Math 2060 Class notes
Spring 2021

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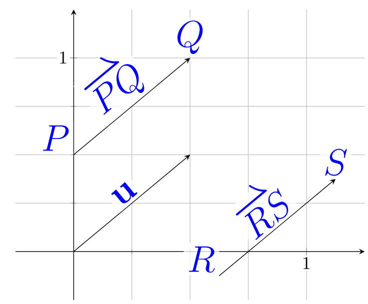
13.1: Vectors and the Geometry of Space

Definition.

- **Vectors**
  - Have a direction and magnitude,
  - vector $\overrightarrow{PQ}$ has a *tail* at P and a *head* at Q,
  - Can be denoted as $\mathbf{u}$ or $\vec{u}$,
  - Equal vectors have the same direction and magnitude (not necessarily the same position)



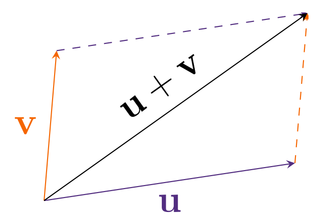
- **Scalars** are quantities with magnitude but no direction (e.g. mass, temperature, price, time, etc.)
- **Zero vector**, denoted $\mathbf{0}$ or $\vec{0}$, has length 0 and no direction

Scalar-vector multiplication:

- Denoted $c\mathbf{v}$ or $c\vec{v}$,
- length of vector multiplied by $|c|$,
- $c\mathbf{v}$ has the same direction as $\mathbf{v}$ if $c > 0$, and has the opposite direction as $\mathbf{v}$ if $c < 0$, (what if $c = 0$?)
- $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if $\mathbf{u} = c\mathbf{v}$. (what vectors are parallel to $\mathbf{0}$?)

Vector Addition and Subtraction:

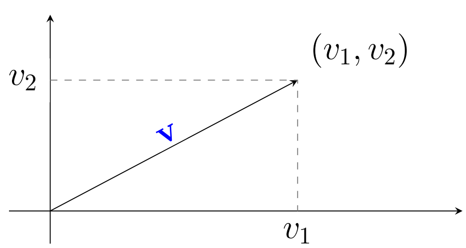
Given two vectors $\mathbf{u}$ and $\mathbf{v}$, their sum, $\mathbf{u}+\mathbf{v}$, can be represented by the parallelogram (triangle) rule: place the tail of $\mathbf{v}$ at the head of $\mathbf{u}$



The difference, denoted $\mathbf{u}-\mathbf{v}$, is the sum of $\mathbf{u}+(-\mathbf{v})$:

Vector Components:

A vector $\mathbf{v}$ whose tail is at the origin $(0,0)$ and head is at (v_1, v_2) is a **position vector** (in **standard position**) and is denoted $\langle v_1, v_2\rangle$. The real numbers v_1 and v_2 are the x- and y-components of $\mathbf{v}$.



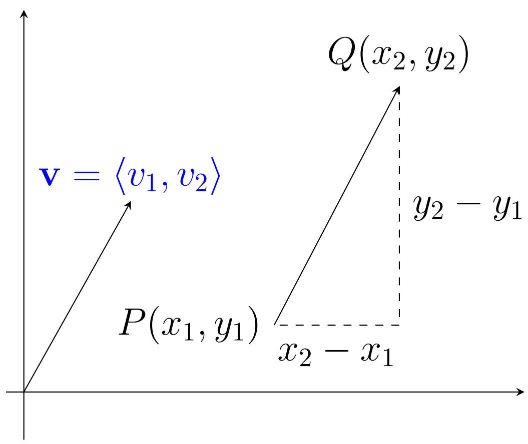
Vectors $\mathbf{u}=\langle u_1, u_2\rangle$ and $\mathbf{v}=\langle v_1, v_2\rangle$ are equal if and only if $u_1 = v_1$ and $u_2 = v_2$.

Magnitude:

Given points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the **magnitude**, or **length**, of vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$, denoted $\left|\overrightarrow{PQ}\right|$, is the distance between points P and Q.

$$\left|\overrightarrow{PQ}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The magnitude of position vector $\mathbf{v} = \langle v_1, v_2 \rangle$ is $|\mathbf{v}|$.
(How do $\left|\overrightarrow{PQ}\right|$ and $\left|\overrightarrow{QP}\right|$ relate to each other?)



Note: The norm, denoted $\|\mathbf{u}\|$ or $\|\mathbf{u}\|_2$, is equivalent to the magnitude of a vector.

Equation of a Circle:

Definition.
A **circle** centered at (a, b) with radius r is the set of points satisfying the equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

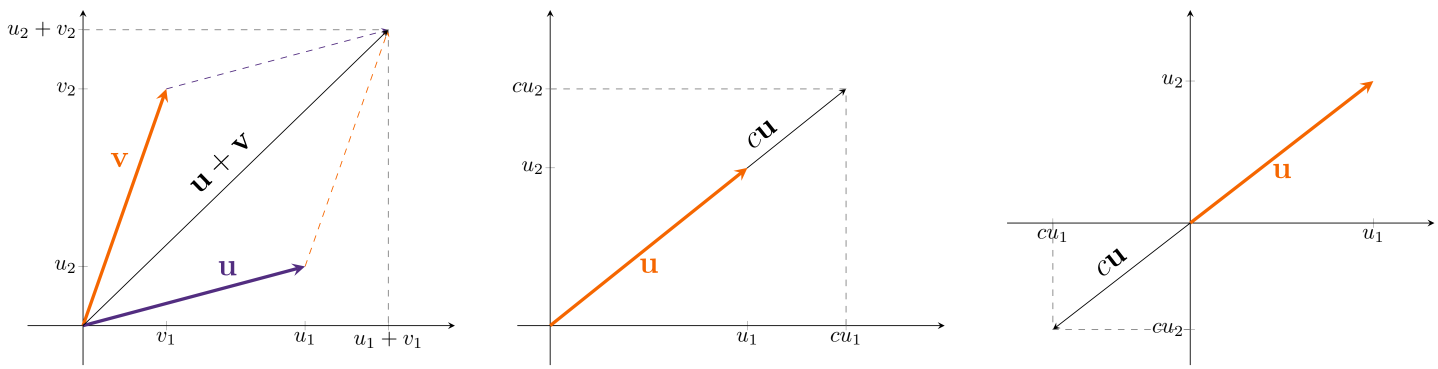
A **disk** centered at (a, b) with radius r is the set of points satisfying the inequality

$$(x - a)^2 + (y - b)^2 \leq r^2.$$

Vector Operations in Terms of Components

Definition. (Vector Operations in $\mathbb{R}^2$)
Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2 \rangle$, and $\mathbf{v} = \langle v_1, v_2 \rangle$.

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \langle u_1 + v_1, u_2 + v_2 \rangle && \text{Vector addition} \\ \mathbf{u} - \mathbf{v} &= \langle u_1 - v_1, u_2 - v_2 \rangle && \text{Vector subtraction} \\ c\mathbf{u} &= \langle cu_1, cu_2 \rangle && \text{Scalar multiplication} \end{aligned}$$



Example. Let $\mathbf{u} = \langle 1, 2 \rangle$, $\mathbf{v} = \langle -2, 3 \rangle$, $c = 2$, and $d = 3$. Find the following:

$\mathbf{u} + \mathbf{v}$ $\qquad$ $c\mathbf{u}$

$c\mathbf{u} + d\mathbf{v}$ $\qquad$ $\mathbf{u} - c\mathbf{v}$

Definition.
A **unit vector** is any vector with length 1.
In $\mathbb{R}^2$, the **coordinate unit vectors** are $\boldsymbol{i} = \langle 1, 0 \rangle$ and $\boldsymbol{j} = \langle 0, 1 \rangle$.

Example. Let $\mathbf{u} = \langle -7, 3 \rangle$. Find two unit vectors parallel to $\mathbf{u}$. Find another vector parallel to $\mathbf{u}$ with a magnitude of 2.

Properties of Vector Operations:

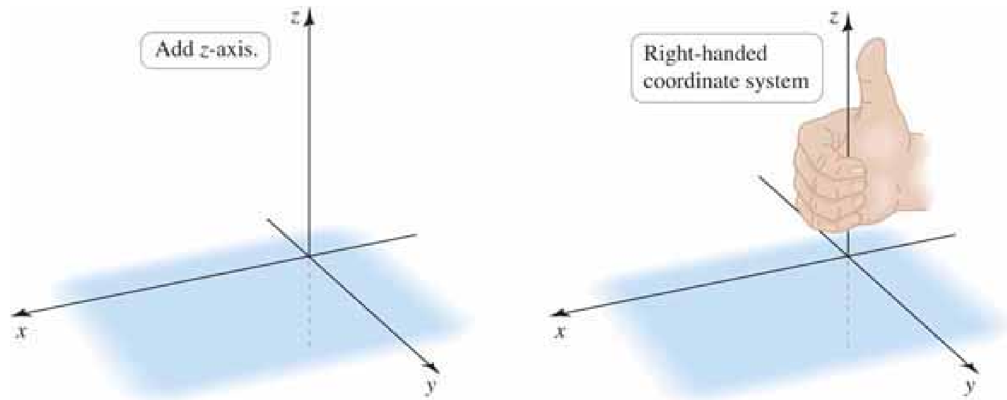
Suppose $\mathbf{u}$, $\mathbf{v}$, and $\boldsymbol{w}$ are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ — Commutative property of addition
2. $(\mathbf{u}+\mathbf{v})+\boldsymbol{w}=\mathbf{u}+(\mathbf{v}+\boldsymbol{w})$ — Associative property of addition
3. $\mathbf{v}+\mathbf{0}=\mathbf{v}$ — Additive identity
4. $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$ — Additive inverse
5. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$ — Distributive property 1
6. $(a+c)\mathbf{v}=a\mathbf{v}+c\mathbf{v}$ — Distributive property 2
7. $0\mathbf{v}=\mathbf{0}$ — Multiplication by zero scalar
8. $c\mathbf{0}=\mathbf{0}$ — Multiplication by zero vector
9. $1\mathbf{v}=\mathbf{v}$ — Multiplicative identity
10. $a(c\mathbf{v})=(ac)\mathbf{v}$ — Associative property of scalar multiplication

13.2: Vectors in Three Dimensions

The xyz- Coordinate System:

The three-dimensional coordinate system is created by adding the z-axis, which is perpendicular to both the x-axis and the y-axis. When looking at the xy-plane, the positive direction of the z-axis protrudes towards the viewer. This can also be shown using the right-hand rule (Figure 13.25 from Briggs):

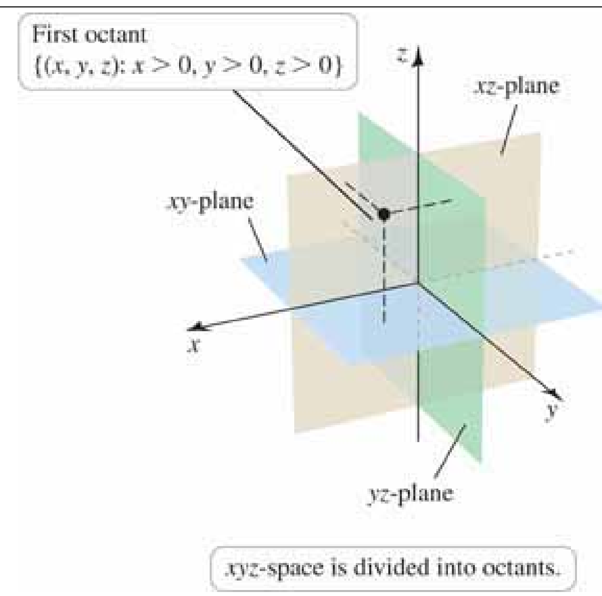


Definition.

