

# Math 125 Class notes

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To accompany  
*Discrete Mathematics with Applications*  
by Epp

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## 1.1: Variables

### Definition.

A **variable** is a placeholder for something which may or may not be unknown.

**Example.** Is there a number with the following property: doubling it and adding 3 gives the same result as squaring it?

- Is there a number  $x$  with the property that  $2x + 3 = x^2$ ?
- Is there a number  $\square$  with the property that  $2 \cdot \square + 3 = \square^2$ ?

**Example.** No matter what number might be chosen, if it is greater than 2, then its square is greater than 4.

- No matter what number  $n$  might be chosen, if  $n$  is greater than 2,  
then  $n^2$  is greater than 4.

**Example.** Use variables to rewrite the following sentences:

Are there numbers with the property that the sum of their squares equals the square of their sum?

Given any real number, its square is nonnegative.

### Definition.

- A **universal statement** says that a certain property is true for all elements in a set.
- A **conditional statement** says that if one thing is true, then some other thing also has to be true.
- Given a property that may or may not be true, an **existential statement** says that there is at least one thing for which the property is true.

**Definition.**

A **universal conditional statement** is both universal and conditional:

For every animal  $a$ , if  $a$  is a dog, then  $a$  is a mammal.

Conditional statements can be rewritten in ways that make them appear more to be purely universal or purely conditional:

If  $a$  is a dog, then  $a$  is a mammal.

All dogs are mammals

**Example.** Rewrite the following universal condition statement:

For every real number  $x$ , if  $x$  is nonzero then  $x^2$  is positive.

If a real number is nonzero, then its square \_\_\_\_\_.

For every nonzero real number  $x$ , \_\_\_\_\_.

If  $x$  \_\_\_\_\_, then \_\_\_\_\_.

The square of any nonzero real number is \_\_\_\_\_.

All nonzero real numbers have \_\_\_\_\_.

### Definition.

A **universal existence statement** is a statement that is universal because its first part says that a certain property is true for all objects of a given type, and it is existential because its second part asserts the existence of something:

Every real number has an additive inverse.

In the above example, note that the particular additive inverse depends on the given real number:

For every real number  $r$ , there is an additive inverse for  $r$ .

**Example.** Rewrite the following universal existence statement:

Every pot has a lid

All pots \_\_\_\_\_.

For ever pot  $P$ , there is \_\_\_\_\_.

For every pot  $P$ , there is a lid  $L$  such that \_\_\_\_\_.

### Definition.

An **existential universal statement** is a statement that is existential because its first part asserts that a certain object exists and is universal because its second part says that the object satisfies a certain property for all things of a certain kind:

There is a positive integer that is less than or equal to every positive integer.

The number one satisfies the above statement, which can also be rewritten:

There is a positive integer  $m$  that is less than or equal to every positive integer.

**Example.** Rewrite the following existence universal statement:

There is a person in my class who is at least as old as every person in my class.

Some \_\_\_\_\_ is at least as old as \_\_\_\_\_.

There is a person  $p$  in my class such that  $p$  is \_\_\_\_\_.

There is a person  $p$  in my class with the property that for every person  $q$  in my class,  $p$  is \_\_\_\_\_.

## 1.2: The Language of Sets

### Definition.

- A **set** is a collection of objects.
- If  $S$  is a set, then we use
  - $x \in S$  to denote that the element  $x$  is in the set  $S$ .
  - $x \notin S$  to denote that the element  $x$  is *not* in the set  $S$ .
- The **set-roster notation** is used to denote all elements in a set between braces:

$$S = \{1, 2, \dots, 100\}$$

Here, we see that  $67 \in S$ , but  $1337 \notin S$ .

- The **axiom of extension** says that a set is completely determined by what its elements are – not the order in which they are listed.

### Example.

Let  $A = \{1, 2, 3\}$ ,  $B = \{3, 1, 2\}$ , and  $C = \{1, 1, 2, 3, 3, 3\}$ . What are the elements of  $A$ ,  $B$ , and  $C$ ? How are  $A$ ,  $B$ , and  $C$  related?

Is  $\{0\} = 0$ ?

How many elements are in the set  $\{1, \{1\}\}$ ?

For each nonnegative integer  $n$ , let  $U_n = \{n, -n\}$ . Find  $U_1$ ,  $U_2$ , and  $U_0$ .

Certain sets of numbers are so frequently referred to that they are given special names and symbols:

**N** or  $\mathbb{N}$  The set of all **natural numbers**

**Z** or  $\mathbb{Z}$  The set of all **integers**

**Q** or  $\mathbb{Q}$  The set of all **rational numbers**, or quotient of integers

**R** or  $\mathbb{R}$  The set of all **real numbers**

*Note:* We may additionally use superscripts to indicate further properties of these sets:

$\mathbb{Z}^+$  or  $\mathbb{Z}^{>0}$  The set of *positive* integers

$\mathbb{Q}^-$  or  $\mathbb{Q}^{<0}$  The set of *negative* rational numbers

$\mathbb{R}^{nonneg}$  or  $\mathbb{R}^{\geq 0}$  The set of *nonnegative* real numbers

*Note:* Different sources denote the natural numbers  $\mathbb{N}$  as  $\mathbb{Z}^+$  or  $\mathbb{Z}^{\geq 0}$ .

### Definition. (Set-Builder Notation)

Let  $S$  be a set and let  $P(x)$  be a property that elements of  $S$  may or may not satisfy. We may define a new set to be **the set of all elements  $x$  in  $S$  such that  $P(x)$  is true**. We denote this set as follows:

$$\{x \in S \mid P(x)\}$$

↑                      ↑  
the set of all            such that

**Example.** Describe each of the following sets:

$$\{x \in \mathbb{R} \mid -2 < x < 5\}$$

$$\{x \in \mathbb{Z} \mid -2 < x < 5\}$$

$$\{x \in \mathbb{Z}^+ \mid -2 < x < 5\}$$

### Definition.

If  $A$  and  $B$  are sets, then  $A$  is called a **subset** of  $B$ , written  $A \subseteq B$ , if, and only if, every element of  $A$  is also an element of  $B$ :

$A \subseteq B$  means that for every element  $x$ , if  $x \in A$ , then  $x \in B$ .

$A \not\subseteq B$  means that there is at least one element  $x$ , such that  $x \in A$  and  $x \notin B$ .

$A$  is a **proper subset** of  $B$  if, and only if, every element of  $A$  is in  $B$ , but there is at least one element of  $B$  that is not in  $A$ :

$A \subsetneq B$  means that for every element  $x$ , if  $x \in A$ , then  $x \in B$ ,  
and there exists  $x \in B$  such that  $x \notin A$ .

**Example.** Let  $A = \mathbb{Z}^+$ ,  $B = \{n \in \mathbb{Z} \mid 0 \leq n \leq 100\}$ , and  $C = \{100, 200, 300, 400, 500\}$ . Evaluate the truth and falsity of each of the following statements.

$$B \subseteq A$$

$$C \text{ is a proper subset of } A$$

$$C \text{ and } B \text{ have at least one element in common}$$

$$C \subseteq B$$

$$C \subseteq C$$

**Example.** Determine which of the following statements are true:

$$2 \in \{1, 2, 3\}$$

$$\{2\} \in \{1, 2, 3\}$$

$$2 \subseteq \{1, 2, 3\}$$

$$\{2\} \subseteq \{1, 2, 3\}$$

$$\{2\} \subseteq \{\{1\}, \{2\}\}$$

$$\{2\} \in \{\{1\}, \{2\}\}$$

### Definition.

Given elements  $a$  and  $b$ , the symbol  $(a, b)$  denotes the **ordered pair** consisting of  $a$  and  $b$  together with the specification that  $a$  is the first element of the pair, and  $b$  is the second element. Two ordered pairs  $(a, b)$  and  $(c, d)$  are equal if, and only if,  $a = c$  and  $b = d$ :

$$(a, b) = (c, d) \text{ means that } a = c \text{ and } b = d.$$

### Example.

Is  $(1, 2) = (2, 1)$ ?

