10.2: Sequences

$\{a_n\} = \{\frac{1}{n}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ $\qquad f(x) = \frac{1}{x} \Rightarrow f(n) = \frac{1}{n} = a_n$

> **Theorem 10.1: Limits of Sequences from Limits of Functions**
> Suppose f is a function such that $f(n) = a_n$, for positive integers n. If $\lim_{x\to\infty} f(x) = L$, then the limit of the sequence $\{a_n\}$ is also L, where L may be $\pm\infty$.

Example. Determine if the following sequences converge or diverge. If the sequence converges, find its limit.

$\left\{e^{2n/(n+2)}\right\}_{n=1}^{\infty}$

$f(x) = e^{\frac{2x}{x+2}}$

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\frac{2x}{x+2}}$ $\quad e^x$ is cont.

$= e^{\lim_{x\to\infty} \frac{2x}{x+2}\left(\frac{1/x}{1/x}\right)} = e^2$

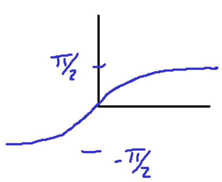
$\lim_{n\to\infty} e^{2n/n+2} = e^2$

$\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$ $\quad f(x) \cancel{=} \frac{(-1)^x}{x}$ $\quad x \neq 0$, x integer, $x \stackrel{?}{=} 3/2$

$f(x) = \frac{1}{x} \qquad \lim_{x\to\infty} f(x) = 0$

$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$

$\left\{\frac{\arctan(n)}{n}\right\}_{n=1}^{\infty}$



$f(x) = \frac{\arctan(x)}{x}$

$\lim_{x\to\infty} \frac{\arctan(x)}{x} = 0$

$\lim_{n\to\infty} \frac{\arctan(n)}{n} = 0$

$\left\{\frac{e^{-n}}{42\sin(e^{-n})}\right\}_{n=1}^{\infty}$

$f(x) = \frac{e^{-x}}{42\sin(e^{-x})}$

$\lim_{x\to\infty} e^{-x} = 0$

Let $u = e^{-x}$

$\lim_{x\to\infty} e^{-x} = \lim_{u\to 0} u$

$\lim_{x\to\infty} \frac{e^{-x}}{42\sin(e^{-x})} = \frac{1}{42}\lim_{u\to 0} \frac{u}{\sin(u)} = \frac{1}{42}(1) = \boxed{\frac{1}{42}}$

$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$

10.2: Limit Laws for Sequences
Assume the sequences $\{a_n\}$ and $\{b_n\}$ have limits A and B, respectively. Then

1. $\lim_{n\to\infty}(a_n \pm b_n) = A \pm B$ ← If $A=\pm\infty$ & $B=\mp\infty$ This doesn't work
2. $\lim_{n\to\infty} ca_n = cA$, where c is a real number
3. $\lim_{n\to\infty} a_n b_n = AB$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$, provided $B \neq 0$.

$a_n = \left\{\frac{2n+1}{n}\right\}$ $\lim_{n\to\infty} a_n = 2$

$3a_n \longrightarrow \lim_{n\to\infty} 3a_n = 6$

Example. Consider the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ where

$$a_n = \frac{1}{n}, \quad b_n = n, \quad c_n = e^n, \quad \text{and } d_n = \sqrt{n}.$$

Compute the following limits.

A. $\lim_{n\to\infty} a_n$

$\lim_{n\to\infty} \frac{1}{n} = 0$

B. $\lim_{n\to\infty} b_n$

$\lim_{n\to\infty} n = \infty$

C. $\lim_{n\to\infty} c_n$

$\lim_{n\to\infty} e^n = \infty$

$\cancel{e^\infty}$

D. $\lim_{n\to\infty} d_n$

$\lim_{n\to\infty} \sqrt{n} = \infty$

E. $\lim_{n\to\infty} a_n b_n$

$\lim_{n\to\infty} \frac{1}{n} n$

$= \lim_{n\to\infty} 1 = 1$

F. $\lim_{n\to\infty} a_n c_n$

$= \lim_{n\to\infty} \frac{1}{n} e^n = \infty$

grows faster

G. $\lim_{n\to\infty} a_n d_n$

$\lim_{n\to\infty} \frac{1}{n}\sqrt{n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

$\frac{n^{1/2}}{n} = n^{-1/2}$

True or False: If for some sequences $\{a_n\}$ and $\{b_n\}$, $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = \infty$, then $\lim_{n\to\infty} a_n b_n = 0$.

