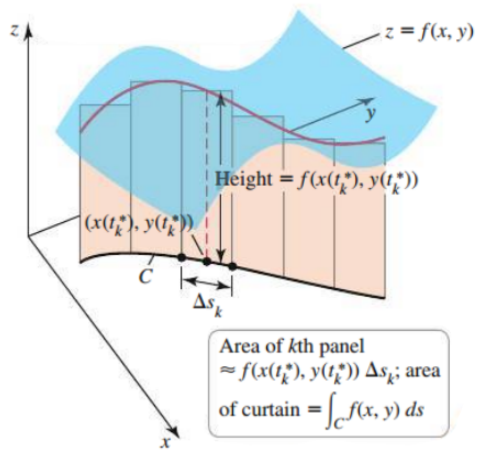
17.2: Line Integrals

Definition. (Scalar Line Integral in the Plane)
Suppose the scalar-valued function f is defined on a region containing the smooth curve C given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, for $a \le t \le b$. The **line integral of f over C** is

$$\int_C f(x(t), y(t))\, ds = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x(t_k^*),\, y(t_k^*))\Delta s_k,$$

provided this limit exists over all partitions of $[a, b]$. When the limit exists, f is said to be **integrable** on C.



Theorem 17.1: Evaluating Scalar Line Integrals in $\mathbb{R}^2$
Let f be continuous on a region containing a smooth curve C: $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, for $a \le t \le b$. Then

$$\begin{aligned}\int_C f\, ds &= \int_a^b f(x(t),\, y(t))|\mathbf{r}'(t)|\, dt\\ &= \int_a^b f(x(t),\, y(t))\sqrt{x'(t)^2 + y'(t)^2}\, dt.\end{aligned}$$

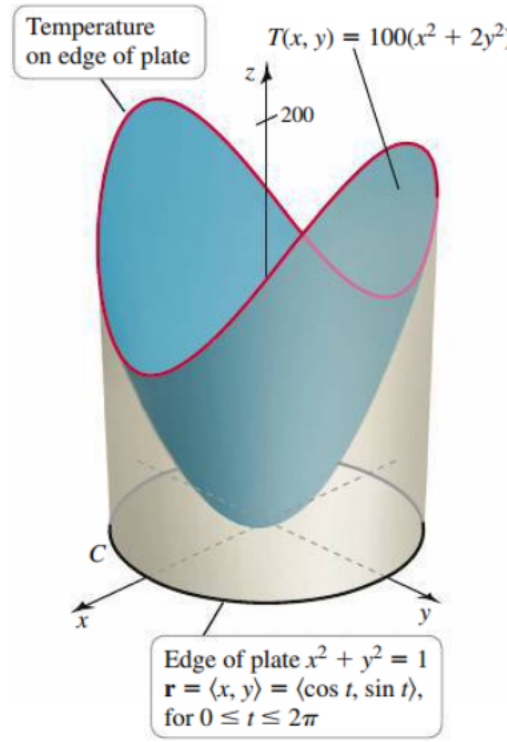
Procedure: Evaluating the Line Integral $\int_C f\,ds$

1. Find a parametric description of C in the form $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$.
2. Compute $|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$.
3. Make substitutions for x and y in the integrand and evaluate an ordinary integral:

$$\int_C f\,ds = \int_a^b f(x(t),\, y(t))|\mathbf{r}'(t)|\,dt.$$

Example. Find the length of the quarter-circle from $(1, 0)$ to $(0, 1)$ with its center at the origin.

Example. The temperature of the circular plate $R = \{(x, y) : x^2 + y^2 \leq 1\}$ is $T(x, y) = 100(x^2 + 2y^2)$. Find the average temperature along the edge of the plate.



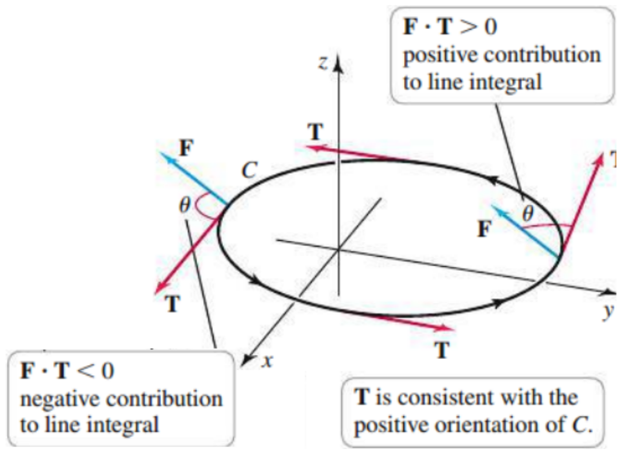
Theorem 17.2: Evaluating Scalar Line Integrals in $\mathbb{R}^3$
Let f be continuous on a region containing a smooth curve $C : \mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$, for $a \le t \le b$. Then

$$\begin{aligned}\int_C f\,ds &= \int_a^b f(x(t),\, y(t),\, z(t))|\mathbf{r}'(t)|\,dt \\ &= \int_a^b f(x(t),\, y(t),\, z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt.\end{aligned}$$

Example. Evaluate $\displaystyle\int_C (x - y + 2z)\,ds$, where C is the circle $\mathbf{r}(t) = \langle 1,\, 3\cos(t),\, 3\sin(t)\rangle$, for $0 \le t \le 2\pi$.