Is  $\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right)$ ?

### Definition.

Let  $n \in \mathbb{N}$  and let  $x_1, x_2, \dots, x_n$  be (not necessarily distinct) elements. The **ordered  $n$ -tuple**,  $(x_1, x_2, \dots, x_n)$ , consists of  $x_1, x_2, \dots, x_n$  together with the ordering: first  $x_1$ , then  $x_2$ , and so forth up to  $x_n$ . An ordered 2-tuple is called an **ordered pair**, and an ordered 3-tuple is called an **ordered triple**.

Two ordered  $n$ -tuples  $(x_1, x_2, \dots, x_n)$  and  $(y_1, y_2, \dots, y_n)$  are **equal** if, and only if,  $x_1 = y_1, x_2 = y_2, \dots$ , and  $x_n = y_n$ :

$$(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n) \iff x_1 = y_1, x_2 = y_2, \dots, x_n = y_n.$$

### Example.

Is  $(1, 2, 3, 4) = (1, 2, 4, 3)$ ?

Is  $\left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right)$ ?

**Definition.**

Given sets  $A_1, A_2, \dots, A_n$ , the **Cartesian product** of  $A_1, A_2, \dots, A_n$ , denoted

$$A_1 \times A_2 \times \cdots \times A_n$$

is the set of all ordered  $n$ -tuples  $(a_1, a_2, \dots, a_n)$  where  $a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n$ :

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n\}$$

**Example.** Let  $A = \{x, y\}$ ,  $B = \{1, 2, 3\}$ , and  $C = \{a, b\}$ . Find the following:

$$A \times B$$

$$B \times A$$

$$A \times A$$

How many elements are in  $A \times B$ ,  $B \times A$ , and  $A \times A$ ?

$$(A \times B) \times C$$

$$A \times B \times C$$

Describe  $\mathbb{R} \times \mathbb{R}$

### Definition.

Let  $n \in \mathbb{N}$ . Given a finite set  $A$ , a **string of length  $n$  over  $A$**  is an ordered  $n$ -tuple of elements of  $A$  written without parentheses or commas. The elements of  $A$  are called the **characters** of the string. The **null string** over  $A$  is defined to be the “string” with no characters, often denoted  $\lambda$ , and is said to have length 0. If  $A = \{0, 1\}$ , then a string over  $A$  is called a **bit string**.

**Example.** Let  $A = \{a, b\}$ . List all strings of length 3 over  $A$  with at least two characters that are the same.

### 1.3: The Language of Relations and Functions

#### Definition.

Let  $A$  and  $B$  be sets. A **relation  $R$  from  $A$  to  $B$**  is a subset of  $A \times B$ . Given an ordered pair  $(x, y)$ ,  $x$  is related to  $y$  by  $R$ , written  $x R y$ , if, and only if,  $(x, y)$  is in  $R$ . The set  $A$  is called the **domain** of  $R$  and the set  $B$  is called its **co-domain**.

The notation for a relation  $R$  may be written symbolically as follows:

$$x R y \text{ means that } (x, y) \in R.$$

The notation  $x \not R y$  means that  $x$  is not related to  $y$  by  $R$ :

$$x \not R y \text{ means that } (x, y) \notin R.$$

**Example.** Let  $A = \{1, 2\}$  and  $B = \{1, 2, 3\}$  and define a relation  $R$  from  $A$  to  $B$  as follows; Given any  $(x, y) \in A \times B$ ,

$$(x, y) \in R \text{ means that } \frac{x - y}{2} \text{ is an integer.}$$

State explicitly which ordered pairs are in  $A \times B$  and which are in  $R$

Is  $1 R 3$ ?

Is  $2 R 3$ ?

Is  $2 R 2$ ?

What are the domain and co-domain of  $R$ ?

**Example.** Define a relation  $C$  from  $\mathbb{R}$  to  $\mathbb{R}$  as follows: For any  $(x, y) \in \mathbb{R} \times \mathbb{R}$ ,

$(x, y) \in C$  means that  $x^2 + y^2 = 1$ .

Is  $(1, 0) \in C$ ?

Is  $(0, 0) \in C$ ?

Is  $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \in C$ ?

Is  $-2 \in C \ 0$ ?

Is  $0 \in C \ (-1)$ ?

Is  $1 \in C \ 1$ ?

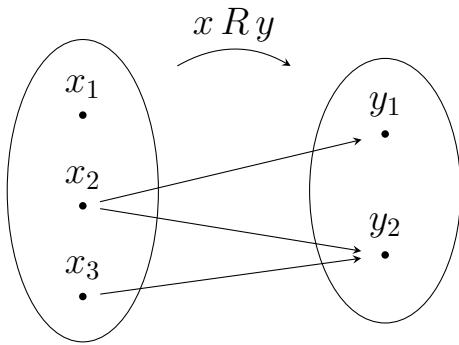
What are the domain and co-domain of  $C$ ?

Draw a graph for  $C$  by plotting the points of  $C$  in the Cartesian plane.

## Definition.

Suppose  $R$  is a relation from set  $A$  to a set  $B$ . The **arrow diagram for  $R$**  is obtained as follows:

1. Represent the elements of  $A$  as points in one region and the elements of  $B$  as points in another region.
2. For each  $x$  in  $A$  and  $y$  in  $B$ , draw an arrow from  $x$  to  $y$  if, and only if,  $x$  is related to  $y$  by  $R$ .

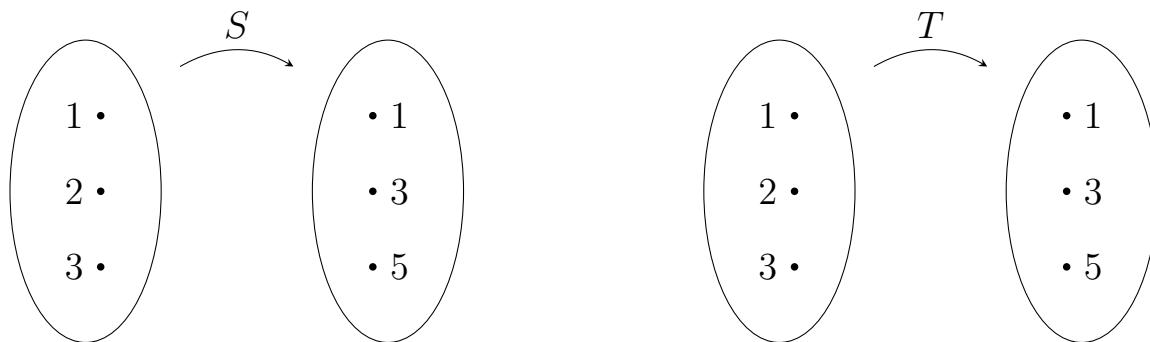


**Example.** Let  $A = \{1, 2, 3\}$  and  $B = \{1, 3, 5\}$  and define relations  $S$  and  $T$  from  $A$  to  $B$  as follows: For every  $(x, y) \in A \times B$ ,

