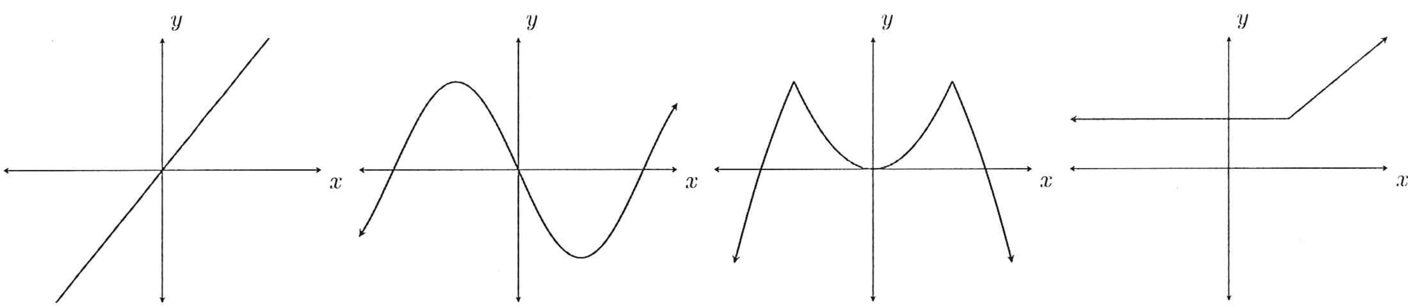
2.6 Continuity

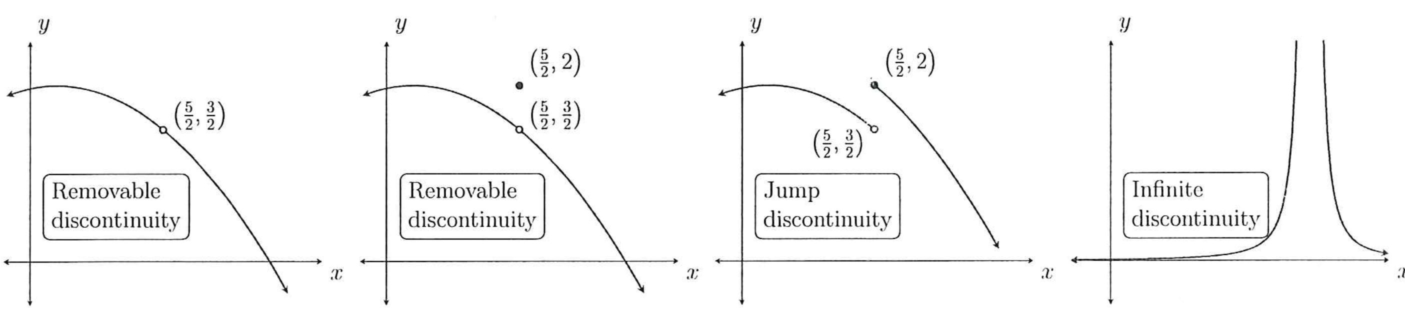
Definition (Continuity at a point). A function f is **continuous** at a if $\lim_{x \to a} f(x) = f(a)$.

> **Continuity Checklist:**
> In order for f to be continuous at a, the following three conditions must hold:
>
> 1. $f(a)$ is defined (a is in the domain of f),
> 2. $\lim_{x \to a} f(x)$ exists,
> 3. $\lim_{x \to a} f(x) = f(a)$ (the value of f equals the limit of f at a).

Graphically:



Types of discontinuity:

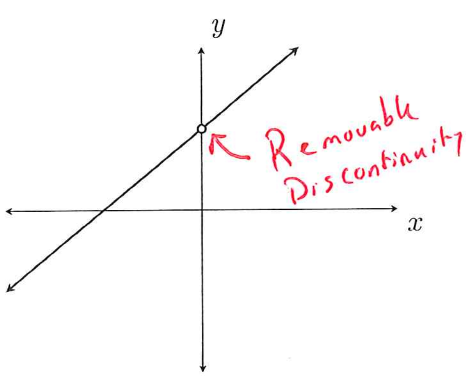


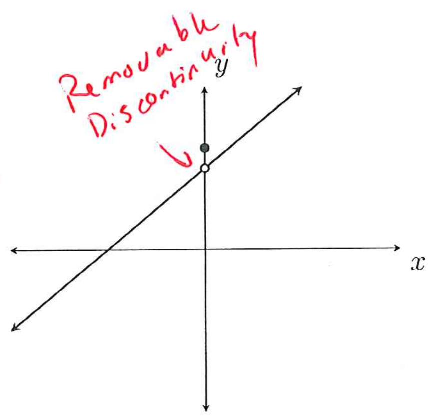
Definition.

