

Math 1080 Class notes Fall 2020

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Table Of Contents

5.5: Substitution Rule 1

5.5: Substitution Rule

Theorem 5.6: Substitution Rule for Indefinite Integrals

Let $u = g(x)$, where g is differentiable on an interval, and let f be continuous on the corresponding range of g . On that interval,

$$\int f(g(x))g'(x) dx = \int f(u) du$$

Example. We know

$$\frac{d}{dx} \left[\frac{(2x+1)^4}{4} \right] = 2(2x+1)^3$$

Thus, if $f(x) = x^3$ and $g(x) = 2x + 1$ then $g'(x) = 2$, so we let $u = 2x + 1$, then

$$\begin{aligned} \int 2(2x+1)^3 dx &= \int f(g(x))g'(x) dx \\ &= \int u^3 du \\ &= \frac{u^4}{4} + C \\ &= \frac{(2x+1)^4}{4} + C \end{aligned}$$

Procedure: Substitution Rule (Change of Variables)

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x) dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u .
4. Write the result in terms of x using $u = g(x)$.

Example. Evaluate the following integrals:

a) $\int 2x(x^2 + 3)^4 dx$

b) $\int (2x + 1)^3 dx$

c) $\int x^2 \sqrt{x^3 + 1} dx$

d) $\int \theta \sqrt[4]{1 - \theta^2} d\theta$

e) $\int \sqrt{4 - t} dt$

f) $\int (2 - x)^6 dx$

Example. Evaluate the following integrals:

a) $\int \sec(2\theta) \tan(2\theta) d\theta$

b) $\int \csc^2\left(\frac{t}{3}\right) dt$

c) $\int \frac{\sin(x)}{1 + \cos^2(x)} dx$

d) $\int \frac{\tan^{-1}(x)}{1 + x^2} dx$

The acceleration of a particle moving back and forth on a line is $a(t) = \frac{d^2s}{dt^2} = \pi^2 \cos(\pi t) \text{ m/s}^2$ for all t . If $s = 0$ and $v = 8 \text{ m/s}$ when $t = 0$, find the value of s when $t = 1$ sec.

Example. Evaluate the following integrals:

a) $\int (6x^2 + 2) \sin(x^3 + x + 1) dx$

b) $\int \frac{\sin(\theta)}{\cos^5(\theta)} d\theta$

c) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

d) $\int \frac{2^t}{2^t + 3} dt$

e) $\int 6x^2 4^{x^3} dx$

f) $\int \frac{dx}{\sqrt{36 - 4x^2}}$

g) $\int \sin(t) \sec^2(\cos(t)) dt$

h) $\int \frac{1}{\sqrt{x}(1 + \sqrt{x})^2} dx$

i) $\int \frac{\sin(\sqrt{x})}{\sqrt{x}} dx$

j) $\int 5 \cos(7x + 5) dx$

k) $\int \frac{3}{\sqrt{1 - 25x^2}} dx$

l) $\int \frac{dx}{\sqrt{1 - 9x^2}}$

Example. Evaluate the following integrals using the recommended substitution:

a) $\int \sec^2(x) \tan(x) \, dx$
where $u = \tan(x)$.

b) $\int \sec^2(x) \tan(x) \, dx$
where $u = \sec(x)$.

Example. Solve the initial value problem: $\frac{dy}{dx} = 4x(x^2 + 8)^{-1/3}, y(0) = 0$.

Example. Evaluate the following integrals:

a) $\int x e^{-x^2} dx$

b) $\int \frac{e^{1/x}}{x^2} dx$

c) $\int \frac{dt}{8-3t}$

d) $\int 5^t \sin(5^t) dt$

e) $\int \frac{e^w}{36 + e^{2w}} dw$

Theorem 5.7: Substitution Rule for Definite Integrals

Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g . Then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

Example. Evaluate the integrals:

a) $\int_0^1 \frac{x^3}{\sqrt{x^4 + 9}} dx$

b) $\int_1^3 \frac{dt}{(t - 4)^2}$

c) $\int_0^3 \frac{v^2 + 1}{\sqrt{v^3 + 3v + 4}} dv$

d) $\int_0^1 2x(4 - x^2) dx$

e) $\int_2^3 \frac{x}{\sqrt[3]{x^2-1}} dx$

f) $\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{1+\cos(x)} dx$

g) $\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos^2(x)} dx$

h) $\int_{-\frac{\pi}{12}}^{\frac{\pi}{8}} \sec^2(2y) dy$

i) $\int_0^1 (1 - 2x^9) dx$

j) $\int_0^1 (1 - 2x)^9 dx$

k) $\int_0^{\frac{1}{2}} \frac{1}{1 + 4x^2} dx$

l) $\int_0^4 \frac{x}{x^2 + 1} dx$

m) $\int_0^\pi 3 \cos^2(x) \sin(x) \, dx$

n) $\int_0^{\frac{\pi}{8}} \sec(2\theta) \tan(2\theta) \, d\theta$

o) $\int_0^1 (3t - 1)^{50} \, dt$

p) $\int_0^3 \frac{1}{5x + 1} \, dx$

q) $\int_0^1 x e^{-x^2} dx$

r) $\int_e^{e^4} \frac{1}{x \sqrt{\ln(x)}} dx$

s) $\int_0^{\frac{1}{2}} \frac{\sin^{-1}(x)}{\sqrt{1-x^2}} dx$

t) $\int_0^1 \frac{e^z + 1}{e^z + z} dz$

$$\text{u)} \int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2}$$

$$\text{v)} \int_{\ln(\frac{\pi}{4})}^{\ln(\frac{\pi}{2})} e^w \cos(e^w) dw$$

$$\text{w)} \int_0^{\frac{1}{8}} \frac{x}{\sqrt{1-16x^2}} dx$$

$$\text{x)} \int_1^{e^2} \frac{\ln(p)}{p} dp$$

$$y) \int_0^{\frac{\pi}{4}} e^{\sin^2(x)} \sin(2x) dx$$

$$z) \int_{-\pi}^{\pi} x^2 \sin(7x^3) dx$$

Example. Average velocity: An object moves in one dimension with a velocity in m/s given by $v(t) = 8 \sin(\pi t) + 2t$. Find its average velocity over the time interval from $t = 0$ to $t = 10$, where t is measured in seconds.

Example. Prove $\int \tan(x) \, dx = \ln |\sec(x)| + C$.

Example. Evaluate the integrals:

a) $\int \frac{x}{(x-2)^3} \, dx$

b) $\int x\sqrt{x-1} \, dx$

c) $\int x^3(1+x^2)^{\frac{3}{2}} dx$

d) $\int \frac{y^2}{(y+1)^4} dy$

e) $\int (z+1)\sqrt{3z+2} dz$

f) $\int_0^1 \frac{x}{(x+2)^3} dx$

Half-Angle Formulas

$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$$

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$$

Example. Evaluate the integrals:

a) $\int \cos^2(x) \, dx$

b) $\int_0^{\frac{\pi}{2}} \cos^2(x) \, dx$

c) $\int \frac{1}{x^2} \cos^2\left(\frac{1}{x}\right) dx$

d) $\int x \sin^2(x^2) dx$

e) $\int \sin^2\left(\theta + \frac{\pi}{6}\right) d\theta$

f) $\int_0^{\frac{\pi}{4}} \cos^2(8\theta) d\theta$

Example. If f is continuous and $\int_0^4 f(x) dx = 10$, find $\int_0^2 f(2x) dx$.

Example. If f is continuous and $\int_0^9 f(x) dx = 4$, find $\int_0^3 xf(x^2) dx$.

Example. Suppose f is an even function with $\int_0^8 f(x) dx = 9$. Evaluate the following:

a) $\int_{-1}^1 xf(x^2) dx$.

b) $\int_{-2}^2 x^2 f(x^3) dx$.

Example. Evaluate the integrals:

a) $\int \sec^2(10x) \, dx$

b) $\int \tan^{10}(4x) \sec^2(4x) \, dx$

c) $\int \left(x^{\frac{3}{2}} + 8\right)^5 \sqrt{x} \, dx$

d) $\int \frac{2x}{\sqrt{3x+2}} \, dx$

e) $\int \frac{7x^2 + 2x}{x} dx$

f) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

g) $\int_0^{\sqrt{3}} \frac{3}{9 + x^2} dx$

h) $\int_0^{\frac{\pi}{6}} \frac{\sin(2y)}{\sin^2(y) + 2} dy$

i) $\int \frac{\sec(z) \tan(z)}{\sqrt{\sec(z)}} dz$

j) $\int \frac{1}{\sin^{-1}(x) \sqrt{1-x^2}} dx$

k) $\int \frac{x}{\sqrt{4-9x^2}} dx$

l) $\int \frac{x}{1+x^4} dx$

m) $\int \frac{\cos \sqrt{\theta}}{\sqrt{\theta} \sin^2 \sqrt{\theta}} d\theta$

n) $\int x^2 \sqrt{2+x} dx$

o) $\int (\sin^5(x) + 3 \sin^3(x) - \sin(x)) \cos(x) dx$

p) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (x^3 + x^4 \tan(x)) \, dx$

q) $\int_0^{\frac{\pi}{2}} \cos(x) \sin(\sin(x)) \, dx$

r) $\int \frac{1+x}{1+x^2} \, dx$

Example. Evaluate these more challenging integrals:

a) $\int \frac{dx}{\sqrt{1 + \sqrt{1 + x}}}$

b) $\int x \sin^4(x^2) \cos(x^2) dx$