5.4: Working with Integrals

Theorem 5.4: Integrals of Even and Odd Functions
Let a be a positive real number and let f be an integrable function on the interval $[-a, a]$.

- If f is even, $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
- If f is odd, $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

Example. Rewrite the following trig functions to determine if it is even or odd:

$\sin(-x) = -\sin(x)$ $\qquad$ $\cos(-x) = \cos(x)$

$\tan(-x) = -\tan(x)$ $\qquad$ $\cot(-x) = -\cot(x)$

$\csc(-x) = -\csc(x)$ $\qquad$ $\sec(-x) = \sec(x)$

Use this to rewrite and evaluate the following integrals:

$$\int_{-\pi}^{\pi} \sin(x)\,dx = 0$$

$$\int_{-\pi}^{\pi} \cos(x)\,dx = 2\int_{0}^{\pi} \cos(x)\,dx$$

$$= 2\sin(x)\Big|_{0}^{\pi}$$

$$= 2[0 - 0] = 0$$

$$\int_{-\pi/4}^{\pi/4} \tan(x)\,dx = 0$$

$$\int_{-\pi/4}^{\pi/4} \sec(x)\,dx$$

$$= 2\int_{0}^{\pi/4} \sec(x)\,dx$$

Note: $\cot(x)$ and $\csc(x)$ are excluded here as they are not continuous on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

Note: $\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$

Example. Use symmetry to evaluate the following integrals:

a) $\displaystyle\int_{-10}^{10} \frac{x}{\sqrt{200-x^2}}\,dx$ $= 0$

$f(x) = \dfrac{x}{\sqrt{200-x^2}}$

$f(-x) = \dfrac{-x}{\sqrt{200-(-x)^2}} = \dfrac{-x}{\sqrt{200-x^2}} = -f(x) \to$ odd

b) $\displaystyle\int_{-2}^{2} (x^9 - 3x^5 + 2x^2 - 10)\,dx$

$= \displaystyle\int_{-2}^{2} x^9\,dx - 3\int_{-2}^{2} x^5\,dx + 2\int_{-2}^{2} x^2\,dx - 10\int_{-2}^{2} dx$

$= 0 + 0 + 4\displaystyle\int_0^2 x^2\,dx - 20\int_0^2 dx$

$= \dfrac{4}{3}x^3\Big|_0^2 - 20x\Big|_0^2$

$= \dfrac{32}{3} - 40$

$= \boxed{\dfrac{-88}{3}}$

c) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^5(x)\,dx = 0$

$f(x) = \sin^5(x)$

$f(-x) = (\sin(-x))^5 = (-\sin(x))^5 = -\sin^5(x) = -f(x)$

$\to$ odd

d) $\displaystyle\int_{-1}^{1} (1-|x|)\,dx = 2\int_0^1 1-x\,dx = 2\left(1-\frac{x^2}{2}\right)\Big|_0^1 = 2\left(\frac{1}{2}-0\right) = 1$

$f(x) = 1-|x|$

$f(-x) = 1-|-x| = 1-|x| = f(x) \to$ even

e) $\displaystyle\int_{-2}^{2} \frac{x^3-4x}{x^2+1}\,dx = 0$

$f(x) = \dfrac{x^3-4x}{x^2+1}$

$f(-x) = \dfrac{(-x)^3-4(-x)}{(-x)^2+1} = \dfrac{-(x^3-4x)}{x^2+1} = -f(x) \to$ odd

Example. Given that $f(x)$ is even and $\int_{-8}^{8} f(x)\,dx = 18$, find

a) $\displaystyle\int_0^8 f(x)\,dx = \frac{1}{2}\int_{-8}^{8} f(x)\,dx$

$= \boxed{9}$

b) $\displaystyle\int_{-8}^{8} x f(x)\,dx = \boxed{0}$

$g(x) = x\,f(x)$

$g(-x) = (-x)\,f(-x)$ f even

$= -x\,f(x)$

$= -g(x) \longrightarrow$ odd

Example. Given that $f(x)$ is odd and $\int_0^4 f(x)\,dx = 3$ and $\displaystyle\int_0^8 f(x)\,dx = 9$, find

a) $\displaystyle\int_{-4}^{8} f(x)\,dx$

$\displaystyle= \int_{-4}^{4} f(x)\,dx + \int_0^8 f(x)\,dx - \int_0^4 f(x)\,dx$

$= 0 + 9 - 3$

$= \boxed{6}$

b) $\displaystyle\int_{-8}^{4} f(x)\,dx = \int_{-8}^{0} f(x)\,dx + \int_0^4 f(x)\,dx$

