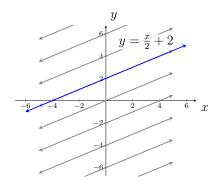
13.5: Lines and Planes in Space

Equation of a Line:

Recall the equation of a line in \mathbb{R}^2 :

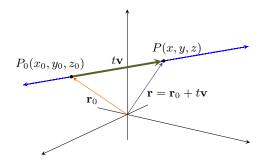
$$y = mx + b$$



where b is the intercept and m is the slope. This idea can be extended into higher dimensions:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

Here, \mathbf{r}_0 is a fixed point, and \mathbf{v} is the position vector that is parallel to the line \mathbf{r} .



Equation of a Line

A vector equation of the line passing through the point $P_0(x_0, y_0, z_0)$ in the direction of the vector $\mathbf{v} = \langle a, b, c \rangle$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, or

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle, \quad \text{for} \quad -\infty < t < \infty$$

Equivalently, the corresponding parametric equations of the line are

$$x = x_0 + at$$
, $y = y_0 + bt$, $z = z_0 + ct$, for $-\infty < t < \infty$

Example. Find the vector equation and parametric equation of the line that

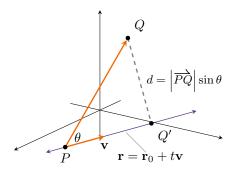
• goes through the points P(-1, -2, 1) and Q(-4, -5, -3) where t = 0 corresponds to P,

• goes through the point P(1, -3, -3) and is parallel to the vector $\mathbf{r} = \langle -4, 1, -1 \rangle$,

• goes through the point P(-2, 5, -2) and is perpendicular to the lines x = 3 - 4t, y = 2 - 3t, z = -1 - t, and x = -2 + 0t, y = 2 - t, z = 3t, where t = 0 corresponds to P.

Distance from a Point to a Line:

Given a point Q and a line ℓ , the shortest distance to the line is the length of $\overrightarrow{QQ'}$.



From the definition of the cross product, we have

$$\left|\mathbf{v} \times \overrightarrow{PQ}\right| = \left|\mathbf{v}\right| \underbrace{\left|\overrightarrow{PQ}\right| \sin \theta}_{d} = \left|\mathbf{v}\right| d$$

From here, solving for d gives us the following:

Distance Between a Point and a Line

The distance d between the point Q and the $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ is

$$d = \frac{\left| \mathbf{v} \times \overline{PQ} \right|}{\left| \mathbf{v} \right|},$$

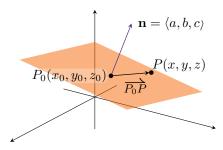
where P is any point on the line and \mathbf{v} is a vector parallel to the line.

Example. Find the distance from the point Q(-4, -1, -3) and the line x = -5 - 5t, y = -5 + t, z = -1 + 4t. (*Hint*: Let P be the point at t = 0)

Equations of Planes:

In \mathbb{R}^2 , two distinct points determine a line.

In \mathbb{R}^3 , three noncollinear points determine a unique plane. Alternatively, a plane is uniquely determined by a point and a vector that is orthogonal to the plane.



Definition. (Plane in \mathbb{R}^3)

Given a fixed point P_0 and a nonzero **normal vector n**, the set of points P in \mathbb{R}^3 for which $\overline{P_0P}$ is orthogonal to **n** is called a **plane**.

Consider the normal vector $\mathbf{n} = \langle a, b, c \rangle$ at the point $P_0(x_0, y_0, z_0)$, and any point P(x, y, z) on the plane. Since \mathbf{n} is orthogonal to the plane, it is also orthogonal to the vector $\overline{P_0P}$, which is also in the plane. Thus,

$$\mathbf{n} \cdot \overrightarrow{P_0 P} = 0$$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = d$$

General Equation of a Plane in \mathbb{R}^3

The plane passing through the point $P_0(x_0, y_0, z_0)$ with a nonzero normal vector $\mathbf{n} = \langle a, b, c \rangle$ is described by the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$
 or $ax + by + cz = d$,

where $d = ax_0 + by_0 + cz_0$.

Example. Find the equation of the plane that

• goes through the point P(-2, 5, 0) and is parallel to the plane x - 5y - 5z = 1,

• goes through the points P(5,-2,1), Q(5,1,3) and R(1,-5,-2)

• that is parallel to the vectors $\langle 4, -2, -3 \rangle$ and $\langle 3, 2, 3 \rangle$, passing through the point P(-2, -2, 5).

Example. Find the location where the line $\langle -3, 1, 4 \rangle + t \langle -1, -4, 2 \rangle$ and the plane 2x - 2y - 4z = 5 intersect.

Definition. (Parallel and Orthogonal Planes)

Two distinct planes are **parallel** if their respective normal vectors are parallel (that is, the normal vectors are scaling multiples of each other). Two plans are **orthogonal** if their respective normal vectors are orthogonal (that is, the dot product of the normal vectors is *zero*).

Example. Find the line of intersection between the planes 3x - y + 4z = -4 and x + 3y - 2z = 0.

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	Find the smallest	t angle betweer	the planes 3	3x - y + 4z = -4	4 and x + 3y - 2z =

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