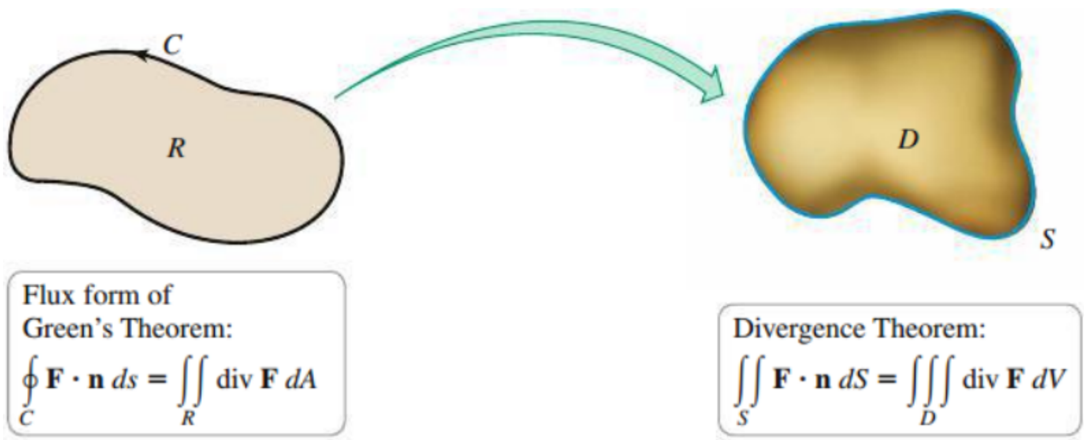
17.8: Divergence Theorem

The Divergence Theorem is the three-dimensional version of the flux form of Green's Theorem. Recall the flux form of Green's Theorem:

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n}\, ds}_{\text{flux across } C} = \iint_R \underbrace{(f_x + g_y)}_{\text{divergence}}\, dA.$$

The above means that the cumulative expansion and contraction throughout R equals the flux across the boundary of R. The Divergence Theorem computes the flux over a surface S in $\mathbb{R}^3$:



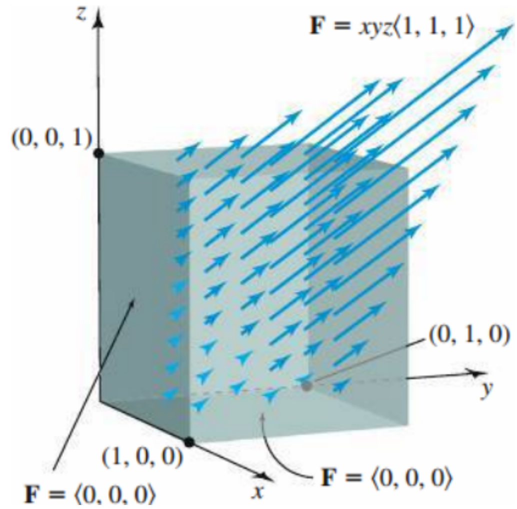
Theorem 17.17: Divergence Theorem
Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives in a connected and simply connected region D in $\mathbb{R}^3$ enclosed by an oriented surface S. Then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV,$$

where $\mathbf{n}$ is the outward unit normal vector on S.

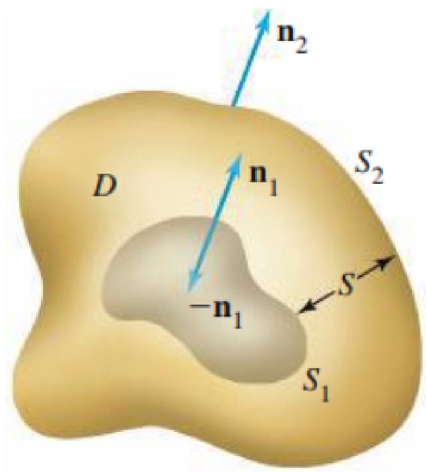
Example. Verify the Divergence Theorem: Consider the radial field $\mathbf{F} = \langle x,\, y,\, z\rangle$ and let S be the sphere $x^2 + y^2 + z^2 = a^2$ that encloses the region D. Assume $\mathbf{n}$ is the outward unit normal vector on the sphere. Evaluate both integrals of the Divergence Theorem.

Example. Find the net outward flux of the field $\mathbf{F} = xyz\langle 1, 1, 1\rangle$ across the boundaries of the cube $D = \{(x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$.



Theorem 17.18: Divergence Theorem for Hollow Regions
Suppose the vector field $\mathbf{F}$ satisfies the conditions of the Divergence Theorem on a region D bounded by two oriented surfaces S_1 and S_2, where S_1 lies within S_2. Let S be the entire boundary of D ($S = S_1 \cup S_2$) and let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the outward unit normal vectors for S_1 and S_2, respectively. Then

$$\iiint\limits_D \nabla \cdot \mathbf{F}\, dV = \iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint\limits_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS - \iint\limits_{S_1} \mathbf{F} \cdot \mathbf{n}_2\, dS.$$



$\mathbf{n}_1$ is the outward unit normal to S_1 and points into D. The outward unit normal to S on S_1 is $-\mathbf{n}_1$.

Example. Consider the inverse square vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{\langle x,\, y,\, z\rangle}{\left(x^2+y^2+z^2\right)^{3/2}}$$

Find the net outward flux of $\mathbf{F}$ across the surface of the region $D = \left\{(x,y,z) : a^2 \leq x^2+y^2+z^2 \leq b^2\right\}$ that lies between concentric spheres with radii a and b.

Find the outward flux of $\mathbf{F}$ across any sphere that encloses the origin.

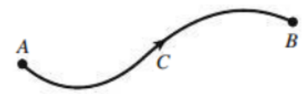
Example. Use the Divergence Theorem to compute the net outward flux of the field $\mathbf{F} = \left\langle x^2,\, y^2,\, z^2 \right\rangle$ across the surface S where S is the sphere $\left\{(x, y, z) : x^2 + y^2 + z^2 = r^2\right\}$.

Fundamental Theorem of Calculus

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

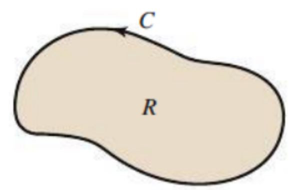
Fundamental Theorem for Line Integrals

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$$



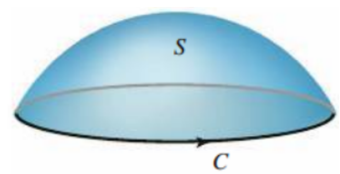
Green's Theorem (Circulation Form)

$$\iint_R (g_x - f_y)\,dA = \oint_C f\,dx + g\,dy$$



Stokes' Theorem

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$$



Divergence Theorem

$$\iiint_D \nabla \cdot \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot \mathbf{n}\,dS$$