1 15.5: Directional Derivatives and the Gradient

Definition. (Directional Derivative)
Let f be differentiable at (a,b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The **directional derivative of f at (a,b) in the direction of u** is

$$D_{\mathbf{u}} f(a,b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a,b)}{h},$$

provided the limit exists.

Theorem 15.10: Directional Derivative
Let f be differentiable at (a,b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The **directional derivative of f at (a,b) in the direction of $\mathbf{u}$** is

$$D_{\mathbf{u}} f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle \cdot \langle u_1, u_2 \rangle.$$

Definition. (Gradient (Two Dimensions))
Let f be differentiable at the point (x,y). The **gradient** of f at (x,y) is the vector-valued function

$$\nabla f(x,y) = \langle f_x(x,y),\ f_y(x,y) \rangle = f_x(x,y)\boldsymbol{i} + f_y(x,y)\boldsymbol{j}.$$

Theorem 15.11: Directions of Change
Let f be differentiable at (a,b) with $\nabla f(a,b) \neq \mathbf{0}$.

1. f has its maximum rate of increase at (a,b) in the direction of the gradient $\nabla f(a,b)$. The rate of change in this direction is $|\nabla f(a,b)|$.
2. f has its maximum rate of decrease at (a,b) in the direction of $-\nabla f(a,b)$. The rate of change in this direction is $-|\nabla f(a,b)|$.
3. The directional derivative is zero in any direction orthogonal to $\nabla f(a,b)$.

Theorem 15.12: The Gradient and Level Curves
Given a function f differentiable at (a, b), the line tangent to the level curve of f at (a, b) is orthogonal to the gradient $\nabla f(a, b)$, provided $\nabla f(a, b) \neq \mathbf{0}$.

Definition. (Directional Derivative and Gradient in Three Dimensions)
Let f be directional at (a, b, c) and let $\mathbf{u} = \langle u_1,\, u_2,\, u_3 \rangle$ be a unit vector. The **directional derivative of f at (a, b, c) in the direction of $\mathbf{u}$** is

$$D_{\mathbf{u}}(a, b, c) = \lim_{h \to 0} \frac{f(a + hu_1,\, b + hu_2,\, c + hu_3) - f(a, b, c)}{h},$$

provided this limit exists.
The **gradient** of f at this point (x, y, z) is the vector-valued function

$$\begin{aligned} \nabla f(x, y, z) &= \langle f_x(x, y, z),\, f_y(x, y, z),\, f_z(x, y, z) \rangle \\ &= f_x(x, y, z)\boldsymbol{i} + f_y(x, y, z)\boldsymbol{j} + f_z(x, y, z)\boldsymbol{k}. \end{aligned}$$

Theorem 15.13: Directional Derivative and Interpreting the Gradient
Let f be differentiable at (a, b, c) and let $\mathbf{u} = \langle u_1,\, u_2,\, u_3 \rangle$ be a unit vector. The directional derivative of f at (a, b, c) in the direction of $\mathbf{u}$ is

$$\begin{aligned} D_{\mathbf{u}} f(a, b, c) &= \nabla f(a, b, c) \cdot \mathbf{u} \\ &= \langle f_x(a, b, c),\, f_y(a, b, c),\, f_z(a, b, c) \rangle \cdot \langle u_1,\, u_2,\, u_3 \rangle. \end{aligned}$$

Assuming $\nabla f(a, b, c) \neq \mathbf{0}$, the gradient in three dimensions has the following properties.

1. f has its maximum rate of increase at (a, b, c) in the direction of the gradient $\nabla f(a, b, c)$ and the rate of change in this direction is $|\nabla f(a, b, c)|$.
2. f has its maximum rate of decrease at (a, b, c) in the direction of $-\nabla f(a, b, c)$ and the rate of change in this direction is $-|\nabla f(a, b, c)|$.
3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b, c)$.