15.1 Trigonometric Identities

Definition. The **Pythagorean Identity for trigonometric functions** is

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\frac{1}{\cos^2\theta}\left(\sin^2(\theta) + \cos^2(\theta) = 1\right) \longrightarrow \tan^2\theta + 1 = \sec^2\theta$$

$$\frac{1}{\sin^2\theta}\left(\sin^2\theta + \cos^2\theta = 1\right) \longrightarrow 1 + \cot^2\theta = \csc^2\theta$$

$$\sin^2\theta = 1 - \cos^2\theta \longrightarrow \sin\theta = \pm\sqrt{1-\cos^2\theta}$$

$$\cos^2\theta = 1 - \sin^2\theta \longrightarrow \cos\theta = \pm\sqrt{1-\sin^2\theta}$$

Definition. The **Angle Sum Formulas** are

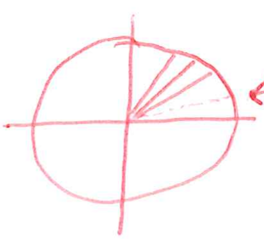
$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$
$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

Note: Since $\cos(\theta)$ is even and $\sin(\theta)$ is odd, we can derive the difference formula from the sum formula.

$$\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right)$$
$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{3+1}{4} = 1$$

$$\boxed{\text{Note: } \sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right) = 1}$$

$$\cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$$
$$= \frac{\sqrt{2}}{2}\,\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\,\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$



$\theta = \frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$

$$\sin\left(x+\frac{\pi}{2}\right) = \sin(x)\underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} + \cos(x)\underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} = \cos(x)$$

$$\cos\left(x-\frac{\pi}{2}\right) = \cos(x)\underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} + \sin(x)\underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} = \sin(x)$$

$$\sin(A-B) = \sin(A)\cos(-B) + \cos(A)\sin(-B)$$
$$= \sin(A)\cos(B) + \cos(A)[-\sin(B)]$$
$$= \sin(A)\cos(B) - \cos(A)\sin(B)$$

Definition. The **double-angle formulas** are a special case of the angle-sum formulas:

$$\begin{aligned}\sin(2\theta) &= \sin(\theta+\theta)\\ &= \sin(\theta)\cos(\theta)+\cos(\theta)\sin(\theta)\\ &= \boxed{2\sin(\theta)\cos(\theta)}\end{aligned}$$

$$\begin{aligned}\cos(2\theta) &= \cos(\theta+\theta)\\ &= \cos(\theta)\cos(\theta)-\sin(\theta)\sin(\theta)\\ &= \boxed{\cos^2(\theta)-\sin^2(\theta)}\end{aligned}$$

Note: Using the Pythagorean Identity, we have 2 additional representations of $\cos(2\theta)$.

$$\cos^2\theta - \sin^2\theta = \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1$$

$$= (1-\sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$$

$$\sin\left(2\cdot\frac{\pi}{6}\right) = 2\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right) = 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$$

$$\boxed{\sin\left(\frac{\pi}{3}\right)}$$

$$\cos\left(2\left(-\frac{\pi}{4}\right)\right) = \cos^2\left(-\frac{\pi}{4}\right) - \sin^2\left(-\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}\right)^2 - \left(-\frac{\sqrt{2}}{2}\right)^2 = 0$$

$$= 2\cos^2\left(-\frac{\pi}{4}\right) - 1 = 2\left(\frac{\sqrt{2}}{2}\right)^2 - 1 = \frac{2\cdot 2}{4} - 1 = 0$$

$$= 1 - 2\sin^2\left(-\frac{\pi}{4}\right) = 1 - 2\left(-\frac{\sqrt{2}}{2}\right)^2 = 1 - \frac{2\cdot 2}{4} = 0$$

$$\boxed{\cos\left(-\frac{\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right)}$$

