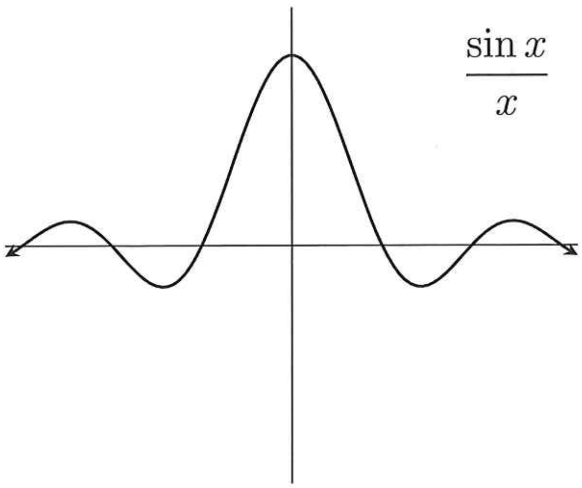
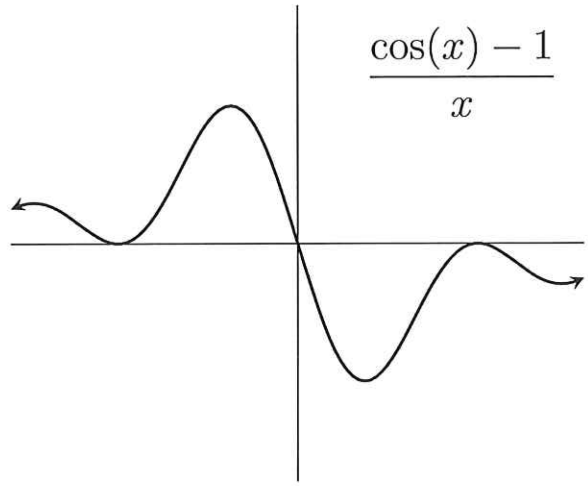
3.5 Derivatives of Trigonometric Functions

Theorem 3.10 Trigonometric Limits

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}\frac{\cos x-1}{x}=0$$





Example. Evaluate the following limits:

$$\lim_{x\to 0}\frac{\sin(3x)}{3x} = \boxed{1}$$

$$\lim_{x\to 0}\frac{\sin(4x)}{x}\left(\frac{4}{4}\right) = 4\lim_{x\to 0}\frac{\sin(4x)}{4x} = \boxed{4}$$

$$\lim_{h\to 0}\frac{5h}{\sin(3h)}\left(\frac{3/5}{3/5}\right) = \frac{1}{3/5}\lim_{h\to 0}\frac{3h}{\sin(3h)} = \boxed{\frac{5}{3}}$$

$$\lim_{t\to\frac{\pi}{2}}\frac{\sin\left(t-\frac{\pi}{2}\right)}{t-\frac{\pi}{2}} = \boxed{1}$$

$$\lim_{x\to 0}\frac{\tan(2x)}{x} = \lim_{x\to 0}\frac{\sin(2x)}{\cos(2x)}\frac{1}{x}\left(\frac{2}{2}\right) = \frac{2}{1}\lim_{x\to 0}\frac{\sin(2x)}{2x} = \boxed{2}$$

$$\lim_{x\to 0}\frac{\sin(7x)}{\sin(5x)} = \lim_{x\to 0}\frac{\sin(7x)}{x}\frac{x}{\sin(5x)}\frac{(5/7)}{(5/7)} = \frac{1}{5/7}\lim_{x\to 0}\frac{\frac{\sin(7x)}{7x}}{\frac{\sin(5x)}{5x}} = \boxed{\frac{7}{5}}$$

Theorem 3.11 Derivatives of Sine and Cosine

$$\frac{d}{dx}[\sin(x)] = \cos(x) \qquad\qquad \frac{d}{dx}[\cos(x)] = -\sin(x)$$

Proof.

$$\begin{aligned}
\frac{d}{dx}[\sin(x)] &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
&= \sin(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h} = \cos(x)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}[\cos(x)] &= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\
&= \cos(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} - \sin(x) \lim_{h\to 0} \frac{\sin(h)}{h} = -\sin(x)
\end{aligned}$$

