2.4 Infinite Limits

An infinite limit occurs when function values increase or decrease without bound near a point.

Limits which have an infinite value are called **infinite limits**. They are a special case of limits that do not exist, but we indicate that they approach infinity.

Example.

Consider the function

$$f(x) = \frac{1}{x}$$

$\lim_{x \to 0^+}$ $f(x) = \frac{1}{\text{sm} +} = \infty$

$\lim_{x \to 0^-}$ $f(x) = \frac{1}{\text{sm} -} = -\infty$

$\lim_{x \to \infty}$ $f(x) = 0$

$\lim_{x \to -\infty}$ $f(x) = 0$

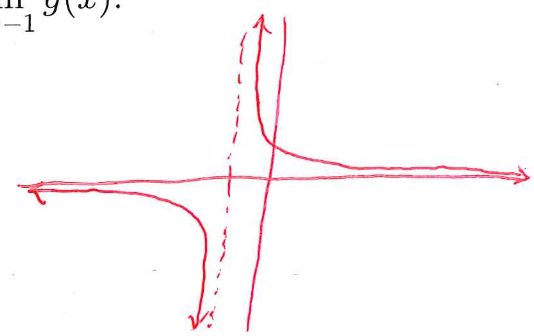
Consider $f(x) = \dfrac{1}{(x-2)^2}$.

Find $\lim_{x \to 2} f(x)$.

$\lim_{x \to 2} f(x) = \infty$

Consider $g(x) = \dfrac{1}{x+1}$.

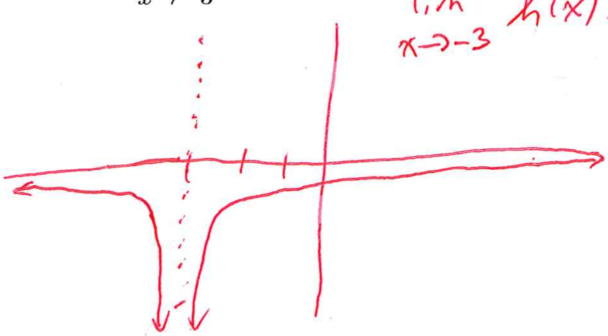
Find $\lim_{x \to -1} g(x)$.



Consider $h(x) = -\dfrac{1}{(x+3)^4}$.

Find $\lim_{x \to -3} h(x)$.

$\lim_{x \to -3} h(x) = -\infty$



$\lim_{x \to -1} g(x)$ DNE

$\because \lim_{x \to -1^-} g(x) \neq \lim_{x \to -1^+}$

Definition. Infinite Limits
Suppose f is defined for all x near a. If $f(x)$ grows arbitrarily large for all x sufficiently close (but not equal) to a, we write

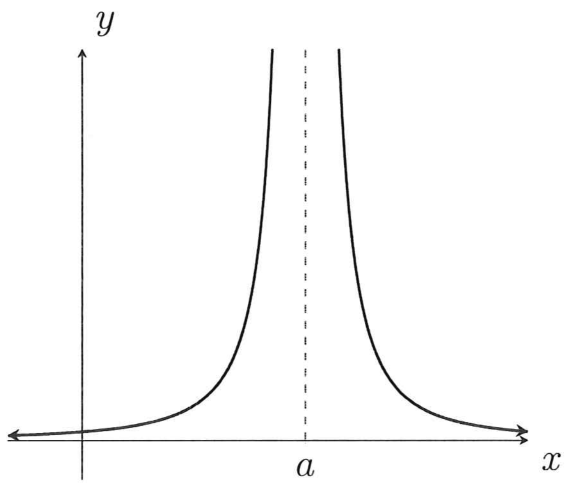
$$\lim_{x \to a} = \infty$$

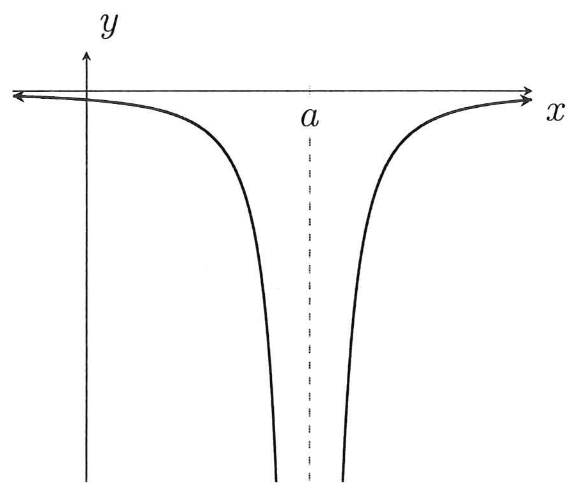
and say the limit of $f(x)$ as x approaches a is infinity.

