Math 123 Class notes
Fall 2025

To accompany
Applied Calculus
by *Tan*

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Math 123 Formula Sheet

Unit 1

Slope of a Line: $$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-Slope form: $$y - y_1 = m(x - x_1)$$

Slope-Intercept form: $$y = mx + b$$

Quadratic formula: $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Derivative: $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Unit 2

Fractional exponents: $$x^{a/b} = \sqrt[b]{x^a}$$

Subtracting exponents: $$x^{a-b} = \frac{x^a}{x^b}$$

Product rule: $$\frac{d}{dx}\Big[f(x) \cdot g(x)\Big] = f'(x)g(x) + f(x)g'(x)$$

Quotient rule: $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{\big[g(x)\big]^2}$$

Chain rule: $$\frac{d}{dx}\Big[f\big(g(x)\big)\Big] = f'\big(g(x)\big) \cdot g'(x)$$

Unit 3

Natural Log: $$e^y = x \longleftrightarrow \ln(x) = y, \qquad x > 0$$

Log of a product: $$\ln(xy) = \ln(x) + \ln(y)$$

Log of a quotient: $$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$

Exponent inside of log: $$\ln(x^y) = y\ln(x)$$

Exponential Rule: $$\frac{d}{dx}\Big[e^{f(x)}\Big] = e^{f(x)} \cdot f'(x)$$

Logarithm Rule: $$\frac{d}{dx}\Big[\ln\big(f(x)\big)\Big] = \frac{f'(x)}{f(x)}$$

Unit 4

Integral Power Rule: $$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \qquad (n \neq -1)$$

u-substitution: $$\int f'\big(g(x)\big) \cdot g'(x)\,dx = \int f(u)\,du$$

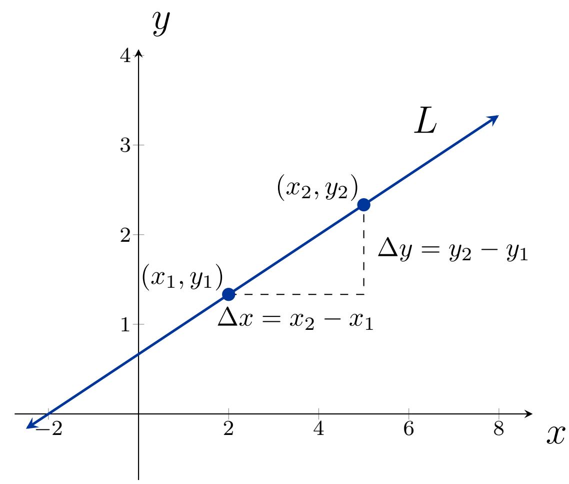
Definite Integral: $$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a), \text{ where } F'(x) = f(x)$$

1.4: Straight Lines

Definition. (Slope of a Nonvertical Line)
If (x_1, y_1) and (x_2, y_2) are any two distinct points on a nonvertical line L, then the slope m of L is given by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example. Compute the slope of the line passing through the points



$(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (4, 2)$

$(x_1, y_1) = (3, 2)$ and $(x_2, y_2) = (-1, 2)$

$(x_1, y_1) = (4, 1)$ and $(x_2, y_2) = (4, 4)$

Definition. (Point-Slope Form of an Equation of a Line)
An equation of the line that has slope m and passes through the point (x_1, y_1) is given by

$$y - y_1 = m(x - x_1)$$

Example. Find the equation of the line going through the points

$(x_1, y_1) = (-2, 1)$ and $(x_2, y_2) = (3, -2)$

$(x_1, y_1) = (3, 4)$ and $(x_2, y_2) = (-1, 4)$

$(x_1, y_1) = (2, 0)$ and $(x_2, y_2) = (2, 1)$

Definition. (Slope-Intercept Form of an Equation of a Line)
An equation of the line that has slope m and intersects the y-axis at the point $(0, b)$ is given by

$$y = mx + b$$

Example. Rewrite the equations in the previous example in slope-intercept form.

Definition. (Parallel and Perpendicular lines)
Let L_1 and L_2 be lines with slopes m_1 and m_2 respectively. If L_1 and L_2 are *parallel*, then

$$m_1 = m_2.$$

If L_1 and L_2 are *perpendicular*, then

$$m_1 = -\frac{1}{m_2}.$$

Example.

Find the line *parallel* to $y = \frac{3}{2}x + 1$ that passes through the point $(-4, 10)$.

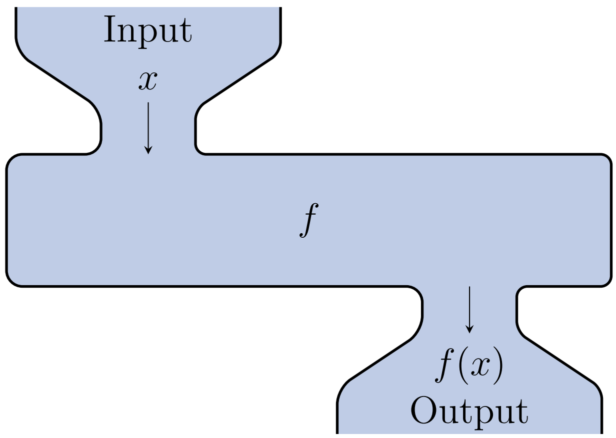
Find the line *perpendicular* to $y = \frac{3}{2}x + 1$ that passes through the point $(-3, 4)$.

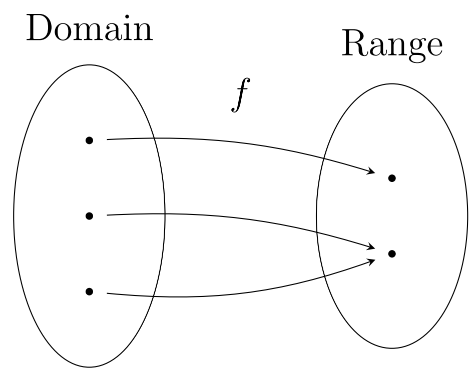
Forms of Linear Equations

$$\begin{aligned} \text{General form: } & Ax + By = C \\ \text{Point-slope form: } & y - y_1 = m(x - x_1) \\ \text{Slope-intercept form: } & y = mx + b \\ \text{Vertical line: } & x = a \\ \text{Horizontal line: } & y = b \end{aligned}$$

2.1: Functions and Their Graphs

> **Definition.**
> A **function** is a rule that assigns to each element in a set A one and only one element in a set B.
> In the context above, the set A is called the **domain**, and the set B is called the **range**.





Example. Let $f(x) = 2x^2 - 2x + 1$. Evaluate the following

$f(1)$

$f(-2)$

$f(a)$

$f(a+h)$

Example. Find the domain and range of the following functions:

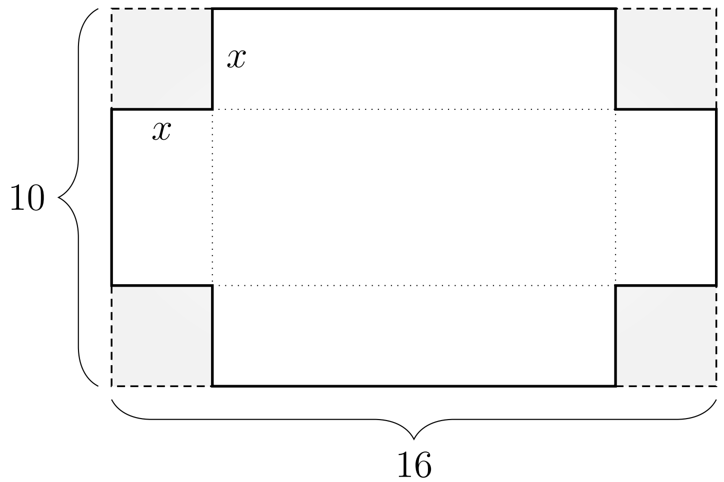
$f(x) = x$

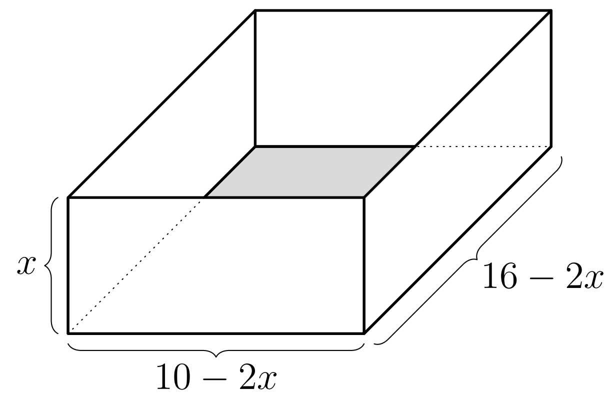
$A = \pi r^2$

$y = \sqrt{x-1}$

$y = \dfrac{1}{x^2 - 4}$

Example. An open box is to be made from a rectangular piece of cardboard 16 inches long and 10 inches wide by cutting away identical squares (x inches by x inches) from each corner and folding up the resulting flaps. Find an expression that gives the volume V of the box as a function of x. What is the domain of the function?





Definition.
A **piecewise** function is a function with different definitions for different portions of the domain.

Example. Rewrite the following as piecewise functions:

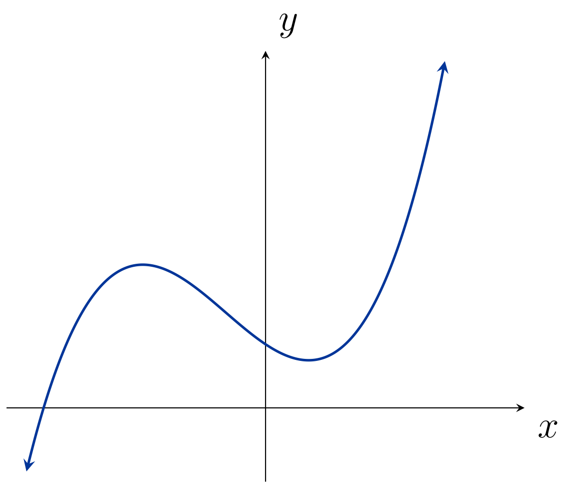
$|x| =$

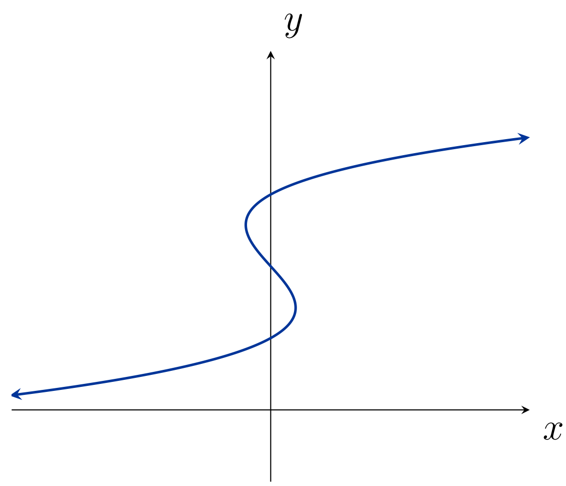
$\dfrac{x}{|x|} =$

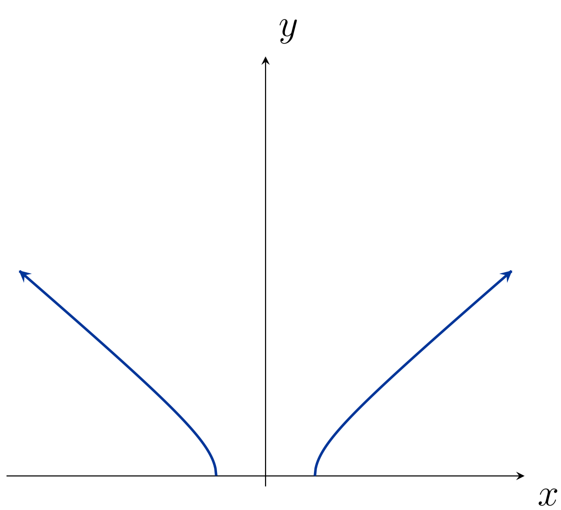
$|x-1| + |4-x| =$

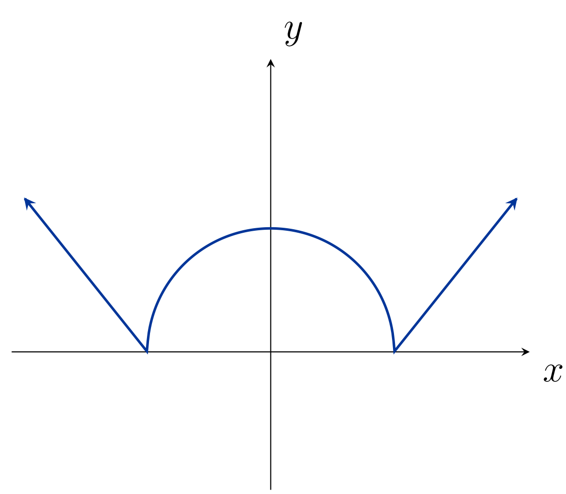
Definition. (Vertical Line Test)
A curve in the xy-plane is the graph of a function $y = f(x)$ (an explicit function) if and only if each vertical line intersects it in at most one point

Example. Use the vertical line test on the following graphs to determine which graphs may represent an explicit function:









2.2: The Algebra of Functions

Definition.
Let f and g be functions with domains A and B, respectively. Then the **sum** $f+g$, **difference** $f-g$, and **product** fg of f and g are functions with domain $A \cap B$.

