11.1 Working with Difference Quotients

Definition. Given a function $f(x)$, the **difference quotient** is

$$\frac{f(x+h)-f(x)}{h}$$

Example. Find and simplify the difference quotient for the following:

$f(x) = 2x^2 - 8x$

$$\frac{f(x+h)-f(x)}{h} = \frac{[2(x+h)^2-8(x+h)]-[2x^2-8x]}{h}$$

$$= \frac{2(x^2+2xh+h^2)-8x-8h-2x^2+8x}{h}$$

$$= \frac{4xh+2h^2-8h}{h}$$

$$= 4x+2h-8$$

$f(x) = \sqrt{x+1}$

$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h+1}-\sqrt{x+1}}{h}\left(\frac{\sqrt{x+h+1}+\sqrt{x+1}}{\sqrt{x+h+1}+\sqrt{x+1}}\right)$$

$$= \frac{(x+h+1)-(x+1)}{h(\sqrt{x+h+1}+\sqrt{x+1})}$$

$$= \frac{h}{h(\sqrt{x+h+1}+\sqrt{x+1})}$$

$$= \frac{1}{\sqrt{x+h+1}+\sqrt{x+1}}$$

$f(x) = x^2 + \dfrac{1}{x}$

$$\frac{f(x+h)-f(x)}{h} = \frac{\left((x+h)^2+\frac{1}{x+h}\right)-\left(x^2+\frac{1}{x}\right)}{h}$$

$$= \frac{2xh+h^2+\frac{(x)-(x+h)}{x(x+h)}}{h}$$

$$= \frac{2xh+h^2-\frac{h}{x(x+h)}}{h}$$

$$= 2x+h-\frac{1}{x(x+h)}$$

$f(x) = \dfrac{x-1}{x+1}$

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{x+h-1}{x+h+1}-\frac{x-1}{x+1}}{h}$$

$$= \frac{(x+1)(x+h-1)-(x-1)(x+h+1)}{(x+1)(x+h+1)}\cdot\frac{1}{h}$$

$$= \frac{x^2+xh-x+x+h-1-x^2-xh-x+x+h+1}{h(x+1)(x+h+1)}$$

$$= \frac{2h}{h(x+1)(x+h+1)}$$

$$= \frac{2}{(x+1)(x+h+1)}$$

3.1 Introducing the Derivative:

Recall that when given a distance function $s(t)$, the average velocity over the interval $[a, t]$ is

$$v_{\text{avg}} = \frac{s(t) - s(a)}{t - a}$$

and the instantaneous velocity is the limit of the average velocities as $t \to a$:

$$v_{\text{inst}} = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$$

Furthermore, the average velocity is the slope of the secant line through the points $(a, s(a))$ and $(t, s(t))$ and the instantaneous velocity is the slope of the tangent line at the point $(a, s(a))$. `https://www.desmos.com/calculator/08syaijrdo`

Definition (Rate of Change and the Slope of the Tangent Line).
The **average rate of change** in f on the interval $[a, x]$ is the slope of the corresponding secant line:

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}$$

The **instantaneous rate of change** in f at a is

$$m_{\text{tan}} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

which is also the **slope of the tangent line** at $(a, f(a))$, provided this limit exists. This **tangent line** is the unique line through $(a, f(a))$ with slope m_{tan}. Its equation is

$$y - f(a) = m_{\text{tan}}(x - a)$$

Example. Find an equation of the line tangent to the graph of $f(x) = \dfrac{3}{x}$ at $\left(2, \dfrac{3}{2}\right)$.

eqn: $y - f(a) = m_{tan}(x-a)$ $\quad (a, f(a)) = (2, 3/2)$

$$m_{tan} = \lim_{x\to 2} \frac{f(x) - f(2)}{x-2} = \lim_{x\to 2} \frac{\frac{3}{x} - \frac{3}{2}}{x-2} = \frac{6-3x}{2x} \cdot \frac{1}{x-2} = \lim_{x\to 2} \frac{-3(x-2)}{2x(x-2)} = \lim_{x\to 2} \frac{-3}{2x}$$

$$= \boxed{\frac{-3}{4}}$$

$\Rightarrow$ The equation of the line tangent to $f(x) = \frac{3}{x}$ at $x = 2$ is