If $f(x)$ is negative and grows arbitrarily large in magnitude for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} = -\infty$$

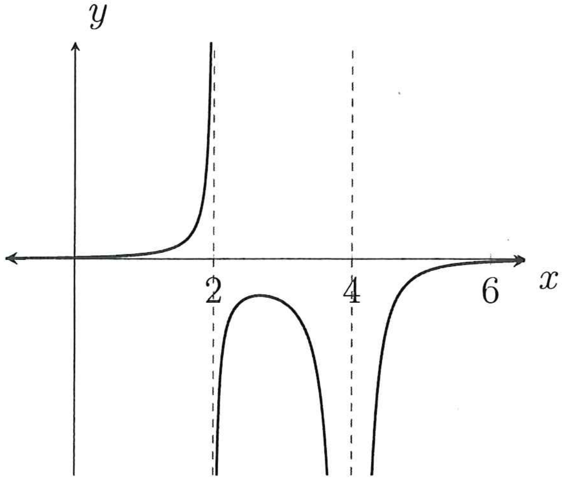
and say the limit of $f(x)$ as x approaches a is negative infinity.
In both cases, the limit does not exist.





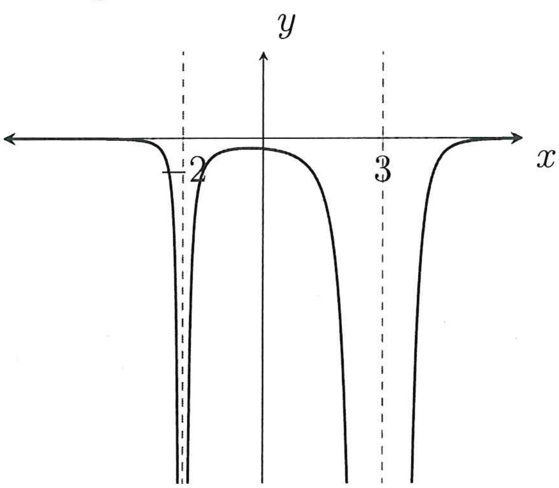
Definition. Vertical Asymptote
If $\lim\limits_{x \to a} f(x) = \pm\infty$, $\lim\limits_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$, the line $x = a$ is called a **vertical asymptote** of f.

Example. The graph of $\ell(x)$ has vertical asymptotes $x = 2$ and $x = 4$. Find the following limits:



1. $\lim_{x\to 2^-} \ell(x)$ $= \infty$
2. $\lim_{x\to 2^+} \ell(x)$ $= -\infty$
3. $\lim_{x\to 2} \ell(x)$ DNE
4. $\lim_{x\to 4^-} \ell(x)$ $= -\infty$
5. $\lim_{x\to 4^-} \ell(x)$ $= -\infty$
6. $\lim_{x\to 4} \ell(x)$ $= -\infty$ ← A special case of DNE

Example. The graph of $p(x)$ has vertical asymptotes $x = -2$ and $x = 3$. Find the following limits:



1. $\lim_{x\to -2^-} p(x)$ $= -\infty$
2. $\lim_{x\to -2^+} p(x)$ $= -\infty$
3. $\lim_{x\to -2} p(x)$ $= -\infty$
4. $\lim_{x\to 3^-} p(x)$ $= -\infty$
5. $\lim_{x\to 3^+} p(x)$ $= -\infty$
6. $\lim_{x\to 3} p(x)$ $= -\infty$

Note: When computing the limit, $\lim_{x\to a} f(x)$ we can try to evaluate $f(a)$.

If $f(a)$ is of the form $\frac{0}{0}$, try factoring, conjugates, etc. (Section 2.3)

If $f(a)$ is of the form $\frac{c}{0}$ where $c \neq 0$, the limit is infinite. Here, we must consider the signs of the numerator and the denominator.

$$\lim_{x\to 3^+} \frac{\overbrace{2-5x}^{-13}}{\underbrace{x-3}_{\text{small pos}}} = -\infty \qquad \lim_{x\to 3^-} \frac{\overbrace{2-5x}^{-13}}{\underbrace{x-3}_{\text{small neg}}} = \infty$$

Example. Evaluate:

a) $\lim_{x\to 3^-} \frac{2}{(x-3)^3} = \frac{2}{\text{sm} -} = -\infty$

b) $\lim_{x\to 3^+} \frac{2}{(x-3)^3} = \frac{2}{\text{sm} +} = \infty$

c) $\lim_{x\to 3} \frac{2}{(x-3)^3}$ DNE

$\lim_{x\to 3^-} \frac{2}{(x-3)^3} \neq \lim_{x\to 3^+} \frac{2}{(x-3)^3}$

Example. For $h(t) = \frac{t^2-4t+3}{t^2-1}$, find $\lim_{t\to 1} h(t)$ and $\lim_{t\to -1} h(t)$.

