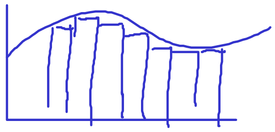
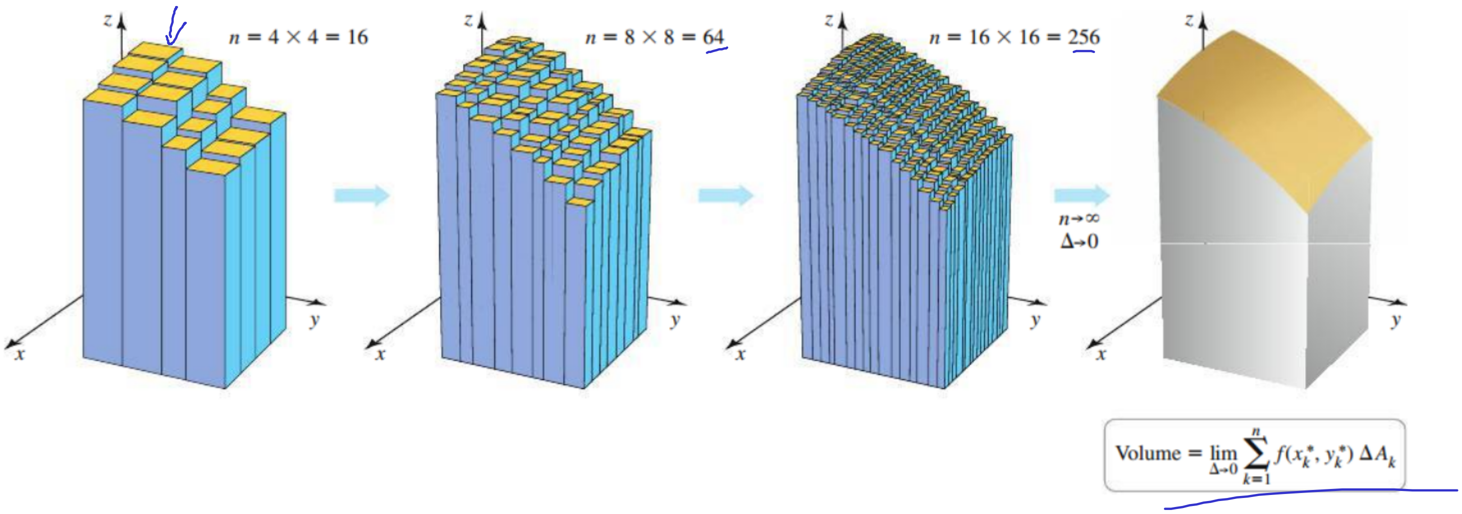


16.1: Double Integrals over Rectangular Regions



Definition. (Double Integrals)
A function f defined on a rectangular region R in the xy-plane is **integrable** on R if $\lim_{\Delta\to 0}\sum_{k=1}^{n} f(x_k^*, y_k^*)\,\Delta A_k$ exists for all partitions of R and for all choices of (x_k^*, y_k^*) within those partitions. The limit is the **double integral of f over R**, which we write

$$\iint_R f(x,y)\,dA = \lim_{\Delta\to 0}\sum_{k=1}^{n} f(x_k^*, y_k^*)\,\Delta A_k.$$

Example. Compute the following integral: $\displaystyle\int_0^1 \left(\int_0^2 (6 - 2x - y)\,dy\right) dx$

$f(x,y)$

$\int_0^1 A(x)\,dx$

$$\int_0^1\int_0^2 6-2x-y\;dy\;dx$$

$$= \int_0^1 6y - 2xy - \frac{y^2}{2}\Bigg|_{y=0}^{y=2} dx = \int_0^1 10-4x\;dx = 10x - 2x^2\Bigg|_{x=0}^{x=1} = \boxed{8}$$

LC #1

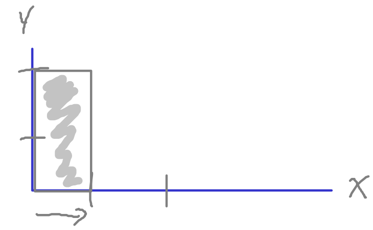
Example. Compute the following integral: $\displaystyle\int_0^2\int_0^1 (6-2x-y)\,dx\,dy$

