3.6 Rate of Change Applications

Definition. (Average and Instantaneous Velocity)

Let $s = f(t)$ be the position function (sometimes referred to as the **displacement** function) of an object moving along a line. The **average velocity** of the object over the time interval $[a, a + \Delta t]$ is the slope of the secant line between $(a, f(a))$ and $(a + \Delta t, f(a + \Delta t))$:

$$v_{avg} = \frac{f(a + \Delta t) - f(a)}{\Delta t}$$

The **instantaneous velocity** at a is the slope of the line tangent to the position curve, which is the derivative of the position function:

$$v(a) = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t} = f'(a).$$

Example (Position and velocity of a patrol car).

Assume a police station is located along a straight east-west freeway. At noon ($t = 0$), a patrol car leaves the station heading east. The position function of the car $s = f(t)$ gives the location of the car in miles east ($s > 0$) or west ($s < 0$) of the station t hours after noon.

a) Describe the location of the patrol car during the first 3.5hr of the trip.

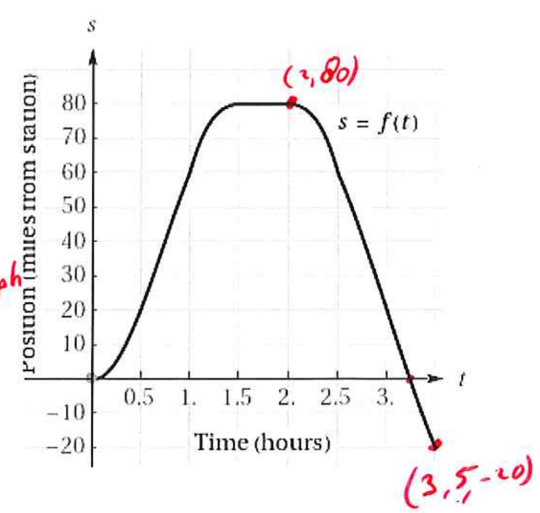
b) Calculate the displacement and average velocity of the car between 2:00 P.M. and 3:30 P.M. ($2 \le t \le 3.5$).

$s(3.5) - s(2) = -100\text{mi}$

$V_{avg} = \frac{s(3.5) - s(2)}{3.5 - 2} = \frac{-100\text{ mi}}{1.5\text{ hr}} = -66.\bar{7}\text{mph}$

c) At what time(s) is the instantaneous velocity greatest *as the car travels east*?

$\frac{1}{2} \le t \le 1 \Rightarrow$ between 12:30 and 1:00



Definition. (Velocity, Speed, and Acceleration)

Suppose an object moves along a line with position $s = f(t)$. Then

the **velocity** at time t is $\quad v = \dfrac{ds}{dt} = f'(t)$

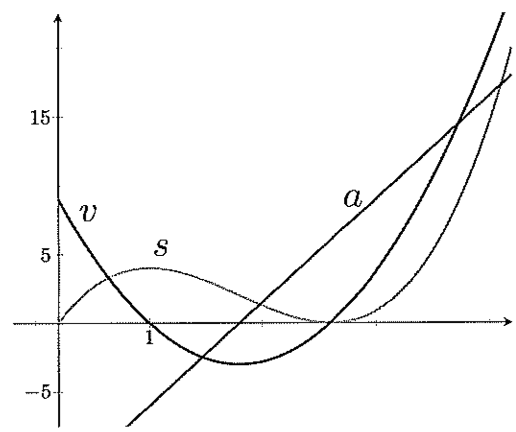
the **speed** at time t is $\quad |v| = |f'(t)|$, and

the **acceleration** at time t is $\quad a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2} = f''(t)$.

- Velocity indicates direction:

  forward is positive, backward is negative
- Speed is direction independent:

  $$v(t) = -30m/s \Rightarrow s(t) = 30m/s.$$
- If velocity changes signs, then velocity was zero. A velocity of zero does not indicate a change in direction.



Example. $s = -t^3 + 3t^2 - 3t,\ 0 \le t \le 3$ gives the position $s = f(t)$ of a body moving on a coordinate line, with s in meters and t in seconds.

1. Find the body's displacement and average velocity for the given time interval.
2. Find the body's speed and acceleration at the endpoints of the interval.
3. When, if ever, during the interval does the body change direction?

1) $s(3) - s(0) = (-27+27-9) + 0 = -9$ meters

$\frac{s(3) - s(0)}{3-0} = -3$ m/sec

2) $v(t) = s'(t) = -3t^2 + 6t - 3$

$v(0) = -3\,\%$, $v(3) = -27 + 18 - 3 = -12$ m/s

$a(t) = v'(t) = s''(t) = -6t + 6$

$a(0) = 6$ m/s^2 , $a(3) = -18 + 6 = -12$ m/s^2

3) When changes direction, $v(t) = 0$

$t=1$

$v(t)$ $\overline{- - - \mid - - -}$

$v(t) = -3t^2 + 6t - 3 \overset{\text{set}}{=} 0$

$-3(t^2 - 2t + 1) = 0$

$-3(t-1)^2 = 0$

$t = 1$

For vertical motion (e.g. an object thrown up in the air), an object's maximum height occurs when velocity is zero and hits the ground at height zero.

Example. A rock is thrown vertically upward from the surface of the moon at a velocity of 24 m/sec (about 86 km/h) reaches a height of $s = 24t - 0.8t^2$ meters in t sec.

1. Find the rock's velocity an acceleration at time t. (The acceleration in this case is the acceleration of gravity on the moon.)
2. How long does it take for the rock to reach it's highest point?
3. How high does the rock go?
4. When does the rock hit the ground?
5. What is the velocity at that instant?

