2.3 Techniques for Computing Limits

Example.

a) $\lim_{x \to 3} \frac{1}{2}x - 7 = \frac{1}{2}(3) - 7 = \frac{3-14}{2} = \boxed{\frac{-11}{2}}$

b) $\lim_{x \to 2} 6 = \boxed{6}$

Definition (Briggs). **Limit Laws:** Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. The following properties hold, where c is a real number, and $n > 0$ is an integer.

1. **Sum:** $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. **Difference:** $\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. **Constant multiple:** $\lim_{x \to a} (cf(x)) = c \lim_{x \to a} f(x)$

4. **Product:** $\lim_{x \to a} (f(x)g(x)) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$

5. **Quotient:** $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$

6. **Power:** $\lim_{x \to a} (f(x))^n = (\lim_{x \to a} f(x))^n$

7. **Root:** $\lim_{x \to a} (f(x))^{1/n} = (\lim_{x \to a} f(x))^{1/n}$

Example. Suppose $\lim_{x\to 2} f(x) = 4$, $\lim_{x\to 2} g(x) = 5$ and $\lim_{x\to 2} h(x) = 8$.

a) $\lim_{x\to 2} \dfrac{f(x) - g(x)}{h(x)}$ $= \dfrac{\lim_{x\to 2}[f(x) - g(x)]}{\lim_{x\to 2} h(x)} = \dfrac{\lim_{x\to 2} f(x) - \lim_{x\to 2} g(x)}{\lim_{x\to 2} h(x)} = \dfrac{4-5}{8} = \boxed{-\dfrac{1}{8}}$

b) $\lim_{x\to 2} (6f(x)g(x) + h(x))$ $= 6 \lim_{x\to 2} f(x) \lim_{x\to 2} g(x) + \lim_{x\to 2} h(x) = \underbrace{6(4)(5)}_{120} + 8 = \boxed{128}$

c) $\lim_{x\to 2} (g(x))^3$ $= \left(\lim_{x\to 2} g(x)\right)^3 = (5)^3 = \boxed{125}$

Example. For $g(x) = \dfrac{x+6}{x^2 - 36}$, find

1. $\lim_{x\to 0}$ $g(x) = \dfrac{0+6}{0^2 - 36} = \dfrac{6}{-36} = \boxed{-\dfrac{1}{6}}$

2. $\lim_{x\to -6}$ $g(x) = \lim_{x\to -6} \dfrac{x+6}{(x+6)(x-6)} = \lim_{x\to -6} \dfrac{1}{x-6} = \boxed{-\dfrac{1}{12}}$

Example. $\lim_{x\to 2} \dfrac{\sqrt{2x^3+9}+3x-1}{4x+1}$

Note: Domain $x \neq -\frac{1}{4}$
so 2 is in domain

$$\lim_{x\to 2} \frac{\sqrt{2x^3+9}+3x-1}{4x+1} = \frac{\sqrt{2(2)^3+9}+3(2)-1}{4(2)+1}$$

$$= \frac{\sqrt{25}+6-1}{8+1} = \frac{5+6-1}{9} = \boxed{\frac{10}{9}}$$

Example. $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} -2x+4 & \text{if } x \leq 1 \\ \sqrt{x-1} & \text{if } x > 1 \end{cases}$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} -2x+4 = \boxed{2}$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \sqrt{x-1} = \boxed{0}$$

$\lim_{x\to 1}$ DNE since $\lim_{x\to 1^-} f(x) = 2 \neq \lim_{x\to 1^+} f(x) = 0$

Example. $\lim_{x\to 2} \frac{x^2-6x+8}{x^2-4} = \lim_{x\to 2} \frac{(x-2)(x-4)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x-4}{x+2}$

$= \frac{2-4}{2+2} = \frac{-2}{4} = \boxed{-\frac{1}{2}}$

Example. $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}+1}\right) = \lim_{x\to 1} \frac{x^2-1}{(x-1)(\sqrt{x}+1)}$

$= \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(\sqrt{x}+1)}$

$= \lim_{x\to 1} \frac{x+1}{\sqrt{x}+1} = \frac{2}{2} = \boxed{1}$

Example. $\lim_{x\to -4} \sqrt{16-x^2}$

<u>Note:</u>

$16-x^2 \geq 0$

$16 \geq x^2$

$4 \geq \pm x$ $\to x \leq 4$, $\to x \geq -4$

$\Rightarrow -4 \leq x \leq 4$

To find $\lim_{x\to -4} \sqrt{16-x^2}$ we must check the left & right limits

$\lim_{x\to 4^+} \sqrt{16-x^2}$ DNE since $\sqrt{16-x^2}$ DNE for $x > 4$

Example. $\lim_{x\to 2} \dfrac{x^3-6x^2+8x}{\sqrt{x-2}} \left(\dfrac{\sqrt{x-2}}{\sqrt{x-2}}\right) \to \lim_{x\to 2^+} \dfrac{x(x-2)(x-4)\sqrt{x-2}}{x-2}$

Domain:

$x-2>0$

$\boxed{x>2}$

$\boxed{\lim_{x\to 2^-} \dfrac{x^3-6x^2+8x}{\sqrt{x-2}} \quad \text{DNE!}}$

$= \lim_{x\to 2^+} x(x-4)\sqrt{x-2}$

$= 2(2-4)\sqrt{2-2} = \boxed{0}$

Example. $\lim_{x\to a} \dfrac{(y-a)^{12}+6y-6a}{y-a} = \lim_{x\to a} \dfrac{(y-a)^{12}+6(y-a)}{y-a}$

Domain:

$y \neq a$

$= \lim_{x\to a} (y-a)^{11}+6$

$= (a-a)^{11}+6 = \boxed{6}$

The Squeeze Theorem: Assume the functions f, g and h satisfy $f(x) \leq g(x) \leq h(x)$ for all values of x near a, except possibly at a. If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

Example. Consider the function $f(x) = x^2 \sin(1/x)$. What is $\lim\limits_{x\to 0} f(x)$?

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$\Rightarrow -x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

$$\Rightarrow \lim_{x\to 0} -x^2 \leq \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) \leq \lim_{x\to 0} x^2$$

$$\Rightarrow 0 \leq \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) \leq 0$$

Thus, by the squeeze theorem, $\lim\limits_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

Example. Use the squeeze theorem on $-|x| \leq x \sin \frac{1}{x} \leq |x|$.

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$\Rightarrow -|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$$

$$\Rightarrow \lim_{x\to 0} -|x| \leq \lim_{x\to 0} x \sin\left(\frac{1}{x}\right) \leq \lim_{x\to 0} |x|$$

$$\Rightarrow 0 \leq \lim_{x\to 0} x \sin\left(\frac{1}{x}\right) \leq 0$$

Thus, by the squeeze theorem, $\lim\limits_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$.

Example. $\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x}$ $= \lim_{x\to 0} \frac{1-\cos^2 x}{1-\cos x}$

$= \lim_{x\to 0} 1+\cos(x) = 1+\cos(0) = 1+1 = \boxed{2}$

Example. $\lim_{x\to 0} \frac{1-\cos 2x}{\sin x}$ $= \lim_{x\to 0} \frac{1-(1-2\sin^2(x))}{\sin(x)}$

$= \lim_{x\to 0} \frac{2\sin^2(x)}{\sin(x)}$

$= \lim_{x\to 0} 2\sin(x) = 2\sin(0) = \boxed{0}$