3.1 Introducing the Derivative:

Recall that when given a distance function $s(t)$, the average velocity over the interval $[a, t]$ is

$$v_{\text{avg}} = \frac{s(t) - s(a)}{t - a}$$

and the instantaneous velocity is the limit of the average velocities as $t \to a$:

$$v_{\text{inst}} = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$$

Furthermore, the average velocity is the slope of the secant line through the points $(a, s(a))$ and $(t, s(t))$ and the instantaneous velocity is the slope of the tangent line at the point $(a, s(a))$. `https://www.desmos.com/calculator/08syaijrdo`

Definition. (Rate of Change and the Slope of the Tangent Line)

The **average rate of change** in f on the interval $[a, x]$ is the slope of the corresponding secant line:

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}$$

The **instantaneous rate of change** in f at a is

$$m_{\text{tan}} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

which is also the **slope of the tangent line** at $(a, f(a))$, provided this limit exists. This **tangent line** is the unique line through $(a, f(a))$ with slope m_{tan}. Its equation is

$$y - f(a) = m_{\text{tan}}(x - a)$$

Example. Find an equation of the line tangent to the graph of $f(x) = \dfrac{3}{x}$ at $\left(2, \dfrac{3}{2}\right)$.

$$m_{tan} = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2} \frac{\frac{3}{x} - \frac{3}{2}}{x - 2}$$

$$= \lim_{x \to 2} \frac{3(2-x)}{2x} \Big/ (x-2)$$

$$= \lim_{x \to 2} \frac{-3}{2x}$$

$$= -3/4$$

tangent line

$$y - 3/2 = -\frac{3}{4}(x-2)$$

$$y = -\frac{3}{4}x + \frac{3}{2} + \frac{3}{2}$$

$$\boxed{y = -\frac{3}{4}x + 3}$$

Definition. (Rate of Change and the Slope of the Tangent Line)

The **average rate of change** in f on the interval $[a, a+h]$ is the slope of the corresponding secant line:

$$m_{\text{sec}} = \frac{f(a+h) - f(a)}{h}.$$

The **instantaneous rate of change** in f at a is

$$m_{\text{tan}} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

which is also the **slope of the tangent line** at $(a, f(a))$, provided this limit exists.

Example. Find an equation of the line tangent to the graph of $f(x) = x^3 + 4x$ at $(1, 5)$.

$$m_{tan} = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{[(1+h)^3 + 4(1+h)] - [5]}{h}$$

$$= \lim_{h\to 0} \frac{1 + 3h + 3h^2 + h^3 + 4 + 4h - 5}{h}$$

$$= \lim_{h\to 0} \frac{h(3 + 3h + h^2 + 4)}{h}$$

$$= \lim_{h\to 0} 3 + 3h + h^2 + 4$$

$$= 3 + 0 + 0 + 4 = \boxed{7}$$

tangent line

$$y - 5 = 7(x-1)$$

$$y = 7x - 7 + 5$$

$$\boxed{y = 7x - 2}$$

Definition. (The Derivative of a Function at a Point)

The **derivative of** f **at** a, denoted $f'(a)$, is given by either of the two following limits, provided the limits exist and a is in the domain of f:

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \quad (1) \quad \text{or } f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \quad (2)$$

If $f'(a)$ exists, we say that f is **differentiable at** a.

Example. Find an equation of the line tangent to the graph of $f(x) = \dfrac{8}{x^2}$ at $(2, 2)$.

$$\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{\frac{8}{(2+h)^2} - \frac{8}{2^2}}{h} = \lim_{h\to 0} \frac{8(4-(2+h)^2)}{4(2+h)^2} \Big/ h$$

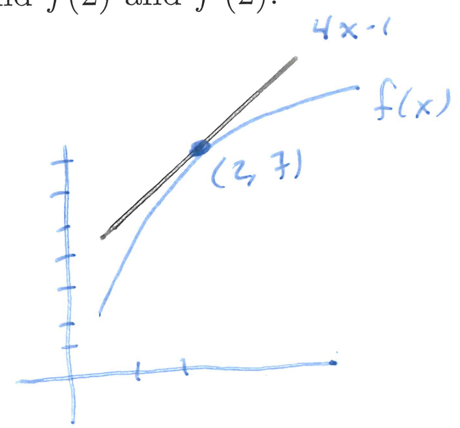
$$= \lim_{h\to 0} \frac{2(4-4-4h-h^2)}{h(2+h)^2} = \lim_{h\to 0} \frac{-8-h}{(2+h)^2} = \frac{-8}{(2+0)^2} = -2$$

$$y - 2 = -2(x-2)$$

$$y = -2x + 4 + 2$$

$$\boxed{y = -2x + 6}$$

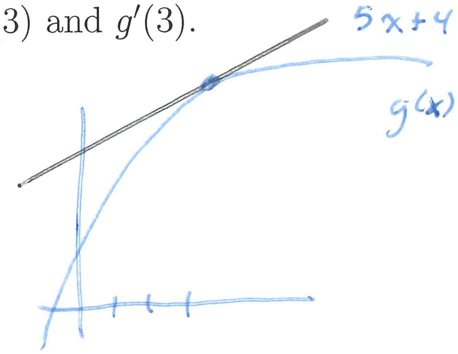
Example. An equation of the line tangent to the graph of f at the $(2, 7)$ is $y = 4x - 1$. Find $f(2)$ and $f'(2)$.



$f(2) = 7$

$f'(2) = 4$

Example. An equation of the line tangent to the graph of g at $x = 3$ is $y = 5x + 4$. Find $g(3)$ and $g'(3)$.



$g(3) = 5(3) + 4 = 19$

$g'(3) = 5$

Example. If $h(1) = 2$ and $h'(1) = 3$, find an equation of the line tangent to the graph of h at $x = 1$.

$y - h(1) = h'(1)(x-1)$

$y = 3(x-1) + 2$

$\boxed{y = 3x - 1}$

Example. If $f'(-2) = 7$, find an equation of the line tangent to the graph of f at the point $(-2, 4)$.

$f(-2) = 4$

$y - 4 = 7(x+2)$

$\boxed{y = 7x + 18}$