Math 1080 Class notes
Fall 2021

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5.5: Substitution Rule

Theorem 5.6: Substitution Rule for Indefinite Integrals
Let $u = g(x)$, where g is differentiable on an interval, and let f be continuous on the corresponding range of g. On that interval,

$$\int f\big(g(x)\big)g'(x)\,dx = \int f(u)\,du$$

Example. We know

$$\frac{d}{dx}\left[\frac{(2x+1)^4}{4}\right] = 2(2x+1)^3$$

Thus, if $f(x) = x^3$ and $g(x) = 2x+1$ then $g'(x) = 2$, so we let $u = 2x+1$, then

$$\begin{aligned}\int 2(2x+1)^3\,dx &= \int f\big(g(x)\big)g'(x)\,dx \\ &= \int u^3\,du \\ &= \frac{u^4}{4} + C \\ &= \frac{(2x+1)^4}{4} + C\end{aligned}$$

Procedure: Substitution Rule (Change of Variables)

1. Given an indefinite integral involving a composite function $f\big(g(x)\big)$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)\,dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Write the result in terms of x using $u = g(x)$.

Example. Evaluate the following integrals:

a) $\int 2x(x^2+3)^4\,dx$

b) $\int (2x+1)^3\,dx$

c) $\int x^2\sqrt{x^3+1}\,dx$

d) $\int \theta\sqrt[4]{1-\theta^2}\,d\theta$

e) $\int \sqrt{4-t}\,dt$

f) $\int (2-x)^6\,dx$

Example. Evaluate the following integrals:

a) $\displaystyle\int \sec(2\theta)\tan(2\theta)\,d\theta$

b) $\displaystyle\int \csc^2\left(\frac{t}{3}\right) dt$

c) $\displaystyle\int \frac{\sin(x)}{1+\cos^2(x)}\,dx$

d) $\displaystyle\int \frac{\tan^{-1}(x)}{1+x^2}\,dx$

The acceleration of a particle moving back and forth on a line is $a(t) = \frac{d^2s}{dt^2} = \pi^2\cos(\pi t)\ m/s^2$ for all t. If $s = 0$ and $v = 8\ m/s$ when $t = 0$, find the value of s when $t = 1$ sec.

Example. Evaluate the following integrals:

a) $\displaystyle\int (6x^2 + 2)\sin(x^3 + x + 1)\,dx$

b) $\displaystyle\int \frac{\sin(\theta)}{\cos^5(\theta)}\,d\theta$

c) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

d) $\displaystyle\int \frac{2^t}{2^t + 3}\,dt$

e) $\int 6x^2 4^{x^3}\,dx$

f) $\int \frac{dx}{\sqrt{36-4x^2}}$

g) $\int \sin(t)\sec^2(\cos(t))\,dt$

h) $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

i) $\displaystyle\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

j) $\displaystyle\int 5\cos(7x+5)\,dx$

k) $\displaystyle\int \frac{3}{\sqrt{1-25x^2}}\,dx$

l) $\displaystyle\int \frac{dx}{\sqrt{1-9x^2}}$

Example. Evaluate the following integrals using the recommended substitution:

a) $\displaystyle\int \sec^2(x)\tan(x)\,dx$
where $u = \tan(x)$.

b) $\displaystyle\int \sec^2(x)\tan(x)\,dx$
where $u = \sec(x)$.

Example. Solve the initial value problem: $\frac{dy}{dx} = 4x(x^2+8)^{-1/3}, y(0) = 0.$

Example. Evaluate the following integrals:

a) $\displaystyle\int xe^{-x^2}\,dx$

b) $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$

c) $\displaystyle\int \frac{dt}{8-3t}$

d) $\displaystyle\int 5^t \sin(5^t)\,dt$

e) $\displaystyle\int \frac{e^w}{36+e^{2w}}\,dw$

Theorem 5.7: Substitution Rule for Definite Integrals
Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g. Then

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Example. Evaluate the integrals:

a) $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$

b) $\displaystyle\int_1^3 \frac{dt}{(t-4)^2}$

c) $\displaystyle\int_0^3 \frac{v^2+1}{\sqrt{v^3+3v+4}}\,dv$

d) $\displaystyle\int_0^1 2x\big(4-x^2\big)\,dx$

e) $\displaystyle\int_2^3 \frac{x}{\sqrt[3]{x^2-1}}\,dx$

f) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{1+\cos(x)}\,dx$

g) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos^2(x)}\,dx$

h) $\displaystyle\int_{-\frac{\pi}{12}}^{\frac{\pi}{8}} \sec^2(2y)\,dy$

i) $\displaystyle\int_0^1 (1-2x^9)\,dx$

j) $\displaystyle\int_0^1 (1-2x)^9\,dx$

k) $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{1+4x^2}\,dx$

l) $\displaystyle\int_0^4 \frac{x}{x^2+1}\,dx$

m) $\displaystyle\int_0^{\pi} 3\cos^2(x)\sin(x)\,dx$

n) $\displaystyle\int_0^{\frac{\pi}{8}} \sec(2\theta)\tan(2\theta)\,d\theta$

o) $\displaystyle\int_0^{1} (3t-1)^{50}\,dt$

p) $\displaystyle\int_0^{3} \frac{1}{5x+1}\,dx$

q) $\displaystyle\int_0^1 xe^{-x^2}\,dx$

r) $\displaystyle\int_e^{e^4} \frac{1}{x\sqrt{\ln(x)}}\,dx$

s) $\displaystyle\int_0^{\frac{1}{2}} \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx$

t) $\displaystyle\int_0^1 \frac{e^z+1}{e^z+z}\,dz$

u) $\displaystyle\int_1^4 \frac{dy}{2\sqrt{y}\left(1+\sqrt{y}\right)^2}$

v) $\displaystyle\int_{\ln\left(\frac{\pi}{4}\right)}^{\ln\left(\frac{\pi}{2}\right)} e^w \cos(e^w)\, dw$

w) $\displaystyle\int_0^{\frac{1}{8}} \frac{x}{\sqrt{1-16x^2}}\, dx$

x) $\displaystyle\int_1^{e^2} \frac{\ln(p)}{p}\, dp$

y) $\displaystyle\int_0^{\frac{\pi}{4}} e^{\sin^2(x)}\sin(2x)\,dx$

z) $\displaystyle\int_{-\pi}^{\pi} x^2\sin(7x^3)\,dx$

Example. Average velocity: An object moves in one dimension with a velocity in m/s given by $v(t) = 8\sin(\pi t) + 2t$. Find its average velocity over the time interval from $t = 0$ to $t = 10$, where t is measured in seconds.

Example. Prove $\int \tan(x)\, dx = \ln|\sec(x)| + C$.

Example. Evaluate the integrals:

a) $\displaystyle\int \frac{x}{(x-2)^3}\, dx$

b) $\displaystyle\int x\sqrt{x-1}\, dx$

c) $\int x^3(1+x^2)^{\frac{3}{2}}\,dx$

d) $\int \frac{y^2}{(y+1)^4}\,dy$

e) $\int (z+1)\sqrt{3z+2}\,dz$

f) $\int_0^1 \frac{x}{(x+2)^3}\,dx$

Half-Angle Formulas

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$
$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$$

Example. Evaluate the integrals:

a) $\displaystyle\int \cos^2(x)\,dx$

b) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2(x)\,dx$

c) $\displaystyle\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx$

d) $\displaystyle\int x\sin^2(x^2)\,dx$

e) $\displaystyle\int \sin^2\left(\theta+\frac{\pi}{6}\right)d\theta$

f) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2(8\theta)\,d\theta$

Example. If f is continuous and $\int_0^4 f(x)\,dx = 10$, find $\int_0^2 f(2x)\,dx$.

Example. If f is continuous and $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 xf(x^2)\,dx$.

Example. Suppose f is an even function with $\int_0^8 f(x)\,dx = 9$. Evaluate the following:

a) $\int_{-1}^1 xf(x^2)\,dx$.

b) $\int_{-2}^2 x^2 f(x^3)\,dx$.

Example. Evaluate the integrals:

a) $\displaystyle\int \sec^2(10x)\,dx$

b) $\displaystyle\int \tan^{10}(4x)\sec^2(4x)\,dx$

c) $\displaystyle\int \left(x^{\frac{3}{2}}+8\right)^5 \sqrt{x}\,dx$

d) $\displaystyle\int \frac{2x}{\sqrt{3x+2}}\,dx$

e) $\displaystyle\int \frac{7x^2+2x}{x}\,dx$

f) $\displaystyle\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

g) $\displaystyle\int_0^{\sqrt{3}} \frac{3}{9+x^2}\,dx$

h) $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\sin(2y)}{\sin^2(y)+2}\,dy$

i) $\displaystyle\int \frac{\sec(z)\tan(z)}{\sqrt{\sec(z)}}\,dz$

j) $\displaystyle\int \frac{1}{\sin^{-1}(x)\sqrt{1-x^2}}\,dx$

k) $\displaystyle\int \frac{x}{\sqrt{4-9x^2}}\,dx$

l) $\displaystyle\int \frac{x}{1+x^4}\,dx$

m) $\displaystyle\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta$

n) $\displaystyle\int x^2\sqrt{2+x}\,dx$

o) $\displaystyle\int \big(\sin^5(x) + 3\sin^3(x) - \sin(x)\big)\cos(x)\,dx$

p) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(x^3 + x^4 \tan(x)\right) dx$

q) $\displaystyle\int_{0}^{\frac{\pi}{2}} \cos(x) \sin\left(\sin(x)\right) dx$

r) $\displaystyle\int \frac{1+x}{1+x^2}\, dx$

Example. Evaluate these more challenging integrals:

