# Math 1080 Class notes
# Fall 2021

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## 5.5: Substitution Rule

**Theorem 5.6: Substitution Rule for Indefinite Integrals**
Let $u = g(x)$, where $g$ is differentiable on an interval, and let $f$ be continuous on the corresponding range of $g$. On that interval,

$$\int f\big(g(x)\big)g'(x)\,dx = \int f(u)\,du$$

**Example.** We know

$$\frac{d}{dx}\left[\frac{(2x+1)^4}{4}\right] = 2(2x+1)^3$$

Thus, if $f(x) = x^3$ and $g(x) = 2x+1$ then $g'(x) = 2$, so we let $u = 2x+1$, then

$$\begin{aligned}\int 2(2x+1)^3\,dx &= \int f\big(g(x)\big)g'(x)\,dx \\ &= \int u^3\,du \\ &= \frac{u^4}{4} + C \\ &= \frac{(2x+1)^4}{4} + C\end{aligned}$$

**Procedure: Substitution Rule (Change of Variables)**

1. Given an indefinite integral involving a composite function $f\big(g(x)\big)$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)\,dx$ in the integral.
3. Evaluate the new indefinite integral with respect to $u$.
4. Write the result in terms of $x$ using $u = g(x)$.

**Example.** Evaluate the following integrals:

a) $\displaystyle\int 2x(x^2+3)^4\,dx$

b) $\displaystyle\int (2x+1)^3\,dx$

c) $\displaystyle\int x^2\sqrt{x^3+1}\,dx$

d) $\displaystyle\int \theta\sqrt[4]{1-\theta^2}\,d\theta$

e) $\displaystyle\int \sqrt{4-t}\,dt$

f) $\displaystyle\int (2-x)^6\,dx$

**Example.** Evaluate the following integrals:

a) $\displaystyle\int \sec(2\theta)\tan(2\theta)\,d\theta$

b) $\displaystyle\int \csc^2\left(\frac{t}{3}\right)\,dt$

c) $\displaystyle\int \frac{\sin(x)}{1+\cos^2(x)}\,dx$

d) $\displaystyle\int \frac{\tan^{-1}(x)}{1+x^2}\,dx$

The acceleration of a particle moving back and forth on a line is $a(t) = \frac{d^2s}{dt^2} = \pi^2\cos(\pi t)\ m/s^2$ for all $t$. If $s = 0$ and $v = 8\ m/s$ when $t = 0$, find the value of $s$ when $t = 1$ sec.

**Example.** Evaluate the following integrals:

a) $\displaystyle\int (6x^2+2)\sin(x^3+x+1)\,dx$

b) $\displaystyle\int \frac{\sin(\theta)}{\cos^5(\theta)}\,d\theta$

c) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

d) $\displaystyle\int \frac{2^t}{2^t+3}\,dt$

e) $\displaystyle\int 6x^2 4^{x^3}\,dx$

f) $\displaystyle\int \frac{dx}{\sqrt{36-4x^2}}$

g) $\displaystyle\int \sin(t)\sec^2(\cos(t))\,dt$

h) $\displaystyle\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

i) $\displaystyle\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

j) $\displaystyle\int 5\cos(7x+5)\,dx$

k) $\displaystyle\int \frac{3}{\sqrt{1-25x^2}}\,dx$

l) $\displaystyle\int \frac{dx}{\sqrt{1-9x^2}}$

**Example.** Evaluate the following integrals using the recommended substitution:

a) $\displaystyle\int \sec^2(x)\tan(x)\,dx$
where $u = \tan(x)$.

b) $\displaystyle\int \sec^2(x)\tan(x)\,dx$
where $u = \sec(x)$.

**Example.** Solve the initial value problem: $\frac{dy}{dx} = 4x(x^2+8)^{-1/3}, y(0) = 0$.

**Example.** Evaluate the following integrals:

a) $\displaystyle\int xe^{-x^2}\,dx$

b) $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$

c) $\displaystyle\int \frac{dt}{8-3t}$

d) $\displaystyle\int 5^t \sin(5^t)\,dt$

e) $\displaystyle\int \frac{e^w}{36+e^{2w}}\,dw$

**Theorem 5.7: Substitution Rule for Definite Integrals**
Let $u = g(x)$, where $g'$ is continuous on $[a, b]$, and let $f$ be continuous on the range of $g$. Then