Example. Evaluate $\int_C xe^{yz}\, ds$, where C is $\mathbf{r}(t) = \langle t, 2t, -2t\rangle$, for $0 \leq t \leq 2$.

Definition. (Line Integral of a Vector Field)
Let $\mathbf{F}$ be a vector field that is continuous on a region containing a smooth oriented curve C parameterized by arc length. Let $\mathbf{T}$ be the unit tangent vector at each point of C consistent with the orientation. The line integral of $\mathbf{F}$ over C is $\displaystyle\int_C \mathbf{F}\cdot\mathbf{T}\,ds$.



Different Forms of Line Integrals of Vector Fields
The line integral $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ may be expressed in the following forms, where $\mathbf{F} = \langle f,\, g,\, h\rangle$ and C has a parameterization $\mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$, for $a \le t \le b$:

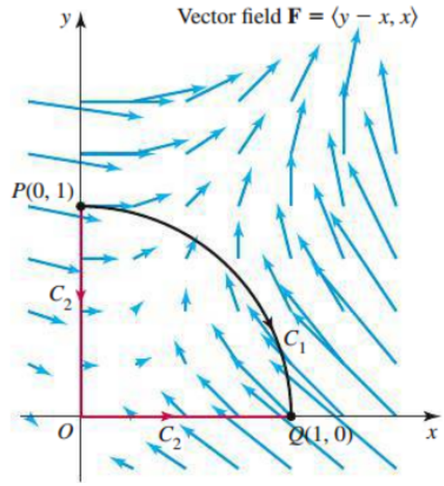
$$\begin{aligned}\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt &= \int_a^b (f(t)x'(t) + g(t)y'(t) + h(t)z'(t))\,dt\\ &= \int_C f\,dx + g\,dy + h\,dz\\ &= \int_C \mathbf{F}\cdot d\mathbf{r}.\end{aligned}$$

For line integrals in the plane, we let $\mathbf{F} = \langle f,\, g\rangle$ and assume C is parameterized in the form $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$. Then

$$\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_a^b (f(t)x'(t) + g(t)y'(t))\,dt = \int_C f\,dx + g\,dy = \int_C \mathbf{F}\cdot d\mathbf{r}.$$

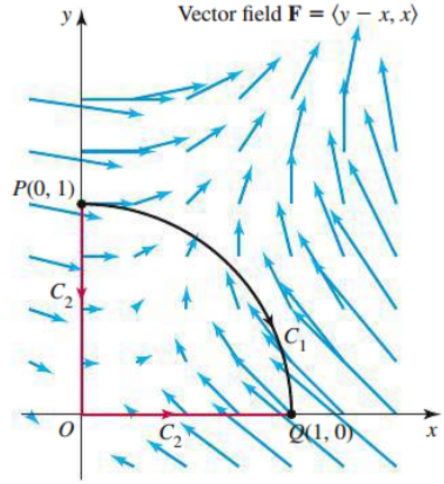
Example. Evaluate $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ with $\mathbf{F} = \langle y - x,\, x \rangle$ on the following oriented paths in $\mathbb{R}^2$.

a) The quarter-circle C_1 from $P(0,1)$ to $Q(1,0)$



b) The quarter-circle $-C_1$ from $Q(1,0)$ to $P(0,1)$

c) the path C_2 from $P(0,1)$ to $Q(1,0)$ via two line segments through $O(0,0)$.



Definition. (Work Done in a Force Field)
Let $\mathbf{F}$ be a continuous force field in a region D of $\mathbb{R}^3$. Let

$$C : \mathbf{r}(t) = \langle x(t),\, y(t),\, z(t) \rangle, \text{ for } a \leq t \leq b,$$

be a smooth curve in D with a unit tangent vector $\mathbf{T}$ consistent with the orientation. The work done in moving an object along C in the positive direction is

