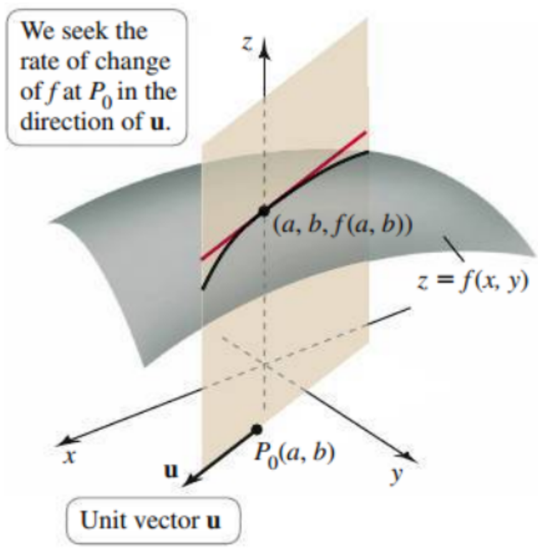
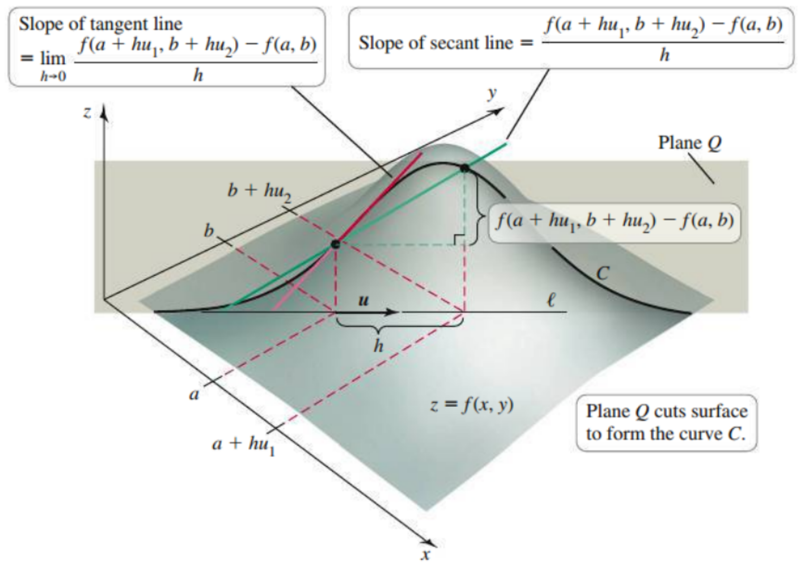
15.5: Directional Derivatives and the Gradient

Directional derivatives allow us to evaluate the rate of change of a function $f(x, y)$ along any direction (not just parallel with the x-axis and y-axis).





> **Definition. (Directional Derivative)**
> Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The **directional derivative of f at (a, b) in the direction of u** is
>
> $$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h},$$
>
> provided the limit exists.

To motivate the formula for the directional derivative, let ℓ be a line going through (a, b) in the direction of the unit vector $\mathbf{u}$. Now, let

$$x = a + su_1, \quad \text{and} \quad y = b + su_2,$$

where $-\infty < s < \infty$ and define

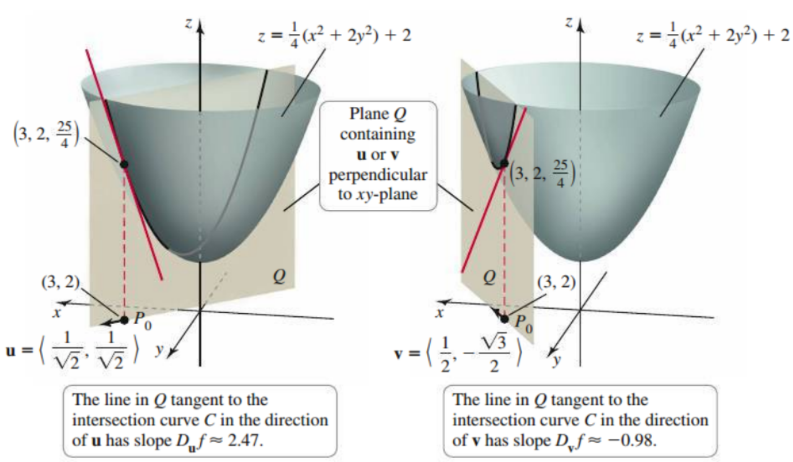
$$g(s) = f(\underbrace{a + su_1}_{x}, \underbrace{b + su_2}_{y}),$$

