Math 2060 Class notes
Spring 2021

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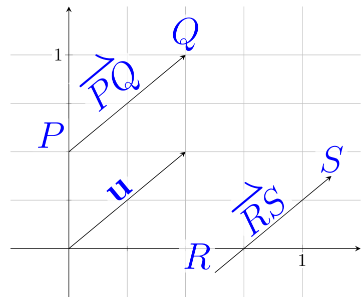
13.1: Vectors and the Geometry of Space

Definition.

- **Vectors**
  - Have a direction and magnitude,
  - vector $\overrightarrow{PQ}$ has a *tail* at P and a *head* at Q,
  - Can be denoted as $\mathbf{u}$ or $\vec{u}$,
  - Equal vectors have the same direction and magnitude (not necessarily the same position)



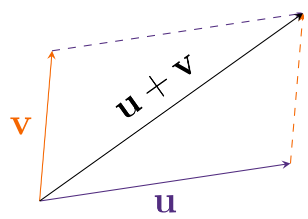
- **Scalars** are quantities with magnitude but no direction (e.g. mass, temperature, price, time, etc.)
- **Zero vector**, denoted $\mathbf{0}$ or $\vec{0}$, has length 0 and no direction

Scalar-vector multiplication:

- Denoted $c\mathbf{v}$ or $c\vec{v}$,
- length of vector multiplied by $|c|$,
- $c\mathbf{v}$ has the same direction as $\mathbf{v}$ if $c > 0$, and has the opposite direction as $\mathbf{v}$ if $c < 0$, (what if $c = 0$?)
- $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if $\mathbf{u} = c\mathbf{v}$. (what vectors are parallel to $\mathbf{0}$?)

Vector Addition and Subtraction:

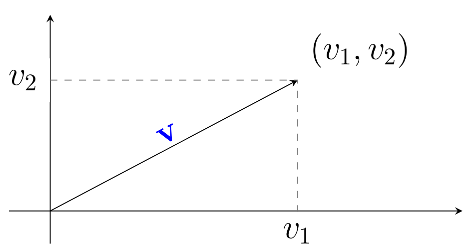
Given two vectors $\mathbf{u}$ and $\mathbf{v}$, their sum, $\mathbf{u}+\mathbf{v}$, can be represented by the parallelogram (triangle) rule: place the tail of $\mathbf{v}$ at the head of $\mathbf{u}$



The difference, denoted $\mathbf{u}-\mathbf{v}$, is the sum of $\mathbf{u}+(-\mathbf{v})$:

Vector Components:

A vector $\mathbf{v}$ whose tail is at the origin $(0,0)$ and head is at (v_1, v_2) is a **position vector** (in **standard position**) and is denoted $\langle v_1, v_2\rangle$. The real numbers v_1 and v_2 are the x- and y-components of $\mathbf{v}$.



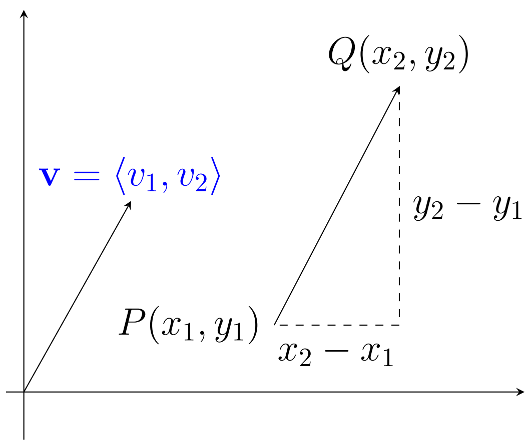
Vectors $\mathbf{u}=\langle u_1, u_2\rangle$ and $\mathbf{v}=\langle v_1, v_2\rangle$ are equal if and only if $u_1 = v_1$ and $u_2 = v_2$.

Magnitude:

Given points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the **magnitude**, or **length**, of vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$, denoted $\left|\overrightarrow{PQ}\right|$, is the distance between points P and Q.

$$\left|\overrightarrow{PQ}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The magnitude of position vector $\mathbf{v} = \langle v_1, v_2 \rangle$ is $|\mathbf{v}|$.
(How do $\left|\overrightarrow{PQ}\right|$ and $\left|\overrightarrow{QP}\right|$ relate to each other?)



