

# Math 125 Class notes Spring 2026

To accompany  
*Discrete Mathematics with Applications*  
by *Epp*

Peter Westerbaan

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# Math 125 Formula Sheet

## 1.1: Variables

**Definition.**

A **variable** is a placeholder for something which may or may not be unknown.

**Example.** Is there a number with the following property: doubling it and adding 3 gives the same result as squaring it?

- Is there a number  $x$  with the property that  $2x + 3 = x^2$ ?
- Is there a number  $\square$  with the property that  $2 \cdot \square + 3 = \square^2$ ?

**Example.** No matter what number might be chosen, if it is greater than 2, then its square is greater than 4.

- No matter what number  $n$  might be chosen, if  $n$  is greater than 2,  
then  $n^2$  is greater than 4.

**Example.** Use variables to rewrite the following sentences:

Are there numbers with the property that the sum of their squares equals the square of their sum?

Given any real number, its square is nonnegative.

**Definition.**

- A **universal statement** says that a certain property is true for all elements in a set.
- A **conditional statement** says that if one thing is true, then some other thing also has to be true.
- Given a property that may or may not be true, an **existential statement** says that there is at least one thing for which the property is true.

**Definition.**

A **universal conditional statement** is both universal and conditional:

For every animal  $a$ , if  $a$  is a dog, then  $a$  is a mammal.

Conditional statements can be rewritten in ways that make them appear more to be purely universal or purely conditional:

If  $a$  is a dog, then  $a$  is a mammal.

All dogs are mammals

**Example.** Rewrite the following universal condition statement:

For every real number  $x$ , if  $x$  is nonzero then  $x^2$  is positive.

If a real number is nonzero, then its square \_\_\_\_\_.

For every nonzero real number  $x$ , \_\_\_\_\_.

If  $x$  \_\_\_\_\_, then \_\_\_\_\_.

The square of any nonzero real number is \_\_\_\_\_.

All nonzero real numbers have \_\_\_\_\_.

**Definition.**

A **universal existence statement** is a statement that is universal because its first part says that a certain property is true for all objects of a given type, and it is existential because its second part asserts the existence of something:

Every real number has an additive inverse.

In the above example, note that the particular additive inverse depends on the given real number:

For every real number  $r$ , there is an additive inverse for  $r$ .

**Example.** Rewrite the following universal existence statement:

Every pot has a lid

All pots \_\_\_\_\_.

For every pot  $P$ , there is \_\_\_\_\_.

For every pot  $P$ , there is a lid  $L$  such that \_\_\_\_\_.

**Definition.**

An **existential universal statement** is a statement that is existential because its first part asserts that a certain object exists and is universal because its second part says that the object satisfies a certain property for all things of a certain kind:

There is a positive integer that is less than or equal to every positive integer.

The number one satisfies the above statement, which can also be rewritten:

There is a positive integer  $m$  that is less than or equal to every positive integer.

**Example.** Rewrite the following existence universal statement:

There is a person in my class who is at least as old as every person in my class.

Some \_\_\_\_\_ is at least as old as \_\_\_\_\_.

There is a person  $p$  in my class such that  $p$  is \_\_\_\_\_.

There is a person  $p$  in my class with the property that for every person  $q$  in my class,  $p$  is \_\_\_\_\_.



## 1.2: The Language of Sets

### Definition.

- A **set** is a collection of objects.
- If  $S$  is a set, then we use
  - $x \in S$  to denote that the element  $x$  is in the set  $S$ .
  - $x \notin S$  to denote that the element  $x$  is *not* in the set  $S$ .
- The **set-roster notation** is used to denote all elements in a set between braces:

$$S = \{1, 2, \dots, 100\}$$

Here, we see that  $67 \in S$ , but  $1337 \notin S$ .

- The **axiom of extension** says that a set is completely determined by what its elements are – not the order in which they are listed.

