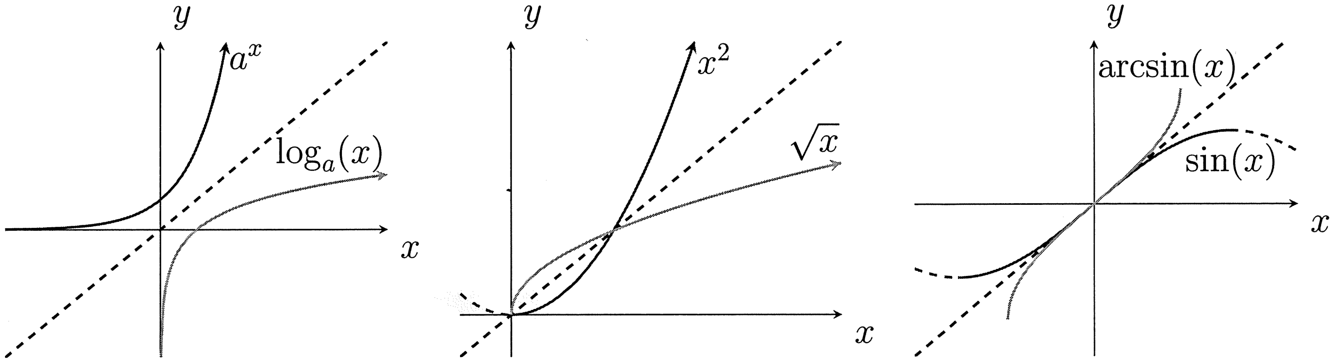
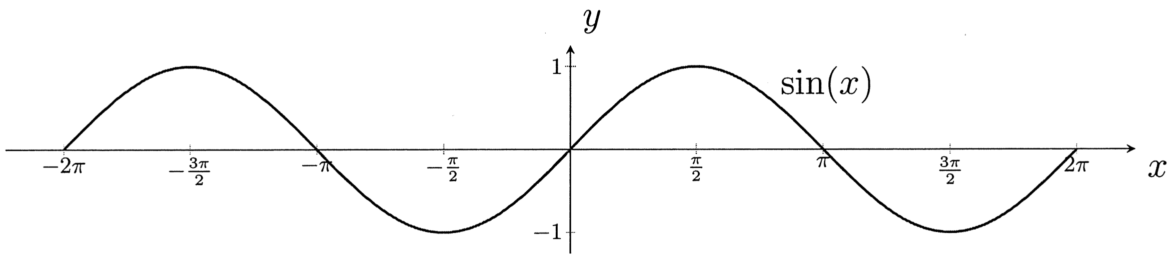
9.1 Definition of $\arcsin x$, the Inverse Sine Function

Recall that a function has an inverse if it is 1-to-1 (e.g. it passes the horizontal line test).

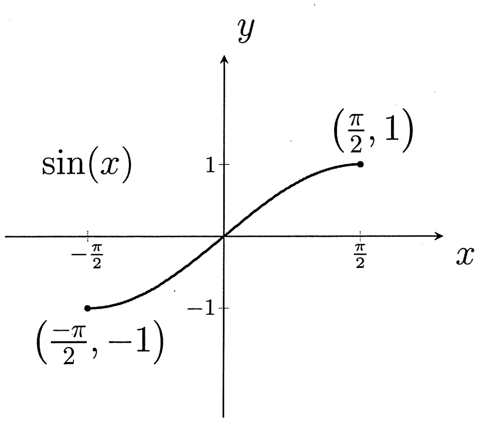


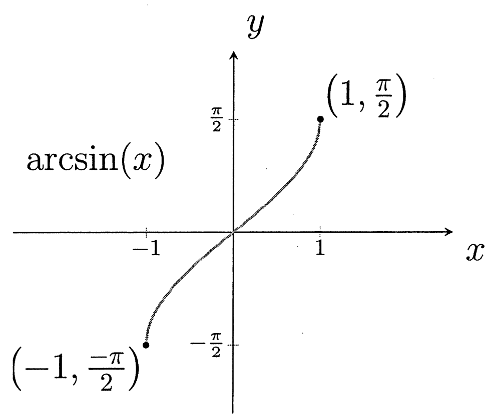
Notice that x^2 and $\sin(x)$ are on restricted domains.