This three-dimensional coordinate system is broken up into eight **octants**, which are separated by

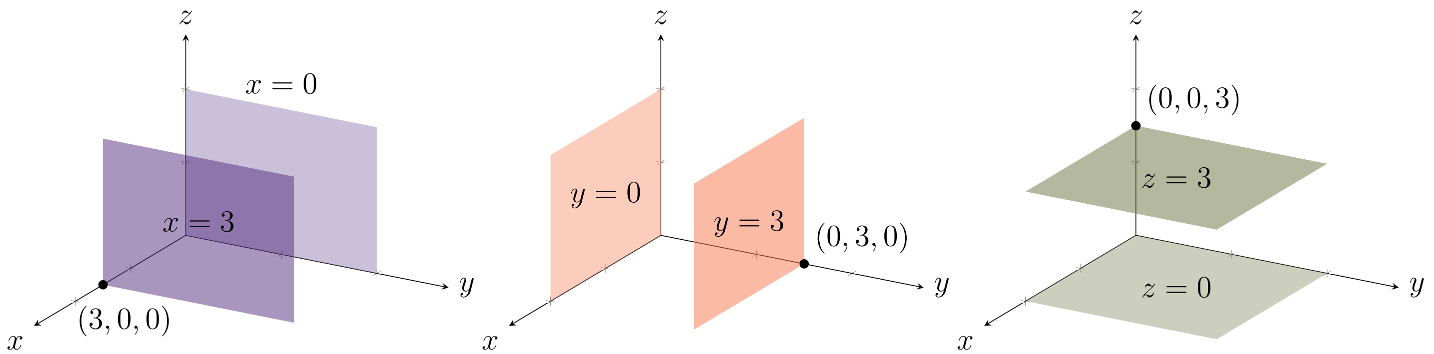
- the xy-**plane** ($z = 0$),
- the xz-**plane** ($y = 0$), and
- the yz-**plane** ($x = 0$).

The **origin** is the location where all three axes intersect.

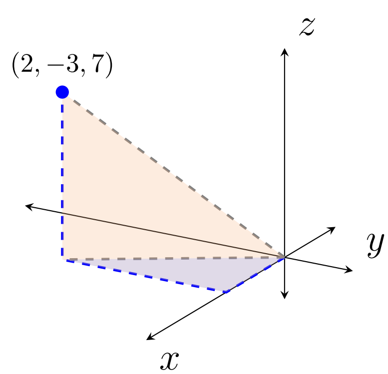


Equations of Simple Planes:

Planes in three-dimensions are analogous to lines in two-dimensions. Below, we see the yz-plane, the xz-plane, and the xy-plane, along with planes that are parallel where x, y, and z are fixed respectively:



Example (Parallel planes)**.** Determine the equation of the plane parallel to the xz-plane passing through the point $(2, -3, 7)$.

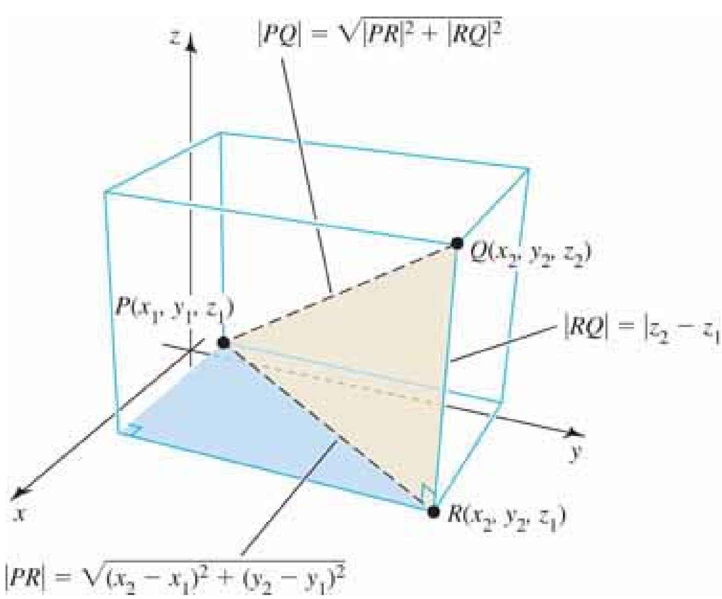


Distances in xyz-Space:

Recall that in $\mathbb{R}^2$, for some vector $\overrightarrow{PR}$, the distance formula is given by

$$|PR| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

where (x_1, y_1) and (x_2, y_2) represent the points P and R respectively. This idea can be further extended into $\mathbb{R}^3$ by considering the two sides of the triangle formed by the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$:



Distance Formula in xyz-Space

The **distance** between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The **midpoint** between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is found by averaging the x-, y-, and z-coordinates:

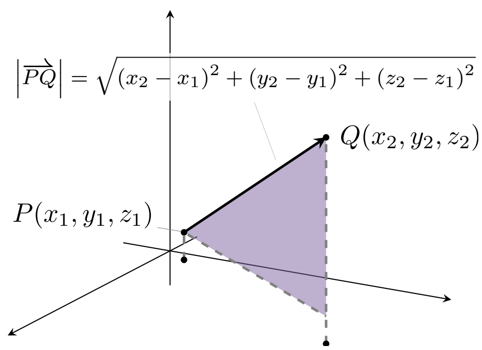
$$\text{Midpoint } = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Magnitude and Unit Vectors:

Definition.
The **magnitude** (or **length**) of the vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ is the distance from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$:

$$\left|\overrightarrow{PQ}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

In $\mathbb{R}^3$, the **coordinate unit vectors** are $\boldsymbol{i} = \langle 1, 0, 0 \rangle$, $\boldsymbol{j} = \langle 0, 1, 0 \rangle$, and $\boldsymbol{k} = \langle 0, 0, 1 \rangle$.



Example. Consider $P(-1, 4, 3)$ and $Q(3, 5, 7)$. Find

- $\left|\overrightarrow{PQ}\right|$

- The midpoint between P and Q

- Two unit vectors parallel to $\overrightarrow{PQ}$

Equation of a Sphere:

Definition.
A **sphere** centered at (a, b, c) with radius r is the set of points satisfying the equation

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

A **ball** centered at (a, b, c) with radius r is the set of points satisfying the inequality

$$(x-a)^2 + (y-b)^2 + (z-c)^2 \leq r^2.$$

Example. Consider $P(-1, 4, 3)$ and $Q(3, 5, 7)$. Find the equation of the sphere centered at the midpoint passing through P and Q

Example. What is the geometry of the intersection between $x^2 + y^2 + z^2 = 50$ and $z = 1$?

Example. Rewrite the following equation into the standard form of a sphere:

$$x^2 + y^2 + z^2 - 2x + 6y - 8z = -1$$

Vector Operations in Terms of Components

Definition. (Vector Operations in $\mathbb{R}^3$)
Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle \qquad \text{Vector addition}$$
$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle \qquad \text{Vector subtraction}$$
$$c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle \qquad \text{Scalar multiplication}$$

Properties of Vector Operations:
Suppose $\mathbf{u}$, $\mathbf{v}$, and $\boldsymbol{w}$ are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

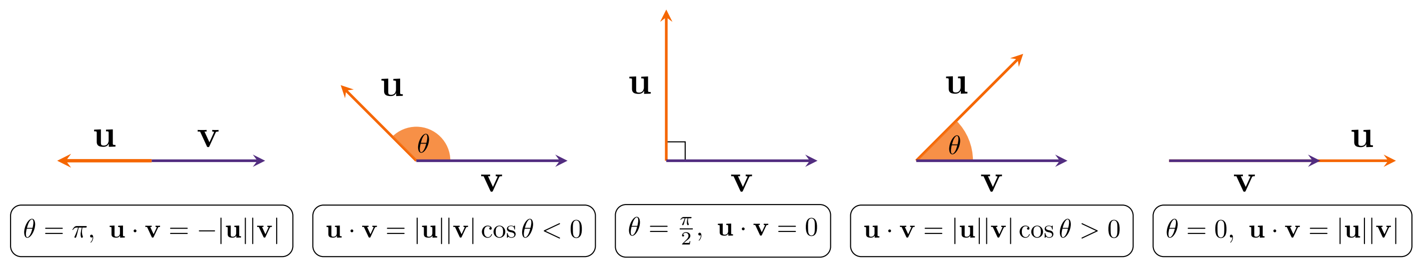
1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ — Commutative property of addition
2. $(\mathbf{u} + \mathbf{v}) + \boldsymbol{w} = \mathbf{u} + (\mathbf{v} + \boldsymbol{w})$ — Associative property of addition
3. $\mathbf{v} + \mathbf{0} = \mathbf{v}$ — Additive identity
4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ — Additive inverse
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ — Distributive property 1
6. $(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$ — Distributive property 2
7. $0\mathbf{v} = \mathbf{0}$ — Multiplication by zero scalar
8. $c\mathbf{0} = \mathbf{0}$ — Multiplication by zero vector
9. $1\mathbf{v} = \mathbf{v}$ — Multiplicative identity
10. $a(c\mathbf{v}) = (ac)\mathbf{v}$ — Associative property of scalar multiplication

13.3: Dot Products

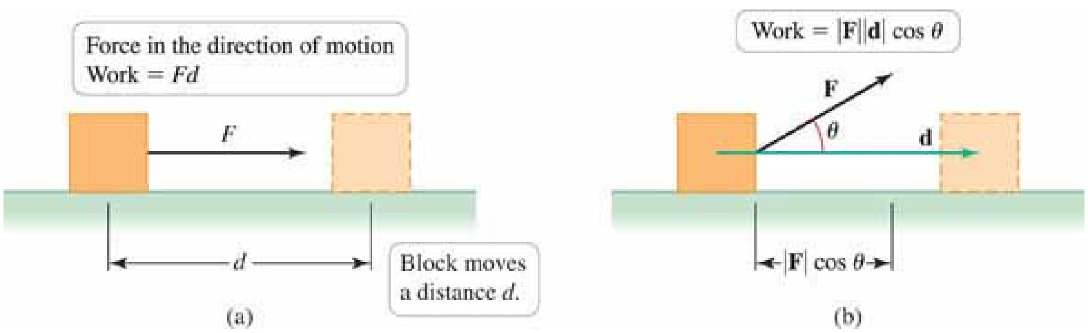
Definition. (Dot Product)
Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in two or three dimensions, their **dot product** is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ with $0 \leq \theta \leq \pi$. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$, and θ is undefined.



A physical example of the dot product is the amount of work done when a force is applied at an angle θ as shown in figure 13.43:



Note: The result of the dot product is a scalar!

Definition. (Orthogonal Vectors)
Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. The zero vector is orthogonal to all vectors. In two or three dimensions, two nonzero orthogonal vectors are perpendicular to each other.

- $\mathbf{u}$ and $\mathbf{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \cdot \mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$.
- $\mathbf{u}$ and $\mathbf{v}$ are perpendicular ($\theta = \frac{\pi}{2}$) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Example. Given $|\mathbf{u}| = 2$ and $|\mathbf{v}| = \sqrt{3}$, compute $\mathbf{u} \cdot \mathbf{v}$ when

- $\theta = \frac{\pi}{4}$
- $\theta = \frac{\pi}{3}$
- $\theta = \frac{5\pi}{6}$

Theorem 31.1: Dot Product
Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$,

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Example. Given vectors $\mathbf{u} = \left\langle \sqrt{3}, 1, 0 \right\rangle$ and $\mathbf{v} = \left\langle 1, \sqrt{3}, 0 \right\rangle$, compute $\mathbf{u} \cdot \mathbf{v}$ and find θ.

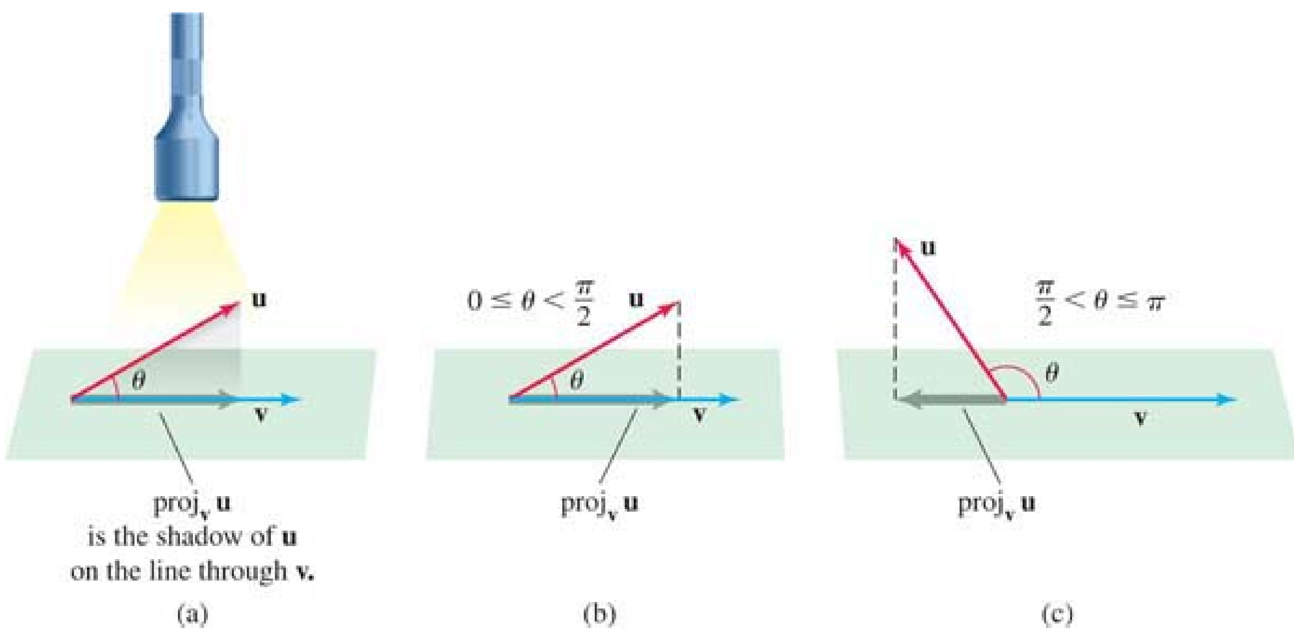
Properties of Dot Products

Theorem 13.2: Properties of the Dot Product
Suppose $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ — Commutative property
2. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$ — Associative property
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ — Distributive property

Orthogonal Projections

Given vectors $\mathbf{u}$ and $\mathbf{v}$, the projection of $\mathbf{u}$ onto $\mathbf{v}$ produces a vector parallel to $\mathbf{v}$ using the "shadow" of $\mathbf{u}$ cast onto $\mathbf{v}$.



Definition. ((Orthogonal) Projection of u onto v)
The **orthogonal projection of u onto v**, denoted $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$, where $\mathbf{v} \neq \mathbf{0}$, is

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \underbrace{|\mathbf{u}| \cos\theta}_{\text{length}} \underbrace{\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)}_{\text{direction}}.$$

The orthogonal projection may also be computed with the formulas

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \operatorname{scal}_{\mathbf{v}} \mathbf{u} \left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v},$$

where the **scalar component of u in the direction of v** is

$$\operatorname{scal}_{\mathbf{v}} \mathbf{u} = |\mathbf{u}| \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}.$$

Example. Find $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$ and $\operatorname{scal}_{\mathbf{v}} \mathbf{u}$ for the following:

- $\mathbf{u} = \langle 1, 1 \rangle$, $\mathbf{v} = \langle -2, 1 \rangle$

- $\mathbf{u} = \langle 7, 1, 7 \rangle$, $\mathbf{v} = \langle 5, 7, 0 \rangle$

Applications of Dot Products

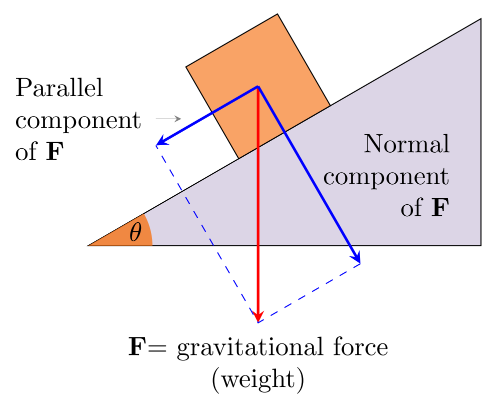
Definition. (Work)
Let a constant force $\mathbf{F}$ be applied to an object, producing a displacement $\mathbf{d}$. If the angle between $\mathbf{F}$ and $\mathbf{d}$ is θ, then the **work** done by the force is

$$W = |\mathbf{F}||\mathbf{d}|\cos\theta = \mathbf{F}\cdot\mathbf{d}$$

Example. A force $\mathbf{F} = \langle 3, 3, 2\rangle$ (in newtons) moves an object along a line segment from $P(1, 1, 0)$ to $Q(6, 6, 0)$ (in meters). What is the work done by the force?

Components of a Force:

Example. A 10-lb block rests on a plane that is inclined at 30° above the horizontal. Find the components of the gravitational force parallel to and normal (perpendicular) to the plane.



13.4: Cross Products

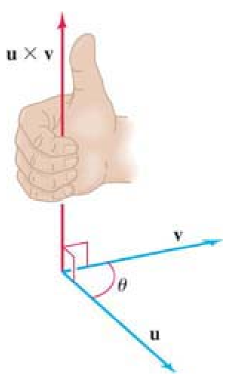
Definition. (Cross Product)
Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the **cross product** $\mathbf{u} \times \mathbf{v}$ is a vector with magnitude

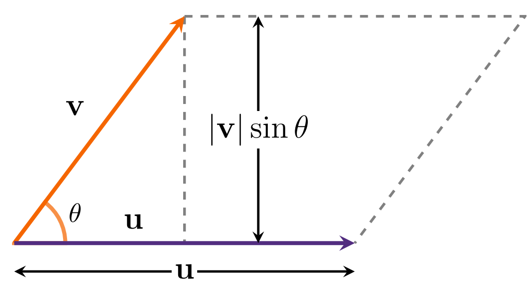
$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta,$$

where $0 \leq \theta \leq \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
The direction of $\mathbf{u} \times \mathbf{v}$ is given by the **right-hand rule**:

> When you put your the vectors tail to tail and let the fingers of your right hand curl from $\mathbf{u}$ to $\mathbf{v}$, the direction of $\mathbf{u} \times \mathbf{v}$ is the direction of your thumb, orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ (Figure 13.56).

When $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, the direction of $\mathbf{u} \times \mathbf{v}$ is undefined.





Theorem 13.3: Geometry of the Cross Product
Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in $\mathbb{R}^3$.

1. The vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
2. If $\mathbf{u}$ and $\mathbf{v}$ are two sides of a parallelogram, then the area of the parallelogram is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$$

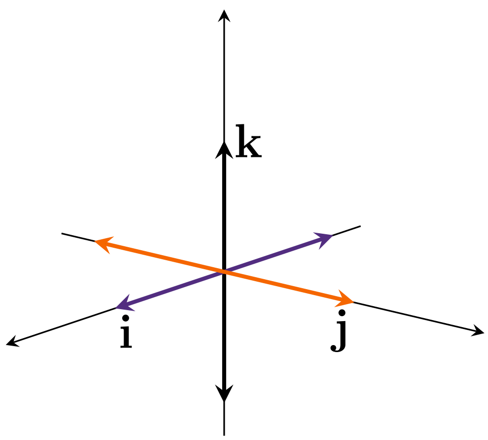
Example. Consider the vectors $\mathbf{u} = \langle 2, 0, 0 \rangle$ and $\mathbf{v} = \left\langle \sqrt{3}, 3, 0 \right\rangle$. The angle between these vectors is $\theta = \frac{\pi}{3}$. Find the area of the parallelogram formed by these vectors.

