15.1: Graphs and Level Curves

In the previous chapter, we considered functions of the form

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle,$$

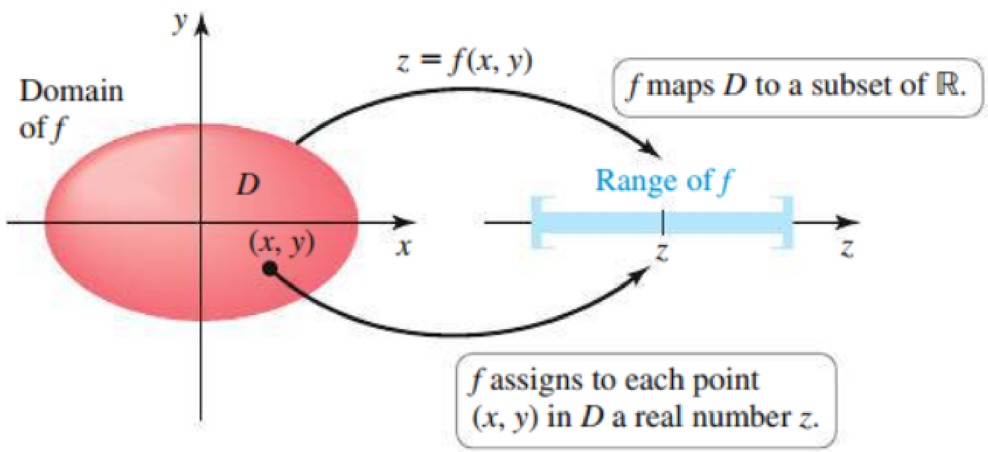
which have one independent variable t and three dependent variables $f(t)$, $g(t)$, and $h(t)$. In this chapter, we consider functions of the form

$$x_{n+1} = f(x_1, x_2, \ldots, x_n),$$

where we have multiple independent variables $x_1, x_2, \ldots, x_n$ and one single dependent variable x_{n+1}. We begin with functions of two variables:

$$z = f(x, y).$$

Definition. (Function, Domain, and Range with 2 Independent Variables) A **function** $z = f(x, y)$ assigns to each point (x, y) in a set D in $\mathbb{R}^2$ a unique real number z in a subset of $\mathbb{R}$. The set D is the **domain** of f. The **range** of f is the set of real numbers z that are assumed as the points (x, y) vary over the domain.



Example. Find the domain of the following functions:

$$f(x, y) = \frac{1}{xy + 2}$$

$xy + 2 \neq 0$

$xy \neq -2$

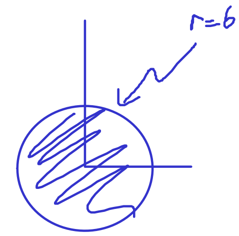
$x \neq \frac{-2}{y}$

$$g(x, y) = \sqrt{108 - 3x^2 - 3y^2}$$

$0 \leq 108 - 3x^2 - 3y^2$

$3x^2 + 3y^2 \leq 108$

$\rightarrow x^2 + y^2 \leq 36 = 6^2$



$$h(x, y) = \log_2 \left(x^3 - y^{1/3} \right)$$

$x^3 - y^{1/3} > 0$

$x^3 > y^{1/3}$

$x^9 > y$

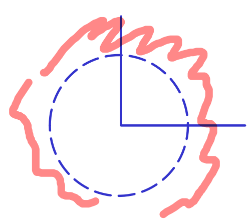
$$j(x, y) = \frac{1}{\sqrt{x^2 + y^2 - 16}}$$

$\sqrt{x^2 + y^2 - 16} \neq 0$

$x^2 + y^2 - 16 \geq 0$

$\Rightarrow x^2 + y^2 - 16 > 0$

$x^2 + y^2 > 4^2$



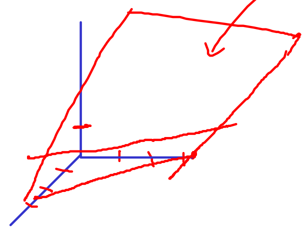
Example. Roughly graph the following functions:

