$1 = (1-x)(1+x+x^2+\dots)$ $|x|<1$

11.2: Properties of Power Series

From the *geometric series*, we have

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots = \frac{1}{1-x}, \quad \text{provided } \boxed{|x| < 1.}$$

Definition. (Power Series)
A **power series** has the general form

$$\sum_{k=0}^{\infty} c_k (x-a)^k,$$

where a and c_k are real numbers, and x is a variable. The c_k's are the **coefficients** of the power series, and a is the **center** of the power series. The set of values of x for which the series converges is its **interval of convergence**. The **radius of convergence** of the power series, denoted R, is the distance from the center of the series to the boundary of the interval of convergence.

Theorem 11.3: Convergence of Power Series
A power series $\sum_{k=0}^{\infty} c_k (x-a)^k$ centered at a converges in one of three ways:

1. The series converges absolutely for all x. It follows, by Theorem 10.19, that the series converges for all x, in which the interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.
2. There is a real number $R > 0$ such that the series converges absolutely (and therefore converges) for $|x - a| < R$ and diverges for $|x - a| > R$, in which case the radius of converge is R.
3. The series converges only at $\boxed{a}$, in which case the radius of convergence is $R = 0$.

Summary: Determining the Radius and Interval of Convergence of $\sum c_k(x-a)^k$

1. Use the Ratio Test or the Root Test to find the interval $\boxed{(a-R, a+R)}$ on which the series converges absolutely; the radius of convergence for the series is R.
2. Use the *radius* of convergence to find the *interval* of convergence:
   - If $R = \infty$, the interval of convergence is $(-\infty, \infty)$.
   - If $R = 0$, the interval of convergence is the single point $x = a$.
   - If $0 < R < \infty$, the interval of convergence consists of the interval $\boxed{(a-R, a+R)}$ and possibly one or both of its endpoints. Determining whether the series $\sum c_k(x-a)^k$ converges at the endpoints $x = a - R$ and $x = a + R$ amounts to analyzing the series $\sum c_k(-R)^k$ and $\sum c_k R^k$.

Example (LC 28.1). Where is the power series $\sum_{k=1}^{\infty} c_k \overbrace{(x-3)^k}^{(x-a)^k}$ centered? $a = 3$

Could it's interval of convergence be $(-2, 8)$? Yes

$(3-R, 3+R) \stackrel{?}{=} (-2, 8)$

$3 - R = -2$, $R = 5$

$3 + R = 8$, $R = 5$

Example (LC 28.2). Where is the power series $\sum_{k=0}^{\infty} \frac{(4x-1)^k}{k^2+3}$ centered?

$a = 1/4$

$\frac{(4x-1)^k}{k^2+3} = \underbrace{\frac{4^k}{k^2+3}}_{c_k}\left(x - \frac{1}{4}\right)^k$

$a = 1/4$

Example (LC 28.3). Where is the power series $\sum_{k=1}^{\infty} c_k(x-1)^k$ centered?

$a = 1$

Could it's interval of convergence be $(-1, 1)$? No

$(1-R, 1+R) \neq (-1, 1)$

Example (LC 28.4-28.5)**.** For the following, determine the radius and interval of convergence.

$|x-a| \leq R$

Power series only converges if $|4x-8| \leq 40$.

$|4x-8| \leq 40$

$|x-2| \leq 10 \longrightarrow R=10$

$-10 \leq x-2 \leq 10$

$+2 \quad\quad +2 \quad\quad +2$

$-8 \leq x \leq 12$

$(2-10, 2+10)$

$a-R \quad\quad a+R$

Power series only converges if $|x-3| < 4$.

$R=4$

$|x-3| < 4$

$-4 < x-3 < 4$

$+3 \quad\quad +3 \quad\quad +3$

$-1 < x < 7 \longrightarrow (-1, 7)$

$$\sum c_k (x-4)^k$$

Example (LC 28.6-28.9). Consider the power series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-4)^k}{9^k\sqrt{k}}$.

