## 16.5: Triple Integrals in Cylindrical and Spherical Coordinates

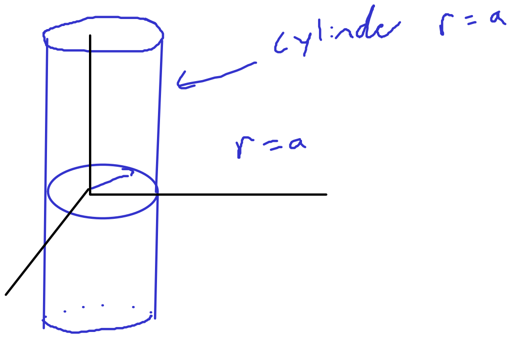
**Cylindrical coordinates:**
The concept of polar coordinates in $\mathbb{R}^2$ from section 16.3 can be extended to $\mathbb{R}^3$. This coordinate system is called *cylindrical coordinates* where every point $P$ in $\mathbb{R}^3$ has coordinates $(r, \theta, z)$, where $0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$, and $-\infty < z < \infty$.

**Transformations between Cylindrical and Rectangular Coordinates**

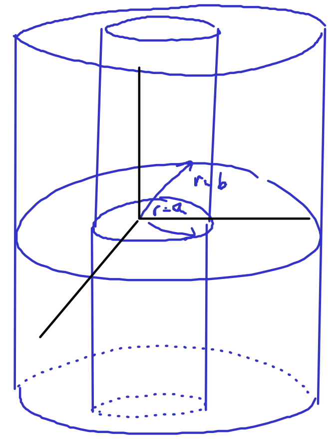
| **Rectangular → Cylindrical** | **Cylindrical → Rectangular** |
|---|---|
| $r^2 = x^2 + y^2$ | $x = r\cos\theta$ |
| $\tan\theta = y/x$ | $y = r\sin\theta$ |
| $z = z$ | $z = z$ |

**Example.** Sketch the following sets represented in cylindrical coordinates:

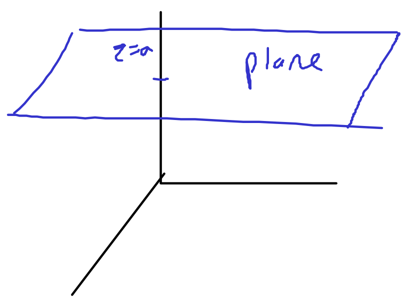
$\{(r, \theta, z) : r = a\}, a > 0$



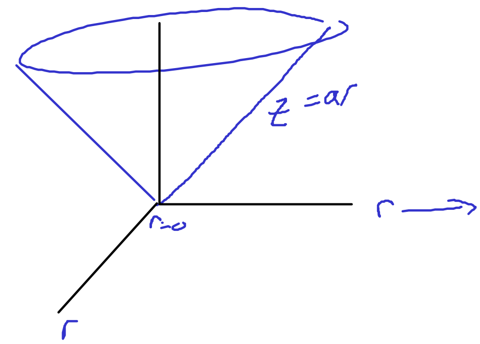

$\{(r, \theta, z) : 0 < a \leq r \leq b\}$



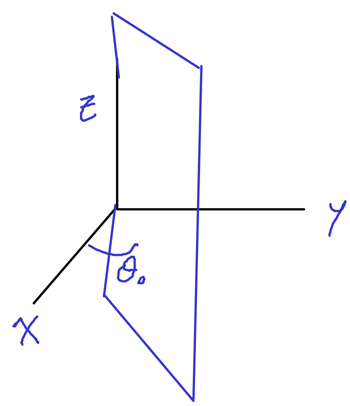
$\{(r, \theta, z) : z = a\}$

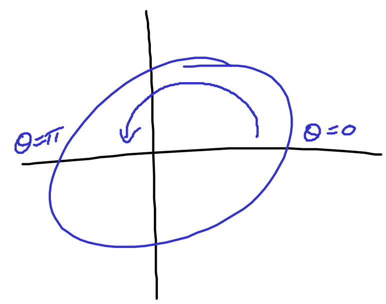



$\{(r, \theta, z) : z = \underline{ar}\}, a \neq 0$




$\{(r, \theta, z) : \theta = \theta_0\}$




**Theorem 16.6: Change of Variables for Triple Integrals in Cylindrical Coordinates**

Let $f$ be continuous over the region $D$, expressed in cylindrical coordinates as

$$D = \{(r, \theta, z) : 0 \leq g(\theta) \leq r \leq h(\theta),\ \alpha \leq \theta \leq \beta,\ G(x, y) \leq z \leq H(x, y)\}$$