$= -9 + 3$

$= \boxed{-6}$

Example. Use symmetry to explain why

$$\int_{-4}^{4} \left(5x^4 + 3x^3 + 2x^2 + x + 1\right) dx = 2\int_{0}^{4} \left(5x^4 + 2x^2 + 1\right) dx$$

$$\int_{-4}^{4} (5x^4 + 3x^3 + 2x^2 + x + 1)\, dx = \int_{-4}^{4} (3x^3 + x)\, dx + \int_{-4}^{4} (5x^4 + 2x^2 + 1)\, dx$$

$$= 2\int_{0}^{4} (5x^4 + 2x^2 + 1)\, dx$$

Example. Evaluate

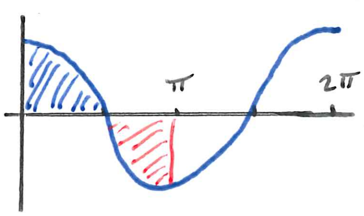
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos(2\theta) + \cos(\theta)\sin(\theta) - 3\sin(\theta^5)\right) d\theta$$

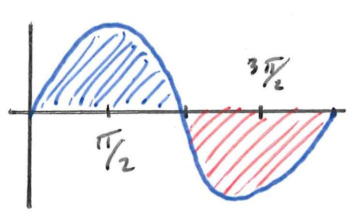
$$= 2\int_{0}^{\pi/2} \cos(2\theta)\, d\theta + \int_{-\pi/2}^{\pi/2} \left(\cos(\theta)\sin(\theta) - 3\sin(\theta^5)\right) d\theta$$

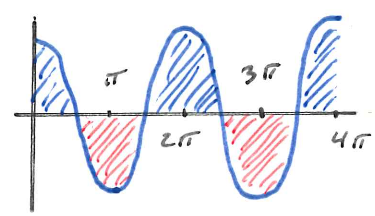
$$= \sin(2\theta)\Big|_0^{\pi/2} = [0 - 0] = \boxed{0}$$

Example. While the following integrals are not on symmetric intervals, symmetry still applies here:

a) $\displaystyle\int_0^{\pi} \cos(x)\, dx$ b) $\displaystyle\int_0^{2\pi} \sin(x)\, dx$ c) $\displaystyle\int_0^{4\pi} \cos(x)\, dx$







Definition. (Average Value of a Function)
The average value of an integrable function f on the interval $[a, b]$ is

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Example. Find the average value of $f(x) = -\frac{x^2}{2}$ on $[0, 3]$.

$$\bar{f} = \frac{1}{3-0} \int_0^3 -\frac{x^2}{2}\, dx = \frac{1}{3}\left(-\frac{x^3}{6}\right)\Big|_0^3 = \frac{1}{3}\left[-\frac{27}{6} + 0\right]$$

$$= \boxed{-\frac{3}{2}}$$

Example. Find the average value of $f(x) = 3x^2 - 3$ on $[0, 1]$.

$$\bar{f} = \frac{1}{1-0} \int_0^1 (3x^2 - 3)\, dx = x^3 - 3x\Big|_0^1 = [-2] - [0] = \boxed{-2}$$

Example. Find the average value of $f(t) = t^2 - t$ on $[-2, 1]$.

$$\bar{f} = \frac{1}{1-(-2)} \int_{-2}^1 (t^2 - t)\, dt = \frac{1}{3}\left(\frac{t^3}{3} - \frac{t^2}{2}\right)\Big|_{-2}^1 = \frac{1}{3}\left(\left[\frac{1}{3} - \frac{1}{2}\right] - \left[\frac{-8}{3} - \frac{4}{2}\right]\right)$$

$$= \frac{1}{3}\left(\frac{9}{3} + \frac{3}{2}\right)$$

$$= \boxed{\frac{3}{2}}$$

Example. Find the average value of $f(x) = \dfrac{1}{x^2+1}$ on $[-1, 1]$.

$$\bar{f} = \frac{1}{1-(-1)} \int_{-1}^{1} \frac{1}{x^2+1}\,dx = \frac{1}{2} \cdot 2 \int_0^1 \frac{1}{x^2+1} = \tan^{-1}(x)\Big|_0^1$$

$$= \boxed{\pi/4}$$

Example. Find the average value of $f(x) = \dfrac{1}{x}$ on $[1, e]$.

$$\bar{f} = \frac{1}{e-1} \int_1^e \frac{1}{x}\,dx = \frac{1}{e-1}\left(\ln|x|\right)\Big|_1^e = \frac{\ln(e)-\ln(1)}{e-1}$$

$$= \boxed{\frac{1}{e-1}}$$

Example. Find the average value of $f(x) = x^{\frac{1}{n}}$ on $[0, 1]$.

$$\bar{f} = \frac{1}{1-0} \int_0^1 x^{1/n}\,dx = \frac{n}{n+1}\, x^{\frac{n+1}{n}}\Big|_0^1 = \frac{n}{n+1}$$

Example. The velocity in m/s of an object moving along a line over the time interval $[0, 6]$ is $v(t) = t^2 + 3t$. Find the average velocity of the object over this time interval.

$$\bar{v} = \frac{1}{6-0}\int_0^6 (t^2+3t)\,dt = \frac{1}{6}\left(\frac{t^3}{3} + \frac{3}{2}t^2\right)\Big|_0^6$$

$$= \frac{1}{6}\left(\frac{6^3}{3} + \frac{3}{2}\cdot 6^2\right)$$

$$= \frac{36}{3} + \frac{18}{2} = \boxed{21\ m/s}$$

Example. A rock is launched vertically upward from the ground with a speed of $64 ft/s$. The height of the rock (in ft) above the ground after t seconds is given by the function $s(t) = -16t^2 + 64t$. Find its average velocity during its flight.