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$

The **quotient** f/g of f and g has domain $A \cap B$ excluding all numbers x such that $g(x) = 0$ and rule given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Example. Let $f(x) = \sqrt{x+1}$ and $g(x) = 4 - x$. Find the domain of the following:

$f(x) + g(x) =$

$f(x) - g(x) =$

$f(x)g(x) =$

$\dfrac{f(x)}{g(x)} =$

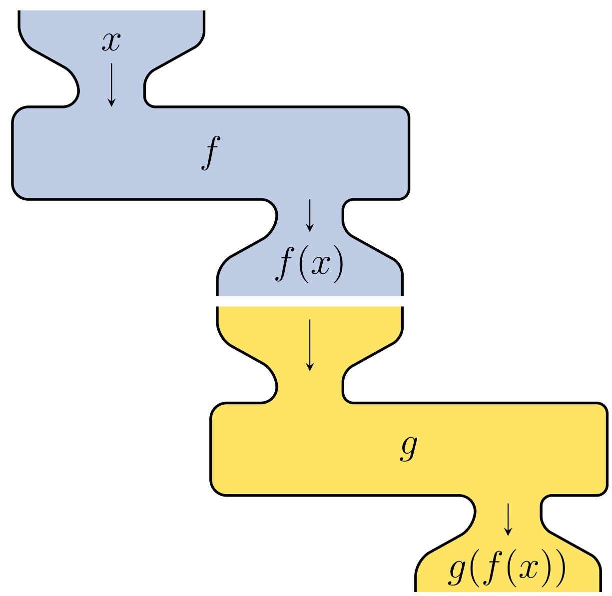
Definition. (The Composition of Two Functions)
Let f and g be functions. Then the composition of g and f is the function $g \circ f$ defined by

$$(g \circ f)(x) = g(f(x))$$

The domain of $g \circ f$ is the set of all x is the domain of f such that $f(x)$ lies in the domain of g.

Example. Let $f(x) = \sqrt{x+1}$ and $g(x) = 4 - x$. Find the domain of the following:

$g(f(x)) =$



$f(g(x)) =$

$f(f(x)) =$

2.4: Limits

Example. Suppose that the position function of a maglev train (in feet) is given by

$$s(t) = 4t^2, \qquad (0 \leq t \leq 30)$$

Using the position function, compute the *average* velocity of the train

on the interval $[t, 2]$

t	1.5	1.9	1.99	1.999	1.9999

on the interval $[2, t]$

t	2.5	2.1	2.01	2.001	2.0001

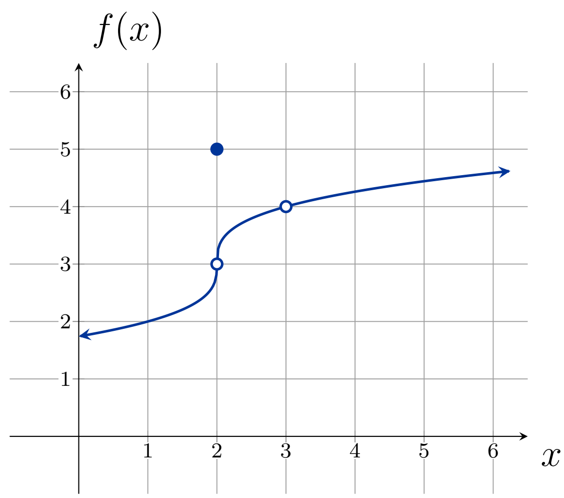
What do the tables above suggest about *instantaneous* velocity of the train at $t = 2$?

Definition. (Limit of a Function)
The function f has the **limit** L as x approaches a, written

$$\lim_{x \to a} f(x) = L$$

if the value of $f(x)$ can be made as close to the number L as we please by taking x sufficiently close to (but not equal to) a.

Example. Using the graph of f, determine the following values:



$f(1)$ and $\lim_{x \to 1} f(x)$

$f(2)$ and $\lim_{x \to 2} f(x)$

$f(3)$ and $\lim_{x \to 3} f(x)$

Example. Find the limit of the following functions at the value specified: Graphs

$f(x) = x^3$ at $x = 2$

$g(x) = \begin{cases} x + 2, & x \neq 1 \\ 1, & x = 1 \end{cases}$ at $x = 1$

$h(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$ at $x = 0$

$j(x) = \dfrac{1}{(x-1)^2}$ at $x = 1$

$k(x) = 4$ at $x = 0$

Theorem 1: Properties of Limits
Suppose

$$\lim_{x \to a} f(x) = L \quad \text{and} \quad \lim_{x \to a} g(x) = M$$

Then

1. $\lim_{x \to a} [f(x)]^r = \left[\lim_{x \to a} f(x)\right]^r$ where r is a positive constant
2. $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$ where c is a real number
3. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M$
4. $\lim_{x \to a} [f(x)g(x)] = \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right] = LM$
5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{L}{M}$ provided $M \neq 0$

Example. Use the above theorem to evaluate the following limits:

$$\lim_{x \to 1} \left(5x^{3/2} - 2\right)$$

$$\lim_{x \to 3} \frac{2x^3\sqrt{x^2 + 7}}{x + 1}$$

Suppose that $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

has an **indeterminate form** of $\frac{0}{0}$. To evaluate such a limit, we replace the given function with a function that's equivalent everywhere except at $x = a$, and then evaluate the limit.

Example. Evaluate the following

$$\lim_{t \to 2} \frac{4t^2 - 16}{t - 2}$$

$$\lim_{h \to 0} \frac{\sqrt{4 + h} - 2}{h}$$

Suppose that $\lim_{x \to a} f(x) = L$ with $L \neq 0$ and $\lim_{x \to a} g(x) = 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

does not exist. We can further specify if this limit tends towards $-\infty$ or ∞.

Example. Evaluate the following Graphs

$$\lim_{x \to 1} \frac{x}{x-1} \qquad \lim_{x \to 3} \frac{1}{(x-3)^2}$$

$$\lim_{x \to -2} \frac{x-2}{x^2-4} \qquad \lim_{x \to 2} \frac{x-2}{x^2-4}$$

Limit of a Function at Infinity
The function f has the limit L as x increases without bound, written

$$\lim_{x \to \infty} f(x) = L$$

if $f(x)$ can be made arbitrarily close to L by taking x large enough.
The function f has the limit M as x decreases without bound, written

$$\lim_{x \to -\infty} f(x) = M$$

if $f(x)$ can be made arbitrarily close to M by taking x to be negative and sufficiently large enough in absolute value.
When the above limits exist, the equations $y = L$ and/or $y = M$ are called **horizontal asymptotes**.

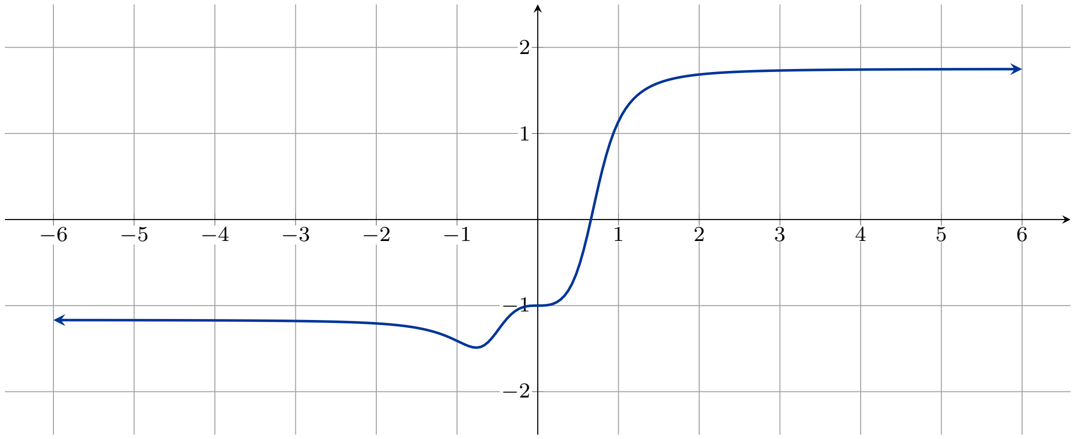
Example. Evaluate the following infinite limits

$$\lim_{x \to \infty} \frac{2x^2 + 3x - 4}{x^2 - 7x + 1}$$

$$\lim_{x \to -\infty} \frac{3x^2 + 4}{2x^3}$$

$$\lim_{x \to \pm\infty} \frac{3x^5 + 2x^3 - 4}{x^4 + 4x^2 - 1}$$

Graph



$$\lim_{x\to-\infty} \frac{7x^3-2}{-x^3+\sqrt{25x^6-4}}$$

$$\lim_{x\to\infty} \frac{7x^3-2}{-x^3+\sqrt{25x^6-4}}$$

Example. The company *Custom Office* makes a line of executive desks. It is estimated that the total cost of making x *Senior Executive Model* desks is

$$C(x) = 100x + 200,000$$

dollars per year. The average cost of making x desks is given by

$$\overline{C}(x) = \frac{C(x)}{x}$$

Compute $\lim_{x \to \infty} \overline{C}(x)$ and interpret the result.

Theorem 2
For all $n > 0$,

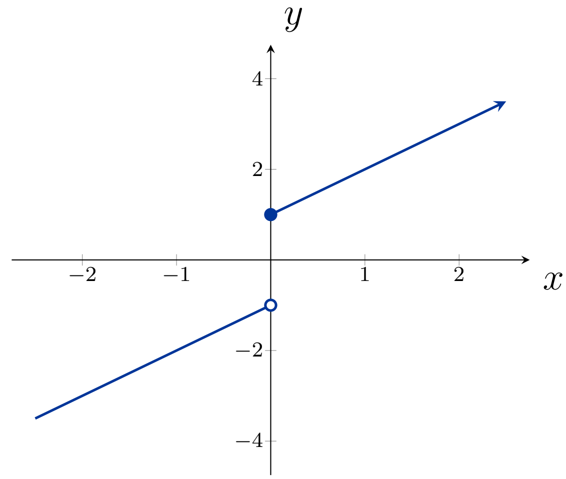
$$\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$$

provided that $\frac{1}{x^n}$ is defined.

2.5: One-Sided Limits and Continuity

Consider the function

$$f(x) = \begin{cases} x - 1, & x < 0 \\ x + 1, & x \geq 0 \end{cases}$$



What is $\lim_{x \to 0} f(x)$?

