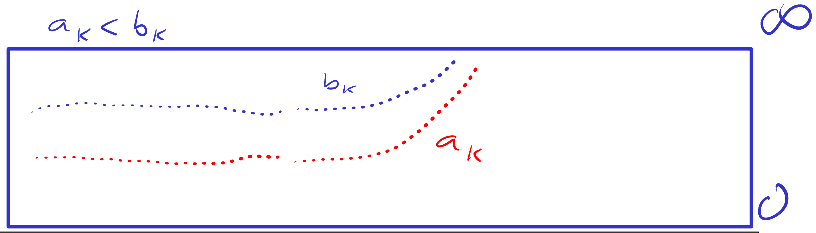
$$\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \sum_{k=4}^{\infty} \frac{1}{k}$$



10.5: Comparison Tests

Direct (DCT)

Theorem 10.14: Comparison Test

Let $\sum a_k$ and $\sum b_k$ be series with positive terms where $a_k \le b_k$.

1. If $\sum b_k$ converges, then $\sum a_k$ converges.
2. If $\sum a_k$ diverges, then $\sum b_k$ diverges.

Example. Use the comparison test to determine if the series $\displaystyle\sum_{k=4}^{\infty} \frac{k^2}{k^3-3}$ converges or diverges.

$$\sum_{k=4}^{\infty} \frac{k^2}{k^3-3} \qquad \sum_{k=4}^{\infty} \frac{k^2}{k^3} = \boxed{\sum_{k=4}^{\infty} \frac{1}{k}} \longrightarrow \text{diverges}$$

$L < 1$

p-series, $p=1$

16, 25, 36, ...

$$\frac{k^2}{k^3-3} \longleftarrow k^2 \ge 0 \longleftarrow k \ge 4$$

$$k^3 - 3 \ge 0 \longleftarrow k \ge 0$$

61, 122, 213, ...

$a_k > b_k$ $\quad L < 2$

①

$$a_k = \frac{k^2}{k^3-3} \overset{?}{\ge} \frac{1}{k} = b_k$$

$$\iff k^3 \ge k^3 - 3$$

$$\iff 0 \ge -3 \quad ✓$$

②

$$\frac{k^2}{k^3-3} \qquad \frac{k^2}{k^3} = \frac{1}{k}$$

smaller denom

e.g. $\frac{1}{10} \ge \frac{1}{100}$

smaller denom

By comparison test, $\displaystyle\sum_{k=4}^{\infty} \frac{k^2}{k^3-3}$ diverges

Let $a_k \geq 0, b_k \geq 0$. Suppose the series $\underbrace{\sum_{k=1}^{\infty} b_k}_{\text{convergence known}}$ diverges and $\lim_{k\to\infty} \frac{a_k}{b_k} = \frac{1}{2}$. Then we can conclude which of the following:

LC#5

from known series

$\sum_{k=1}^{\infty} a_k$ diverges $\quad 0 < \frac{1}{2} < \infty$

Theorem 10.15: Limit Comparison Test (LCT)

Let $\sum a_k$ and $\sum b_k$ be series with positive terms and let

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L.$$

$\sum b_k$ is known

1. If $\boxed{0 < L < \infty}$ (that is, L is a finite positive number), then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.
2. If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.
3. If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Example. Using either the Direct Comparison Test or the Limit Comparison Test, determine if the series

$$\sum_{k=1}^{\infty} \frac{4k^2 - k}{k^3 + 9}$$

converges or diverges.

$$\sum_{k=1}^{\infty} \frac{4k^2}{k^3} = \sum_{k=1}^{\infty} \frac{4}{k}$$

p-series $p = 1$ diverges

Direct comparison: $\boxed{\text{If}}$ $\frac{4}{k} \leq \frac{4k^2-k}{k^3+9}$

then $\sum_{k=1}^{\infty} \frac{4k^2-k}{k^3+9}$ also diverges

$$b_k := \frac{4}{k} = \frac{4k^2}{k^3} > \frac{4k^2 - k}{\underbrace{k^3+9}_{a_k}}$$

smaller, larger } smaller term

Can't use direct comp.

$a_k < b_k$ $\quad \boxed{\text{LC\#4}}$

$\frac{4}{k} \overset{?}{\cancel{\leq}} \frac{4k^2-k}{k^3+9}$

$\Leftrightarrow 4k^3 + 9 \cancel{\leq} 4k^3 - k^2 \Leftrightarrow 9 \cancel{\leq} -k^2$

Pos: $\frac{4k^2-k}{k^3+9}$ $\quad 4k^2 - k > 0 \Leftarrow k \geq 1$ $\quad k^3 + 9 > 0 \Leftarrow k \geq 1$

$$\lim_{k\to\infty} \frac{\frac{4k^2-k}{k^3+9}}{1/k} \left(\frac{k/1}{k/1}\right) = \lim_{k\to\infty} \frac{4k^2-k}{k^3+9} \cdot \frac{k}{1} = \lim_{k\to\infty} \frac{4k^3-k^2}{k^3+9} = 4 \Leftarrow 0 < 4 < \infty$$

Both diverge

Example. Determine if the following series converge or diverge.

$$\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}}$$

$\longleftarrow \quad \sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2}} = \sum_{k=1}^{\infty} \frac{1}{k^2} \longleftarrow$ convergent p-series