### Example.

Let  $A = \{1, 2, 3\}$ ,  $B = \{3, 1, 2\}$ , and  $C = \{1, 1, 2, 3, 3, 3\}$ . What are the elements of  $A$ ,  $B$ , and  $C$ ? How are  $A$ ,  $B$ , and  $C$  related?

Is  $\{0\} = 0$ ?

How many elements are in the set  $\{1, \{1\}\}$ ?

For each nonnegative integer  $n$ , let  $U_n = \{n, -n\}$ . Find  $U_1$ ,  $U_2$ , and  $U_0$ .

Certain sets of numbers are so frequently referred to that they are given special names and symbols:

- N** or  $\mathbb{N}$  The set of all **natural numbers**
- Z** or  $\mathbb{Z}$  The set of all **integers**
- Q** or  $\mathbb{Q}$  The set of all **rational numbers**, or quotient of integers
- R** or  $\mathbb{R}$  The set of all **real numbers**

*Note:* We may additionally use superscripts to indicate further properties of these sets:

- $\mathbb{Z}^+$  or  $\mathbb{Z}^{>0}$  The set of *positive* integers
- $\mathbb{Q}^-$  or  $\mathbb{Q}^{<0}$  The set of *negative* rational numbers
- $\mathbb{R}^{nonneg}$  or  $\mathbb{R}^{\geq 0}$  The set of *nonnegative* real numbers

*Note:* Different sources denote the natural numbers  $\mathbb{N}$  as  $\mathbb{Z}^+$  or  $\mathbb{Z}^{\geq 0}$ .

**Definition. (Set-Builder Notation)**

Let  $S$  be a set and let  $P(x)$  be a property that elements of  $S$  may or may not satisfy. We may define a new set to be **the set of all elements  $x$  in  $S$  such that  $P(x)$  is true**. We denote this set as follows:

$$\{x \in S \mid P(x)\}$$

the set of all                      such that

**Example.** Describe each of the following sets:

$$\{x \in \mathbb{R} \mid -2 < x < 5\}$$

$$\{x \in \mathbb{Z} \mid -2 < x < 5\}$$

$$\{x \in \mathbb{Z}^+ \mid -2 < x < 5\}$$

**Definition.**

If  $A$  and  $B$  are sets, then  $A$  is called a **subset** of  $B$ , written  $A \subseteq B$ , if, and only if, every element of  $A$  is also an element of  $B$ :

$A \subseteq B$  means that for every element  $x$ , if  $x \in A$ , then  $x \in B$ .

$A \not\subseteq B$  means that there is at least one element  $x$ , such that  $x \in A$  and  $x \notin B$ .

$A$  is a **proper subset** of  $B$  if, and only if, every element of  $A$  is in  $B$ , but there is at least one element of  $B$  that is not in  $A$ :

$A \subsetneq B$  means that for every element  $x$ , if  $x \in A$ , then  $x \in B$ ,  
and there exists  $x \in B$  such that  $x \notin A$ .

**Example.** Let  $A = \mathbb{Z}^+$ ,  $B = \{n \in \mathbb{Z} \mid 0 \leq n \leq 100\}$ , and  $C = \{100, 200, 300, 400, 500\}$ . Evaluate the truth and falsity of each of the following statements.

$$B \subseteq A$$

$C$  is a proper subset of  $A$

$C$  and  $B$  have at least one element in common

$$C \subseteq B$$

$$C \subseteq C$$

**Example.** Determine which of the following statements are true:

$$2 \in \{1, 2, 3\}$$

$$\{2\} \in \{1, 2, 3\}$$

$$2 \subseteq \{1, 2, 3\}$$

$$\{2\} \subseteq \{1, 2, 3\}$$

$$\{2\} \subseteq \{\{1\}, \{2\}\}$$

$$\{2\} \in \{\{1\}, \{2\}\}$$

**Definition.**

Given elements  $a$  and  $b$ , the symbol  $(a, b)$  denotes the **ordered pair** consisting of  $a$  and  $b$  together with the specification that  $a$  is the first element of the pair, and  $b$  is the second element. Two ordered pairs  $(a, b)$  and  $(c, d)$  are equal if, and only if,  $a = c$  and  $b = d$ :

