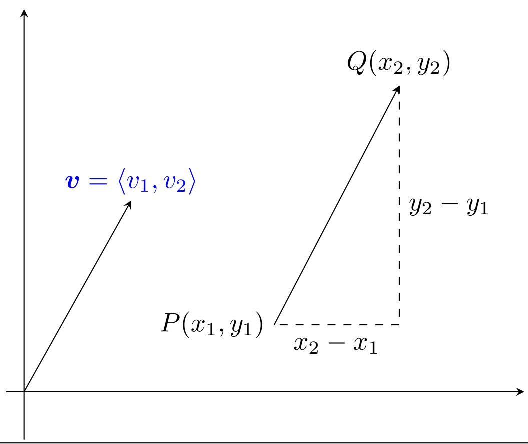
Vectors $\boldsymbol{u}=\langle u_1, u_2\rangle$ and $\boldsymbol{v}=\langle v_1, v_2\rangle$ are equal if and only if $u_1=v_1$ and $u_2=v_2$.

Magnitude:

Given points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the **magnitude**, or **length**, of vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$, denoted $\left|\overrightarrow{PQ}\right|$, is the distance between points P and Q.

$$\left|\overrightarrow{PQ}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The magnitude of position vector $\boldsymbol{v} = \langle v_1, v_2 \rangle$ is $|\boldsymbol{v}|$.
(How do $\left|\overrightarrow{PQ}\right|$ and $\left|\overrightarrow{QP}\right|$ relate to each other?)



<u>Note:</u> The norm, denoted $\|\boldsymbol{u}\|$ or $\|\boldsymbol{u}\|_2$, is equivalent to the magnitude of a vector.

Equation of a Circle:

Definition.
A **circle** centered at (a, b) with radius r is the set of points satisfying the equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

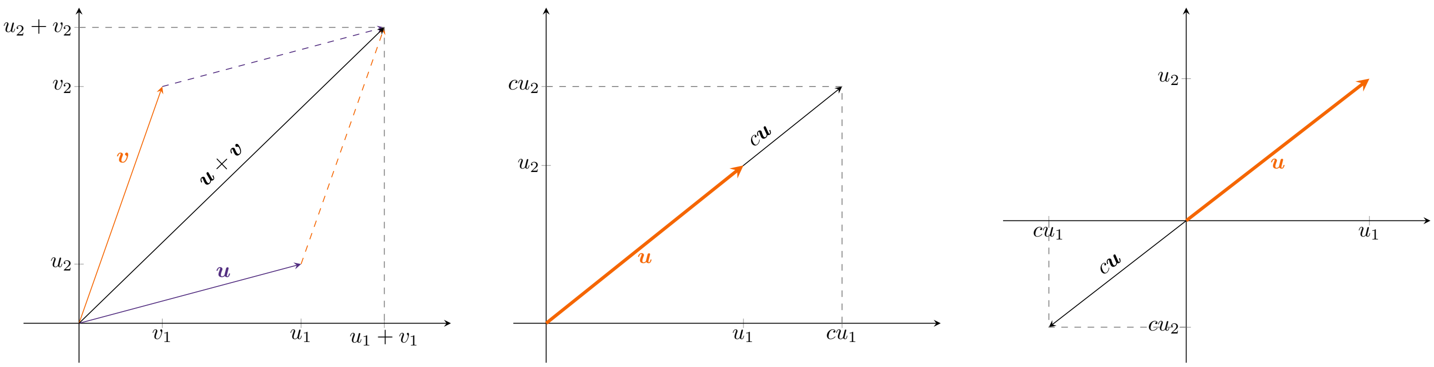
A **disk** centered at (a, b) with radius r is the set of points satisfying the inequality

$$(x - a)^2 + (y - b)^2 \leq r^2.$$

Vector Operations in Terms of Components

Definition. (Vector Operations in $\mathbb{R}^2$)
Suppose c is a scalar, $\boldsymbol{u} = \langle u_1, u_2 \rangle$, and $\boldsymbol{v} = \langle v_1, v_2 \rangle$.

$$\begin{aligned} \boldsymbol{u} + \boldsymbol{v} &= \langle u_1 + v_1, u_2 + v_2 \rangle && \text{Vector addition} \\ \boldsymbol{u} - \boldsymbol{v} &= \langle u_1 - v_1, u_2 - v_2 \rangle && \text{Vector subtraction} \\ c\boldsymbol{u} &= \langle cu_1, cu_2 \rangle && \text{Scalar multiplication} \end{aligned}$$



Definition.
A **unit vector** is any vector with length 1.
In $\mathbb{R}^2$, the **coordinate unit vectors** are $\boldsymbol{i} = \langle 1, 0 \rangle$ and $\boldsymbol{j} = \langle 0, 1 \rangle$.

Properties of Vector Operations:
Suppose $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

1. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$ — Commutative property of addition
2. $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$ — Associative property of addition
3. $\boldsymbol{v} + \mathbf{0} = \boldsymbol{v}$ — Additive identity
4. $\boldsymbol{v} + (-\boldsymbol{v}) = \mathbf{0}$ — Additive inverse
5. $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$ — Distributive property 1
6. $(a + c)\boldsymbol{v} = a\boldsymbol{v} + c\boldsymbol{v}$ — Distributive property 2
7. $0\boldsymbol{v} = \mathbf{0}$ — Multiplication by zero scalar
8. $c\mathbf{0} = \mathbf{0}$ — Multiplication by zero vector
9. $1\boldsymbol{v} = \boldsymbol{v}$ — Multiplicative identity
10. $a(c\boldsymbol{v}) = (ac)\boldsymbol{v}$ — Associative property of scalar multiplication

Applications of Vectors: