2.7 Precise Definition of Limits

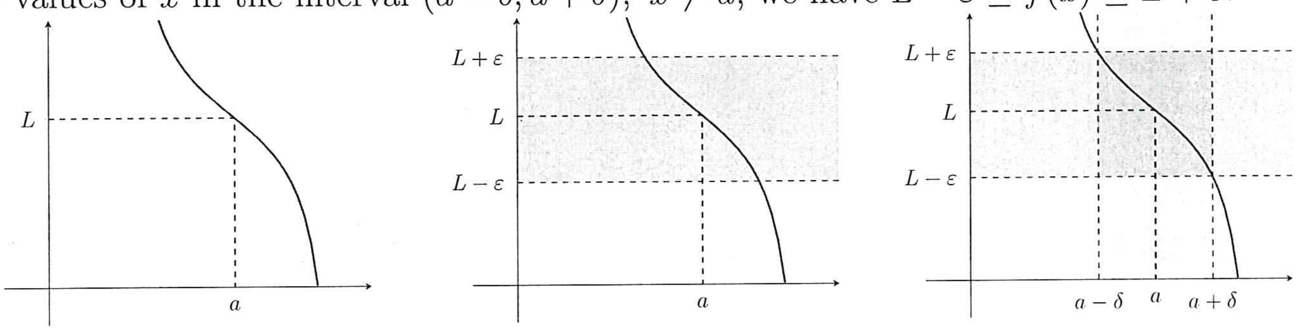
Definition (Limit of a Function)**.** Assume $f(x)$ is defined for all x in some open interval containing a, except possibly at a. We say **the limit of $f(x)$ as x approaches a is L**, written

$$\lim_{x \to a} f(x) = L$$

if for *any* number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

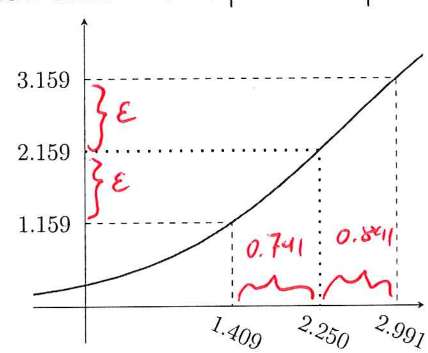
$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta$$

If we know L and $\varepsilon > 0$ is given, we can draw horizontal lines $L - \varepsilon$ and $L + \varepsilon$. Using the intersections of the graph and the horizontal lines, we can solve for $\delta > 0$ such that for values of x in the interval $(a - \delta, a + \delta)$, $x \neq a$, we have $L - \varepsilon \leq f(x) \leq L + \varepsilon$.



Note: As ε becomes smaller, δ will become smaller as well.

Example. Use the graph of f below to find a number δ such that if $0 < |x - 2.25| < \delta$ then $|f(x) - 2.159| < 1$.

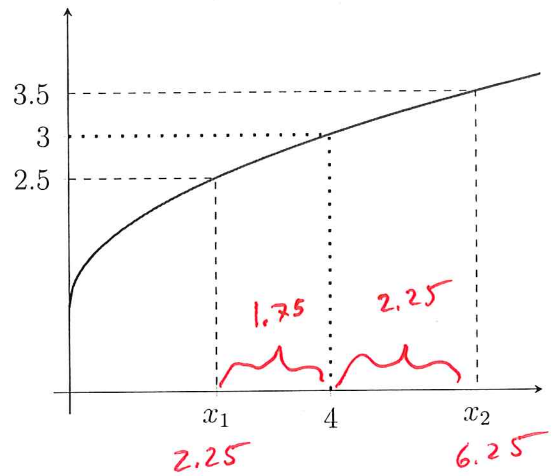


$$\begin{array}{r} 2.250 \\ -\ 1.409 \\ \hline 0.841 \end{array} \qquad \begin{array}{r} 2.991 \\ -\ 2.250 \\ \hline 0.741 \end{array}$$

Pick smaller

$\delta = 0.741$

Example. Use the graph of $g(x) = \sqrt{x}+1$ to help find a number δ such that if $|x-4| < \delta$ then $\left|(\sqrt{x}+1)-3\right| < \dfrac{1}{2}$.



$$|(\sqrt{x}+1)-3| < \frac{1}{2}$$

$$-\frac{1}{2} < \sqrt{x}-2 < \frac{1}{2}$$

$$1.5 < \sqrt{x} < 2.5$$

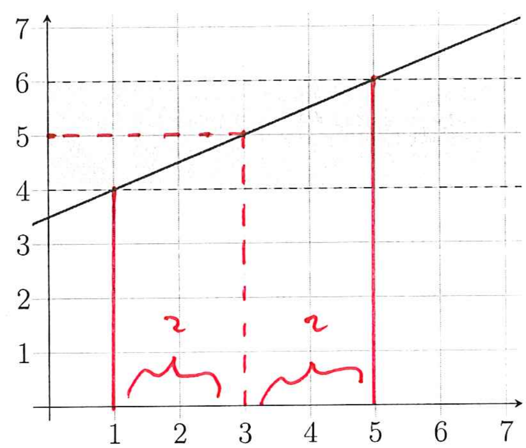
$$2.25 < x < 6.25$$

$$\Rightarrow \delta = \min\{4-2.25,\ 6.25-4\}$$

$$= \min\{1.75,\ 2.25\}$$

$$\Rightarrow \delta = 1.75$$

Example. Use the graph of the following linear function where $\lim_{x\to 3} h(x) = 5$ to find $\delta > 0$ such that $|h(x)-5| < 1$ whenever $0 < |x-3| < \delta$.