$$y - f(2) = -\frac{3}{4}(x-2)$$

$$y - \frac{3}{2} = -\frac{3}{4}x + \frac{3}{2}$$

$$\boxed{y = -\frac{3}{4}x + 3}$$

Definition (Rate of Change and the Slope of the Tangent Line).
The **average rate of change** in f on the interval $[a, a+h]$ is the slope of the corresponding secant line:

$$m_{\text{sec}} = \frac{f(a+h) - f(a)}{h}.$$

The **instantaneous rate of change** in f at a is

$$m_{\text{tan}} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

which is also the **slope of the tangent line** at $(a, f(a))$, provided this limit exists.

Example. Find an equation of the line tangent to the graph of $f(x) = x^3 + 4x$ at $(1, 5)$.

$\boxed{y - f(a) = m_{tan}(x-a)}$ $\quad (a, f(a)) = (1,5)$

$$m_{tan} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{[(1+h)^3 + 4(1+h)] - [1^3 + 4(1)]}{h}$$

$$= \lim_{h \to 0} \frac{1^3 + 3(1)^2 h + 3(1)h^2 + h^3 + 4(1) - 4h - 1^3 - 4(1)}{h}$$

$$= \lim_{h \to 0} \frac{3h + 3h^2 + h^3 - 4h}{h}$$

$$= \lim_{h \to 0} 3 + 3h + h^2 + 4 = \boxed{7}$$

$\Rightarrow$ The equation of the line tangent to $f(x) = x^3 + 4x$ at $x = 1$ is

$$y - f(a) = m_{tan}(x-a)$$
$$y - 5 = 7(x-1)$$
$$\boxed{y = 7x - 2}$$

Definition (The Derivative of a Function at a Point).
The **derivative of f at a**, denoted $f'(a)$, is given by either of the two following limits, provided the limits exist and a is in the domain of f:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad (1) \quad \text{or } f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad (2)$$

If $f'(a)$ exists, we say that f is **differentiable at** a.

Example. Find an equation of the line tangent to the graph of $f(x) = \dfrac{8}{x^2}$ at $(2, 2)$.

$$m_{tan} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\frac{8}{(2+h)^2} - \frac{8}{2^2}\left(\frac{(2+h)^2}{(2+h)^2}\right)}{h}$$

$\frac{8}{4} = 2$

$$= \lim_{h\to 0} \frac{8-2(2+h)^2}{h(2+h)^2}$$

$$= \lim_{h\to 0} \frac{8-2(4+4h+h^2)}{h(2+h)^2}$$

$$\Rightarrow \quad y-(2) = -1(x-2)$$

$$\boxed{y = -x+4}$$

$$= \lim_{h\to 0} \frac{8-8-4h-h^2}{h(2+h)^2}$$

$$= \lim_{h\to 0} \frac{-4-h}{(2+h)^2} = \frac{-4-0}{(2+0)^2} = \boxed{-1}$$

Example. An equation of the line tangent to the graph of f at the $(2, 7)$ is $y = 4x - 1$. Find $f(2)$ and $f'(2)$.

Note: The tangent line and $f(x)$ meet at $(a, f(a))$

$$\Rightarrow \boxed{f(2) = 4(2)-1 = 7}$$

Note: The slope of the tangent line at $x=a$ is the slope of $f(x)$ at $x=a$

$$\Rightarrow \boxed{f'(2) = 4}$$

Example. An equation of the line tangent to the graph of g at $x = 3$ is $y = 5x + 4$. Find $g(3)$ and $g'(3)$.

$$g(3) = 5(3) + 4 = 19$$

$$g'(3) = 5$$

Example. If $h(1) = 2$ and $h'(1) = 3$, find an equation of the line tangent to the graph of h at $x = 1$.

$$y - h(a) = h'(a)(x - a)$$

$$\Rightarrow \quad y - 2 = 3(x - 1)$$

$$\boxed{y = 3x - 1}$$

Example. If $f'(-2) = 7$, find an equation of the line tangent to the graph of f at the point $(-2, 4)$.

$\uparrow$
$f(-2)$

$$\Rightarrow \quad y - f(-2) = f'(-2)(x - (-2))$$

$$y - 4 = 7(x + 2)$$

$$\boxed{y = 7x + 18}$$