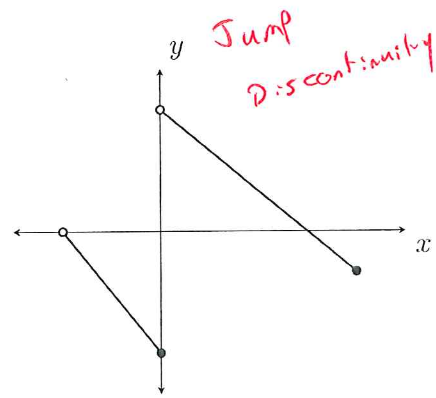
A **removable discontinuity** at $x = a$ is one that disappears when the function becomes continuous after defining $f(a) = \lim_{x \to a} f(x)$.

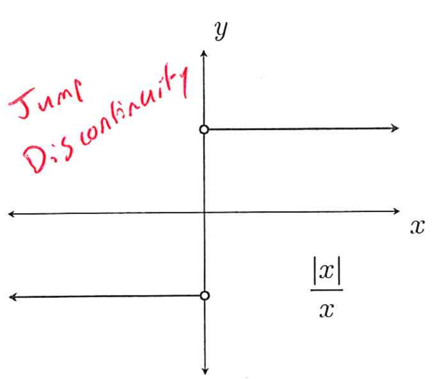
A **jump discontinuity** is one that occurs whenever $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$.

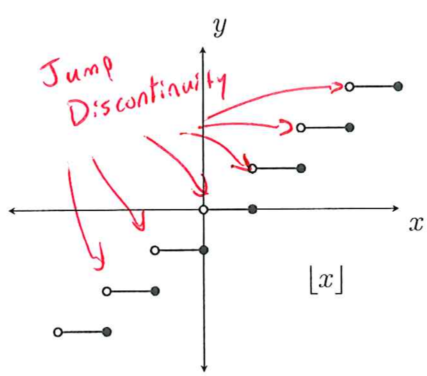
A **vertical discontinuity** occurs whenever $f(x)$ has a vertical asymptote.