Without restriction on its domain, $\sin(x)$ is NOT 1-to-1:



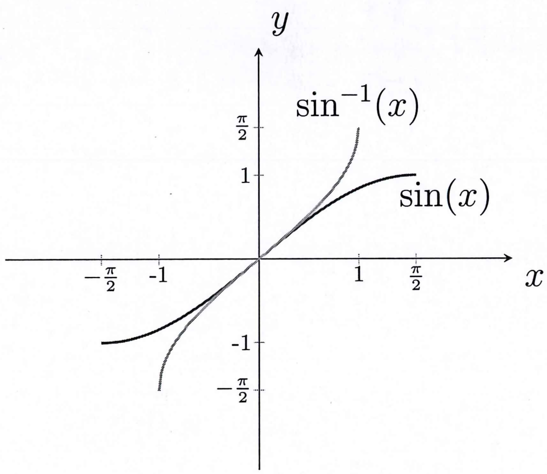
The range of $\sin(x)$ is $[-1, 1]$ and all of these values are attained on a restricted domain of $[-\pi/2, \pi/2]$:

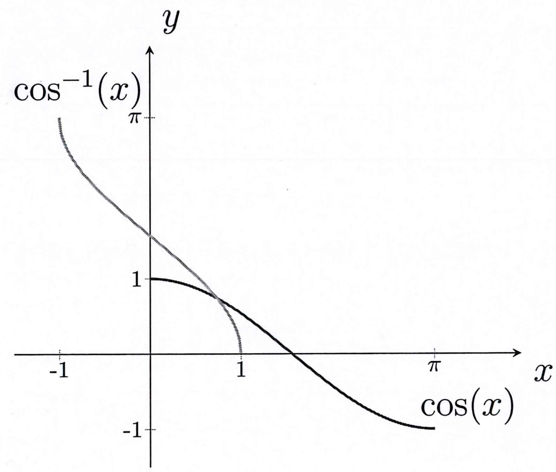




> **Definition. (Inverse Sine and Cosine)**
>
> $y = \sin^{-1}(x)$ is the value of y such that $x = \sin(y)$, where $-\pi/2 \le y \le \pi/2$.
>
> $y = \cos^{-1}(x)$ is the value of y such that $x = \cos(y)$, where $0 \le y \le \pi$.
>
> The domain of both $\sin^{-1}(x)$ and $\cos^{-1}(x)$ is $\{x \mid -1 \le x \le 1\}$.

Note: The inverse sine function can be denoted as $\arcsin(x)$ or $\sin^{-1}(x)$.
This means that $\sin^{-1}(x) \neq \dfrac{1}{\sin(x)}$.
Similarly, $\arccos(x)$ and $\cos^{-1}(x)$ denote the inverse cosine functions.





Example. Solve the following:

$\sin^{-1}(0) = \boxed{0}$
$\sin(x) = 0$

$\arcsin(1) = \boxed{\pi/2}$
$\sin(x) = 1$

$\sin(x) = \frac{\sqrt{3}}{2}$
$\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \boxed{\pi/3}$

$\cos^{-1}(-1) = \boxed{\pi}$
$\cos(x) = -1$

$\cos^{-1}\left(-\dfrac{1}{2}\right) = \boxed{2\pi/3}$
$\cos(x) = -\frac{1}{2}$

$\arccos\left(-\dfrac{\sqrt{3}}{2}\right) = \boxed{\frac{5\pi}{6}}$
$\cos(x) = -\frac{\sqrt{3}}{2}$

9.2 The Functions $\arctan x$ and $\operatorname{arcsec} x$

Similar to $\sin^{-1}(x)$, we also have inverse functions for the restricted $\tan(x)$ and $\sec(x)$ functions.

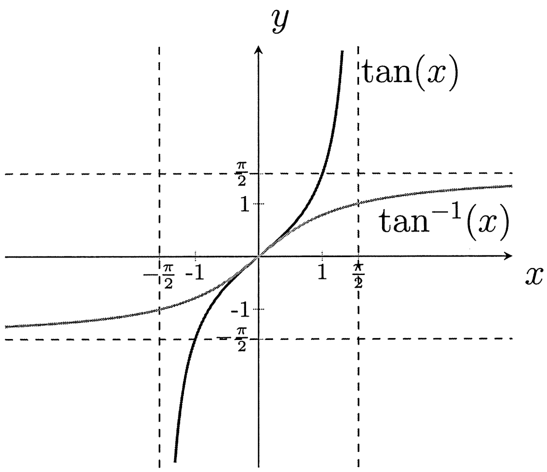
Definition. (Inverse Tangent and Secant)

