14.5: Curvature and Normal Vectors:

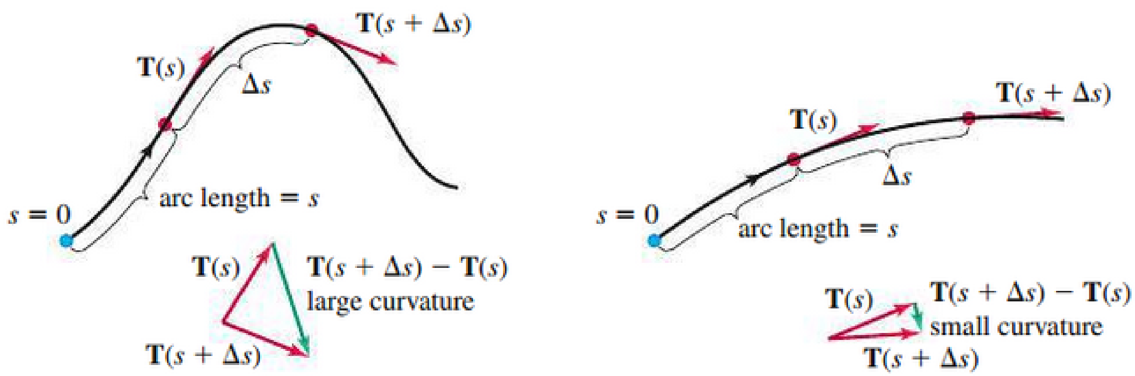
There are two ways to change the velocity, or in other words, to accelerate:

- change in speed
- change in direction

The change in direction is referred to as *curvature*. Recall that if we have a smooth curve $\mathbf{r}(t)$, the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} \implies |\vec{T}(t)| = 1$$

Specifically, *curvature* of the curve is the magnitude of the rate at which $\mathbf{T}$ changes with respect to arc length.



Definition. (Curvature)
Let $\mathbf{r}$ describe a smooth parameterized curve. If s denotes arc length and $\mathbf{T} = \mathbf{r}'/|\mathbf{r}'|$ is the unit tangent vector, the **curvature** is $\kappa(s) = \left|\frac{d\mathbf{T}}{ds}\right|$. ≥ 0 change unit tangent vector w.r.t. arc length

Kappa

$\vec{v}(t)$

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| \qquad \left|\frac{d\vec{T}}{dt}\right| = \left|\frac{d\vec{T}}{ds}\cdot\frac{ds}{dt}\right| \Rightarrow \left|\frac{d\vec{T}}{ds}\right| = \frac{|d\vec{T}/dt|}{|ds/dt|}$$

Theorem 14.4: Curvature Formula
Let $\mathbf{r}(t)$ describe a smooth parameterized curve, where t is any parameter. If $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{T}$ is the unit tangent vector, then the curvature is

$$\kappa(t) = \frac{1}{\underbrace{|\mathbf{v}|}_{\text{speed}}}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}. \longleftarrow \quad \cancel{\left|\frac{\vec{T}'(t)}{\vec{r}'(t)}\right|}$$

$$\cancel{\frac{\sqrt{\quad}}{\sqrt{\quad}} = \sqrt{\frac{0}{0} + \frac{5}{6} + \vdots}}$$

- κ is a non-negative scalar-valued function
- Curvature of zero corresponds to a straight line
- A relatively flat curve has a small curvature
- A tight curve has a larger curvature

Example. Consider the line

$$\mathbf{r}(t) = \langle x_0 + at,\ y_0 + bt,\ z_0 + ct\rangle, \text{ for } -\infty < t < \infty.$$

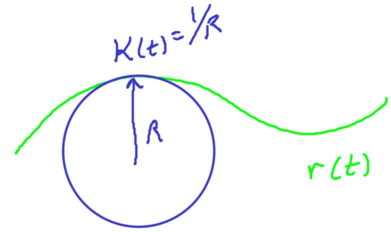
Compute κ.

$$\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\langle 0,0,0\rangle|}{\sqrt{a^2+b^2+c^2}} = 0$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\langle a,b,c\rangle}{\sqrt{a^2+b^2+c^2}} \longrightarrow \vec{T}'(t) = \langle 0,0,0\rangle$$

$R \to \infty \Rightarrow K = 0$

Example. Consider the circle



$$\mathbf{r}(t) = \langle R\cos(t),\ R\sin(t)\rangle$$

for $0 \le t \le 2\pi$, where $R > 0$. Show that $\kappa = 1/R$.