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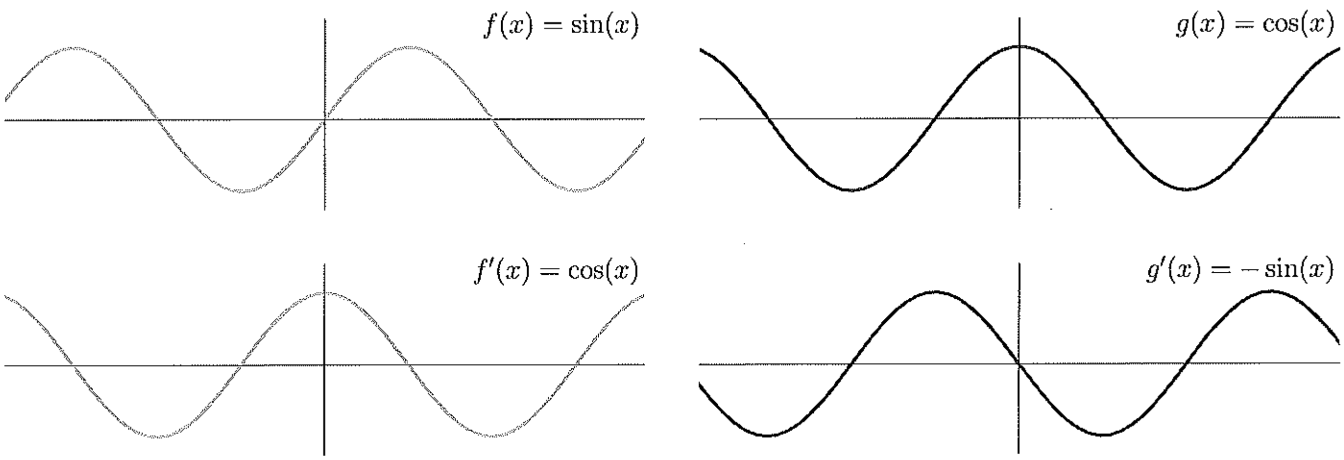

Example. Find the derivative of the following functions:

$$y = 3\cos(x) - 2x^{\frac{3}{2}}$$

$$y' = -3\sin(x) - 3x^{1/2}$$

$$z = \frac{\sin(x)}{x}$$

$$z' = \frac{x\cos(x) - \sin(x)(1)}{x^2}$$

$$w = \frac{x}{\cos(x)}$$

$$w' = \frac{\cos(x)(1) - x(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos(x) + \sin(x)}{\cos^2(x)}$$

$$\ell = e^x\cos(x)$$

$$\ell' = e^x(-\sin(x)) + e^x\cos(x)$$

$$= e^x[\cos(x) - \sin(x)]$$

$$m = \frac{\cos(x)}{\sin(x)}$$

$$m' = \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin^2(x)}$$

$$= \frac{-(\sin^2(x) + \cos^2(x))}{\sin^2(x)}$$

$$= \frac{-1}{\sin^2(x)}$$

$$= -\csc^2(x)$$

$$n = \sin^2(x) + \cos^2(x) = 1$$

$$n' = 0$$

W/o simplification, this problem requires the product rule twice.

Example. Find the equation of the line tangent to $y = \cos(x)$ at $x = \frac{\pi}{4}$.

$$y' = -\sin(x)$$

$$y\left(\tfrac{\pi}{4}\right) = \cos\left(\tfrac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$y'\left(\tfrac{\pi}{4}\right) = -\sin\left(\tfrac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$y - y_1 = m(x - x_1)$$

$$y - \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$$

$$y = -\frac{\sqrt{2}}{2}x + \frac{\pi\sqrt{2}}{8} + \frac{\sqrt{2}}{2}$$

Example. Find the derivative of $y = \dfrac{x\cos(x)}{1+\sin(x)}$ and simplify. $\boxed{\text{Frx}}$

$$y' = \frac{(1+\sin(x))\frac{d}{dx}[x\cos(x)] - (x\cos(x))[\cos(x)]}{(1+\sin(x))^2}$$

