17.5: Divergence and Curl

The idea behind Green's Theorem can be extended from $\mathbb{R}^2$ to $\mathbb{R}^3$. The following tools are needed to accomplish this:

- three-dimensional divergence and curl (17.5)
- *surface integrals* (17.6)
- *Stokes' Theorem* (17.7): relates line integrals over a simple closed oriented curve in $\mathbb{R}^3$ to a double integral over a surface whose boundary is that curve
- *Divergence Theorem* (17.8): relates integrals over a closed oriented surface in $\mathbb{R}^3$ to triple integrals over the corresponding region

Divergence:

Recall the *del operator* ∇:

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$$

When f is a scalar valued function, we obtain the gradient:

$f(x,y,z)$

$$\nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k} = \langle f_x, f_y, f_z \rangle$$

The dot product of ∇ and a vector field $\mathbf{F}$, produces the three dimensional divergence:

$$\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f,\, g,\, h \rangle = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

Definition. (Divergence of a Vector Field)
The **divergence** of a vector field $\mathbf{F} = \langle f,\, g,\, h \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

If $\nabla \cdot \mathbf{F} = 0$, the vector field is **source free**.

Example. Compute the divergence of the following vector fields

$\mathbf{F} = \langle x, -2y, 3z \rangle$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(3z)$$

$$= 1 - 2 + 3 = \boxed{2}$$

LC #1

$\langle -y, x \rangle$

(rotational field)

$\mathbf{F} = \langle -y, x - z, y \rangle$

$$\nabla \cdot \vec{F} = 0 + 0 + 0 = \boxed{0}$$

LC #2

$\mathbf{F} = \langle 4yz\cos(x),\ 3xz\tan(y),\ -5xy\csc(z) \rangle$

$$\nabla \cdot \vec{F} = -4yz\sin(x) + 3xz\sec^2(y) + 5xy\csc(z)\cot(z)$$

Example. Compute the divergence of the radial vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\langle x,\, y,\, z\rangle}{\sqrt{x^2+y^2+z^2}}.$$

$$\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{1/2}}\right) = \frac{(x^2+y^2+z^2)^{1/2}(1) - x\,\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x)}{x^2+y^2+z^2}$$

$$= \frac{|\vec{r}| - x^2\,|\vec{r}|^{-1}}{|\vec{r}|^2}\left(\frac{|\vec{r}|}{|\vec{r}|}\right)$$

$$= \frac{|\vec{r}|^2 - x^2}{|\vec{r}|^3}$$

$$\Rightarrow \nabla\cdot\vec{F} = \frac{|\vec{r}|^2 - x^2}{|\vec{r}|^3} + \frac{|\vec{r}|^2 - y^2}{|\vec{r}|^3} + \frac{|\vec{r}|^2 - z^2}{|\vec{r}|^3}$$

$$= \frac{3|\vec{r}|^2 - (x^2+y^2+z^2)}{|\vec{r}|^3}$$

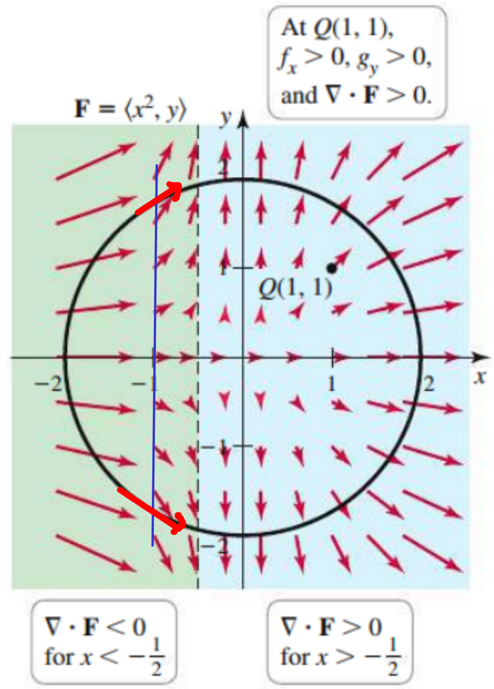
$$= \frac{3|\vec{r}|^2 - |\vec{r}|^2}{|\vec{r}|^3} = \frac{2}{|\vec{r}|}$$

LC # 3
$a = 2$

Theorem 17.10: Divergence of Radial Vector Fields
For a real number p, the divergence of the radial vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x,\, y,\, z\rangle}{(x^2+y^2+z^2)^{p/2}} \quad \text{is} \quad \nabla\cdot\mathbf{F} = \frac{3-p}{|\mathbf{r}|^p}.$$

Example. Consider the two-dimensional vector field $\mathbf{F} = \langle x^2, y \rangle$ and a circle C of radius 2 centered at the origin.