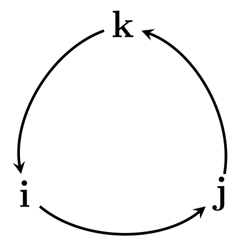
Theorem 13.4: Properties of the Cross Product Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^3$, and let a and b be scalars.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ Anticommutative property
2. $(a\mathbf{u}) \times (b\mathbf{v}) = ab(\mathbf{u} \times \mathbf{v})$ Associative property
3. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ Distributive property
4. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$ Distributive property

Theorem 13.5: Cross Products of Coordinate Unit Vectors

$$\boldsymbol{i} \times \boldsymbol{j} = -(\boldsymbol{j} \times \boldsymbol{i}) = \boldsymbol{k} \qquad \boldsymbol{j} \times \boldsymbol{k} = -(\boldsymbol{k} \times \boldsymbol{j}) = \boldsymbol{i}$$
$$\boldsymbol{k} \times \boldsymbol{i} = -(\boldsymbol{i} \times \boldsymbol{k}) = \boldsymbol{j} \qquad \boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = \boldsymbol{0}$$





$\mathbf{i} \times \mathbf{j} = \mathbf{k}$
$\mathbf{j} \times \mathbf{k} = \mathbf{i}$
$\mathbf{k} \times \mathbf{i} = \mathbf{j}$

Using the unit vectors, we can compute $\mathbf{u} \times \mathbf{v}$:

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}) \times (v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}) \\
&= u_1v_1 \underbrace{(\boldsymbol{i} \times \boldsymbol{i})}_{\boldsymbol{0}} + u_1v_2 \underbrace{(\boldsymbol{i} \times \boldsymbol{j})}_{\boldsymbol{k}} + u_1v_3 \underbrace{(\boldsymbol{i} \times \boldsymbol{k})}_{-\boldsymbol{j}} \\
&+ u_2v_1 \underbrace{(\boldsymbol{j} \times \boldsymbol{i})}_{-\boldsymbol{k}} + u_2v_2 \underbrace{(\boldsymbol{j} \times \boldsymbol{j})}_{\boldsymbol{0}} + u_2v_3 \underbrace{(\boldsymbol{j} \times \boldsymbol{k})}_{\boldsymbol{i}} \\
&+ u_3v_1 \underbrace{(\boldsymbol{k} \times \boldsymbol{i})}_{\boldsymbol{j}} + u_3v_2 \underbrace{(\boldsymbol{k} \times \boldsymbol{j})}_{-\boldsymbol{i}} + u_3v_3 \underbrace{(\boldsymbol{k} \times \boldsymbol{k})}_{\boldsymbol{0}} \\
&= (u_2v_3 - u_3v_2)\boldsymbol{i} - (u_1v_3 - u_3v_1)\boldsymbol{j} + (u_1v_2 - u_2v_1)\boldsymbol{k}
\end{aligned}$$

Theorem 13.6: Evaluating the Cross Product
Let $\mathbf{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$ and $\mathbf{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$. Then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \boldsymbol{k}$$

Note:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\boldsymbol{i} - (u_1v_3 - u_3v_1)\boldsymbol{j} + (u_1v_2 - u_2v_1)\boldsymbol{k}$$

Alternative approach:

$$\left|\begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ u_1 & u_2 & u_3 & u_1 & u_2 \\ v_1 & v_2 & v_3 & v_1 & v_2 \end{array}\right.$$

Example. Compute $\mathbf{u} \times \mathbf{v}$ for $\mathbf{u} = \langle 3, 5, 4 \rangle$ and $\mathbf{v} = \langle 1, -1, 9 \rangle$.

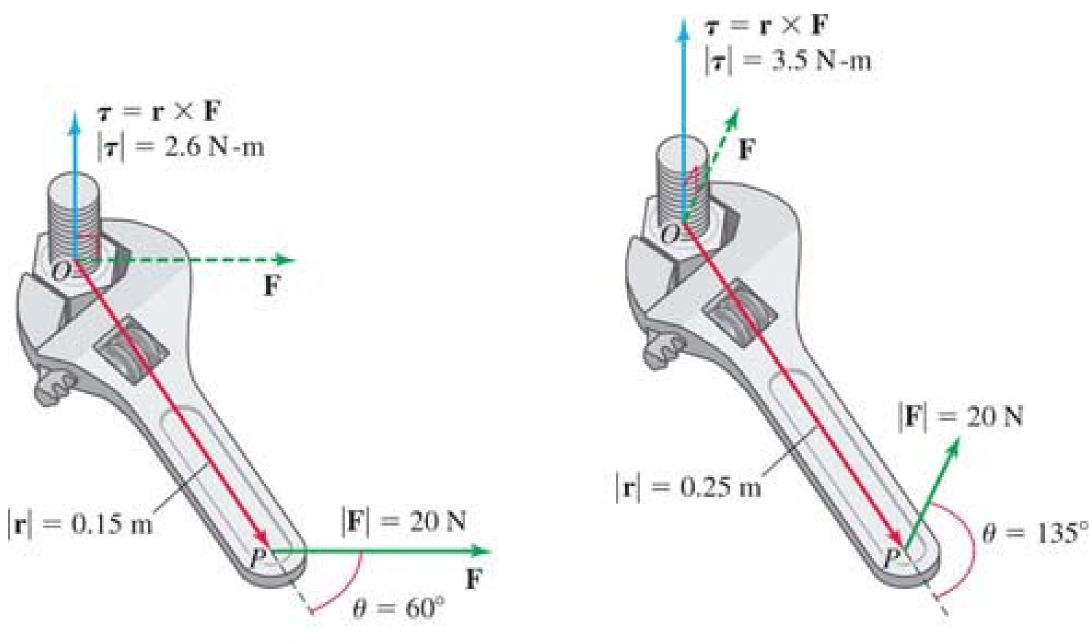
Example. Consider the vectors $\mathbf{u} = \left\langle \sqrt{3}, 1, 0 \right\rangle$ and $\mathbf{v} = \left\langle -\sqrt{3}, 1, 0 \right\rangle$. From the unit circle, we know the angle between these two vectors is $\theta = \frac{2\pi}{3}$. Use the definition of the cross product to show this.

Example. Find the area of the triangle formed by $\mathbf{u} = \langle 1, 2, 3 \rangle$ and $\mathbf{v} = \langle 3, -1, 1 \rangle$.

Example. Given a force $\mathbf{F}$ applied to a point P at the head of the vector $\mathbf{r} = \overrightarrow{OP}$, the **torque** produced at point O is given by $\tau = \mathbf{r} \times \mathbf{F}$ with magnitude

$$|\tau| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin\theta.$$

Now suppose a force of $20N$ is applied to a wrench attached to a bolt in a direction perpendicular to the bolt. Which produces more torque: applying the force at an angle of 60° on a wrench that is $0.15m$ long or applying the force at an angle of 135° on a wrench that is $0.25m$ long?



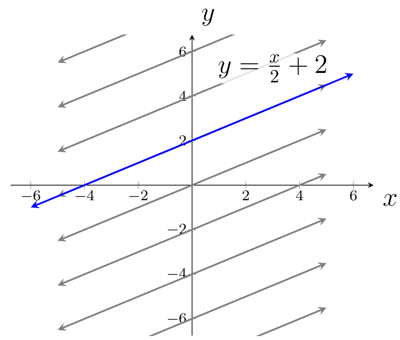
(a) (b)

13.5: Lines and Planes in Space

Equation of a Line:

Recall the equation of a line in $\mathbb{R}^2$:

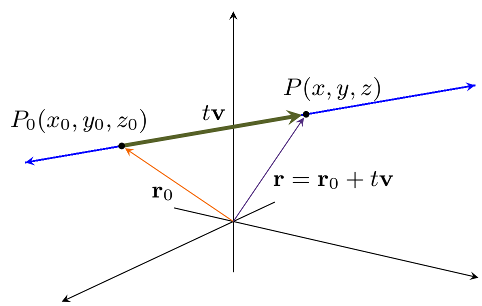
$$y = mx + b$$



where b is the intercept and m is the slope. This idea can be extended into higher dimensions:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

Here, $\mathbf{r}_0$ is a fixed point, and $\mathbf{v}$ is the position vector that is parallel to the line $\mathbf{r}$.



Equation of a Line

A **vector equation of the line** passing through the point $P_0(x_0, y_0, z_0)$ in the direction of the vector $\mathbf{v} = \langle a, b, c\rangle$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, or

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle, \quad \text{for} \quad -\infty < t < \infty$$

Equivalently, the corresponding **parametric equations of the line** are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \text{for} \quad -\infty < t < \infty$$

Example. Find the vector equation and parametric equation of the line that

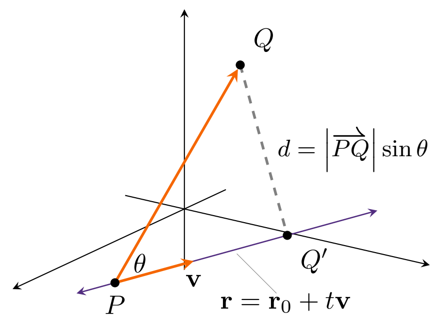
- goes through the points $P(-1,-2,1)$ and $Q(-4,-5,-3)$ where $t=0$ corresponds to P,

- goes through the point $P(1,-3,-3)$ and is parallel to the vector $\mathbf{r} = \langle -4, 1, -1 \rangle$,

- goes through the point $P(-2,5,-2)$ and is perpendicular to the lines $x = 3-4t$, $y = 2-3t$, $z = -1-t$, and $x = -2+0t$, $y = 2-t$, $z = 3t$, where $t=0$ corresponds to P.

Distance from a Point to a Line:

Given a point Q and a line ℓ, the shortest distance to the line is the length of $\overrightarrow{QQ'}$.