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

**Example.** Evaluate the integrals:

a) $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$

b) $\displaystyle\int_1^3 \frac{dt}{(t-4)^2}$

c) $\displaystyle\int_0^3 \frac{v^2+1}{\sqrt{v^3+3v+4}}\,dv$

d) $\displaystyle\int_0^1 2x\big(4-x^2\big)\,dx$

e) $\displaystyle\int_2^3 \frac{x}{\sqrt[3]{x^2-1}}\,dx$

f) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{1+\cos(x)}\,dx$

g) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos^2(x)}\,dx$

h) $\displaystyle\int_{-\frac{\pi}{12}}^{\frac{\pi}{8}} \sec^2(2y)\,dy$

i) $\displaystyle\int_0^1 (1-2x^9)\,dx$

j) $\displaystyle\int_0^1 (1-2x)^9\,dx$

k) $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{1+4x^2}\,dx$

l) $\displaystyle\int_0^4 \frac{x}{x^2+1}\,dx$

m) $\displaystyle\int_0^{\pi} 3\cos^2(x)\sin(x)\,dx$

n) $\displaystyle\int_0^{\frac{\pi}{8}} \sec(2\theta)\tan(2\theta)\,d\theta$

o) $\displaystyle\int_0^{1} (3t-1)^{50}\,dt$

p) $\displaystyle\int_0^{3} \frac{1}{5x+1}\,dx$

q) $\displaystyle\int_0^1 xe^{-x^2}\,dx$

r) $\displaystyle\int_e^{e^4} \frac{1}{x\sqrt{\ln(x)}}\,dx$

s) $\displaystyle\int_0^{\frac{1}{2}} \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx$

t) $\displaystyle\int_0^1 \frac{e^z+1}{e^z+z}\,dz$

u) $\displaystyle\int_1^4 \frac{dy}{2\sqrt{y}\left(1+\sqrt{y}\right)^2}$

v) $\displaystyle\int_{\ln\left(\frac{\pi}{4}\right)}^{\ln\left(\frac{\pi}{2}\right)} e^w \cos(e^w)\,dw$

w) $\displaystyle\int_0^{\frac{1}{8}} \frac{x}{\sqrt{1-16x^2}}\,dx$

x) $\displaystyle\int_1^{e^2} \frac{\ln(p)}{p}\,dp$

y) $\displaystyle\int_0^{\frac{\pi}{4}} e^{\sin^2(x)} \sin(2x)\, dx$

z) $\displaystyle\int_{-\pi}^{\pi} x^2 \sin(7x^3)\, dx$

**Example. Average velocity:** An object moves in one dimension with a velocity in $m/s$ given by $v(t) = 8\sin(\pi t) + 2t$. Find its average velocity over the time interval from $t = 0$ to $t = 10$, where $t$ is measured in seconds.

**Example.** Prove $\displaystyle\int \tan(x)\,dx = \ln|\sec(x)| + C$.

**Example.** Evaluate the integrals:

a) $\displaystyle\int \frac{x}{(x-2)^3}\,dx$

b) $\displaystyle\int x\sqrt{x-1}\,dx$

c) $\displaystyle\int x^3(1+x^2)^{\frac{3}{2}}\,dx$

d) $\displaystyle\int \frac{y^2}{(y+1)^4}\,dy$

e) $\displaystyle\int (z+1)\sqrt{3z+2}\,dz$

f) $\displaystyle\int_0^1 \frac{x}{(x+2)^3}\,dx$

**Half-Angle Formulas**

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$
$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$$

**Example.** Evaluate the integrals:

a) $\displaystyle\int \cos^2(x)\,dx$

b) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2(x)\,dx$

c) $\displaystyle\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx$

d) $\displaystyle\int x\sin^2(x^2)\,dx$

e) $\displaystyle\int \sin^2\left(\theta+\frac{\pi}{6}\right)d\theta$

f) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2(8\theta)\,d\theta$

**Example.** If $f$ is continuous and $\int_0^4 f(x)\,dx = 10$, find $\int_0^2 f(2x)\,dx$.

**Example.** If $f$ is continuous and $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 xf(x^2)\,dx$.

**Example.** Suppose $f$ is an even function with $\int_0^8 f(x)\,dx = 9$. Evaluate the following:

a) $\int_{-1}^1 xf(x^2)\,dx$.

b) $\int_{-2}^2 x^2 f(x^3)\,dx$.