1) $v(t) = s'(t) = 24 - 1.6t$

$a(t) = s''(t) = -1.6$

2) Solve $v(t) = 0 \Rightarrow 24 - 1.6t = 0$

$24 = 1.6t$

$t = 15\,s$

3) $s(15) = 24(15) - 0.8(15)^2 = 360 - 180 = 180\ m$

4) Solve $s(t) = 0 \Rightarrow 24t - 0.8t^2 = 0$

$0.8t(30 - t) = 0 \Rightarrow$ $t = 0$ sec, $\boxed{t = 30 \text{ sec}}$

5) $v(30) = 24 - 1.6(30) = -24$ m/s

Example. Suppose a stone is thrown vertically upward from the edge of a cliff on Earth with an initial velocity of 32 ft/s from a height of 48 ft above the ground. The height (in feet) of the stone above the ground t seconds after it is thrown is $s(t) = -16t^2 + 32t + 48$.

1. Determine the velocity v of the stone after t seconds.
2. When does the stone reach its highest point?
3. What is the height of the stone at the highest point?
4. When does the stone strike the ground?
5. With what velocity does the stone strike the ground?
6. On what intervals is the speed increasing?

1) $v(t) = s'(t) = -32t + 32$

2) Solve $v(t) = 0 \Rightarrow -32t + 32 = 0$

$\boxed{t = 1\ s}$

3) $s(1) = -16 + 32 + 48 = \boxed{64\ m.}$

4) Solve $s(t) = 0 \Rightarrow -16t^2 + 32t + 48 = 0$

$-16(t^2 - 2t - 3) = 0$

$-16(t+1)(t-3) = 0 \Rightarrow$ $t = -1$ $\boxed{t = 3\ secs}$

5) $v(3) = -32(3) + 32 = \boxed{-64\ m/s}$

6) Find where $a(t) = -32$ and $v(t) = -32t + 32$ have the same sign

	1.5	
$a(t)$	neg	neg
$v(t)$	pos	neg

$\Rightarrow$ speed is increasing on interval $1.5 \le t \le 3$

Example (Velocity of a bullet). A bullet is fired vertically into the air at an initial velocity of 1200 ft/s. On Mars, the height s (in feet) of the bullet above the ground after t seconds is $1200t - 6t^2$ and on Earth, $s = 1200t - 16t^2$. How much higher will the bullet travel on Mars than on Earth?

Let $s_M(t) = 1200t - 6t^2$ be the displacement function on Mars

and $s_E(t) = 1200t - 16t^2$ be the displacement function on Earth.

1. Find $v_M(t)$ and $v_E(t)$ and solve both for zero

$v_M(t) = 1200 - 12t \overset{set}{=} 0$

$1200 = 12t$

$\boxed{100 = t}$

$V_E(t) = 1200 - 32t \overset{set}{=} 0$

$1200 = 32t$

$\boxed{37.5 = t}$

2. Use this time to find the heights and compare.

$s_M(100) = 1200(100) - 6(100)^2$

$= 60{,}000$ ft

$s_E(37.5) = 1200(37.5) - 16(37.5)^2$

$= 22{,}500$ ft

$\boxed{\text{Thus, the bullet traveled } 60{,}000 - 22{,}500 = 37{,}500 \text{ ft higher on Mars}}$.

Definition. (Average and Marginal Cost)

The **cost function** $C(x)$ gives the cost to produce the first x items in a manufacturing process. The **average cost** to produce x items is $\bar{C}(x) = C(x)/x$. The **marginal cost** $C'(x)$ is the approximate cost to produce one additional item after producing x items.

Example. Suppose $C(x) = 10,000+5x+0.01x^2$ dollars is the estimated cost of producing x items. The marginal cost at the production level of 500 items is:

$C'(x) = 5 + 0.02x$

$C'(500) = 5 + 10 = \$15$ per items

v.s.

$C(501) - C(500) = 15015.01 - 15,000 = 15.01$

Example. The cost function for production of a commodity is

$$C(x) = 339 + 25x - 0.09x^2 + 0.0004x^3$$

1. Find and interpret $C'(100)$.
2. Compare $C'(100)$ with the cost of producing the 101st item.

1) $C'(x) = 25 - 0.18x + 0.0012x^2$
This gives the approximate cost, in dollars, of producing the $x+1^{st}$ additional item.

2) $C'(100) = 25 - 18 + 12 = \$19$ v.s. $C(101) - C(100)$
$= 2358.0304 - 2339$
$= \$19.03$

Example. For the following cost functions,

a) Find the average cost and marginal cost functions.

b) Determine the average cost and the marginal cost when $x = a$.

c) Interpret the values obtained in part (b)

1. $C(x) = 500 + 0.02x,\ 0 \leq x \leq 2000,\ a = 1000.$

a) $\bar{C}(x) = \frac{500}{x} + 0.02$, $C'(x) = 0.02$

b) $\bar{C}(1000) = \frac{500}{1000} + 0.02 = \$0.52$ per item

$C'(1000) = \$\ 0.02$

c) It costs, on average, \$0.52 to create each item when making 1000 items and approximately \$0.02 to create the 1001st item.

2. $C(x) = -0.01x^2 + 40x + 100,\ 0 \leq x \leq 1500,\ a = 1000.$

a) $\bar{C}(x) = -0.01x + 40 + \frac{100}{x}$

$C'(x) = -0.02x + 40$

b) $\bar{C}(1000) = -10 + 40 + 0.1 = \$\ 30.1$ per item

$C'(1000) = -20 + 40 = \$\ 20$

c) It costs, on average, \$ 30.10 to create each item when 1000 items are produced and costs approximately \$20 to create the 1001st item.