## 2.4: Limits

**Example.** Suppose that the position function of a maglev train (in feet) is given by

$$s(t) = 4t^2, \qquad (0 \leq t \leq 30)$$

Using the position function, compute the *average* velocity of the train

on the interval $[t, 2]$

| $t$ | 1.5 | 1.9 | 1.99 | 1.999 | 1.9999 |
|---|---|---|---|---|---|
| | | | | | |

on the interval $[2, t]$

| $t$ | 2.5 | 2.1 | 2.01 | 2.001 | 2.0001 |
|---|---|---|---|---|---|
| | | | | | |

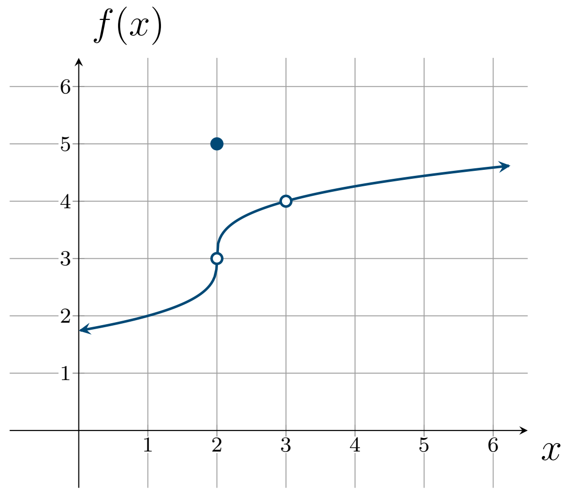
What do the tables above suggest about *instantaneous* velocity of the train at $t = 2$?

**Definition. (Limit of a Function)**
The function $f$ has the **limit** $L$ as $x$ approaches $a$, written

$$\lim_{x \to a} f(x) = L$$

if the value of $f(x)$ can be made as close to the number $L$ as we please by taking $x$ sufficiently close to (but not equal to) $a$.

**Example.** Using the graph of $f$, determine the following values:




$f(1)$ and $\lim_{x \to 1} f(x)$

$f(2)$ and $\lim_{x \to 2} f(x)$

$f(3)$ and $\lim_{x \to 3} f(x)$

**Example.** Find the limit of the following functions at the value specified: Graphs

$$f(x) = x^3 \quad \text{at } x = 2$$

$$g(x) = \begin{cases} x+2, & x \neq 1 \\ 1, & x = 1 \end{cases} \quad \text{at } x = 1$$

$$h(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases} \quad \text{at } x = 0$$

$$j(x) = \frac{1}{(x-1)^2} \quad \text{at } x = 1$$

$$k(x) = 4 \quad \text{at } x = 0$$

**Theorem 1: Properties of Limits**
Suppose

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M$$

Then

1. $\lim\limits_{x\to a}[f(x)]^r = \left[\lim\limits_{x\to a} f(x)\right]^r$ where $r$ is a positive constant
2. $\lim\limits_{x\to a} cf(x) = c\lim\limits_{x\to a} f(x)$ where $c$ is a real number
3. $\lim\limits_{x\to a}[f(x) \pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x) = L \pm M$
4. $\lim\limits_{x\to a}[f(x)g(x)] = \left[\lim\limits_{x\to a} f(x)\right]\left[\lim\limits_{x\to a} g(x)\right] = LM$
5. $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{L}{M}$ provided $M \neq 0$

**Example.** Use the above theorem to evaluate the following limits:

$$\lim_{x\to 1}\left(5x^{3/2} - 2\right)$$

$$\lim_{x\to 3}\frac{2x^3\sqrt{x^2+7}}{x+1}$$

Suppose that $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

has an **indeterminate form** of $\frac{0}{0}$. To evaluate such a limit, we replace the given function with a function that's equivalent everywhere except at $x = a$, and then evaluate the limit.

