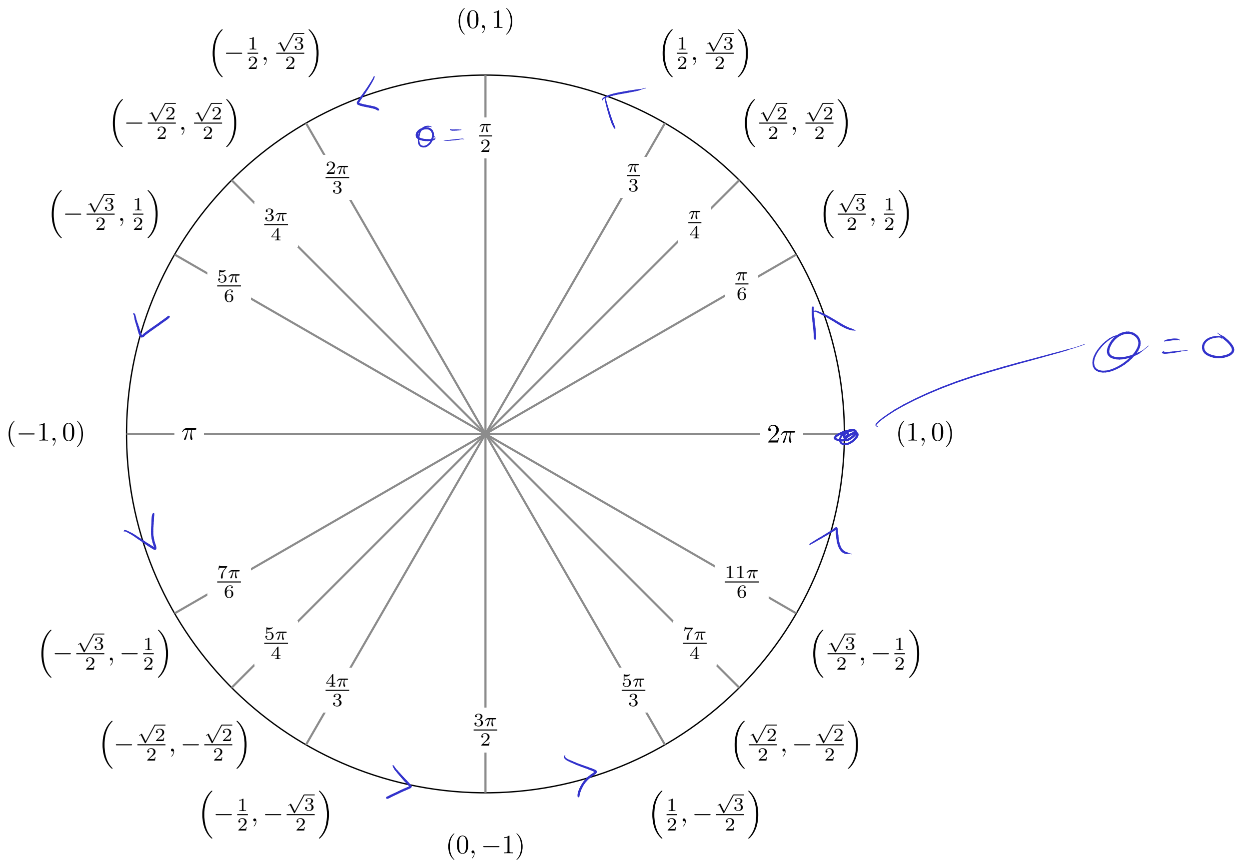
12.1: Parametric Equations

We've already seen a parametric equation represented by the unit circle. Here, we have

$$x(\theta) = \cos(\theta) \text{ and } y(\theta) = \sin(\theta), \text{ where } 0 \leq \theta \leq 2\pi$$



Definition. (Positive Orientation)
The direction in which a parametric curve is generated as the parameter increases is called the **positive orientation** of the curve (and is indicated by arrows on the curve).

