

## 2.4: Limits

**Example.** Suppose that the position function of a maglev train (in feet) is given by

$$s(t) = 4t^2, \quad (0 \leq t \leq 30)$$

Using the position function, compute the *average* velocity of the train

$$g(t) = \frac{s(t) - s(2)}{t - 2} = \frac{4t^2 - 16}{t - 2} = \frac{4(t^2 - 4)}{t - 2} = \frac{4(t-2)(t+2)}{t-2} = 4(t+2)$$

on the interval  $[t, 2]$

$t$	1.5	1.9	1.99	1.999	1.9999
	14	15.6	15.96	15.996	15.9996

on the interval  $[2, t]$

$t$	2.5	2.1	2.01	2.001	2.0001
	18	16.4	16.04	16.004	16.0004

What do the tables above suggest about *instantaneous* velocity of the train at  $t = 2$ ?

The train is going 16 ft/s at  $t = 2$  secs.

**Definition. (Limit of a Function)**

The function  $f$  has the **limit**  $L$  as  $x$  approaches  $a$ , written

$$\lim_{x \rightarrow a} f(x) = L$$

if the value of  $f(x)$  can be made as close to the number  $L$  as we please by taking  $x$  sufficiently close to (but not equal to)  $a$ .

**Example.** Using the graph of  $f$ , determine the following values:

$$f(1) \text{ and } \lim_{x \rightarrow 1} f(x)$$

$$f(1) = 2$$

$$\lim_{x \rightarrow 1} f(x) = 2$$

$$f(2) \text{ and } \lim_{x \rightarrow 2} f(x)$$

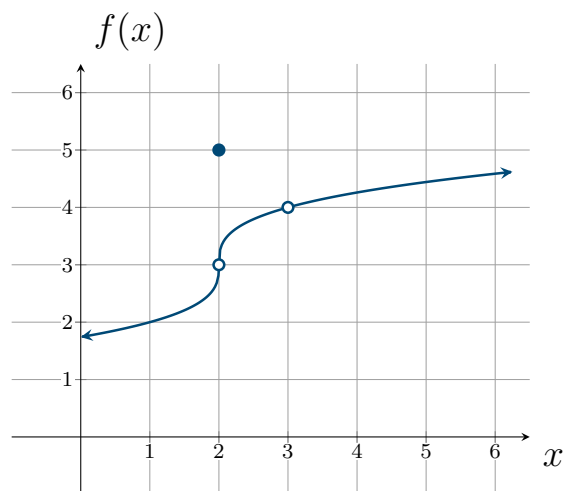
$$f(2) = 5$$

$$\lim_{x \rightarrow 2} f(x) = 3$$

$$f(3) \text{ and } \lim_{x \rightarrow 3} f(x)$$

$$f(3) \text{ DNE}$$

$$\lim_{x \rightarrow 3} f(x) = 4$$



**Example.** Find the limit of the following functions at the value specified:

[Graphs](#)

$$f(x) = x^3 \quad \text{at } x = 2$$

$$\lim_{x \rightarrow 2} f(x) = 2^3 = 8$$

$$g(x) = \begin{cases} x + 2, & x \neq 1 \\ 1, & x = 1 \end{cases} \quad \text{at } x = 1$$

$$\lim_{x \rightarrow 1} g(x) = 1 + 2 = 3$$

$$h(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases} \quad \text{at } x = 0$$

$$\lim_{x \rightarrow 0} h(x) \text{ DNE}$$

$$j(x) = \frac{1}{(x-1)^2} \quad \text{at } x = 1$$

$$\lim_{x \rightarrow 1} j(x) = \infty$$

$$k(x) = 4 \quad \text{at } x = 0$$

$$\lim_{x \rightarrow 0} k(x) = 4$$

## Theorem 1: Properties of Limits

Suppose

$$\lim_{x \rightarrow a} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = M$$

Then

1.  $\lim_{x \rightarrow a} [f(x)]^r = \left[ \lim_{x \rightarrow a} f(x) \right]^r$  where  $r$  is a positive constant
2.  $\lim_{x \rightarrow a} c f(x) = c \lim_{x \rightarrow a} f(x)$  where  $c$  is a real number
3.  $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = L \pm M$
4.  $\lim_{x \rightarrow a} [f(x)g(x)] = \left[ \lim_{x \rightarrow a} f(x) \right] \left[ \lim_{x \rightarrow a} g(x) \right] = LM$
5.  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{M}$  provided  $M \neq 0$

