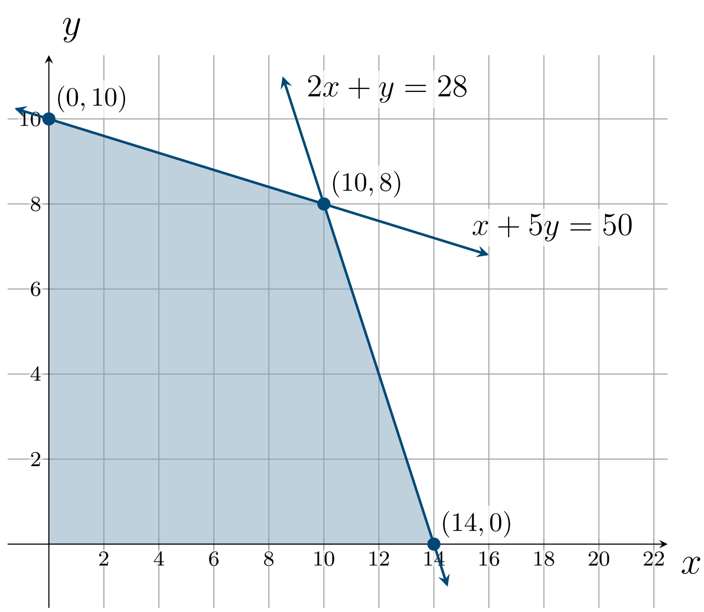
4.2: Linear Programming: Graphical Methods

Definition.
Linear programming is an optimization technique that can be used to solve linearly constrained problems:

$$\begin{aligned} \max F &= 3x + y \\ \text{subject to} \quad x + 5y &\le 50 \\ 2x + y &\le 28 \\ x \ge 0, y &\ge 0 \end{aligned}$$

The **constraints** of a linear program (LP) may be limitations or requirements of the variables. The **objective function** is the function that we wish to optimize (e.g. maximize profit *or* minimize cost).



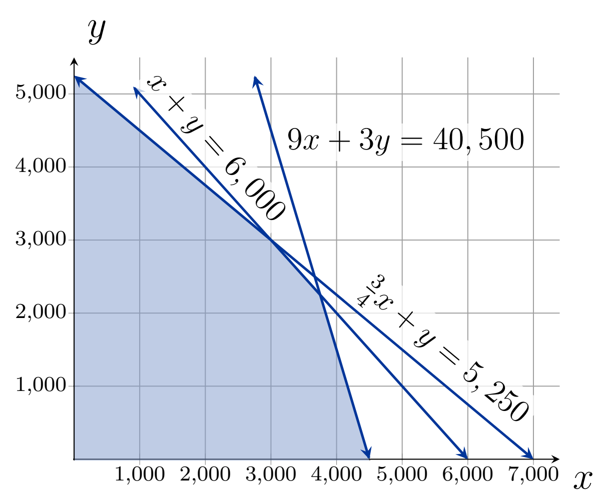
Linear programming (graphical method)

1. Write the objective function and constraint inequalities from the problem.
2. Graph the solution of the constraint system.
3. Find the corners of the resulting feasible region.
4. Evaluate the objective function at each corner.
5. If two corners give the optimal value, then the entire boundary joining these two points optimizes the function.

Example. A farm co-op has 6000 acres available on which to plant corn and soybeans. The following table summarizes each crop's requirement for fertilizer/herbicide, harvesting labor hours, and the available amounts of these resources.

	Corn	**Soybeans**	**Available**
Fertilizer/herbicide	9 gal/acre	3 gal/acre	40,500 gal
Harvesting labor	3/4 hr/acre	1 hr/acre	5,250 hr

Setup the system of inequalities that represents the constraints.

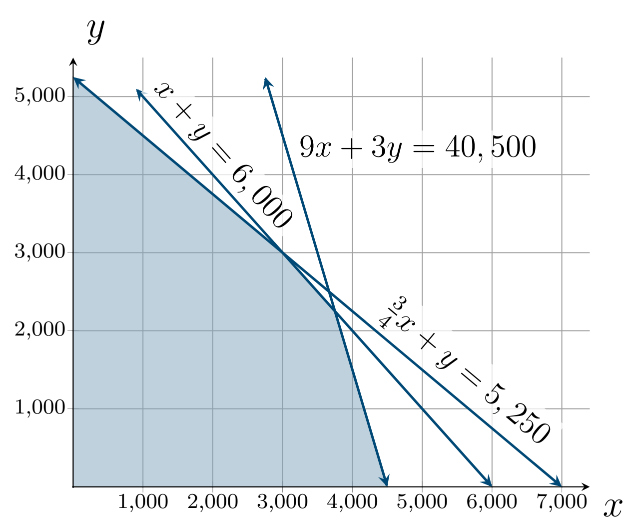


Example. Using the linear constraints from above, suppose the co-ops profits per acre are \$240 for corn and \$160 for soybeans. This gives us the following linear program:

$$\begin{aligned}
\max P = 240x + 160y \\
\text{subject to} \qquad x + \quad y &\le 6,000 \\
9x + 3y &\le 40,500 \\
\frac{3}{4}x + \quad y &\le 5,250 \\
x \ge 0, y &\ge 0
\end{aligned}$$

1. Find the "corners" of the feasible region
2. Evaluate the profit function at the corners

(x, y)	$P = 240x + 160y$
$(0, 0)$	\$0
$(0, 5250)$	\$840,000
$(3000, 3000)$	\$1,200,000
$(3750, 2250)$	\$1,260,000
$(4500, 0)$	\$1,080,000

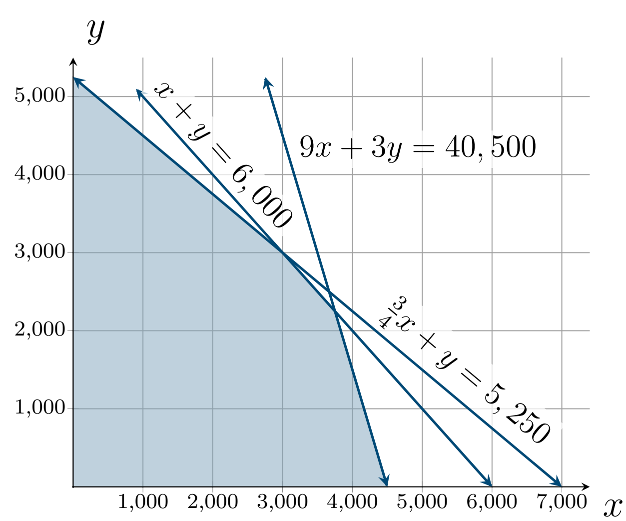


Example. Suppose the profits per acre are instead \$300 for corn, and \$100 for soybeans. This gives us the following linear program:

$$\begin{aligned} \max P = 300x + 100y \\ \text{subject to} \quad x + y &\leq 6,000 \\ 9x + 3y &\leq 40,500 \\ \frac{3}{4}x + y &\leq 5,250 \\ x \geq 0, y &\geq 0 \end{aligned}$$

Evaluate the profit function at the corners. What combination of corn and soy-beans maximizes the profit?

(x, y)	$P = 300x + 100y$
$(0, 0)$	\$0
$(0, 5250)$	\$525,000
$(3000, 3000)$	\$1,200,000
$(3750, 2250)$	\$1,350,000
$(4500, 0)$	\$1,350,000



Example. Two chemical plants, one at Macon and one at Jonesboro, produce three types of fertilizer: low phosphorus (LP), medium phosphorus (MP), and high phosphorus (HP). At each plant, the fertilizer is produced in a single production run, so the three types are produced in fixed proportions. The Macon plant produces 1 ton of LP, 2 tons of MP, and 3 tons of HP in a single operation and charges \$600 for what is produced in one operation. On the other hand, one operation of the Jonesboro plant produces 1 ton of LP, 5 tons of MP, and 1 ton of HP, and it charges \$1,000 for what it produces in one operation. If a customer needs 100 tons of LP, 260 tons of MP, and 180 tons of HP, how many production runs should be ordered from each plant to minimize costs?

Organize the information from the problem in the following table:

	Macon	Jonesboro	Requirements
Units of LP			
Units of MP			
Units of HP			

What is the objective function?

Write the linear program that we aim to solve below:

Example. From above, we get the following linear program

$$\begin{aligned} \min C = 600x + 1000y & \\ \text{subject to} \quad x + \quad y &\geq 100 \\ 2x + \quad 5y &\geq 260 \\ 3x + \quad y &\geq 180 \\ x \geq 0, y &\geq 0 \end{aligned}$$

Graph the solution region of the constraints and evaluate the objective function at the corners of the feasible region above.

(x, y)	$P = 600x + 1,000y$
$(0, 180)$	$\$180,000$
$(40, 60)$	$\$84,000$
$(80, 20)$	$\$68,000$
$(130, 0)$	$\$78,000$