$(x, y) \in S$  means that  $x < y$

$$T = \{(2, 1), (2, 5)\}.$$

Draw arrow diagrams for  $S$  and  $T$



## Definition.

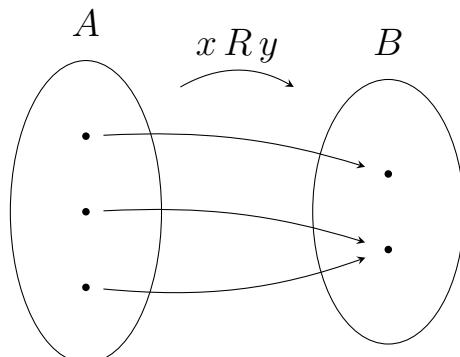
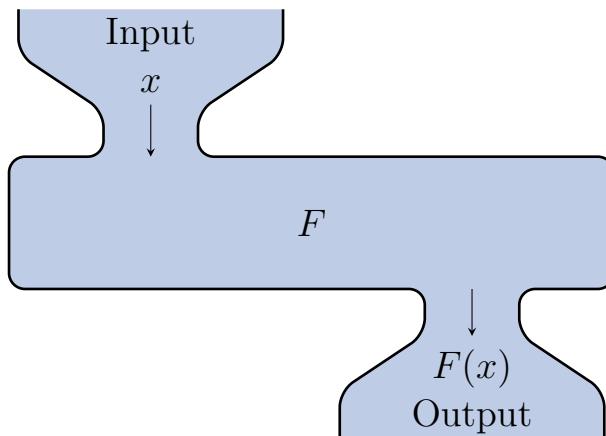
A **function  $F$  from a set  $A$  to a set  $B$**  is a relation with domain  $A$  and co-domain  $B$  that satisfies the following two properties:

1. For every element  $x$  in  $A$ , there is an element  $y$  in  $B$  such that  $(x, y) \in F$ .
2. For all elements  $x$  in  $A$  and  $y$  and  $z$  in  $B$ ,  
if  $(x, y) \in F$  and  $(x, z) \in F$ , then  $y = z$ .

*Note:* A relation from  $A$  to  $B$  is a function if, and only if,

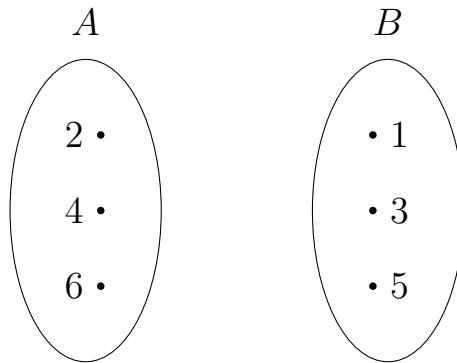
1. Every element of  $A$  is the first element of an ordered pair of  $F$
2. No two distinct ordered pairs in  $F$  have the same first element.

*Note:* If  $A$  and  $B$  are sets and  $F$  is a function from  $A$  to  $B$ , then given any element  $x$  in  $A$ , the unique element in  $B$  that is related to  $x$  by  $F$  is denoted  $F(x)$ , which is read “ $F$  of  $x$ ”.

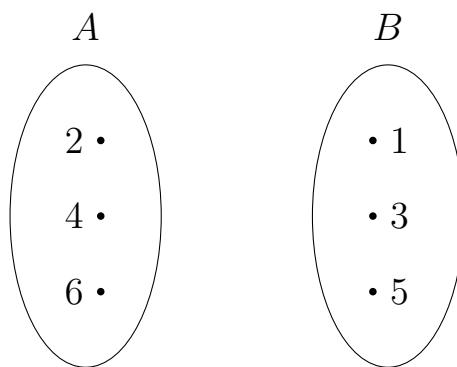


**Example.** Let  $A = \{2, 4, 6\}$  and  $B = \{1, 3, 5\}$ . Which of the relations  $R$ ,  $S$ , and  $T$  defined below are functions from  $A$  to  $B$ ?

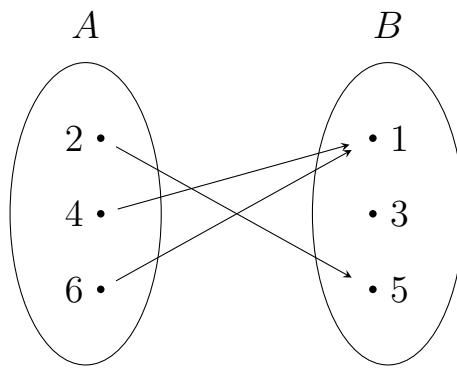
$$R = \{(2, 5), (4, 1), (4, 3), (6, 5)\}$$



For every  $(x, y) \in A \times B$ ,  $(x, y) \in S$  means that  $y = x + 1$ .



$T$  is defined by the arrow diagram



## 2.1: Logical Form and Logical Equivalence

### Definition.

A **statement** (or **proposition**) is a sentence that is true or false, but not both.

**Example.** Determine which of the following are statements:

$$2 + 2 = 4$$

$$2 + 2 = 5$$

$$x^2 + 2 = 11$$

Today is Saturday.

She is a computer science major.

Jane is a computer science major.

## Definition. (Compound Statements)

Let  $p$  and  $q$  be statement variables.

- The **negation** of  $p$  is “not  $p$ ”, and is denoted as  $\sim p$  (or  $\neg p$ )
- The **conjunction** of  $p$  and  $q$  is “ $p$  and  $q$ ”, and is denoted at  $p \wedge q$
- The **disjunction** of  $p$  and  $q$  is “ $p$  or  $q$ ”, and is denoted  $p \vee q$ .
- The **exclusive or** of  $p$  and  $q$  is “ $p$  x-or  $q$ ”, and is denoted  $p \oplus q$  (or  $p$  XOR  $q$ )

The **order of operations** specifies that  $\sim$  is performed first.

**Example.** Consider the following statements:

$$\begin{aligned} p &: \text{It is raining.} \\ q &: \text{It is sunny.} \\ r &: \text{It is cloudy.} \end{aligned}$$

Rewrite the following compound statements in words:

$$\sim p$$

$$p \vee q$$

$$q \wedge r$$

$$q \wedge \sim r$$

$$p \wedge (q \vee r)$$

$$p \oplus q$$

**Definition.**

A **statement form** (or **propositional form**) is an expression made up of statement variables (e.g.,  $p$ ,  $q$ , and  $r$ ), and logical connectives (e.g.  $\sim$ ,  $\wedge$ ,  $\vee$ , and  $\oplus$ ).

The **truth table** for a given statement form displays the truth values that correspond to all possible combinations of truth values for its component statement variables.

**Example.** Let  $p$  and  $q$  be statement variables. Fill out the following truth tables:

$p$	$\sim p$
T	
F	

$p$	$q$	$p \wedge q$	$p \vee q$	$p \oplus q$
T	T			
T	F			
F	T			
F	F			

$p$	$q$	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$	$(p \vee q) \wedge \sim(p \wedge q)$
T	T				
T	F				
F	T				
F	F				

**Example.** Construct a truth table for the statement form  $(p \wedge q) \vee \sim r$ .

**Definition.**

Two *statement forms* are called **logically equivalent** if, and only if, they have identical true values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms  $P$  and  $Q$  is denoted  $P \equiv Q$ .

**Example.** Use truth tables to test if the following statement forms are equivalent:

$$p \text{ and } \sim(\sim p)$$

$$\sim(p \wedge q) \text{ and } \sim p \wedge \sim q$$

**Definition. (De Morgan's Laws)**

The negation of an *and* statement is logically equivalent to the *or* statement in which each component is negated.

The negation of an *or* statement is logically equivalent to the *and* statement in which each component is negated.

**Example.** Use truth tables to show that the following statement forms are equivalent:

$$\sim(p \wedge q) \text{ and } \sim p \vee \sim q$$

$$\sim(p \vee q) \text{ and } \sim p \wedge \sim q$$

**Example.** Using De Morgan's law to write the negation of the following statements:

Jim is at least 6 feet tall and weighs at least 200 pounds.

The bus was late or Tom's watch was slow.

$$-1 < x \leq 4$$

**Definition.**

A **tautology** is a statement form that is always true.

A **contradiction** is a statement form that is always false.