**Example.** Evaluate the integrals:

a) $\displaystyle\int \sec^2(10x)\,dx$

b) $\displaystyle\int \tan^{10}(4x)\sec^2(4x)\,dx$

c) $\displaystyle\int \left(x^{\frac{3}{2}}+8\right)^5 \sqrt{x}\,dx$

d) $\displaystyle\int \frac{2x}{\sqrt{3x+2}}\,dx$

e) $\displaystyle\int \frac{7x^2 + 2x}{x}\, dx$

f) $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$

g) $\displaystyle\int_0^{\sqrt{3}} \frac{3}{9 + x^2}\, dx$

h) $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\sin(2y)}{\sin^2(y) + 2}\, dy$

i) $\displaystyle\int \frac{\sec(z)\tan(z)}{\sqrt{\sec(z)}}\,dz$

j) $\displaystyle\int \frac{1}{\sin^{-1}(x)\sqrt{1-x^2}}\,dx$

k) $\displaystyle\int \frac{x}{\sqrt{4-9x^2}}\,dx$

l) $\displaystyle\int \frac{x}{1+x^4}\,dx$

m) $\displaystyle\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta$

n) $\displaystyle\int x^2\sqrt{2+x}\,dx$

o) $\displaystyle\int \left(\sin^5(x) + 3\sin^3(x) - \sin(x)\right)\cos(x)\,dx$

p) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(x^3 + x^4 \tan(x)\right) dx$

q) $\displaystyle\int_{0}^{\frac{\pi}{2}} \cos(x) \sin\left(\sin(x)\right) dx$

r) $\displaystyle\int \frac{1+x}{1+x^2}\, dx$

**Example.** Evaluate these more challenging integrals:

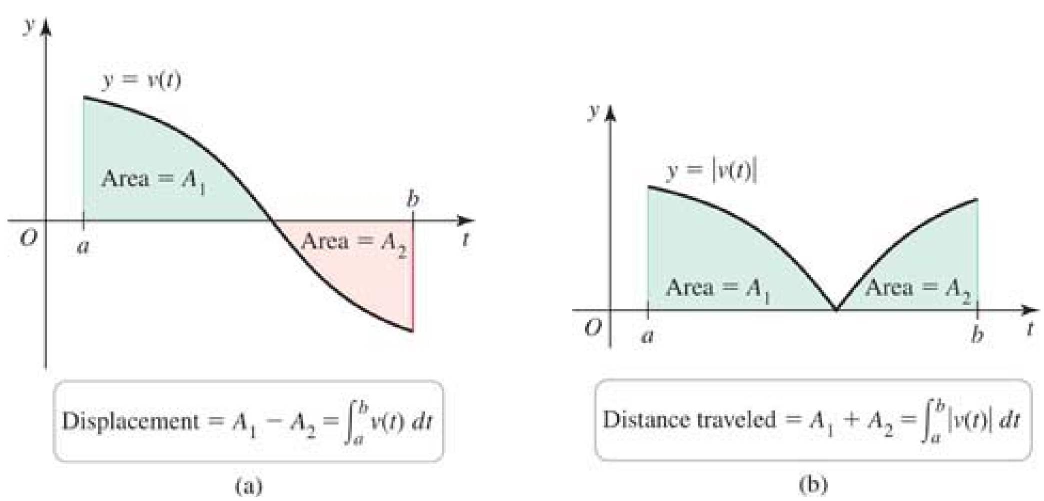
a) $\displaystyle\int \frac{dx}{\sqrt{1+\sqrt{1+x}}}$

b) $\displaystyle\int x\sin^4(x^2)\cos(x^2)\,dx$

## 6.1: Velocity and Net Change

**Definition. (Position, Velocity, Displacement, and Distance)**

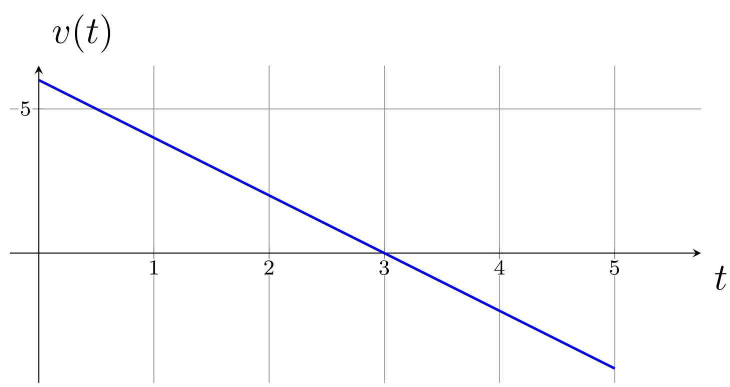
1. The **position** of an object moving along a line at time $t$, denoted $s(t)$, is the location of the object relative to the origin.
2. The **velocity** of an object at time $t$ is $v(t) = s'(t)$.
3. The **displacement** of the object between $t = a$ and $t = b > a$ is
$$s(b) - s(a) = \int_a^b v(t)\, dt.$$
4. The **distance traveled** by the object between $t = a$ and $t = b > a$ is
$$\int_a^b |v(t)|\, dt$$
where $|v(t)|$ is the **speed** of the object at time $t$.



