10.7: The Ratio and Root Tests

Theorem 10.20: Ratio Test

Let $\sum a_k$ be an infinite series, and let $\boxed{r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|}$

1. If $r < 1$, the series converges absolutely, and therefore it converges (by Theorem 10.19)
2. If $r > 1$ (including $r = \infty$), the series diverges.
3. If $r = 1$, the test is inconclusive.

Note: The ratio test is used to determine if a series converges or diverges and indicates nothing about the *value* of the series.

Example. Use the ratio test on the harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ and the alternating harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$.

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ $\quad r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{1/(k+1)}{1/k}\right| = \lim_{k\to\infty} \left|\frac{1}{k+1}\cdot\frac{k}{1}\right| = \lim_{k\to\infty}\left|\frac{k}{k+1}\right| = 1$

Diverges
Harmonic series
& p-series w/ $p = 1 \leq 1$

Inconclusive

$\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ $\quad r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{1/(k+1)}{1/k}\right| = \lim_{k\to\infty} \left|\frac{1}{k+1}\cdot\frac{k}{1}\right| = \lim_{k\to\infty}\left|\frac{k}{k+1}\right| = 1$

Converges
by AST

Inconclusive

$0! = 1$ $\qquad$ $n=3 \to \frac{(2n)!}{(2n-1)!} = \frac{6!}{5!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1} = 6$

Example. *Note:* $n! = n\cdot(n-1)\cdots3\cdot2\cdot1$

$(2n)! = 2n\cdot\underbrace{(2n-1)\cdot\ldots\cdot1}_{(2n-1)!}$

Rewrite $n!n! = (n!)^2 \neq (2n)!$ and $\frac{(2n)!}{(2n-1)!} = \frac{2n\cdot(2n-1)!}{(2n-1)!} = 2n$

$n\cdot(n-1)\cdot(n-2)\cdots1 \times n\cdot(n-1)\cdots1$ $\qquad$ $2n\cdot(2n-1)\cdot(2n-2)\cdots1$

Example. Consider the series below. Use the ratio test, if appropriate, to show if each of the series converges or diverges.

$$\sum_{k=1}^{\infty}\frac{k^2}{2^k} \qquad r = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)^2}{\underbrace{2^{k+1}}_{2^k\cdot 2}}\cdot\frac{2^k}{k^2}\right| = \lim_{k\to\infty}\left|\frac{\overbrace{(k+1)^2}^{k^2}}{2\,k^2}\right| = \frac{1}{2}$$

Since $r = \frac{1}{2} < 1$, the series $\underbrace{\text{converges absolutely}}_{\text{implies convergence}}$ by the Ratio Test

Ratio test helpful

$$\sum_{k=1}^{\infty}\frac{(-1)^k k}{k^3+1} \qquad r = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\cancel{(-1)^{k+1}}(k+1)}{(k+1)^3+1}\cdot\frac{k^3+1}{\cancel{(-1)^k}k}\right| = \lim_{k\to\infty}\left|\frac{\overbrace{(k+1)(k^3+1)}^{k^4}}{\underbrace{[(k+1)^3+1]k}_{k^4}}\right| = 1$$

The $\underbrace{\text{ratio test is inconclusive}}_{\text{not helpful}}$ since $r=1$

$$\boxed{a_k \xrightarrow{r} a_{k+1} \quad \text{Geometric Behavior}}$$

Divergence test is inconclusive $\lim_{k\to\infty}\frac{(-1)^k k}{k^3+1} = 0$

$\lim_{k\to\infty}\frac{k}{k^3+1} = 0$ $\qquad$ AST

① $\lim_{k\to\infty} a_k = 0$
② $0 < a_{k+1} \le a_k$
} Converges by AST

ew

~~$(k^3+1)(k+1) < k((k+1)^3+1)$~~

~~$\frac{k+1}{(k+1)^3+1} < \frac{k}{k^3+1}$~~ ← want

$f(x) = \frac{x}{x^3+1}$

$f'(x) = \frac{(x^3+1) - x(3x^2)}{(x^3+1)^2} = \frac{1-2x^3}{(x^3+1)^2} < 0$ when $1-2x^3 < 0$, $\sqrt[3]{\frac{1}{2}} < x$ $\qquad$ $k \ge 1$, this works positive a_k w/ $\lim_{k\to\infty} a_k = 0$ Does not imply non increasing