From the definition of the cross product, we have

$$\left|\mathbf{v} \times \overrightarrow{PQ}\right| = |\mathbf{v}| \underbrace{\left|\overrightarrow{PQ}\right| \sin\theta}_{d} = |\mathbf{v}| d$$

From here, solving for d gives us the following:

Distance Between a Point and a Line

The distance d between the point Q and the $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ is

$$d = \frac{\left|\mathbf{v} \times \overrightarrow{PQ}\right|}{|\mathbf{v}|},$$

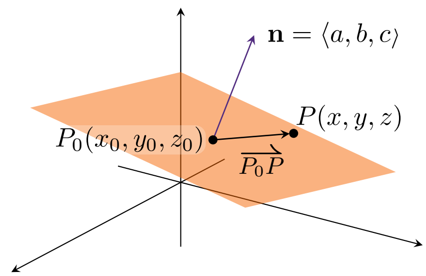
where P is any point on the line and $\mathbf{v}$ is a vector parallel to the line.

Example. Find the distance from the point $Q(-4, -1, -3)$ and the line $x = -5 - 5t$, $y = -5 + t$, $z = -1 + 4t$. (*Hint:* Let P be the point at $t = 0$)

Equations of Planes:

In $\mathbb{R}^2$, two distinct points determine a line.

In $\mathbb{R}^3$, three noncollinear points determine a unique plane. Alternatively, a plane is uniquely determined by a point and a vector that is orthogonal to the plane.



> **Definition. (Plane in $\mathbb{R}^3$)**
> Given a fixed point P_0 and a nonzero **normal vector n**, the set of points P in $\mathbb{R}^3$ for which $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$ is called a **plane**.

Consider the normal vector $\mathbf{n} = \langle a, b, c \rangle$ at the point $P_0(x_0, y_0, z_0)$, and any point $P(x, y, z)$ on the plane. Since $\mathbf{n}$ is orthogonal to the plane, it is also orthogonal to the vector $\overrightarrow{P_0P}$, which is also in the plane. Thus,

$$\begin{aligned} \mathbf{n} \cdot \overrightarrow{P_0P} &= 0 \\ \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \\ ax + by + cz &= d \end{aligned}$$

> **General Equation of a Plane in $\mathbb{R}^3$**
> The plane passing through the point $P_0(x_0, y_0, z_0)$ with a nonzero normal vector $\mathbf{n} = \langle a, b, c \rangle$ is described by the equation
>
> $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{or} \quad ax + by + cz = d,$$
>
> where $d = ax_0 + by_0 + cz_0$.

Example. Find the equation of the plane that

- goes through the point $P(-2, 5, 0)$ and is parallel to the plane $x - 5y - 5z = 1$,

- goes through the points $P(5, -2, 1)$, $Q(5, 1, 3)$ and $R(1, -5, -2)$

- that is parallel to the vectors $\langle 4, -2, -3 \rangle$ and $\langle 3, 2, 3 \rangle$, passing through the point $P(-2, -2, 5)$.

Example. Find the location where the line $\langle -3, 1, 4\rangle + t\langle -1, -4, 2\rangle$ and the plane $2x - 2y - 4z = 5$ intersect.

> **Definition. (Parallel and Orthogonal Planes)**
> Two distinct planes are **parallel** if their respective normal vectors are parallel (that is, the normal vectors are scaling multiples of each other). Two planes are **orthogonal** if their respective normal vectors are orthogonal (that is, the dot product of the normal vectors is *zero*).

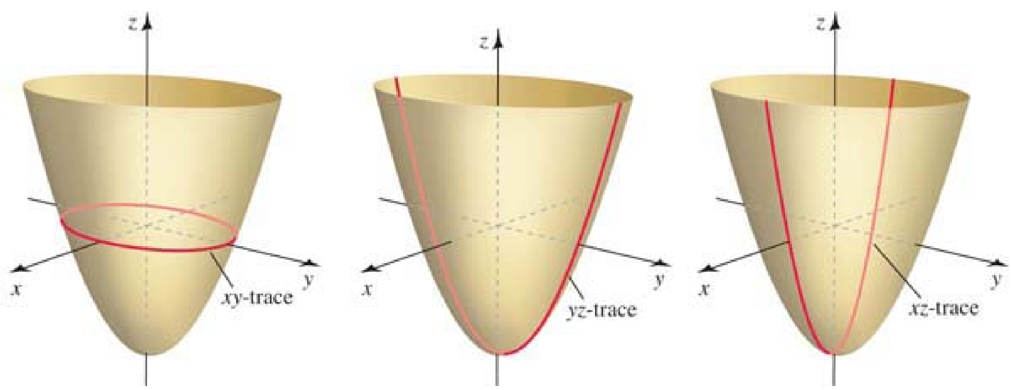
Example. Find the line of intersection between the planes $3x - y + 4z = -4$ and $x + 3y - 2z = 0$.

Example. Find the smallest angle between planes $3x - y + 4z = -4$ and $x + 3y - 2z = 0$.

13.6: Cylinders and Quadric Surfaces

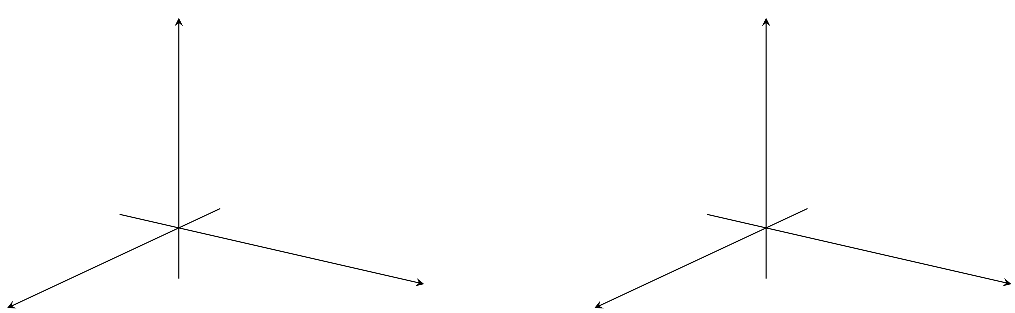
Cylinders and Traces:
When talking about three-dimensional surfaces, a *cylinder* refers to a surface that is parallel to a line. When considering surfaces that is parallel to one of the coordinate axes, that the associated variable is missing (e.g. $3y^2 + z^2 = 8$ is parallel to the x-axis).

Definition. (Trace)
A **trace** of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes. The traces in the coordinate planes are called the xy**-trace**, the yz**-trace**, and the xz**-trace** (Figure 13.80).



Example. Roughly sketch the following functions:

1. $x^2 + 4y^2 = 16$
2. $x - \sin(z) = 0$



Quadric Surfaces:

Quadric surfaces are described by the general quadratic (second-degree) equation in three variables,

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0,$$

Where the coefficients $A, \ldots, J$ and not all zero. To sketch quadric surfaces, keep the following ideas in mind:

1. **Intercepts** Determine the points, if any, where the surface intersects the coordinate axes. To find these intercepts, set x, y, and z equal to zero in pairs in the equation of the surface, and solve for the third coordinate.

2. **Traces** Finding traces of the surface helps visualize the surface. Setting x, y, and z equal to zero in pairs gives the planes parallel in that pair's plane.

3. **Completing the figure** Sketch some traces in parallel planes, then draw smooth curves that pass through the traces to fill out the surface.

Example (An ellipsoid)**.** The surface defined by the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. Graph $a = 3$, $b = 4$ and $c = 5$.

Example (An elliptic parabaloid). The surface defined by the equation $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$. Graph the elliptic paraboloid with $a = 4$ and $b = 2$.

Example (A hyperboloid of one sheet)**.**

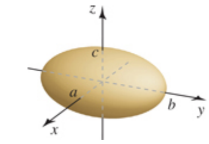
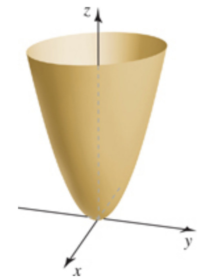
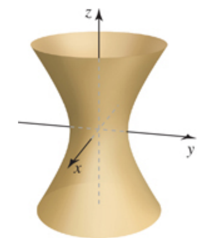
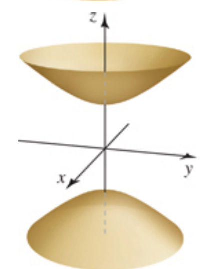
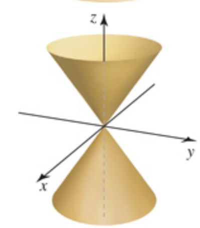
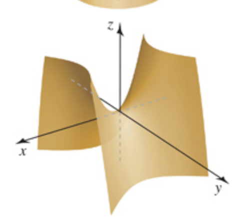
Graph the surface defined by the equation $\dfrac{x^2}{4} + \dfrac{y^2}{9} - z^2 = 1$.

Example (A hyperboloid of two sheets)**.** Graph the surface defined by the equation $-16x^2 - 4y^2 + z^2 + 64x - 80 = 0$.

Example (Elliptic cones). Graph the surface defined by the equation $\dfrac{y^2}{4} + z^2 = 4x^2$.

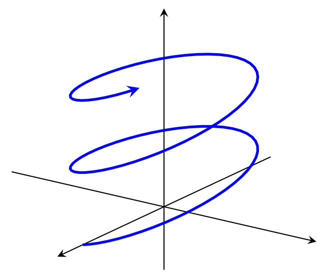
Example (A hyperbolic paraboloid).

Graph the surface defined by the equation $z = x^2 - \dfrac{y^2}{4}$.

Name	Standard Equation	Features	Graph
Ellipsoid	$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$	All traces are ellipses.	
Elliptic paraboloid	$z=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$	Traces with $z=z_0>0$ are ellipses. Traces with $x=x_0$ or $y=y_0$ are parabolas.	
Hyperboloid of one sheet	$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$	Traces with $z=z_0$ are ellipses for all z_0. Traces with $x=x_0$ or $y=y_0$ are hyperbolas.	
Hyperboloid of two sheets	$-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$	Traces with $z=z_0$ with $\lvert z_0\rvert>\lvert c\rvert$ are ellipses. Traces with $x=x_0$ and $y=y_0$ are hyperbolas.	
Elliptic cone	$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\dfrac{z^2}{c^2}$	Traces with $z=z_0\neq 0$ are ellipses. Traces with $x=x_0$ or $y=y_0$ are hyperbolas or intersecting lines.	
Hyperbolic paraboloid	$z=\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}$	Traces with $z=z_0\neq 0$ are hyperbolas. Traces with $x=x_0$ or $y=y_0$ are parabolas.	