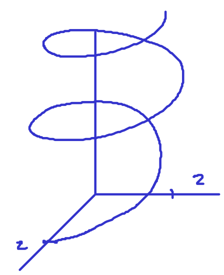
$$K = \frac{|d\vec{T}/dt|}{|ds/dt|} = \frac{|\langle -\cos(t), -\sin(t)\rangle|}{R} = \frac{1}{R}\sqrt{\underbrace{\cos^2(t) + \sin^2(t)}_{1}} = \frac{1}{R}$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\langle -R\sin(t), R\cos(t)\rangle}{\sqrt{(-R\sin(t))^2 + (R\cos(t))^2}} = \frac{\langle -R\sin(t), R\cos(t)\rangle}{\sqrt{R^2(\underbrace{\sin^2(t) + \cos^2(t)}_{1})}} = \frac{\langle -R\sin(t), R\cos(t)\rangle}{R}$$

$|ds/dt|$

$$= \langle -\sin(t), \cos(t)\rangle$$

Example. Consider the curve



$$\mathbf{r}(t) = \left\langle 2\cos(t),\ 2\sin(t),\ \underbrace{\sqrt{5}}_{3}t \right\rangle$$

Compute κ. $\quad K \overset{?}{=} \frac{1}{2}$

$$K = \frac{|d\vec{T}/dt|}{|ds/dt|} = \frac{|\frac{1}{3}\langle -2\cos(t), -2\sin(t), 0\rangle|}{3} = \frac{1}{9}\sqrt{4(\underbrace{\cos^2(t) + \sin^2(t)}_{1})} = \frac{2}{9}$$

$$T(t) = \frac{\langle -2\sin(t), 2\cos(t), \sqrt{5}\rangle}{\sqrt{2^2(\underbrace{\sin^2(t) + \cos^2(t)}_{1}) + 5}} = \frac{1}{3}\langle -2\sin(t), 2\cos(t), \sqrt{5}\rangle$$

An Alternative Curvature Formula:

Consider a smooth function $\mathbf{r}(t)$ with non-zero velocity $\mathbf{v}(t) = \mathbf{r}'(t)$ and non-zero acceleration $\mathbf{a}(t) = \mathbf{v}'(t)$.

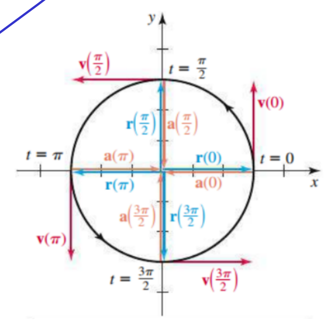
$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} \;\Rightarrow\; \mathbf{v} = |\mathbf{v}|\,\mathbf{T}.$$

$\times$ $\vec{T}(t)$ $\vec{T}'(t)$

Thus

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}[|\mathbf{v}|\,\mathbf{T}] = \frac{d}{dt}[|\mathbf{v}|]\,\mathbf{T} + |\mathbf{v}|\,\frac{d\mathbf{T}}{dt}.$$

Now we form $\mathbf{v} \times \mathbf{a}$:



$$\begin{aligned}\mathbf{v} \times \mathbf{a} &= |\mathbf{v}|\,\mathbf{T} \times \left(\frac{d}{dt}[|\mathbf{v}|]\,\mathbf{T} + |\mathbf{v}|\,\frac{d\mathbf{T}}{dt}\right)\\ &= \underbrace{|\mathbf{v}|\,\mathbf{T} \times \frac{d}{dt}[|\mathbf{v}|]\,\mathbf{T}}_{\mathbf{0}} + |\mathbf{v}|\,\mathbf{T} \times |\mathbf{v}|\,\frac{d\mathbf{T}}{dt}\end{aligned}$$