**Example.** Use the above theorem to evaluate the following limits:

$$\begin{aligned} \lim_{x \rightarrow 1} (5x^{3/2} - 2) &= \lim_{x \rightarrow 1} 5x^{3/2} - \lim_{x \rightarrow 1} 2 = 5 \lim_{x \rightarrow 1} x^{3/2} - 2 \\ &= 5(1) - 2 = \boxed{3} \end{aligned}$$

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$$\begin{aligned} \lim_{x \rightarrow 3} \frac{2x^3 \sqrt{x^2 + 7}}{x + 1} &= \frac{\lim_{x \rightarrow 3} 2x^3 \sqrt{x^2 + 7}}{\lim_{x \rightarrow 3} x + 1} = \frac{2(3)^3 \lim_{x \rightarrow 3} \sqrt{x^2 + 7}}{3 + 1} \\ &= \frac{54 \sqrt{\lim_{x \rightarrow 3} x^2 + \lim_{x \rightarrow 3} 7}}{4} = \frac{54 \sqrt{3^2 + 7}}{4} = \frac{54 \sqrt{16}}{4} = \boxed{54} \end{aligned}$$

Suppose that  $\lim_{x \rightarrow a} f(x) = 0$  and  $\lim_{x \rightarrow a} g(x) = 0$ . Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

has an **indeterminate form** of  $\frac{0}{0}$ . To evaluate such a limit, we replace the given function with a function that's equivalent everywhere except at  $x = a$ , and then evaluate the limit.

**Example.** Evaluate the following

$$\begin{aligned} \lim_{t \rightarrow 2} \frac{4t^2 - 16}{t - 2} &= \lim_{t \rightarrow 2} \frac{4(t^2 - 4)}{t - 2} = \lim_{t \rightarrow 2} \frac{4(t - 2)(t + 2)}{t - 2} \\ &= \lim_{t \rightarrow 2} 4(t + 2) \\ &= \boxed{16} \end{aligned}$$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\sqrt{4+h} - 2}{h} \left( \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} \right) &= \lim_{h \rightarrow 0} \frac{4+h-4}{h(\sqrt{4+h} + 2)} \\ &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{4+h} + 2)} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{4+h} + 2} = \boxed{\frac{1}{4}} \end{aligned}$$

## Limit of a Function at Infinity

The function  $f$  has the limit  $L$  as  $x$  increases without bound, written

$$\lim_{x \rightarrow \infty} f(x) = L$$

if  $f(x)$  can be made arbitrarily close to  $L$  by taking  $x$  large enough.

The function  $f$  has the limit  $M$  as  $x$  decreases without bound, written

$$\lim_{x \rightarrow -\infty} f(x) = M$$

if  $f(x)$  can be made arbitrarily close to  $M$  by taking  $x$  to be negative and sufficiently large enough in absolute value.

When the above limits exist, the equations  $y = L$  and/or  $y = M$  are called **horizontal asymptotes**.

**Example.** Evaluate the following infinite limits

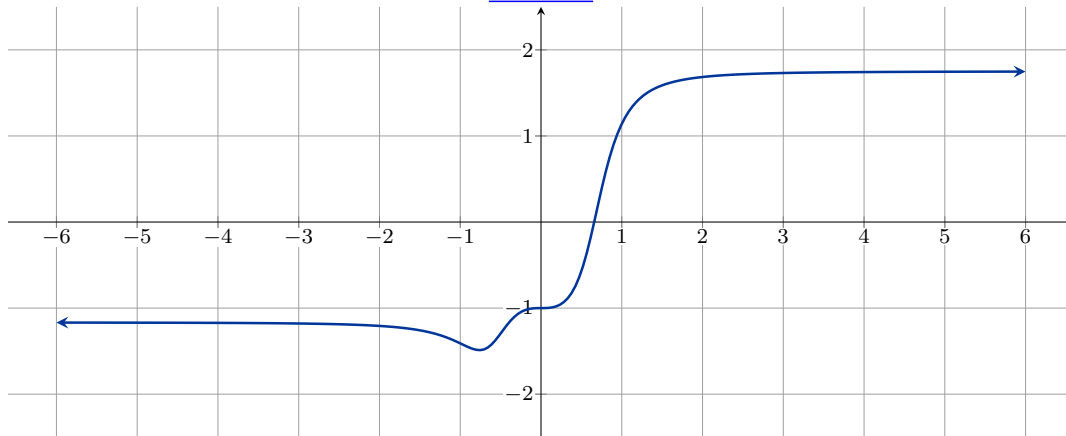
$$\lim_{x \rightarrow \infty} \frac{2x^2 + 3x - 4}{x^2 - 7x + 1} \left( \frac{1/x^2}{1/x^2} \right) = \lim_{x \rightarrow \infty} \frac{\frac{2x^2}{x^2} + \frac{3x}{x^2} - \frac{4}{x^2}}{\frac{x^2}{x^2} - \frac{7x}{x^2} + \frac{1}{x^2}} = \lim_{x \rightarrow \infty} \frac{2 + \frac{3}{x} - \frac{4}{x^2}}{1 - \frac{7}{x} + \frac{1}{x^2}} = \boxed{2}$$

