14.5: Curvature and Normal Vectors:

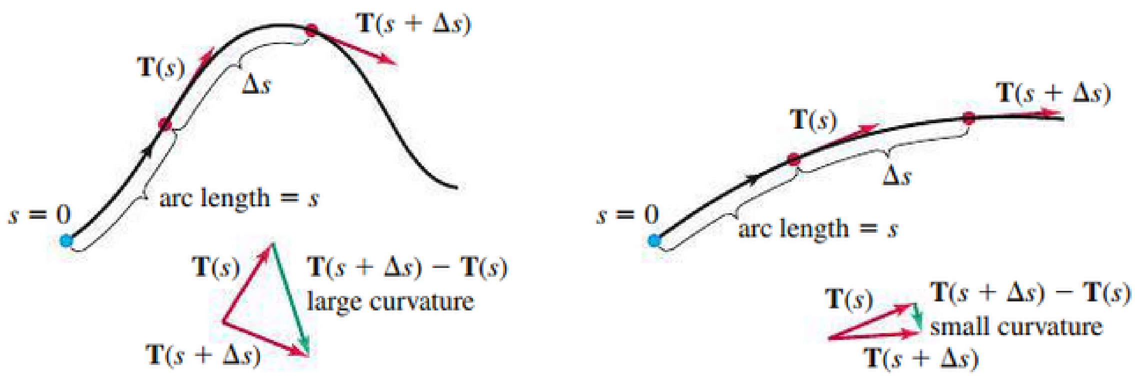
There are two ways to change the velocity, or in other words, to accelerate:

- change in speed
- change in direction

The change in direction is referred to as *curvature*. Recall that if we have a smooth curve $\mathbf{r}(t)$, the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

Specifically, *curvature* of the curve is the magnitude of the rate at which $\mathbf{T}$ changes with respect to arc length.



Definition. (Curvature)
Let $\mathbf{r}$ describe a smooth parameterized curve. If s denotes arc length and $\mathbf{T} = \mathbf{r}'/|\mathbf{r}'|$ is the unit tangent vector, the **curvature** is $\kappa(s) = \left|\dfrac{d\mathbf{T}}{ds}\right|$.

Theorem 14.4: Curvature Formula
Let $\mathbf{r}(t)$ describe a smooth parameterized curve, where t is any parameter. If $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{T}$ is the unit tangent vector, then the curvature is

$$\kappa(t) = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

- κ is a non-negative scalar-valued function
- Curvature of zero corresponds to a straight line
- A relatively flat curve has a small curvature
- A tight curve has a larger curvature

Example. Consider the line

$$\mathbf{r}(t) = \langle x_0 + at,\, y_0 + bt,\, z_0 + ct\rangle, \text{ for } -\infty < t < \infty.$$

Compute κ.

Example. Consider the circle

$$\mathbf{r}(t) = \langle R\cos(t),\ R\sin(t)\rangle$$

for $0 \le t \le 2\pi$, where $R > 0$. Show that $\kappa = 1/R$.

Example. Consider the curve

$$\mathbf{r}(t) = \left\langle 2\cos(t),\ 2\sin(t),\ \sqrt{5}t \right\rangle$$

Compute κ.

An Alternative Curvature Formula:

Consider a smooth function $\mathbf{r}(t)$ with non-zero velocity $\mathbf{v}(t) = \mathbf{r}'(t)$ and non-zero acceleration $\mathbf{a}(t) = \mathbf{v}'(t)$.

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} \;\Rightarrow\; \mathbf{v} = |\mathbf{v}|\,\mathbf{T}.$$

Thus

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}[|\mathbf{v}|\,\mathbf{T}] = \frac{d}{dt}[|\mathbf{v}|]\,\mathbf{T} + |\mathbf{v}|\,\frac{d\mathbf{T}}{dt}.$$

Now we form $\mathbf{v} \times \mathbf{a}$:

$$\begin{aligned}\mathbf{v} \times \mathbf{a} &= |\mathbf{v}|\,\mathbf{T} \times \left(\frac{d}{dt}[|\mathbf{v}|]\,\mathbf{T} + |\mathbf{v}|\,\frac{d\mathbf{T}}{dt}\right)\\ &= \underbrace{|\mathbf{v}|\,\mathbf{T} \times \frac{d}{dt}[|\mathbf{v}|]\,\mathbf{T}}_{\mathbf{0}} + |\mathbf{v}|\,\mathbf{T} \times |\mathbf{v}|\,\frac{d\mathbf{T}}{dt}\end{aligned}$$

Since $\mathbf{T}$ is a unit vector, $\mathbf{T}$ and $d\mathbf{T}/dt$ are orthogonal (Theorem 14.2). Thus

$$|\mathbf{v} \times \mathbf{a}| = \left| |\mathbf{v}|\,\mathbf{T} \times |\mathbf{v}|\,\frac{d\mathbf{T}}{dt} \right| = |\mathbf{v}|\,\underbrace{|\mathbf{T}|}_{1}\,\left| |\mathbf{v}| \frac{d\mathbf{T}}{dt} \right| \underbrace{\sin\theta}_{1} = |\mathbf{v}|^2 \left| \frac{d\mathbf{T}}{dt} \right|$$

Now, using Theorem 14.4, where $\left| \dfrac{d\mathbf{T}}{dt} \right| = \kappa |\mathbf{v}|$, we have

$$|\mathbf{v} \times \mathbf{a}| = |\mathbf{v}|^2 \left| \frac{d\mathbf{T}}{dt} \right| = |\mathbf{v}|^2\,\kappa |\mathbf{v}| = \kappa |\mathbf{v}|^3.$$

Theorem 14.5: Alternative Curvature Formula

Let $\mathbf{r}$ be the position of an object moving on a smooth curve. The **curvature** at a point on the curve is

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3},$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{a} = \mathbf{v}'$ is the acceleration.

Example. Consider the curve

$$\mathbf{r}(t) = \langle -16\cos(t),\, 16\sin(t),\, 0\rangle.$$

Compute the curvature κ using both methods.

Principal Unit Normal Vector

Curvature indicates how quickly a curve turns. The principal unit normal vector determines the *direction* in which a curve turns.

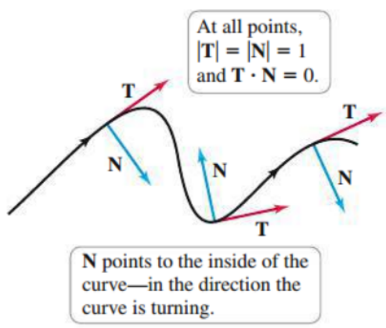
Definition. (Principal Unit Normal Vector)
Let $\mathbf{r}$ describe a smooth curve parameterized by arc length. The **principal unit normal vector** at a point P on the curve at which $\kappa \neq 0$ is

$$\mathbf{N}(s) = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

For other parameters, we use the equivalent formula

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|},$$

evaluated at the value of t corresponding to P.



Theorem 14.6: Properties of the Principal Unit Normal Vector
Let $\mathbf{r}$ describe a smooth parameterized curve with unit tangent vector $\mathbf{T}$ and principal unit normal vector $\mathbf{N}$.

1. $\mathbf{T}$ and $\mathbf{N}$ are orthogonal at all points of the curve; that is, $\mathbf{T} \cdot \mathbf{N} = 0$ at all points where $\mathbf{N}$ is defined.
2. The principal unit normal vector points to the inside of the curve – in the direction that the curve is turning.

Example. For the curve $\mathbf{r}(t) = \langle a\cos(t), a\cos(t), bt\rangle$, find the unit tangent vector $\mathbf{T}$ and the principal unit normal vector $\mathbf{N}$. Verify $|\mathbf{T}| = |\mathbf{N}| = 1$ and $\mathbf{T} \cdot \mathbf{N} = 0$.

Components of the Acceleration

Recall that the change in velocity, or acceleration, of an object can change in *speed* (in the direction of $\mathbf{T}$) and in *direction* (in the direction of $\mathbf{N}$). $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|} \Longrightarrow \mathbf{v} = \mathbf{T}|\mathbf{v}| = \mathbf{T}\,\dfrac{ds}{dt}$.

$$
\begin{aligned}
\mathbf{a} = \frac{d\mathbf{v}}{dt} &= \frac{d}{dt}\left(\mathbf{T}\frac{ds}{dt}\right) \\
&= \frac{d\mathbf{T}}{dt}\,\frac{ds}{dt} + \mathbf{T}\frac{d^2s}{dt^2} \\
&= \underbrace{\frac{d\mathbf{T}}{ds}}_{\kappa\mathbf{N}}\,\underbrace{\frac{ds}{dt}}_{|\mathbf{v}|}\,\underbrace{\frac{ds}{dt}}_{|\mathbf{v}|} + \mathbf{T}\frac{d^2s}{dt^2} \\
&= \kappa|\mathbf{v}|^2\,\mathbf{N} + \frac{d^2s}{dt^2}\,\mathbf{T}.
\end{aligned}
$$

Theorem 14.7: Tangential and Normal Components of the Acceleration
The acceleration vector of an object moving in space along a smooth curve has the following representation in terms of its **tangential component** a_T (in the direction of $\mathbf{T}$) and its **normal component** a_N (in the direction of $\mathbf{N}$):

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T},$$

where $a_N = \kappa|\mathbf{v}|^2 = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|}$ and $a_T = \dfrac{d^2s}{dt^2}$.

Example. Consider the function

$$\mathbf{r}(t) = \langle -2t+2,\, -2t+3,\, -2t+2 \rangle.$$

Find the tangential and normal components of the acceleration.

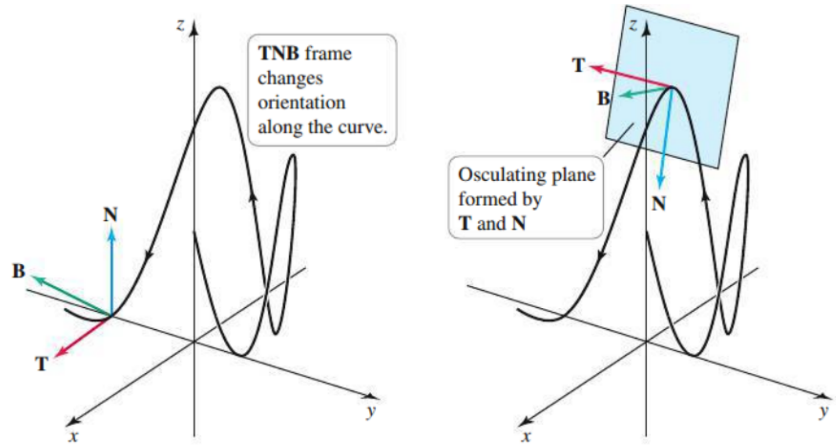
Example. Find the components of the acceleration on the circular trajectory

$$\mathbf{r}(t) = \langle R\cos(\omega t),\, R\sin(\omega t) \rangle.$$

Example. The driver of a car follows the parabolic trajectory $\mathbf{r}(t) = \left\langle t,\, t^2 \right\rangle$, for $-2 \leq t \leq 2$, through a sharp bend. Find the tangential and normal components of the acceleration of the car.

The Binormal Vector and Torsion

On a smooth parameterized curve C, $\mathbf{T}$ and $\mathbf{N}$ determine a plane called the *osculating plane.*



The coordinate system defined by these vectors is called the **TNB frame**. The rate at which the curve C twists out of the plane is the rate at which $\mathbf{B}$ changes as we move along C, which is $\dfrac{d\mathbf{B}}{ds}$.

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T}\times\mathbf{N}) = \underbrace{\frac{d\mathbf{T}}{ds}\times\mathbf{N}}_{\mathbf{0}} + \mathbf{T}\times\frac{d\mathbf{N}}{ds} = \mathbf{T}\times\frac{d\mathbf{N}}{ds}$$

$\dfrac{d\mathbf{B}}{ds}$ is:

- orthogonal to both $\mathbf{T}$ and $\dfrac{d\mathbf{N}}{ds}$,
- orthogonal to $\mathbf{B}$ (Theorem 14.2),
- parallel with $\mathbf{N}$.

Since $\dfrac{d\mathbf{B}}{ds}$ is parallel to $\mathbf{N}$, we write

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

where τ is the *torsion* (the negative sign is conventional). We can solve for τ via the dot product:

$$\frac{d\mathbf{B}}{ds}\cdot\mathbf{N} = -\tau\underbrace{\mathbf{N}\cdot\mathbf{N}}_{1} \quad\Longrightarrow\quad \frac{d\mathbf{B}}{ds}\cdot\mathbf{N} = -\tau$$

Definition. (Unit Binormal Vector and Torsion)
Let C be a smooth parameterized curve with unit tangent and principal unit normal vectors $\mathbf{T}$ and $\mathbf{N}$, respectively. Then at each point of the curve at which the curvature is nonzero, the **unit binomial vector** is

$$\mathbf{B} = \mathbf{T}\times\mathbf{N},$$

and the **torsion** is

$$\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

Example. Consider the circle C defined by

$$\mathbf{r}(t) = \langle R\cos(t),\ R\sin(t)\rangle,\ \text{for } 0 \le t \le 2\pi,\ \text{with } R > 0.$$

Find the unit binormal vector $\mathbf{B}$ and determine the torsion.

Example. Compute the torsion of the helix

$$\mathbf{r}(t) = \langle a\cos(t),\, a\sin(t),\, bt\rangle,\ \text{for } t \ge 0,\ \text{and } b > 0.$$

Summary: Formula for Curves in Space

Position function: $\mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$

Velocity: $\mathbf{v} = \mathbf{r}'$

Acceleration: $\mathbf{a} = \mathbf{v}'$

Unit tangent vector: $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

Principal unit normal vector: $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$ (provided $d\mathbf{T}/dt \neq \mathbf{0}$)

Curvature: $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{1}{|\mathbf{v}|}\left|\dfrac{d\mathbf{T}}{dt}\right| = \dfrac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$

Components of acceleration: $\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}$, where

$$a_N = \kappa|\mathbf{v}|^2 = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|} \text{ and } a_T = \frac{d^2s}{dt^2} = \frac{\mathbf{v}\cdot\mathbf{a}}{\mathbf{v}}$$

Unit binormal vector: $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \dfrac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|}$

Torsion: $\tau = -\dfrac{d\mathbf{B}}{ds}\cdot\mathbf{N} = \dfrac{(\mathbf{v}\times\mathbf{a})\cdot\mathbf{a}'}{|\mathbf{v}\times\mathbf{a}|^2} = \dfrac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}'\times\mathbf{r}''|^2}$