$\delta = 2$

$$\boxed{\text{Note: Slope is } 1/2 \quad \text{so } \delta = \frac{\varepsilon}{m}}$$

Steps for proving that $\lim_{x\to a} f(x) = L$

1. **Find** $\boldsymbol{\delta}$. Let ε be an arbitrary positive number. Use the inequality $|f(x) - L| < \varepsilon$ to find a condition of the form $|x - a| < \delta$, where δ depends only on the value of ε.
2. **Write a proof.** For any $\varepsilon > 0$, assume $0 < |x - a| < \delta$ and use the relationship between ε and δ found in Step 1 to prove that $|f(x) - L| < \varepsilon$.

Example. Use the $\varepsilon - \delta$ definition of a limit to prove $\lim_{x\to 4}(2x - 5) = 3$.

① Find δ

Want $|f(x) - 3| < \varepsilon$

$|(2x-5)-3| < \varepsilon$

$|2x-8| < \varepsilon$

$2|x-4| < \varepsilon$

$\Rightarrow |x-4| < \frac{\varepsilon}{2} =: \delta$

② Show this works

Let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{2}$, then for $|x-4| < \delta$, we have

$$|(2x-5)-3| = |2x-8| = 2|x-4| < 2\delta = 2\left(\frac{\varepsilon}{2}\right) = \varepsilon$$

$|x-4| < \delta$

$\Rightarrow \boxed{|f(x) - 3| < \varepsilon}$

Example. Use the $\varepsilon - \delta$ definition of a limit to prove $\lim_{x\to 2} \frac{x}{5} = \frac{2}{5}$.

① Find δ

$|f(x) - \frac{2}{5}| < \varepsilon$

$|\frac{x}{5} - \frac{2}{5}| < \varepsilon$

$\frac{1}{5}|x-2| < \varepsilon$

$\Rightarrow |x-2| < 5\varepsilon =: \delta$

② Show this works

Let $\varepsilon > 0$ and $\delta = 5\varepsilon$, then for $|x-2| < \delta$, we have

$$|f(x) - \frac{2}{5}| = |\frac{x}{5} - \frac{2}{5}| = \frac{1}{5}|x-2| < \frac{1}{5}\delta = \frac{1}{5}(5\varepsilon) = \varepsilon$$

$|x-2| < \delta$

Example. Use the $\varepsilon - \delta$ definition of a limit to prove $\lim_{x\to 2} \dfrac{x^2+x-6}{x-2} = 5.$

① Find δ

Want $\left| \dfrac{x^2+x-6}{x-2} - 5 \right| < \varepsilon$

$\left| \dfrac{(x+3)(x-2)}{x-2} - 5 \right| < \varepsilon$

$|(x+3) - 5| < \varepsilon$

$\Longrightarrow \quad |x-2| < \varepsilon =: \delta$

② Show it works

Let $\varepsilon > 0$ and $\delta = \varepsilon$, then for $|x-2|<\delta$, we have

$$\left| \frac{x^2+x-6}{x-2} - 5 \right| = \left| \frac{(x+3)(x-2)}{x-2} - 5 \right|$$

$$= |x+3-5|$$

$$= |x-2| \quad |x-2|<\delta$$

$$< \delta = \varepsilon$$

$\Rightarrow \quad |f(x) - 5| < \varepsilon$

Example. Use the $\varepsilon - \delta$ definition of a limit to prove $\lim_{x\to 3} \dfrac{x^2+2x-15}{2x-6} = 4.$

① Find δ

Want $\left| \dfrac{x^2+2x-15}{2x-6} - 4 \right| < \varepsilon$

$\left| \dfrac{(x+5)(x-3)}{2(x-3)} - 4 \right| < \varepsilon$

$\left| \frac{1}{2}(x+5) - \frac{1}{2}(8) \right| < \varepsilon$

$\frac{1}{2} |x-3| < \varepsilon$

$\Rightarrow \quad |x-3| < 2\varepsilon =: \delta$

② Show it works

Let $\varepsilon > 0$ and $\delta = 2\varepsilon$, then for $|x-3| < \delta$, we have

$$\left| \frac{x^2+2x-15}{2x-6} - 4 \right| = \left| \frac{(x+5)(x-3)}{2(x-3)} - 4 \right|$$

$$= \left| \tfrac{1}{2}(x+5) - \tfrac{1}{2}(8) \right|$$

$$= \tfrac{1}{2} |x-3| \quad |x-3|<\delta$$

$$< \tfrac{1}{2}\delta$$

$$= \tfrac{1}{2}(2\varepsilon) = \varepsilon$$

$\Rightarrow \quad \left| \dfrac{x^2+2x-15}{2x-6} - 4 \right| < \varepsilon.$