$$= \frac{(1+\sin(x))[(1)\cos(x) + x(-\sin(x))] - x\cos^2(x)}{(1+\sin(x))^2}$$

$$= \frac{\cos(x) - x\sin(x) + \cos(x)\sin(x) - x\sin^2(x) - x\cos^2(x)}{(1+\sin(x))^2}$$

$$-x(\sin^2(x) + \cos^2(x))$$

$$= \frac{\cos(x) - x\sin(x) + \cos(x)\sin(x) - x}{(1+\sin(x))^2}$$

$$= \frac{\cos(x)(1+\sin(x)) - x(1+\sin(x))}{(1+\sin(x))^2}$$

$$= \boxed{\frac{\cos(x) - x}{1+\sin(x)}}$$

Theorem 3.12 Derivatives of the Trigonometric Functions

$$\frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\cos(x) = -\sin(x)$$
$$\frac{d}{dx}\tan(x) = \sec^2(x) \qquad \frac{d}{dx}\cot(x) = -\csc^2(x)$$
$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x) \qquad \frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

Example. Find the derivatives of the following:

$y = \frac{4}{x} - \frac{9}{13}\tan(x) = 4x^{-1} - \frac{9}{13}\tan(x)$

$$y' = \frac{-4}{x^2} - \frac{9}{13}\sec^2(x)$$

$f(x) = -4x^3\cot(x)$

$$f'(x) = \frac{d}{dx}\left[-4x^3\right]\cot(x) - 4x^3\frac{d}{dx}\left[\cot(x)\right]$$
$$= -12x^2\cot(x) + 4x^3\csc^2(x)$$

$g(\theta) = \dfrac{\sec(\theta)}{1+\sec(\theta)}$

$$g'(\theta) = \frac{\left[1+\sec(\theta)\right]\left[\sec(\theta)\tan(\theta)\right] - \sec(\theta)\left[\sec(\theta)\tan(\theta)\right]}{(1+\sec\theta)^2}$$
$$= \frac{\sec(\theta)\tan(\theta)}{(1+\sec\theta)^2}$$

$h(w) = e^w\csc(w)$

$$h'(w) = e^w\csc(w) - e^w\csc(w)\cot(w)$$

Example. Evaluate

$$\frac{d}{dx}[\tan(x)]\bigg|_{x=\frac{\pi}{4}}$$

$$= \sec^2(x)\Big|_{x=\frac{\pi}{4}}$$

$$= \frac{1}{\cos^2(\pi/4)} = \frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2}$$

$$= \frac{1}{\frac{2}{4}} = \boxed{2}$$

$$\frac{d}{dx}\Big[(\sin(x) + \cos(x))\csc(x)\Big]$$

$$= \frac{d}{dx}\left[\sin(x) + \cos(x)\right]\csc(x) + (\sin(x) + \cos(x))\frac{d}{dx}\left[\csc(x)\right]$$

$$= (\cos(x) - \sin(x))\csc(x) - (\sin(x) + \cos(x))\csc(x)\tan(x)$$

$$\frac{d}{d\theta}\left[\theta^2 \sin(\theta)\tan(\theta)\right] = \frac{d}{d\theta}\left[\theta^2\right]\sin\theta\tan\theta$$

$$+ \theta^2 \frac{d}{d\theta}\left[\sin\theta\right]\tan\theta$$

$$+ \theta^2 \sin\theta \frac{d}{d\theta}\left[\tan\theta\right]$$

$$= 2\theta \sin\theta \tan\theta$$

$$+ \theta^2 \cos\theta \tan\theta$$

$$+ \theta^2 \sin\theta \sec^2\theta$$

Example. Find the following higher order derivatives:

y'' when $y = \cos(x)$

$y' = -\sin(x)$

$y'' = -\cos(x)$

$f''(x)$ when $f(x) = \sin(x)$

$f'(x) = \cos(x)$

$f''(x) = -\sin(x)$

$y^{(42)}$ when $y = \cos(x)$

$y^{(1)} = -\sin(x)$

$y^{(2)} = -\cos(x)$

$y^{(3)} = \sin(x)$

$\vdots$

$y^{(40)} = \cos(x)$

$y^{(41)} = -\sin(x) \rightarrow y^{(42)} = -\cos(x)$

$y'' = \dfrac{d^2}{d\theta^2}[\sin(\theta)\cos(\theta)]$

$\dfrac{d}{d\theta}[\sin\theta]\cos\theta + \sin\theta\dfrac{d}{d\theta}[\cos\theta]$