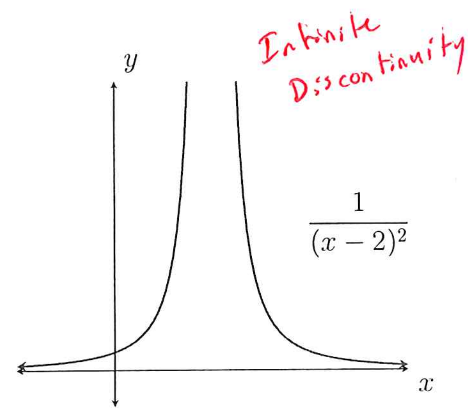


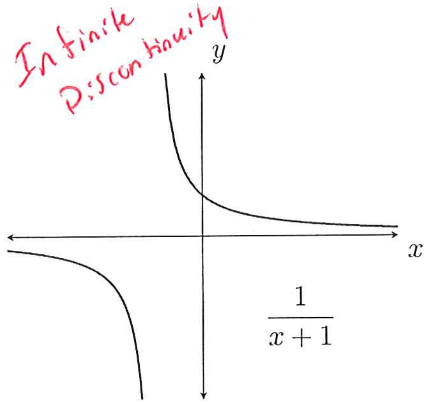


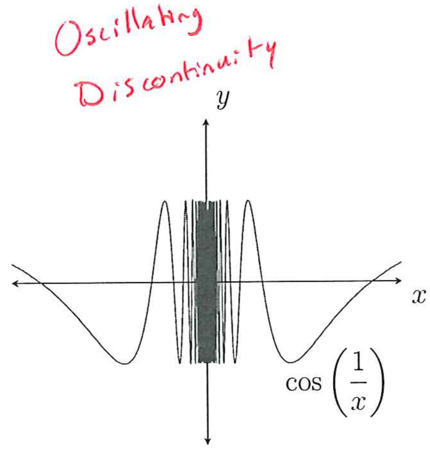


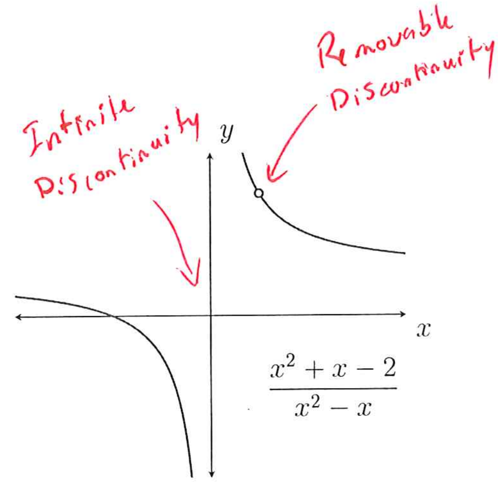


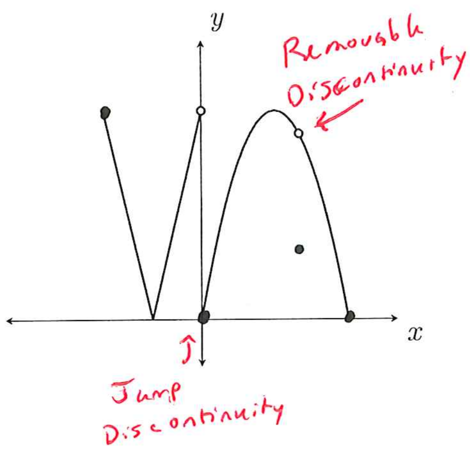


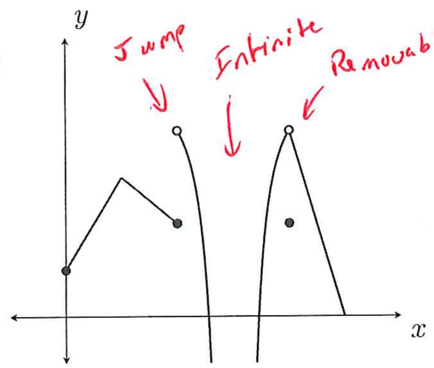


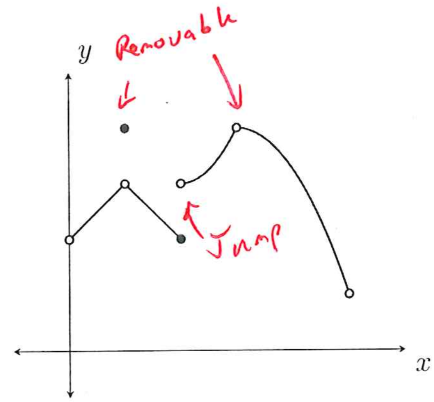








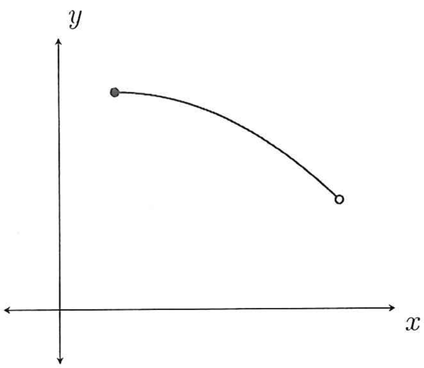




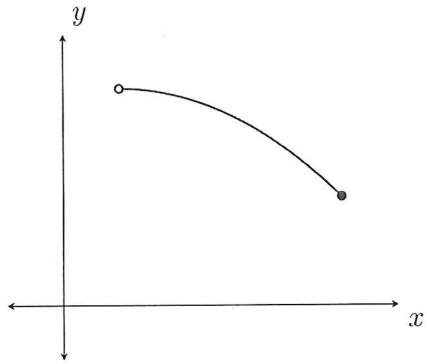
Definition (Continuity at Endpoints). A function f is **continuous from the right** (or **right-continuous**) at a if $\lim_{x \to a^+} f(x) = f(a)$, and f is **continuous from the left** (or **left-continuous**) at b if $\lim_{x \to b^-} f(x) = f(b)$.

