Math 1080 Class notes
Fall 2021

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Table Of Contents

5.5: Substitution Rule

Theorem 5.6: Substitution Rule for Indefinite Integrals
Let $u = g(x)$, where g is differentiable on an interval, and let f be continuous on the corresponding range of g. On that interval,

$$\int f\big(g(x)\big)g'(x)\,dx = \int f(u)\,du$$

Example. We know

$$\frac{d}{dx}\left[\frac{(2x+1)^4}{4}\right] = 2(2x+1)^3$$

Thus, if $f(x) = x^3$ and $g(x) = 2x+1$ then $g'(x) = 2$, so we let $u = 2x+1$, then

$$\begin{aligned}\int 2(2x+1)^3\,dx &= \int f\big(g(x)\big)g'(x)\,dx \\ &= \int u^3\,du \\ &= \frac{u^4}{4} + C \\ &= \frac{(2x+1)^4}{4} + C\end{aligned}$$

Procedure: Substitution Rule (Change of Variables)

1. Given an indefinite integral involving a composite function $f\big(g(x)\big)$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)\,dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Write the result in terms of x using $u = g(x)$.

Example. Evaluate the following integrals:

a) $\displaystyle\int 2x(x^2+3)^4\,dx$

b) $\displaystyle\int (2x+1)^3\,dx$

c) $\displaystyle\int x^2\sqrt{x^3+1}\,dx$

d) $\displaystyle\int \theta\sqrt[4]{1-\theta^2}\,d\theta$

e) $\displaystyle\int \sqrt{4-t}\,dt$

f) $\displaystyle\int (2-x)^6\,dx$

Example. Evaluate the following integrals:

a) $\displaystyle\int \sec(2\theta)\tan(2\theta)\,d\theta$

b) $\displaystyle\int \csc^2\left(\frac{t}{3}\right)dt$

c) $\displaystyle\int \frac{\sin(x)}{1+\cos^2(x)}\,dx$

d) $\displaystyle\int \frac{\tan^{-1}(x)}{1+x^2}\,dx$

The acceleration of a particle moving back and forth on a line is $a(t) = \frac{d^2s}{dt^2} = \pi^2\cos(\pi t)\ m/s^2$ for all t. If $s = 0$ and $v = 8\ m/s$ when $t = 0$, find the value of s when $t = 1$ sec.

Example. Evaluate the following integrals:

a) $\displaystyle\int (6x^2 + 2)\sin(x^3 + x + 1)\,dx$

b) $\displaystyle\int \frac{\sin(\theta)}{\cos^5(\theta)}\,d\theta$

c) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

d) $\displaystyle\int \frac{2^t}{2^t + 3}\,dt$

e) $\int 6x^2 4^{x^3}\, dx$

f) $\int \frac{dx}{\sqrt{36-4x^2}}$

g) $\int \sin(t)\sec^2(\cos(t))\, dt$

h) $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\, dx$

i) $\displaystyle\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

j) $\displaystyle\int 5\cos(7x+5)\,dx$

k) $\displaystyle\int \frac{3}{\sqrt{1-25x^2}}\,dx$

l) $\displaystyle\int \frac{dx}{\sqrt{1-9x^2}}$

Example. Evaluate the following integrals using the recommended substitution:

a) $\displaystyle\int \sec^2(x)\tan(x)\,dx$
where $u = \tan(x)$.

b) $\displaystyle\int \sec^2(x)\tan(x)\,dx$
where $u = \sec(x)$.

Example. Solve the initial value problem: $\frac{dy}{dx} = 4x(x^2+8)^{-1/3}, y(0) = 0$.

Example. Evaluate the following integrals:

a) $\displaystyle\int xe^{-x^2}\,dx$

b) $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$

c) $\displaystyle\int \frac{dt}{8-3t}$

d) $\displaystyle\int 5^t \sin(5^t)\,dt$

e) $\displaystyle\int \frac{e^w}{36+e^{2w}}\,dw$

Theorem 5.7: Substitution Rule for Definite Integrals
Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g. Then

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Example. Evaluate the integrals:

a) $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$

b) $\displaystyle\int_1^3 \frac{dt}{(t-4)^2}$

c) $\displaystyle\int_0^3 \frac{v^2+1}{\sqrt{v^3+3v+4}}\,dv$

d) $\displaystyle\int_0^1 2x\big(4-x^2\big)\,dx$

e) $\displaystyle\int_2^3 \frac{x}{\sqrt[3]{x^2-1}}\,dx$

f) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{1+\cos(x)}\,dx$

g) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos^2(x)}\,dx$

h) $\displaystyle\int_{-\frac{\pi}{12}}^{\frac{\pi}{8}} \sec^2(2y)\,dy$

i) $\displaystyle\int_0^1 (1-2x^9)\,dx$

j) $\displaystyle\int_0^1 (1-2x)^9\,dx$

k) $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{1+4x^2}\,dx$

l) $\displaystyle\int_0^4 \frac{x}{x^2+1}\,dx$

m) $\displaystyle\int_0^{\pi} 3\cos^2(x)\sin(x)\,dx$

n) $\displaystyle\int_0^{\frac{\pi}{8}} \sec(2\theta)\tan(2\theta)\,d\theta$

o) $\displaystyle\int_0^{1} (3t-1)^{50}\,dt$

p) $\displaystyle\int_0^{3} \frac{1}{5x+1}\,dx$

q) $\displaystyle\int_0^1 xe^{-x^2}\,dx$

r) $\displaystyle\int_e^{e^4} \frac{1}{x\sqrt{\ln(x)}}\,dx$

s) $\displaystyle\int_0^{\frac{1}{2}} \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx$

t) $\displaystyle\int_0^1 \frac{e^z+1}{e^z+z}\,dz$

u) $\displaystyle\int_1^4 \frac{dy}{2\sqrt{y}\left(1+\sqrt{y}\right)^2}$

v) $\displaystyle\int_{\ln\left(\frac{\pi}{4}\right)}^{\ln\left(\frac{\pi}{2}\right)} e^w \cos(e^w)\,dw$

w) $\displaystyle\int_0^{\frac{1}{8}} \frac{x}{\sqrt{1-16x^2}}\,dx$

x) $\displaystyle\int_1^{e^2} \frac{\ln(p)}{p}\,dp$

y) $\displaystyle\int_0^{\frac{\pi}{4}} e^{\sin^2(x)} \sin(2x)\, dx$

z) $\displaystyle\int_{-\pi}^{\pi} x^2 \sin(7x^3)\, dx$

Example. Average velocity: An object moves in one dimension with a velocity in m/s given by $v(t) = 8\sin(\pi t) + 2t$. Find its average velocity over the time interval from $t = 0$ to $t = 10$, where t is measured in seconds.

Example. Prove $\int \tan(x)\,dx = \ln|\sec(x)| + C$.

Example. Evaluate the integrals:

a) $\displaystyle\int \frac{x}{(x-2)^3}\,dx$

b) $\displaystyle\int x\sqrt{x-1}\,dx$

c) $\int x^3(1+x^2)^{\frac{3}{2}}\,dx$

d) $\int \frac{y^2}{(y+1)^4}\,dy$

e) $\int (z+1)\sqrt{3z+2}\,dz$

f) $\int_0^1 \frac{x}{(x+2)^3}\,dx$

Half-Angle Formulas

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$
$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$$

Example. Evaluate the integrals:

a) $\displaystyle\int \cos^2(x)\,dx$

b) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2(x)\,dx$

c) $\displaystyle\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx$

d) $\displaystyle\int x\sin^2(x^2)\,dx$

e) $\displaystyle\int \sin^2\left(\theta+\frac{\pi}{6}\right)d\theta$

f) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2(8\theta)\,d\theta$

Example. If f is continuous and $\displaystyle\int_0^4 f(x)\,dx = 10$, find $\displaystyle\int_0^2 f(2x)\,dx$.

Example. If f is continuous and $\displaystyle\int_0^9 f(x)\,dx = 4$, find $\displaystyle\int_0^3 xf(x^2)\,dx$.

Example. Suppose f is an even function with $\displaystyle\int_0^8 f(x)\,dx = 9$. Evaluate the following:

a) $\displaystyle\int_{-1}^1 xf(x^2)\,dx$.

b) $\displaystyle\int_{-2}^2 x^2 f(x^3)\,dx$.

Example. Evaluate the integrals:

a) $\displaystyle\int \sec^2(10x)\,dx$

b) $\displaystyle\int \tan^{10}(4x)\sec^2(4x)\,dx$

c) $\displaystyle\int \left(x^{\frac{3}{2}}+8\right)^5\sqrt{x}\,dx$

d) $\displaystyle\int \frac{2x}{\sqrt{3x+2}}\,dx$

e) $\displaystyle\int \frac{7x^2+2x}{x}\,dx$

f) $\displaystyle\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

g) $\displaystyle\int_0^{\sqrt{3}} \frac{3}{9+x^2}\,dx$

h) $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\sin(2y)}{\sin^2(y)+2}\,dy$

i) $\displaystyle\int \frac{\sec(z)\tan(z)}{\sqrt{\sec(z)}}\,dz$

j) $\displaystyle\int \frac{1}{\sin^{-1}(x)\sqrt{1-x^2}}\,dx$

k) $\displaystyle\int \frac{x}{\sqrt{4-9x^2}}\,dx$

l) $\displaystyle\int \frac{x}{1+x^4}\,dx$

m) $\displaystyle\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta$

n) $\displaystyle\int x^2\sqrt{2+x}\,dx$

o) $\displaystyle\int \big(\sin^5(x) + 3\sin^3(x) - \sin(x)\big)\cos(x)\,dx$

p) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(x^3 + x^4 \tan(x)\right) dx$

q) $\int_0^{\frac{\pi}{2}} \cos(x) \sin\left(\sin(x)\right) dx$

r) $\int \frac{1+x}{1+x^2}\, dx$

Example. Evaluate these more challenging integrals:

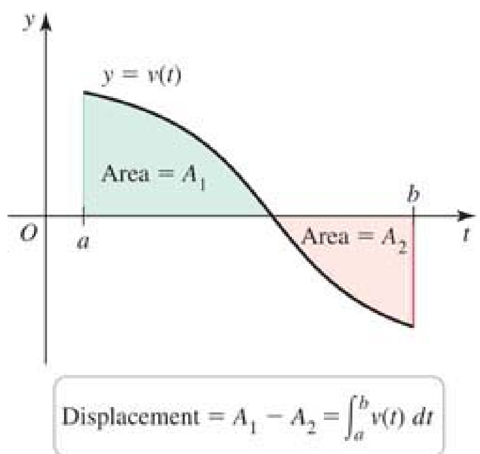
a) $\displaystyle\int \frac{dx}{\sqrt{1+\sqrt{1+x}}}$

b) $\int x \sin^4(x^2) \cos(x^2)\, dx$

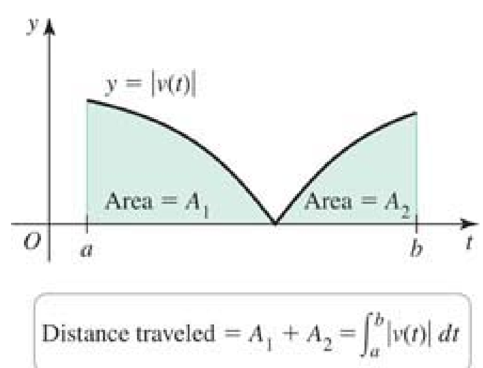
6.1: Velocity and Net Change

Definition. (Position, Velocity, Displacement, and Distance)

1. The **position** of an object moving along a line at time t, denoted $s(t)$, is the location of the object relative to the origin.
2. The **velocity** of an object at time t is $v(t) = s'(t)$.
3. The **displacement** of the object between $t = a$ and $t = b > a$ is
$$s(b) - s(a) = \int_a^b v(t)\, dt.$$
4. The **distance traveled** by the object between $t = a$ and $t = b > a$ is
$$\int_a^b |v(t)|\, dt$$
where $|v(t)|$ is the **speed** of the object at time t.



(a)

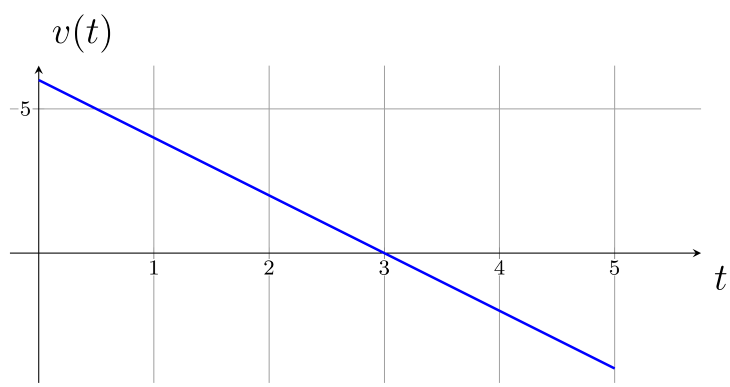


(b)

Example. Suppose an object moves along a line with velocity (in ft/s) $v(t) = 6 - 2t$, for $0 \leq t \leq 5$, where t is measured in seconds.

