2.7 Precise Definition of Limits

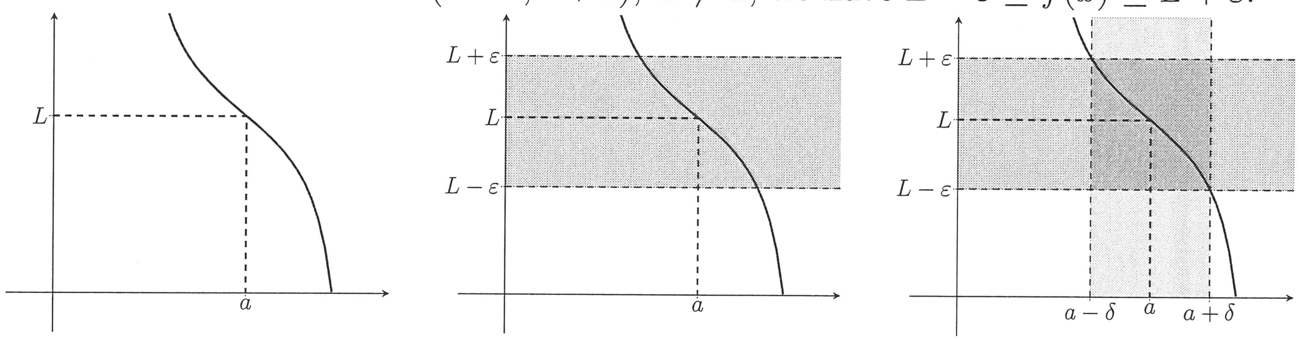
Definition. (Limit of a Function)
Assume $f(x)$ is defined for all x in some open interval containing a, except possibly at a. We say **the limit of $f(x)$ as x approaches a is L**, written

$$\lim_{x\to a} f(x) = L$$

if for *any* number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

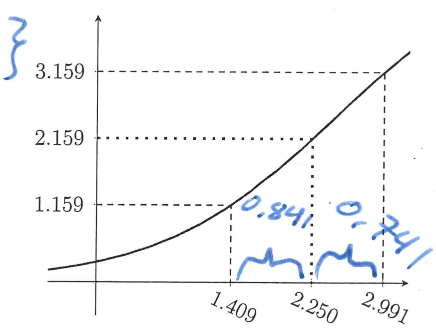
$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta$$

If we know L and $\varepsilon > 0$ is given, we can draw horizontal lines $L - \varepsilon$ and $L + \varepsilon$. Using the intersections of the graph and the horizontal lines, we can solve for $\delta > 0$ such that for values of x in the interval $(a - \delta, a + \delta)$, $x \neq a$, we have $L - \varepsilon \leq f(x) \leq L + \varepsilon$.



Note: As ε becomes smaller, δ will become smaller as well.

Example. Use the graph of f below to find a number δ such that if $0 < |x - 2.25| < \delta$ then $|f(x) - 2.159| < 1$.



Let $\delta = \min\{|1.409 - 2.250|, |2.250 - 2.991|\}$

$= \min\{0.841, 0.741\}$

$= 0.741$

Example. Use the graph of $g(x) = \sqrt{x}+1$ to help find a number δ such that if $|x-4| < \delta$ then $\left|\left(\sqrt{x}+1\right)-3\right| < \dfrac{1}{2}$. $\varepsilon = 1/2$

$-\frac{1}{2} < (\sqrt{x}+1) - 3 < \frac{1}{2}$

$-1/2 < \sqrt{x} - 2 < \frac{1}{2}$

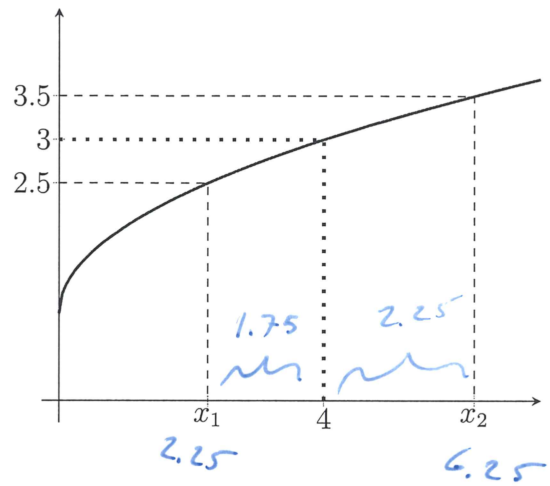
$\frac{3}{2} < \sqrt{x} < \frac{5}{2}$

$2.25 = \frac{9}{4} < x < \frac{25}{4} = 6.25$

$\Rightarrow \delta = \min\{|2.25-4|, |6.25-4|\}$

$= \min\{1.75, 2.25\}$

$= 1.75$

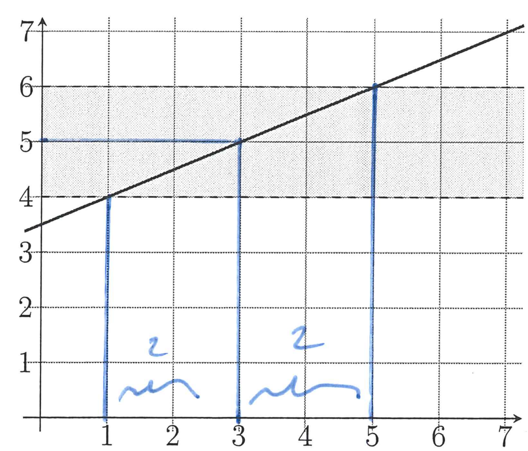


Example. Use the graph of the following linear function where $\lim_{x\to 3} h(x) = 5$ to find $\delta > 0$ such that $|h(x) - 5| < 1$ whenever $0 < |x-3| < \delta$. $\varepsilon = 1$

