1 5.5: Substitution Rule

Theorem 5.6: Substitution Rule for Indefinite Integrals
Let $u = g(x)$, where g is differentiable on an interval, and let f be continuous on the corresponding range of g. On that interval,

$$\int f\big(g(x)\big)g'(x)\,dx = \int f(u)\,du$$

Example. We know

$$\frac{d}{dx}\left[\frac{(2x+1)^4}{4}\right] = 2(2x+1)^3$$

Thus, if $f(x) = x^3$ and $g(x) = 2x+1$ then $g'(x) = 2$, so we let $u = 2x+1$, then

$$\begin{aligned}\int 2(2x+1)^3\,dx &= \int f\big(g(x)\big)g'(x)\,dx \\ &= \int u^3\,du \\ &= \frac{u^4}{4} + C \\ &= \frac{(2x+1)^4}{4} + C\end{aligned}$$

Procedure: Substitution Rule (Change of Variables)

1. Given an indefinite integral involving a composite function $f\big(g(x)\big)$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)\,dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Write the result in terms of x using $u = g(x)$.

Example. Evaluate the following integrals:

a) $\displaystyle\int 2x(x^2+3)^4\,dx$

b) $\displaystyle\int (2x+1)^3\,dx$

c) $\displaystyle\int x^2\sqrt{x^3+1}\,dx$

d) $\displaystyle\int \theta\,\sqrt[4]{1-\theta^2}\,d\theta$

e) $\displaystyle\int \sqrt{4-t}\,dt$

f) $\displaystyle\int (2-x)^6\,dx$

Example. Evaluate the following integrals:

a) $\displaystyle\int \sec(2\theta)\tan(2\theta)\,d\theta$

b) $\displaystyle\int \csc^2\left(\frac{t}{3}\right)\,dt$

c) $\displaystyle\int \frac{\sin(x)}{1+\cos^2(x)}\,dx$

d) $\displaystyle\int \frac{\tan^{-1}(x)}{1+x^2}\,dx$

The acceleration of a particle moving back and forth on a line is $a(t) = \frac{d^2s}{dt^2} = \pi^2\cos(\pi t)\ m/s^2$ for all t. If $s = 0$ and $v = 8\ m/s$ when $t = 0$, find the value of s when $t = 1$ sec.

Example. Evaluate the following integrals:

a) $\displaystyle\int (6x^2+2)\sin(x^3+x+1)\,dx$

b) $\displaystyle\int \frac{\sin(\theta)}{\cos^5(\theta)}\,d\theta$

c) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

d) $\displaystyle\int \frac{2^t}{2^t+3}\,dt$

e) $\int 6x^2 4^{x^3}\, dx$

f) $\int \frac{dx}{\sqrt{36-4x^2}}$

g) $\int \sin(t)\sec^2(\cos(t))\, dt$

h) $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\, dx$

i) $\displaystyle\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

j) $\displaystyle\int 5\cos(7x+5)\,dx$

k) $\displaystyle\int \frac{3}{\sqrt{1-25x^2}}\,dx$

l) $\displaystyle\int \frac{dx}{\sqrt{1-9x^2}}$

Example. Evaluate the following integrals using the recommended substitution:

a) $\int \sec^2(x)\tan(x)\,dx$
where $u = \tan(x)$.

b) $\int \sec^2(x)\tan(x)\,dx$
where $u = \sec(x)$.

Example. Solve the initial value problem: $\frac{dy}{dx} = 4x(x^2+8)^{-1/3}, y(0) = 0$.

Example. Evaluate the following integrals:

a) $\displaystyle\int xe^{-x^2}\,dx$

b) $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$

c) $\displaystyle\int \frac{dt}{8-3t}$

d) $\displaystyle\int 5^t \sin(5^t)\,dt$

e) $\displaystyle\int \frac{e^w}{36+e^{2w}}\,dw$

Theorem 5.7: Substitution Rule for Definite Integrals
Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g. Then

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Example. Evaluate the integrals:

a) $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$

b) $\displaystyle\int_1^3 \frac{dt}{(t-4)^2}$

c) $\displaystyle\int_0^3 \frac{v^2+1}{\sqrt{v^3+3v+4}}\,dv$

d) $\displaystyle\int_0^1 2x\big(4-x^2\big)\,dx$

e) $\displaystyle\int_2^3 \frac{x}{\sqrt[3]{x^2-1}}\,dx$

f) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{1+\cos(x)}\,dx$

g) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos^2(x)}\,dx$

h) $\displaystyle\int_{-\frac{\pi}{12}}^{\frac{\pi}{8}} \sec^2(2y)\,dy$

i) $\displaystyle\int_0^1 (1-2x^9)\,dx$

j) $\displaystyle\int_0^1 (1-2x)^9\,dx$

k) $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{1+4x^2}\,dx$

l) $\displaystyle\int_0^4 \frac{x}{x^2+1}\,dx$

m) $\displaystyle\int_0^{\pi} 3\cos^2(x)\sin(x)\,dx$

n) $\displaystyle\int_0^{\frac{\pi}{8}} \sec(2\theta)\tan(2\theta)\,d\theta$

o) $\displaystyle\int_0^{1} (3t-1)^{50}\,dt$

p) $\displaystyle\int_0^{3} \frac{1}{5x+1}\,dx$

q) $\displaystyle\int_0^1 xe^{-x^2}\,dx$

r) $\displaystyle\int_e^{e^4} \frac{1}{x\sqrt{\ln(x)}}\,dx$

s) $\displaystyle\int_0^{\frac{1}{2}} \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx$

