12.3: Calculus in Polar Coordinates

Theorem 12.2: Slope of a Tangent Line
Let f be a differentiable function at θ_0. The slope of the line tangent to the curve $r = f(\theta)$ at the point $(f(\theta_0), \theta_0)$ is

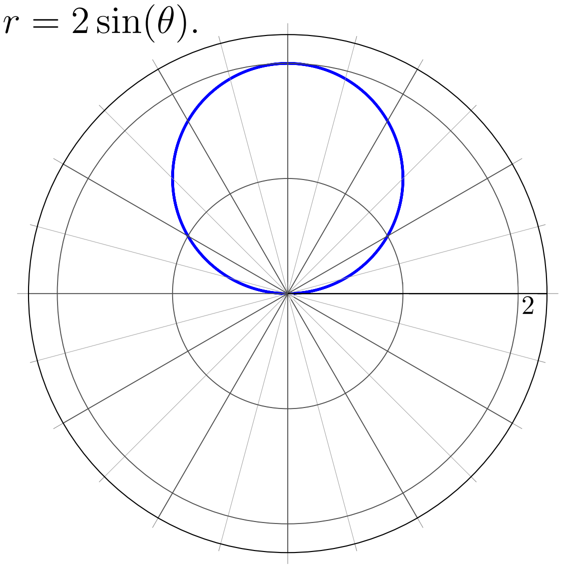
$$\left.\frac{dy}{dx}\right|_{\theta=\theta_0} = \frac{f'(\theta_0)\sin(\theta_0) + f(\theta_0)\cos(\theta_0)}{f'(\theta_0)\cos(\theta_0) - f(\theta_0)\sin(\theta_0)},$$

provided the denominator is nonzero at the point. At angles θ_0 for which $f(\theta_0) = 0$, $f'(\theta_0) \neq 0$, and $\cos(\theta_0) \neq 0$, the tangent line is $\theta = \theta_0$ with slope $\tan(\theta_0)$.

Example. Compute the slope of the line tangent to the polar curve $r = e^{-\theta}$ at $\theta = \pi$.

Example (LC 34.1-34.3)**.** Consider the polar curve $r = 2\sin(\theta)$.

Express this polar curve in Cartesian coordinates



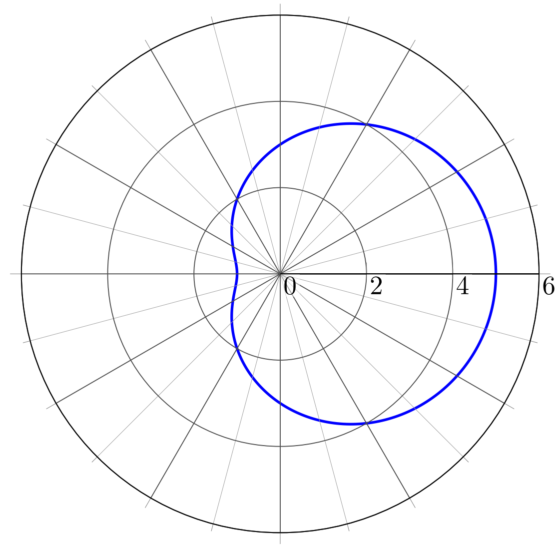
Locate all points (r, θ), where this curve has a horizontal tangent line

Locate all points (r, θ), where this curve has a vertical tangent line

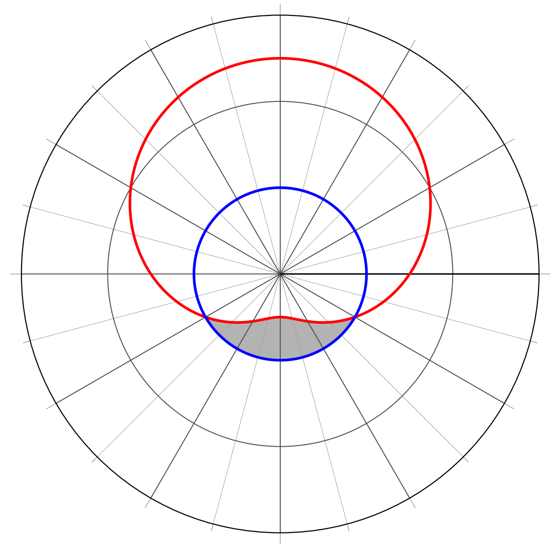
Definition. (Area of Regions in Polar Coordinates)
Let R be the region bounded by the graphs of $r = f(\theta)$ and $r = g(\theta)$, between $\theta = \alpha$ and $\theta = \beta$, where f and g are continuous and $f(\theta) \geq g(\theta) \geq 0$ on $[\alpha, \beta]$. The area of R is

$$\int_{\alpha}^{\beta} \frac{1}{2}\big(f(\theta)^2 - g(\theta)^2\big)\, d\theta.$$

Example (LC 34.5-34.6)**.** Find the area enclosed by the polar curve $r = 3 + 2\cos(\theta)$.

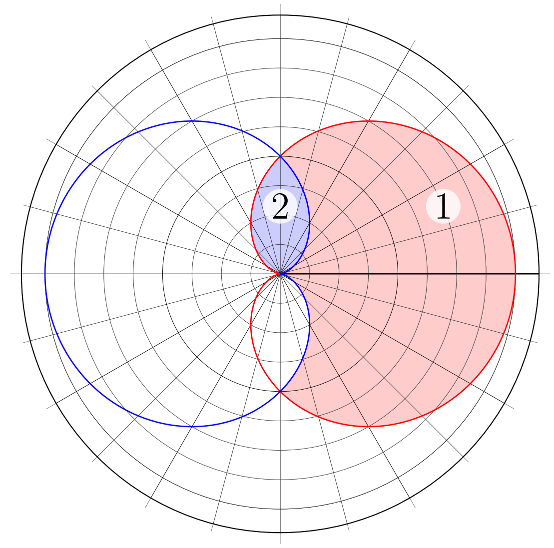


Example (LC 34.7-34.9)**.** Find the area of the region inside the polar curve $r = 2$ and outside of the polar curve $r = 3 + 2\sin(\theta)$.



Example. Consider the polar curves $r = 1 + \cos(\theta)$ and $r = 1 - \cos(\theta)$.

Setup the integral(s) that finds the area of area 1.



Setup and solve the integral(s) that finds the area of area 2.

Arc Length of a Polar Curve

Let f have a continuous derivative on the interval $[\alpha, \beta]$. The **arc length** of the polar curve $r = f(\theta)$ on $[\alpha, \beta]$ is

$$L = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta.$$

Example (LC 34.13)**.** Find the length of the polar curve $r = e^{-a\theta}$ for $\theta \geq 0$ and $a > 0$ (a is constant).