Note: The norm, denoted $\|\mathbf{u}\|$ or $\|\mathbf{u}\|_2$, is equivalent to the magnitude of a vector.

Equation of a Circle:

Definition.
A **circle** centered at (a, b) with radius r is the set of points satisfying the equation

$$(x-a)^2 + (y-b)^2 = r^2.$$

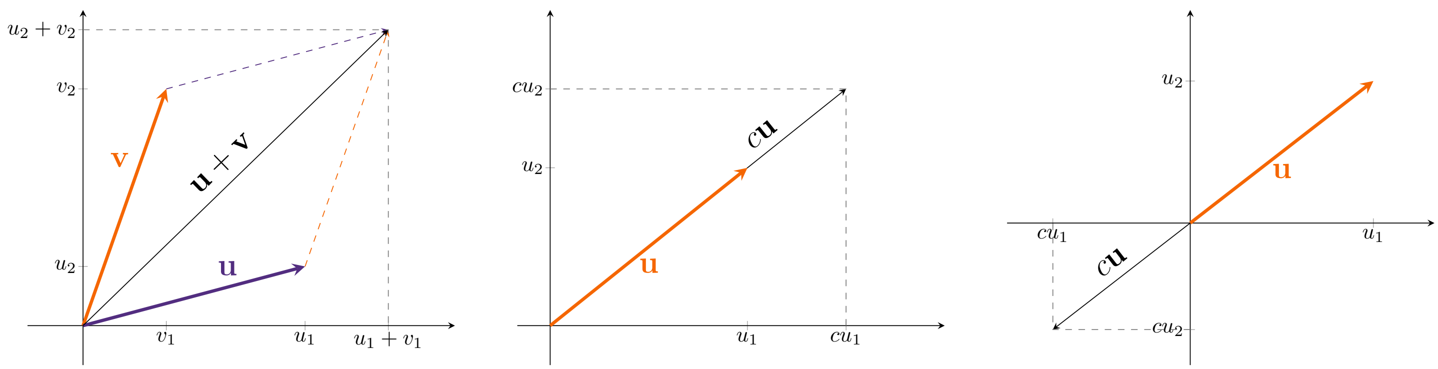
A **disk** centered at (a, b) with radius r is the set of points satisfying the inequality

$$(x-a)^2 + (y-b)^2 \le r^2.$$

Vector Operations in Terms of Components

Definition. (Vector Operations in $\mathbb{R}^2$)
Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2 \rangle$, and $\mathbf{v} = \langle v_1, v_2 \rangle$.

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle \qquad \text{Vector addition}$$
$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2 \rangle \qquad \text{Vector subtraction}$$
$$c\mathbf{u} = \langle cu_1, cu_2 \rangle \qquad \text{Scalar multiplication}$$



Example. Let $\mathbf{u} = \langle 1, 2 \rangle$, $\mathbf{v} = \langle -2, 3 \rangle$, $c = 2$, and $d = 3$. Find the following:

$\mathbf{u} + \mathbf{v}$ $\qquad$ $c\mathbf{u}$

$c\mathbf{u} + d\mathbf{v}$ $\qquad$ $\mathbf{u} - c\mathbf{v}$

Definition.
A **unit vector** is any vector with length 1.
In $\mathbb{R}^2$, the **coordinate unit vectors** are $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.

Example. Let $\mathbf{u} = \langle -7, 3 \rangle$. Find two unit vectors parallel to $\mathbf{u}$. Find another vector parallel to $\mathbf{u}$ with a magnitude of 2.

