2.4: Limits

Example. Suppose that the position function of a maglev train (in feet) is given by

$$s(t) = 4t^2, \qquad (0 \leq t \leq 30)$$

Using the position function, compute the *average* velocity of the train

on the interval $[t, 2]$

t	1.5	1.9	1.99	1.999	1.9999

on the interval $[2, t]$

t	2.5	2.1	2.01	2.001	2.0001

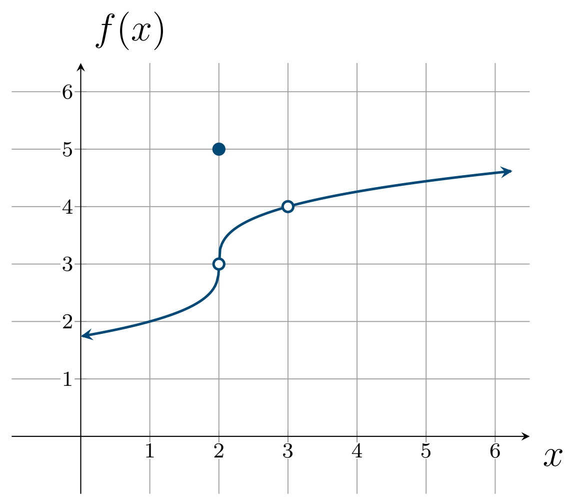
What do the tables above suggest about *instantaneous* velocity of the train at $t = 2$?

Definition. (Limit of a Function)
The function f has the **limit** L as x approaches a, written

$$\lim_{x \to a} f(x) = L$$

if the value of $f(x)$ can be made as close to the number L as we please by taking x sufficiently close to (but not equal to) a.

Example. Using the graph of f, determine the following values:



$f(1)$ and $\lim_{x \to 1} f(x)$

$f(2)$ and $\lim_{x \to 2} f(x)$

$f(3)$ and $\lim_{x \to 3} f(x)$

Example. Find the limit of the following functions at the value specified: Graphs

$f(x) = x^3$ at $x = 2$

$g(x) = \begin{cases} x+2, & x \neq 1 \\ 1, & x = 1 \end{cases}$ at $x = 1$

$h(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$ at $x = 0$

$j(x) = \dfrac{1}{(x-1)^2}$ at $x = 1$

$k(x) = 4$ at $x = 0$

Theorem 1: Properties of Limits

Suppose

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M$$

Then

1. $\lim\limits_{x\to a}[f(x)]^r = \left[\lim\limits_{x\to a} f(x)\right]^r$ where r is a positive constant
2. $\lim\limits_{x\to a} cf(x) = c\lim\limits_{x\to a} f(x)$ where c is a real number
3. $\lim\limits_{x\to a}[f(x) \pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x) = L \pm M$
4. $\lim\limits_{x\to a}[f(x)g(x)] = \left[\lim\limits_{x\to a} f(x)\right]\left[\lim\limits_{x\to a} g(x)\right] = LM$
5. $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{L}{M}$ provided $M \neq 0$

Example. Use the above theorem to evaluate the following limits:

$$\lim_{x\to 1}\left(5x^{3/2} - 2\right)$$

$$\lim_{x\to 3} \frac{2x^3\sqrt{x^2+7}}{x+1}$$

Suppose that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

has an **indeterminate form** of $\frac{0}{0}$. To evaluate such a limit, we replace the given function with a function that's equivalent everywhere except at $x = a$, and then evaluate the limit.

Example. Evaluate the following

$$\lim_{t\to 2} \frac{4t^2 - 16}{t - 2}$$

$$\lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h}$$

Limit of a Function at Infinity
The function f has the limit L as x increases without bound, written

$$\lim_{x\to\infty} f(x) = L$$

if $f(x)$ can be made arbitrarily close to L by taking x large enough.
The function f has the limit M as x decreases without bound, written

$$\lim_{x\to-\infty} f(x) = M$$

if $f(x)$ can be made arbitrarily close to M by taking x to be negative and sufficiently large enough in absolute value.
When the above limits exist, the equations $y = L$ and/or $y = M$ are called **horizontal asymptotes**.

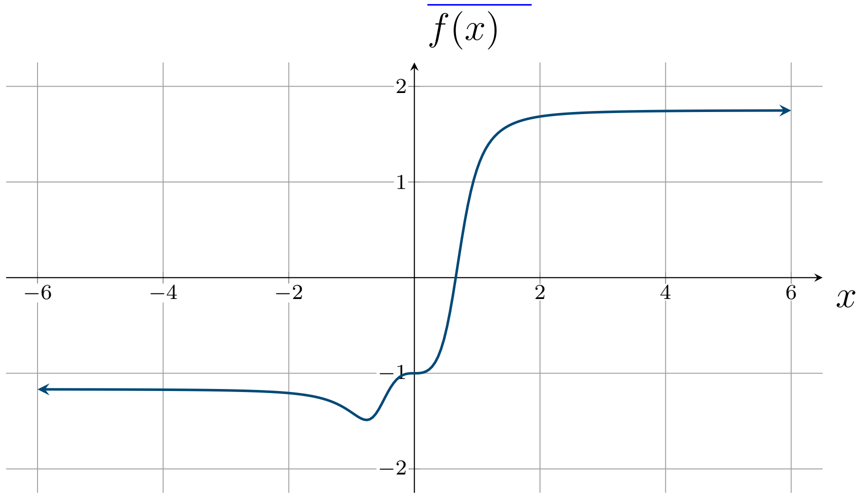
Example. Evaluate the following infinite limits

$$\lim_{x\to\infty} \frac{2x^2+3x-4}{x^2-7x+1}$$

$$\lim_{x\to-\infty} \frac{3x^2+4}{2x^3}$$

$$\lim_{x\to\pm\infty} \frac{3x^5+2x^3-4}{x^4+4x^2-1}$$

Graph



$$\lim_{x\to-\infty} \frac{7x^3-2}{-x^3+\sqrt{25x^6-4}}$$

$$\lim_{x\to\infty} \frac{7x^3-2}{-x^3+\sqrt{25x^6-4}}$$

Example. The company *Custom Office* makes a line of executive desks. It is estimated that the total cost of making x *Senior Executive Model* desks is

$$C(x) = 100x + 200,000$$

dollars per year. The average cost of making x desks is given by

$$\overline{C}(x) = \frac{C(x)}{x}$$

Compute $\lim\limits_{x\to\infty} \overline{C}(x)$ and interpret the result.

Theorem 2
For all $n > 0$,

$$\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$$

provided that $\dfrac{1}{x^n}$ is defined.