10.2: Sequences

Theorem 10.1: Limits of Sequences from Limits of Functions
Suppose f is a function such that $f(n) = a_n$, for positive integers n. If $\lim_{x\to\infty} f(x) = L$, then the limit of the sequence $\{a_n\}$ is also L, where L may be $\pm\infty$.

Example. Determine if the following sequences converge or diverge. If the sequence converges, find its limit.

$$\left\{e^{2n/(n+2)}\right\}_{n=1}^{\infty}$$

$$\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$$

$$\left\{\frac{\arctan(n)}{n}\right\}_{n=1}^{\infty}$$

$$\left\{\frac{e^{-n}}{42\sin(e^{-n})}\right\}_{n=1}^{\infty}$$

10.2: Limit Laws for Sequences
Assume the sequences $\{a_n\}$ and $\{b_n\}$ have limits A and B, respectively. Then

1. $\lim\limits_{n\to\infty} (a_n \pm b_n) = A \pm B$
2. $\lim\limits_{n\to\infty} ca_n = cA$, where c is a real number
3. $\lim\limits_{n\to\infty} a_n b_n = AB$
4. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$, provided $B \neq 0$.

Example. Consider the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ where

$$a = \frac{1}{n}, \quad b_n = n, \quad c_n = e^n, \quad \text{and } d_n = \sqrt{n}.$$

Compute the following limits.

A. $\lim\limits_{n\to\infty} a_n$

B. $\lim\limits_{n\to\infty} b_n$

C. $\lim\limits_{n\to\infty} c_n$

D. $\lim\limits_{n\to\infty} d_n$

E. $\lim\limits_{n\to\infty} a_n b_n$

F. $\lim\limits_{n\to\infty} a_n c_n$

G. $\lim\limits_{n\to\infty} a_n d_n$

True or False: If for some sequence $\{a_n\}$ and $\{b_n\}$, $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} b_n = \infty$, then $\lim\limits_{n\to\infty} a_n b_n = 0$.

Definition. (Terminology for Sequences)

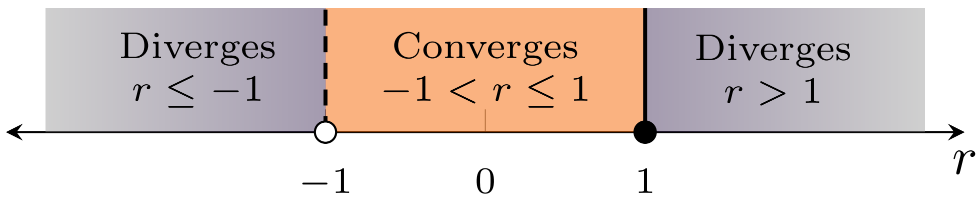
- $\{a_n\}$ is **increasing** if $a_{n+1} > a_n$
- $\{a_n\}$ is **nondecreasing** if $a_{n+1} \geq a_n$
- $\{a_n\}$ is **decreasing** if $a_{n+1} < a_n$
- $\{a_n\}$ is **nonincreasing** if $a_{n+1} \leq a_n$
- $\{a_n\}$ is **monotonic** if it is either nonincreasing or nondecreasing (it moves in one direction)
- $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \leq M$, for all relevant values of n
- $\{a_n\}$ is **bounded below** if there is a number N such that $a_n \geq N$, for all relevant values of n.
- If $\{a_n\}$ is bounded above and bounded below, then we say that $\{a_n\}$ is a **bounded** sequence.

Example. Consider the sequence $\left\{-n^2\right\}_{n=1}^{\infty}$. What can we say about this sequence?

Theorem 10.3: Geometric Sequences
Let r be a real number. Then

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist} & \text{if } r \leq -1 \text{ or } r > 1. \end{cases}$$

If $r > 0$, then $\{r^n\}$ is a monotonic sequence. If $r < 0$, then $\{r^n\}$ oscillates.



Example. Determine if the following sequences converge

$$\left\{\frac{3^{n+1}+3}{3^n}\right\}$$

$$\left\{2^{n+1}3^{-n}\right\}$$

$$\left\{\frac{(-1)^n}{2^n}\right\}$$

$$\left\{\frac{75^{n-1}}{99^n} + \frac{5^n \sin(n)}{8^n}\right\}$$

Theorem 10.4: Squeeze Theorem for Sequences
Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences with $a_n \leq b_n \leq c_n$, for all integers n greater than some index N. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Example. Find the limit of the sequence $b_n = \dfrac{9\cos(n)}{n^2+1}$.

Theorem 10.5: Bounded Monotonic Sequence
A bounded monotonic sequence converges.

Theorem 10.6: Growth Rates of Sequences
The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, if $\{a_n\}$ appears before $\{b_n\}$ in the list, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$:

$$\{(\ln n)^q\} \ll \{n^p\} \ll \{n^p(\ln n)^r\} \ll \left\{n^{p+s}\right\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\}$$

Example. Use growth rates to determine which of the following sequences converge.

$$\left\{\frac{\ln(n^{10})}{0.00001n}\right\}$$

$$\left\{\frac{n^8 \ln(n)}{n^{8.001}}\right\}$$

$$\left\{\frac{n!}{10^n}\right\}$$

Definition. (Limit of a Sequence)
The sequence $\{a_n\}$ converges to L provided the terms of a_n can be made arbitrarily close to L by taking n sufficiently large. More precisely, $\{a_n\}$ has the unique limit L if, given any $\varepsilon > 0$, it is possible to find a positive integer N (depending only on ε) such that

$$|a_n - L| < \varepsilon \qquad \text{whenever } n > N.$$

If the **limit of a sequence** is L, we say the sequence **converges** to L, written

$$\lim_{n\to\infty} a_n = L.$$

A sequence that does not converge is said to **diverge**.