Definition. (One-Sided Limits)
The function f has a **right-hand limit** L as x approaches a from the right, written

$$\lim_{x \to a^+} f(x) = L$$

if the values of $f(x)$ can be made as close to L as we please by taking x sufficiently close to (but not equal to) a and to the right of a.
The function f has a **left-hand limit** L as x approaches a from the left, written

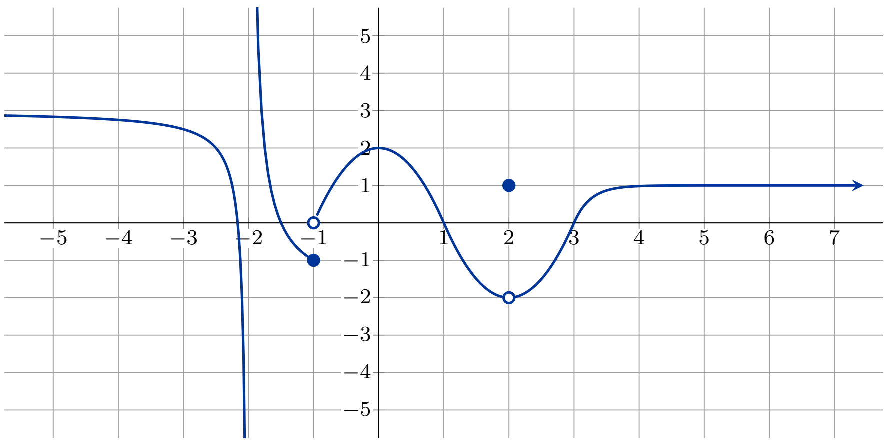
$$\lim_{x \to a^-} f(x) = M$$

if the values of $f(x)$ can be made as close to L as we please by taking x sufficiently close to (but not equal to) a and to the left of a.

Theorem 3
Let f be a function that is defined for all values of x close to $x = a$ with the possible exception of a itself. Then

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

Example. Using the graph below, evaluate the following limits:



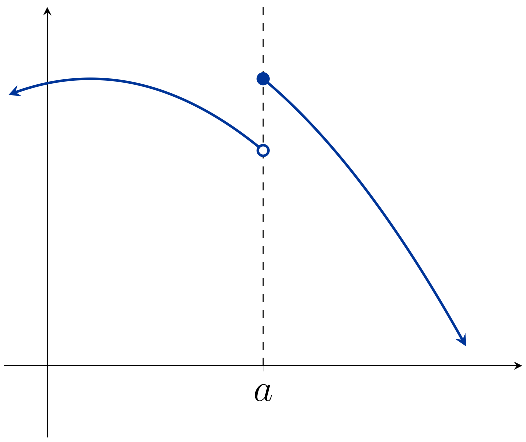
$\lim_{x\to -2^-} f(x)$ $\qquad$ $\lim_{x\to -2^+} f(x)$ $\qquad$ $\lim_{x\to -2} f(x)$

$\lim_{x\to -1^-} f(x)$ $\qquad$ $\lim_{x\to -1^+} f(x)$ $\qquad$ $\lim_{x\to -1} f(x)$

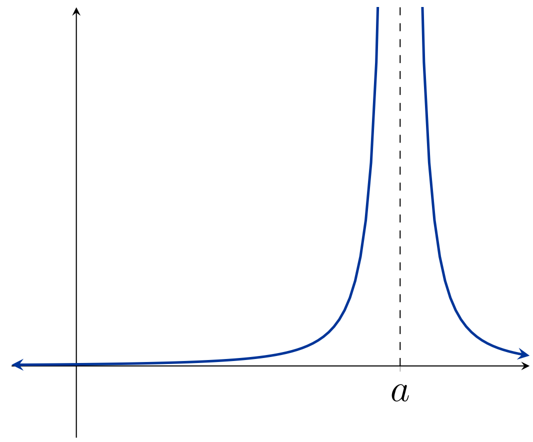
$\lim_{x\to 1} f(x)$ $\qquad$ $\lim_{x\to 2} f(x)$ $\qquad$ $\lim_{x\to \infty} f(x)$

Below are examples where the limit does not exist: [Graph]

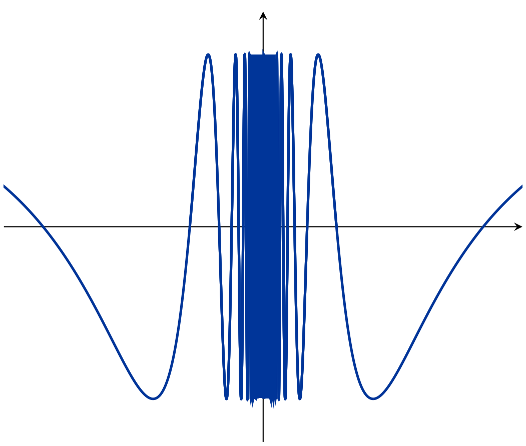
$$\lim_{x \to a} f(x)$$



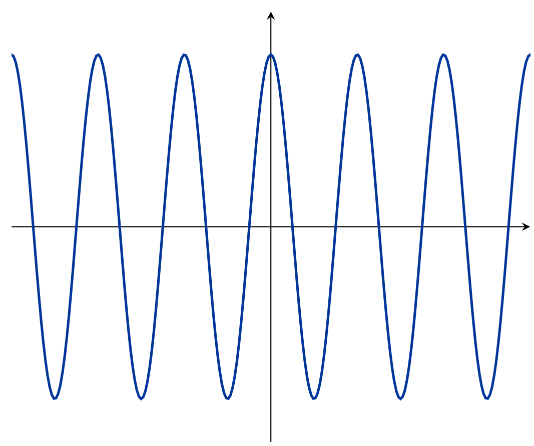
$$\lim_{x \to a} f(x)$$



$$\lim_{x \to 0} f(x)$$



$$\lim_{x \to \infty} f(x)$$



Definition. (Continuity of a Function at a Number)
A function f is **continuous** at a if $\lim_{x \to a} f(x) = f(a)$.

Continuity Checklist:
In order for f to be continuous at a, the following three conditions must hold:

1. $f(a)$ is defined (a is in the domain of f),
2. $\lim_{x \to a} f(x)$ exists,
3. $\lim_{x \to a} f(x) = f(a)$ (the value of f equals the limit of f at a).

Example. Determine the values of x for which the following functions are continuous:

$$f(x) = 3x^3 + 2x^2 - x + 10$$

$$g(x) = \frac{8x^{10} - 4x + 1}{x^2 + 1}$$

$$h(x) = \frac{4x^3 - 3x^2 + 1}{x^2 - 3x + 1}$$

Example. Determine whether the following are continuous at a:

$$f(x) = x^2 + \sqrt{7-x},\ a = 4$$

$$g(x) = \frac{1}{x-3},\ a = 3$$

$$h(x) = \begin{cases} \dfrac{x^2+x}{x+1}, & x \neq -1 \\ 0, & x = -1 \end{cases},\ a = -1$$

$$j(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases},\ a = 0$$

$$k(x) = \begin{cases} \dfrac{x^2+x-6}{x^2-x}, & x \neq 2 \\ -1, & x = 2 \end{cases},\ a = 2$$

Properties of Continuous Functions

1. The constant function $f(x) = c$ is continuous everywhere.
2. The identify function $f(x) = x$ is continuous everywhere.

If f and g are continuous at $x = a$, then

> $[f(x)]^n$, where n is a real number, is continuous at $x = a$ whenever it is defined at that number
>
> $f \pm g$ is continuous at $x = a$
>
> fg is continuous at $x = a$
>
> f/g is continuous at $x = a$ provided that $g(a) \neq 0$

Polynomial and Rational Functions

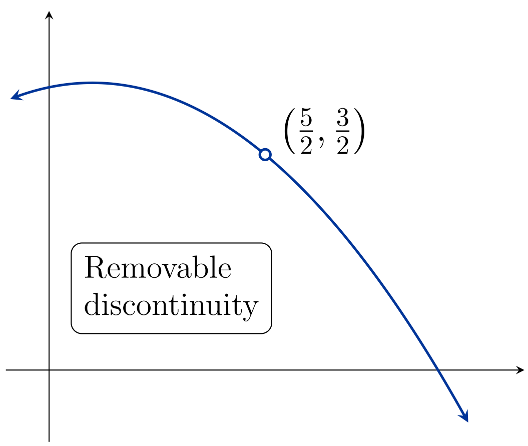
1. A polynomial function is continuous for all x.
2. A rational function (a function of the form $\frac{p}{q}$, where p and q are polynomials) is continuous for all x for which $q(x) \neq 0$.

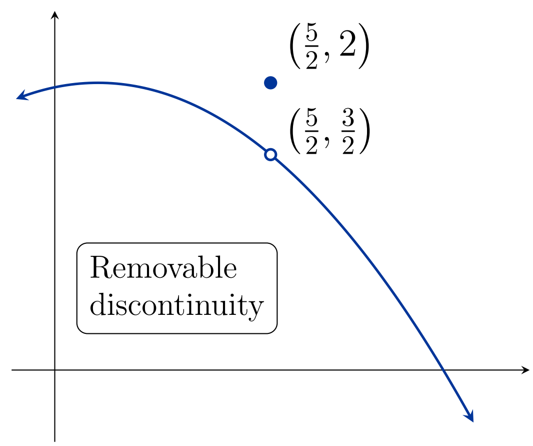
Definition.

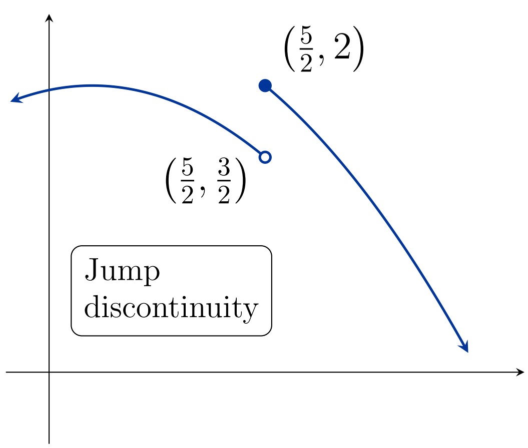
A **removable discontinuity** at $x = a$ is one that disappears when the function becomes continuous after defining $f(a) = \lim_{x \to a} f(x)$.

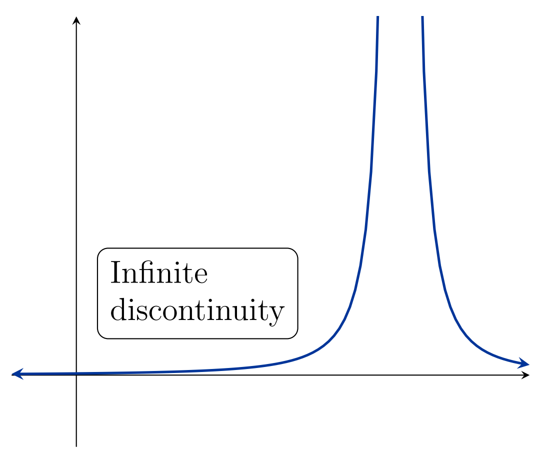
A **jump discontinuity** is one that occurs whenever $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$.

A **vertical discontinuity** occurs whenever $f(x)$ has a vertical asymptote.

