

17.6: Surface Integrals

Imagine a sphere with a known temperature distribution. How would we find the average temperature over the sphere?

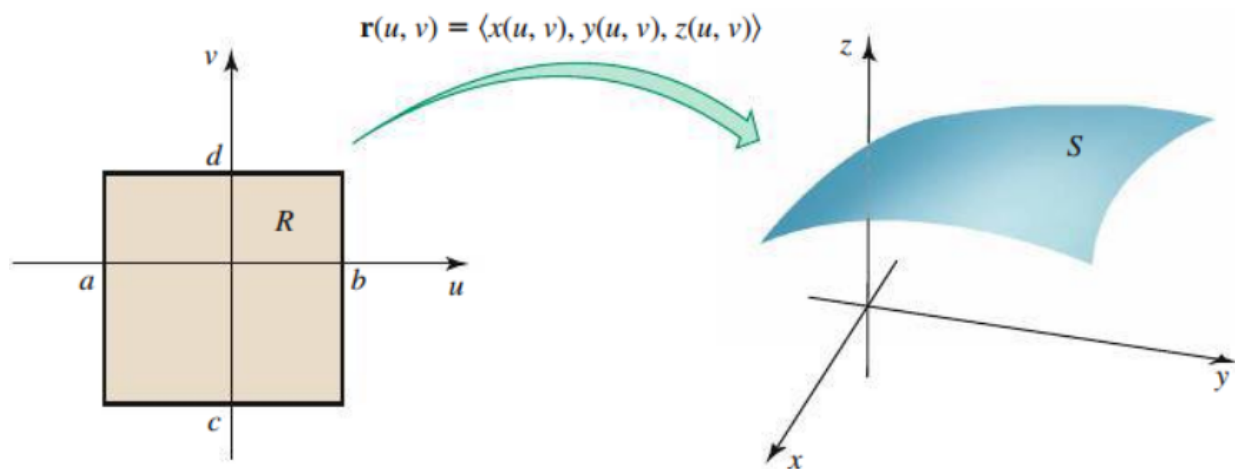
Parallel Concepts	
Curves	Surfaces
Arc length	Surface area
Line integrals	Surface integrals
One-parameter description	Two-parameter description

Parameterized Surfaces

Recall that in \mathbb{R}^2 , we parameterized a curve by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ where $a \leq t \leq b$. In \mathbb{R}^3 , we parameterize a surface by

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$

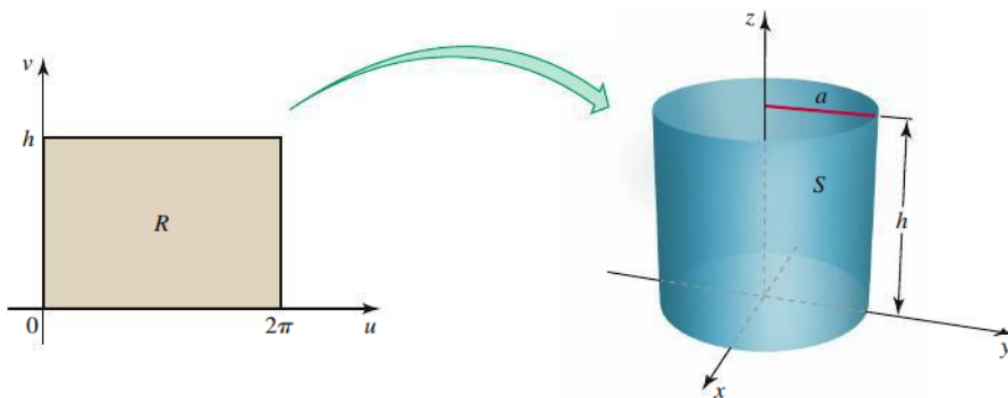
where the parameters are over $R = \{(u, v) : a \leq u \leq b, c \leq v \leq d\}$



Cylinders:

$$\{(x, y, z) : x = a \cos(\theta), y = a \sin(\theta), 0 \leq \theta \leq 2\pi, 0 \leq z \leq h\}$$

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle = \langle a \cos(u), a \sin(u), v \rangle$$



Cones:

