$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

$f_1 \; f_2 \; \ldots$

$\underbrace{f_{k+2} = f_{k+1} + f_k}_{\text{recurrence relation}}$

10.1: An Overview of Sequences and Infinite Series

Definition. (Sequence)
A **sequence** $\{a_n\}$ is an ordered list of numbers of the form

$$\{a_1, a_2, a_3, \ldots, a_n, \ldots\}.$$

A sequence may be generated by a **recurrence relation** of the form $a_{n+1} = f(a_n)$, for $n = 1, 2, 3, \ldots$, where a_1 is given. A sequence may also be defined with an **explicit formula** of the form $a_n = f(n)$, for $n = 1, 2, 3, \ldots$.

Example. Consider the sequence $a_n = \frac{2^{n+1}}{2^n+1}$; Compute a_1, a_2, a_3, and a_4.

$$a_1 = \frac{2^{1+1}}{2^1+1} = \frac{2^2}{2+1} = 4/3$$

$$a_2 = \frac{2^{2+1}}{2^2+1} = \frac{8}{5}$$

$$a_3 = \frac{2^{3+1}}{2^3+1} = \frac{16}{9}$$

$$a_4 = \frac{2^{4+1}}{2^4+1} = \frac{32}{17}$$

$$\lim_{n\to\infty} a_n = 2$$

$\{1, -1, 1, -1, 1, -1, \ldots\} \rightarrow$ Does not converge

$\Rightarrow$ Diverges

Definition. (Limit of a Sequence)
If the terms of a sequence $\{a_n\}$ approach a unique number L as n increases— that is, if a_n can be made arbitrarily close to L by taking n sufficiently large— then we say $\lim_{n\to\infty} a_n = L$ exists, and the sequence **converges** to L. If the terms of the sequence do not approach a single number as n increases, the sequence has no limit, and the sequence **diverges**.

Example. Determine if the sequence given by

$$a_n = \frac{3+5n^2}{n+n^2}$$

converges or diverges. If it converges, find the value that the sequence converges to.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3+5n^2}{n+n^2} = 5$$

continuous

$$f(x) = \frac{3+5x^2}{x+x^2} \longrightarrow \lim_{x\to\infty} \frac{3+5x^2}{x+x^2} = 5$$

Example. Determine if the sequence given by

$$a_n = (-1)^n \frac{3+5n^2}{n+n^2}$$

$$(-1)^n = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$$

converges or diverges. If it converges, find the value that the sequence converges to.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \frac{3+5n^2}{n+n^2} \Rightarrow \text{Diverges}$$

n	1	2	3	
a_n	$-\frac{8}{2}$	$23/6$	$-48/12$	...

$\leftarrow$ oscillating $\Rightarrow$ Don't converge

Example. A ball is thrown upward to a height of 10 meters. After each bounce, the ball rebounds to 2/3 of its previous height. Let h_n be the height after the nth bounce. Find an explicit formula for the nth term of the sequence $\{h_n\}$.

$a_0 = 10$

$a_1 = \frac{2}{3} a_0 = \frac{20}{3}$

$a_2 = \frac{2}{3} a_1 = \frac{2}{3}\left(\frac{2}{3} a_0\right) = \frac{40}{9}$

$a_3 = \frac{2}{3} a_2 = \frac{2}{3}\left(\left(\frac{2}{3}\right)^2 a_0\right) = \left(\frac{2}{3}\right)^3 a_0 = \left(\frac{2}{3}\right)^3 10$

$\vdots$

$a_n = \left(\frac{2}{3}\right)^n 10 \longrightarrow \lim_{n\to\infty} a_n = 0$

Definition. (Infinite series)
Given a sequence $\{a_1, a_2, a_3, \dots\}$, the sum of its terms

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$$

Summation from $k=1$ to infinity of a_k

is called an **infinite series**. The **sequence of partial sums** $\{S_n\}$ associated with this series has the terms

$$\begin{aligned}
S_1 &= a_1\\
S_2 &= a_1 + a_2\\
S_3 &= a_1 + a_2 + a_3\\
&\ \vdots\\
S_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k, \qquad \text{for } n = 1, 2, 3, \dots.
\end{aligned}$$

$\{S_1, S_2, S_3, \dots\dots, S_n, \dots\}$

If the sequence of partial sums $\{S_n\}$ has a limit L, the infinite series **converges** to that limit, and we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \underbrace{\sum_{k=1}^{\overset{n}{\cancel{\infty}}} a_k}_{S_n} = \lim_{n\to\infty} S_n = L.$$

If the sequence of partial sums diverges, the infinite series also **diverges**.

Example. Consider the infinite series $4 + 0.9 + 0.09 + 0.009 + \ldots$. Compute S_1, S_2, S_3, and S_4. What is the value of this series?

$S_1 = 4$

$S_2 = 4 + 0.9 = 4.9$

$S_3 = 4 + 0.9 + 0.09 = 4.99$

$S_4 = 4 + 0.9 + 0.09 + 0.009 = 4.999$

$4.\bar{9} = 5$

$\lim_{n \to \infty} S_n \quad 5$

Example. A sequence $\{a_n\}$ has partial sums given by the formula $S_n = 5 - \frac{1}{\sqrt{n}}$.

What is the value of the series $\sum_{n=1}^{\infty} a_n$?

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 5 - \frac{1}{\sqrt{n}} = 5 - 0 = 5$$

What is the formula for a_n?

$$S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$$
$$- \quad S_{n-1} = a_1 + a_2 + \cdots + a_{n-1}$$
$$S_n - S_{n-1} = a_n$$

$$S_n = 5 - \frac{1}{\sqrt{n}}$$
$$S_{n-1} = 5 - \frac{1}{\sqrt{n-1}}$$
$$a_n = S_n - S_{n-1} = \left(5 - \frac{1}{\sqrt{n}}\right) - \left(5 - \frac{1}{\sqrt{n-1}}\right)$$
$$= \frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}$$

What is the limit $\lim_{n\to\infty} a_n$?

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}} = 0 - 0 = 0$$