which evaluates f along ℓ. Thus, $g'(s)$ gives us the derivative along this line, and $g'(0)$ gives us the directional derivative of f at (a, b):

$$\begin{aligned} D_{\mathbf{u}} f(a, b) = g'(0) &= \left(\frac{\partial f}{\partial x} \underbrace{\frac{dx}{ds}}_{u_1} + \frac{\partial f}{\partial y} \underbrace{\frac{dy}{ds}}_{u_2} \right) \Bigg|_{s=0} \\ &= f_x(a, b)u_1 + f_y(a, b)u_2 \\ &= \langle f_x(a, b),\ f_y(a, b) \rangle \cdot \langle u_1, u_2 \rangle. \end{aligned}$$

Theorem 15.10: Directional Derivative
Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The **directional derivative of f at (a, b) in the direction of $\mathbf{u}$** is

$$D_{\mathbf{u}} f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle \cdot \langle u_1, u_2 \rangle.$$



Example. Compute the directional derivatives of the following functions at the given point along the given direction.

$f(x, y) = \sqrt{4 - x^2 - 2y}$; $P(2, -2)$; and $\mathbf{u} = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$,

$g(x, y) = \tan^{-1}(xy)$; $P(\pi, 1/3)$; along $\mathbf{u} = \langle 1, 1 \rangle$,

$h(x, y) = 2x^2 - xy + 3y^2$; $P(1, -3)$; along $\mathbf{u} = \langle 1, -1 \rangle$ and $\mathbf{v} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$.

The Gradient Vector:

The vector of derivatives used in the directional derivative is called the *gradient* of f.

> **Definition. (Gradient (Two Dimensions))**
> Let f be differentiable at the point (x, y). The **gradient** of f at (x, y) is the vector-valued function
> $$\nabla f(x,y) = \langle f_x(x,y),\ f_y(x,y)\rangle = f_x(x,y)\boldsymbol{i} + f_y(x,y)\boldsymbol{j}.$$

Example. For $f(x, y) = 3 - \dfrac{x^2}{10} + \dfrac{xy^2}{10}$, compute $\nabla f(3, -1)$, then compute $D_{\mathbf{u}} f(3, -1)$, where $\mathbf{u} = \left\langle \dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}} \right\rangle$.

> **Theorem 15.11: Directions of Change**
> Let f be differentiable at (a, b) with $\nabla f(a, b) \neq \mathbf{0}$.
>
> 1. f has its maximum rate of increase at (a, b) in the direction of the gradient $\nabla f(a, b)$. The rate of change in this direction is $|\nabla f(a, b)|$.
> 2. f has its maximum rate of decrease at (a, b) in the direction of $-\nabla f(a, b)$. The rate of change in this direction is $-|\nabla f(a, b)|$.
> 3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$.

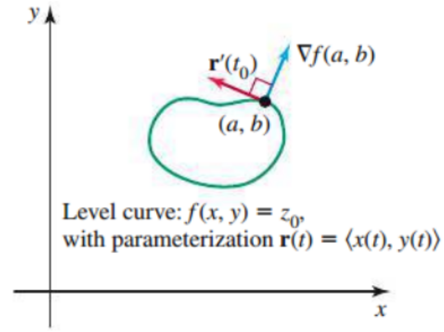
Example. For $f = 4 + x^2 + 3y^2$:

What direction is the greatest ascent at $P(2, -\frac{1}{2}, \frac{35}{4})$? What is the rate of change in this direction?

