6.4: Volume by Shells

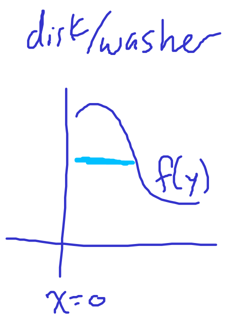
Volume by the Shell Method

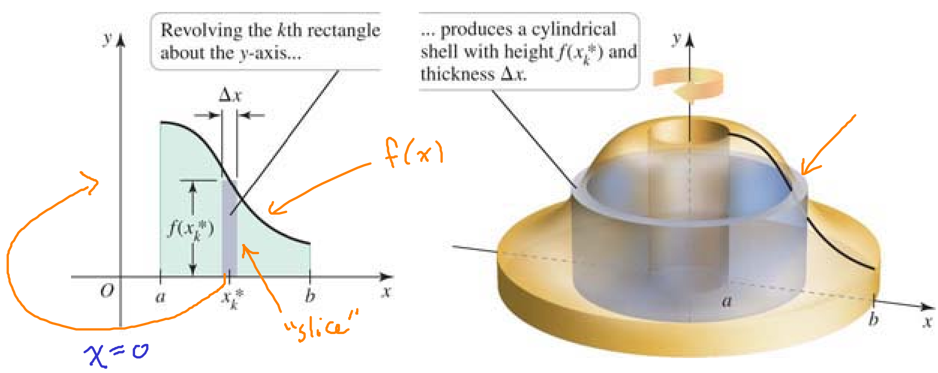
Let f and g be continuous functions with $f(x) \geq g(x)$ on $[a, b]$. If R is the region bounded by the curves $y = f(x)$ and $y = g(x)$ between the lines $x = a$ and $x = b$, the volume of the solid generated when R is revolved about the y-axis is (x=0)

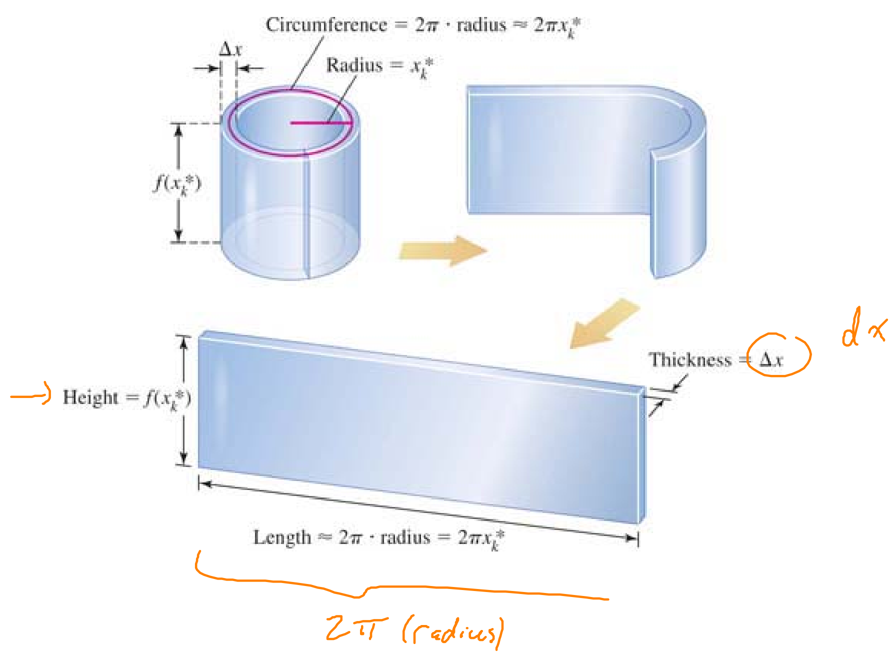
$$V = \int_a^b \underbrace{2\pi \overset{\text{radius}}{x}}_{\substack{\text{shell}\\ \text{circumference}}} \underbrace{(f(x) - g(x))}_{\substack{\text{shell}\\ \text{height}}}\, dx.$$

revolve around x=c

radius $\underbrace{x-c}_{\text{want pos}}$

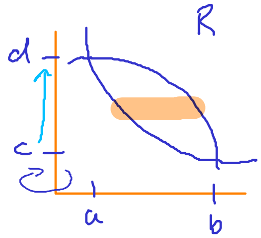






Example. Consider a general region R revolved around the y-axis.
$(x=0)$

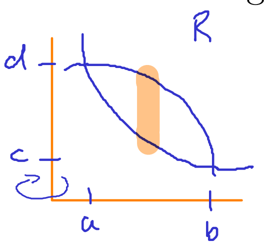
When using the **disk/washer** method, we integrate with respect to $\underline{\quad y \quad}$



$$\int_c^d \pi\left((f-0)^2-(g-0)^2\right)\,dy$$

When using the **shell** method, we integrate with respect to $\underline{\quad x \quad}$

