17.6: Surface Integrals

Imagine a sphere with a known temperature distribution. How would we find the average temperature over the sphere?

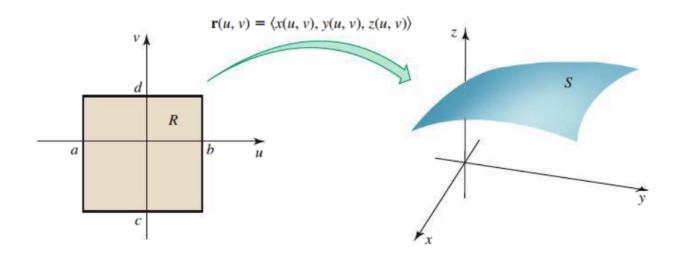
Parallel Concepts				
Curves	Surfaces			
Arc length	Surface area			
Line integrals	Surface integrals			
One-parameter description	Two-parameter description			

Parameterized Surfaces

Recall that in \mathbb{R}^2 , we parameterized a curve by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ where $a \leq t \leq b$. In \mathbb{R}^3 , we parameterize a surface by

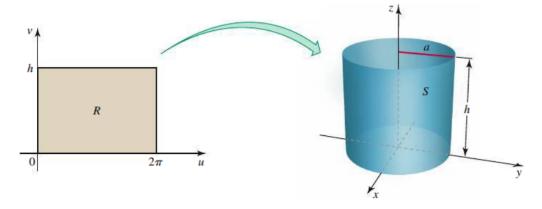
$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

where the parameters are over $R = \{(u, v) : a \le u \le b, c \le v \le d\}$



Cylinders:

$$\{(x, y, z) : x = a\cos(\theta), y = a\sin(\theta), 0 \le \theta \le 2\pi, 0 \le z \le h\}$$
$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle = \langle a\cos(u), a\sin(u), v \rangle$$



Cones:

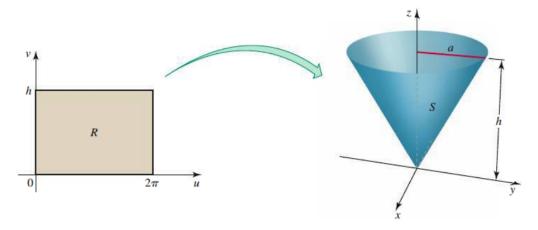
$$\{(r, \, \theta, \, z) : 0 \le r \le a, \, 0 \le \theta \le 2\pi, \, z = rh/a\}$$

For a fixed value of z, r = az/h:

$$x = r\cos(\theta) = \frac{az}{h}\cos(\theta)$$
 and $y = r\sin(\theta) = \frac{az}{h}\sin(\theta)$

Now, let $u = \theta$ and v = z, then

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle = \left\langle \frac{av}{h} \cos(u), \frac{av}{h} \sin(u), v \right\rangle$$



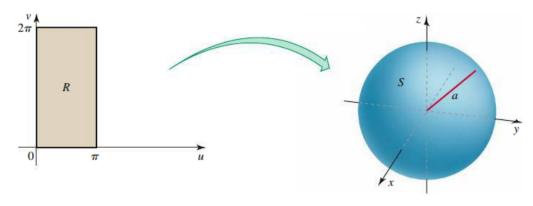
Spheres:

$$\{(\rho, \, \varphi, \, \theta) : \rho = a, 0 \le \varphi \le \pi, \, 0 \le \theta \le 2\pi\}$$

$$x = a\sin(\varphi)\cos(\theta), \qquad y = a\sin(\varphi)\sin(\theta), \qquad z = a\cos(\varphi)$$

Now, let $u = \theta$ and v = z, then

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle = \langle a\sin(u)\cos(v), a\sin(u)\cos(v), a\cos(u) \rangle$$

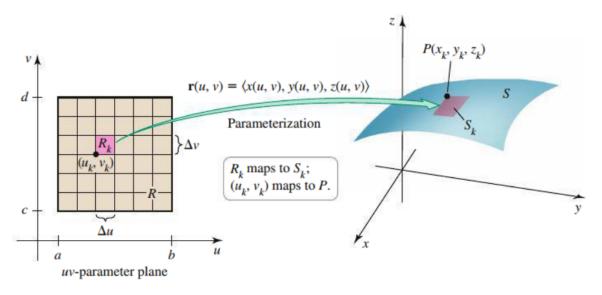


Example. Find parametric descriptions for the following surfaces

The plane 3x - 2y + z = 2

The paraboloid $z = x^2 + y^2$, for $0 \le z \le 9$

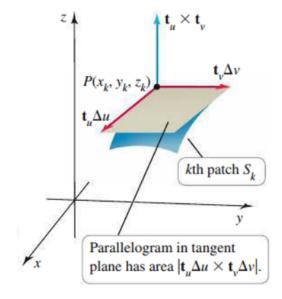
Surface Integrals of Scalar-Valued Functions



Using the parameterization

$$r(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$

over the region $R = \{(u, v) : a \le u \le b, c \le v \le d\}$, it is important that we know ΔS_k , which is the area of S_k .



