16.4: Triple Integrals

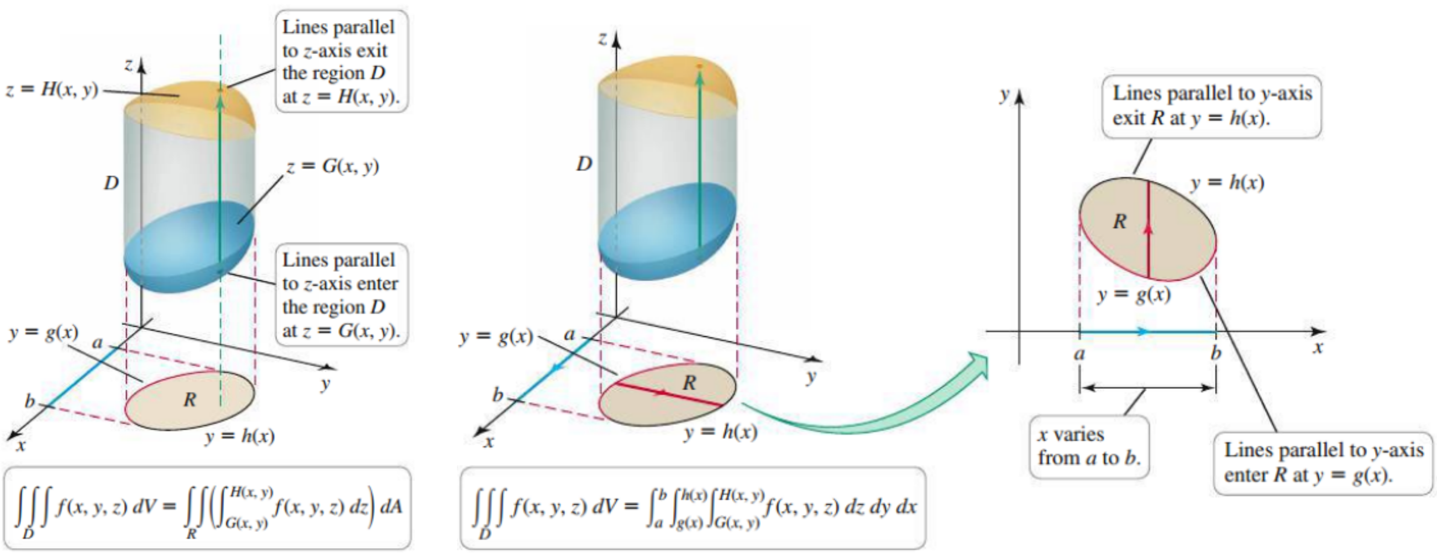
$$\int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx$$

Theorem 16.5: Triple Integrals
Let f be continuous over the region

$$D = \{(x,y,z) : a \le x \le b,\ g(x) \le y \le h(x),\ G(x,y) \le z \le H(x,y)\},$$

where g, h, G, and H are continuous functions. Then f is integrable over D and the triple integral is evaluated as the iterated integral

$$\iiint\limits_D f(x,y,z)\,dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

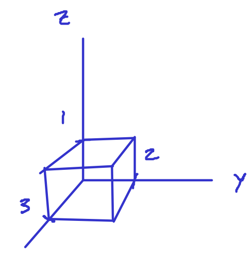


Integral	Variable	Interval
Inner	z	$G(x,y) \le z \le H(x,y)$
Middle	y	$g(x) \le y \le h(x)$
Outer	x	$a \le x \le b$

$0 \le x \le 3$ $\quad 0 \le y \le 2$ $\quad 0 \le z \le 1$

Example. A solid box D is bounded by the planes $x = 0$, $x = 3$, $y = 0$, $y = 2$, $z = 0$, and $z = 1$. The density of the box decreases linearly in the positive z-direction and is given by $f(x, y, z) = 2 - z$. Find the mass of the box.

$$\int_0^3 \int_0^2 \int_0^1 (2-z)\, dz\, dy\, dx$$



$$= \int_0^3 \int_0^2 2z - \frac{z^2}{2} \Big|_{z=0}^{z=1} dy\, dx \quad = \int_0^3 \int_0^2 \frac{3}{2}\, dy\, dx$$