Compute the two-dimensional divergence at Q.

$$\operatorname{div} F = \nabla \cdot \vec{F} = 2x + 1$$

$$\nabla \cdot \vec{F}\Big|_{(1,1)} = 3$$

Where is the divergence positive? Negative?

Solve $\nabla \cdot \vec{F} = 2x+1 > 0$ $\quad x > -1/2$

$\nabla \cdot \vec{F} < 0$ $\quad x < -1/2$

Where on C is the flux outward? Inward?

$$\left.\begin{array}{ll} \text{outward} & x > -1 \\ \text{inward} & x < -1 \end{array}\right\} \text{approx}$$

Is the net flux across C positive or negative?

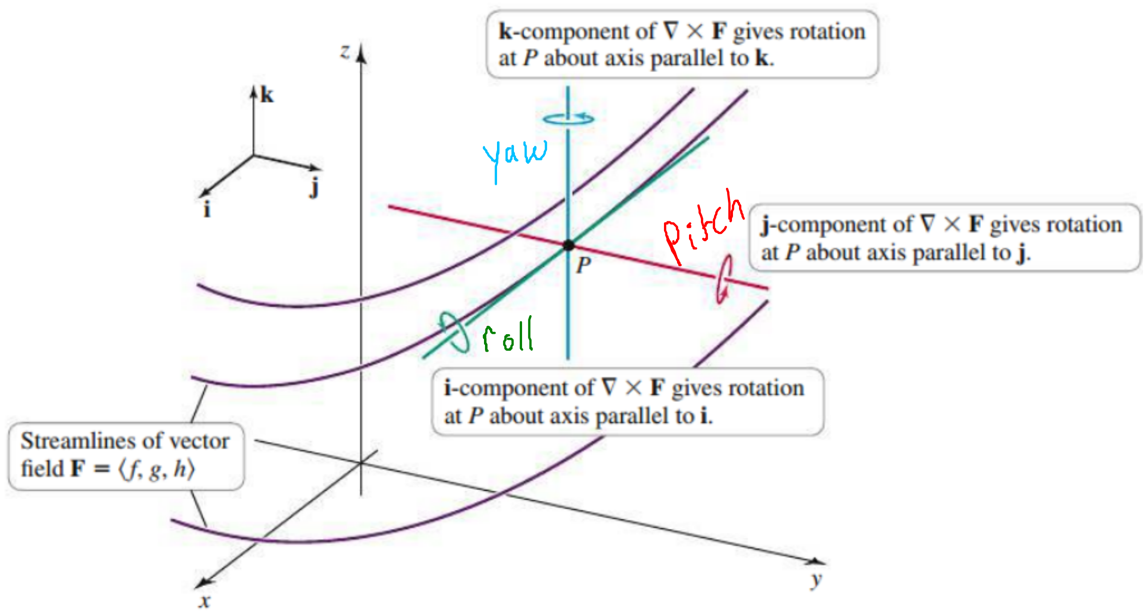
looks positive

$$\text{flux} = \int_C \underbrace{f\,dy - g\,dx}_{\vec{F}\cdot\vec{n}\,ds} = \iint_R \underbrace{(f_x + g_y)}_{\operatorname{div}\vec{F} = \nabla\cdot\vec{F}}\,dA = \int_0^{2\pi}\int_0^2 (2r\cos\theta + 1)\, r\,dr\,d\theta$$

$$= \boxed{4\pi}$$

Curl:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle$$



> **Definition. (Curl of a Vector Field)**
> The **curl** of a vector field $\mathbf{F} = \langle f, g, h \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is
>
> $$\begin{aligned} \nabla \times \mathbf{F} &= \operatorname{curl} \mathbf{F} \\ &= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\boldsymbol{k} \end{aligned}$$
>
> If $\nabla \times \mathbf{F} = \mathbf{0}$, the vector field is **irrotational**.

