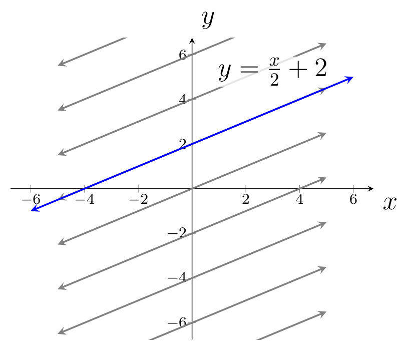
13.5: Lines and Planes in Space

Equation of a Line:

Recall the equation of a line in $\mathbb{R}^2$:



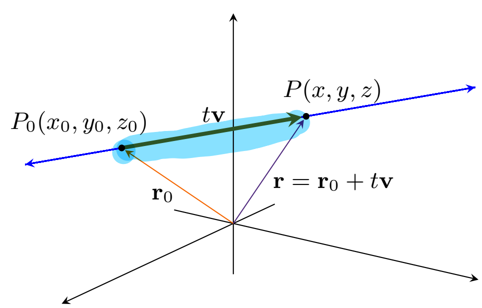
slope-intercept $\quad y = mx + b \quad \rightarrow \quad -mx + y = b$

standard form $\quad Ax + By = C$

where b is the intercept and m is the slope. This idea can be extended into higher dimensions:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

Here, $\mathbf{r}_0$ is a fixed point, and $\mathbf{v}$ is the position vector that is parallel to the line $\mathbf{r}$.



Equation of a Line

A **vector equation of the line** passing through the point $P_0(x_0, y_0, z_0)$ in the direction of the vector $\mathbf{v} = \langle a, b, c\rangle$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, or

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle, \quad \text{for} \quad -\infty < t < \infty$$

Equivalently, the corresponding **parametric equations of the line** are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \text{for} \quad -\infty < t < \infty$$

$$\langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle = \langle x_0 + at,\ y_0 + bt,\ z_0 + ct\rangle$$

$\vec{r} = \vec{r}_0 + t\vec{v}$

Example. Find the vector equation and parametric equation of the line that

- goes through the points $P(\underbrace{-1,-2,1}_{\vec{r}_0})$ and $Q(-4,-5,-3)$ where $t = 0$ corresponds to P,

$\vec{r}_0 = \langle -1,-2,1\rangle$

$\vec{v} = \cancel{QR} = \langle -4-(-1), -5-(-2), -3-1\rangle = \langle -3,-3,-4\rangle$

$\vec{PQ} = \langle 3,3,4\rangle$ $\quad \vec{r} = \langle -1, \underset{y_0}{-2}, \underset{z_0}{1}\rangle + t\langle \underset{a}{3}, 3, \underset{c}{4}\rangle$

$\vec{r} = \langle -1,-2,1\rangle + t\langle -3,-3,-4\rangle$

same line
diff directions

$x = -1+3t,\ y = -2+3t,\ z = 1+4t$

- goes through the point $P(1,-3,-3)$ and is parallel to the vector $\mathbf{r} = \langle -4, 1, -1\rangle$,

$\vec{r} = \vec{r}_0 + t\vec{v}$

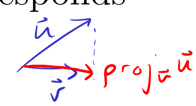
$\vec{l} = \langle 1,-3,-3\rangle + t\langle -4,1,-1\rangle$

$x = 1-4t,\ y = -3+t,\ z = -3-t$

- goes through the point $P(-2,5,-2)$ and is perpendicular to the lines $x = 3-4t$, $y = 2-3t$, $z = -1-t$, and $x = -2+0t$, $y = 2-t$, $z = 3t$, where $t = 0$ corresponds to P.

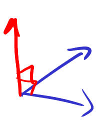
$\vec{r} = \vec{r}_0 + t\vec{v}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & -3 & -1 \\ 0 & -1 & 3 \end{vmatrix} \begin{matrix} \hat{i} & \hat{j} \\ -4 & -3 \\ 0 & -1 \end{matrix}$$



$\vec{r}_0 = \langle -2,5,-2\rangle$

$\vec{v} = \underbrace{\langle -4,-3,-1\rangle}_{\vec{u}} \times \underbrace{\langle 0,-1,3\rangle}_{\vec{v}} = \langle -10, 12, 4\rangle$



$\vec{r} = \langle -2,5,-2\rangle + t\langle -10,12,4\rangle$

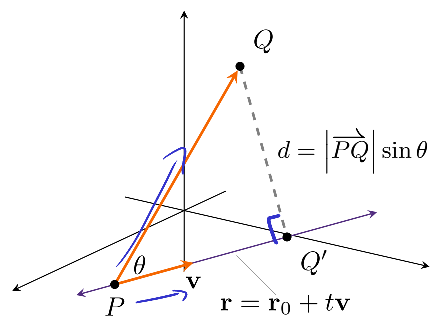
$x = -2-10t$
$y = 5+12t$
$z = -2+4t$

LC

$z = -2 + \underset{\substack{\uparrow \\ 4}}{c}\,t$

Distance from a Point to a Line:

Given a point Q and a line ℓ, the shortest distance to the line is the length of $\overrightarrow{QQ'}$.



$$\sin\theta = \frac{d}{|\overrightarrow{PQ}|}$$

From the definition of the cross product, we have

$$\left|\mathbf{v}\times\overrightarrow{PQ}\right| = |\mathbf{v}|\underbrace{\left|\overrightarrow{PQ}\right|\sin\theta}_{d} = |\mathbf{v}|d$$

From here, solving for d gives us the following:

Distance Between a Point and a Line

The distance d between the point Q and the $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ is

$$d = \frac{\left|\mathbf{v}\times\overrightarrow{PQ}\right|}{|\mathbf{v}|},$$

where P is any point on the line and $\mathbf{v}$ is a vector parallel to the line.

Example. Find the distance from the point $Q(-4,-1,-3)$ and the line $x = -5 - 5t$, $y = -5 + t$, $z = -1 + 4t$. (*Hint:* Let P be the point at $t = 0$)

$P = (-5, -5, -1)$ $\quad \overrightarrow{PQ} = \langle -4-(-5), -1-(-5), -3-(-1)\rangle = \langle 1, 4, -2\rangle$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & 1 & 4 \\ 1 & 4 & -2 \end{vmatrix}$$

$\vec{v}\times\overrightarrow{PQ} = \langle -2-16, -(10-4), -20-1\rangle = \langle -18, -6, -21\rangle = 3\langle -6, -2, -7\rangle$

$|\vec{v}\times\overrightarrow{PQ}| = 3\sqrt{36+4+49} = 3\sqrt{89}$

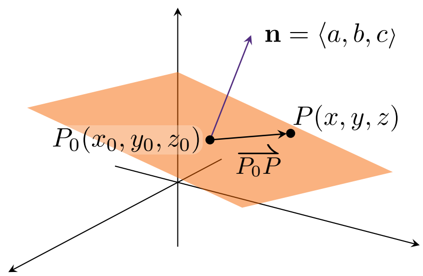
$|\vec{v}| = \sqrt{42}$

$$d = \frac{3\sqrt{89}}{\sqrt{42}}$$

Equations of Planes:

In $\mathbb{R}^2$, two distinct points determine a line.

In $\mathbb{R}^3$, three noncollinear points determine a unique plane. Alternatively, a plane is uniquely determined by a point and a vector that is orthogonal to the plane.



> **Definition. (Plane in $\mathbb{R}^3$)**
> Given a fixed point P_0 and a nonzero **normal vector n**, the set of points P in $\mathbb{R}^3$ for which $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$ is called a **plane**.

Consider the normal vector $\mathbf{n} = \langle a, b, c\rangle$ at the point $P_0(x_0, y_0, z_0)$, and any point $P(x, y, z)$ on the plane. Since $\mathbf{n}$ is orthogonal to the plane, it is also orthogonal to the vector $\overrightarrow{P_0P}$, which is also in the plane. Thus,

$$\begin{aligned} \mathbf{n} \cdot \overrightarrow{P_0P} &= 0 \\ \langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \\ ax + by + cz &= d \end{aligned}$$

> **General Equation of a Plane in $\mathbb{R}^3$**
> The plane passing through the point $P_0(x_0, y_0, z_0)$ with a nonzero normal vector $\mathbf{n} = \langle a, b, c\rangle$ is described by the equation
>
> $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{or} \quad ax + by + cz = d,$$
>
> where $d = ax_0 + by_0 + cz_0$.

Example. Find the equation of the plane that

- goes through the point $P(-2, 5, 0)$ and is parallel to the plane $x - 5y - 5z = 1$,

- goes through the points $P(5, -2, 1)$, $Q(5, 1, 3)$ and $R(1, -5, -2)$

- that is parallel to the vectors $\langle 4, -2, -3 \rangle$ and $\langle 3, 2, 3 \rangle$, passing through the point $P(-2, -2, 5)$.

Example. Find the location where the line $\langle -3, 1, 4\rangle + t\langle -1, -4, 2\rangle$ and the plane $2x - 2y - 4z = 5$ intersect.

> **Definition. (Parallel and Orthogonal Planes)**
> Two distinct planes are **parallel** if their respective normal vectors are parallel (that is, the normal vectors are scaling multiples of each other). Two plans are **orthogonal** if their respective normal vectors are orthogonal (that is, the dot product of the normal vectors is *zero*).

Example. Find the line of intersection between the planes $3x - y + 4z = -4$ and $x + 3y - 2z = 0$.

Example. Find the smallest angle between the planes $3x - y + 4z = -4$ and $x + 3y - 2z = 0$.