6.1: Set Theory: Definitions and the Element Method of Proof

Element Argument: The Basic Method for Proving that One set is a Subset of Another
Let sets X and Y be given. To prove that $X \subseteq Y$,

1. **suppose** that x is a particular but arbitrarily chosen element of X,
2. **show** that x is an element of Y

Example. Define sets A and B as follows:

$$A = \{m \in \mathbb{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbb{Z}\}$$
$$B = \{n \in \mathbb{Z} \mid n = 3s \text{ for some } s \in \mathbb{Z}\}$$

Outline a proof that $A \subseteq B$

Prove that $A \subseteq B$

Disprove that $B \subseteq A$

Definition.
Given sets A and B, A **equals** B, written $\mathbf{A = B}$, if, and only if, every element of A is in B and every element of B is in A:

$$A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A.$$

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Is $A = B$?

Definition.
Given an integer n and a positive integer d, when n is divided by d, then

$$\begin{aligned} n \operatorname{div} d &= \text{ the integer quotient} \\ n \bmod d &= \text{ the nonnegative integer remainder} \end{aligned}$$

If n and d are integers and $d > 0$, then

$$n \operatorname{div} d = q \quad \text{and} \quad n \bmod d = r \quad \Leftrightarrow \quad n = dq + r$$

Example. Compute the following:

$32 \operatorname{div} 9,\ \ 32 \bmod 9$

$365 \operatorname{div} 7,\ \ 365 \bmod 7$

Example. If it is currently 11:00 am, what time will it be in

37 hours?

51 hours?

11 hours?

-1 hours?

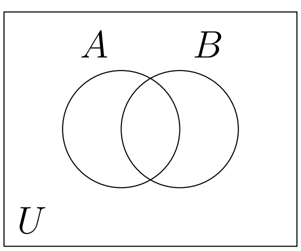
Example. Let $A = \left\{4, \sqrt{16}, 19 \bmod 15\right\}$ and $B = \{12 \bmod 8\}$. Is $A \subseteq B$? Is $B \subseteq A$?

Definition.
Let A and B be subsets of a universal set U.

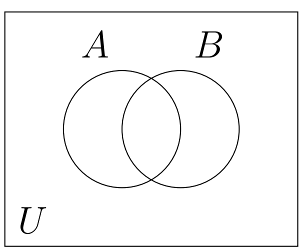
1. The **union** of A and B is the set of all elements that are in at least one of A or B.
$$A \cup B = \{x \in U \,|\, x \in A \text{ or } x \in B\}$$
2. The **intersection** of A and B is the set of all elements that are common to both A and B.
$$A \cap B = \{x \in U \,|\, x \in A \text{ and } x \in B\}$$
3. The **difference** of A and B is the set of all elements that are in B and not A.
$$B - A = \{x \in U \,|\, x \in B \text{ and } x \notin A\}$$
4. The **complement** of A is the set of all elements in U that are not in A.
$$A^c = \{x \in U \,|\, x \notin A\}$$

Example. Represent the following sets using the Venn diagrams below:

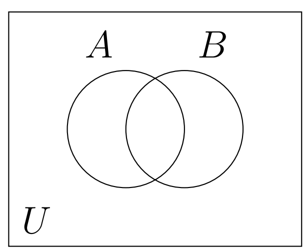
$A \cup B$



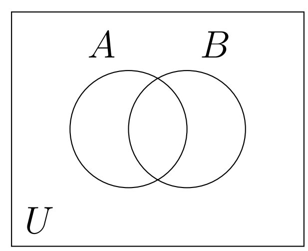
$A \cap B$



$B - A$



A^c



Example. Let the universal set be the set $U = \{a, b, c, d, e, f, g\}$, and let $A = \{a, c, e, g\}$ and $B = \{d, e, f, g\}$. Find

$A \cup B$ $\qquad$ $A \cap B$

$B - A$ $\qquad$ A^c

Definition.
Given real numbers a and b with $a \leq b$:

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\} \qquad (a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$
$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\} \qquad [a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

Example. Let the universal set be $\mathbb{R}$, and let $A = (-1, 0]$ and $B = [0, 1)$. Find

$A \cup B$ $\qquad$ $A \cap B$

$B - A$ $\qquad$ A^c

Definition.
Given sets $A_0, A_1, A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n,

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i, \text{ for at least one } i = 0, 1, 2, \ldots, n\}$$
$$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i, \text{ for every } i = 0, 1, 2, \ldots, n\}$$

Example. For each positive integer i, let $A_i = \left\{x \in \mathbb{R} \,\middle|\, -\frac{1}{i} < x < \frac{1}{i}\right\} = \left(-\frac{1}{i}, \frac{1}{i}\right)$. Find

$A_1 \cup A_2 \cup A_3$

$A_1 \cap A_2 \cap A_3$

$\displaystyle\bigcup_{i=1}^{\infty} A_i$

$\displaystyle\bigcap_{i=1}^{\infty} A_i$

Definition.
The **empty set** (or **null set**), denoted $\varnothing$, is the set with no elements.

$$\{1,3\} \cap \{2,4\} = \varnothing$$

Two sets are called **disjoint** if, and only if, they have no elements in common:

$$A \cap B = \varnothing.$$

Sets $A_1, A_2, A_3, \ldots$ are **mutually disjoint** (or **pairwise disjoint**) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common:

$$A_i \cap A_j = \varnothing \text{ whenever } i \neq j.$$

Example.
Let $A_1 = \{3,5\}$, $A_2 = \{1,4,6\}$, and $A_3 = \{2\}$. Are A_1, A_2, and A_3 mutually disjoint?

Let $B_1 = \{2,4,6\}$, $B_2 = \{3,7\}$, and $B_3 = \{4,5\}$. Are B_1, B_2, B_3 mutually disjoint?

Definition.
A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, \ldots\}$ is a **partition** of a set A if, and only if,

1. A is the union of all the A_i;
2. the sets $A_1, A_2, A_3, \ldots$ are mutually disjoint.

Example.
Let $A = \{1, 2, 3, 4, 5, 6\}$, $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$, and $A_3 = \{5, 6\}$. Is $\{A_1, A_2, A_3\}$ a partition of A?

Let $\mathbb{Z}$ be the set of all integers and let

$$T_i = \{n \in \mathbb{Z} \,|\, n = 3k + i, \text{ for some integer } k\}.$$

Is $\{T_0, T_1, T_3\}$ a partition of $\mathbb{Z}$?

Definition.
Given a set A, the **power set** of A, denoted $\mathscr{P}(A)$, is the set of all subsets of A.

Example. Find $\mathscr{P}(\{x, y\})$.