Definition (Continuity on an Interval). A function f is **continuous on an interval I** if it is continuous at all points of I. If I contains its endpoints, continuity on I means continuous from the right or left at the endpoints.

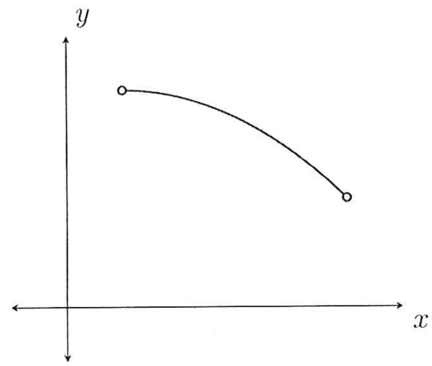
Continuous on $[a, b)$



Continuous on $(a, b]$



Continuous on (a, b)

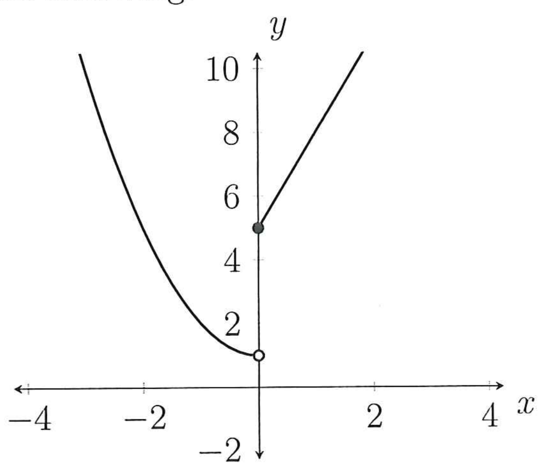


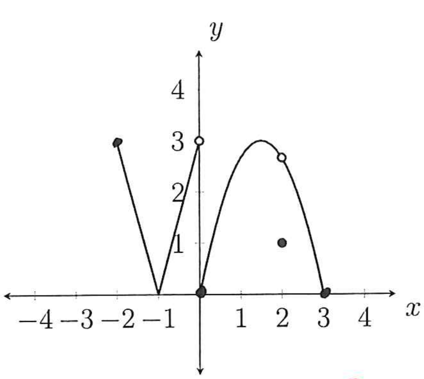
Example. Determine the interval of continuity for the following:

$$f(x) = \begin{cases} x^2 + 1, & x \leq 0 \\ 3x + 5, & x > 0 \end{cases}$$

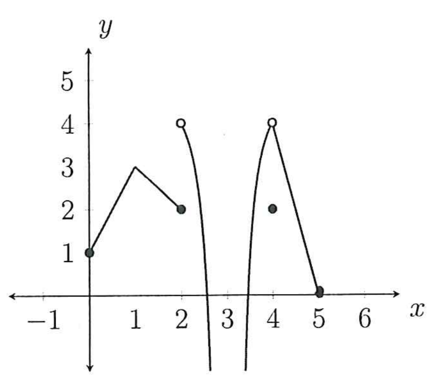
$(-\infty, 0]$ and $(0, \infty)$

$\boxed{\text{Don't union these intervals}}$

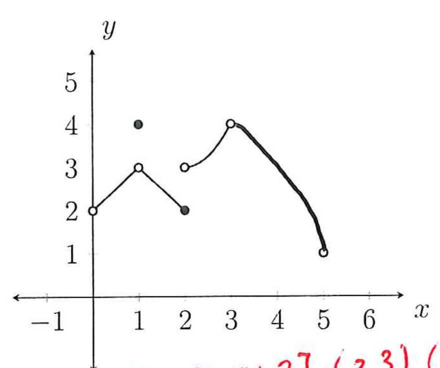




$[-2, 0)$, $[0, 2)$, $(2, 3]$



$[0, 2]$, $(2, 3)$, $(3, 4)$, $(4, 5]$



$(0, 1)$, $(1, 2]$, $(2, 3)$, $(3, 5)$

Example. Determine whether the following are continuous at a:

$f(x) = x^2 + \sqrt{7-x}, \ a = 4$

$f(4) = 4^2 - \sqrt{7-4} = 16 - \sqrt{3}$

Domain:
$7 - x \geq 0$
$7 \geq x$

$\lim_{x \to 4} f(x) = 16 - \sqrt{3}$

$\boxed{\text{Continuous at } x = 4}$

$g(x) = \dfrac{1}{x-3}, \ a = 3$

$g(3)$ DNE $\rightarrow$ $\boxed{\text{Discontinuous at } x = 3}$

$h(x) = \begin{cases} \dfrac{x^2 - x}{x+1}, & x \neq -1 \\ 0, & x = -1 \end{cases}, \ a = -1$

$h(-1) = 0$

$\lim_{x \to -1} h(x) = \lim_{x \to -1} \dfrac{x^2 - x}{x+1}$ DNE

because $\lim_{x \to -1} h(x) = -\infty$ and $\lim_{x \to -1^+} h(x) = \infty$.

$\boxed{\text{Thus, } h(x) \text{ is discontinuous at } x = -1}$

$j(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases}, \ a = 0$

$j(0) = 0$

$\lim_{x \to 0^-} j(x) = \lim_{x \to 0^-} -x = 0$

$\lim_{x \to 0^+} j(x) = \lim_{x \to 0^+} x = 0$

$\lim_{x \to 0} j(x) = 0$

$\boxed{\text{Thus, } j(x) \text{ is continuous at } x = 0}$

$k(x) = \begin{cases} \dfrac{x^2 + x - 6}{x^2 - x}, & x \neq 2 \\ -1, & x = 2 \end{cases}, \ a = 2$

$K(2) = -1$

$\lim_{x \to 2} \dfrac{x^2 + x - 6}{x^2 - x} = \lim_{x \to 2} \dfrac{(x+3)(x-2)}{x(x-1)} = \dfrac{0}{2(1)} = 0$

$\boxed{\text{Thus } K(x) \text{ is discontinuous at } x = 2 \text{ b/c } K(2) \neq \lim_{x \to 2} K(x)}$

Theorem 2.9: Continuity Rules
If f and g are continuous at a, then the following functions are also continuous at a. Assume c is a constant and $n > 0$ is an integer.

a) $f + g$

b) $f - g$

c) cf

d) fg

e) f/g, provided that $g(a) \neq 0$.

f) $(f(x))^n$

Theorem 2.10: Polynomial and Rational Functions

a) A polynomial function is continuous for all x.

b) A rational function (a function of the form $\frac{p}{q}$, where p and q are polynomials) is continuous for all x for which $q(x) \neq 0$.

Theorem 2.11: Continuity of Composite Functions at a Point
If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

Theorem 2.12: Limits of Composite Functions

1. If g is continuous at a and f is continuous at $g(a)$, then

$$\lim_{x \to a} f(g(x)) = f\Big(\lim_{x \to a} g(x)\Big).$$

2. If $\lim_{x \to a} g(x) = L$ and f is continuous at L, then

$$\lim_{x \to a} f(g(x)) = f\Big(\lim_{x \to a} g(x)\Big).$$

Theorem 2.13: Continuity of Functions with Roots
Assume n is a positive integer. If n is an odd integer, then $(f(x))^{1/n}$ is continuous at all points at which f is continuous.
If n is even, then $(f(x))^{1/n}$ is continuous at all points a at which f is continuous at $f(a) > 0$.

Theorem 2.14: Continuity of Inverse Functions
If a function f is continuous on an interval I and has an inverse on I, then its inverse f^{-1} is also continuous (on the interval consisting of the points $f(x)$, where x is in I).