**Example.** Suppose an object moves along a line with velocity (in ft/s) $v(t) = 6 - 2t$, for $0 \le t \le 5$, where $t$ is measured in seconds.

- Find the displacement of the object on the interval $0 \le t \le 5$.

- Find the distance traveled by the object on the interval $0 \le t \le 5$.



**Example.** A cyclist rides down a long straight road at a velocity (in m/min) given by $v(t) = 400 - 20t$, for $0 \leq t \leq 10$.

- How far does the cyclists travel in the first 5 minutes?

- How far does the cyclists travel in the first 10 minutes?

- How far has the cyclist traveled when her velocity is 250 m/min?

**Example.** The population of a community of foxes is observed to fluctuate on a 10-year cycle due to variations in the availability of prey. When population measurements began ($t = 0$), the population was 35 foxes. The growth rate in units of foxes/year was observed to be:

$$P'(t) = 5 + 10\sin\left(\frac{\pi t}{5}\right)$$

- Find $P(t)$.

- Find the population of foxes after the first 5 years, rounded to the nearest whole number of foxes.

**Theorem 6.1: Position from Velocity**
Given the velocity $v(t)$ of an object moving along a line and its initial position $s(0)$, the position function of the object for future times $t \geq 0$ is

$$\underbrace{s(t)}_{\substack{\text{position}\\ \text{at } t}} = \underbrace{s(0)}_{\substack{\text{initial}\\ \text{position}}} + \underbrace{\int_0^t v(x)\,dx}_{\substack{\text{displacement}\\ \text{over } [0,t]}}.$$

**Theorem 6.2: Velocity from Acceleration**
Given the acceleration $a(t)$ of an object moving along a line and its initial velocity $v(0)$, the velocity of the object for future times $t \geq 0$ is

$$v(t) = v(0) + \int_0^t a(x)\,dx.$$

**Example.** At $t = 0$, a train approaching a station begins decelerating from a speed of 80 miles/hour according to the acceleration function $a(t) = -1280(1 + 8t)^{-3}$, where $t \geq 0$ is measured in hours. The units of acceleration are mi/hr$^2$.

- Find the velocity of the train at $t = 0.25$.

- How far does the train travel in the first 15 minutes (1/4 hour)?

- How long does it take the train to travel 9 miles?

**Theorem 6.3: Net Change and Future Value**

Suppose a quantity $Q$ changes over time at a known rate $Q'$. Then the **net change** in $Q$ between $t = a$ and $t = b > a$ is

$$\underbrace{Q(b) - Q(a)}_{\text{net change in } Q} = \int_a^b Q'(t)\,dt.$$

Given the initial value $Q(0)$, the **future value** of $Q$ at time $t \geq 0$ is

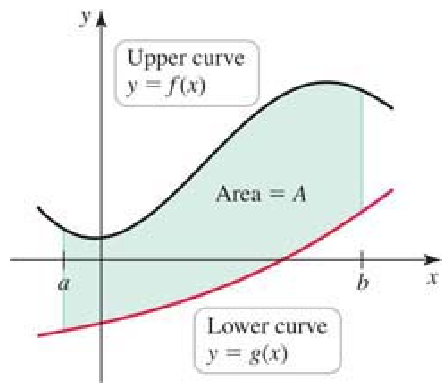
$$Q(t) = Q(0) + \int_0^t Q'(x)\,dx.$$

| **Velocity-Displacement Problems** | **General Problems** |
|---|---|
| Position $s(t)$ | Quantity $Q(t)$ (such as volume or population) |
| Velocity: $s'(t) = v(t)$ | Rate of change: $Q'(t)$ |
| Displacement: $s(b) - s(a) = \int_a^b v(t)\,dt$ | Net change: $Q(b) - Q(a) = \int_a^b Q'(t)\,dt$ |
| Future position: $s(t) = s(0) + \int_0^t v(x)\,dx$ | Future value of $Q$: $Q(t) = Q(0) + \int_0^t Q'(x)\,dx$ |

