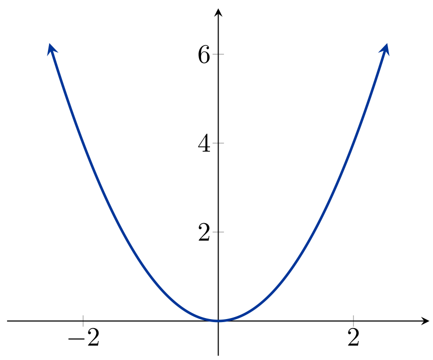
4.4: Optimization I

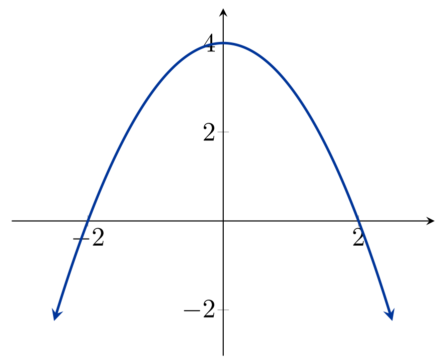
Definition. (Absolute Extrema)
Let f be defined on a set D containing c. If

- $f(c) \geq f(x)$ for every x in D, then $f(c)$ is an **absolute maximum** value of f
- $f(c) \leq f(x)$ for every x in D, then $f(c)$ is an **absolute minimum** value of f

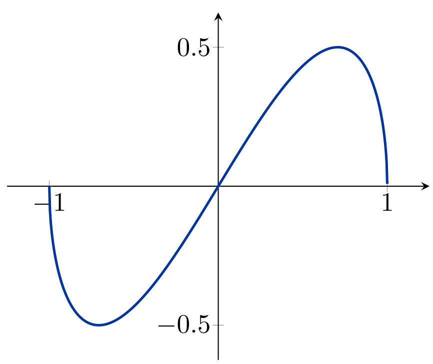
$f(0) = 0$ is the absolute minimum;
No absolute maximum



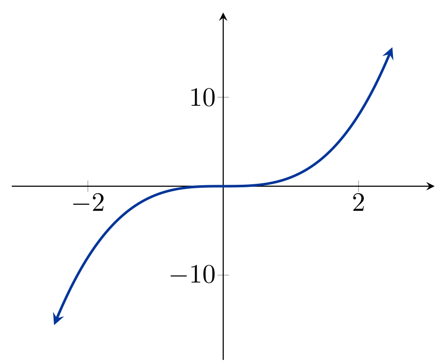
No absolute minimum;
$f(0) = 4$ is the absolute maximum



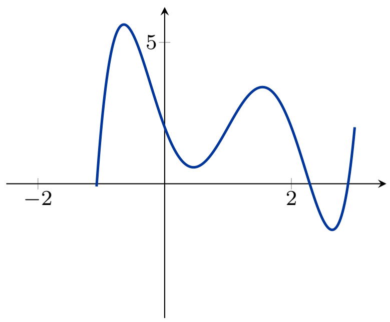
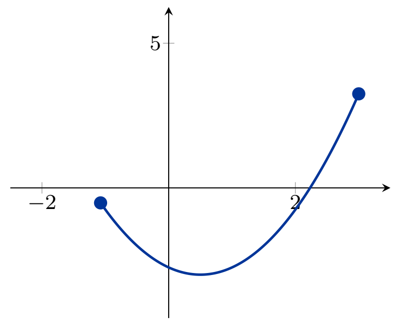
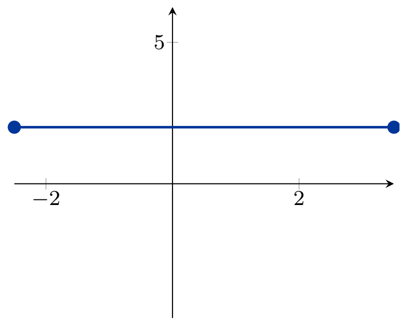
$f(-\sqrt{2}) = -\frac{1}{2}$ is the absolute minimum;
$f(\sqrt{2}) = \frac{1}{2}$ is the absolute maximum