$$a_k = \frac{1-\sin(k)}{k} \geq 0, \text{ oscillating sequence}$$

$$\frac{k+x}{k} \xrightarrow{x=1} \frac{k+1}{k}$$

$$\sum_{k=1}^{\infty} \frac{5^k k!}{k^k} \qquad \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left| \frac{\overbrace{5^{k+1}}^{5^k 5}\overbrace{(k+1)!}^{(k+1)k!}}{\underbrace{(k+1)^{k+1}}_{(k+1)^k(k+1)}} \cdot \frac{k^k}{5^k k!} \right|$$

$$\lim_{k\to\infty} \left(1 + \frac{x}{k}\right)^k = e^x$$

L'Hopital's Rule

$$= \lim_{k\to\infty} \left| \frac{5\,(k+1)\,k^k}{(k+1)^k\,(k+1)} \right| = \lim_{k\to\infty} \left| 5 \left(\frac{k}{k+1}\right)^k \right|$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$= 5 \lim_{k\to\infty} \left| \left(\frac{k+1}{k}\right)^{-k} \right| = 5 \lim_{k\to\infty} \left| \left(1 + \frac{1}{k}\right)^{-k} \right| = 5 \cdot e^{-1} > 1$$

Diverges by ratio test

Ratio Test helpful

$$\sum_{k=1}^{\infty} \frac{(-7)^k}{(2k+1)!} \quad r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left| \frac{\overbrace{7^{k+1}}^{7^k \cdot 7}}{\underbrace{(2(k+1)+1)!}_{\substack{(2k+3)! \\ (2k+3)(2k+2)(2k+1)!}}} \cdot \frac{(2k+1)!}{7^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{7}{(2k+3)(2k+2)} \right| = 0 < 1$$

→ Converges by Ratio Test

Ratio Test helpful

$$\sum_{k=1}^{\infty} \frac{(-1)^k \ln(k)}{k} \quad r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left| \frac{\ln(k+1)}{k+1} \cdot \frac{k}{\ln(k)} \right| = \lim_{k\to\infty} \left| \frac{k}{k+1} \frac{\ln(k+1)}{\ln(k)} \right|$$

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$$

Ratio Test not helpful

Example. Use the ratio test to determine if the series $\sum_{k=1}^{\infty} \underbrace{k\left(\frac{2}{3}\right)^k}_{a_k}$ converges or diverges. LC #9

$$r = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)\left(\frac{2}{3}\right)^{k+1}}{k\left(\frac{2}{3}\right)^k}\right| \qquad \left(\frac{2}{3}\right)^k\left(\frac{2}{3}\right)$$

$$= \lim_{k\to\infty}\left|\frac{(k+1)\cancel{\left(\frac{2}{3}\right)^k}\left(\frac{2}{3}\right)}{k\cancel{\left(\frac{2}{3}\right)^k}}\right|$$

$$= \frac{2}{3}\underbrace{\lim_{k\to\infty}\frac{k+1}{k}}_{1} = \frac{2}{3}$$

$r = 2/3 < 1$ so the series converges (absolutely) by the ratio test

Example. Use the ratio test to determine if the series $\sum_{k=1}^{\infty} \frac{(-1)^k k}{(2k)!}$ converges or diverges.

$$r = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \left| \frac{k+1}{(2(k+1))!} \cdot \frac{(2k)!}{k} \right|$$

$(2k+2)!$

$$= \lim_{k \to \infty} \left| \underbrace{\frac{k+1}{k}}_{\to 1} \cdot \underbrace{\frac{\cancel{(2k)!}}{(2k+2)(2k+1)\cancel{(2k)!}}}_{\to 0} \right|$$

$$= \lim_{k \to \infty} \left| \frac{k+1}{k(2k+2)(2k+1)} \right|$$

$$= 0$$

Since $r = 0 < 1$, the series converges (absolutely) by the ratio test

Note: Recall series diverges when $r > 1$

Ratio test inconclusive when $r = 1$

Example. Use the ratio test to determine if the series $\displaystyle\sum_{k=1}^{\infty} \frac{(2k)!}{(k!)^2}$ converges or diverges.