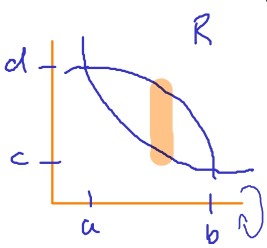
$(x=0)$



$$\int_a^b 2\pi\,\underbrace{(x-0)}_{\substack{\text{want}\\ \geq 0}}\,(f-g)\,dx$$

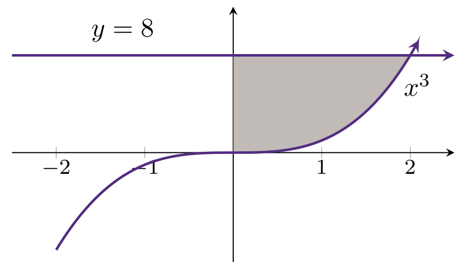
Example. Consider a general region R revolved around the x-axis.
$(y=0)$

When using the **disk/washer** method, we integrate with respect to $\underline{\quad x \quad}$



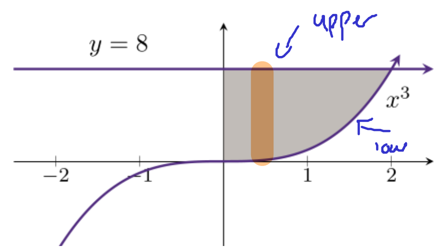
When using the **shell** method, we integrate with respect to $\underline{\quad y \quad}$

Example. Consider the region bounded between $y = x^3$, $y = 8$ and $x = 0$.



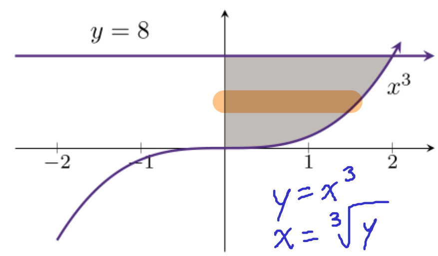
Perpendicular to axis of revolution

Use the disk/washer method to setup the integral that represents the volume of the solid generated by rotating the region about the x-axis. $(y=0)$



$$V = \int_0^2 \pi\left(8^2 - (x^3)^2\right) dx$$

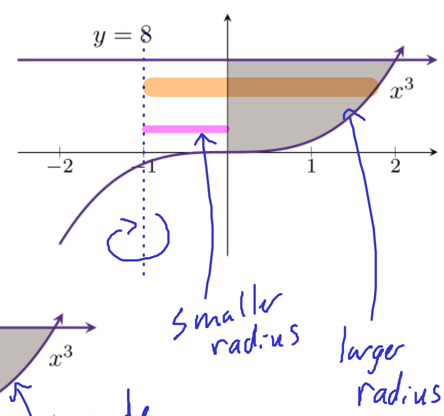
about the y-axis.



$$V = \int_0^8 \pi\left[\left(\sqrt[3]{y}\right)^2 - 0^2\right] dy = \int_0^8 \pi\left(y^{1/3}\right)^2 dy$$

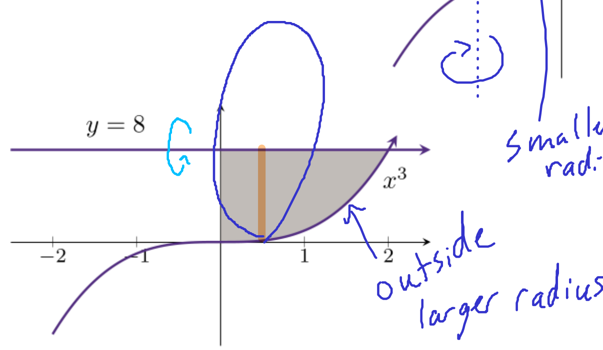
Use the disk/washer method to setup the integral that represents the volume of the solid generated by rotating the region about the line $x = -1$.

$$V = \int_0^8 \pi\left(\left(\underset{y^{1/3}+1}{y^{1/3} - (-1)}\right)^2 - \left(\underset{1}{0 - (-1)}\right)^2\right) dy$$



about the line $y = 8$.

$$V = \int_0^2 \pi\left(\left(\underset{(f-c)}{8 - x^3}\right)^2 - \left(\underset{(g-c)}{8 - 8}\right)^2\right) dx$$



