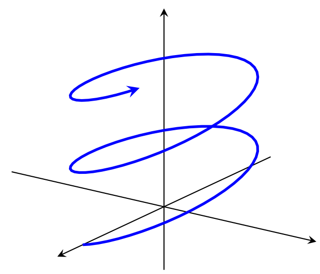
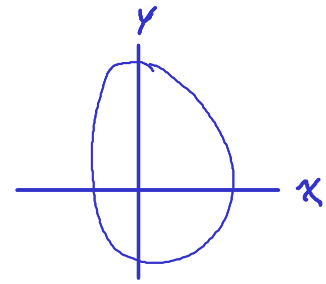
14.1: Vector-Valued Functions

Vector-valued functions are functions of the form $\mathbf{r}(t) = \langle \underline{x(t)}, \underline{y(t)}, \underline{z(t)} \rangle$, where $x(t)$, $y(t)$, and $z(t)$ are parametric equations dependent on t.





Curves in Space

Consider

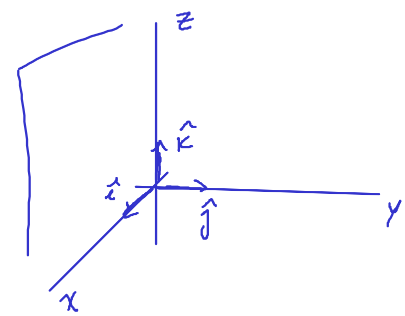
$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k},$$

where f, g, and h are defined for $a \le t \le b$. The **domain** of $\underline{\mathbf{r}}$ is the largest set of t for which all of f, g, and h are defined.

$= \langle t, 0, 6t^3 \rangle$

Example. What plane does the curve $\mathbf{r}(t) = t\boldsymbol{i} + 6t^3\boldsymbol{k}$ lie?

xz-plane



Example (Lines as vector-valued functions)**.** Find a vector function for the line that passes through the points $P(5, 2, -4)$ and $Q(5, 5, -2)$. What about the line segment that connects P and Q?

$$\ell = \vec{r}_0 + t\vec{r} = \langle 5, 2, -4 \rangle + t \underbrace{\langle 5-5, 5-2, -2+4 \rangle}_{\overrightarrow{PQ}}$$

$$= \langle 5, 2, -4 \rangle + t\langle 0, 3, 2 \rangle = \langle 5, 2+3t, -4+2t \rangle$$

Example. Find the domain of

$$\mathbf{r}(t) = \sqrt{16-t^2}\,\boldsymbol{i} + \sqrt{t}\,\boldsymbol{j} + \frac{4}{\sqrt{3+t}}\boldsymbol{k}$$

$\sqrt{16-t^2}$

$16-t^2 \ge 0$

$16 \ge t^2$

$4 \ge t \qquad -4 \le t$

$t \in [-4,4]$

$-4 \le t \le 4$

$\sqrt{t}$

$t \ge 0$

$t \in [0,\infty)$

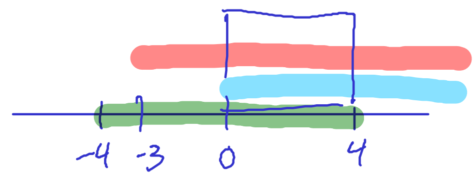
$\dfrac{4}{\sqrt{3+t}}$

$\sqrt{3+t} \ne 0$

$\left.\begin{array}{l} 3+t \ne 0 \\ 3+t \ge 0 \end{array}\right\} \quad 3+t > 0$

$t > -3$

$t \in (-3,\infty)$



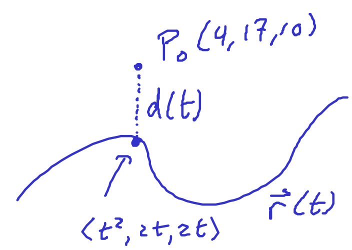
$0 \le t \le 4$

$t \in [0,4]$

Example. Find the point P on

$$\mathbf{r}(t) = t^2\boldsymbol{i} + 2t\boldsymbol{j} + 2t\boldsymbol{k}, \quad = \langle t^2, 2t, 2t\rangle$$

closest to $P_0(4, 17, 10)$. What is the distance between P and P_0?



