16.5: Triple Integrals in Cylindrical and Spherical Coordinates

Cylindrical coordinates:

The concept of polar coordinates in $\mathbb{R}^2$ from section 16.3 can be extended to $\mathbb{R}^3$. This coordinate system is called *cylindrical coordinates* where every point P in $\mathbb{R}^3$ has coordinates (r, θ, z), where $0 \le r < \infty$, $0 \le \theta \le 2\pi$, and $-\infty < z < \infty$.

Transformations between Cylindrical and Rectangular Coordinates

Rectangular → Cylindrical	**Cylindrical → Rectangular**
$r^2 = x^2 + y^2$	$x = r\cos\theta$
$\tan\theta = y/x$	$y = r\sin\theta$
$z = z$	$z = z$

Example. Sketch the following sets represented in cylindrical coordinates:

$$\{(r, \theta, z) : r = a\}, a > 0 \qquad \{(r, \theta, z) : 0 < a \le r \le b\}$$

$\{(r,\theta,z) : z = a\}$

$\{(r,\theta,z) : z = ar\}, a \neq 0$

$\{(r,\theta,z) : \theta = \theta_0\}$

Theorem 16.6: Change of Variables for Triple Integrals in Cylindrical Coordinates

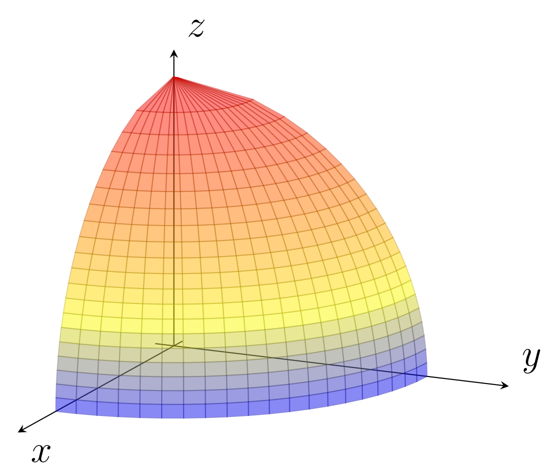
Let f be continuous over the region D, expressed in cylindrical coordinates as

$$D = \{(r,\theta,z) : 0 \leq g(\theta) \leq r \leq h(\theta),\ \alpha \leq \theta \leq \beta,\ G(x,y) \leq z \leq H(x,y)\}$$

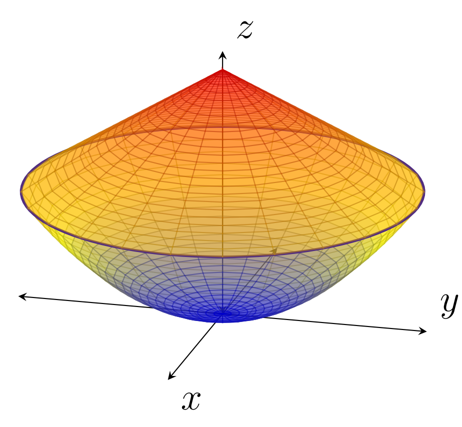
Then f is integrable over D, and the triple integral of f over D is

$$\iiint\limits_D f(x,y,z)\,dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta,\ r\sin\theta)}^{H(r\cos\theta,\ r\sin\theta)} f(r\cos\theta,\ r\sin\theta)\,dz\,r\,dr\,d\theta.$$

Example. Evaluate the following integral using cylindrical coordinates:
$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{x^2+y^2}} \left(x^2+y^2\right)^{-1/2} \, dz \, dy \, dx$$



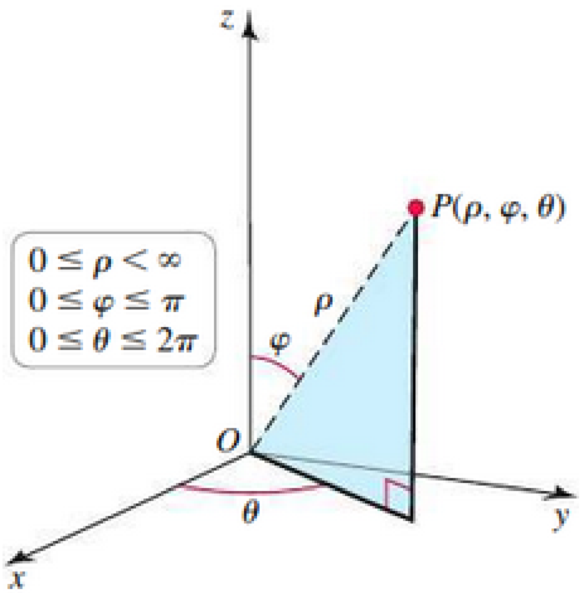
Example. Find the volume of the solid bounded below by the paraboloid $z = x^2 + y^2$ and bounded above by the cone $z = 2 - \sqrt{x^2 + y^2}$.

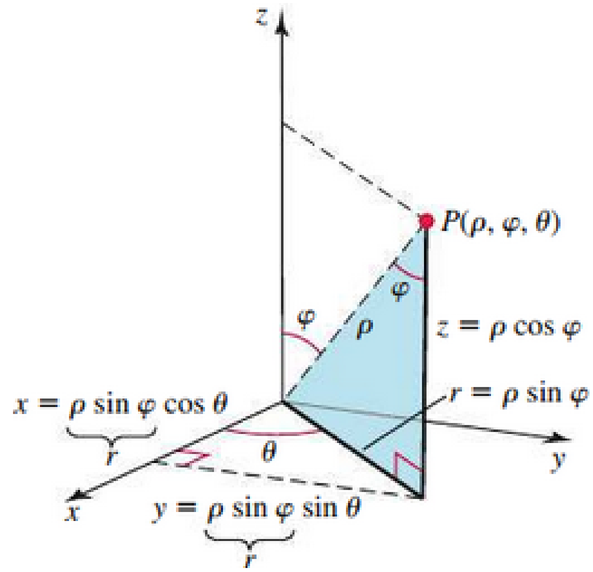


Spherical Coordinates:

Spherical coordinates can represent a point P in $\mathbb{R}^3$ as (ρ, φ, θ) where

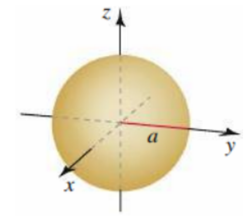
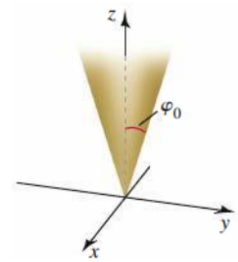
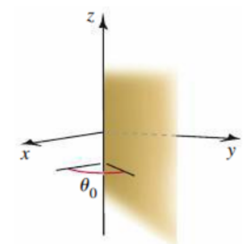
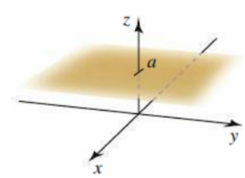
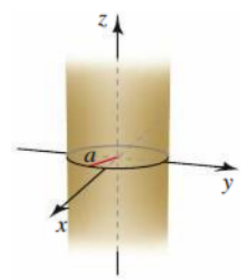
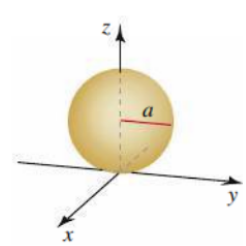
- ρ is the distance from the origin to P,
- φ is the angle between the positive z-axis and the line OP, and
- θ is the same angle as in cylindrical coordinates.





Transformations between Spherical and Rectangular Coordinates

Rectangular → Spherical	**Spherical → Rectangular**
$\rho^2 = x^2 + y^2 + z^2$	$x = \rho \sin(\varphi)\cos(\theta)$
Use trigonometry to find	$y = \rho \sin(\varphi)\sin(\theta)$
φ and θ.	$z = \rho \cos(\varphi)$

Name	Description	Example
Sphere, radius a, center $(0,0,0)$	$\{(\rho,\varphi,\theta) : \rho = a\}, a > 0$	
Cone	$\{(\rho,\varphi,\theta) : \varphi = \varphi_0\}, \varphi_0 \neq 0, \pi/2, \pi$	
Vertical half-plane	$\{(\rho,\varphi,\theta) : \theta = \theta_0\}$	
Horizontal plane, $z = a$	$a > 0 : \{(\rho,\varphi,\theta) : \rho = a\sec(\varphi),\ 0 \leq \varphi < \pi/2\}$ $a < 0 : \{(\rho,\varphi,\theta) : \rho = a\sec(\varphi),\ \pi/2 < \varphi \leq \pi\}$	
Cylinder, radius $a > 0$	$\{(\rho,\varphi,\theta) : \rho = \alpha\csc(\varphi),\ 0 < \varphi < \pi\}$	
Sphere, radius $a > 0$ center $(0,0,a)$	$\{(\rho,\varphi,\theta) : \rho = 2a\cos(\varphi),\ 0 \leq \varphi \leq \pi/2\}$	

Theorem 16.7: Change of Variables for Triple Integrals in Spherical Coordinates
Let f be continuous over the region D, expressed in spherical coordinates as

$$D = \{(\rho, \varphi, \theta) : 0 \leq g(\varphi, \theta) \leq \rho \leq h(\varphi, \theta),\ a \leq \varphi \leq b,\ \alpha \leq \theta \leq \beta\}.$$

Then f is integrable over D, and the triple integral of f over D is

$$\iiint\limits_D f(x, y, z)\, dV$$
$$= \int_\alpha^\beta \int_a^b \int_{g(\varphi,\theta)}^{h(\varphi,\theta)} f(\rho \sin(\varphi) \cos(\theta),\ \rho \sin(\varphi) \sin(\theta),\ \rho \cos(\varphi))\, \rho^2 \sin(\varphi)\ d\rho\, d\varphi\, d\theta.$$

Example. Evaluate $\iiint\limits_D \left(x^2 + y^2 + z^2\right)^{-3/2} dV$, where D is the region in the first octant between two spheres of radius 1 and 2 centered at the origin

Example. Find the volume of the solid region D that lies inside the cone $\varphi = \pi/6$ and inside the sphere $\rho = 4$.