

# Math 2060 Class notes Spring 2021

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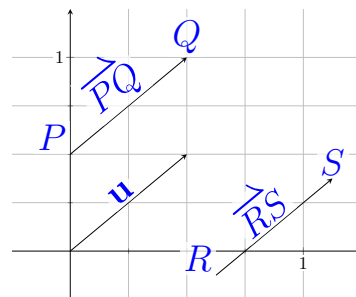
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## 13.1: Vectors and the Geometry of Space

### Definition.

- **Vectors**

- Have a direction and magnitude,
- vector  $\overrightarrow{PQ}$  has a *tail* at  $P$  and a *head* at  $Q$ ,
- Can be denoted as  $\mathbf{u}$  or  $\vec{u}$ ,
- Equal vectors have the same direction and magnitude (not necessarily the same position)



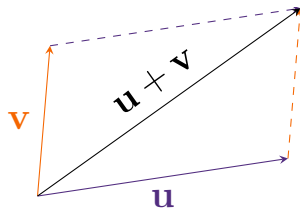
- **Scalars** are quantities with magnitude but no direction (e.g. mass, temperature, price, time, etc.)
- **Zero vector**, denoted  $\mathbf{0}$  or  $\vec{0}$ , has length 0 and no direction

### Scalar-vector multiplication:

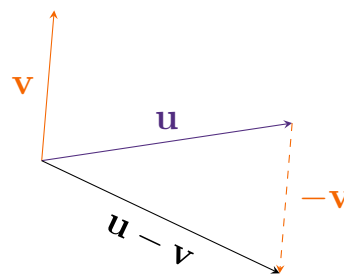
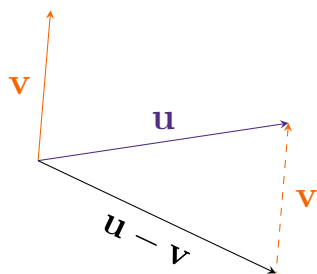
- Denoted  $c\mathbf{v}$  or  $c\vec{v}$ ,
- length of vector multiplied by  $|c|$ ,
- $c\mathbf{v}$  has the same direction as  $\mathbf{v}$  if  $c > 0$ , and has the opposite direction as  $\mathbf{v}$  if  $c < 0$ , (what if  $c = 0$ ?)
- $\mathbf{u}$  and  $\mathbf{v}$  are **parallel** if  $\mathbf{u} = c\mathbf{v}$ . (what vectors are parallel to  $\mathbf{0}$ ?)

### Vector Addition and Subtraction:

Given two vectors  $\mathbf{u}$  and  $\mathbf{v}$ , their sum,  $\mathbf{u} + \mathbf{v}$ , can be represented by the parallelogram (triangle) rule: place the tail of  $\mathbf{v}$  at the head of  $\mathbf{u}$

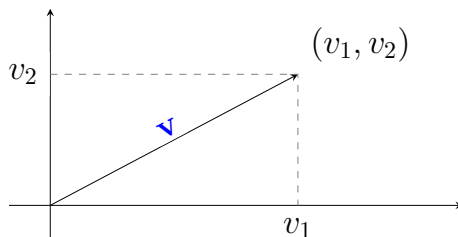


The difference, denoted  $\mathbf{u} - \mathbf{v}$ , is the sum of  $\mathbf{u} + (-\mathbf{v})$ :



### Vector Components:

A vector  $\mathbf{v}$  whose tail is at the origin  $(0, 0)$  and head is at  $(v_1, v_2)$  is a **position vector** (in **standard position**) and is denoted  $\langle v_1, v_2 \rangle$ . The real numbers  $v_1$  and  $v_2$  are the  $x$ - and  $y$ -components of  $\mathbf{v}$ .



Vectors  $\mathbf{u} = \langle u_1, u_2 \rangle$  and  $\mathbf{v} = \langle v_1, v_2 \rangle$  are equal if and only if  $u_1 = v_1$  and  $u_2 = v_2$ .

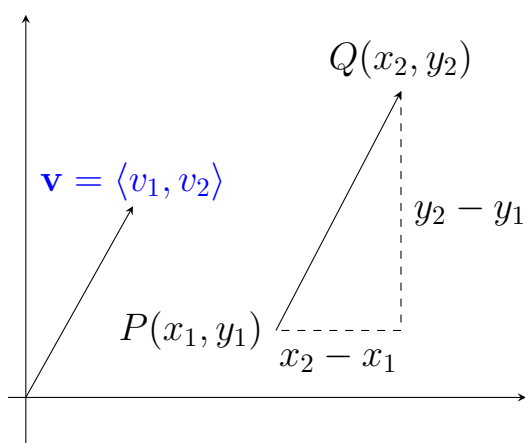
### Magnitude:

Given points  $P(x_1, y_1)$  and  $Q(x_2, y_2)$ , the **magnitude**, or **length**, of vector  $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$ , denoted  $|\overrightarrow{PQ}|$ , is the distance between points  $P$  and  $Q$ .

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The magnitude of position vector  $\mathbf{v} = \langle v_1, v_2 \rangle$  is  $|\mathbf{v}|$ .

(How do  $|\overrightarrow{PQ}|$  and  $|\overrightarrow{QP}|$  relate to each other?)



Note: The norm, denoted  $\|\mathbf{u}\|$  or  $\|\mathbf{u}\|_2$ , is equivalent to the magnitude of a vector.

### Equation of a Circle:

#### Definition.

A **circle** centered at  $(a, b)$  with radius  $r$  is the set of points satisfying the equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

A **disk** centered at  $(a, b)$  with radius  $r$  is the set of points satisfying the inequality

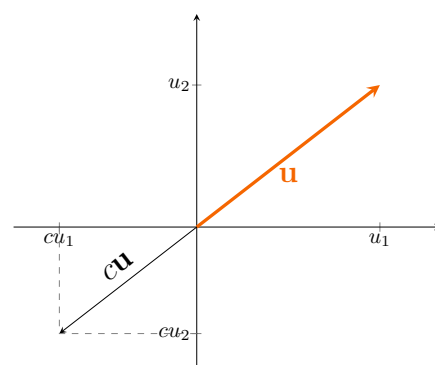
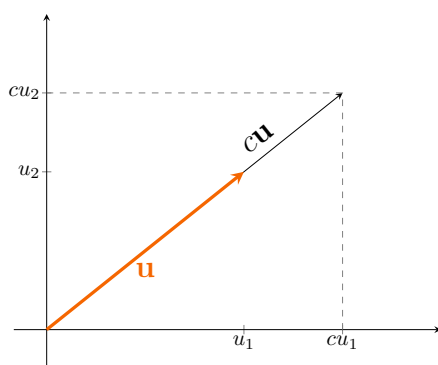
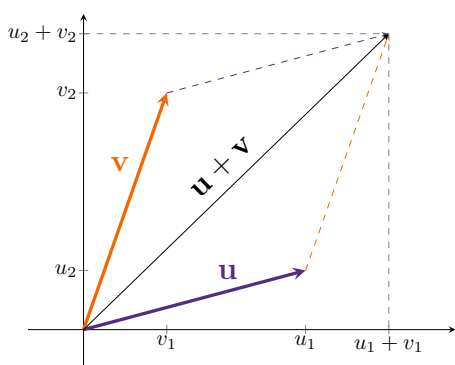
$$(x - a)^2 + (y - b)^2 \leq r^2.$$

## Vector Operations in Terms of Components

### Definition. (Vector Operations in $\mathbb{R}^2$ )

Suppose  $c$  is a scalar,  $\mathbf{u} = \langle u_1, u_2 \rangle$ , and  $\mathbf{v} = \langle v_1, v_2 \rangle$ .

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= \langle u_1 + v_1, u_2 + v_2 \rangle && \text{Vector addition} \\ \mathbf{u} - \mathbf{v} &= \langle u_1 - v_1, u_2 - v_2 \rangle && \text{Vector subtraction} \\ c\mathbf{u} &= \langle cu_1, cu_2 \rangle && \text{Scalar multiplication}\end{aligned}$$



**Example.** Let  $\mathbf{u} = \langle 1, 2 \rangle$ ,  $\mathbf{v} = \langle -2, 3 \rangle$ ,  $c = 2$ , and  $d = 3$ . Find the following:

$$\mathbf{u} + \mathbf{v}$$

$$c\mathbf{u}$$

$$c\mathbf{u} + d\mathbf{v}$$

$$\mathbf{u} - c\mathbf{v}$$

### Definition.

A **unit vector** is any vector with length 1.

In  $\mathbb{R}^2$ , the **coordinate unit vectors** are  $\mathbf{i} = \langle 1, 0 \rangle$  and  $\mathbf{j} = \langle 0, 1 \rangle$ .

**Example.** Let  $\mathbf{u} = \langle -7, 3 \rangle$ . Find two unit vectors parallel to  $\mathbf{u}$ . Find another vector parallel to  $\mathbf{u}$  with a magnitude of 2.

### Properties of Vector Operations:

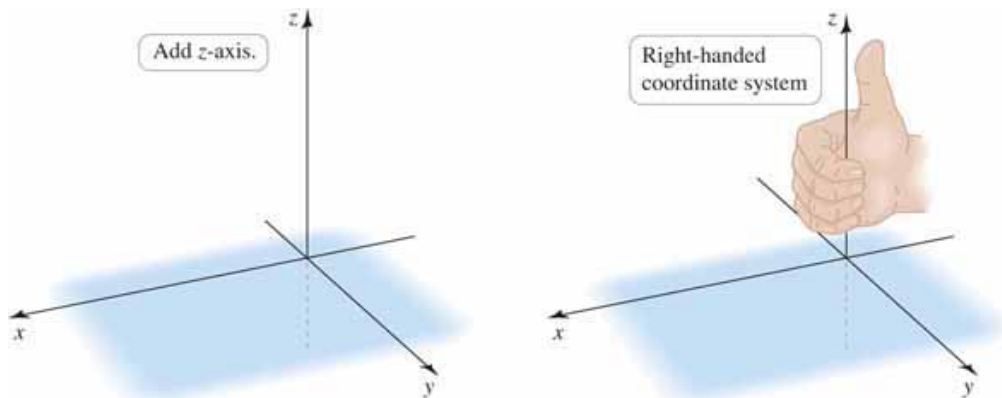
Suppose  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  are vectors and  $a$  and  $c$  are scalars. Then the following properties hold (for vectors in any number of dimensions).

- |  |   |
|--|---|
| 1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$                               | Commutative property of addition              |
| 2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | Associative property of addition              |
| 3. $\mathbf{v} + \mathbf{0} = \mathbf{v}$  | Additive identity                             |
| 4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$   | Additive inverse                              |
| 5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$                          | Distributive property 1                       |
| 6. $(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$                                   | Distributive property 2                       |
| 7. $0\mathbf{v} = \mathbf{0}$  | Multiplication by zero scalar                 |
| 8. $c\mathbf{0} = \mathbf{0}$  | Multiplication by zero vector                 |
| 9. $1\mathbf{v} = \mathbf{v}$  | Multiplicative identity                       |
| 10. $a(c\mathbf{v}) = (ac)\mathbf{v}$  | Associative property of scalar multiplication |

## 13.2: Vectors in Three Dimensions

### The $xyz$ - Coordinate System:

The three-dimensional coordinate system is created by adding the  $z$ -axis, which is perpendicular to both the  $x$ -axis and the  $y$ -axis. When looking at the  $xy$ -plane, the positive direction of the  $z$ -axis protrudes towards the viewer. This can also be shown using the right-hand rule (Figure 13.25 from Briggs):

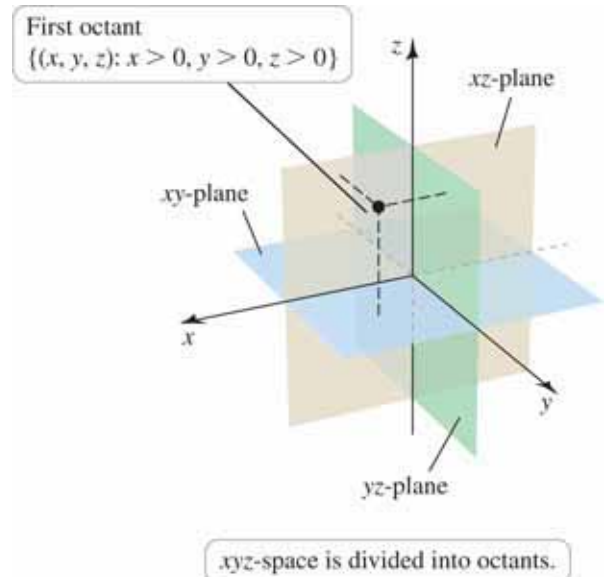


### Definition.

This three-dimensional coordinate system is broken up into eight **octants**, which are separated by

- the  $xy$ -**plane** ( $z = 0$ ),
- the  $xz$ -**plane** ( $y = 0$ ), and
- the  $yz$ -**plane** ( $x = 0$ ).

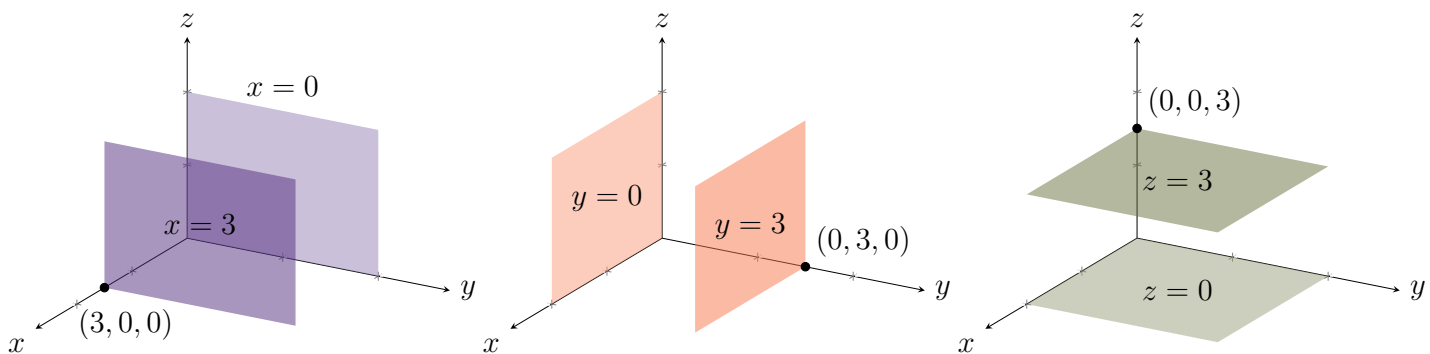
The **origin** is the location where all three axes intersect.



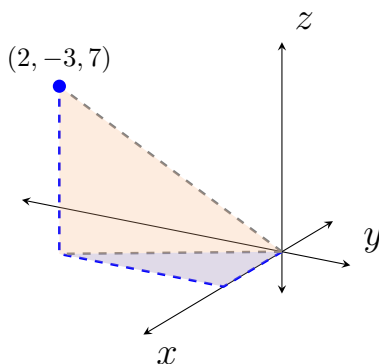


### Equations of Simple Planes:

Planes in three-dimensions are analogous to lines in two-dimensions. Below, we see the  $yz$ -plane, the  $xz$ -plane, and the  $xy$ -plane, along with planes that are parallel where  $x$ ,  $y$ , and  $z$  are fixed respectively:



**Example** (Parallel planes). Determine the equation of the plane parallel to the  $xz$ -plane passing through the point  $(2, -3, 7)$ .



### Distances in $xyz$ -Space:

Recall that in  $\mathbb{R}^2$ , for some vector  $\overrightarrow{PR}$ , the distance formula is given by

$$|PR| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

where  $(x_1, y_1)$  and  $(x_2, y_2)$  represent the points  $P$  and  $R$  respectively. This idea can be further extended into  $\mathbb{R}^3$  by considering the two sides of the triangle formed by the points  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$ :



### Distance Formula in $xyz$ -Space

The **distance** between points  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$  is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The **midpoint** between points  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$  is found by averaging the  $x$ -,  $y$ -, and  $z$ -coordinates:

$$\text{Midpoint} = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

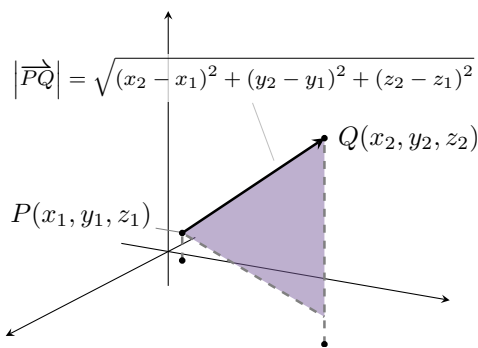
## Magnitude and Unit Vectors:

### Definition.

The **magnitude** (or **length**) of the vector  $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$  is the distance from  $P(x_1, y_1, z_1)$  to  $Q(x_2, y_2, z_2)$ :

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

In  $\mathbb{R}^3$ , the **coordinate unit vectors** are  $\mathbf{i} = \langle 1, 0, 0 \rangle$ ,  $\mathbf{j} = \langle 0, 1, 0 \rangle$ , and  $\mathbf{k} = \langle 0, 0, 1 \rangle$ .