Definition. (Surface Integral of Scalar-Valued Functions on Parameterized Surfaces)

Let f be a continuous scalar-valued function on a smooth surface S given parametrically by $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$, where u and v vary over $R = \{(u,v) : a \le u \le b, c \le v \le d\}$. Assume also that the tangent vectors

$$\mathbf{t}_{u} = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \text{ and } \mathbf{t}_{v} = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

are continuous on R and the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R. Then the surface integral of f over S is

$$\iint\limits_{S} f(x, y, z) dS = \iint\limits_{R} f(x(u, v), y(u, v), z(u, v)) |\mathbf{t}_{u} \times \mathbf{t}_{v}| dA$$

If f(x, y, z) = 1, this integral equals the surface area of S.

Example. Find the surface area of the following surfaces

A cylinder with radius a > 0 and height h (open ends)

A sphere of radius a

Example. The temperature on the surface of a sphere of radius a varies with latitude according to the function $T(\varphi, \theta) = 10 + 50\sin(\varphi)$, for $0 \le \varphi \le \pi$ and $0 \le \theta \le 2\pi$. Find the average temperature over the sphere.

Surface Integrals on Explicitly Defined Surfaces

Suppose a smooth surface S is defined explicitly as z = g(x, y). Here, we let u = x and v = y. This gives us

$$\mathbf{t}_u = \mathbf{t}_x = \langle 1, 0, z_x \rangle, \quad \mathbf{t}_v = \mathbf{t}_y = \langle 0, 1, z_y \rangle$$

thus

$$\mathbf{t}_x \times \mathbf{t}_y = \langle -z_x, -z_y, 1 \rangle$$

and

$$|\mathbf{t}_x \times \mathbf{t}_y| = \sqrt{z_x^2 + z_y^2 + 1}$$

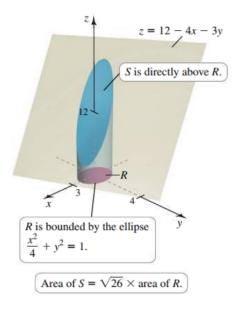
Theorem 17.14: Evaluation of Surface Integrals of Scalar-Valued Functions on Explicitly Defined Surfaces

Let f be a continuous function on a smooth surface S given by z = g(x, y), for (x, y) in a region R. The surface integral of f over S is

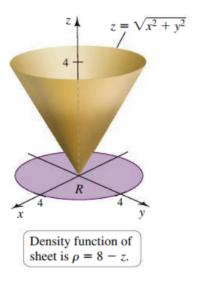
$$\iint_{S} f(x, y, z) dS = \iint_{S} f(x, y, g(x, y)) \sqrt{z_x^2 + z_y^2 + 1} dA.$$

If f(x, y, z) = 1, the surface integral equals the area of the surface.

Example. Find the area of the surface S that lies in the plane z = 12 - 4x - 3y directly above the region R bounded by the ellipse $x^2/4 + y^2 = 1$



Example. A thin conical sheet is described by the surface $z = (x^2 + y^2)^{\frac{1}{2}}$, for $0 \le z \le 4$. The density of the sheet in g/ cm² is $\rho = f(x, y, z) = (8 - z)$. What is the mass of the cone?



Explicit Description $z = g(x, y)$			Parametric Description			
Surface	Equation	Normal vector $\pm \langle -z_x, -z_y, 1 \rangle$	$egin{aligned} \mathbf{magnitude} \ \langle -z_x, -z_y, 1 angle \end{aligned}$	Equation	$egin{aligned} \mathbf{Normal} & \mathbf{vector} \ \mathbf{t}_u imes \mathbf{t}_v \end{aligned}$	$egin{aligned} \mathbf{magnitude} \ \mathbf{t}_u imes \mathbf{t}_v \end{aligned}$
Cylinder	$x^2 + y^2 = a^2,$ $0 \le z \le h$	$\langle x,y,0 \rangle$	a	$\mathbf{r} = \langle a\cos(u), a\sin(u), v \rangle, 0 \le u \le 2\pi, 0 \le v \le h$	$\langle a\cos(u), a\sin(u), 0\rangle$	a
Cone	$z^2 = x^2 + y^2,$ $0 \le z \le h$	$\langle x/z, y/z, -1 \rangle$	$\sqrt{2}$	$\mathbf{r} = \langle v \cos(u), v \sin(u), v \rangle, 0 \le u \le 2\pi, 0 \le v \le h$	$\langle v\cos(u), v\sin(u), -v\rangle$	$\sqrt{2}v$
Sphere	$x^2 + y^2 + z^2 = a^2$	$\langle x/z, y/z, 1 \rangle;$	a/z	$\mathbf{r} = \langle a \sin(u) \cos(v), \\ a \sin(u) \sin(v), \\ a \cos(u) \rangle \\ 0 \le u \le \pi, \ 0 \le v \le 2\pi$	$\langle a^2 \sin^2(u) \cos(v),$ $a^2 \sin^2(u) \sin(v),$ $a^2 \sin(u) \cos(u) \rangle$	$a^2\sin(u)$
Paraboloid	$z = x^2 + y^2,$ $0 \le z \le h$	$\langle 2x, 2y, -1 \rangle$	$\sqrt{1+4(x^2+y^2)}$	$\mathbf{r} = \langle v \cos(u), v \sin(u), v^2 \rangle, 0 \le u \le 2\pi, 0 \le v \le \sqrt{h}$	$\langle 2v^2\cos(u), 2v^2\sin(u), -v \rangle$	$v\sqrt{1+4v^2}$