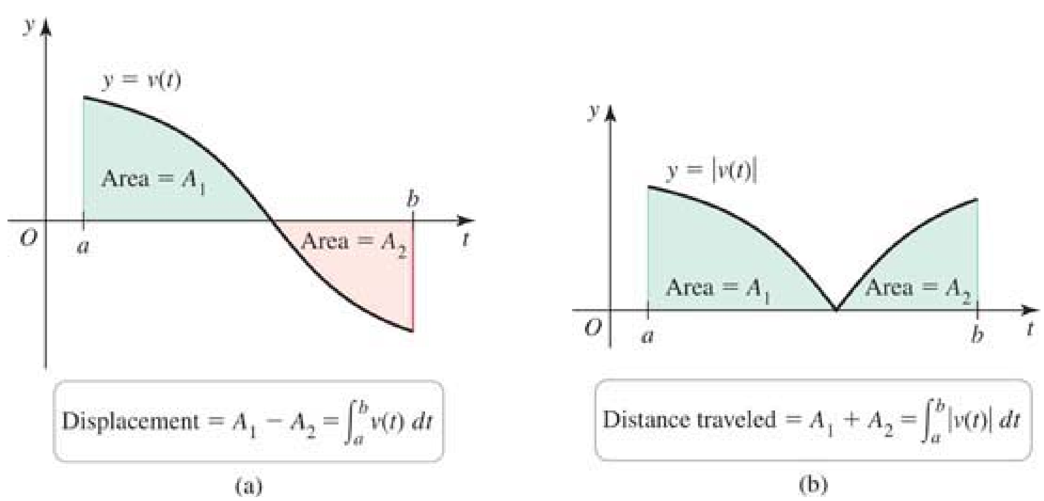
a) $\displaystyle\int \frac{dx}{\sqrt{1+\sqrt{1+x}}}$

b) $\int x \sin^4(x^2) \cos(x^2)\, dx$

6.1: Velocity and Net Change

Definition. (Position, Velocity, Displacement, and Distance)

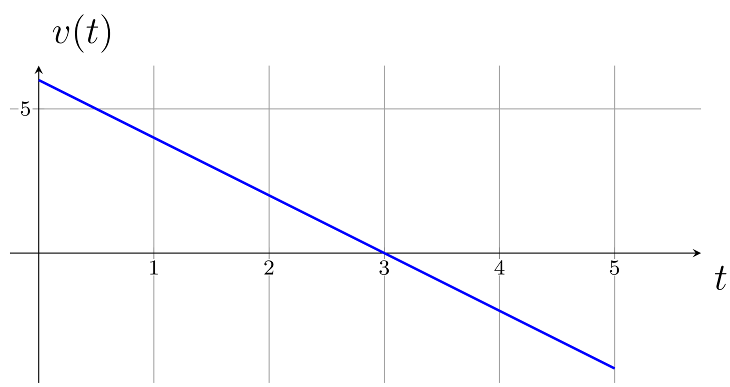
1. The **position** of an object moving along a line at time t, denoted $s(t)$, is the location of the object relative to the origin.
2. The **velocity** of an object at time t is $v(t) = s'(t)$.
3. The **displacement** of the object between $t = a$ and $t = b > a$ is
$$s(b) - s(a) = \int_a^b v(t)\, dt.$$
4. The **distance traveled** by the object between $t = a$ and $t = b > a$ is
$$\int_a^b |v(t)|\, dt$$
where $|v(t)|$ is the **speed** of the object at time t.



Example. Suppose an object moves along a line with velocity (in ft/s) $v(t) = 6 - 2t$, for $0 \le t \le 5$, where t is measured in seconds.

- Find the displacement of the object on the interval $0 \le t \le 5$.

- Find the distance traveled by the object on the interval $0 \le t \le 5$.



Example. A cyclist rides down a long straight road at a velocity (in m/min) given by $v(t) = 400 - 20t$, for $0 \leq t \leq 10$.

- How far does the cyclists travel in the first 5 minutes?

- How far does the cyclists travel in the first 10 minutes?

- How far has the cyclist traveled when her velocity is 250 m/min?

Example. The population of a community of foxes is observed to fluctuate on a 10-year cycle due to variations in the availability of prey. When population measurements began ($t = 0$), the population was 35 foxes. The growth rate in units of foxes/year was observed to be:

$$P'(t) = 5 + 10\sin\left(\frac{\pi t}{5}\right)$$

- Find $P(t)$.

- Find the population of foxes after the first 5 years, rounded to the nearest whole number of foxes.