**Example.** Consider  $P(-1, 4, 3)$  and  $Q(3, 5, 7)$ . Find

- $|\overrightarrow{PQ}|$
- The midpoint between  $P$  and  $Q$
- Two unit vectors parallel to  $\overrightarrow{PQ}$

## Equation of a Sphere:

**Definition.**

A **sphere** centered at  $(a, b, c)$  with radius  $r$  is the set of points satisfying the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

A **ball** centered at  $(a, b, c)$  with radius  $r$  is the set of points satisfying the inequality

$$(x - a)^2 + (y - b)^2 + (z - c)^2 \leq r^2.$$

**Example.** Consider  $P(-1, 4, 3)$  and  $Q(3, 5, 7)$ . Find the equation of the sphere centered at the midpoint passing through  $P$  and  $Q$

**Example.** What is the geometry of the intersection between  $x^2 + y^2 + z^2 = 50$  and  $z = 1$ ?

**Example.** Rewrite the following equation into the standard form of a sphere:

$$x^2 + y^2 + z^2 - 2x + 6y - 8z = -1$$

## Vector Operations in Terms of Components

### Definition. (Vector Operations in $\mathbb{R}^3$ )

Suppose  $c$  is a scalar,  $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ , and  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ .

$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$	Vector addition
$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$	Vector subtraction
$c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle$	Scalar multiplication

### Properties of Vector Operations:

Suppose  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  are vectors and  $a$  and  $c$  are scalars. Then the following properties hold (for vectors in any number of dimensions).

- |  |   |
|--|---|
| 1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$                               | Commutative property of addition              |
| 2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | Associative property of addition              |
| 3. $\mathbf{v} + \mathbf{0} = \mathbf{v}$  | Additive identity                             |
| 4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$   | Additive inverse                              |
| 5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$                          | Distributive property 1                       |
| 6. $(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$                                   | Distributive property 2                       |
| 7. $0\mathbf{v} = \mathbf{0}$  | Multiplication by zero scalar                 |
| 8. $c\mathbf{0} = \mathbf{0}$  | Multiplication by zero vector                 |
| 9. $1\mathbf{v} = \mathbf{v}$  | Multiplicative identity                       |
| 10. $a(c\mathbf{v}) = (ac)\mathbf{v}$  | Associative property of scalar multiplication |

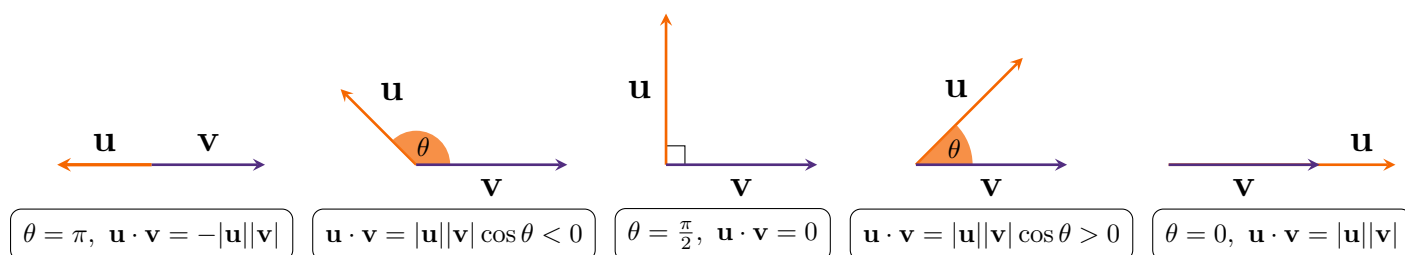
### 13.3: Dot Products

#### Definition. (Dot Product)

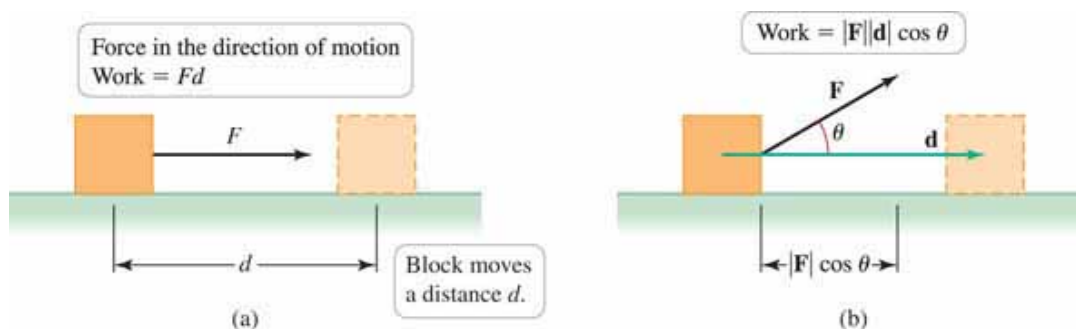
Given two nonzero vectors  $\mathbf{u}$  and  $\mathbf{v}$  in two or three dimensions, their **dot product** is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta,$$

where  $\theta$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$  with  $0 \leq \theta \leq \pi$ . If  $\mathbf{u} = \mathbf{0}$  or  $\mathbf{v} = \mathbf{0}$ , then  $\mathbf{u} \cdot \mathbf{v} = 0$ , and  $\theta$  is undefined.



A physical example of the dot product is the amount of work done when a force is applied at an angle  $\theta$  as shown in figure 13.43:



*Note:* The result of the dot product is a scalar!

**Definition. (Orthogonal Vectors)**

Two vectors  $\mathbf{u}$  and  $\mathbf{v}$  are **orthogonal** if and only if  $\mathbf{u} \cdot \mathbf{v} = 0$ . The zero vector is orthogonal to all vectors. In two or three dimensions, two nonzero orthogonal vectors are perpendicular to each other.

- $\mathbf{u}$  and  $\mathbf{v}$  are parallel ( $\theta = 0$  or  $\theta = \pi$ ) if and only if  $\mathbf{u} \cdot \mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$ .
- $\mathbf{u}$  and  $\mathbf{v}$  are perpendicular ( $\theta = \frac{\pi}{2}$ ) if and only if  $\mathbf{u} \cdot \mathbf{v} = 0$ .

**Example.** Given  $|\mathbf{u}| = 2$  and  $|\mathbf{v}| = \sqrt{3}$ , compute  $\mathbf{u} \cdot \mathbf{v}$  when

- $\theta = \frac{\pi}{4}$
- $\theta = \frac{\pi}{3}$
- $\theta = \frac{5\pi}{6}$

**Theorem 31.1: Dot Product**

Given two vectors  $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$  and  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ ,

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

**Example.** Given vectors  $\mathbf{u} = \langle \sqrt{3}, 1, 0 \rangle$  and  $\mathbf{v} = \langle 1, \sqrt{3}, 0 \rangle$ , compute  $\mathbf{u} \cdot \mathbf{v}$  and find  $\theta$ .



## Properties of Dot Products

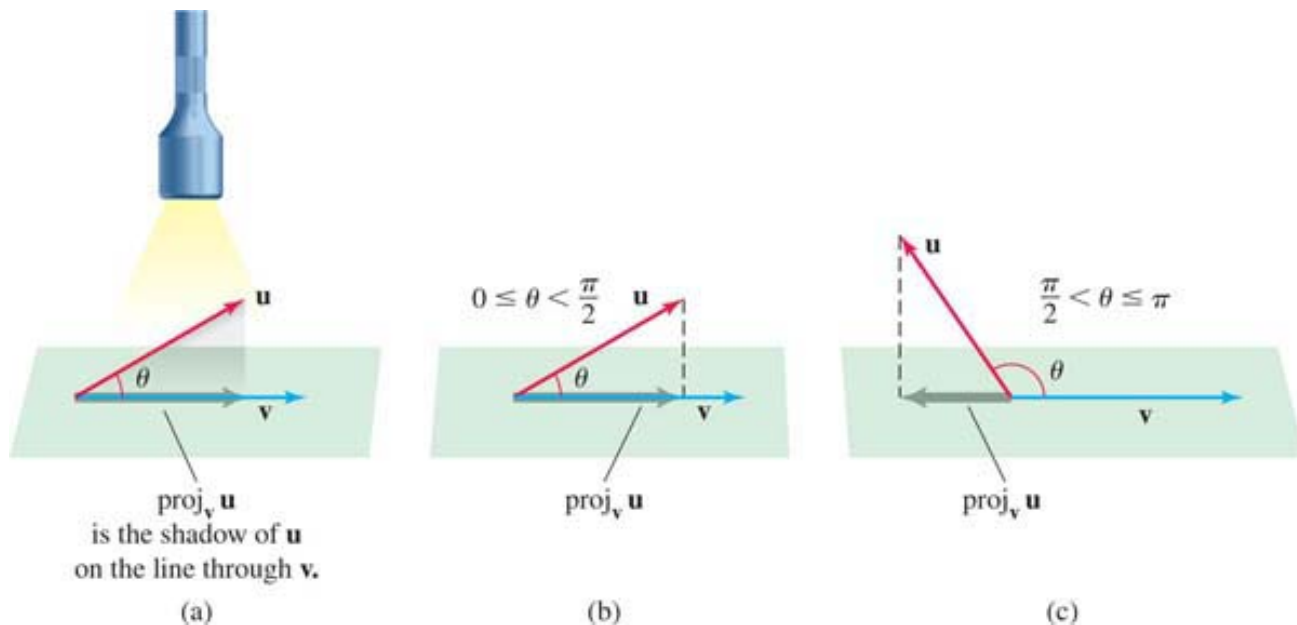
### Theorem 13.2: Properties of the Dot Product

Suppose  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  are vectors and let  $c$  be a scalar.

