2.2 Definitions of Limits

Definition.
(Briggs) Suppose the function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close (but not equal) to a, we write

$$\lim_{x\to a} f(x) = L$$

and say the limit of $f(x)$ as x approaches a equals L.

Note– Most of the time, we can think of the limit as the value of the function if it could be evaluated at a specific point.

Example. Using the graph of f, determine the following values:

- $f(1)$ and $\lim_{x\to 1} f(x)$

  $f(1) = 2$

  $\lim_{x\to 1} f(x) = 2$

- $f(2)$ and $\lim_{x\to 2} f(x)$

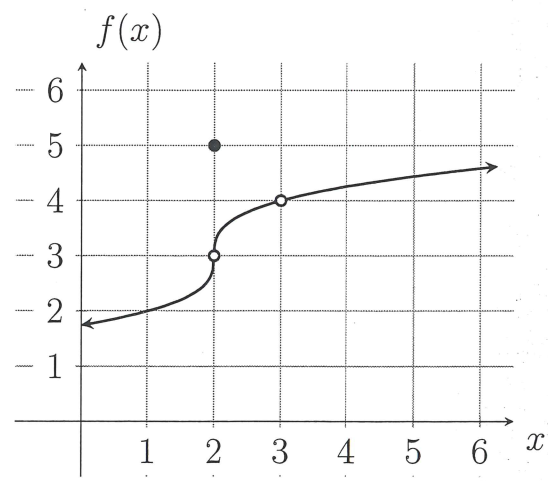
  $f(2) = 5$

  $\lim_{x\to 2} f(x) = 3$

- $f(3)$ and $\lim_{x\to 3} f(x)$

  $f(3)$ DNE

  $\lim_{x\to 3} f(x) = 4$



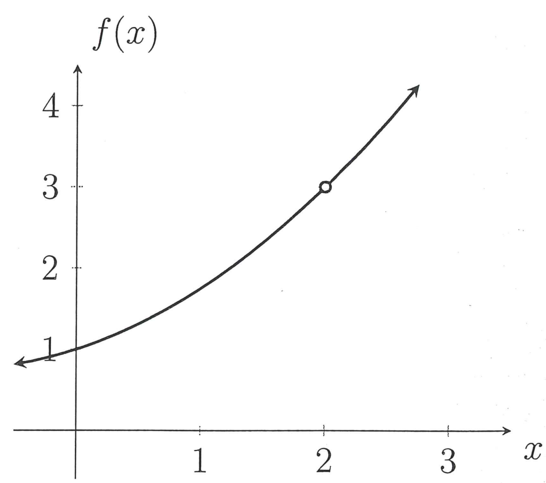
Definition.
(Briggs)

1. **Right-sided limit** Suppose f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$, we write
$$\lim_{x \to a^+} f(x) = L$$
and say the limit of $f(x)$ as x approaches a from the right equals L.
2. **Left-sided limit** Suppose f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$, we write
$$\lim_{x \to a^-} f(x) = L$$
and say the limit of $f(x)$ as x approaches a from the left equals L.

$$f(x) = \frac{(x-2)(x^2+2x+4)}{4(x-2)} = \frac{x^2+2x+4}{4}$$

Example. For $f(x) = \dfrac{x^3-8}{4(x-2)}$, find

- $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \dfrac{x^2+2x+4}{4} = \dfrac{12}{4} = 3$
- $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \dfrac{x^2+2x+4}{4} = \dfrac{12}{4} = 3$



Definition.
(Briggs) **Relationship Between One-Sided and Two-Sided Limits**
Assume f is defined for all x near a except possibly at a. Then $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$.

Example. For $f(x)$ above, is $\lim_{x\to 2} f(x)$ defined? If so, what is it? What is $f(2)$?

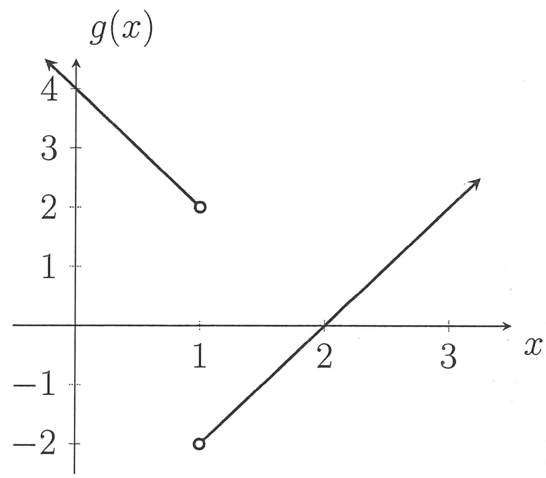
$\lim_{x\to 2^+} f(x) = 3$ and $\lim_{x\to 2^-} f(x) = 3$ so $\lim_{x\to 2} f(x) = 3$