Definition. The **half-angle formulas** are derived from the double angle formula:

$$\sin(\theta) = \pm\sqrt{\frac{1-\cos(2\theta)}{2}}$$
$$\cos(\theta) = \pm\sqrt{\frac{1+\cos(2\theta)}{2}}$$

$$\sin\left(\frac{\pi}{6}\right) = \sqrt{\frac{1-\cos\left(2\left(\frac{\pi}{6}\right)\right)}{2}} = \sqrt{\frac{1-\cos\left(\frac{\pi}{3}\right)}{2}} = \sqrt{\frac{1-\frac{1}{2}}{2}} = \sqrt{\frac{\frac{1}{2}}{2}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$\cos\left(\frac{5\pi}{6}\right) = \underbrace{-\sqrt{\frac{1+\cos\left(2\left(\frac{5\pi}{6}\right)\right)}{2}} = -\sqrt{\frac{1+\cos\left(\frac{5\pi}{3}\right)}{2}} = -\sqrt{\frac{1+\frac{1}{2}}{2}} = -\sqrt{\frac{3/2}{2}} = -\sqrt{\frac{3}{4}} = \frac{-\sqrt{3}}{2}}_{\text{negative because } \frac{\pi}{2} \leq \theta \leq \pi.}$$

$$\cos\left(\frac{\pi}{12}\right) = \sqrt{\frac{1+\cos\left(2\,\frac{\pi}{12}\right)}{2}} = \sqrt{\frac{1+\cos\left(\frac{\pi}{6}\right)}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2}$$

$\boxed{\text{compare with } \cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)}$

Example. Solve all the following on $[0, 2\pi]$.

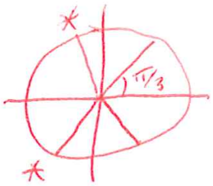
a) $2\theta\cos(\theta) + \theta = 0$

$\theta(2\cos\theta + 1) = 0$

$\Rightarrow \theta = 0$

$2\cos\theta + 1 = 0$

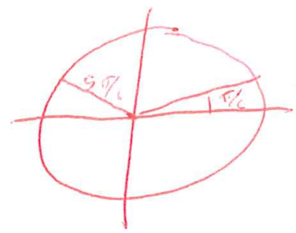
$\cos\theta = -1/2$



$\Rightarrow \theta = \frac{2\pi}{3}, \theta = \frac{4\pi}{3}$

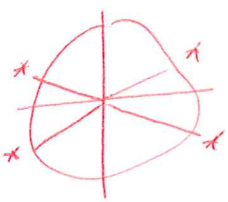
b) $\sin(\theta) = \frac{1}{2}$ $\rightarrow \theta = \pi/6$

$\theta = 5\pi/6$



c) $4\cos^2(x) - 3 = 0$

$\cos^2(x) = \frac{3}{4}$



$\cos(x) = \pm\frac{\sqrt{3}}{2}$

$x = \pi/6, x = 5\pi/6$

$x = \frac{7\pi}{6}, x = \frac{11\pi}{6}$

d) $2\underbrace{\sin^2(x)}_{y^2} - \underbrace{\sin(x)}_{y} - 1 = 0$

$2y^2 - y - 1 = 0$

$(2y+1)(y-1) = 0$

$\rightarrow 2y+1 = 0 \rightarrow y = -1/2$

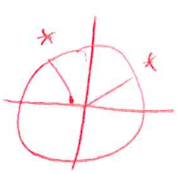
$\sin(x) = -\frac{1}{2}$

$x = \pi/6, x = \frac{11\pi}{6}$

$\rightarrow y - 1 = 0 \rightarrow \sin x = 1$

$x = \pi/2$

e) $\sin(3x) = \dfrac{\sqrt{2}}{2}$



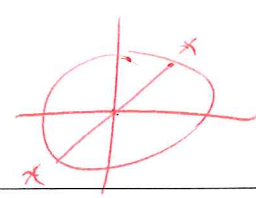
$\Rightarrow 3x = \frac{\pi}{4}$ or $3x = \frac{3\pi}{4}$

$x = \frac{\pi}{12}$ $\quad$ $x = \frac{\pi}{4}$

f) $\cos(3x) = \sin(3x)$

$\frac{\cos(3x)}{\sin(3x)} = 1 \Rightarrow \frac{\cos(3x)}{\sin(3x)} = \frac{\pm\frac{\sqrt{2}}{2}}{\pm\frac{\sqrt{2}}{2}}$

$\Rightarrow 3x = \frac{\pi}{4}, 3x = \frac{5\pi}{4}$



$x = \frac{\pi}{12}, x = \frac{5\pi}{12}$