1. Follow the steps below to show $9.\bar{9} = 10$.   $= 0.\bar{9} = 1$

   (a) (__/1 pts.) Write $9.\bar{9} = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \dots$ as a geometric series.

$$9 + \frac{9}{10} + \frac{9}{10^2} + \dots$$

$a = 9$
$\frac{9}{10} \quad \frac{9}{10}$
$r = \frac{9}{10}$

$$9.\bar{9} = \sum_{k=0}^{\infty} 9\left(\frac{1}{10}\right)^k$$

Geometric series
Converge
$|r| < 1$
$S_n = a + ar + ar^2 + ar^3 + \dots + ar^n$

$$9.\bar{9} = \sum_{k=0}^{\infty} 9\left(\frac{1}{10}\right)^k = \lim_{n \to \infty} \sum_{k=0}^{n} 9\left(\frac{1}{10}\right)^k = \lim_{n \to \infty} S_n$$

2. For the following infinite series,

   - Find a formula for the partial sum $S_n$
   - Evaluate the infinite series

   (a) (__/3 pts.) $\frac{1}{16} + \frac{3}{64} + \frac{9}{256} + \dots = \sum_{k=0}^{\infty} \frac{1}{16}\left(\frac{3}{4}\right)^k$

$$\sum_{k=0}^{n} ar^n = S_n = a\frac{1 - r^n}{1 - r}$$

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}, \quad |r| < 1$$

Geometric

$a = \frac{1}{16} \quad r = \frac{3}{4}$

(b) (__/3 pts.) $\sum_{k=2}^{\infty} \frac{6}{k^2 + 2k}$ (Hint: Use PFD)

Telescoping

$$S_n = \sum_{k=2}^{n} \frac{3}{k} - \frac{3}{k+2} = \left[\frac{3}{2} - \frac{3}{4}\right] + \left[\frac{3}{3} - \frac{3}{5}\right] + \left[\frac{3}{4} - \frac{3}{6}\right] + \left[\frac{3}{5} - \frac{3}{7}\right] + \dots$$

$k=2$  $k=3$  $k=4$  $k=5$

$$+ \left[\frac{3}{n-1} - \frac{3}{n+1}\right] + \left[\frac{3}{n} - \frac{3}{n+2}\right] = \frac{5}{2} - \frac{3}{n+1} - \frac{3}{n+2}$$

(c) (__/3 pts.) $\sum_{k=1}^{\infty} \left(\frac{-4}{3}\right)^{-k} = \sum_{k=1}^{\infty} \left(-\frac{3}{4}\right)^k$

$a = -\frac{3}{4}$
$r = -\frac{3}{4}$

Geometric

$$\sum_{k=1}^{n} \left(\frac{4}{\sqrt{k+5}} - \frac{4}{\sqrt{k+7}}\right) = \left(\frac{4}{\sqrt{6}} - \frac{4}{\sqrt{8}}\right) + \left(\frac{4}{\sqrt{7}} - \frac{4}{\sqrt{9}}\right) + \left(\frac{4}{\sqrt{8}} - \frac{4}{\sqrt{10}}\right) + \dots$$

$k=1$  $k=2$  $k=3$

$$\sum_{k=1}^{\infty}\left(4 - \frac{4}{3}\right)\frac{1}{3^k}$$

$$\dots + \left(\frac{4}{\sqrt{n+4}} - \frac{4}{\sqrt{n+6}}\right) + \left(\frac{4}{\sqrt{n+5}} - \frac{4}{\sqrt{n+7}}\right)$$

$$= \frac{4}{\sqrt{6}} + \frac{4}{\sqrt{7}} - \frac{4}{\sqrt{n+6}} - \frac{4}{\sqrt{n+7}}$$

$$\sum_{k=1}^{\infty}\left(\frac{4}{3^k} - \frac{4}{3^{k+1}}\right)$$

Geometric
Telescoping

$$\sum_{k=1}^{\infty} \frac{4}{3^k} - \sum_{k=1}^{\infty} \frac{4}{3^{k+1}}$$

$$s \sum_{k=1}^{\infty} \frac{4}{3^k} = \frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \cdots = \frac{4/3}{1 - 1/3} = \frac{4}{3} \cdot \frac{1}{2/3} = 2$$

