8.2: Reflexivity, Symmetry, and Transitivity

Definition.
Let R be a relation on a set A.

1. R is **reflexive** if, and only if, for every $x \in A$, $x\,R\,x$.

$$\forall x \in A, (x, x) \in R$$

2. R is **symmetric** if, and only if, for every $x, y \in A$, if $x\,R\,y$ then $y\,R\,x$.

$$\forall x, y \in A, \text{ if } (x, y) \in R \text{ then } (y, x) \in R$$

3. R is **transitive** if, and only if, for every $x, y, z \in A$, if $x\,R\,y$ and $y\,R\,z$, then $x\,R\,z$.

$$\forall x, y, z \in A, \text{ if } (x, y) \in R \text{ and } (y, z) \in R \text{ then } (x, z) \in R$$

Note: A relation R is

not reflexive $\Leftrightarrow \exists x \in A$ such that $x \not R\, x$
or $(x, x) \notin R$.

not symmetric $\Leftrightarrow \exists x, y \in A$ such that $x\,R\,y$ but $y \not R\, x$
or $(x, y) \in R$ but $(y, x) \notin R$.

not transitive $\Leftrightarrow \exists x, y, z \in A$ such that $x\,R\,y$ and $y\,R\,z$, but $x \not R\, z$
or $(x, y) \in R$ and $(y, z) \in R$, but $(x, z) \notin R$.

irreflexive $\Leftrightarrow \forall x \in A, x \not R\, x$

asymmetric $\Leftrightarrow \forall x, y \in A$, if $x\,R\,y$ then $y \not R\, x$

intransitive $\Leftrightarrow \forall x, y, z \in A$, if $x\,R\,y$ and $y\,R\,z$, then $x \not R\, z$

Example. Define a relation R on $\mathbb{R}$ as follows:

$$x\,R\,y \Leftrightarrow x = y.$$

Is R reflexive?

Yes

$\forall x \in \mathbb{R}$, $x = x$

$x R x$

Is R symmetric?

Yes

$\forall x, y \in \mathbb{R}$

$x = y \Leftrightarrow y = x$

$x R y \quad y R x$

Is R transitive?

Yes

$\forall x, y, z$

$x = y$ & $y = z \Rightarrow x = z$

$x R y \quad y R z \quad x R z$

Example. Define a relation R on $\mathbb{R}$ as follows:

$$x\,R\,y \Leftrightarrow x < y.$$

Is R reflexive?

No

$0 \not< 0$

Is R symmetric?

No

$1 < 2$, but $2 \not< 1$

Is R transitive?

Yes

$\forall x, y, z$

$x < y$ & $y < z \Rightarrow x < z$

$x R y \quad y R z \quad x R z$

Example. Let $A = \{0, 1, 2, 3\}$ and define relations R, S, and T on A as follows:

$$R = \{(0,0), (0,1), (0,3), (1,0), (1,1), (2,2), (3,0), (3,3)\}$$
$$S = \{(0,0), (0,2), (0,3), (2,3)\}$$
$$T = \{(0,1), (2,3)\}$$

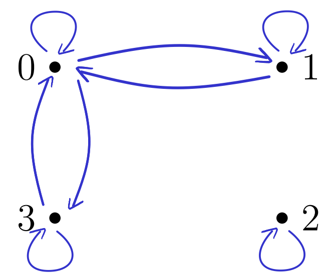
For each relation, draw the directed graph, then identify if it is reflexive, symmetric, and/or transitive.

R

Reflexive - Yes; each node maps to itself

Symmetric - Yes; both directions exist for each arrow

Transitive - No; (1,0) and (0,3) exist, but (1,3) does not exist

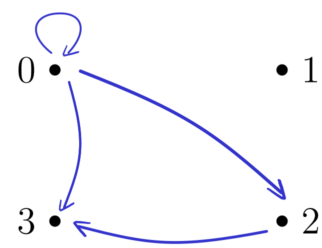


S

Reflexive - No; only 0 maps to itself

Symmetric - No; connections are only one way

Transitive - Yes; (0,2) and (2,3) exist, and so does (0,3)
No other pairs exist for the hypothesis

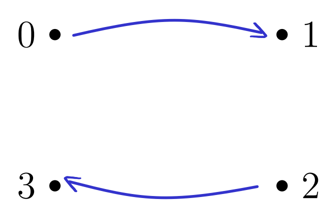


T

Reflexive - No (This is irreflexive)

Symmetric - No; connections are only one way

Transitive - Yes; vacuously true since no pairs exist for hypothesis



Example. Define a relation T on $\mathbb{Z}$ as follows:

$$\forall m, n \in \mathbb{Z},\ m\,T\,n \Leftrightarrow 3 \mid (m-n).$$

This relation is called **congruence modulo 3.**

Is T reflexive?

Yes

Let $m \in \mathbb{Z}$

If $m T m$

$\Leftrightarrow 3 \mid m-m$

$\Leftrightarrow 3 \mid 0$

which is always true

$(0 = 3 \cdot 0 + 0)$

Is T symmetric?

Yes

Let $m, n \in \mathbb{Z}$

If $m T n$

$\Leftrightarrow 3 \mid (m-n)$

$\Leftrightarrow m-n = 3k,\ k \in \mathbb{Z}$

$\Leftrightarrow -(m-n) = 3(-k),\ k \in \mathbb{Z}$

$\Leftrightarrow 3 \mid -(m-n)$

$\Leftrightarrow 3 \mid n-m$

$\Leftrightarrow n T m$

Is T transitive?

Yes

Let $l, m, n \in \mathbb{Z}$

If $l T m$ and $m T n$

$\Leftrightarrow \begin{cases} l-m = 3k, & k \in \mathbb{Z} \\ m-n = 3j, & j \in \mathbb{Z} \end{cases}$

$\Leftrightarrow (l-m) + (m-n) = 3k + 3j$

$\Leftrightarrow l-n = 3(k+j)$

Since $k+j \in \mathbb{Z}$, then

$3 \mid l-n \Leftrightarrow l T n$

Example. Define a relation S on $\mathbb{R}$ as follows:

$$\forall x, y \in \mathbb{R},\ x\,S\,y \Leftrightarrow |x| + |y| = 1.$$

Is S reflexive?

No

Consider $x = 2$

$2 \not S 2$ since $|2| + |2| \neq 1$

Note: There are values of x where this works

e.g. $x = \pm\frac{1}{2}$

Is S symmetric?

Yes

$$x\,S\,y \Leftrightarrow |x| + |y| = 1$$
$$\Leftrightarrow |y| + |x| = 1$$
$$\Leftrightarrow y\,S\,x$$

Is S transitive?

No

Consider $(x, y) = (1, 0)$ and $(y, z) = (0, 1)$

$1\,S\,0$ and $0\,S\,1$

but $1 \not S 1$