Example (LC 32.1-32.2)**.** Consider the parametric equations

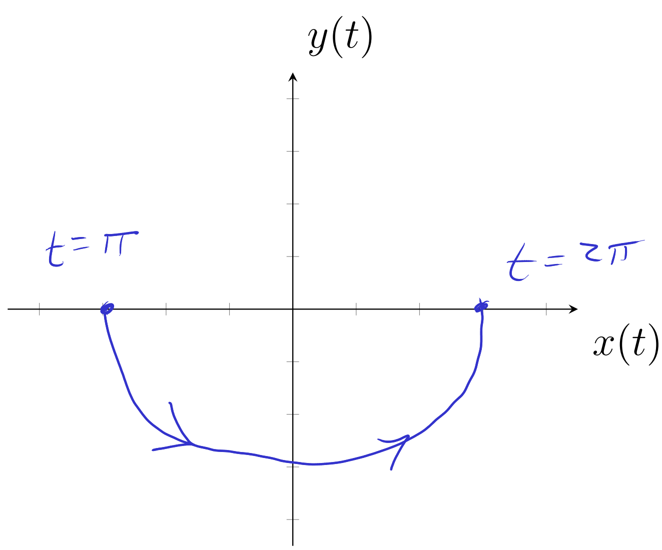
$$x = 3\cos(t),\ \ y = 3\sin(t); \pi \le t \le 2\pi$$

Eliminate the parameter t and rewrite as a function of x and y.

$$x^2 + y^2$$
$$= (3\cos(t))^2 + (3\sin(t))^2$$
$$= 9\cos^2(t) + 9\sin^2(t)$$
$$= 9(\cos^2(t) + \sin^2(t)) = 9$$

$$\left.\right\} \quad x^2 + y^2 = 9 \quad ; \quad -3 \le y \le 0$$

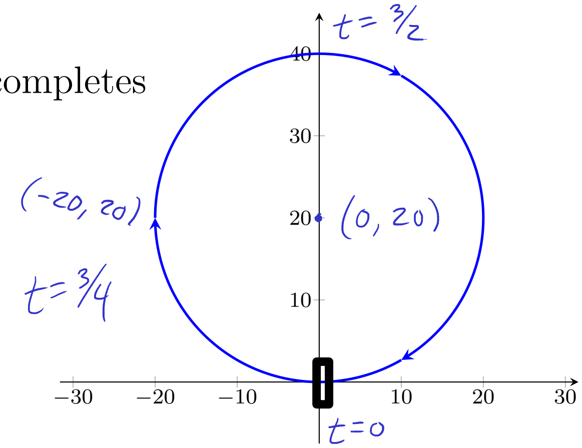
Graph the equation found above indicating the positive orientation.



Example (LC 32.3-32.4)**.** A Ferris wheel has a radius of 20 m and completes a revolution in the **clockwise** direction at constant speed in 3 minutes. Assume x and y measure the horizontal and vertical positions of a seat on the Ferris wheel relative to a coordinate system whose origin is at the low point of the wheel. Assume the seat begins moving at the origin.

What is the domain of t such that the Ferris wheel completes one revolution?

$$0 \le t \le 3$$



$x(t)$ and $y(t)$ will be parameterized using $\sin(bt)$ and $\cos(bt)$. What is b?

Pick b such that t going from 0 to 3 corresponds to a single revolution.

$$\begin{aligned} 0 &= b(0) \\ 2\pi &= b(3) \end{aligned} \longrightarrow b = \frac{2\pi}{3}$$

What parametric equations describe the path of the seat on the Ferris wheel?

$$x(t) = -20\sin\left(\frac{2\pi}{3}t\right)$$

$$y(t) = 20 - 20\cos\left(\frac{2\pi}{3}t\right)$$

t	(x, y)
0	(0, 0)
3/4	(−20, 20)
3/2	(0, 40)

$$-20\sin\left(\frac{2\pi}{3}\cdot\frac{3}{2}\right) = -20\sin(\pi) = 0$$

$$20 - 20\cos(\pi) = 20 + 20 = 40$$

$$\sin(-\theta) = -\sin(\theta)$$

Summary: Parametric Equations of a Line
The equations

$$x = x_0 + at, \; y = y_0 + bt, \quad \text{for } -\infty < t < \infty,$$

where x_0, y_0, a, and b are constants with $a \neq 0$, describe a line with slope $\frac{b}{a}$ passing through the point (x_0, y_0). If $a = 0$ and $b \neq 0$, the line is vertical.

