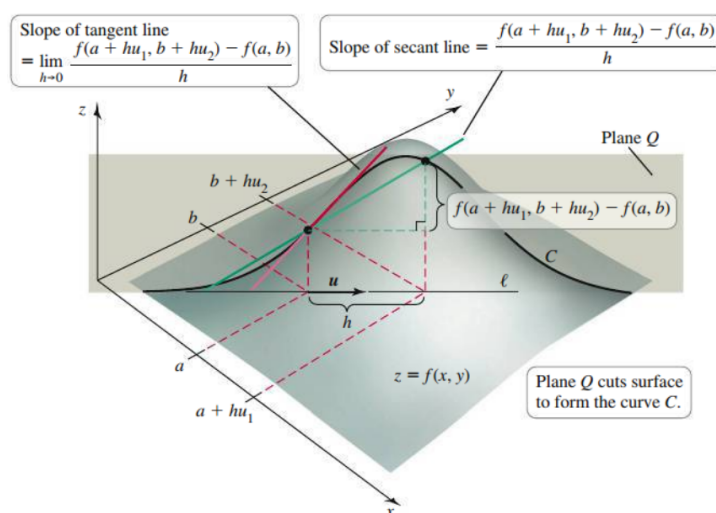
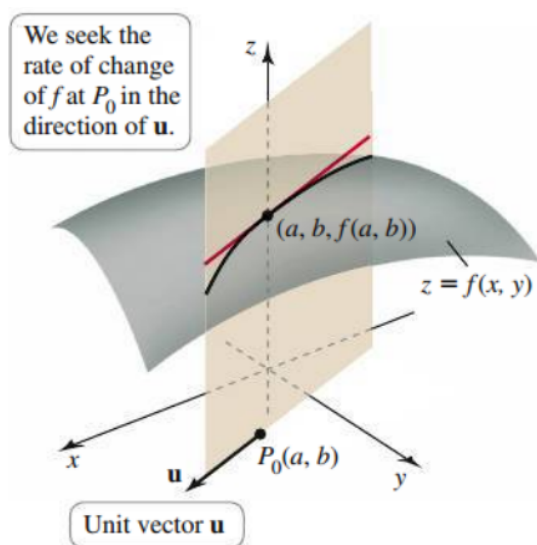


1 15.5: Directional Derivatives and the Gradient

Directional derivatives allow us to evaluate the rate of change of a function $f(x, y)$ along any direction (not just parallel with the x -axis and y -axis).



Definition. (Directional Derivative)

Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy -plane. The **directional derivative of f at (a, b) in the direction of \mathbf{u}** is

$$D_{\mathbf{u}}f(a, b) = \lim_{h \rightarrow 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h},$$

provided the limit exists.

Theorem 15.10: Directional Derivative

Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy -plane. The **directional derivative of f at (a, b) in the direction of \mathbf{u}** is

$$D_{\mathbf{u}}f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle \cdot \langle u_1, u_2 \rangle.$$

Definition. (Gradient (Two Dimensions))

Let f be differentiable at the point (x, y) . The **gradient** of f at (x, y) is the vector-valued function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}.$$

Theorem 15.11: Directions of Change

Let f be differentiable at (a, b) with $\nabla f(a, b) \neq \mathbf{0}$.

1. f has its maximum rate of increase at (a, b) in the direction of the gradient $\nabla f(a, b)$. The rate of change in this direction is $|\nabla f(a, b)|$.
2. f has its maximum rate of decrease at (a, b) in the direction of $-\nabla f(a, b)$. The rate of change in this direction is $-|\nabla f(a, b)|$.
3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$.

Theorem 15.12: The Gradient and Level Curves

Given a function f differentiable at (a, b) , the line tangent to the level curve of f at (a, b) is orthogonal to the gradient $\nabla f(a, b)$, provided $\nabla f(a, b) \neq \mathbf{0}$.

Definition. (Directional Derivative and Gradient in Three Dimensions)

Let f be directional at (a, b, c) and let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ be a unit vector. The **directional derivative of f at (a, b, c) in the direction of \mathbf{u}** is

$$D_{\mathbf{u}}f(a, b, c) = \lim_{h \rightarrow 0} \frac{f(a + hu_1, b + hu_2, c + hu_3) - f(a, b, c)}{h},$$

provided this limit exists.

The **gradient** of f at this point (x, y, z) is the vector-valued function

$$\begin{aligned}\nabla f(x, y, z) &= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \\ &= f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}.\end{aligned}$$

Theorem 15.13: Directional Derivative and Interpreting the Gradient

Let f be differentiable at (a, b, c) and let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ be a unit vector. The directional derivative of f at (a, b, c) in the direction of \mathbf{u} is

$$\begin{aligned}D_{\mathbf{u}}f(a, b, c) &= \nabla f(a, b, c) \cdot \mathbf{u} \\ &= \langle f_x(a, b, c), f_y(a, b, c), f_z(a, b, c) \rangle \cdot \langle u_1, u_2, u_3 \rangle.\end{aligned}$$

Assuming $\nabla f(a, b, c) \neq \mathbf{0}$, the gradient in three dimensions has the following properties.

1. f has its maximum rate of increase at (a, b, c) in the direction of the gradient $\nabla f(a, b, c)$ and the rate of change in this direction is $|\nabla f(a, b, c)|$.
2. f has its maximum rate of decrease at (a, b, c) in the direction of $-\nabla f(a, b, c)$ and the rate of change in this direction is $-|\nabla f(a, b, c)|$.
3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b, c)$.