

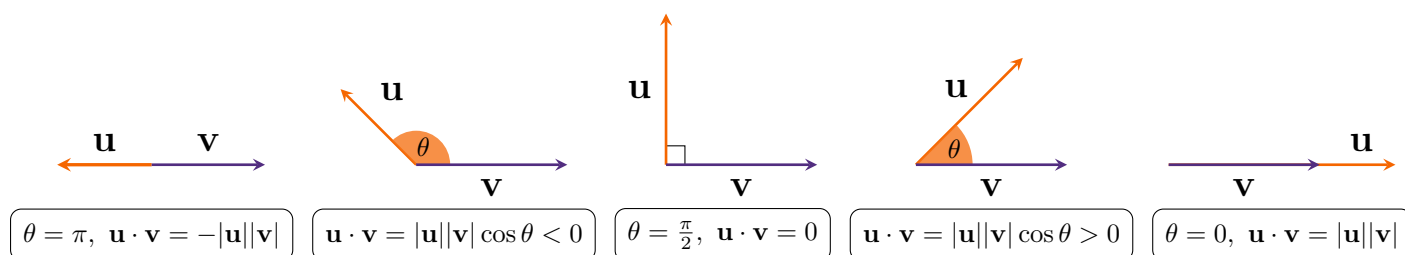
1 13.3: Dot Products

Definition. (Dot Product)

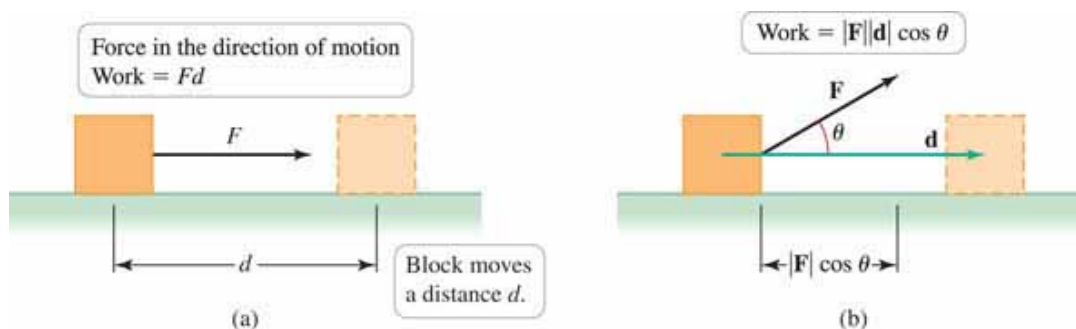
Given two nonzero vectors \mathbf{u} and \mathbf{v} in two or three dimensions, their **dot product** is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta,$$

where θ is the angle between \mathbf{u} and \mathbf{v} with $0 \leq \theta \leq \pi$. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$, and θ is undefined.



A physical example of the dot product is the amount of work done when a force is applied at an angle θ as shown in figure 13.43:



Note: The result of the dot product is a scalar!

Definition. (Orthogonal Vectors)

Two vectors \mathbf{u} and \mathbf{v} are **orthogonal** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. The zero vector is orthogonal to all vectors. In two or three dimensions, two nonzero orthogonal vectors are perpendicular to each other.

- \mathbf{u} and \mathbf{v} are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \cdot \mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$.
- \mathbf{u} and \mathbf{v} are perpendicular ($\theta = \frac{\pi}{2}$) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Example. Given $|\mathbf{u}| = 2$ and $|\mathbf{v}| = \sqrt{3}$, compute $\mathbf{u} \cdot \mathbf{v}$ when

- $\theta = \frac{\pi}{4}$
- $\theta = \frac{\pi}{3}$
- $\theta = \frac{5\pi}{6}$

Theorem 31.1: Dot Product

Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$,

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

Example. Given vectors $\mathbf{u} = \langle \sqrt{3}, 1, 0 \rangle$ and $\mathbf{v} = \langle 1, \sqrt{3}, 0 \rangle$, compute $\mathbf{u} \cdot \mathbf{v}$ and find θ .

