

1 14.2: Calculus of Vector-Valued Functions

Definition. (Derivative and Tangent Vector)

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g , and h are differentiable functions on (a, b) . Then \mathbf{r} has a **derivative** (or is **differentiable**) on (a, b) and

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

Provided $\mathbf{r}'(t) \neq \mathbf{0}$, $\mathbf{r}'(t)$ is a **tangent vector** at the point corresponding to $\mathbf{r}(t)$.

Definition. (Unit Tangent Vector)

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a smooth parameterized curve, for $a \leq t \leq b$. The **unit tangent vector** for a particular value of t is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Derivative Rules

Let \mathbf{u} and \mathbf{v} be differentiable vector-valued functions, and let f be a differentiable scalar-valued function, all at a point t . Let \mathbf{c} be a constant vector. The following rules apply.

1. $\frac{d}{dt}(\mathbf{c}) = \mathbf{0}$ Constant Rule
2. $\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$ Sum Rule
3. $\frac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ Product Rule
4. $\frac{d}{dt}(\mathbf{u}(f(t))) = \mathbf{u}'(f(t))f'(t)$ Chain Rule
5. $\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ Dot Product Rule
6. $\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ Cross Product Rule

Definition. (Indefinite Integral of a Vector-Valued Function)

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function, and let

$\mathbf{R}(t) = F(t)\mathbf{i} + G(t)\mathbf{j} + H(t)\mathbf{k}$, where F , G , and H are antiderivatives of f , g , and h , respectively. The **indefinite integral** of \mathbf{r} is

$$\int \mathbf{r}(t) dt = \mathbf{R}(t) + \mathbf{C},$$

where \mathbf{C} is an arbitrary constant vector. Alternatively, in component form,

$$\int \langle f(t), g(t), h(t) \rangle dt = \langle F(t), G(t), H(t) \rangle + \langle C_1, C_2, C_3 \rangle.$$

Definition. (Definite Integral of a Vector-Valued Function)

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f , g , and h are integrable on the interval $[a, b]$. The **definite integral** of \mathbf{r} on $[a, b]$ is

$$\int_a^b \mathbf{r}(t) dt = \left(\int_a^b f(t) dt \right) \mathbf{i} + \left(\int_a^b g(t) dt \right) \mathbf{j} + \left(\int_a^b h(t) dt \right) \mathbf{k}$$