Example. Find 2 parameterized equations of the line that goes through the points $\underbrace{(3,-4)}_{t=0}$ and $\underbrace{(-2,3)}_{t=5}$.

$x(5) = -2 = 3 + a(5) \longrightarrow a = -1$

$y(5) = 3 = -4 + b(5) \longrightarrow b = 7/5$

① $x(t) = 3 + at \qquad 3 - t$
$y(t) = -4 + bt = -4 + \frac{7}{5}t$

$-\infty < t < \infty$

② $x(t) = 3 + t$
$y(t) = -4 - \frac{7}{5}t$ $\Big\} \longrightarrow$ $(3,-4)\; t=0$, $(-2,3)\; t=-5$

$x(t) = 3 + t(-2-3) = 3 - 5t \qquad x(0) = 3,\; x(1) = -2$

$y(t) = -4 + t(3-(-4)) = -4 + 7t \qquad y(0) = -4,\; y(1) = 3$

$-\infty \leq t \leq \infty$

Example. Find a parameterized equation for the line segment that connects the points $(3, 0)$ and $(-1, 3)$.

Let $0 \leq t \leq 1$

$x(1) = 3 + a(1) = -1$
$a = -4$

$x(t) = 3 + at = 3 - 4t \qquad 0 \leq t \leq 1$
$y(t) = 0 + bt = 3t$

$y(1) = 0 + b(1) = 3$
$b = 3$

$x(t) = 3 + t(-1-3) = 3 - 4t$

$x(0) = 3$
$x(1) = -1$

$x(0) = 3$
$x(10) = -1$

$x(t) = 3 + \frac{t}{10}(-1-3) = 3 - \frac{4}{10}t \qquad 0 \leq t \leq 10$

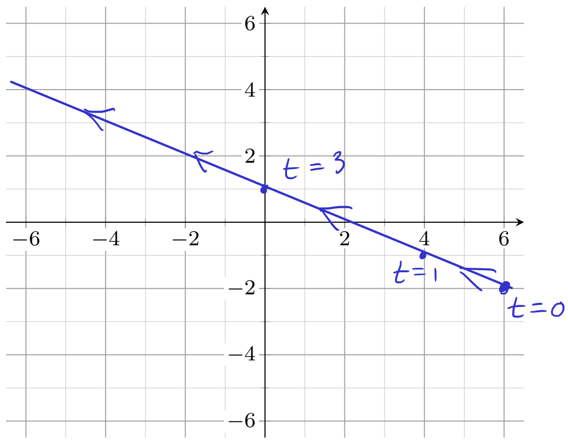
Example. Consider the parametric equations

$$x(t) = 6 - 2t \text{ and } y(t) = -2 + t,$$

Graph the curve indicating the positive orientation

t	(x, y)
0	$(6, -2)$
1	$(4, -1)$
3	$(0, 1)$

$y = -\frac{1}{2}x + 1$



Eliminate the parameter to find an equation in x and y.

$x = 6 - 2t \longrightarrow x = 6 - 2(y+2)$

$y = -2 + t \longrightarrow t = y + 2$

$2(y+2) = 6 - x$

$y + 2 = 3 - \frac{1}{2}x$

$y = -\frac{1}{2}x + 1$

Example (LC 32.5-32.7). Consider the parametric equations

$$x = 1 + e^{2t} \text{ and } y = e^t,$$

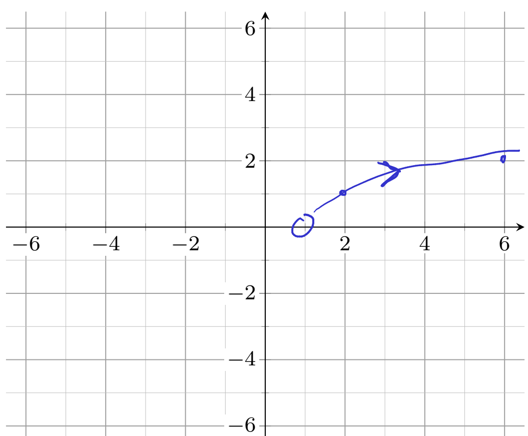
Eliminate the parameter to find an equation in x and y

