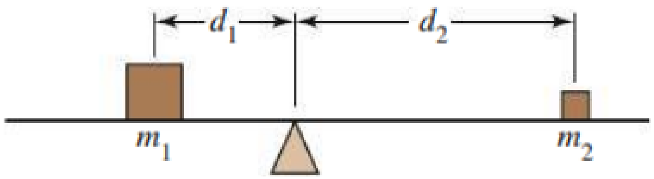
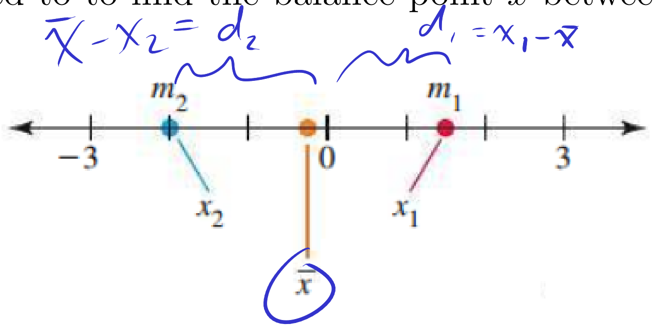
16.6: Integrals for Mass Calculations

Suppose we have two masses m_1 and m_2 on a beam (with no mass) that are distances d_1 and d_2 away from a pivot point. This beam will be balanced when $m_1 d_1 = m_2 d_2$.



This concept can be used to to find the balance point $\bar{x}$ between 2 objects with masses m_1 and m_2:



$$m_1(x_1 - \bar{x}) = m_2(\bar{x} - x_2) \;\Rightarrow\; m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) = 0.$$

$$\Rightarrow \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Next, we can generalize this to n objects with masses $m_1, \ldots, m_n$:

$$m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) + \cdots + m_n(x_n - \bar{x}) = \sum_{k=1}^{n} m_k(x_k - \bar{x}) = 0.$$

$$\Rightarrow \bar{x} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k}$$

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Definition. (Center of Mass in One Dimension)
Let ρ be an integrable density function on the interval $[a, b]$ (which represents a thin rod or wire). The **center of mass** is located at the point $\bar{x} = \frac{M}{m}$, where the **total moment** M and mass m are

$$M = \int_a^b x\rho(x)\,dx \quad \text{and} \quad m = \int_a^b \rho(x)\,dx.$$

Example. Find the mass and center of mass of the thin rods with the following density functions:

$\rho(x) = 2 + \cos(x)$, for $0 \le x \le \pi$

$$\bar{x} = \frac{M}{m}$$

$$m = \int_0^{\pi} \rho(x)\,dx = \int_0^{\pi} 2 + \cos(x)\,dx = 2x + \sin(x)\Big|_{x=0}^{x=\pi}$$

$$= (2\pi + 0) - (0 + 0) = 2\pi$$

$$M = \int_0^{\pi} x(2 + \cos(x))\,dx = \int_0^{\pi} 2x + x\cos(x)\,dx$$

$u = x$ $\quad$ $v = \sin(x)$

$du = dx$ $\quad$ $dv = \cos(x)\,dx$

$$\frac{d}{dx}[u \cdot v] = u'v + uv'$$

$$uv = \int u\,dv + \int v\,du$$

$$\int u\,dv = uv - \int v\,du$$

$$= x^2\Big|_{x=0}^{x=\pi} + x\sin(x)\Big|_0^{\pi} - \int_0^{\pi} \sin(x)\,dx$$

$$= \pi^2 + (0 - 0) + \cos(x)\Big|_{x=0}^{x=\pi}$$

$$M = \pi^2 - 2$$

$$\bar{x} = \frac{M}{m} = \boxed{\frac{\pi^2 - 2}{2\pi}}$$

$$\rho(x) = \begin{cases} x^2 & \text{if } 0 \le x \le 1 \\ x(2-x) & \text{if } 1 < x \le 2 \end{cases}$$

$$m = \int_0^2 \rho(x)\,dx$$

$$= \int_0^1 x^2\,dx + \int_1^2 x(2-x)\,dx$$

$$= \frac{x^3}{3}\Big|_{x=0}^{x=1} + \left(x^2 - \frac{x^3}{3}\right)\Big|_{x=1}^{x=2}$$

$$= \frac{1}{3} + \left(4 - \frac{8}{3}\right) - \left(1 - \tfrac{1}{3}\right)$$

$$= 4 - 1 - 6/3 = 1$$

Lc #1

$$M = \int_0^2 x\,\rho(x)\,dx$$

$$= \int_0^1 x \cdot x^2\,dx + \int_1^2 x \cdot x(2-x)\,dx$$

$$= \frac{x^4}{4}\Big|_{x=0}^{x=1} + \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)\Big|_{x=1}^{x=2}$$

$$= \frac{1}{4} + \left(\frac{16}{3} - \frac{16}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right)$$

4

$$= \frac{1}{4} + \frac{14}{3} - \frac{15}{4}$$

$$= \frac{14}{3} - \frac{14}{4}$$

$$= \frac{14}{12} = \frac{7}{6}$$

Lc #2

$$\bar{x} = 7/6$$

$$\bar{x} = \frac{M}{m} = \frac{7/6}{1} = \boxed{7/6}$$

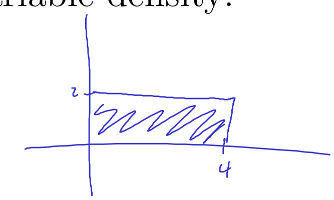
Definition. (Center of Mass in Two Dimensions)
Let ρ be an integrable area density function defined over a closed bounded region R in $\mathbb{R}^2$. The coordinates of the center of mass of the object represented by R are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint\limits_R x\rho(x,y)\,dA \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint\limits_R y\rho(x,y)\,dA,$$

where $m = \iint_R \rho(x,y)\,dA$ is the mass, and M_y and M_x are the moments with respect to the y-axis and x-axis, respectively. If ρ is constant, the center of mass is called the **centroid** and is independent of the density.

