1 15.6: Tangent Planes and Linear Approximation

Definition. (Equation of the Tangent Plane for $F(x, y, z) = 0$)
Let F be differentiable at the point $P_0(a, b, c)$ with $\nabla F(a, b, c) \neq \mathbf{0}$. The plane tangent to the surface $F(x, y, z) = 0$ at P_0, called the **tangent plane**, is the plane passing through P_0 orthogonal to $\nabla F(a, b, c)$. An equation of the tangent plane is

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

Example. Consider the ellipsoid

$$F(x, y, z) = \frac{x^2}{9} + \frac{y^2}{5} + z^2 - 1 = 0.$$

a) Find an equation of the plane tangent to the ellipsoid at $(0, 4, \frac{3}{5})$.

b) At what points on the ellipsoid is the tangent plane horizontal?

Surfaces of the form $z = f(x, y)$ are a special case of $F(x, y, z) = 0$: Define $F(x, y, z) = z - f(x, y) = 0$, then

$$\nabla F(a, b, f(a, b)) = \langle -f_x(a, b),\ -f_y(a, b),\ 1\rangle$$

so the tangent plane is

$$-f_x(a, b)(x - a) - f_y(a, b)(y - b) + 1(z - f(a, b)) = 0$$

Tangent Plane for $z = f(x, y)$
Let f be differentiable at the point (a, b). An equation of the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

Example. Find an equation of the plane tangent to $f(x, y) = 4e^{xy^2}$ at $(3, 0, 4)$ and $(0, 2, 4)$.

Example. Find an equation of the plane tangent to $f(x, y) = \tan^{-1}(xy)$ at $\left(\sqrt{3},\, 1,\, \frac{\pi}{3}\right)$ and $\left(\frac{\sqrt{3}}{3}, 1, \frac{\pi}{6}\right)$.

Definition. (Linear Approximation)
Let f be differentiable at (a, b). The linear approximation to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is the tangent plane at that point, given by the equation

$$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b),$$

For a function of three variables, the linear approximation to $w = f(x, y, z)$ at the point $(a, b, c, f(a, b, c))$ is given by

$$L(x, y, z) = f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) + f(a, b, c).$$

Example. Let $f(x, y) = \dfrac{5}{x^2 + y^2}$. Find the linear approximation to the function at the point $(-1, 2, 1)$. Use this to approximate $f(-1.05, 2.1)$.

Example. Let $f(x, y) = \sqrt{x^2 + y^2}$. Find the linear approximation to the function at the point $(-8, 15)$. Use this to approximate $f(-7.91, 14.96)$.

Definition. (The differential dz)
Let f be differentiable at the point (x, y). The change in $z = f(x, y)$ as the independent variables change from (x, y) to $(x+dx, y+dy)$ is denoted Δz and is approximated by the differential dz:

$$\Delta z \approx dz = f_x(x, y)\, dx + f_y(x, y)\, dy.$$

Example. Let $z = f(x, y) = \dfrac{5}{x^2 + y^2}$. Approximate the change in z when the variables change from $(-1, 2)$ to $(-0.93, 1.94)$.

Example. A company manufactures cylindrical aluminum tubes to rigid specifications. The tubes are designed to have an outside radius of $r = 10$ cm, a height of $h = 50$ cm, and a thickness of $t = 0.1$ cm. The manufacturing process produces tubes with a maximum error of ± 0.05 cm in the radius and height, and a maximum error of ± 0.0005 cm in the thickness. The volume of the cylindrical tube is $V(r, h, t) = \pi h t(2r - t)$. Use differentials to estimate the maximum error in the volume of a tube.