- Find the displacement of the object on the interval $0 \leq t \leq 5$.

- Find the distance traveled by the object on the interval $0 \leq t \leq 5$.



Example. A cyclist rides down a long straight road at a velocity (in m/min) given by $v(t) = 400 - 20t$, for $0 \leq t \leq 10$.

- How far does the cyclists travel in the first 5 minutes?

- How far does the cyclists travel in the first 10 minutes?

- How far has the cyclist traveled when her velocity is 250 m/min?

Example. The population of a community of foxes is observed to fluctuate on a 10-year cycle due to variations in the availability of prey. When population measurements began ($t = 0$), the population was 35 foxes. The growth rate in units of foxes/year was observed to be:

$$P'(t) = 5 + 10\sin\left(\frac{\pi t}{5}\right)$$

- Find $P(t)$.

- Find the population of foxes after the first 5 years, rounded to the nearest whole number of foxes.

Theorem 6.1: Position from Velocity
Given the velocity $v(t)$ of an object moving along a line and its initial position $s(0)$, the position function of the object for future times $t \geq 0$ is

$$\underbrace{s(t)}_{\substack{\text{position}\\ \text{at } t}} = \underbrace{s(0)}_{\substack{\text{initial}\\ \text{position}}} + \underbrace{\int_0^t v(x)\, dx}_{\substack{\text{displacement}\\ \text{over } [0,t]}}.$$

Theorem 6.2: Velocity from Acceleration
Given the acceleration $a(t)$ of an object moving along a line and its initial velocity $v(0)$, the velocity of the object for future times $t \geq 0$ is

$$v(t) = v(0) + \int_0^t a(x)\, dx.$$

Example. At $t = 0$, a train approaching a station begins decelerating from a speed of 80 miles/hour according to the acceleration function $a(t) = -1280(1+8t)^{-3}$, where $t \geq 0$ is measured in hours. The units of acceleration are mi/hr^2.

- Find the velocity of the train at $t = 0.25$.

- How far does the train travel in the first 15 minutes (1/4 hour)?

- How long does it take the train to travel 9 miles?

Theorem 6.3: Net Change and Future Value
Suppose a quantity Q changes over time at a known rate Q'. Then the **net change** in Q between $t = a$ and $t = b > a$ is

$$\underbrace{Q(b) - Q(a)}_{\text{net change in } Q} = \int_a^b Q'(t)\,dt.$$

Given the initial value $Q(0)$, the **future value** of Q at time $t \geq 0$ is

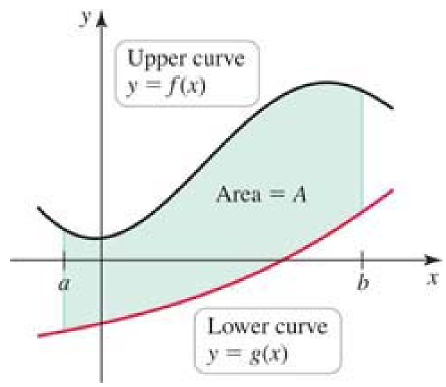
$$Q(t) = Q(0) + \int_0^t Q'(x)\,dx.$$

Velocity-Displacement Problems	**General Problems**
Position $s(t)$	Quantity $Q(t)$ (such as volume or population)
Velocity: $s'(t) = v(t)$	Rate of change: $Q'(t)$
Displacement: $s(b) - s(a) = \int_a^b v(t)\,dt$	Net change: $Q(b) - Q(a) = \int_a^b Q'(t)\,dt$
Future position: $s(t) = s(0) + \int_0^t v(x)\,dx$	Future value of Q: $Q(t) = Q(0) + \int_0^t Q'(x)\,dx$

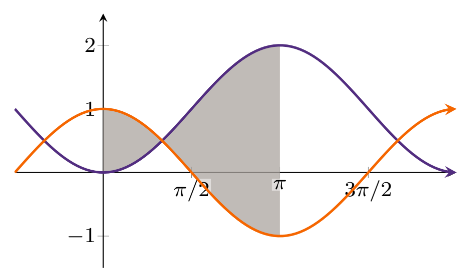
6.2: Regions Between Curves

Definition. (Area of a Region Between Two Curves)
Suppose f and g are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$. The area of the region bounded by the graphs of f and g on $[a, b]$ is

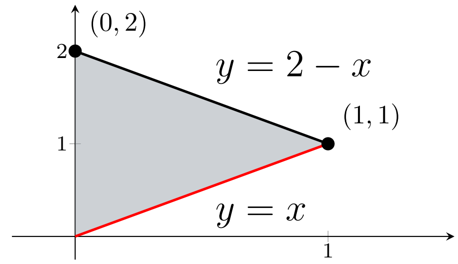
$$A = \int_a^b (f(x) - g(x))\, dx.$$



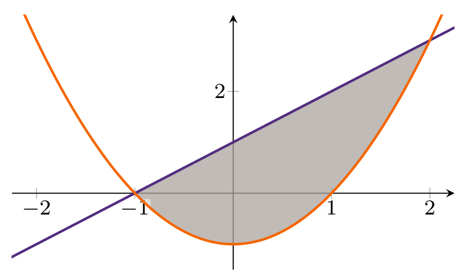
Example. Consider the region bounded by the curves $y = \cos(x)$ and $y = 1 - \cos(x)$, $0 \leq x \leq \pi$. Set up the integral(s) representing the area of this region.



Example. Find the area of the region by integrating with respect to x.



Example. Find the volume of the solid whose base is bounded by the graphs of $y = x + 1$ and $y = x^2 - 1$, with the cross sections in the shape of rectangles of height 2 taken perpendicular to the x-axis.



Definition. (Area of a Region Between Two Curves with Respect to y) Suppose f and g are continuous functions with $f(y) \geq g(y)$ on the interval $[c, d]$. The area of the region bounded by the graphs $x = f(y)$ and $x = g(y)$ on $[c, d]$ is

$$A = \int_c^d (f(y) - g(y))\, dy.$$

Example. Find the area of the region bounded by $x = 3y$, and $x = y^2 - 10$

by integrating with respect to x

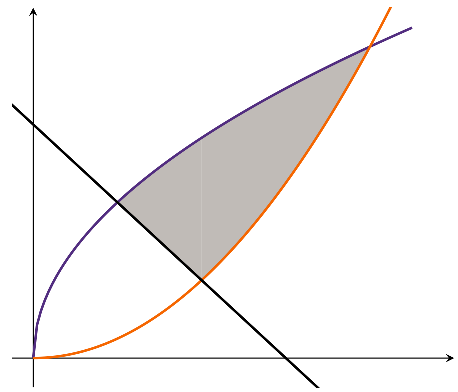
by integrating with respect to y

Example. Find the area of the region bounded by $y = x^3$, and $y = \sqrt{x}$

by integrating with respect to x

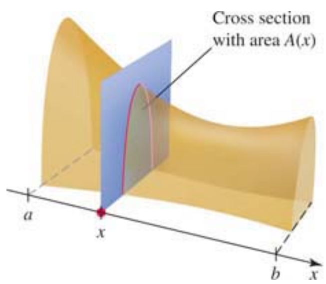
by integrating with respect to y

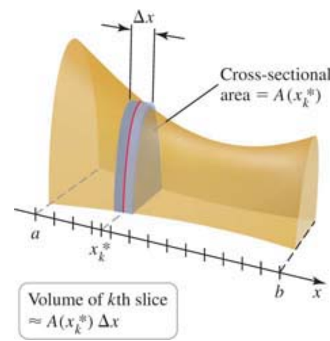
Example. Find the area of the region bounded by $y = 4\sqrt{2x}$, $y = 2x^2$, and $y = -4x + 6$



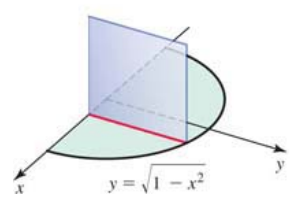
6.3: Volume by Slicing

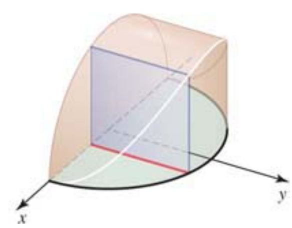
> **General Slicing Method**
> Suppose a solid object extends from $x = a$ to $y = b$, and the cross section of the solid perpendicular to the x-axis has an area given by a function A that is integrable on $[a, b]$. The volume of the solid is
> $$V = \int_a^b A(x)\,dx.$$





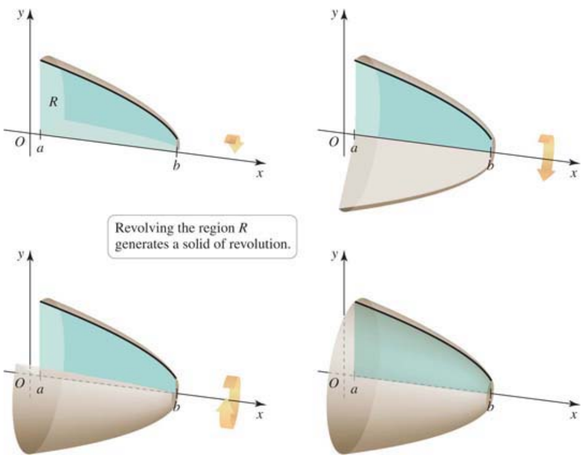
Example. Use the general slicing method to find the volume of the solid whose base is the region bounded by the semicircle $y = \sqrt{1 - x^2}$ and the x-axis, and whose cross sections through the solid perpendicular to the x-axis are squares.

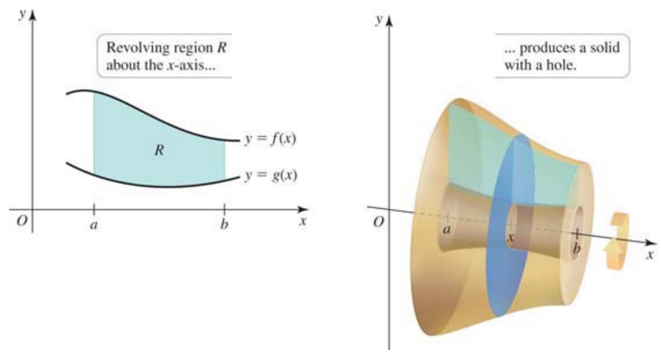




Disk Method about the x-Axis
Let f be continuous with $f(x) \geq 0$ on the interval $[a, b]$. If the region R bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi \underbrace{f(x)}_{\text{disk radius}}{}^2 dx.$$

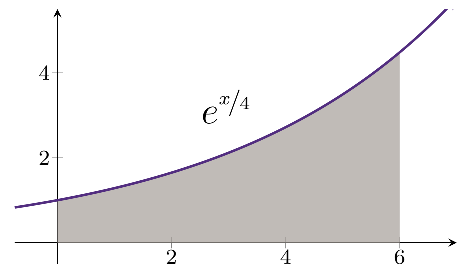




Washer Method about the x-Axis
Let f and g be continuous functions with $f(x) \geq g(x) \geq 0$ on $[a, b]$. Let R be the region bounded by $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$. When R is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi(\underbrace{f(x)}_{\text{outer radius}}{}^2 - \underbrace{g(x)}_{\text{inner radius}}{}^2)\, dx.$$

Example. Consider the region bounded by $y = e^{x/4}$, $y = 0$, $x = 0$, and $x = 6$. Find the volume of the solid generated by rotating the region about the x-axis.

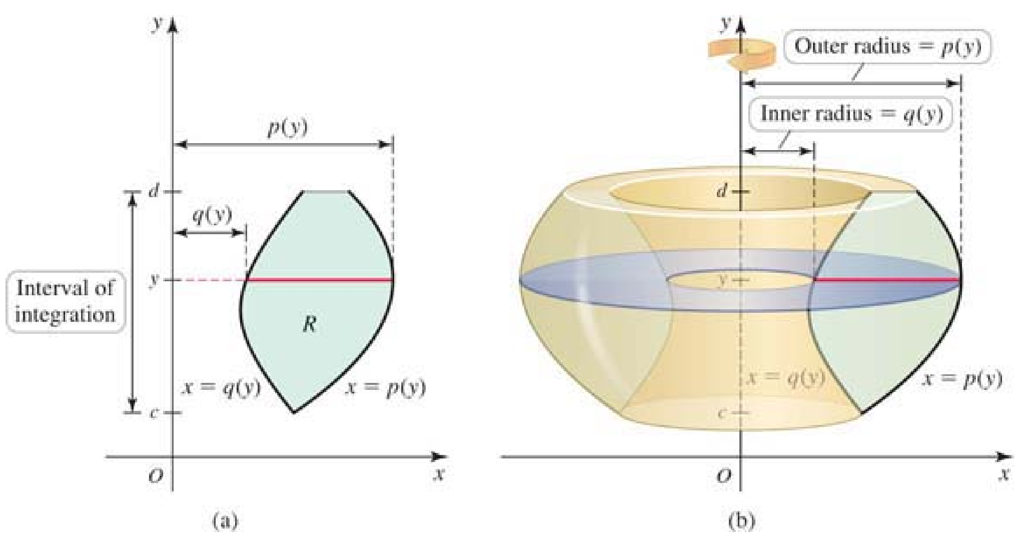


Disk and Washer Methods about the y-Axis
Let p and q be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let R be the region bounded by $x = p(y)$, $x = q(y)$, and the lines $y = c$ and $y = d$. When R is revolved around the y-axis, the volume of the resulting solid of revolution is given by

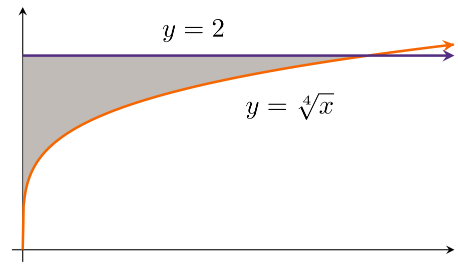
$$V = \int_c^d \pi(\underbrace{p(y)}_{\substack{\text{outer}\\\text{radius}}}^2 - \underbrace{q(y)}_{\substack{\text{inner}\\\text{radius}}}^2)\, dy.$$

If $q(y) = 0$, the disk method results:

$$V = \int_c^d \pi\, \underbrace{p(y)}_{\substack{\text{disk}\\\text{radius}}}^2 dy.$$



Example. Consider the region bounded between $y = \sqrt[4]{x}$, $y = 2$, and $x = 0$.