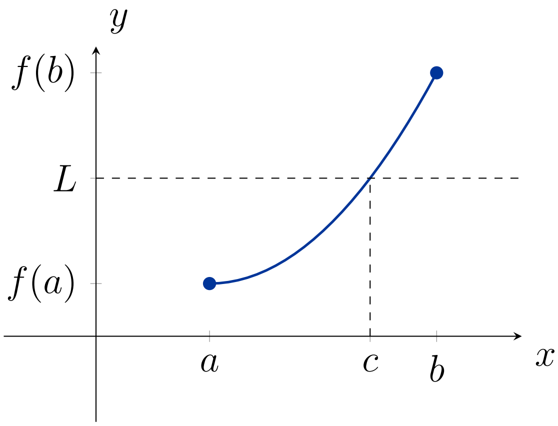


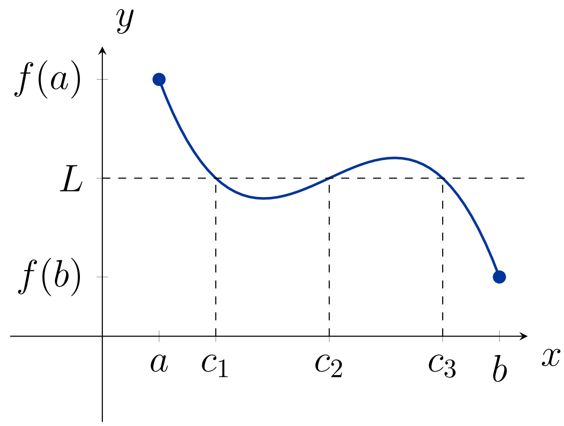




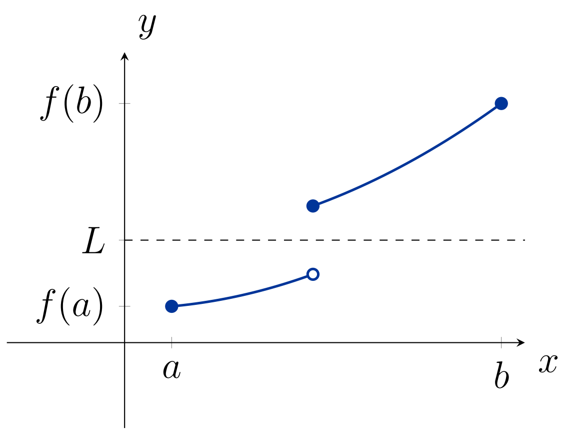


Theorem 4: Intermediate Value Theorem
Suppose f is continuous on the interval $[a, b]$ and L is a number strictly between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) satisfying $f(c) = L$.





Note: It is important that the function be continuous on the interval $[a, b]$:



Theorem 5: Existence of Zeros of a Continuous Function
If f is a continuous function on a closed interval $[a, b]$, and if $f(a)$ and $f(b)$ have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval (a, b).

Example. Check the conditions of the Intermediate Value Theorem to see if there exists a value c on the interval (a, b) such that the following equations hold: Graph

$x^x - x^2 = \frac{1}{2}$ on $[0, 2]$

$\sqrt{x^4 + 25x^3 + 10} = 5$ on $[0, 1]$

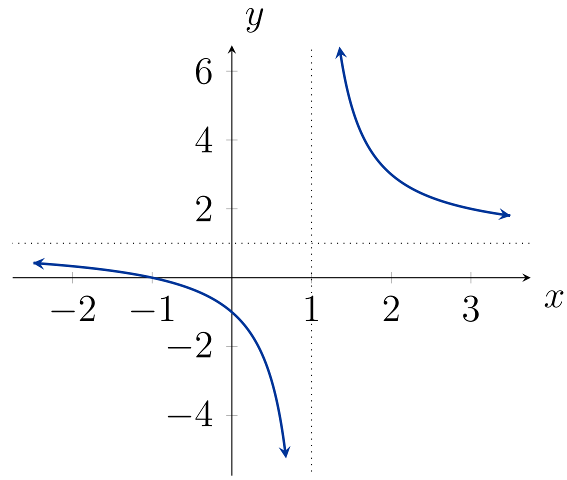
$x + \sqrt{1 - x^2} = 0$ on $[-1, 0]$

$$\frac{x^2}{x^2 + 1} = 0 \qquad \text{on } [-1, 1]$$

Example. Consider the function

$$f(x) = \frac{x+1}{x-1}$$

on the interval $[0, 2]$. Does there exist a c on the interval $[0, 2]$ such that $f(c) = 1$?

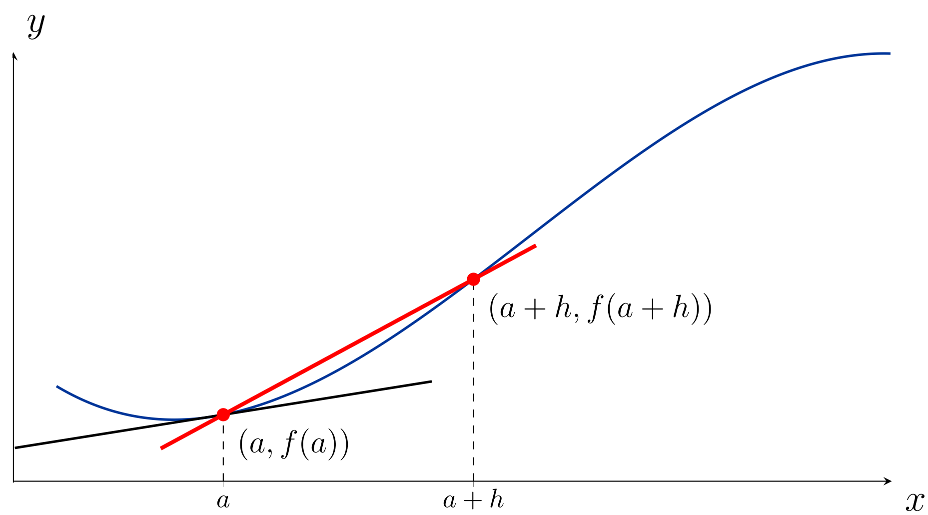


2.6: The Derivative

Definition.
Given a function $f(x)$:

- the **secant line** is the line that passes through two *distinct* points lying on the graph of $f(x)$,
- the **tangent line** is the line that intersects $f(x)$ in exactly one place (locally) and matches the slope of the graph at that point.

Graph



Definition. (Slope of a Tangent Line)
The slope of the tangent line to the graph of f at the point $P(x, f(x))$ is given by

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if it exists.

Definition. (Average and Instantaneous Rates of Change)
The **average rate of change** of f over the interval $[x, x+h]$ or **slope of the secant line** to the graph of f through the points $(x, f(x))$ and $(x+h, f(x+h))$ is

$$\frac{f(x+h)-f(x)}{h}$$

The above fraction is referred to as the **difference quotient**.

The **instantaneous rate of change** of f at x or **slope of the tangent line** to the graph of f at $(x, f(x))$ is

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

Definition. (Derivative of a Function)
The derivative of a function f with respect to x is the function f' (read "f prime"),

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

The domain of f' is the set of all x for which the limit exists.

Some other notations for the derivative are

$$D_x f(x) \qquad \frac{dy}{dx} \qquad y'$$

Example. Find the slope of the line tangent to the graph $f(x) = 3x + 5$ at any point $(x, f(x))$

Example. Let $f(x) = x^2$.

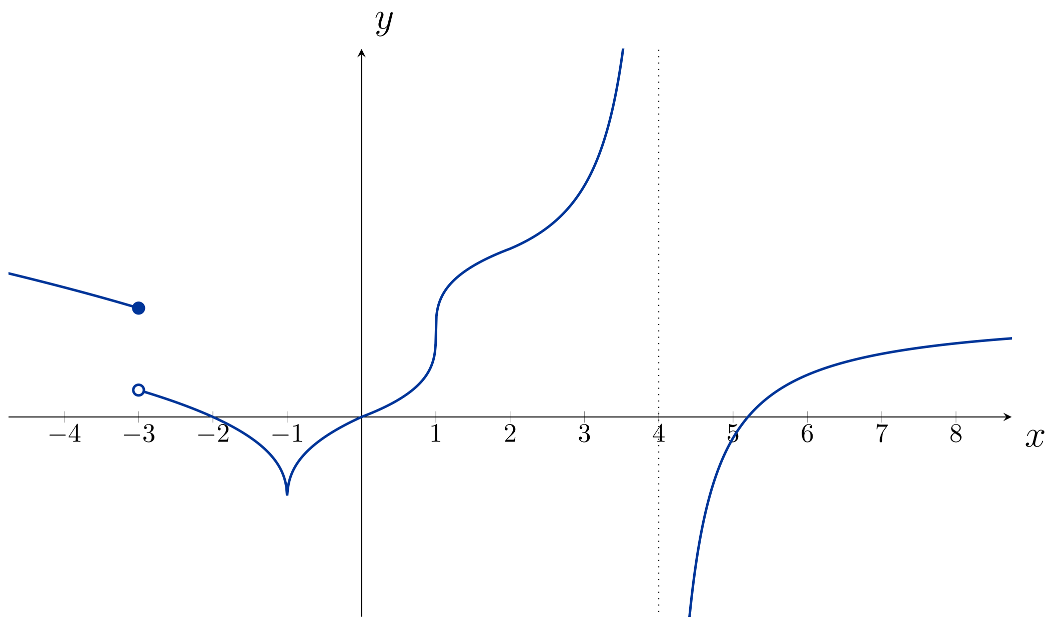
- Find $f'(x)$.
- Compute $f'(2)$ and interpret your result.

Example. Let $f(x) = x^2 - 4x$. Find the point on the graph where the tangent line is horizontal.

Example. Let $f(x) = \dfrac{1}{x}$. Find the equation of the tangent line at $x = 2$.

Differentiability and Continuity
If a function is differentiable at $x = a$, then it is continuous at $x = a$.

Example. For the graph below, identify each point where the derivative is undefined.



3.1: Basic Rules of Differentiation

Rule 1: Derivative of a Constant

$$\frac{d}{dx}[c] = 0$$

Rule 2: The Power Rule

If n is any real number, then

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

Example. Find the derivative of the following functions

$f(x) = x$

$g(x) = x^8$

$h(x) = x^{\frac{5}{2}}$

$j(x) = \sqrt{x}$

$k(x) = \dfrac{1}{\sqrt[3]{x}}$

$\ell(x) = \pi^4$

Rule 3: Derivative of a Constant Multiple of a Function

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

Rule 4: The Sum Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

Example. Find the derivative of the following functions

$f(x) = 5x^3$

$g(x) = \dfrac{3}{\sqrt{x}}$

$h(x) = 4x^5 + 3x^4 - 8x^2 + x + 3$

$j(t) = \dfrac{t^2}{5} + \dfrac{5}{t^2} + \pi$

Example. Find the line tangent to the curve

$$f(x) = 2x + \frac{1}{\sqrt{x}}$$

at the point $(1, 3)$ Graph

Example. An experimental rocket lifts off vertically. Its altitude (in feet) t seconds into flight is given by

$$s = f(t) = -t^3 + 96t^2 + 5, \qquad (t \geq 0)$$

Find an expression v for the rocket's velocity at any time t.

Compute the rocket's velocity when $t = 0, 30, 50, 64$, and 70. Interpret your results.

Using the results from above and the observation that at the highest point in its trajectory the rocket's velocity is zero, find the maximum altitude attained by the rocket.

3.2: The Product and Quotient Rules

Rule 5: The Product Rule

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Note:

$$\frac{d}{dx}[f(x) \cdot g(x)] \neq f'(x) \cdot g'(x)$$

Example. Find the derivative of the following functions

- by expanding
- by using the product rule

$f(x) = \left(2x^2 - 1\right)(x + 3)$ $\qquad$ $g(x) = x^3(\sqrt{x} + 1)$

Note:

$$\frac{d}{dx}[fghj] = f'ghj + fg'hj + fgh'j + fghj'$$

Rule 6: The Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

"Lo De Hi, minus Hi De Lo, over the square of what's below"

Example. Find the derivative of the following functions

$$f(x) = \frac{3x^2 - 4x + 7}{x}$$

$$g(x) = \frac{x}{2x - 4}$$

$$h(x) = \frac{x^2 + 1}{x^2 - 1}$$

$$j(x) = \frac{\sqrt{x}}{x^2 + 1}$$

$$k(x) = \frac{3x(x^2+1)}{x^2-1}$$

3.3: The Chain Rule

Example. Let $f(x) = \left(x^3 + x + 1\right)^2$. Find $f'(x)$

using the product rule

by expanding

What about $\dfrac{d}{dx}\left[\left(x^3 + x + 1\right)^{100}\right]$?

