

5.2: Logarithmic Functions and Their Properties

Definition.

For $a > 0$ and $a \neq 1$, the **logarithmic function**

$$y = \log_a(x) \quad (\text{logarithmic form})$$

has domain $x > 0$, base a , and is defined by

$$a^y = x \quad (\text{exponential form})$$

Example. Rewrite the following in exponential form

$$4 = \log_2(16)$$

$$2^4 = 16$$

$$5 = \log_{10}(100,000)$$

$$10^5 = 100,000$$

$$\frac{1}{2} = \log_{16}(4)$$

$$16^{1/2} = 4$$

$$\boxed{x^{1/m} = \sqrt[m]{x}}$$

$$-4 = \log_3\left(\frac{1}{81}\right)$$

$$3^{-4} = \frac{1}{81}$$

$$\boxed{x^{-m} = \frac{1}{x^m}}$$

$$-\frac{1}{4} = \log_{625}\left(\frac{1}{5}\right)$$

$$625^{-1/4} = \frac{1}{5}$$

$$-\frac{5}{3} = \log_{1/8}(32)$$

$$\left(\frac{1}{8}\right)^{-5/3} = 32$$

Example. Simplify the following:

$$y = \log_3(9)$$

$$3^y = 9 \Rightarrow y = 2$$

$$\Rightarrow \boxed{\log_3(9) = 2}$$

$$y = \log_4(2)$$

$$4^y = 2 \Rightarrow y = \frac{1}{2}$$

$$\Rightarrow \boxed{\log_4(2) = \frac{1}{2}}$$

Example. Solve the following:

$$\log_5(x) = 4$$

$$x = 5^4$$

$$\boxed{x = 625}$$

$$\log_8(x) = 1$$

$$x = 8^1$$

$$\boxed{x = 8}$$

$$\log_{81}(x) = -\frac{1}{4}$$

$$x = 81^{-1/4}$$

$$= \frac{1}{81^{1/4}}$$

$$= \frac{1}{\sqrt[4]{81}}$$

$$\boxed{x = \frac{1}{3}}$$

$$\log_{10}(x + 4) = 3$$

$$x + 4 = 10^3$$

$$-4 + x + 4 = 1000 - 4$$

$$\boxed{x = 996}$$

Common logarithms: $\log(x) = \log_{10}(x)$ Natural logarithms: $\ln(x) = \log_e(x)$

Example (The Rule of 70). If $\$P$ is invested for t years at interest rate r , compounded continuously, then the future value of the investment is given by

$$S = Pe^{rt}.$$

Find the value of t when the investment doubles.

$$S = 2P$$

$$\frac{2P}{P} = \frac{Pe^{rt}}{P}$$

$$2 = e^{rt} \longrightarrow \log_e(2) = rt$$

$$\frac{\ln(2)}{r} = \frac{rt}{r}$$

$$\Rightarrow t = \frac{\ln(2)}{r} \approx \frac{0.69314\dots}{r}$$

If we represent the interest rate as a percentage (e.g. 5% instead of 0.05), then we have

$$t \approx \frac{70}{r}$$

Suppose $r=5\%$:

$$t \approx \frac{70}{5} = 14$$

$$t = \frac{\ln(2)}{0.05} \approx 13.863$$

Change of base formula:

If $a > 0$, $b > 0$ with $a \neq 1$ and $b \neq 1$, then

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}.$$

Note: This works for any valid base!

Base e : $\log_b(x) = \frac{\ln(x)}{\ln(b)}$

Base 10 : $\log_b(x) = \frac{\log(x)}{\log(b)}$

Example. Solve the following

$$3^x = 10$$

$$\begin{aligned} x &= \log_3(10) \\ &= \frac{\ln(10)}{\ln(3)} \end{aligned}$$

$$x \approx 2.0959$$

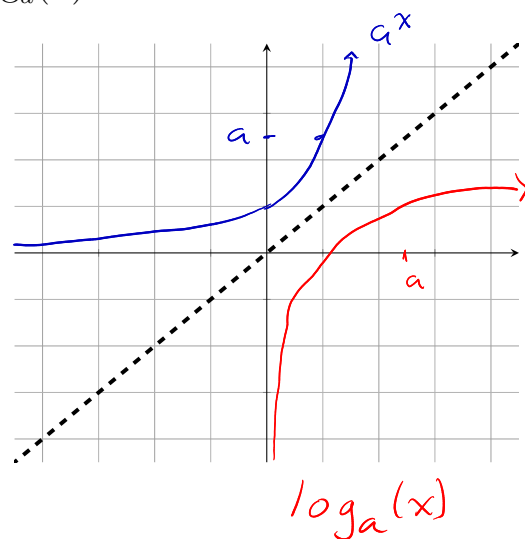
$$6.5^x = 5$$

$$\begin{aligned} x &= \log_{6.5}(5) \\ &= \frac{\ln(5)}{\ln(6.5)} \end{aligned}$$