Theorem 6.1: Position from Velocity
Given the velocity $v(t)$ of an object moving along a line and its initial position $s(0)$, the position function of the object for future times $t \geq 0$ is

$$\underbrace{s(t)}_{\substack{\text{position}\\ \text{at } t}} = \underbrace{s(0)}_{\substack{\text{initial}\\ \text{position}}} + \underbrace{\int_0^t v(x)\,dx}_{\substack{\text{displacement}\\ \text{over } [0,t]}}.$$

Theorem 6.2: Velocity from Acceleration
Given the acceleration $a(t)$ of an object moving along a line and its initial velocity $v(0)$, the velocity of the object for future times $t \geq 0$ is

$$v(t) = v(0) + \int_0^t a(x)\,dx.$$

Example. At $t = 0$, a train approaching a station begins decelerating from a speed of 80 miles/hour according to the acceleration function $a(t) = -1280(1 + 8t)^{-3}$, where $t \geq 0$ is measured in hours. The units of acceleration are mi/hr^2.

- Find the velocity of the train at $t = 0.25$.

- How far does the train travel in the first 15 minutes (1/4 hour)?

- How long does it take the train to travel 9 miles?

Theorem 6.3: Net Change and Future Value
Suppose a quantity Q changes over time at a known rate Q'. Then the **net change** in Q between $t = a$ and $t = b > a$ is

$$\underbrace{Q(b) - Q(a)}_{\text{net change in } Q} = \int_a^b Q'(t)\,dt.$$

Given the initial value $Q(0)$, the **future value** of Q at time $t \geq 0$ is

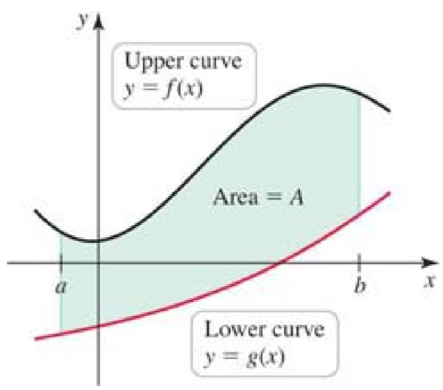
$$Q(t) = Q(0) + \int_0^t Q'(x)\,dx.$$

Velocity-Displacement Problems	**General Problems**
Position $s(t)$	Quantity $Q(t)$ (such as volume or population)
Velocity: $s'(t) = v(t)$	Rate of change: $Q'(t)$
Displacement: $s(b) - s(a) = \int_a^b v(t)\,dt$	Net change: $Q(b) - Q(a) = \int_a^b Q'(t)\,dt$
Future position: $s(t) = s(0) + \int_0^t v(x)\,dx$	Future value of Q: $Q(t) = Q(0) + \int_0^t Q'(x)\,dx$

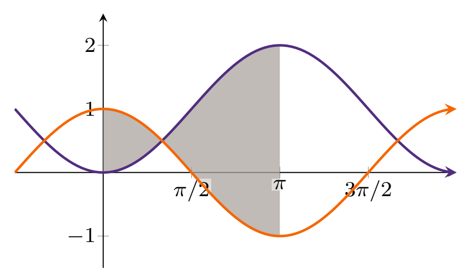
6.2: Regions Between Curves

Definition. (Area of a Region Between Two Curves)
Suppose f and g are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$. The area of the region bounded by the graphs of f and g on $[a, b]$ is

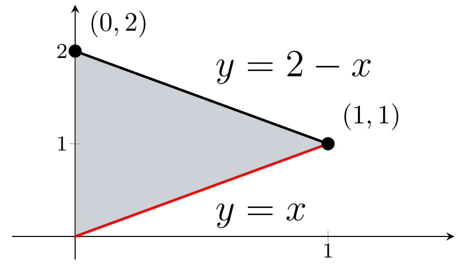
$$A = \int_a^b (f(x) - g(x))\,dx.$$



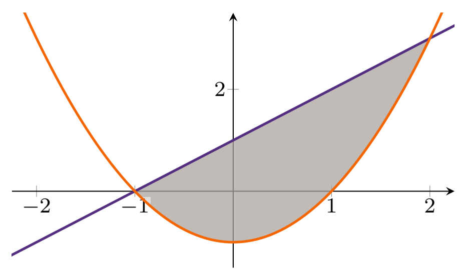
Example. Consider the region bounded by the curves $y = \cos(x)$ and $y = 1 - \cos(x)$, $0 \leq x \leq \pi$. Set up the integral(s) representing the area of this region.



Example. Find the area of the region by integrating with respect to x.



Example. Find the volume of the solid whose base is bounded by the graphs of $y = x+1$ and $y = x^2 - 1$, with the cross sections in the shape of rectangles of height 2 taken perpendicular to the x-axis.



