3.3 Rules of Differentiation

Theorem 3.2 Constant Rule
If c is a real number, then $\frac{d}{dx}(c) = 0$.

Example. Find the derivatives of

$f(x) = 3$ $\qquad$ $g(x) = \pi$ $\qquad$ $h(x) = e^{\pi}$

$f'(x) = 0$ $\qquad$ $g'(x) = 0$ $\qquad$ $h'(x) = 0$

Theorem 3.3 Power Rule
If n is a nonnegative integer, then $\frac{d}{dx}(x^n) = nx^{n-1}$

Example. Find the derivative of

$j(x) = x^3$ $\qquad$ $\ell(x) = x^{\pi}$ $\qquad$ $m(x) = \pi^{42\cos(e)}$

$j'(x) = 3x^2$ $\qquad$ $\ell'(x) = \pi x^{\pi - 1}$ $\qquad$ $m'(x) = 0$

CONSTANT!

Note! Power Rule ONLY works for a variable raised to a number

Proof. (Briggs, p153)
Let $f(x) = x^n$ and use the definition of the derivative in the form

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

With $n = 1$ and $f(x) = x$, we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x - a}{x - a} = 1$$

as given by the Power Rule.
With $n \geq 2$ and $f(x) = x^n$, note that $f(x) - f(a) = x^n - a^n$. A factoring formula gives

$$x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}).$$

Therefore,

$$\begin{aligned}
f'(a) &= \lim_{x \to a} \frac{x^n - a^n}{x - a} \\
&= \lim_{x \to a} \frac{(x - a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{x - a} \\
&= \lim_{x \to a} (x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}) \\
&= \underbrace{a^{n-1} + a^{n-2} \cdot a + \cdots + a \cdot a^{n-2} + a^{n-1}}_{n \text{ terms}} = na^{n-1}
\end{aligned}$$

$\square$

Theorem 3.4 Constant Multiple Rule
If f is differentiable at x and c is a constant, then

$$\frac{d}{dx}(cf(x)) = cf'(x)$$

Example.

$$\frac{d}{dx}(-4x^9) = -4\frac{d}{dx}(x^9) = -4(9x^8) = -36x^8$$

$$\frac{d}{dx}\left(-\frac{7x^{11}}{8}\right) = -\frac{7}{8}\frac{d}{dx}(x^{11}) = -\frac{7}{8}(11x^{10}) = -\frac{77}{8}x^{10}$$

$$\frac{d}{dx}\left(\tfrac{1}{3}x^3\right) = \frac{1}{3}\frac{d}{dx}(x^3) = \frac{1}{3}(3x^2) = x^2$$

Theorem 3.5 Sum Rule
If f and g are differentiable at x, then

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

Example. Find the derivative of the following:

$p(x) = 3x^{100} + 4x^e - 17x + 24 - \pi^{\cos(e)}$ $\qquad$ $t(w) = 2\ ^3 + 9w^2 - 6w + 4$

$p'(x) = 300x^{99} + 4e\,x^{e-1} - 17 + 0 - 0$ $\qquad$ $t'(w) = 6 \cdot w + 18w - 6$

Definition. (The Number e)

The number $e = 2.718281828459\ldots$ satisfies

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

It is the base of the natural exponential function $f(x) = e^x$

Note: One way to show the above result is to recall that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.

Theorem 3.6 The Derivative of e^x

The function $f(x) = e^x$ is differentiable for all real numbers x, and

$$\frac{d}{dx}(e^x) = e^x$$

Proof.

$$\frac{d}{dx}(e^x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h} = e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x$$

□

Example. Find the derivatives of the following

e^x $\qquad$ $42e^x$ $\qquad$ $7e^x - 14x^e$

$\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(42e^x) = 42e^x$

$\frac{d}{dx}(7e^x - 14x^e)$

$= 7e^x - 14ex^{e-1}$

Example. *Note:* Simplify the expression before taking the derivative

a) $\dfrac{d}{ds}\left(\dfrac{12s^3 - 8s^2 + 12s}{4s}\right)$

$= \dfrac{d}{ds}\left(3s^2 - 2s + 3\right)$

$= 6s - 2$

b) $h(x) = \dfrac{x^3 - 6x^2 + 8x}{x^2 - 2x} = \dfrac{x(x-2)(x-4)}{x(x-2)} = (x-4)$

$h'(x) = 1$

c) $\dfrac{d}{dx}\left(\dfrac{x-a}{\sqrt{x}-\sqrt{a}}\right)$

$= \dfrac{d}{dx}\left(\dfrac{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}{\sqrt{x}-\sqrt{a}}\right)$

$= \dfrac{d}{dx}\left(\sqrt{x}+\sqrt{a}\right) = \dfrac{1}{2\sqrt{x}}$

$\sqrt{x} = x^{1/2}$

$\dfrac{d}{dx}\left(\sqrt{x}\right) = \dfrac{d}{dx}\left(x^{1/2}\right) = \dfrac{1}{2}x^{-1/2}$

d) $g(w) = \begin{cases} w + 5e^w, & \text{if } w \le 1 \\ 2w^3 + 4w + 5, & \text{if } w > 1 \end{cases}$

$g'(w) = \begin{cases} 1 + 5e^w, & w \le 1 \\ 6w + 4, & w > 1 \end{cases}$

Not differentiable at $w = 1$.

Example. Use the table to find the following derivatives:

x	1	2	3	4	5
$f'(x)$	3	4	2	1	4
$g'(x)$	2	4	3	1	5

a) $\left.\dfrac{d}{dx}[f(x)+g(x)]\right|_{x=1}$

$= f'(1) + g'(1)$

$= 3 + 2 = \boxed{5}$

b) $\left.\dfrac{d}{dx}[1.5f(x)]\right|_{x=2}$

$= 1.5\ f'(2)$

$= 1.5\ (4) = \boxed{6}$

c) $\left.\dfrac{d}{dx}[2x - 3g(x)]\right|_{x=4}$

$= 2\left.\dfrac{d}{dx}[x]\right|_{x=4} - 3g'(4)$

$= 2 \quad - 3(1)$

$= \boxed{-1}$

Example. Find the equation of the tangent line to $y = x^3 - 4x^2 + 2x - 1$ at $a = 2$

$y' = 3x^2 - 8x + 2 \longrightarrow y'(2) = 3(2)^2 - 8(2) + 2 = -2$

$y(2) = 2^3 - 4(2)^2 + 2(2) - 1 = -5$

tangent line $y - (-5) = -2(x-2)$

$y = -2x + 4 - 5$

$\boxed{y = -2x - 1}$

Example. Find the equation of the tangent line to $y = \dfrac{e^x}{4} - x$ at $a = 0$.

$y(0) = \frac{1}{4}$

$y' = \frac{e^x}{4} - 1$

$y'(0) = -\frac{3}{4}$

tangent line

$y - \frac{1}{4} = -\frac{3}{4}(x - 0)$

$\boxed{y = -\frac{3}{4}x - \frac{1}{2}}$

Example. Find the equation of the normal line to $f(x) = 1 - x^2$ at $x = 2$.

$f'(x) = -2x$ $\quad f(2) = -3$ $\quad m_{normal} = -\frac{1}{f'(2)} = \frac{1}{4}$

$f'(2) = -4$

$y - (-3) = \frac{1}{4}(x-2)$

$y = \frac{1}{4} - \frac{1}{2} - 3$

$\boxed{y = \frac{1}{4} - \frac{7}{2}}$

Example. Find the equations of the tangent line and normal line to $y = \frac{1}{2}x^4$ at $a = 2$.

$y' = 2x^3$ $\quad y(2) = 8$

$y'(2) = 16$

tangent line

$y - 8 = 16(x-2)$

$\boxed{y = 16x - 24}$

normal line

$y - 8 = -\frac{1}{16}(x-2)$

$y = -\frac{1}{16}x + \frac{1}{8} + 8$

$\boxed{y = -\frac{1}{16}x + \frac{65}{8}}$

Example. At what x-values does $f(x) = x - 2x^2$ have horizontal tangents?