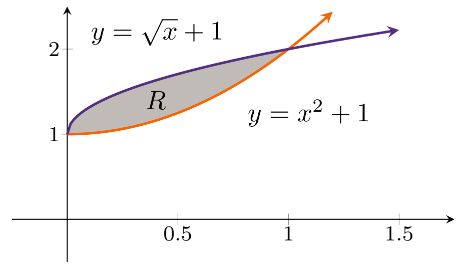


Setup the integral with respect to x that gives the area of the region.

Setup the integral with respect to y that gives the area of the region.

Use the disk/washer method to setup the that represents the volume of the solid generated by rotating the region about the x-axis.

Example. Consider the region R between $y = \sqrt{x}+1$ and $y = x^2+1$. Setup the integrals which find the volume of the solid obtained by rotating the region R as indicated below.



about the y-axis

about the x-axis

about the line $x = 1$

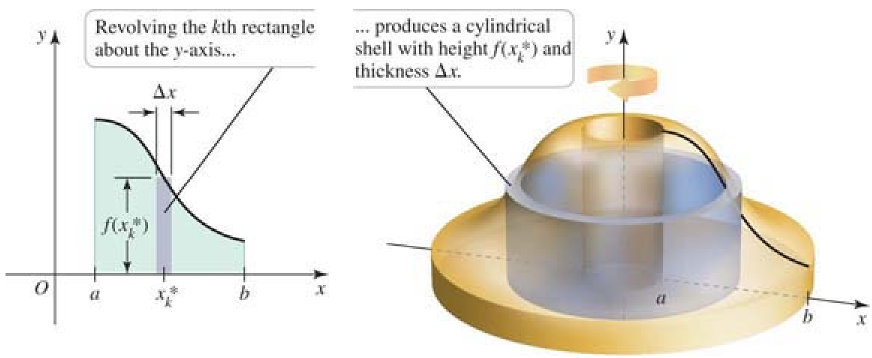
about the line $y = -1$

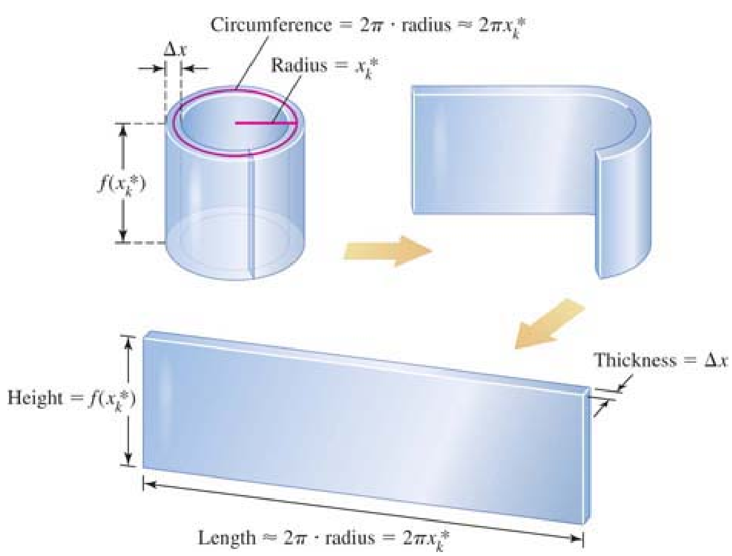
6.4: Volume by Shells

Volume by the Shell Method

Let f and g be continuous functions with $f(x) \geq g(x)$ on $[a, b]$. If R is the region bounded by the curves $y = f(x)$ and $y = g(x)$ between the lines $x = a$ and $x = b$, the volume of the solid generated when R is revolved about the y-axis is

$$V = \int_a^b \underbrace{2\pi x}_{\substack{\text{shell}\\ \text{circumference}}} \underbrace{(f(x) - g(x))}_{\substack{\text{shell}\\ \text{height}}} \, dx.$$





Example. Consider a general region R revolved around the y-axis.

When using the **disk/washer** method, we integrate with respect to __________

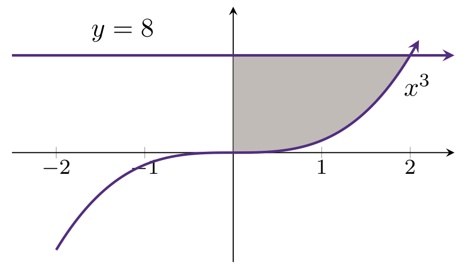
When using the **shell** method, we integrate with respect to __________

Example. Consider a general region R revolved around the x-axis.

When using the **disk/washer** method, we integrate with respect to __________

When using the **shell** method, we integrate with respect to __________

Example. Consider the region bounded between $y = x^3$, $y = 8$ and $x = 0$.



Use the disk/washer method to setup the integral that represents the volume of the solid generated by rotating the region about the x-axis.

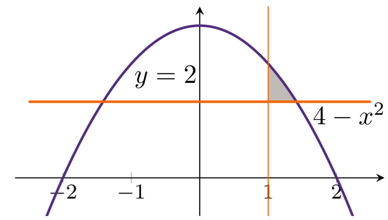
about the y-axis.

Use the disk/washer method to setup the integral that represents the volume of the solid generated by rotating the region about the line $x = -1$.

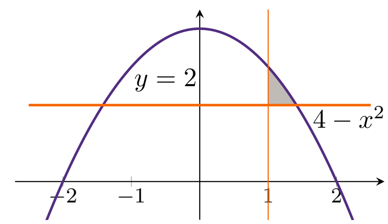
about the line $y = 8$.

Example. Consider the region R bounded by $y = 4 - x^2$, $y = 2$, and $x = 1$. Use the shell method to setup the integral that represents the volume of the solid generated by rotating the region R about the indicated axis of rotation.

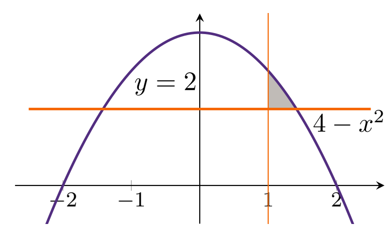
about x-axis,



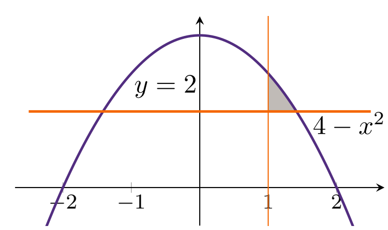
about y-axis,



about the line $x = -2$,



about the line $y = 2$.



Example. Consider the region bounded by $y = \dfrac{1}{x+1}$ and $y = 1 - \dfrac{x}{3}$. Use both the disk/washer method and shell method to find the volume of the solid generated when R is rotated about the x-axis.

Example. Determine if the following statements are true.

When using the shell method, the axis of the cylindrical shells is parallel to the axis of revolution.

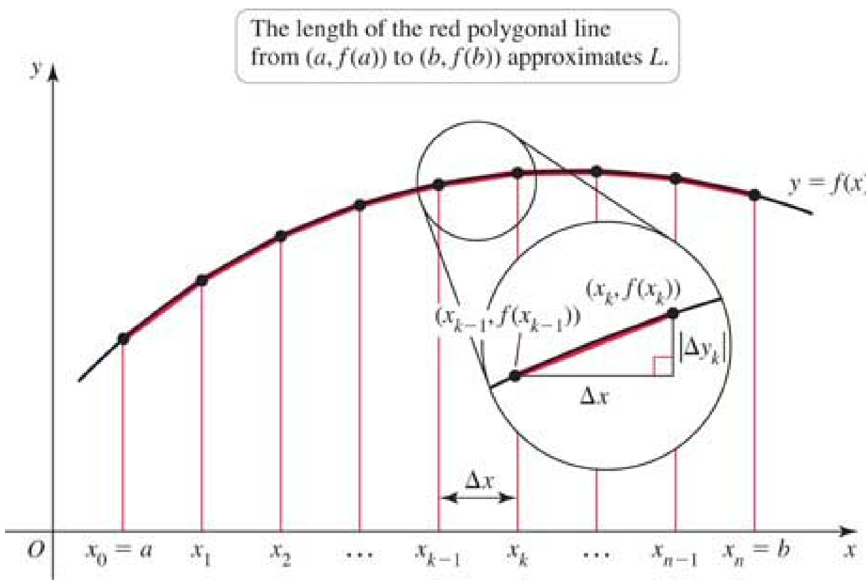
If a region is revolved about the y-axis, then the shell method must be used.

If a region is revolved about the x-axis, it is possible to use the disk/washer method and integrate with respect to x.

6.5: Length of Curves

Definition. (Arc Length for $y = f(x)$)
Let f have a continuous first derivative on the interval $[a, b]$. The length of the curve from $(a, f(a))$ to $(b, f(b))$ is

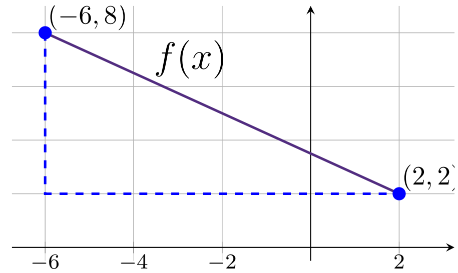
$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$



Definition. (Arc Length for $x = g(y)$)
Let g have a continuous first derivative on the interval $[c, d]$. The length of the curve from $(g(c), c)$ to $(g(d), d)$ is

$$L = \int_c^d \sqrt{1 + g'(y)^2}\, dy.$$

Example. Using a geometric argument, we can see that the length of $f(x) = -\frac{3}{4}x + \frac{7}{2}$ on the interval $[-6, 2]$ is $L = 10$. Compute this using the arc-length formula.



Example. Find the arc length of the curve $y = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$, for $1 \le x \le 2$.

Example. Find the arc length of the curve $y = \frac{1}{3}x^{3/2}$ on $[0, 12]$.

Example. Find a curve that passes through $(1, 2)$ on $[2, 6]$ whose arc length is computed using

$$\int_2^6 \sqrt{1 + 16x^{-2}}\, dx.$$

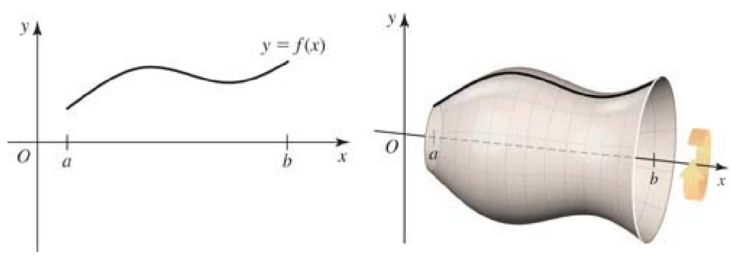
Example. Suppose f has length L on $[a, b]$. Evaluate

$$\int_{a/c}^{b/c} \sqrt{1 + f'(cx)^2}\, dx.$$

6.6: Surface Area

Definition. (Area of a Surface of Revolution)
Let f be a nonnegative function with a continuous first derivative on the interval $[a, b]$. The area of the surface generated when the graph of f on the interval $[a, b]$ is revolved around the x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx.$$



Example. Find the exact area of the surface obtained by rotating the curve $y = x^3$, $0 \le x \le 2$ about the x-axis.

Example. Find the exact area of the surface obtained by rotating the curve $y = \sqrt{8x - x^2}$, $1 \leq x \leq 7$ about the x-axis.

Example. Find the exact area of the surface obtained by rotating the curve $y = \dfrac{1}{2}(e^x + e^{-x})$, $-\ln(2) \le x \le \ln(2)$ about the x-axis.

6.7: Physical Applications

Definition. (Mass of a One-Dimensional Object)
Suppose a thin bar or wire is represented by the interval $a \leq x \leq b$ with a density function ρ (with units of mass per length). The **mass** of the object is

$$m = \int_a^b \rho(x)\, dx.$$

Example. A thin bar, represented by the interval $0 \leq x \leq 4$, has density in units of kg/m given by $\rho(x) = 5e^{-2x}$. What is the mass of the bar?

