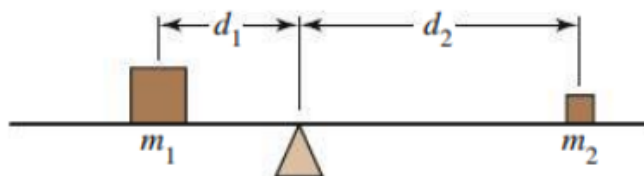
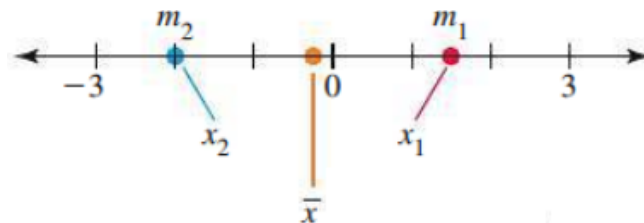


## 16.6: Integrals for Mass Calculations

Suppose we have two masses  $m_1$  and  $m_2$  on a beam (with no mass) that are distances  $d_1$  and  $d_2$  away from a pivot point. This beam will be balanced when  $m_1d_1 = m_2d_2$ .



This concept can be used to find the balance point  $\bar{x}$  between 2 objects with masses  $m_1$  and  $m_2$ :



$$m_1(x_1 - \bar{x}) = m_2(\bar{x} - x_2) \Rightarrow m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) = 0.$$

$$\Rightarrow \bar{x} =$$

Next, we can generalize this to  $n$  objects with masses  $m_1, \dots, m_n$ :

$$m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) + \cdots + m_n(x_n - \bar{x}) = \sum_{k=1}^n m_k(x_k - \bar{x}) = 0.$$

$$\Rightarrow \bar{x} =$$

**Definition. (Center of Mass in One Dimension)**

Let  $\rho$  be an integrable density function on the interval  $[a, b]$  (which represents a thin rod or wire). The **center of mass** is located at the point  $\bar{x} = \frac{M}{m}$ , where the **total moment**  $M$  and mass  $m$  are

$$M = \int_a^b x\rho(x) dx \quad \text{and} \quad m = \int_a^b \rho(x) dx.$$

**Example.** Find the mass and center of mass of the thin rods with the following density functions:

$$\rho(x) = 2 + \cos(x), \text{ for } 0 \leq x \leq \pi$$

$$\rho(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1 \\ x(2-x) & \text{if } 1 < x \leq 2 \end{cases}$$

**Definition. (Center of Mass in Two Dimensions)**

Let  $\rho$  be an integrable area density function defined over a closed bounded region  $R$  in  $\mathbb{R}^2$ . The coordinates of the center of mass of the object represented by  $R$  are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_R x\rho(x, y) dA \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_R y\rho(x, y) dA,$$

where  $m = \iint_R \rho(x, y) dA$  is the mass, and  $M_y$  and  $M_x$  are the moments with respect to the  $y$ -axis and  $x$ -axis, respectively. If  $\rho$  is constant, the center of mass is called the **centroid** and is independent of the density.

**Example.** Find the center of mass of the following plane regions with variable density:

$$R = \{(x, y) : 0 \leq x \leq 4, 0 \leq y \leq 2\}; \quad \rho(x, y) = 1 + x/2.$$

The quarter disk in the first quadrant bounded by  $x^2 + y^2 = 4$  with  $\rho(x, y) = 1 + x^2 + y^2$ .

**Definition. (Center of Mass in Two Dimensions)**

Let  $\rho$  be an integrable area density function defined over a closed bounded region  $D$  in  $\mathbb{R}^3$ . The coordinates of the center of mass of the region are

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m} \iiint_D x \rho(x, y, z) dV$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1}{m} \iiint_D y \rho(x, y, z) dV$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m} \iiint_D z \rho(x, y, z) dV$$

where  $m = \iint_D \rho(x, y, z) dA$  is the mass, and  $M_{yz}$ ,  $M_{xz}$ , and  $M_{xy}$  are the moments with respect to the coordinate planes.