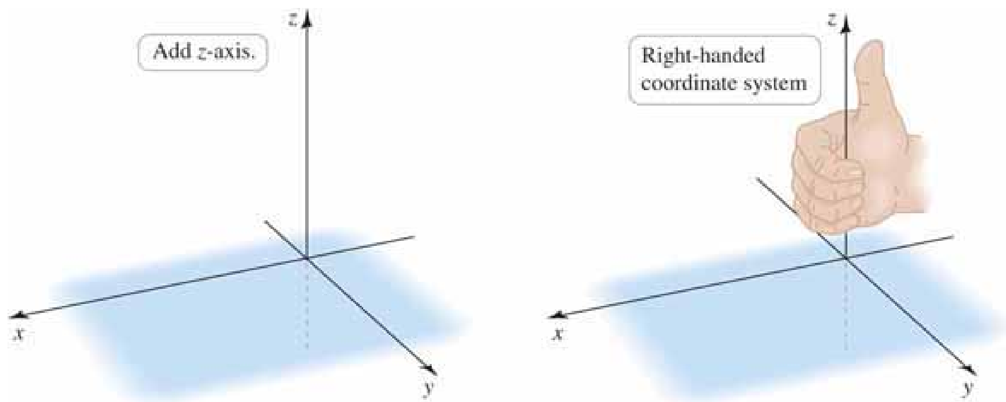
Properties of Vector Operations:
Suppose $\mathbf{u}$, $\mathbf{v}$, and $\boldsymbol{w}$ are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ Commutative property of addition
2. $(\mathbf{u} + \mathbf{v}) + \boldsymbol{w} = \mathbf{u} + (\mathbf{v} + \boldsymbol{w})$ Associative property of addition
3. $\mathbf{v} + \mathbf{0} = \mathbf{v}$ Additive identity
4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ Additive inverse
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ Distributive property 1
6. $(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$ Distributive property 2
7. $0\mathbf{v} = \mathbf{0}$ Multiplication by zero scalar
8. $c\mathbf{0} = \mathbf{0}$ Multiplication by zero vector
9. $1\mathbf{v} = \mathbf{v}$ Multiplicative identity
10. $a(c\mathbf{v}) = (ac)\mathbf{v}$ Associative property of scalar multiplication

13.2: Vectors in Three Dimensions

The xyz- Coordinate System:

The three-dimensional coordinate system is created by adding the z-axis, which is perpendicular to both the x-axis and the y-axis. When looking at the xy-plane, the positive direction of the z-axis protrudes towards the viewer. This can also be shown using the right-hand rule (Figure 13.25 from Briggs):

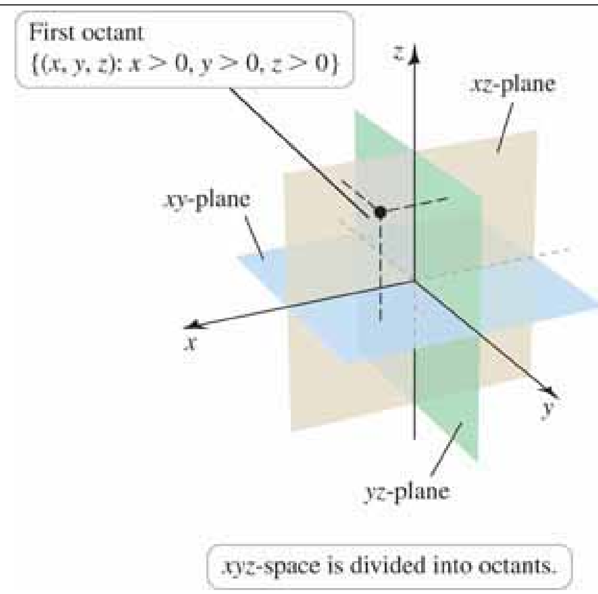


Definition.

This three-dimensional coordinate system is broken up into eight **octants**, which are separated by

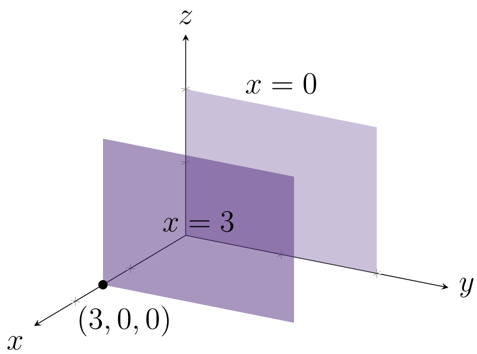
- the xy-**plane** ($z = 0$),
- the xz-**plane** ($y = 0$), and
- the yz-**plane** ($x = 0$).