Then $f$ is integrable over $D$, and the triple integral of $f$ over $D$ is

$$\iiint_D f(x, y, z)\, dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{\underline{G(r\cos\theta,\ r\sin\theta)}}^{H(r\cos\theta,\ r\sin\theta)} f(r\cos\theta,\ r\sin\theta)\, \underline{dz}\, r\, dr\, d\theta.$$

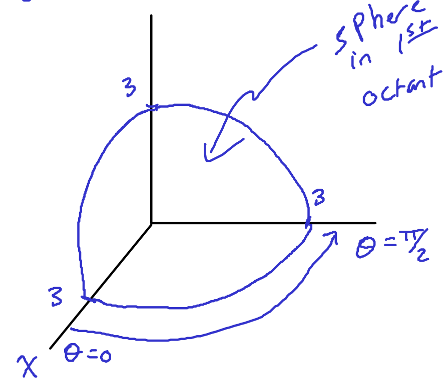
$r^2 = x^2 + y^2$

$x = r\cos\theta$
$y = r\sin\theta$

**Example.** Evaluate the following integral using cylindrical coordinates:

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{x^2+y^2}} (x^2+y^2)^{-1/2}\, dz\, dy\, dx$$

$\sqrt{x^2+y^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = \sqrt{r^2} = r$




$0 \le x \le 3$

$0 \le y \le \sqrt{9-x^2} \longrightarrow x^2+y^2 = 3^2$

$0 \le z \le \sqrt{x^2+y^2}$ $\quad z^2 \le x^2+y^2$

$x^2+y^2 = r^2$

$0 \le r \le 3$

$0 \le \theta \le \pi/2$

$$\int_0^{\pi/2}\int_0^3\int_0^r (r^2)^{-1/2}\, dz\; r\, dr\, d\theta$$

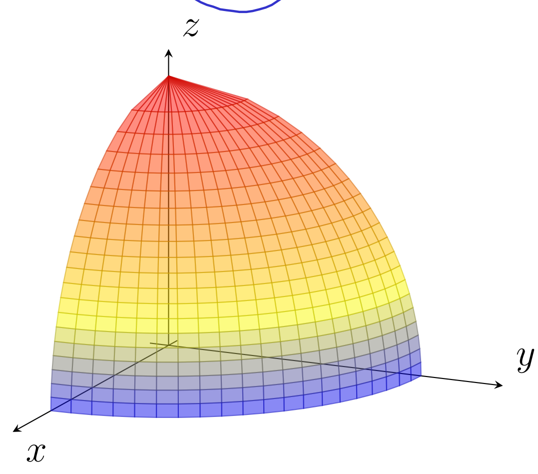
$r^{-1}\cdot r = 1$

LC #2

$\frac{9\pi}{4}$

$$= \int_0^{\pi/2}\int_0^3 z\Big|_{z=0}^{z=r} dr\, d\theta = \int_0^{\pi/2}\int_0^3 r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \frac{r^2}{2}\Big|_{r=0}^{r=3} d\theta = \frac{9}{2}\int_0^{\pi/2} d\theta \quad \frac{9}{2}\,\theta\Big|_0^{\pi/2} = \frac{9\pi}{4}$$



**Example.** Find the volume of the solid bounded below by the paraboloid $z = x^2 + y^2$ and bounded above by the cone $z = 2 - \sqrt{x^2 + y^2}$.

