

6.4: Boolean Algebras, Russell's Paradox, and the Halting Problem

Definition.

A **Boolean algebra** is a set B together with two operations, generally denoted $+$ and \cdot , such that for all a and b in B both $a + b$ and $a \cdot b$ are in B and the following axioms are assumed to hold:

1. *Commutative Laws:* For all a and b in B ,

$$a + b = b + a \text{ and } a \cdot b = b \cdot a$$

2. *Associative Laws:* For all a and b in B ,

$$(a + b) + c = a + (b + c) \text{ and } (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

3. *Distributive Laws:* For all a and b in B ,

$$a + (b \cdot c) = (a + b) \cdot (a + c) \text{ and } a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

4. *Identity Laws:* There exist distinct elements 0 and 1 in B such that for each a in B ,

$$a + 0 = a \text{ and } a \cdot 1 = a$$

5. *Complement Laws:* For each a in B , there exists an element in B , denoted \bar{a} and called the **complement** or **negation** of a , such that

$$a + \bar{a} = 1 \text{ and } a \cdot \bar{a} = 0$$

Properties of a Boolean Algebra

Let B be any Boolean algebra.

1. *Uniqueness of the Complement Laws:* For all a and x in B , if $a + x = 1$ and $a \cdot x = 0$, then $x = \bar{a}$.
2. *Uniqueness of 0 and 1:* If there exists x in B such that $a + x = a$ for every a in B , then $x = 0$, and if there exists y in B such that $a \cdot y = a$ for every a in B , then $y = 1$.
3. *Double Complement Law:* For every $a \in B$, $\overline{\overline{a}} = a$.
4. *Idempotent Laws:* For every $a \in B$,

$$a + a = a \text{ and } a \cdot a = a.$$

5. *Universal Bound Laws:* For every $a \in B$,

$$a + 1 = 1 \text{ and } a \cdot 0 = 0.$$

6. *De Morgan's Laws:* For all a and $b \in B$,

$$\overline{a + b} = \bar{a} \cdot \bar{b} \text{ and } \overline{a \cdot b} = \bar{a} + \bar{b}.$$

7. *Absorption Laws:* For all a and $b \in B$,

$$(a + b) \cdot a = a \text{ and } (a \cdot b) + a = a.$$

8. *Complements of 0 and 1:*

$$\overline{0} = 1 \text{ and } \overline{1} = 0.$$

Example. Prove that for all elements a in a Boolean algebra B :

$$\overline{(\bar{a})} = a.$$

$$\text{Let } x = \bar{a} \rightarrow (\bar{\bar{a}}) = \bar{x} \rightarrow \bar{x} + x = x + \bar{x} \quad \text{Commutative law for +}$$

$$= 1 \quad \text{Complement law for +}$$

$$\rightarrow (\bar{\bar{a}}) + \bar{a} = 1$$

$$\rightarrow a = \overline{(\bar{a})} \quad \text{Uniqueness of Complement Laws}$$

$$a + a = a.$$

$$a = a + 0 \quad \text{Identity law for +}$$

$$= a + (a \cdot \bar{a}) \quad \text{Complement laws for \cdot}$$

$$= (a+a) \cdot (a+\bar{a}) \quad \text{Distributive law for + over \cdot}$$

$$= (a+a) \cdot 1 \quad \text{Complement law for +}$$

$$= a + a \quad \text{Identity law for \cdot}$$

Example. Prove that for all elements a in a Boolean algebra B :

$$a \cdot a = a.$$

$$\begin{aligned}
 a &= a \cdot 1 && \text{Identity law for } \cdot \\
 &= a \cdot (a + \bar{a}) && \text{Complement law for } + \\
 &= (a \cdot a) + (a \cdot \bar{a}) && \text{Distributive law for } \cdot \text{ over } + \\
 &= (a \cdot a) + 0 && \text{Complement law for } + \\
 &= a \cdot a && \text{Identity law for } +
 \end{aligned}$$

$$(a + b) \cdot a = a.$$

$$\begin{aligned}
 (a + b) \cdot a &= a \cdot (a + b) && \text{Associative law for } \cdot \\
 &= a \cdot a + a \cdot b && \text{Distributive law for } \cdot \text{ over } + \\
 &= a + a \cdot b && \text{From above} \\
 &= a \cdot 1 + a \cdot b && \text{Identity law for } \cdot \\
 &= a \cdot (1 + b) && \text{Distributive law for } \cdot \text{ over } + \\
 &= a \cdot (b + 1) && \text{Commutative law for } + \\
 &= a \cdot 1 && \text{WebAssign question 6.4.002} \\
 &= a && \text{Identity law for } \cdot
 \end{aligned}$$

Russell's Paradox

Define the following set S :

$$S = \{A \mid A \text{ is a set and } A \notin A\}.$$

Is S an element of itself?

The Barber Puzzle: In a certain town, there is a male barber who shaves all those men, and only those men, who do not shave themselves.

Does the barber shave himself?

Yes and no; this is a contradiction.

Is the sentence “The barber shaves himself” a statement?

This is not a statement because it is neither true or false.

Example. Determine whether each sentence is a statement:

If $1 + 1 = 3$, then $1 = 0$.

This is a statement because it is false

This sentence is false and $1 + 1 = 3$.

This is not a statement since it is neither true or false

The Halting Problem (Alan M. Turing)

There is no computer algorithm that will accept any algorithm X and data set D as input and then will output “halts” or “loops forever” to indicate whether or not X terminates in a finite number of steps when X is run with data set D .

Logical Equivalences

For all statement variables p , q , and r :

$$p \vee q \equiv q \vee p$$

$$p \wedge q \equiv q \wedge p$$

$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$$

$$p \vee (q \vee r) \equiv (p \vee q) \vee r$$

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

$$p \vee \mathbf{c} \equiv p$$

$$p \wedge \mathbf{t} \equiv p$$

$$p \vee \sim p \equiv \mathbf{t}$$

$$p \wedge \sim p \equiv \mathbf{c}$$

$$\sim (\sim p) \equiv p$$

$$p \wedge p \equiv p$$

$$p \vee p \equiv p$$

$$p \vee \mathbf{t} \equiv \mathbf{t}$$

$$p \wedge \mathbf{c} \equiv \mathbf{c}$$

$$\sim (p \vee q) \equiv \sim p \wedge \sim q$$

$$\sim (p \wedge q) \equiv \sim p \vee \sim q$$

$$p \vee (p \wedge q) \equiv p$$

$$p \wedge (p \vee q) \equiv p$$

$$\sim \mathbf{t} \equiv \mathbf{c}$$

$$\sim \mathbf{c} \equiv \mathbf{t}$$

Set Properties

For all sets A , B , and C :

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cup \emptyset = A$$

$$A \cap U = A$$

$$A \cup A^c = U$$

$$A \cap A^c = \emptyset$$

$$(A^c)^c = A$$

$$A \cup A = A$$

$$A \cap A = A$$

$$A \cup U = U$$

$$A \cap \emptyset = \emptyset$$

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

$$U^c = \emptyset$$

$$\emptyset^c = U$$