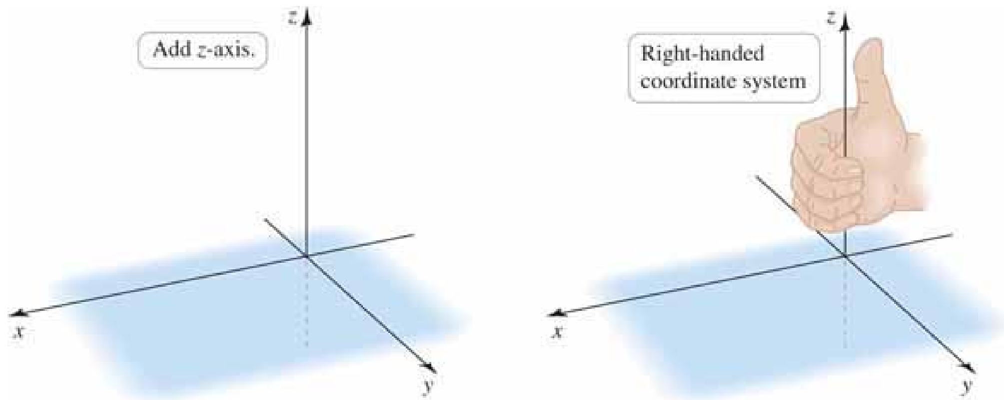
13.2: Vectors in Three Dimensions

The xyz- Coordinate System:

The three-dimensional coordinate system is created by adding the z-axis, which is perpendicular to both the x-axis and the y-axis. When looking at the xy-plane, the positive direction of the z-axis protrudes towards the viewer. This can also be shown using the right-hand rule (Figure 13.25 from Briggs):

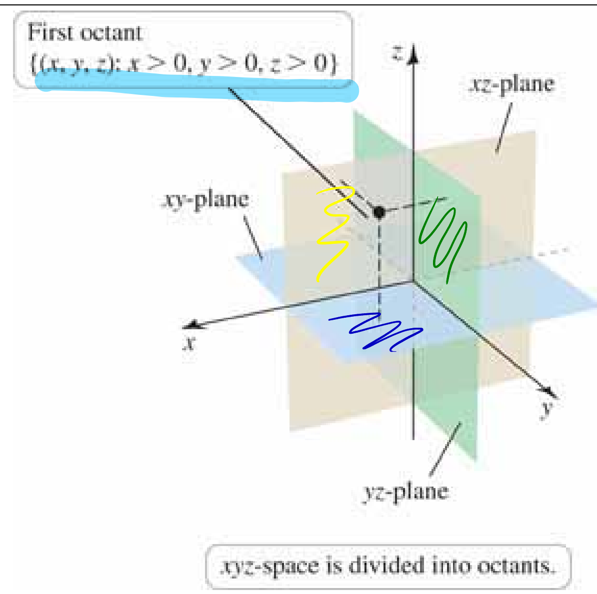


Definition.

This three-dimensional coordinate system is broken up into eight **octants**, which are separated by

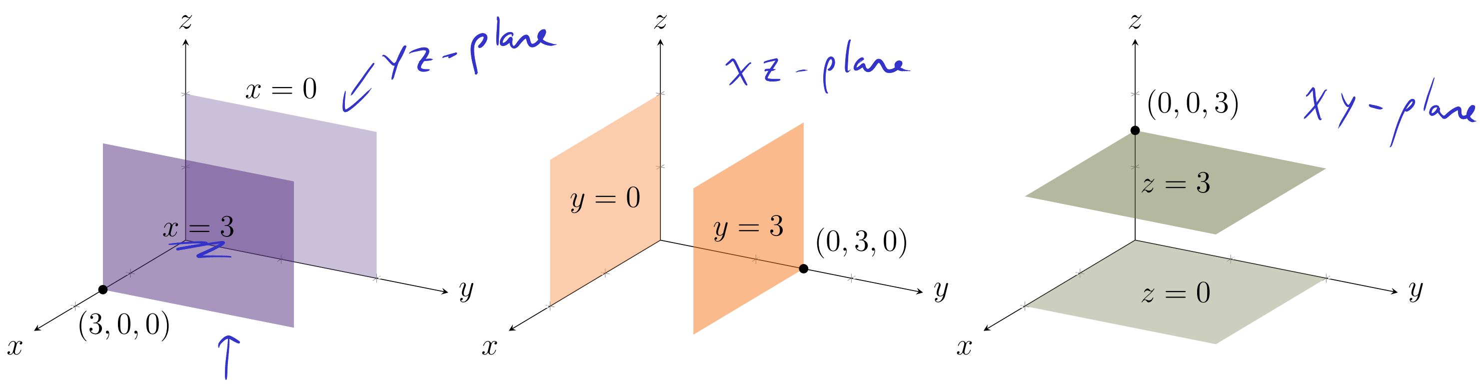
- the xy**-plane** ($z = 0$), ← z fixed
- the xz**-plane** ($y = 0$), and
- the yz**-plane** ($x = 0$).