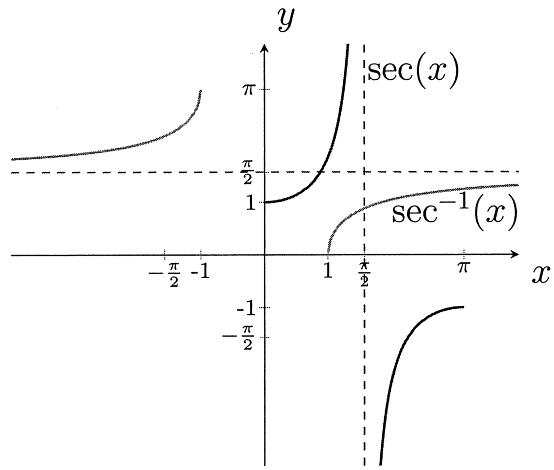
$y = \tan^{-1}(x)$ is the value of y such that $x = \tan(y)$, where $-\pi/2 < y < \pi/2$.

The domain of $\tan^{-1}(x)$ is $\{x \mid -\infty < x < \infty\}$.

$y = \sec^{-1}(x)$ is the value of y such that $x = \sec(y)$, where $0 \le y \le \pi,\ y \ne \pi/2$.

The domain of $\sec^{-1}(x)$ is $(-\infty, -1] \cup [1, \infty)$





The *Just In Time* book does not include information on $\cos^{-1}(x)$, $\cot^{-1}(x)$ and $\csc^{-1}(x)$. Section 1.4 of *Briggs* provides a very nice reference:

Definition. (Other Inverse Trigonometric Functions)

$y = \cot^{-1}(x)$ is the value of y such that $x = \cot(y)$, where $0 < y < \pi$.

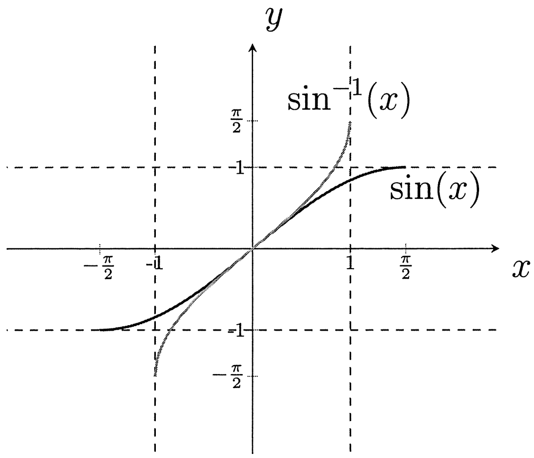
The domain of $\cot^{-1}(x)$ is $\{x \mid -\infty < x < \infty\}$.

$y = \csc^{-1}(x)$ is the value of y such that $x = \csc(y)$, where $-\pi/2 \le y \le \pi/2,\ y \ne 0$.

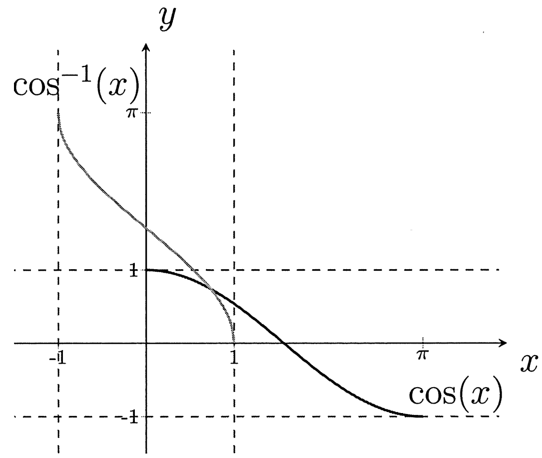
The domain of $\csc^{-1}(x)$ is $(-\infty, -1] \cup [1, \infty)$

Function	Restricted Domain	Range
$\sin(x)$	$[-\pi/2, \pi/2]$	$[-1, 1]$
$\cos(x)$	$[0, \pi]$	$[-1, 1]$
$\tan(x)$	$(-\pi/2, \pi/2)$	$(-\infty, \infty)$
$\cot(x)$	$(0, \pi)$	$(-\infty, \infty)$
$\sec(x)$	$[0, \pi/2) \cup (\pi/2, \pi]$	$(-\infty, -1] \cup [1, \infty)$
$\csc(x)$	$[-\pi/2, 0) \cup (0, \pi/2]$	$(-\infty, -1] \cup [1, \infty)$

