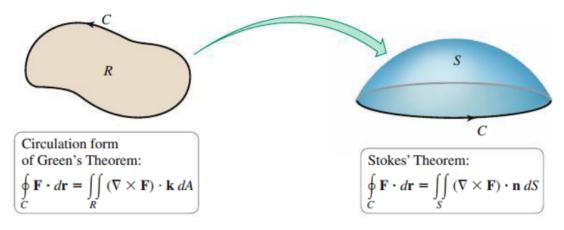
17.7: Stokes' Theorem

Stokes' Theorem is the three-dimensional version of the circulation form of Green's Theorem. Recall the circulation form of Green's Theorem:

$$\oint_{C} \mathbf{F} \cdot d\mathbf{r} = \iint_{R} \underbrace{(g_x - f_y)}_{\text{curl or rotation}} dA.$$

The above means that the cumulative rotation within R equals the circulation along the boundary of R. Stokes' Theorem computes the circulation over a surface S in \mathbb{R}^3 :



Theorem 17.15: Stokes' Theorem

Let S be an oriented surface in \mathbb{R}^3 with a piecewise-smooth closed boundary C whose orientation is consistent with that of S. Assume $\mathbf{F} = \langle f, g, h \rangle$ is a vector field whose components have continuous first partial derivatives on S. Then

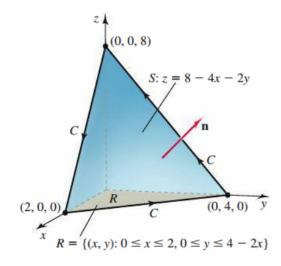
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS,$$

where \mathbf{n} is the unit vector normal to S determined by the orientation of S.

Example. Verify Stokes' Theorem: Confirm that Stokes' Theorem holds for the vector field $\mathbf{F} = \langle z - y, x, -x \rangle$, where S is the hemisphere $x^2 + y^2 + z^2 = 4$, for $z \ge 0$, and C is the circle $x^2 + y^2 = 4$, oriented counterclockwise.

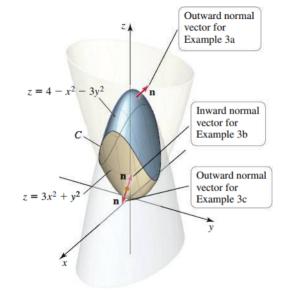
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Example. Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle z, -z, x^2 - y^2 \rangle$ and C consists of the three line segments that bound the plane z = 8 - 4x - 2y in the first octant.

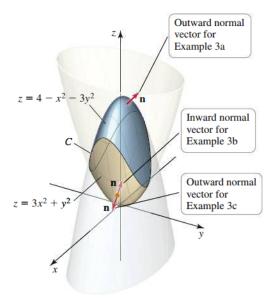


Example. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS$, where $\mathbf{F} = \langle -y, \, x, \, z \rangle$, where:

S is the part of the paraboloid $z = 4 - x^2 - 3y^2$ contained within $z = 3x^2 + y^2$, with **n** pointing upwards.



• S is the part of the paraboloid $z = 3x^2 + y^2$ contained within $z = 4 - x^2 - 3y^2$ with n pointing upwards.



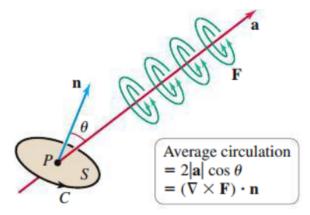
• S is the part of the paraboloid $z = 3x^2 + y^2$ contained within $z = 4 - x^2 - 3y^2$ with n pointing downwards.

Interpreting the Curl:

The average circulation is

$$\frac{1}{\text{area of } S} \oint_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{\text{area of } S} \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS.$$

Consider a general rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{r} = \langle x, y, z \rangle$. Now, let S be a small circular disk centered at a point P, whose normal vector \mathbf{n} makes an angle θ with the axis \mathbf{a} :



The average circulation of this vector field on S is

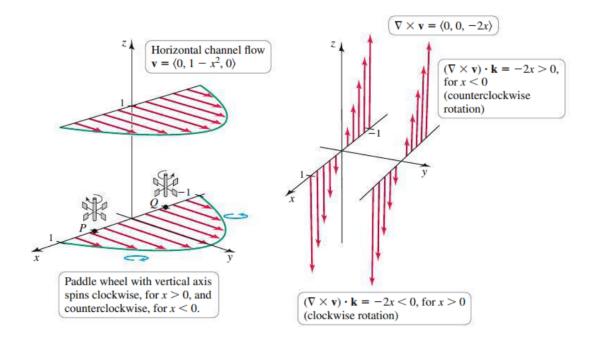
$$\frac{1}{\text{area of } S} \iint_{S} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \frac{(\nabla \times \mathbf{F}) \cdot \mathbf{n}}{\text{area of } S} \text{ (area of } S)$$
$$= 2\mathbf{a} \cdot \mathbf{n}$$
$$= 2|\mathbf{a}|\cos(\theta)$$

From this, we see

- The scalar component of $\nabla \times \mathbf{F}$ at P in the direction of \mathbf{n} is the average circulation of \mathbf{F} on S.
- The direction of $\nabla \times \mathbf{F}$ at P is the direction that maximizes the average circulation of \mathbf{F} on S.

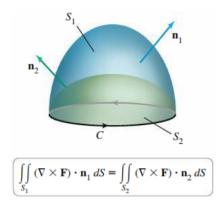
A similar argument for the curl can be applied to more general vector fields.

Example. Consider the vector field $\mathbf{v} = \langle 0, 1 - x^2, 0 \rangle$ for $|x| \leq 1$ and $|z| \leq 1$. Compute the curl of \mathbf{v} .



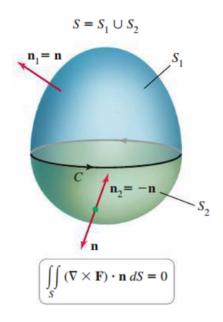
Since, using Stokes' Theorem, we evaluate the surface integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS$ using only the boundary C, then for any two smooth oriented surfaces \tilde{S}_1 and S_2 both with a consistent orientation with that of C,

$$\iint\limits_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_1 \, dS = \iint\limits_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, dS$$



Furthermore, if S is a closed surface consisting of S_1 and S_2 , with $\mathbf{n} = \mathbf{n}_1$ and $\mathbf{n} = -n_2$, then

$$\iint\limits_{S} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \iint\limits_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS + \iint\limits_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = 0$$



Theorem 17.11 (Section 17.5) states that if **F** is conservative, then $\nabla \times \mathbf{F} = \mathbf{0}$. Now, the converse follows using Stokes' Theorem:

Theorem 17.16: Curl F = 0 implies F Is Conservative

Suppose $\nabla \times \mathbf{F} = \mathbf{0}$ throughout an open simply connected region D of \mathbb{R}^3 . Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all closed simple smooth curves C in D, and \mathbf{F} is a conservative vector field on D.

Proof. Given a closed simple smooth curve C, it can be shown that C is the boundary of at least one smooth oriented surface S in D. By Stokes' Theorem,

$$\oint_{C} \mathbf{F} \cdot d\mathbf{r} = \iint_{S} \underbrace{(\nabla \times \mathbf{F})}_{0} \cdot \mathbf{n} \, dS = 0$$

Since the line integral equals zero over all such curves in D, the vector field is conservative on D.