Composite Functions:

Let f and g be functions of x. Then, the **composite functions** g of f (denoted $g \circ f$) and f of g (denoted $f \circ g$) are defined as:

$$(g \circ f)(x) = g(f(x))$$
$$(f \circ g)(x) = f(g(x))$$

Example. 'Break-down' the following composite functions:

$$\frac{1}{x+3}$$

$$\left(x^4 + 3x - 8\right)^3$$

$$\left(\frac{1-x}{x^3+1}\right)^4$$

$$\frac{3}{\sqrt{(x+1)^2 - 1}}$$

Rule 7: The Chain Rule

$$\frac{d}{dx}\left[f\big(g(x)\big)\right] = f'\big(g(x)\big) \cdot g'(x)$$

If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Note:

$$\frac{d}{dx}\left[f\Bigg(g\bigg(h\Big(j(x)\Big)\bigg)\Bigg)\right] = f'\Bigg(g\bigg(h\Big(j(x)\Big)\bigg)\Bigg) \cdot g'\bigg(h\Big(j(x)\Big)\bigg) \cdot h'\Big(j(x)\Big) \cdot j'(x)$$

The General Power Rule

$$\frac{d}{dx}[(f(x))^n] = n(f(x))^{n-1}f'(x)$$

Example. Use the chain rule to show

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Example. Find the derivative of the following functions

$$F(x) = \left(x^3 + x + 1\right)^{100}$$

$$G(t) = (3t + 1)^2$$

$$H(u) = \sqrt{u^2 + 1} - 3$$

$$J(\nu) = \nu^2 (2\nu + 3)^5$$

$$\kappa(x) = \left(2x^2+3\right)^4 (3x-1)^5$$

$$\tau(x) = \frac{1}{\left(4x^2-7\right)^2}$$

Example. Find the equation of the line tangent to $f(x)$ at $\left(0, \dfrac{1}{8}\right)$

$$f(x) = \left(\frac{2x+1}{3x+2}\right)^3$$

Example. The membership of The Fitness Center, which opened a few years ago, is approximated by the function

$$N(t) = 100(64 + 4t)^{2/3} \qquad (0 \leq t \leq 52)$$

where $N(t)$ gives the number of members at the beginning of week t.

Find $N'(t)$

How fast was the center's membership increasing initially ($t = 0$)?

How fast was the membership increasing at the beginning of the 40th week?

What was the membership when the center first opened? At the beginning of the 40th week?

Rule 1: Derivative of a Constant

$$\frac{d}{dx}[c] = 0$$

Rule 2: The Power Rule

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

Rule 3: Derivative of a Constant Multiple of a Function

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

Rule 4: The Sum Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

Rule 5: The Product Rule

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Rule 6: The Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Rule 7: The Chain Rule

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

3.6: Implicit Differentiation and Related Rates

Implicit Functions	Explicit Functions	Graph
$x^2y + y - x^2 + 1 = 0$	$y = \dfrac{x^2 - 1}{x^2 + 1}$	
$x^2 + y^2 = 4$	$y = \pm\sqrt{4 - x^2}$	
$y^3 + y^2 - xy + \dfrac{x^4}{4} = y$		

Implicit Differentiation:

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with ${}^{dy}\!/_{dx}$ on one side of the equation.
3. Solve for ${}^{dy}\!/_{dx}$.

Example. Find the derivatives of the following by rewriting each function explicitly before taking the derivative, and by using implicit differentiation. Compare the results.

$y^2 = x$

$\sqrt{x} + \sqrt{y} = 4$

Example. Find $\dfrac{dy}{dx}$ given the equation

$$y^3 - y + 2x^3 - x = 8$$

Example. Consider the equation $x^2 + y^2 = 4$.

Find $\dfrac{dy}{dx}$ by implicit differentiation.

Find the slope of the tangent line to the graph of the function $y = f(x)$ at the point $(1, \sqrt{3})$.

Find an equation of the tangent line.

Related Rates:
Related rates are problems that use a mathematical relationship between two or more objects under specific constraints. From this, we can differentiate this relationship and examine how each variable changes with respect to time.

The volume of a cone with radius r and height h is given by

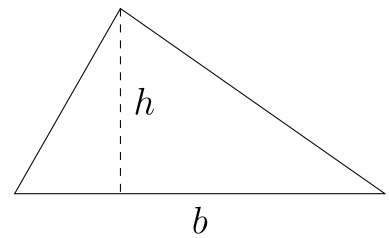
$$V = \frac{1}{3}\pi r^2 h$$

Find dV/dt when r and h are changing.

Find dV/dt when r is constant and h is changing.

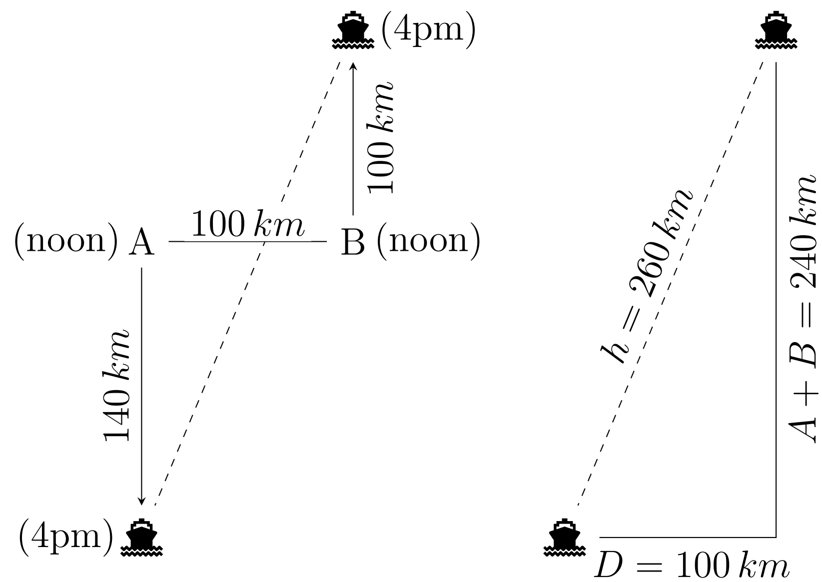
Find dV/dt when r is changing and h is constant.

Example. The altitude of a triangle is increasing at a rate of $1\,cm/min$ while the area of the triangle is increasing at a rate of $2\,cm^2/min$. How fast is the base of the triangle changing when the altitude is $10\,cm$ and the area is $100\,cm^2$.



Example. The base of a 13-ft ladder leaning against a wall begins to slide away from the wall. At the instant of time when the base is 12 ft from the wall, the base is moving at a rate of 8 ft/sec. How fast is the top of the ladder sliding down the wall at that instant of time?

Example. At noon, ship A is 100 km west of ship B. Ship A is sailing south at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4pm?



3.5: Higher–Order Derivatives

Definition.
The **second derivative** of f is

$$f''(x) = \frac{d}{dx}[f'(x)] = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

We can repeatedly take the derivative of $f(x)$:

$$f'(x), f''(x), f'''(x), \ldots, f^{(n)}(x)$$

Example. Find all derivatives of

$$f(x) = x^5 - 7x^4 - 5x^3 - 2x^2 + 6x - 6$$

Example. Let $f(x) = x^{2/3}$. Find $f'''(x)$.

Example. Find the second derivative of $y = \left(2x^2 + 3\right)^{3/2}$

Example. The position function of a maglev train (in feet) is given by

$$s(t) = 4t^2, \qquad (0 \le t \le 30).$$

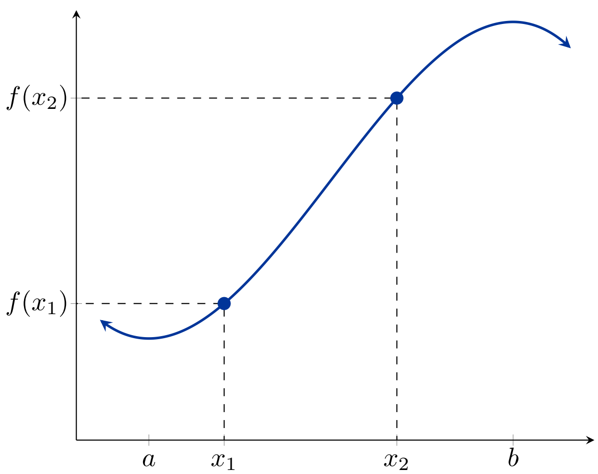
Find the velocity and the acceleration of the maglev train at time t

Example. Find $\dfrac{d^2y}{dx^2}$ if $x^2 + y^2 = 4$.

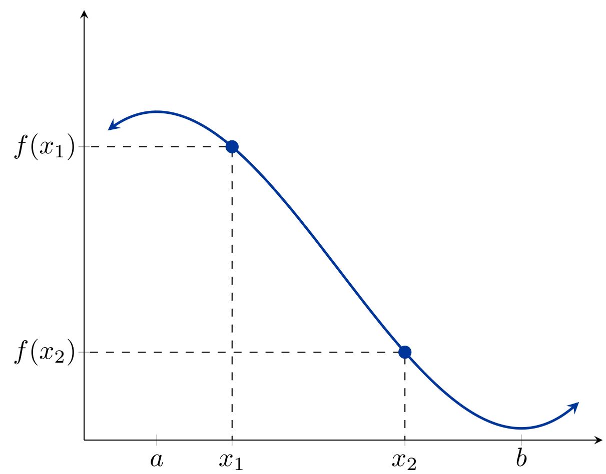
4.1: Applications of the First Derivative

Definition.
Consider the function $f(x)$ on the interval (a, b). Given *any* two numbers x_1 and x_2 in (a, b) where $x_1 < x_2$, we say f is

increasing if $f(x_1) < f(x_2)$



decreasing if $f(x_1) > f(x_2)$



Thus, for every value of x on the interval (a, b), if

– $f'(x) > 0$, then f is increasing on (a, b).

– $f'(x) < 0$, then f is decreasing on (a, b).

– $f'(x) = 0$, then f is constant on (a, b).

Example. Find the intervals where $f(x) = x^2$ is increasing and decreasing.

Determining intervals where a function is increasing or decreasing.

1. Find all values of x such that $f'(x) = 0$ or $f'(x)$ is undefined.
2. Determine the sign of $f'(x)$ on each open interval.

Example. Suppose that f is continuous everywhere and

$$f'(x) = \frac{(x-1)(x+2)}{(x-4)^2(x+5)}.$$

We see that $f'(-2) = f(1) = 0$ and $f(-5)$ and $f(4)$ are undefined. Complete a sign chart to show where $f(x)$ is increasing and decreasing.

Example. Find the intervals where the following functions are increasing and decreasing:

$f(x) = x^3 - 3x^2 - 24x + 32$ Graph

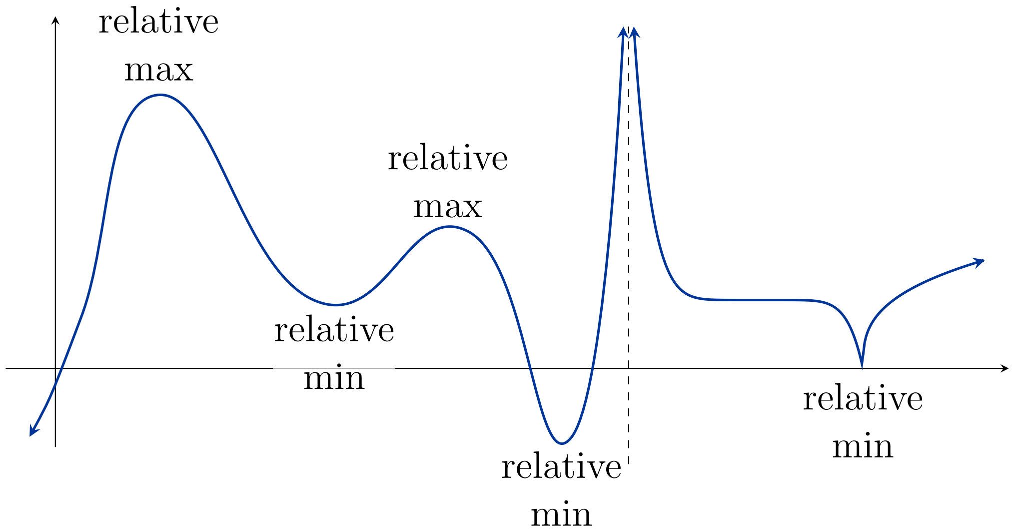
$g(x) = (x + 1)^{2/3}$

$$h(x) = x + \frac{1}{x}$$

$$j(x) = \frac{x^2}{1 - x^2}$$

Definition. (Relative Extrema)
A function f has a

- **relative maximum** at $x = c$ if $f(c) \geq f(x)$ for every x in (a, b)
- **relative minimum** at $x = c$ if $f(c) \leq f(x)$ for every x in (a, b)



Definition.
A **critical point** of a function f is any number x in the domain of f such that $f'(x) = 0$ or $f'(x)$ does not exist.