- |   |                       |
|---|-----------------------|
| 1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$  | Commutative property  |
| 2. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$       | Associative property  |
| 3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ | Distributive property |

## Orthogonal Projections

Given vectors  $\mathbf{u}$  and  $\mathbf{v}$ , the projection of  $\mathbf{u}$  onto  $\mathbf{v}$  produces a vector parallel to  $\mathbf{v}$  using the “shadow” of  $\mathbf{u}$  cast onto  $\mathbf{v}$ .



**Definition. ((Orthogonal) Projection of  $\mathbf{u}$  onto  $\mathbf{v}$ )**

The **orthogonal projection of  $\mathbf{u}$  onto  $\mathbf{v}$** , denoted  $\text{proj}_{\mathbf{v}} \mathbf{u}$ , where  $\mathbf{v} \neq \mathbf{0}$ , is

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \underbrace{|\mathbf{u}|}_{\text{length}} \cos \theta \underbrace{\left( \frac{\mathbf{v}}{|\mathbf{v}|} \right)}_{\text{direction}}.$$

The orthogonal projection may also be computed with the formulas

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \text{scal}_{\mathbf{v}} \mathbf{u} \left( \frac{\mathbf{v}}{|\mathbf{v}|} \right) = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v},$$

where the **scalar component of  $\mathbf{u}$  in the direction of  $\mathbf{v}$**  is

$$\text{scal}_{\mathbf{v}} \mathbf{u} = |\mathbf{u}| \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}.$$

**Example.** Find  $\text{proj}_{\mathbf{v}} \mathbf{u}$  and  $\text{scal}_{\mathbf{v}} \mathbf{u}$  for the following:

- $\mathbf{u} = \langle 1, 1 \rangle$ ,  $\mathbf{v} = \langle -2, 1 \rangle$

- $\mathbf{u} = \langle 7, 1, 7 \rangle$ ,  $\mathbf{v} = \langle 5, 7, 0 \rangle$

## Applications of Dot Products

### Definition. (Work)

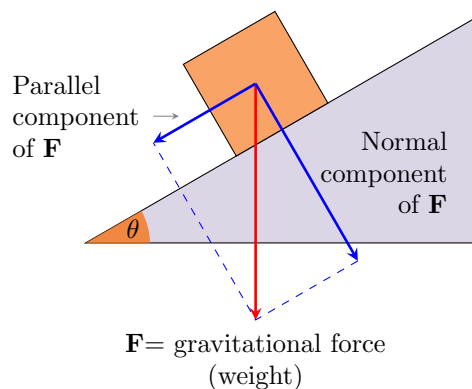
Let a constant force  $\mathbf{F}$  be applied to an object, producing a displacement  $\mathbf{d}$ . If the angle between  $\mathbf{F}$  and  $\mathbf{d}$  is  $\theta$ , then the **work** done by the force is

$$W = |\mathbf{F}||\mathbf{d}| \cos \theta = \mathbf{F} \cdot \mathbf{d}$$

**Example.** A force  $\mathbf{F} = \langle 3, 3, 2 \rangle$  (in newtons) moves an object along a line segment from  $P(1, 1, 0)$  to  $Q(6, 6, 0)$  (in meters). What is the work done by the force?

### Parallel and Normal Forces:

**Example.** A 10-lb block rests on a plane that is inclined at  $30^\circ$  above the horizontal. Find the components of the gravitational force parallel to and normal (perpendicular) to the plane.



## 13.4: Cross Products

### Definition. (Cross Product)

Given two nonzero vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbb{R}^3$ , the **cross product**  $\mathbf{u} \times \mathbf{v}$  is a vector with magnitude

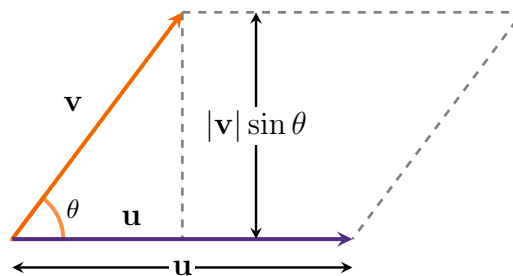
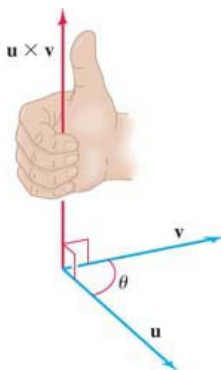
$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta,$$

where  $0 \leq \theta \leq \pi$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$ .

The direction of  $\mathbf{u} \times \mathbf{v}$  is given by the **right-hand rule**:

When you put your the vectors tail to tail and let the fingers of your right hand curl from  $\mathbf{u}$  to  $\mathbf{v}$ , the direction of  $\mathbf{u} \times \mathbf{v}$  is the direction of your thumb, orthogonal to both  $\mathbf{u}$  and  $\mathbf{v}$  (Figure 13.56).

When  $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ , the direction of  $\mathbf{u} \times \mathbf{v}$  is undefined.



### Theorem 13.3: Geometry of the Cross Product

Let  $\mathbf{u}$  and  $\mathbf{v}$  be two nonzero vectors in  $\mathbb{R}^3$ .

1. The vectors  $\mathbf{u}$  and  $\mathbf{v}$  are parallel ( $\theta = 0$  or  $\theta = \pi$ ) if and only if  $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ .
2. If  $\mathbf{u}$  and  $\mathbf{v}$  are two sides of a parallelogram, then the area of the parallelogram is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta$$

**Example.** Consider the vectors  $\mathbf{u} = \langle 2, 0, 0 \rangle$  and  $\mathbf{v} = \langle \sqrt{3}, 3, 0 \rangle$ . The angle between these vectors is  $\theta = \frac{\pi}{3}$ . Find the area of the parallelogram formed by these vectors.

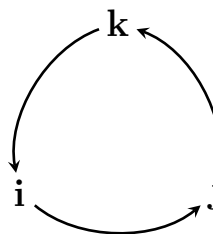
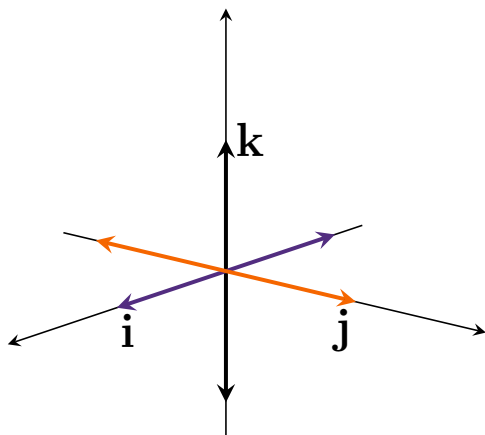
**Theorem 13.4: Properties of the Cross Product** Let  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  be nonzero vectors in  $\mathbb{R}^3$ , and let  $a$  and  $b$  be scalars.

- |  |                          |
|--|--------------------------|
| 1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$  | Anticommutative property |
| 2. $(a\mathbf{u}) \times (b\mathbf{v}) = ab(\mathbf{u} \times \mathbf{v})$   | Associative property     |
| 3. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ | Distributive property    |
| 4. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$ | Distributive property    |

### Theorem 13.5: Cross Products of Coordinate Unit Vectors

$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= -(\mathbf{j} \times \mathbf{i}) = \mathbf{k} \\ \mathbf{k} \times \mathbf{i} &= -(\mathbf{i} \times \mathbf{k}) = \mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{j} \times \mathbf{k} &= -(\mathbf{k} \times \mathbf{j}) = \mathbf{i} \\ \mathbf{i} \times \mathbf{i} &= \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}\end{aligned}$$



$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j}\end{aligned}$$

Using the unit vectors, we can compute  $\mathbf{u} \times \mathbf{v}$ :

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1 \underbrace{(\mathbf{i} \times \mathbf{i})}_{\mathbf{0}} + u_1v_2 \underbrace{(\mathbf{i} \times \mathbf{j})}_{\mathbf{k}} + u_1v_3 \underbrace{(\mathbf{i} \times \mathbf{k})}_{-\mathbf{j}} \\ &\quad + u_2v_1 \underbrace{(\mathbf{j} \times \mathbf{i})}_{-\mathbf{k}} + u_2v_2 \underbrace{(\mathbf{j} \times \mathbf{j})}_{\mathbf{0}} + u_2v_3 \underbrace{(\mathbf{j} \times \mathbf{k})}_{\mathbf{i}} \\ &\quad + u_3v_1 \underbrace{(\mathbf{k} \times \mathbf{i})}_{\mathbf{j}} + u_3v_2 \underbrace{(\mathbf{k} \times \mathbf{j})}_{-\mathbf{i}} + u_3v_3 \underbrace{(\mathbf{k} \times \mathbf{k})}_{\mathbf{0}} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}\end{aligned}$$

**Theorem 13.6: Evaluating the Cross Product**

Let  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$  and  $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ . Then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

*Note:*

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

**Alternative approach:**

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \begin{vmatrix} \mathbf{i} & \mathbf{j} \\ u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

**Example.** Compute  $\mathbf{u} \times \mathbf{v}$  for  $\mathbf{u} = \langle 3, 5, 4 \rangle$  and  $\mathbf{v} = \langle 1, -1, 9 \rangle$ .