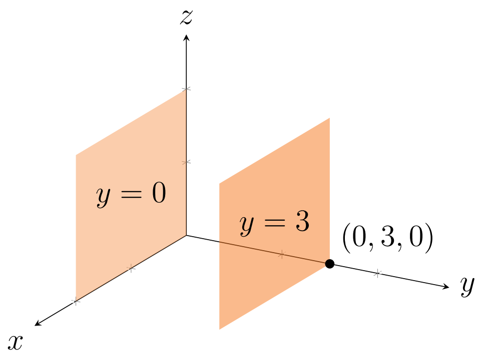
The **origin** is the location where all three axes intersect.

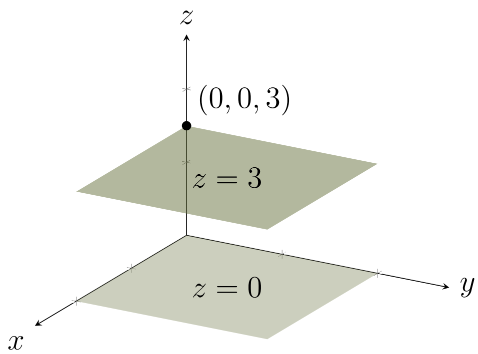


Equations of Simple Planes:

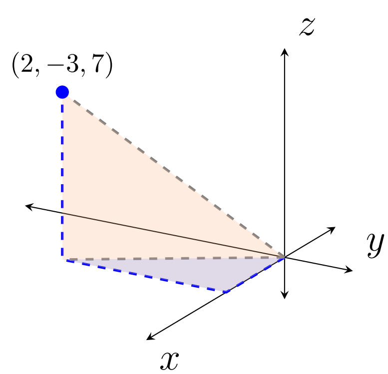
Planes in three-dimensions are analogous to lines in two-dimensions. Below, we see the yz-plane, the xz-plane, and the xy-plane, along with planes that are parallel where x, y, and z are fixed respectively:







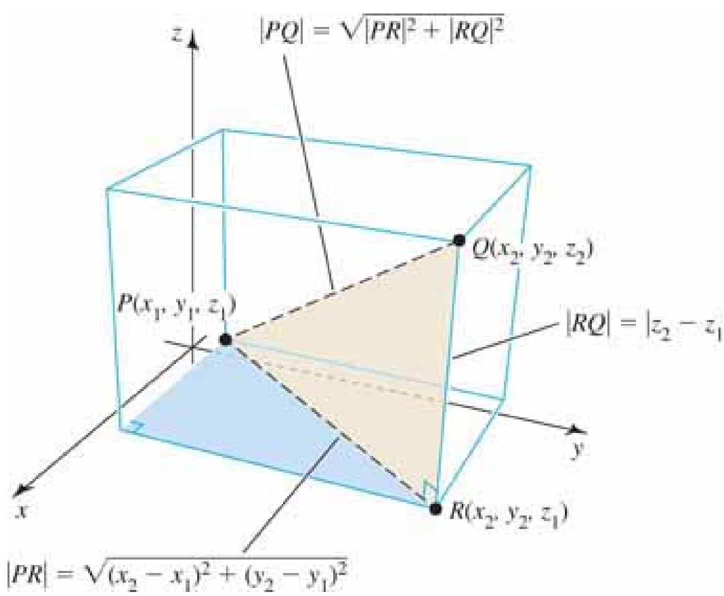
Example (Parallel planes)**.** Determine the equation of the plane parallel to the xz-plane passing through the point $(2, -3, 7)$.



Distances in xyz-Space:
Recall that in $\mathbb{R}^2$, for some vector $\overrightarrow{PR}$, the distance formula is given by

$$|PR| = \sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2}$$

where (x_1, y_1) and (x_2, y_2) represent the points P and R respectively. This idea can be further extended into $\mathbb{R}^3$ by considering the two sides of the triangle formed by the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$:



Distance Formula in xyz-Space
The **distance** between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The **midpoint** between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is found by averaging the x-, y-, and z-coordinates:

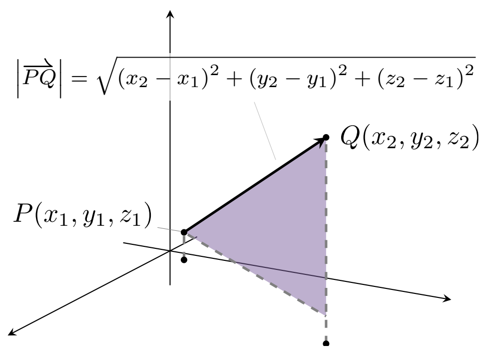
$$\text{Midpoint } = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Magnitude and Unit Vectors:

Definition.
The **magnitude** (or **length**) of the vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ is the distance from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$:

$$\left|\overrightarrow{PQ}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

In $\mathbb{R}^3$, the **coordinate unit vectors** are $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$.



Example. Consider $P(-1, 4, 3)$ and $Q(3, 5, 7)$. Find

- $\left|\overrightarrow{PQ}\right|$

- The midpoint between P and Q

- Two unit vectors parallel to $\overrightarrow{PQ}$

Equation of a Sphere:

Definition.
A **sphere** centered at (a, b, c) with radius r is the set of points satisfying the equation

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

A **ball** centered at (a, b, c) with radius r is the set of points satisfying the inequality

$$(x-a)^2 + (y-b)^2 + (z-c)^2 \leq r^2.$$

Example. Consider $P(-1, 4, 3)$ and $Q(3, 5, 7)$. Find the equation of the sphere centered at the midpoint passing through P and Q

Example. What is the geometry of the intersection between $x^2+y^2+z^2 = 50$ and $z = 1$?

Example. Rewrite the following equation into the standard form of a sphere:

$$x^2 + y^2 + z^2 - 2x + 6y - 8z = -1$$

Vector Operations in Terms of Components

Definition. (Vector Operations in $\mathbb{R}^3$)
Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$.

$$\begin{aligned} \mathbf{u}+\mathbf{v} &= \langle u_1+v_1, u_2+v_2, u_3+v_3\rangle && \text{Vector addition} \\ \mathbf{u}-\mathbf{v} &= \langle u_1-v_1, u_2-v_2, u_3-v_3\rangle && \text{Vector subtraction} \\ c\mathbf{u} &= \langle cu_1, cu_2, cu_3\rangle && \text{Scalar multiplication} \end{aligned}$$

Properties of Vector Operations:
Suppose $\mathbf{u}$, $\mathbf{v}$, and $\boldsymbol{w}$ are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

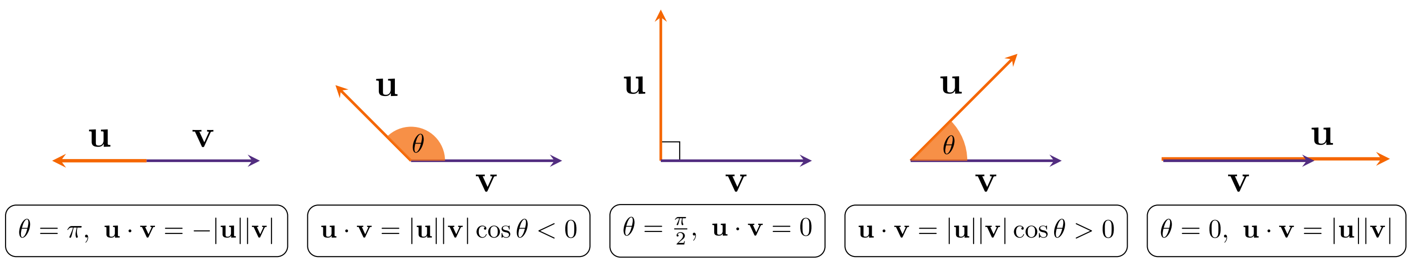
1. $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ — Commutative property of addition
2. $(\mathbf{u}+\mathbf{v})+\boldsymbol{w} = \mathbf{u}+(\mathbf{v}+\boldsymbol{w})$ — Associative property of addition
3. $\mathbf{v}+\mathbf{0} = \mathbf{v}$ — Additive identity
4. $\mathbf{v}+(-\mathbf{v}) = \mathbf{0}$ — Additive inverse
5. $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}$ — Distributive property 1
6. $(a+c)\mathbf{v} = a\mathbf{v}+c\mathbf{v}$ — Distributive property 2
7. $0\mathbf{v} = \mathbf{0}$ — Multiplication by zero scalar
8. $c\mathbf{0} = \mathbf{0}$ — Multiplication by zero vector
9. $1\mathbf{v} = \mathbf{v}$ — Multiplicative identity
10. $a(c\mathbf{v}) = (ac)\mathbf{v}$ — Associative property of scalar multiplication