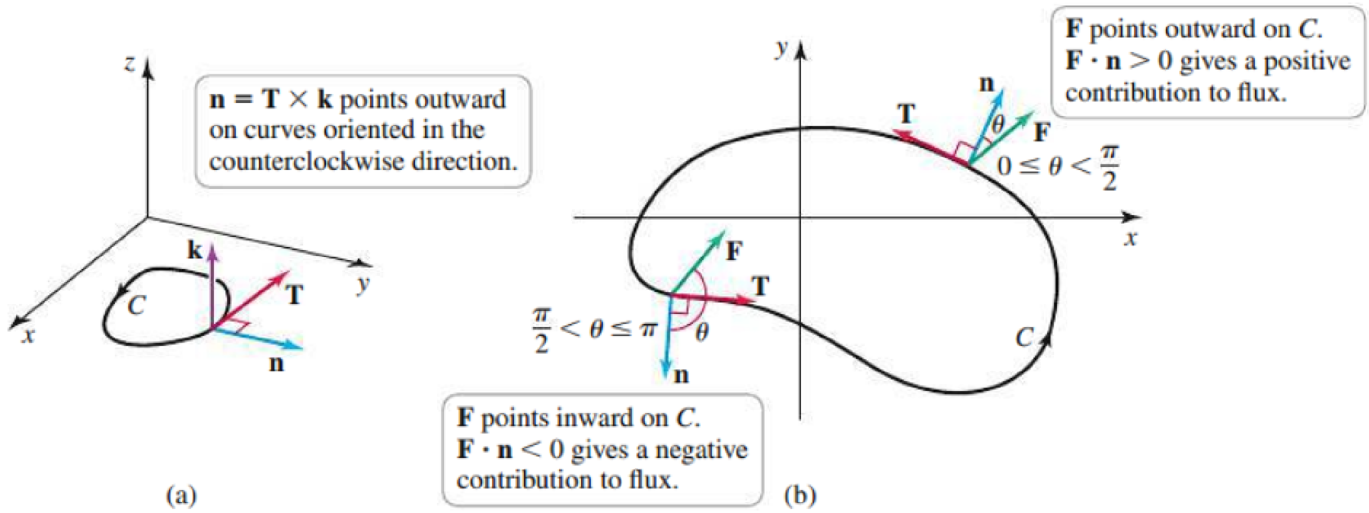
$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt.$$

Example. For the force field $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}$, calculate the work required to move an object from $(1, 1, 1)$ to $(10, 10, 10)$.

Definition. (Circulation)
Let $\mathbf{F}$ be a continuous vector field on a region D of $\mathbb{R}^3$, and let C be a closed smooth oriented curve in D. The **circulation** of $\mathbf{F}$ on C is $\displaystyle\int_C \mathbf{F}\cdot\mathbf{T}\,ds$, where $\mathbf{T}$ is the unit vector tangent to C consistent with the orientation.

Example. Compute the circulation in the vector field $\mathbf{F} = \dfrac{\langle y, -2x\rangle}{\sqrt{4x^2+y^2}}$ along the curve C given by $\mathbf{r}(t) = \langle 2\cos(t),\, 4\sin(t)\rangle$, for $0 \le t \le 2\pi$.

Flux of the vector field is the total forces orthogonal to each point on the curve C. Let $\mathbf{F} = \langle f, g\rangle$ be a continuous vector field in a region R of $\mathbb{R}^2$. Using $\mathbf{n}$ to represent a unit vector normal to C, the component of $\mathbf{F}$ that is normal to C is $\mathbf{F}\cdot\mathbf{n}$.



Since C is in the xy-plane, the unit tangent vector $\mathbf{T} = \langle T_x, T_y, 0\rangle$ is also in the xy-plane. We let $\mathbf{n}$ be in the xy-plane as well, but using the cross product of $\mathbf{T}$ and $\boldsymbol{k}$:

$$\mathbf{n} = \mathbf{T}\times\boldsymbol{k} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ T_x & T_y & 0 \\ 0 & 0 & 1 \end{vmatrix} = T_y\boldsymbol{i} - T_x\boldsymbol{j}.$$

Since $\mathbf{T} = \dfrac{\mathbf{r}(t)}{|\mathbf{r}(t)|}$, we have

$$\mathbf{n} = T_y\boldsymbol{i} - T_x\boldsymbol{j} = \frac{y'(t)}{|\mathbf{r}'(t)|}\boldsymbol{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\boldsymbol{j} = \frac{\langle y'(t), -x'(t)\rangle}{|\mathbf{r}'(t)|}.$$

Thus, we have the flux integral

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_a^b \mathbf{F}\cdot\frac{\langle y'(t), -x'(t)\rangle}{|\mathbf{r}'(t)|}|\mathbf{r}'(t)|\,dt = \int_a^b (f(t)y'(t) - g(t)x'(t))\,dt = \int_C f\,dy - g\,dx.$$

Definition. (Flux)
Let $\mathbf{F} = \langle f, g\rangle$ be a continuous vector field on a region R of $\mathbb{R}^2$. Let $C : \mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, $a \le t \le b$, be a smooth orientated curve in R that does not intersect itself. The **flux** of the vector field $\mathbf{F}$ across C is

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_a^b \left(f(t)y'(t) - g(t)x'(t)\right)dt,$$

where $\mathbf{n} = \mathbf{T}\times\boldsymbol{k}$ is the unit normal vector and $\mathbf{T}$ is the unit tangent vector consistent with the orientation. If C is a closed curve with counterclockwise orientation, $\mathbf{n}$ is the outward normal vector, and the flux integral gives the **outward flux** across C.

Example. Compute the flux in the vector field $\mathbf{F} = \dfrac{\langle y, -2x\rangle}{\sqrt{4x^2+y^2}}$ along the curve C given by $\mathbf{r}(t) = \langle 2\cos(t),\, 4\sin(t)\rangle$, for $0 \le t \le 2\pi$.