**Example.** Complete the truth tables for  $p \wedge \sim p$  and  $p \vee \sim p$

**Example.** Let **t** be a tautology, and **c** be a contradiction. Show that  $p \wedge \mathbf{t} \equiv p$  and  $p \wedge \mathbf{c} \equiv \mathbf{c}$

### Theorem 2.1.1 Logical Equivalences (p 49)

Given any statement variables  $p$ ,  $q$ , and  $r$ , a tautology  $\mathbf{t}$  and a contradiction  $\mathbf{c}$ , the following logical equivalences hold:

1. Commutative laws:

$$p \wedge q \equiv q \wedge p$$

$$p \vee q \equiv q \vee p$$

2. Associative laws:

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

3. Distributive laws:

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

4. Identity laws:

$$p \wedge \mathbf{t} \equiv p$$

$$p \vee \mathbf{c} \equiv p$$

5. Negation laws:

$$p \vee \sim p \equiv \mathbf{t}$$

$$p \wedge \sim p \equiv \mathbf{c}$$

6. Double negative law:

$$\sim (\sim p) \equiv p$$

7. Idempotent laws:

$$p \wedge p \equiv p$$

$$p \vee p \equiv p$$

8. Universal bound laws:

$$p \vee \mathbf{t} \equiv \mathbf{t}$$

$$p \wedge \mathbf{c} \equiv \mathbf{c}$$

9. De Morgan's laws:

$$\sim (p \wedge q) \equiv \sim p \vee \sim q$$

$$\sim (p \vee q) \equiv \sim p \wedge \sim q$$

10. Absorption laws:

$$p \wedge (p \vee q) \equiv p$$

$$p \wedge (p \vee q) \equiv p$$

11. Negations of  $\mathbf{t}$  and  $\mathbf{c}$ :

$$\sim \mathbf{t} \equiv \mathbf{c}$$

$$\sim \mathbf{c} \equiv \mathbf{t}$$

## 2.2: Conditional Statements

### Definition.

If  $p$  and  $q$  are statement variables, the **conditional** of  $q$  by  $p$  is “If  $p$  then  $q$ ”, or “ $p$  implies  $q$ ” and is denoted by  $p \rightarrow q$ . It is false when  $p$  is true and  $q$  is false; otherwise it is true. We call  $p$  the **hypothesis** (or **antecedent**) of the conditional and  $q$  the **conclusion** (or **consequent**).

A conditional statement that is always true because the hypothesis is false is called **vacuously true**.

If  $\underbrace{4,686 \text{ is divisible by } 6}_{\text{hypothesis}}$ , then  $\underbrace{4,686 \text{ is divisible by } 3}_{\text{conclusion}}$

**Example.** Consider the following statement:

If Lander is open, then we will have class.

Create the truth table for  $p \rightarrow q$

$p$	$q$	$p \rightarrow q$
T	T	
T	F	
F	T	
F	F	

*Note:* The **order of operations** states that  $\rightarrow$  is performed last

**Example.** Create the truth table for  $p \vee \sim q \rightarrow \sim p$ .

$p$	$q$	$\sim q$	$p \vee \sim q$	$\sim p$	$p \vee \sim q \rightarrow \sim p$
T	T	F	T	F	F
T	F	T	T	F	F
F	T	F	F	T	T
F	F	T	T	T	T

**Example.** Use a truth table to show that  $p \vee q \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$

$p$	$q$	$r$	$p \vee q$	$p \vee q \rightarrow r$	$p \rightarrow r$	$q \rightarrow r$	$(p \rightarrow r) \wedge (q \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	F	F
T	F	T	T	T	F	T	F
T	F	F	T	T	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F
F	F	T	F	F	T	T	T
F	F	F	F	F	T	F	F

**Definition.**

The **negation** of “if  $p$  then  $q$ ” is logically equivalent to “ $p$  and not  $q$ ”:

$$\sim(p \rightarrow q) \equiv p \wedge \sim q$$

**Example.** Write negations for each of the following statements:

If my car is in the repair shop, then I cannot get to class.

If Sara lives in Athens, then she lives in Greece.

**Definition.**

The **contrapositive** of a conditional statement of the form “If  $p$  then  $q$ ” is

$$\text{If } \sim q \text{ then } \sim p : \sim q \rightarrow \sim p$$

A conditional statement is logically equivalent to its contrapositive.

**Example.** Write each of the following statements in its equivalent contrapositive form:

If Howard can swim across the lake, then Howard can swim to the island.

If today is Easter, then tomorrow is Monday.

### **Definition.**

Suppose a conditional statement of the form “If  $p$  then  $q$ ” is given.

- The **converse** is “If  $q$  then  $p$ ”:  $q \rightarrow p$
- The **inverse** is “If  $\sim p$  then  $\sim q$ ”:  $\sim p \rightarrow \sim q$

**Example.** Write the converse and inverse of each of the following statements:

If Howard can swim across the lake, then Howard can swim to the island.

**Converse:**

**Inverse:**

If today is Easter, then tomorrow is Monday.

**Converse:**

**Inverse:**

*Note:*

1. A conditional statement and its converse are *not* logically equivalent.
2. A conditional statement and its inverse are *not* logically equivalent.
3. The converse and the inverse of a conditional statement are logically equivalent to each other.

**Definition.**

If  $p$  and  $q$  are statements,  $p$  **only if**  $q$  means “if not  $q$  then not  $p$ ”:

$$\sim q \rightarrow \sim p \equiv p \rightarrow q$$

**Example.** Rewrite the following statement in if-then form in two ways, one of which is the contrapositive of the other:

John will break the world’s record for the mile run only if he runs the mile in under four minutes.

$$\sim q \rightarrow \sim p$$

$$p \rightarrow q$$

*Note:*

1. “ $p$  only if  $q$ ” does *not* mean  $p$  if  $q$
2. It is possible for “ $p$  only if  $q$ ” to be true at the same time that “ $p$  if  $q$ ” is false.

e.g.: If John runs a mile in under four minutes, he still might not be fast enough to break the record.

### Definition.

Given statement variables  $p$  and  $q$ , the **biconditional of  $p$  and  $q$**  is “ $p$  if, and only if,  $q$ ” and is denoted  $p \leftrightarrow q$ . It is true if both  $p$  and  $q$  have the same truth values and is false otherwise. The words *if and only if* are sometimes abbreviated **iff**.

*Note:* The **order of operations** states that  $\leftrightarrow$  is coequal with  $\rightarrow$

**Example.** Create the truth table for  $p \leftrightarrow q$

$p$	$q$	$p \leftrightarrow q$
T	T	
T	F	
F	T	
F	F	

### Order of Operations for Logical Operators

- $\sim$  Evaluate negations first
- $\wedge, \vee$  Evaluate  $\wedge$  and  $\vee$  second. When both present, parentheses may be needed.
- $\rightarrow, \leftrightarrow$  Evaluate  $\rightarrow$  and  $\leftrightarrow$  third. When both present, parentheses may be needed.

### Definition.

If  $r$  and  $s$  are statements:

1.  $r$  is a **sufficient condition** for  $s$  means “if  $r$  then  $s$ ”.  $r \rightarrow s$
2.  $r$  is a **necessary condition** for  $s$  means “if not  $r$  then not  $s$ ”.  $\sim r \rightarrow \sim s$

By property of the contrapositive:

3.  $r$  is a *necessary and sufficient condition* for  $s$  means “ $r$  if, and only if  $s$ . $r \leftrightarrow s$

**Example.** Rewrite the following statement in the form “If  $A$  then  $B$ ”:

Having two  $45^\circ$  angles is a sufficient condition for this triangle to be a right triangle.

**Example.** Use the contrapositive to rewrite the following statement in two ways:

George’s attaining age 35 is a necessary condition for his being president of the United States.

## 2.5: Application: Number Systems and Circuits for Addition

Recall our how we write numbers in base 10:

$$\begin{aligned}5,049 &= 5 \cdot 1000 + 0 \cdot 100 + 4 \cdot 10 + 9 \cdot 1 \\&= 5 \cdot 10^3 + 0 \cdot 10^2 + 4 \cdot 10^1 + 9 \cdot 10^0\end{aligned}$$

### Definition.

Any integer  $b > 1$  can be used as a base for a numbering system. A numbering system of base  $b$  has the digits  $0, 1, \dots, b - 1$ .

A **base 2 notation** or **binary notation**, uses the digits 0, 1. In binary, every integer is represented as sum of products of the form

$$d \cdot 2^n$$

where  $n \in \mathbb{Z}$  and  $d \in \{0, 1\}$ .