$$(a, b) = (c, d) \text{ means that } a = c \text{ and } b = d.$$

**Example.**

$$\text{Is } (1, 2) = (2, 1)?$$

$$\text{Is } \left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right)?$$

**Definition.**

Let  $n \in \mathbb{N}$  and let  $x_1, x_2, \dots, x_n$  be (not necessarily distinct) elements. The **ordered  $n$ -tuple**,  $(x_1, x_2, \dots, x_n)$ , consists of  $x_1, x_2, \dots, x_n$  together with the ordering: first  $x_1$ , then  $x_2$ , and so forth up to  $x_n$ . and ordered 2-tuple is called an **ordered pair**, and ordered 3-tuple is called an **ordered triple**.

Two ordered  $n$ -tuples  $(x_1, x_2, \dots, x_n)$  and  $(y_1, y_2, \dots, y_n)$  are **equal** if, and only if,  $x_1 = y_1, x_2 = y_2, \dots$ , and  $x_n = y_n$ :

$$(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n) \iff x_1 = y_1, x_2 = y_2, \dots, x_n = y_n.$$

**Example.**

$$\text{Is } (1, 2, 3, 4) = (1, 2, 4, 3)?$$

$$\text{Is } \left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right)?$$

**Definition.**

Given sets  $A_1, A_2, \dots, A_n$ , the **Cartesian product** of  $A_1, A_2, \dots, A_n$ , denoted

$$A_1 \times A_2 \times \cdots \times A_n$$

is the set of all ordered  $n$ -tuples  $(a_1, a_2, \dots, a_n)$  where  $a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n$ :

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n\}$$

**Example.** Let  $A = \{x, y\}$ ,  $B = \{1, 2, 3\}$ , and  $C = \{a, b\}$ . Find the following:

$$A \times B$$

$$B \times A$$

$$A \times A$$

How many elements are in  $A \times B$ ,  $B \times A$ , and  $A \times A$ ?

$$(A \times B) \times C$$

$$A \times B \times C$$

Describe  $\mathbb{R} \times \mathbb{R}$

**Definition.**

Let  $n \in \mathbb{N}$ . Given a finite set  $A$ , a **string of length  $n$  over  $A$**  is an ordered  $n$ -tuple of elements of  $A$  written without parentheses or commas. The elements of  $A$  are called the **characters** of the string. The **null string** over  $A$  is defined to be the “string” with no characters, often denoted  $\lambda$ , and is said to have length 0. If  $A = \{0, 1\}$ , then a string over  $A$  is called a **bit string**.

**Example.** Let  $A = \{a, b\}$ . List all strings of length 3 over  $A$  with at least two characters that are the same.



### 1.3: The Language of Relations and Functions

#### Definition.

Let  $A$  and  $B$  be sets. A **relation**  $R$  **from**  $A$  **to**  $B$  is a subset of  $A \times B$ . Given an ordered pair  $(x, y)$ ,  $x$  **is related to**  $y$  **by**  $R$ , written  $x R y$ , if, and only if,  $(x, y)$  is in  $R$ . The set  $A$  is called the **domain** of  $R$  and the set  $B$  is called its **co-domain**.

The notation for a relation  $R$  may be written symbolically as follows:

$$x R y \text{ means that } (x, y) \in R.$$

The notation  $x \not R y$  means that  $x$  is not related to  $y$  by  $R$ :

$$x \not R y \text{ means that } (x, y) \notin R.$$

**Example.** Let  $A = \{1, 2\}$  and  $B = \{1, 2, 3\}$  and define a relation  $R$  from  $A$  to  $B$  as follows; Given any  $(x, y) \in A \times B$ ,

$$(x, y) \in R \text{ means that } \frac{x - y}{2} \text{ is an integer.}$$

State explicitly which ordered pairs are in  $A \times B$  and which are in  $R$

Is  $1 R 3$ ?

Is  $2 R 3$ ?