Definition. (Work)
The work done by a variable force F moving an object along a line from $x = a$ to $x = b$ in the direction of the force is

$$W = \int_a^b F(x)\,dx.$$

Example. According to **Hooke's Law**, the force required to keep a spring in a compressed or stretched position x units from the equilibrium position is $F(x) = kx$, where the positive spring constant k measures the stiffness of the spring.

Suppose a force of 40 N is required to stretch a spring 0.1 m from its equilibrium position. Assuming the spring obeys Hooke's Law, how much work is required to stretch the spring 0.4 m beyond is equilibrium position?

Example (Work from force)**.** How much work is required to move an object from $x = 1$ to $x = 3$ (measured in meters) in the presence of a force (in N) given by $F(x) = \dfrac{2}{x^2}$ acting along the x-axis?

Example. Imagine a chain of length L meters with constant density ρ kg/m is hanging vertically. Using g to represent the acceleration due to gravity, the work required to lift the chain is

$$W = \int_0^L \rho g(L - y)\, dy$$

A 50 meter long chain hangs vertically from a cylinder attached to a winch. Assume there is no friction in the system and the chain has a density of 3 kg/m. How much work is required to wind the entire chain onto the cylinder if a 60-kg load is attached to the end of the chain? Use g for the acceleration due to gravity.

Example. A 30-meter long rope hangs freely from a ledge. The rope has a density of 5 kg/m. How much work is done if the top 1/3 of the rope is pulled up to the ledge? Use g for the acceleration due to gravity.

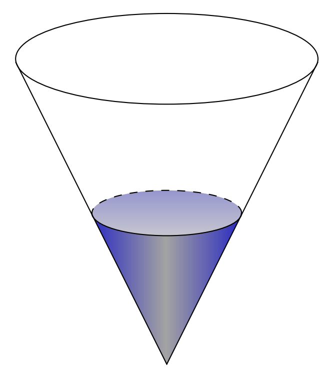
Procedure: Solving Pumping Problems

1. Draw a y-axis in the vertical direction (parallel to gravity) and choose a convenient origin. Assume the interval $[a, b]$ corresponds to the vertical extent of the fluid.
2. For $a \leq y \leq b$, find the cross-sectional area $A(y)$ of the horizontal slices and the distance $D(y)$ the slices must be lifted.
3. The work required to lift the water is

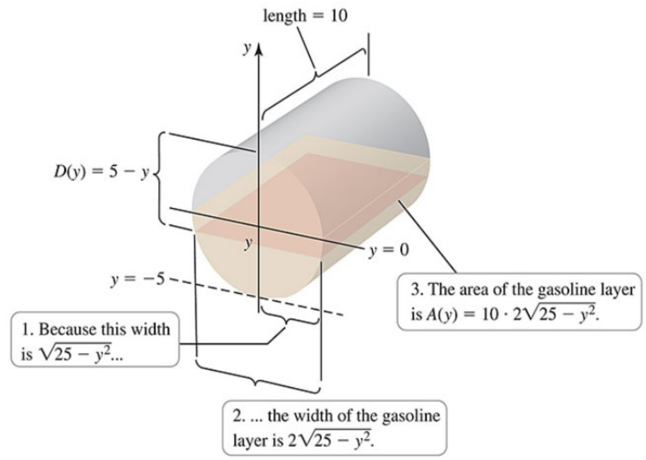
$$W = \int_a^b \rho g A(y) D(y)\, dy.$$

Note: Lifting problems are a special case of pumping problems where $A(y) = 1$.

Example. A water tank is shaped like an inverted cone with height 6 meters and base radius 1.5 meters. If the tank is full, how much work is required to pump the water to the level of the top of the tank and out of the tank? Use g for the acceleration due to gravity and note that the density of water is 1000 kg/m^3.



Example. (Pumping gasoline) A cylindrical tank with a length of 10 m and a radius of 5 m is on its side and half full of gasoline. How much work is required to empty the tank through an outlet pipe at the top of the tank? The density of gasoline is $\rho = 737\,\mathrm{kg/m^3}$

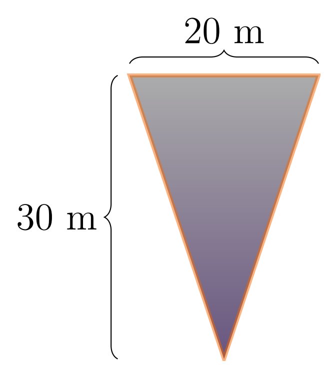


Procedure: Solving Force-on-Dam Problems

1. Draw a y-axis on the face of the dam in the vertical direction and choose a convenient origin (often taken to be the base of the dam).
2. Find the width function $w(y)$ for each value of y on the face of the dam.
3. If the base of the dam is at $y = 0$ and the top of the dam is at $y = a$, then the total force on the dam is

$$F = \int_0^a \rho g \underbrace{(a-y)}_{\text{depth}} \underbrace{w(y)}_{\text{width}} \, dy.$$

Example. The figure to the right shows the shape and dimensions of a small dam. Assuming the water level is at the top of the dam, find the total force on the face of the dam. Use ρ for the density of the water and g for the acceleration due to gravity.



Example. Force on a building A large building shaped like a box is 50 m high with a face that is 80 m wide. A strong wind blows directly at the face of the building, exerting a pressure of 150 $\mathrm{N/m^2}$ at the ground and increasing with height according to $P(y) = 150 + 2y$, where y is the height above the ground. Calculate the total force on the building, which is a measure of the resistance that must be included in the design of the building.

8.1: Basic Approaches (to Integration)

Example. Derive the integral formula $\int \sec(ax)\,dx = \frac{1}{a}\ln|\sec(ax)+\tan(ax)|+C$.

Example. Evaluate $\int \frac{dx}{e^{3x}+e^{-3x}}$.

Example. Evaluate $\displaystyle\int \frac{\sin(x) + \cos^4(x)}{\csc(x)}\,dx.$

Note: $\begin{cases} \cos^2(x) = \dfrac{1+\cos(2x)}{2} \\ \sin^2(x) = \dfrac{1-\cos(2x)}{2} \end{cases}$

Example. Evaluate $\displaystyle\int \frac{2x^2 + 3x - 4}{x - 2}\,dx.$

Example. Evaluate $\displaystyle\int \frac{dx}{\sqrt{7-6x-x^2}}$.

8.2: Integration by Parts

Integraton by Parts
Suppose u and v are differentiable functions. Then

$$\int u\,dv = uv - \int v\,du.$$

A good mnemonic is ILATE.

Example. Evaluate $\int xe^{-\frac{x}{2}}\,dx$.

Integration by Parts for Definite Integrals
Let u and v be differentiable. Then

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx$$

Example. Find the area of the region between the x-axis and $f(x) = \dfrac{\ln(x)}{x^2}$ on $[1, e]$.

Example. Evaluate $\int x^2 \cos(2x)\, dx$.

Example. Evaluate $\int e^{-x} \sin(3x)\, dx$.

Example. Evaluate $\int e^{4x} \cos(3x)\, dx$.

Example. Derive the integral formula

$$\int \ln(x)\,dx = x\ln(x) - x + C$$

Example. Evaluate $\displaystyle\int 10\cos(\sqrt{x})\,dx$

Example. Evaluate $\displaystyle\int_1^e \ln(2x)\,dx$.

8.3: Trigonometric Integrals

Important trigonometric identities

Pythagorean Identities	$\sin^2(\theta) + \cos^2(\theta) = 1$
Angle sum formulas	$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$ $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
Double angle formulas	$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
Half angle formulas	$\sin^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$ $\cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$

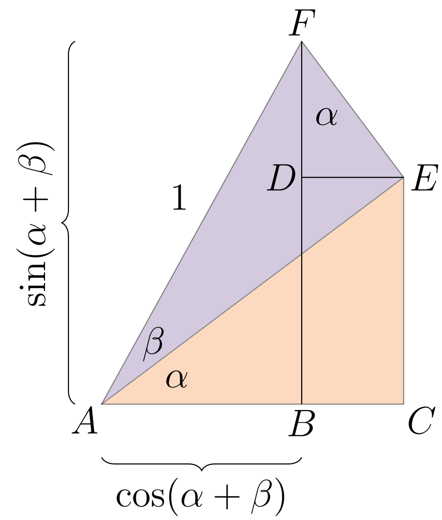
Derivation of angle sum formulas

$$\sin(\alpha) = \frac{\overline{DE}}{\overline{EF}} = \frac{\overline{DE}}{\sin(\beta)} \quad \Rightarrow \overline{DE} = \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha) = \frac{\overline{DF}}{\overline{EF}} = \frac{\overline{DF}}{\sin(\beta)} \quad \Rightarrow \overline{DF} = \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha) = \frac{\overline{CE}}{\overline{AE}} = \frac{\overline{CE}}{\cos(\beta)} \quad \Rightarrow \overline{CE} = \sin(\alpha)\cos(\beta)$$

$$\cos(\alpha) = \frac{\overline{AC}}{\overline{AE}} = \frac{\overline{AC}}{\cos(\beta)} \quad \Rightarrow \overline{AC} = \cos(\alpha)\cos(\beta)$$



Derivation of the double angle formulas

$$\sin(2\theta) = \sin(\theta + \theta) = \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos(\theta + \theta) = \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta) = \cos^2(\theta) - \sin^2(\theta)$$

Derivation of the half angle formulas

Start with the cosine double angle formula:

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \boxed{2\cos^2(\theta) - 1} = \boxed{1 - 2\sin^2(\theta)}$$

Solve for either $\sin^2(\theta)$ or $\cos^2(\theta)$:

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2} \qquad\qquad \cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$$

Example. Evaluate the integral $\int \cos^5(x)\, dx$.

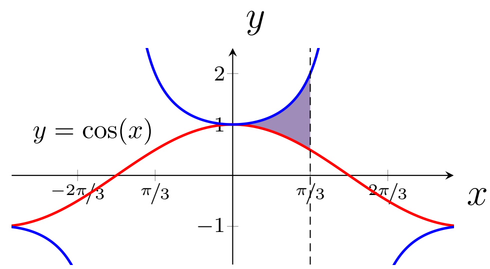
Example. Evaluate the integral $\int \sin^3(x) \cos^{3/2}(x)\, dx$.

Example. Evaluate the integral $\int 20\sin^2(x)\cos^2(x)\,dx$

Example. Evaluate the integral $\displaystyle\int \sec^6(x)\tan^4(x)\,dx$.

Example. Evaluate the integral $\displaystyle\int 35\tan^5(x)\sec^4(x)\,dx$.

Example. Consider the region bounded by $y = \sec(x)$ and $y = \cos(x)$ for $0 \le x \le {}^{\pi}\!/_{3}$. Find the volume of the solid generated when rotating this region about the line $y = -1$.

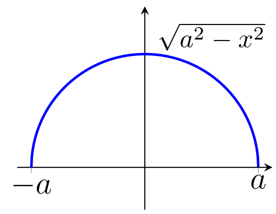


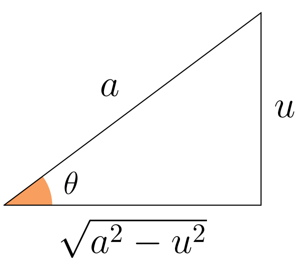
Example. Find the length of the curve $y = \ln(2\sec(x))$ on the interval $[0, \pi/6]$.

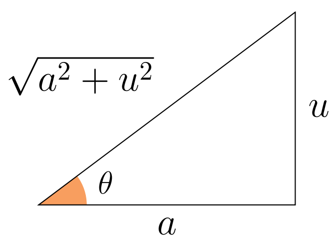
$\int \sin^m(x)\cos^n(x)\,dx$	**Strategy**
m odd and positive, n real	Split off $\sin(x)$, rewrite the resulting even power of $\sin(x)$ in terms of $\cos(x)$, and then use $u = \cos(x)$.
n odd and positive, m real	Split off $\cos(x)$, rewrite the resulting even power of $\cos(x)$ in terms of $\sin(x)$, and then use $u = \sin(x)$.
m and n both even, nonnegative integers	Use half-angle formulas to transform the integrand into a polynomial in $\cos(2x)$, and apply the preceding strategies once again to powers of $\cos(2x)$ greater than 1.
$\int \tan^m(x)\sec^n(x)\,dx$	
n even and positive, m real	Split off $\sec^2(x)$, rewrite the remaining even power of $\sec(x)$ in terms of $\tan(x)$, and use $u = \tan(x)$.
m odd and positive, n real	Split off $\sec(x)\tan(x)$, rewrite the remaining even power of $\tan(x)$ in terms of $\sec(x)$, and use $u = \sec(x)$.
m even and positive, n odd and positive	Rewrite $\tan^m(x)$ in terms of $\sec(x)$
$\int \sec^n(x)\,dx$	
n odd	Use integration by parts with $u = \sec^{n-2}(x)$ and $dv = \sec^2(x)\,dx$
n even	Split off $\sec^2(x)$, rewrite the remaining powers of $\sec(x)$ in terms of $\tan(x)$, and use $u = \tan(x)$.
$\int \tan^m(x)\,dx$	Split off $\tan^2(x)$ and rewrite in terms of $\sec(x)$. Expand into difference of integrals substituting $u = \tan(x)$. Repeat the process as needed for remaining powers of $\tan(x)$.