$a = 4/3$

$r = 1/3$

$$\sum_{k=1}^{\infty} \frac{4}{3^k} = \sum_{k=0}^{\infty} \frac{4}{3^{k+1}} = \sum_{k=0}^{\infty} \frac{4}{3} \left(\frac{1}{3}\right)^k$$

$a$  $r$

## 10.4: The Divergence and Integral Tests

Harmonic Series

$\sum_{k=1}^{\infty} \frac{1}{k}$  Diverges

$4 \sum_{k=1}^{\infty} \frac{1}{3^k}$

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**Theorem 10.9: Divergence Test**

If $\sum a_k$ converges, then $\lim_{k\to\infty} a_k = 0$. Equivalently, if $\lim_{k\to\infty} a_k \neq 0$, then the series diverges.

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**Example.** If $\lim_{k\to\infty} a_k = 1$, what can we conclude about $\sum_{k=1}^{\infty} a_k$?

$$\sum_{k=1}^{\infty} a_k \quad \text{Diverges}$$

**Example.** If $\sum_{k=1}^{\infty} a_k = 42$, what can we conclude about $\lim_{k\to\infty} a_k$?

$$\lim_{k\to\infty} a_k = 0$$

**Example.** If $\lim_{k\to\infty} a_k = 0$, what can we conclude about $\sum_{k=1}^{\infty} a_k$?

Nothing

**Example.** Determine which of the following series diverge by the divergence test.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}}$$

$\lim\limits_{k \to \infty} \frac{1}{\sqrt{k+1}} = 0$    Don't know

*Diverges*

$$\sum_{k=1}^{\infty} \frac{k^3 + 100}{3k^3 + k + 1}$$

$\lim\limits_{k \to \infty} \frac{k^3 + 100}{3k^3 + k + 1} \left( \frac{1/3k^3}{1/3k^3} \right) = \frac{1}{3} \neq 0$

$\implies$ Series diverges

$$\sum_{k=1}^{\infty} \frac{e^k}{k^2}$$

$\lim\limits_{k \to \infty} \frac{e^k}{k^2} = \infty$

grows faster than $k^2$

$\ln(x) \ll x^n \ll b^m \ll n! \ll n^n$

diverges $\implies$ series diverges

$\lim_{n \to \infty} a_n = 4$

$\lim_{n \to \infty} a_n = 0$

$\lim_{n \to \infty} a_n = \infty$

**Table 1  Series Convergence**

| Scenario | Sequence of Terms $\{a_1, a_2, a_3, \dots\}$ | Sequence of Partial Sums $\{s_1, s_2, s_3, \dots\}$ | Series $\sum_{n=1}^{\infty} a_n$ | Possible or Impossible? |
|---|---|---|---|---|
| A | * Converges | Diverges | Diverges | Possible |
| B | Converges | Diverges | Converges | Impossible |
| C | Converges | Converges | Diverges | Impossible |
| * D | Converges | Converges | Converges | Possible |
| E | Diverges | Converges | Diverges | Impossible |
| F | Diverges | Converges | Converges | Impossible |
| G | Diverges | Diverges | Diverges | Possible |
| H | Diverges | Diverges | Converges | Impossible |

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**Theorem 10.10: Harmonic Series**

The harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$ diverges—even though the terms of the series approach zero.