13.3: Dot Products

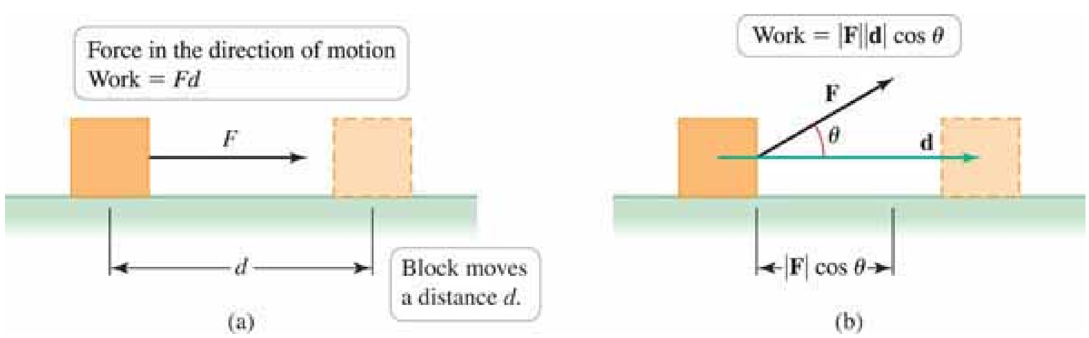
Definition. (Dot Product)
Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in two or three dimensions, their **dot product** is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ with $0 \leq \theta \leq \pi$. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$, and θ is undefined.



A physical example of the dot product is the amount of work done when a force is applied at an angle θ as shown in figure 13.43:



Note: The result of the dot product is a scalar!

Definition. (Orthogonal Vectors)
Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. The zero vector is orthogonal to all vectors. In two or three dimensions, two nonzero orthogonal vectors are perpendicular to each other.

- $\mathbf{u}$ and $\mathbf{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \cdot \mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$.
- $\mathbf{u}$ and $\mathbf{v}$ are perpendicular ($\theta = \frac{\pi}{2}$) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Theorem 31.1: Dot Product
Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$,