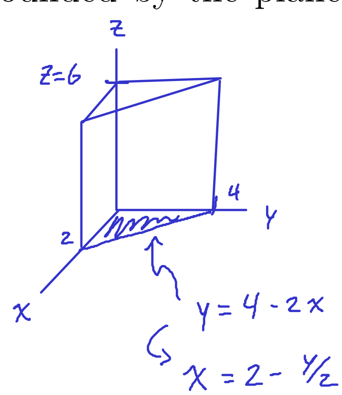
$$= \int_0^3 \frac{3}{2} y \Big|_{y=0}^{y=2} dx \quad = \int_0^3 3\, dx \quad = 3x \Big|_{x=0}^{x=3} \quad = \boxed{9}$$

$\boxed{LC\#1}$

Example. Find the volume of the prism D in the first octant bounded by the planes $y = 4 - 2x$ and $z = 6$.

$0 \le z \le 6$
$0 \le y \le 4$
$0 \le x \le 2 - y/2$

$0 \le x \le 2$
$0 \le y \le 4 - 2x$



$$\int_0^4 \int_0^{2-y/2} \int_0^6 1\, dz\, dx\, dy$$

$$= \int_0^4 \int_0^{2-y/2} z \Big|_{z=0}^{z=6} dx\, dy = 6 \int_0^4 x \Big|_{x=0}^{x=2-y/2} dy = 6 \int_0^4 \left(2 - y/2\right) dy$$

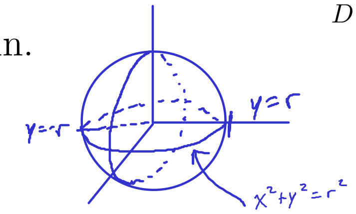
$$= 6 \left(2y - \frac{y^2}{4}\right) \Big|_{y=0}^{y=4}$$

$$= 6(8-4) = 24$$

$\boxed{LC\#2}$

Example. Write the triple integral for $\iiint\limits_D f(x,y,z)\,dV$ where D is a sphere of radius r centered at the origin.

$0 \le x^2+y^2+z^2 \le r^2$



$-r \le y \le r$

$-\sqrt{r^2-y^2} \le x \le \sqrt{r^2-y^2}$

$-\sqrt{r^2-x^2-y^2} \le z \le \sqrt{r^2-x^2-y^2}$

$z = \pm\sqrt{r^2-x^2-y^2}$

$$\int_{-r}^{r}\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}}\int_{-\sqrt{r^2-x^2-y^2}}^{\sqrt{r^2-x^2-y^2}} f(x,y,z)\,dz\,dx\,dy$$

Example. Find the volume of the solid D bounded by the paraboloids $y = x^2+3z^2+1$ and $y = 5-3x^2-z^2$.

$x^2+3z^2+1 \le y \le 5-3x^2-z^2$

$-\sqrt{1-x^2} \le z \le \sqrt{1-x^2}$

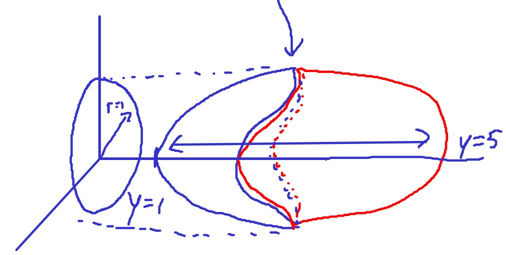
$-1 \le x \le 1$

$x^2+3z^2+1 = 5-3x^2-z^2$

$4x^2+4z^2=4$

$x^2+z^2=1$

$\hookrightarrow \quad z = \pm\sqrt{1-x^2}$



LC #3

2π

$\Rightarrow c=2$

The concept of changing the order of integration for double integrals also extends to triple integrals:

Example. Consider the integral

$$\int_0^{\sqrt[4]{\pi}} \int_0^z \int_y^z 12y^2 z^3 \sin\left(x^4\right) dx\, dy\, dz.$$

Sketch the region of integration, then evaluate the integral by changing the order of integration.

Definition. (Average Value of a Function of Three Variables)
If f is continuous on a region D of $\mathbb{R}^3$, then the **average value** of f over D is

$$\bar{f} = \frac{1}{\text{volume of } D} \iiint\limits_D f(x,y,z)\,dV.$$

Example. Find the average y-coordinate of the points in the standard simplex $D = \{(x,y,z) : x+y+z \leq 1,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$.