1. Find interval of flight:

Solve $s(t) = 0$

$-16t^2 + 64t = 0$

$-16t(t-4) = 0$

$t = 0,\ t = 4$

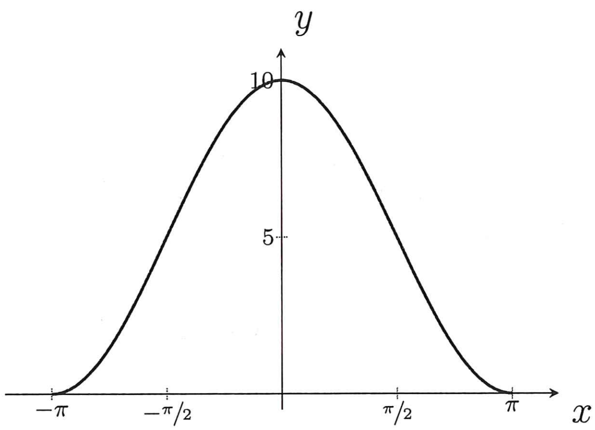
2. Use $\int v(t)\,dt = s(t)$ to find average velocity

$$\bar{v} = \frac{1}{4-0}\int_0^4 v(t)\,dt$$

$$= \frac{1}{4}\left(s(4) - s(0)\right)$$

$$= \boxed{0\ ft/s}$$

Example. The surface of a water wave is described by $y = 5(1+\cos(x))$, for $-\pi \le x \le \pi$, where $y = 0$ corresponds to a trough of the wave. Find the average height of the wave above the trough on $[-\pi, \pi]$.



$$\bar{h} = \frac{1}{\pi-(-\pi)}\int_{-\pi}^{\pi} \overbrace{5(1+\cos(x))}^{\text{even}}\,dx$$

$$= \frac{2\cdot 5}{2\pi}\int_{0}^{\pi} 1+\cos(x)\,dx$$

$$= \frac{5}{\pi}\Big[x+\sin(x)\Big]_0^{\pi}$$

$$= \frac{5}{\pi}\Big(\big[\pi+0\big]-\big[0+0\big]\Big)$$

$$= \boxed{5}$$

Theorem 5.5: Mean Value Theorem for Integrals
Let f be continuous on the interval $[a,b]$. There exists a point c in (a,b) such that

$$f(c) = \bar{f} = \frac{1}{b-a}\int_a^b f(t)\,dt$$

Example. For the following problems, find the point(s) that satisfy the Mean Value Theorem for Integrals.

a) $f(x) = \dfrac{1}{x^2}$ on $[1,4]$.

① $\bar{f} = \frac{1}{4-1}\int_1^4 x^{-2}\,dx = \frac{1}{3}\left(-x^{-1}\right)\Big|_1^4 = \frac{1}{3}\left(-\frac{1}{4}-(-1)\right)$
$= \frac{1}{3}\left(\frac{3}{4}\right) = \boxed{\frac{1}{4}}$

② solve $f(x) = \bar{f}$
$\frac{1}{x^2} = \frac{1}{4} \Rightarrow \boxed{x = 2}$

b) $f(x) = e^x$ on $[0,2]$.

① $\bar{f} = \frac{1}{2}\int_0^2 e^x\,dx = \frac{1}{2}\left(e^x\right)\Big|_0^2 = \frac{1}{2}(e^2-1)$

② solve $f(x) = \bar{f} \rightarrow e^x = \frac{1}{2}(e^2-1) \Rightarrow \boxed{x = \ln\left(\frac{e^2-1}{2}\right)} \approx 1.1614$

c) $f(x) = \cos(x)$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

cos(x) even

① $\bar{f} = \frac{1}{\frac{\pi}{2}-(-\frac{\pi}{2})}\int_{-\pi/2}^{\pi/2}\cos(x)\,dx = \frac{2}{\pi}\sin(x)\Big|_0^{\pi/2} = \boxed{\frac{2}{\pi}}$

② Solve $f(x) = \bar{f} \rightarrow \boxed{x = \cos^{-1}(2/\pi)} \approx 0.8807$

d) $f(x) = 1-|x|$ on $[-1,1]$.

① $\bar{f} = \frac{1}{1-(-1)}\int_{-1}^{1} 1-|x|\,dx = \frac{2}{2}\left(x - \frac{x^2}{2}\right)\Big|_0^1 = \boxed{\frac{1}{2}}$

② solve $f(x) = \bar{f} \rightarrow 1-|x| = \frac{1}{2}$
$|x| = \frac{1}{2} \rightarrow \boxed{x = \pm 1/2}$