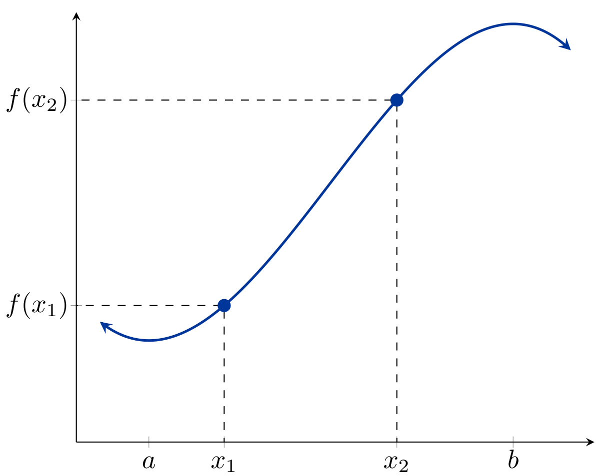
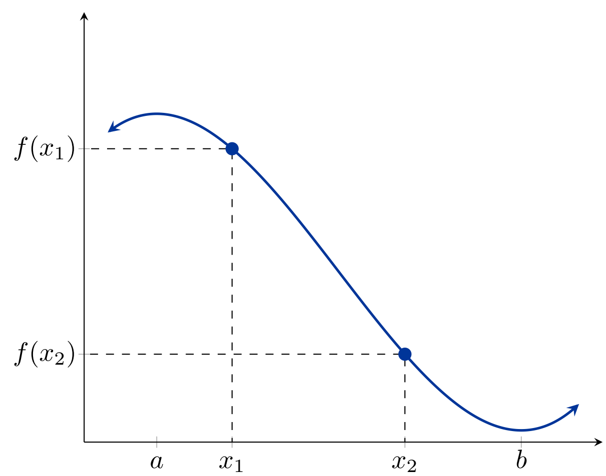
4.1: Applications of the First Derivative

Definition.
Consider the function $f(x)$ on the interval (a, b). Given *any* two numbers x_1 and x_2 in (a, b) where $x_1 < x_2$, we say f is

increasing if $f(x_1) < f(x_2)$



decreasing if $f(x_1) > f(x_2)$



Thus, for every value of x on the interval (a, b), if

- $f'(x) > 0$, then f is increasing on (a, b).
- $f'(x) < 0$, then f is decreasing on (a, b).
- $f'(x) = 0$, then f is constant on (a, b).

Example. Find the intervals where $f(x) = x^2$ is increasing and decreasing.

$f'(x) = 2x$

Solve $f'(x) = 0$

$2x = 0$

$x = 0$

0

$2x$ $-$ $+$

Inc: $(0, \infty)$

Dec: $(-\infty, 0)$

Determining intervals where a function is increasing or decreasing.

1. Find all values of x such that $f'(x) = 0$ or $f'(x)$ is undefined.
2. Determine the sign of $f'(x)$ on each open interval.

Example. Suppose that f is continuous everywhere and

$$f'(x) = \frac{(x-1)(x+2)}{(x-4)^2(x+5)}.$$

We see that $f'(-2) = f(1) = 0$ and $f(-5)$ and $f(4)$ are undefined. Complete a sign chart to show where $f(x)$ is increasing and decreasing.

-5 $\quad -2$ $\quad 1$ $\quad 4$

	-10	-3	0	2	10
$x-1$	$-$	$-$	$-$	$+$	$+$
$x+2$	$-$	$-$	$+$	$+$	$+$
$(x-4)^2$	$+$	$+$	$+$	$+$	$+$
$x+5$	$-$	$+$	$+$	$+$	$+$
	$-$	$+$	$-$	$+$	$+$
	↘	↗	↘	↗	↗

$\boxed{\begin{array}{l} \text{Inc: } (-5,-2) \cup (1,4) \cup (4,\infty) \\ \text{Dec: } (-\infty,-5) \cup (-2,1) \end{array}}$

Example. Find the intervals where the following functions are increasing and decreasing:

$$f(x) = x^3 - 3x^2 - 24x + 32$$

Solve $f'(x) = 0$

& $f'(x)$ DNE

Graph

$f'(x) = 3x^2 - 6x - 24 = 3(x+2)(x-4)$

Quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \dfrac{6 \pm \sqrt{(-6)^2 - 4(3)(-24)}}{2(3)} = \dfrac{6 \pm 18}{6}$ $\to x = 4$, $\to x = -2$