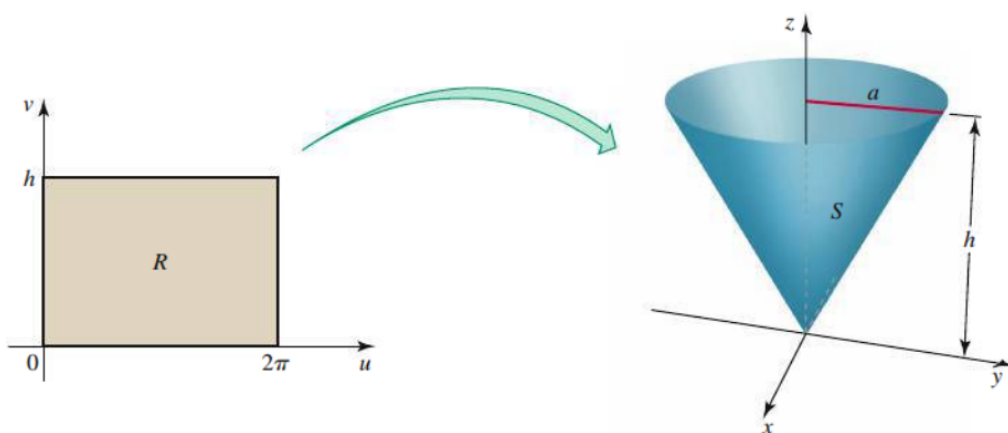
$$\{(r, \theta, z) : 0 \leq r \leq a, 0 \leq \theta \leq 2\pi, z = rh/a\}$$

For a fixed value of z , $r = az/h$:

$$x = r \cos(\theta) = \frac{az}{h} \cos(\theta) \text{ and } y = r \sin(\theta) = \frac{az}{h} \sin(\theta)$$

Now, let $u = \theta$ and $v = z$, then

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle = \left\langle \frac{av}{h} \cos(u), \frac{av}{h} \sin(u), v \right\rangle$$



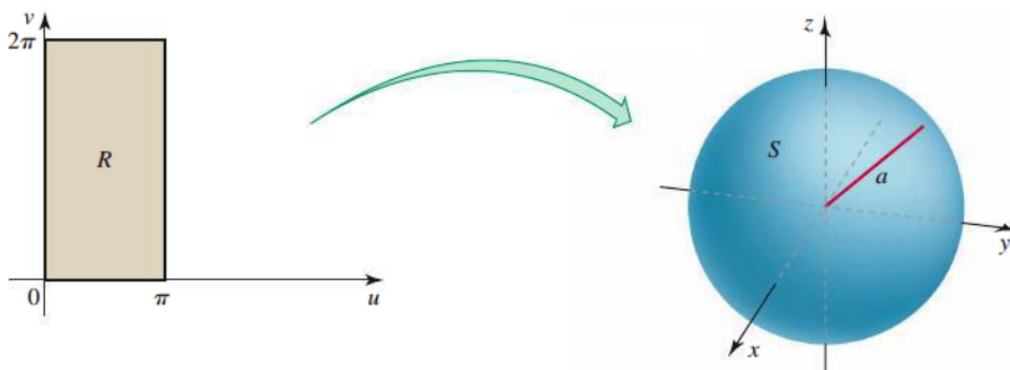
Spheres:

$$\{(\rho, \varphi, \theta) : \rho = a, 0 \leq \varphi \leq \pi, 0 \leq \theta \leq 2\pi\}$$

$$x = a \sin(\varphi) \cos(\theta), \quad y = a \sin(\varphi) \sin(\theta), \quad z = a \cos(\varphi)$$

Now, let $u = \theta$ and $v = \varphi$, then

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle = \langle a \sin(v) \cos(u), a \sin(v) \sin(u), a \cos(v) \rangle$$

