

10.6: Alternating Series

Theorem 10.16: Alternating Series Test

The alternating series $\sum (-1)^{k+1} a_k$ converges provided

1. the terms of the series are nonincreasing in magnitude ($0 < a_{k+1} \leq a_k$, for k greater than some index N) and

2. $\lim_{k \rightarrow \infty} a_k = 0$.

show $f(k) = a_k$
 $\Delta f'(x) < 0$

Example. Which of the following are considered alternating series?

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k+2}$$

✓

$$\sum_{k=4}^{\infty} \left(\frac{-3}{2}\right)^k$$

✓

$$\sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{2}\right)^k$$

✗

$$\sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{1}{2}\right)^k$$

✓

$$\sum_{k=0}^{\infty} (-1)^{k+1} a_k$$

$$a_k = \frac{1}{k+2}$$

$$\sum_{k=4}^{\infty} (-1)^k \left(\frac{3}{2}\right)^k$$

$$a_k = \left(\frac{3}{2}\right)^k$$

Diverges
 $\lim_{k \rightarrow \infty} a_k = \infty$

$$(*) \sum_{k=-3}^{\infty} \frac{\cos(k\pi)}{(k+4)^2}$$

$$\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$$

✗

$$\sum_{k=0}^{\infty} (-1)^{k+1} \left(\frac{1}{-2}\right)^k$$

✗

$$= \sum_{k=0}^{\infty} \underbrace{(-1)^{k+1} (-1)^k}_{(-1)^{2k+1} = -1} \left(\frac{1}{2}\right)^k$$

k	cos(kπ)
-3	-1
-2	1
-1	-1
0	1
1	-1

✓

$$(*) = \sum_{k=-3}^{\infty} \frac{(-1)^k}{(k+4)^2}$$

Can't rewrite
 $\sin(k)$ as $(-1)^k$

Example. Consider the series $\sum_{k=1}^{\infty} (-1)^k \frac{\sqrt{k}}{2k+3}$. Let a_k represent that magnitude of the terms of the given series.

- What is $\lim_{k \rightarrow \infty} a_k$?

$$\lim_{k \rightarrow \infty} \frac{\sqrt{k}}{2k+3} \left(\frac{1/k}{1/k} \right) = \lim_{k \rightarrow \infty} \frac{1/\sqrt{k}}{2 + 3/k} = 0$$

LC#4

$$a_{k+1} = \frac{\sqrt{k+1}}{2(k+1)+3} = \frac{\sqrt{k+1}}{2k+5}$$

LC#3

- Compute $f'(x)$ where $f(k) = a_k$.

$$f'(x) = \frac{(2x+3) \frac{1}{2\sqrt{x}} - 2\sqrt{x}}{(2x+3)^2}$$

$$= \frac{2x+3 - 4x}{2\sqrt{x}(2x+3)^2}$$

$$= \frac{-2x+3}{2\sqrt{x}(2x+3)^2}$$

$$f(x) = \frac{\sqrt{x}}{2x+3}$$

$$\text{Want } f'(x) < 0$$

$$\frac{-2x+3}{2\sqrt{x}(2x+3)^2} < 0$$

$$\Rightarrow -2x+3 < 0$$

$$x > \frac{3}{2} \Rightarrow$$

Since $f'(x)$ is negative for $x > 3/2$, the end behavior of the function is decreasing. We can have finitely many terms that are increasing.

a_k decreasing for $k \geq 2$

LC#5

- Use the Alternating Series Test to determine if the given series converges.

(1) a_k are non-increasing (decreasing)

(2) $\lim_{k \rightarrow \infty} a_k = 0$

\Rightarrow By the AST, $\sum_{k=1}^{\infty} (-1)^k \frac{\sqrt{k}}{2k+3}$ converges

Example. Does the series $\sum_{k=0}^{\infty} (-1)^{k+1} \left(\frac{4}{3}\right)^k$ converge?

$$\lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} \left(\frac{4}{3}\right)^k = \infty \rightarrow \text{Diverges by the divergence test}$$

Example. Does the series $\sum_{k=1}^{\infty} \cos(\pi k) e^{-k}$ converge? *Use table for correct index*

$$= \sum_{k=1}^{\infty} (-1)^k e^{-k}$$

k	$\cos(\pi k)$
1	-1
2	1
3	-1

① $f(x) = e^{-x}$
 $f'(x) = -e^{-x} < 0$, for all x $-\infty < x < \infty \rightarrow$ decreasing a_k for all k

② $\lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} e^{-k} = \lim_{k \rightarrow \infty} \frac{1}{e^k} = 0$

\Rightarrow Converges by the AST

Theorem 10.17: Alternating Harmonic Series

The alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges (even though the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges).