The **origin** is the location where all three axes intersect.

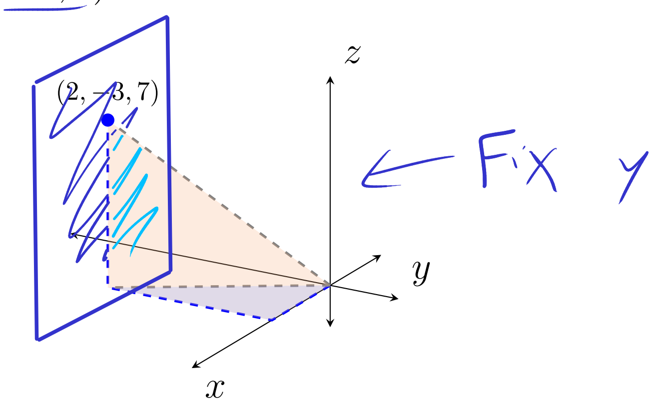


Equations of Simple Planes:

Planes in three-dimensions are analogous to lines in two-dimensions. Below, we see the yz-plane, the xz-plane, and the xy-plane, along with planes that are parallel where x, y, and z are fixed respectively:



Example (Parallel planes)**.** Determine the equation of the plane parallel to the xz-plane passing through the point $(2, -3, 7)$.



$y = -3$

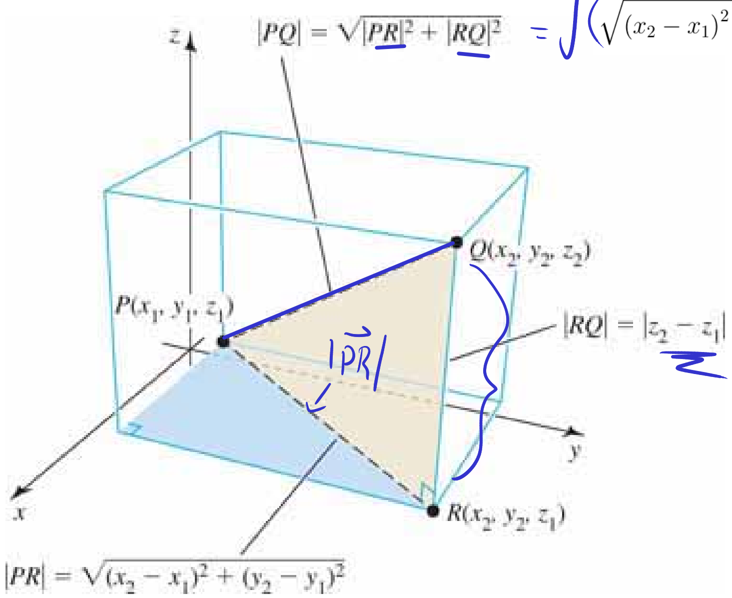
Distances in xyz-Space:

Recall that in $\mathbb{R}^2$, for some vector $\overrightarrow{PR}$, the distance formula is given by

$$|PR| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

where (x_1, y_1) and (x_2, y_2) represent the points P and R respectively. This idea can be further extended into $\mathbb{R}^3$ by considering the two sides of the triangle formed by the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$:

$$= \sqrt{\left(\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right)^2 + (z_2 - z_1)^2}$$



Distance Formula in xyz-Space

The **distance** between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The **midpoint** between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is found by averaging the x-, y-, and z-coordinates:

$$\text{Midpoint } = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

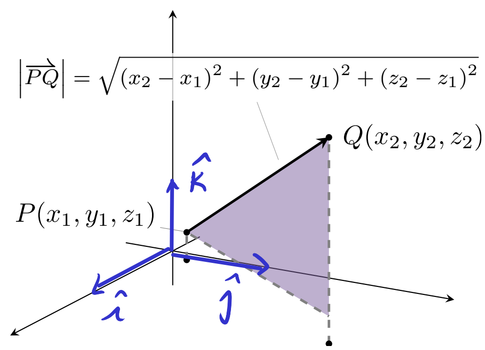
Magnitude and Unit Vectors:

$\vec{u} = \langle u_1, u_2, u_3 \rangle$ $\qquad |\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$

Definition.
The **magnitude** (or **length**) of the vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ is the distance from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$:

$$\left|\overrightarrow{PQ}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

In $\mathbb{R}^3$, the **coordinate unit vectors** are $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$.



