# 1 15.7: Maximum/Minimum Problems

**Definition. (Local Maximum/Minimum Values)**
Suppose $(a, b)$ is a point in a region $R$ on which $f$ is defined.

- If $f(x, y) \leq f(a, b)$ for all $(x, y)$ in the domain of $f$ and in some open disk centered at $(a, b)$, then $f(a, b)$ is a **local maximum value** of $f$.
- If $f(x, y) \geq f(a, b)$ for all $(x, y)$ in the domain of $f$ and in some open disk centered at $(a, b)$, then $f(a, b)$ is a **local minimum value** of $f$.
- Local maximum and local minimum values are also called **local extreme values** or **local extrema**.

**Theorem 15.14: Derivatives and Local Maximum/Minimum Values**
If $f$ has a local maximum or minimum value at $(a, b)$ and the partial derivatives $f_x$ and $f_y$ exist at $(a, b)$, then $f_x(a, b) = f_y(a, b) = 0$.

**Definition. (Critical Point)**
An interior point $(a, b)$ in the domain of $f$ is a **critical point** of $f$ if either

1. $f_x(a, b) = f_y(a, b) = 0$, or
2. at least one of the partial derivatives $f_x$ and $f_y$ does not exist at $(a, b)$.

**Definition. (Saddle Point)**
Consider a function $f$ that is differentiable at a critical point $(a, b)$. Then $f$ has a **saddle point** at $(a, b)$ if, in every open disk centered at $(a, b)$, there are points $(x, y)$ for which $f(x, y) > f(a, b)$ and points for which $f(x, y) < f(a, b)$.

**Theorem 15.15: Second Derivative Test**

Suppose the second partial derivatives of $f$ are continuous throughout an open disk centered at the point $(a, b)$, where $f_x(a, b) = f_y(a, b) = 0$. Let

$$D(x, y) = f_{xx}(xy)f_{yy}(x, y) - (f_{xy}(x, y))^2.$$

1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f$ has a local maximum value at $(a, b)$.
2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f$ has a local minimum value at $(a, b)$.
3. If $D(a, b) < 0$, then $f$ has a saddle point at $(a, b)$.
4. If $D(a, b) = 0$, then the test is inconclusive.

**Definition. (Absolute Maximum/Minimum Values)**

Let $f$ be defined on a set $R$ in $\mathbb{R}^2$ containing the point $(a, b)$.

- If $f(a, b) \geq f(x, y)$ for every $(x, y)$ in $R$, then $f(a, b)$ is an **absolute maximum value** of $f$ on $R$.
- If $f(a, b) \leq f(x, y)$ for every $(x, y)$ in $R$, then $f(a, b)$ is an **absolute minimum value** of $f$ on $R$.

**Procedure: Finding Absolute Maximum/Minimum Values on Closed Bounded Sets**

Let $f$ be continuous on a closed bounded set $R$ in $\mathbb{R}^2$. To find the absolute maximum and minimum values of $f$ on $R$:

1. Determine the values of $f$ at all critical points in $R$.
2. Find the maximum and minimum values of $f$ on the boundary of $R$.
3. The greatest function value found in Steps 1 and 2 is the absolute maximum value of $f$ on $R$, and the least function value found in Steps 1 and 2 is the absolute minimum value of $f$ on $R$.