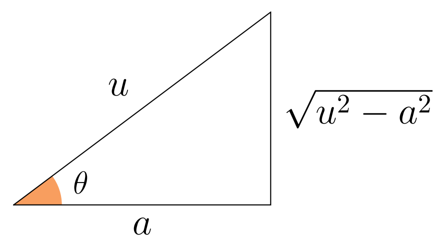
8.4: Trigonometric Substitutions

Example. Verify the formula for the area of a circle with radius a by finding the area under $f(x) = \sqrt{a^2 - x^2}$.









$a^2 - u^2$	$u = a\sin(\theta)$,	$-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$, for $\lvert u \rvert \leq a$	$a^2 - a^2\sin^2(\theta) = a^2\cos^2(\theta)$
$a^2 + u^2$	$u = a\tan(\theta)$,	$-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$,	$a^2 + a^2\tan^2(\theta) = a^2\sec^2(\theta)$
$u^2 - a^2$	$u = a\sec(\theta)$,	$\begin{cases} 0 \leq \theta < \frac{\pi}{2}, & \text{for } u \geq a \\ \frac{\pi}{2} < \theta \leq \pi, & \text{for } u \leq -a \end{cases}$	$a^2\sec^2(\theta) - a^2 = a^2\tan^2(\theta)$

Example. $\displaystyle\int \frac{\sqrt{x^2-4}}{x^3}\,dx$

Example. $\displaystyle\int \frac{\sqrt{16-x^2}}{x}\,dx$

Example. $\displaystyle\int \frac{x^2}{\left(25-4x^2\right)^{3/2}}\,dx$

Example. $\displaystyle\int_0^{1/3} \frac{dx}{(9x^2+1)^{3/2}}$

Example. $\displaystyle\int \frac{x}{\sqrt{x^2 - 2x + 10}}$

8.5: Partial Fractions

Example. Simplify $f(x) = \dfrac{1}{x-2} + \dfrac{2}{x+4}$ by finding a common denominator.

Procedure: Partial Fractions with Simple Linear Factors
Suppose $f(x) = p(x)/q(x)$, where p and q are polynomials with no common factors and with the degree of P less than the degree of q. Assume q is the product of simple linear factors. The partial fraction decomposition is obtained as follows.

Step 1: Factor the denominator q in the form $(x - r_1)(x - r_2)\dots(x - r_n)$

Step 2: Partial fraction decomposition

$$\frac{p(x)}{q(x)} = \frac{A_1}{(x - r_1)} + \frac{A_2}{(x - r_2)} + \cdots + \frac{A_n}{(x - r_n)}.$$

Step 3: Clear denominators Multiply both sides of the equation in Step 2 by $q(x) = (x - r_1)(x - r_2)\dots(x - r_n)$

Step 4: Solve for coefficients Equate like powers of x in Step 3 to solve for the undetermined coefficients $A_1, \dots, A_n$.

Example. Perform partial fraction decomposition on $f(x) = \dfrac{3x}{x^2 + 2x - 8}$.

Example. $\displaystyle\int \frac{28x^3 - 56x^2 + 9}{x^2 - 2x}\,dx$

Procedure: Partial Fractions for Repeated Linear Factors
Suppose the repeated linear factor $(x-r)^m$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition has a partial fraction for each power of $(x-r)$ up to and including the mth power; that is, the partial fraction decomposition contains the sum

$$\frac{A_1}{(x-r)} + \frac{A_3}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}$$

where $A_1, \ldots, A_m$ are constants to be determined.

Example. Setup the partial fraction decomposition for $f(x) = \dfrac{x^3 - 8x + 19}{x^4 + 3x^3}$.

Example. Setup the partial fraction decomposition for $g(x) = \dfrac{2}{x^5 - 6x^4 + 9x^3}$.

Example. Evaluate $\displaystyle\int \frac{x^2+1}{(2x-3)(x-2)^2}\,dx.$

Example. Evaluate $\int \frac{8}{3x^3 + 7x^2 + 4x}\,dx$.

Procedure: Partial Fractions with Simple Irreducible Quadratic Factors
Suppose a simple irreducible factor ax^2+bx+c appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition contains a term of the form
$$\frac{Ax+B}{ax^2+bx+c},$$
where A and B are unknown coefficients to be determined.

Example. Perform partial fraction decomposition on the following fractions or identify them as irreducible.

$$\frac{1}{x^2-13x+43}$$

$$\frac{x^2}{(x-4)(x+5)}$$

Example. Perform partial fraction decomposition on the following fractions or identify them as irreducible.

$$\frac{7}{(x^2+1)^2}$$

$$\frac{1}{x^2+11x+28}$$

Example. Evaluate $\displaystyle\int \frac{4x}{(x+1)(x^2+1)}\,dx$

Example. Evaluate $\displaystyle\int \frac{3x^2 + 2x + 12}{(x^2+4)^2}\,dx$

Example. Evaluate $\int \frac{1}{x\sqrt{1+2x}}\,dx$ using the substitution $u = \sqrt{1+2x}$.

Summary: Partial Fraction Decomposition
Let $f(x) = p(x)/q(x)$ be a proper rational function in reduced form. Assume the denominator q has been factored completely over the real numbers and m is a positive integer.

1. **Simple linear factor:** A factor $x - r$ in the denominator requires the partial fraction $\dfrac{A}{x-r}$.
2. **Repeated linear factor:** A factor $(x-r)^m$ with $m > 1$ in the denominator requires the partial fractions
$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}.$$
3. **Simple irreducible quadratic factor:** An irreducible factor $ax^2 + bx + c$ in the denominator requires the partial fraction
$$\frac{Ax+B}{ax^2+bx+c}.$$
4. **Repeated irreducible quadratic factor:** An irreducible factor $(ax^2+bx+c)^m$ with $m > 1$ in the denominator requires the partial fractions
$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}.$$

8.6: Integration Strategies

Example. What integration methods can be used to evaluate the functions below? (No need to evaluate the integral)

$$\int \frac{1}{1-x^2}\,dx$$

$$\int x\sec^2(x)\,dx$$

$$\int \frac{x}{\sqrt{64-x^2}}\,dx$$

$$\int \frac{x^3}{\sqrt{64-x^2}}\,dx$$

Example. Identify two integration techniques which can be used to evaluate

$$\int \frac{4-3x^2}{x(x^2-4)}\,dx.$$

Example. Perform a substitution of variables to rewrite $\displaystyle\int x\sin(\sqrt{x})\,dx$.

Example. $\displaystyle\int_1^3 \frac{\tan^{-1}\left(\sqrt{x}\right)}{x^{1/2}+x^{3/2}}\,dx$

8.9: Improper Integrals

Definition. (Improper Integrals over Infinite Intervals)

1. If f is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If f is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. If f is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx.$$

   where c is any real number. It can be shown that the choice of c does not affect the value or convergence of the original integral.

If the limits in cases 1.– 3. exist, then the improper integrals **converge**; otherwise they **diverge**.

Example. Evaluate $\displaystyle\int_1^{\infty} \frac{\ln(x)}{x}\,dx$ and determine if the integral converges or diverges.

Example. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{e^{3x}}{1+e^{6x}}\,dx$.

Example. For what values of p does $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converge?

Definition. (Improper Integrals with Unbounded Integrand)

1. Suppose f is continuous on $(a, b]$ with $\lim_{x\to a^+} f(x) = \pm\infty$. Then
$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$
2. Suppose f is continuous on $[a, b)$ with $\lim_{x\to b^-} f(x) = \pm\infty$. Then
$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$
3. Suppose f is continuous on $[a, b]$ except at the interior point p where f is unbounded. Then
$$\int_a^b f(x)\,dx = \lim_{c\to p^-} \int_a^c f(x)\,dx + \lim_{d\to p^+} \int_d^b f(x)\,dx.$$

If the limits in cases 1.– 3. exist, then the improper integrals **converge**; otherwise, they **diverge**.

Example. Determine which of the following integrals are improper integrals

$$\int_0^1 \sec(x)\,dx$$

$$\int_{\pi/2}^{3\pi/4} \tan(x)\,dx$$

$$\int_1^e \ln(x)\,dx$$

$$\int_0^1 \arctan(x)\,dx$$

$$\int_0^{0.5} \ln(x)\,dx$$

$$\int_{-10}^{-1} \frac{1}{x^{1/3}}\,dx$$

Example. Evaluate $\int_1^9 \frac{dx}{(x-1)^{2/3}}$. Does this integral converge or diverge?

Example. Evaluate $\int_{-1}^{1} \frac{e^{2/x}}{x^2}\, dx$. Does this integral converge or diverge?

Theorem 8.2: Comparison Test for Improper Integrals
Suppose the functions f and g are continuous on the interval $[a, \infty)$, with $f(x) \geq g(x) \geq 0$, for $x \geq a$.

1. If $\displaystyle\int_a^\infty f(x)\,dx$ converges, then $\displaystyle\int_a^\infty g(x)\,dx$ converges.
2. If $\displaystyle\int_a^\infty g(x)\,dx$ diverges, then $\displaystyle\int_a^\infty f(x)\,dx$ diverges.

Example. Determine if the integral $\displaystyle\int_2^\infty \frac{x^3}{x^4 - x^3 - 1}\,dx$ converges or diverges.

Example (Gabriel's Horn)**.** Let R be the region bounded by the graph of $y = 1/x$ and the x-axis for $x \geq 1$.

What is the volume of the solid generated when R is revolved around the x-axis?

What is the surface area of the solid generated when R is revolved about the x-axis?

10.1: An Overview of Sequences and Infinite Series

Definition. (Sequence)
A **sequence** $\{a_n\}$ is an ordered list of numbers of the form

$$\{a_1, a_2, a_3, \ldots, a_n, \ldots\}.$$

A sequence may be generated by a **recurrence relation** of the form $a_{n+1} = f(a_n)$, for $n = 1, 2, 3, \ldots$, where a_1 is given. A sequence may also be defined with an **explicit formula** of the form $a_n = f(n)$, for $n = 1, 2, 3, \ldots$.

Example. Consider the sequence $a_n = \frac{2^{n+1}}{2^n+1}$; Compute a_1, a_2, a_3, and a_4.

Definition. (Limit of a Sequence)
If the terms of a sequence $\{a_n\}$ approach a unique number L as n increases— that is, if a_n can be made arbitrarily close to L by taking n sufficiently large— then we say $\lim_{n\to\infty} a_n = L$ exists, and the sequence **converges** to L. If the terms of the sequence do not approach a single number as n increases, the sequence has no limit, and the sequence **diverges**.

Example. Determine if the sequence given by

$$a_n = \frac{3+5n^2}{n+n^2}$$

converges or diverges. If it converges, find the value that the sequence converges to.

Example. Determine if the sequence given by

$$a_n = (-1)^n \frac{3+5n^2}{n+n^2}$$

converges or diverges. If it converges, find the value that the sequence converges to.

Example. A ball is thrown upward to a height of 10 meters. After each bounce, the ball rebounds to $2/3$ of its previous height. Let h_n be the height after the nth bounce. Find an explicit formula for the nth term of the sequence $\{h_n\}$.

Definition. (Infinite series)
Given a sequence $\{a_1, a_2, a_3, \dots\}$, the sum of its terms

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$$

is called an **infinite series**. The **sequence of partial sums** $\{S_n\}$ associated with this series has the terms

$$\begin{aligned}
S_1 &= a_1\\
S_2 &= a_1 + a_2\\
S_3 &= a_1 + a_2 + a_3\\
&\vdots\\
S_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k, \qquad \text{for } n = 1, 2, 3, \dots.
\end{aligned}$$

If the sequence of partial sums $\{S_n\}$ has a limit L, the infinite series **converges** to that limit, and we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} a_k}_{S_n} = \lim_{n\to\infty} S_n = L.$$

If the sequence of partial sums diverges, the infinite series also **diverges**.

Example. Consider the infinite series $4 + 0.9 + 0.09 + 0.009 + \ldots$. Compute S_1, S_2, S_3, and S_4. What is the value of this series?

Example. A sequence $\{a_n\}$ has partial sums given by the formula $S_n = 5 - \frac{1}{\sqrt{n}}$.

What is the value of the series $\displaystyle\sum_{n=1}^{\infty} a_n$?

What is the formula for a_n?

What is the limit $\displaystyle\lim_{n\to\infty} a_n$?

10.2: Sequences

Theorem 10.1: Limits of Sequences from Limits of Functions
Suppose f is a function such that $f(n) = a_n$, for positive integers n. If $\lim_{x\to\infty} f(x) = L$, then the limit of the sequence $\{a_n\}$ is also L, where L may be $\pm\infty$.

Example. Determine if the following sequences converge or diverge. If the sequence converges, find its limit.

