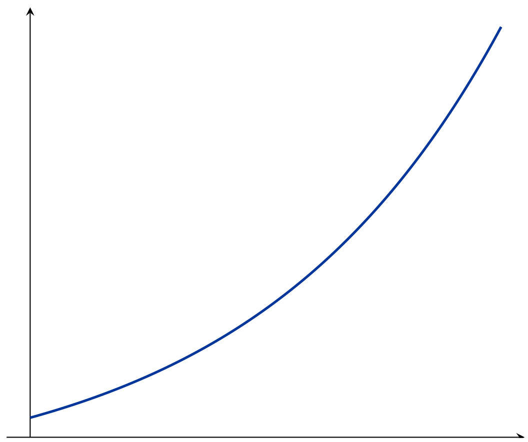
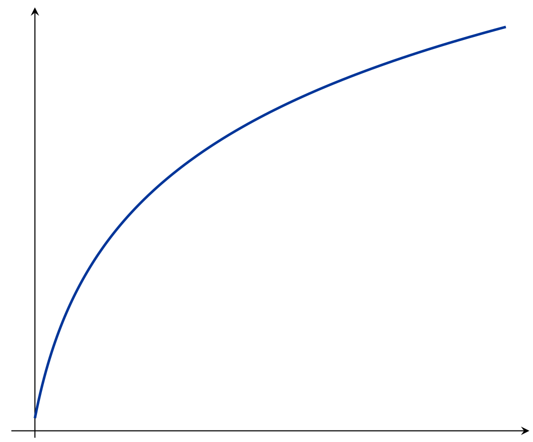
4.2: Applications of the Second Derivative

Definition.
Consider any differentiable function $f(x)$ on the interval (a, b). We say f is

concave up if $f'(x)$ is increasing



concave down if $f'(x)$ is decreasing

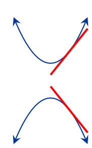


Thus, for every value of x on the interval (a, b), if

- $f''(x) > 0$, then f' is increasing, and f is concave *up* on (a, b).
- $f''(x) < 0$, then f' is decreasing, and f is concave *down* on (a, b).
- If f is continuous at c and f changes concavity at c, then f has an **inflection point** at c.

Note: $f(x)$ is

- concave up if its tangent lines lie below the curve
- concave down if its tangent lines lie above the curve



Determining the Intervals of Concavity of the Graph of f

1. Determine the values of x for which f'' is zero or undefined.
2. Determine the sign of $f''(x)$ to the left and right of each point from above:
   Let c be a convenient test point on the interval of interest. Then,
   a) if $f''(c) > 0$, then f is concave up on that interval.
   b) if $f''(c) < 0$, then f is concave down on that interval.

Example. Find the intervals where the following functions are concave up and concave down:

$$f(x) = x^3 - 3x^2 - 24x + 32$$

Graph

$f'(x) = 3x^2 - 6x - 24$

$f''(x) = 6x - 6$

$f''(x) = 0$ $\quad$ $f''(x)$ DNE

$6x - 6 = 0$ $\quad$ —

$x = 1$

	0	6	10
$6x-6$	$-$		$+$
	$\cap$		$\cup$

$$\boxed{\begin{array}{ll} \text{Concave down:} & (-\infty, 6) \\ \text{Concave up:} & (6, \infty) \end{array}}$$

$$g(x) = (x+1)^{2/3}$$

$g'(x) = \frac{2}{3}(x+1)^{-1/3}$

$g''(x) = \frac{-2}{9}(x+1)^{-4/3} = \dfrac{-2}{3(x+1)^{4/3}}$

$g''(x) = 0$ $\quad$ $g''(x)$ DNE

— $\quad$ $3(x+1)^{4/3} \neq 0$

$x \neq -1$

	-10	-1	0
-2	$-$		$-$
3	$+$		$+$
$(x+1)^{4/3}$	$+$		$+$
	$-$		$-$
	$\cap$		$\cap$

$$\boxed{\text{Concave down: } (-\infty, -1) \cup (-1, \infty)}$$

$$h(x) = x + \frac{1}{x} = x + x^{-1}$$

$$h'(x) = 1 - x^{-2}$$

$$h''(x) = 2x^{-3} = \frac{2}{x^3}$$

Solve $h''(x) = 0$ & $h''(x)$ DNE

$x \neq 0$

	-1	0	1
$\frac{2}{x^3}$	$-$		$+$
	$\cap$		$\cup$

Concave down: $(-\infty, \ \ )$
Concave up: $(0, \infty)$

$$j(x) = \frac{x^2}{1-x^2}$$

$$j'(x) = \frac{(1-x^2)\,2x - x^2(-2x)}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}$$

$$j''(x) = \frac{(1-x^2)^2 \cdot 2 - 2x \cdot 2(1-x^2)\cdot(-2x)}{(1-x^2)^4} = \frac{6x^2+2}{(1-x^2)^3}$$

