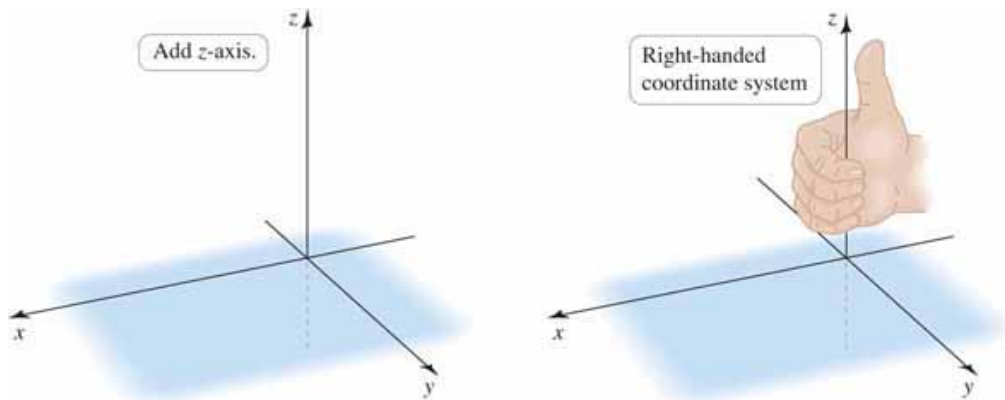


1 13.2: Vectors in Three Dimensions

The xyz - Coordinate System:

The three-dimensional coordinate system is created by adding the z -axis, which is perpendicular to both the x -axis and the y -axis. When looking at the xy -plane, the positive direction of the z -axis protrudes towards the viewer. This can also be shown using the right-hand rule (Figure 13.25 from Briggs):

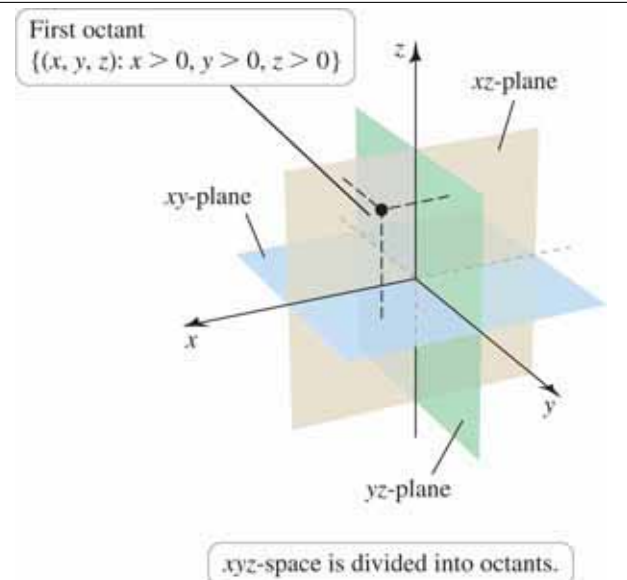


Definition.

This three-dimensional coordinate system is broken up into eight **octants**, which are separated by

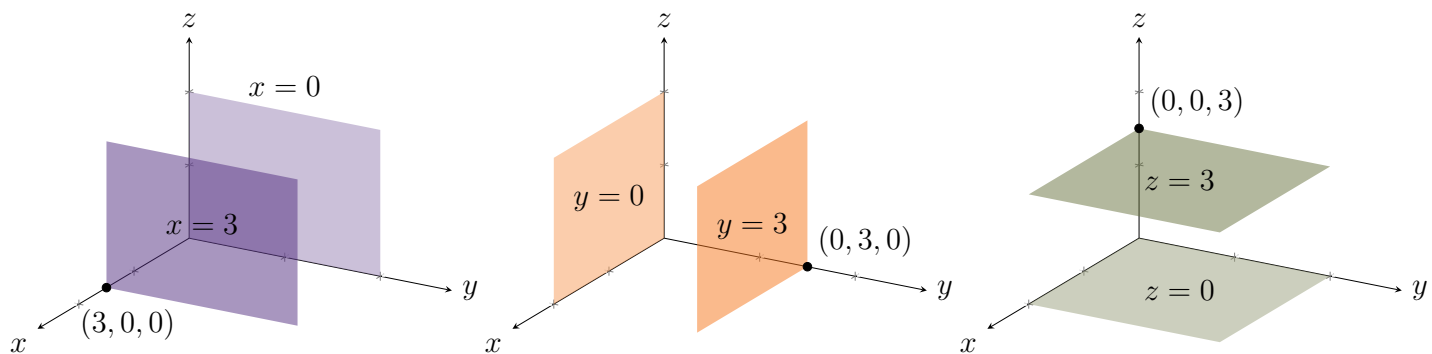
- the **xy -plane** ($z = 0$),
- the **xz -plane** ($y = 0$), and
- the **yz -plane** ($x = 0$).

The **origin** is the location where all three axes intersect.