$\left\{e^{2n/(n+2)}\right\}_{n=1}^{\infty}$

$\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

$\left\{\frac{\arctan(n)}{n}\right\}_{n=1}^{\infty}$

$\left\{\frac{e^{-n}}{42\sin(e^{-n})}\right\}_{n=1}^{\infty}$

10.2: Limit Laws for Sequences
Assume the sequences $\{a_n\}$ and $\{b_n\}$ have limits A and B, respectively. Then

1. $\lim\limits_{n\to\infty} (a_n \pm b_n) = A \pm B$
2. $\lim\limits_{n\to\infty} ca_n = cA$, where c is a real number
3. $\lim\limits_{n\to\infty} a_n b_n = AB$
4. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$, provided $B \neq 0$.

Example. Consider the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ where

$$a = \frac{1}{n}, \quad b_n = n, \quad c_n = e^n, \quad \text{ and } d_n = \sqrt{n}.$$

Compute the following limits.

A. $\lim\limits_{n\to\infty} a_n$ B. $\lim\limits_{n\to\infty} b_n$ C. $\lim\limits_{n\to\infty} c_n$ D. $\lim\limits_{n\to\infty} d_n$

E. $\lim\limits_{n\to\infty} a_n b_n$ F. $\lim\limits_{n\to\infty} a_n c_n$ G. $\lim\limits_{n\to\infty} a_n d_n$

True or False: If for some sequences $\{a_n\}$ and $\{b_n\}$, $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} b_n = \infty$, then $\lim\limits_{n\to\infty} a_n b_n = 0$.

Definition. (Terminology for Sequences)

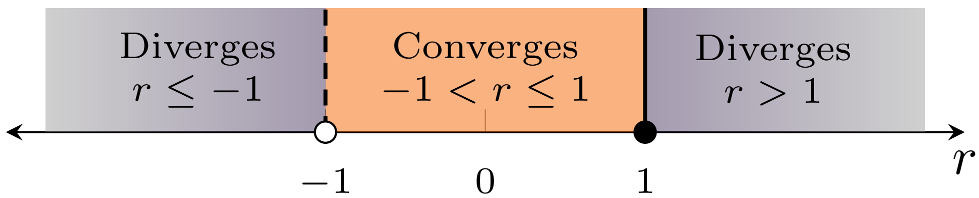
- $\{a_n\}$ is **increasing** if $a_{n+1} > a_n$
- $\{a_n\}$ is **nondecreasing** if $a_{n+1} \geq a_n$
- $\{a_n\}$ is **decreasing** if $a_{n+1} < a_n$
- $\{a_n\}$ is **nonincreasing** if $a_{n+1} \leq a_n$
- $\{a_n\}$ is **monotonic** if it is either nonincreasing or nondecreasing (it moves in one direction)
- $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \leq M$, for all relevant values of n
- $\{a_n\}$ is **bounded below** if there is a number N such that $a_n \geq N$, for all relevant values of n.
- If $\{a_n\}$ is bounded above and bounded below, then we say that $\{a_n\}$ is a **bounded** sequence.

Example. Consider the sequence $\left\{-n^2\right\}_{n=1}^{\infty}$. What can we say about this sequence?

Theorem 10.3: Geometric Sequences
Let r be a real number. Then

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist} & \text{if } r \leq -1 \text{ or } r > 1. \end{cases}$$

If $r > 0$, then $\{r^n\}$ is a monotonic sequence. If $r < 0$, then $\{r^n\}$ oscillates.



Example. Determine if the following sequences converge

$$\left\{\frac{3^{n+1}+3}{3^n}\right\}$$

$$\left\{2^{n+1}3^{-n}\right\}$$

$$\left\{\frac{(-1)^n}{2^n}\right\}$$

$$\left\{\frac{75^{n-1}}{99^n} + \frac{5^n \sin(n)}{8^n}\right\}$$

Theorem 10.4: Squeeze Theorem for Sequences
Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences with $a_n \leq b_n \leq c_n$, for all integers n greater than some index N. If $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.

Example. Find the limit of the sequence $b_n = \dfrac{9\cos(n)}{n^2+1}$.

Theorem 10.5: Bounded Monotonic Sequence
A bounded monotonic sequence converges.

Theorem 10.6: Growth Rates of Sequences
The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, if $\{a_n\}$ appears before $\{b_n\}$ in the list, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$:

$$\{(\ln n)^q\} \ll \{n^p\} \ll \{n^p(\ln n)^r\} \ll \left\{n^{p+s}\right\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\}$$

Example. Use growth rates to determine which of the following sequences converge.

$$\left\{\frac{\ln(n^{10})}{0.00001n}\right\}$$

$$\left\{\frac{n^8\ln(n)}{n^{8.001}}\right\}$$

$$\left\{\frac{n!}{10^n}\right\}$$

Definition. (Limit of a Sequence)
The sequence $\{a_n\}$ converges to L provided the terms of a_n can be made arbitrarily close to L by taking n sufficiently large. More precisely, $\{a_n\}$ has the unique limit L if, given any $\varepsilon > 0$, it is possible to find a positive integer N (depending only on ε) such that

$$|a_n - L| < \varepsilon \qquad \text{whenever } n > N.$$

If the **limit of a sequence** is L, we say the sequence **converges** to L, written

$$\lim_{n\to\infty} a_n = L.$$

A sequence that does not converge is said to **diverge**.

10.3: Infinite Series

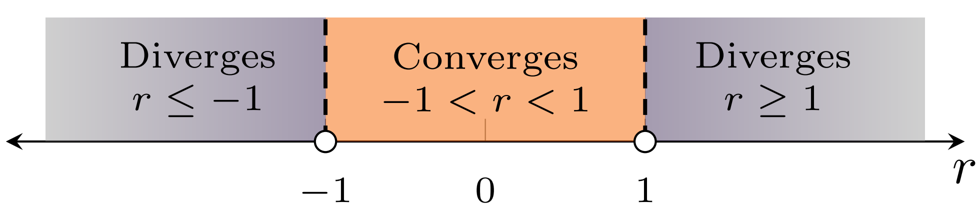
A **Geometric sum** with n terms has the form

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k$$

Derivation of partial sum formula:

Theorem 10.7: Geometric Series

Let $a \neq 0$ and r be real numbers. If $|r| < 1$, then $\displaystyle\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. If $|r| \geq 1$, then the series diverges.



Example. Evaluate the following geometric series or state that the series diverges

$$\sum_{k=0}^{\infty} 1.1^k$$

$$\sum_{k=0}^{\infty} e^{-k}$$

$$\sum_{k=2}^{\infty} 3(-0.75)^k$$

$$\sum_{k=1}^{\infty} \frac{7}{10^k}$$

Telescoping Series:

Example. Evaluate the following series

$$\sum_{k=1}^{\infty} \cos\left(\frac{1}{k^2}\right) - \cos\left(\frac{1}{(k+1)^2}\right)$$

$$\sum_{k=3}^{\infty} \frac{1}{(k-2)(k-1)}$$

Theorem 10.8: Properties of Convergent Series

1. Suppose $\sum a_k$ converges to A and c is a real number. The series $\sum ca_k$ converges, and $\sum ca_k = c\sum a_k = cA$.
2. Suppose $\sum a_k$ diverges. Then $\sum ca_k$ also diverges, for any real number $c \neq 0$.
3. Suppose $\sum a_k$ converges to A and $\sum b_k$ converges to B. The series $\sum (a_k \pm b_k)$ converges and $\sum (a_k \pm b_k) = \sum a_k \pm \sum b_k = A \pm B$.
4. Suppose $\sum a_k$ diverges and $\sum b_k$ converges. Then $\sum (a_k \pm b_k)$ diverges.
5. If M is a positive integer, then $\displaystyle\sum_{k=1}^{\infty} a_k$ and $\displaystyle\sum_{k=M}^{\infty} a_k$ either both converge or both diverge. In general, *whether* a series converges does not depend on a finite number of terms added to or removed from the series. However, the *value* of a convergent series does change if nonzero terms are added or removed.

Example. Evaluate

$$\sum_{k=1}^{\infty}\left[\frac{1}{2}\left(\frac{2}{5}\right)^k + \frac{2}{3}\left(\frac{1}{6}\right)^k\right]$$

10.4: The Divergence and Integral Tests

Theorem 10.9: Divergence Test
If $\sum a_k$ converges, then $\lim_{k\to\infty} a_k = 0$. Equivalently, if $\lim_{k\to\infty} a_k \neq 0$, then the series diverges.

Example. If $\lim_{k\to\infty} a_k = 1$, what can we conclude about $\sum_{k=1}^{\infty} a_k$?

Example. If $\sum_{k=1}^{\infty} a_k = 42$, what can we conclude about $\lim_{k\to\infty} a_k$?

Example. If $\lim_{k\to\infty} a_k = 0$, what can we conclude about $\sum_{k=1}^{\infty} a_k$?

Example. Determine which of the following series diverge by the divergence test.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}}$$

$$\sum_{k=1}^{\infty} \frac{k^3+100}{3k^3+k+1}$$

$$\sum_{k=1}^{\infty} \frac{e^k}{k^2}$$

Table 1 Series Convergence				
Scenario	Sequence of Terms $\{a_1, a_2, a_3, \ldots\}$	Sequence of Partial Sums $\{s_1, s_2, s_3, \ldots\}$	Series $\sum_{n=1}^{\infty} a_n$	Possible or Impossible?
A	Converges	Diverges	Diverges	
B	Converges	Diverges	Converges	
C	Converges	Converges	Diverges	
D	Converges	Converges	Converges	
E	Diverges	Converges	Diverges	
F	Diverges	Converges	Converges	
G	Diverges	Diverges	Diverges	
H	Diverges	Diverges	Converges	

Theorem 10.10: Harmonic Series

The harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$ diverges—even though the terms of the series approach zero.

Theorem 10.11: Integral Test
Suppose f is a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Then

$$\sum_{k=1}^{\infty} a_k \text{ and } \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge. In the case of convergence, the value of the integral is *not* equal to the value of the series.

Example. Which of the following series below satisfy all the conditions to use the Integral Test?

$$\sum_{k=1}^{\infty} \arctan(k)$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{e^k}$$

Example. Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

Use the integral test to show that the Harmonic Series diverges. For what values of p does this series converge?

Theorem 10.12: Convergence of the p-series

The p**-series** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p \le 1$.

Example. Determine if the following p-series converge or diverge.

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$\sum_{k=1}^{\infty} k^{-1/3}$$

$$\sum_{k=1}^{\infty} \frac{k^2}{k^{\pi}}$$

$$\sum_{k=1}^{\infty} \frac{2}{k}$$

$$\sum_{k=1}^{\infty} \frac{-3}{\sqrt[3]{k^4}}$$

$$\sum_{k=1}^{\infty} \frac{k^3+1}{k^5}$$

Example. Apply the Integral Test to determine if the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}}$ converges or diverges.

Theorem 10.13: Estimating Series with Positive Terms
Let f be a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Let $S = \sum_{k=1}^{\infty} a_k$ be a convergent series and let $S_n = \sum_{k=1}^{n} a_k$ be the sum of the first n terms of the series. The remainder $R_n = S - S_n$ satisfies

$$R_n < \int_n^{\infty} f(x)\,dx.$$

Furthermore, the exact value of the series is bounded as follows:

$$L_n = S_n + \int_{n+1}^{\infty} f(x)\,dx < \sum_{k=1}^{\infty} a_k < S_n + \int_n^{\infty} f(x)\,dx = U_n.$$

Example. How many terms of the convergent p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ must be summed to obtain an approximation that is within 10^{-3} of the exact value of the series?

10.5: Comparison Tests

Theorem 10.14: Comparison Test
Let $\sum a_k$ and $\sum b_k$ be series with positive terms where $a_k \leq b_k$.

1. If $\sum b_k$ converges, then $\sum a_k$ converges.
2. If $\sum a_k$ diverges, then $\sum b_k$ diverges.

Example. Use the comparison test to determine if the series $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{k^3 - 3}$ converges or diverges.

Theorem 10.15: Limit Comparison Test
Let $\sum a_k$ and $\sum b_k$ be series with positive terms and let

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L.$$

1. If $0 < L < \infty$ (that is, L is a finite positive number), then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.
2. If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.
3. If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Example. Using either the Comparison Test or the Limit Comparison Test, determine if the series

$$\sum_{k=1}^{\infty} \frac{4k^2 - k}{k^3 + 9}$$

converges or diverges.

Example. Determine if the following series converge or diverge.

$$\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}}$$

$$\sum_{k=1}^{\infty} \frac{\ln(k)}{k^2}$$

$$\sum_{k=1}^{\infty} \left(1 + \frac{2}{k}\right)^k$$

$$\frac{1}{14^3} + \frac{2}{15^3} + \frac{3}{16^3} + \dots$$

$$\sum_{k=1}^{\infty} \frac{\sin\left(\frac{\pi}{k}\right)}{k^3}$$

$$\sum_{k=1}^{\infty} \frac{\sqrt[3]{k^2+4}}{\sqrt{k^3+9}}$$

10.6: Alternating Series

Theorem 10.16: Alternating Series Test
The alternating series $\sum(-1)^{k+1}a_k$ converges provided

1. the terms of the series are nonincreasing in magnitude ($0 < a_{k+1} \leq a_k$, for k greater than some index N) and
2. $\lim\limits_{k\to\infty} a_k = 0$.