Theorem 2.15: Continuity of Transcendental Functions
The following functions are continuous at all points of their domains.

Trigonometric		**Inverse Trigonometric**		**Exponential**	
$\sin x$	$\cos x$	$\sin^{-1} x$	$\cos^{-1} x$	b^x	e^x
$\tan x$	$\cot x$	$\tan^{-1} x$	$\cot^{-1} x$	**Logarithmic**	
$\sec x$	$\csc x$	$\sec^{-1} x$	$\csc^{-1} x$	$\log_b x$	$\ln x$

Example. Determine the intervals of continuity for the following functions:

a) $g(x) = \dfrac{3x^2 - 6x + 7}{x^2 + x + 1}$

$x^2 + x + 1 \neq 0$

$x \neq \dfrac{-1 \pm \sqrt{1 - 4(1)(1)}}{2} = \dfrac{-1 \pm \sqrt{-3}}{2}$

$\Rightarrow$ Denom never zero
$\Rightarrow$ Domain: $(-\infty, \infty)$

b) $h(x) = \dfrac{3x^2 - 6x + 7}{x^2 - x - 1}$ $\neq 0$

$x \neq \dfrac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

Domain: $\left(-\infty, \frac{1-\sqrt{5}}{2}\right) \cup \left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right) \cup \left(\frac{1+\sqrt{5}}{2}, \infty\right)$

c) $s(x) = \dfrac{x^2 - 4x + 3}{x^2 - 1}$

$x^2 - 1 \neq 0$
$x \neq \pm 1$

$(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

d) $t(x) = \dfrac{x^2 - 4x + 3}{x^2 + 1}$

$x^2 + 1 \neq 0$
$x \neq \sqrt{-1}$

$(-\infty, \infty)$

e) $q(x) = \sqrt[3]{x^2 - 2x - 3}$

Odd powered root $\Rightarrow (-\infty, \infty)$

f) $r(x) = \sqrt{x^2 - 2x - 3}$

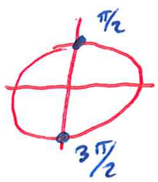
$x^2 - 2x - 3 \geq 0$

$(x-3)(x+1) \geq 0$

		-1		3	
$(x+1)$	$-$		$+$		$+$
$(x-3)$	$-$		$-$		$+$
	$+$		$-$		$+$

$\Rightarrow (-\infty, -1) \cup (3, \infty)$

g) $a(x) = \sec x = \frac{1}{\cos x}$



$\Rightarrow \cos(x) \neq 0$

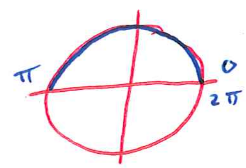
$x \neq \frac{\pi}{2} + k\pi$, k an integer

h) $b(x) = \sqrt{\sin x}$

$\sin x \geq 0$

$[0, \pi], [2\pi, 3\pi], \ldots,$

$[2k\pi, 2(k+1)\pi]$



i) $\ell(x) = \begin{cases} x^3 + 4x + 1, & x \leq 0 \\ 2x^3, & x > 0 \end{cases}$

$\lim_{x \to 0^-} \ell(x) = 1$

$\lim_{x \to 0^+} \ell(x) = 0$

$\Rightarrow (-\infty, 0], (0, \infty)$

j) $m(x) = \begin{cases} \sin x, & x < \frac{\pi}{4} \\ \cos x, & x \geq \frac{\pi}{4} \end{cases}$

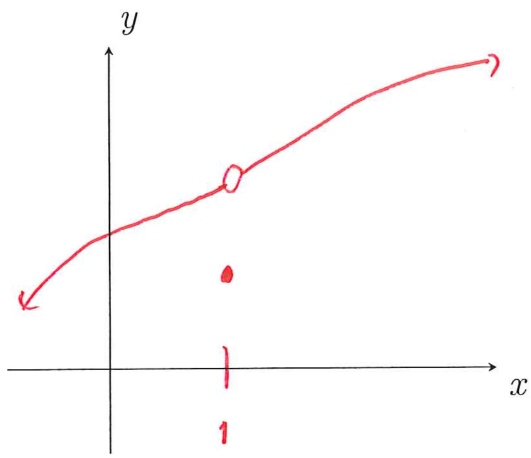
$\lim_{x \to \pi/4^-} m(x) = \lim_{x \to \pi/4^-} \sin(x) = \frac{\sqrt{2}}{2}$