Solve $j''(x) = 0$ & $j''(x)$ DNE

$(1-x^2)^3 \neq 0$

$\pm 1 \neq x$

	-10	-1		1	10
$6x^2+2$	$+$		$+$		$+$
$(1-x^2)^3$	$-$		$+$		$-$
	$-$		$+$		$-$
	$\cap$		$\cup$		$\cap$

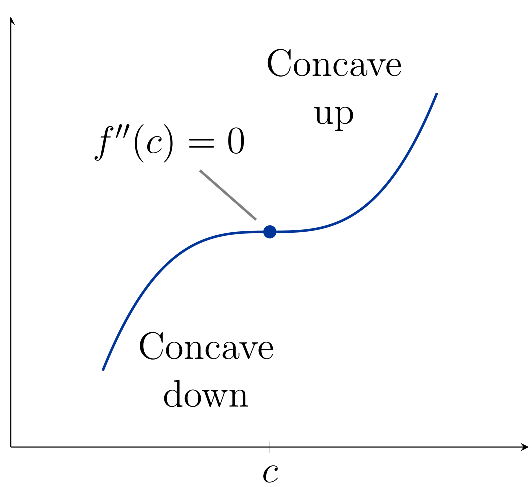
c. down: $(-\infty, -1) \cup (1, \infty)$

c. up $(-1, 1)$

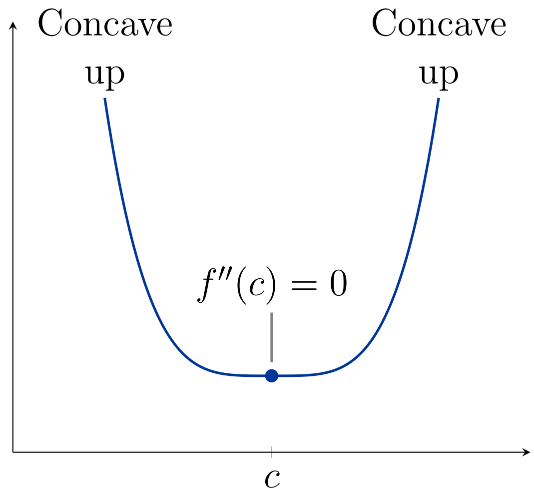
Finding inflection points

1. Compute $f''(x)$.
2. Locate where $f''(x) = 0$ or $f''(x)$ does not exist.
3. Determine if the sign of $f''(x)$ changes at the points found above.

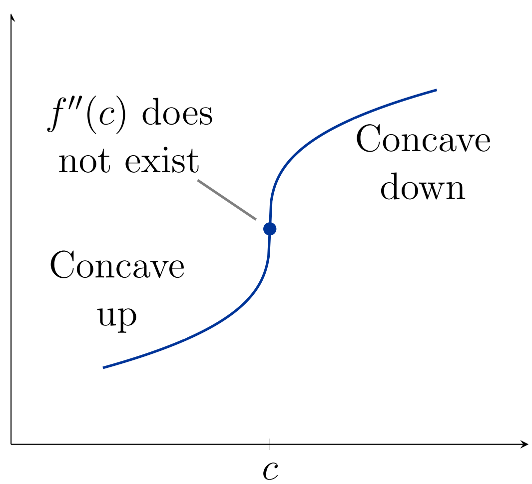
Inflection point at $x = c$



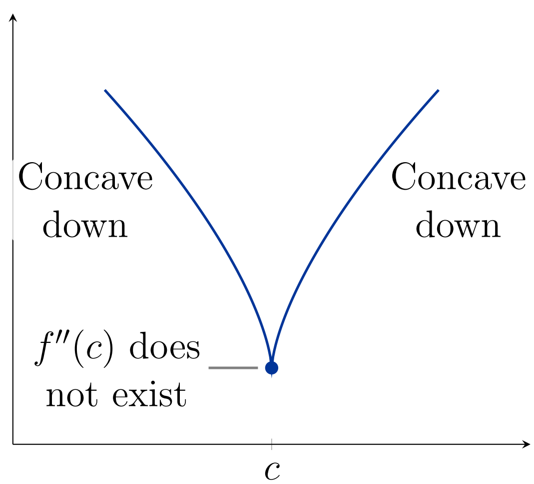
No inflection point at $x = c$



Inflection point at $x = c$



No inflection point at $x = c$



Example. For the following functions, determine the intervals of concavity and find any inflection points.