$f'(x) = 0$

$f'(x) = 1 - 4x$

Solve

$1 - 4x = 0$

$1 = 4x$

$\boxed{\frac{1}{4} = x}$

Example. Find an equation of the line having slope $\frac{1}{4}$ that is tangent to the curve $y = \sqrt{x}$. $= x^{1/2}$

$y' = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$

Solve

$\frac{1}{2\sqrt{x}} = \frac{1}{4}$

$2 = \sqrt{x}$

$4 = x$

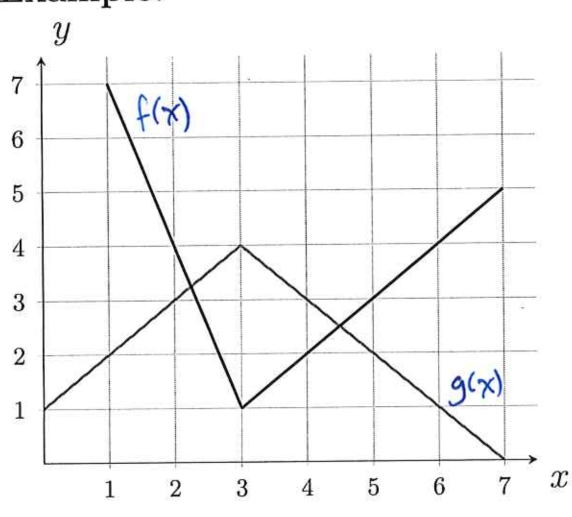
$\Rightarrow$ tangent line

$y - \sqrt{4} = \frac{1}{4}(x-4)$

$y = \frac{1}{4}x - 1 + 2$

$\boxed{y = \frac{1}{4}x + 1}$

Example.



a) $F'(2)$ $= -3$

b) $G'(2)$ $= 1$

c) $F'(5)$ $= 1$

d) $G'(5)$ $= -1$

Example. The line tangent to the graph of f at $x = 5$ is $y = \frac{1}{10}x - 2$. Find $\left.\frac{d}{dx}(4f(x))\right|_{x=5}$

$\Rightarrow f'(5) = \frac{1}{10}$

$\Rightarrow \left.\frac{d}{dx}(4f(x))\right|_{x=5} = 4f'(5) = \frac{4}{10} = \boxed{\frac{2}{5}}$

Example. At what point on the curve $y = 1 + 2e^x - 3x$ is the tangent line parallel to the line $\underbrace{3x - y = 5}$.

$y = 3x - 5 \rightarrow$ slope $= 3$

$y' = 2e^x - 3$

Solve $2e^x - 3 = 3$

$2e^x = 6$

$e^x = 3$

$\boxed{x = \ln(x)}$

Example. Find equations of both lines that are tangent to the curve $y = 1 + x^3$ and parallel to the line $\underbrace{12x - y = 1}$.

$y = 12x - 1 \rightarrow$ slope $= 12$

$y' = 3x^2$

Solve

$3x^2 = 12$

$x^2 = 4$

$\boxed{x = \pm 2}$

Definition. Higher-Order Derivatives

Assuming $y = f(x)$ can be differentiated as often as necessary, the **second derivative** of f is

$$f''(x) = \frac{d}{dx}(f'(x))$$

For integers $n \geq 1$, the **nth derivative** of f is

$$f^{(n)}(x) = \frac{d}{dx}\left(f^{(n-1)}(x)\right)$$

Example. Find all the derivatives of $y = \dfrac{x^5}{120}$

$y' = \dfrac{5x^4}{120} = \dfrac{x^4}{24}$

$y'' = \dfrac{4x^3}{24} = \dfrac{x^3}{6}$

$y''' = \dfrac{3x^2}{6} = \dfrac{x^2}{2}$

$y^{(4)} = \dfrac{2x}{2} = x$

$y^{(5)} = 1$

$y^{(6)} = 0$

$y^{(k)} = 0, \quad k \geq 6$

Example. Find the first, second and third derivatives of $f(x) = 5x^4 + 10x^3 + 3x + 6$

$f'(x) = 20x^3 + 30x^2 + 3$

$f''(x) = 60x^2 + 60x$

$f'''(x) = 120x + 60$

Example. Find the first, second and third derivatives of $f(x) = x^2(2 + x^{-3})$. $= 2x^2 + x^{-1}$

$f'(x) = 4x - x^{-2}$

$f''(x) = 4 + 2x^{-3}$

$f'''(x) = -6x^{-4}$