LC #6 LC #7

Positive $\frac{1}{k\sqrt{k^2+1}}$ $\leftarrow \sqrt{k^2+1} > 0$, $k \geq 0$

$k > 0$, $k \geq 1$

$$\lim_{k\to\infty} \frac{\frac{1}{k\sqrt{k^2+1}}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{1}{k\sqrt{k^2+1}} \cdot \frac{k^2}{1} = \lim_{k\to\infty} \frac{k^2}{\underbrace{k\sqrt{k^2+1}}_{\sqrt{k^2(k^2+1)}}}\left(\frac{1/k^2}{1/k^2}\right) = \lim_{k\to\infty} \frac{1}{\frac{1}{k^2}\sqrt{k^4+k^2}}$$

$$= \lim_{k\to\infty} \frac{1}{\sqrt{1+\frac{1}{k^2}}} = 1$$

LC #8

Both converge by limit Comparison test (LCT)

LC #9

$$\sum_{k=1}^{\infty} \frac{\ln(k)}{\underbrace{k^2}_{a_k}}$$

$\sum_{k=1}^{\infty} \frac{1}{\underbrace{k^2}_{b_k}} \longleftarrow$ converges b/c p-series with $p = 2 > 1$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{\ln(k)}{k^2}}{1/k^2} = \lim_{k\to\infty} \frac{\ln(k)}{k^2} \cdot \frac{k^2}{1} = \lim_{k\to\infty} \ln(k) = \infty$$

$\sum \frac{1}{k^2}$ converges

$\lim_{k\to\infty} \frac{a_k}{b_k} = \infty$

Not one of options for LCT

$\ln(x)^p << x^q$

$\sum_{k=1}^{\infty} \frac{\ln(k)}{\underbrace{k^2}_{a_k}}$, $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{\underbrace{k^2}_{b_k}} = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \rightarrow$ converges b/c p-series w/ $p = 3/2$

Using growth rates, $\ln(k) \ll \sqrt{k}$ $\Big\}$ Converges by DCT

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty}\left(1+\frac{2}{k}\right)^k = \lim_{k\to\infty} e^{k\ln\left(1+\frac{2}{k}\right)} = e^{\lim_{k\to\infty}\frac{\ln\left(1+\frac{2}{k}\right)}{1/k}} \overset{LR}{=} e^{\lim_{k\to\infty}\frac{\frac{1}{1+2/k}\left(-2/k^2\right)}{-1/k^2}}$$

(1^∞) Indeterminant form

$(\infty \cdot 0)$ Indet Form

$(\frac{0}{0})$ Indet Form

$$= e^{\lim_{k\to\infty}\frac{2}{1+2/k}} = e^2$$

$$\sum_{k=1}^{\infty} \underbrace{\left(1+\frac{2}{k}\right)^k}_{a_k}$$

Divergence test $\lim_{k\to\infty} a_k = e^2 \neq 0 \Rightarrow$ diverges by the divergence test

$$\sum_{k=1}^{\infty} \underbrace{e^2}_{b_k}$$ diverges

$$\lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\frac{\left(1+\frac{2}{k}\right)^k}{e^2} = \frac{e^2}{e^2} = 1$$

Both series diverge

$$S_n = \underbrace{e^2+e^2+\dots+e^2}_{n\text{-times}} = n\cdot e^2$$

$$\frac{1}{14^3}+\frac{2}{15^3}+\frac{3}{16^3}+\dots = \sum_{k=1}^{\infty}\underbrace{\frac{k}{(k+13)^3}}_{a_k}$$

Divergence test $\lim_{k\to\infty}\frac{k}{(k+13)^3} = 0$ Inconclusive

$$\sum_{k=1}^{\infty}\frac{k}{k^3} = \sum_{k=1}^{\infty}\underbrace{\frac{1}{k^2}}_{b_k} \leftarrow$$ converges, p-series w/ $p=2>1$

$$\lim_{k\to\infty}\frac{\frac{k}{(k+13)^3}}{\frac{1}{k^2}} = \lim_{k\to\infty}\frac{k}{(k+13)^3}\cdot\frac{k^2}{1} = \lim_{k\to\infty}\frac{k^3}{(k+13)^3} = 1$$

Converges by LCT

Direct comparison can not be done because $\sin\left(\frac{\pi}{k}\right)$ is not nonnegative

$$\sum_{k=1}^{\infty} \underbrace{\frac{\sin\left(\frac{\pi}{k}\right)}{k^3}}_{a_k} \qquad \sum_{k=1}^{\infty} \underbrace{\frac{1}{k^3}}_{b_k} \leftarrow \text{Converges}$$

$$\lim_{k\to\infty} \frac{\sin\left(\frac{\pi}{k}\right)}{k^3} \cdot \frac{k^3}{1} = \lim_{k\to\infty} \sin\left(\frac{\pi}{k}\right) = \sin(0) = 0$$

Converges by LCT ②

$$\frac{3}{2} - \frac{2}{3} = \frac{5}{6}$$

$$\sum_{k=1}^{\infty} \frac{\sqrt[3]{k^2+4}}{\sqrt{k^3+9}} \qquad \sum_{k=1}^{\infty} \frac{k^{2/3}}{k^{3/2}} = \sum_{k=1}^{\infty} \frac{1}{k^{5/6}} \leftarrow \text{diverges}$$

p-series w/ $p = 5/6 \le 1$

$$\lim_{k\to\infty} \frac{\frac{\sqrt[3]{k^2+4}}{\sqrt{k^3+9}}}{k^{2/3}/k^{3/2}} = \lim_{k\to\infty} \frac{\sqrt[3]{k^2+4}}{\sqrt{k^3+9}} \cdot \frac{k^{3/2}}{k^{2/3}} \left(\frac{\frac{1}{k^{2/3}k^{3/2}}}{\frac{1}{k^{3/2}k^{2/3}}}\right)$$

$$= \lim_{k\to\infty} \frac{\sqrt[3]{1+4/k^{2/3}}}{\sqrt{1+9/k^{3/2}}} \cdot \frac{1}{1} = 1$$

Diverges by LCT