**Example.** Consider the vectors  $\mathbf{u} = \langle \sqrt{3}, 1, 0 \rangle$  and  $\mathbf{v} = \langle -\sqrt{3}, 1, 0 \rangle$ . From the unit circle, we know the angle between these two vectors is  $\theta = \frac{2\pi}{3}$ . Use the definition of the cross product to show this.

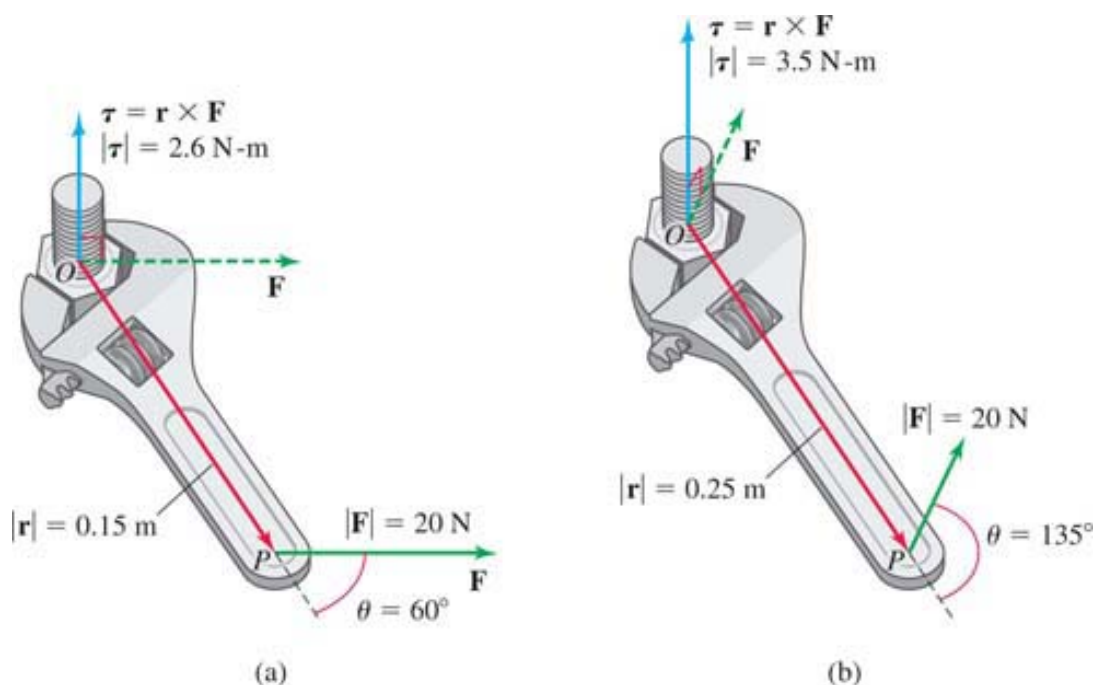
**Example.** Find the area of the triangle formed by  $\mathbf{u} = \langle 1, 2, 3 \rangle$  and  $\mathbf{v} = \langle 3, -1, 1 \rangle$ .



**Example.** Given a force  $\mathbf{F}$  applied to a point  $P$  at the head of the vector  $\mathbf{r} = \overrightarrow{OP}$ , the **torque** produced at point  $O$  is given by  $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$  with magnitude

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin \theta.$$

Now suppose a force of  $20\text{N}$  is applied to a wrench attached to a bolt in a direction perpendicular to the bolt. Which produces more torque: applying the force at an angle of  $60^\circ$  on a wrench that is  $0.15\text{m}$  long or applying the force at an angle of  $135^\circ$  on a wrench that is  $0.25\text{m}$  long?

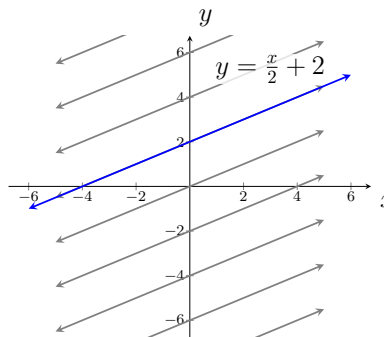


## 13.5: Lines and Planes in Space

### Equation of a Line:

Recall the equation of a line in  $\mathbb{R}^2$ :

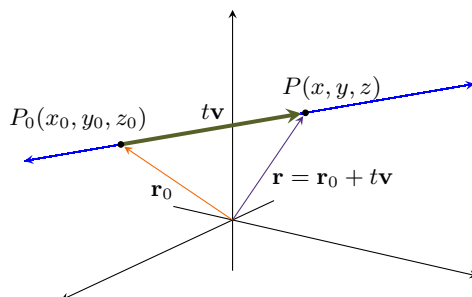
$$y = mx + b$$



where  $b$  is the intercept and  $m$  is the slope. This idea can be extended into higher dimensions:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

Here,  $\mathbf{r}_0$  is a fixed point, and  $\mathbf{v}$  is the position vector that is parallel to the line  $\mathbf{r}$ .



### Equation of a Line

A **vector equation of the line** passing through the point  $P_0(x_0, y_0, z_0)$  in the direction of the vector  $\mathbf{v} = \langle a, b, c \rangle$  is  $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ , or

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle, \quad \text{for } -\infty < t < \infty$$

Equivalently, the corresponding **parametric equations of the line** are

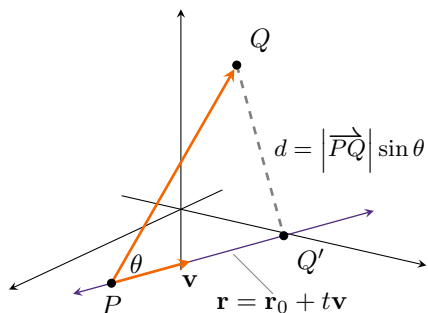
$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \text{for } -\infty < t < \infty$$

**Example.** Find the vector equation and parametric equation of the line that

- goes through the points  $P(-1, -2, 1)$  and  $Q(-4, -5, -3)$  where  $t = 0$  corresponds to  $P$ ,
- goes through the point  $P(1, -3, -3)$  and is parallel to the vector  $\mathbf{r} = \langle -4, 1, -1 \rangle$ ,
- goes through the point  $P(-2, 5, -2)$  and is perpendicular to the lines  $x = 3 - 4t$ ,  $y = 2 - 3t$ ,  $z = -1 - t$ , and  $x = -2 + 0t$ ,  $y = 2 - t$ ,  $z = 3t$ , where  $t = 0$  corresponds to  $P$ .

### Distance from a Point to a Line:

Given a point  $Q$  and a line  $\ell$ , the shortest distance to the line is the length of  $\overrightarrow{QQ'}$ .



From the definition of the cross product, we have

$$|\mathbf{v} \times \overrightarrow{PQ}| = |\mathbf{v}| \underbrace{|\overrightarrow{PQ}| \sin \theta}_d = |\mathbf{v}| d$$

From here, solving for  $d$  gives us the following:

#### Distance Between a Point and a Line

The distance  $d$  between the point  $Q$  and the  $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$  is

$$d = \frac{|\mathbf{v} \times \overrightarrow{PQ}|}{|\mathbf{v}|},$$

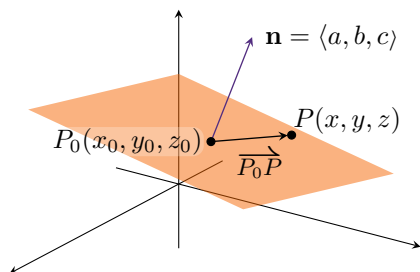
where  $P$  is any point on the line and  $\mathbf{v}$  is a vector parallel to the line.

