## 16.5: Triple Integrals in Cylindrical and Spherical Coordinates

### Cylindrical coordinates:

The concept of polar coordinates in $\mathbb{R}^2$ from section 16.3 can be extended to $\mathbb{R}^3$. This coordinate system is called *cylindrical coordinates* where every point $P$ in $\mathbb{R}^3$ has coordinates $(r, \theta, z)$, where $0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$, and $-\infty < z < \infty$.

**Transformations between Cylindrical and Rectangular Coordinates**

| **Rectangular → Cylindrical** | **Cylindrical → Rectangular** |
|---|---|
| $r^2 = x^2 + y^2$ | $x = r\cos\theta$ |
| $\tan\theta = y/x$ | $y = r\sin\theta$ |
| $z = z$ | $z = z$ |

**Example.** Sketch the following sets represented in cylindrical coordinates:

$\{(r, \theta, z) : r = a\}, a > 0$

$\{(r, \theta, z) : 0 < a \leq r \leq b\}$

$\{(r, \theta, z) : z = a\}$

$\{(r, \theta, z) : z = ar\}, a \neq 0$

$\{(r, \theta, z) : \theta = \theta_0\}$

**Theorem 16.6: Change of Variables for Triple Integrals in Cylindrical Coordinates**

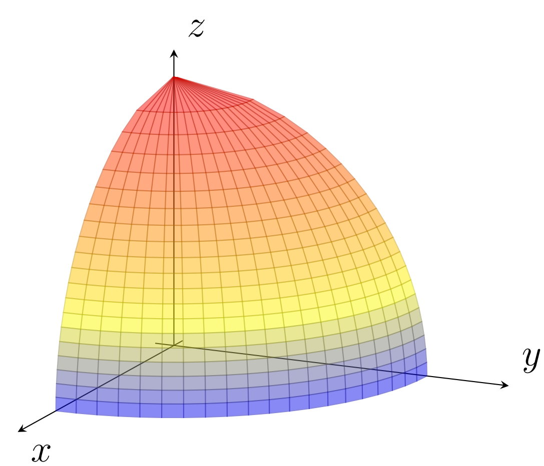
Let $f$ be continuous over the region $D$, expressed in cylindrical coordinates as

$$D = \{(r, \theta, z) : 0 \leq g(\theta) \leq r \leq h(\theta),\ \alpha \leq \theta \leq \beta,\ G(x, y) \leq z \leq H(x, y)\}$$

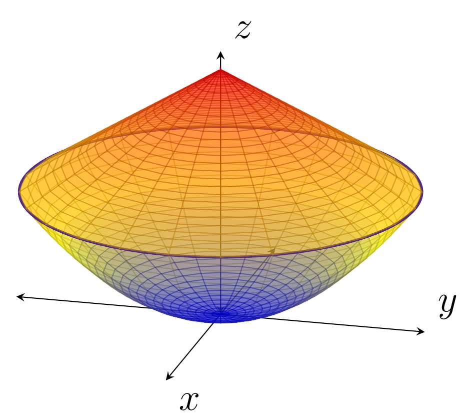
Then $f$ is integrable over $D$, and the triple integral of $f$ over $D$ is

$$\iiint\limits_D f(x, y, z)\, dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta,\ r\sin\theta)}^{H(r\cos\theta,\ r\sin\theta)} f(r\cos\theta,\ r\sin\theta)\, dz\, r\, dr\, d\theta.$$

**Example.** Evaluate the following integral using cylindrical coordinates:
$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{x^2+y^2}} \left(x^2+y^2\right)^{-1/2}\, dz\, dy\, dx$$



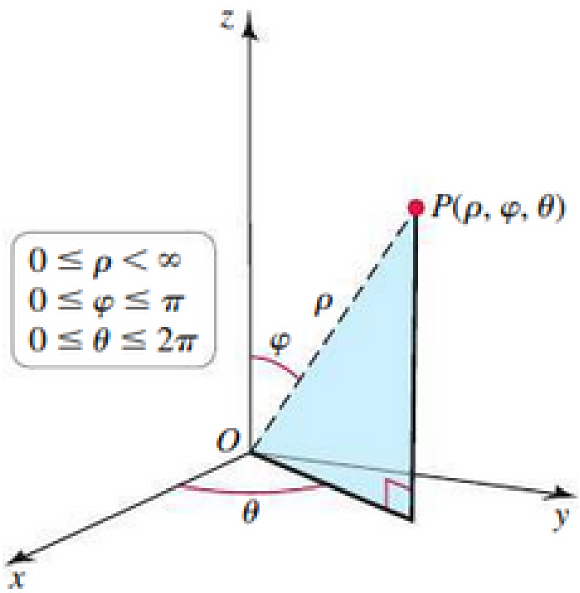
**Example.** Find the volume of the solid bounded below by the paraboloid $z = x^2 + y^2$ and bounded above by the cone $z = 2 - \sqrt{x^2 + y^2}$.