Handwritten notes: $\sum_{k=1}^{\infty} |a_k|$ where $a_k = \frac{(-1)^{k+1}}{k}$

Example. Use the Alternating Series Test to show that the alternating harmonic series converges.

Handwritten work:

(1) $a_{k+1} = \frac{1}{k+1} < \frac{1}{k} = a_k$

(2) $\lim_{k \rightarrow \infty} \frac{1}{k} = 0$

By AST, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges

Theorem 10.18: Remainder in Alternating Series

Let $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ be a convergent alternating series with terms that are nonincreasing in magnitude. Let $R_n = S - S_n$ be the remainder in approximating the value of that series by the sum of its first n terms. Then $|R_n| \leq a_{n+1}$. In other words, the magnitude of the remainder is less than or equal to the magnitude of the first neglected term.

Example. Find the minimum value of n such that $|R_n| < 10^{-4}$ for the following series:

$$\ln(2) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

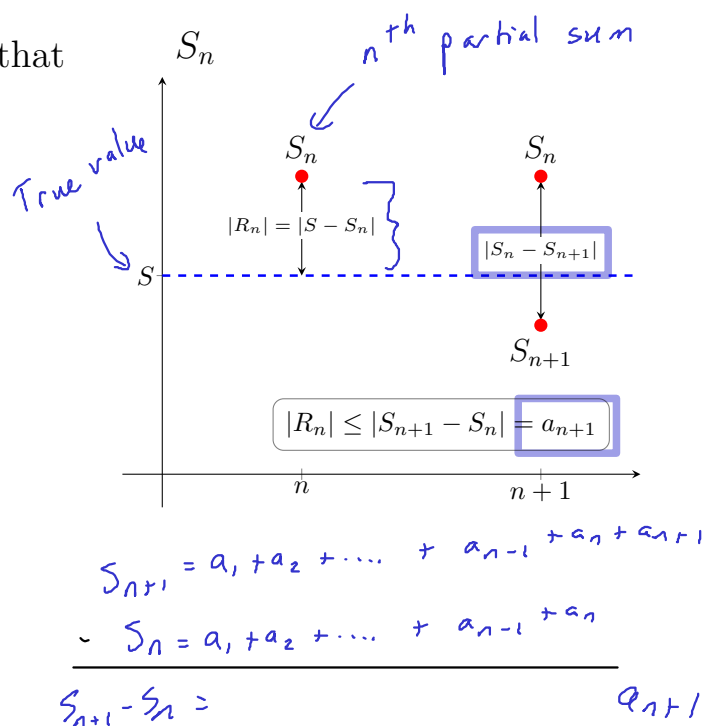
$$|R_n| \leq a_{n+1} < 10^{-4}$$

$$\frac{1}{n+1} < 10^{-4}$$

$$10^4 < n+1$$

$$9,999 < n$$

$$10,000 \leq n$$



Quiz 8

$$\sum \frac{(-1)^{k+1}}{k^2} \rightarrow \sum |a_k| \rightarrow \sum \frac{1}{k^2} \leftarrow \text{Still converges, converges abs.}$$

$$\sum \frac{(-1)^{k+1}}{k} \xrightarrow{\text{Converge}} \sum |a_k| \rightarrow \sum \frac{1}{k} \leftarrow \text{diverges, converges cond.}$$

Definition. (Absolute and Conditional Convergence)

If $\sum |a_k|$ converges, then $\sum a_k$ **converges absolutely**.

If $\sum |a_k|$ diverges and $\sum a_k$ converges, then $\sum a_k$ **converges conditionally**.