**Example.** Find the distance from the point  $Q(-4, -1, -3)$  and the line  $x = -5 - 5t$ ,  $y = -5 + t$ ,  $z = -1 + 4t$ . (*Hint:* Let  $P$  be the point at  $t = 0$ )

## Equations of Planes:

In  $\mathbb{R}^2$ , two distinct points determine a line.

In  $\mathbb{R}^3$ , three noncollinear points determine a unique plane. Alternatively, a plane is uniquely determined by a point and a vector that is orthogonal to the plane.



### Definition. (Plane in $\mathbb{R}^3$ )

Given a fixed point  $P_0$  and a nonzero **normal vector**  $\mathbf{n}$ , the set of points  $P$  in  $\mathbb{R}^3$  for which  $\overrightarrow{P_0P}$  is orthogonal to  $\mathbf{n}$  is called a **plane**.

Consider the normal vector  $\mathbf{n} = \langle a, b, c \rangle$  at the point  $P_0(x_0, y_0, z_0)$ , and any point  $P(x, y, z)$  on the plane. Since  $\mathbf{n}$  is orthogonal to the plane, it is also orthogonal to the vector  $\overrightarrow{P_0P}$ , which is also in the plane. Thus,

$$\begin{aligned}\mathbf{n} \cdot \overrightarrow{P_0P} &= 0 \\ \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \\ ax + by + cz &= d\end{aligned}$$

### General Equation of a Plane in $\mathbb{R}^3$

The plane passing through the point  $P_0(x_0, y_0, z_0)$  with a nonzero normal vector  $\mathbf{n} = \langle a, b, c \rangle$  is described by the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{or} \quad ax + by + cz = d,$$

where  $d = ax_0 + by_0 + cz_0$ .

**Example.** Find the equation of the plane that

- goes through the point  $P(-2, 5, 0)$  and is parallel to the plane  $x - 5y - 5z = 1$ ,
- goes through the points  $P(5, -2, 1)$ ,  $Q(5, 1, 3)$  and  $R(1, -5, -2)$
- that is parallel to the vectors  $\langle 4, -2, -3 \rangle$  and  $\langle 3, 2, 3 \rangle$ , passing through the point  $P(-2, -2, 5)$ .

**Example.** Find the location where the line  $\langle -3, 1, 4 \rangle + t\langle -1, -4, 2 \rangle$  and the plane  $2x - 2y - 4z = 5$  intersect.

**Definition. (Parallel and Orthogonal Planes)**

Two distinct planes are **parallel** if their respective normal vectors are parallel (that is, the normal vectors are scaling multiples of each other). Two planes are **orthogonal** if their respective normal vectors are orthogonal (that is, the dot product of the normal vectors is *zero*).

**Example.** Find the line of intersection between the planes  $3x - y + 4z = -4$  and  $x + 3y - 2z = 0$ .

**Example.** Find the smallest angle between planes  $3x - y + 4z = -4$  and  $x + 3y - 2z = 0$ .



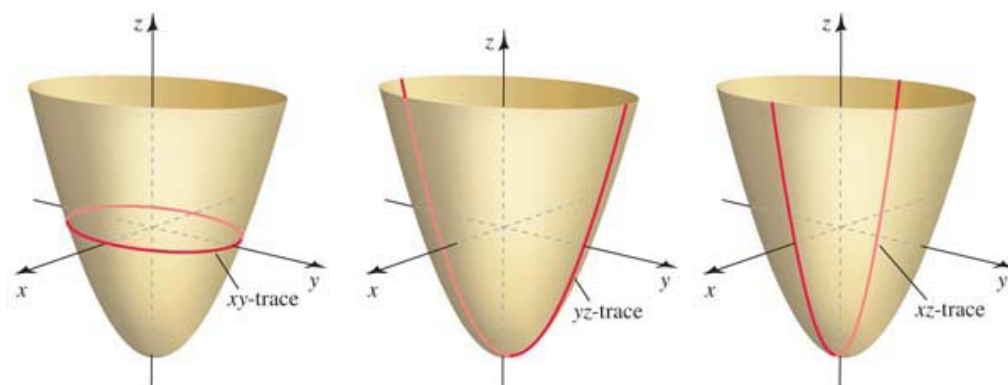
## 13.6: Cylinders and Quadric Surfaces

### Cylinders and Traces:

When talking about three-dimensional surfaces, a *cylinder* refers to a surface that is parallel to a line. When considering surfaces that is parallel to one of the coordinate axes, that the associated variable is missing (e.g.  $3y^2 + z^2 = 8$  is parallel to the  $x$ -axis).

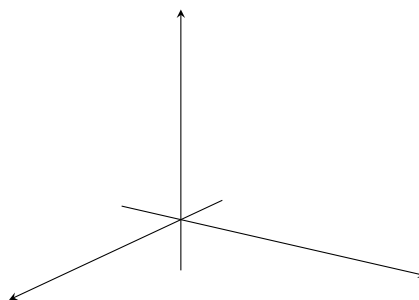
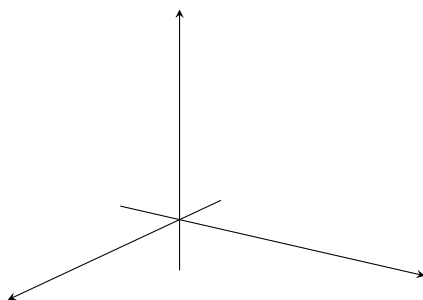
#### Definition. (Trace)

A **trace** of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes. The traces in the coordinate planes are called the  **$xy$ -trace**, the  **$yz$ -trace**, and the  **$xz$ -trace** (Figure 13.80).



**Example.** Roughly sketch the following functions:

1.  $x^2 + 4y^2 = 16$
2.  $x - \sin(z) = 0$



### Quadric Surfaces:

**Quadric surfaces** are described by the general quadratic (second-degree) equation in three variables,

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0,$$

Where the coefficients  $A, \dots, J$  and not all zero. To sketch quadric surfaces, keep the following ideas in mind:

1. **Intercepts** Determine the points, if any, where the surface intersects the coordinate axes. To find these intercepts, set  $x$ ,  $y$ , and  $z$  equal to zero in pairs in the equation of the surface, and solve for the third coordinate.
2. **Traces** Finding traces of the surface helps visualize the surface. Setting  $x$ ,  $y$ , and  $z$  equal to zero in pairs gives the planes parallel in that pair's plane.
3. **Completing the figure** Sketch some traces in parallel planes, then draw smooth curves that pass through the traces to fill out the surface.

**Example** (An ellipsoid). The surface defined by the equation  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ . Graph  $a = 3$ ,  $b = 4$  and  $c = 5$ .

**Example** (An elliptic paraboloid). The surface defined by the equation  $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ . Graph the elliptic paraboloid with  $a = 4$  and  $b = 2$ .

**Example** (A hyperboloid of one sheet).

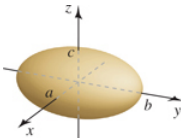
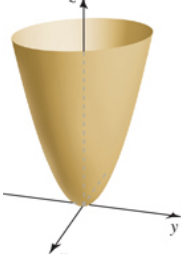
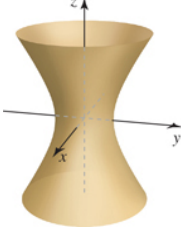
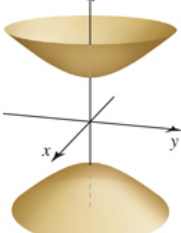
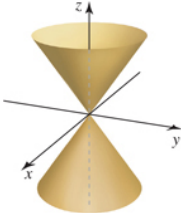
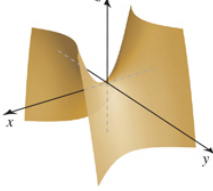
Graph the surface defined by the equation  $\frac{x^2}{4} + \frac{y^2}{9} - z^2 = 1$ .

**Example** (A hyperboloid of two sheets). Graph the surface defined by the equation  $-16x^2 - 4y^2 + z^2 + 64x - 80 = 0$ .

**Example** (Elliptic cones). Graph the surface defined by the equation  $\frac{y^2}{4} + z^2 = 4x^2$ .

**Example** (A hyperbolic paraboloid).

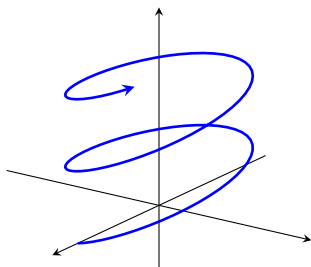
Graph the surface defined by the equation  $z = x^2 - \frac{y^2}{4}$ .

Name	Standard Equation	Features	Graph
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	All traces are ellipses.	
Elliptic paraboloid	$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	Traces with $z = z_0 > 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are parabolas.	
Hyperboloid of one sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	Traces with $z = z_0$ are ellipses for all $z_0$ . Traces with $x = x_0$ or $y = y_0$ are hyperbolas.	
Hyperboloid of two sheets	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Traces with $z = z_0$ with $ z_0  >  c $ are ellipses. Traces with $x = x_0$ and $y = y_0$ are hyperbolas.	
Elliptic cone	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$	Traces with $z = z_0 \neq 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are hyperbolas or intersecting lines.	
Hyperbolic paraboloid	$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$	Traces with $z = z_0 \neq 0$ are hyperbolas. Traces with $x = x_0$ or $y = y_0$ are parabolas.	



