6.4: Boolean Algebras, Russell's Paradox, and the Halting Problem

Definition.
A **Boolean algebra** is a set B together with two operations, generally denoted $+$ and $\cdot$, such that for all a and b in B both $a + b$ and $a \cdot b$ are in B and the following axioms are assumed to hold:

1. *Commutative Laws:* For all a and b in B,
$$a + b = b + a \text{ and } a \cdot b = b \cdot a$$
2. *Associative Laws:* For all a and b in B,
$$(a + b) + c = a + (b + c) \text{ and } (a \cdot b) \cdot c = a \cdot (b \cdot c)$$
3. *Distributive Laws:* For all a and b in B,
$$a + (b \cdot c) = (a + b) \cdot (a + c) \text{ and } a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$
4. *Identity Laws:* There exist distinct elements 0 and 1 in B such that for each a in B,
$$a + 0 = a \text{ and } a \cdot 1 = a$$
5. *Complement Laws:* For each a in B, there exists an element in B, denoted $\overline{a}$ and called the **complement** or **negation** of a, such that
$$a + \overline{a} = 1 \text{ and } a \cdot \overline{a} = 0$$

Properties of a Boolean Algebra

Let B be any Boolean algebra.

1. *Uniqueness of the Complement Laws:* For all a and x in B, if $a + x = 1$ and $a \cdot x = 0$, then $x = \overline{a}$.
2. *Uniqueness of* 0 *and* 1*:* If there exists x in B such that $a + x = a$ for every a in B, then $x = 0$, and if there exists y in B such that $a \cdot y = a$ for every a in B, then $y = 1$.
3. *Double Complement Law:* For every $a \in B$, $\overline{(\overline{a})} = a$.
4. *Idempotent Laws:* For every $a \in B$,
$$a + a = a \text{ and } a \cdot a = a.$$
5. *Universal Bound Laws:* For every $a \in B$,
$$a + 1 = 1 \text{ and } a \cdot 0 = 0.$$
6. *De Morgan's Laws:* For all a and $b \in B$,
$$\overline{a + b} = \overline{a} \cdot \overline{b} \text{ and } \overline{a \cdot b} = \overline{a} + \overline{b}.$$
7. *Absorption Laws:* For all a and $b \in B$,
$$(a + b) \cdot a = a \text{ and } (a \cdot b) + a = a.$$
8. *Complements of* 0 *and* 1*:*
$$\overline{0} = 1 \text{ and } \overline{1} = 0.$$

Example. Prove that for all elements a in a Boolean algebra B:

$\overline{(\overline{a})} = a$.

Let $x = \bar{a}$ $\rightarrow$ $\overline{(\bar{a})} = \bar{x}$ $\rightarrow$ $\bar{x} + x = x + \bar{x}$ Commutative law for +

$= 1$ Complement law for +

$\rightarrow \overline{(\bar{a})} + \bar{a} = 1$

$\rightarrow a = \overline{(\bar{a})}$ Uniqueness of Complement Laws

$a + a = a$.

$a = a + 0$ Identity law for +

$= a + (a \cdot \bar{a})$ Complement laws for •

$= (a + a) \cdot (a + \bar{a})$ Distributive law for + over •

$= (a + a) \cdot 1$ Complement law for +

$= a + a$ Identity law for •

Example. Prove that for all elements a in a Boolean algebra B:

$a \cdot a = a.$

$a = a \cdot 1$	Identity law for •
$= a \cdot (a + \bar{a})$	Complement law for +
$= (a \cdot a) + (a \cdot \bar{a})$	Distributive law for • over +
$= (a \cdot a) + 0$	Complement law for •
$= a \cdot a$	Identity law for +

$(a + b) \cdot a = a.$

$(a+b) \cdot a = a \cdot (a+b)$	Associative law for •
$= a \cdot a + a \cdot b$	Distributive law for • over +
$= a + a \cdot b$	From above
$= a \cdot 1 + a \cdot b$	Identity law for •
$= a \cdot (1 + b)$	Distributive law for • over +
$= a \cdot (b + 1)$	Commutative law for +
$= a \cdot 1$	WebAssign question 6.4.002
$= a$	Identity law for •

Russell's Paradox
Define the following set S:

$$S = \{A \mid A \text{ is a set and } A \notin A\}.$$

Is S an element of itself?

The Barber Puzzle: In a certain town, there is a male barber who shaves all those men, and only those men, who do not shave themselves.
Does the barber shave himself?

Yes and no; this is a contradiction.

Is the sentence "The barber shaves himself" a statement?

This is not a statement because it is neither true or false.

Example. Determine whether each sentence is a statement:

If $1 + 1 = 3$, then $1 = 0$.

This is a statement because it is false

This sentence is false and $1 + 1 = 3$.

This is not a statement since it is neither true or false

The Halting Problem (Alan M. Turing)
There is no computer algorithm that will accept any algorithm X and data set D as input and then will output "halts" or "loops forever" to indicate whether or not X terminates in a finite number of steps when X is run with data set D.

Example boolean algebras:

Logical Equivalences For all statement variables p, q, and r:	**Set Properties** For all sets A, B, and C:
$p \vee q \equiv q \vee p$	$A \cup B = B \cup A$
$p \wedge q \equiv q \wedge p$	$A \cap B = B \cap A$
$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	$A \cap (B \cap C) = (A \cap B) \cap C$
$p \vee (q \vee r) \equiv (p \vee q) \vee r$	$A \cup (B \cup C) = (A \cup B) \cup C$
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$p \vee \mathbf{c} \equiv p$	$A \cup \varnothing = A$
$p \wedge \mathbf{t} \equiv p$	$A \cap U = A$
$p \vee \sim p \equiv \mathbf{t}$	$A \cup A^c = U$
$p \wedge \sim p \equiv \mathbf{c}$	$A \cap A^c = \varnothing$
$\sim (\sim p) \equiv p$	$(A^c)^c = A$
$p \wedge p \equiv p$	$A \cup A = A$
$p \vee p \equiv p$	$A \cap A = A$
$p \vee \mathbf{t} \equiv \mathbf{t}$	$A \cup U = U$
$p \wedge \mathbf{c} \equiv \mathbf{c}$	$A \cap \varnothing = \varnothing$
$\sim (p \vee q) \equiv \sim p \wedge \sim q$	$(A \cup B)^c = A^c \cap B^c$
$\sim (p \wedge q) \equiv \sim p \vee \sim q$	$(A \cap B)^c = A^c \cup B^c$
$p \vee (p \wedge q) \equiv p$	$A \cup (A \cap B) = A$
$p \wedge (p \vee q) \equiv p$	$A \cap (A \cup B) = A$
$\sim \mathbf{t} \equiv \mathbf{c}$	$U^c = \varnothing$
$\sim \mathbf{c} \equiv \mathbf{t}$	$\varnothing^c = U$