Example. Can a series of **strictly positive** terms **converge conditionally**?

$$\Rightarrow \sum a_k \text{ converges, } \sum |a_k| \text{ diverges}$$

$$a_k > 0 \rightarrow |a_k| = a_k$$

impossible

Converges absolutely

Example. Consider the series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{4+k}{k^2}$. Determine if this series converges absolutely, converges conditionally, or diverges.

$$\sum_{k=1}^{\infty} \frac{4}{k^2} + \sum_{k=1}^{\infty} \frac{1}{k} \leftarrow \text{diverges by p-series}$$

Direct Comparison Test

$$\sum_{k=1}^{\infty} \left| (-1)^{k+1} \frac{4+k}{k^2} \right| = \sum_{k=1}^{\infty} \underbrace{\frac{4+k}{k^2}}_{a_k}$$

$$\sum_{k=1}^{\infty} \underbrace{\frac{1}{k}}_{b_k} = \sum_{k=1}^{\infty} \frac{k}{k^2} \leftarrow \text{divergent}$$

$$4k + k^2 > k^2$$

$$\frac{4+k}{k^2} > \frac{1}{k}$$

Since $b_k \leq a_k$, $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

$\sum_{k=1}^{\infty} \frac{4+k}{k^2}$ also diverges by DCT

Does not converge abs.

Cond conti $\sum |a_k|$ diverges
 $\sum a_k$ converges

$$\textcircled{1} 0 < a_{k+1} \leq a_k$$

$$\frac{4+k}{k^2} = \frac{4}{k^2} + \frac{1}{k}$$

both decreasing

$$\textcircled{2} \lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} \frac{4+k}{k^2} = 0$$

\Rightarrow converges by AST

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{4+k}{k^2}$$

\Rightarrow converges conditionally.

Example. Determine if the following series converge absolutely, converge conditionally, or diverge.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2\sqrt{k}-1}$$

absolute
convergence

LCT w/ $\frac{1}{\sqrt{k}}$

Divergent p-series w/ $p = 1/2$

$$\sum_{k=1}^{\infty} \underbrace{\frac{1}{2\sqrt{k}-1}}_{a_k}$$

$$\sum_{k=1}^{\infty} \underbrace{\frac{1}{\sqrt{k}}}_{b_k}$$

$$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \lim_{k \rightarrow \infty} \frac{\sqrt{k}}{2\sqrt{k}-1} = \frac{1}{2} \Rightarrow \text{series diverges by LCT}$$

AST

conditional
convergence

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2\sqrt{k}-1}$$

$$\textcircled{1} f(x) = \frac{1}{2\sqrt{x}-1}$$

$$f'(x) = \frac{-1}{\sqrt{x}(2\sqrt{x}-1)^2} < 0, \text{ when } x \neq \frac{1}{4} \Rightarrow k \geq 1$$

converges
by AST

$$\textcircled{2} \lim_{k \rightarrow \infty} \frac{1}{2\sqrt{k}-1} = 0$$

→ conditionally convergent

$$\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k$$

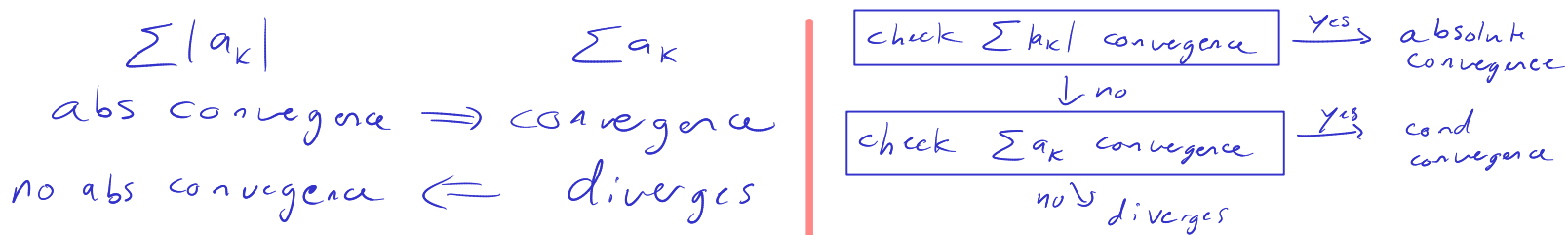
absolute
convergence

$$\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k$$

← convergent geometric
series w/ $r = 3/4$

Same as
original series

⇒ absolutely convergent



Theorem 10.19: Absolute Convergence Implies Convergence

If $\sum |a_k|$ converges, then $\sum a_k$ converges (absolute convergence implies convergence).
 Equivalently, if $\sum a_k$ diverges, then $\sum |a_k|$ diverges.