Since $\mathbf{T}$ is a unit vector, $\mathbf{T}$ and $d\mathbf{T}/dt$ are orthogonal (Theorem 14.2). Thus

$$|\mathbf{v} \times \mathbf{a}| = \left||\mathbf{v}|\,\mathbf{T} \times |\mathbf{v}|\,\frac{d\mathbf{T}}{dt}\right| = |\mathbf{v}|\,\underbrace{|\mathbf{T}|}_{1}\,\left||\mathbf{v}|\frac{d\mathbf{T}}{dt}\right|\underbrace{\sin\theta}_{1} = |\mathbf{v}|^2\left|\frac{d\mathbf{T}}{dt}\right|$$

Now, using Theorem 14.4, where $\left|\dfrac{d\mathbf{T}}{dt}\right| = \kappa|\mathbf{v}|$, we have

$$\kappa = \frac{|d\vec{T}/dt|}{|ds/dt|} = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

$$|\mathbf{v} \times \mathbf{a}| = |\mathbf{v}|^2\left|\frac{d\mathbf{T}}{dt}\right| = |\mathbf{v}|^2\,\kappa|\mathbf{v}| = \kappa|\mathbf{v}|^3.$$

Theorem 14.5: Alternative Curvature Formula

Let $\mathbf{r}$ be the position of an object moving on a smooth curve. The **curvature** at a point on the curve is

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3},$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{a} = \mathbf{v}'$ is the acceleration.

$$\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3} \qquad \kappa = \frac{1}{16}$$

Example. Consider the curve

$$\kappa = \frac{1}{a} \qquad a = 16$$

$$\mathbf{r}(t) = \langle -16\cos(t),\, 16\sin(t),\, 0\rangle.$$

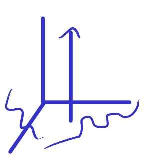
Compute the curvature κ using both methods.

$$\vec{v}(t) = \langle 16\sin(t),\ 16\cos(t),\ 0\rangle \qquad |\vec{v}(t)| = \sqrt{\underbrace{16^2\sin^2(t) + 16^2\cos^2(t)}_{16^2}} = 16$$

$$\vec{a}(t) = \langle 16\cos(t),\ -16\sin(t),\ 0\rangle$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\langle 16\sin(t),\ 16\cos(t),\ 0\rangle}{16} = \langle \sin(t), \cos(t), 0\rangle$$

$$\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{16}\left|\langle \cos(t), -\sin(t), 0\rangle\right| = \frac{1}{16}\sqrt{\underbrace{\cos^2(t)+\sin^2(t)}_{1}} = \frac{1}{16}$$



$$\kappa = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3} = \frac{|\langle 0, -0, -16^2\sin^2(t) - 16^2\cos^2(t)\rangle|}{16^3} = \frac{16^2}{16^3} = \frac{1}{16}$$

Principal Unit Normal Vector

Curvature indicates how quickly a curve turns. The principal unit normal vector determines the *direction* in which a curve turns.

Definition. (Principal Unit Normal Vector)
Let $\mathbf{r}$ describe a smooth curve parameterized by arc length. The **principal unit normal vector** at a point P on the curve at which $\kappa \neq 0$ is

$$\mathbf{N}(s) = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

Unit change in unit tangent vector wrt arc length

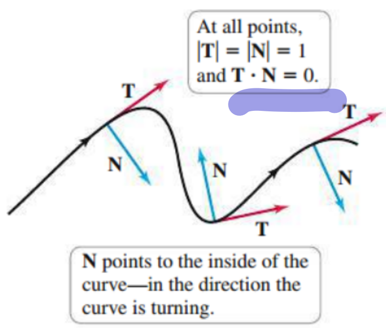
For other parameters, we use the equivalent formula

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|},$$

evaluated at the value of t corresponding to P.

$|\vec{T}| = 1$

$|\vec{N}| = 1$



Theorem 14.6: Properties of the Principal Unit Normal Vector
Let $\mathbf{r}$ describe a smooth parameterized curve with unit tangent vector $\mathbf{T}$ and principal unit normal vector $\mathbf{N}$.