	-10	-2 0	4 10
3	+	+	+
$x+2$	−	+	+
$x-4$	−	−	+
	+	−	+
	↗	↘	↗

$\boxed{\text{Inc: } (-\infty, -2) \cup (4, \infty) \quad \text{Dec: } (-2, 4)}$

$$g(x) = (x+1)^{2/3}$$

$g'(x) = \frac{2}{3}(x+1)^{-1/3} = \dfrac{2}{3(x+1)^{1/3}}$

Solve $g'(x) = 0$ ✗

& $g'(x)$ DNE $\to$ $3(x+1)^{1/3} \neq 0$

$\Rightarrow$ $x \neq -1$

	-10	-1 0
2	+	+
3	+	+
$(x+1)^{1/3}$	−	+
	−	+
	↘	↗

$\boxed{\text{Inc: } (-1, \infty) \quad \text{Dec: } (-\infty, -1)}$

$$h(x) = x + \frac{1}{x} = x + x^{-1}$$

$$h'(x) = 1 - x^{-2} = 1 - \frac{1}{x^2}$$

Solve $h'(x) = 0$ & $h'(x)$ DNE

$$1 - \frac{1}{x^2} = 0$$

$$1 = \frac{1}{x^2}$$

$$x^2 = 1$$

$$x = \pm 1$$

$$x \neq 0$$

	-1		0		1	
	-10		-0.5	0.5		10
$1 - \frac{1}{x^2}$	$+$		$-$	$-$		$+$
	↗		↘	↘		↗

$$\boxed{\begin{array}{l} \text{Inc: } (-\infty, -1) \cup (1, \infty) \\ \text{Dec: } (-1, 0) \cup (0, 1) \end{array}}$$

$$j(x) = \frac{x^2}{1 - x^2}$$

$$j'(x) = \frac{(1-x^2)\,2x - x^2(-2x)}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}$$

Solve $j'(x) = 0$ & $j'(x)$ DNE

$$\frac{2x}{(1-x^2)^2} = 0$$

$$2x = 0$$

$$x = 0$$

$$(1-x^2)^2 \neq 0$$

$$1 - x^2 \neq 0$$

$$1 \neq x^2$$

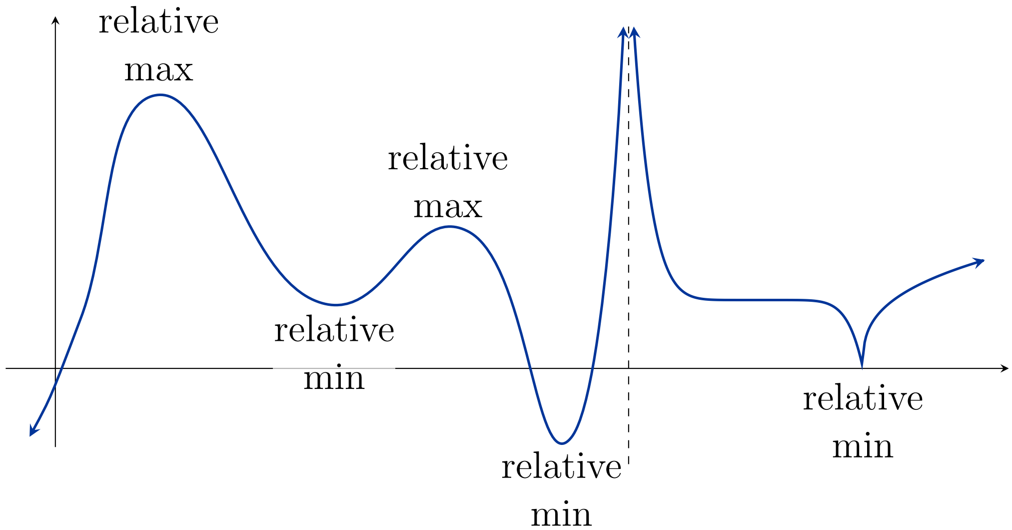
$$\pm 1 \neq x$$

	-1	0	1	
	-10	-0.5	0.5	10
$2x$	$-$	$-$	$+$	$+$
$(1-x^2)^2$	$+$	$+$	$+$	$+$
	$-$	$-$	$+$	$+$
	↘	↘	↗	↗

$$\boxed{\begin{array}{l} \text{Inc: } (0, 1) \cup (1, \infty) \\ \text{Dec: } (-\infty, -1) \cup (-1, 0) \end{array}}$$