Definition. (Terminology for Sequences)

- $\{a_n\}$ is **increasing** if $a_{n+1} > a_n$ $\{1, 2, 5, 10, 42, 79, \ldots\}$
- $\{a_n\}$ is **nondecreasing** if $a_{n+1} \geq a_n$ $\{1, 2, 2, 2, 5, 7, \ldots\}$
- $\{a_n\}$ is **decreasing** if $a_{n+1} < a_n$ $\{1, 1/2, 1/4, 1/8, \ldots\}$
- $\{a_n\}$ is **nonincreasing** if $a_{n+1} \leq a_n$ $\{1, 1/2, 1/4, 1/4, 1/4, \ldots\}$
- $\{a_n\}$ is **monotonic** if it is either nonincreasing or nondecreasing (it moves in one direction)

  monotonic $\{1, 2, 3, 4, \ldots\}$, $\{1, 1, 2, 3, 5, 8, \ldots\}$

  not monotonic $\{-1, 1/2, -1/4, 1/8, \ldots\}$ $\{4, 3, 3.5, \ldots\}$
- $\{a_n\}$ is **bounded above** if there is a $\underbrace{\text{number } M}_{\text{largest}}$ such that $a_n \leq M$, for all relevant values of n

  $\{a_n\} = \{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots\} \rightarrow a_n \leq 1$
- $\{a_n\}$ is **bounded below** if there is a number N such that $a_n \geq N$, for all relevant values of n.

  $\{b_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \longrightarrow b_n \geq 0$
- If $\{a_n\}$ is bounded above and bounded below, then we say that $\{a_n\}$ is a **bounded** sequence.

Example. Consider the sequence $\{-n^2\}_{n=1}^{\infty}$. What can we say about this sequence?

$\{-n\}_{n=1}^{\infty} = \{-1, -4, -9, -16, -25, \ldots\}$

decreasing (non-increasing), monotonic

not a bounded sequence

bounded above by 0 (by -1)

Not bounded below

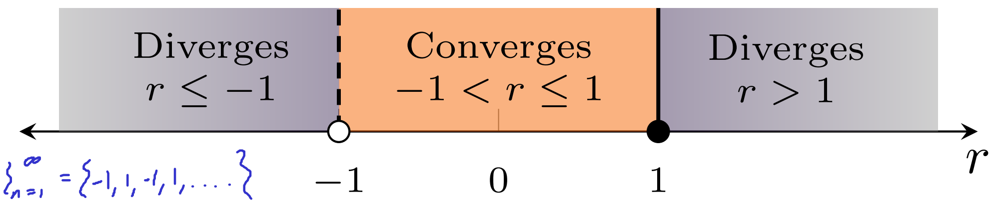
C. The sequence is not bounded and is monotonic decreasing.

Theorem 10.3: Geometric Sequences
Let r be a real number. Then

$\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist} & \text{if } r \leq -1 \text{ or } r > 1. \end{cases}$$

If $r > 0$, then $\{r^n\}$ is a monotonic sequence. If $r < 0$, then $\{r^n\}$ oscillates.



$\{(-1)^n\}_{n=1}^{\infty} = \{-1, 1, -1, 1, \dots\}$

Example. Determine if the following sequences converge

$\left\{\frac{3^{n+1}+3}{3^n}\right\}$ Converges

$$\lim_{n\to\infty} \frac{3^{n+1}+3}{3^n} = \lim_{n\to\infty} \frac{3^{n+1}}{3^n} + \underbrace{\lim_{n\to\infty}\frac{3}{3^n}}_{\to 0}$$

$$= \lim_{n\to\infty} \frac{3\cdot 3^n}{3^n} = 3\lim_{n\to\infty}\frac{3^n}{3^n} = 3\lim_{n\to\infty} 1 = \boxed{3}$$