Surface Integrals of Vector Fields:

The surfaces we consider must be

- two-sided or orientable
- oriented

Definition. (Surface Integral of a Vector Field)

Suppose $\mathbf{F} = \langle f, g, h \rangle$ is a continuous vector field on a region of \mathbb{R}^3 containing a smooth oriented surface S. If S is defined parametrically as $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, for (u, v) in a region R, then

$$\iint\limits_{S} \mathbf{F} \cdot \mathbf{n} \, ds = \iint\limits_{R} \mathbf{F} \cdot (\mathbf{t}_{u} \times \mathbf{t}_{v}) \, dA,$$

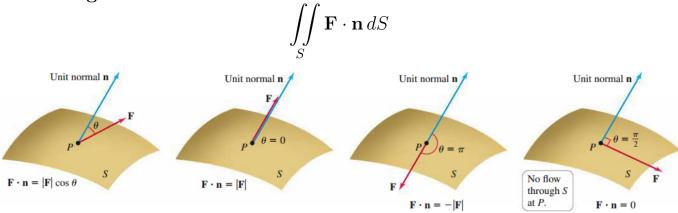
where

$$\mathbf{t}_{u} = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \text{ and } \mathbf{t}_{v} = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

and continuous on R, the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R, and the direction of the normal vector is consistent with the orientation of S. If S is defined in the form z = s(x, y), for (x, y) in a region R, then

$$\iint\limits_{S} \mathbf{F} \cdot \mathbf{n} \, dS = \iint\limits_{S} \left(-f z_x - g z_y + h \right) dA.$$

Flux Integrals:



The unit normal vector we use is

$$\mathbf{n} = rac{\mathbf{t}_u imes \mathbf{t}_v}{|\mathbf{t}_u imes \mathbf{t}_v|}$$

giving us

$$\iint_{S} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{R} \mathbf{F} \cdot \mathbf{n} |\mathbf{t}_{u} \times \mathbf{t}_{v}| \, dA$$

$$= \iint_{R} \mathbf{F} \cdot \frac{\mathbf{t}_{u} \times \mathbf{t}_{v}}{|\mathbf{t}_{u} \times \mathbf{t}_{v}|} |\mathbf{t}_{u} \times \mathbf{t}_{v}| \, dA$$

$$= \iint_{R} \mathbf{F} \cdot (\mathbf{t}_{u} \times \mathbf{t}_{v}) \, dA$$

When the surface S is explicitly given as z = s(x, y), then

$$\mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v) = -fz_x - gz_y + h$$

Definition. (Surface Integral of a Vector Field)

Suppose $\mathbf{F} = \langle f, g, h \rangle$ is a continuous vector field on a region of \mathbb{R}^3 containing a smooth oriented surface S. If S is defined parametrically as $\mathbf{r}(u,v)$ = $\langle x(u,v), y(u,v), z(u,v) \rangle$, for (u,v) in a region R, then

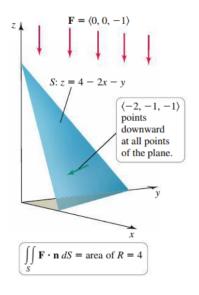
$$\iint_{S} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{R} \mathbf{F} \cdot (\mathbf{t}_{u} \times \mathbf{t}_{v}) \, dA,$$

where $\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$ and $\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$ are continuous on R, the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R, and the direction of the normal vector is consistent with the orientation of S. If S is defined in the form z = s(x, y), for (x, y)in a region R, then

$$\iint\limits_{S} \mathbf{F} \cdot \mathbf{n} \, dS = \iint\limits_{R} \left(-f z_x - g z_y + h \right) dA.$$

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Example. Consider the vertical field $\mathbf{F} = \langle 0, 0, -1 \rangle$. Find the flux in the downward direction across the surface S, which is the plane z = 4 - 2y - y in the first octant.



Example. Consider the radial vector field $\mathbf{F} = \langle f, g, h \rangle = \langle x, y, z \rangle$.