$\lim_{k \to \infty} \frac{1}{k} = 0$    But  $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

**Theorem 10.11: Integral Test**

Suppose $f$ is a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Then
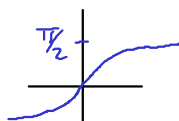
$$\sum_{k=1}^{\infty} a_k \text{ and } \int_{1}^{\infty} f(x)\, dx$$

either both converge or both diverge. In the case of convergence, the value of the integral is *not* equal to the value of the series.
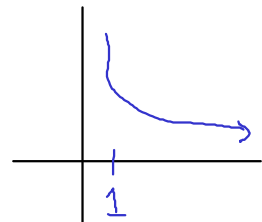
**Example.** Which of the following series below satisfy all the conditions to use the Integral Test?

$$\sum_{k=1}^{\infty} \arctan(k)$$

Cont. on $[1, \infty)$
Pos. on $(0, \infty)$
dec ?

Continuous
Positive
decreasing



$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$
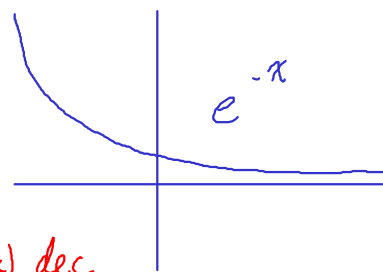
cont?

$\dfrac{(-1)^{3/2}}{(3/2)^2}$  DNÉ

$$\sum_{k=1}^{\infty} \frac{1}{e^k}$$

$e^{-k}$

cont on $[1, \infty)$
pos on $[1, \infty)$
dec $\longrightarrow$

$f(x) = e^{-x} > 0$

$f'(x) = -e^{-x} < 0$

$\longrightarrow f(x)$ dec

**Example.** Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

Use the <u>integral test</u> to show that the Harmonic Series diverges. For what values of $p$ does this series converge?

cont, pos, dec $[1, \infty)$

$$f(x) = \frac{1}{x^p} \qquad x \neq 0 \qquad 1 \leq x < \infty \checkmark$$

$$\frac{1}{x^p} > 0 \; , \; x > 0 \qquad 1 \leq x < \infty \checkmark$$

$$f'(x) = -p \, x^{-p-1} = \frac{-p}{x^{p+1}} < 0 \quad \begin{array}{l} \text{when } x > 0 \\ \text{and } p > 0 \end{array} \qquad \frac{1}{k}, \frac{1}{k^2}, \frac{1}{k^3}...$$

$$p = 1 \qquad \int_1^{\infty} \frac{1}{x} \, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x} \, dx = \lim_{b \to \infty} \ln|x| \Big|_1^b = \lim_{b \to \infty} \ln(b) - \underbrace{\ln(1)}_{0} = \infty$$

Diverges

By the integral test $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ also diverges

$$p > 1 \qquad \int_1^{\infty} \frac{1}{x^p} \, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^p} \, dx = \lim_{b \to \infty} \frac{x^{-p+1}}{-p+1} \Big|_1^b = \lim_{b \to \infty} \underbrace{\frac{1}{(1-p) b^{p-1}}}_{0} - \underbrace{\frac{1}{(1-p) 1^{p-1}}}_{\frac{1}{1-p}}$$

$\hookrightarrow p - 1 > 0$

$\longrightarrow$ By the integral test, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ $= -\frac{1}{1-p} = \frac{1}{p-1}$

Converges when $p > 1$

$$\int \frac{1}{x^p} \, dx = \int x^{-p} dx = \frac{x^{-p+1}}{-p+1} + c$$

$p > 1 \qquad \lim_{b \to \infty} \frac{1}{b^{p-1}} = 0$

$p < 1 \qquad \lim_{b \to \infty} \frac{1}{b^{p-1}} = \infty$

**Theorem 10.12: Convergence of the $p$-series**

The $p$-**series** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

Geometric $\sum a \, r^k$

P-series $\sum \frac{1}{k^p}$

**Example.** Determine if the following $p$-series converge or diverge.

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$   $p = 2$

$\Rightarrow$ Converges

$$\sum_{k=1}^{\infty} k^{-1/3} = \sum_{k=1}^{\infty} \frac{1}{k^{1/3}}$$   $p = \frac{1}{3} < 1$

diverges

$$\sum_{k=1}^{\infty} \frac{k^2}{k^\pi} = \sum_{k=1}^{\infty} \frac{1}{k^{\pi - 2}}$$   $p = \pi - 2 > 1$