$\lim_{x \to \pi/4^+} m(x) = \lim_{x \to \pi/4^+} \cos(x) = \frac{\sqrt{2}}{2}$

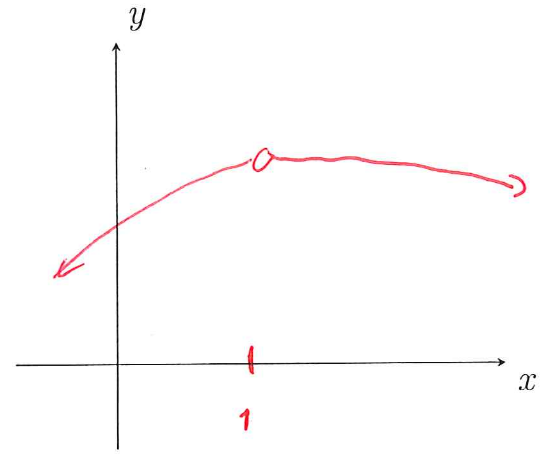
$\Rightarrow (-\infty, \infty)$

Example. Sketch a function that:

Is defined, but not continuous at $x = 1$,



Has a limit, but not continuous at $x = 1$.



Example. Determine the value of a for which $f(x)$ is continuous: $f(c) = \lim_{x \to c} f(x)$

1. $f(x) = \begin{cases} \dfrac{x^3 - 1}{x - 1}, & x \neq 1 \\ a, & x = 1 \end{cases}$

$\lim_{x \to 1} f(x) = \lim_{x \to 1} \dfrac{(x-1)(x^2+x+1)}{x-1} = 3$

$\Rightarrow a = 3$

2. $f(x) = \begin{cases} \dfrac{t^2 + 3t - 10}{t - 2}, & t \neq 2 \\ a, & t = 2 \end{cases}$

$\lim_{t \to 2} f(x) = \lim_{t \to 2} \dfrac{(t+5)(t-2)}{t-2} = 7$

$\Rightarrow a = 7$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = 4$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2} ax^2 - bx + 3 = 4a - 2b + 3$$

3. $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x < 2 \\ ax^2 - bx + 3, & 2 \le x < 3 \\ 2x - a + b, & x \ge 3 \end{cases}$

Since $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) \Rightarrow \underline{4a - 2b + 3 = 4}$ (1)

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3} ax^2 - bx + 3 = 9a - 3b + 3$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3} 2x - a + b = 6 - a + b$

$9a - 3b + 3 = 6 - a + b$

$\Rightarrow \underline{10a - 4b = 3}$ (2)

(1) $(4a - 2b = 1)(-2)$

(2) $10a - 4b = 3$

$2a + 0b = 1 \Rightarrow \boxed{a = 1/2}$

(1) $\Rightarrow 4(\frac{1}{2}) - 2b = 1$

$\Rightarrow \boxed{b = \frac{1}{2}}$

Example. Redefine the following functions so that they are continuous everywhere:

1. $g(x) = \dfrac{x^3 - x^2 - 2x}{x-2} = \dfrac{x(x-2)(x+1)}{x-2} = x(x+1), \quad x \neq 2$

$g(2)$ DNE

Redefine

$$\boxed{g(x) = x(x+1)}$$

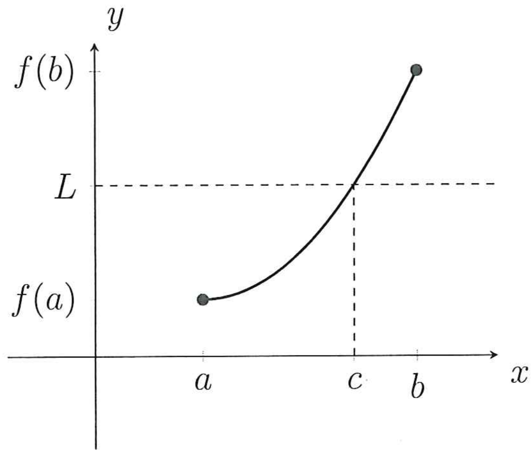
2. $g(x) = \dfrac{x^2 + x - 6}{x - 2} = \dfrac{(x-2)(x+3)}{x-2} = x+3, \quad x \neq 2$