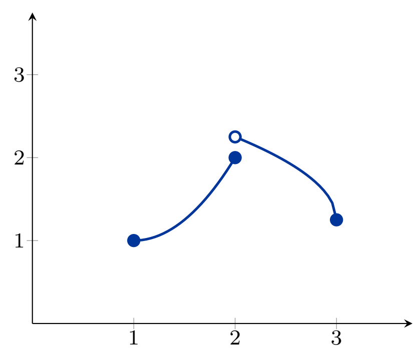
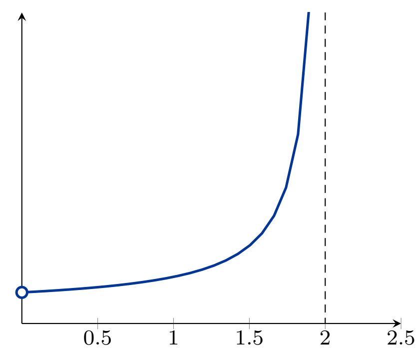
No absolute minimum;
No absolute maximum



Theorem 3
A function that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value on that interval.

Note: It is important that the function is both continuous *and* the interval is closed:

Finding the Absolute Extrema of f on a Closed Interval

1. Find the critical points of f within the interval (a, b).
2. Compute $f(x)$ at $x = a$, $x = b$, and at each of the critical points found above.
3. The absolute maximum and absolute minimum will correspond to the largest and smallest values found above.

Example. Find the absolute extrema of the following functions on the intervals indicated

$f(x) = x^2$ on $[-1, 2]$ Graphs

① $f'(x) = 2x$

$f'(x) = 0$ & $f'(x)$ DNE

$2x = 0$

$x = 0$

②

x	$f(x)$
-1	1
0	0
2	4

③ Abs max @ $x = 2$
(2, 4)

Abs min @ $x = 0$
(0, 0)

$$g(x) = x^3 - 2x^2 - 4x + 4 \text{ on } [0,3]$$

1 $g'(x) = 3x^2 - 4x - 4$

$g'(x) = 0$ $\quad$ $g'(x)$ DNE

$3x^2 - 4x - 4 = 0$

Quadratic formula

$x = \dfrac{4 \pm \sqrt{(-4)^2 - 4(3)(-4)}}{2(3)}$

$\rightarrow \boxed{x = -\frac{2}{3}}$, $\boxed{x = 2}$

Outside [0,3]

2

x	$g(x)$
0	4
2	-4
3	1

3

Abs max @ $x = 0$
(0,4)

Abs min @ $x = 2$
(2,-4)

$$h(x) = x^{2/3} \text{ on } [-1,8]$$

1 $h'(x) = \frac{2}{3} x^{-1/3} = \dfrac{2}{3x^{1/3}}$

$h'(x) = 0$ $\quad$ $h'(x)$ DNE

$3x^{1/3} \neq 0$

$x \neq 0$

2

x	$h(x)$
-1	1
0	0
8	4

3

Abs max @ $x = 8$
(8,4)

Abs min @ $x = 0$
(0,0)

Example. The daily average cost function (in dollars per unit) of Elektra Electronics is given by

$$\overline{C}(x) = 0.0001x^2 - 0.08x + 40 + \frac{5000}{x} \qquad (x > 0)$$

$5000x^{-1}$

where x stands for the number of graphing calculators that Elektra produces. Show that a production level of 500 units per day results in a minimum average cost for the company.

① $\bar{C}'(x) = 0.0002x - 0.08 - 5000x^{-2}$

$\bar{C}'(x) = 0$ & $\bar{C}'(x)$ DNE

Difficult to solve

$x \neq 0$

$\bar{C}'(500) = 0.0002(500) - 0.08 - \frac{5000}{(500)^2} = 0$

②

x	$\bar{C}(x)$
500	35

How to verify this is a minimum?

	1	500	1,000,000
$\bar{C}'(x)$	$-$		$+$
	↘		↗

⇒ Relative min and absolute min

Example. The altitude (in feet) of a rocket t seconds into flight is given by

$$s = f(t) = -t^3 + 96t^2 + 5 \qquad (t \geq 0)$$

Find the maximum altitude attained by the rocket.

① $v = f'(t) = -3t^2 + 192t$

$f'(t) = 0$ & $f'(t)$ DNE

$-3t^2 + 192t = 0$

$-3t(t-64) = 0$

$t = 0$ $\quad$ $t = 64$

②

t	$f(t)$
0	5
64	131,077

③ Max altitude of 131,077 ft. at $t = 64$

Find the maximum velocity attained by the rocket.

① $a = f''(t) = -6t + 192$

$f''(t) = 0$ & $f''(t)$ DNE

$-6t + 192 = 0$

$t = 32$

②

t	$f'(t)$
0	0
32	3072

③ Max velocity of 3072 ft./sec at $t = 32$ sec.