8.2: Reflexivity, Symmetry, and Transitivity

Definition.
Let R be a relation on a set A.

1. R is **reflexive** if, and only if, for every $x \in A$, $x \, R \, x$.
$$\forall x \in A, (x, x) \in R$$
2. R is **symmetric** if, and only if, for every $x, y \in A$, if $x \, R \, y$ then $y \, R \, x$.
$$\forall x, y \in A, \text{ if } (x, y) \in R \text{ then } (y, x) \in R$$
3. R is **transitive** if, and only if, for every $x, y, z \in A$, if $x \, R \, y$ and $y \, R \, z$, then $x \, R \, z$.
$$\forall x, y, z \in A, \text{ if } (x, y) \in R \text{ and } (y, z) \in R \text{ then } (x, z) \in R$$

Note: A relation R is

not reflexive $\Leftrightarrow \exists x \in A$ such that $x \not R \, x$
or $(x, x) \notin R$.

not symmetric $\Leftrightarrow \exists x, y \in A$ such that $x \, R \, y$ but $y \not R \, x$
or $(x, y) \in R$ but $(y, x) \notin R$.

not transitive $\Leftrightarrow \exists x, y, z \in A$ such that $x \, R \, y$ and $y \, R \, z$, but $x \not R \, z$
or $(x, y) \in R$ and $(y, z) \in R$, but $(x, z) \notin R$.

irreflexive $\Leftrightarrow \forall x \in A, x \not R \, x$

asymmetric $\Leftrightarrow \forall x, y \in A$, if $x \, R \, y$ then $y \not R \, x$

intransitive $\Leftrightarrow \forall x, y, z \in A$, if $x \, R \, y$ and $y \, R \, z$, then $x \not R \, z$

Example. Define a relation R on $\mathbb{R}$ as follows:

$$x \, R \, y \Leftrightarrow x = y.$$

Is R reflexive? Is R symmetric? Is R transitive?

Example. Define a relation R on $\mathbb{R}$ as follows:

$$x \, R \, y \Leftrightarrow x < y.$$

Is R reflexive? Is R symmetric? Is R transitive?

Example. Let $A = \{0, 1, 2, 3\}$ and define relations R, S, and T on A as follows:

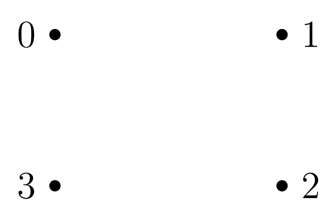
$$\begin{aligned} R &= \{(0,0),(0,1),(0,3),(1,0),(1,1),(2,2),(3,0),(3,3)\} \\ S &= \{(0,0),(0,2),(0,3),(2,3)\} \\ T &= \{(0,1),(2,3)\} \end{aligned}$$

For each relation, draw the directed graph, then identify if it is reflexive, symmetric, and/or transitive.

R

S

T



Example. Define a relation T on $\mathbb{Z}$ as follows:

$$\forall m, n \in \mathbb{Z},\ m\,T\,n \Leftrightarrow 3 \mid (m-n).$$

This relation is called **congruence modulo 3.**
Is T reflexive, symmetric, and/or transitive?

Example. Define a relation S on $\mathbb{R}$ as follows:

$$\forall x, y \in \mathbb{R},\ x\,S\,y \Leftrightarrow |x| + |y| = 1.$$

Is S reflexive, symmetric, and/or transitive?