11.4: Working with Taylor Series

Limits by Taylor Series

Example (LC 31.1-31.2)**.** Evaluate the following limit using its Taylor series:

$$\lim_{x\to 0} \frac{12x - 8x^3 - 6\sin(2x)}{x^5}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \ldots \qquad = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!},$$

$$L = \lim_{x\to 0} \frac{12x - 8x^3 - 6\left((2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \frac{(2x)^9}{9!} \cdots\right)}{x^5}$$

$$= \lim_{x\to 0} \frac{\boxed{12x}\boxed{-8x^3}\boxed{-12x}\boxed{+\frac{48}{6}x^3} - 6\left(\frac{32x^5}{5!} + \frac{(2x)^7}{7!} - \frac{(2x)^9}{9!} \cdots\right)}{x^5}$$

$$= \lim_{x\to 0} \; -\frac{6\cdot 2^5}{5!} + 6\left(\underbrace{\frac{128x^2}{7!}}_{0} - \underbrace{\frac{512x^4}{9!}}_{0} + \underbrace{\ldots}_{0}\right) = -\frac{6\cdot 2^5}{5!}$$

$$= -\frac{\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdot 8}{5\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}$$

$$= \boxed{\frac{-8}{5}}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \ldots \qquad = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

Example. Evaluate the following limit using its Taylor series:

$$\lim_{x\to\infty} 2x^2\left(e^{-2/x^2} - 1\right)$$

$\infty \cdot 0$

$$\lim_{x\to\infty} 2x^2\left(-1 + e^{-2/x^2}\right) = \lim_{x\to\infty} 2x^2\Big(\cancel{-1} + \cancel{1} + \underbrace{\left(\frac{-2}{x^2}\right) + \frac{\left(-2/x^2\right)^2}{2!} + \frac{\left(-2/x^2\right)^3}{3!} + \ldots}_{e^{-2/x^2}}\Big)$$

$$= \lim_{x\to\infty} -4 + \underbrace{\frac{2x^2\left(-2/x^2\right)^2}{2!}}_{\frac{2x^2(-8)}{2(x^2)^2}} + \frac{2x^2\left(-2/x^2\right)^3}{3!} + \ldots$$

$$= \lim_{x\to\infty} -4 \underbrace{- \frac{8}{x^2} - \frac{16}{6x^2} + \ldots}_{\to 0}$$

$$= -4$$

Differentiating Power Series

Example (LC 31.3-31.4)**.** The differential equation

$$y'(t) + 4y = 8; \quad y(0) = 0$$

is satisfied by the function

$$y(t) = \sum_{k=1}^{\infty} \frac{8(-4)^{k-1}t^k}{k!}$$

Find $y'(t)$ as a power series.

$$y'(t) = \sum_{k=1}^{\infty} \frac{8(-4)^{k-1} k t^{k-1}}{\underbrace{k!}_{k\cdot(k-1)!}} = \sum_{k=1}^{\infty} \frac{8(-4)^{k-1} t^{k-1}}{(k-1)!}$$

Identify the function $y(t)$ represented by this power series.

$$e^x = 1 + \underbrace{\sum_{k=1}^{\infty} \frac{x^k}{k!}}_{e^x - 1}$$

$$\sum_{k=1}^{\infty} \frac{8(-4)^{k-1} t^k}{k!} = \sum_{k=1}^{\infty} \frac{8(-4t)^k}{-4k!} = -2 \underbrace{\sum_{k=1}^{\infty} \frac{(-4t)^k}{k!}}_{e^{-4t} - 1}$$

$$= \boxed{2 - 2e^{-4t}}$$

Integrating Power Series

Example (LC 31.5-31.6)**.** Given that

$$x\cos\left(x^3\right) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{6k+1}}{(2k)!}, \text{ for } |x| < \infty$$

Evalute $\displaystyle\int_0^1 x\cos\left(x^3\right) dx$ as an infinite series

$$\int_0^1 x\cos(x^3)\,dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{6k+1}}{(2k)!}\,dx = \sum_{k=0}^{\infty} \frac{(-1)^k x^{6k+2}}{(2k)!(6k+2)}\Bigg|_0^1$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!(6k+2)}$$

Using the Alternating Series Estimation Theorem, what is the bound on $|R_3|$?

$$|R_n| \le a_{n+1} \qquad a_{n+1} = \frac{1}{(2(n+1))!\,(6(n+1)+2)} = \frac{1}{(2n+2)!\,(6n+8)}$$

$$|R_3| \le a_4 = \frac{1}{6!\,(20)} \qquad \frac{1}{8!\,24}$$

Representing Real Numbers

Example (LC 31.7)**.** Given that $\tan^{-1}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$, for $|x| \leq 1$, can we approximate $\frac{\pi}{3}$ using $x = \sqrt{3}$?

$$\frac{\pi}{3} = \tan^{-1}(\sqrt{3}) \neq \sum_{k=0}^{\infty} \frac{(-1)^k (\sqrt{3})^{2k+1}}{2k+1}$$

False because $|x| \leq 1$

$1 < 3$

$1 = \sqrt{1} < \sqrt{3}$

Example (LC 31.8)**.** Evaluate $\sum_{k=0}^{\infty} \frac{(\ln(2))^k}{k!}$.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad |x| < \infty$$

$$\sum_{k=0}^{\infty} \frac{(\ln(2))^k}{k!} = e^{\ln(2)} = 2$$

Example. Let $f(x) = \begin{cases} \frac{e^x-1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$. Using $f(x)$ and $f'(x)$, evaluate

$$\sum_{k=1}^{\infty} \frac{k\,2^{k-1}}{(k+1)!}$$

$$f(x) = \frac{e^x - 1}{x} = \frac{-1 + \sum_{k=0}^{\infty} \frac{x^k}{k!}}{x} = \frac{-1 + 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}{x} = \frac{x}{2} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots$$

$$= \sum_{k=1}^{\infty} \frac{x^k}{(k+1)!}$$

$$f'(x) = \frac{d}{dx}\left[\sum_{k=1}^{\infty} \frac{x^k}{(k+1)!}\right] = \sum_{k=1}^{\infty} \frac{k x^{k-1}}{(k+1)!}$$

$$\sum_{k=1}^{\infty} \frac{k\,2^{k-1}}{(k+1)!} = f'(2)$$

$$f'(x) = \frac{d}{dx}\left[\frac{e^x - 1}{x}\right] = \frac{x e^x - (e^x - 1)}{x^2} = \frac{e^x(x-1)+1}{x^2}$$

$$\sum_{k=1}^{\infty} \frac{k\,2^{k-1}}{(k+1)!} = f'(2) = \frac{e^2(2-1)+1}{2^2} = \boxed{\frac{e^2+1}{4}}$$

Representing Functions as Power Series

Example (LC 31.9-31.10). Consider the following Taylor series:

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{3^k}{k\,5^k}$$

What function is being represented by this power series?

$$\boxed{\ln(1+x)} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$$

What does the sum of the series equal?

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} (3/5)^k}{k} = \ln\left(1+3/5\right) = \ln\left(8/5\right)$$

Example. Identify the function represented by

$$\sum_{k=0}^{\infty} \boxed{(-1)}^k \boxed{\frac{x^{5k}}{3^k}}$$

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$$

$$\sum_{k=0}^{\infty} (-1)^k \left(\frac{x^5}{3}\right)^k = \frac{1}{1 + x^5/3}\left(\frac{3}{3}\right) = \frac{3}{3+x^5}$$