$f(x) = (x-1)^{5/3}$ [Graph]

$f'(x) = \frac{5}{3}(x-1)^{2/3}$

$f''(x) = \frac{10}{9}(x-1)^{-1/3} = \dfrac{10}{9(x-1)^{1/3}}$

$f''(x) = 0$ $f''(x)$ DNE

— $9(x-1)^{1/3} \neq 0$

$x \neq 1$

1

	0	2
10	+	+
9	+	+
$(x-1)^{1/3}$	−	+
	−	+
	$\cap$	$\cup$

Concave down: $(-\infty, 1)$

Concave up: $(1, \infty)$

Inflection point: $(1, f(1))$

$\to (1, 0)$

$f(x)$ must be continuous in order for the inflection point to exist!!

$$g(x) = \frac{1}{x^2+1} = (x^2+1)^{-1}$$

$$g'(x) = -(x^2+1)^{-2} \cdot 2x = \frac{-2x}{(x^2+1)^2}$$

$$g''(x) = \frac{(x^2+1)^2(-2) - (-2x)\, 2(x^2+1)\cdot 2x}{(x^2+1)^4} = \frac{-2(x^2-4x+1)}{(x^2+1)^3}$$

$g''(x) = 0$

$x^2 - 4x + 1 = 0$

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(1)}}{2}$$

$$= \frac{4 \pm \sqrt{12}}{2}$$

$$= 2 \pm \sqrt{3}$$

$g''(x)$ DNE

$(x^2+1)^3 \neq 0$

	0	$2-\sqrt{3}$	2	$2+\sqrt{3}$	10
-2	$-$		$-$		$-$
x^2-4x+1	$+$		$-$		$+$
$(x^2+1)^3$	$+$		$+$		$+$
	$-$		$+$		$-$
	$\cap$		$\cup$		$\cap$

Concave down: $(-\infty, 2-\sqrt{3}) \cup (2+\sqrt{3}, \infty)$

Concave up: $(2-\sqrt{3}, 2+\sqrt{3})$

Inflection points: $(2-\sqrt{3}, g(2-\sqrt{3}))$ $\quad (2+\sqrt{3}, g(2+\sqrt{3}))$

$\rightarrow \left(2-\sqrt{3}, \frac{1}{8-4\sqrt{3}}\right.$ $\quad \rightarrow \left(2+\sqrt{3}, \frac{1}{8+4\sqrt{3}}\right)$

Second Derivative Test for Local Extrema

Suppose f'' is continuous on an open interval containing c with $f'(c) = 0$.

- If $f''(c) > 0$, then f has a local minimum at c.
- If $f''(c) < 0$, then f has a local maximum at c.
- If $f''(c) = 0$, then the test is inconclusive; f may have a local maximum, local minimum, or neither at c.

Example. Find the relative extrema of

$$f(x) = x^3 - 3x^2 - 24x + 32$$

Graph

$f'(x) = 3x^2 - 6x - 24 = 3(x+2)(x-4)$

Quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{6 \pm \sqrt{(-6)^2 - 4(3)(-24)}}{2(3)} = \frac{6 \pm 18}{6}$ $\nearrow x = 4$, $\searrow x = -2$

$f''(x) = 6x - 6$

$f''(-2) = -18 \longrightarrow$ concave down $\frown$ $\Rightarrow$ Rel. max.

$f''(4) = 18 \longrightarrow$ concave up $\smile$ $\Rightarrow$ Rel. min.

$f(x)$	$f'(x)$	$f''(x)$
increasing	positive	—
decreasing	negative	—
max/min	crit. pt. & changes sign	—
concave up	increasing	positive
concave down	decreasing	negative
Inflection point	max/min	crit. pt. & changes sign