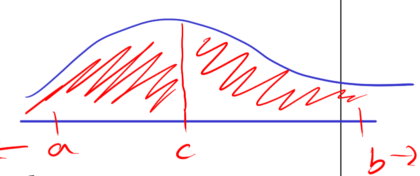
8.9: Improper Integrals

Definition. (Improper Integrals over Infinite Intervals)

1. If f is continuous on $[a, \infty)$, then
$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If f is continuous on $(-\infty, b]$, then
$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

$$\lim_{a\to-\infty} \lim_{b\to\infty} \int_a^b f(x)\,dx$$



3. If f is continuous on $(-\infty, \infty)$, then
$$\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to-\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx.$$
where c is any real number. It can be shown that the choice of c does not affect the value or convergence of the original integral.

If the limits in cases 1.– 3. exist, then the improper integrals **converge**; otherwise they **diverge**.

Example. Evaluate $\displaystyle\int_1^\infty \frac{\ln(x)}{x}\,dx$ and determine if the integral converges or diverges.

$$= \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x}\,dx$$

$u = \ln(x)$

$du = \frac{1}{x}\,dx$

$x = 1, \quad u = 0$

$x = b, \quad u = \ln(b)$

$$= \lim_{b\to\infty} \int_0^{\ln(b)} u\,du = \lim_{b\to\infty} \frac{u^2}{2}\Big|_0^{\ln(b)} = \lim_{b\to\infty} \frac{(\ln(b))^2}{2} = \infty$$

Diverges

Example. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{e^{3x}}{1+e^{6x}}\,dx$. arbitrary (e.g. c=0)

$$\lim_{a\to-\infty}\int_a^c \frac{e^{3x}}{1+e^{6x}}\,dx + \lim_{b\to\infty}\int_c^b \frac{e^{3x}}{1+e^{6x}}\,dx$$

$u=e^{3x}$

$du = 3e^{3x}\,dx$

$\frac{du}{3} = e^{3x}\,dx$

$x=a,\ u=e^{3a}$

$x=c,\ u=e^{3c}$ $\quad c=0 \to u=1$

$x=b,\ u=e^{3b}$

$$= \lim_{a\to-\infty}\frac{1}{3}\int_{e^{3a}}^{1}\frac{du}{1+u^2} + \lim_{b\to\infty}\frac{1}{3}\int_{1}^{e^{3b}}\frac{du}{1+u^2}$$

$$= \lim_{a\to-\infty}\frac{1}{3}\tan^{-1}(u)\Big|_{e^{3a}}^{1} + \lim_{b\to\infty}\frac{1}{3}\tan^{-1}(u)\Big|_{1}^{e^{3b}}$$

$$= \lim_{a\to-\infty}\frac{1}{3}\left(\tan^{-1}(1) - \underbrace{\tan^{-1}(e^{3a})}_{0}\right) + \lim_{b\to\infty}\frac{1}{3}\left(\underbrace{\tan^{-1}(e^{3b})}_{\pi/2} - \tan^{-1}(1)\right)$$

$$= \boxed{\pi/6} \longrightarrow \boxed{\text{converges}}$$

Example. For what values of p does $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converge?

$p \neq 1$

$$\lim_{b\to\infty}\int_1^b x^{-p}\,dx = \lim_{b\to\infty}\left.\frac{x^{1-p}}{1-p}\right|_1^b = \lim_{b\to\infty}\frac{b^{1-p}-1}{1-p}$$

$p<1,\ 1-p>0$

$$\lim_{b\to\infty}\frac{1}{1-p}\left(b^{1-p}-1\right)=\infty$$

$p>1,\ 1-p<0$

$$\lim_{b\to\infty}\frac{1}{1-p}\left(\underset{0}{b^{1-p}}-1\right) = \frac{-1}{1-p} = \frac{1}{p-1}$$

$p=1$

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\left.\ln|x|\right|_1^b$$

$$= \lim_{b\to\infty}\ln(b) = \infty$$

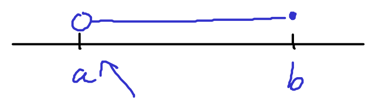
$\boxed{\text{Converges when } p>1}$

Definition. (Improper Integrals with Unbounded Integrand)

