

## 1 1.1: Solutions of Linear Equations and Inequalities in One Variable

### Definition.

A **function**  $f$  is a special relation between  $x$  and  $y$  such that each input  $x$  results in *at most* one  $y$ . The symbol  $f(x)$  is read “ $f$  of  $x$ ” and is called the **value of  $f$  at  $x$**

**Example.** Let  $f(x) = \frac{x^2}{2} + x$ . Evaluate the following:

$$f(1)$$

$$f\left(\frac{1}{2}\right)$$

$$f(-2)$$

$$f(0)$$

$$f(f(x))$$

### Composite Functions:

Let  $f$  and  $g$  be functions of  $x$ . Then, the **composite functions**  $g$  of  $f$  (denoted  $g \circ f$ ) and  $f$  of  $g$  (denoted  $f \circ g$ ) are defined as:

$$(g \circ f)(x) = g(f(x))$$

$$(f \circ g)(x) = f(g(x))$$

**Example.** Let  $g = x - 1$ . Find:

$$(g \circ f)(x)$$

$$(f \circ g)(x)$$

## Operations with Functions:

Let  $f$  and  $g$  be functions of  $x$  and define the following:

Sum	$(f + g)(x) = f(x) + g(x)$
Difference	$(f - g)(x) = f(x) - g(x)$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$
Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ if $g(x) \neq 0$

### Definition.

An **expression** is a meaningful string of numbers, variables and operations:

$$3x - 2$$

An **equation** is a statement that two quantities or algebraic expressions are equal:

$$3x - 2 = 7$$

A **solution** is a value of the variable that makes the equation true:

$$3(3) - 2 = 7$$

$$9 - 2 = 7$$

$$7 = 7$$

A **solution set** is the set of ALL possible solutions of an equation:

$3x - 2 = 7$  only has the solution  $x = 3$ ,

$2(x - 1) = 2x - 2$  is true for all possible values of  $x$ .

## Properties of Equality:

**Substitution Property:** The equation formed by substituting one expression for an equal expression is equivalent to the original equation:

$$\begin{aligned}3(x - 3) - \frac{1}{2}(4x - 18) &= 4 \\3x - 9 - 2x + 9 &= 4 \\x &= 4\end{aligned}$$

**Addition Property:** The equation formed by adding the same quantity to both sides of an equation is equivalent to the original equation:

$$\begin{array}{ll}x - 4 = 6 & x + 5 = 12 \\x - 4 + 4 = 6 + 4 & x + 5 + (-5) = 12 + (-5) \\x = 10 & x = 7\end{array}$$

**Multiplication Property:** The equation formed by multiplying both sides of an equation by the same *nonzero* quantity is equivalent to the original equation:

$$\begin{array}{ll}\frac{1}{3}x = 6 & 5x = 20 \\3\left(\frac{1}{3}x\right) = 3(6) & \frac{5x}{5} = \frac{20}{5} \\x = 18 & x = 4\end{array}$$

## Solving a linear equation:

Using the properties of equality above, we can solve any linear equation in 1 variable:

**Example.** Solve  $\frac{3x}{4} + 3 = \frac{x-1}{3}$

1. Eliminate fractions:

$$12\left(\frac{3x}{4} + 3\right) = 12\left(\frac{x-1}{3}\right)$$

2. Remove/evaluate parenthesis:

$$9x + 36 = 4x - 4$$

3. Use addition property to isolate the variable to one side:

$$9x + 36 \text{--}36 \text{--}4x = 4x - 4 \text{--}36 \text{--}4x$$

4. Use multiplication property to isolate variable:

$$\frac{5x}{5} = \frac{-40}{5}$$

5. Verify solution via substitution:

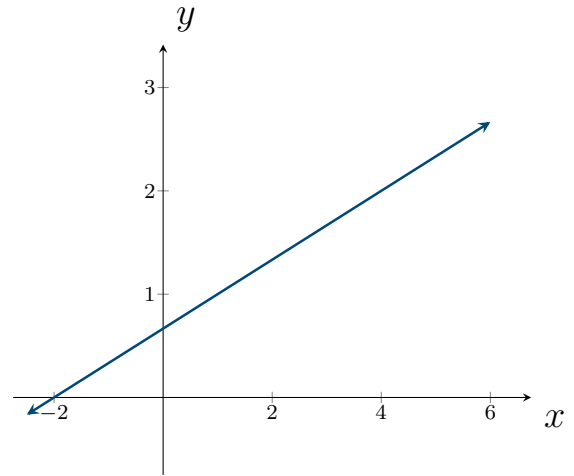
$$\underbrace{\frac{3(-8)}{4} + 3}_{-6 + 3 = -3} \stackrel{?}{=} \underbrace{\frac{(-8) - 1}{3}}_{\frac{-9}{3} = -3}$$

**Example.** Solve the following:

$$\frac{3x+1}{2} = \frac{x}{3} - 3$$

$$\frac{2x-1}{x-3} = 4 + \frac{5}{x-3}$$

**Example.** Solve  $-2x + 6y = 4$  for  $y$



**Example.** Suppose that the relationship between a firm's profit,  $P$ , and the number of items sold,  $x$ , can be described by the equation

$$5x - 4P = 1200$$

a) How many units must be produced and sold for the firm to make a profit of \$150?

b) Solve this equation for  $P$  in terms of  $x$ . Then, find the profit when 240 units are sold.

**Definition.**

An **inequality** is a statement that one quantity is greater than (or less than) another quantity.

**Properties of Inequalities**

**Substitution Property:** The inequality formed by substituting one expression for an equal expression is equivalent to the original inequality:

$$5x - 4x + 2 < 6$$

$$x < 4 \Rightarrow \text{The solution set is } \{x : x < 6\}$$

**Addition Property:** The inequality formed by adding the same quantity to both sides of an inequality is equivalent to the original inequality:

$$x - 4 < 6$$

$$x - 4 + 4 < 6 + 4$$

$$x < 10$$

$$x + 5 \geq 12$$

$$x + 5 + (-5) \geq 12 + (-5)$$

$$x \geq 7$$

**Multiplication Property** The inequality formed by multiplying both sides of an inequality by the same *positive* quantity is equivalent to the original inequality. The direction of the inequality is flipped when multiplying by a *negative* quantity:

$$\frac{1}{3}x > 6$$

$$3\left(\frac{1}{3}x\right) > 3(6)$$

$$x > 18$$

$$5x - 5 \leq 6x + 20$$

$$-x \leq 25$$

$$x \geq -25$$

**Example.** Solve

$$-x + 8 \leq 2x - 4$$

first by gathering the  $x$  variable on the left, then again on the right. See that the multiplication property holds in both cases.