**Example.** Below is the binary representation for the integers 1 to 9:

$$\begin{array}{lllll}1_{10} & = & 1 \cdot 2^0 & = & 1_2 \\2_{10} & = & 1 \cdot 2^1 + 0 \cdot 2^0 & = & 10_2 \\3_{10} & = & 1 \cdot 2^1 + 1 \cdot 2^0 & = & 11_2 \\4_{10} & = & 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 & = & 100_2 \\5_{10} & = & 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 & = & 101_2 \\6_{10} & = & 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 & = & 110_2 \\7_{10} & = & 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 & = & 111_2 \\8_{10} & = & 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 & = & 1000_2 \\9_{10} & = & 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 & = & 1001_2\end{array}$$

**Converting binary → decimal:**

To convert from binary to decimal, multiply each digit by its corresponding power of 2 and sum the results.

**Example.** Represent the following in decimal notation (base-10):

$$110_2$$

$$1011_2$$

$$11110_2$$

$$101011_2$$

### Converting decimal → binary:

To convert from decimal to binary, we repeated divide by 2, and record the remainders.

#### Example.

$$\begin{aligned}27_{10} &= 16 + 8 + 2 + 1 \\&= 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 \\&= 11011_2\end{aligned}$$

**Example.** Represent the following in binary notation:

$$243_{10}$$

$$587_{10}$$

$$990_{10}$$

$$531_{10}$$

**Binary arithmetic:**

In binary arithmetic,  $10_2$  behaves similarly to 10 in decimal arithmetic.

**Example.** Add  $1101_2$  and  $111_2$  using binary notation.

**Example.** Subtract  $1011_2$  from  $11000_2$  using binary notation.

**Definition.**

The **8-bit two's complement** for an integer  $a$  between  $-128$  and  $127$  is the 8-bit binary representation for

$$\begin{cases} a, & \text{if } a \geq 0 \\ 2^8 - |a|, & \text{if } a < 0. \end{cases}$$

Two's complement allows maximum representation for  $2^8$  integers with 8 binary digits.

**Example.** Below are a few integers represented in binary using 8-bit two's complement:

$$-128 \rightarrow 2^8 - |-128| = 128_{10} = 10000000_2 \quad 0 \rightarrow 0_{10} = 00000000_2$$

$$-127 \rightarrow 2^8 - |-127| = 129_{10} = 10000001_2 \quad 1 \rightarrow 1_{10} = 00000001_2$$

$$\vdots \quad 2 \rightarrow 2_{10} = 00000010_2$$

$$-2 \rightarrow 2^8 - |-2| = 254_{10} = 10000000_2 \quad \vdots$$

$$-1 \rightarrow 2^8 - |-1| = 255_{10} = 11111111_2 \quad 127 \rightarrow 127_{10} = 01111111_2$$

**Example.** Find the 8-bit two's complement for the following:

$-46$

$42$

$120$

$-82$

### **Two's complement of a negative integer:**

To find the decimal representation of the negative integer with a given 8-bit two's complement:

- Flip the bits
- Add 1
- Convert to base-10 and swap the sign

**Example.** Find the decimal representation of the integers with the following 8-bit two's complement:

$11100101_2$

$11000000_2$

**Addition and Subtraction with Integers in Two's Complement Form:**

When performing binary addition on integers written in Two's Complement form, we discard any “carry” bit.

**Example.** Perform binary addition using the Two's Complement form of the following:

83 and  $-55$

$-87$  and  $-46$

## Definition.

**Hexadecimal notation** uses a **base 16 notation**. In hexadecimal, every integer is represented as sum of products of the form

$$d \cdot 16^n$$

where  $n \in \mathbb{Z}$  and  $d \in \{0, 1, \dots, 9, A, B, C, D, E, F\}$ .

Decimal	Hexadecimal	4-Bit Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
10	A	1010
11	B	1011
12	C	1100
13	D	1101
14	E	1110
15	F	1111

**Example.** Convert  $3CF_{16}$  to decimal notation.

**Example.** Convert  $B09F_{16}$  to binary notation.

**Example.** Convert  $100110110101001_2$  to hexadecimal notation.

### 3.1: Predicates and Quantified Statements I

## Definition.

A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.

The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

**Example.** Let  $P(x)$  be the predicate “ $x^2 > x$ ” with domain the set  $\mathbb{R}$ . Write  $P(2)$ ,  $P\left(\frac{1}{2}\right)$ , and  $P\left(-\frac{1}{2}\right)$ , and indicate which of these statements are true and which are false.

## Definition.

If  $P(x)$  is a predicate and  $x$  has domain  $D$ , the **truth set** of  $P(x)$  is the set of all elements of  $D$  that make  $P(x)$  true when they are substituted for  $x$ . The truth set of  $P(x)$  is denoted

$$\{x \in D \mid P(x)\}$$

**Example.** Let  $Q(n)$  be the predicate “ $n$  is a factor of 8”. Find the truth set of  $Q(n)$  if

the domain of  $n$  is  $\mathbb{Z}^+$

the domain of  $n$  is  $\mathbb{Z}$

## Definition.

Let  $Q(x)$  be a predicate and  $D$  the domain of  $x$ .

- **Quantifiers** are words that refer to quantities such as “some” or “all” and tell for how many elements a given predicate is true.
- The **universal quantifier** is represented by the symbol “ $\forall$ ”.
- A **universal statement** is a statement of the form “ $\forall x \in D, Q(x)$ ”.
  - It is defined to be true if, and only if,  $Q(x)$  is true for *each* individual  $x$  in  $D$ .
  - It is defined to be false if, and only if,  $Q(x)$  is false for *at least one*  $x$  in  $D$ .
- A value for  $x$  for which  $Q(x)$  is false is called a **counterexample** to the universal statement.

**Example.** Let  $D = \{1, 2, 3, 4, 5\}$ , and consider the statement

$$\forall x \in D, x^2 \geq x.$$

Write one way to read this statement out loud, and show that it is true.

The above example uses the **method of exhaustion**.

**Example.** Consider the statement

$$\forall x \in \mathbb{R}, x^2 \geq x.$$

Find a counter example to show that this statement is false.

**Definition.**

Let  $Q(x)$  be a predicate and  $D$  the domain of  $x$ .

- The **existential quantifier** is represented by the symbol “ $\exists$ ”.
- An **existential statement** is a statement of the form “ $\exists x \in D$  such that  $Q(x)$ ”.
  - It is defined to be true if, and only if,  $Q(x)$  is true for *at least one*  $x$  in  $D$ .
  - It is false if, and only if,  $Q(x)$  is false *for all*  $x$  in  $D$ .

**Example.** Consider the statement

$$\exists m \in \mathbb{Z}^+ \text{ such that } m^2 = m.$$

Write one way to read this statement out loud, and show that it is true.

**Example.** Let  $E = \{5, 6, 7, 8\}$  and consider the statement

$$\exists m \in E \text{ such that } m^2 = m.$$

Show that this statement is false.

**Example.** Rewrite the following statements formally using quantifiers and variables:

All triangles have three sides.

No dogs have wings.

Some programs are structured.

**Definition.**

A **universal conditional statement** is of the form:

$$\forall x, \text{ if } P(x) \text{ then } Q(x).$$

**Example.** Rewrite each of the following statements in the form

$$\forall \underline{\quad}, \text{ if } \underline{\quad} \text{ then } \underline{\quad}$$

If a real number is an integer, then it is a rational number.

All bytes have eight bits.

No fire trucks are green.