t) $\displaystyle\int_0^1 \frac{e^z+1}{e^z+z}\,dz$

u) $\displaystyle\int_1^4 \frac{dy}{2\sqrt{y}\left(1+\sqrt{y}\right)^2}$

v) $\displaystyle\int_{\ln\left(\frac{\pi}{4}\right)}^{\ln\left(\frac{\pi}{2}\right)} e^w \cos(e^w)\, dw$

w) $\displaystyle\int_0^{\frac{1}{8}} \frac{x}{\sqrt{1-16x^2}}\, dx$

x) $\displaystyle\int_1^{e^2} \frac{\ln(p)}{p}\, dp$

y) $\displaystyle\int_0^{\frac{\pi}{4}} e^{\sin^2(x)} \sin(2x)\, dx$

z) $\displaystyle\int_{-\pi}^{\pi} x^2 \sin(7x^3)\, dx$

Example. Average velocity: An object moves in one dimension with a velocity in m/s given by $v(t) = 8\sin(\pi t) + 2t$. Find its average velocity over the time interval from $t = 0$ to $t = 10$, where t is measured in seconds.

Example. Prove $\int \tan(x)\,dx = \ln|\sec(x)| + C$.

Example. Evaluate the integrals:

a) $\displaystyle\int \frac{x}{(x-2)^3}\,dx$

b) $\displaystyle\int x\sqrt{x-1}\,dx$

c) $\int x^3(1+x^2)^{\frac{3}{2}}\,dx$

d) $\int \frac{y^2}{(y+1)^4}\,dy$

e) $\int (z+1)\sqrt{3z+2}\,dz$

f) $\int_0^1 \frac{x}{(x+2)^3}\,dx$

Half-Angle Formulas

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$
$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$$

Example. Evaluate the integrals:

a) $\displaystyle\int \cos^2(x)\,dx$

b) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2(x)\,dx$

c) $\int \frac{1}{x^2} \cos^2\left(\frac{1}{x}\right) dx$

d) $\int x \sin^2(x^2)\, dx$

e) $\int \sin^2\left(\theta + \frac{\pi}{6}\right) d\theta$

f) $\int_0^{\frac{\pi}{4}} \cos^2(8\theta)\, d\theta$

Example. If f is continuous and $\int_0^4 f(x)\,dx = 10$, find $\int_0^2 f(2x)\,dx$.

Example. If f is continuous and $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 xf(x^2)\,dx$.

Example. Suppose f is an even function with $\int_0^8 f(x)\,dx = 9$. Evaluate the following:

a) $\int_{-1}^1 xf(x^2)\,dx$.

b) $\int_{-2}^2 x^2 f(x^3)\,dx$.

Example. Evaluate the integrals:

a) $\displaystyle\int \sec^2(10x)\,dx$

b) $\displaystyle\int \tan^{10}(4x)\sec^2(4x)\,dx$

c) $\displaystyle\int \left(x^{\frac{3}{2}}+8\right)^5 \sqrt{x}\,dx$

d) $\displaystyle\int \frac{2x}{\sqrt{3x+2}}\,dx$

e) $\displaystyle\int \frac{7x^2 + 2x}{x}\, dx$

f) $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$

g) $\displaystyle\int_0^{\sqrt{3}} \frac{3}{9 + x^2}\, dx$

h) $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\sin(2y)}{\sin^2(y) + 2}\, dy$

i) $\displaystyle\int \frac{\sec(z)\tan(z)}{\sqrt{\sec(z)}}\,dz$

j) $\displaystyle\int \frac{1}{\sin^{-1}(x)\sqrt{1-x^2}}\,dx$

k) $\displaystyle\int \frac{x}{\sqrt{4-9x^2}}\,dx$

l) $\displaystyle\int \frac{x}{1+x^4}\,dx$

m) $\displaystyle\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta$

n) $\displaystyle\int x^2\sqrt{2+x}\,dx$

o) $\displaystyle\int \left(\sin^5(x) + 3\sin^3(x) - \sin(x)\right)\cos(x)\,dx$

p) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(x^3 + x^4 \tan(x)\right) dx$

q) $\displaystyle\int_0^{\frac{\pi}{2}} \cos(x) \sin\left(\sin(x)\right) dx$

r) $\displaystyle\int \frac{1+x}{1+x^2}\, dx$

Example. Evaluate these more challenging integrals:

a) $\displaystyle\int \frac{dx}{\sqrt{1+\sqrt{1+x}}}$

b) $\int x \sin^4(x^2) \cos(x^2)\, dx$