## 15.1: Graphs and Level Curves

In the previous chapter, we considered functions of the form

$$\mathbf{r}(t) = \langle f(t),\, g(t),\, h(t)\rangle,$$

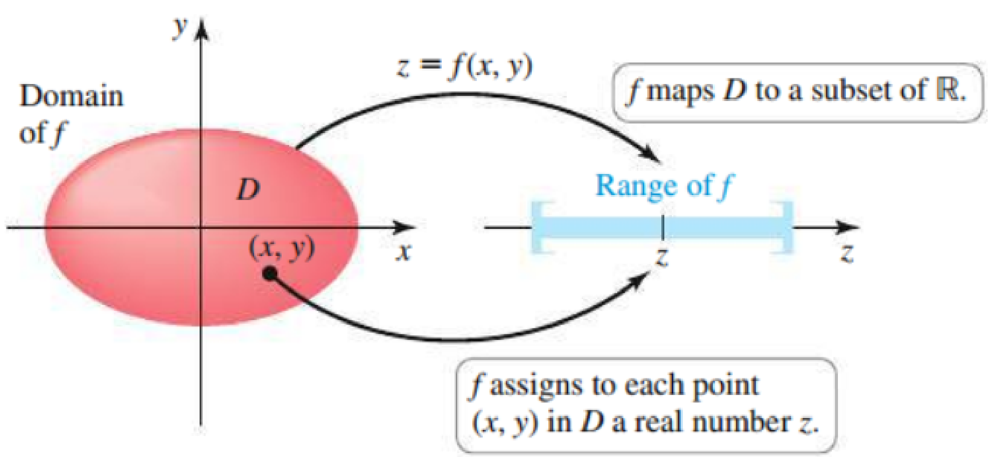
which have one independent variable $t$ and three dependent variables $f(t)$, $g(t)$, and $h(t)$. In this chapter, we consider functions of the form

$$x_{n+1} = f(x_1, x_2, \ldots, x_n),$$

where we have multiple independent variables $x_1$, $x_2, \ldots,$ $x_n$ and one single dependent variable $x_{n+1}$. We begin with functions of two variables:

$$z = f(x, y).$$

**Definition. (Function, Domain, and Range with 2 Independent Variables)** A **function** $z = f(x, y)$ assigns to each point $(x, y)$ in a set $D$ in $\mathbb{R}^2$ a unique real number $z$ in a subset of $\mathbb{R}$. The set $D$ is the **domain** of $f$. The **range** of $f$ is the set of real numbers $z$ that are assumed as the points $(x, y)$ vary over the domain.



