

# Math 1080 Class notes Fall 2020

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## 5.5: Substitution Rule

### Theorem 5.6: Substitution Rule for Indefinite Integrals

Let  $u = g(x)$ , where  $g$  is differentiable on an interval, and let  $f$  be continuous on the corresponding range of  $g$ . On that interval,

$$\int f(g(x))g'(x) dx = \int f(u) du$$

**Example.** We know

$$\frac{d}{dx} \left[ \frac{(2x+1)^4}{4} \right] = 2(2x+1)^3$$

Thus, if  $f(x) = x^3$  and  $g(x) = 2x + 1$  then  $g'(x) = 2$ , so we let  $u = 2x + 1$ , then

$$\begin{aligned} \int 2(2x+1)^3 dx &= \int f(g(x))g'(x) dx \\ &= \int u^3 du \\ &= \frac{u^4}{4} + C \\ &= \frac{(2x+1)^4}{4} + C \end{aligned}$$

### Procedure: Substitution Rule (Change of Variables)

1. Given an indefinite integral involving a composite function  $f(g(x))$ , identify an inner function  $u = g(x)$  such that a constant multiple of  $g'(x)$  appears in the integrand.
2. Substitute  $u = g(x)$  and  $du = g'(x) dx$  in the integral.
3. Evaluate the new indefinite integral with respect to  $u$ .
4. Write the result in terms of  $x$  using  $u = g(x)$ .

**Example.** Evaluate the following integrals:

a)  $\int 2x(x^2 + 3)^4 dx$

b)  $\int (2x + 1)^3 dx$

c)  $\int x^2 \sqrt{x^3 + 1} dx$

d)  $\int \theta \sqrt[4]{1 - \theta^2} d\theta$

e)  $\int \sqrt{4 - t} dt$

f)  $\int (2 - x)^6 dx$

**Example.** Evaluate the following integrals:

a)  $\int \sec(2\theta) \tan(2\theta) d\theta$

b)  $\int \csc^2\left(\frac{t}{3}\right) dt$

c)  $\int \frac{\sin(x)}{1 + \cos^2(x)} dx$

d)  $\int \frac{\tan^{-1}(x)}{1 + x^2} dx$

The acceleration of a particle moving back and forth on a line is  $a(t) = \frac{d^2s}{dt^2} = \pi^2 \cos(\pi t) \text{ m/s}^2$  for all  $t$ . If  $s = 0$  and  $v = 8 \text{ m/s}$  when  $t = 0$ , find the value of  $s$  when  $t = 1$  sec.

**Example.** Evaluate the following integrals:

a)  $\int (6x^2 + 2) \sin(x^3 + x + 1) dx$

b)  $\int \frac{\sin(\theta)}{\cos^5(\theta)} d\theta$

c)  $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

d)  $\int \frac{2^t}{2^t + 3} dt$

e)  $\int 6x^2 4^{x^3} dx$

f)  $\int \frac{dx}{\sqrt{36 - 4x^2}}$

g)  $\int \sin(t) \sec^2(\cos(t)) dt$

h)  $\int \frac{1}{\sqrt{x}(1 + \sqrt{x})^2} dx$

i)  $\int \frac{\sin(\sqrt{x})}{\sqrt{x}} dx$

j)  $\int 5 \cos(7x + 5) dx$

k)  $\int \frac{3}{\sqrt{1 - 25x^2}} dx$

l)  $\int \frac{dx}{\sqrt{1 - 9x^2}}$



**Example.** Evaluate the following integrals using the recommended substitution:

a)  $\int \sec^2(x) \tan(x) \, dx$   
where  $u = \tan(x)$ .

b)  $\int \sec^2(x) \tan(x) \, dx$   
where  $u = \sec(x)$ .

**Example.** Solve the initial value problem:  $\frac{dy}{dx} = 4x(x^2 + 8)^{-1/3}, y(0) = 0$ .

**Example.** Evaluate the following integrals:

a)  $\int x e^{-x^2} dx$

b)  $\int \frac{e^{1/x}}{x^2} dx$

c)  $\int \frac{dt}{8-3t}$

d)  $\int 5^t \sin(5^t) dt$

e)  $\int \frac{e^w}{36 + e^{2w}} dw$

**Theorem 5.7: Substitution Rule for Definite Integrals**

Let  $u = g(x)$ , where  $g'$  is continuous on  $[a, b]$ , and let  $f$  be continuous on the range of  $g$ . Then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

