11.3: Taylor Series

$e^x = \underbrace{1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots}_{\text{(infinite) polynomial}}$

Definition. (Taylor/Maclaurin Series for a Function)
Suppose the function f has derivatives of all orders on an interval centered at the point a. The **Taylor series for f centered at a is**

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots = \sum_{k=0}^{\infty} \boxed{\frac{f^{(k)}(a)}{k!}}(x-a)^k.$$

A Taylor series centered at 0 is called a **Maclaurin series**. $\sum c_k (x-a)^k$

Example (LC 30.1). Can we find a Taylor series centered at $a = 0$ for $f(x) = \sqrt{x}$?

$f(x) = \sqrt{x}$ $\quad f(0) = 0$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\quad f'(0)$ DNE

No

Example (LC 30.2-30.5). Consider the function $f(x) = \sin(\pi x)$ and the Taylor series representation centered at $a = 0$.

Find the first four nonzero terms

$f(x) = \sin(\pi x)$ $\quad f(0) = 0$

$f'(x) = \pi\cos(\pi x)$ $\quad f'(0) = \pi$

$f''(x) = -\pi^2\sin(\pi x)$ $\quad f''(0) = 0$

$f^{(3)}(x) = -\pi^3\cos(\pi x)$ $\quad f'''(0) = -\pi^3$

$f^{(4)}(x) = \pi^4\sin(\pi x)$ $\quad f^{(4)}(0) = 0$

$f^{(5)}(x) = \pi^5\cos(\pi x)$ $\quad f^{(5)}(0) = \pi^5$

$f^{(6)}(0) = 0$

$f^{(7)}(0) = -\pi^7$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k$$

$$= 0 + \frac{\pi}{1!}(x)^1 + 0 + \frac{-\pi^3}{3!}x^3 + 0 + \frac{\pi^5}{5!}x^5 + 0 + \frac{-\pi^7}{7!}x^7 + \cdots$$

$$= \pi x - \frac{\pi^3}{3!}x^3 + \frac{\pi^5}{5!}x^5 - \frac{\pi^7}{7!}x^7 + \cdots$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(\pi x)^{2k-1}}{(2k-1)!}$$

Write this Taylor series using summation notation

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k$$

$$= 0 + \frac{\pi}{1!}(x)^1 + 0 + \frac{-\pi^3}{3!}x^3 + 0$$

$$+ \frac{\pi^5}{5!}x^5 + 0 + \frac{-\pi^7}{7!}x^7 + \ldots$$

$$= \pi x - \frac{\pi^3}{3!}x^3 + \frac{\pi^5}{5!}x^5 - \frac{\pi^7}{7!}x^7 + \ldots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (\pi x)^{2k+1}}{(2k+1)!}$$

Theorem 11.7: Convergence of Taylor Series
Let f have derivatives of all orders on an open interval I containing a. The Taylor series for f centered at a converges to f, for all x in I, if and only if $\lim_{n\to\infty} R_n(x) = 0$, for all x in I, where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

is the remainder at x, with c between x and a.

$a = 2$, $f(1/2)$ } $\frac{1}{2} \le c \le 2$

$$\lim \quad = r < 1 \qquad \frac{x-a}{R} < 1$$

$$x - a < R$$

What is the interval of convergence?

$$\sum_{k=0}^{\infty} \frac{(-1)^k (\pi x)^{2k+1}}{(2k+1)!}$$

$$r = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$$

$$= \lim_{k\to\infty} \frac{(\pi x)^{2k+3}}{(2k+3)!} \cdot \frac{(2k+1)!}{(\pi x)^{2k+1}}$$

$$= \lim_{k\to\infty} \frac{(\pi x)^2}{(2k+3)(2k+2)}$$

$$= (\pi x)^2 \lim_{k\to\infty} \frac{1}{(2k+3)(2k+2)} = 0 < 1$$

$$\Rightarrow R = \infty$$

$$I = (-\infty, \infty)$$

What is the upper bound on $|R_n(x)|$?

$f(x) = \sin(\pi x)$

$f'(x) = \pi \cos(\pi x)$

$f''(x) = -\pi^2 \sin(\pi x)$

$f^{(3)}(x) = -\pi^3 \cos(\pi x)$

$f^{(4)}(x) = \pi^4 \sin(\pi x)$

$f^{(5)}(x) = \pi^5 \cos(\pi x)$

$\vdots$

$$|R_n(x)| \le M \frac{|x-a|^{n+1}}{(n+1)!} \quad \text{where } |f^{(n+1)}(c)| \le M$$

$$|f^{(n+1)}(c)| = \begin{cases} \pi^{n+1} |\sin(\pi c)|, & n \text{ odd} \\ \pi^{n+1} |\cos(\pi c)|, & n \text{ even} \end{cases} \Rightarrow M = \pi^{n+1}$$

$$\Rightarrow |R_n(x)| \le \frac{\pi^{n+1} |x|^{n+1}}{(n+1)!} = \boxed{\frac{|\pi x|^{n+1}}{(n+1)!}}$$

Example (LC 30.6). If a Taylor series only converges on $(-2, 2)$, does $f(x^2)$ have a Taylor series that also only converges on $(-2, 2)$?

Example (LC 30.7)**.** Use the definition of a Taylor series to find the Taylor series for $f(x) = e^{2x}$ at $a = 3$.

Example (LC 30.8)**.** Given that $\ln(1+x) = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k}$, for $-1 < x \leq 1$, find the first nonzero terms of the Taylor series centered at $a = 0$ for the function $\ln(1+2x)$.

Example (LC 30.9). Given that $\cos(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$, for $|x| < \infty$, find the Taylor series centered at $a = 0$ for the function $x\cos(x^3)$.

Common Taylor Series:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \ldots \qquad = \sum_{k=0}^{\infty} x^k, \qquad \text{for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^k x^k + \ldots \qquad = \sum_{k=0}^{\infty} (-1)^k x^k, \qquad \text{for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \ldots \qquad = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad \text{for } |x| < \infty$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \ldots \qquad = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \qquad \text{for } |x| < \infty$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \ldots \qquad = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \qquad \text{for } |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k} + \ldots \qquad = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \qquad \text{for } -1 < x \le 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \ldots \qquad = \sum_{k=1}^{\infty} \frac{x^k}{k}, \qquad \text{for } -1 \le x < 1$$

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \ldots \qquad = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \qquad \text{for } |x| \le 1$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \ldots \qquad = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \qquad \text{for } |x| < \infty$$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2k}}{(2k)!} + \ldots \qquad = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \qquad \text{for } |x| < \infty$$

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k, \text{ for } |x| < 1 \text{ and } \binom{p}{k} = \frac{p(p-1)(p-2)\ldots(p-k+1)}{k!}, \quad \binom{p}{0} = 1$$