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Properties of Dot Products

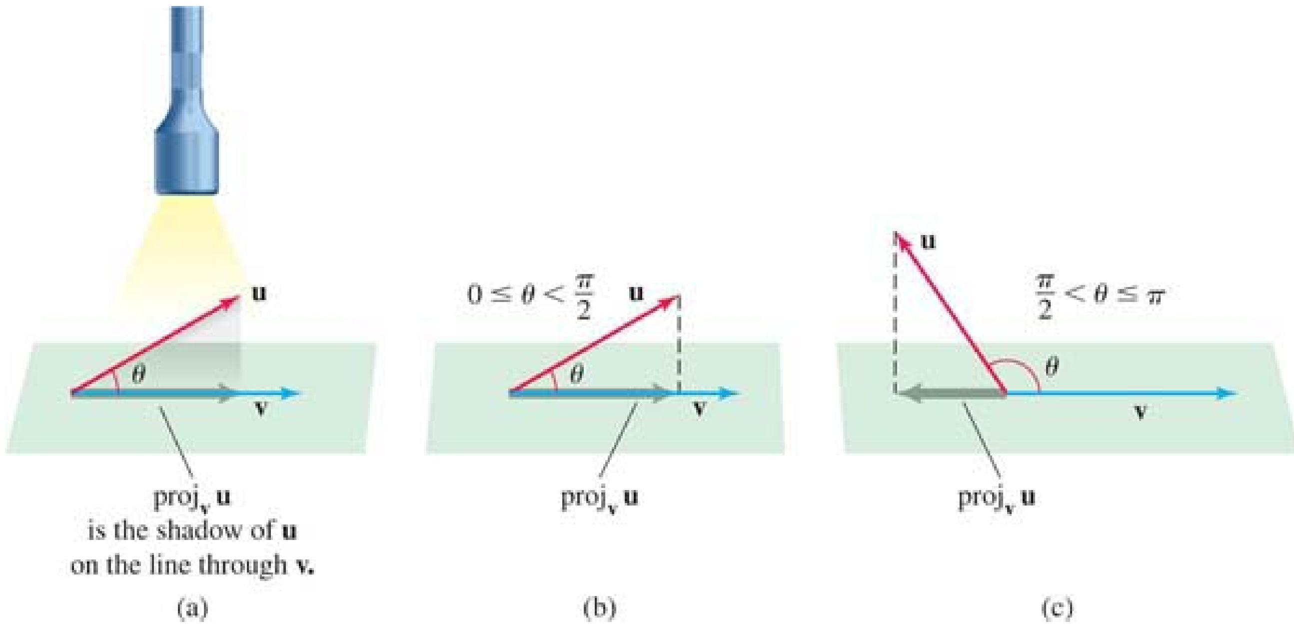
Theorem 13.2: Properties of the Dot Product

Suppose $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ — Commutative property
2. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$ — Associative property
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ — Distributive property

Orthogonal Projections

Given vectors $\mathbf{u}$ and $\mathbf{v}$, the projection of $\mathbf{u}$ onto $\mathbf{v}$ produces a vector parallel to $\mathbf{v}$ using the "shadow" of $\mathbf{u}$ cast onto $\mathbf{v}$.



Definition. **((Orthogonal) Projection of u onto v)**
The **orthogonal projection of u onto v**, denoted $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$, where $\mathbf{v} \neq \mathbf{0}$, is

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \underbrace{|\mathbf{u}| \cos\theta}_{\text{length}} \underbrace{\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)}_{\text{direction}}.$$

The orthogonal projection may also be computed with the formulas

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \operatorname{scal}_{\mathbf{v}} \mathbf{u} \left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v},$$

where the **scalar component of u in the direction of v** is

$$\operatorname{scal}_{\mathbf{v}} \mathbf{u} = |\mathbf{u}| \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}.$$

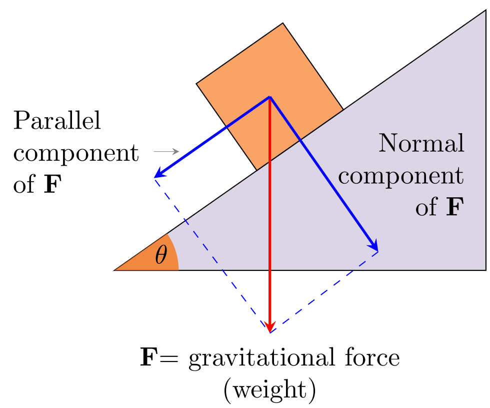
Applications of Dot Products

Definition. (Work)
Let a constant force $\mathbf{F}$ be applied to an object, producing a displacement $\mathbf{d}$. If the angle between $\mathbf{F}$ and $\mathbf{d}$ is θ, then the **work** done by the force is

$$W = |\mathbf{F}||\mathbf{d}|\cos\theta = \mathbf{F}\cdot\mathbf{d}$$

Example.

<u>Parallel and Normal Forces:</u>
Example.



