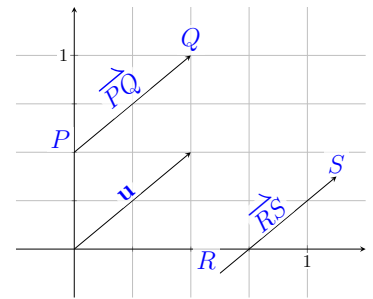


## 1 13.1: Vectors and the Geometry of Space

### Definition.

- **Vectors**

- Have a direction and magnitude,
- vector  $\overrightarrow{PQ}$  has a *tail* at  $P$  and a *head* at  $Q$ ,
- Can be denoted as  $\mathbf{u}$  or  $\vec{u}$ ,
- Equal vectors have the same direction and magnitude (not necessarily the same position)



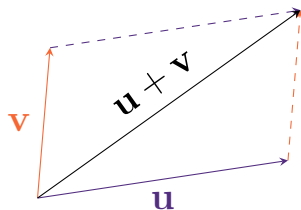
- **Scalars** are quantities with magnitude but no direction (e.g. mass, temperature, price, time, etc.)
- **Zero vector**, denoted  $\mathbf{0}$  or  $\vec{0}$ , has length 0 and no direction

### Scalar-vector multiplication:

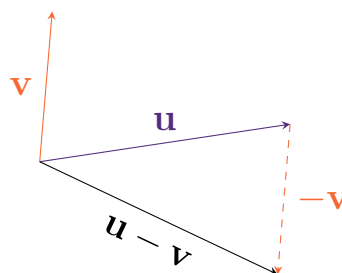
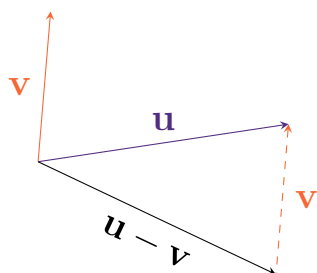
- Denoted  $c\mathbf{v}$  or  $c\vec{v}$ ,
- length of vector multiplied by  $|c|$ ,
- $c\mathbf{v}$  has the same direction as  $\mathbf{v}$  if  $c > 0$ , and has the opposite direction as  $\mathbf{v}$  if  $c < 0$ , (what if  $c = 0$ ?)
- $\mathbf{u}$  and  $\mathbf{v}$  are **parallel** if  $\mathbf{u} = c\mathbf{v}$ . (what vectors are parallel to  $\mathbf{0}$ ?)

### Vector Addition and Subtraction:

Given two vectors  $\mathbf{u}$  and  $\mathbf{v}$ , their sum,  $\mathbf{u} + \mathbf{v}$ , can be represented by the parallelogram (triangle) rule: place the tail of  $\mathbf{v}$  at the head of  $\mathbf{u}$

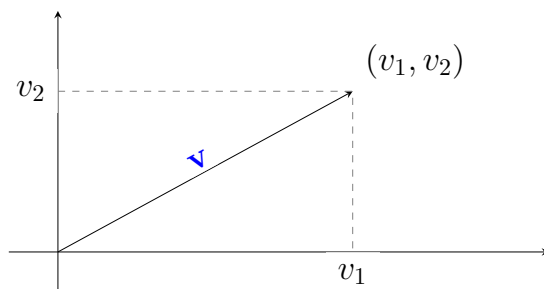


The difference, denoted  $\mathbf{u} - \mathbf{v}$ , is the sum of  $\mathbf{u} + (-\mathbf{v})$ :



### Vector Components:

A vector  $\mathbf{v}$  whose tail is at the origin  $(0, 0)$  and head is at  $(v_1, v_2)$  is a **position vector** (in **standard position**) and is denoted  $\langle v_1, v_2 \rangle$ . The real numbers  $v_1$  and  $v_2$  are the  $x$ - and  $y$ -components of  $\mathbf{v}$ .



Vectors  $\mathbf{u} = \langle u_1, u_2 \rangle$  and  $\mathbf{v} = \langle v_1, v_2 \rangle$  are equal if and only if  $u_1 = v_1$  and  $u_2 = v_2$ .

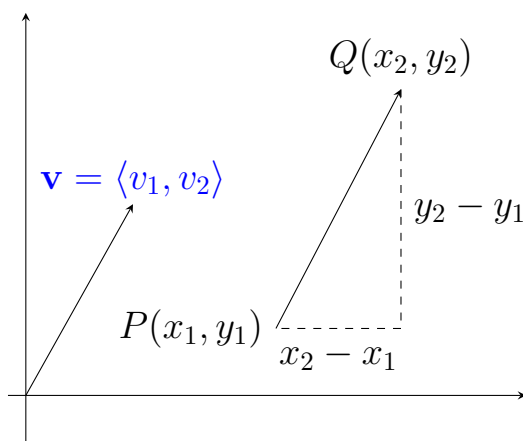
### Magnitude:

Given points  $P(x_1, y_1)$  and  $Q(x_2, y_2)$ , the **magnitude**, or **length**, of vector  $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$ , denoted  $|\overrightarrow{PQ}|$ , is the distance between points  $P$  and  $Q$ .

