

## 1.1: Solutions of Linear Equations and Inequalities in One Variable

### Definition.

A **function**  $f$  is a special relation between  $x$  and  $y$  such that each input  $x$  results in *at most* one  $y$ . The symbol  $f(x)$  is read “ $f$  of  $x$ ” and is called the **value of  $f$  at  $x$**

**Example.** Let  $f(x) = 4x - 1$ . Evaluate the following:

$$\begin{aligned} f(1) &= 4(1) - 1 \\ &= 4 - 1 \\ &= \boxed{3} \end{aligned}$$

$$\begin{aligned} f\left(\frac{1}{2}\right) &= 4\left(\frac{1}{2}\right) - 1 \\ &= \frac{4}{2} - 1 \\ &= 2 - 1 = \boxed{1} \end{aligned}$$

$$\begin{aligned} f(-2) &= 4(-2) - 1 \\ &= -8 - 1 \\ &= \boxed{-9} \end{aligned}$$

$$\begin{aligned} f(0) &= 4(0) - 1 \\ &= 0 - 1 \\ &= \boxed{-1} \end{aligned}$$

$$f(\odot) = \boxed{4\odot - 1}$$

$$\begin{aligned} f(f(x)) &= 4(f(x)) - 1 \\ &= 4(4x - 1) - 1 \\ &= 16x - 4 - 1 = \boxed{16x - 5} \end{aligned}$$

### Composite Functions:

Let  $f$  and  $g$  be functions of  $x$ . Then, the **composite functions**  $g$  of  $f$  (denoted  $g \circ f$ ) and  $f$  of  $g$  (denoted  $f \circ g$ ) are defined as:

$$(g \circ f)(x) = g(f(x))$$

$$(f \circ g)(x) = f(g(x))$$

**Example.** Let  $g(x) = x - 1$ . Find:

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= (f(x)) - 1 \\ &= (4x - 1) - 1 \\ &= \boxed{4x - 2} \end{aligned}$$

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= 4(g(x)) - 1 \\ &= 4(x - 1) - 1 \\ &= 4x - 4 - 1 \\ &= \boxed{4x - 5} \end{aligned}$$

## Operations with Functions:

Let  $f$  and  $g$  be functions of  $x$  and define the following:

Sum	$(f + g)(x) = f(x) + g(x)$
Difference	$(f - g)(x) = f(x) - g(x)$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$
Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ if $g(x) \neq 0$

### Definition.

An **expression** is a meaningful string of numbers, variables and operations:

$$3x - 2$$

An **equation** is a statement that two quantities or algebraic expressions are equal:

$$3x - 2 = 7$$

A **solution** is a value of the variable that makes the equation true:

$$3(3) - 2 = 7$$

$$9 - 2 = 7$$

$$7 = 7$$

A **solution set** is the set of ALL possible solutions of an equation:

$3x - 2 = 7$  only has the solution  $x = 3$ ,

$2(x - 1) = 2x - 2$  is true for all possible values of  $x$ .

## Properties of Equality:

**Substitution Property:** The equation formed by substituting one expression for an equal expression is equivalent to the original equation:

$$\begin{aligned}3(x - 3) - \frac{1}{2}(4x - 18) &= 4 \\3x - 9 - 2x + 9 &= 4 \\x &= 4\end{aligned}$$

**Addition Property:** The equation formed by adding the same quantity to both sides of an equation is equivalent to the original equation:

$$\begin{array}{ll}x - 4 = 6 & x + 5 = 12 \\x - 4 + 4 = 6 + 4 & x + 5 + (-5) = 12 + (-5) \\x = 10 & x = 7\end{array}$$

**Multiplication Property:** The equation formed by multiplying both sides of an equation by the same *nonzero* quantity is equivalent to the original equation:

$$\begin{array}{ll}\frac{1}{3}x = 6 & 5x = 20 \\3\left(\frac{1}{3}x\right) = 3(6) & \frac{5x}{5} = \frac{20}{5} \\x = 18 & x = 4\end{array}$$

## Solving a linear equation:

Using the properties of equality above, we can solve any linear equation in 1 variable:

**Example.** Solve  $\frac{3x}{4} + 3 = \frac{x-1}{3}$

1. Eliminate fractions:
2. Remove/evaluate parenthesis:
3. Use addition property to isolate the variable to one side:
4. Use multiplication property to isolate variable:
5. Verify solution via substitution:

$$12\left(\frac{3x}{4} + 3\right) = 12\left(\frac{x-1}{3}\right)$$

$$9x + 36 = 4x - 4$$

$$9x + 36 - 36 - 4x = 4x - 4 - 36 - 4x$$

$$\frac{5x}{5} = \frac{-40}{5}$$

$$\underbrace{\frac{3(-8)}{4} + 3}_{-6 + 3 = -3} \stackrel{?}{=} \underbrace{\frac{(-8) - 1}{3}}_{\frac{-9}{3} = -3}$$

**Example.** Solve the following:

$$\begin{aligned} (2)(3)\left(\frac{3x+1}{2}\right) &= \left(\frac{x}{3} - 3\right)(2)(3) \\ 3(3x+1) &= 2x - (2)(3)3 \\ 9x + 3 - 2x &= 2x - 18 - 2x \\ 7x + 3 - 3 &= -18 - 3 \\ 7x &= -21 \\ \frac{7x}{7} &= \frac{-21}{7} \\ \boxed{x = -3} \end{aligned}$$

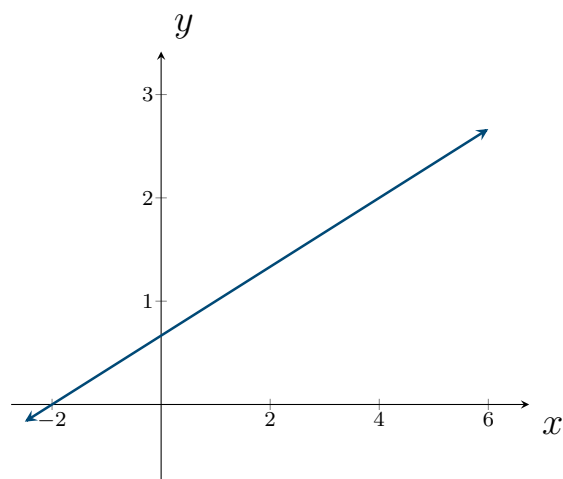
Verification of solution omitted

$$\begin{aligned} (x-3)\left(\frac{2x-1}{x-3}\right) &= \left(4 + \frac{5}{x-3}\right)(x-3) \\ 2x-1 &= 4(x-3) + 5 \\ 2x-1 - 4x &= 4x-12+5-4x \\ -2x - 1 + 1 &= -7 + 1 \\ \frac{-2x}{-2} &= \frac{-6}{-2} \\ \boxed{x = 3} \end{aligned}$$

When verifying this solution, we see that both sides are undefined at  $x=3$ .

**Example.** Solve  $-2x + 6y = 4$  for  $y$

$$\begin{aligned} \underline{-2x} + 6y &= 4 \\ \underline{+2x} &+ 2x \\ 6y &= 2x + 4 \\ \underline{6} &\quad \underline{6} \\ y &= \frac{2x}{6} + \frac{4}{6} \\ y &= \frac{x}{3} + \frac{2}{3} \end{aligned}$$



**Example.** Suppose that the relationship between a firm's profit,  $P$ , and the number of items sold,  $x$ , can be described by the equation

$$5x - 4P = 1200$$

a) How many units must be produced and sold for the firm to make a profit of \$150?

$$\begin{aligned} P &= \$150, \text{ Find } x \\ 5x - 4(150) &= 1200 \\ 5x - 600 &+ 600 = 1200 + 600 \\ \underline{5x} &= \underline{1800} \\ x &= 360 \end{aligned}$$

b) Solve this equation for  $P$  in terms of  $x$ . Then, find the profit when 240 units are sold.

$$\begin{aligned} \underline{5x} - 4P &= 1200 \\ \underline{-5x} &+ 5x \\ -4P &= -5x + 1200 \\ \underline{-4} &\quad \underline{-4} \\ P &= \frac{-5x}{-4} + \frac{1200}{-4} \\ P &= \frac{5x}{4} - 300 \end{aligned}$$

$$\begin{aligned} x &= 240 \\ P &= \frac{5(240)}{4} - 300 \\ &= 5(60) - 300 \\ &= 300 - 300 \\ P &= 0 \end{aligned}$$

**Break-even point**

**Example.** Jill Ball has \$90,000 to invest. She has chosen one relatively safe investment fund that has an annual yield of 10% and another riskier one that has a 15% annual yield. How much should she invest in each fund if she would like to earn \$10,000 in one year from her investments?