1. $\mathbf{T}$ and $\mathbf{N}$ are orthogonal at all points of the curve; that is, $\mathbf{T}\cdot\mathbf{N} = 0$ at all points where $\mathbf{N}$ is defined.

2. The principal unit normal vector points to the inside of the curve – in the direction that the curve is turning.

Example. For the curve $\mathbf{r}(t) = \langle a\cos(t),\, a\sin(t),\, bt\rangle$, find the unit tangent vector $\mathbf{T}$ and the principal unit normal vector $\mathbf{N}$. Verify $|\mathbf{T}| = |\mathbf{N}| = 1$ and $\mathbf{T}\cdot\mathbf{N} = 0$.

$$\vec{r}\,'(t) = \langle -a\sin(t), a\cos(t), b\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{\underbrace{(-a\sin(t))^2 + (a\cos(t))^2}_{a^2} + b^2} = \sqrt{a^2+b^2} \quad \text{constant}$$

$$\vec{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\langle -a\sin(t), a\cos(t), b\rangle}{\sqrt{a^2+b^2}}$$

$$|\vec{T}| = \left|\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\right| = \frac{1}{|\vec{r}\,'(t)|}\,|\vec{r}\,'(t)| = 1$$

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} = \frac{\frac{1}{\sqrt{a^2+b^2}}\langle -a\cos(t), -a\sin(t), 0\rangle}{\underbrace{\sqrt{\frac{(-a\cos(t))^2 + (-a\sin(t))^2}{\sqrt{a^2+b^2}}}}_{\frac{\sqrt{a^2}}{\sqrt{a^2+b^2}}}} = \frac{1}{a}\langle -a\cos(t), -a\sin(t), 0\rangle$$

$$= \langle -\cos(t), -\sin(t), 0\rangle$$

$$|\vec{N}| = 1$$

$$\vec{T}\cdot\vec{N} = \frac{1}{\sqrt{a^2+b^2}}\langle -a\sin(t), a\cos(t), b\rangle \cdot \langle -\cos(t), -\sin(t), 0\rangle$$

$$= \frac{1}{\sqrt{a^2+b^2}}\left(a\sin(t)\cos(t) - a\sin(t)\cos(t) + 0\right) = 0$$

Components of the Acceleration

Recall that the change in velocity, or acceleration, of an object can change in *speed* (in the direction of $\mathbf{T}$) and in *direction* (in the direction of $\mathbf{N}$). $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|} \implies \mathbf{v} = \mathbf{T}|\mathbf{v}| = \mathbf{T}\,\dfrac{ds}{dt}$.

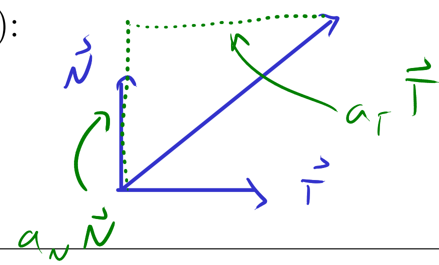
$$\frac{d}{dt}[\vec{v}] = \vec{a}$$

$$\begin{aligned}
\mathbf{a} = \frac{d\mathbf{v}}{dt} &= \frac{d}{dt}\left(\mathbf{T}\frac{ds}{dt}\right) \qquad \text{product rule} \\
&= \frac{d\mathbf{T}}{dt}\frac{ds}{dt} + \mathbf{T}\frac{d^2s}{dt^2} \\
&= \underbrace{\frac{d\mathbf{T}}{ds}}_{\kappa\mathbf{N}}\ \underbrace{\frac{ds}{dt}}_{|\mathbf{v}|}\ \underbrace{\frac{ds}{dt}}_{|\mathbf{v}|} + \mathbf{T}\frac{d^2s}{dt^2} \\
&= \kappa|\mathbf{v}|^2\,\mathbf{N} + \frac{d^2s}{dt^2}\,\mathbf{T}.
\end{aligned}$$

Theorem 14.7: Tangential and Normal Components of the Acceleration
The acceleration vector of an object moving in space along a smooth curve has the following representation in terms of its **tangential component** a_T (in the direction of $\mathbf{T}$) and its **normal component** a_N (in the direction of $\mathbf{N}$):

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T},$$

where $a_N = \kappa|\mathbf{v}|^2 = \dfrac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|}$ and $a_T = \dfrac{d^2s}{dt^2}$.



$$K = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3}$$

Example. Consider the function

linear

$$\mathbf{r}(t) = \langle -2t + 2,\ -2t + 3,\ -2t + 2\rangle. \quad \to \vec{a}(t) = \langle 0,0,0\rangle$$

Find the tangential and normal components of the acceleration.