Use the ratio test to compute the radius of convergence.

$$r = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\overbrace{(x-4)^{k+1}}^{(x-4)(x-4)^k}}{\underbrace{9^{k+1}}_{9\cdot 9^k}\sqrt{k+1}}\cdot\frac{9^k\sqrt{k}}{(x-4)^k}\right|$$

$$= \lim_{k\to\infty}\left|\frac{x-4}{9}\sqrt{\frac{k}{k+1}}\right| = \left|\frac{x-4}{9}\right|\underbrace{\lim_{k\to\infty}\left|\sqrt{\frac{k}{k+1}}\right|}_{1} = \underbrace{\frac{|x-4|}{9}}_{r} < 1$$

$a=4$

want

$a=4$

$|x-4| < 9$

$R=9$

What is the interval of convergence?

What about when $r=1$

Ratio test is inconclusive

$r=1$ at $x=-5$, $x=13$

$|x-4| < 9$

$-9 < x-4 < 9$

$+4 \quad +4 \quad +4$

$-5 < x < 13$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-4)^k}{9^k\sqrt{k}}$$

$x=-5$: $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(-9)^k}{9^k\sqrt{k}} = \sum_{k=1}^{\infty}\frac{-1}{k^{1/2}}$

exclude

p-series w/ $p=1/2 \le 1$ diverges

Interval of convergence

$x=13$: $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}9^k}{9^k\sqrt{k}}$

include

$\displaystyle= \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sqrt{k}}$

Converges by AST

$-5 < x \le 13$

$(-5, 13]$

1. ratio/root test
2. Find open interval
3. check end points

$a=2$

Example (LC 28.10-28.13)**.** Consider the power series $\displaystyle\sum_{k=1}^{\infty} \frac{(x-2)^k}{k^k}$.

Use the root test to compute the radius of convergence.

Root test: $\rho = \lim_{k\to\infty} \left| \frac{(x-2)^k}{k^k} \right|^{1/k} = \lim_{k\to\infty} \left| \frac{(x-2)}{k} \right| = |x-2| \lim_{k\to\infty} \left| \frac{1}{k} \right| = 0$

$\Longrightarrow R = \infty$

What is the interval of convergence?

$(-\infty, \infty)$ $\rightarrow$ Converges regardless of the value of x

Quiz 10

Theorem 11.4: Combining Power Series
Suppose the power series $\sum c_k x^k$ and $\sum d_k x^k$ converge to $f(x)$ and $g(x)$, respectively, on an interval I.

1. **Sum and difference:** The power series $\sum (c_k \pm d_k)x^k$ converges to $f(x) \pm g(x)$ on I
2. **Multiplication by a power:** Suppose m is an integer such that $k + m \geq 0$, for all terms of the power series $x^m \sum c_k x^k = \sum c_k x^{k+m}$. This series converges to $x^m f(x)$, for all $x \neq 0$ in I. When $x = 0$, the series converges to $\lim_{x\to 0} x^m f(x)$.
3. **Composition:** If $h(x) = bx^m$, where m is a positive integer and b is a nonzero real number, the power series $\sum c_k (h(x))^k$ converges to the composite function $f(h(x))$, for all x such that $h(x)$ is in I.

Example (LC 29.1)**.** Using the power series representation of

$$\rightarrow \quad f(x) = \ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k},$$

where $-1 \leq x < 1$, find the power series centered at 0 for $g(x) = x\ln(1-x^3)$.

$$g(x) = x \ln(1-x^3) = x f(x^3) = x\left(-\sum_{k=1}^{\infty} \frac{(x^3)^k}{k}\right)$$

$$= -\sum_{k=1}^{\infty} \frac{x^{3k+1}}{k}$$

Example (LC 29.2-29.3). Recall the geometric series:

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots = \frac{1}{1-x}, \quad \text{provided } |x| < 1.$$

Find the function represented by the power series $\sum_{k=0}^{\infty} \left(\sqrt{x} - 2\right)^k$.