$x = r\cos\theta$ above $z = 2 - r$ $\quad 2 - r \le z \le r^2$

$y = r\sin\theta$ below $z = r^2$ $\quad 0 \le r \le 1$

$z = z$ $\quad 0 \le \theta \le 2\pi$

above & below have the same z-values at the intersection

$r^2 = 2 - r \longrightarrow r^2 + r - 2 = 0$

$(r+2)(r-1) = 0 \longrightarrow r = 1,\ r = -2$

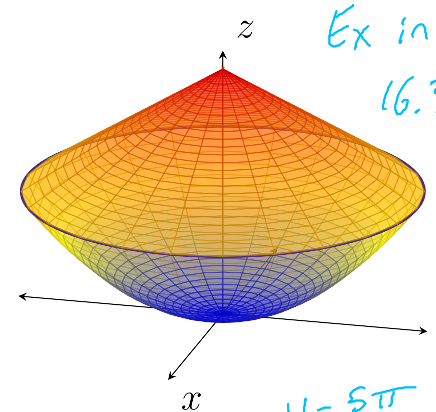
$$V = \int_0^{2\pi}\int_0^1\int_{r^2}^{2-r} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 zr\Big|_{z=r^2}^{z=2-r} dr\, d\theta$$

LC #3 $\frac{5\pi}{6}$

$$= \int_0^{2\pi}\int_0^1 r(2-r-r^2)\, dr\, d\theta = \int_0^{2\pi} r^2 - \frac{r^3}{3} - \frac{r^4}{4}\Big|_{r=0}^{r=1} d\theta$$

$$= \frac{5}{12}\int_0^{2\pi} d\theta = \frac{5}{12}\theta\Big|_0^{2\pi} = \boxed{\frac{5\pi}{6}}$$

Ex in 16.3




$V = \frac{5\pi}{6}$

rho phi theta

### Spherical Coordinates:

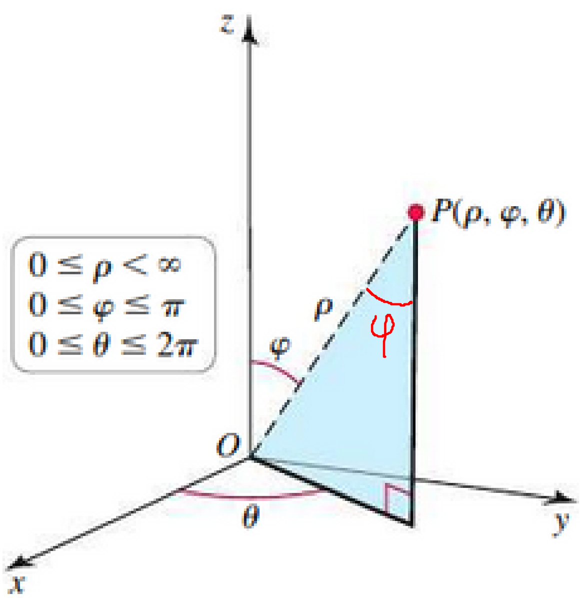
Spherical coordinates can represent a point $P$ in $\mathbb{R}^3$ as $(\rho, \varphi, \theta)$ where

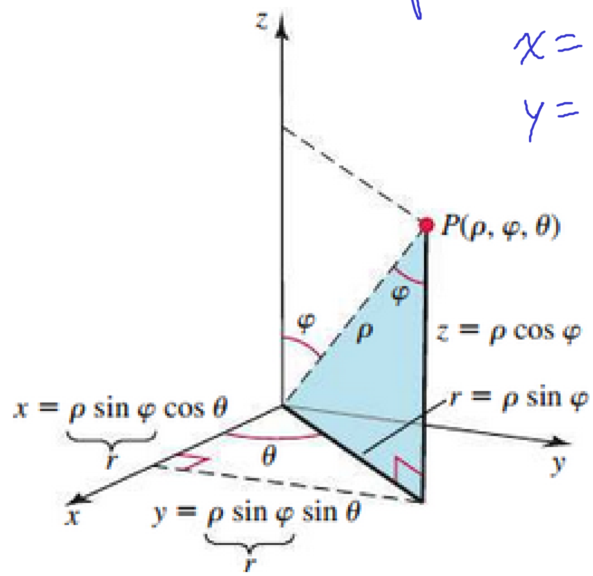
- $\rho$ is the distance from the origin to $P$, $0 \le \rho$
- $\varphi$ is the angle between the positive $z$-axis and the line $OP$, and $0 \le \varphi \le \pi$
- $\theta$ is the same angle as in cylindrical coordinates. $0 \le \theta \le 2\pi$

