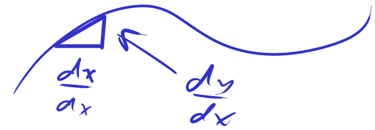
arc length

$y = f(x)$

$$L = \int_a^b \sqrt{1^2 + (f'(x))^2}\, dx$$



14.4: Length of Curves

Definition. (Arc Length for Vector Functions)
Consider the parameterized curve $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where f', g', and h' are continuous, and the curve is traversed once for $a \leq t \leq b$. The **arc length** of the curve between $(f(a), g(a), h(a))$ and $(f(b), g(b), h(b))$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt = \int_a^b |\mathbf{r}'(t)|\, dt.$$

Example (Flight of an eagle)**.** Suppose an eagle rises at a rate of 100 vertical ft/min on a helical path given by

$$\mathbf{r}(t) = \langle 250\cos(t),\ 250\sin(t),\ 100t \rangle$$

where $\mathbf{r}$ is measured in feet and t is measured in minutes. How far does it travel in 10 minutes?

$$L = \int_0^{10} \sqrt{(-250\sin(t))^2 + (250\cos(t))^2 + 100^2}\, dt$$

$$(-250)^2 = (-1)^2(250)^2$$

$$= \int_0^{10} \sqrt{250^2(\underbrace{\sin^2(t) + \cos^2(t)}_{1}) + 100^2}\, dt$$

$$= \sqrt{50^2(5^2 + 2^2)} \int_0^{10} dt$$

$$= 50\sqrt{29}\ t\Big|_0^{10} = 500\sqrt{29}$$

Example. Suppose a particle has a trajectory given by

$$\mathbf{r}(t) = \langle 10\cos(3t),\, 10\sin(3t) \rangle$$

where $0 \le t \le \pi$. How far does this particle travel?

$$L = \int_0^{\pi} \sqrt{(-30\sin(3t))^2 + (30\cos(3t))^2}\, dt$$

$$= 30 \int_0^{\pi} \underbrace{\sqrt{\sin^2(3t) + \cos^2(3t)}}_{1}\, dt$$

$$= 30\, t \Big|_0^{\pi} = 30\pi$$

Example. Find the length of the curve

$$\mathbf{r}(t) = \left\langle 3t^2 - 5,\, 4t^2 + 5 \right\rangle$$

where $0 \le t \le 1$.

$$L = \int_0^1 \sqrt{(6t)^2 + (8t)^2}\, dt$$

$$= \int_0^1 \sqrt{100t^2}\, dt$$

$$= 10 \int_0^1 t\, dt$$

$$= 5t^2 \Big|_0^1 = 5$$

Example. Find the length of $\mathbf{r}(t) = \left\langle t^2, \dfrac{(4t+1)^{\frac{3}{2}}}{6} \right\rangle$ where $0 \leq t \leq 6$.

$$\int_0^6 \sqrt{(2t)^2 + \left((4t+1)^{1/2}\right)^2}\, dt$$

$$\cancel{\frac{3}{2}} \frac{(4t+1)^{1/2}}{\cancel{6}} \cancel{(4)}$$

$$= \int_0^6 \sqrt{4t^2 + 4t + 1}\, dt$$

$$\left(\frac{1}{2}\right)^2$$

$$4\left(t^2 + t + \frac{1}{4} - \frac{1}{4}\right) + 1$$

$$4\left(t + \frac{1}{2}\right)^2 - 1 + 1$$

$$= \int_0^6 \sqrt{4(t+1/2)^2}\, dt$$

$$(2t+1)^2$$

$$= \int_0^6 2(t + 1/2)\, dt = 2\left(\frac{t^2}{2} + \frac{t}{2}\right)\Big|_0^6 = 2\left(\frac{36}{2} + \frac{6}{2}\right) = \boxed{42}$$

Example. Find the length of $\mathbf{r}(t) = \langle 2\sqrt{2}, \sin(t), \cos(t) \rangle$ where $0 \leq t \leq 5$.

$$L = \int_0^5 \underbrace{\sqrt{0^2 + (\cos(t))^2 + (-\sin(t))^2}}_{1}\, dt$$

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

$$= \int_0^5 dt = t\Big|_0^5 = \boxed{5}$$

LC

Q5 & Q6

Theorem 14.3: Arc Length as a Function of a Parameter
Let $\mathbf{r}(t)$ describe a smooth curve, for $t \geq a$. The arc length is given by

$$s(t) = \int_a^t |\mathbf{v}(u)|\, du,$$

$$\int_a^t \underbrace{|v(u)|}_{1}\, du = u\Big|_a^t = t - a$$

where $|\mathbf{v}| = |\mathbf{r}'|$. Equivalently, $\dfrac{ds}{dt} = |\mathbf{v}(t)|$. If $|\mathbf{v}(t)| = 1$, for all $t \geq a$, then the parameter t corresponds to arc length.

Example. For the following functions, determine if $\mathbf{r}(t)$ uses arc length as a parameter. If not, find a description that uses arc length as a parameter.

$s(t) = 5t$

a) $\mathbf{r}(t) = \langle -4t + 1, 3t - 1 \rangle$, $0 \leq t \leq 4$.

$$s(t) = \int_0^t \sqrt{(-4)^2 + (3)^2}\, du = 5\int_0^t du = 5u\Big|_0^t = 5t$$

$\rightarrow s = 5t$

$t = \frac{s}{5}$

Not using arc length as a parameter.

$$\vec{r}\left(\frac{s}{5}\right) = \left\langle -\frac{4s}{5} + 1, \frac{3s}{5} - 1 \right\rangle \qquad L = \int_0^t \left|\vec{r}\,'\left(\frac{s}{5}\right)\right| ds = t$$

$\vec{r}(4/5) \longrightarrow$ length 4

b) $\mathbf{r}(t) = \left\langle \frac{1}{\sqrt{10}} \cos(t), \frac{3}{\sqrt{10}} \cos(t), \sin(t) \right\rangle$, $0 \leq t \leq 2\pi$.

$$s(t) = \int_0^t \sqrt{\left(\frac{-\sin(x)}{\sqrt{10}}\right)^2 + \left(\frac{-3\sin(x)}{\sqrt{10}}\right)^2 + (\cos(x))^2}\, dx$$

$$= \int_0^t \sqrt{\underbrace{\underbrace{\left(\frac{1}{10} + \frac{9}{10}\right)}_{1} \sin^2(x) + \cos^2(x)}_{1}}\, dx = \int_0^t dx = x\Big|_0^t = t$$

$L = s(t) = t$

$s(42)$
Length 42