

10.8: Choosing a Convergence Test

Example. Consider the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$. Is this series conditionally convergent, absolutely convergent, or divergent? Which test do you use?

abs convergence: $\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges b/c p-series w/ $p=1 \leq 1$

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \Rightarrow$ Not abs convergent

① $a_k = \frac{1}{k}$
 $a_{k+1} = \frac{1}{k+1}$ } $\hookrightarrow \frac{1}{k+1} < \frac{1}{k} \Rightarrow$ non increasing } AST \Rightarrow convergent

② $\lim_{k \rightarrow \infty} a_k = 0$

Conditionally convergent

Example. Consider the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$. Is this series conditionally convergent, absolutely convergent, or divergent? Which test do you use?

abs convergent $\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k^2} \right| = \sum_{k=1}^{\infty} \frac{1}{k^2}$

p-series w/ $p=2 > 1$
so series converges

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$ AST ① $k^2 < (k+1)^2 \Rightarrow \frac{1}{(k+1)^2} < \frac{1}{k^2}$
② $\lim_{k \rightarrow \infty} \frac{1}{k^2} = 0$

Converges by AST

2. many choice

Consider the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$. Select all statements that are TRUE about this series.

A. This series is absolutely convergent.

B. This series is conditionally convergent.

C. This series is absolutely convergent by the Alternating Series Test.

D. This series is conditionally convergent by the Alternating Series Test.

E. This series is convergent by the Alternating Series Test.

F. This series is divergent by the Alternating Series Test.

← No

Example. Which of the following series can be rewritten as a p -series?

$$\sum_{k=1}^{\infty} \frac{(-1)^{2k}}{k\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$$

p -series w/ $p=3/2$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^5}$$

Not a p -series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

$$\sum_{k=1}^{\infty} \frac{k^2 + k + 1}{k^4 + 2}$$

Not a p -series

$$\sum_{k=1}^{\infty} \frac{3^k}{k^4}$$

Not a p -series

Note: can compare to
a p -series $\sum_{k=1}^{\infty} \frac{1}{k^2}$

(LCR)

$$\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$$

p -series w/ $p=3/2$

$$k^{1/2} k^{-2} = k^{1/2 - 2} = k^{-3/2}$$

LC #3

Example. Which test *cannot* be used to determine the convergence of $\sum_{k=1}^{\infty} \frac{k^2 2^{k-1}}{(-5)^k}$?

✓ AST

✓ Ratio Test

✓ Root Test

$$\lim \left| \frac{(k+1)^2 2^k}{(-5)^{k+1}} \cdot \frac{(-5)^k}{k^2 2^{k-1}} \right|$$

$$\lim \left| \frac{k^2 2^{k-1}}{5^k} \right|^{1/k}$$

✗ LCT

because

a_k is negative

LC #4

Example. For the following series, which test should be used to determine if the series converges or diverges? Use your selected test to show convergence or divergence.

$$\sum_{k=1}^{\infty} (-1)^k \frac{k}{k+2}$$

✗ AST

$$\lim_{k \rightarrow \infty} a_k = 1$$

LC #5, #6

✓ Divergence Test $\lim_{k \rightarrow \infty} a_k = 1 \neq 0 \rightarrow \text{Diverges}$

✗ Root $\rightarrow \rho = 1$

✗ p-series

✗ Integral Test $\rightarrow a_k$ not strictly positive

✗ Ratio $\rightarrow r = 1$

✗ Geometric series

$$\sum_{k=1}^{\infty} (-1)^k \frac{k}{k+2} \quad \checkmark \text{ Divergence Test } \lim_{k \rightarrow \infty} a_k = 1 \neq 0 \rightarrow \text{Diverges}$$

$$\sum_{k=1}^{\infty} \frac{k!}{2^k (k+2)!}$$

Ratio Test

$$r = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{\overset{(k+1) \cdot k!}{(k+1)!}}{\underset{\substack{\uparrow \\ 2^k \cdot 2}}{2^{k+1}} \cdot \underline{(k+3)!}} \cdot \frac{2^k \cdot \underline{(k+2)!}}{k!} \right|$$

LC #8

$$= \lim_{k \rightarrow \infty} \left| \frac{k+1}{k+3} \cdot \frac{1}{2} \right| = \frac{1}{2} < 1$$

Converges (absolutely) by the Ratio Test

DCT
 $a_k \geq 0, b_k \geq 0$ $\sum b_k$ converges, $\sum a_k$ converges
 $a_k \leq b_k$ $\sum a_k$ diverges, $\sum b_k$ diverges
 LCT
 $0 < L < \infty$, both converge or diverge
 $\lim \frac{a_k}{b_k} = L \xrightarrow{L=\infty} \sum a_k$ diverges if $\sum b_k$ diverges
 $\xrightarrow{L=0} \sum a_k$ converges if $\sum b_k$ converges

Comparison	$\sum_{k=1}^{\infty} a_k$	$0 < a_k \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges	$0 < b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges
Limit Comparison	$\sum_{k=1}^{\infty} a_k$	$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = c$ $0 \leq c < \infty$ $\sum_{k=1}^{\infty} b_k$ converges	$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = c$ $0 < c \leq \infty$ $\sum_{k=1}^{\infty} b_k$ diverges

$$\sum_{k=1}^{\infty} \frac{|\sin(2k)|}{1+2^k}$$

$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$ convergent Geometric series w/ $r=1/2$

LC #9

Converges by DCT

$$\frac{|\sin(2k)|}{1+2^k} \leq \frac{1}{1+2^k} \leq \frac{1}{2^k}$$

$2^k \leq 1+2^k$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\sqrt{k}-1}$$

AST

$\cos(k\pi)$
 ~~$\cos(k)$~~

① $\frac{1}{\sqrt{k+1}-1} < \frac{1}{\sqrt{k}-1}$

② $\lim_{k \rightarrow \infty} \frac{1}{\sqrt{k}-1} = 0$

Converges by AST

LC #11 & 12

$$\sum_{k=2}^{\infty} \frac{1}{k\sqrt{\ln(k)}}$$

$$f(x) = \frac{1}{x\sqrt{\ln(x)}} > 0 \text{ for } x \geq 2$$

$$f'(x) = -\left(x\sqrt{\ln(x)}\right)^{-2} \left(\sqrt{\ln(x)} - \frac{1}{2\sqrt{\ln(x)}}\right) = -\frac{\left(\sqrt{\ln(x)} - \frac{1}{2\sqrt{\ln(x)}}\right)}{x^2 \ln(x)} < 0$$

$$u = \ln(x)$$

$$du = \frac{1}{x} dx$$

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln(x)}} dx = \lim_{b \rightarrow \infty} \int_{\ln(2)}^{\ln(b)} \frac{1}{\sqrt{u}} du = \lim_{b \rightarrow \infty} 2\sqrt{u} \Big|_{\ln(2)}^{\ln(b)}$$

$$= \lim_{b \rightarrow \infty} \underbrace{2\sqrt{\ln(b)}}_{\infty} - 2\sqrt{\ln(2)} = \infty \Rightarrow \text{series diverges}$$

$$\sum_{k=1}^{\infty} (2^{1/k} - 1)^k$$

Root test, Converge

$$\rho = \lim_{k \rightarrow \infty} \left| (2^{1/k} - 1)^k \right|^{1/k} = \lim_{k \rightarrow \infty} (2^{1/k} - 1) = 1 - 1 = 0 < 1$$

Converges

$$\sum_{k=3}^{\infty} \frac{1}{k^{2/5} \ln(k)} = \sum_{k=3}^{\infty} a_k$$

DCT: $b_k = \frac{1}{k \ln(k)} \rightarrow \sum_{k=3}^{\infty} \frac{1}{k \ln(k)}$ diverges by Integral test
(details omitted)

$$k^{2/5} \ln(k) > 0 \Rightarrow a_k = \frac{1}{k^{2/5} \ln(k)} > 0$$

$$k \ln(k) > 0 \Rightarrow b_k = \frac{1}{k \ln(k)} > 0$$

$$b_k \leq a_k : k^{2/5} \leq k \rightarrow \frac{1}{k \ln(k)} \leq \frac{1}{k^{2/5} \ln(k)}$$

