2.2: Conditional Statements

Definition.
If p and q are statement variables, the **conditional** of q by p is "If p then q", or "p implies q" and is denoted by $p \to q$. It is false when p is true and q is false; otherwise it is true. We call p the **hypothesis** (or **antecedent**) of the conditional and q the **conclusion** (or **consequent**).
A conditional statement that is always true because the hypothesis is false is called **vacuously true**.

$$\text{If } \underbrace{4{,}686 \text{ is divisible by } 6,}_{\text{hypothesis}} \text{ then } \underbrace{4{,}686 \text{ is divisible by } 3}_{\text{conclusion}}$$

Example. Consider the following statement:

If Lander is open, then we will have class.

Create the truth table for $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Note: The **order of operations** states that $\rightarrow$ is performed last

Example. Create the truth table for $p \lor \sim q \rightarrow \sim p$.

p	q	$\sim q$	$p \lor \sim q$	$\sim p$	$p \lor \sim q \rightarrow \sim p$
T	T	F	F	F	T
T	F	T	T	F	F
F	T	F	F	T	T
F	F	T	F	T	T

Example. Use a truth table to show that $p \lor q \rightarrow r \equiv (p \rightarrow r) \land (q \rightarrow r)$

p	q	r	$p \lor q$	$p \lor q \rightarrow r$	$p \rightarrow r$	$q \rightarrow r$	$(p \rightarrow r) \land (q \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Definition.
The **negation** of "if p then q" is logically equivalent to "p and not q":

$$\sim(p \to q) \equiv p \wedge \sim q$$

Example. Write negations for each of the following statements:

If my car is in the repair shop, then I cannot get to class.

My car is in the shop and I can get to class

If Sara lives in Athens, then she lives in Greece.

Sara lives in Athens and she does not live in Greece

Definition.
The **contrapositive** of a conditional statement of the form "If p then q" is

$$\text{If } \sim q \text{ then } \sim p : \quad \sim q \to \sim p$$

A conditional statement is logically equivalent to its contrapositive.

Example. Write each of the following statements in its equivalent contrapositive form:

If Howard can swim across the lake, then Howard can swim to the island.

If Howard cannot swim to the island, then Howard cannot swim across the lake

If today is Easter, then tomorrow is Monday.

If tomorrow is not Monday, then today is not Easter

Definition.
Suppose a conditional statement of the form "If p then q" is given.

- The **converse** is "If q then p": $q \to p$
- The **inverse** is "If $\sim p$ then $\sim q$": $\sim p \to \sim q$

Example. Write the converse and inverse of each of the following statements:

If Howard can swim across the lake, then Howard can swim to the island.

Converse:

If Howard can swim to the island, then Howard can swim across the lake

Inverse:

If Howard can NOT swim to the island, then Howard can NOT swim across the lake

If today is Easter, then tomorrow is Monday.

Converse:

If tomorrow is Monday, then today is Easter

Inverse:

If tomorrow is NOT Monday, then today is NOT Easter

Note:

1. A conditional statement and its converse are *not* logically equivalent.
2. A conditional statement and its inverse are *not* logically equivalent.
3. The converse and the inverse of a conditional statement are logically equivalent to each other.

Definition.
If p and q are statements, p **only if** q means "if not q then not p":

$$\sim q \to \sim p \equiv p \to q$$

Example. Rewrite the following statement in if-then form in two ways, one of which is the contrapositive of the other:

> John will break the world's record for the mile run only if he runs the mile in under four minutes.

$\sim q \to \sim p$

If John does NOT run the mile in under four minutes,
then John will NOT break the world's record for the mile.

$p \to q$

If John will break the world's record for the mile,
then he runs the mile in under four minutes.

Note:

1. "p only if q" does *not* mean p if q
2. It is possible for "p only if q" to be true at the same time that "p if q" is false.

   e.g.: If John runs a mile in under four minutes, he still might not be fast enough to break the record.

Definition.
Given statement variables p and q, the **biconditional of p and q** is "p if, and only if, q" and is denoted $p \leftrightarrow q$. It is true if both p and q have the same truth values and is false otherwise. The words *if and only if* are sometimes abbreviated **iff**.

Note: The **order of operations** states that $\leftrightarrow$ is coequal with $\rightarrow$

Example. Create the truth table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Order of Operations for Logical Operators

$\sim$	Evaluate negations first
$\wedge, \vee$	Evaluate $\wedge$ and $\vee$ second. When both present, parentheses may be needed.
$\rightarrow, \leftrightarrow$	Evaluate $\rightarrow$ and $\leftrightarrow$ third. When both present, parentheses may be needed.

Definition.
If r and s are statements:

1. r is a **sufficient condition** for s means "if r then s". $r \rightarrow s$

2. r is a **necessary condition** for s means "if not r then not s". $\sim r \rightarrow \sim s$

By property of the contrapositive:

3. r is a *necessary and sufficient condition* for s means "r if, and only if s." $r \leftrightarrow s$

Example. Rewrite the following statement in the form "If A then B":

Having two $45°$ angles is a sufficient condition for this triangle to be a right triangle.

IF a triangle is a right triangle, THEN it has two 45° angles

Example. Use the contrapositive to rewrite the following statement in two ways:

George's attaining age 35 is a necessary condition for his being president of the United States.

If George is president of the United States, then George is at least 35 years old.

If George is NOT at least 35 years old, then George is NOT the president of the United States.