**Example.** Evaluate the following

$$\lim_{t \to 2} \frac{4t^2 - 16}{t - 2}$$

$$\lim_{h \to 0} \frac{\sqrt{4 + h} - 2}{h}$$

Suppose that $\lim_{x \to a} f(x) = L$ with $L \neq 0$ and $\lim_{x \to a} g(x) = 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

does not exist. We can further specify if this limit tends towards $-\infty$ or $\infty$.

**Example.** Evaluate the following  Graphs

$$\lim_{x \to 1} \frac{x}{x-1}$$

$$\lim_{x \to 3} \frac{1}{(x-3)^2}$$

$$\lim_{x \to -2} \frac{x-2}{x^2-4}$$

$$\lim_{x \to 2} \frac{x-2}{x^2-4}$$

**Limit of a Function at Infinity**
The function $f$ has the limit $L$ as $x$ increases without bound, written

$$\lim_{x\to\infty} f(x) = L$$

if $f(x)$ can be made arbitrarily close to $L$ by taking $x$ large enough.
The function $f$ has the limit $M$ as $x$ decreases without bound, written

$$\lim_{x\to-\infty} f(x) = M$$

if $f(x)$ can be made arbitrarily close to $M$ by taking $x$ to be negative and sufficiently large enough in absolute value.
When the above limits exist, the equations $y = L$ and/or $y = M$ are called **horizontal asymptotes**.

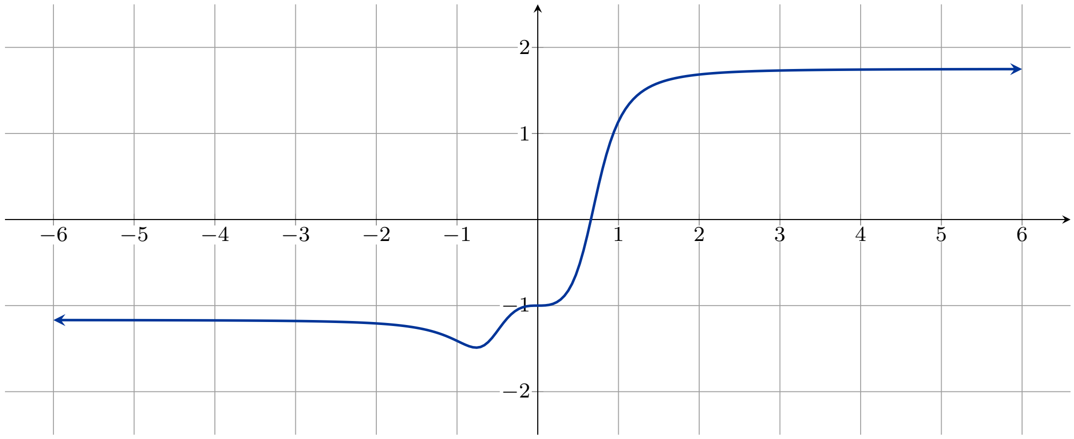
**Example.** Evaluate the following infinite limits

$$\lim_{x\to\infty} \frac{2x^2 + 3x - 4}{x^2 - 7x + 1}$$

$$\lim_{x\to-\infty} \frac{3x^2 + 4}{2x^3}$$

$$\lim_{x\to\pm\infty} \frac{3x^5 + 2x^3 - 4}{x^4 + 4x^2 - 1}$$

Graph




$$\lim_{x\to-\infty} \frac{7x^3-2}{-x^3+\sqrt{25x^6-4}}$$

$$\lim_{x\to\infty} \frac{7x^3-2}{-x^3+\sqrt{25x^6-4}}$$

**Example.** The company *Custom Office* makes a line of executive desks. It is estimated that the total cost of making $x$ *Senior Executive Model* desks is

$$C(x) = 100x + 200,000$$

dollars per year. The average cost of making $x$ desks is given by

$$\overline{C}(x) = \frac{C(x)}{x}$$

Compute $\lim_{x\to\infty} \overline{C}(x)$ and interpret the result.

**Theorem 2**
For all $n > 0$,

$$\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$$

provided that $\frac{1}{x^n}$ is defined.