$a_N = \dfrac{|\vec{v} \times \vec{a}|}{|\vec{v}|}$ $\quad a_T = \dfrac{d^2 s}{dt^2} = \dfrac{d}{dt}\left[|\vec{v}(t)|\right] = 0$ $\quad$ LC 3

$\vec{v}(t) = \langle -2,-2,-2\rangle$

$|\vec{v}(t)| = 2\sqrt{3}$ constant

$\vec{a}(t) = \langle 0,0,0\rangle$

$a_N = \dfrac{|\langle 0,0,0\rangle|}{2\sqrt{3}} = 0$ $\quad$ LC 2

Example. Find the components of the acceleration on the circular trajectory

$$\mathbf{r}(t) = \langle R\cos(\omega t),\ R\sin(\omega t)\rangle, 0\rangle \qquad K = \frac{1}{R}$$

$\vec{v}(t) = \langle -R\omega \sin(\omega t), R\omega\cos(\omega t), 0\rangle$ $\quad |\vec{v}(t)| = \sqrt{(-R\omega\sin(\omega t))^2 + (R\omega\cos(\omega t))^2}$

$\vec{a}(t) = \langle -R\omega^2\cos(\omega t), -R\omega^2\sin(\omega t), 0\rangle$ $\quad = R\omega$

$a_N = \dfrac{|\vec{v} \times \vec{a}|}{|\vec{v}|} = \dfrac{|\langle 0,0, R^2\omega^3\sin^2(\omega t) + R^2\omega^3\cos^2(\omega t)\rangle|}{R\omega^2} = \dfrac{R^2\omega^3}{R\omega}$

$R=2, \omega=3$

LC 5 $\to 2\cdot 3^2 = 18$ $\quad = R\omega^2$

$a_T = \dfrac{d}{dt}\left[|\vec{v}(t)|\right] = 0$ $\quad$ LC 4 $\to$ 0

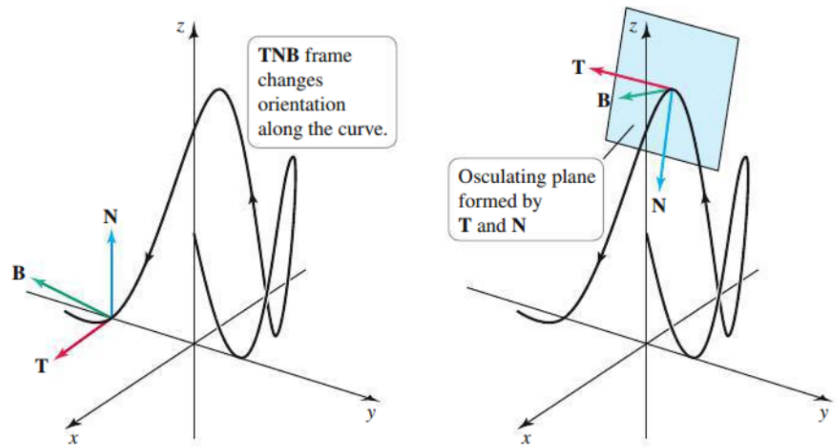
LC

1. 16
2. 0
3. 0
4. 0
5. 18

Example. The driver of a car follows the parabolic trajectory $\mathbf{r}(t) = \left\langle t,\, t^2 \right\rangle$, for $-2 \le t \le 2$, through a sharp bend. Find the tangential and normal components of the acceleration of the car.