$$x \approx 0.8598$$

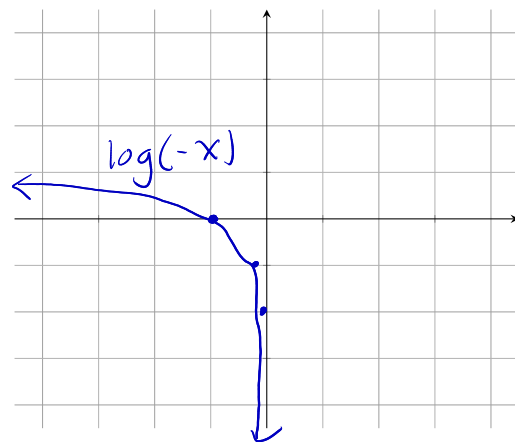
Example. Fill in the tables below and graph a^x and $\log_a(x)$ on the same axes.

x	$y = a^x$	x	$y = \log_a(x)$
-2	$1/a^2$	$1/a^2$	-2
-1	$1/a$	$1/a$	-1
0	1	1	0
1	a	a	1
2	a^2	a^2	2



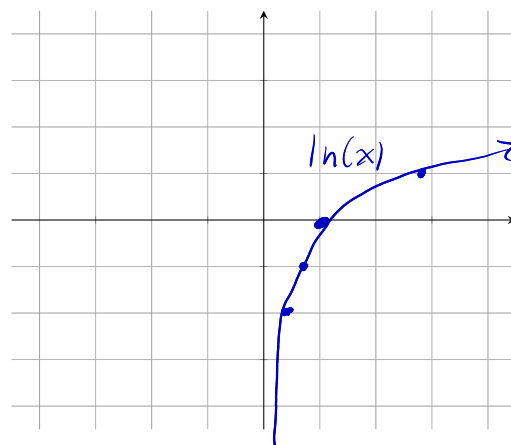
Example. Graph $\log(-x)$

x	$\log(-x)$
$-1/10^2$	-2
$-1/10$	-1
-1	0
-10	1
-100	2



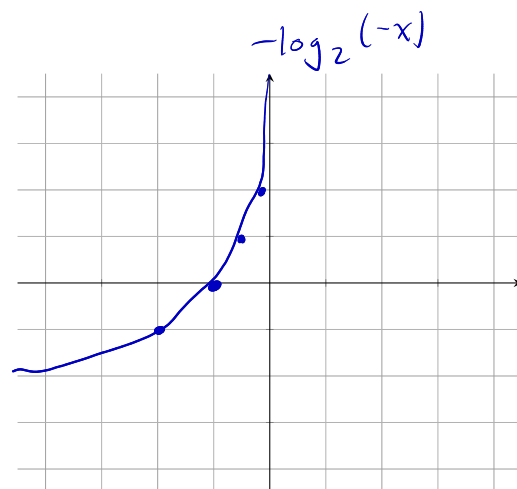
Example. Graph $\ln(x)$

x	$\ln(x)$
$0.368 \approx e^{-2}$	-2
$0.35 \approx e^{-1}$	-1
1	0
e	1
e^2	2



Example. Graph $-\log_2(-x)$

x	$-\log_2(-x)$
-2^2	-2
-2	-1
-1	0
$-1/2$	1
$-1/2^2$	2



Example. Evaluate the following:

$$f(x) = \ln(x); \quad f(e^{-3x})$$

$$f(e^{-3x}) = \ln(e^{-3x})$$

$$= -3x$$

$$f(x) = 5^x; \quad f(\log_5(10))$$

$$f(\log_5(10)) = 5^{\log_5(10)}$$

$$= 10$$

Properties of exponents and logarithms: Assume $a > 0$:

$$a^y = x$$

$$\log_a(x) = y$$

$$a^1 = a$$

$$\log_a(a) = 1$$

$$a^0 = 1$$

$$\log_a(1) = 0$$

$$a^x a^y = a^{x+y}$$

$$\log_a(xy) = \log_a(x) + \log_a(y)$$

$$\frac{a^x}{a^y} = a^{x-y}$$

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

$$a^{xy} = (a^x)^y$$

$$\log_a(x^y) = y \log_a(x)$$

$$a^{\log_a(x)} = x$$

$$\log_a(a^x) = x$$