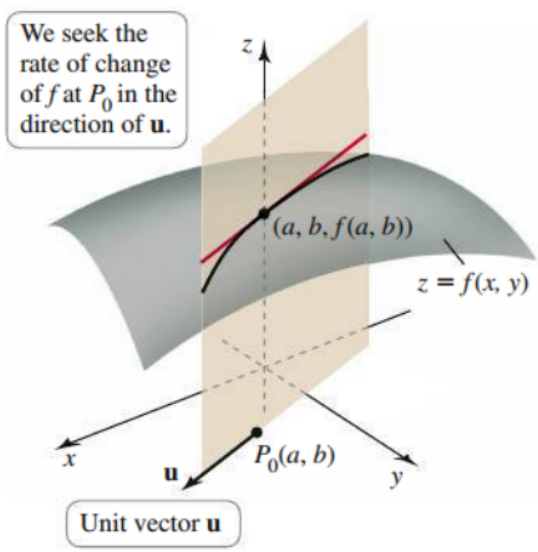
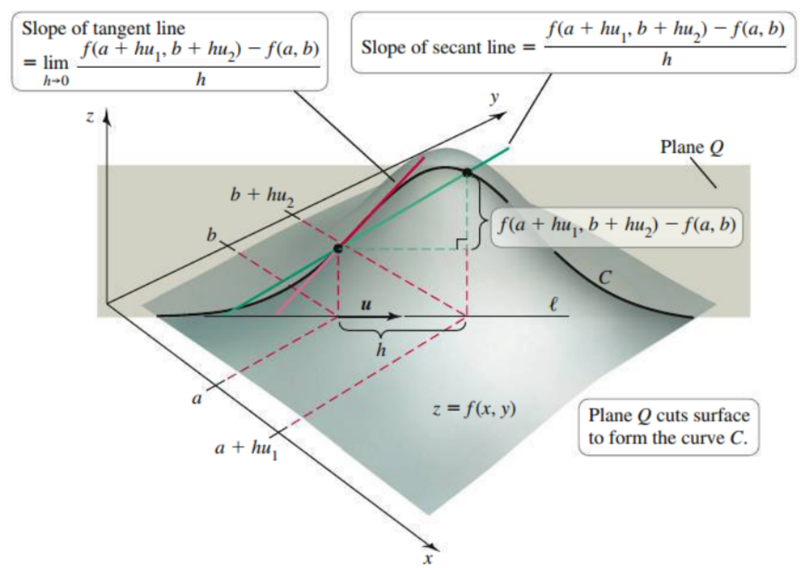
## 15.5: Directional Derivatives and the Gradient

Directional derivatives allow us to evaluate the rate of change of a function $f(x, y)$ along any direction (not just parallel with the $x$-axis and $y$-axis).




> **Definition. (Directional Derivative)**
> Let $f$ be differentiable at $(a, b)$ and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the $xy$-plane. The **directional derivative of $f$ at $(a, b)$ in the direction of $u$** is
> $$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h},$$
> provided the limit exists.

To motivate the formula for the directional derivative, let $\ell$ be a line going through $(a, b)$ in the direction of the unit vector $\mathbf{u}$. Now, let

$$x = a + su_1, \qquad \text{and} \qquad y = b + su_2,$$

where $-\infty < s < \infty$ and define

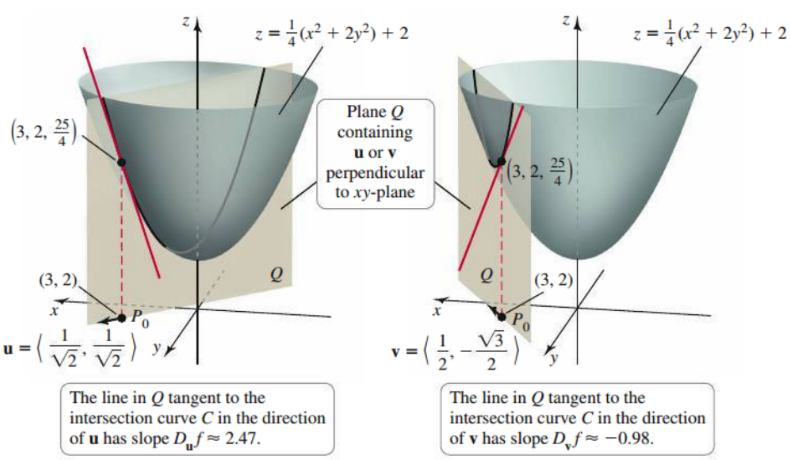
$$g(s) = f(\underbrace{a + su_1}_{x}, \underbrace{b + su_2}_{y}),$$