$\delta = 2$

Note! slope $m = 1/2$

$\delta = \dfrac{\varepsilon}{|m|}$



Steps for proving that $\lim_{x \to a} f(x) = L$

1. **Find δ.** Let ε be an arbitrary positive number. Use the inequality $|f(x) - L| < \varepsilon$ to find a condition of the form $|x - a| < \delta$, where δ depends only on the value of ε.
2. **Write a proof.** For any $\varepsilon > 0$, assume $0 < |x - a| < \delta$ and use the relationship between ε and δ found in Step 1 to prove that $|f(x) - L| < \varepsilon$.

Example. Use the $\varepsilon - \delta$ definition of a limit to prove $\lim_{x \to 4}(2x - 5) = 3$.

① Find δ:

Want $|f(x) - L| < \varepsilon$

$|(2x-5) - 3| < \varepsilon$

$|2x - 8| < \varepsilon$

$2|x-4| < \varepsilon$

$|x-4| < \varepsilon/2$

$\rightarrow \delta = \varepsilon/2$

② Let $\varepsilon > 0$ be given and let $\delta = \frac{\varepsilon}{2}$, then when $|x-4| < \delta$ we have

$$|(2x-5)-3| = |2x-8| = 2|x-4| < 2\delta = 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.$$

Example. Use the $\varepsilon - \delta$ definition of a limit to prove $\lim_{x \to 2} \frac{x}{5} = \frac{2}{5}$.

① Find δ

Want $|f(x) - L| < \varepsilon$

$\left|\frac{x}{5} - \frac{2}{5}\right| < \varepsilon$

$\frac{1}{5}|x-2| < \varepsilon$

$|x-2| < 5\varepsilon$

$\rightarrow \delta = 5\varepsilon$

② Let $\varepsilon > 0$ be given and let $\delta = 5\varepsilon$, then when $|x-2| < \delta$, we have

$$\left|\left(\frac{x}{5}\right) - \frac{2}{5}\right| = \frac{1}{5}|x-2| < \frac{1}{5}\delta = \frac{1}{5}(5\varepsilon) = \varepsilon$$

Example. Use the $\varepsilon - \delta$ definition of a limit to prove $\lim_{x \to 2} \frac{x^2+x-6}{x-2} = 5$.

① Find δ

Want $|f(x) - L| < \varepsilon$

$$\left| \frac{x^2+x-6}{x-2} - 5 \right| < \varepsilon$$

$$\left| \frac{(x+3)(x-2)}{x-2} - 5 \right| < \varepsilon$$

$$| x+3 - 5 | < \varepsilon$$

$$|x-2| < \varepsilon$$

$\rightarrow \delta = \varepsilon$

② Let $\varepsilon > 0$ be given and let $\delta = \varepsilon$, then when $|x-2| < \delta$, we have

$$\left| \frac{x^2+x-6}{x-2} - 5 \right| = \left| \frac{(x+3)(x-2)}{x-2} - 5 \right|$$
$$= |x+3-5|$$
$$= |x-2|$$
$$< \delta$$
$$= \varepsilon$$

Example. Use the $\varepsilon - \delta$ definition of a limit to prove $\lim_{x \to 3} \frac{x^2+2x-15}{2x-6} = 4$.

① Find δ

Want $|f(x) - L| < \varepsilon$

$$\left| \frac{x^2+2x-15}{2x-6} - 4 \right| < \varepsilon$$

$$\left| \frac{(x+5)(x-3)}{2(x-3)} - 4 \right| < \varepsilon$$

$$\left| \frac{x+5}{2} - 4 \right| < \varepsilon$$

$$\left| \frac{x}{2} - \frac{3}{2} \right| < \varepsilon$$

$$\frac{1}{2} |x-3| < \varepsilon$$

$$|x-3| < 2\varepsilon$$

$\rightarrow \delta = 2\varepsilon$

② Let $\varepsilon > 0$ be given and let $\delta = 2\varepsilon$, then when $|x-3| < \delta$, we have

$$\left| \frac{x^2+2x-15}{2x-6} - 4 \right| = \left| \frac{(x+5)(x-2)}{2(x-2)} - 4 \right|$$
$$= \left| \frac{x+5}{2} - 4 \right|$$
$$= \frac{1}{2} |x-3|$$
$$< \frac{1}{2} \delta$$
$$= \frac{1}{2}(2\varepsilon) = \varepsilon$$