Example. Find the center of mass of the following plane regions with variable density:

$R = \{(x,y) : 0 \le x \le 4,\ 0 \le y \le 2\};\ \rho(x,y) = 1 + x/2.$



$$m = \iint\limits_R \rho(x,y)\,dA = \int_0^2\int_0^4 1 + x/2 \, dx\, dy = \int^2 x + \frac{x^2}{4}\Big|_{x=0}^{x=4} dy$$

$$= \int_0^2 4 + 4\,dy = 8\int_0^2 dy = 8\,y\Big|_{y=0}^{y=2} = 16$$

LC #3

$$M_y = \iint\limits_R x\,\rho(x,y)\,dA = \int_0^4\int_0^2 x(1+x/2)\,dy\,dx = \int_0^4 y\,x(1+x/2)\Big|_{y=0}^{y=2} dx$$

$$= \int_0^4 2x + x^2\,dx = x^2 + \frac{x^3}{3}\Big|_{x=0}^{x=4} = 16 + \frac{64}{3} = 16\left(\frac{3}{3} + \frac{4}{3}\right) = 16(7/3)$$

$$\Rightarrow \bar{x} = \frac{M_y}{m} = \frac{16(7/3)}{16} = 7/3$$

LC #4

$$M_x = \iint\limits_R y\,\rho(x,y)\,dA = \int_0^4\int_0^2 y(1+x/2)\,dy\,dx = \int_0^4 \frac{y^2}{2}(1+x/2)\Big|_{y=0}^{y=2} dx = \int_0^4 2 + x\,dx$$

$$= 2x + \frac{x^2}{2}\Big|_{x=0}^{x=4} = 8 + 8 = 16 \Longrightarrow \bar{y} = \frac{M_x}{m} = \frac{16}{16} = 1$$

LC #5

The quarter disk in the first quadrant bounded by $x^2+y^2=4$ with $\rho(x,y)=1+x^2+y^2$.

Polar coordinates

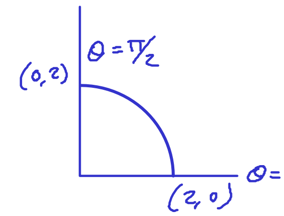
$x = r\cos\theta$

$y = r\sin\theta$

$0 \le r \le 2$

$0 \le \theta \le \pi/2$

$$\rho(r,\theta) = 1 + \underbrace{r^2\cos^2\theta}_{x^2} + \underbrace{r^2\sin^2\theta}_{y^2} = 1 + r^2$$



$$m = \iint_R \rho(x,y)\,dA = \int_0^{\pi/2}\int_0^2 (1+r^2)\,r\,dr\,d\theta = \int_0^{\pi/2} \frac{r^2}{2} + \frac{r^4}{4}\Big|_{r=0}^{r=2} d\theta$$

$$= \int_0^{\pi/2} 6\,d\theta = 3\pi$$

$$M_y = \iint_R x\,\rho(x,y)\,dA = \int_0^2\int_0^{\pi/2} \underbrace{r\cos\theta}_{x}(1+r^2)\,r\,d\theta\,dr = \int_0^2 \sin\theta\,(r^2+r^4)\Big|_{\theta=0}^{\theta=\pi/2} dr = \int_0^2 (r^2+r^4)\,dr$$

$$= \frac{r^3}{3} + \frac{r^5}{5}\Big|_{r=0}^{r=2} = \frac{8}{3} + \frac{32}{5} = 8\left(\frac{1}{3}+\frac{4}{5}\right) = 8\left(\frac{5+12}{15}\right) = 8\left(\frac{17}{15}\right) = \boxed{\frac{136}{15}}$$

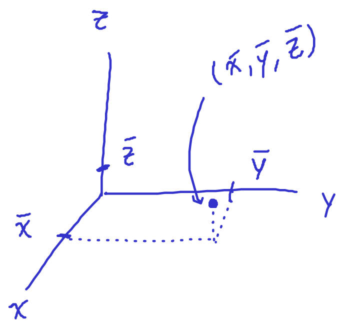
$$\bar{x} = \frac{M_y}{m} = \frac{136/15}{3\pi} = \frac{136}{45\pi}$$

$$M_x = \iint_R y\,\rho(x,y)\,dA = \int_0^{\pi/2}\int_0^2 \underbrace{r\sin\theta}_{y}(1+r^2)\,r\,dr\,d\theta = \int_0^{\pi/2} \sin\theta\left(\frac{r^3}{3}+\frac{r^5}{5}\right)\Big|_{r=0}^{r=2} d\theta$$

$$= -\left(\frac{8}{3}+\frac{32}{5}\right)\cos\theta\Big|_{\theta=0}^{\theta=\pi/2} = \frac{136}{15}$$

$$\longrightarrow \quad \bar{y} = \frac{M_x}{m} = \frac{136}{45\pi}$$

Definition. (Center of Mass in Three Dimensions)
Let ρ be an integrable area density function defined over a closed bounded region D in $\mathbb{R}^3$. The coordinates of the center of mass of the region are



$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m}\iiint\limits_D x\rho(x,y,z)\,dV$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1}{m}\iiint\limits_D y\rho(x,y,z)\,dV$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m}\iiint\limits_D z\rho(x,y,z)\,dV$$

$(\bar{x}, \bar{y}, \bar{z})$

where $m = \iiint_D \rho(x,y,z)\,dA$ is the mass, and M_{yz}, M_{xz}, and M_{xy} are the moments with respect to the coordinate planes.