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The magnitude of position vector  $\mathbf{v} = \langle v_1, v_2 \rangle$  is  $|\mathbf{v}|$ .

(How do  $|\overrightarrow{PQ}|$  and  $|\overrightarrow{QP}|$  relate to each other?)



Note: The norm, denoted  $\|\mathbf{u}\|$  or  $\|\mathbf{u}\|_2$ , is equivalent to the magnitude of a vector.

### Equation of a Circle:

#### Definition.

A **circle** centered at  $(a, b)$  with radius  $r$  is the set of points satisfying the equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

A **disk** centered at  $(a, b)$  with radius  $r$  is the set of points satisfying the inequality

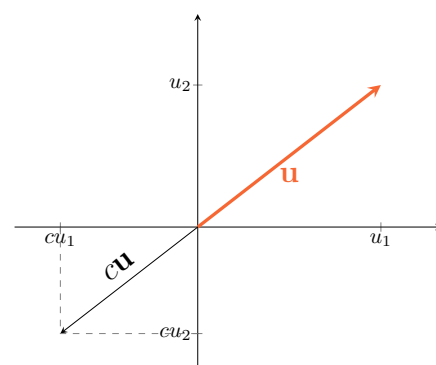
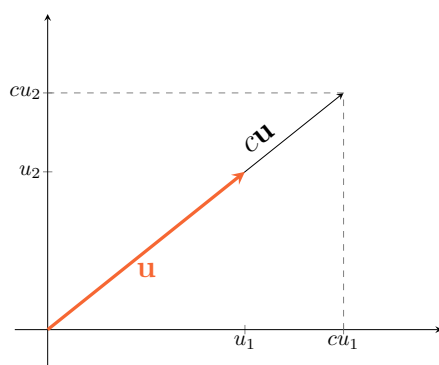
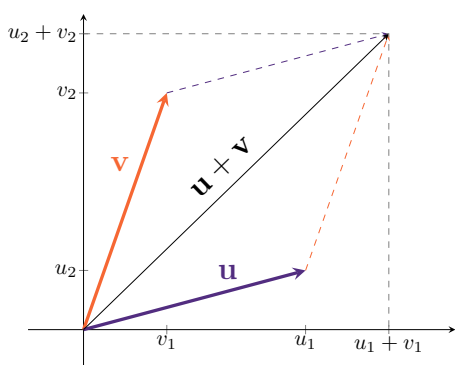
$$(x - a)^2 + (y - b)^2 \leq r^2.$$

## Vector Operations in Terms of Components

### Definition. (Vector Operations in $\mathbb{R}^2$ )

Suppose  $c$  is a scalar,  $\mathbf{u} = \langle u_1, u_2 \rangle$ , and  $\mathbf{v} = \langle v_1, v_2 \rangle$ .

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= \langle u_1 + v_1, u_2 + v_2 \rangle && \text{Vector addition} \\ \mathbf{u} - \mathbf{v} &= \langle u_1 - v_1, u_2 - v_2 \rangle && \text{Vector subtraction} \\ c\mathbf{u} &= \langle cu_1, cu_2 \rangle && \text{Scalar multiplication}\end{aligned}$$



**Example.** Let  $\mathbf{u} = \langle 1, 2 \rangle$ ,  $\mathbf{v} = \langle -2, 3 \rangle$ ,  $c = 2$ , and  $d = 3$ . Find the following:

$$\mathbf{u} + \mathbf{v}$$

$$c\mathbf{u}$$

$$c\mathbf{u} + d\mathbf{v}$$

$$\mathbf{u} - c\mathbf{v}$$

### Definition.

A **unit vector** is any vector with length 1.

In  $\mathbb{R}^2$ , the **coordinate unit vectors** are  $\mathbf{i} = \langle 1, 0 \rangle$  and  $\mathbf{j} = \langle 0, 1 \rangle$ .

**Example.** Let  $\mathbf{u} = \langle -7, 3 \rangle$ . Find two unit vectors parallel to  $\mathbf{u}$ . Find another vector parallel to  $\mathbf{u}$  with a magnitude of 2.

### Properties of Vector Operations:

Suppose  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  are vectors and  $a$  and  $c$  are scalars. Then the following properties hold (for vectors in any number of dimensions).

- |  |   |
|--|---|
| 1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$                               | Commutative property of addition              |
| 2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | Associative property of addition              |
| 3. $\mathbf{v} + \mathbf{0} = \mathbf{v}$  | Additive identity                             |
| 4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$   | Additive inverse                              |
| 5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$                          | Distributive property 1                       |
| 6. $(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$                                   | Distributive property 2                       |
| 7. $0\mathbf{v} = \mathbf{0}$  | Multiplication by zero scalar                 |
| 8. $c\mathbf{0} = \mathbf{0}$  | Multiplication by zero vector                 |
| 9. $1\mathbf{v} = \mathbf{v}$  | Multiplicative identity                       |
| 10. $a(c\mathbf{v}) = (ac)\mathbf{v}$  | Associative property of scalar multiplication |