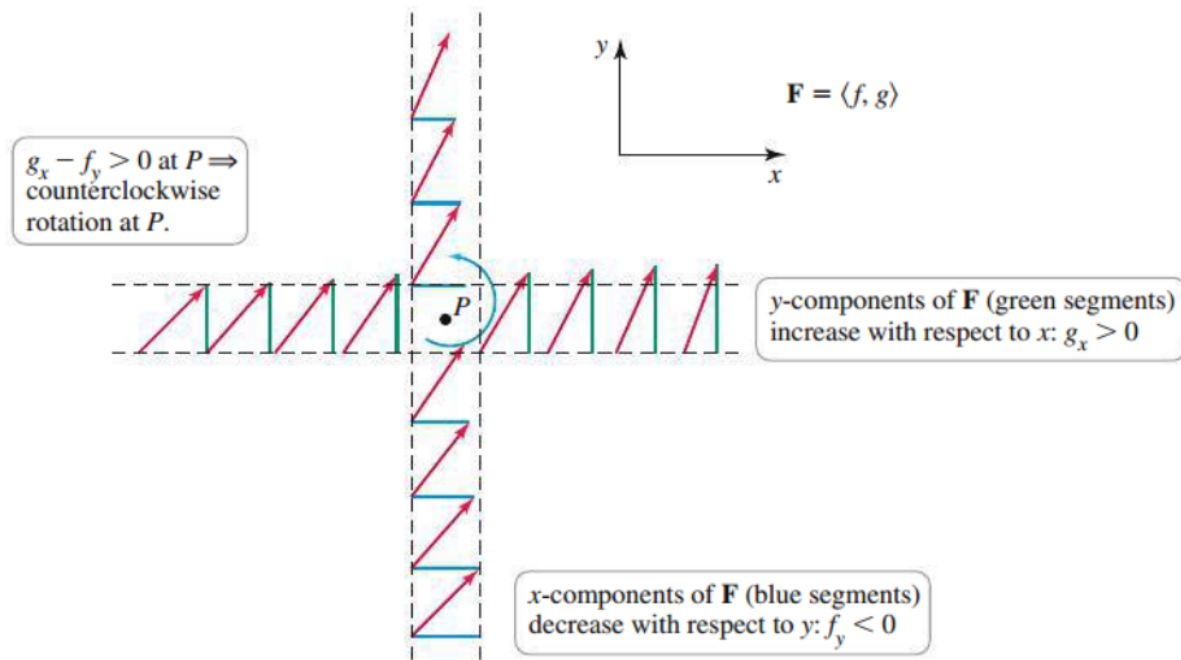


17.4: Green's Theorem

Green's Theorem — Circulation Form

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R . Then

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{circulation}} = \underbrace{\oint_C f dx + g dy}_{\text{circulation}} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA.$$

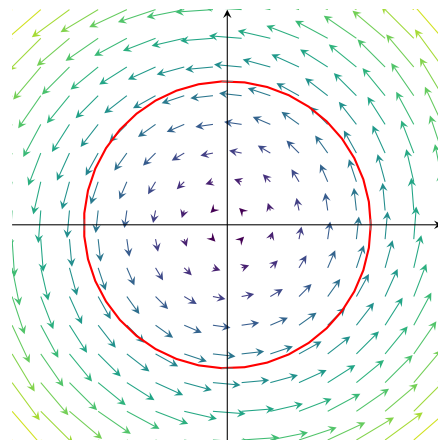


Definition. (Two-Dimensional Curl)

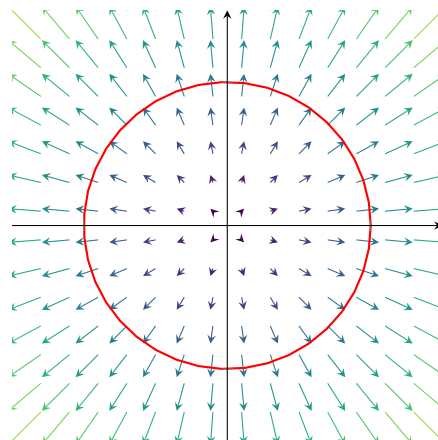
The **two-dimensional curl** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$. If the curl is zero throughout a region, the vector field is **irrotational** on the region.

