shell method

$$V = \int_c^d 2\pi \underbrace{y}_{\text{radius}} \underbrace{(f-g)}_{\text{height}}\, dy$$

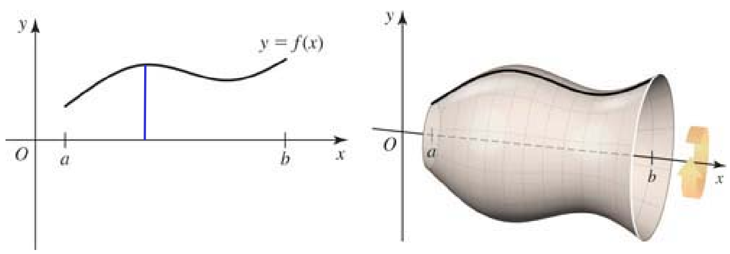
Arc length

$$L = \int_a^b \sqrt{\underbrace{1 + f'(x)^2}_{A^2+B^2=C^2}}\, dx$$

6.6: Surface Area

Definition. (Area of a Surface of Revolution)
Let f be a nonnegative function with a continuous first derivative on the interval $[a, b]$. The area of the surface generated when the graph of f on the interval $[a, b]$ is revolved around the x-axis is

$$S = \int_a^b 2\pi \overbrace{f(x)}^{\text{radius}} \sqrt{\underline{1 + f'(x)^2}}\, dx.$$



Example. Find the exact area of the surface obtained by rotating the curve $y = x^3$, $0 \le x \le 2$ about the x-axis.

$y' = 3x^2$

$$SA = \int_0^2 2\pi\, x^3 \sqrt{1 + (3x^2)^2}\, dx = \int_0^2 2\pi\, x^3 \sqrt{1 + 9x^4}\, dx$$

$u = 1 + 9x^4$

$du = 36x^3\, dx$

$\frac{du}{36} = x^3\, dx$

$$= \frac{2\pi}{36} \int_1^{145} u^{1/2}\, du$$

$$= \frac{\pi}{18} \frac{2}{3} u^{3/2} \Big|_1^{145} = \boxed{\frac{\pi}{27}\left(145^{3/2} - 1\right)}$$

Example. Find the exact area of the surface obtained by rotating the curve $y = \sqrt{8x - x^2}$, $1 \leq x \leq 7$ about the x-axis.

$$y' = \frac{8-2x}{2\sqrt{8x-x^2}} = \frac{4-x}{\sqrt{8x-x^2}}$$

$$SA = \int_1^7 2\pi \sqrt{8x-x^2} \sqrt{1 + \left(\frac{4-x}{\sqrt{8x-x^2}}\right)^2}\, dx$$

$$= 2\pi \int_1^7 \boxed{\sqrt{8x-x^2}} \sqrt{\frac{8x-x^2}{\boxed{8x-x^2}} + \frac{16-8x+x^2}{\boxed{8x-x^2}}}\, dx$$

$$= 2\pi \int_1^7 \sqrt{16}\, dx = 8\pi x \Big|_1^7 = 48\pi$$

Example. Find the exact area of the surface obtained by rotating the curve $y = \frac{1}{2}(e^x + e^{-x})$, $-\ln(2) \le x \le \ln(2)$ about the x-axis. $y' = \frac{1}{2}(e^x - e^{-x})$

$$SA = \int_{-\ln(2)}^{\ln(2)} 2\pi\, \frac{1}{2}(e^x + e^{-x})\sqrt{1 + \left(\frac{1}{2}(e^x - e^{-x})\right)^2}\, dx$$

$$= \pi \int_{-\ln(2)}^{\ln(2)} (e^x + e^{-x})\sqrt{1 + \frac{1}{4}(e^{2x} - 2 + e^{-2x})}\, dx$$

$$= \pi \int_{-\ln(2)}^{\ln(2)} (e^x + e^{-x})\sqrt{1 + \frac{e^{2x}}{4} - \frac{1}{2} + \frac{e^{-2x}}{4}}\, dx$$

$$= \pi \int_{-\ln(2)}^{\ln(2)} (e^x + e^{-x})\sqrt{\frac{e^{2x}}{4} + \frac{1}{2} + \frac{e^{-2x}}{4}}\, dx = \pi \int_{-\ln(2)}^{\ln(2)} (e^x + e^{-x})\sqrt{\frac{1}{4}(e^x + e^{-x})^2}\, dx$$

$$= \pi \int_{-\ln(2)}^{\ln(2)} \frac{(e^x + e^{-x})^2}{2}\, dx = \frac{\pi}{2}\int_{-\ln(2)}^{\ln(2)} e^{2x} + 2 + e^{-2x}\, dx$$

$$= \frac{\pi}{2}\left[\frac{e^{2x}}{2} + 2x - \frac{e^{-2x}}{2}\right]_{-\ln(2)}^{\ln(2)} = \frac{\pi}{2}\left[\left(\frac{4}{2} + 2\ln(2) - \frac{1}{8}\right) - \left(\frac{1}{8} - 2\ln(2) - \frac{4}{2}\right)\right]$$

$$e^{2\ln(2)} = e^{\ln(2^2)} = 4$$

$$e^{-2\ln(2)} = \frac{1}{4}$$

$$= \frac{\pi}{2}\left(4 + 4\ln(2) - \frac{1}{4}\right)$$