Definition. (Relative Extrema)
A function f has a

- **relative maximum** at $x = c$ if $f(c) \geq f(x)$ for every x in (a, b)
- **relative minimum** at $x = c$ if $f(c) \leq f(x)$ for every x in (a, b)



Definition.
A **critical point** of a function f is any number x in the domain of f such that $f'(x) = 0$ or $f'(x)$ does not exist.

Procedure for Finding the Relative Extrema of a Continuous Function f
The First Derivative Test:

1. Determine the critical points of f.
2. Determine the sign change of $f'(x)$ to the left and right of each critical point:
   If, at $x = c$, $f'(x)$ ...

   a) changes sign from *positive* to *negative*, then f has a *relative maximum*

   b) changes sign from *negative* to *positive*, then f has a *relative minimum*

   c) does not change sign, then f does *not* have a relative extremum

   at $x = c$.

Example. Consider the function $f(x) = 6x - x^3$. Graph

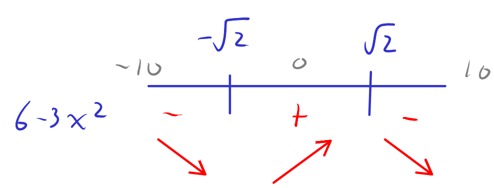
Use $f'(x)$ to find the intervals on which the function is increasing and decreasing.

$f'(x) = 6 - 3x^2$

$f'(x) = 0$ $\quad$ $f'(x)$ DNE

$6 - 3x^2 = 0$ $\quad$ —

$x = \pm\sqrt{2}$



Inc: $(-\sqrt{2}, \sqrt{2})$

Dec: $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

Identify the function's local extreme values (e.g. "local max of __ at $x =$ __")

Local max of $f(\sqrt{2}) = 4\sqrt{2}$ at $x = \sqrt{2}$

Local min of $f(-\sqrt{2}) = 4\sqrt{2}$ at $x = -\sqrt{2}$

Example. Find the relative maximums/relative minimums of the following:

$$f(x) = x^3 - 3x^2 - 24x + 32$$

Graph

$f'(x) = 3x^2 - 6x - 24 = 3(x+2)(x-4)$

Quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \dfrac{6 \pm \sqrt{(-6)^2 - 4(3)(-24)}}{2(3)} = \dfrac{6 \pm 18}{6}$ → $x = 4$, $x = -2$

	-10	-2	0	4	10
3	$+$		$+$		$+$
$x+2$	$-$		$+$		$+$
$x-4$	$-$		$-$		$+$
	$+$ ↗		$-$ ↘		$+$ ↗

$$\boxed{\begin{array}{l} \text{max of } f(-2) = 60 \text{ at } x = -2 \\ \text{min of } f(4) = -48 \text{ at } x = 4 \end{array}}$$

$$g(x) = (x+1)^{2/3}$$

$g'(x) = \dfrac{2}{3}(x+1)^{-1/3} = \dfrac{2}{3(x+1)^{1/3}}$

Solve $g'(x) = 0$ ✗

& $g'(x)$ DNE → $3(x+1)^{1/3} \neq 0$

$\Rightarrow x \neq -1$

	-10	-1	0
2	$+$		$+$
3	$+$		$+$
$(x+1)^{1/3}$	$-$		$+$
	$-$ ↘		$+$ ↗

$$\boxed{\text{Min of } g(-1) = 0 \text{ at } x = -1}$$

$$h(x) = x + \frac{1}{x} = x + x^{-1}$$

$$h'(x) = 1 - x^{-2} = 1 - \frac{1}{x^2}$$

Solve $h'(x) = 0$ & $h'(x)$ DNE

$$1 - \frac{1}{x^2} = 0$$

$$1 = \frac{1}{x^2}$$

$$x^2 = 1$$

$$x = \pm 1$$

$$x \neq 0$$

	-10	-1	-0.5	0	0.5	1	10
$1 - \frac{1}{x^2}$	$+$	|	$-$	|	$-$	|	$+$
	↗		↘		↘		↗

$$\boxed{\begin{array}{l} \text{max of } h(-1) = -2 \text{ at } x = -1 \\ \text{min of } h(1) = 2 \text{ at } x = 1 \end{array}}$$

$$j(x) = \frac{x^2}{1 - x^2}$$

$$j'(x) = \frac{(1-x^2)\,2x - x^2(-2x)}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}$$