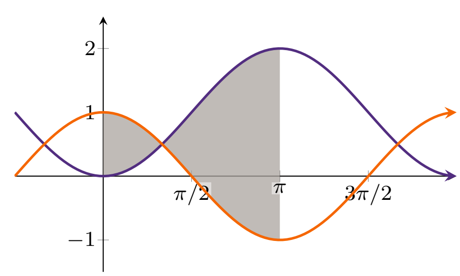
## 6.2: Regions Between Curves

**Definition. (Area of a Region Between Two Curves)**
Suppose $f$ and $g$ are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$. The area of the region bounded by the graphs of $f$ and $g$ on $[a, b]$ is

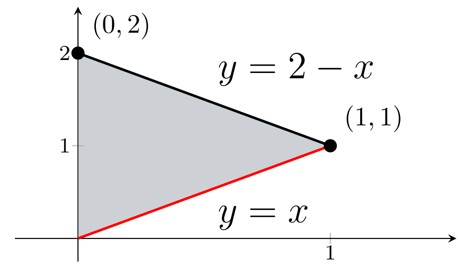
$$A = \int_a^b (f(x) - g(x))\, dx.$$



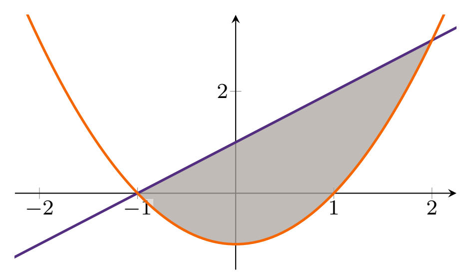

**Example.** Consider the region bounded by the curves $y = \cos(x)$ and $y = 1 - \cos(x)$, $0 \leq x \leq \pi$. Set up the integral(s) representing the area of this region.



**Example.** Find the area of the region by integrating with respect to $x$.




**Example.** Find the volume of the solid whose base is bounded by the graphs of $y = x + 1$ and $y = x^2 - 1$, with the cross sections in the shape of rectangles of height 2 taken perpendicular to the $x$-axis.



**Definition. (Area of a Region Between Two Curves with Respect to $y$)** Suppose $f$ and $g$ are continuous functions with $f(y) \geq g(y)$ on the interval $[c, d]$. The area of the region bounded by the graphs $x = f(y)$ and $x = g(y)$ on $[c, d]$ is

$$A = \int_c^d (f(y) - g(y))\, dy.$$

**Example.** Find the area of the region bounded by $x = 3y$, and $x = y^2 - 10$

by integrating with respect to $x$

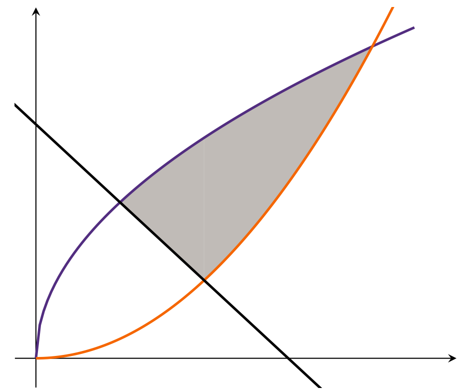
by integrating with respect to $y$

**Example.** Find the area of the region bounded by $y = x^3$, and $y = \sqrt{x}$

by integrating with respect to $x$

by integrating with respect to $y$

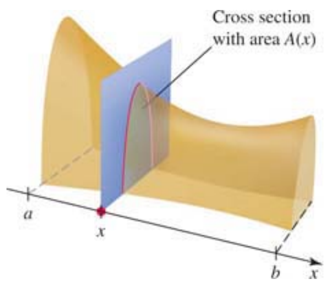
**Example.** Find the area of the region bounded by $y = 4\sqrt{2x}$, $y = 2x^2$, and $y = -4x + 6$

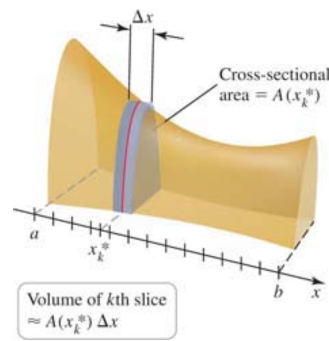


## 6.3: Volume by Slicing

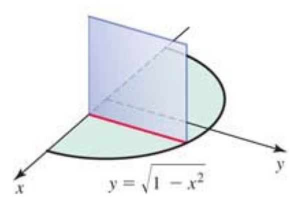
**General Slicing Method**
Suppose a solid object extends from $x = a$ to $y = b$, and the cross section of the solid perpendicular to the $x$-axis has an area given by a function $A$ that is integrable on $[a, b]$. The volume of the solid is

