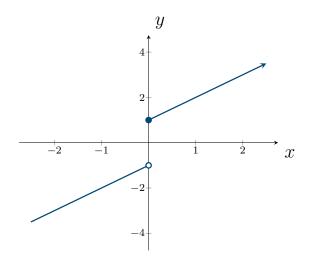
2.5: One-Sided Limits and Continuity

Consider the function

$$f(x) = \begin{cases} x - 1, & x < 0 \\ x + 1, & x \ge 0 \end{cases}$$

What is $\lim_{x\to 0} f(x)$?



Definition. (One-Sided Limits)

The function f has a **right-hand limit** L as x approaches a from the right, written

$$\lim_{x \to a^+} f(x) = L$$

if the values of f(x) can be made as close to L as we please by taking x sufficiently close to (but not equal to) a and to the right of a.

The function f has a **left-hand limit** L as x approaches a from the left, written

$$\lim_{x \to a^{-}} f(x) = M$$

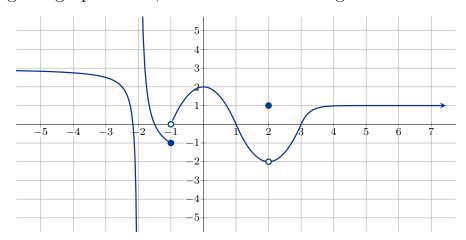
if the values of f(x) can be made as close to L as we please by taking x sufficiently close to (but not equal to) a and to the left of a.

Theorem 3

Let f be a function that is defined for all values of x close to x = a with the possible exception of a itself. Then

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^{-}} f(x) = \lim_{x \to a^{+}} f(x) = L$$

Example. Using the graph below, evaluate the following limits:



$$\lim_{x \to -2^-} f(x)$$

$$\lim_{x \to -2^+} f(x)$$

$$\lim_{x \to -2} f(x)$$

$$\lim_{x \to -1^-} f(x)$$

$$\lim_{x \to -1^+} f(x)$$

$$\lim_{x \to -1} f(x)$$

$$\lim_{x \to 1} f(x)$$

$$\lim_{x \to 2} f(x)$$

$$\lim_{x \to \infty} f(x)$$

Definition. (Continuity of a Function at a Number)

A function f is **continuous** at a if $\lim_{x\to a} f(x) = f(a)$.

Continuity Checklist:

In order for f to be continuous at a, the following three conditions must hold:

- 1. f(a) is defined (a is in the domain of f),
- 2. $\lim_{x \to a} f(x)$ exists,
- 3. $\lim_{x\to a} f(x) = f(a)$ (the value of f equals the limit of f at a).

Example. Determine the values of x for which the following functions are continuous:

$$f(x) = 3x^3 + 2x^2 - x + 10$$

$$g(x) = \frac{8x^{10} - 4x + 1}{x^2 + 1}$$

$$h(x) = \frac{4x^3 - 3x^2 + 1}{x^2 - 3x + 1}$$

Example. Determine whether the following are continuous at a:

$$f(x) = x^2 + \sqrt{7 - x}, \ a = 4$$

$$g(x) = \frac{1}{x - 3}, \ a = 3$$

$$h(x) = \begin{cases} \frac{x^2 + x}{x+1}, & x \neq -1 \\ 0, & x = -1 \end{cases}, \quad a = -1 \qquad j(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}, \quad a = 0$$

$$j(x) = |x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}, \ a = 0$$

$$k(x) = \begin{cases} \frac{x^2 + x - 6}{x^2 - x}, & x \neq 2 \\ -1, & x = 2 \end{cases}, \ a = 2$$

Properties of Continuous Functions

- 1. The constant function f(x) = c is continuous everywhere.
- 2. The identify function f(x) = x is continuous everywhere.