14.1: Vector-Valued Functions

Vector-valued functions are functions of the form $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, where $x(t)$, $y(t)$, and $z(t)$ are parametric equations dependent on t.



Curves in Space

Consider

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k},$$

where f, g, and h are defined for $a \leq t \leq b$. The **domain** of $\mathbf{r}$ is the largest set of t for which all of f, g, and h are defined.

Example. What plane does the curve $\mathbf{r}(t) = t\boldsymbol{i} + 6t^3\boldsymbol{k}$ lie?

Example (Lines as vector-valued functions)**.** Find a vector function for the line that passes through the points $P(5, 2, -4)$ and $Q(5, 5, -2)$. What about the line segment that connects P and Q?

Example. Find the domain of

$$\mathbf{r}(t) = \sqrt{16 - t^2}\boldsymbol{i} + \sqrt{t}\boldsymbol{j} + \frac{4}{\sqrt{3+t}}\boldsymbol{k}$$

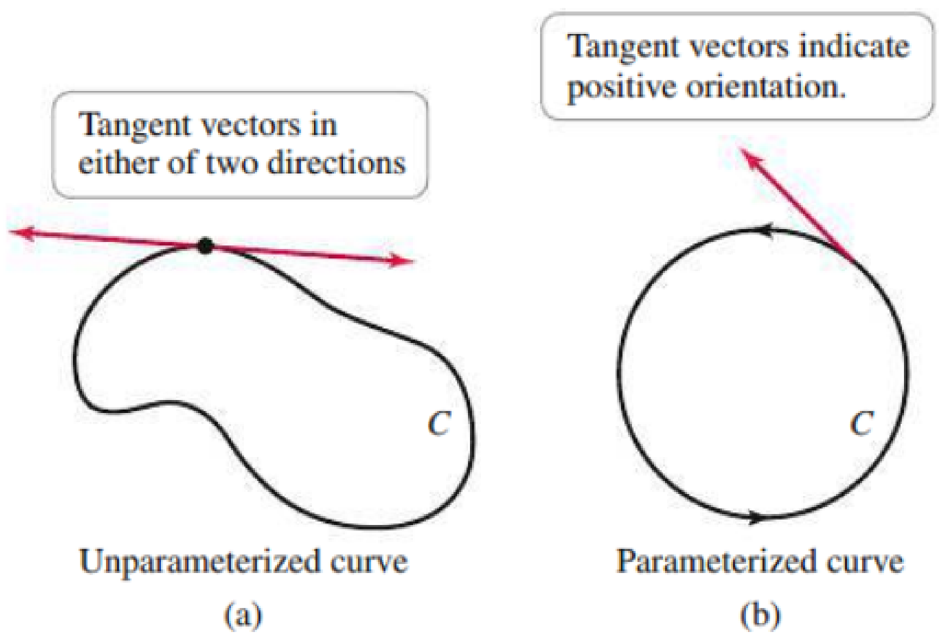
Example. Find the point P on

$$\mathbf{r}(t) = t^2\boldsymbol{i} + 2t\boldsymbol{j} + 2t\boldsymbol{k},$$

closest to $P_0(4, 17, 10)$. What is the distance between P and P_0?

Orientation of Curves

- A **unparameterized curve** is a smooth curve C with no specified direction and the tangent vector can be drawn in two directions.
- A **parameterized curve** is a smooth curve C described by a function $\mathbf{r}(t)$ for $a \leq t \leq b$ and has a direction referred to as its **orientation**.
- The *positive* orientation is the direction of the curve generated when t increases from a to b.
- The tangent vector of a parameterized curve points in the positive orientation of the curve.



Example. Graph the curve described by the equation

$$\mathbf{r}(t) = 4\cos(t)\boldsymbol{i} + \sin(t)\boldsymbol{j} + \frac{t}{2\pi}\boldsymbol{k},$$

where $0 \leq t \leq 2\pi$. Indicate the positive orientation of this curve.

Limits and Continuity for Vector-Valued Functions

The properties of limits extend to vector-valued functions naturally. In particular, for $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, if

$$\lim_{t\to a} f(t) = L_1, \qquad \lim_{t\to a} g(t) = L_2, \qquad \lim_{t\to a} h(t) = L_3$$

then

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} f(t),\ \lim_{t\to a} g(t),\ \lim_{t\to a} h(t) \right\rangle = \langle L_1, L_2, L_3\rangle.$$

Definition. (Limit of a Vector-Valued Function)
A vector-valued function $\mathbf{r}$ approaches the limit $\mathbf{L}$ as t approaches a, written $\lim\limits_{t\to a} \mathbf{r}(t) = \mathbf{L}$, provided $\lim\limits_{t\to a} |\mathbf{r}(t) - \mathbf{L}| = 0$.

A function $\mathbf{r}(t)$ is **continuous** at $t = a$, provided $\lim\limits_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$.

Example. Evaluate the following limits:

$$\lim_{t\to\pi} \left(\cos(t)\boldsymbol{i} - 7\sin\left(-\frac{t}{2}\right)\boldsymbol{j} + \frac{t}{\pi}\boldsymbol{k} \right)$$

$$\lim_{t\to\infty} \left(\frac{t}{t-3}\boldsymbol{i} + \frac{40}{1+19e^{-t}}\boldsymbol{j} + \frac{1}{2t}\boldsymbol{k} \right)$$

14.2: Calculus of Vector-Valued Functions

Definition. (Derivative and Tangent Vector)
Let $\mathbf{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$, where f,g, and h are differentiable functions on (a, b). Then $\mathbf{r}$ has a **derivative** (or is **differentiable**) on (a, b) and

$$\mathbf{r}'(t) = f'(t)\boldsymbol{i} + g'(t)\boldsymbol{j} + h'(t)\boldsymbol{k}.$$

Provided $\mathbf{r}'(t) \neq \mathbf{0}$, $\mathbf{r}'(t)$ is a **tangent vector** at the point corresponding to $\mathbf{r}(t)$.

Example. For the following functions below, find $\mathbf{r}'(t)$

a) $\mathbf{r}(t) = \left\langle e^{-t^2},\, \log_2(t-4),\, \sin(t) \right\rangle$

b) $\mathbf{r}(t) = 3\boldsymbol{i} - 2\tan(t)\boldsymbol{j} + e^t\boldsymbol{k}$

Example. For $\mathbf{r}(t) = \langle 3t,\, \sec(2t),\, \cos(t)\rangle$ compute $\mathbf{r}'(t)$ at $t = \frac{\pi}{4}$.

Definition. (Unit Tangent Vector)
Let $\mathbf{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$ be a smooth parameterized curve, for $a \leq t \leq b$. The **unit tangent vector** for a particular value of t is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Example. For $\mathbf{r}(t) = \langle 3\sin(t),\, -2\cos(2t),\, 3\cos(t)\rangle$, find the unit tangent vector.

Example. For $\mathbf{r}(t) = \langle \sin(6t),\, 3t,\, \cos(3t)\rangle$, compute $\mathbf{T}\left(\frac{\pi}{3}\right)$.

Derivative Rules

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector-valued functions, and let f be a differentiable scalar-valued function, all at a point t. Let $\mathbf{c}$ be a constant vector. The following rules apply.

1. $\dfrac{d}{dt}(\mathbf{c}) = \mathbf{0}$ — Constant Rule
2. $\dfrac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$ — Sum Rule
3. $\dfrac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ — Product Rule
4. $\dfrac{d}{dt}(\mathbf{u}(f(t))) = \mathbf{u}'(f(t))f'(t)$ — Chain Rule
5. $\dfrac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ — Dot Product Rule
6. $\dfrac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ — Cross Product Rule

Example. Given $\mathbf{u}(t) = \left\langle 4t^2,\, 1,\, 3t \right\rangle$ and $\mathbf{v}(t) = \left\langle e^{-2t},\, -2e^t,\, e^t \right\rangle$, find $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$.

Definition. (Indefinite Integral of a Vector-Valued Function)
Let $\mathbf{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$ be a vector function, and let $\mathbf{R}(t) = F(t)\boldsymbol{i} + G(t)\boldsymbol{j} + H(t)\boldsymbol{k}$, where F, G, and H are antiderivatives of f, g, and h, respectively. The **indefinate integral** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C},$$

where $\mathbf{C}$ is an arbitrary constant vector. Alternatively, in component form,

$$\int \langle f(t),\, g(t),\, h(t)\rangle\,dt = \langle F(t),\, G(t),\, H(t)\rangle + \langle C_1,\, C_2,\, C_3\rangle.$$

Example. Find $\mathbf{r}(t)$ such that $\mathbf{r}'(t) = \left\langle \frac{t}{t^2+1},\, t^2e^{-t^3},\, \frac{-2t}{\sqrt{t^2+16}} \right\rangle$ and $\mathbf{r}(0) = \left\langle 3,\, \frac{5}{3},\, -5 \right\rangle$.

Definition. (Definite Integral of a Vector-Valued Function)
Let $\mathbf{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$, where f, g, and h are integrable on the interval $[a, b]$. The **definite integral** of $\mathbf{r}$ on $[a, b]$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\boldsymbol{i} + \left(\int_a^b g(t)\,dt\right)\boldsymbol{j} + \left(\int_a^b h(t)\,dt\right)\boldsymbol{k}$$

Example. $\displaystyle\int_{-\pi}^{\pi} \langle \sin(t),\, \cos(t),\, 8t \rangle\,dt$

14.3: Motion in Space

Definition.
Let the **position** of an object moving in three-dimensional space be given by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, for $t \geq 0$. The **velocity** of the object is

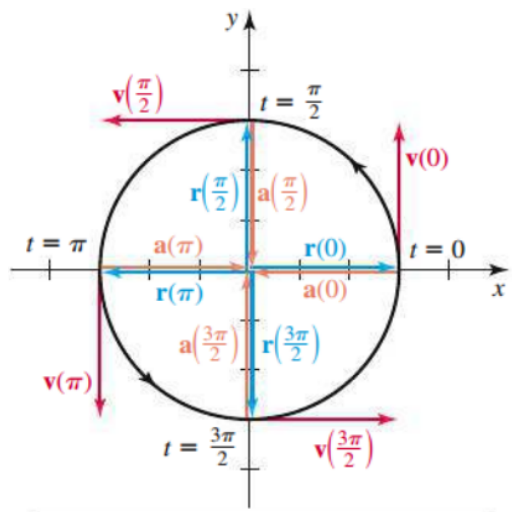
$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle.$$

The **speed** of the object is the scalar function

$$|\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

The **acceleration** of the object is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

Example. Given $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t)\rangle$ for $0 \leq t \leq 2\pi$, find the velocity, speed, and acceleration.