$h(t) = \frac{(t-1)(t-3)}{(t-1)(t+1)} = \frac{t-3}{t+1}, \; t \neq 1$

$\to \lim_{t\to 1} h(t) = \lim_{t\to 1} \frac{t-3}{t+1} = \frac{-2}{2} = \boxed{-1}$

$\to \lim_{t\to -1^-} h(t) = \lim_{t\to -1^-} \frac{t-3}{t+1} = \frac{-4}{\text{sm} -} = \infty$

$\lim_{t\to -1^+} h(t) = \lim_{t\to -1^+} \frac{t-3}{t+1} = \frac{-4}{\text{sm} +} = -\infty$

$\lim_{t\to -1} h(t)$ DNE

Are these infinite limits or limits at infinity?

$\boxed{\lim_{x\to a} f(x) = \pm\infty}$ $\qquad$ $\boxed{\lim_{x\to \pm\infty} f(x) = L}$

Example. Evaluate $\lim_{\nu \to 7} \frac{4}{(\nu - 7)^2}$. $= \frac{4}{\text{sm} +} = \infty$

Note

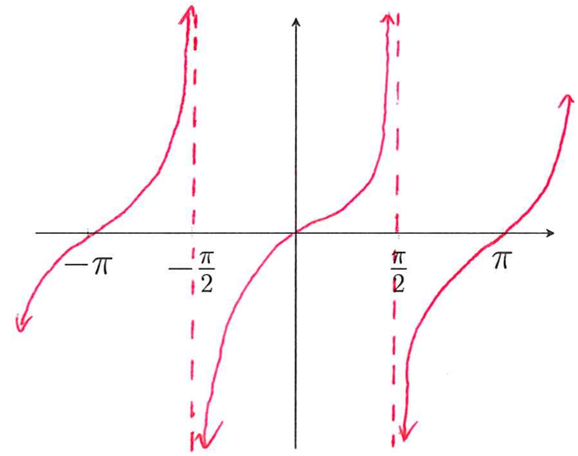
$(\nu - 7)^2 \geq 0$

Example. Evaluate $\lim_{r \to 1} \frac{r}{|r - 1|}$. $= \frac{1}{\text{sm} +} = \infty$

Note

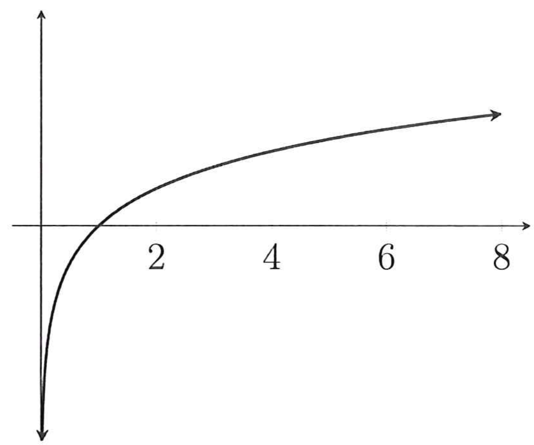
$|r-1| \geq 0$

Example. Evaluate



- $\lim_{x\to\pi/2^-} \tan x = \infty$
- $\lim_{x\to\pi/2^+} \tan x = -\infty$
- $\lim_{x\to-\pi/2^-} \tan x = \infty$
- $\lim_{x\to-\pi/2^+} \tan x = -\infty$

Example. Below is the graph of $\ln(x)$. Use this to evaluate the following limits:



- $\lim_{x\to 0^+} \ln(x) = -\infty$
- $\lim_{x\to\infty} \ln(x) = \infty$

Example. Find all vertical asymptotes, $x = a$, for $f(x) = \dfrac{\cos x}{x^2 + 2x}$.

$x^2 + 2x \neq 0$

$x(x+2) \neq 0$

$\Rightarrow x \neq 0$

$\Rightarrow x+2 \neq 0 \rightarrow x \neq -2$

$$\lim_{x\to 0^-} \frac{\cos(x)}{x(x+2)} = \frac{1}{(\text{sm}-)(2)} = -\infty$$

$$\lim_{x\to 0^+} \frac{\cos(x)}{x(x+2)} = \frac{1}{(\text{sm}+)(2)} = \infty$$

Note:

$1.57 \approx \pi/2 < 2 < \pi \approx 3.14$

$\Rightarrow \cos(2) < 0$

$$\lim_{x\to -2^-} \frac{\cos(x)}{x(x+2)} = \frac{\text{"neg"}}{(-2)(\text{sm}-)} = -\infty$$

$$\lim_{x\to -2^+} \frac{\cos(x)}{x(x+2)} = \frac{\text{"neg"}}{(-2)(\text{sm}+)} = \infty$$

Thus, $x=-2$ and $x=0$ are the equations of vertical asymptotes.