$$d(t) = \sqrt{(t^2-4)^2 + (2t-17)^2 + (2t-10)^2}$$

$$= \sqrt{t^4 - 8t^2 + 16 + 4t^2 - 68t + 289 + 4t^2 - 40t + 100}$$

$$= \sqrt{t^4 - 108t + 405}$$

$$d'(t) = \frac{4t^3 - 108}{2\sqrt{t^4 - 108t + 405}} \overset{\text{set}}{=} 0 \longrightarrow \quad 4t^3 = 108$$

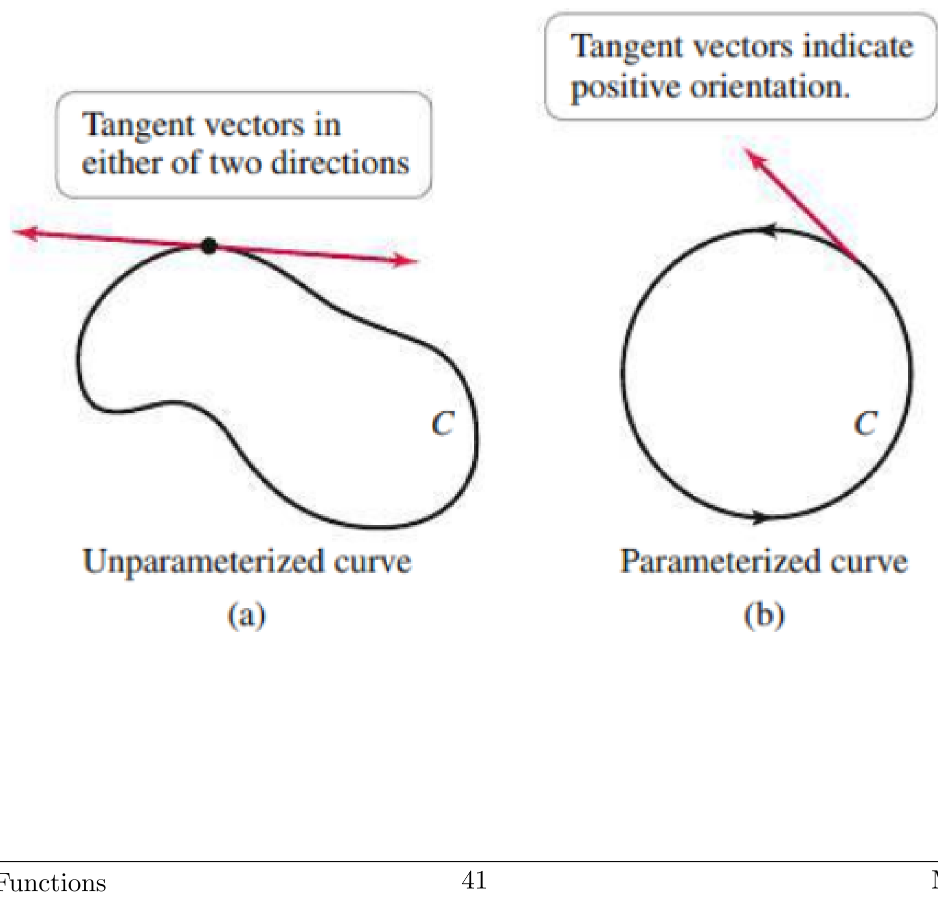
$t^3 = 27$

$t = 3$

$P = (3^2, 2\cdot 3, 2\cdot 3) = (9,6,6) \qquad$ distance $d(3)$

Orientation of Curves

- A **unparameterized curve** is a smooth curve C with no specified direction and the tangent vector can be drawn in two directions.
- A **parameterized curve** is a smooth curve C described by a function $\mathbf{r}(t)$ for $a \le t \le b$ and has a direction referred to as its **orientation**.
- The *positive* orientation is the direction of the curve generated when t increases from a to b.
- The tangent vector of a parameterized curve points in the positive orientation of the curve.