$$f(x,y) = -4x + 3y - 10$$

$z = -4x + 3y - 10$

$z = 0 \longrightarrow \quad y = \frac{1}{3}(4x+10)$

$z = 1 \longrightarrow \quad y = \frac{1}{3}(4x+11)$



$$\underbrace{g(x,y)}_{z} = x^2 + y^2 + 4$$

$z = 0 \longrightarrow \quad x^2 + y^2 = -4$
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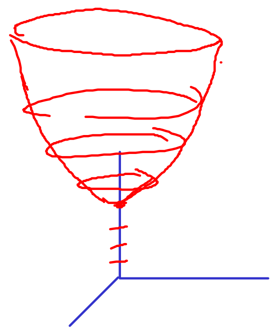
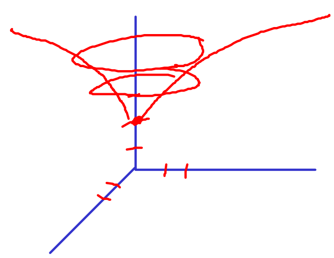


$z = 4 \longrightarrow \quad x^2 + y^2 = 0$

$z = 5 \longrightarrow \quad x^2 + y^2 = 1$

$z = 8 \longrightarrow \quad x^2 + y^2 = 4$

$$\underbrace{h(x,y)}_{z} = \sqrt{4 + x^2 + y^2}$$



$z = 0 = \sqrt{4 + x^2 + y^2}$

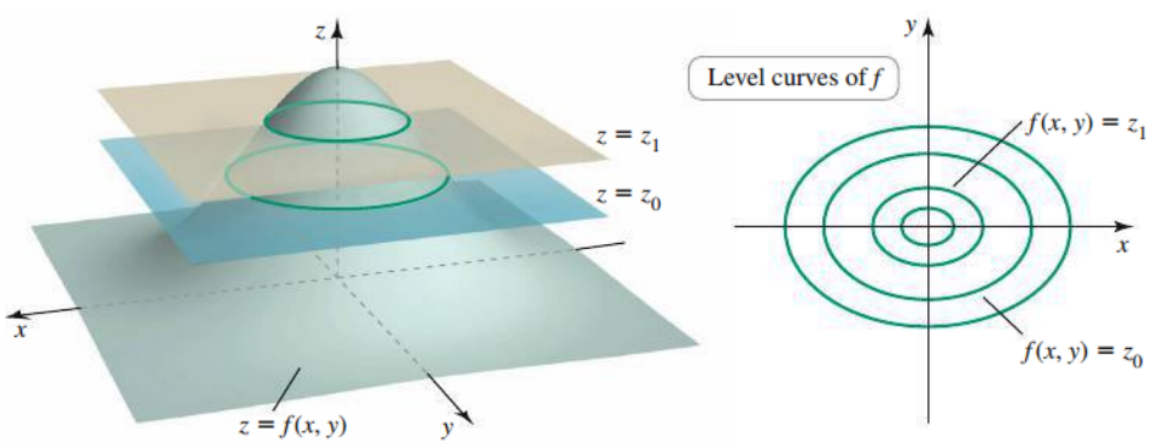
$z = 2 = \sqrt{4 + x^2 + y^2}$

$z = 3 = \sqrt{4 + x^2 + y^2} \longrightarrow \quad x^2 + y^2 = 5$

$z = 4 = \sqrt{4 + x^2 + y^2} \longrightarrow \quad x^2 + y^2 = 12$

Level Curves:

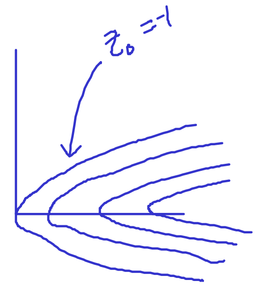
A **contour curve** is formed by tracing a three-dimensional surface at a constant height. A **level curve** is formed when a contour curve is projected to the xy-plane.



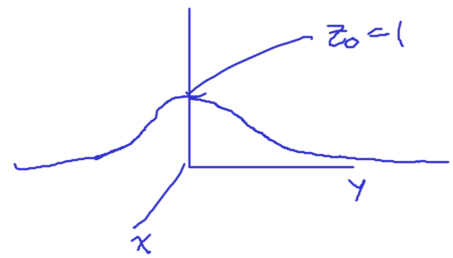
Example. Find the level curves of the following functions:

$$f(x,y) = y - x^2 - 1$$

$z_0 = y - x^2 - 1 \longrightarrow y = x^2 + (1 + z_0)$



e^{-x^2}



$$g(x,y) = e^{-x^2-y^2}$$

$z_0 = e^{-x^2-y^2}$

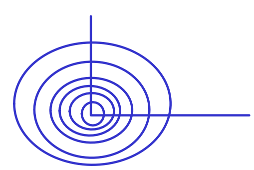
$-\ln(z_0) = (x^2+y^2) \longrightarrow -\ln(z_0) \geq 0$