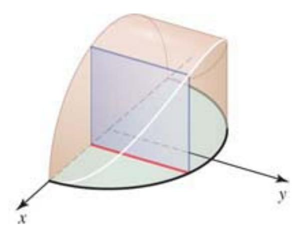
$$V = \int_a^b A(x)\,dx.$$




**Example.** Use the general slicing method to find the volume of the solid whose base is the region bounded by the semicircle $y = \sqrt{1 - x^2}$ and the $x$-axis, and whose cross sections through the solid perpendicular to the $x$-axis are squares.

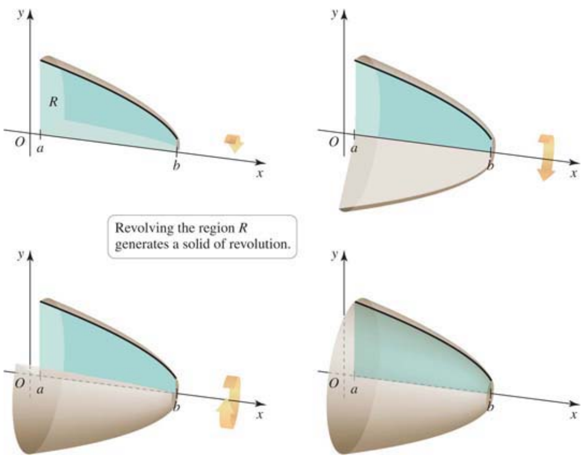


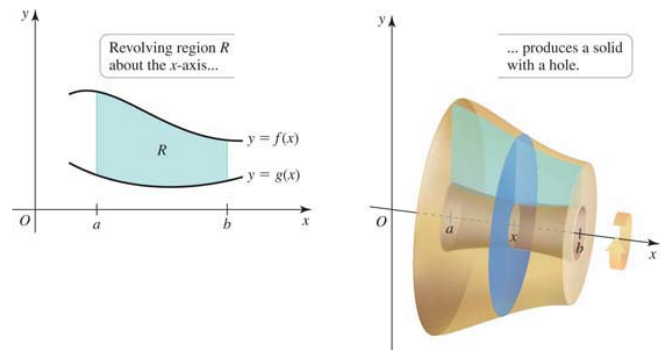



**Disk Method about the $x$-Axis**

Let $f$ be continuous with $f(x) \geq 0$ on the interval $[a, b]$. If the region $R$ bounded by the graph of $f$, the $x$-axis, and the lines $x = a$ and $x = b$ is revolved about the $x$-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi \underbrace{f(x)}_{\substack{\text{disk} \\ \text{radius}}}{}^2 dx.$$



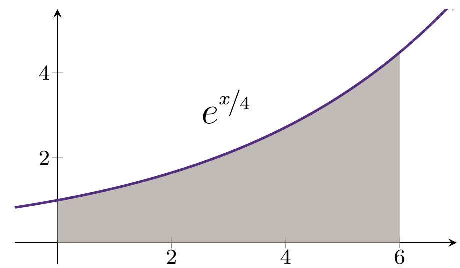




**Washer Method about the $x$-Axis**

Let $f$ and $g$ be continuous functions with $f(x) \geq g(x) \geq 0$ on $[a, b]$. Let $R$ be the region bounded by $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$. When $R$ is revolved about the $x$-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi\big(\underbrace{f(x)}_{\substack{\text{outer} \\ \text{radius}}}{}^2 - \underbrace{g(x)}_{\substack{\text{inner} \\ \text{radius}}}{}^2\big)\, dx.$$

**Example.** Consider the region bounded by $y = e^{x/4}$, $y = 0$, $x = 0$, and $x = 6$. Find the volume of the solid generated by rotating the region about the $x$-axis.

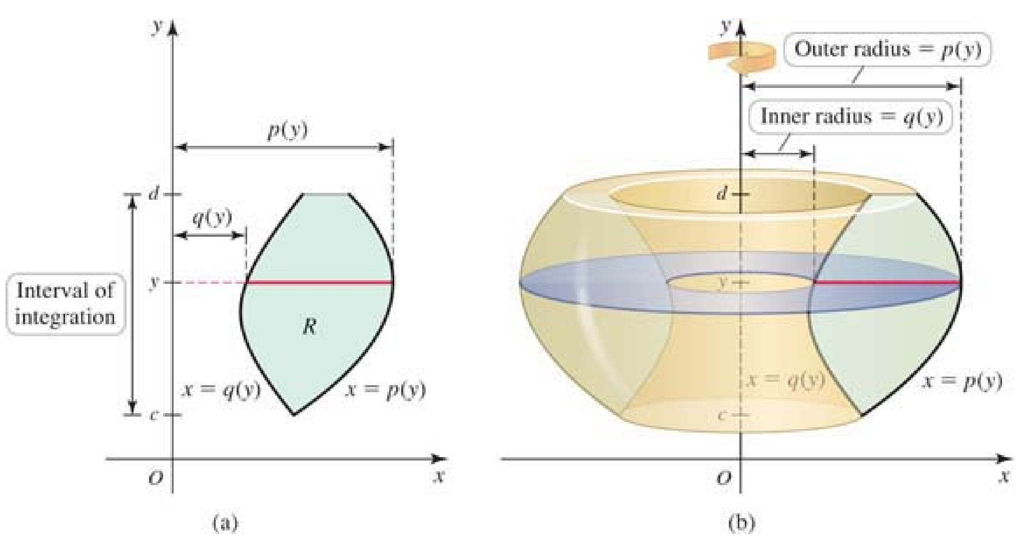


**Disk and Washer Methods about the $y$-Axis**
Let $p$ and $q$ be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let $R$ be the region bounded by $x = p(y)$, $x = q(y)$, and the lines $y = c$ and $y = d$. When $R$ is revolved around the $y$-axis, the volume of the resulting solid of revolution is given by

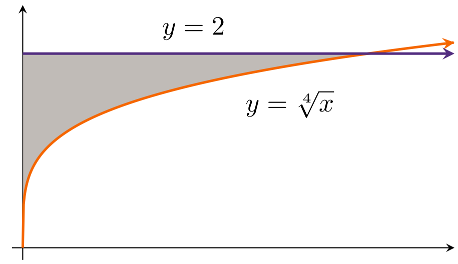
$$V = \int_c^d (\underbrace{p(y)}_{\substack{\text{outer}\\\text{radius}}}{}^2 - \underbrace{q(y)}_{\substack{\text{inner}\\\text{radius}}}{}^2\,)\,dy.$$

If $q(y) = 0$, the disk method results:

$$V = \int_c^d \underbrace{p(y)}_{\substack{\text{disk}\\\text{radius}}}{}^2 dy.$$



**Example.** Consider the region bounded between $y = \sqrt[4]{x}$, $y = 2$, and $x = 0$.

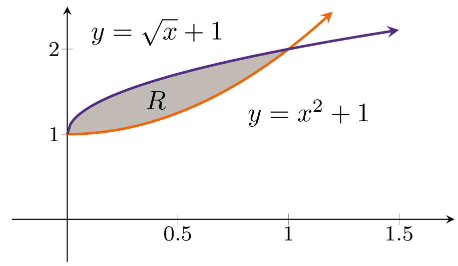



Setup the integral with respect to $x$ that gives the area of the region.

Setup the integral with respect to $y$ that gives the area of the region.

Use the disk/washer method to setup the that represents the volume of the solid generated by rotating the region about the $x$-axis.

**Example.** Consider the region $R$ between $y = \sqrt{x}+1$ and $y = x^2+1$. Setup the integrals which find the volume of the solid obtained by rotating the region $R$ as indicated below.




