11.1: Approximating Functions with Polynomials

A *power series* is an infinite series of the form

$$\sum_{k=0}^{\infty} c_k(x-a)^k = \underbrace{c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n}_{n\text{th-degree polynomial}} + c_{n-1}(x-a)^{n-1} + \ldots,$$

Example. The tangent line of a function $f(x)$ at $x = a$ is a linear function $p_1(x)$ that can approximate $f(x)$ for values of x 'close' to a:

$$p_1(x) = f(a) + f'(a)(x-a)$$

Find a quadratic function $p_2(x)$ that can approximate $f(x)$ near $x = a$,

Find a cubic function $p_2(x)$ that can approximate $f(x)$ near $x = a$,

Find an nth degree polynomial $p_n(x)$ that can approximate $f(x)$ near $x = a$.

Definition. (Taylor Polynomials)
Let f be a function with $f', f'', \ldots,$ and $f^{(n)}$ defined at a. The n**th-order Taylor polynomial** for f with its **center** at a, denoted p_n, has the property that it matches f in value, slope, and all derivatives up to the nth derivative at a; that is,

$$p_n(a) = f(a),\ p_n'(a) = f'(a), \ldots, \text{ and } p_n^{(n)}(a) = f^{(n)}(a).$$

The nth-order Taylor polynomial centered at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

More compactly, $p_n(x) = \displaystyle\sum_{k=0}^{\infty} c_k(x-a)^k$, where the **coefficients** are

$$c_k = \frac{f^{(k)}(a)}{k!}, \qquad \text{for } k = 0, 1, 2, \ldots, n.$$

Example (LC 26.1)**.** Suppose $f(4) = 3$, $f'(4) = -1$, $f''(4) = 6$, and $f^{(3)}(4) = 16$. Find the third-order Taylor polynomial $p_3(x)$ for f centered at $a = 4$.

Example (LC 26.2)**.** For the following functions, find $p_2(x)$, the 2nd degree Taylor polynomial, centered at $a = 0$.

$y = \sqrt{1 + 2x}$

$y = \dfrac{1}{\sqrt{1 + 2x}}$

$$y = \frac{1}{1+2x}$$

$$y = \frac{1}{(1+2x)^3}$$

$y = e^{2x}$

$y = e^{-2x}$

Example (LC 26.3)**.** Find the Taylor polynomial $p_3(x)$ centered at $a = \frac{\pi}{4}$ for $f(x) = \sin(x)$.

Example (LC 26.4)**.** Use the 4th degree Taylor polynomial of $y = \ln(x)$ centered at $a = 1$ to approximate $\ln(1.1)$.

Definition. (Remainder in a Taylor Polynomial)
Let p_n be the Taylor polynomial of order n for f. The **remainder** in using p_n to approximate f at the point x is

$$R_n(x) = f(x) - p_n(x).$$

Theorem 11.1: Taylor's Theorem (Remainder Theorem)
Let f have continuous derivatives up to $f^{(n+1}$ on an open interval I containing a. For all x in I,

$$f(x) = p_n(x) + R_n(x),$$

where p_n is the nth-order Taylor polynomial for f centered at a and the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some point c between x and a.

Theorem 11.2: Estimate of the Remainder
Let n be a fixed positive integer. Suppose there exists a number M such that $\left|f^{(n+1)}(n)\right| \leq M$, for all c between a and x inclusive. The remainder in the nth-order Taylor polynomial for f centered at a satisfies

$$|R_n(x)| = |f(x) - p_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}.$$

Example (LC 27.1-27.2)**.** The third-order Taylor polynomial centered at $a = 1$ for $f(x) = x\ln(x)$ is

$$p_3(x) = (x-1) + \frac{(x-1)^2}{2} - \frac{(x-1)^3}{6}.$$

Find the smallest number M such that $\left|f^{(4)}(x)\right| \le M$ for $\frac{1}{2} \le x \le \frac{3}{2}$.

Compute the upper bound for $|R_3(x)|$.

Example (LC 27.3-27.5)**.** Consider $f(x) = e^x$.

Find the Taylor polynomial $p_4(x)$ centered at $a = 0$.

What is the smallest *integer* M such that $\left|f^{(5)}(x)\right| \leq M$ for $0 \leq x \leq {}^1\!/\!_4$?

Compute the upper bound for $|R_4(x)|$ when $p_4(x)$ is used to compute $e^{1/4}$.

Example (LC 27.6-27.7)**.** We want to approximate $\sin(0.2)$ with an absolute error no greater than 10^{-3} by using a nth degree Taylor polynomial for $f(x) = \sin(x)$ centered at $a = 0$. We want to determine the minimum order of the Taylor polynomial that is required to meet this condition.

What is the smallest *integer* number M that bounds $f^{(n+1)}(x)$ on $0 \leq x \leq 0.2$?

Apply Taylor's Estimate of the Remainder Theorem to find the minimum value of n such that $|R_n(x)| \leq \frac{1}{10^3}$.