Definition. (Area of a Region Between Two Curves with Respect to y) Suppose f and g are continuous functions with $f(y) \geq g(y)$ on the interval $[c, d]$. The area of the region bounded by the graphs $x = f(y)$ and $x = g(y)$ on $[c, d]$ is

$$A = \int_c^d (f(y) - g(y))\, dy.$$

Example. Find the area of the region bounded by $x = 3y$, and $x = y^2 - 10$

by integrating with respect to x

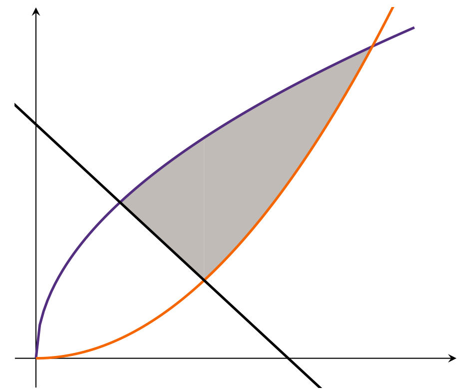
by integrating with respect to y

Example. Find the area of the region bounded by $y = x^3$, and $y = \sqrt{x}$

by integrating with respect to x

by integrating with respect to y

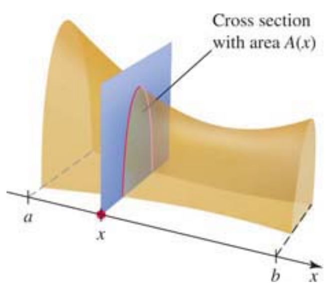
Example. Find the area of the region bounded by $y = 4\sqrt{2x}$, $y = 2x^2$, and $y = -4x + 6$

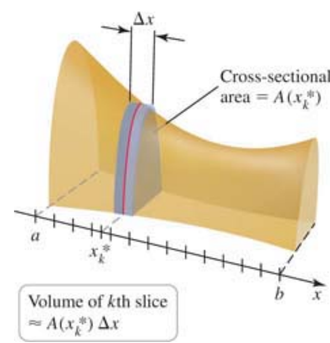


6.3: Volume by Slicing

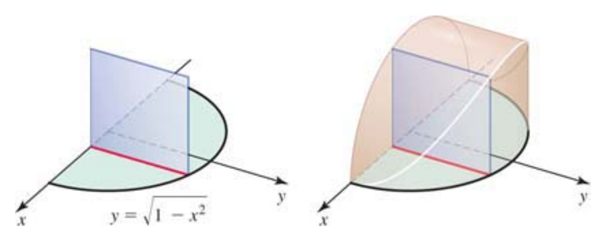
General Slicing Method
Suppose a solid object extends from $x = a$ to $y = b$, and the cross section of the solid perpendicular to the x-axis has an area given by a function A that is integrable on $[a, b]$. The volume of the solid is

$$V = \int_a^b A(x)\,dx.$$





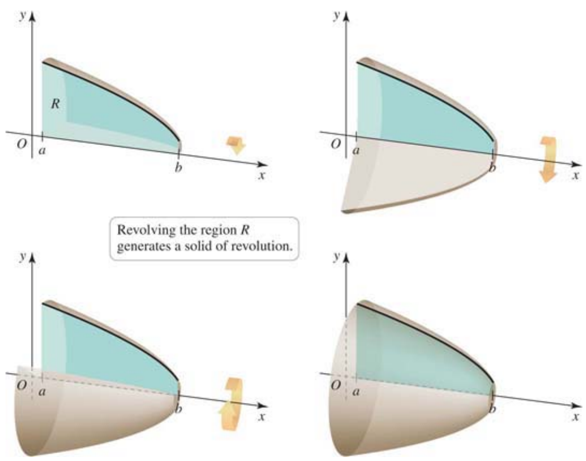
Example. Use the general slicing method to find the volume of the solid whose base is the region bounded by the semicircle $y = \sqrt{1 - x^2}$ and the x-axis, and whose cross sections through the solid perpendicular to the x-axis are squares.

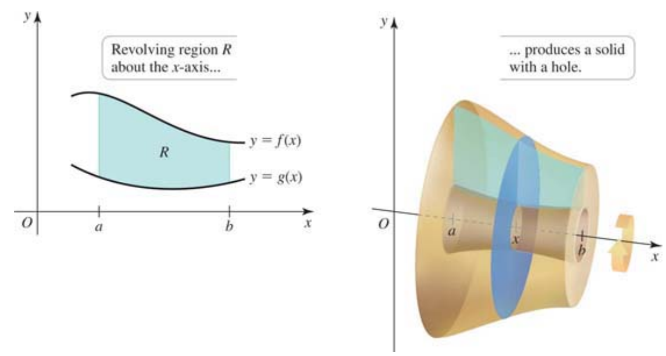


Disk Method about the x-Axis

Let f be continuous with $f(x) \geq 0$ on the interval $[a, b]$. If the region R bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi \underbrace{f(x)}_{\substack{\text{disk} \\ \text{radius}}}{}^2 dx.$$



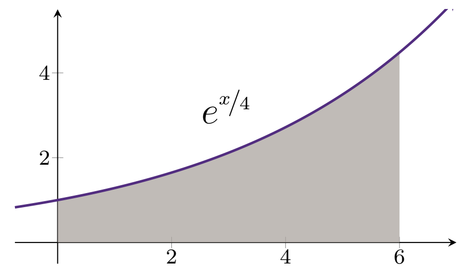


Washer Method about the x-Axis

Let f and g be continuous functions with $f(x) \geq g(x) \geq 0$ on $[a, b]$. Let R be the region bounded by $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$. When R is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi(\underbrace{f(x)}_{\substack{\text{outer} \\ \text{radius}}}{}^2 - \underbrace{g(x)}_{\substack{\text{inner} \\ \text{radius}}}{}^2)\, dx.$$

Example. Consider the region bounded by $y = e^{x/4}$, $y = 0$, $x = 0$, and $x = 6$. Find the volume of the solid generated by rotating the region about the x-axis.