which evaluates $f$ along $\ell$. Thus, $g'(s)$ gives us the derivative along this line, and $g'(0)$ gives us the directional derivative of $f$ at $(a, b)$:

$$\begin{aligned} D_{\mathbf{u}}f(a,b) = g'(0) &= \left.\left(\frac{\partial f}{\partial x}\underbrace{\frac{dx}{ds}}_{u_1} + \frac{\partial f}{\partial y}\underbrace{\frac{dy}{ds}}_{u_2}\right)\right|_{s=0} \\ &= f_x(a,b)u_1 + f_y(a,b)u_2 \\ &= \langle f_x(a,b),\ f_y(a,b)\rangle \cdot \langle u_1, u_2\rangle. \end{aligned}$$

**Theorem 15.10: Directional Derivative**
Let $f$ be differentiable at $(a, b)$ and let $\mathbf{u} = \langle u_1, u_2\rangle$ be a unit vector in the $xy$-plane. The **directional derivative of $f$ at $(a, b)$ in the direction of $\mathbf{u}$** is

$$D_{\mathbf{u}}f(a,b) = \langle f_x(a,b), f_y(a,b)\rangle \cdot \langle u_1, u_2\rangle.$$



**Example.** Compute the directional derivatives of the following functions at the given point along the given direction.

$f(x, y) = \sqrt{4 - x^2 - 2y}$; $P(2, -2)$; and $\mathbf{u} = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$,

$g(x, y) = \tan^{-1}(xy)$; $P(\pi, 1/3)$; along $\mathbf{u} = \langle 1, 1 \rangle$,

$h(x, y) = 2x^2 - xy + 3y^2$; $P(1, -3)$; along $\mathbf{u} = \langle 1, -1 \rangle$ and $\mathbf{v} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$.

**The Gradient Vector:**

The vector of derivatives used in the directional derivative is called the *gradient* of $f$.

> **Definition. (Gradient (Two Dimensions))**
> Let $f$ be differentiable at the point $(x, y)$. The **gradient** of $f$ at $(x, y)$ is the vector-valued function
> $$\nabla f(x,y) = \langle f_x(x,y),\, f_y(x,y)\rangle = f_x(x,y)\boldsymbol{i} + f_y(x,y)\boldsymbol{j}.$$

**Example.** For $f(x, y) = 3 - \dfrac{x^2}{10} + \dfrac{xy^2}{10}$, compute $\nabla f(3, -1)$, then compute $D_{\mathbf{u}} f(3, -1)$, where $\mathbf{u} = \left\langle \dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}} \right\rangle$.

> **Theorem 15.11: Directions of Change**
> Let $f$ be differentiable at $(a, b)$ with $\nabla f(a, b) \neq \mathbf{0}$.
>
> 1. $f$ has its maximum rate of increase at $(a, b)$ in the direction of the gradient $\nabla f(a, b)$. The rate of change in this direction is $|\nabla f(a, b)|$.
> 2. $f$ has its maximum rate of decrease at $(a, b)$ in the direction of $-\nabla f(a, b)$. The rate of change in this direction is $-|\nabla f(a, b)|$.
> 3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$.

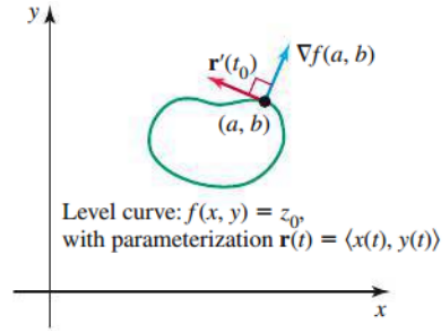
**Example.** For $f = 4 + x^2 + 3y^2$:

What direction is the greatest ascent at $P(2, -\frac{1}{2}, \frac{35}{4})$? What is the rate of change in this direction?