What is the interval of convergence?

$$\sum_{k=0}^{\infty} \left(\sqrt{x} - 2\right)^k = \frac{1}{1 - (\sqrt{x} - 2)} = \frac{1}{3 - \sqrt{x}}$$

Geometric series $\Rightarrow$ $|\sqrt{x} - 2| < 1$

$$-1 < \sqrt{x} - 2 < 1$$

$$+2 \qquad +2 \qquad +2$$

$$1 < \sqrt{x} < 3$$

$$1 < x < 9$$

$$(1, 9)$$

$x = 1$

$$\sum_{k=0}^{\infty} \left(\sqrt{1} - 2\right)^k = \sum_{k=0}^{\infty} (-1)^k$$

diverges

Example. Find the function represented by the power series $\sum_{k=0}^{\infty} \left(\frac{x^2+3}{7} \right)^{k}$.

What is the interval of convergence?

Geometric Series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad |x|<1$$

$$\sum_{k=0}^{\infty} \left(\frac{x^2+3}{7} \right)^k = \frac{1}{1-\frac{x^2+3}{7}}\left(\frac{7}{7}\right) = \frac{7}{7-(x^2+3)} = \frac{7}{4-x^2}$$

$$\left| \frac{x^2+3}{7} \right| < 1$$

$$-1 < \frac{x^2+3}{7} < 1$$

$$-7 < x^2+3 < 7$$

$$-3 \qquad -3 \qquad -3$$

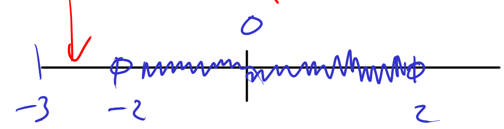
$$-10 < x^2 < 4$$

$$-2 < x < 2$$

$$0 < x^2 < 4$$

$$-2 < x < 2$$

$$(-2.5)^2 = 6.25 \not< 4$$



Theorem 11.5: Differentiating and Integrating Power Series
Suppose the power series $\sum c_k(x-a)^k$ converges for $|x-a|<R$ and defines a function f on that interval.

1. Then f is differentiable (which implies continuous) for $|x-a|<R$, and f' is found by differentiating the power series for f term by term; that is

$$f'(x) = \sum kc_k(x-a)^{k-1},$$

for $|x-a|<R$.

2. The indefinite integral of f is found by integrating the power series for f term by term; that is

$$\int f(x)\,dx = \sum c_k \frac{(x-a)^{k+1}}{k+1} + C,$$

for $|x-a|<R$, where C is an arbitrary constant.

Note: (LC 29.4) Differentiating or integrating a power series does not change the radius of convergence.

True → Endpoints need to be checked!

Example (LC 29.5). Evaluate $\int xe^{-x^3}\,dx$ by integrating the power series representation:

$$f(x) = xe^{-x^3} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k+1}}{k!}, \quad \text{for } -\infty < x < \infty.$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$xe^{-x^3} = \sum_{k=0}^{\infty} \frac{(-x^3)^k}{k!}x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k+1}}{k!}$$

$$\int xe^{-x^3}\,dx = \int \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k+1}}{k!}\,dx = C + \sum_{k=0}^{\infty} \frac{(-1)^k x^{\boxed{3k+2}}}{k!\boxed{(3k+2)}}$$

Example (LC 29.6)**.** Compute $f'(x)$ given that

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{2k+1}, \text{ for } |x| \le 1.$$

$$f'(x) = \frac{d}{dx}\left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{2k+1}\right]$$

$$= \sum_{k=0}^{\infty} \frac{(4k+2)(-1)^k x^{4k+1}}{2k+1}$$

$$= \sum_{k=0}^{\infty} 2(-1)^k x^{4k+1}$$

$x=-1$: $\sum_{k=0}^{\infty} 2(-1)^k(-1)^{4k+1} = \sum_{k=0}^{\infty} 2$ Diverges $\rightarrow$ Exclude $x=-1$