Procedure for Finding the Relative Extrema of a Continuous Function f
The First Derivative Test:

1. Determine the critical points of f.
2. Determine the sign change of $f'(x)$ to the left and right of each critical point:
   If, at $x = c$, $f'(x)$ ...
   a) changes sign from *positive* to *negative*, then f has a *relative maximum*
   b) changes sign from *negative* to *positive*, then f has a *relative minimum*
   c) does not change sign, then f does *not* have a relative extremum

   at $x = c$.

Example. Consider the function $f(x) = 6x - x^3$. Graph

Use $f'(x)$ to find the intervals on which the function is increasing and decreasing.

Identify the function's local extreme values (e.g. "local max of ___ at $x =$ ___")

Example. Find the relative maximums/relative minimums of the following:

$f(x) = x^3 - 3x^2 - 24x + 32$ Graph

$g(x) = (x+1)^{2/3}$

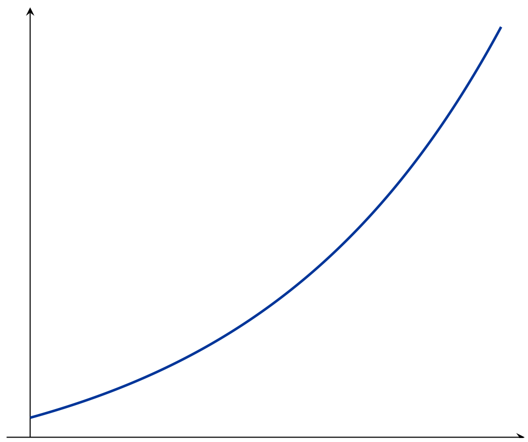
$$h(x) = x + \frac{1}{x}$$

$$j(x) = \frac{x^2}{1 - x^2}$$

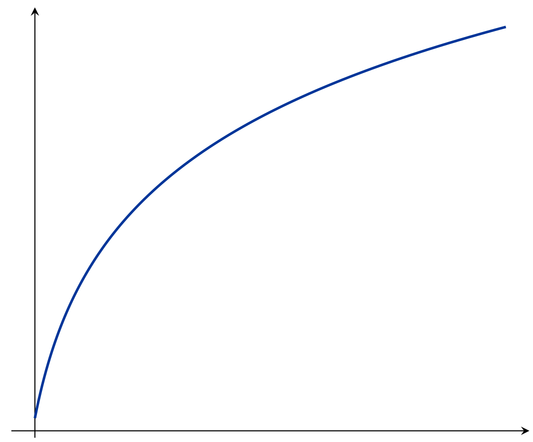
4.2: Applications of the Second Derivative

Definition.
Consider any differentiable function $f(x)$ on the interval (a,b). We say f is

concave up if $f'(x)$ is increasing



concave down if $f'(x)$ is decreasing



Thus, for every value of x on the interval (a,b), if

– $f''(x) > 0$, then f' is increasing, and f is concave *up* on (a,b).

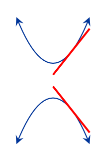
– $f''(x) < 0$, then f' is decreasing, and f is concave *down* on (a,b).

– If f is continuous at c and f changes concavity at c, then f has an **inflection point** at c.

Note: $f(x)$ is

– concave up if its tangent lines lie below the curve

– concave down if its tangent lines lie above the curve



Determining the Intervals of Concavity of the Graph of f

1. Determine the values of x for which f'' is zero or undefined.

2. Determine the sign of $f''(x)$ to the left and right of each point from above:
   Let c be a convenient test point on the interval of interest. Then,

   a) if $f''(c) > 0$, then f is concave up on that interval. 

   b) if $f''(c) < 0$, then f is concave down on that interval. 

Example. Find the intervals where the following functions are concave up and concave down:

$f(x) = x^3 - 3x^2 - 24x + 32$ Graph

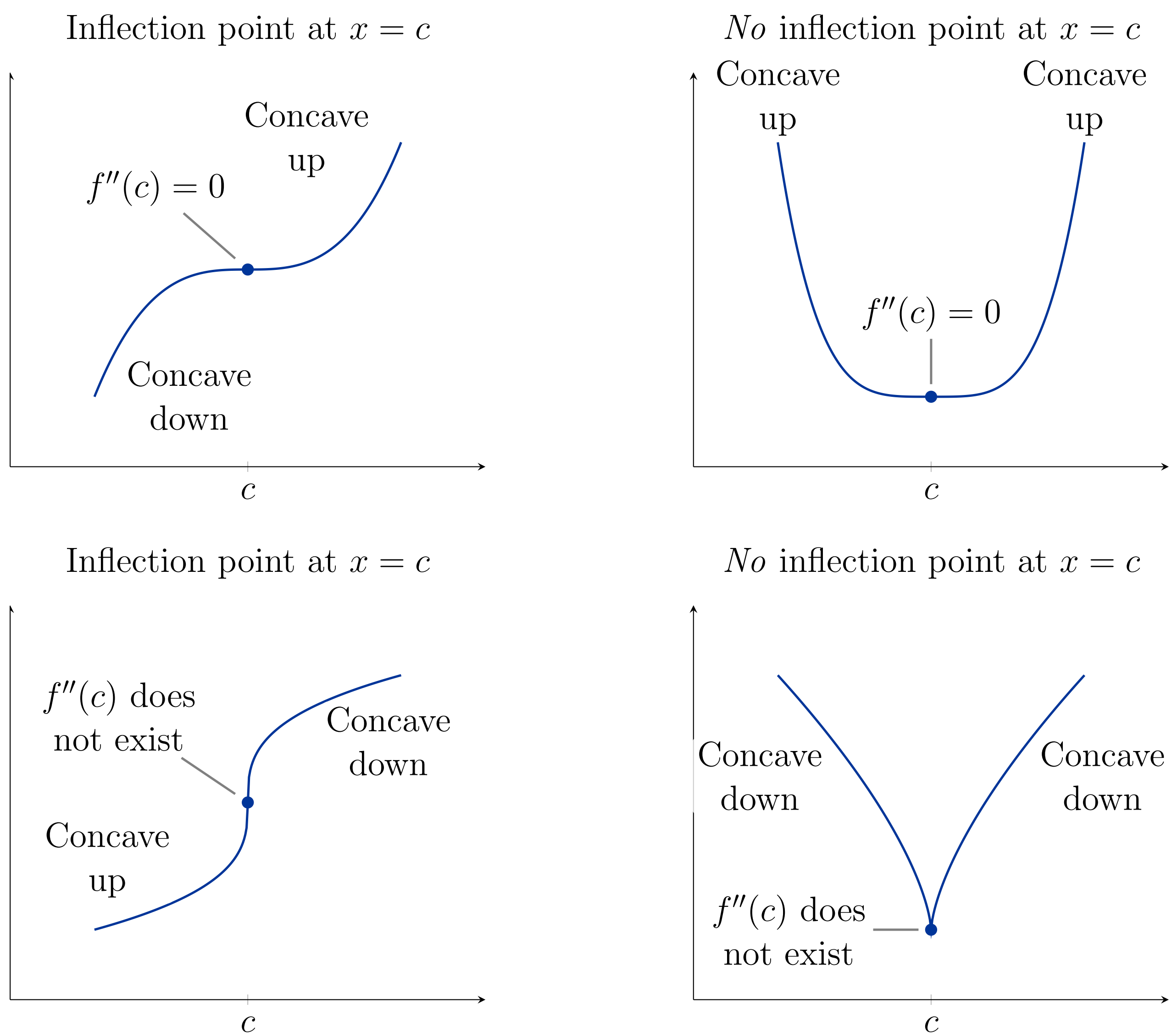
$g(x) = (x + 1)^{2/3}$

$$h(x) = x + \frac{1}{x}$$

$$j(x) = \frac{x^2}{1 - x^2}$$

Finding inflection points

1. Compute $f''(x)$.
2. Locate where $f''(x) = 0$ or $f''(x)$ does not exist.
3. Determine if the sign of $f''(x)$ changes at the points found above.



Example. For the following functions, determine the intervals of concavity and find any inflection points.

$f(x) = (x-1)^{5/3}$ Graph

$$g(x) = \frac{1}{x^2 + 1}$$

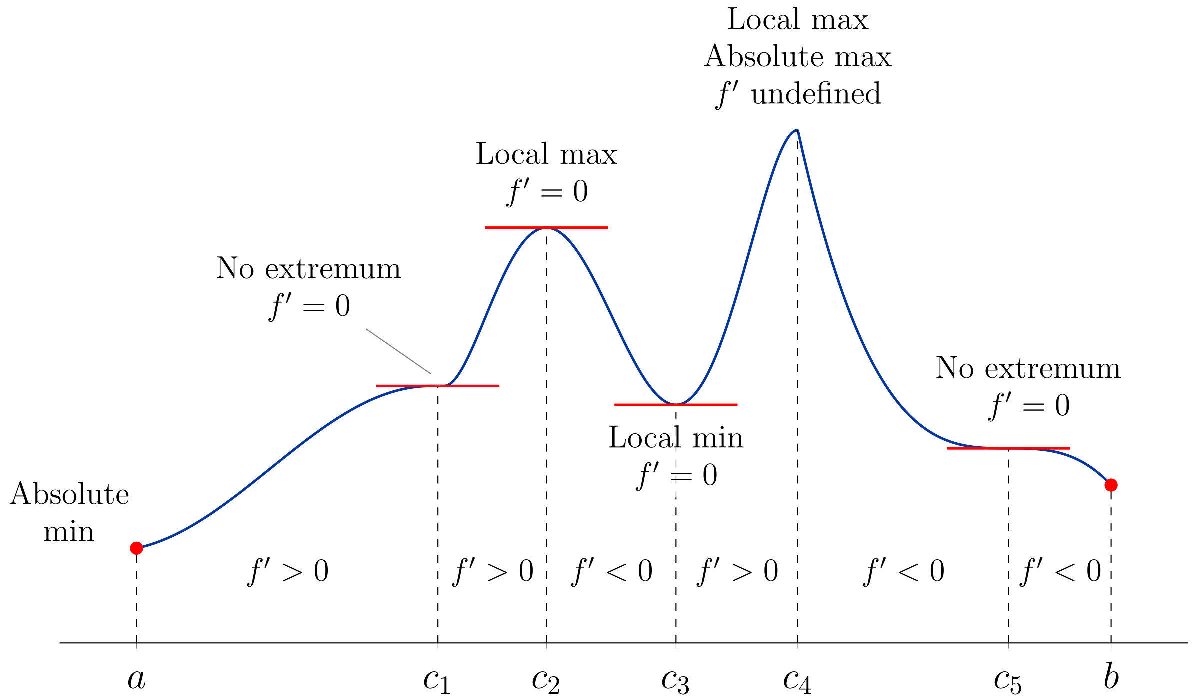
Second Derivative Test for Local Extrema
Suppose f'' is continuous on an open interval containing c with $f'(c) = 0$.

- If $f''(c) > 0$, then f has a local minimum at c.
- If $f''(c) < 0$, then f has a local maximum at c.
- If $f''(c) = 0$, then the test is inconclusive; f may have a local maximum, local minimum, or neither at c.