Since $\sum b_k$ diverges, $\sum a_k$ diverges by DCT

$$\sum_{k=1}^{\infty} \frac{8(3k)!}{(k!)^3}$$

Ratio test:

$$r = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{8(3k+3)!}{((k+1)!)^3} \cdot \frac{(k!)^3}{8(3k)!} \right| = \lim_{k \rightarrow \infty} \left| \frac{(3k+3)(3k+2)(3k+1)(3k)!}{(3k)!} \cdot \left(\frac{k!}{(k+1)k!} \right)^3 \right|$$

$$= \lim_{k \rightarrow \infty} \left| \frac{(3k+3)(3k+2)(3k+1)}{(k+1)(k+1)(k+1)} \right| = 9 > 1 \Rightarrow \text{series diverges by ratio test.}$$

$$\sum_{k=1}^{\infty} \sin\left(\frac{9}{k^{12}}\right)$$

$$a_k = \sin\left(\frac{9}{k^{12}}\right) \geq 0 \quad \text{for } k \geq 1$$

$$b_k = \frac{9}{k^{12}} \geq 0 \quad \text{for } k \geq 1$$

$$\text{LCT w/ } \sum_{k=1}^{\infty} \frac{9}{k^{12}}$$

convergent p-series w/ $p=12 > 1$

$$L = \lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \lim_{k \rightarrow \infty} \frac{\sin\left(\frac{9}{k^{12}}\right)}{9/k^{12}} \stackrel{\text{LR}}{=} \lim_{k \rightarrow \infty} \frac{\frac{-\log}{k^{13}} \cos\left(\frac{9}{k^{12}}\right)}{\frac{-\log}{k^{13}}} = \lim_{k \rightarrow \infty} \cos\left(\frac{9}{k^{12}}\right) = 1$$

Since $L=1 < \infty$, then both $\sum a_k$ and $\sum b_k$ converge by LCT

Series or Test	Form of Series	Condition for Convergence	Condition for Divergence	Comments
Geometric series	$\sum_{k=0}^{\infty} ar^k, a \neq 0$	$ r < 1$	$ r \geq 1$	If $ r < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$.
Divergence Test	$\sum_{k=1}^{\infty} a_k$	Does not apply	$\lim_{k \rightarrow \infty} a_k \neq 0$	Cannot be used to prove convergence.
Integral Test	$\sum_{k=1}^{\infty} a_k$, where $a_k = f(k)$ and f is continuous, positive, and decreasing.	$\int_1^{\infty} f(x) dx$ converges.	$\int_1^{\infty} f(x) dx$ diverges.	The value of the integral is not the value of the series.
p -series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	Useful for comparison tests.
Ratio Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k \rightarrow \infty} \left \frac{a_{k+1}}{a_k} \right < 1$	$\lim_{k \rightarrow \infty} \left \frac{a_{k+1}}{a_k} \right > 1$	Inconclusive if $\lim_{k \rightarrow \infty} \left \frac{a_{k+1}}{a_k} \right = 1$
Root Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k \rightarrow \infty} \sqrt[k]{ a_k } < 1$	$\lim_{k \rightarrow \infty} \sqrt[k]{ a_k } > 1$	Inconclusive if $\lim_{k \rightarrow \infty} \sqrt[k]{ a_k } = 1$
Comparison Test (DCT)	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$a \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges.	$b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Limit Comparison Test (LCT)	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0, b_k > 0$	$0 \leq \lim_{k \rightarrow \infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges.	$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} > 0$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Alternating Series Test (AST)	$\sum_{k=1}^{\infty} (-1)^k a_k$, where $a_k > 0$	$\lim_{k \rightarrow \infty} a_k$ and $0 < a_{k+1} \leq a_k$	$\lim_{k \rightarrow \infty} a_k \neq 0$	Remainder R_n satisfies $ R_n \leq a_{n+1}$
Absolute Convergence	$\sum_{k=1}^{\infty} a_k, a_k$ arbitrary	$\sum_{k=1}^{\infty} a_k $ converges.		Applies to arbitrary series