polar/cylindrical

$x = r\cos\theta$

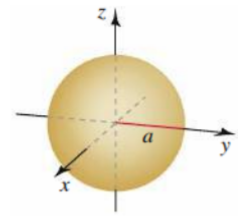
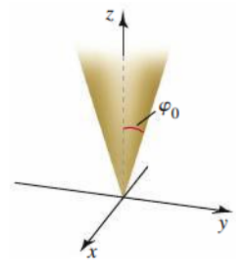
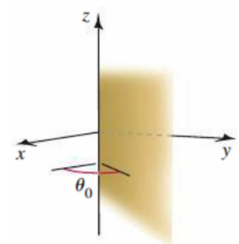
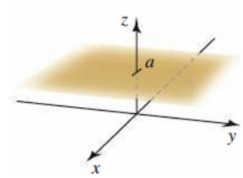
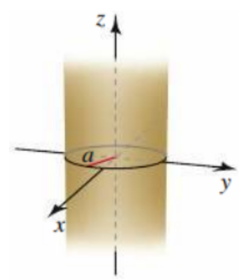
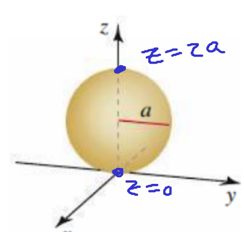
$y = r\sin\theta$




**Transformations between Spherical and Rectangular Coordinates**

| **Rectangular → Spherical** | **Spherical → Rectangular** |
|---|---|
| $\rho^2 = x^2 + y^2 + z^2$ | $x = \rho\sin(\varphi)\cos(\theta)$ |
| Use trigonometry to find | $y = \rho\sin(\varphi)\sin(\theta)$ |
| $\varphi$ and $\theta$. | $z = \rho\cos(\varphi)$ |

| **Name** | **Description** | **Example** |
|---|---|---|
| Sphere, radius $a$, center $(0,0,0)$ | $\{(\rho,\varphi,\theta) : \rho = a\}, a > 0$ |  |
| Cone | $\{(\rho,\varphi,\theta) : \varphi = \varphi_0\}, \varphi_0 \neq 0, \pi/2, \pi$ |  |
| Vertical half-plane | $\{(\rho,\varphi,\theta) : \theta = \theta_0\}$ |  |
| Horizontal plane, $z = a$ | $a = \rho\cos\varphi = z$<br>$a > 0 : \{(\rho,\varphi,\theta) : \rho = a\sec(\varphi),\ 0 \le \varphi < \pi/2\}$<br>$a < 0 : \{(\rho,\varphi,\theta) : \rho = a\sec(\varphi),\ \pi/2 < \varphi \le \pi\}$ |  |
| Cylinder, radius $a > 0$ | $\{(\rho,\varphi,\theta) : \rho = \alpha\csc(\varphi),\ 0 < \varphi < \pi\}$<br>$a = \rho\sin(\varphi)$<br>fix the radius for $x$ & $y$<br>$x = \rho\sin\varphi\cos\theta$<br>$y = \rho\sin\varphi\sin\theta$ |  |
| Sphere, radius $a > 0$ center $(0,0,a)$ | $\{(\rho,\varphi,\theta) : \rho = 2a\cos(\varphi),\ 0 \le \varphi \le \pi/2\}$<br>$z = \rho\cos(\varphi) = 2a\cos(\varphi)\cos(\varphi) = 2a\cos^2(\varphi)$ |  |



$0 \le 2a\cos^2(\varphi) \le 2a$

**Theorem 16.7: Change of Variables for Triple Integrals in Spherical Coordinates**

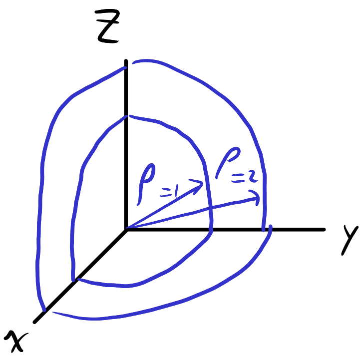
Let $f$ be continuous over the region $D$, expressed in spherical coordinates as

$$D = \{(\rho, \varphi, \theta) : 0 \le g(\varphi, \theta) \le \rho \le h(\varphi, \theta),\ a \le \varphi \le b,\ \alpha \le \theta \le \beta\}.$$