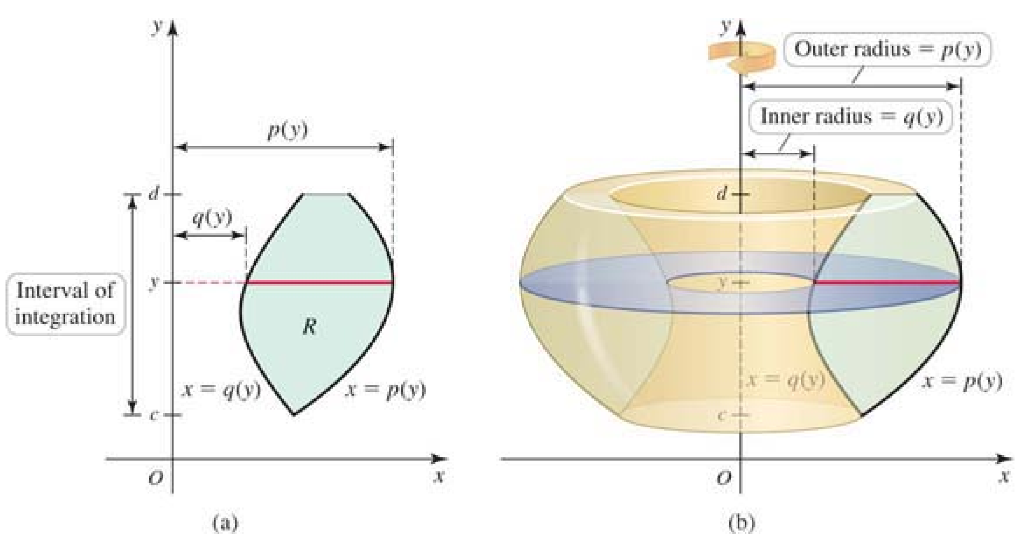
Disk and Washer Methods about the y-Axis

Let p and q be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let R be the region bounded by $x = p(y)$, $x = q(y)$, and the lines $y = c$ and $y = d$. When R is revolved around the y-axis, the volume of the resulting solid of revolution is given by

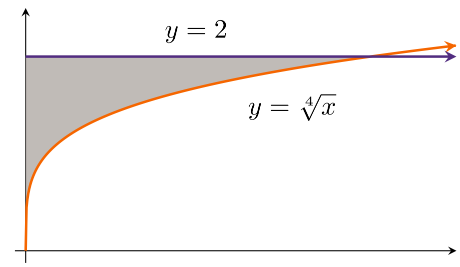
$$V = \int_c^d \pi(\underbrace{p(y)}_{\substack{\text{outer}\\\text{radius}}}{}^2 - \underbrace{q(y)}_{\substack{\text{inner}\\\text{radius}}}{}^2\ )\,dy.$$

If $q(y) = 0$, the disk method results:

$$V = \int_c^d \pi\, \underbrace{p(y)}_{\substack{\text{disk}\\\text{radius}}}{}^2 dy.$$



Example. Consider the region bounded between $y = \sqrt[4]{x}$, $y = 2$, and $x = 0$.

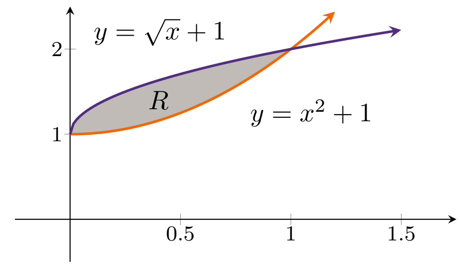


Setup the integral with respect to x that gives the area of the region.

Setup the integral with respect to y that gives the area of the region.

Use the disk/washer method to setup the that represents the volume of the solid generated by rotating the region about the x-axis.