$x = 1 + y^2 \longrightarrow y = \cancel{\pm}\sqrt{x-1}$

$y = e^t > 0$

t	(x, y)
0	$(2, 1)$
1	$(1+e, e)$

Graph the curve indicating the positive orientation



$x = 1 + e^{2t}$

$y = e^t$

$(x, y) = (1, 0)$

$\Rightarrow 1 + e^{2t} = 1$

$e^{2t} = 0$

$\Rightarrow e^t = 0$

Which of the following parametric equations are equivalent?

$$\boxed{x = 2t^2, \qquad y = 4 + t; \qquad -4 \le t \le 4}$$

$$x = 2t^4, \qquad y = 4 + t^2; \qquad -2 \le t \le 2$$

$$\boxed{x = 2t^{2/3}, \qquad y = 4 + t^{1/3}; \qquad -64 \le t \le 64}$$

Theorem 12.1: Derivative for Parametric Curves
Let $x = f(t)$ and $y = g(t)$, where f and g are differentiable on an interval $[a, b]$. Then the slope of the line tangent to the curve at the point corresponding to t is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\boxed{g'(t)}}{\boxed{f'(t)}},$$

provided $f'(t) \neq 0$.

Example (LC 32.8-32.9)**.** Consider the parametric equations

$$x = \sqrt{t}, \qquad y = 2t,$$

Find $\frac{dy}{dt}$. $= 2$

$\frac{dx}{dt} = \frac{1}{2\sqrt{t}}$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2}{\frac{1}{2\sqrt{t}}} = 4\sqrt{t}$

Find the equation of the line tangent to the curve at $t = 4$.

$x = \sqrt{t}; \quad y = 2t$

$y = f(x) \longrightarrow y = f(a) + f'(a)(x - a)$

$y = y(4) + \left.\frac{dy}{dx}\right|_{t=4} (x - x(4))$

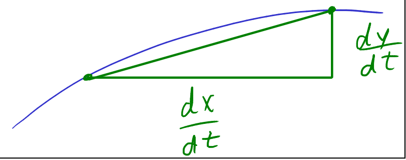
$= 8 + 4\sqrt{4}(x - \sqrt{4})$

$= 8x - 8$

$$L = \int_a^b \sqrt{\underset{\uparrow \atop \left(\frac{dx}{dx}\right)^2}{1} + \underset{\uparrow \atop \left(\frac{dy}{dx}\right)^2}{f'(x)^2}}\, dx$$

Definition. (Arc Length for Curves Defined by Parametric Equations) Consider the curve described by the parametric equations $x = f(t)$, $y = g(t)$, where f' and g' are continuous, and the curve is traversed once for $a \leq t \leq b$. The **arc length** of the curve between $(f(a), g(a))$ and $(f(b), g(b))$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt.$$



Example (LC 33.1-33.2)**.** Find the arc length of the curve given by $x = 6t^2$, $y = 2t^3$, for $0 \leq t \leq 4$.

$$L = \int_0^4 \sqrt{(12t)^2 + (6t^2)^2}\, dt$$

$$= \int_0^4 \sqrt{144t^2 + 36t^4}\, dt$$

$$= \int_0^4 6t\sqrt{4 + t^2}\, dt$$

$u = 4 + t^2$

$du = 2t\, dt \longrightarrow 3du = 6t\,dt$

$t = 0, \quad u = 4$

$t = 4, \quad u = 20$

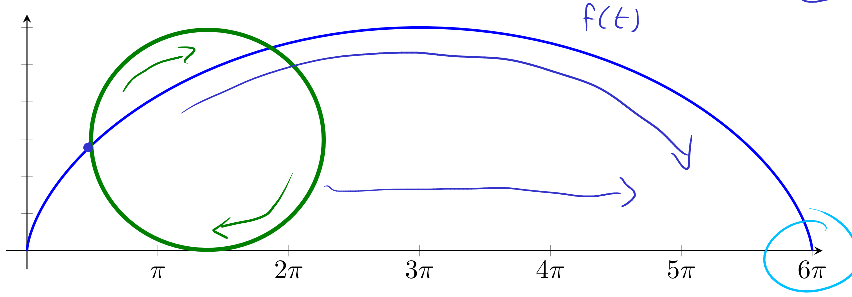
$$= 3\int_4^{20} u^{1/2}\, du$$

$$= 3\left[\frac{2}{3}u^{3/2}\right]_4^{20} = 2\left[20^{3/2} - 4^{3/2}\right] = 2\left[20^{3/2} - 8\right]$$