**Example.** Evaluate the integrals:

a)  $\int_0^1 \frac{x^3}{\sqrt{x^4 + 9}} dx$

b)  $\int_1^3 \frac{dt}{(t - 4)^2}$

c)  $\int_0^3 \frac{v^2 + 1}{\sqrt{v^3 + 3v + 4}} dv$

d)  $\int_0^1 2x(4 - x^2) dx$

e)  $\int_2^3 \frac{x}{\sqrt[3]{x^2-1}} dx$

f)  $\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{1+\cos(x)} dx$

g)  $\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos^2(x)} dx$

h)  $\int_{-\frac{\pi}{12}}^{\frac{\pi}{8}} \sec^2(2y) dy$

i)  $\int_0^1 (1 - 2x^9) dx$

j)  $\int_0^1 (1 - 2x)^9 dx$

k)  $\int_0^{\frac{1}{2}} \frac{1}{1 + 4x^2} dx$

l)  $\int_0^4 \frac{x}{x^2 + 1} dx$

m)  $\int_0^\pi 3 \cos^2(x) \sin(x) \, dx$

n)  $\int_0^{\frac{\pi}{8}} \sec(2\theta) \tan(2\theta) \, d\theta$

o)  $\int_0^1 (3t - 1)^{50} \, dt$

p)  $\int_0^3 \frac{1}{5x + 1} \, dx$

q)  $\int_0^1 x e^{-x^2} dx$

r)  $\int_e^{e^4} \frac{1}{x \sqrt{\ln(x)}} dx$

s)  $\int_0^{\frac{1}{2}} \frac{\sin^{-1}(x)}{\sqrt{1-x^2}} dx$

t)  $\int_0^1 \frac{e^z + 1}{e^z + z} dz$

$$\text{u)} \int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2}$$

$$\text{v)} \int_{\ln(\frac{\pi}{4})}^{\ln(\frac{\pi}{2})} e^w \cos(e^w) dw$$

$$\text{w)} \int_0^{\frac{1}{8}} \frac{x}{\sqrt{1-16x^2}} dx$$

$$\text{x)} \int_1^{e^2} \frac{\ln(p)}{p} dp$$



$$\text{y) } \int_0^{\frac{\pi}{4}} e^{\sin^2(x)} \sin(2x) \, dx$$

$$\text{z) } \int_{-\pi}^{\pi} x^2 \sin(7x^3) \, dx$$

**Example. Average velocity:** An object moves in one dimension with a velocity in  $m/s$  given by  $v(t) = 8 \sin(\pi t) + 2t$ . Find its average velocity over the time interval from  $t = 0$  to  $t = 10$ , where  $t$  is measured in seconds.

**Example.** Prove  $\int \tan(x) \, dx = \ln |\sec(x)| + C$ .

**Example.** Evaluate the integrals:

a)  $\int \frac{x}{(x-2)^3} \, dx$

b)  $\int x\sqrt{x-1} \, dx$

c)  $\int x^3(1+x^2)^{\frac{3}{2}} dx$

d)  $\int \frac{y^2}{(y+1)^4} dy$

e)  $\int (z+1)\sqrt{3z+2} dz$

f)  $\int_0^1 \frac{x}{(x+2)^3} dx$

### Half-Angle Formulas

$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$$

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$$

**Example.** Evaluate the integrals:

a)  $\int \cos^2(x) \, dx$

b)  $\int_0^{\frac{\pi}{2}} \cos^2(x) \, dx$

c)  $\int \frac{1}{x^2} \cos^2\left(\frac{1}{x}\right) dx$

d)  $\int x \sin^2(x^2) dx$

e)  $\int \sin^2\left(\theta + \frac{\pi}{6}\right) d\theta$

f)  $\int_0^{\frac{\pi}{4}} \cos^2(8\theta) d\theta$

**Example.** If  $f$  is continuous and  $\int_0^4 f(x) dx = 10$ , find  $\int_0^2 f(2x) dx$ .

**Example.** If  $f$  is continuous and  $\int_0^9 f(x) dx = 4$ , find  $\int_0^3 xf(x^2) dx$ .

**Example.** Suppose  $f$  is an even function with  $\int_0^8 f(x) dx = 9$ . Evaluate the following:

a)  $\int_{-1}^1 xf(x^2) dx$ .

b)  $\int_{-2}^2 x^2 f(x^3) dx$ .

**Example.** Evaluate the integrals:

a)  $\int \sec^2(10x) \, dx$

b)  $\int \tan^{10}(4x) \sec^2(4x) \, dx$

c)  $\int \left(x^{\frac{3}{2}} + 8\right)^5 \sqrt{x} \, dx$

d)  $\int \frac{2x}{\sqrt{3x+2}} \, dx$

e)  $\int \frac{7x^2 + 2x}{x} dx$

f)  $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

g)  $\int_0^{\sqrt{3}} \frac{3}{9 + x^2} dx$

h)  $\int_0^{\frac{\pi}{6}} \frac{\sin(2y)}{\sin^2(y) + 2} dy$



i)  $\int \frac{\sec(z) \tan(z)}{\sqrt{\sec(z)}} dz$

j)  $\int \frac{1}{\sin^{-1}(x) \sqrt{1-x^2}} dx$

k)  $\int \frac{x}{\sqrt{4-9x^2}} dx$

l)  $\int \frac{x}{1+x^4} dx$

$$\text{m)} \int \frac{\cos \sqrt{\theta}}{\sqrt{\theta} \sin^2 \sqrt{\theta}} d\theta$$

$$\text{n)} \int x^2 \sqrt{2+x} dx$$

$$\text{o)} \int (\sin^5(x) + 3 \sin^3(x) - \sin(x)) \cos(x) dx$$

p)  $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (x^3 + x^4 \tan(x)) \, dx$

q)  $\int_0^{\frac{\pi}{2}} \cos(x) \sin(\sin(x)) \, dx$

r)  $\int \frac{1+x}{1+x^2} \, dx$

**Example.** Evaluate these more challenging integrals:

a)  $\int \frac{dx}{\sqrt{1 + \sqrt{1 + x}}}$

b)  $\int x \sin^4(x^2) \cos(x^2) dx$