Then $f$ is integrable over $D$, and the triple integral of $f$ over $D$ is

$$\iiint_D f(x, y, z)\, dV$$
$$= \int_\alpha^\beta \int_a^b \int_{g(\varphi,\theta)}^{h(\varphi,\theta)} f(\rho\sin(\varphi)\cos(\theta),\ \rho\sin(\varphi)\sin(\theta),\ \rho\cos(\varphi))\, \rho^2\sin(\varphi)\ d\rho\, d\varphi\, d\theta.$$

**Example.** Evaluate $\iiint_D \left(x^2+y^2+z^2\right)^{-3/2} dV$, where $D$ is the region in the first octant between two spheres of radius 1 and 2 centered at the origin




$(\rho, \varphi, \theta)$

$1 \le \rho \le 2$

$0 \le \varphi \le \pi/2$

$0 \le \theta \le \pi/2$

$x^2+y^2+z^2 = (\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2 + (\rho\cos\varphi)^2$

$= \rho^2(\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\varphi)$

$= \rho^2(\sin^2\varphi(\cos^2\theta + \sin^2\theta) + \cos^2\varphi)$ $\quad 2$

$= \rho$

$$\iiint_D (x^2+y^2+z^2)^{-3/2}\, dV = \int_0^{\pi/2}\int_1^2\int_0^{\pi/2} (\rho^2)^{-3/2} \rho^2 \sin(\varphi)\, d\theta\, d\rho\, d\varphi$$

$$= \int_0^{\pi/2}\int_1^2\int_0^{\pi/2} \rho^{-1}\sin\varphi \, d\theta\, d\rho\, d\varphi$$

$$= \int_0^{\pi/2}\int_1^2 \theta\rho^{-1}\sin\varphi \Big|_{\theta=0}^{\theta=\pi/2} d\rho\, d\varphi$$

$$= \frac{\pi}{2}\int_0^{\pi/2}\int_1^2 \rho^{-1}\sin\varphi\, d\rho\, d\varphi$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \ln(\rho)\sin\varphi \Big|_{\rho=1}^{\rho=2} d\varphi$$

$$= \frac{\pi}{2}\ln(2)\int_0^{\pi/2} \sin\varphi\, d\varphi$$

$$= -\frac{\pi}{2}\ln(2)\cos(\varphi)\Big|_{\varphi=0}^{\varphi=\pi/2}$$

LC #4

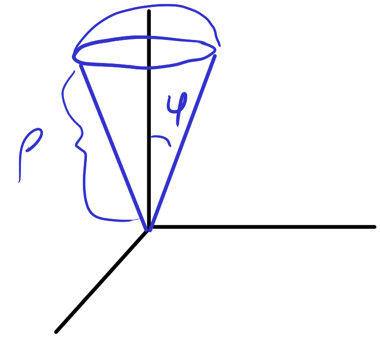
$$= -\frac{\pi}{2}\ln(2)(0-1) = \frac{\pi}{2}\ln(2) \approx 1.0888$$

**Example.** Find the volume of the solid region $D$ that lies inside the cone $\varphi = \pi/6$ and inside the sphere $\rho = 4$.

$0 \le \rho \le 4$

$0 \le \varphi \le \pi/6$

$0 \le \theta \le 2\pi$




$$V = \iiint_D dV = \int_0^{2\pi} \int_0^4 \int_0^{\pi/6} \rho^2 \sin\varphi \, d\varphi \, d\rho \, d\theta$$

$$= \int_0^{2\pi} \int_0^4 -\rho^2 \cos\varphi \Big|_{\varphi=0}^{\varphi=\pi/6} d\rho \, d\theta$$

$$= \int_0^{2\pi} \int_0^4 -\rho^2 \left( \frac{\sqrt{3}}{2} - 1 \right) d\rho \, d\theta$$

$$= \left(1 - \frac{\sqrt{3}}{2}\right) \int_0^{2\pi} \frac{\rho^3}{3} \Big|_{\rho=0}^{\rho=4} d\theta$$

$$= \frac{64}{3}\left(1 - \frac{\sqrt{3}}{2}\right) \int_0^{2\pi} d\theta = \frac{64}{3}\left(1 - \frac{\sqrt{3}}{2}\right) \theta \Big|_{\theta=0}^{\theta=2\pi}$$

LCS

$$= \frac{64\pi}{3}\left(2 - \sqrt{3}\right) \approx 26.9372$$