7.4: Estimating the Population Proportion with Confidence Intervals

Definition.
Suppose that we wish to estimate a population proportion p based on a sample proportion $\hat{p}$.

- A **confidence interval** is an interval about the point estimate $\hat{p}$ that we can be confident contains the true population proportion p:

$$\hat{p} \pm m$$

- The **margin of error (ME)** is half the width of the confidence interval. When estimating a population proportion, the margin of error is

$$m = z^* SE$$

- The **confidence level** measures how often the estimation method is successful. A larger confidence level results in a larger margin of error.

Recall the standard error (SE) for population proportions is

$$SE = \sqrt{\frac{p(1-p)}{n}}$$

Some common values for the margin of error:

Confidence Level	Margin of Error
99.7%	$3.0 \cdot SE$
99%	$2.58 \cdot SE$
95%	$1.96 \cdot SE$
90%	$1.645 \cdot SE$
80%	$1.28 \cdot SE$

Example. In 2018, Gallup took a poll of 497 randomly selected adults who teach K–12 students and 42% of them said that digital devices (smartphones, tablets, computers) had “mostly helpful” effects on students’ education.

$\hat{p}$ n

Check that the conditions of the CLT apply.

1. SRS – Gallup polls are random & independent
2. $n\hat{p} \geq 10$ $\quad 497(0.42) = 208.74 \geq 10$

   $n(1-\hat{p}) \geq 10$ $\quad 497(1-0.42) = 288.26 \geq 10$
3. $N \geq 10n$ $\quad N \geq 10(497) = 4970$

It's safe to assume there are AT LEAST 4,970 K-12 teachers

Estimate the standard error.

$$SE = \sqrt{\frac{0.42(1-0.42)}{497}}$$

$$= 0.0221$$

Give the 95% confidence interval and interpret the result.

$\#\text{ successes} = 0.42(497) = 208.74 \approx 209$

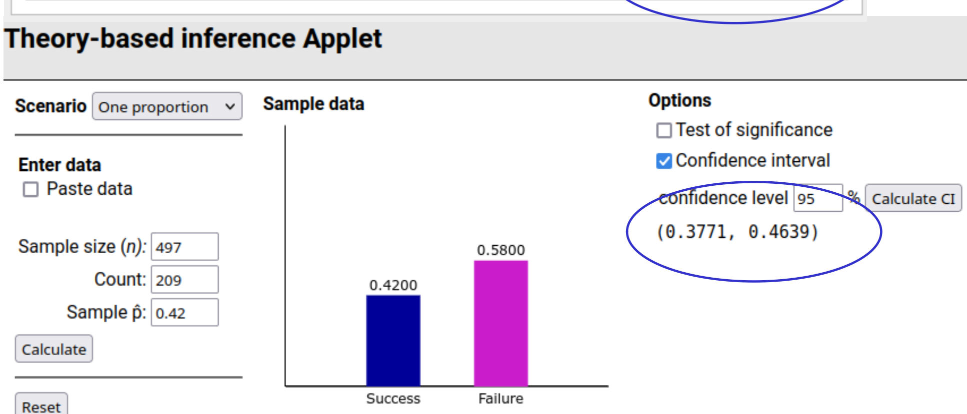
Options

One sample proportion summary confidence interval:
p : Proportion of successes
Method: Standard-Wald

95% confidence interval results:

Proportion	Count	Total	Sample Prop.	Std. Err.	Critical Z	L. Limit	U. Limit
p	209	497	0.42052314	0.022142916	1.959964	0.37712382	0.46392246

We can assert with 95% confidence that the true population proportion is contained within the interval (0.3771,0.4639).



Example. After the Great Recession, the Pew Research Center noted there seemed to be a decline in households that rented their homes and were looking to purchase homes. However, Pew reported that in 2016 "a solid 72%" of renters reported that they wished to buy their own home. Pew reports that the "margin of error at 95% confidence level is plus-or-minus 5.4 points."