Example. Find the relative extrema of

$$f(x) = x^3 - 3x^2 - 24x + 32$$

Graph



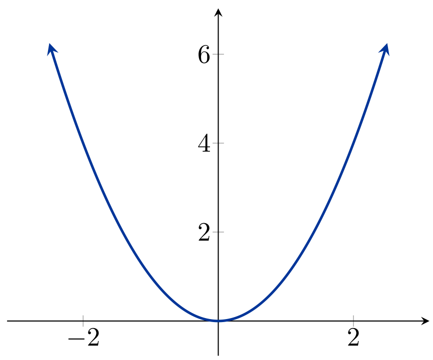
$f(x)$	$f'(x)$	$f''(x)$
increasing	positive	—
decreasing	negative	—
max/min	crit. pt. & changes sign	—
concave up	increasing	positive
concave down	decreasing	negative
Inflection point	max/min	crit. pt. & changes sign

4.4: Optimization I

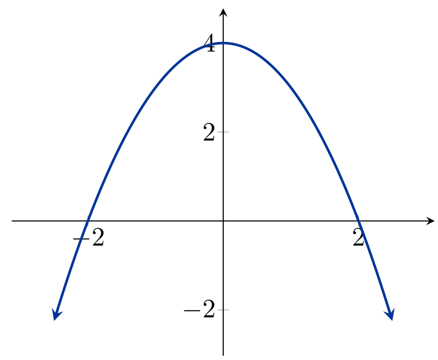
Definition. (Absolute Extrema)
Let f be defined on a set D containing c. If

- $f(c) \geq f(x)$ for every x in D, then $f(c)$ is an **absolute maximum** value of f
- $f(c) \leq f(x)$ for every x in D, then $f(c)$ is an **absolute minimum** value of f

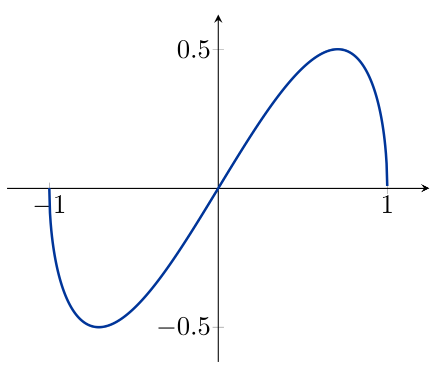
$f(0) = 0$ is the absolute minimum;
No absolute maximum



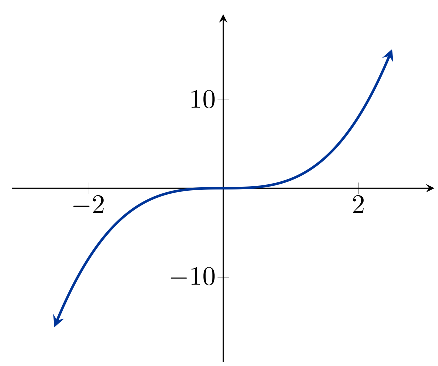
No absolute minimum;
$f(0) = 4$ is the absolute maximum



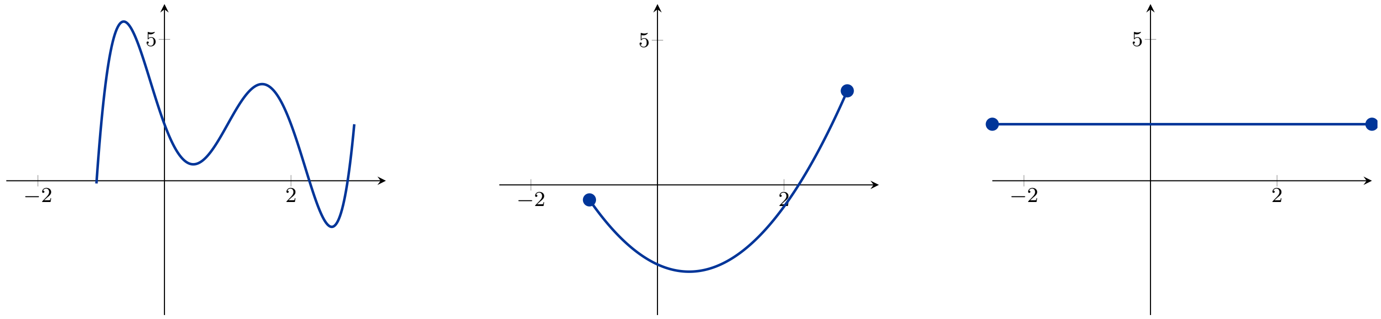
$f(-\sqrt{2}) = -\frac{1}{2}$ is the absolute minimum;
$f(\sqrt{2}) = \frac{1}{2}$ is the absolute maximum



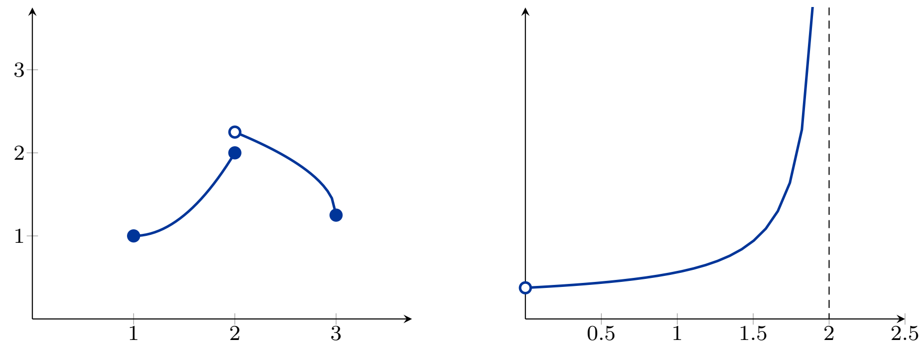
No absolute minimum;
No absolute maximum



Theorem 3
A function that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value on that interval.



Note: It is important that the function is both continuous *and* the interval is closed:



Finding the Absolute Extrema of f on a Closed Interval

1. Find the critical points of f within the interval (a, b).
2. Compute $f(x)$ at $x = a$, $x = b$, and at each of the critical points found above.
3. The absolute maximum and absolute minimum will correspond to the largest and smallest values found above.

Example. Find the absolute extrema of the following functions on the intervals indicated

$f(x) = x^2$ on $[-1, 2]$ Graphs

$g(x) = x^3 - 2x^2 - 4x + 4$ on $[0, 3]$

$h(x) = x^{2/3}$ on $[-1, 8]$

Example. The daily average cost function (in dollars per unit) of Elektra Electronics is given by

$$\overline{C}(x) = 0.0001x^2 - 0.08x + 40 + \frac{5000}{x} \qquad (x > 0)$$

where x stands for the number of graphing calculators that Elektra produces. Show that a production level of 500 units per day results in a minimum average cost for the company.

Example. The altitude (in feet) of a rocket t seconds into flight is given by

$$s = f(t) = -t^3 + 96t^2 + 5 \qquad (t \geq 0)$$

Find the maximum altitude attained by the rocket.

Find the maximum velocity attained by the rocket.

4.5: Optimization II

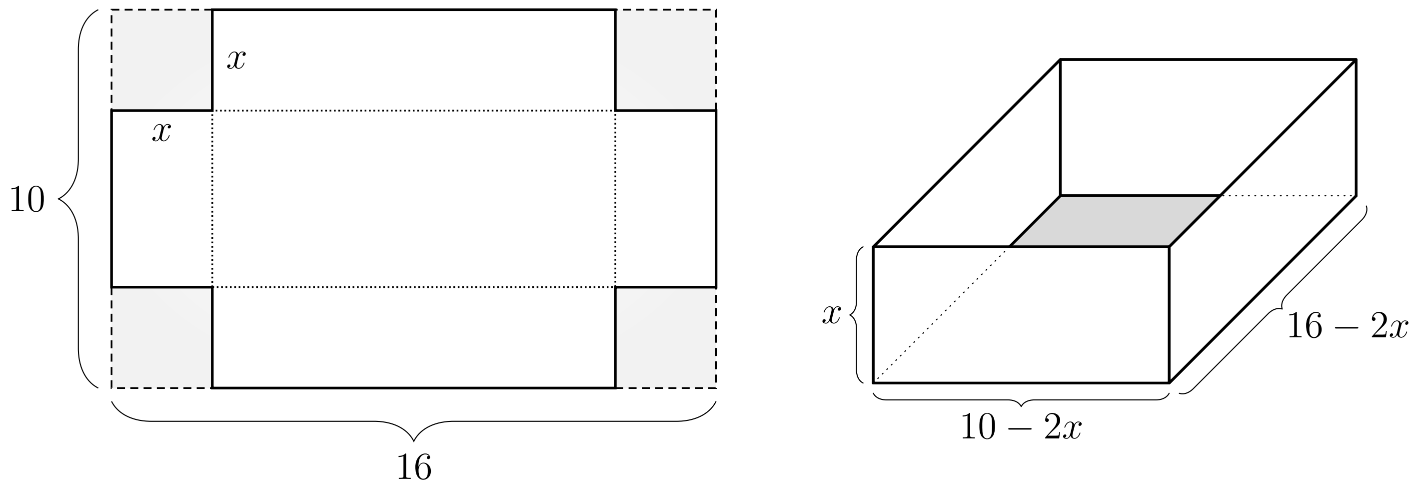
Guidelines for Solving Optimization Problems

1. Assign a letter to each variable mentioned in the problem. If appropriate, draw and label a figure.
2. Find an expression for the quantity to be optimized.
3. Use the conditions given in the problem to write the quantity to be optimized as a function f of *one* variable. Note any restrictions to be placed on the domain of f from physical considerations of the problem.
4. Optimize the function f over its domain.

Example. Show among all rectangles with an 8 meter perimeter, the one with the *largest* area is a square.

Example. A man wishes to have a rectangular-shaped garden in his backyard. He has 50 feet of fencing with which to enclose his garden. Find the dimensions for the largest garden he can have if he uses all of his fencing.

Example. By cutting away identical squares from each corner of a rectangular piece of cardboard and folding up the resulting flaps, the cardboard may be turned into an open box. If the cardboard is 16" long and 10" wide, find the dimensions of the box that will yield the maximum volume.



Example. A cylindrical can is to be made to hold $1\,L$ $(1000\,cm^3)$ of oil. Find the dimensions of the can that will minimize the cost of the metal to manufacture the can.

Example. Of all boxes with a square base and a volume of $8\,\mathrm{m}^3$, which one has the minimum surface area?

Example. Find the point on the line $y = 2x + 3$ that is closest to the origin.

Example. A pencil cup with a capacity of $36\,\text{in}^3$ is to be constructed in the shape of a rectangular box with a square base and an open top. If the material for the sides cost $\$0.15/\text{in}^2$ and the material for the base costs $\$0.40/\text{in}^2$, what should the dimensions of the cup be to minimize the construction cost?

5.4: Differentiation of Exponential Functions

Rule 1: Derivative of the Exponential Function

$$\frac{d}{dx}[e^x] = e^x$$

Example. Find the derivative of the following functions

$f(x) = x^2 e^x$

$g(t) = (e^t + 2)^{3/2}$

Rule 2: The Chain Rule for Exponential Functions
If $f(x)$ is a differentiable function, then

$$\frac{d}{dx}\left[e^{f(x)}\right] = e^{f(x)}\, f'(x)$$

Example. Find the derivative of the following functions

$f(x) = e^{2x}$

$y = e^{-3x}$

$g(t) = e^{2t^2+t}$

$y = xe^{-2x}$

Example. Find the inflection points of the function $f(x) = e^{-x^2}$.