**Example.** Find the domain of the following functions:

$$f(x,y) = \frac{1}{xy+2}$$

$$g(x,y) = \sqrt{108 - 3x^2 - 3y^2}$$

$$h(x,y) = \log_2\left(x^3 - y^{1/3}\right)$$

$$j(x,y) = \frac{1}{\sqrt{x^2+y^2-16}}$$

**Example.** Roughly graph the following functions:

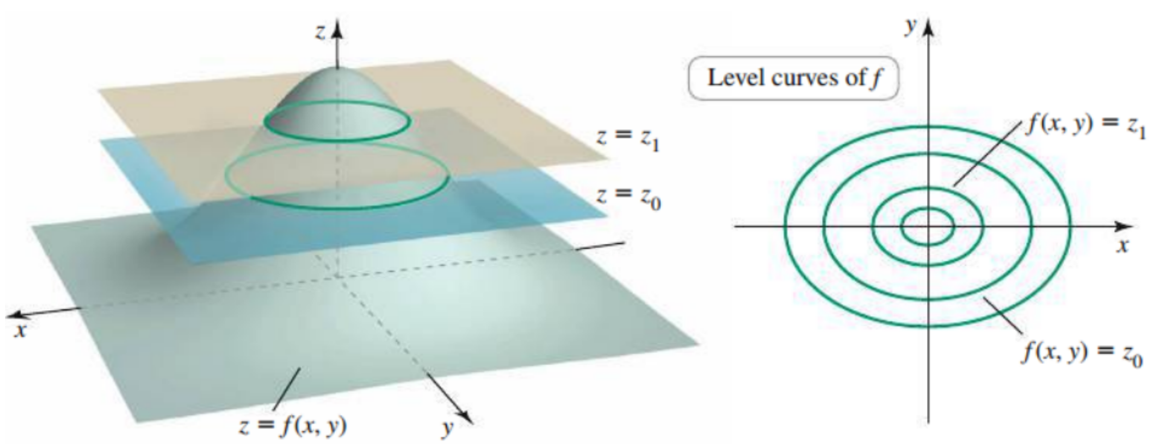
$$f(x, y) = -4x + 3y - 10$$

$$g(x, y) = x^2 + y^2 + 4$$

$$h(x, y) = \sqrt{4 + x^2 + y^2}$$

**Level Curves:**

A **contour curve** is formed by tracing a three-dimensional surface at a constant height. A **level curve** is formed when a contour curve is projected to the $xy$-plane.

**Example.** Find the level curves of the following functions:

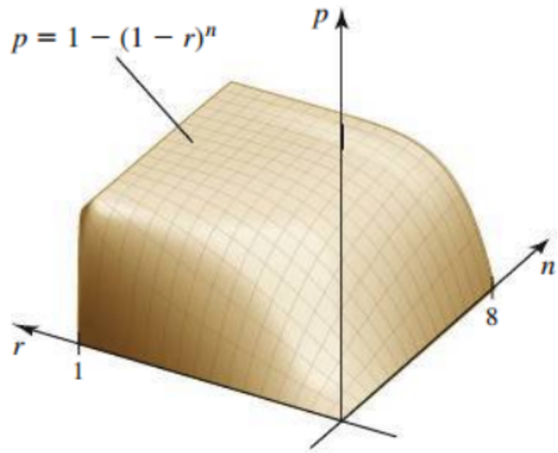
$$f(x, y) = y - x^2 - 1$$

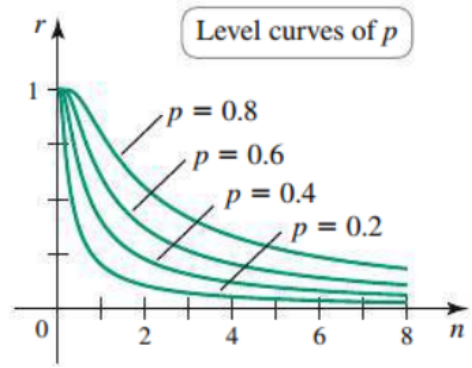
$$g(x, y) = e^{-x^2-y^2}$$

$$h(x, y) = x^2 + y^2$$

**Applications of Functions of Two Variables:**

**Example. A probability function of two variables:** Suppose on a particular day, the fraction of students on campus infected with COVID-19 is $r$, where $0 \le r \le 1$. If you have $n$ random (possibly repeated) encounters with students during the day, the probability of meeting *at least* one infected person is $p(n, r) = 1 - (1 - r)^n$.




**Functions of More than Two Variables:**

| Number of Independent Variables | Explicit Form | Implicit Form | Graph Resides In... |
|---|---|---|---|
| 1 | $y=f(x)$ | $F(x,y)=0$ | $\mathbb{R}^2(xy - \text{plane})$ |
| 2 | $z=f(x,y)$ | $F(x,y,z)=0$ | $\mathbb{R}^3(xyz - \text{space})$ |
| 3 | $w=f(x,y,z)$ | $F(x,y,z,w)=0$ | $\mathbb{R}^4$ |
| $n$ | $x_{n+1}=f(x_1,x_2,\dots,x_n)$ | $F(x_1,x_2,\dots,x_n,x_{n+1})=0$ | $\mathbb{R}^{n+1}$ |

**Definition. (Function, Domain, and Range with $n$ Independent Variables)** The **function** $x_{n+1} = f(x_1, x_2, \ldots, x_n)$ assigns a unique real number $x_{n+1}$ to each point $(x_1, x_2, \ldots, x_n)$ in a set $D$ in $\mathbb{R}^4$. The set $D$ is the **domain** of $f$. The **range** is the set of real numbers $x_{n+1}$ that are assumed as the points $(x_1, x_2, \ldots, x_n)$ vary over the domain.

**Example.** Find the domain of the following functions:

$$f(x, y, z) = 4xyz - 2xz + 5yz$$

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 9}$$

**Graphs of Functions of More Than Two Variables:**

The idea of level curves can be extended to **level surfaces**. Level surfaces can be used to represent functions of three variables: