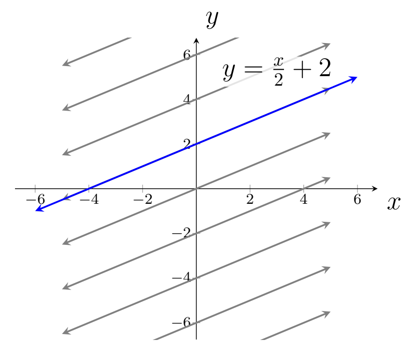
13.5: Lines and Planes in Space

Equation of a Line:

Recall the equation of a line in $\mathbb{R}^2$:

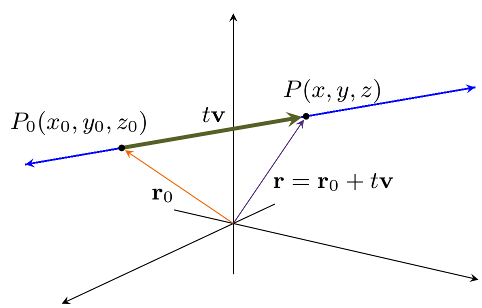
$$y = mx + b$$



where b is the intercept and m is the slope. This idea can be extended into higher dimensions:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

Here, $\mathbf{r}_0$ is a fixed point, and $\mathbf{v}$ is the position vector that is parallel to the line $\mathbf{r}$.



Equation of a Line

A **vector equation of the line** passing through the point $P_0(x_0, y_0, z_0)$ in the direction of the vector $\mathbf{v} = \langle a, b, c\rangle$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, or

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle, \quad \text{for} \quad -\infty < t < \infty$$

Equivalently, the corresponding **parametric equations of the line** are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \text{for} \quad -\infty < t < \infty$$

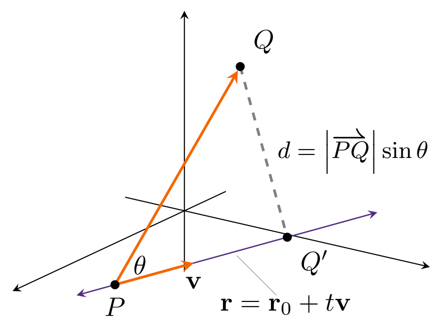
Example. Find the vector equation and parametric equation of the line that

- goes through the points $P(-1,-2,1)$ and $Q(-4,-5,-3)$ where $t=0$ corresponds to P,

- goes through the point $P(1,-3,-3)$ and is parallel to the vector $\mathbf{r}=\langle -4,1,-1\rangle$,

- goes through the point $P(-2,5,-2)$ and is perpendicular to the lines $x=3-4t$, $y=2-3t$, $z=-1-t$, and $x=-2+0t$, $y=2-t$, $z=3t$, where $t=0$ corresponds to P.

Distance from a Point to a Line:
Given a point Q and a line ℓ, the shortest distance to the line is the length of $\overrightarrow{QQ'}$.



From the definition of the cross product, we have

$$\left|\mathbf{v} \times \overrightarrow{PQ}\right| = |\mathbf{v}| \underbrace{\left|\overrightarrow{PQ}\right| \sin\theta}_{d} = |\mathbf{v}| d$$

From here, solving for d gives us the following:

Distance Between a Point and a Line
The distance d between the point Q and the $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ is

$$d = \frac{\left|\mathbf{v} \times \overrightarrow{PQ}\right|}{|\mathbf{v}|},$$

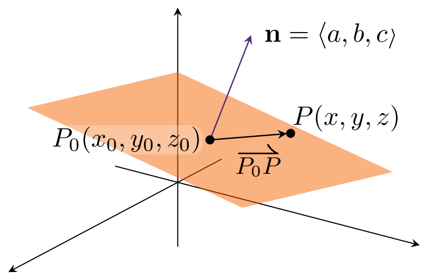
where P is any point on the line and $\mathbf{v}$ is a vector parallel to the line.

Example. Find the distance from the point $Q(-4, -1, -3)$ and the line $x = -5 - 5t$, $y = -5 + t$, $z = -1 + 4t$. (*Hint:* Let P be the point at $t = 0$)

Equations of Planes:

In $\mathbb{R}^2$, two distinct points determine a line.

In $\mathbb{R}^3$, three noncollinear points determine a unique plane. Alternatively, a plane is uniquely determined by a point and a vector that is orthogonal to the plane.



> **Definition. (Plane in $\mathbb{R}^3$)**
> Given a fixed point P_0 and a nonzero **normal vector n**, the set of points P in $\mathbb{R}^3$ for which $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$ is called a **plane**.

Consider the normal vector $\mathbf{n} = \langle a, b, c\rangle$ at the point $P_0(x_0, y_0, z_0)$, and any point $P(x, y, z)$ on the plane. Since $\mathbf{n}$ is orthogonal to the plane, it is also orthogonal to the vector $\overrightarrow{P_0P}$, which is also in the plane. Thus,

$$\begin{aligned} \mathbf{n} \cdot \overrightarrow{P_0P} &= 0 \\ \langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \\ ax + by + cz &= d \end{aligned}$$

> **General Equation of a Plane in $\mathbb{R}^3$**
> The plane passing through the point $P_0(x_0, y_0, z_0)$ with a nonzero normal vector $\mathbf{n} = \langle a, b, c\rangle$ is described by the equation
>
> $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{or} \quad ax + by + cz = d,$$
>
> where $d = ax_0 + by_0 + cz_0$.

Example. Find the equation of the plane that

- goes through the point $P(-2, 5, 0)$ and is parallel to the plane $x - 5y - 5z = 1$,

- goes through the points $P(5, -2, 1)$, $Q(5, 1, 3)$ and $R(1, -5, -2)$

- that is parallel to the vectors $\langle 4, -2, -3 \rangle$ and $\langle 3, 2, 3 \rangle$, passing through the point $P(-2, -2, 5)$.

Example. Find the location where the line $\langle -3, 1, 4\rangle + t\langle -1, -4, 2\rangle$ and the plane $2x - 2y - 4z = 5$ intersect.

> **Definition. (Parallel and Orthogonal Planes)**
> Two distinct planes are **parallel** if their respective normal vectors are parallel (that is, the normal vectors are scaling multiples of each other). Two plans are **orthogonal** if their respective normal vectors are orthogonal (that is, the dot product of the normal vectors is *zero*).

Example. Find the line of intersection between the planes $3x - y + 4z = -4$ and $x + 3y - 2z = 0$.

Example. Find the smallest angle between the planes $3x - y + 4z = -4$ and $x + 3y - 2z = 0$.