

5.5: Differentiation of Logarithmic Functions

Definition. (Natural log)

The inverse of the exponential function is the logarithm:

$$e^y = x \iff \ln(x) = y, \quad x > 0$$

We may also denote this as $\log_e(x)$

Example. Solve the following exponential equations

[Graphs](#)

$$y = e^{-3}$$

$$e^{x^2-x} = e^2$$

$$e^x = 10$$

$$4e^{1-x^2} = 6$$

Rule 3: Derivative of $\ln(x)$

$$\frac{d}{dx} [\ln|x|] = \frac{1}{x}, \quad x \neq 0$$

Rule 4: Derivative of $\ln(f(x))$

$$\frac{d}{dx} [\ln(f(x))] = \frac{f'(x)}{f(x)}$$

Example. Find the derivative of the following functions

$$y = \ln(\sqrt[3]{x})$$

$$f(x) = \sqrt{x} \ln(x)$$

$$g(x) = \ln(x^2 + 1)$$

$$h(t) = \frac{1 + \ln(t)}{1 - \ln(t)}$$

Additional properties of logarithms

$$e^y = x$$

$$\ln(x) = y$$

$$e^1 = a$$

$$\ln(a) = 1$$

$$e^0 = 1$$

$$\ln(1) = 0$$

$$e^x e^y = e^{x+y}$$

$$\ln(xy) = \ln(x) + \ln(y)$$

$$\frac{e^x}{e^y} = e^{x-y}$$

$$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$

$$e^{xy} = (e^x)^y$$

$$\ln(x^y) = y \ln(x)$$

$$e^{\ln(x)} = x$$

$$\ln(e^x) = x$$

Definition. (Other logarithms)

Logarithms are defined for any base $a > 0$:

$$a^y = x \quad \longleftrightarrow \quad \log_a(x) = y, \quad x > 0$$

Example. Find the derivative of the following functions

$$y = \ln(\sqrt[3]{x})$$

$$f(x) = \ln\left((x^2 + 1)(x^3 + 2)^6\right)$$

$$g(x) = \ln\left(\frac{1-x}{1+x}\right)$$

$$h(x) = \ln(x + \sqrt{x^2 - 1})$$

$$j(t) = 2(\ln(5t))^{3/2}$$

$$k(u) = \ln(\sqrt{u^2 - 4})$$

Logarithmic Differentiation

Using properties of logs, we can transform a function before taking its derivative:

1. Take the natural log of both sides rewriting products and quotients as sums and differences.
2. Differentiate both sides.
3. Solve for $\frac{dy}{dx}$

Example. Differentiate

$$y = x^2(x - 1)(x^2 + 4)^3$$