## 2.4 Infinite Limits

An infinite limit occurs when function values increase or decrease without bound near a point.

Limits which have an infinite value are called **infinite limits**. They are a special case of limits that do not exist, but we indicate that they approach infinity.

**Example.** Consider the function

$f(x) = \frac{1}{x}$

$\lim_{x\to 0^+} f(x) = \frac{1}{\text{sm}+} = \infty$

$\lim_{x\to 0^-} f(x) = \frac{1}{\text{sm}-} = -\infty$

$\lim_{x\to \infty} f(x) = 0$

$\lim_{x\to -\infty} f(x) = 0$

Consider $f(x) = \dfrac{1}{(x-2)^2}$.

Find $\lim\limits_{x\to 2} f(x)$. $= \boxed{\infty}$

$\lim\limits_{x\to 2^-} f(x) = \frac{1}{(\text{sm}-)^2} = \frac{1}{\text{sm}+} = \infty$

$x \to 2^+$ $(\text{sm}+)$ $\text{sm}+$ $\infty$

Consider $h(x) = -\dfrac{1}{(x+3)^4}$.

Find $\lim\limits_{x\to -3} h(x)$.

$\lim\limits_{x\to -3} -\frac{1}{(x+3)^4} = -\frac{1}{\text{sm}+} = \boxed{-\infty}$

Consider $g(x) = \dfrac{1}{x+1}$.

Find $\lim\limits_{x\to -1} g(x)$.

$\lim\limits_{x\to -1^-} \frac{1}{x+1} = \frac{1}{\text{sm}-} = -\infty$

$\lim\limits_{x\to -1^+} \frac{1}{x+1} = \frac{1}{\text{sm}+} = \infty$

$\Rightarrow \boxed{\lim\limits_{x\to -1} g(x) \text{ DNE}}$

**Definition.**
**Infinite Limits**
Suppose $f$ is defined for all $x$ near $a$. If $f(x)$ grows arbitrarily large for all $x$ sufficiently close (but not equal) to $a$, we write

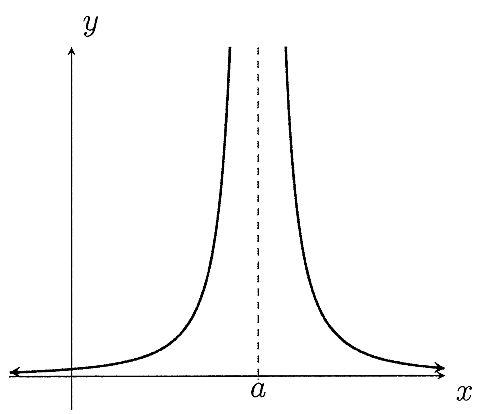
$$\lim_{x \to a} f(x) = \infty$$

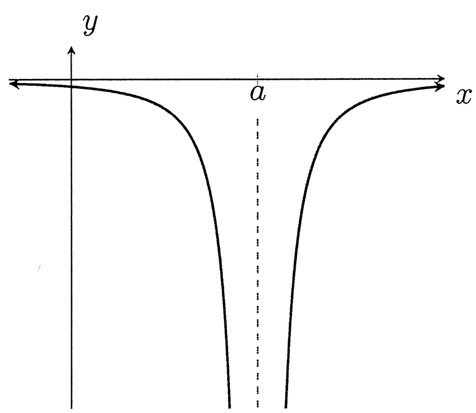
and say the limit of $f(x)$ as $x$ approaches $a$ is infinity.

