11.3: Taylor Series

Definition. (Taylor/Maclaurin Series for a Function)
Suppose the function f has derivatives of all orders on an interval centered at the point a. The **Taylor series for f centered at a is**

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

A Taylor series centered at 0 is called a **Maclaurin series**.

Example (LC 30.1)**.** Can we find a Taylor series centered at $a = 0$ for $f(x) = \sqrt{x}$?

Example (LC 30.2-30.5)**.** Consider the function $f(x) = \sin(\pi x)$ and the Taylor series representation centered at $a = 0$.

Find the first four nonzero terms

Write this Taylor series using summation notation

Theorem 11.7: Convergence of Taylor Series
Let f have derivatives of all orders on an open interval I containing a. The Taylor series for f centered at a converges to f, for all x in I, if and only if $\lim_{n\to\infty} R_n(x) = 0$, for all x in I, where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

is the remainder at x, with c between x and a.

What is the interval of convergence?

What is the upper bound on $|R_n(x)|$?

Example (LC 30.6)**.** If a Taylor series only converges on $(-2, 2)$, does $f(x^2)$ have a Taylor series that also only converges on $(-2, 2)$?

Example (LC 30.7)**.** Use the definition of a Taylor series to find the Taylor series for $f(x) = e^{2x}$ at $a = 3$.

Example (LC 30.8)**.** Given that $\ln(1+x) = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k}$, for $-1 < x \leq 1$, find the first nonzero terms of the Taylor series centered at $a = 0$ for the function $\ln(1+2x)$.

Example (LC 30.9)**.** Given that $\cos(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$, for $|x| < \infty$, find the Taylor series centered at $a = 0$ for the function $x\cos(x^3)$.

Common Taylor Series:

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^k + \dots = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \dots + (-1)^k x^k + \dots = \sum_{k=0}^{\infty} (-1)^k x^k, \quad \text{for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^k}{k!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^k x^{2k}}{(2k)!} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^{k+1} x^k}{k} + \dots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \quad \text{for } -1 < x \leq 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^k}{k} + \dots = \sum_{k=1}^{\infty} \frac{x^k}{k}, \quad \text{for } -1 \leq x < 1$$

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + \frac{(-1)^k x^{2k+1}}{2k+1} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \quad \text{for } |x| \leq 1$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2k+1}}{(2k+1)!} + \dots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2k}}{(2k)!} + \dots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k, \text{ for } |x| < 1 \text{ and } \binom{p}{k} = \frac{p(p-1)(p-2)\dots(p-k+1)}{k!}, \ \binom{p}{0} = 1$$