Solve $j'(x) = 0$ & $j'(x)$ DNE

$$\frac{2x}{(1-x^2)^2} = 0$$

$$2x = 0$$

$$x = 0$$

$$(1-x^2)^2 \neq 0$$

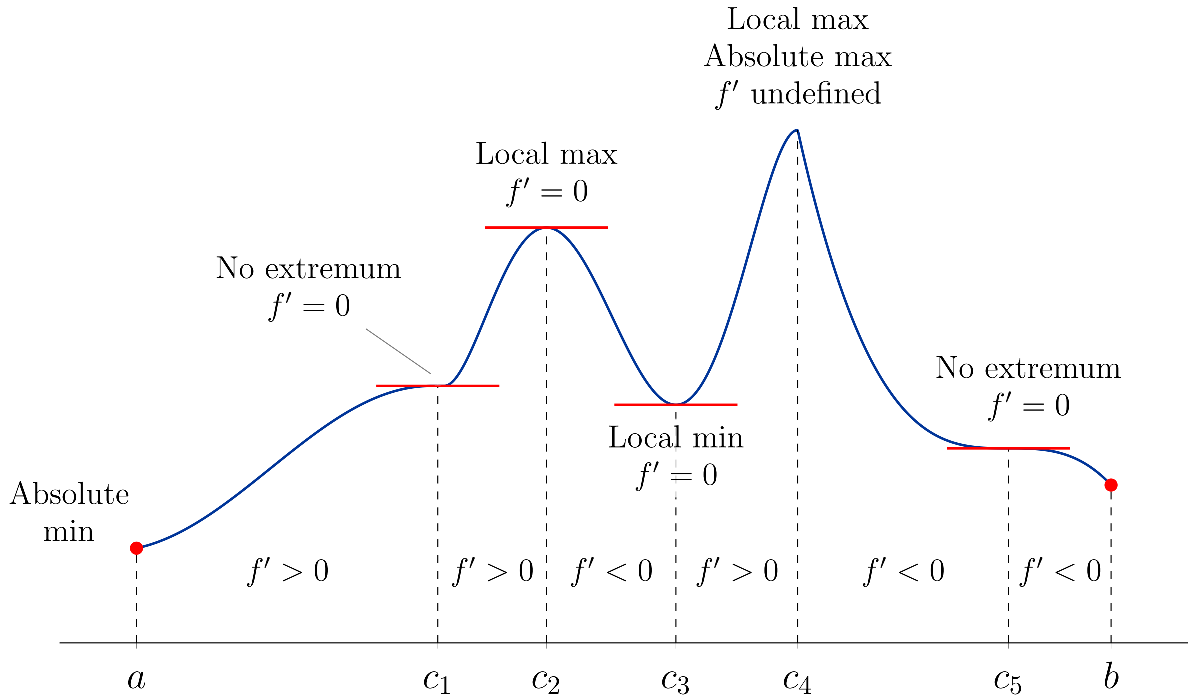
$$1 - x^2 \neq 0$$

$$1 \neq x^2$$

$$\pm 1 \neq x$$

	-10	-1	-0.5	0	0.5	1	10
$2x$	$-$	|	$-$	|	$+$	|	$+$
$(1-x^2)^2$	$+$	|	$+$	|	$+$	|	$+$
	$-$		$-$		$+$		$+$
	↘		↘		↗		↗

$$\boxed{\text{max of } j(0) = 0 \text{ at } x = 0}$$


Local max
Absolute max
f' undefined
Local max
$f' = 0$
No extremum
$f' = 0$
Local min
$f' = 0$
No extremum
$f' = 0$
Absolute
min
$f' > 0$
$f' > 0$
$f' < 0$
$f' > 0$
$f' < 0$
$f' < 0$
a
c_1
c_2
c_3
c_4
c_5
b