Example. Which of the following are considered alternating series?

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k+2} \qquad \sum_{k=4}^{\infty} \left(\frac{-3}{2}\right)^k \qquad \sum_{k=0}^{\infty} (-1)\left(\frac{1}{2}\right)^k \qquad \sum_{k=1}^{\infty} (-1)^{k+1}\left(\frac{1}{2}\right)^k$$

$$\sum_{k=-3}^{\infty} \frac{\cos(k\pi)}{(k+4)^2} \qquad \sum_{k=1}^{\infty} \frac{\sin(k)}{k^2} \qquad \sum_{k=0}^{\infty} (-1)^{k+1}\left(\frac{1}{-2}\right)^k$$

Example. Consider the series $\sum_{k=1}^{\infty}(-1)^k\frac{\sqrt{k}}{2k+3}$. Let a_k represent that magnitude of the terms of the given series.

- What is $\lim_{k\to\infty} a_k$?

- Compute $f'(x)$ where $f(k) = a_k$.

- Use the Alternating Series Test to determine if the given series converges.

Example. Does the series $\displaystyle\sum_{k=0}^{\infty}(-1)^{k+1}\left(\frac{4}{3}\right)^{k}$ converge?

Example. Does the series $\displaystyle\sum_{k=1}^{\infty}\cos(\pi k)e^{-k}$ converge?

Theorem 10.17: Alternating Harmonic Series

The alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges (even though the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges).

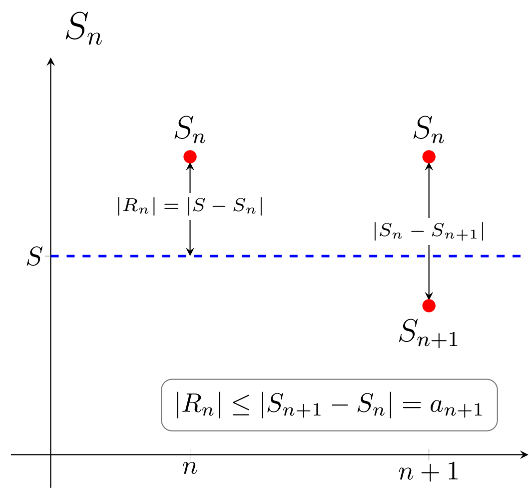
Example. Use the Alternating Series Test to show that the alternating harmonic series converges.

Theorem 10.18: Remainder in Alternating Series

Let $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ be a convergent alternating series with terms that are nonincreasing in magnitude. Let $R_n = S - S_n$ be the remainder in approximating the value of that series by the sum of its first n terms. Then $|R_n| \leq a_{n+1}$. In other words, the magnitude of the remainder is less than or equal to the magnitude of the first neglected term.

Example. Find the minimum value of n such that $|R_n| < 10^{-4}$ for the following series:

$$\ln(2) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$



Definition. (Absolute and Conditional Convergence)
If $\sum |a_k|$ converges, then $\sum a_k$ **converges absolutely**.
If $\sum |a_k|$ diverges and $\sum a_k$ converges, then $\sum a_k$ **converges conditionally**.

Example. Can a series of strictly positive terms converge conditionally?

Example. Consider the series $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{4+k}{k^2}$. Determine if this series converges absolutely, converges conditionally, or diverges.

Example. Determine if the following series converge absolutely, converge conditionally, or diverge.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2\sqrt{k}-1}$$

$$\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^{k}$$

Theorem 10.19: Absolute Convergence Implies Convergence
If $\sum |a_k|$ converges, then $\sum a_k$ converges (absolute convergence implies convergence). Equivalently, if $\sum a_k$ diverges, then $\sum |a_k|$ diverges.

Example. Determine whether each of the following series converges absolutely, converges conditionally or diverges.

$$\sum_{k=1}^{\infty} (-1)^k e^{1/k}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^6}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$$

$$\sum_{k=1}^{\infty} \frac{(-5)^k}{3^k}$$

$$\sum_{k=1}^{\infty} \frac{(-2)^{k-1}}{3^k}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{3^k}$$

Example. Does the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2\sqrt{k}-1}$ converge conditionally, converge absolutely, or diverge?

10.7: The Ratio and Root Tests

Theorem 10.20: Ratio Test
Let $\sum a_k$ be an infinite series, and let $r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$

1. If $r < 1$, the series converges absolutely, and therefore it converges (by Theorem 10.19)
2. If $r > 1$ (including $r = \infty$), the series diverges.
3. If $r = 1$, the test is inconclusive.

Note: The ratio test is used to determine if a series converges or diverges and indicates nothing about the *value* of the series.

Example. Use the ratio test on the harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ and the alternating harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$.

Example. *Note:* $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$

Rewrite $n!n!$ and $\dfrac{(2n)!}{(2n-1)!}$

Example. Consider the series below. Use the ratio test, if appropriate, to show if each of the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{k^2}{2^k}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k k}{k^3+1}$$

$$\sum_{k=1}^{\infty} \frac{5^k k!}{k^k}$$

$$\lim_{k\to\infty} \left(1 + \frac{x}{k}\right)^k = e^x$$

$$\sum_{k=1}^{\infty} \frac{(-7)^k}{(2k+1)!}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k \ln(k)}{k}$$

Example. Use the ratio test to determine if the series $\displaystyle\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$ converges or diverges.

Example. Use the ratio test to determine if the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k}{(2k)!}$ converges or diverges.

Example. Use the ratio test to determine if the series $\sum_{k=1}^{\infty} \frac{(2k)!}{(k!)^2}$ converges or diverges.

10.21: Root Test
Let $\sum a_k$ be an infinite series, and let $\rho = \lim_{k\to\infty} \sqrt[k]{|a_k|}$.

1. If $\rho < 1$, the series converges absolutely, and therefore it converges (by Theorem 10.19)
2. If $\rho > 1$ (including $\rho = \infty$), the series diverges.
3. If $\rho = 1$, the test is inconclusive.

Note: The root test is used to determine if a series converges or diverges and indicates nothing about the *value* of the series.

Example. Use the root test to determine if the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k^k}{3^{k^2}}$ converges.

Example. Consider the series below. Use the root test to show if each of the series converges or diverges.

$$\sum_{k=1}^{\infty} \left(\frac{1}{\ln(k+1)} \right)^k$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{3k^2+1}{k-2k^2} \right)^k$$

$$\sum_{k=1}^{\infty} \left(\frac{k+3}{k+1} \right)^{2k}$$

Example. Use the root test to determine if the series $\displaystyle\sum_{k=1}^{\infty}\left(1-\frac{3}{k}\right)^{k^2}$ converges.

Example. Determine whether each of the series below converges conditionally, converges absolutely, or diverges.

$$\sum_{k=1}^{\infty} (-1)^k k^{-1/3}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\arctan(k)}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k!}$$

Example. Determine if the series $\displaystyle\sum_{k=1}^{\infty}\left(\frac{k}{k+5}\right)^{3k^2}$ converges.

Example. Determine a condition for $x \geq 0$ such that $\displaystyle\sum_{k=1}^{\infty} \frac{4x^k}{5k^2}$ converges.

10.8: Choosing a Convergence Test

Example. Consider the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$. Is this series conditionally convergent, absolutely convergent, or divergent? Which test do you use?

Example. Consider the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$. Is this series conditionally convergent, absolutely convergent, or divergent? Which test do you use?

Example. Which of the following series can be rewritten as a p-series?

$$\sum_{k=1}^{\infty} \frac{(-1)^{2k}}{k\sqrt{k}}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k}}{k^5}$$

$$\sum_{k=1}^{\infty} \frac{k^2+k+1}{k^4+2}$$

$$\sum_{k=1}^{\infty} \frac{3^k}{k^4}$$

$$\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2}$$

Example. Which test *cannot* be used to determine the convergence of $\sum_{k=1}^{\infty} \frac{k^2\, 2^{k-1}}{(-5)^k}$?

Example. For the following series, which test should be used to determine if the series converges or diverges? Use your selected test to show convergence or divergence.

$$\sum_{k=1}^{\infty} (-1)^k \frac{k}{k+2}$$

$$\sum_{k=1}^{\infty}(-1)^k\frac{k}{k+2}$$

$$\sum_{k=1}^{\infty}\frac{k!}{2^k(k+2)!}$$

$$\sum_{k=1}^{\infty} \frac{|\sin(2k)|}{1+2^k}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\sqrt{k}-1}$$

$$\sum_{k=2}^{\infty} \frac{1}{k\sqrt{\ln(k)}}$$

$$\sum_{k=1}^{\infty} \left(2^{1/k} - 1\right)^{k}$$

$$\sum_{k=3}^{\infty} \frac{1}{k^{2/5}\ln(k)}$$

$$\sum_{k=1}^{\infty} \frac{8(3k)!}{(k!)^3}$$

$$\sum_{k=1}^{\infty} \sin\left(\frac{9}{k^{12}}\right)$$

Series or Test	Form of Series	Condition for Convergence	Condition for Divergence	Comments
Geometric series	$\sum_{k=0}^{\infty} ar^k$, $a \neq 0$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	If $\lvert r\rvert < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$.
Divergence Test	$\sum_{k=1}^{\infty} a_k$	Does not apply	$\lim_{k\to\infty} a_k \neq 0$	Cannot be used to prove convergence.
Integral Test	$\sum_{k=1}^{\infty} a_k$, where $a_k = f(k)$ and f is continuous, positive, and decreasing.	$\int_1^{\infty} f(x)\,dx$ converges.	$\int_1^{\infty} f(x)\,dx$ diverges.	The value of the integral is not the value of the series.
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	Useful for comparison tests.
Ratio Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert < 1$	$\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert > 1$	Inconclusive if $\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert = 1$
Root Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} < 1$	$\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} > 1$	Inconclusive if $\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} = 1$
Comparison Test (DCT)	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$a \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges.	$b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Limit Comparison Test (LCT)	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$, $b_k > 0$	$0 \leq \lim_{k\to\infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges.	$\lim_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Alternating Series Test (AST)	$\sum_{k=1}^{\infty} (-1)^k a_k$, where $a_k > 0$	$\lim_{k\to\infty} a_k$ and $0 < a_{k+1} \leq a_k$	$\lim_{k\to\infty} a_k \neq 0$	Remainder R_n satisfies $\lvert R_n\rvert \leq a_{n+1}$
Absolute Convergence	$\sum_{k=1}^{\infty} a_k$, a_k arbitrary	$\sum_{k=1}^{\infty} \lvert a_k\rvert$ converges.		Applies to arbitrary series

11.1: Approximating Functions with Polynomials

A *power series* is an infinite series of the form

$$\sum_{k=0}^{\infty} c_k(x-a)^k = \underbrace{c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n}_{n\text{th-degree polynomial}} + c_{n-1}(x-a)^{n-1} + \ldots,$$

Example. The tangent line of a function $f(x)$ at $x = a$ is a linear function $p_1(x)$ that can approximate $f(x)$ for values of x 'close' to a:

$$p_1(x) = f(a) + f'(a)(x-a)$$

Find a quadratic function $p_2(x)$ that can approximate $f(x)$ near $x = a$,

Find a cubic function $p_2(x)$ that can approximate $f(x)$ near $x = a$,

Find an nth degree polynomial $p_n(x)$ that can approximate $f(x)$ near $x = a$.

Definition. (Taylor Polynomials)
Let f be a function with $f', f'', \ldots,$ and $f^{(n)}$ defined at a. The n**th-order Taylor polynomial** for f with its **center** at a, denoted p_n, has the property that it matches f in value, slope, and all derivatives up to the nth derivative at a; that is,

$$p_n(a) = f(a),\ p_n'(a) = f'(a), \ldots, \text{ and } p_n^{(n)}(a) = f^{(n)}(a).$$

The nth-order Taylor polynomial centered at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

More compactly, $p_n(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$, where the **coefficients** are

$$c_k = \frac{f^{(k)}(a)}{k!}, \qquad \text{for } k = 0, 1, 2, \ldots, n.$$

Example (LC 26.1)**.** Suppose $f(4) = 3$, $f'(4) = -1$, $f''(4) = 6$, and $f^{(3)}(4) = 16$. Find the third-order Taylor polynomial $p_3(x)$ for f centered at $a = 4$.

Example (LC 26.2)**.** For the following functions, find $p_2(x)$, the 2nd degree Taylor polynomial, centered at $a = 0$.

$y = \sqrt{1 + 2x}$

$$y = \frac{1}{\sqrt{1 + 2x}}$$

$$y = \frac{1}{1+2x}$$

$$y = \frac{1}{(1+2x)^3}$$

$y = e^{2x}$

$y = e^{-2x}$

Example (LC 26.3)**.** Find the Taylor polynomial $p_3(x)$ centered at $a = \frac{\pi}{4}$ for $f(x) = \sin(x)$.