### 3.2: Predicates and Quantified Statements II

#### Definition.

- The negation of a statement of the form

$$\forall x \text{ in } D, Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \text{ in } D \text{ such that } \sim Q(x).$$

$$\sim (\forall x \in D, Q(x)) \equiv \exists x \in D \text{ such that } \sim Q(x).$$

- The negation of a statement of the form

$$\exists x \text{ in } D \text{ such that } Q(x)$$

is logically equivalent to a statement of the form

$$\forall x \text{ in } D, \sim Q(x).$$

$$\sim (\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \sim Q(x)$$

**Example.** Negate the following statements:

$$\forall \text{ primes } p, p \text{ is odd}$$

$$\exists \text{ a triangle } T \text{ such that the sum of the angles of } T \text{ equals } 200^\circ$$

**Example.** Rewrite the following statements formally, then write the formal and informal negations.

No politicians are honest

The number 1,357 is not divisible by any integer between 1 and 37.

**Example.** Write informal negations for the following statements:

All computer programs are finite.

Some computer hackers are over 40.

## Negation of a Universal Conditional Statement

$$\sim (\forall x, \text{ if } P(x) \text{ then } Q(x)) \equiv \exists x \text{ such that } P(x) \text{ and } \sim Q(x)$$

### Definition.

A statement of the form

$$\forall x \text{ in } D, \text{ if } P(x) \text{ then } Q(x)$$

is called **vacuously true** or **true by default** if, and only if,  $P(x)$  is false for every  $x$  in  $D$ .

**Example.** The following statement is vacuously true since it's negation is false:

All kangaroos enrolled in my class are passing.

**Definition.**

Consider a statement of the form  $\forall x \in D$ , if  $P(x)$  then  $Q(x)$ .

1. Its **contrapositive** is the statement  $\forall x \in D$ , if  $\sim Q(x)$  then  $\sim P(x)$ .
2. Its **converse** is the statement  $\forall x \in D$ , if  $Q(x)$  then  $P(x)$ .
3. Its **inverse** is the statement  $\forall x \in D$ , if  $\sim P(x)$  then  $\sim Q(x)$ .

**Example.** Write a formal and informal contrapositive, converse, and inverse for the following statement:

If a real number is greater than 2, then its square is greater than 4.

## Definition.

- “ $\forall x, r(x)$  is a **sufficient condition** for  $s(x)$ ”       $\rightarrow$  “ $\forall x, \text{if } r(x) \text{ then } s(x)$ ”
- “ $\forall x, r(x)$  is a **necessary condition** for  $s(x)$ ”       $\rightarrow$  “ $\forall x, \text{if } \sim r(x) \text{ then } \sim s(x)$ ”  
 $\rightarrow$  “ $\forall x, \text{if } s(x) \text{ then } r(x)$ ”
- “ $\forall x, r(x)$  **only if**  $s(x)$ ”       $\rightarrow$  “ $\forall x, \text{if } \sim s(x), \text{ then } \sim r(x)$ ”  
 $\rightarrow$  “ $\forall x, \text{if } r(x) \text{ then } s(x)$ ”

**Example.** Rewrite each of the following as a universal conditional statement, quantified either explicitly or implicitly. Do not use the word *necessary* or *sufficient*.

Squareness is a sufficient condition for rectangularity.

Being at least 35 years old is a necessary condition for being president of the United States.

**Example.** Rewrite the following as a universal conditional statement:

A product of two numbers is 0 only if one of the numbers is 0.

## 6.1: Set Theory: Definitions and the Element Method of Proof

### Element Argument: The Basic Method for Proving that One set is a Subset of Another

Let sets  $X$  and  $Y$  be given. To prove that  $X \subseteq Y$ ,

1. suppose that  $x$  is a particular but arbitrarily chosen element of  $X$ ,
2. show that  $x$  is an element of  $Y$

**Example.** Define sets  $A$  and  $B$  as follows:

$$A = \{m \in \mathbb{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbb{Z}\}$$

$$B = \{n \in \mathbb{Z} \mid n = 3s \text{ for some } s \in \mathbb{Z}\}$$

Prove that  $A \subseteq B$

Disprove that  $B \subseteq A$

**Definition.**

Given sets  $A$  and  $B$ ,  $A$  **equals**  $B$ , written  $\mathbf{A} = \mathbf{B}$ , if, and only if, every element of  $A$  is in  $B$  and every element of  $B$  is in  $A$ :

$$A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A.$$

**Example.** Define sets  $A$  and  $B$  as follows:

$$A = \{m \in \mathbb{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbb{Z}\}$$

$$B = \{n \in \mathbb{Z} \mid n = 3s \text{ for some } s \in \mathbb{Z}\}$$

Is  $A = B$ ?

**Definition.**

Given an integer  $n$  and a positive integer  $d$ , when  $n$  is divided by  $d$ , then

$$n \text{ div } d = \text{ the integer quotient}$$
$$n \text{ mod } d = \text{ the nonnegative integer remainder}$$

If  $n$  and  $d$  are integers and  $d > 0$ , then

$$n \text{ div } d = q \quad \text{and} \quad n \text{ mod } d = r \quad \Leftrightarrow \quad n = dq + r$$

**Example.** Compute the following:

$$32 \text{ div } 9, \quad 32 \text{ mod } 9$$

$$365 \text{ div } 7, \quad 365 \text{ mod } 7$$

**Example.** If it is currently 11:00, what time will it be in

$$51 \text{ hours?}$$

$$121 \text{ hours?}$$

$$11 \text{ hours?}$$

$$-1 \text{ hours?}$$

**Example.** Let  $A = \{4, \sqrt{16}, 19 \text{ mod } 15\}$  and  $B = \{12 \text{ mod } 8\}$ . Is  $A \subseteq B$ ? Is  $B \subseteq A$ ?

## Definition.

Let  $A$  and  $B$  be subsets of a universal set  $U$ .

1. The **union** of  $A$  and  $B$  is the set of all elements that are in at least one of  $A$  or  $B$ .

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$$

2. The **intersection** of  $A$  and  $B$  is the set of all elements that are common to both  $A$  and  $B$ .

$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$

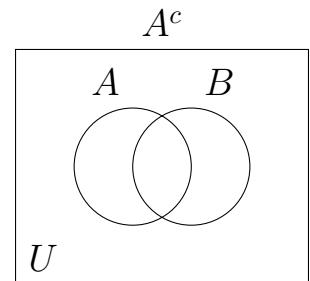
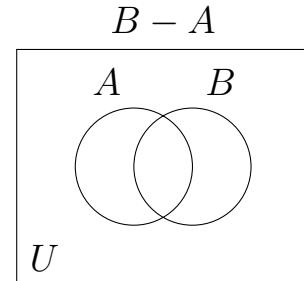
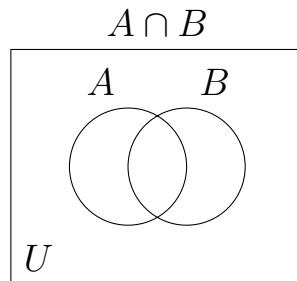
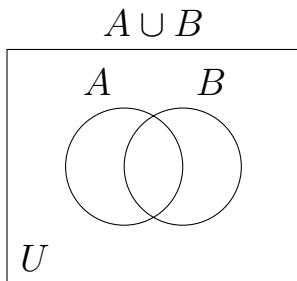
3. The **difference** of  $A$  and  $B$  is the set of all elements that are in  $B$  and not  $A$ .

$$B - A = \{x \in U \mid x \in B \text{ and } x \notin A\}$$

4. The **complement** of  $A$  is the set of all elements in  $U$  that are not in  $A$ .

$$A^c = \{x \in U \mid x \notin A\}$$

**Example.** Represent the following sets using the Venn diagrams below:



**Example.** Let the universal set be the set  $U = \{a, b, c, d, e, f, g\}$ , and let  $A = \{a, c, e, g\}$  and  $B = \{d, e, f, g\}$ . Find

$$A \cup B$$

$$A \cap B$$

$$B - A$$

$$A^c$$

### Definition.

Given real numbers  $a$  and  $b$  with  $a \leq b$ :