Example. Consider the following vector fields \mathbf{F} over the region $R = \{(x, y) : x^2 + y^2 \leq 1\}$. Compute the circulation using Green's Theorem.

$$\mathbf{F} = \langle -y, x \rangle$$



$$\mathbf{F} = \langle x, y \rangle$$



Area of a Plane Region by Line Integrals

Under the conditions of Green's Theorem, the area of a region R enclosed by a curve C is

$$\oint_C x \, dy = - \oint_C y \, dx = \frac{1}{2} \oint_C (x \, dy - y \, dx).$$

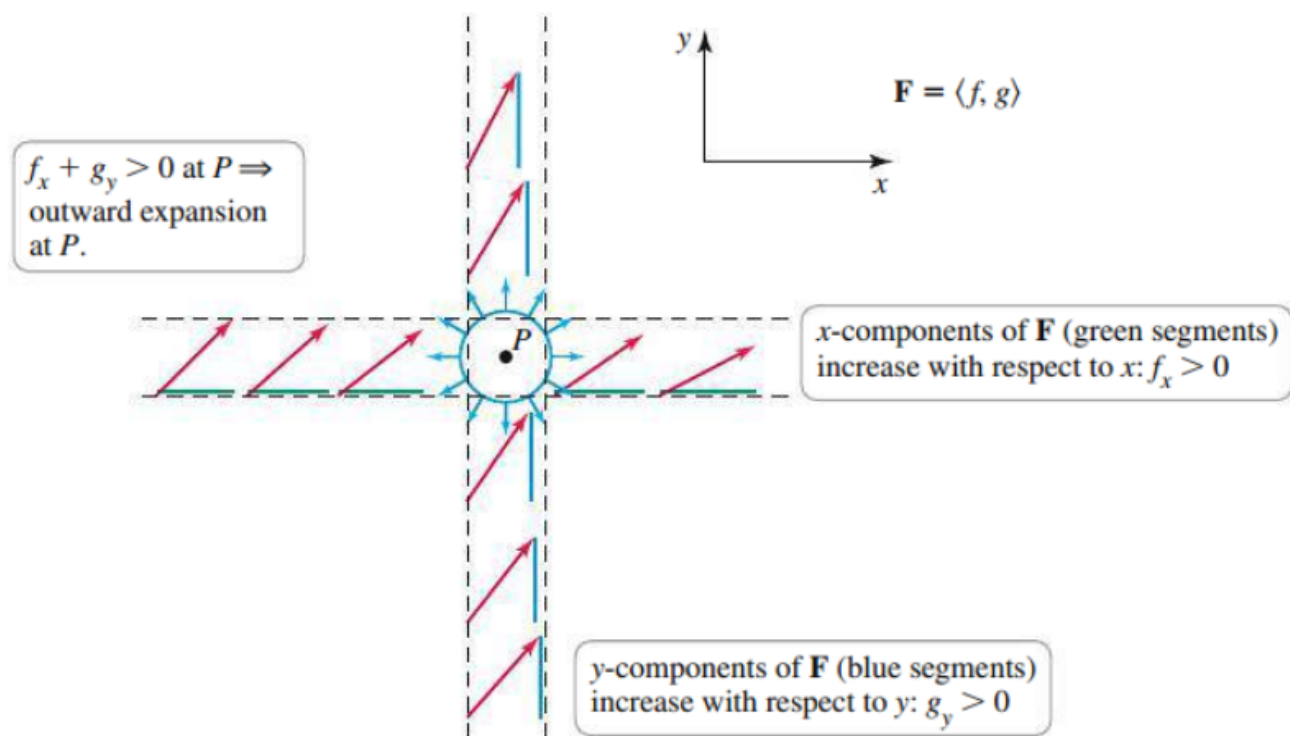
Example. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Green's Theorem — Flux Form

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R . Then

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n} ds}_{\text{outward flux}} = \underbrace{\oint_C f dy - g dx}_{\text{outward flux}} = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA,$$

where \mathbf{n} is the outward unit normal vector on the curve.