1. Suppose f is continuous on $(a, b]$ with $\lim\limits_{x\to a^+} f(x) = \pm\infty$. Then

f(x) DNE

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

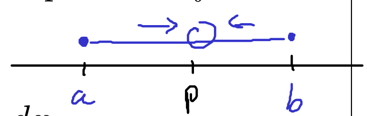


$\lim\limits_{x\to a^+}$

2. Suppose f is continuous on $[a, b)$ with $\lim\limits_{x\to b^-} f(x) = \pm\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$

3. Suppose f is continuous on $[a, b]$ except at the interior point p where f is unbounded. Then



$$\int_a^b f(x)\,dx = \lim_{c\to p^-} \int_a^c f(x)\,dx + \lim_{d\to p^+} \int_d^b f(x)\,dx.$$

If the limits in cases 1.– 3. exist, then the improper integrals **converge**; otherwise, they **diverge**.

Example. Determine which of the following integrals are improper integrals

$$\int_0^1 \sec(x)\,dx$$

Not Improper

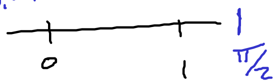
Domain $\sec(x)$

$\sec(x) = \frac{1}{\cos(x)}$

$\sec(x)$ DNE at $x = \frac{\pi}{2} + n\pi$

$\pi \approx 3.14$

$\pi/2 \approx 1.57\ldots$



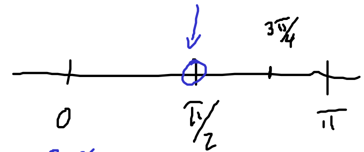
$$\int_{\pi/2}^{3\pi/4} \tan(x)\,dx$$

$\tan(x) = \frac{\sin(x)}{\cos(x)}$

Domain $x \neq \frac{\pi}{2} + n\pi$

Improper Integral

$\tan(x)$ DNE



$$\lim_{a \to \pi/2^+} \int_a^{3\pi/4} \tan(x)\,dx$$

$$\int_1^e \ln(x)\,dx$$

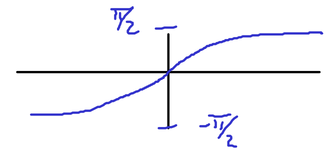
$x \in (0, \infty)$

Not improper

$$\int_0^1 \arctan(x)\,dx$$

Domain $(-\infty, \infty)$

Not improper



$$\int_0^{0.5} \ln(x)\,dx$$

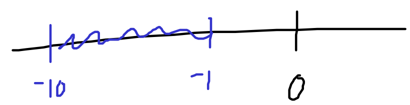
$x \in (0, \infty)$

Improper

$$\lim_{a \to 0^+} \int_a^{0.5} \ln(x)\,dx$$

$$\int_{-10}^{-1} \frac{1}{x^{1/3}}\,dx$$

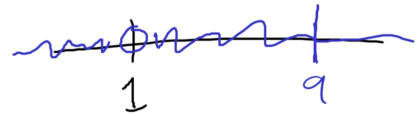
Domain: $x \neq 0$



Not Improper

Example. Evaluate $\displaystyle\int_1^9 \frac{dx}{(x-1)^{2/3}}$. Does this integral converge or diverge?

$x \neq 1$



$$= \lim_{a \to 1^+} \int_a^9 \frac{dx}{(x-1)^{2/3}}$$

$u = x - 1$ $\qquad$ $x = a,\ u = a - 1$

$du = dx$ $\qquad$ $x = 9,\ u = 8$

$$= \lim_{a \to 1^+} \int_{a-1}^{8} u^{-2/3}\, du$$

$$= \lim_{a \to 1^+} 3u^{1/3} \Big|_{a-1}^{8} = \lim_{a \to 1^+} 3\left(8^{1/3} - \underbrace{(a-1)^{1/3}}_{0}\right) = 3(2 - 0) = \boxed{6}$$

$\boxed{\text{Converges}}$

Example. Evaluate $\displaystyle\int_{-1}^{1} \frac{e^{2/x}}{x^2}\,dx$. Does this integral converge or diverge?

$$x \neq 0 \rightarrow \lim_{c \to 0^-} \int_{-1}^{c} \frac{e^{2/x}}{x^2}\,dx + \lim_{d \to 0^+} \int_{d}^{1} \frac{e^{2/x}}{x^2}\,dx$$