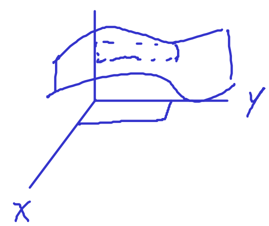
$$\int_0^2\int_0^1 6-2x-y\,dx\,dy = \int_0^2 6x-x^2-xy\Big|_{x=0}^{x=1} dy$$

$$= \int_0^2 5-y\,dy$$

$$= 5y-\frac{y^2}{2}\Big|_{y=0}^{y=2}$$

$$= 10-2 = \boxed{8}$$





question 3

The set such that

Theorem 16.1: (Fubini) Double Integrals over Rectangular Regions
Let f be continuous on the rectangular region $R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$. The double integral of f over R may be evaluated by either of the two iterated integrals:

$$\iint\limits_R f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx.$$

Example. Find the volume of the solid bounded by the surface $f(x,y) = 4 + 9x^2y^2$ over the region $R = \{(x,y) : -1 \le x \le 1,\ 0 \le y \le 2\}$. Integrate with respect to x first, then with respect to y first.

x 1st

$$\int_0^2 \int_{-1}^1 4 + 9x^2y^2 \, dx \, dy = \int_0^2 4x + 3x^3y^2 \Big|_{x=-1}^{x=1} dy$$

$$= \int_0^2 (4 + 3y^2) - (-4 - 3y^2) \, dy$$

$$= \int_0^2 8 + 6y^2 \, dy = 8y + 2y^3 \Big|_{y=0}^{y=2} = 16 + 16 = \boxed{32}$$

y 1st

$$\int_{-1}^1 \int_0^2 4 + 9x^2y^2 \, dy \, dx = \int_{-1}^1 4y + 3x^2y^3 \Big|_{y=0}^{y=2} dx$$

$$= \int_{-1}^1 8 + 24x^2 \, dx$$

$$= 8x + 8x^3 \Big|_{x=-1}^{x=1}$$

$$= (8 + 8) - (-8 - 8) = \boxed{32}$$

$\boxed{\text{LC \#3}}$

Example. Evaluate $\displaystyle\iint_R ye^{xy}\,dA$, where $R = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq \ln(2)\}$.

$$\int_0^{\ln(2)}\int_0^1 ye^{xy}\,dx\,dy = \int_0^{\ln(2)}\int_1^{e^y} du\,dy$$

$u = e^{xy}$

$du = ye^{xy}\,dx$

$x=0 \Rightarrow u=1$

$x=1 \Rightarrow u=e^y$

$$= \int_0^{\ln(2)} u \Big|_{u=1}^{u=e^y} dy$$

$$= \int_0^{\ln(2)} e^y - 1\,dy = e^y - y\Big|_{y=0}^{\ln(2)} = (2-\ln(2)) - (1-0)$$

$$= \boxed{1-\ln(2)}$$

LC #4

$$\int_0^1\int_0^{\ln(2)} ye^{xy}\,dy\,dx$$

$u = y$

$du = dy$

$v = \frac{e^{xy}}{x}$

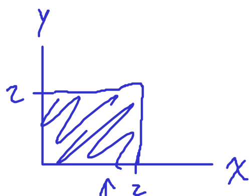
$dv = e^{xy}\,dy$

Average $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

Definition. (Average Value of a Function over a Plane Region)
The **average value** of an integrable function f over a region R is

$$\bar{f} = \frac{1}{\text{area of } R}\iint\limits_R f(x,y)\,dA.$$

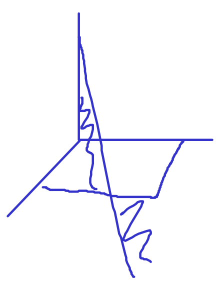
Example. Find the average value of $f(x,y) = 2 - x - y$ over the region $R = \{(x,y) : 0 \le x \le 2,\ 0 \le y \le 2\}$.



area of $R = 4$

$$\bar{f} = \frac{1}{4}\int_0^2\int_0^2 2 - x - y \; dx\; dy$$

$$= \frac{1}{4}\int_0^2 2x - \frac{x^2}{2} - xy \Big|_{x=0}^{x=2} dy$$



$$= \frac{1}{4}\int_0^2 2 - 2y \; dy$$

$$= \frac{1}{4}\left(2y - y^2\right)\Big|_{y=0}^{y=2} = \frac{1}{4}(4-4) = 0$$

$R = \{(x,y) : 0 \le x \le 2,\ 0 \le y \le 2\}$

$$\text{area of } R = \iint\limits_R dA = \int_0^2\int_0^2 dy\; dx = \int_0^2 y\Big|_{y=0}^{y=2} dx$$

$$= \int_0^2 2\; dx = 2x\Big|_{x=0}^{x=2} = 4$$