5.5: Differentiation of Logarithmic Functions

Definition. (Natural log)
The inverse of the exponential function is the logarithm:

$$e^{y} = x \longleftrightarrow \ln(x) = y, \qquad x > 0$$

We may also denote this as $\log_e(x)$

Example. Solve the following exponential equations [Graphs]

$y = e^{-3}$

$e^{x^2-x} = e^2$

$e^x = 10$

$4e^{1-x^2} = 6$

Rule 3: Derivative of $\ln(x)$

$$\frac{d}{dx}\Big[\ln|x|\Big] = \frac{1}{x}, \qquad x \neq 0$$

Rule 4: Derivative of $\ln\left(f(x)\right)$

$$\frac{d}{dx}\Big[\ln\left(f(x)\right)\Big] = \frac{f'(x)}{f(x)}$$

Example. Find the derivative of the following functions

$y = \ln\left(\sqrt[3]{x}\right)$

$f(x) = \sqrt{x}\ln(x)$

$g(x) = \ln\left(x^2 + 1\right)$

$h(t) = \dfrac{1 + \ln(t)}{1 - \ln(t)}$

Additional properties of logarithms

$$e^y = x \qquad\qquad \ln(x) = y$$

$$e^1 = e \qquad\qquad \ln(e) = 1$$

$$e^0 = 1 \qquad\qquad \ln(1) = 0$$

$$e^x e^y = e^{x+y} \qquad\qquad \ln(xy) = \ln(x) + \ln(y)$$

$$\frac{e^x}{e^y} = e^{x-y} \qquad\qquad \ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$

$$e^{xy} = (e^x)^y \qquad\qquad \ln(x^y) = y\ln(x)$$

$$e^{\ln(x)} = x \qquad\qquad \ln(e^x) = x$$

Definition. (Other logarithms)
Logarithms are defined for any base $a > 0$:

$$a^y = x \quad \longleftrightarrow \quad \log_a(x) = y, \qquad x > 0$$

Example. Find the derivative of the following functions

$$y = \ln\left(\sqrt[3]{x}\right)$$

$$f(x) = \ln\left(\left(x^2+1\right)\left(x^3+2\right)^6\right)$$

$$g(x) = \ln\left(\frac{1-x}{1+x}\right)$$

$$h(x) = \ln\left(x + \sqrt{x^2-1}\right)$$

$$j(t) = 2\big(\ln(5t)\big)^{3/2}$$

$$k(u) = \ln\left(\sqrt{u^2-4}\right)$$

Logarithmic Differentiation
Using properties of logs, we can transform a function before taking it's derivative:

1. Take the natural log of both sides rewriting products and quotients as sums and differences.
2. Differentiate both sides.
3. Solve for $\frac{dy}{dx}$

Example. Differentiate
$$y = x^2(x-1)\left(x^2+4\right)^3$$

5.6: Exponential Functions As Mathematical Models

Example. Consider the exponential function

$$Q(t) = Q_0 e^{kt}$$

What does Q_0 represent?

What does k represent?

Show that the rate of increase of $Q(t)$ is proportional to the quantity $Q(t)$.

> **Definition.**
> $Q(t)$ is said to exhibit **Exponential Growth**.

Example. Under ideal laboratory conditions, the number of bacteria in a culture grows in accordance with the law $Q(t) = Q_0 e^{kt}$, where Q_0 denotes the number of bacteria initially present in the culture, k is a constant determined by the strain of bacteria under consideration and other factors, and t is the elapsed time measured in hours. Suppose $10,000$ bacteria are present initially in the culture and $60,000$ are present 2 hours later.

How many bacteria will there be in the culture at the end of 4 hours?

What is the rate of growth of the population after 4 hours?

Example. Radioactive substances decay exponentially. For example, the amount of radium present at any time t obeys the law $Q(t) = Q_0e^{-kt}$, where Q_0 is the initial amount present and k is a specific positive constant. The **half-life of a radioactive substance** is the time required for a given amount to be reduced by one-half. It is known that the half-life of radium is approximately 1600 years. Suppose initially there are 200 milligrams of pure radium.

What is the amount left after t years? What about 800 years?

How fast is the amount of radium decaying after t years? What about 800 years?

Example. Carbon 14, a radioactive isotope of carbon, has a half-life of 5730 years. What is its decay constant?

Example. The Camera Division of Eastman Optical produces a 35-mm single-lens reflex camera. Eastman's training department determines that after completing the basic training program, a new, previously inexperienced employee will be able to assemble

$$Q(t) = 50 - 30e^{-0.5t}$$

model F cameras per day t months after the employee starts work on the assembly line.

How many model F cameras can a new employee assemble per day after basic training?

How many model F cameras can an employee with 1 month of experience assemble per day? What about 2 months? 6 months?

How many model F cameras can the average experienced employee ultimately be expected to assemble per day?

Example. The number of soldiers at Fort MacArthur who contracted influenza after t days during a flu epidemic is approximated by the *logistic model*

$$Q(t) = \frac{5000}{1 + 1249e^{-kt}}$$

If 40 soldiers contracted the flu by day 7, find how many soldiers contracted the flu by day 15.

At what rate is the number of soldiers contracting the flu changing on day 15?

6.1: Antiderivatives and the Rules of Integration

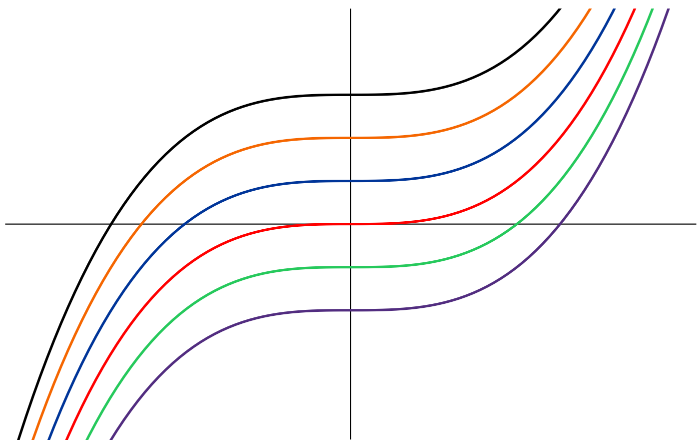
Definition. (Antiderivatives)
A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Example. Show that $F(x) = \dfrac{1}{3}x^3 - 2x^2 + x - 1$ is an antiderivative of $f(x) = x^2 - 4x + 1$.

Example. Let $F(x) = x$, $G(x) = x + 2$, and $H(x) = x + C$, where C is a constant. Show that F, G, and H are all antiderivatives of $f(x) = 1$.

Theorem 1
Let G be an antiderivative of a function f on an interval I. Then, every antiderivative of F of f on I must be of the form $F(x) = G(x) + C$, where C is a constant.

Example. If $f'(x) = x^2$, then $f(x) = \dfrac{x^3}{3} + C$ is the family of antiderivatives of $f'(x)$.



Definition. (Integration)
The process of finding the antiderivative is called **integration**:

$$\int f(x)\,dx = F(x) + C$$

The **indefinite integral** of f is the family of functions given by $F(x) + C$ where $F'(x) = f(x)$. The function to be integrated, f, is called the **integrand**. C is the **constant of integration**.

Rule 1: The Indefinite Integral of a Constant

$$\int k\,dx = kx + C \qquad (k,\ \text{a constant})$$

Rule 2: The Power Rule

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1} + C \qquad (n \neq -1)$$

Example. Find each of the following indefinite integrals

$$\int 2\,dx$$

$$\int \pi^2\,dx$$

$$\int x^3\,dx$$

$$\int \frac{1}{x^{3/2}}\,dx$$

Rule 3: The Indefinite Integral of a Constant Multiple of a Function

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad (c,\ \text{a constant})$$

Rule 4: The Sum Rule

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Example. Find each of the following indefinite integrals

$$\int \frac{1}{5} - \frac{2}{t^3} + 2t\,dt$$

$$\int 3x^5 + 4x^{3/2} - 2x^{-1/2}\,dx$$

Rule 5: The Indefinite Integral of the Exponential Function

$$\int e^x \, dx = e^x + C$$

Rule 6: The Indefinite Integral of the Function $f(x) = x^{-1}$

$$\int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln|x| + C \qquad (x \neq 0)$$

Example. Find each of the following indefinite integrals

$$\int 2e^x - x^3 + x^e - e^e \, dx$$

$$\int 2x + \frac{3}{x} + \frac{4}{x^2} \, dx$$

$$\int \frac{2}{\sqrt{x}} - \frac{2}{x} \, dx$$

$$\int \frac{1}{4e^x} - \frac{4}{x} + e^x \, dx$$

Rule 1: The Indefinite Integral of a Constant

$$\int k\,dx = kx + C \qquad (k,\ \text{a constant})$$

Rule 2: The Power Rule

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1} + C \qquad (n \neq -1)$$

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$$\int cf(x)\,dx = c\int f(x)\,dx \qquad (c,\ \text{a constant})$$

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Rule 6: The Indefinite Integral of the Function $f(x) = x^{-1}$

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C \qquad (x \neq 0)$$

6.2: Integration by Substitution

Example. Find the derivative of the following functions:

$$f(x) = \frac{(2x+1)^4}{4}$$

$$g(x) = \frac{1}{x+3}$$

Let $u = g(x)$, where g is differentiable on an interval, and let f be continuous on the corresponding range of g. On that interval,

$$\int f\big(g(x)\big) \cdot g'(x)\, dx = \int f(u)\, du$$

Procedure: Substitution Rule (Change of Variables)

1. Let $u = g(x)$, where $g(x)$ is part of the integrand, usually the "inside function" of a composite function $f\big(g(x)\big)$.
2. Find $du = g'(x)\,dx$.
3. Use the substitution $u = g(x)$ and $du = g'(x)\,dx$ to convert the *entire* integral into one involving only u.
4. Find the resulting integral
5. Replace u by $g(x)$ to obtain the final solution as a function of x.

Example. Evaluate the following integrals:

$$\int 2x(x^2+3)^4\,dx \qquad\qquad \int (2x+1)^3\,dx$$

$$\int x^2\sqrt{x^3+1}\,dx$$

$$\int t\,\sqrt[4]{1-t^2}\,dt$$

$$\int \sqrt{4-t}\,dt$$

$$\int (2-x)^6\,dx$$

$$\int e^{-3x}\,dx$$

$$\int \frac{t}{3t^2+1}\,dt$$

$$\int \frac{\big(\ln(x)\big)^2}{2x}\,dx$$

$$\int u^3\big(u^2+1\big)^{3/2}\,du$$

6.5: Evaluating Definite Integrals

The Fundamental Theorem of Calculus
Let f be continuous on $[a, b]$. Then,

$$\int_a^b f(x)\, dx = F(x)\Big|_a^b = F(b) - F(a)$$

where F is any antiderivative of f; that is, $F'(x) = f(x)$.

Example. Let R be the region under the graph of $f(x) = x$ on the interval $[1, 3]$. Find the area of R Graph

using geometry

using the Fundamental Theorem of Calculus

Properties of the Definite Integral

Let f and g be integrable functions; then,

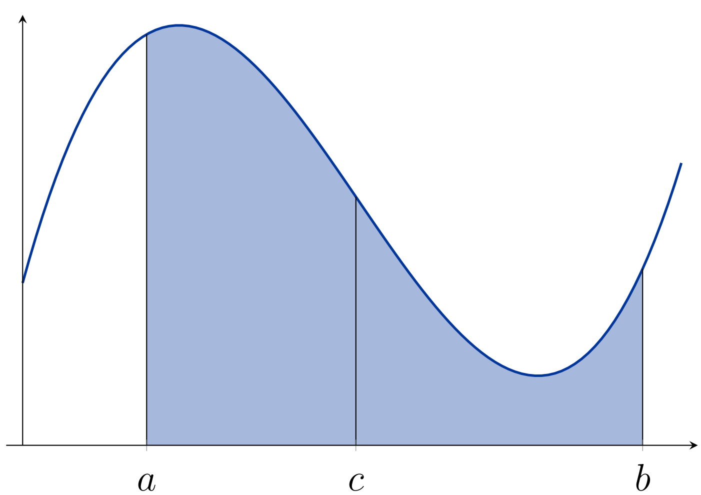
1. $\displaystyle\int_a^a f(x)\,dx = 0$

2. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ (c constant)

4. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ ($a < c < b$)



Example. Evaluate the following definite integrals

$$\int_0^4 x\sqrt{9+x^2}\,dx$$

$$\int_0^2 xe^{2x^2}\,dx$$

Example. Find the area of each region R described below: Graphs

Under $f(x) = \sqrt{x}$ from $x = 1$ to $x = 4$

Under $f(x) = e^{x/2}$ from $x = -1$ to $x = 1$