Example. Consider $P(-1, 4, 3)$ and $Q(3, 5, 7)$. Find

$\overrightarrow{PQ} = \langle 3-(-1), 5-4, 7-3 \rangle$

- $\left|\overrightarrow{PQ}\right| = \sqrt{(3-(-1))^2 + (5-4)^2 + (7-3)^2}$

  $= \sqrt{4^2 + 1^2 + 4^2} = \sqrt{33}$

  $\left|\overrightarrow{PQ}\right|^2 = 33$

- The midpoint between P and Q

  $M_p = \left(\frac{3-1}{2}, \frac{5+4}{2}, \frac{7+3}{2} \right) = \left(1, \frac{9}{2}, 5 \right)$

- Two unit vectors parallel to $\overrightarrow{PQ}$

  $\overrightarrow{PQ} = \langle 4, 1, 4 \rangle$ $\qquad \vec{a} = \frac{1}{\sqrt{33}} \langle 4, 1, 4 \rangle$

  $\vec{b} = -\frac{1}{\sqrt{33}} \langle 4, 1, 4 \rangle$

Equation of a Sphere:

Definition.
A **sphere** centered at (a, b, c) with radius r is the set of points satisfying the equation

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

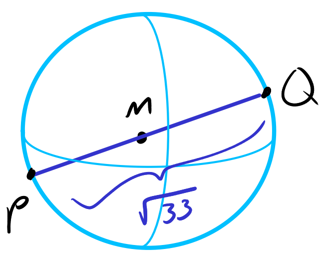
A **ball** centered at (a, b, c) with radius r is the set of points satisfying the inequality

$$(x-a)^2 + (y-b)^2 + (z-c)^2 \leq r^2.$$

Example. Consider $P(-1, 4, 3)$ and $Q(3, 5, 7)$. Find the equation of the sphere centered at the midpoint passing through P and Q

$|\vec{PQ}| = \sqrt{33}$

$M_p = (1, \frac{9}{2}, 5)$



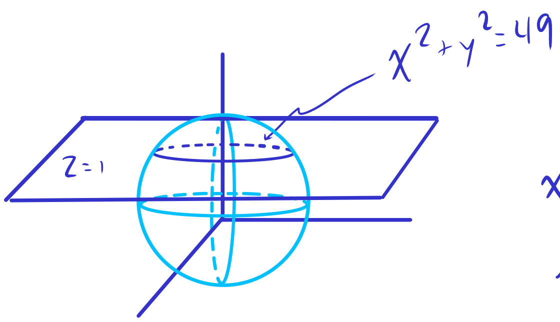
$r = \dfrac{\sqrt{33}}{2}$

$(x-1)^2 + (y-\frac{9}{2})^2 + (z-5)^2 = \dfrac{33}{4}$

Example. What is the geometry of the intersection between $x^2+y^2+z^2 = 50$ and $z = 1$?

sphere centered @ origin

$r = \sqrt{50}$



$x^2+y^2+(1)^2 = 50$

$x^2+y^2 = 49 \longrightarrow$ circle $r = 7$

$(x-3)^2 + (y+1)^2 + (z+\frac{5}{2})^2 = \frac{165}{4} \longrightarrow r = \dfrac{\sqrt{165}}{2}$

$(3, -1, -5/2)$

Example. Rewrite the following equation into the standard form of a sphere:

$$x^2 + y^2 + z^2 - 2x + 6y - 8z = -1$$

$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$

$x^2 - 2x + \underline{1} - \underline{1} + y^2 + 6y + \underline{9} - \underline{9} + z^2 - 8z + \underline{16} - \underline{16} = -1$

$\left(\frac{-2}{2}\right)^2 = 1$ $\quad\left(\frac{6}{2}\right)^2 = 9$ $\quad\left(\frac{-8}{2}\right)^2 = 16$

$(x-1)^2 + (y+3)^2 + (z-4)^2 = 25$

$(x-1)^2 + (y+3)^2 + (z-4)^2 = 5^2$

Vector Operations in Terms of Components

Definition. (Vector Operations in $\mathbb{R}^3$)
Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$	Vector addition
$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$	Vector subtraction
$c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle$	Scalar multiplication

Properties of Vector Operations:
Suppose $\mathbf{u}$, $\mathbf{v}$, and $\boldsymbol{w}$ are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ — Commutative property of addition
2. $(\mathbf{u} + \mathbf{v}) + \boldsymbol{w} = \mathbf{u} + (\mathbf{v} + \boldsymbol{w})$ — Associative property of addition
3. $\mathbf{v} + \mathbf{0} = \mathbf{v}$ — Additive identity
4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ — Additive inverse
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ — Distributive property 1
6. $(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$ — Distributive property 2
7. $0\mathbf{v} = \mathbf{0}$ — Multiplication by zero scalar
8. $c\mathbf{0} = \mathbf{0}$ — Multiplication by zero vector
9. $1\mathbf{v} = \mathbf{v}$ — Multiplicative identity
10. $a(c\mathbf{v}) = (ac)\mathbf{v}$ — Associative property of scalar multiplication