Circular motion: At all times $\mathbf{a}(t) = -\mathbf{r}(t)$ and $\mathbf{v}(t)$ is orthogonal to $\mathbf{r}(t)$ and $\mathbf{a}(t)$.

Theorem 14.2: Motion with constant $|\mathbf{r}|$
Let $\mathbf{r}$ describe a path on which $|\mathbf{r}|$ is constant (motion on a circle or sphere centered at the origin). Then $\mathbf{r} \cdot \mathbf{v} = 0$, which means the position vector and the velocity vector are orthogonal at all times for which the functions are defined.

Example (Path on a sphere)**.** Consider

$$\mathbf{r}(t) = \langle 3\cos(t),\, 5\sin(t),\, 4\cos(t)\rangle, \quad \text{for } 0 \le t \le 2\pi.$$

a) Show that an object with this trajectory moves on a sphere and find the radius.

b) Find the velocity and speed of the above trajectory.

c) Show that $\mathbf{r}(t) = \langle 5\cos(t),\, 5\sin(t),\, 5\sin(2t)\rangle$ does not lie on a sphere. How could this function be modified so that it does lie on a sphere?

Example. Given $\mathbf{a}(t) = \langle \cos(t), 4\sin(t) \rangle$, with an initial velocity $\langle \mathbf{u}_0, \mathbf{v}_0 \rangle = \langle 0, 4 \rangle$ and an initial position $\langle x_0, y_0 \rangle = \langle 5, 0 \rangle$ where $t \geq 0$, find the velocity and position vector.

Summary: Two-Dimensional Motion in a Gravitational Field
Consider an object moving in a plane with a horizontal x-axis and a vertical y-axis, subject only to the force of gravity. Given the initial velocity $\mathbf{v}(0) = \langle u_0, v_0 \rangle$ and the initial position $\mathbf{r}(0) = \langle x_0, y_0 \rangle$, the velocity of the object, for $t \geq 0$, is

$$\mathbf{v}(t) = \langle x'(t), y'(t) \rangle = \langle u_0, -gt + v_0 \rangle$$

and the position is

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \left\langle u_0 t + x_0, -\frac{1}{2}gt^2 + v_0 t + y_0 \right\rangle.$$

Example. Consider a ball with an initial position of $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$ m and an initial velocity of $\langle u_0, v_0 \rangle = \langle 25, 4 \rangle$ m/s.

a) Find the position and velocity of the ball while it is in the air

Summary: Two-Dimensional Motion
Assume an object traveling over horizontal ground, acted on only by the gravitational force, has an initial position $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$ and initial velocity $\langle u_0, v_0 \rangle = \langle |\mathbf{v}_0| \cos\alpha, |\mathbf{v}_0| \sin\alpha \rangle$. The trajectory, which is a segment of a parabola, has the following properties.

$$\text{time of flight} = T = \frac{2|\mathbf{v}_0| \sin\alpha}{g}$$

$$\text{range} = \frac{|\mathbf{v}_0|^2 \sin(2\alpha)}{g}$$

$$\text{maximum height} = y\left(\frac{T}{2}\right) = \frac{(|\mathbf{v}_0| \sin\alpha)^2}{2g}$$

Example. Consider a ball with an initial position of $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$ m and an initial velocity of $\langle u_0, v_0 \rangle = \langle 25, 4 \rangle$ m/s. Assuming the ground is flat and level:

b) How long is the ball in the air?

c) How far does the ball travel horizontally?

d) What is the maximum height that the ball reaches?

14.4: Length of Curves

Definition. (Arc Length for Vector Functions)
Consider the parameterized curve $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where f', g', and h' are continuous, and the curve is traversed once for $a \le t \le b$. The **arc length** of the curve between $(f(a), g(a), h(a))$ and $(f(b), g(b), h(b))$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt = \int_a^b |\mathbf{r}'(t)|\, dt.$$

Example (Flight of an eagle)**.** Suppose an eagle rises at a rate of 100 vertical ft/min on a helical path given by

$$\mathbf{r}(t) = \langle 250\cos(t),\ 250\sin(t),\ 100t \rangle$$

where $\mathbf{r}$ is measured in feet and t is measured in minutes. How far does it travel in 10 minutes?

Example. Suppose a particle has a trajectory given by

$$\mathbf{r}(t) = \langle 10\cos(3t),\, 10\sin(3t)\rangle$$

where $0 \leq t \leq \pi$. How far does this particle travel?

Example. Find the length of the curve

$$\mathbf{r}(t) = \left\langle 3t^2 - 5,\, 4t^2 + 5 \right\rangle$$

where $0 \leq t \leq 1$.

Example. Find the length of $\mathbf{r}(t) = \left\langle t^2, \dfrac{(4t+1)^{\frac{3}{2}}}{6} \right\rangle$ where $0 \leq t \leq 6$.

Example. Find the length of $\mathbf{r}(t) = \langle 2\sqrt{2}, \sin(t), \cos(t) \rangle$ where $0 \leq t \leq 5$.

Theorem 14.3: Arc Length as a Function of a Parameter
Let $\mathbf{r}(t)$ describe a smooth curve, for $t \geq a$. The arc length is given by

$$s(t) = \int_a^t |\mathbf{v}(u)|\, du,$$

where $|\mathbf{v}| = |\mathbf{r}'|$. Equivalently, $\dfrac{ds}{dt} = |\mathbf{v}(t)|$. If $|\mathbf{v}(t)| = 1$, for all $t \geq a$, then the parameter t corresponds to arc length.

Example. For the following functions, determine if $\mathbf{r}(t)$ uses arc length as a parameter. If not, find a description that uses arc length as a parameter.

a) $\mathbf{r}(t) = \langle -4t+1, 3t-1 \rangle$, $0 \leq t \leq 4$.

b) $\mathbf{r}(t) = \left\langle \frac{1}{\sqrt{10}} \cos(t),\ \frac{3}{\sqrt{10}} \cos(t),\ \sin(t) \right\rangle$, $0 \leq t \leq 2\pi$.

14.5: Curvature and Normal Vectors:

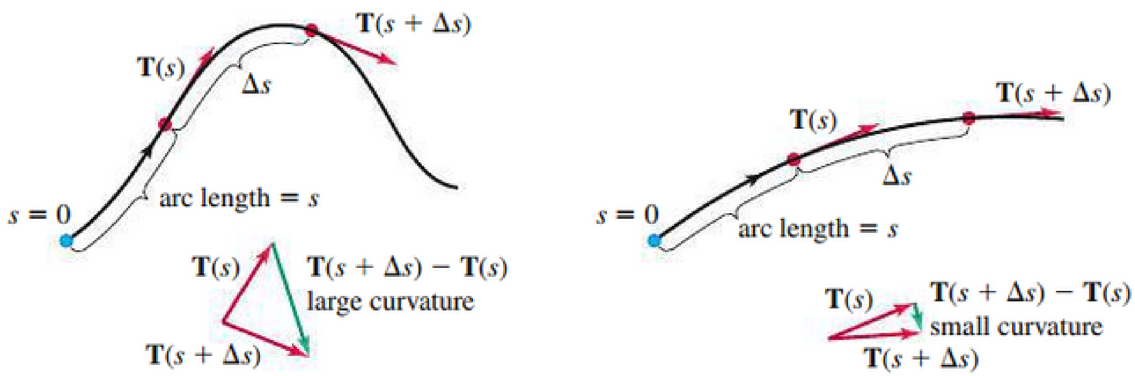
There are two ways to acceleration:

- change in speed
- change in direction

The change in direction is referred to as *curvature*. Recall that if we have a smooth curve $\mathbf{r}(t)$, the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

Specifically, *curvature* of the curve is the magnitude of the rate at which $\mathbf{T}$ changes with respect to arc length.



Definition. (Curvature)
Let $\mathbf{r}$ describe a smooth parameterized curve. If s denotes arc length and $\mathbf{T} = \mathbf{r}'/|\mathbf{r}'|$ is the unit tangent vector, the **curvature** is $\kappa(s) = \left|\frac{d\mathbf{T}}{ds}\right|$.

Theorem 14.4: Curvature Formula
Let $\mathbf{r}(t)$ describe a smooth parameterized curve, where t is any parameter. If $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{T}$ is the unit tangent vector, then the curvature is

$$\kappa(t) = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

- κ is a non-negative scalar-valued function
- Curvature of zero corresponds to a straight line
- A relatively flat curve has a small curvature
- A tight curve has a larger curvature

Example. Consider the line

$$\mathbf{r}(t) = \langle x_0 + at,\, y_0 + bt,\, z_0 + ct\rangle, \text{ for } -\infty < t < \infty.$$

Compute κ.

Example. Consider the circle

$$\mathbf{r}(t) = \langle R\cos(t),\ R\sin(t)\rangle$$

for $0 \leq t \leq 2\pi$, where $R > 0$. Show that $\kappa = 1/R$.

Example. Consider the curve

$$\mathbf{r}(t) = \left\langle 2\cos(t),\ 2\sin(t),\ \sqrt{5}t \right\rangle$$

Compute κ.

An Alternative Curvature Formula:

Consider a smooth function $\mathbf{r}(t)$ with non-zero velocity $\mathbf{v}(t) = \mathbf{r}'(t)$ and non-zero acceleration $\mathbf{a}(t) = \mathbf{v}'(t)$.