Example. Consider the region R between $y = \sqrt{x}+1$ and $y = x^2+1$. Setup the integrals which find the volume of the solid obtained by rotating the region R as indicated below.



about the y-axis

about the x-axis

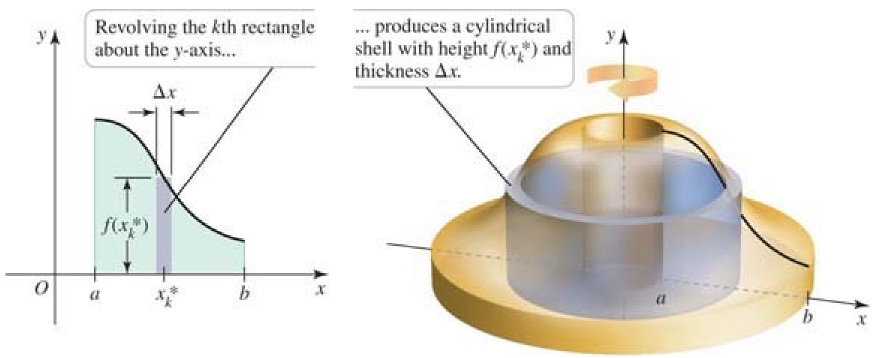
about the line $x = 1$

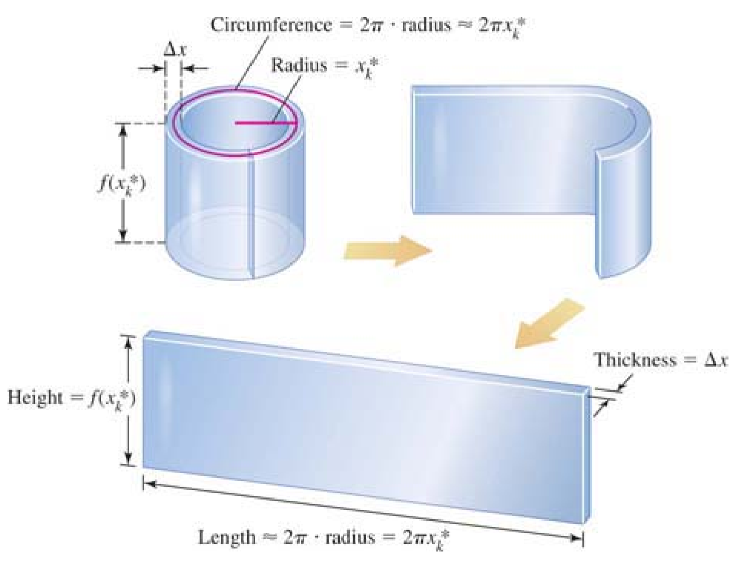
about the line $y = -1$

6.4: Volume by Shells

Volume by the Shell Method
Let f and g be continuous functions with $f(x) \geq g(x)$ on $[a, b]$. If R is the region bounded by the curves $y = f(x)$ and $y = g(x)$ between the lines $x = a$ and $x = b$, the volume of the solid generated when R is revolved about the y-axis is

$$V = \int_a^b \underbrace{2\pi x}_{\substack{\text{shell}\\\text{circumference}}} \underbrace{(f(x) - g(x))}_{\substack{\text{shell}\\\text{height}}} \, dx.$$





Example. Consider a general region R revolved around the y-axis.

When using the **disk/washer** method, we integrate with respect to __________

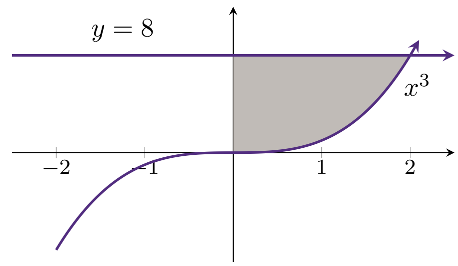
When using the **shell** method, we integrate with respect to __________

Example. Consider a general region R revolved around the x-axis.

When using the **disk/washer** method, we integrate with respect to __________

When using the **shell** method, we integrate with respect to __________

Example. Consider the region bounded between $y = x^3$, $y = 8$ and $x = 0$.



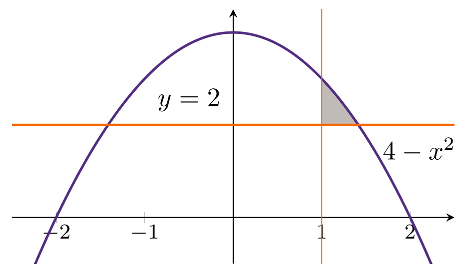
Use the disk/washer method to setup the integral that represents the volume of the solid generated by rotating the region about the x-axis.

about the y-axis.

Use the disk/washer method to setup the integral that represents the volume of the solid generated by rotating the region about the line $x = -1$.

about the line $y = 8$.

Example. Consider the region R bounded by $y = 4 - x^2$, $y = 2$, and $x = 1$. Use the shell method to setup the integral that represents the volume of the solid generated by rotating the region R about the indicated axis of rotation.



about x-axis,

about y-axis,

about the line $x = -2$,

about the line $y = 2$.

Example. Consider the region bounded by $y = \dfrac{1}{x+1}$ and $y = 1 - \dfrac{x}{3}$. Use both the disk/washer method and shell method to find the volume of the solid generated when R is rotated about the x-axis.

Example. Determine if the following statements are true.

When using the shell method, the axis of the cylindrical shells is parallel to the axis of revolution.