If f and g are continuous at x = a, then

 $[f(x)]^n$, where n is a real number, is continuous at x = a whenever it is defined at that number

 $f \pm g$ is continuous at x = a

fg is continuous at x = a

f/g is continuous at x=a provided that $g(a)\neq 0$

Polynomial and Rational Functions

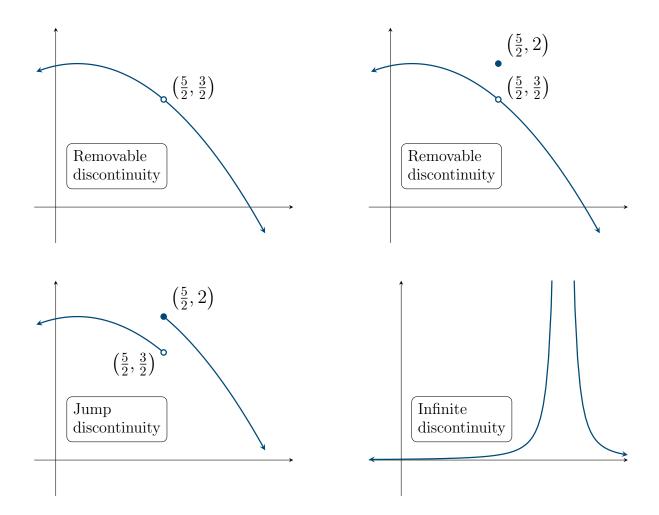
- 1. A polynomial function is continuous for all x.
- 2. A rational function (a function of the form $\frac{p}{q}$, where p and q are polynomials) is continuous for all x for which $q(x) \neq 0$.

Definition.

A **removable discontinuity** at x = a is one that disappears when the function becomes continuous after defining $f(a) = \lim_{x \to a} f(x)$.

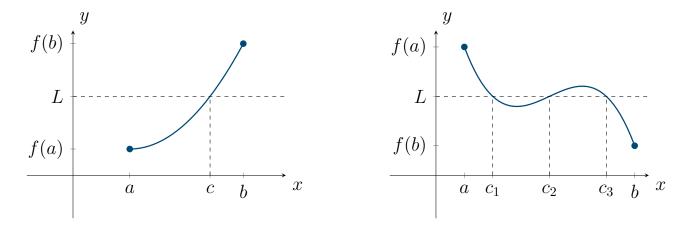
A **jump discontinuity** is one that occurs whenever $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist, but $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$.

A **vertical discontinuity** occurs whenever f(x) has a vertical asymptote.

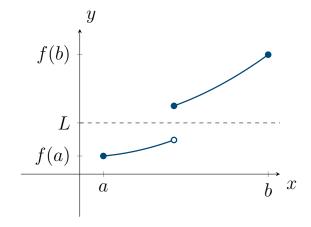


Theorem 4: Intermediate Value Theorem

Suppose f is continuous on the interval [a, b] and L is a number strictly between f(a) and f(b). Then there exists at least one number c in (a, b) satisfying f(c) = L.



Note: It is important that the function be continuous on the interval [a, b]:



Theorem 5: Existence of Zeros of a Continuous Function

If f is a continuous function on a closed interval [a, b], and if f(a) and f(b) have opposite signs, then there is at least one solution of the equation f(x) = 0 in the interval (a, b).

Example. Check the conditions of the Intermediate Value Theorem to see if there exists a value c on the interval (a, b) such that the following equations hold: Graph

$$x^x - x^2 = \frac{1}{2}$$

on
$$[0, 2]$$

$$\sqrt{x^4 + 25x^3 + 10} = 5 \quad \text{on } [0, 1]$$

$$x + \sqrt{1 - x^2} = 0$$
 on $[-1, 0]$

on
$$[-1, 0]$$

$$\frac{x^2}{x^2 + 1} = 0 on [-1, 1]$$

on
$$[-1, 1]$$

Example. Consider the function

$$f(x) = \frac{x+1}{x-1}$$

on the interval [0,2]. Does there exist a c on the interval [0,2] such that f(c)=1?