Example (LC 26.4)**.** Use the 4th degree Taylor polynomial of $y = \ln(x)$ centered at $a = 1$ to approximate $\ln(1.1)$.

Definition. (Remainder in a Taylor Polynomial)
Let p_n be the Taylor polynomial of order n for f. The **remainder** in using p_n to approximate f at the point x is

$$R_n(x) = f(x) - p_n(x).$$

Theorem 11.1: Taylor's Theorem (Remainder Theorem)
Let f have continuous derivatives up to $f^{(n+1}$ on an open interval I containing a. For all x in I,

$$f(x) = p_n(x) + R_n(x),$$

where p_n is the nth-order Taylor polynomial for f centered at a and the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some point c between x and a.

Theorem 11.2: Estimate of the Remainder
Let n be a fixed positive integer. Suppose there exists a number M such that $\left|f^{(n+1)}(c)\right| \leq M$, for all c between a and x inclusive. The remainder in the nth-order Taylor polynomial for f centered at a satisfies

$$|R_n(x)| = |f(x) - p_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}.$$

Example (LC 27.1-27.2)**.** The third-order Taylor polynomial centered at $a = 1$ for $f(x) = x\ln(x)$ is

$$p_3(x) = (x-1) + \frac{(x-1)^2}{2} - \frac{(x-1)^3}{6}.$$

Find the smallest number M such that $\left|f^{(4)}(x)\right| \leq M$ for $\frac{1}{2} \leq x \leq \frac{3}{2}$.

Compute the upper bound for $|R_3(x)|$.

Example (LC 27.3-27.5)**.** Consider $f(x) = e^x$.

Find the Taylor polynomial $p_4(x)$ centered at $a = 0$.

What is the smallest *integer* M such that $\left|f^{(5)}(x)\right| \leq M$ for $0 \leq x \leq {}^1\!/\!_4$?

Compute the upper bound for $|R_4(x)|$ when $p_4(x)$ is used to compute $e^{1/4}$.

Example (LC 27.6-27.7)**.** We want to approximate $\sin(0.2)$ with an absolute error no greater than 10^{-3} by using a nth degree Taylor polynomial for $f(x) = \sin(x)$ centered at $a = 0$. We want to determine the minimum order of the Taylor polynomial that is required to meet this condition.

What is the smallest *integer* number M that bounds $f^{(n+1)}(x)$ on $0 \leq x \leq 0.2$?

Apply Taylor's Estimate of the Remainder Theorem to find the minimum value of n such that $|R_n(x)| \leq \frac{1}{10^3}$.

11.2: Properties of Power Series

From the *geometric series*, we have

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots = \frac{1}{1-x}, \quad \text{provided } |x| < 1.$$

Definition. (Power Series)
A **power series** has the general form

$$\sum_{k=0}^{\infty} c_k (x-a)^k,$$

where a and c_k are real numbers, and x is a variable. The c_k's are the **coefficients** of the power series, and a is the **center** of the power series. The set of values of x for which the series converges is its **interval of convergence**. The **radius of convergence** of the power series, denoted R, is the distance from the center of the series to the boundary of the interval of convergence.

Theorem 11.3: Convergence of Power Series
A power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ centered at a converges in one of three ways:

1. The series converges absolutely for all x. It follows, by Theorem 10.19, that the series converges for all x, in which the interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.
2. There is a real number $R > 0$ such that the series converges absolutely (and therefore converges) for $|x - a| < R$ and diverges for $|x - a| > R$, in which case the radius of converge is R.
3. The series converges only at a, in which case the radius of convergence is $R = 0$.

Summary: Determining the Radius and Interval of Convergence of $\sum c_k(x-a)^k$

1. Use the Ratio Test or the Root Test to find the interval $(a-R, a+R)$ on which the series converges absolutely; the radius of convergence for the series is R.
2. Use the *radius* of convergence to find the *interval* of convergence:
   - If $R = \infty$, the interval of convergence is $(-\infty, \infty)$.
   - If $R = 0$, the interval of convergence is the single point $x = a$.
   - If $0 < R < \infty$, the interval of convergence consists of the interval $(a-R, a+R)$ and possibly one or both of its endpoints. Determining whether the series $\sum c_k(x-a)^k$ converges at the endpoints $x = a - R$ and $x = a + R$ amounts to analyzing the series $\sum c_k(-R)^k$ and $\sum c_k R^k$.

Example (LC 28.1)**.** Where is the power series $\displaystyle\sum_{k=1}^{\infty} c_k(x-3)^k$ centered?

Could it's interval of convergence be $(-2, 8)$?

Example (LC 28.2)**.** Where is the power series $\displaystyle\sum_{k=0}^{\infty} \frac{(4x-1)^k}{k^2+3}$ centered?

Example (LC 28.3)**.** Where is the power series $\displaystyle\sum_{k=1}^{\infty} c_k(x-1)^k$ centered?

Could it's interval of convergence be $(-1, 1)$?

Example (LC 28.4-28.5)**.** For the following, determine the radius and interval of convergence.

Power series only converges if $|4x - 8| \leq 40$.

Power series only converges if $|x - 3| < 4$.

Example (LC 28.6-28.9)**.** Consider the power series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-4)^k}{9^k\sqrt{k}}$.

Use the ratio test to compute the radius of convergence.

What is the interval of convergence?

Example (LC 28.10-28.13)**.** Consider the power series $\displaystyle\sum_{k=1}^{\infty} \frac{(x-2)^k}{k^k}$.

Use the root test to compute the radius of convergence.

What is the interval of convergence?

Theorem 11.4: Combining Power Series
Suppose the power series $\sum c_k x^k$ and $\sum d_k x^k$ converge to $f(x)$ and $g(x)$, respectively, on an interval I.

1. **Sum and difference:** The power series $\sum (c_k \pm d_k)x^k$ converges to $f(x) \pm g(x)$ on I
2. **Multiplication by a power:** Suppose m is an integer such that $k + m \geq 0$, for all terms of the power series $x^m \sum c_k x^k = \sum c_k x^{k+m}$. This series converges to $x^m f(x)$, for all $x \neq 0$ in I. When $x = 0$, the series converges to $\lim_{x\to 0} x^m f(x)$.
3. **Composition:** If $h(x) = bx^m$, where m is a positive integer and b is a nonzero real number, the power series $\sum c_k (h(x))^k$ converges to the composite function $f(h(x))$, for all x such that $h(x)$ is in I.

Example (LC 29.1)**.** Using the power series representation of

$$f(x) = \ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k},$$

where $-1 \leq x < 1$, find the power series centered at 0 for $g(x) = x\ln(1-x^3)$.

Example (LC 29.2-29.3)**.** Recall the geometric series:

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots = \frac{1}{1-x}, \quad \text{provided } |x| < 1.$$

Find the function represented by the power series $\displaystyle\sum_{k=0}^{\infty} \left(\sqrt{x} - 2\right)^k$.

What is the interval of convergence?

Example. Find the function represented by the power series $\displaystyle\sum_{k=0}^{\infty}\left(\frac{x^2+3}{7}\right)^k$.

What is the interval of convergence?

Theorem 11.5: Differentiating and Integrating Power Series
Suppose the power series $\sum c_k(x-a)^k$ converges for $|x-a| < R$ and defines a function f on that interval.

1. Then f is differentiable (which implies continuous) for $|x-a| < R$, and f' is found by differentiating the power series for f term by term; that is

   $$f'(x) = \sum kc_k(x-a)^{k-1},$$

   for $|x-a| < R$.

2. The indefinite integral of f is found by integrating the power series for f term by term; that is

   $$\int f(x)\,dx = \sum c_k \frac{(x-a)^{k+1}}{k+1} + C,$$

   for $|x-a| < R$, where C is an arbitrary constant.

Note: (LC 29.4) Differentiating or integrating a power series does not change the radius of convergence.

Example (LC 29.5)**.** Evaluate $\int xe^{-x^3}\,dx$ by integrating the power series representation:

$$f(x) = xe^{-x^3} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k+1}}{k!}, \quad \text{for } -\infty < x < \infty.$$

Example (LC 29.6)**.** Compute $f'(x)$ given that

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{2k+1}, \text{ for } |x| \leq 1.$$

Example (LC 29.7)**.** Find the power series representation of $g(x) = \dfrac{2}{(1-2x)^2}$ by using $f(x) = \dfrac{1}{1-2x}$.

Example (LC 29.8-29.10)**.** Find the power series representation of $g(x) = \ln(1 - 3x)$ by using $f(x) = \dfrac{1}{1 - 3x}$. What is the interval of convergence of this power series?

11.3: Taylor Series

Definition. (Taylor/Maclaurin Series for a Function)
Suppose the function f has derivatives of all orders on an interval centered at the point a. The **Taylor series for f centered at a is**

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

A Taylor series centered at 0 is called a **Maclaurin series**.

Example (LC 30.1)**.** Can we find a Taylor series centered at $a = 0$ for $f(x) = \sqrt{x}$?

Example (LC 30.2-30.5)**.** Consider the function $f(x) = \sin(\pi x)$ and the Taylor series representation centered at $a = 0$.

Find the first four nonzero terms

Write this Taylor series using summation notation

Theorem 11.7: Convergence of Taylor Series
Let f have derivatives of all orders on an open interval I containing a. The Taylor series for f centered at a converges to f, for all x in I, if and only if $\lim_{n\to\infty} R_n(x) = 0$, for all x in I, where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

is the remainder at x, with c between x and a.

What is the interval of convergence?

What is the upper bound on $|R_n(x)|$?

Example (LC 30.6)**.** If a Taylor series only converges on $(-2, 2)$, does $f(x^2)$ have a Taylor series that also only converges on $(-2, 2)$?

Example (LC 30.7)**.** Use the definition of a Taylor series to find the Taylor series for $f(x) = e^{2x}$ at $a = 3$.

Example (LC 30.8)**.** Given that $\ln(1+x) = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k}$, for $-1 < x \leq 1$, find the first nonzero terms of the Taylor series centered at $a = 0$ for the function $\ln(1+2x)$.

Example (LC 30.9)**.** Given that $\cos(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$, for $|x| < \infty$, find the Taylor series centered at $a = 0$ for the function $x\cos(x^3)$.

Common Taylor Series:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \ldots = \sum_{k=0}^{\infty} x^k, \qquad \text{for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^k x^k + \ldots = \sum_{k=0}^{\infty} (-1)^k x^k, \qquad \text{for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad \text{for } |x| < \infty$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \text{ for } |x| < \infty$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \qquad \text{for } |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k} + \ldots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \text{ for } -1 < x \le 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \ldots = \sum_{k=1}^{\infty} \frac{x^k}{k}, \qquad \text{for } -1 \le x < 1$$

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \text{ for } |x| \le 1$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \ldots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \qquad \text{for } |x| < \infty$$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2k}}{(2k)!} + \ldots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \qquad \text{for } |x| < \infty$$

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k, \text{ for } |x| < 1 \text{ and } \binom{p}{k} = \frac{p(p-1)(p-2)\ldots(p-k+1)}{k!}, \ \binom{p}{0} = 1$$

11.4: Working with Taylor Series

Limits by Taylor Series

Example (LC 31.1-31.2)**.** Evaluate the following limit using its Taylor series:

$$\lim_{x \to 0} \frac{12x - 8x^3 - 6\sin(2x)}{x^5}$$

Example. Evaluate the following limit using its Taylor series:

$$\lim_{x\to\infty} 2x^2\left(e^{-2/x^2} - 1\right)$$

Differentiating Power Series

Example (LC 31.3-31.4)**.** The differential equation

$$y'(t) + 4y = 8; \quad y(0) = 0$$

is satisfied by the function

$$y(t) = \sum_{k=1}^{\infty} \frac{8(-4)^{k-1} t^k}{k!}$$

Find $y'(t)$ as a power series.

Identify the function $y(t)$ represented by this power series.

$$e^x = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}$$

Integrating Power Series

Example (LC 31.5-31.6)**.** Given that

$$x\cos\left(x^3\right) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{6k+1}}{(2k)!},\ \text{for } |x| < \infty$$

Evalute $\displaystyle\int_0^1 x\cos\left(x^3\right)dx$ as an infinite series

Using the Alternating Series Estimation Theorem, what is the bound on $|R_3|$?

Representing Real Numbers

Example (LC 31.7)**.** Given that $\tan^{-1}(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$, for $|x| \leq 1$, can we approximate $\dfrac{\pi}{3}$ using $x = \sqrt{3}$?

Example (LC 31.8)**.** Evaluate $\displaystyle\sum_{k=0}^{\infty} \frac{(\ln(2))^k}{k!}$.

Example. Let $f(x) = \begin{cases} \frac{e^x-1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$. Using $f(x)$ and $f'(x)$, evaluate

$$\sum_{k=1}^{\infty} \frac{k\,2^{k-1}}{(k+1)!}$$

Representing Functions as Power Series

Example (LC 31.9-31.10)**.** Consider the following Taylor series:

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{3^k}{k\,5^k}$$

What function is being represented by this power series?

What does the sum of the series equal?

Example. Identify the function represented by

$$\sum_{k=0}^{\infty}(-1)^k\frac{x^{5k}}{3^k}$$