

10.3: Infinite Series

A **Geometric sum** with n terms has the form

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k$$

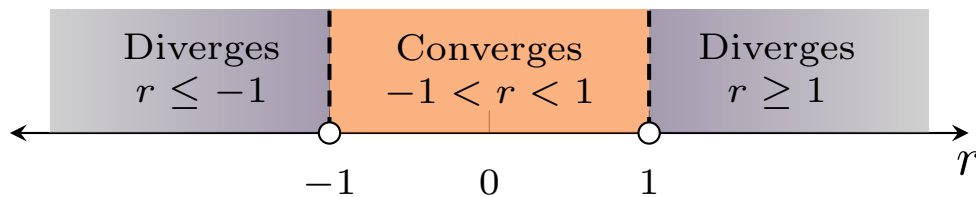
initial term
↓
ratio ↑

Derivation of partial sum formula:

$$\begin{aligned} S_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ - rS_n &= -(ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n) \\ \hline S_n - rS_n &= a - ar^n \\ S_n(1-r) &= a(1-r^n) \\ S_n &= \frac{a(1-r^n)}{1-r} \quad \leftarrow \text{Partial sum} \end{aligned}$$

Theorem 10.7: Geometric Series

Let $a \neq 0$ and r be real numbers. If $|r| < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. If $|r| \geq 1$, then the series diverges.



Example. Evaluate the following geometric series or state that the series diverges

$$\sum_{k=0}^{\infty} 1.1^k = 1 + 1.1 + 1.1^2 + 1.1^3 + \dots$$

\uparrow
 a

$$a = 1$$

$$r = 1.1$$

$$S_n = \frac{1(1-1.1^n)}{1-1.1}$$

$$|r| = |1.1| > 1 \text{ diverges}$$

$$\sum_{k=0}^{\infty} e^{-k} = 1 + e^{-1} + e^{-2} + e^{-3} + \dots$$

$$a = 1$$

$$S_n = \frac{1(1-\frac{1}{e^n})}{1-\frac{1}{e}}$$

$$r = e^{-1} = \frac{1}{e}$$

$$|r| < 1 \longrightarrow \text{Converges}$$

$$\sum_{k=2}^{\infty} 3(-0.75)^k = 3\left(-\frac{3}{4}\right)^2 + 3\left(-\frac{3}{4}\right)^3 + \dots$$

$$= \frac{27}{16} - \frac{81}{64} + \dots$$

$$a = \frac{27}{16}$$

$$S_n = \frac{27}{16} \frac{(1-(-\frac{3}{4})^n)}{1-(-\frac{3}{4})}$$

$$r = -\frac{3}{4}$$

$$= \frac{27}{28} (1-(-\frac{3}{4})^n)$$

$$\lim_{n \rightarrow \infty} S_n = \boxed{\frac{27}{28}}$$

$$\sum_{k=1}^{\infty} \frac{7}{10^k} = \sum_{k=0}^{\infty} \frac{7}{10^{k+1}} = \sum_{k=0}^{\infty} \underbrace{\frac{7}{10}}_a \left(\underbrace{\frac{1}{10}}_r\right)^k$$

$$= \frac{7}{10} + \frac{7}{10^2} + \frac{7}{10^3} + \frac{7}{10^4} + \dots$$

a

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{7}{10} \frac{(1-(\frac{1}{10})^n)}{1-\frac{1}{10}}$$

$$= \frac{7}{9} (1-(\frac{1}{10})^n)$$

$$\sum_{k=2}^{\infty} 3(-0.75)^k = \sum_{k=0}^{\infty} 3(-0.75)^{k+2} = \sum_{k=0}^{\infty} \underbrace{3(-0.75)^2}_{a=\frac{27}{16}} \underbrace{(-0.75)^k}_{r=-0.75}$$

Telescoping Series:

Example. Evaluate the following series

$$\sum_{k=1}^{\infty} \cos\left(\frac{1}{k^2}\right) - \cos\left(\frac{1}{(k+1)^2}\right)$$

$$S_n = \left[\underbrace{\cos\left(\frac{1}{1}\right) - \cos\left(\frac{1}{4}\right)}_{k=1} \right] + \left[\underbrace{\cos\left(\frac{1}{4}\right) - \cos\left(\frac{1}{9}\right)}_{k=2} \right] + \dots + \left[\cos\left(\frac{1}{n^2}\right) - \cos\left(\frac{1}{(n+1)^2}\right) \right]$$

$\left[\cos\left(\frac{1}{4}\right) - \cos\left(\frac{1}{16}\right) \right]$
↓

$$= \cos(1) - \cos\left(\frac{1}{(n+1)^2}\right)$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \cos(1) - \underbrace{\cos\left(\frac{1}{(n+1)^2}\right)}_{\rightarrow 0} = \boxed{\cos(1) - 1}$$

converges

$$\sum_{k=3}^{\infty} \frac{1}{(k-2)(k-1)}$$

Theorem 10.8: Properties of Convergent Series

1. Suppose $\sum a_k$ converges to A and c is a real number. The series $\sum ca_k$ converges, and $\sum ca_k = c \sum a_k = cA$.
2. Suppose $\sum a_k$ diverges. Then $\sum ca_k$ also diverges, for any real number $c \neq 0$.
3. Suppose $\sum a_k$ converges to A and $\sum b_k$ converges to B . The series $\sum (a_k \pm b_k)$ converges and $\sum (a_k \pm b_k) = \sum a_k \pm \sum b_k = A \pm B$.
4. Suppose $\sum a_k$ diverges and $\sum b_k$ converges. Then $\sum (a_k \pm b_k)$ diverges.
5. If M is a positive integer, then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=M}^{\infty} a_k$ either both converge or both diverge. In general, *whether* a series converges does not depend on a finite number of terms added to or removed from the series. However, the *value* of a convergent series does change if nonzero terms are added or removed.

Example. Evaluate

$$\sum_{k=1}^{\infty} \left[\frac{1}{2} \left(\frac{2}{5} \right)^k + \frac{2}{3} \left(\frac{1}{6} \right)^k \right]$$