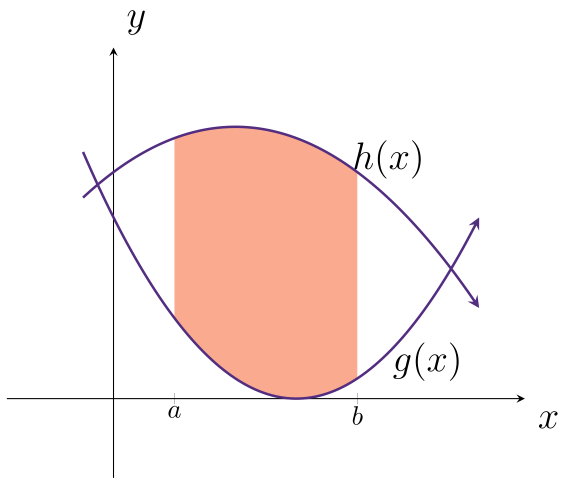
16.2: Double Integrals over General Regions

In this section, we consider double integrals over non-rectangular regions. For instance, my domain for x and y can be constrained where $a \leq x \leq b$ and $g(x) \leq y \leq h(x)$:



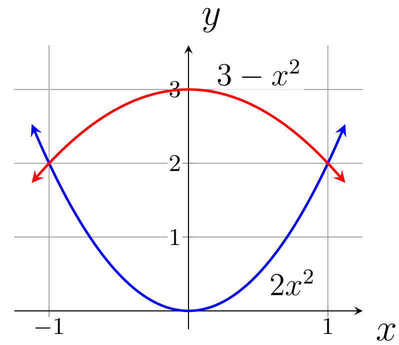
Theorem 16.2: Double Integrals over Nonrectangular Regions
Let R be a region bounded below and above by the graphs of the continuous functions $y = g(x)$ and $y = h(x)$, respectively, and by the lines $x = a$ and $x = b$. If f is continuous on R, then

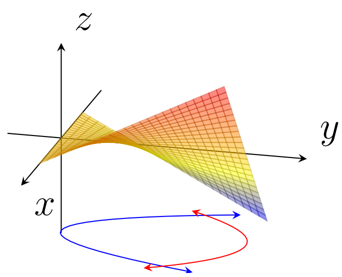
$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx.$$

Let R be a region bounded on the left and right by the graphs of the continuous functions $x = g(y)$ and $x = h(y)$, respectively, and the lines $y = c$ and $y = d$. If f is continuous on R, then

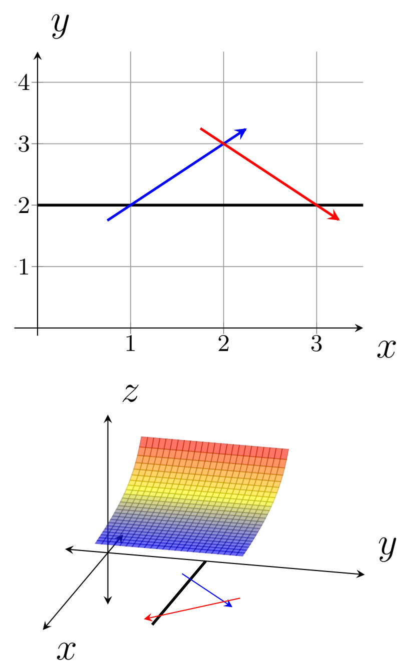
$$\iint\limits_R f(x, y)\, dA = \int_c^d \int_{g(y)}^{h(y)} f(x, y)\, dx\, dy.$$

Example. Consider the surface generated by the function $f(x, y) = 3xy$. Find the volume of the solid generated by $f(x, y)$ over the region bounded by $2x^2$ and $3 - x^2$.





Example. Find the area under $f(x, y) = \frac{1}{x} + 1$ over the region formed by the lines $x = 2$, $y = 1 + x$, and $y = 5 - x$.



Example. Find the volume of the tetrahedron in the first octant bounded by the plane $z = c - ax - by$ and the coordinate planes ($x = 0$, $y = 0$, and $z = 0$). Assume a, b, and c are positive real numbers.

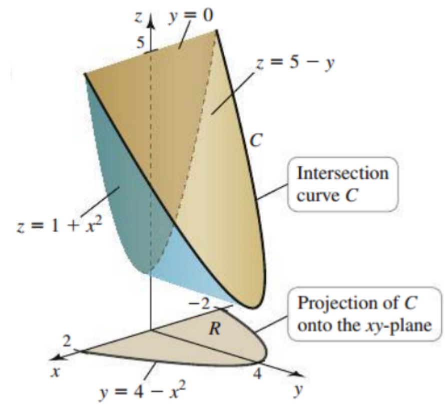
Example. For the following problems, reverse the order of integration

- $\displaystyle\int_0^2 \int_0^{2x} f(x,y)\,dy\,dx$

- $\displaystyle\int_0^1 \int_{x^3}^{\sqrt{x}} f(x,y)\,dy\,dx$

- $\displaystyle\int_{-3}^4 \int_{2x^2}^{2x+24} f(x,y)\,dy\,dx$

Example. Find the volume between $f(x, y) = 5 - y$ and $g(x, y) = 1 + x^2$ over the region $R = \{(x, y) : 0 \le y \le 4 - x^2,\ -2 \le x \le 2\}$.



Areas of Regions by Double Integrals

Let R be a region in the xy-plane. Then

$$\text{area of } R = \iint\limits_R dA.$$

Example. Find the area of the region R bounded by $y = x^2$, $y = 6 - x$, and $y = 6 + 5x$ where $x \geq 0$.