$\left\{2^{n+1}3^{-n}\right\}$ Converges

$2 \cdot 2^n$

$$\lim_{n\to\infty} 2^{n+1}3^{-n} = \lim_{n\to\infty}\frac{2^{n+1}}{3^n}$$

$$= 2\lim_{n\to\infty}\frac{2^n}{3^n} = 2\underbrace{\lim_{n\to\infty}\left(\frac{2}{3}\right)^n}_{r = \frac{2}{3} < 1}$$

$$= 2 \cdot 0 = \boxed{0}$$

$\left\{\frac{(-1)^n}{2^n}\right\}$

$$\underbrace{\lim_{n\to\infty}\frac{-1}{2^n}}_{} \leq \lim_{n\to\infty}\frac{(-1)^n}{2^n} \leq \underbrace{\lim_{n\to\infty}\frac{1}{2^n}}_{}$$

$$0 \leq \lim_{n\to\infty}\frac{(-1)^n}{2^n} \leq 0$$

$$\Rightarrow \lim_{n\to\infty}\frac{(-1)^n}{2^n} = 0$$

$\left\{\frac{75^{n-1}}{99^n} + \frac{5^n \sin(n)}{8^n}\right\}$

$75^{n-1} = \frac{75^n}{75}$

$$= \lim_{n\to\infty} \frac{75^{n-1}}{99^n} + \frac{5^n\sin(n)}{8^n}$$

Geometric

$$= \lim_{n\to\infty}\frac{1}{75}\underbrace{\left(\frac{75}{99}\right)^n}_{0} + \lim_{n\to\infty}\underbrace{\left(\frac{5}{8}\right)^n}_{0}\underbrace{\sin(n)}_{-1\leq\sin(n)\leq 1}$$

$$= \frac{1}{75}(0) + 0 = \boxed{0}$$

> **Theorem 10.4: Squeeze Theorem for Sequences**
> Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences with $a_n \le b_n \le c_n$, for all integers n greater than some index N. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Example. Find the limit of the sequence $b_n = \dfrac{9\cos(n)}{n^2+1}$.

$-1 \le \cos(n) \le 1$

$$a_n = -\frac{9}{n^2+1}$$

$$c_n = \frac{9}{n^2+1}$$

$$\underbrace{\lim_{n\to\infty} \frac{-9}{n^2+1}} \le \lim_{n\to\infty} \frac{9\cos(n)}{n^2+1} \le \underbrace{\lim_{n\to\infty} \frac{9}{n^2+1}}$$

$$0 \le \lim_{n\to\infty} \frac{9\cos(n)}{n^2+1} \le 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{9\cos(n)}{n^2+1} = 0$$

> **Theorem 10.5: Bounded Monotonic Sequence**
> A bounded $\underbrace{\text{monotonic}}_{\text{one direction}}$ sequence converges.

between upper & lower bound

Theorem 10.6: Growth Rates of Sequences
The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, if $\{a_n\}$ appears before $\{b_n\}$ in the list, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$:

$$\{(\ln n)^q\} \ll \{n^p\} \ll \{n^p(\ln n)^r\} \ll \underset{x^2}{\{n^{p+s}\}} \ll \underset{2^x}{\{b^n\}} \ll \{n!\} \ll \{n^n\}$$

Example. Use growth rates to determine which of the following sequences converge.

$\left\{\dfrac{\ln(n^{10})}{0.00001n}\right\}$ $\quad \lim_{n\to\infty} \dfrac{\ln(n^{10})}{0.00001n} = \lim_{n\to\infty} \dfrac{10\ln(n)}{\frac{1}{100000}n} = 0$ grows slower

$\left\{\dfrac{n^8\ln(n)}{n^{8.001}}\right\}$ $\quad \lim_{n\to\infty} \dfrac{n^8 \ln(n)}{n^{8.001}} = \lim_{n\to\infty} \dfrac{n^8 \ln(n)}{n^{8+1/1000}} = 0$

$n^p(\ln(n))^r$

faster $\quad n^{p+s}$

$\left\{\dfrac{n!}{10^n}\right\}$ $\quad \lim_{n\to\infty} \dfrac{n!}{10^n} = \infty$ faster

$\left\{\dfrac{n^{1000}}{2^n}\right\}$ $\quad \lim_{n\to\infty} \dfrac{n^{1000}}{2^n} = 0$ $\quad$ Converges $\quad$ LC #4

Definition. (Limit of a Sequence)
The sequence $\{a_n\}$ converges to L provided the terms of a_n can be made arbitrarily close to L by taking n sufficiently large. More precisely, $\{a_n\}$ has the unique limit L if, given any $\varepsilon > 0$, it is possible to find a positive integer N (depending only on ε) such that

$$|a_n - L| < \varepsilon \qquad \text{whenever } n > N.$$

If the **limit of a sequence** is L, we say the sequence **converges** to L, written

$$\lim_{n\to\infty} a_n = L.$$

A sequence that does not converge is said to **diverge**.