If $f(x)$ is negative and grows arbitrarily large in magnitude for all $x$ sufficiently close (but not equal) to $a$, we write

$$\lim_{x \to a} f(x) = -\infty$$

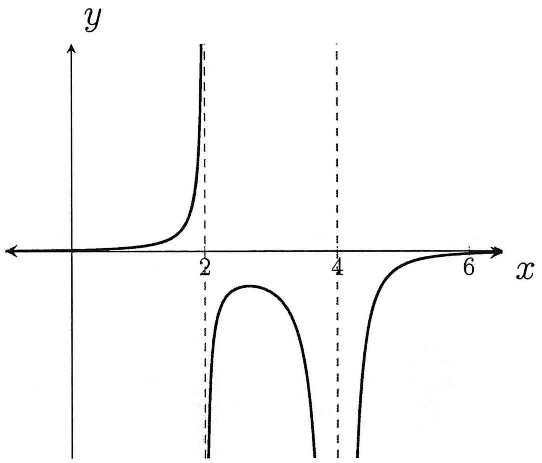
and say the limit of $f(x)$ as $x$ approaches $a$ is negative infinity.
*In both cases, the limit does not exist.*



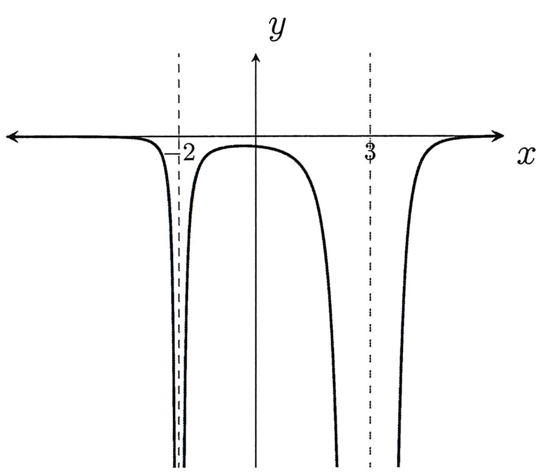

**Definition.**
**Vertical Asymptote**
If $\lim_{x \to a} f(x) = \pm\infty$, $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$, the line $x = a$ is called a **vertical asymptote** of $f$.

**Example.** The graph of $\ell(x)$ has vertical asymptotes $x = 2$ and $x = 4$. Find the following limits:




1. $\lim_{x\to 2^-} \ell(x) = \infty$
2. $\lim_{x\to 2^+} \ell(x) = -\infty$
3. $\lim_{x\to 2} \ell(x)$ DNE
4. $\lim_{x\to 4^-} \ell(x) = -\infty$
5. $\lim_{x\to 4^-} \ell(x) = -\infty$
6. $\lim_{x\to 4} \ell(x) = -\infty$

**Example.** The graph of $p(x)$ has vertical asymptotes $x = -2$ and $x = 3$. Find the following limits:




1. $\lim_{x\to -2^-} p(x) = -\infty$
2. $\lim_{x\to -2^+} p(x) = -\infty$
3. $\lim_{x\to -2} p(x) = -\infty$
4. $\lim_{x\to 3^-} p(x) = -\infty$
5. $\lim_{x\to 3^+} p(x) = -\infty$
6. $\lim_{x\to 3} p(x) = -\infty$

*Note:* When computing the limit, $\lim_{x\to a} f(x)$ we can try to evaluate $f(a)$.

If $f(a)$ is of the form $\frac{0}{0}$, try factoring, conjugates, etc. (Section 2.3)

If $f(a)$ is of the form $\frac{c}{0}$ where $c \neq 0$, the limit is infinite. Here, we must consider the signs of the numerator and the denominator.

$$\lim_{x\to 3^+} \frac{\overbrace{2-5x}^{-13}}{\underbrace{x-3}_{\text{small pos}}} = -\infty \qquad \lim_{x\to 3^-} \frac{\overbrace{2-5x}^{-13}}{\underbrace{x-3}_{\text{small neg}}} = \infty$$

**Example.** Evaluate:

a) $\lim_{x\to 3^-} \frac{2}{(x-3)^3}$ $= \frac{2}{(\text{sm } -)^3}$ $= -\infty$

b) $\lim_{x\to 3^+} \frac{2}{(x-3)^3}$ $= \frac{2}{(\text{sm } +)^3}$ $= \infty$

c) $\lim_{x\to 3} \frac{2}{(x-3)^3}$ DNE

**Example.** For $h(t) = \frac{t^2-4t+3}{t^2-1}$, find $\lim_{t\to 1} h(t)$ and $\lim_{t\to -1} h(t)$. $h(t) = \frac{(t-3)(t-1)}{(t-1)(t+1)}$