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

**Example.** Let the universal set be  $\mathbb{R}$ , and let  $A = (-1, 0]$  and  $B = [0, 1)$ . Find

$$A \cup B$$

$$A \cap B$$

$$B - A$$

$$A^c$$

**Definition.**

Given sets  $A_0, A_1, A_2, \dots$  that are subsets of a universal set  $U$  and given a nonnegative integer  $n$ ,

$$\bigcup_{i=0}^n A_i = \{x \in U \mid x \in A_i, \text{ for at least one } i = 0, 1, 2, \dots, n\}$$

$$\bigcap_{i=0}^n A_i = \{x \in U \mid x \in A_i, \text{ for every } i = 0, 1, 2, \dots, n\}$$

**Example.** For each positive integer  $i$ , let  $A_i = \left\{x \in \mathbb{R} \mid -\frac{1}{i} < x < \frac{1}{i}\right\} = \left(-\frac{1}{i}, \frac{1}{i}\right)$ .  
Find

$$A_1 \cup A_2 \cup A_3$$

$$A_1 \cap A_2 \cap A_3$$

$$\bigcup_{i=1}^{\infty} A_i$$

$$\bigcap_{i=1}^{\infty} A_i$$

### Definition.

The **empty set** (or **null set**), denoted  $\emptyset$ , is the set with no elements.

$$\{1, 3\} \cap \{2, 4\} = \emptyset$$

Two sets are called **disjoint** if, and only if, they have no elements in common:

$$A \cap B = \emptyset.$$

Sets  $A_1, A_2, A_3, \dots$  are **mutually disjoint** (or **pairwise disjoint**) if, and only if, no two sets  $A_i$  and  $A_j$  with distinct subscripts have any elements in common:

$$A_i \cap A_j = \emptyset \text{ whenever } i \neq j.$$

### Example.

Let  $A_1 = \{3, 5\}$ ,  $A_2 = \{1, 4, 6\}$ , and  $A_3 = \{2\}$ . Are  $A_1$ ,  $A_2$ , and  $A_3$  mutually disjoint?

Let  $B_1 = \{2, 4, 6\}$ ,  $B_2 = \{3, 7\}$ , and  $B_3 = \{4, 5\}$ . Are  $B_1$ ,  $B_2$ ,  $B_3$  mutually disjoint?

**Definition.**

A finite or infinite collection of nonempty sets  $\{A_1, A_2, A_3, \dots\}$  is a **partition** of a set  $A$  if, and only if,

1.  $A$  is the union of all the  $A_i$ ;
2. the sets  $A_1, A_2, A_3, \dots$  are mutually disjoint.

**Example.**

Let  $A = \{1, 2, 3, 4, 5, 6\}$ ,  $A_1 = \{1, 2\}$ ,  $A_2 = \{3, 4\}$ , and  $A_3 = \{5, 6\}$ . Is  $\{A_1, A_2, A_3\}$  a partition of  $A$ ?

Let  $\mathbb{Z}$  be the set of all integers and let

$$T_i = \{n \in \mathbb{Z} \mid n = 3k + i, \text{ for some integer } k\}.$$

Is  $\{T_0, T_1, T_2\}$  a partition of  $\mathbb{Z}$ ?

**Definition.**

Given a set  $A$ , the **power set** of  $A$ , denoted  $\mathcal{P}(A)$ , is the set of all subsets of  $A$ .

**Example.** Find  $\mathcal{P}(\{x, y\})$ .

## 6.4: Boolean Algebras, Russell's Paradox, and the Halting Problem

### Definition.

A **Boolean algebra** is a set  $B$  together with two operations, generally denoted  $+$  and  $\cdot$ , such that for all  $a$  and  $b$  in  $B$  both  $a + b$  and  $a \cdot b$  are in  $B$  and the following axioms are assumed to hold:

1. *Commutative Laws:* For all  $a$  and  $b$  in  $B$ ,

$$a + b = b + a \text{ and } a \cdot b = b \cdot a$$

2. *Associative Laws:* For all  $a$  and  $b$  in  $B$ ,

$$(a + b) + c = a + (b + c) \text{ and } (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

3. *Distributive Laws:* For all  $a$  and  $b$  in  $B$ ,

$$a + (b \cdot c) = (a + b) \cdot (a + c) \text{ and } a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

4. *Identity Laws:* There exist distinct elements  $0$  and  $1$  in  $B$  such that for each  $a$  in  $B$ ,

$$a + 0 = a \text{ and } a \cdot 1 = a$$

5. *Complement Laws:* For each  $a$  in  $B$ , there exists an element in  $B$ , denoted  $\bar{a}$  and called the **complement** or **negation** of  $a$ , such that

$$a + \bar{a} = 1 \text{ and } a \cdot \bar{a} = 0$$

## Properties of a Boolean Algebra

Let  $B$  be any Boolean algebra.

1. *Uniqueness of the Complement Laws:* For all  $a$  and  $x$  in  $B$ , if  $a + x = 1$  and  $a \cdot x = 0$ , then  $x = \bar{a}$ .
2. *Uniqueness of 0 and 1:* If there exists  $x$  in  $B$  such that  $a + x = a$  for every  $a$  in  $B$ , then  $x = 0$ , and if there exists  $y$  in  $B$  such that  $a \cdot y = a$  for every  $a$  in  $B$ , then  $y = 1$ .
3. *Double Complement Law:* For every  $a \in B$ ,  $\overline{\overline{a}} = a$ .
4. *Idempotent Laws:* For every  $a \in B$ ,

$$a + a = a \text{ and } a \cdot a = a.$$

5. *Universal Bound Laws:* For every  $a \in B$ ,

$$a + 1 = 1 \text{ and } a \cdot 0 = 0.$$

6. *De Morgan's Laws:* For all  $a$  and  $b \in B$ ,

$$\overline{a + b} = \bar{a} \cdot \bar{b} \text{ and } \overline{a \cdot b} = \bar{a} + \bar{b}.$$

7. *Absorption Laws:* For all  $a$  and  $b \in B$ ,

$$(a + b) \cdot a = a \text{ and } (a \cdot b) + a = a.$$

8. *Complements of 0 and 1:*

$$\overline{0} = 1 \text{ and } \overline{1} = 0.$$

**Example.** Prove that for all elements  $a$  in a Boolean algebra  $B$ :

$$\overline{(\bar{a})} = a.$$

$$a + a = a.$$

**Example.** Prove that for all elements  $a$  in a Boolean algebra  $B$ :

$$a \cdot a = a.$$

$$(a + b) \cdot a = a.$$

## Russell's Paradox

Define the following set  $S$ :

$$S = \{A \mid A \text{ is a set and } A \notin A\}.$$

Is  $S$  an element of itself?

**The Barber Puzzle:** In a certain town, there is a male barber who shaves all those men, and only those men, who do not shave themselves.

Does the barber shave himself?

Is the sentence “The barber shaves himself” a statement?

**Example.** Determine whether each sentence is a statement:

If  $1 + 1 = 3$ , then  $1 = 0$ .

This sentence is false and  $1 + 1 = 3$ .

### The Halting Problem (Alan M. Turing)

There is no computer algorithm that will accept any algorithm  $X$  and data set  $D$  as input and then will output “halts” or “loops forever” to indicate whether or not  $X$  terminates in a finite number of steps when  $X$  is run with data set  $D$ .