	Investment Amt	Return
10%	$90,000 - x$	$0.10(90,000 - x) = 9,000 - 0.1x$
15%	$x$	$0.15x$

$$-9000 + 9000 - 0.1x + 0.15x = 10000 - 9000$$

$$\frac{0.05x}{0.05} = \frac{1000}{0.05} \Rightarrow x = 20,000$$

$\$70,000 @ 10\%$   
 $\$20,000 @ 15\%$

**Example.** A woman making \$2,000 per month has her salary reduced by 10% because of sluggish sales. One year later, after a dramatic improvement in sales, she is given a 20% raise over her reduced salary. Find her salary after the raise. What percent change is this from the \$2,000 per month?

Salary after raise  
 $2000 (1 - 0.10) (1 + 0.20) = 2000 (0.9) (1.2) = \$2,160$

20% increase  
 10% decrease

Percent change

$$\frac{2160 - 2000}{2000} = 0.08 = 8\%$$

or  $(0.9)(1.2) = 1.08$   
 100% of original salary  
 8% increase

**Definition.**

An **inequality** is a statement that one quantity is greater than (or less than) another quantity.

**Properties of Inequalities**

**Substitution Property:** The inequality formed by substituting one expression for an equal expression is equivalent to the original inequality:

$$5x - 4x + 2 < 6$$

$$x < 4 \Rightarrow \text{The solution set is } \{x : x < 6\}$$

**Addition Property:** The inequality formed by adding the same quantity to both sides of an inequality is equivalent to the original inequality:

$$x - 4 < 6$$

$$x - 4 + 4 < 6 + 4$$

$$x < 10$$

$$x + 5 \geq 12$$

$$x + 5 + (-5) \geq 12 + (-5)$$

$$x \geq 7$$

**Multiplication Property** The inequality formed by multiplying both sides of an inequality by the same *positive* quantity is equivalent to the original inequality. The direction of the inequality is flipped when multiplying by a *negative* quantity:

$$\frac{1}{3}x > 6$$

$$3\left(\frac{1}{3}x\right) > 3(6)$$

$$x > 18$$

$$5x - 5 + 5 \leq 6x + 20 + 5$$

$$\frac{-x}{-1} \leq \frac{25}{-1}$$

$$x \geq -25$$

**Example.** Solve

$$-x + 8 \leq 2x - 4$$

first by gathering the  $x$  variable on the left, then again on the right. See that the multiplication property holds in both cases. Plot the solution set on a numberline.

$$-x + 8 + x \leq 2x - 4 + x$$

$$8 + 4 \leq 3x - 4 + 4$$

$$\frac{12}{3} \leq \frac{3x}{3}$$

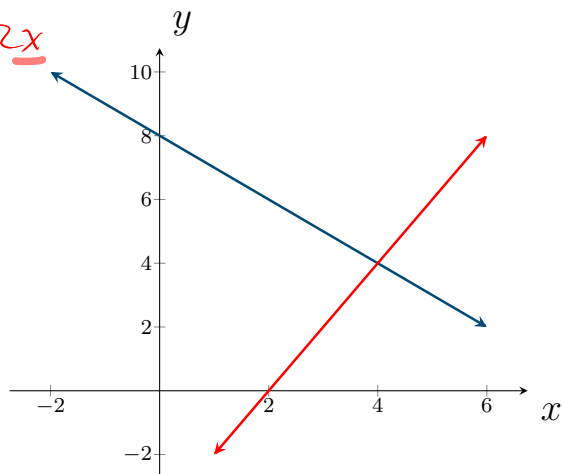
$$4 \leq x$$

$$-x + 8 - 2x \leq 2x - 4 - 2x$$

$$-3x + 8 - 8 \leq -4 - 8$$

$$\frac{-3x}{-3} \leq \frac{-12}{-3}$$

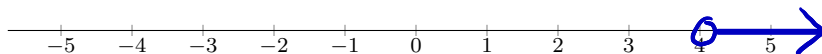
$$x \geq 4$$



\* Note:

-The inequality on the right changed direction in the last step because we divided by a negative number

-The answers above are equivalent. They are only formatted differently.



**Example.** Plot the following inequalities:

$$x \leq 2$$

$$x > -3$$