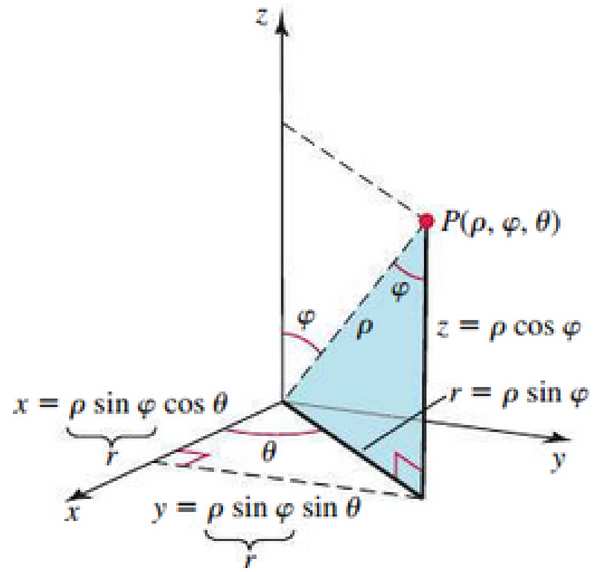


**Spherical Coordinates:**

Spherical coordinates can represent a point $P$ in $\mathbb{R}^3$ as $(\rho, \varphi, \theta)$ where

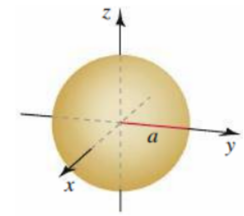
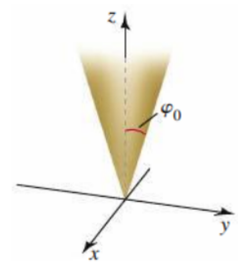
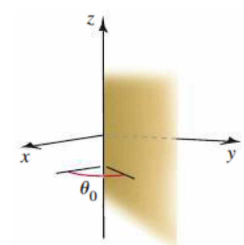
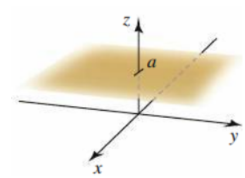
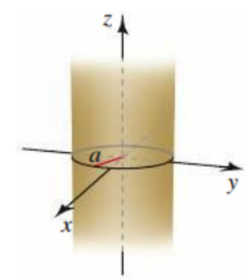
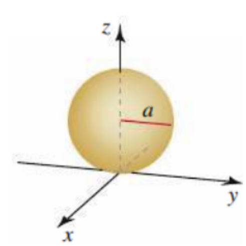
- $\rho$ is the distance from the origin to $P$,
- $\varphi$ is the angle between the positive $z$-axis and the line $OP$, and
- $\theta$ is the same angle as in cylindrical coordinates.




**Transformations between Spherical and Rectangular Coordinates**

| **Rectangular → Spherical** | **Spherical → Rectangular** |
|---|---|
| $\rho^2 = x^2 + y^2 + z^2$ | $x = \rho \sin(\varphi) \cos(\theta)$ |
| Use trigonometry to find | $y = \rho \sin(\varphi) \sin(\theta)$ |
| $\varphi$ and $\theta$. | $z = \rho \cos(\varphi)$ |

| Name | Description | Example |
| --- | --- | --- |
| Sphere, radius $a$, center $(0,0,0)$ | $\{(\rho,\varphi,\theta) : \rho = a\}, a > 0$ |  |
| Cone | $\{(\rho,\varphi,\theta) : \varphi = \varphi_0\}, \varphi_0 \neq 0, \pi/2, \pi$ |  |
| Vertical half-plane | $\{(\rho,\varphi,\theta) : \theta = \theta_0\}$ |  |
| Horizontal plane, $z = a$ | $a > 0 : \{(\rho,\varphi,\theta) : \rho = a\sec(\varphi),\ 0 \leq \varphi < \pi/2\}$<br>$a < 0 : \{(\rho,\varphi,\theta) : \rho = a\sec(\varphi),\ \pi/2 < \varphi \leq \pi\}$ |  |
| Cylinder, radius $a > 0$ | $\{(\rho,\varphi,\theta) : \rho = \alpha\csc(\varphi),\ 0 < \varphi < \pi\}$ |  |
| Sphere, radius $a > 0$ center $(0,0,a)$ | $\{(\rho,\varphi,\theta) : \rho = 2a\cos(\varphi),\ 0 \leq \varphi \leq \pi/2\}$ |  |

**Theorem 16.7: Change of Variables for Triple Integrals in Spherical Coordinates**
Let $f$ be continuous over the region $D$, expressed in spherical coordinates as

$$D = \{(\rho, \varphi, \theta) : 0 \le g(\varphi, \theta) \le \rho \le h(\varphi, \theta),\ a \le \varphi \le b,\ \alpha \le \theta \le \beta\}.$$

Then $f$ is integrable over $D$, and the triple integral of $f$ over $D$ is

$$\iiint\limits_D f(x, y, z)\, dV$$
$$= \int_\alpha^\beta \int_a^b \int_{g(\varphi,\theta)}^{h(\varphi,\theta)} f(\rho \sin(\varphi)\cos(\theta),\ \rho \sin(\varphi)\sin(\theta),\ \rho\cos(\varphi))\, \rho^2 \sin(\varphi)\ d\rho\, d\varphi\, d\theta.$$

**Example.** Evaluate $\iiint\limits_D \left(x^2 + y^2 + z^2\right)^{-3/2} dV$, where $D$ is the region in the first octant between two spheres of radius 1 and 2 centered at the origin

**Example.** Find the volume of the solid region $D$ that lies inside the cone $\varphi = \pi/6$ and inside the sphere $\rho = 4$.