$f(2)$ DNE

$$g(x) = \frac{2(x-2)(x-1)}{|x-1|}$$

Example. Consider the graph of

$$g(x) = \frac{2x^2 - 6x + 4}{|x-1|} = \begin{cases} -2(x-2) & x < 1 \\ 2(x-2) & x > 1 \end{cases}$$



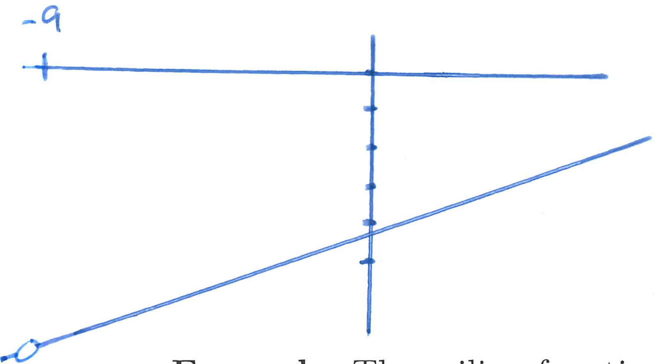
Find

- $\lim_{x\to 1^-} g(x)$ $= 2$
- $\lim_{x\to 1^+} g(x)$ $= -2$
- $\lim_{x\to 1} g(x)$ DNE

Example. Consider the function

$$h(x) = \frac{x^2 - 81}{2x + 18} = \frac{(x-9)(x+9)}{2(x+9)} = \frac{x-9}{2}, \quad x \neq -9$$

What does this function look like? What is $h(-9)$? What is $\lim_{x \to -9} h(x)$?



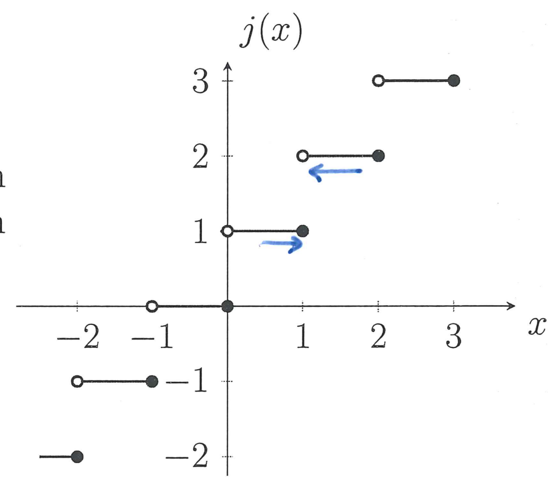
$h(-9)$ DNE

$$\lim_{x \to -9} h(x) = \frac{-9-9}{2} = \frac{-18}{2} = \boxed{-9}$$

Example. The ceiling function is

$$j(x) = \lceil x \rceil$$

where $\lceil x \rceil$ returns the smallest integer greater than or equal to x. In other words, the ceiling function always rounds up. Find the following:



- $\lim_{x \to 1^-} j(x) = 1$
- $\lim_{x \to 1^+} j(x) = 2$
- $\lim_{x \to 1} j(x)$ DNE
- $j(1) = 1$

- $\lim_{x \to 1.5^-} j(x)$ $= 2$
- $\lim_{x \to 1.5^+} j(x)$ $= 2$
- $\lim_{x \to 1.5} j(x)$ $= 2$
- $j(1.5)$ $= 2$

Example. Consider the function

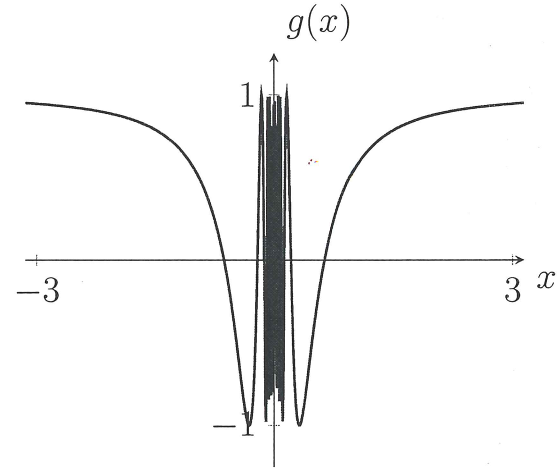
$$h(x) = \cos\left(\frac{1}{x}\right)$$

What is $\lim_{x\to 0} h(x)$?

DNE

Consider $x = 1/(n\pi)$. As $n \to \infty, x \to 0$, then,

$$\cos\left(\frac{1}{x}\right) = \cos(n\pi) = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$$



Example. Graph an example with the following characteristics:

$$\lim_{x\to -2^-} f(x) = -4 \qquad \lim_{x\to -2^+} f(x) = 2 \qquad f(-2) = 0$$

$$\lim_{x\to 4} f(x) = 2 \qquad f(4) \textbf{ DNE}$$

$$\lim_{x\to 8} f(x) = -2 \qquad f(8) = -2$$