$\rightarrow \lim_{t\to 1} h(t) = \lim_{t\to 1} \frac{(t-3)(t-1)}{(t-1)(t+1)} = \lim_{t\to 1} \frac{t-3}{t+1} = \frac{-2}{2} = \boxed{-1}$

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$\rightarrow \lim_{t\to -1^-} h(t) = \lim_{t\to -1^-} \frac{t-3}{t+1} = \frac{-4}{\text{sm}-} = \infty$

$\rightarrow \lim_{t\to -1^+} h(t) = \lim_{t\to -1^+} \frac{t-3}{t+1} = \frac{-4}{\text{sm}+} = -\infty$

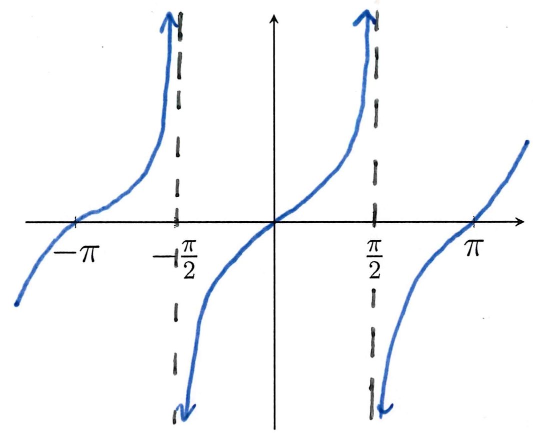
Are these infinite limits or limits at infinity?

$\Rightarrow \lim_{t\to -1} h(t)$ DNE

**Example.** Evaluate $\lim_{\nu \to 7} \frac{4}{(\nu-7)^2}$. $= \frac{4}{\text{sml}} = \infty$

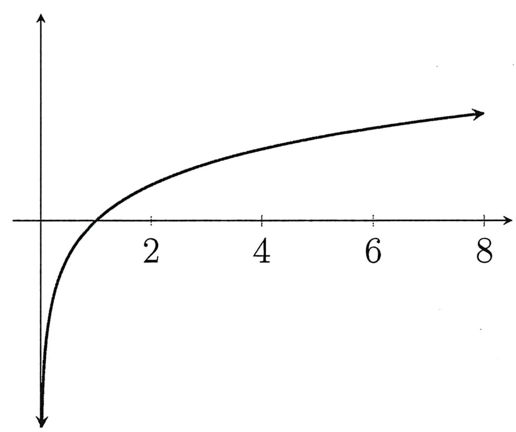
**Example.** Evaluate $\lim_{r \to 1} \frac{r}{|r-1|}$. $= \frac{1}{\text{sml}} = \infty$

**Example.** Evaluate




- $\lim_{x\to\pi/2^-} \tan x$ $= \infty$
- $\lim_{x\to\pi/2^+} \tan x$ $= -\infty$
- $\lim_{x\to-\pi/2^-} \tan x$ $= \infty$
- $\lim_{x\to-\pi/2^+} \tan x$ $= -\infty$

**Example.** Below is the graph of $\ln(x)$. Use this to evaluate the following limits:




- $\lim_{x\to 0^+} \ln(x)$ $= -\infty$
- $\lim_{x\to\infty} \ln(x)$ $= \infty$

**Example.** Find all vertical asymptotes, $x = a$, for $f(x) = \dfrac{\cos x}{x^2 + 2x}$.

where $x^2 + 2x = 0$
and $\cos(x) \neq 0$

Find $x^2 + 2x = 0$

$x(x+2) = 0 \rightarrow x = 0, x = -2$

So our vertical asymptotes are $x = 0$, $x = 2$.