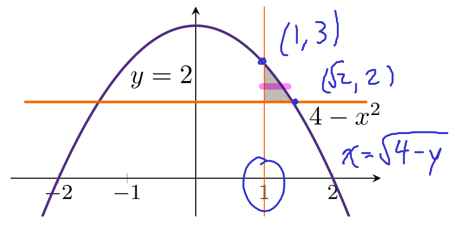
$\int 2\pi r\,(\text{height})\,d$

Example. Consider the region R bounded by $y = 4 - x^2$, $y = 2$, and $x = 1$. Use the shell method to setup the integral that represents the volume of the solid generated by rotating the region R about the indicated axis of rotation.

about x-axis, $(y=0)$

$$V = \int_2^3 2\pi (y-0)\left(\sqrt{4-y}\ -1\right) dy$$

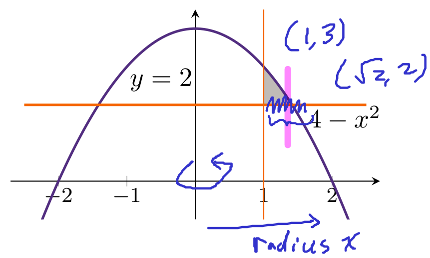
$2\pi\, y\left(\sqrt{4-y}\ -1\right)$



about y-axis, $(x=0)$

$$V = \int_1^{\sqrt{2}} 2\pi\, x\left(4-x^2-2\right) dx$$

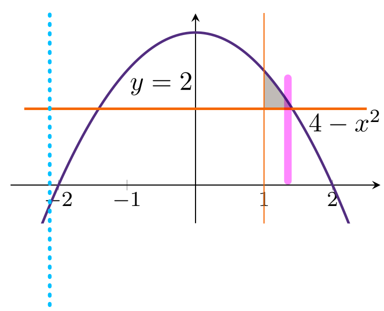
$2\pi\, x\left(2-x^2\right)$



about the line $x = -2$,

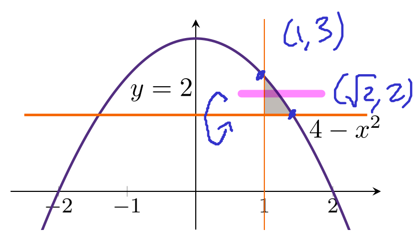
$$V = \int_1^{\sqrt{2}} 2\pi\left(x-(-2)\right)\left(4-x^2-2\right) dx$$

$2\pi(x+2)(2-x^2)$

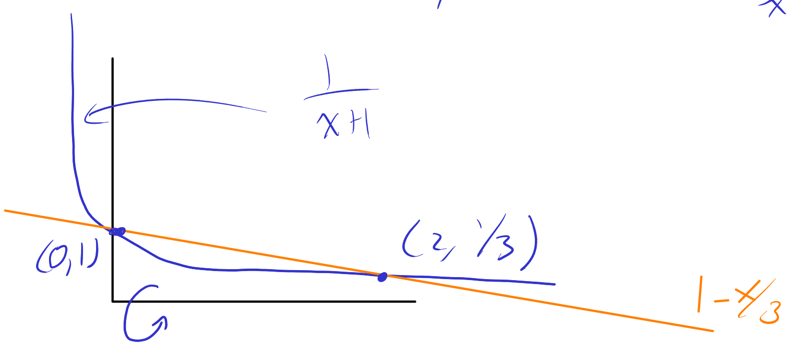


about the line $y = 2$.

$$V = \int_2^3 2\pi\left(y-2\right)\left(\sqrt{4-y}\ -1\right) dy$$



Example. Consider the region bounded by $y = \dfrac{1}{x+1}$ and $y = 1 - \dfrac{x}{3}$. Use both the disk/washer method and shell method to find the volume of the solid generated when R is rotated about the x-axis.

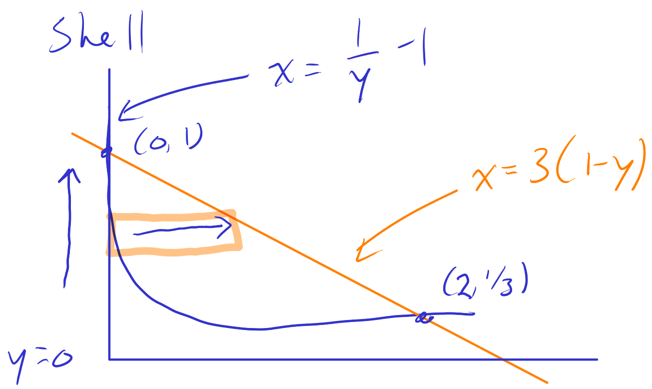


$$\frac{1}{x+1} = 1 - \frac{x}{3}$$
$$3 = (3-x)(x+1)$$
$$3 = 3x + 3 - x^2 - x$$
$$x^2 - 2x = 0$$
$$x(x-2) = 0$$
$$x=0 \qquad x=2$$