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} \;\Rightarrow\; \mathbf{v} = |\mathbf{v}|\,\mathbf{T}.$$

Thus

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}[|\mathbf{v}|\,\mathbf{T}] = \frac{d}{dt}[|\mathbf{v}|]\,\mathbf{T} + |\mathbf{v}|\,\frac{d\mathbf{T}}{dt}.$$

Now we form $\mathbf{v} \times \mathbf{a}$:

$$\begin{aligned}\mathbf{v} \times \mathbf{a} &= |\mathbf{v}|\,\mathbf{T} \times \left(\frac{d}{dt}[|\mathbf{v}|]\,\mathbf{T} + |\mathbf{v}|\,\frac{d\mathbf{T}}{dt}\right) \\ &= \underbrace{|\mathbf{v}|\,\mathbf{T} \times \frac{d}{dt}[|\mathbf{v}|]\,\mathbf{T}}_{\mathbf{0}} + |\mathbf{v}|\,\mathbf{T} \times |\mathbf{v}|\,\frac{d\mathbf{T}}{dt}\end{aligned}$$

Since $\mathbf{T}$ is a unit vector, $\mathbf{T}$ and $d\mathbf{T}/dt$ are orthogonal (Theorem 14.2). Thus

$$|\mathbf{v} \times \mathbf{a}| = \left| |\mathbf{v}|\,\mathbf{T} \times |\mathbf{v}|\,\frac{d\mathbf{T}}{dt}\right| = |\mathbf{v}|\,\underbrace{|\mathbf{T}|}_{1}\,\left||\mathbf{v}|\frac{d\mathbf{T}}{dt}\right|\underbrace{\sin\theta}_{1} = |\mathbf{v}|^2\left|\frac{d\mathbf{T}}{dt}\right|$$

Now, using Theorem 14.4, where $\left|\dfrac{d\mathbf{T}}{dt}\right| = \kappa|\mathbf{v}|$, we have

$$|\mathbf{v} \times \mathbf{a}| = |\mathbf{v}|^2\left|\frac{d\mathbf{T}}{dt}\right| = |\mathbf{v}|^2\,\kappa|\mathbf{v}| = \kappa|\mathbf{v}|^3.$$

Theorem 14.5: Alternative Curvature Formula

Let $\mathbf{r}$ be the position of an object moving on a smooth curve. The **curvature** at a point on the curve is

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3},$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{a} = \mathbf{v}'$ is the acceleration.

Example. Consider the curve

$$\mathbf{r}(t) = \langle -16\cos(t),\, 16\sin(t),\, 0\rangle.$$

Compute the curvature κ using both methods.

Principal Unit Normal Vector

Curvature indicates how quickly a curve turns. The principal unit normal vector determines the *direction* in which a curve turns.

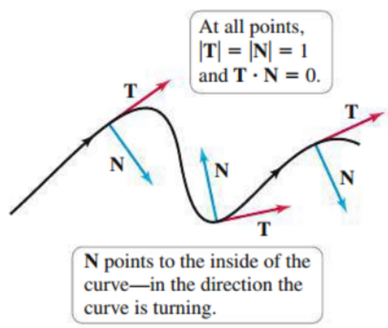
Definition. (Principal Unit Normal Vector)
Let $\mathbf{r}$ describe a smooth curve parameterized by arc length. The **principal unit normal vector** at a point P on the curve at which $\kappa \neq 0$ is

$$\mathbf{N}(s) = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

For other parameters, we use the equivalent formula

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|},$$

evaluated at the value of t corresponding to P.



Theorem 14.6: Properties of the Principal Unit Normal Vector
Let $\mathbf{r}$ describe a smooth parameterized curve with unit tangent vector $\mathbf{T}$ and principal unit normal vector $\mathbf{N}$.

1. $\mathbf{T}$ and $\mathbf{N}$ are orthogonal at all points of the curve; that is, $\mathbf{T} \cdot \mathbf{N} = 0$ at all points where $\mathbf{N}$ is defined.
2. The principal unit normal vector points to the inside of the curve – in the direction that the curve is turning.

Example. For the curve $\mathbf{r}(t) = \langle a\cos(t), a\cos(t), bt\rangle$, find the unit tangent vector $\mathbf{T}$ and the principal unit normal vector $\mathbf{N}$. Verify $|\mathbf{T}| = |\mathbf{N}| = 1$ and $\mathbf{T} \cdot \mathbf{N} = 0$.

Components of the Acceleration

Recall that the change in velocity, or acceleration, of an object can change in *speed* (in the direction of $\mathbf{T}$) and in *direction* (in the direction of $\mathbf{N}$). $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|} \Longrightarrow \mathbf{v} = \mathbf{T}|\mathbf{v}| = \mathbf{T}\,\dfrac{ds}{dt}$.

$$
\begin{aligned}
\mathbf{a} = \frac{d\mathbf{v}}{dt} &= \frac{d}{dt}\left(\mathbf{T}\frac{ds}{dt}\right) \\
&= \frac{d\mathbf{T}}{dt}\,\frac{ds}{dt} + \mathbf{T}\frac{d^2s}{dt^2} \\
&= \underbrace{\frac{d\mathbf{T}}{ds}}_{\kappa\mathbf{N}}\,\underbrace{\frac{ds}{dt}}_{|\mathbf{v}|}\,\underbrace{\frac{ds}{dt}}_{|\mathbf{v}|} + \mathbf{T}\frac{d^2s}{dt^2} \\
&= \kappa|\mathbf{v}|^2\,\mathbf{N} + \frac{d^2s}{dt^2}\,\mathbf{T}.
\end{aligned}
$$

Theorem 14.7: Tangential and Normal Components of the Acceleration

The acceleration vector of an object moving in space along a smooth curve has the following representation in terms of its **tangential component** a_T (in the direction of $\mathbf{T}$) and its **normal component** a_N (in the direction of $\mathbf{N}$):

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T},$$

where $a_N = \kappa|\mathbf{v}|^2 = \dfrac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|}$ and $a_T = \dfrac{d^2s}{dt^2}$.

Example. Consider the function

$$\mathbf{r}(t) = \langle -2t+2,\, -2t+3,\, -2t+2 \rangle.$$

Find the tangential and normal components of the acceleration.

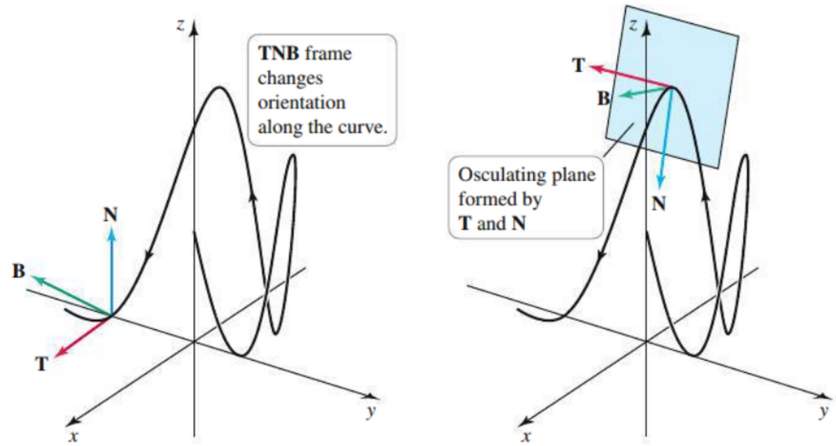
Example. Find the components of the acceleration on the circular trajectory

$$\mathbf{r}(t) = \langle R\cos(\omega t),\, R\sin(\omega t) \rangle.$$

Example. The driver of a car follows the parabolic trajectory $\mathbf{r}(t) = \left\langle t,\, t^2 \right\rangle$, for $-2 \leq t \leq 2$, through a sharp bend. Find the tangential and normal components of the acceleration of the car.

The Binormal Vector and Torsion

On a smooth parameterized curve C, $\mathbf{T}$ and $\mathbf{N}$ determine a plane called the *osculating plane.*



The coordinate system defined by these vectors is called the **TNB frame**. The rate at which the curve C twists out of the plane is the rate at which $\mathbf{B}$ changes as we move along C, which is $\dfrac{d\mathbf{B}}{ds}$.

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T} \times \mathbf{N}) = \underbrace{\frac{d\mathbf{T}}{ds} \times \mathbf{N}}_{\mathbf{0}} + \mathbf{T} \times \frac{d\mathbf{N}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds}$$

$\dfrac{d\mathbf{B}}{ds}$ is:

- orthogonal to both $\mathbf{T}$ and $\dfrac{d\mathbf{N}}{ds}$,
- orthogonal to $\mathbf{B}$ (Theorem 14.2),
- parallel with $\mathbf{N}$.

Since $\dfrac{d\mathbf{B}}{ds}$ is parallel to $\mathbf{N}$, we write

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

where τ is the *torsion* (the negative sign is conventional). We can solve for τ via the dot product:

$$\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\tau \underbrace{\mathbf{N} \cdot \mathbf{N}}_{1} \quad \Longrightarrow \quad \frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\tau$$

Definition. (Unit Binormal Vector and Torsion)
Let C be a smooth parameterized curve with unit tangent and principal unit normal vectors $\mathbf{T}$ and $\mathbf{N}$, respectively. Then at each point of the curve at which the curvature is nonzero, the **unit binomial vector** is

$$\mathbf{B} = \mathbf{T} \times \mathbf{N},$$

and the **torsion** is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

Example. Consider the circle C defined by

$$\mathbf{r}(t) = \langle R\cos(t),\ R\sin(t)\rangle,\ \text{for } 0 \le t \le 2\pi,\ \text{with } R > 0.$$

Find the unit binormal vector $\mathbf{B}$ and determine the torsion.

Example. Compute the torsion of the helix

$$\mathbf{r}(t) = \langle a\cos(t),\, a\sin(t),\, bt\rangle,\ \text{for } t \ge 0,\ \text{and } b > 0.$$

Summary: Formula for Curves in Space

Position function: $\mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$

Velocity: $\mathbf{v} = \mathbf{r}'$

Acceleration: $\mathbf{a} = \mathbf{v}'$

Unit tangent vector: $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

Principal unit normal vector: $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$ (provided $d\mathbf{T}/dt \neq \mathbf{0}$)

Curvature: $\kappa = \left|\dfrac{d\mathbf{T}}{dt}\right| = \dfrac{1}{|\mathbf{v}|}\left|\dfrac{d\mathbf{T}}{dt}\right| = \dfrac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$

Components of acceleration: $\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}$, where

$$a_N = \kappa|\mathbf{v}|^2 = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|} \text{ and } a_T = \frac{d^2s}{dt^2} = \frac{\mathbf{v}\cdot\mathbf{a}}{\mathbf{v}}$$

Unit binormal vector: $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \dfrac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|}$

Torsion: $\tau = -\dfrac{d\mathbf{B}}{ds}\cdot\mathbf{N} = \dfrac{(\mathbf{v}\times\mathbf{a})\cdot\mathbf{a}'}{|\mathbf{v}\times\mathbf{a}|^2} = \dfrac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}'\times\mathbf{r}''|^2}$