State the confidence interval in interval form and interpret the result.

$$\hat{p} \pm ME$$

$$(0.72 - 0.054,\ 0.72 + 0.054)$$

$$(0.666,\ 0.774)$$

We can assert with 95% confidence that the true population proportion is contained within the interval (0.666,0.774).

8.1: The Essential Ingredients of Hypothesis Testing

Definition.

- A **hypothesis test** is a procedure that enables us to choose between two claims.
- The **null hypothesis**, H_0, represents the current belief, or status quo.
- The **alternative hypothesis**, H_a, is what we wish to test.

A hypothesis test has 4 steps:

1. Formulate your null and alternative hypotheses
2. Examine or collect data
3. Compare data to our expectations; is the result significant?
4. Interpret the results

Two-Sided	One-Sided (Left)	One-Sided (Right)
$H_0 : p = p_0$	$H_0 : p = p_0$	$H_0 : p = p_0$
$H_a : p \neq p_0$	$H_a : p < p_0$	$H_a : p > p_0$

Example. When flipping a coin, it is considered fair if both sides of the coin have an equally likely chance of appearing face up. Suppose we have a coin that we believe might be unfair. Let p be the proportion of times where heads appears face up. Formulate the null and alternative hypotheses.

$$H_0 : p = \overbrace{0.5}^{p_0}$$

$$H_a : p \neq 0.5$$

Example. Historically, about 70% of all U.S. adults were married. A sociologist who asks whether marriage rates in the United States have declined will take a random sample of U.S. adults and record whether or not they are married.

Write the null and alternative hypotheses.

$H_0: p = 0.70$

$H_a: p < 0.70$

Example. An Internet retail business is trying to decide whether to pay a search engine company to upgrade its advertising. In the past 15% of customers who visited the company's web page by clicking on the advertisement bought something. The search engine company offers to do an experiment: for one day a random sample of customers will see the retail business's ad in a more prominent position to try and increase the proportion of customers who make a purchase.

Write the null and alternative hypotheses.

$H_0: p = 0.15$

$H_a: p > 0.15$

Definition.

- The **significance level**, denoted by α, is the probability of rejecting the null hypothesis when it is actually true (false positive).
- A **test statistic** is similar to a z score comparing the alternative hypothesis to the null hypothesis:

$$z = \frac{\hat{p} - p_0}{SE}, \quad \text{where} \quad SE = \sqrt{\frac{p_0(1-p_0)}{n}}$$

- The p**-value** is the probability that the null hypothesis is true. When the p-value is
  - greater than α, we fail to reject the null hypothesis
  - less than or equal to α, we reject the null hypothesis

Note: Hypothesis tests don't prove the null hypothesis!

8.2: Hypothesis Testing in Four Steps

1. **Hypothesize:** formulate your hypotheses
2. **Check conditions:**
   - **Random and Independent:** The sample must be randomly collected from the population, and observations are independent of each other
   - **Large Sample:** The sample size must be large enough for at least 10 successes, $np_0 \geq 10$, and 10 failures, $n(1-p_0) \geq 10$.
   - **Large Population:** If the sample is collected without replacement, the population of size N must be at least 10 times bigger than the sample: $N \geq 10n$

   If these conditions are met, we compute the test statistic for the One-Proportion z-Test which follows a z-distribution:

   $$z = \frac{\hat{p} - p_0}{SE}, \qquad \text{where} \qquad SE = \sqrt{\frac{p_0(1-p_0)}{n}}$$

3. **Compute:** Stating a significance level, compute the observed test statistic z and/or p-value.
4. **Interpret:** Decide whether to reject or fail to reject the null hypothesis.

Example. Unlike flipping a coin, spinning a coin leads to a biased outcome. Suppose we spun a coin 60 times, and saw a sample proportion of $\hat{p} = 0.35$.

Formulate the null and alternative hypotheses

$H_0: p = 0.50$

$H_a: p \neq 0.50$

Check the conditions required to perform a hypothesis test.

1. & 4. SRS ✓

3. $N \geq 10n$

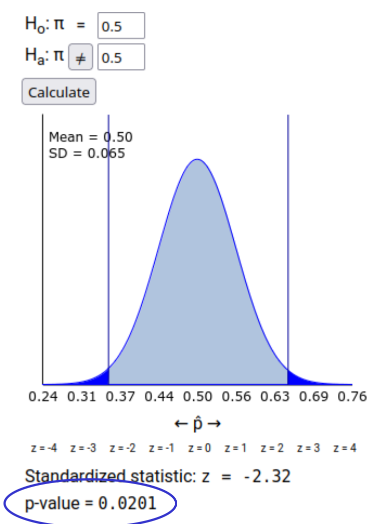
2. $np_0 \geq 10$ ✓ $\quad 60(0.50) = 30$

$n(1-p_0) \geq 10$ ✓ $\quad 60(1-0.50) = 30$

$N \geq 10(60) = 600$

It is safe to assume that the total number of coin spins is at least 600.

Find the test statistic and p-value



$$Z = \frac{0.35 - 0.5}{SE} = -2.324$$

$$SE = \sqrt{\frac{0.5(1-0.5)}{60}} = 0.0645$$

Stat Crunch

One sample proportion summary hypothesis test:

p : Proportion of successes

H_0 : p = 0.5

H_A : p ≠ 0.5

Hypothesis test results:

Proportion	Count	Total	Sample Prop.	Std. Err.	Z-Stat	P-value
p	21	60	0.35	0.064549722	-2.32379	0.0201

Using a significance level of $\alpha = 0.05$, decide whether to reject or fail to reject the null hypothesis.

Since p-value $= 0.0201 < \alpha = 0.05$, we reject the null hypothesis and assert that the true population proportion $p \neq 0.5$.

Example. A group of medical researchers knew from previous studies that in the past, about 39% of all men between the ages of 45 and 59 were regularly active. Researchers were concerned that this percentage had declined over time. For this reason, they did selected a random sample, without replacement, of 1927 men in this age group and interviewed them. Out of this sample, 680 said they were regularly active.

n

$\hat{p} = \frac{680}{1927} = 0.353$

Formulate the null and alternative hypotheses

$H_0: p = 0.39$

$H_a: p < 0.39$

Check the conditions required to perform a hypothesis test.

1. & 4. SRS ✓

2. $np_0 \geq 10$ ✓ $1927(0.39) = 751.53$

$n(1-p_0) \geq 10$ ✓ $1927(1-0.39) = 1175.47$

3. $N \geq 10n$

$N \geq 10(1927) = 19{,}270$

It is safe to assume that the total number of men between the ages of 45 and 59 is at least 19,270.

Find the test statistic and p-value

$$Z = \frac{0.353 - 0.39}{SE} = -3.341$$

$$SE = \sqrt{\frac{0.39(1-0.39)}{1927}} = 0.0111$$

One sample proportion summary hypothesis test:

p : Proportion of successes

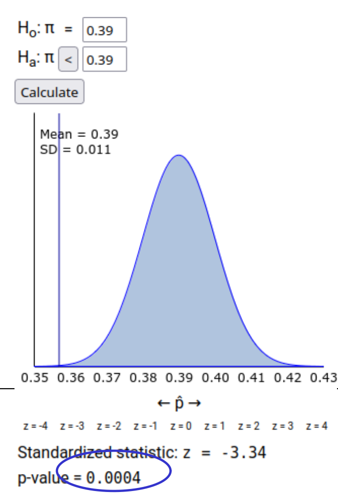
H_0 : p = 0.39

H_A : p < 0.39

Hypothesis test results:

Proportion	Count	Total	Sample Prop.	Std. Err.	Z-Stat	P-value
p	680	1927	0.35288012	0.011111082	-3.3407975	0.0004

Using a significance level of $\alpha = 0.05$, decide whether to reject or fail to reject the null hypothesis.



Since p-value $= 0.0004 < \alpha = 0.05$, we reject the null hypothesis and assert that the true population proportion $p < 0.39$