2D curl

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \bullet & \bullet & \bullet \\ F_1 & F_2 & 0 \end{vmatrix}$$

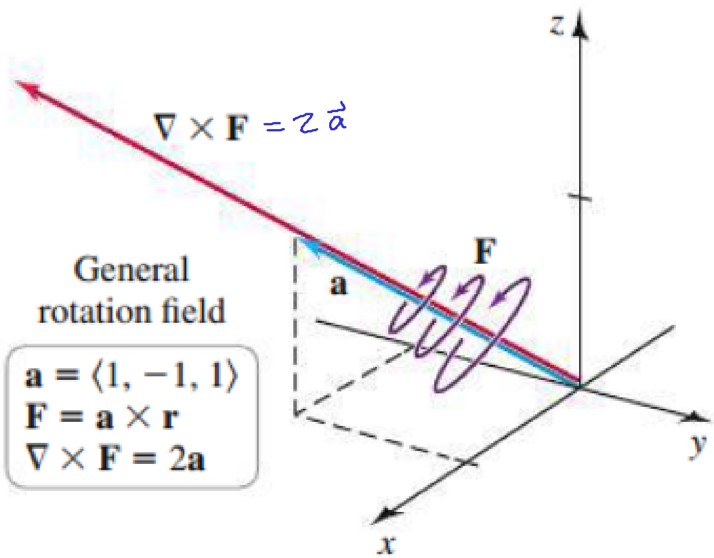
Curl of a General Rotation Vector Field

Let $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$. Then

$\langle x, y, z \rangle$

$$\nabla \cdot \mathbf{F} = 0 \qquad\qquad \nabla \times \mathbf{F} = 2\mathbf{a}$$

divergence curl



$\nabla \times \vec{F} = 2\vec{a}$ $\qquad$ $|\nabla \times \vec{F}| = 2|\vec{a}|$

General Rotation Vector Field

The **general rotation vector field** is $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, when the nonzero constant vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is the axis of rotation and $\mathbf{r} = \langle x, y, z \rangle$. For all nonzero choices of a, $|\nabla \times \mathbf{F}| = 2|\mathbf{a}|$ and $\nabla \cdot \mathbf{F} = 0$. If $\mathbf{F}$ is a vector field, then the constant angular speed of the field is

$$\omega = |\mathbf{a}| = \frac{1}{2}|\nabla \times \mathbf{F}|.$$

Example. Compute the curl of the rotational field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a} = \langle -3,\, 2,\, 1\rangle$ and $\mathbf{r} = \langle x,\, y,\, z\rangle$. What are the direction and magnitude of the curl?

$$\vec{F} = \vec{a} \times \vec{r} = \langle \underline{a_2 z - a_3 y},\ \underline{a_3 x - a_1 z},\ \underline{a_1 y - a_2 x} \rangle$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$\nabla \times \vec{F} = \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z},\ \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x},\ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle$$

$$= \langle a_1 + a_1,\ a_2 + a_2,\ a_3 + a_3 \rangle$$

$$= \langle 2a_1,\ 2a_2,\ 2a_3 \rangle$$

$$= 2\vec{a}$$

LC #4

$a = 14$

$$= \langle -6, 4, 2 \rangle \longleftarrow \text{Direction}$$

$$|\nabla \times \vec{F}| = |2\vec{a}| = 2\sqrt{9+4+1} = 2\sqrt{14} \leftarrow \text{magnitude}$$

Properties of Divergence and Curl:

Divergence Properties	**Curl Properties**
$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$	$\nabla \times (\mathbf{F} + \mathbf{G}) = (\nabla \times \mathbf{F}) + (\nabla \times \mathbf{G})$
$\nabla \cdot (c\mathbf{F}) = c(\nabla \cdot \mathbf{F})$	$\nabla \times (c\mathbf{F}) = c(\nabla \times \mathbf{F})$

Theorem 17.11: Curl of a Conservative Vector Field
Suppose $\mathbf{F}$ is a conservative vector field on an open region D of $\mathbb{R}^3$. Let $\mathbf{F} = \nabla\varphi$, where φ is a potential function with continuous second partial derivatives on D. Then $\nabla \times \mathbf{F} = \nabla \times \nabla_{\varphi} = \mathbf{0}$: The curl of the gradient is the zero vector and $\mathbf{F}$ is irrotational.