$$\lim_{x \rightarrow -\infty} \frac{3x^2 + 4}{2x^3} \left( \frac{1/x^3}{1/x^3} \right) = \lim_{x \rightarrow -\infty} \frac{\frac{3x^2}{x^3} + \frac{4}{x^3}}{\frac{2x^3}{x^3}} = \lim_{x \rightarrow -\infty} \frac{\frac{3}{x} + \frac{4}{x^3}}{2} = \frac{0}{2} = \boxed{0}$$

$$\lim_{x \rightarrow \pm\infty} \frac{3x^5 + 2x^3 - 4}{x^4 + 4x^2 - 1} \left( \frac{1/x^4}{1/x^4} \right) = \lim_{x \rightarrow \pm\infty} \frac{\frac{3x^5}{x^4} + \frac{2x^3}{x^4} - \frac{4}{x^4}}{\frac{x^4}{x^4} + \frac{4x^2}{x^4} - \frac{1}{x^4}} = \lim_{x \rightarrow \pm\infty} \frac{3x + \frac{2}{x} - \frac{4}{x^4}}{1 + \frac{4}{x^2} - \frac{1}{x^4}} = \lim_{x \rightarrow \pm\infty} 3x = \boxed{\infty}$$

$$\sqrt{x^6} = \begin{cases} x^3, & x \rightarrow \infty \\ -x^3, & x \rightarrow -\infty \end{cases}$$

Graph



$$\lim_{x \rightarrow -\infty} \frac{7x^3 - 2}{-x^3 + \sqrt{25x^6 - 4}} \quad \left( \begin{array}{l} 1/x^3 \\ 1/x^3 \end{array} \right)$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{7x^3}{x^3} - \frac{2}{x^3}}{-\frac{x^3}{x^3} + \frac{1}{x^3} \sqrt{25x^6 - 4}}$$

$$= \lim_{x \rightarrow \infty} \frac{7 - \frac{2}{x^3}}{-1 + \sqrt{\frac{1}{x^6} \sqrt{25x^6 - 4}}}$$

$$= \lim_{x \rightarrow \infty} \frac{7}{-1 + \sqrt{\frac{25x^6}{x^6} - \frac{4}{x^6}}}$$

$$= \lim_{x \rightarrow \infty} \frac{7}{-1 + \sqrt{25}} = \boxed{-\frac{7}{6}}$$

$$\lim_{x \rightarrow \infty} \frac{7x^3 - 2}{-x^3 + \sqrt{25x^6 - 4}} \quad \left( \begin{array}{l} 1/x^3 \\ 1/x^3 \end{array} \right)$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{7x^3}{x^3} - \frac{2}{x^3}}{-\frac{x^3}{x^3} + \frac{1}{x^3} \sqrt{25x^6 - 4}}$$

$$= \lim_{x \rightarrow \infty} \frac{7 - \frac{2}{x^3}}{-1 + \sqrt{\frac{1}{x^6} \sqrt{25x^6 - 4}}}$$

$$= \lim_{x \rightarrow \infty} \frac{7}{-1 + \sqrt{\frac{25x^6}{x^6} - \frac{4}{x^6}}}$$

$$= \lim_{x \rightarrow \infty} \frac{7}{-1 + \sqrt{25}} = \boxed{\frac{7}{4}}$$

**Example.** The company *Custom Office* makes a line of executive desks. It is estimated that the total cost of making  $x$  *Senior Executive Model* desks is

$$C(x) = 100x + 200,000$$

dollars per year. The average cost of making  $x$  desks is given by

$$\overline{C}(x) = \frac{C(x)}{x}$$

Compute  $\lim_{x \rightarrow \infty} \overline{C}(x)$  and interpret the result.

$$\lim_{x \rightarrow \infty} \frac{100x + 200,000}{x} = \lim_{x \rightarrow \infty} 100 + \frac{200,000}{x} = 100$$

As the number of units,  $x$ , increases, the average cost per unit approaches (decreases) to \$100 per unit

### Theorem 2

For all  $n > 0$ ,

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x^n} = 0$$

provided that  $\frac{1}{x^n}$  is defined.