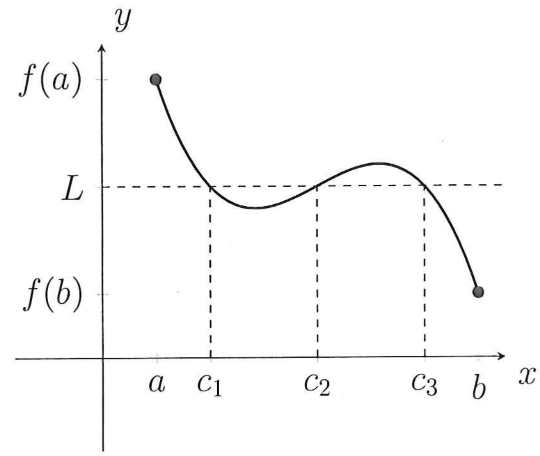
$g(2)$ DNE

Redefine

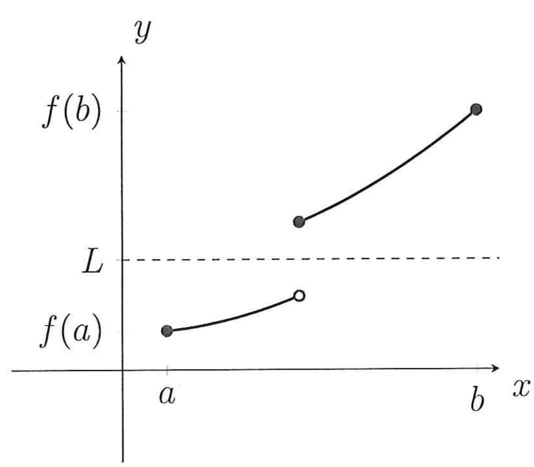
$$\boxed{g(x) = x+3}$$

Theorem 2.16: Intermediate Value Theorem
Suppose f is continuous on the interval $[a, b]$ and L is a number strictly between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) satisfying $f(c) = L$.





Note: It is important that the function be continuous on the interval $[a, b]$:



Example. Show that $f(x)$ has a root using the IVT: $f(x) = x^3 + 4x + 4$

$f(0) = 4$

$f(-1) = (-1)^3 + 4(-1) + 4 = -1$

Since $f(x)$ is continuous on $(-1, 0)$ and $f(-1) = -1$ and $f(0) = 4$ then by the IVT, there exists a c such that $-1 \leq c \leq 0$ and $f(c) = 0$.

Example. Show that $\sqrt{x^4 + 25x^3 + 10} = 5$ on the interval $(0, 1)$.

$\sqrt{0^4 + 25(0)^3 + 10} = \sqrt{10}$

Note $\left.\begin{matrix} \sqrt{9} = 3 \\ \sqrt{16} = 4 \end{matrix}\right\}$ $3 \leq \sqrt{10} \leq 4$

$\sqrt{1^4 + 25(1)^3 + 10} = \sqrt{36} = 6$

Since $\sqrt{x^4 + 25x^3 + 10}$ is continuous on $(0, 1)$ and $\sqrt{10} < 5 < \sqrt{36}$, then there exists some value c such that $0 < c < 1$ and $\sqrt{c^4 + 25c^3 + 10} = 5$.

Example. Show that $\underbrace{-x^5 - 4x^2 + 2\sqrt{x} + 5}_{h(x)} = 0$ on $(0, 3)$.

$h(0) = 5$

$h(3) = -243 - 4(9) + 2\sqrt{3} + 5 = -319 + 2\sqrt{3} < 0$

Since $h(x)$ is continuous for all $x \geq 0$ and $h(3) < 0 < h(0)$, then there exists c such that $0 < c < 3$ and $h(c) = 0$