$$r = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(2(k+1))!}{((k+1)!)^2} \cdot \frac{(k!)^2}{(2k)!} \right|$$

$$= \lim_{k\to\infty} \left| \left(\frac{k!}{(k+1)!} \right)^2 \frac{(2k+2)(2k+1)(2k)!}{(2k)!} \right|$$

$$\left(\frac{1}{k+1}\right)^2$$

$$= \lim_{k\to\infty} \left| \left(\frac{1}{k+1}\right)^2 (2k+2)(2k+1) \right|$$

$$= \lim_{k\to\infty} \left| \frac{(2k+2)(2k+1)}{(k+1)\ (k+1)} \right| = 4$$

$r = 4 > 1$, so the series diverges by the ratio test

What is $\sqrt[k]{|k!|}$? !! Not a good candidate for the root test.

10.21: Root Test
Let $\sum a_k$ be an infinite series, and let $\rho = \lim\limits_{k\to\infty} \sqrt[k]{|a_k|}$.

1. If $\rho < 1$, the series converges absolutely, and therefore it converges (by Theorem 10.19)
2. If $\rho > 1$ (including $\rho = \infty$), the series diverges.
3. If $\rho = 1$, the test is inconclusive.

Note: The root test is used to determine if a series converges or diverges and indicates nothing about the *value* of the series.

Example. Use the root test to determine if the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}k^k}{3^{k^2}}$ converges.

$$\rho = \lim_{k\to\infty} \underbrace{\sqrt[k]{|a_k|}}_{|a_k|^{1/k}} = \lim_{k\to\infty} \left|\frac{k^k}{3^{k^2}}\right|^{1/k}$$

$$= \lim_{k\to\infty} \left|\frac{k^{k/k}}{3^{k^2/k}}\right|$$

$k << 3^k$

$$= \lim_{k\to\infty} \left|\frac{k}{3^k}\right| = 0$$

Since $\rho = 0 < 1$, the series converges (absolutely) by the root test.

Example. Consider the series below. Use the root test to show if each of the series converges or diverges.

$$\sum_{k=1}^{\infty}\left(\frac{1}{\ln(k+1)}\right)^k$$

$\rho = \lim_{k\to\infty} \left| \left(\frac{1}{\ln(k+1)}\right)^k \right|^{1/k} = \lim_{k\to\infty} \frac{1}{\ln(k+1)} = 0 < 1$

$\boxed{\text{Converges}}$

$k\to\infty \Rightarrow k+1\to\infty$

$\Rightarrow \ln(k+1)\to\infty$

$$\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{3k^2+1}{k-2k^2}\right)^k$$

$\rho = \lim_{k\to\infty} \left| \left(\frac{3k^2+1}{k-2k^2}\right)^k \right|^{1/k} = \lim_{k\to\infty} \left| \frac{3k^2+1}{k-2k^2} \right| = \frac{3}{2} > 1$

$|(-1)^{k+1}| = 1$

$\boxed{\text{Diverges}}$

$$\sum_{k=1}^{\infty}\left(\frac{k+3}{k+1}\right)^{2k}$$

$\rho = \lim_{k\to\infty} \left| \left(\frac{k+3}{k+1}\right)^{2k} \right|^{1/k} = \lim_{k\to\infty} \left(\frac{k+3}{k+1}\right)^2 = 1 = 1$

$\boxed{\text{Inconclusive}}$

True or False: If $\lim_{k\to\infty}|a_k|^{1/k} = \frac{1}{4}$, then $\sum_{k=1}^{\infty} 10a_k$ converges absolutely.