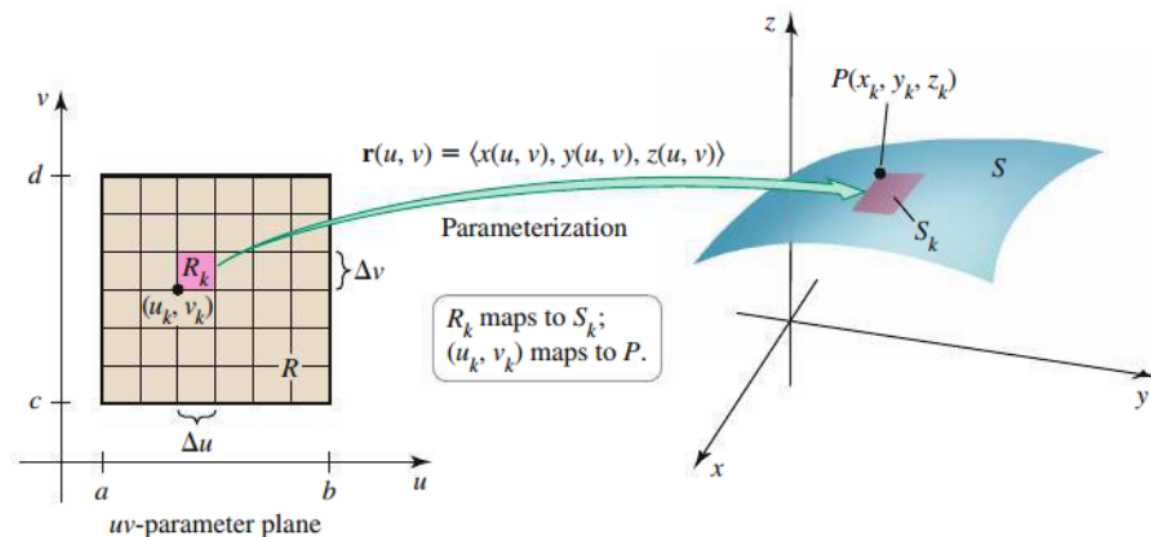


Example. Find parametric descriptions for the following surfaces

The plane $3x - 2y + z = 2$

The paraboloid $z = x^2 + y^2$, for $0 \leq z \leq 9$

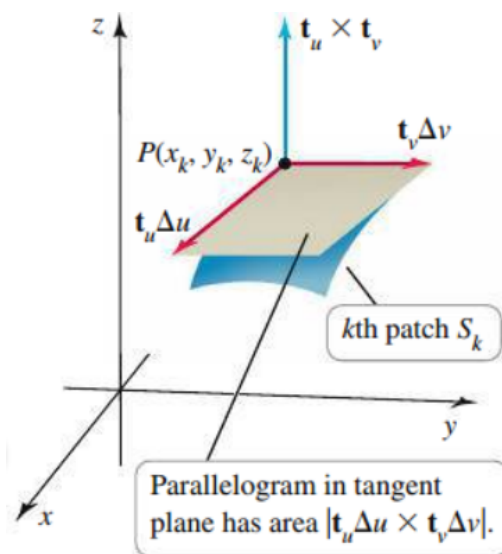
Surface Integrals of Scalar-Valued Functions



Using the parameterization

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$

over the region $R = \{(u, v) : a \leq u \leq b, c \leq v \leq d\}$, it is important that we know ΔS_k , which is the area of S_k .



Definition. (Surface Integral of Scalar-Valued Functions on Parameterized Surfaces)

Let f be a continuous scalar-valued function on a smooth surface S given parametrically by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, where u and v vary over

$R = \{(u, v) : a \leq u \leq b, c \leq v \leq d\}$. Assume also that the tangent vectors

$$\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \text{ and } \mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

are continuous on R and the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R . Then the **surface integral of f over S** is

$$\iint_S f(x, y, z) dS = \iint_R f(x(u, v), y(u, v), z(u, v)) |\mathbf{t}_u \times \mathbf{t}_v| dA$$

If $f(x, y, z) = 1$, this integral equals the surface area of S .

Example. Find the surface area of the following surfaces

A cylinder with radius $a > 0$ and height h (open ends)

A sphere of radius a

Example. The temperature on the surface of a sphere of radius a varies with latitude according to the function $T(\varphi, \theta) = 10 + 50 \sin(\varphi)$, for $0 \leq \varphi \leq \pi$ and $0 \leq \theta \leq 2\pi$. Find the average temperature over the sphere.

Surface Integrals on Explicitly Defined Surfaces

Suppose a smooth surface S is defined explicitly as $z = g(x, y)$. Here, we let $u = x$ and $v = y$. This gives us

$$\mathbf{t}_u = \mathbf{t}_x = \langle 1, 0, z_x \rangle, \quad \mathbf{t}_v = \mathbf{t}_y = \langle 0, 1, z_y \rangle$$

thus

$$\mathbf{t}_x \times \mathbf{t}_y = \langle -z_x, -z_y, 1 \rangle$$

and

$$|\mathbf{t}_x \times \mathbf{t}_y| = \sqrt{z_x^2 + z_y^2 + 1}$$

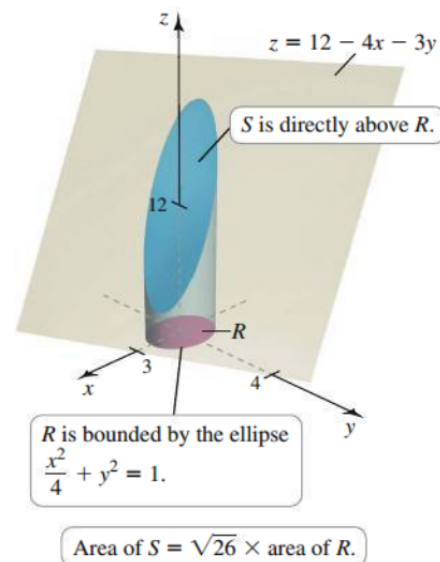
Theorem 17.14: Evaluation of Surface Integrals of Scalar-Valued Functions on Explicitly Defined Surfaces