Definition. (Two-Dimensional Divergence)

The **two-dimensional divergence** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$. If the divergence is zero throughout a region, the vector field is **source free** on that region.

Example. Compute the curl of $\mathbf{F} = \langle x^2, 2y^2 \rangle$ where C is the upper half of the unit circle and the line segment $-1 \leq x \leq 1$.

Example. Integrate $\oint_C (2x + e^{y^2}) dy - (4y^2 + e^{x^2}) dx$, where C is the boundary of the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$.

Example. Compute the circulation and outward flux across the boundary of the given regions:

$\mathbf{F} = \langle x, y \rangle$; R is the half-annulus $\{(r, \theta) : 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$,

$\mathbf{F} = \langle -y, x \rangle$; R is the annulus $\{(r, \theta) : 1 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$.

Stream functions:

In the same way that a vector field is conservative if there exists a potential function φ , a vector field is source free if a **stream function** ψ exists such that

$$\frac{\partial \psi}{\partial y} = f, \quad \frac{\partial \psi}{\partial x} = -g.$$

If such a function exists, then the divergence is zero:

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = \underbrace{\frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial y} \left(-\frac{\partial \psi}{\partial x} \right)}_{\psi_{yx} = \psi_{xy}} = 0$$

If a vector field is both conservative and source-free, then it has both a potential function and a stream function. Furthermore, the level curves of the potential and stream functions form orthogonal families. These vector fields have zero divergence

$$0 = f_x + g_y = \varphi_{xx} + \varphi_{yy},$$

and zero curl

$$0 = g_x - f_y = -\psi_{xx} - \psi_{yy}.$$

Thus, conservative, source-free vector fields satisfy **Laplace's equation**:

$$\varphi_{xx} + \varphi_{yy} = 0 \quad \text{and} \quad \psi_{xx} + \psi_{yy} = 0.$$

Example. For $\mathbf{F} = \langle -e^{-x} \sin(y), e^{-x} \cos(y) \rangle$

Show \mathbf{F} is conservative and source-free field

Find the potential function φ and the stream function ψ

Conservative Fields $\mathbf{F} = \langle f, g \rangle$

$$\text{curl} = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0$$

Potential function φ with
 $\mathbf{F} = \nabla \varphi$ or $f = \frac{\partial \varphi}{\partial x}, \quad g = \frac{\partial \varphi}{\partial y}$

Circulation = $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all
closed curves C .

Evaluation of the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$$

Source-Free Fields $\mathbf{F} = \langle f, g \rangle$

$$\text{divergence} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0$$

Stream function ψ with
 $f = \frac{\partial \psi}{\partial y}, \quad g = -\frac{\partial \psi}{\partial x}$

Flux = $\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = 0$ on all closed
curves C .

Evaluation of the line integral

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \psi(B) - \psi(A)$$

Circulation/work integrals: $\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C f \, dx + g \, dy$

	C closed	C not closed
F conservative ($\mathbf{F} = \nabla\varphi$)	$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$	$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$
F not conservative	Green's Theorem $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (g_x - f_y) \, dA$	Direct evaluation $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b (fx' + gy') \, dt$

Flux integrals: $\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_C f \, dy - g \, dx$

	C closed	C not closed
F source free ($f = \psi_y, g = -\psi_x$)	$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = 0$	$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \psi(B) - \psi(A)$
F not source free	Green's Theorem $\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R (f_x + g_y) \, dA$	Direct evaluation $\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_a^b (fy' - gx') \, dt$

Example. Suppose C is a circle centered at the origin, oriented counter-clockwise, that encloses disk R in the plane. For $\mathbf{F} = \langle 4x^3y, xy^2 + x^4 \rangle$

a) Calculate the two-dimensional curl of \mathbf{F}

b) Calculate the two-dimensional divergence of \mathbf{F}

c) Is \mathbf{F} irrotational on R ?

d) Is \mathbf{F} source free on R ?

Proof.