Example. Suppose the function $y = h(x)$ is nonnegative and continuous on $[\alpha, \beta]$, which implies that the area bounded by the graph of h and the x-axis on $[\alpha, \beta]$ equals $\int_\alpha^\beta h(x)\,dx$ or $\int_\alpha^\beta y\,dx$. If the graph of $y = h(x)$ on $[\alpha, \beta]$ is traced exactly once by the parametric equations $x = f(t)$, $y = g(t)$, for $a \le t \le b$, then it follows by substitution that the area bounded by h is

$$\int_\alpha^\beta \overset{y}{h(x)}\,dx = \int_a^b \overset{y=g(t)}{g(t)}\,\overset{\frac{dy}{dx}}{f'(t)}\,dt \text{ if } \alpha = f(a) \text{ and } \beta = f(b)$$

Find the area under one arch of the cycloid $x = 3(\underbrace{t - \sin(t)}_{f(t)})$, $y = 3\underbrace{(1 - \cos(t))}_{g(t)}$.



solve $y(t)=0$

$3(1-\cos(t))=0$

$t = 0, 2\pi, \dots$

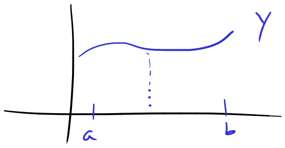
$$\int_0^{2\pi} \underbrace{3(1-\cos(t))}_{g(t)}\,\underbrace{3(1-\cos(t))}_{f'(t)}\,dt = 9\int_0^{2\pi} (1-\cos(t))^2\,dt$$

$$= 9\int_0^{2\pi} \Big[1 - 2\cos(t) + \underbrace{\cos^2(t)}_{\frac{1+\cos(2t)}{2}}\Big]\,dt$$

$$= 9\left[\frac{3}{2}t - 2\sin(t) + \frac{\cos(4t)}{4}\right]_0^{2\pi}$$

$$= 9\left[\left(3\pi - 0 + \frac{1}{4}\right) - \left(0 - 0 + \frac{1}{4}\right)\right]$$

$$= 27\pi$$

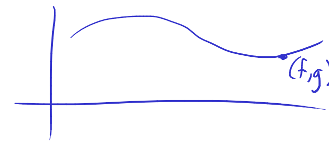


$$S = \int_a^b 2\pi\, y \sqrt{1+(y')^2}\, dx$$

Example (33.3 Surface area)**.** Let C be the curve $x = f(t)$, $y = g(t)$, for $a \le t \le b$, where f' and g' are continuous on $[a, b]$ and C does not intersect itself, except possibly at its endpoints. If g is nonnegative on $[a, b]$, then the area of the surface obtained by revolving C about the x-axis is

y $\quad \left(\frac{dx}{dx}\right)^2 \quad (y')^2$

$$S = \int_a^b 2\pi g(t)\sqrt{f'(t)^2 + g'(t)^2}\, dt.$$



Setup the integral used to find the area of the surface obtained by revolving the curve $x = \underbrace{t\sin(t)}_{f(t)}$, $y = \underbrace{t\cos(t)}_{g(t)}$, for $0 \le t \le \pi/2$, about the x-axis.

$$S = \int_0^{\pi/2} 2\pi\, t\cos(t)\sqrt{(\sin(t)+t\cos(t))^2 + (\cos(t) - t\sin(t))^2}\, dt$$

$f'(t) = \sin(t) + t\cos(t)$

$g'(t) = \cos(t) - t\sin(t)$

$$(f'(t))^2 + (g'(t))^2 = \left.\begin{matrix} \sin^2(t) + 2t\sin(t)\cos(t) + t^2\cos^2(t) \\ + \cos^2(t) - 2t\sin(t)\cos(t) + t^2\sin^2(t) \end{matrix}\right\} = 1 + t^2$$

$$= \int_0^{\pi/2} 2\pi\, t\cos(t)\sqrt{1+t^2}\, dt$$