Is  $2 R 2$ ?

What are the domain and co-domain of  $R$ ?

**Example.** Define a relation  $C$  from  $\mathbb{R}$  to  $\mathbb{R}$  as follows: For any  $(x, y) \in \mathbb{R} \times \mathbb{R}$ ,

$$(x, y) \in C \text{ means that } x^2 + y^2 = 1.$$

Is  $(1, 0) \in C$ ?

Is  $(0, 0) \in C$ ?

Is  $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \in C$ ?

Is  $-2 \in C$ ?

Is  $0 \in C$ ?

Is  $1 \in C$ ?

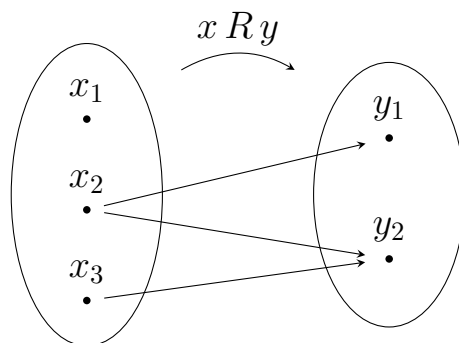
What are the domain and co-domain of  $C$ ?

Draw a graph for  $C$  by plotting the points of  $C$  in the Cartesian plane.

**Definition.**

Suppose  $R$  is a relation from set  $A$  to a set  $B$ . The **arrow diagram for  $R$**  is obtained as follows:

1. Represent the elements of  $A$  as points in one region and the elements of  $B$  as points in another region.
2. For each  $x$  in  $A$  and  $y$  in  $B$ , draw an arrow from  $x$  to  $y$  if, and only if,  $x$  is related to  $y$  by  $R$ .



**Example.** Let  $A = \{1, 2, 3\}$  and  $B = \{1, 3, 5\}$  and define relations  $S$  and  $T$  from  $A$  to  $B$  as follows: For every  $(x, y) \in A \times B$ ,

$(x, y) \in S$  means that  $x < y$

$T = \{(2, 1), (2, 5)\}$ .

Draw arrow diagrams for  $S$  and  $T$



**Definition.**

A **function**  $F$  from a set  $A$  to a set  $B$  is a relation with domain  $A$  and co-domain  $B$  that satisfies the following two properties:

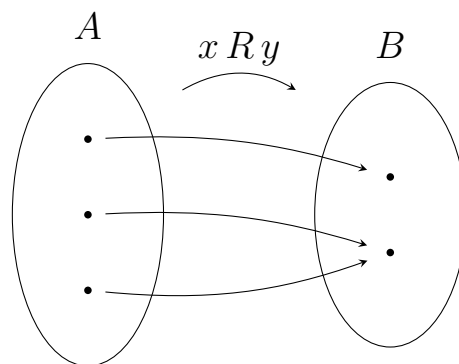
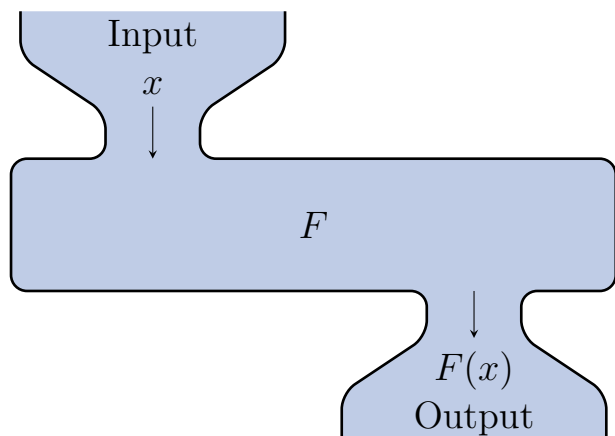
1. For every element  $x$  in  $A$ , there is an element  $y$  in  $B$  such that  $(x, y) \in F$ .
2. For all elements  $x$  in  $A$  and  $y$  and  $z$  in  $B$ ,

if  $(x, y) \in F$  and  $(x, z) \in F$ , then  $y = z$ .