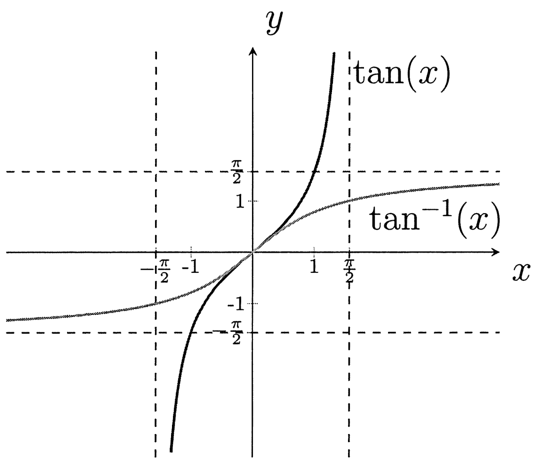
Function	Domain	Range
$\sin^{-1}(x)$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}(x)$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1}(x)$	$(-\infty, \infty)$	$(0, \pi)$
$\sec^{-1}(x)$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$
$\csc^{-1}(x)$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$



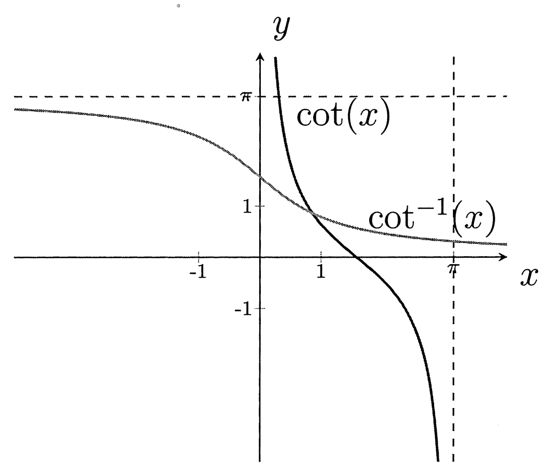
y
$\sin^{-1}(x)$
$\sin(x)$
x



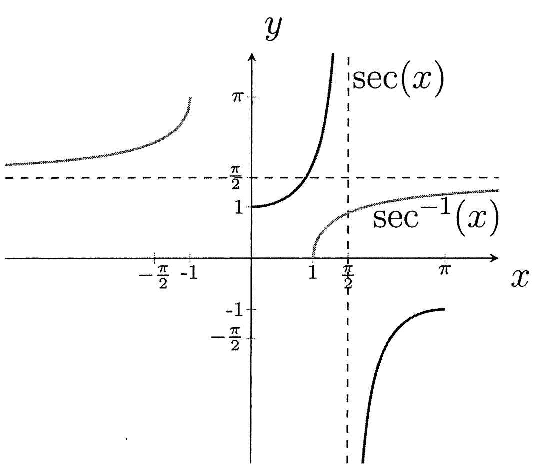
y
$\cos^{-1}(x)$
$\cos(x)$
x



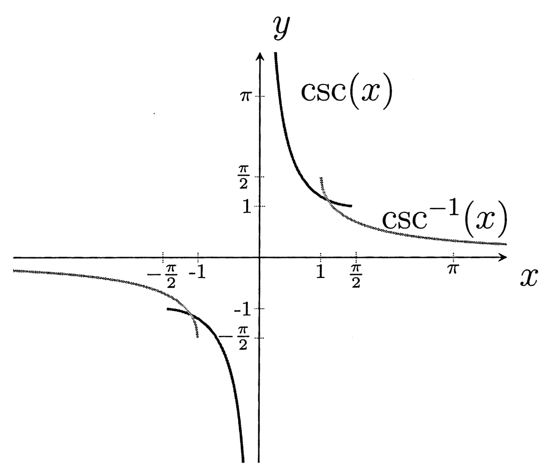
y
$\tan(x)$
$\tan^{-1}(x)$
x



y
$\cot(x)$
$\cot^{-1}(x)$
x

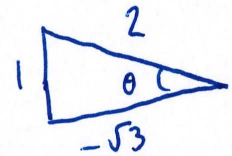


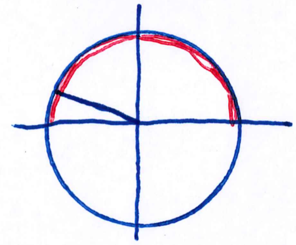
y
$\sec(x)$
$\sec^{-1}(x)$
x



y
$\csc(x)$
$\csc^{-1}(x)$
x

$$-\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$





Example. Solve the following:

$\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \boxed{-\pi/4}$ $\qquad \sin(x) = -\frac{\sqrt{2}}{2}$

$\sec^{-1}(2) = \boxed{\pi/3}$ $\qquad \sec(x) = 2$ $\qquad \cos(x) = \frac{1}{2}$

$\cot^{-1}(-\sqrt{3}) = 5\pi/6$ $\qquad \cot(x) = \frac{-\frac{\sqrt{3}}{2}}{1/2}$

9.3 Inverse Trigonometric Identities

While $\sin(x)$ and $\sin^{-1}(x)$ are inverse functions, the inverse relationship only holds when working in the correct domains:

$$\sin^{-1}(\sin(\pi)) = \sin^{-1}(0) = 0 \neq \pi \qquad\qquad \sin\left(\sin^{-1}(-1)\right) = \sin(-\pi/2) = -1$$

Example. Solve the following:

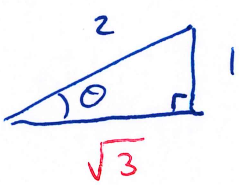
$\tan(\tan^{-1}(5)) = \boxed{5}$

$\because$ Domain of $\tan^{-1}(x)$ is $(-\infty, \infty)$

$\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}(-1) = \boxed{-\frac{\pi}{4}}$

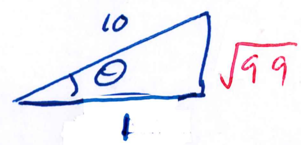
$\because$ Range of $\tan^{-1}(x)$ is $(-\pi/2, \pi/2)$

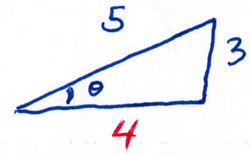
$\cos\left(\underbrace{\arcsin\frac{1}{2}}_{\theta}\right) = \boxed{\frac{\sqrt{3}}{2}}$



$\cos^{-1}(\cos(5\pi)) = \cos^{-1}(-1) = \pi$

$\because$ $\cos(x)$ is restricted to $[0, \pi]$ and $\cos^{-1}(x)$ has a range of $[0, \pi]$





$$\sin^{-1}\left(\sin\left(\frac{7\pi}{3}\right)\right)$$

$= \sin^{-1}(\sin(\pi/3))$

$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$= \boxed{\pi/3}$

$$\tan(\underbrace{\sec^{-1}(10)}_{\theta}) = \boxed{\sqrt{99}}$$

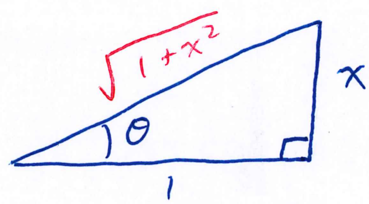
$$\sin\left(\underbrace{2\sin^{-1}\left(\frac{3}{5}\right)}_{2\theta}\right)$$

$\sin(2\theta) = 2\sin\theta\cos\theta$

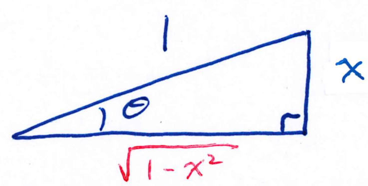
$= 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \boxed{\frac{24}{25}}$

Example. Simplify the following using triangles.

$$\cos(\underbrace{\tan^{-1}(x)}_{\theta}) = \frac{1}{\sqrt{1+x^2}}$$



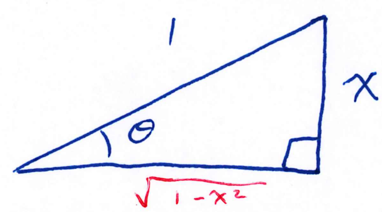
$$\sec(\underbrace{\sin^{-1}(x)}_{\theta}) = \sqrt{1-x^2}$$



$$\cos(2\underbrace{\sin^{-1}(x)}_{\theta})$$

$= \cos(2\theta)$

$= \cos^2\theta - \sin^2\theta = \left(\sqrt{1-x^2}\right)^2 - (x)^2 = \boxed{1-2x^2}$



$$\sin(2\underbrace{\tan^{-1}(x)}_{\theta})$$

$= \sin(2\theta) = 2\sin\theta\cos\theta$

$= 2\,\frac{x}{\sqrt{1+x^2}}\,\frac{1}{\sqrt{1+x^2}}$

$= \boxed{\frac{2x}{1+x^2}}$