Converges

$$\sum_{k=1}^{\infty} \frac{2}{k} = 2\sum_{k=1}^{\infty} \frac{1}{k^1}$$   $p = 1$

diverges

$$\sum_{k=1}^{\infty} \frac{-3}{\sqrt[3]{k^4}} = -3\sum_{k=1}^{\infty} \frac{1}{k^{4/3}}$$   $p = \frac{4}{3} > 1$

Converges

$$\sum_{k=1}^{\infty} \frac{k^3 + 1}{k^5} = \sum_{k=1}^{\infty} \frac{k^3}{k^5} + \sum_{k=1}^{\infty} \frac{1}{k^5}$$

$\uparrow$ $p = 2$      $\uparrow$ $p = 5$

Converges

**Example.** Apply the Integral Test to determine if the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}}$ converges or diverges.

$$f(x) = \frac{1}{\sqrt{x+1}} \qquad x \neq -1 \qquad 1 \leq x < \infty$$

Cont
pos
dec

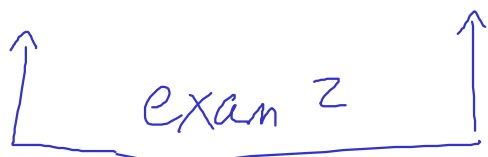$$(x+1)^{-\frac{1}{2}} = \frac{1}{\sqrt{x+1}} > 0 \quad , \quad x > -1 \qquad 1 \leq x < \infty$$

$$f'(x) = -\frac{1}{2}(x+1)^{-\frac{3}{2}} = \frac{-1}{2\sqrt{(x+1)^3}} < 0 \qquad \rightarrow f(x) \text{ dec}$$
$$x > -1$$

$$\lim_{b \to \infty} \int_{1}^{b} \frac{1}{\sqrt{x+1}} \, dx = \lim_{b \to \infty} \int_{2}^{b+1} u^{-\frac{1}{2}} \, du = \lim_{b \to \infty} 2u^{\frac{1}{2}} \Big|_{2}^{b+1}$$

$$u = x+1$$
$$du = dx$$
$$x=1, \ u=2$$
$$x=b, \ u=b+1$$

$$= \lim_{b \to \infty} 2\left(\underbrace{\sqrt{b+1}}_{\infty} - \sqrt{2}\right)$$

$$= \infty$$

By the integral test, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}}$ diverges

exam 2

**Theorem 10.13: Estimating Series with Positive Terms**

Let $f$ be a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$ Let $S = \sum_{k=1}^{\infty} a_k$ be a convergent series and let $S_n = \sum_{k=1}^{n} a_k$ be the sum of the first $n$ terms of the series. The remainder $R_n = S - S_n$ satisfies

$$\boxed{R_n < \int_n^{\infty} f(x)\, dx.}$$

Furthermore, the exact value of the series is bounded as follows:

$$L_n = S_n + \int_{n+1}^{\infty} f(x)\, dx < \sum_{k=1}^{\infty} a_k < S_n + \int_n^{\infty} f(x)\, dx = U_n.$$

**Example.** How many terms of the convergent $p$-series $\sum_{k=1}^{\infty} \dfrac{1}{k^2}$ must be summed to obtain an approximation that is within $10^{-3}$ of the exact value of the series?

$$R_n < \lim_{b \to \infty} \int_n^{b} \frac{1}{x^2}\, dx$$

$$= \lim_{b \to \infty} \left. -x^{-1} \right|_n^{b}$$

$$= \lim_{b \to \infty} -\frac{1}{b} - \frac{-1}{n} = \frac{1}{n} < 10^{-3}$$

$$10^3 < n$$

$$1000 \leq n-1 \longrightarrow 1001 \leq n$$