*Note:* A relation from  $A$  to  $B$  is a function if, and only if,

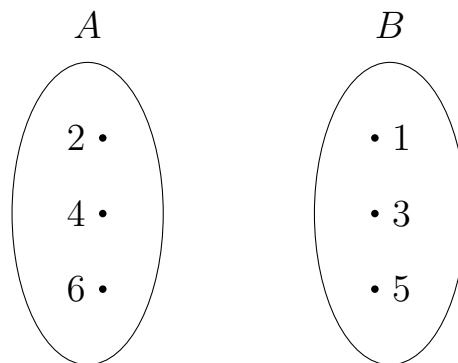
1. Every element of  $A$  is the first element of an ordered pair of  $F$
2. No two distinct ordered pairs in  $F$  have the same first element.

*Note:* If  $A$  and  $B$  are sets and  $F$  is a function from  $A$  to  $B$ , then given any element  $x$  in  $A$ , the unique element in  $B$  that is related to  $x$  by  $F$  is denoted  $F(x)$ , which is read “ $F$  of  $x$ ”.

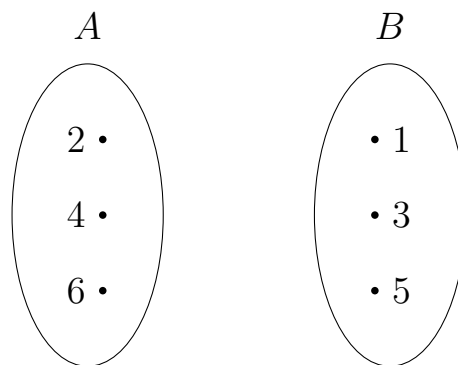


**Example.** Let  $A = \{2, 4, 6\}$  and  $B = \{1, 3, 5\}$ . Which of the relations  $R$ ,  $S$ , and  $T$  defined below are functions from  $A$  to  $B$ ?

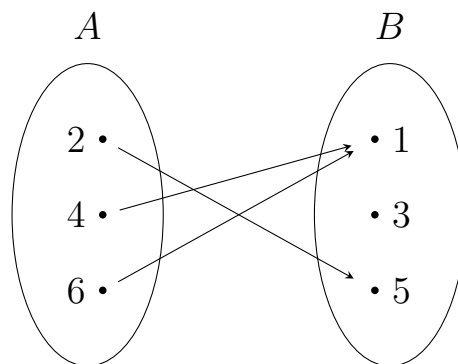
$$R = \{(2, 5), (4, 1), (4, 3), (6, 5)\}$$



For every  $(x, y) \in A \times B$ ,  $(x, y) \in S$  means that  $y = x + 1$ .



$T$  is defined by the arrow diagram



## 2.1: Logical Form and Logical Equivalence

**Definition.**

A **statement** (or **proposition**) is a sentence that is true or false, but not both.

**Example.** Determine which of the following are statements:

$$2 + 2 = 4$$

$$2 + 2 = 5$$

$$x^2 + 2 = 11$$

Today is Saturday.

She is a computer science major.

Jane is a computer science major.

**Definition. (Compound Statements)**

Let  $p$  and  $q$  be statement variables.

- The **negation** of  $p$  is “not  $p$ ”, and is denoted as  $\sim p$  (or  $\neg p$ )
- The **conjunction** of  $p$  and  $q$  is “ $p$  and  $q$ ”, and is denoted at  $p \wedge q$
- The **disjunction** of  $p$  and  $q$  is “ $p$  or  $q$ ”, and is denoted  $p \vee q$ .
- The **exclusive or** of  $p$  and  $q$  is “ $p$  x-or  $q$ ”, and is denoted  $p \oplus q$  (or  $p \text{ XOR } q$ )

The **order of operations** specifies that  $\sim$  is performed first.

**Example.** Consider the following statements:

$p$  : It is raining.

$q$  : It is sunny.

$r$  : It is cloudy.