## 14.1: Vector-Valued Functions

Vector-valued functions are functions of the form  $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ , where  $x(t)$ ,  $y(t)$ , and  $z(t)$  are parametric equations dependent on  $t$ .



### Curves in Space

Consider

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k},$$

where  $f$ ,  $g$ , and  $h$  are defined for  $a \leq t \leq b$ . The **domain** of  $\mathbf{r}$  is the largest set of  $t$  for which all of  $f, g$ , and  $h$  are defined.

**Example.** What plane does the curve  $\mathbf{r}(t) = t\mathbf{i} + 6t^3\mathbf{k}$  lie?

**Example** (Lines as vector-valued functions). Find a vector function for the line that passes through the points  $P(5, 2, -4)$  and  $Q(5, 5, -2)$ . What about the line segment that connects  $P$  and  $Q$ ?

**Example.** Find the domain of

$$\mathbf{r}(t) = \sqrt{16 - t^2}\mathbf{i} + \sqrt{t}\mathbf{j} + \frac{4}{\sqrt{3 + t}}\mathbf{k}$$

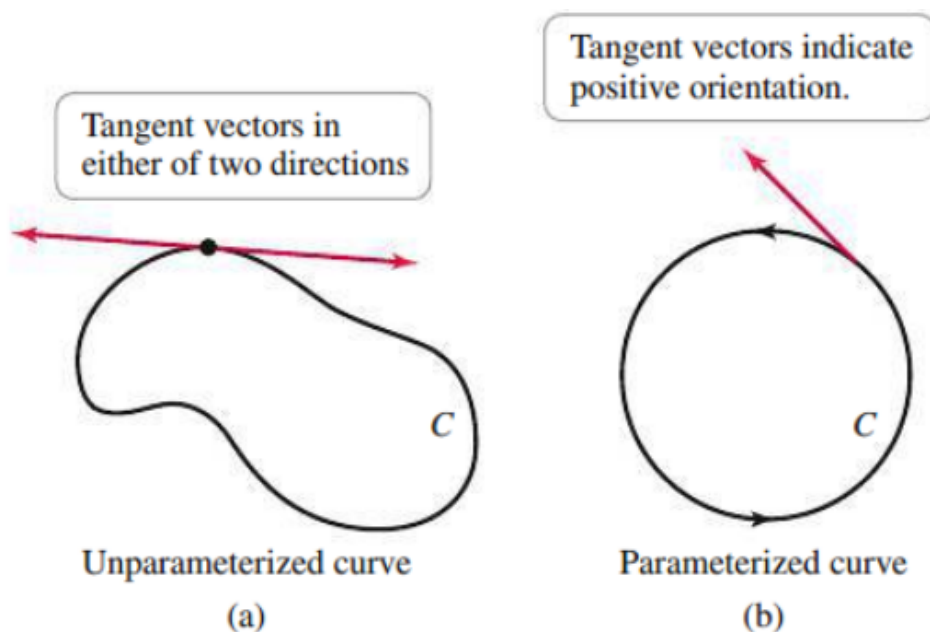
**Example.** Find the point  $P$  on

$$\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j} + 2t\mathbf{k},$$

closest to  $P_0(4, 17, 10)$ . What is the distance between  $P$  and  $P_0$ ?

## Orientation of Curves

- A **unparameterized curve** is a smooth curve  $C$  with no specified direction and the tangent vector can be drawn in two directions.
- A **parameterized curve** is a smooth curve  $C$  described by a function  $\mathbf{r}(t)$  for  $a \leq t \leq b$  and has a direction referred to as its **orientation**.
- The *positive* orientation is the direction of the curve generated when  $t$  increases from  $a$  to  $b$ .
- The tangent vector of a parameterized curve points in the positive orientation of the curve.



**Example.** Graph the curve described by the equation

$$\mathbf{r}(t) = 4 \cos(t) \mathbf{i} + \sin(t) \mathbf{j} + \frac{t}{2\pi} \mathbf{k},$$

where  $0 \leq t \leq 2\pi$ . Indicate the positive orientation of this curve.

## Limits and Continuity for Vector-Valued Functions

The properties of limits extend to vector-valued functions naturally. In particular, for  $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ , if

$$\lim_{t \rightarrow a} f(t) = L_1, \quad \lim_{t \rightarrow a} g(t) = L_2, \quad \lim_{t \rightarrow a} h(t) = L_3$$

then

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \left\langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \right\rangle = \langle L_1, L_2, L_3 \rangle.$$

### Definition. (Limit of a Vector-Valued Function)

A vector-valued function  $\mathbf{r}$  approaches the limit  $\mathbf{L}$  as  $t$  approaches  $a$ , written  $\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{L}$ , provided  $\lim_{t \rightarrow a} |\mathbf{r}(t) - \mathbf{L}| = 0$ .

A function  $\mathbf{r}(t)$  is **continuous** at  $t = a$ , provided  $\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{r}(a)$ .

**Example.** Evaluate the following limits:

$$\lim_{t \rightarrow \pi} \left( \cos(t) \mathbf{i} - 7 \sin \left( -\frac{t}{2} \right) \mathbf{j} + \frac{t}{\pi} \mathbf{k} \right)$$

$$\lim_{t \rightarrow \infty} \left( \frac{t}{t-3} \mathbf{i} + \frac{40}{1+19e^{-t}} \mathbf{j} + \frac{1}{2t} \mathbf{k} \right)$$

## 14.2: Calculus of Vector-Valued Functions

### Definition. (Derivative and Tangent Vector)

Let  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ , where  $f, g$ , and  $h$  are differentiable functions on  $(a, b)$ . Then  $\mathbf{r}$  has a **derivative** (or is **differentiable**) on  $(a, b)$  and

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

Provided  $\mathbf{r}'(t) \neq \mathbf{0}$ ,  $\mathbf{r}'(t)$  is a **tangent vector** at the point corresponding to  $\mathbf{r}(t)$ .

**Example.** For the following functions below, find  $\mathbf{r}'(t)$

a)  $\mathbf{r}(t) = \langle e^{-t^2}, \log_2(t - 4), \sin(t) \rangle$

b)  $\mathbf{r}(t) = 3\mathbf{i} - 2\tan(t)\mathbf{j} + e^t\mathbf{k}$

**Example.** For  $\mathbf{r}(t) = \langle 3t, \sec(2t), \cos(t) \rangle$  compute  $\mathbf{r}'(t)$  at  $t = \frac{\pi}{4}$ .

**Definition. (Unit Tangent Vector)**

Let  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$  be a smooth parameterized curve, for  $a \leq t \leq b$ . The **unit tangent vector** for a particular value of  $t$  is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

**Example.** For  $\mathbf{r}(t) = \langle 3 \sin(t), -2 \cos(2t), 3 \cos(t) \rangle$ , find the unit tangent vector.

**Example.** For  $\mathbf{r}(t) = \langle \sin(6t), 3t, \cos(3t) \rangle$ , compute  $\mathbf{T}\left(\frac{\pi}{3}\right)$ .

## Derivative Rules

Let  $\mathbf{u}$  and  $\mathbf{v}$  be differentiable vector-valued functions, and let  $f$  be a differentiable scalar-valued function, all at a point  $t$ . Let  $\mathbf{c}$  be a constant vector. The following rules apply.

- |   |                    |
|---|--------------------|
| 1. $\frac{d}{dt}(\mathbf{c}) = \mathbf{0}$  | Constant Rule      |
| 2. $\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$  | Sum Rule           |
| 3. $\frac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$  | Product Rule       |
| 4. $\frac{d}{dt}(\mathbf{u}(f(t))) = \mathbf{u}'(f(t))f'(t)$  | Chain Rule         |
| 5. $\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$    | Dot Product Rule   |
| 6. $\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ | Cross Product Rule |

**Example.** Given  $\mathbf{u}(t) = \langle 4t^2, 1, 3t \rangle$  and  $\mathbf{v}(t) = \langle e^{-2t}, -2e^t, e^t \rangle$ , find  $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$ .