Example. Graph the curve described by the equation

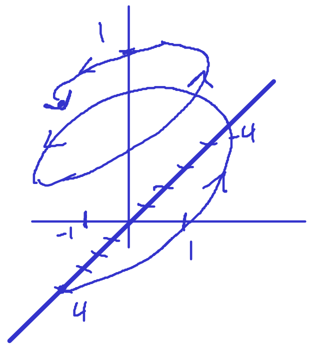
$$\mathbf{r}(t) = \underbrace{4\cos(t)}_{[-4,4]}\boldsymbol{i} + \underbrace{\sin(t)}_{[-1,1]}\boldsymbol{j} + \frac{t}{2\pi}\boldsymbol{k}, \quad \frac{t}{2\pi} \leftarrow [0,1]$$

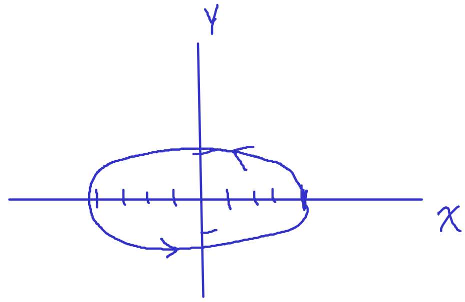
where $0 \le t \le 2\pi$. Indicate the positive orientation of this curve.

<u>Int</u>

$$\left.\begin{array}{l} \sin(t) = 0 \\ \frac{t}{2\pi} = 0 \end{array}\right\} 4\cos(t) \Rightarrow 4\cos(0) = 4$$

$\langle 4, 0, 0 \rangle$





$\vec{r}(2\pi) = \langle 4, 0, 1 \rangle$

$$\lim_{x \to a} g(x) = c \quad \left.\begin{array}{l} \\ f(c)\ \cancel{\text{defined}}\ \text{cont.} \end{array}\right\} \quad \lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Limits and Continuity for Vector-Valued Functions

The properties of limits extend to vector-valued functions naturally. In particular, for $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, if

$$\lim_{t \to a} f(t) = L_1, \qquad \lim_{t \to a} g(t) = L_2, \qquad \lim_{t \to a} h(t) = L_3$$

then

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t),\ \lim_{t \to a} g(t),\ \lim_{t \to a} h(t) \right\rangle = \langle L_1, L_2, L_3 \rangle.$$

Definition. (Limit of a Vector-Valued Function)
A vector-valued function $\mathbf{r}$ approaches the limit $\mathbf{L}$ as t approaches a, written $\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}$, provided $\lim_{t \to a} |\mathbf{r}(t) - \mathbf{L}| = 0$.

A function $\mathbf{r}(t)$ is **continuous** at $t = a$, provided $\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$.

Example. Evaluate the following limits:

$$\lim_{t \to \pi} \left(\cos(t)\boldsymbol{i} - 7\sin\left(-\frac{t}{2}\right)\boldsymbol{j} + \frac{t}{\pi}\boldsymbol{k} \right) = \lim_{t \to \pi} \cos(t)\,\hat{\imath} - 7\lim_{t \to \pi} \sin\left(-\frac{t}{2}\right)\hat{\jmath} + \lim_{t \to \pi} \frac{t}{\pi}\, k$$

$$= -1\,\hat{\imath} - 7(-1)\,\hat{\jmath} + 1\,\hat{k} = -\hat{\imath} + 7\hat{\jmath} + \hat{k}$$

logistic ↓

$$\lim_{t \to \infty} \left(\frac{t}{t-3}\boldsymbol{i} + \frac{40}{1 + 19e^{-t}}\boldsymbol{j} + \frac{1}{2t}\boldsymbol{k} \right) = 1\,\hat{\imath} + \lim_{t \to \infty} \frac{40}{1 + 19e^{-t}}\,\hat{\jmath} + 0\,\hat{k}$$

$$= \hat{\imath} + 40\,\hat{\jmath}$$

$$\lim_{t \to \infty} \frac{t}{t-3}\left(\frac{1/t}{1/t}\right)$$

$$\lim_{t \to \infty} \frac{40}{1 + 19e^{-t}} = \frac{40}{1 + 19\lim_{t \to \infty} e^{-t}} = \frac{40}{1+0}$$