Proof.

$$\nabla \times \mathbf{F} = \nabla \times \nabla\varphi = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \varphi_x & \varphi_y & \varphi_z \end{vmatrix} = \langle \varphi_{zy} - \varphi_{yz},\, \varphi_{xz} - \varphi_{zx},\, \varphi_{yx} - \varphi_{xy} \rangle = \mathbf{0}$$

$\square$

Theorem 17.12: Divergence of the Curl
Suppose $\mathbf{F} = \langle f,\, g,\, h \rangle$, where f, g, and h have continuous second partial derivatives. Then $\nabla \cdot (\nabla \times \mathbf{F}) = 0$: The divergence of the curl is zero.

Proof.

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{F}) &= \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \\ &= (h_{yx} - h_{xy}) + (g_{xz} - g_{zx}) + (f_{zy} - f_{yz}) = 0 \end{aligned}$$

$\square$

The **Laplacian**, denoted $\nabla^2 u$ or Δu, arises from $\nabla \cdot \nabla u$:

$$\nabla \cdot \nabla u = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial}{\partial z}\frac{\partial u}{\partial z} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Theorem 17.13: Product Rule for the Divergence
Let u be a scalar-valued function that is differentiable on a region D and let $\mathbf{F}$ be a vector field that is differentiable on D. Then

$$\nabla \cdot (u\mathbf{F}) = \underbrace{\overbrace{\nabla u}^{\text{scalar}}}_{\text{vector}} \cdot \mathbf{F} + \underbrace{u}_{\text{scalar}}(\overbrace{\nabla \cdot \mathbf{F}}^{\text{scalar}}).$$

Example. Let $\mathbf{r} = \langle x, y, z\rangle$ and let $\varphi = \frac{1}{|\mathbf{r}|}$ be a potential function.

Find the associated gradient field $\mathbf{F} = \nabla\left(\frac{1}{|\mathbf{r}|}\right) = \nabla\left((x^2+y^2+z^2)^{-1/2}\right)$

$$\frac{\partial \varphi}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x) = \frac{-x}{(x^2+y^2+z^2)^{3/2}} = \frac{-x}{|\vec{r}|^3}$$

$$\vec{F} = \frac{-\langle x, y, z\rangle}{|\vec{r}|^3} = \frac{-\vec{r}}{|\vec{r}|^3}$$

Compute $\nabla \cdot \mathbf{F}$

$$\nabla \cdot \vec{F} = \nabla \cdot \left(\frac{-\vec{r}}{|\vec{r}|^3}\right) = -\nabla \frac{1}{|\vec{r}|^3} \cdot \vec{r} - \frac{1}{|\vec{r}|^3}\underbrace{(\nabla \cdot \vec{r})}_{3}$$

$$= \frac{3\vec{r}}{|\vec{r}|^5} \cdot \vec{r} - \frac{3}{|\vec{r}|^3}$$

$$\frac{\vec{r}\cdot\vec{r}}{|\vec{r}|^5} = \frac{|\vec{r}|^2}{|\vec{r}|^5} = \frac{1}{|\vec{r}|^3}$$

$$= \frac{3}{|\vec{r}|^3} - \frac{3}{|\vec{r}|^3} = 0$$

$$\boxed{\begin{matrix}\text{LC \#5} \\ 0\end{matrix}}$$

Properties of a Conservative Vector Field

Let $\mathbf{F}$ be a conservative vector field whose components have continuous second partial derivatives on an open connected region D in $\mathbb{R}^3$. Then $\mathbf{F}$ has the following equivalent properties.

1. There exists a potential function φ such that $\mathbf{F} = \nabla_{\varphi}$ (definition).
2. $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$ for all points A and B in D and all piecewise smooth oriented curves C in D from A to B.
3. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple piecewise-smooth closed oriented curves C in D.
4. $\nabla \times \mathbf{F} = \mathbf{0}$ at all points of D.