**Definition. (Indefinite Integral of a Vector-Valued Function)**

Let  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$  be a vector function, and let

$\mathbf{R}(t) = F(t)\mathbf{i} + G(t)\mathbf{j} + H(t)\mathbf{k}$ , where  $F$ ,  $G$ , and  $H$  are antiderivatives of  $f$ ,  $g$ , and  $h$ , respectively. The **indefinite integral** of  $\mathbf{r}$  is

$$\int \mathbf{r}(t) dt = \mathbf{R}(t) + \mathbf{C},$$

where  $\mathbf{C}$  is an arbitrary constant vector. Alternatively, in component form,

$$\int \langle f(t), g(t), h(t) \rangle dt = \langle F(t), G(t), H(t) \rangle + \langle C_1, C_2, C_3 \rangle.$$

**Example.** Find  $\mathbf{r}(t)$  such that  $\mathbf{r}'(t) = \left\langle \frac{t}{t^2+1}, t^2 e^{-t^3}, \frac{-2t}{\sqrt{t^2+16}} \right\rangle$  and  $\mathbf{r}(0) = \left\langle 3, \frac{5}{3}, -5 \right\rangle$ .

**Definition. (Definite Integral of a Vector-Valued Function)**

Let  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ , where  $f$ ,  $g$ , and  $h$  are integrable on the interval  $[a, b]$ . The **definite integral** of  $\mathbf{r}$  on  $[a, b]$  is

$$\int_a^b \mathbf{r}(t) dt = \left( \int_a^b f(t) dt \right) \mathbf{i} + \left( \int_a^b g(t) dt \right) \mathbf{j} + \left( \int_a^b h(t) dt \right) \mathbf{k}$$

**Example.**  $\int_{-\pi}^{\pi} \langle \sin(t), \cos(t), 8t \rangle dt$

## 14.3: Motion in Space

### Definition.

Let the **position** of an object moving in three-dimensional space be given by  $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ , for  $t \geq 0$ . The **velocity** of the object is

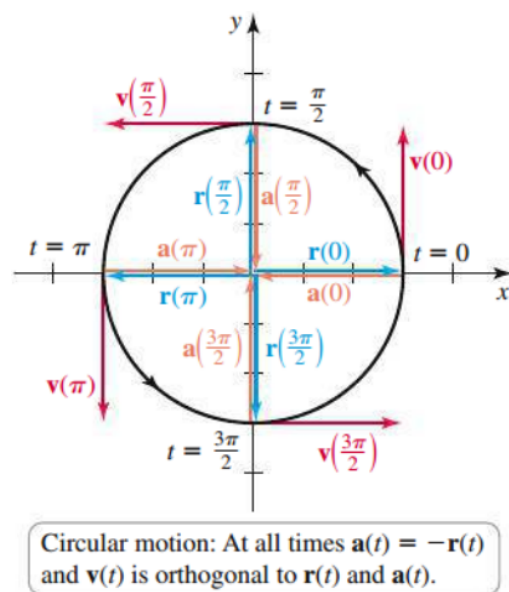
$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.$$

The **speed** of the object is the scalar function

$$|\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

The **acceleration** of the object is  $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$ .

**Example.** Given  $\mathbf{r}(t) = \langle 3 \cos(t), 3 \sin(t) \rangle$  for  $0 \leq t \leq 2\pi$ , find the velocity, speed, and acceleration.



**Theorem 14.2: Motion with constant  $|\mathbf{r}|$** 

Let  $\mathbf{r}$  describe a path on which  $|\mathbf{r}|$  is constant (motion on a circle or sphere centered at the origin). Then  $\mathbf{r} \cdot \mathbf{v} = 0$ , which means the position vector and the velocity vector are orthogonal at all times for which the functions are defined.

**Example** (Path on a sphere). Consider

$$\mathbf{r}(t) = \langle 3 \cos(t), 5 \sin(t), 4 \cos(t) \rangle, \quad \text{for } 0 \leq t \leq 2\pi.$$

a) Show that an object with this trajectory moves on a sphere and find the radius.

b) Find the velocity and speed of the above trajectory.

c) Show that  $\mathbf{r}(t) = \langle 5 \cos(t), 5 \sin(t), 5 \sin(2t) \rangle$  does not lie on a sphere. How could this function be modified so that it does lie on a sphere?

**Example.** Given  $\mathbf{a}(t) = \langle \cos(t), 4 \sin(t) \rangle$ , with an initial velocity  $\langle \mathbf{u}_0, \mathbf{v}_0 \rangle = \langle 0, 4 \rangle$  and an initial position  $\langle x_0, y_0 \rangle = \langle 5, 0 \rangle$  where  $t \geq 0$ , find the velocity and position vector.

**Summary: Two-Dimensional Motion in a Gravitational Field**

Consider an object moving in a plane with a horizontal  $x$ -axis and a vertical  $y$ -axis, subject only to the force of gravity. Given the initial velocity  $\mathbf{v}(0) = \langle u_0, v_0 \rangle$  and the initial position  $\mathbf{r}(0) = \langle x_0, y_0 \rangle$ , the velocity of the object, for  $t \geq 0$ , is

$$\mathbf{v}(t) = \langle x'(t), y'(t) \rangle = \langle u_0, -gt + v_0 \rangle$$

and the position is

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \left\langle u_0 t + x_0, -\frac{1}{2}gt^2 + v_0 t + y_0 \right\rangle.$$

**Example.** Consider a ball with an initial position of  $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$  m and an initial velocity of  $\langle u_0, v_0 \rangle = \langle 25, 4 \rangle$  m/s.

a) Find the position and velocity of the ball while it is in the air

**Summary: Two-Dimensional Motion**

Assume an object traveling over horizontal ground, acted on only by the gravitational force, has an initial position  $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$  and initial velocity  $\langle u_0, v_0 \rangle = \langle |\mathbf{v}_0| \cos \alpha, |\mathbf{v}_0| \sin \alpha \rangle$ . The trajectory, which is a segment of a parabola, has the following properties.

$$\text{time of flight} = T = \frac{2|\mathbf{v}_0| \sin \alpha}{g}$$

$$\text{range} = \frac{|\mathbf{v}_0|^2 \sin(2\alpha)}{g}$$

$$\text{maximum height} = y\left(\frac{T}{2}\right) = \frac{(|\mathbf{v}_0| \sin \alpha)^2}{2g}$$

**Example.** Consider a ball with an initial position of  $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$  m and an initial velocity of  $\langle u_0, v_0 \rangle = \langle 25, 4 \rangle$  m/s.

b) Assuming the ground is flat and level, how far does the ball travel horizontally?

c) What is the maximum height that the ball reaches?

## 14.4: Length of Curves

### Definition. (Arc Length for Vector Functions)

Consider the parameterized curve  $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ , where  $f'$ ,  $g'$ , and  $h'$  are continuous, and the curve is traversed once for  $a \leq t \leq b$ . The **arc length** of the curve between  $(f(a), g(a), h(a))$  and  $(f(b), g(b), h(b))$  is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} dt = \int_a^b |\mathbf{r}'(t)| dt.$$

**Example** (Flight of an eagle). Suppose an eagle rises at a rate of 100 vertical ft/min on a helical path given by

$$\mathbf{r}(t) = \langle 250 \cos(t), 250 \sin(t), 100t \rangle$$

where  $\mathbf{r}$  is measured in feet and  $t$  is measured in minutes. How far does it travel in 10 minutes?



**Example.** Suppose a particle has a trajectory given by

$$\mathbf{r}(t) = \langle 10 \cos(3t), 10 \sin(3t) \rangle$$

where  $0 \leq t \leq \pi$ . How far does this particle travel?

**Example.** Find the length of the curve

$$\mathbf{r}(t) = \langle 3t^2 - 5, 4t^2 + 5 \rangle$$

where  $0 \leq t \leq 1$ .

**Example.** Find the length of  $\mathbf{r}(t) = \left\langle t^2, \frac{(4t+1)^{\frac{3}{2}}}{6} \right\rangle$  where  $0 \leq t \leq 6$ .

**Example.** Find the length of  $\mathbf{r}(t) = \langle 2\sqrt{2}, \sin(t), \cos(t) \rangle$  where  $0 \leq t \leq 5$ .

**Theorem 14.3: Arc Length as a Function of a Parameter**

Let  $\mathbf{r}(t)$  describe a smooth curve, for  $t \geq a$ . The arc length is given by

$$s(t) = \int_a^t |\mathbf{v}(u)| \, du,$$

where  $|\mathbf{v}| = |\mathbf{r}'|$ . Equivalently,  $\frac{ds}{dt} = |\mathbf{v}(t)|$ . If  $|\mathbf{v}(t)| = 1$ , for all  $t \geq a$ , then the parameter  $t$  corresponds to arc length.

**Example.** For the following functions, determine if  $\mathbf{r}(t)$  uses arc length as a parameter. If not, find a description that uses arc length as a parameter.

a)  $\mathbf{r}(t) = \langle -4t + 1, 3t - 1 \rangle, 0 \leq 4$ .

b)  $\mathbf{r}(t) = \left\langle \frac{1}{\sqrt{10}} \cos(t), \frac{3}{\sqrt{10}} \sin(t) \right\rangle, 0 \leq t \leq 2\pi$ .