The Binormal Vector and Torsion

On a smooth parameterized curve C, $\mathbf{T}$ and $\mathbf{N}$ determine a plane called the *osculating plane.*



The coordinate system defined by these vectors is called the **TNB frame**. The rate at which the curve C twists out of the plane is the rate at which $\mathbf{B}$ changes as we move along C, which is $\dfrac{d\mathbf{B}}{ds}$.

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T} \times \mathbf{N}) = \underbrace{\frac{d\mathbf{T}}{ds} \times \mathbf{N}}_{\mathbf{0}} + \mathbf{T} \times \frac{d\mathbf{N}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds}$$

$\dfrac{d\mathbf{B}}{ds}$ is:

- orthogonal to both $\mathbf{T}$ and $\dfrac{d\mathbf{N}}{ds}$,
- orthogonal to $\mathbf{B}$ (Theorem 14.2),
- parallel with $\mathbf{N}$.

Since $\dfrac{d\mathbf{B}}{ds}$ is parallel to $\mathbf{N}$, we write

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

where τ is the *torsion* (the negative sign is conventional). We can solve for τ via the dot product:

$$\frac{d\mathbf{B}}{ds}\cdot\mathbf{N} = -\tau\underbrace{\mathbf{N}\cdot\mathbf{N}}_{1} \quad\Longrightarrow\quad \frac{d\mathbf{B}}{ds}\cdot\mathbf{N} = -\tau$$

Definition. (Unit Binormal Vector and Torsion)
Let C be a smooth parameterized curve with unit tangent and principal unit normal vectors $\mathbf{T}$ and $\mathbf{N}$, respectively. Then at each point of the curve at which the curvature is nonzero, the **unit binomial vector** is

$$\mathbf{B} = \mathbf{T}\times\mathbf{N},$$

and the **torsion** is

$$\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

Example. Consider the circle C defined by

$$\mathbf{r}(t) = \langle R\cos(t),\, R\sin(t)\rangle, \text{ for } 0 \le t \le 2\pi, \text{ with } R > 0.$$

Find the unit binormal vector $\mathbf{B}$ and determine the torsion.

Example. Compute the torsion of the helix

$$\mathbf{r}(t) = \langle a\cos(t),\, a\sin(t),\, bt\rangle, \text{ for } t \ge 0, \text{ and } b > 0.$$

Summary: Formula for Curves in Space

Position function:	$\mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$
Velocity:	$\mathbf{v} = \mathbf{r}'$
Acceleration:	$\mathbf{a} = \mathbf{v}'$
Unit tangent vector:	$\mathbf{T} = \dfrac{\mathbf{v}}{\lvert\mathbf{v}\rvert}$
Principal unit normal vector:	$\mathbf{N} = \dfrac{d\mathbf{T}/dt}{\lvert d\mathbf{T}/dt\rvert}$ (provided $d\mathbf{T}/dt \neq \mathbf{0}$)
Curvature:	$\kappa = \left\lvert\dfrac{d\mathbf{T}}{ds}\right\rvert = \dfrac{1}{\lvert\mathbf{v}\rvert}\left\lvert\dfrac{d\mathbf{T}}{dt}\right\rvert = \dfrac{\lvert\mathbf{v}\times\mathbf{a}\rvert}{\lvert\mathbf{v}\rvert^3}$
Components of acceleration:	$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}$, where
	$a_N = \kappa\lvert\mathbf{v}\rvert^2 = \dfrac{\lvert\mathbf{v}\times\mathbf{a}\rvert}{\lvert\mathbf{v}\rvert}$ and $a_T = \dfrac{d^2s}{dt^2} = \dfrac{\mathbf{v}\cdot\mathbf{a}}{\mathbf{v}}$
Unit binormal vector:	$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \dfrac{\mathbf{v}\times\mathbf{a}}{\lvert\mathbf{v}\times\mathbf{a}\rvert}$
Torsion:	$\tau = -\dfrac{d\mathbf{B}}{ds}\cdot\mathbf{N} = \dfrac{(\mathbf{v}\times\mathbf{a})\cdot\mathbf{a}'}{\lvert\mathbf{v}\times\mathbf{a}\rvert^2} = \dfrac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{\lvert\mathbf{r}'\times\mathbf{r}''\rvert^2}$