### Example boolean algebras:

Logical Equivalences	Set Properties
For all statement variables $p, q$ , and $r$ :	For all sets $A, B$ , and $C$ :
$p \vee q \equiv q \vee p$	$A \cup B = B \cup A$
$p \wedge q \equiv q \wedge p$	$A \cap B = B \cap A$
$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	$A \cap (B \cap C) = (A \cap B) \cap C$
$p \vee (q \vee r) \equiv (p \vee q) \vee r$	$A \cup (B \cup C) = (A \cup B) \cup C$
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$p \vee \mathbf{c} \equiv p$	$A \cup \emptyset = A$
$p \wedge \mathbf{t} \equiv p$	$A \cap U = A$
$p \vee \sim p \equiv \mathbf{t}$	$A \cup A^c = U$
$p \wedge \sim p \equiv \mathbf{c}$	$A \cap A^c = \emptyset$
$\sim(\sim p) \equiv p$	$(A^c)^c = A$
$p \wedge p \equiv p$	$A \cup A = A$
$p \vee p \equiv p$	$A \cap A = A$
$p \vee \mathbf{t} \equiv \mathbf{t}$	$A \cup U = U$
$p \wedge \mathbf{c} \equiv \mathbf{c}$	$A \cap \emptyset = \emptyset$
$\sim(p \vee q) \equiv \sim p \wedge \sim q$	$(A \cup B)^c = A^c \cap B^c$
$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$(A \cap B)^c = A^c \cup B^c$
$p \vee (p \wedge q) \equiv p$	$A \cup (A \cap B) = A$
$p \wedge (p \vee q) \equiv p$	$A \cap (A \cup B) = A$
$\sim \mathbf{t} \equiv \mathbf{c}$	$U^c = \emptyset$
$\sim \mathbf{c} \equiv \mathbf{t}$	$\emptyset^c = U$

## 8.1: Relations on Sets

### Definition.

A relation  $R$  from  $A$  to  $B$  is called a **binary relation** because it is a subset of a Cartesian product of two sets.

**Example.** Define a relation  $L$  from  $\mathbb{R}$  to  $\mathbb{R}$ :

$$\forall x, y \in \mathbb{R}, x L y \Leftrightarrow x < y.$$

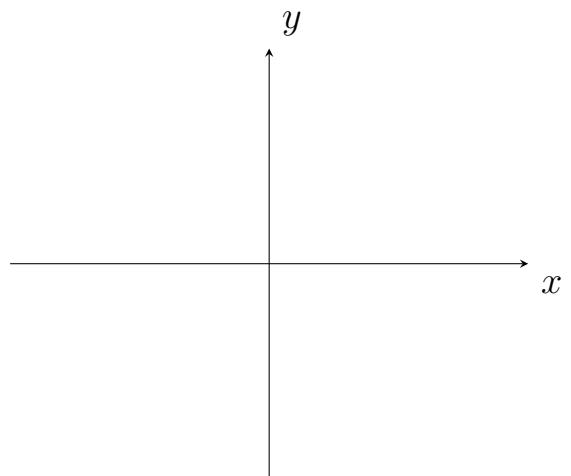
Is  $57 L 53$ ?

Is  $(-17) L (-14)$ ?

Is  $143 L 143$ ?

Is  $(-35) L 1$ ?

Draw the graph of  $L$  as a subset of the Cartesian plane  $\mathbb{R} \times \mathbb{R}$ .



**Definition.**

Two integers  $m$  and  $n$  are **congruent modulo 2** if, and only if,  $m \bmod 2 = n \bmod 2$ .

**Example.** Define a relation  $E$  from  $\mathbb{Z}$  to  $\mathbb{Z}$ :

$$\forall (m, n) \in \mathbb{Z} \times \mathbb{Z}, m E n \Leftrightarrow m - n \text{ is even.}$$

Is  $4 E 0$ ? Is  $2 E 6$ ? Is  $3 E (-3)$ ? Is  $5 E 2$ ?

Prove that if  $n$  is any odd integer, then  $n E 1$ .

**Example.** Let  $X = \{a, b, c\}$ . Define a relation  $\mathbf{S}$  from  $\mathcal{P}(X)$  to  $\mathcal{P}(X)$  as follows:

$$\forall A, B \in \mathcal{P}(X), A \mathbf{S} B \Leftrightarrow A \text{ has at least as many elements as } B.$$

Is  $\{a, b\} \mathbf{S} \{b, c\}$ ?

Is  $\{a\} \mathbf{S} \emptyset$ ?

Is  $\{b, c\} \mathbf{S} \{a, b, c\}$ ?

Is  $\{c\} \mathbf{S} \{a\}$ ?

**Definition.**

Let  $R$  be a relation from  $A$  to  $B$ . Define the inverse relation  $R^{-1}$  from  $B$  to  $A$  as follows:

$$R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}.$$

**Example.** Let  $A = \{2, 3, 4\}$  and  $B = \{2, 6, 8\}$ , and let  $R$  be the “divides” relation from  $A$  to  $B$ :

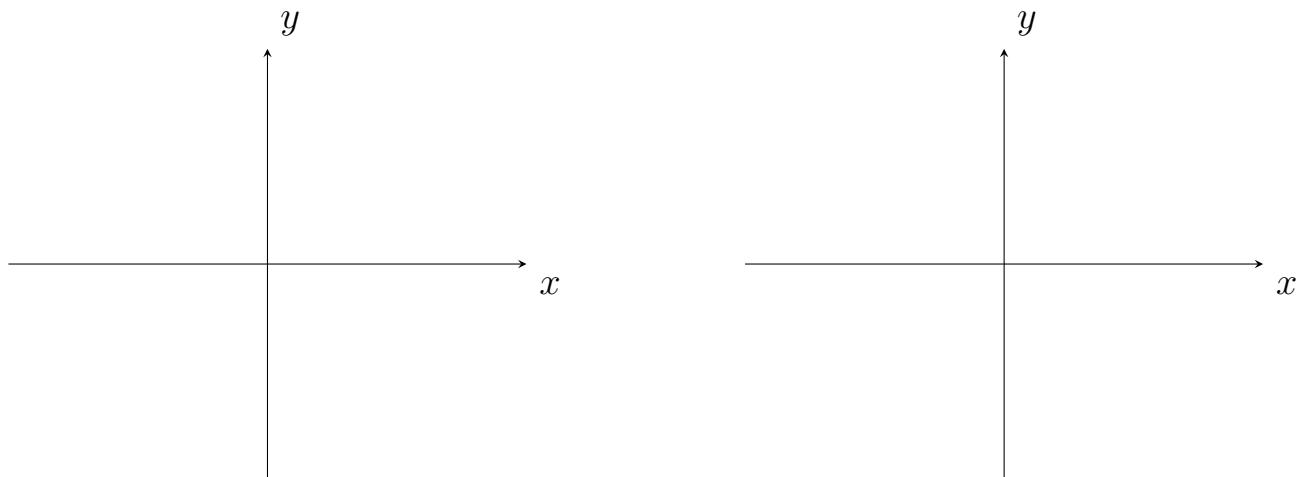
$$\forall (x, y) \in A \times B, x R y \Leftrightarrow x \mid y$$

Explicitly state which ordered pairs are in  $R$  and  $R^{-1}$ . Draw arrow diagrams for both.

**Example.** Define a relation  $R$  from  $\mathbb{R}$  to  $\mathbb{R}$  as follows:

$$\forall(x, y) \in \mathbb{R} \times \mathbb{R}, x R y \Leftrightarrow y = 2|x|.$$

Draw the graphs of  $R$  and  $R^{-1}$  in the Cartesian plane. Is  $R^{-1}$  a function?



**Definition.**

A **relation on a set**  $A$  is a relation from  $A$  to  $A$ .

A **graph**  $G$  consists of two finite sets:

- a nonempty set  $V(G)$  of **vertices** and
- a set  $E(G)$  of **edges**,

where each edge is associated with a set consisting of either one or two vertices called its endpoints.

A **directed graph** is a graph whose edges are directional.

**Example.** Let  $A = \{3, 4, 5, 6, 7, 8\}$  and define a relation  $R$  on  $A$  as follows

$$\forall x, y \in A, x R y \Leftrightarrow 2 \mid (x - y).$$

Draw the directed graph of  $R$ .

**Definition.**

Given sets  $A_1, A_2, \dots, A_n$ , an **n-ary relation**  $R$  on  $A_1 \times A_2 \times \dots \times A_n$  is a subset of  $A_1 \times A_2 \times \dots \times A_n$ . The special case of 2-ary, 3-ary, and 4-ary relations are called **binary**, **ternary**, and **quaternary relations**, respectively.