What direction is the greatest descent at $P(\frac{5}{2}, -2, \frac{89}{4})$? What is the rate of change in this direction?

What direction results in no change in function values at $P(3, 1, 16)$?

**Theorem 15.12: The Gradient and Level Curves**
Given a function $f$ differentiable at $(a, b)$, the line tangent to the level curve of $f$ at $(a, b)$ is orthogonal to the gradient $\nabla f(a, b)$, provided $\nabla f(a, b) \neq \mathbf{0}$.




*Note:* From Theorem 15.12, we get an equation for the line tangent to the curve $z = f(x, y)$ at $(a, b)$:

$$\nabla f(a, b) \cdot \langle x - a,\ y - b \rangle = 0.$$

**Example.** Consider the upper sheet $z = f(x, y) = \sqrt{1 + 2x^2 + y^2}$ of a hyperboloid of two sheets.

Verify that the gradient at $(1, 1)$ is orthogonal to the corresponding level curve at that point.

Find an equation of the line tangent to the level curve at $(1, 1)$.

**Example.** Consider $z = f(x, y) = 15 - \dfrac{x^2}{25} - \dfrac{y^2}{9}$:

Compute the slope of the tangent line at $P(5\sqrt{5}, -6, 6)$.

Verify the gradient is orthogonal to the tangent line.

**Definition. (Directional Derivative and Gradient in Three Dimensions)** Let $f$ be directional at $(a, b, c)$ and let $\mathbf{u} = \langle u_1,\, u_2,\, u_3\rangle$ be a unit vector. The **directional derivative of $f$ at $(a, b, c)$ in the direction of u** is

$$D_{\mathbf{u}}(a,b,c) = \lim_{h\to 0} \frac{f(a+hu_1,\, b+hu_2,\, c+hu_3) - f(a,b,c)}{h},$$

provided this limit exists.
The **gradient** of $f$ at this point $(x, y, z)$ is the vector-valued function

$$\begin{aligned}\nabla f(x,y,z) &= \langle f_x(x,y,z),\, f_y(x,y,z),\, f_z(x,y,z)\rangle \\ &= f_x(x,y,z)\boldsymbol{i} + f_y(x,y,z)\boldsymbol{j} + f_z(x,y,z)\boldsymbol{k}.\end{aligned}$$

**Theorem 15.13: Directional Derivative and Interpreting the Gradient**
Let $f$ be differentiable at $(a, b, c)$ and let $\mathbf{u} = \langle u_1,\, u_2,\, u_3\rangle$ be a unit vector. The directional derivative of $f$ at $(a, b, c)$ in the direction of $\mathbf{u}$ is

$$\begin{aligned}D_{\mathbf{u}}f(a,b,c) &= \nabla f(a,b,c)\cdot\mathbf{u} \\ &= \langle f_x(a,b,c),\, f_y(a,b,c),\, f_z(a,b,c)\rangle\cdot\langle u_1,\, u_2,\, u_3\rangle.\end{aligned}$$

Assuming $\nabla f(a, b, c) \neq \mathbf{0}$, the gradient in three dimensions has the following properties.

1. $f$ has its maximum rate of increase at $(a, b, c)$ in the direction of the gradient $\nabla f(a, b, c)$ and the rate of change in this direction is $|\nabla f(a, b, c)|$.
2. $f$ has its maximum rate of decrease at $(a, b, c)$ in the direction of $-\nabla f(a, b, c)$ and the rate of change in this direction is $-|\nabla f(a, b, c)|$.
3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b, c)$.

**Example.** Consider $f(x, y, z) = x^2+2y^2+4z^2-1$ and the level surface $f(x, y, z) = 3$. Find the gradient and the corresponding rate of change at the points $P(2, 0, 0)$, $Q(0, \sqrt{2}, 0)$, $R(0, 0, 1)$, and $S(1, 1, 1/2)$ on the level surface.