$$y = \frac{1}{x+1}$$
$$x+1 = \frac{1}{y}$$
$$x = \frac{1}{y} - 1$$

$$y = 1 - \frac{x}{3}$$
$$\frac{x}{3} = 1-y$$
$$x = 3(1-y)$$

Shell

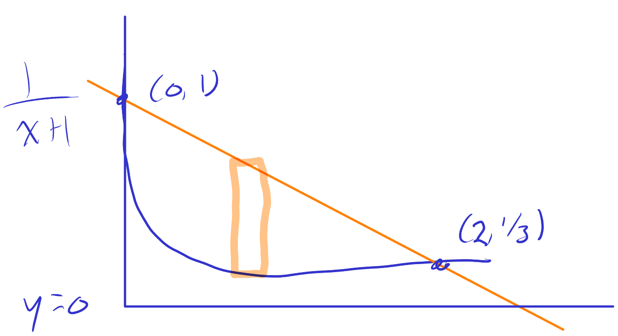


$$V = \int_{1/3}^{1} 2\pi (y-0)\Big(\underbrace{3(1-y)}_{\text{right}} - \underbrace{\left(\tfrac{1}{y}-1\right)}_{\text{left}}\Big)\,dy$$

$$= 2\pi \int_{1/3}^{1} -3y^2 + 4y - 1\,dy$$

$$= 2\pi \Big[-y^3 + 2y^2 - y\Big]_{1/3}^{1} = 2\pi\left(\frac{4}{27}\right)$$

disk/washer



$$V = \int_0^2 \pi\Big(\underbrace{(1-x/3)^2}_{\text{top}} - \underbrace{\left(\tfrac{1}{x+1}\right)^2}_{\text{bottom}}\Big)\,dx$$

1-x/3

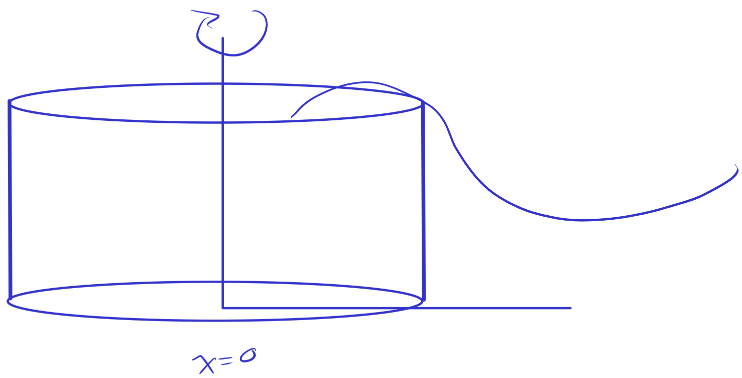
$u = 1 - x/3$ $\quad x=0, u=1$
$du = -1/3\,dx$ $\quad x=2, u=1/3$
$-3du = dx$

$v = x+1$ $\quad x=0, v=1$
$dv = dx$ $\quad x=2, v=3$

$$= \pi \int_1^{1/3} u^2\,du - \pi \int_1^3 v^{-2}\,dv = \frac{8\pi}{27}$$

Example. Determine if the following statements are true.

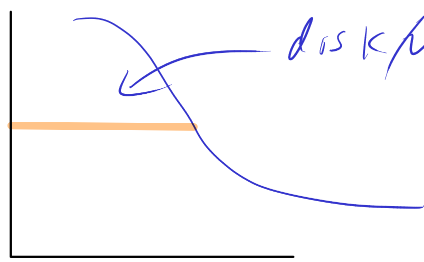
When using the shell method, the axis of the cylindrical shells is parallel to the axis of revolution.



True

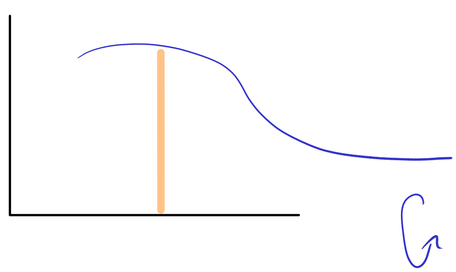
If a region is revolved about the y-axis, then the shell method must be used.

False



disk/washer w.r.t. y

If a region is revolved about the x-axis, it is possible to use the disk/washer method and integrate with respect to x.



True