$u = 2/x$

$du = -2/x^2\,dx$

$x=-1,\quad u=-2$

$x=c,\quad u=2/c$

$x=d,\quad u=2/d$

$x=1,\quad u=2$

$$= -\lim_{c \to 0^-} \frac{1}{2} \int_{-2}^{2/c} e^u\,du - \lim_{d \to 0^+} \frac{1}{2} \int_{2/d}^{2} e^u\,du$$

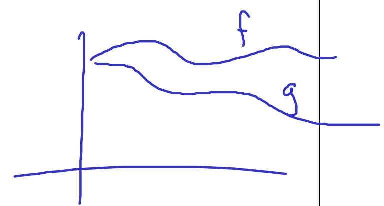
$$= -\lim_{c \to 0^-} \frac{1}{2} e^u \Big|_{-2}^{2/c} - \lim_{d \to 0^+} \frac{1}{2} e^u \Big|_{2/d}^{2}$$

$$= -\frac{1}{2}\left[\lim_{c \to 0^-} \underbrace{e^{2/c}}_{0} - e^{-2}\right] - \frac{1}{2}\left[\lim_{d \to 0^+} e^{2} - \underbrace{e^{2/d}}_{\infty}\right]$$

$= \infty \longrightarrow$ diverges

Theorem 8.2: Comparison Test for Improper Integrals
Suppose the functions f and g are continuous on the interval $[a, \infty)$, with $f(x) \geq g(x) \geq 0$, for $x \geq a$.

1. If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.
2. If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.



Example. Determine if the integral $\int_2^\infty \frac{x^3}{x^4 - x^3 - 1}\,dx$ converges or diverges.

$$f(x) = \frac{x^3}{x^4 - x^3 - 1} \geq 0 \qquad g(x) = \frac{x^3}{x^4} = \frac{1}{x} \geq 0$$

which is larger

smaller denominator $\Rightarrow$ bigger fraction

$\frac{1}{3}$ vs. $\frac{1}{2}$

$$\frac{x^3}{x^4 - x^3 - 1} > \frac{1}{x}$$

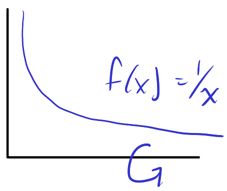
$$x^4 > x^4 - x^3 - 1$$

$$\int_2^\infty g(x)\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln|x| \Big|_2^b = \infty$$

$$\int_2^\infty g(x)\,dx \text{ diverges} \Rightarrow \int_2^\infty f(x)\,dx \text{ diverges}$$

Example (Gabriel's Horn). Let R be the region bounded by the graph of $y = 1/x$ and the x-axis for $x \geq 1$.

What is the volume of the solid generated when R is revolved around the x-axis?



$$V = \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx = \lim_{b \to \infty} \pi \int_1^b x^{-2}\, dx$$

$$= \lim_{b \to \infty} \left. \frac{-\pi}{x} \right|_1^b$$

$$= \lim_{b \to \infty} \underbrace{\frac{-\pi}{b}}_{0} - \frac{-\pi}{1} = \boxed{\pi}$$

What is the surface area of the solid generated when R is revolved about the x-axis?

$f(x) = \frac{1}{x}$

$f'(x) = \frac{-1}{x^2}$

$$SA = \int_1^\infty 2\pi \frac{1}{x} \sqrt{1 + \left(\frac{-1}{x^2}\right)^2}\, dx = \lim_{b \to \infty} 2\pi \int_1^b \frac{1}{x} \sqrt{\frac{x^2+1}{(x^2)^2}}\, dx$$

$$= \lim_{b \to \infty} 2\pi \int_1^b \frac{\sqrt{x^4+1}}{x^3}\, dx$$

Note: $\sqrt{x^4+1} > \sqrt{x^4} = x^2$

$$\geq \lim_{b \to \infty} 2\pi \int_1^b \frac{x^2}{x^3}\, dx = \lim_{b \to \infty} 2\pi \int_1^b \frac{1}{x}\, dx$$

$$= \lim_{b \to \infty} 2\pi \left. \ln(x) \right|_1^b = \infty$$

Since the smaller function diverges then the surface area of this "horn" also diverges