What direction is the greatest descent at $P(\frac{5}{2}, -2, \frac{89}{4})$? What is the rate of change in this direction?

What direction results in no change in function values at $P(3, 1, 16)$?

Theorem 15.12: The Gradient and Level Curves
Given a function f differentiable at (a,b), the line tangent to the level curve of f at (a,b) is orthogonal to the gradient $\nabla f(a,b)$, provided $\nabla f(a,b) \neq \mathbf{0}$.



Note: From Theorem 15.12, we get an equation for the line tangent to the curve $z = f(x,y)$ at (a,b):

$$\nabla f(a,b) \cdot \langle x-a,\, y-b \rangle = 0.$$

Example. Consider the upper sheet $z = f(x,y) = \sqrt{1+2x^2+y^2}$ of a hyperboloid of two sheets.

Verify that the gradient at $(1,1)$ is orthogonal to the corresponding level curve at that point.

Find an equation of the line tangent to the level curve at $(1,1)$.

Example. Consider $z = f(x, y) = 15 - \dfrac{x^2}{25} - \dfrac{y^2}{9}$:

Compute the slope of the tangent line at $P(5\sqrt{5}, -6, 6)$.

Verify the gradient is orthogonal to the tangent line.

Definition. (Directional Derivative and Gradient in Three Dimensions)
Let f be directional at (a, b, c) and let $\mathbf{u} = \langle u_1,\, u_2,\, u_3\rangle$ be a unit vector. The **directional derivative of f at (a, b, c) in the direction of u** is

$$D_{\mathbf{u}}(a,b,c) = \lim_{h\to 0} \frac{f(a+hu_1,\, b+hu_2,\, c+hu_3) - f(a,b,c)}{h},$$

provided this limit exists.
The **gradient** of f at this point (x, y, z) is the vector-valued function

$$\begin{aligned} \nabla f(x,y,z) &= \langle f_x(x,y,z),\, f_y(x,y,z),\, f_z(x,y,z)\rangle \\ &= f_x(x,y,z)\boldsymbol{i} + f_y(x,y,z)\boldsymbol{j} + f_z(x,y,z)\boldsymbol{k}. \end{aligned}$$

Theorem 15.13: Directional Derivative and Interpreting the Gradient
Let f be differentiable at (a, b, c) and let $\mathbf{u} = \langle u_1,\, u_2,\, u_3\rangle$ be a unit vector. The directional derivative of f at (a, b, c) in the direction of $\mathbf{u}$ is

$$\begin{aligned} D_{\mathbf{u}} f(a,b,c) &= \nabla f(a,b,c)\cdot \mathbf{u} \\ &= \langle f_x(a,b,c),\, f_y(a,b,c),\, f_z(a,b,c)\rangle \cdot \langle u_1,\, u_2,\, u_3\rangle. \end{aligned}$$

Assuming $\nabla f(a, b, c) \neq \mathbf{0}$, the gradient in three dimensions has the following properties.

1. f has its maximum rate of increase at (a, b, c) in the direction of the gradient $\nabla f(a, b, c)$ and the rate of change in this direction is $|\nabla f(a, b, c)|$.
2. f has its maximum rate of decrease at (a, b, c) in the direction of $-\nabla f(a, b, c)$ and the rate of change in this direction is $-|\nabla f(a, b, c)|$.
3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b, c)$.

Example. Consider $f(x,y) = x^2 + 2y^2 + 4z^2 - 1$ and the level surface $f(x,y,z) = 3$. Find the gradient and the corresponding rate of change at the points $P(2,0,0)$, $Q(0,\sqrt{2},0)$, $R(0,0,1)$, and $S(1,1,1/2)$ on the level surface.