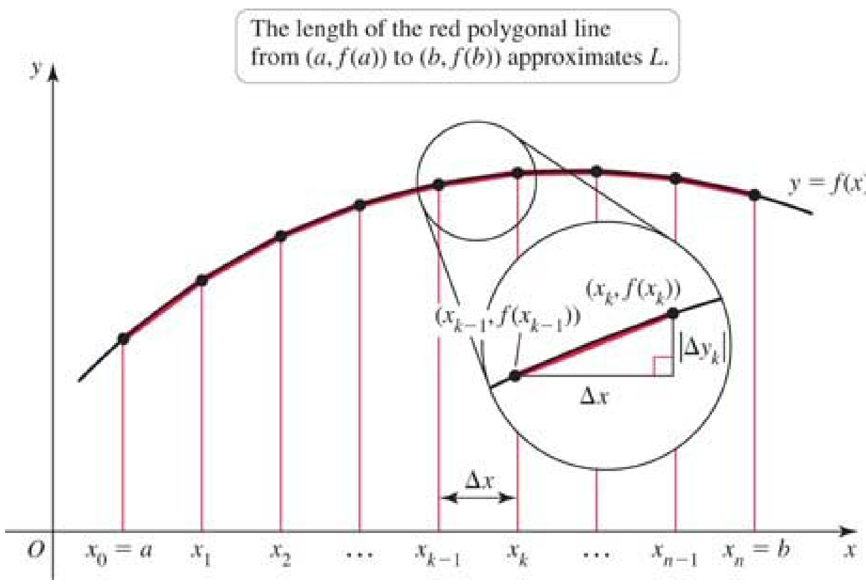
If a region is revolved about the y-axis, then the shell method must be used.

If a region is revolved about the x-axis, it is possible to use the disk/washer method and integrate with respect to x.

6.5: Length of Curves

Definition. (Arc Length for $y = f(x)$)
Let f have a continuous first derivative on the interval $[a, b]$. The length of the curve from $(a, f(a))$ to $(b, f(b))$ is

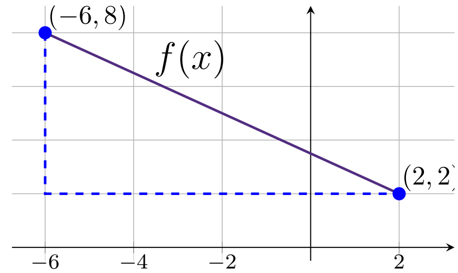
$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$



Definition. (Arc Length for $x = g(y)$)
Let g have a continuous first derivative on the interval $[c, d]$. The length of the curve from $(g(c), c)$ to $(g(d), d)$ is

$$L = \int_c^d \sqrt{1 + g'(y)^2}\, dy.$$

Example. Using a geometric argument, we can see that the length of $f(x) = -\dfrac{3}{4}x + \dfrac{7}{2}$ on the interval $[-6, 2]$ is $L = 10$. Compute this using the arc-length formula.



Example. Find the arc length of the curve $y = \dfrac{1}{4}x^2 - \dfrac{1}{2}\ln(x)$, for $1 \leq x \leq 2$.

Example. Find the arc length of the curve $y = \dfrac{1}{3}x^{3/2}$ on $[0, 12]$.

Example. Find a curve that passes through $(1, 2)$ on $[2, 6]$ whose arc length is computed using

$$\int_2^6 \sqrt{1 + 16x^{-2}}\, dx.$$

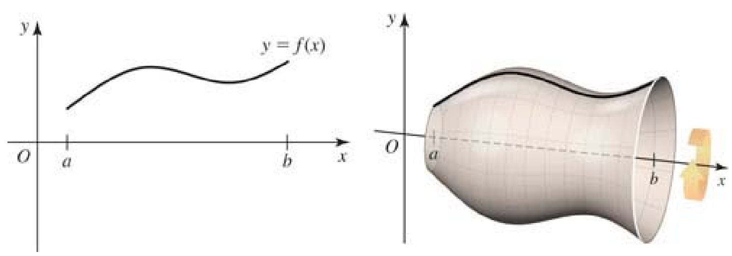
Example. Suppose f has length L on $[a, b]$. Evaluate

$$\int_{a/c}^{b/c} \sqrt{1 + f'(cx)^2}\, dx.$$

6.6: Surface Area

Definition. (Area of a Surface of Revolution)
Let f be a nonnegative function with a continuous first derivative on the interval $[a, b]$. The area of the surface generated when the graph of f on the interval $[a, b]$ is revolved around the x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx.$$



Example. Find the exact area of the surface obtained by rotating the curve $y = x^3$, $0 \le x \le 2$ about the x-axis.

Example. Find the exact area of the surface obtained by rotating the curve $y = \sqrt{8x - x^2}$, $1 \leq x \leq 7$ about the x-axis.

Example. Find the exact area of the surface obtained by rotating the curve $y = \dfrac{1}{2}(e^x + e^{-x})$, $-\ln(2) \le x \le \ln(2)$ about the x-axis.