$\hookrightarrow 10 \underbrace{\sum_{k=1}^{\infty} a_k}$

$\lim_{k\to\infty} |10 a_k|^{1/k} = \lim_{k\to\infty} 10^{1/k} |a_k|^{1/k}$

Converges

$= \lim_{k\to\infty} 10^0 |a_k|^{1/k}$

$= \lim_{k\to\infty} |a_k|^{1/k}$

$$\lim_{k\to\infty}\left(1+\frac{x}{k}\right)^k = e^x$$

Example. Use the root test to determine if the series $\displaystyle\sum_{k=1}^{\infty}\left(1-\frac{3}{k}\right)^{k^2}$ converges.

$$\rho = \lim_{k\to\infty}\left|\left(1-\frac{3}{k}\right)^{k^2}\right|^{1/k}$$

$$= \lim_{k\to\infty}\left(1-\frac{3}{k}\right)^{k} = e^{-3} < 1$$

$\Rightarrow$ converges (absolutely) by the root test

$\cancel{\rho = \lim_{k\to\infty} \left| k^{-1/3} \right|^{1/k}}$ root test

$\boxed{\text{Inconclusive}}$

Example. Determine whether each of the series below converges conditionally, converges absolutely, or diverges.

$$\sum_{k=1}^{\infty} (-1)^k k^{-1/3}$$

$r = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(k+1)^{-1/3}}{k^{-1/3}} \right| = \lim_{k\to\infty} \left| \left(\frac{k+1}{k}\right)^{-1/3} \right| = 1$!!

AST ① Non increasing: $k < k+1$, $k^{1/3} < (k+1)^{1/3}$, $(k+1)^{-1/3} < k^{-1/3}$

② $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k^{1/3}} = 0$

Converges

$\sum_{k=1}^{\infty} \left|(-1)^k k^{-1/3}\right| = \sum_{k=1}^{\infty} \frac{1}{k^{1/3}} \Leftarrow$ diverges b/c p-series w/ $p = 1/3 \le 1$

$\boxed{\text{Conditionally convergent}}$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\arctan(k)}$$

$\lim_{k\to\infty} \frac{1}{\arctan(k)} = \frac{1}{\pi/2} = \frac{2}{\pi} \neq 0$

$\Rightarrow$ diverges by $\boxed{\text{divergence test}}$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k!}$$

$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{k!}{(k+1)!} \right| = \lim_{k\to\infty} \frac{1}{k+1} = 0$

$\Rightarrow$ Converges absolutely by ratio test

$$\lim_{k\to\infty}\left(1+\frac{x}{k}\right)^k = e^x$$

Example. Determine if the series $\displaystyle\sum_{k=1}^{\infty}\left(\frac{k}{k+5}\right)^{3k^2}$ converges.

Root test:

$$\rho = \lim_{k\to\infty}|a_k|^{1/k} = \lim_{k\to\infty}\left|\left(\frac{k}{k+5}\right)^{3k^2}\right|^{1/k}$$

$$= \lim_{k\to\infty}\left(\frac{k}{k+5}\right)^{3k}$$

$$= \lim_{k\to\infty}\left(\frac{k+5}{k}\right)^{-3k}$$

$$= \lim_{k\to\infty}\left[\left(1+\frac{5}{k}\right)^k\right]^{-3} = \left(e^5\right)^{-3} = e^{-15}$$

Since $\rho = e^{-15} < 1$, the series converges (absolutely) by the root test

Example. Determine a condition for $x \geq 0$ such that $\displaystyle\sum_{k=1}^{\infty} \frac{4x^k}{5k^2}$ converges.

Ratio Test

$$r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left| \frac{4x^{k+1}}{5(k+1)^2} \cdot \frac{5(k)^2}{4x^k} \right|$$

$$= \lim_{k\to\infty} \left| x \left(\frac{k}{k+1}\right)^2 \right| = |x|$$

Note that $r = |x|$ converges for $|x| < 1$.

When $r = |x| = 1$, the ratio test is inconclusive. Thus, we consider these cases separately:

$x = 1 \longrightarrow \displaystyle\sum_{k=1}^{\infty} \frac{4}{5k^2} = \frac{4}{5}\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series w/ $p=2$.

$x = -1 \longrightarrow \displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{4}{5k^2} = \frac{4}{5}\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$

1. $0 < a_{k+1} \leq a_k$ Since $k < k+1 \Rightarrow k^2 < (k+1)^2 \Rightarrow \frac{1}{(k+1)^2} < \frac{1}{k^2}$

2. $\displaystyle\lim_{k\to\infty} \frac{1}{k^2} = 0$

$\Rightarrow$ Converges by AST

So $\displaystyle\sum_{k=1}^{\infty} \frac{4x^k}{5k^2}$ converges for $-1 \leq x \leq 1$