

6.1: Set Theory: Definitions and the Element Method of Proof

Element Argument: The Basic Method for Proving that One set is a Subset of Another

Let sets X and Y be given. To prove that $X \subseteq Y$,

1. **suppose** that x is a particular but arbitrarily chosen element of X ,
2. **show** that x is an element of Y

Example. Define sets A and B as follows:

$$A = \{m \in \mathbb{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbb{Z}\}$$

$$B = \{n \in \mathbb{Z} \mid n = 3s \text{ for some } s \in \mathbb{Z}\}$$

Prove that $A \subseteq B$

Disprove that $B \subseteq A$

Definition.

Given sets A and B , A **equals** B , written $\mathbf{A} = \mathbf{B}$, if, and only if, every element of A is in B and every element of B is in A :

$$A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A.$$

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Is $A = B$?

Definition.

Given an integer n and a positive integer d , when n is divided by d , then

$n \operatorname{div} d =$ the integer quotient

$n \bmod d =$ the nonnegative integer remainder

If n and d are integers and $d > 0$, then

$$n \operatorname{div} d = q \quad \text{and} \quad n \bmod d = r \quad \Leftrightarrow \quad n = dq + r$$

Example. Compute the following:

$$32 \operatorname{div} 9, \quad 32 \bmod 9$$

$$365 \operatorname{div} 7, \quad 365 \bmod 7$$

Example. If it is currently 11:00, what time will it be in

51 hours?

121 hours?

11 hours?

−1 hours?

Example. Let $A = \{4, \sqrt{16}, 19 \bmod 15\}$ and $B = \{12 \bmod 8\}$. Is $A \subseteq B$? Is $B \subseteq A$?

Definition.

Let A and B be subsets of a universal set U .

1. The **union** of A and B is the set of all elements that are in at least one of A or B .

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$$

2. The **intersection** of A and B is the set of all elements that are common to both A and B .

$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$

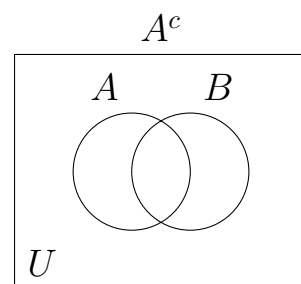
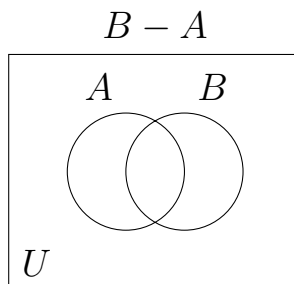
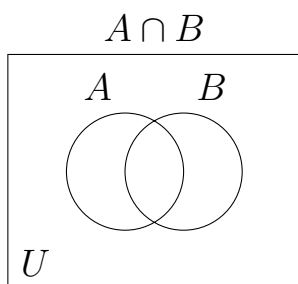
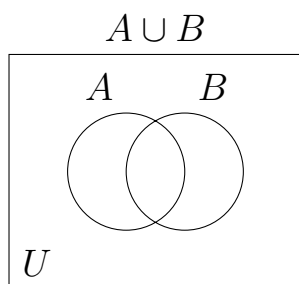
3. The **difference** of A and B is the set of all elements that are in B and not A .

$$B - A = \{x \in U \mid x \in B \text{ and } x \notin A\}$$

4. The **complement** of A is the set of all elements in U that are not in A .

$$A^c = \{x \in U \mid x \notin A\}$$

Example. Represent the following sets using the Venn diagrams below:



Example. Let the universal set be the set $U = \{a, b, c, d, e, f, g\}$, and let $A = \{a, c, e, g\}$ and $B = \{d, e, f, g\}$. Find

$$A \cup B$$

$$A \cap B$$

$$B - A$$

$$A^c$$

Definition.

Given real numbers a and b with $a \leq b$:

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

Example. Let the universal set be \mathbb{R} , and let $A = (-1, 0]$ and $B = [0, 1)$. Find

$$A \cup B$$

$$A \cap B$$

$$B - A$$

$$A^c$$

Definition.

Given sets A_0, A_1, A_2, \dots that are subsets of a universal set U and given a nonnegative integer n ,

$$\bigcup_{i=0}^n A_i = \{x \in U \mid x \in A_i, \text{ for at least one } i = 0, 1, 2, \dots, n\}$$

$$\bigcap_{i=0}^n A_i = \{x \in U \mid x \in A_i, \text{ for every } i = 0, 1, 2, \dots, n\}$$

Example. For each positive integer i , let $A_i = \left\{x \in \mathbb{R} \mid -\frac{1}{i} < x < \frac{1}{i}\right\} = \left(-\frac{1}{i}, \frac{1}{i}\right)$. Find

$$A_1 \cup A_2 \cup A_3$$

$$A_1 \cap A_2 \cap A_3$$

$$\bigcup_{i=1}^{\infty} A_i$$

$$\bigcap_{i=1}^{\infty} A_i$$

Definition.

The **empty set** (or **null set**), denoted \emptyset , is the set with no elements.

$$\{1, 3\} \cap \{2, 4\} = \emptyset$$

Two sets are called **disjoint** if, and only if, they have no elements in common:

$$A \cap B = \emptyset.$$

Sets A_1, A_2, A_3, \dots are **mutually disjoint** (or **pairwise disjoint**) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common:

$$A_i \cap A_j = \emptyset \text{ whenever } i \neq j.$$

Example.

Let $A_1 = \{3, 5\}$, $A_2 = \{1, 4, 6\}$, and $A_3 = \{2\}$. Are A_1 , A_2 , and A_3 mutually disjoint?

Let $B_1 = \{2, 4, 6\}$, $B_2 = \{3, 7\}$, and $B_3 = \{4, 5\}$. Are B_1 , B_2 , B_3 mutually disjoint?

Definition.

A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, \dots\}$ is a **partition** of a set A if, and only if,

1. A is the union of all the A_i ;
2. the sets A_1, A_2, A_3, \dots are mutually disjoint.

Example.

Let $A = \{1, 2, 3, 4, 5, 6\}$, $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$, and $A_3 = \{5, 6\}$. Is $\{A_1, A_2, A_3\}$ a partition of A ?

Let \mathbb{Z} be the set of all integers and let

$$T_i = \{n \in \mathbb{Z} \mid n = 3k + i, \text{ for some integer } k\}.$$

Is $\{T_0, T_1, T_2\}$ a partition of \mathbb{Z} ?

Definition.

Given a set A , the **power set** of A , denoted $\mathcal{P}(A)$, is the set of all subsets of A .

Example. Find $\mathcal{P}(\{x, y\})$.