Let f be a continuous function on a smooth surface S given by $z = g(x, y)$, for (x, y) in a region R . The surface integral of f over S is

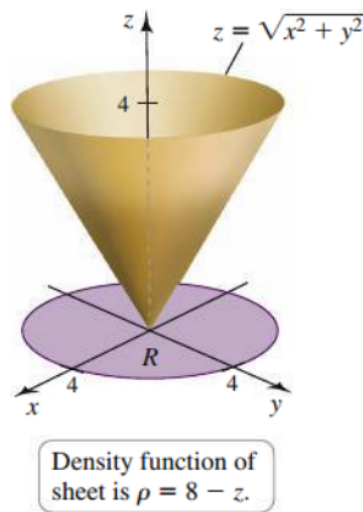
$$\iint_S f(x, y, z) \, dS = \iint_S f(x, y, g(x, y)) \sqrt{z_x^2 + z_y^2 + 1} \, dA.$$

If $f(x, y, z) = 1$, the surface integral equals the area of the surface.

Example. Find the area of the surface S that lies in the plane $z = 12 - 4x - 3y$ directly above the region R bounded by the ellipse $x^2/4 + y^2 = 1$



Example. A thin conical sheet is described by the surface $z = (x^2 + y^2)^{\frac{1}{2}}$, for $0 \leq z \leq 4$. The density of the sheet in g/ cm² is $\rho = f(x, y, z) = (8 - z)$. What is the mass of the cone?



Explicit Description $z = g(x, y)$				Parametric Description		
Surface	Equation	Normal vector $\pm \langle -z_x, -z_y, 1 \rangle$	magnitude $ \langle -z_x, -z_y, 1 \rangle $	Equation	Normal vector $\mathbf{t}_u \times \mathbf{t}_v$	magnitude $ \mathbf{t}_u \times \mathbf{t}_v $
Cylinder	$x^2 + y^2 = a^2,$ $0 \leq z \leq h$	$\langle x, y, 0 \rangle$	a	$\mathbf{r} = \langle a \cos(u), a \sin(u), v \rangle,$ $0 \leq u \leq 2\pi, 0 \leq v \leq h$	$\langle a \cos(u), a \sin(u), 0 \rangle$	a
Cone	$z^2 = x^2 + y^2,$ $0 \leq z \leq h$	$\langle x/z, y/z, -1 \rangle$	$\sqrt{2}$	$\mathbf{r} = \langle v \cos(u), v \sin(u), v \rangle,$ $0 \leq u \leq 2\pi, 0 \leq v \leq h$	$\langle v \cos(u), v \sin(u), -v \rangle$	$\sqrt{2}v$
Sphere	$x^2 + y^2 + z^2 = a^2$	$\langle x/z, y/z, 1 \rangle;$	a/z	$\mathbf{r} = \langle a \sin(u) \cos(v),$ $a \sin(u) \sin(v),$ $a \cos(u) \rangle$ $0 \leq u \leq \pi, 0 \leq v \leq 2\pi$	$\langle a^2 \sin^2(u) \cos(v),$ $a^2 \sin^2(u) \sin(v),$ $a^2 \sin(u) \cos(u) \rangle$	$a^2 \sin(u)$
Paraboloid	$z = x^2 + y^2,$ $0 \leq z \leq h$	$\langle 2x, 2y, -1 \rangle$	$\sqrt{1 + 4(x^2 + y^2)}$	$\mathbf{r} = \langle v \cos(u), v \sin(u), v^2 \rangle,$ $0 \leq u \leq 2\pi, 0 \leq v \leq \sqrt{h}$	$\langle 2v^2 \cos(u), 2v^2 \sin(u), -v \rangle$	$v\sqrt{1 + 4v^2}$

Surface Integrals of Vector Fields:

The surfaces we consider must be

- **two-sided** or **orientable**
- **oriented**

Definition. (Surface Integral of a Vector Field)

Suppose $\mathbf{F} = \langle f, g, h \rangle$ is a continuous vector field on a region of \mathbb{R}^3 containing a smooth oriented surface S . If S is defined parametrically as $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, for (u, v) in a region R , then

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v) \, dA,$$

where

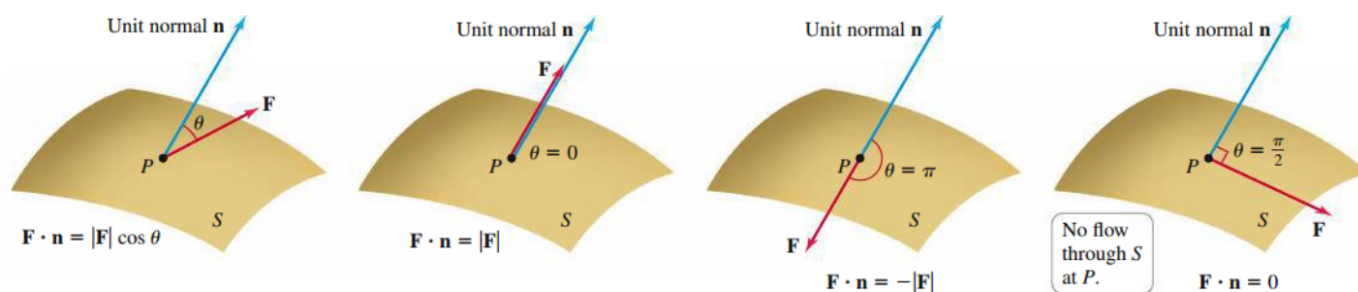
$$\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \text{ and } \mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

and continuous on R , the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R , and the direction of the normal vector is consistent with the orientation of S . If S is defined in the form $z = s(x, y)$, for (x, y) in a region R , then

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_R (-fz_x - gz_y + h) \, dA.$$

Flux Integrals:

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$



The unit normal vector we use is

$$\mathbf{n} = \frac{\mathbf{t}_u \times \mathbf{t}_v}{|\mathbf{t}_u \times \mathbf{t}_v|}$$

giving us

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \iint_R \mathbf{F} \cdot \mathbf{n} |\mathbf{t}_u \times \mathbf{t}_v| \, dA \\ &= \iint_R \mathbf{F} \cdot \frac{\mathbf{t}_u \times \mathbf{t}_v}{|\mathbf{t}_u \times \mathbf{t}_v|} |\mathbf{t}_u \times \mathbf{t}_v| \, dA \\ &= \iint_R \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v) \, dA \end{aligned}$$

When the surface S is explicitly given as $z = s(x, y)$, then

$$\mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v) = -fz_x - gz_y + h$$

Definition. (Surface Integral of a Vector Field)

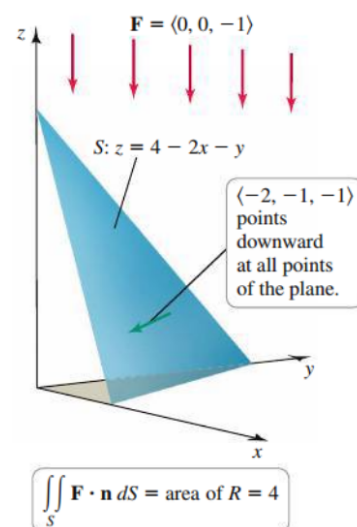
Suppose $\mathbf{F} = \langle f, g, h \rangle$ is a continuous vector field on a region of \mathbb{R}^3 containing a smooth oriented surface S . If S is defined parametrically as $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, for (u, v) in a region R , then

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_R \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v) \, dA,$$

where $\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$ and $\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$ are continuous on R , the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R , and the direction of the normal vector is consistent with the orientation of S . If S is defined in the form $z = s(x, y)$, for (x, y) in a region R , then

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_R (-f z_x - g z_y + h) \, dA.$$

Example. Consider the vertical field $\mathbf{F} = \langle 0, 0, -1 \rangle$. Find the flux in the downward direction across the surface S , which is the plane $z = 4 - 2x - y$ in the first octant.



Example. Consider the radial vector field $\mathbf{F} = \langle f, g, h \rangle = \langle x, y, z \rangle$. Compute the upward flux of the field across the:

hemisphere $x^2 + y^2 + z^2 = 1$, for $z \geq 0$,

paraboloid $z = 1 - x^2 - y^2$, for $z \geq 0$.