$= \cos^2\theta - \sin^2\theta$

$y'' = \dfrac{d}{d\theta}[\cos\theta]\cos\theta + \dfrac{d}{d\theta}[\cos\theta]\cos(\theta)$

$- \dfrac{d}{d\theta}[\sin\theta]\sin\theta - \dfrac{d}{d\theta}[\sin\theta]\sin\theta$

$= -4\sin\theta\cos\theta$

$\dfrac{d^2}{dx^2}\left[\frac{1}{2}e^x\cos(x)\right]$

$y' = \frac{1}{2}\dfrac{d}{dx}[e^x]\cos(x) + \frac{1}{2}e^x\dfrac{d}{dx}[\cos(x)]$

$= \frac{1}{2}e^x[\cos(x) - \sin(x)]$

$y'' = \frac{1}{2}e^x[\cos(x) - \sin(x)] + \frac{1}{2}e^x[-\sin(x) - \cos(x)]$

$\dfrac{d^2}{dx^2}[\cot x]$

$y' = -\csc^2(x)$

$y'' = -\dfrac{d}{dx}[\csc(x)]\csc(x)$

$- \csc(x)\dfrac{d}{dx}[\csc(x)]$

$= \csc(x)\cot(x)\csc(x)$

$+ \csc(x)\csc(x)\cot(x)$

$= 2\csc^2(x)\cot(x)$

Example. For

$$f = \begin{cases} \frac{3\sin(x)}{x}, & x \neq 0 \\ a, & x = 0 \end{cases}$$

Find a such that f is continuous.

$$\lim_{x\to 0} \frac{3\sin(x)}{x} = 3 \underbrace{\lim_{x\to 0} \frac{\sin(x)}{x}}_{1} = 3 \quad \Rightarrow \quad \boxed{a=3}$$

Example. Find the equation of the line tangent to $y = \dfrac{\cos(x)}{1-\cos(x)}$ at $x = \frac{\pi}{3}$.

$$y' = \frac{(1-\cos(x))[-\sin(x)] - \cos(x)[\sin(x)]}{[1-\cos(x)]^2}$$

$$= \frac{-\sin(x)}{(1-\cos(x))^2}$$

$$y(\pi/3) = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

$$y'(\pi/3) = \frac{-\sqrt{3}/2}{(1-\frac{1}{2})^2} = -2\sqrt{3}$$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -2\sqrt{3}(x - \pi/3)$$

$$\boxed{y = -2\sqrt{3}\,x + \frac{2\sqrt{3}\pi}{3} + 1}$$

Example. For what values of x does $x - \sin(x)$ have a horizontal tangent line?

$$\frac{d}{dx}[x - \sin(x)] = 1 - \cos(x)$$

Solve $1 - \cos(x) = 0$

$1 = \cos(x) \Rightarrow x = 2k\pi$

Example. Evaluate the following limits

$$\lim_{x\to\pi/4} \frac{\tan(x)-1}{x-\pi/4} = \lim_{x\to\pi/4} \frac{\tan(x)-\tan(\pi/4)}{x-\pi/4} = \frac{d}{dx}\left[\tan(x)\right]\Big|_{x=\pi/4}$$

$$= \sec^2(x)\Big|_{x=\pi/4}$$

$$= \frac{1}{\cos^2(\pi/4)} = \frac{1}{(\sqrt{2}/2)^2} = \boxed{2}$$

$$\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{6}+h\right)-\frac{1}{2}}{h} = \frac{d}{dx}\left[\sin(x)\right]\Big|_{x=\pi/6} = \cos(x)\Big|_{x=\pi/6} = \boxed{\frac{\sqrt{3}}{2}}$$

$$\lim_{x\to\pi/4} \frac{\cot x-1}{x-\frac{\pi}{4}} = \frac{d}{dx}\left[\cot(x)\right]\Big|_{x=\pi/4} = -\csc^2(x)\Big|_{x=\pi/4}$$

$$= -\frac{1}{\sin^2(\pi/4)} = \boxed{-2}$$