13.4: Cross Products

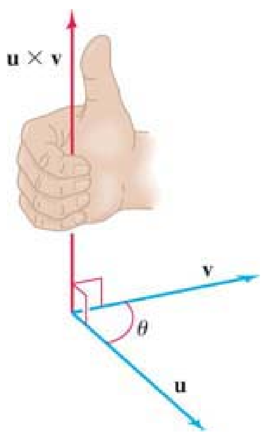
Definition. (Cross Product)
Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the **cross product** $\mathbf{u} \times \mathbf{v}$ is a vector with magnitude

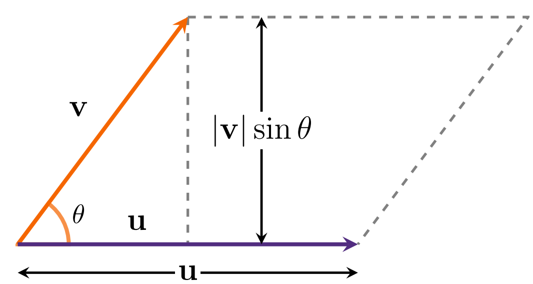
$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta,$$

where $0 \leq \theta \leq \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
The direction of $\mathbf{u} \times \mathbf{v}$ is given by the **right-hand rule**:

> When you put your the vectors tail to tail and let the fingers of your right hand curl from $\mathbf{u}$ to $\mathbf{v}$, the direction of $\mathbf{u} \times \mathbf{v}$ is the direction of your thumb, orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ (Figure 13.56).

When $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, the direction of $\mathbf{u} \times \mathbf{v}$ is undefined.





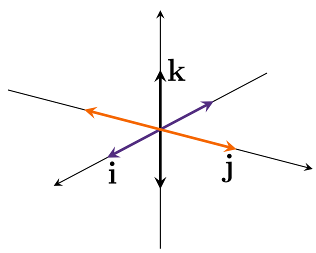
Theorem 13.3: Geometry of the Cross Product
Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in $\mathbb{R}^3$.

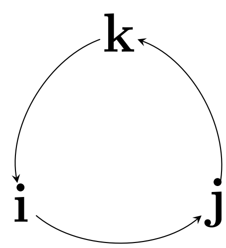
1. The vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
2. If $\mathbf{u}$ and $\mathbf{v}$ are two sides of a parallelogram, then the area of the parallelogram is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$$

Theorem 13.4: Properties of the Cross Product Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^3$, and let a and b be scalars.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ — Anticommutative property
2. $(a\mathbf{u}) \times (b\mathbf{v}) = ab(\mathbf{u} \times \mathbf{v})$ — Associative property
3. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ — Distributive property
4. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$ — Distributive property





$\mathbf{i} \times \mathbf{j} = \mathbf{k}$
$\mathbf{j} \times \mathbf{k} = \mathbf{i}$
$\mathbf{k} \times \mathbf{i} = \mathbf{j}$

Theorem 13.5: Cross Products of Coordinate Unit Vectors

$$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}) = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j}) = \mathbf{i}$$
$$\mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k}) = \mathbf{j} \qquad \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

Using the unit vectors, we can compute $\mathbf{u} \times \mathbf{v}$:

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= u_1v_1 \underbrace{(\mathbf{i} \times \mathbf{i})}_{\mathbf{0}} + u_1v_2 \underbrace{(\mathbf{i} \times \mathbf{j})}_{\mathbf{k}} + u_1v_3 \underbrace{(\mathbf{i} \times \mathbf{k})}_{-\mathbf{j}} \\
&+ u_2v_1 \underbrace{(\mathbf{j} \times \mathbf{i})}_{-\mathbf{k}} + u_2v_2 \underbrace{(\mathbf{j} \times \mathbf{j})}_{\mathbf{0}} + u_2v_3 \underbrace{(\mathbf{j} \times \mathbf{k})}_{\mathbf{i}} \\
&+ u_3v_1 \underbrace{(\mathbf{k} \times \mathbf{i})}_{\mathbf{j}} + u_3v_2 \underbrace{(\mathbf{k} \times \mathbf{j})}_{-\mathbf{i}} + u_3v_3 \underbrace{(\mathbf{k} \times \mathbf{k})}_{\mathbf{0}} \\
&= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}
\end{aligned}$$

Theorem 13.6: Evaluating the Cross Product
Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

Note:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Alternative approach:

$$\left|\begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ u_1 & u_2 & u_3 & u_1 & u_2 \\ v_1 & v_2 & v_3 & v_1 & v_2 \end{array}\right.$$