about the $y$-axis

about the $x$-axis

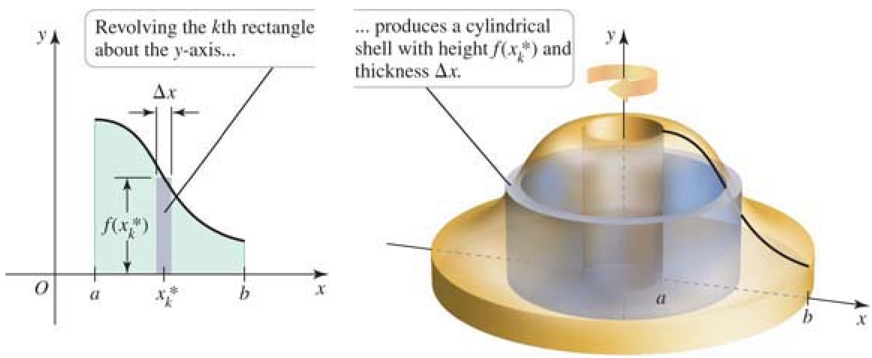
about the line $x = 1$

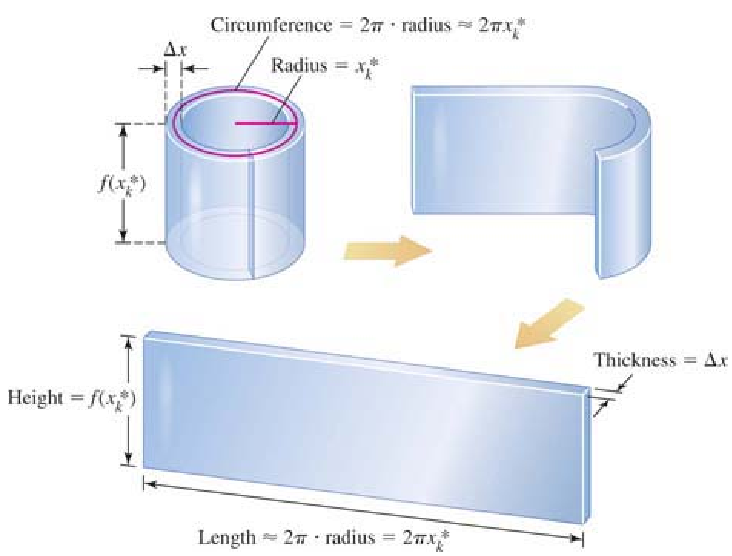
about the line $y = -1$

## 6.4: Volume by Shells

**Volume by the Shell Method**
Let $f$ and $g$ be continuous functions with $f(x) \geq g(x)$ on $[a,b]$. If $R$ is the region bounded by the curves $y = f(x)$ and $y = g(x)$ between the lines $x = a$ and $x = b$, the volume of the solid generated when $R$ is revolved about the $y$-axis is

$$V = \int_a^b \underbrace{2\pi x}_{\substack{\text{shell} \\ \text{circumference}}} \underbrace{\big(f(x) - g(x)\big)}_{\substack{\text{shell} \\ \text{height}}}\, dx.$$



**Example.** Consider a general region $R$ revolved around the $y$-axis.

When using the **disk/washer** method, we integrate with respect to __________

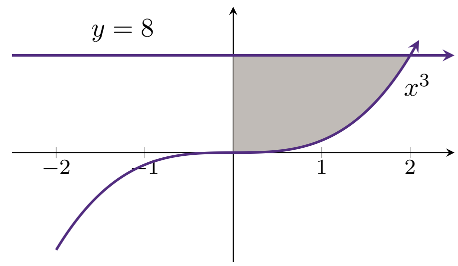
When using the **shell** method, we integrate with respect to __________

**Example.** Consider a general region $R$ revolved around the $x$-axis.

When using the **disk/washer** method, we integrate with respect to __________

When using the **shell** method, we integrate with respect to __________

**Example.** Consider the region bounded between $y = x^3$, $y = 8$ and $x = 0$.




Use the disk/washer method to setup the integral that represents the volume of the solid generated by rotating the region about the $x$-axis.

about the $y$-axis.

Use the disk/washer method to setup the integral that represents the volume of the solid generated by rotating the region about the line $x = -1$.

about the line $y = 8$.

**Example.** Consider the region bounded by $y = \dfrac{1}{x+1}$ and $y = 1 - \dfrac{x}{3}$. Use both the disk/washer method and shell method to find the volume of the solid generated when $R$ is rotated about the $x$-axis.

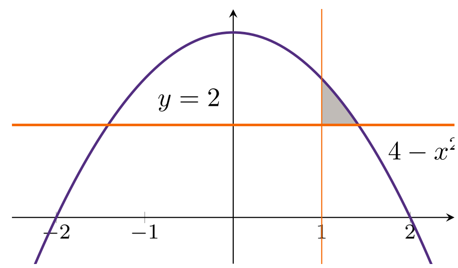
**Example.** Determine if the following statements are true.

When using the shell method, the axis of the cylindrical shells is parallel to the axis of revolution.

If a region is revolved about the $y$-axis, then the shell method must be used.

If a region is revolved about the $x$-axis, it is possible to use the disk/washer method and integrate with respect to $x$.

**Example.** Consider the region $R$ bounded by $y = 4 - x^2$, $y = 2$, and $x = 1$. Use the shell method to setup the integral that represents the volume of the solid generated by rotating the region $R$ about the indicated axis of rotation.




about $x$-axis,

about $y$-axis,

about the line $x = -2$,

about the line $y = 2$.