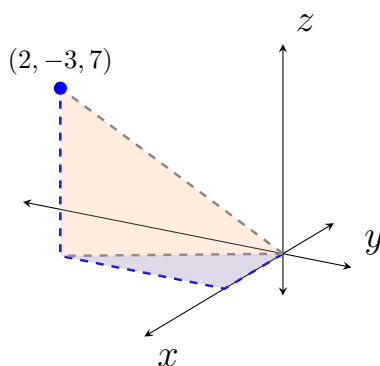


Equations of Simple Planes:

Planes in three-dimensions are analogous to lines in two-dimensions. Below, we see the yz -plane, the xz -plane, and the xy -plane, along with planes that are parallel where x , y , and z are fixed respectively:



Example (Parallel planes). Determine the equation of the plane parallel to the xz -plane passing through the point $(2, -3, 7)$.

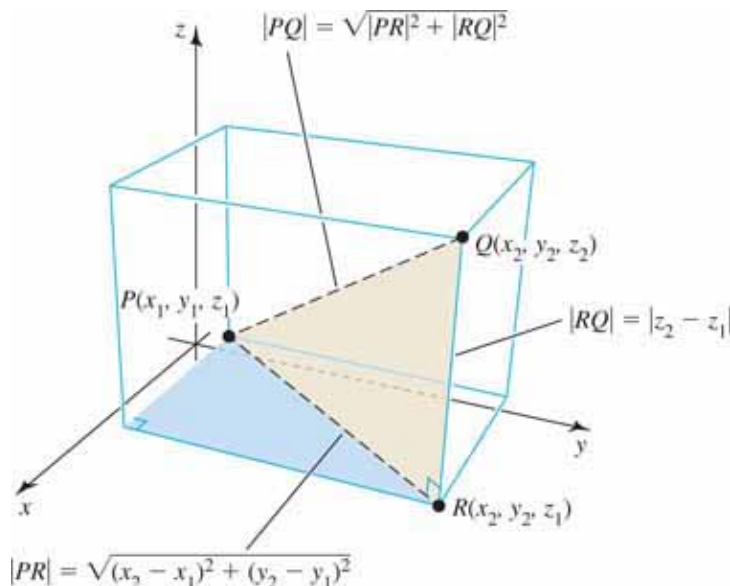


Distances in xyz -Space:

Recall that in \mathbb{R}^2 , for some vector \overrightarrow{PR} , the distance formula is given by

$$|PR| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

where (x_1, y_1) and (x_2, y_2) represent the points P and R respectively. This idea can be further extended into \mathbb{R}^3 by considering the two sides of the triangle formed by the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$:



Distance Formula in xyz -Space

The **distance** between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The **midpoint** between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is found by averaging the x -, y -, and z -coordinates:

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

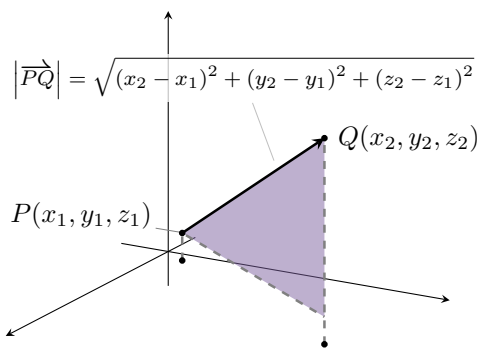
Magnitude and Unit Vectors:

Definition.

The **magnitude** (or **length**) of the vector $\vec{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ is the distance from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$:

$$|\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

In \mathbb{R}^3 , the **coordinate unit vectors** are $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$.



Example. Consider $P(-1, 4, 3)$ and $Q(3, 5, 7)$. Find

- $|\vec{PQ}|$
- The midpoint between P and Q
- Two unit vectors parallel to \vec{PQ}

Equation of a Sphere:

Definition.

A **sphere** centered at (a, b, c) with radius r is the set of points satisfying the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

A **ball** centered at (a, b, c) with radius r is the set of points satisfying the inequality

$$(x - a)^2 + (y - b)^2 + (z - c)^2 \leq r^2.$$

Example. Consider $P(-1, 4, 3)$ and $Q(3, 5, 7)$. Find the equation of the sphere centered at the midpoint passing through P and Q

Example. What is the geometry of the intersection between $x^2 + y^2 + z^2 = 50$ and $z = 1$?

Example. Rewrite the following equation into the standard form of a sphere:

$$x^2 + y^2 + z^2 - 2x + 6y - 8z = -1$$

Vector Operations in Terms of Components

Definition. (Vector Operations in \mathbb{R}^3)

Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$	Vector addition
$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$	Vector subtraction
$c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle$	Scalar multiplication

Properties of Vector Operations:

Suppose \mathbf{u} , \mathbf{v} , and \mathbf{w} are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

- | | |
|--|---|
| 1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | Commutative property of addition |
| 2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | Associative property of addition |
| 3. $\mathbf{v} + \mathbf{0} = \mathbf{v}$ | Additive identity |
| 4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ | Additive inverse |
| 5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ | Distributive property 1 |
| 6. $(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$ | Distributive property 2 |
| 7. $0\mathbf{v} = \mathbf{0}$ | Multiplication by zero scalar |
| 8. $c\mathbf{0} = \mathbf{0}$ | Multiplication by zero vector |
| 9. $1\mathbf{v} = \mathbf{v}$ | Multiplicative identity |
| 10. $a(c\mathbf{v}) = (ac)\mathbf{v}$ | Associative property of scalar multiplication |