$x=1$: $\sum_{k=0}^{\infty} 2(-1)^k(1)^{4k+1} = \sum_{k=0}^{\infty} 2(-1)^k$ Diverges $\rightarrow$ Exclude $x=1$

Interval of convegence

$|x|<1$

$-1<x<1$

Example (LC 29.7). Find the power series representation of $g(x) = \dfrac{2}{(1-2x)^2}$ by using $f(x) = \dfrac{1}{1-2x}$. $= (1-2x)^{-1}$ $\qquad \dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad |x|<1$

$$f(x) = \sum_{k=0}^{\infty} (2x)^k, \quad |2x|<1 \longrightarrow |x|<\tfrac{1}{2}$$

$$f'(x) = -1(1-2x)^{-2}(-2) = \frac{2}{(1-2x)^2} = g(x)$$

$$\Rightarrow g(x) = \frac{d}{dx}\left[\sum_{k=0}^{\infty}(2x)^k\right] = \sum_{k=0}^{\infty} k(2x)^{k-1}\underbrace{\cdot 2}_{\text{chain rule}} = \sum_{k=0}^{\infty} 2k(2x)^{k-1}$$

$|x|<1/2$

$-1/2 < x < 1/2$

Example (LC 29.8-29.10). Find the power series representation of $g(x) = \ln(1-3x)$ by using $f(x) = \dfrac{1}{1-3x}$. What is the interval of convergence of this power series?

$$\int f(x)\,dx = \int \frac{1}{1-3x}\,dx = -\frac{1}{3}\int \frac{1}{u}\,du = -\frac{1}{3}\ln|u| + C = -\frac{1}{3}\ln|1-3x| + C$$

$$= -\frac{1}{3}g(x)$$

$u = 1-3x$

$du = -3\,dx$

$g(0) = \ln(1) = 0$

$$\underbrace{-\frac{1}{3}\ln(1) + C}_{=0} \Rightarrow C = 0$$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad |x|<1$$

$|3x|<1$

$\to (-1/3, 1/3)$

Check endpoints

$$\Rightarrow -3\int f(x)\,dx = g(x)$$

$$f(x) = \frac{1}{1-(3x)} = \sum_{k=0}^{\infty} (3x)^k$$

$$= -3\int \sum_{k=0}^{\infty} 3^k x^k\,dx$$

$$g(x) = -3\int f(x)\,dx = -3\int \sum_{k=0}^{\infty} (3x)^k\,dx$$

$$= -3\sum_{k=0}^{\infty} \frac{3^k x^{k+1}}{k+1} + C$$

$$= -3\int 1 + 3x + (3x)^2 + (3x)^3 + (3x)^4 + \dots\,dx$$

$$= -\sum_{k=0}^{\infty} \frac{(3x)^{k+1}}{k+1} + C$$

$$= -3\left[x + \frac{3x^2}{2} + \frac{3^2x^3}{3} + \frac{3^3x^4}{4} + \frac{3^4x^5}{5} + \dots + C\right]$$

$$= -\sum_{k=1}^{\infty} \frac{(3x)^k}{k} + C$$

$$= -\left[3x + \frac{(3x)^2}{2} + \frac{(3x)^3}{3} + \frac{(3x)^4}{4} + \frac{(3x)^5}{5} + \dots + C\right]$$

$$= -\sum_{k=1}^{\infty} \frac{(3x)^k}{k} + C$$

$g(0) = \ln(1) = 0$

$$-\sum_{k=1}^{\infty} \frac{(0)^k}{k} + C = 0 \Rightarrow C = 0$$

$$g(x) = -\sum_{k=1}^{\infty} \frac{(3x)^k}{k}$$

$$\left[-\frac{1}{3}, \frac{1}{3}\right)$$

$x = -\frac{1}{3}$: $-\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges by AST

$x = \frac{1}{3}$: $-\sum_{k=1}^{\infty} \frac{1}{k}$ diverges p-series w/ $p = 1 \leq 1$