Example. Determine whether each of the following series converges absolutely, converges conditionally or diverges.

$\sum_{k=1}^{\infty} (-1)^k e^{1/k}$
 abs. conv. $\sum_{k=1}^{\infty} |(-1)^k e^{1/k}| = \sum_{k=1}^{\infty} e^{1/k}$ Divergence Test $\lim_{k \rightarrow \infty} e^{1/k} = e^0 = 1$
 \Rightarrow diverges by div test
 cond. conv. $\sum_{k=1}^{\infty} (-1)^k e^{1/k}$

① $0 < a_{k+1} \leq a_k$ $f(x) = e^{1/x}$
 $f'(x) = -\frac{e^{1/x}}{x^2} < 0, x > 0$

② $\lim_{k \rightarrow \infty} e^{1/k} = e^0 = 1$
 \Rightarrow diverges by divergence test

diverges

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^6}$
 abs conv $\rightarrow \sum_{k=1}^{\infty} \frac{1}{k^6}$ converge? \rightarrow convergent p-series w/ $p=6 > 1$
 \rightarrow converges abs. \Rightarrow converges

$\sum |a_k|$

$\sum a_k$

convergent of $\sum \frac{(-1)^{k+1}}{k^6} \rightarrow \sum |a_k| \rightarrow$ p-series $\rightarrow \sum a_k$ converges
 as + $\left. \begin{array}{l} \text{① } a_{k+1} < a_k \\ \text{② } \lim_{k \rightarrow \infty} a_k = 0 \end{array} \right\} \Rightarrow \text{converges}$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$$

abs conv.

$$\sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k$$

Geometric w/ $r = 1/3 < 1$

Converges absolutely
 \Rightarrow converges

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$$

abs conv $\sum 1/k^{1/2}$ divergent p-series $p = 1/2 \leq 1$

cond conv: $\left. \begin{array}{l} \textcircled{1} a_{k+1} \leq a_k \because \frac{1}{\sqrt{k+1}} < \frac{1}{\sqrt{k}} \\ \textcircled{2} \lim_{k \rightarrow \infty} \frac{1}{\sqrt{k}} = 0 \end{array} \right\} \text{converges by AST}$

$$\sum_{k=1}^{\infty} \frac{(-5)^k}{3^k}$$

\rightarrow converges conditionally

Divergence test $\lim_{k \rightarrow \infty} \left(-\frac{5}{3}\right)^k$ diverges

oscillates
b/w $\pm \infty$

$$\sum_{k=1}^{\infty} \frac{(-2)^{k-1}}{3^k}$$

Diverges \rightarrow not abs convergent

$$\sum a_k$$

$$\sum |a_k|$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{3^k}$$

abs conv

$$\sum_{k=1}^{\infty} \frac{2^{k-1}}{3^k} = \sum_{k=1}^{\infty} \frac{1}{2} \left(\frac{2}{3}\right)^k$$

convergent geometric
series w/ $r = 2/3$

\Rightarrow abs conv $\Rightarrow \sum a_k$ converges

Example. Does the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2\sqrt{k}-1}$ converge conditionally, converge absolutely, or diverge?

LCT w/ $\frac{1}{\sqrt{k}}$

Divergent p-series w/ $p = 1/2$

absolute
convergence

$$\sum_{k=1}^{\infty} \underbrace{\frac{1}{2\sqrt{k}-1}}_{a_k}$$

$$\sum_{k=1}^{\infty} \underbrace{\frac{1}{\sqrt{k}}}_{b_k}$$

$$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \lim_{k \rightarrow \infty} \frac{\sqrt{k}}{2\sqrt{k}-1} = \frac{1}{2} \Rightarrow \text{series diverges by LCT}$$

conditional
convergence

AST

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2\sqrt{k}-1}$$

$$\textcircled{1} f(x) = \frac{1}{2\sqrt{x}-1}$$

$$f'(x) = \frac{-1}{\sqrt{x}(2\sqrt{x}-1)^2} < 0, \text{ when } x \neq \frac{1}{4} \Rightarrow k \geq 1$$

$$\textcircled{2} \lim_{k \rightarrow \infty} \frac{1}{2\sqrt{k}-1} = 0$$

converges
by AST

→ conditionally convergent