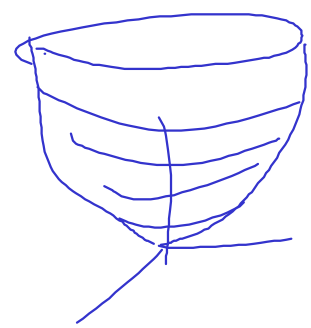
$\ln(z_0) \leq 0$

$z_0 \in (0,1]$

$$h(x,y) = x^2 + y^2$$

z_0

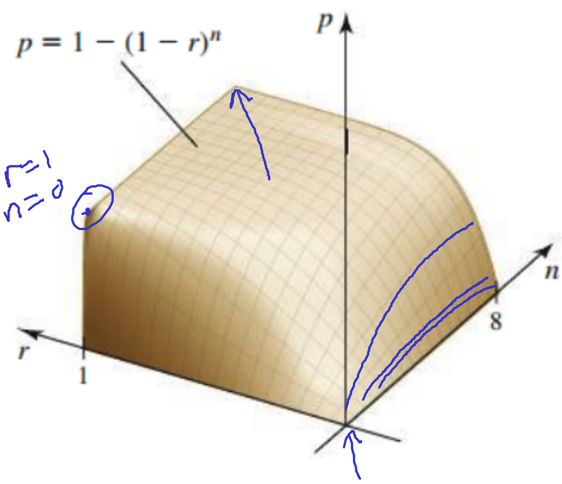


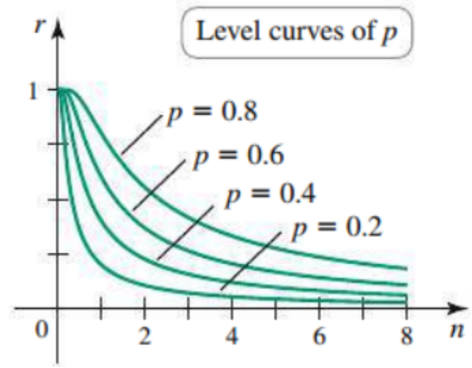


Applications of Functions of Two Variables:

Example. A probability function of two variables: Suppose on a particular day, the fraction of students on campus infected with COVID-19 is r, where $0 \le r \le 1$. If you have n random (possibly repeated) encounters with students during the day, the probability of meeting *at least* one infected person is $p(n, r) = 1 - (1 - r)^n$.

$p=1 \Leftarrow r=1,\ n=0$





Functions of More than Two Variables:

Number of Independent Variables	Explicit Form	Implicit Form	Graph Resides In...
1	$y = f(x)$	$F(x, y) = 0$	$\mathbb{R}^2(xy - \text{plane})$
2	$z = f(x, y)$	$F(x, y, z) = 0$	$\mathbb{R}^3(xyz - \text{space})$
3	$w = f(x, y, z)$	$F(x, y, z, w) = 0$	$\mathbb{R}^4$
n	$x_{n+1} = f(x_1, x_2, \ldots, x_n)$	$F(x_1, x_2, \ldots, x_n, x_{n+1}) = 0$	$\mathbb{R}^{n+1}$

Definition. (Function, Domain, and Range with n Independent Variables) The **function** $x_{n+1} = f(x_1, x_2, \ldots, x_n)$ assigns a unique real number x_{n+1} to each point $(x_1, x_2, \ldots, x_n)$ in a set D in $\mathbb{R}^4$. The set D is the **domain** of f. The **range** is the set of real numbers x_{n+1} that are assumed as the points $(x_1, x_2, \ldots, x_n)$ vary over the domain.

Example. Find the domain of the following functions:

$$f(x, y, z) = 4xyz - 2xz + 5yz$$

$x \in \mathbb{R}$ $\quad -\infty < x < \infty$

$y \in \mathbb{R}$ $\quad -\infty < y < \infty$

$z \in \mathbb{R}$ $\quad -\infty < z < \infty$

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 9}$$

$x^2 + y^2 + z^2 - 9 \geq 0$

$x^2 + y^2 + z^2 \geq 3^2$

Graphs of Functions of More Than Two Variables:

The idea of level curves can be extended to **level surfaces**. Level surfaces can be used to represent functions of three variables: