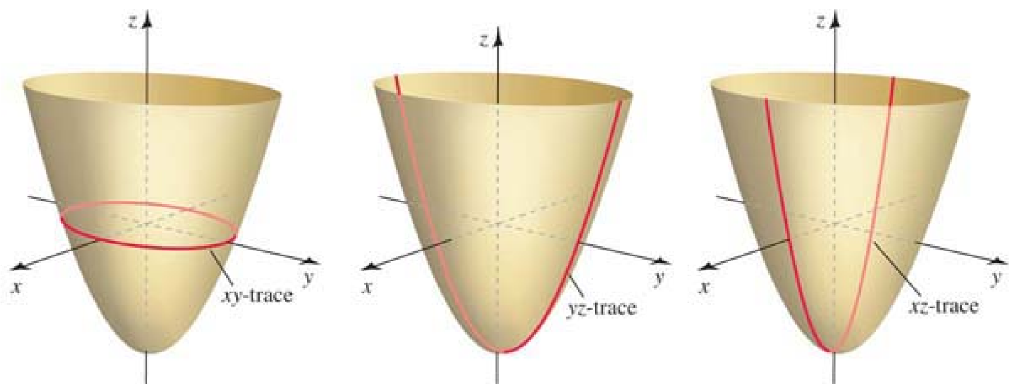
13.6: Cylinders and Quadric Surfaces

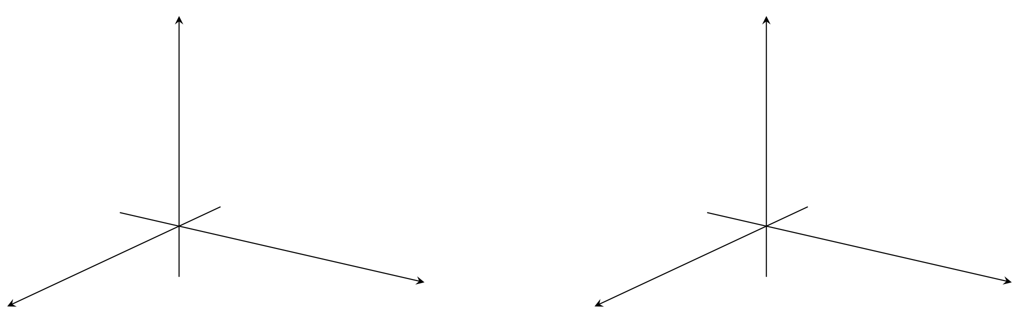
Cylinders and Traces:
When talking about three-dimensional surfaces, a *cylinder* refers to a surface that is parallel to a line. When considering surfaces that is parallel to one of the coordinate axes, that the associated variable is missing (e.g. $3y^2 + z^2 = 8$ is parallel to the x-axis).

Definition. (Trace)
A **trace** of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes. The traces in the coordinate planes are called the xy**-trace**, the yz**-trace**, and the xz**-trace** (Figure 13.80).



Example. Roughly sketch the following functions:

1. $x^2 + 4y^2 = 16$
2. $x - \sin(z) = 0$



Quadric Surfaces:

Quadric surfaces are described by the general quadratic (second-degree) equation in three variables,

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0,$$

Where the coefficients $A, \dots, J$ and not all zero. To sketch quadric surfaces, keep the following ideas in mind:

1. **Intercepts** Determine the points, if any, where the surface intersects the coordinate axes. To find these intercepts, set x, y, and z equal to zero in pairs in the equation of the surface, and solve for the third coordinate.
2. **Traces** Finding traces of the surface helps visualize the surface. Setting x, y, and z equal to zero in pairs gives the planes parallel in that pair's plane.
3. **Completing the figure** Sketch some traces in parallel planes, then draw smooth curves that pass through the traces to fill out the surface.

Example (An ellipsoid)**.** The surface defined by the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. Graph $a = 3$, $b = 4$ and $c = 5$.

Example (An elliptic parabaloid)**.** The surface defined by the equation $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$. Graph the elliptic paraboloid with $a = 4$ and $b = 2$.

Example (A hyperboloid of one sheet)**.**

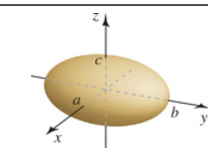
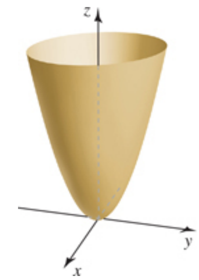
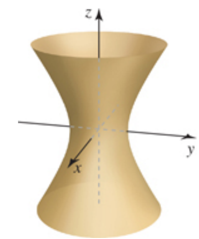
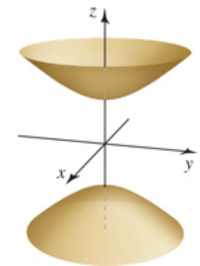
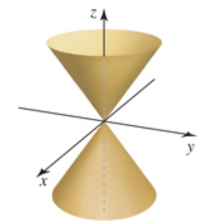
Graph the surface defined by the equation $\dfrac{x^2}{4} + \dfrac{y^2}{9} - z^2 = 1$.

Example (A hyperboloid of two sheets)**.** Graph the surface defined by the equation $-16x^2 - 4y^2 + z^2 + 64x - 80 = 0$.

Example (Elliptic cones)**.** Graph the surface defined by the equation $\frac{y^2}{4} + z^2 = 4x^2$.

Example (A hyperbolic paraboloid).

Graph the surface defined by the equation $z = x^2 - \dfrac{y^2}{4}$.

Name	**Standard Equation**	**Features**	**Graph**
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	All traces are ellipses.	
Elliptic paraboloid	$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$	Traces with $z=z_0>0$ are ellipses. Traces with $x=x_0$ or $y=y_0$ are parabolas.	
Hyperboloid of one sheet	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	Traces with $z=z_0$ are ellipses for all z_0. Traces with $x=x_0$ or $y=y_0$ are hyperbolas.	
Hyperboloid of two sheets	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	Traces with $z=z_0$ with $\lvert z_0\rvert>\lvert c\rvert$ are ellipses. Traces with $x=x_0$ and $y=y_0$ are hyperbolas.	
Elliptic cone	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$	Traces with $z=z_0\neq 0$ are ellipses. Traces with $x=x_0$ or $y=y_0$ are hyperbolas or intersecting lines.	
Hyperbolic paraboloid	$z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$	Traces with $z=z_0\neq 0$ are hyperbolas. Traces with $x=x_0$ or $y=y_0$ are parabolas.