Properties of Dot Products

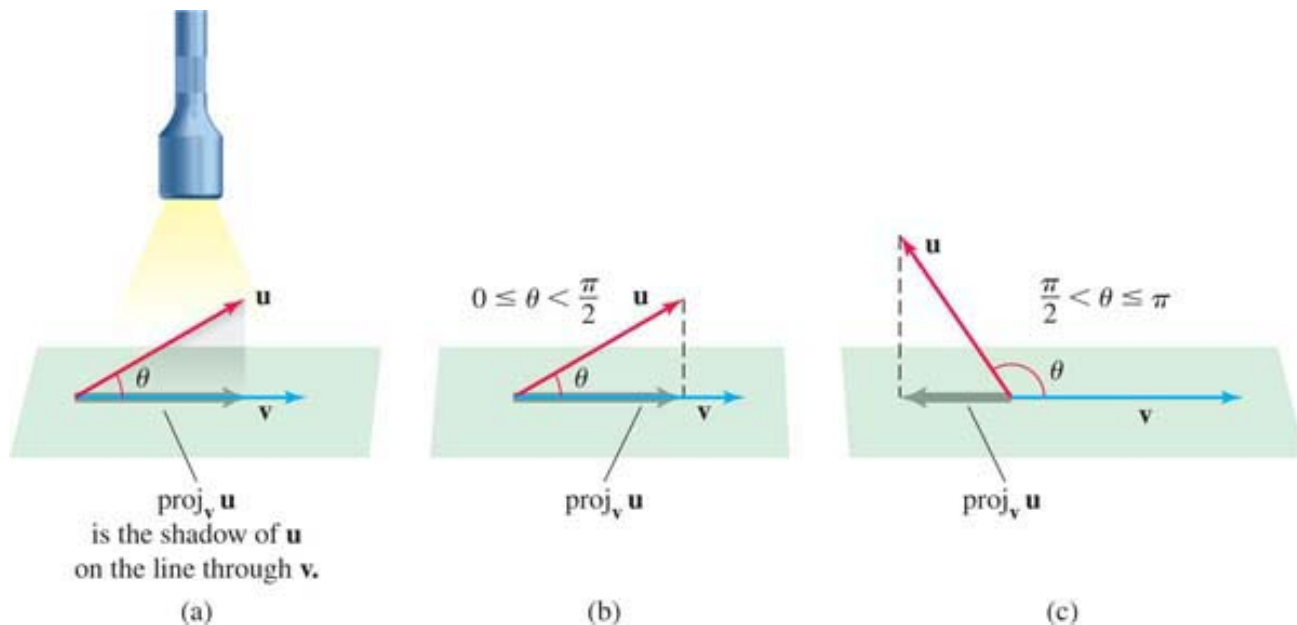
Theorem 13.2: Properties of the Dot Product

Suppose \mathbf{u} , \mathbf{v} and \mathbf{w} are vectors and let c be a scalar.

- | | |
|---|-----------------------|
| 1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ | Commutative property |
| 2. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$ | Associative property |
| 3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ | Distributive property |

Orthogonal Projections

Given vectors \mathbf{u} and \mathbf{v} , the projection of \mathbf{u} onto \mathbf{v} produces a vector parallel to \mathbf{v} using the “shadow” of \mathbf{u} cast onto \mathbf{v} .



Definition. ((Orthogonal) Projection of \mathbf{u} onto \mathbf{v})

The **orthogonal projection of \mathbf{u} onto \mathbf{v}** , denoted $\text{proj}_{\mathbf{v}} \mathbf{u}$, where $\mathbf{v} \neq \mathbf{0}$, is

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \underbrace{|\mathbf{u}| \cos \theta}_{\text{length}} \underbrace{\left(\frac{\mathbf{v}}{|\mathbf{v}|} \right)}_{\text{direction}}.$$

The orthogonal projection may also be computed with the formulas

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \text{scal}_{\mathbf{v}} \mathbf{u} \left(\frac{\mathbf{v}}{|\mathbf{v}|} \right) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v},$$

where the **scalar component of \mathbf{u} in the direction of \mathbf{v}** is

$$\text{scal}_{\mathbf{v}} \mathbf{u} = |\mathbf{u}| \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}.$$

Example. Find $\text{proj}_{\mathbf{v}} \mathbf{u}$ and $\text{scal}_{\mathbf{v}} \mathbf{u}$ for the following:

- $\mathbf{u} = \langle 1, 1 \rangle$, $\mathbf{v} = \langle -2, 1 \rangle$

- $\mathbf{u} = \langle 7, 1, 7 \rangle$, $\mathbf{v} = \langle 5, 7, 0 \rangle$

Applications of Dot Products

Definition. (Work)

Let a constant force \mathbf{F} be applied to an object, producing a displacement \mathbf{d} . If the angle between \mathbf{F} and \mathbf{d} is θ , then the **work** done by the force is

$$W = |\mathbf{F}||\mathbf{d}| \cos \theta = \mathbf{F} \cdot \mathbf{d}$$

Example. A force $\mathbf{F} = \langle 3, 3, 2 \rangle$ (in newtons) moves an object along a line segment from $P(1, 1, 0)$ to $Q(6, 6, 0)$ (in meters). What is the work done by the force?

Components of a Force:

Example. A 10-lb block rests on a plane that is inclined at 30° above the horizontal. Find the components of the gravitational force parallel to and normal (perpendicular) to the plane.