Rewrite the following compound statements in words:

$$\sim p$$

$$p \vee q$$

$$q \wedge r$$

$$q \wedge \sim r$$

$$p \wedge (q \vee r)$$

$$p \oplus q$$

**Definition.**

A **statement form** (or **propositional form**) is an expression made up of statement variables (e.g.,  $p$ ,  $q$ , and  $r$ ), and logical connectives (e.g.  $\sim$ ,  $\wedge$ ,  $\vee$ , and  $\oplus$ ).

The **truth table** for a given statement form displays the truth values that correspond to all possible combinations of truth values for its component statement variables.

**Example.** Let  $p$  and  $q$  be statement variables. Fill out the following truth tables:

$p$	$\sim p$
T	
F	

$p$	$q$	$p \wedge q$	$p \vee q$	$p \oplus q$
T	T			
T	F			
F	T			
F	F			

$p$	$q$	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$(p \vee q) \wedge \sim (p \wedge q)$
T	T				
T	F				
F	T				
F	F				



**Example.** Construct a truth table for the statement form  $(p \wedge q) \vee \sim r$ .

**Definition.**

Two *statement forms* are called **logically equivalent** if, and only if, they have identical true values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms  $P$  and  $Q$  is denoted  $P \equiv Q$ .

**Example.** Use truth tables to test if the following statement forms are equivalent:

$p$  and  $\sim (\sim p)$

$\sim (p \wedge q)$  and  $\sim p \wedge \sim q$

**Definition. (De Morgan's Laws)**

The negation of an *and* statement is logically equivalent to the *or* statement in which each component is negated.

The negation of an *or* statement is logically equivalent to the *and* statement in which each component is negated.

**Example.** Use truth tables to show that the following statement forms are equivalent:

$$\sim (p \wedge q) \text{ and } \sim p \vee \sim q$$

$$\sim (p \vee q) \text{ and } \sim p \wedge \sim q$$

**Example.** Using De Morgan's law to write the negation of the following statements:

Jim is at least 6 feet tall and weighs at least 200 pounds.

The bus was late or Tom's watch was slow

$$-1 < x \leq 4$$

**Definition.**

A **tautology** is a statement form that is always true.

A **contradiction** is a statement form that is always false.

**Example.** Complete the truth tables for  $p \wedge \sim p$  and  $p \vee \sim p$

**Example.** Let **t** be a tautology, and **c** be a contradiction. Show that  $p \wedge \mathbf{t} \equiv p$  and  $p \wedge \mathbf{c} \equiv \mathbf{c}$

### Theorem 2.1.1 Logical Equivalences (p 49)

Given any statement variables  $p$ ,  $q$ , and  $r$ , a tautology  $\mathbf{t}$  and a contradiction  $\mathbf{c}$ , the following logical equivalences hold:

1. Commutative laws:

$$p \wedge q \equiv q \wedge p$$

$$p \vee q \equiv q \vee p$$

2. Associative laws:

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

3. Distributive laws:

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

4. Identity laws:

$$p \wedge \mathbf{t} \equiv p$$

$$p \vee \mathbf{c} \equiv p$$

5. Negation laws:

$$p \vee \sim p \equiv \mathbf{t}$$

$$p \wedge \sim p \equiv \mathbf{c}$$

6. Double negative law:

$$\sim(\sim p) \equiv p$$

7. Idempotent laws:

$$p \wedge p \equiv p$$

$$p \vee p \equiv p$$

8. Universal bound laws:

$$p \vee \mathbf{t} \equiv \mathbf{t}$$

$$p \wedge \mathbf{c} \equiv \mathbf{c}$$

9. De Morgan's laws:

$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$

$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$

10. Absorption laws:

$$p \wedge (p \vee q) \equiv p$$

$$p \vee (p \wedge q) \equiv p$$

11. Negations of  $\mathbf{t}$  and  $\mathbf{c}$ :

$$\sim \mathbf{t} \equiv \mathbf{c}$$

$$\sim \mathbf{c} \equiv \mathbf{t}$$