

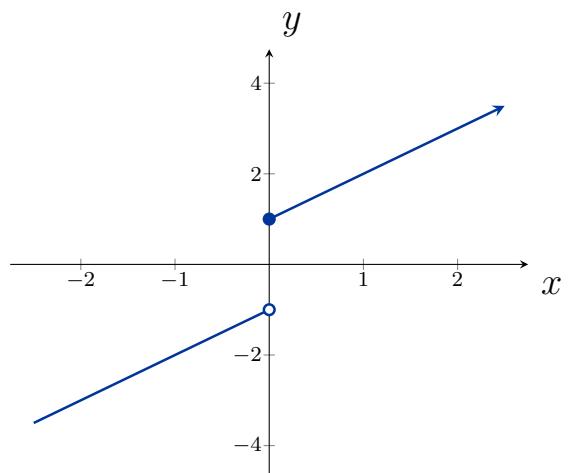
## 2.5: One-Sided Limits and Continuity

Consider the function

$$f(x) = \begin{cases} x - 1, & x < 0 \\ x + 1, & x \geq 0 \end{cases}$$

What is  $\lim_{x \rightarrow 0} f(x)$ ?

$$\lim_{x \rightarrow 0} f(x) \text{ DNE}$$



### Definition. (One-Sided Limits)

The function  $f$  has a **right-hand limit**  $L$  as  $x$  approaches  $a$  from the right, written

$$\lim_{x \rightarrow a^+} f(x) = L$$

if the values of  $f(x)$  can be made as close to  $L$  as we please by taking  $x$  sufficiently close to (but not equal to)  $a$  and to the right of  $a$ .

The function  $f$  has a **left-hand limit**  $M$  as  $x$  approaches  $a$  from the left, written

$$\lim_{x \rightarrow a^-} f(x) = M$$

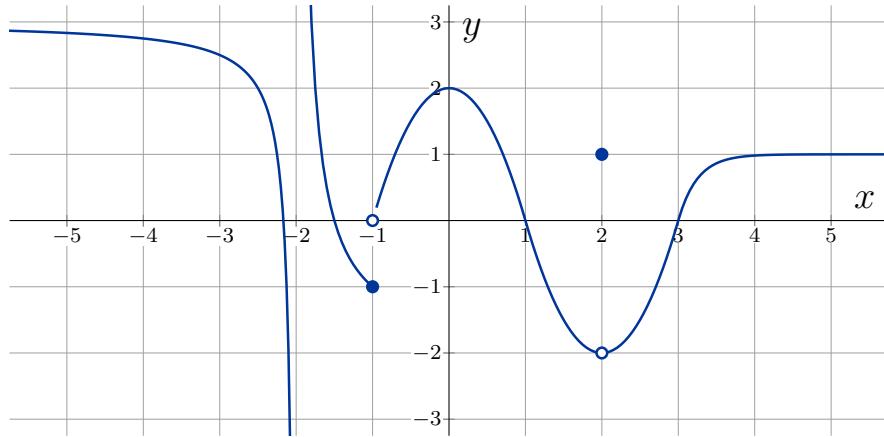
if the values of  $f(x)$  can be made as close to  $M$  as we please by taking  $x$  sufficiently close to (but not equal to)  $a$  and to the left of  $a$ .

### Theorem 3

Let  $f$  be a function that is defined for all values of  $x$  close to  $x = a$  with the possible exception of  $a$  itself. Then

$$\lim_{x \rightarrow a} f(x) = L \quad \text{if and only if} \quad \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L$$

**Example.** Using the graph below, evaluate the following limits:



$$\lim_{x \rightarrow -2^-} f(x) = -\infty$$

$$\lim_{x \rightarrow -2^+} f(x) = \infty$$

$$\lim_{x \rightarrow -2} f(x) \quad \text{DNE}$$

$$\lim_{x \rightarrow -1^-} f(x) = -1$$

$$f(-1) = -1$$

$$\lim_{x \rightarrow -1^+} f(x) = 0$$

$$\lim_{x \rightarrow -1} f(x) \quad \text{DNE}$$

$$\lim_{x \rightarrow 1} f(x) = 0$$

$$\lim_{x \rightarrow 2} f(x) = -2$$

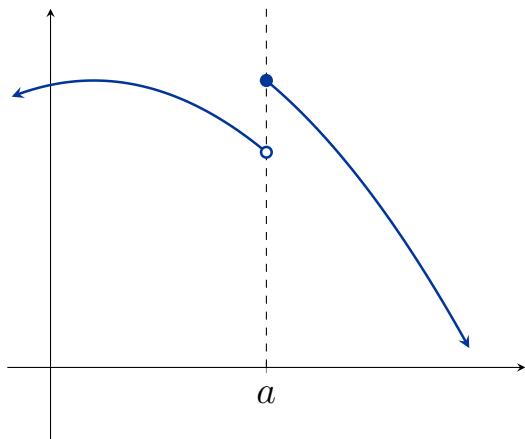
$$\lim_{x \rightarrow \infty} f(x) = 1$$

$$f(2) = 1$$

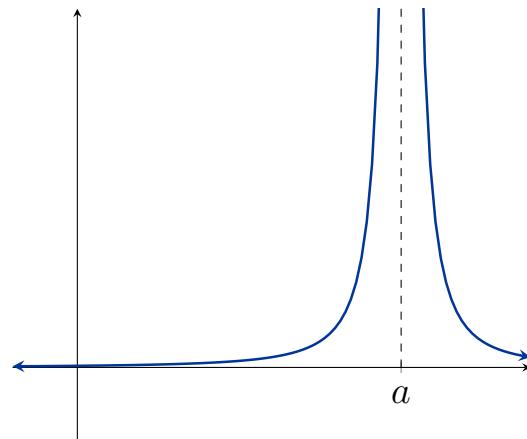
Below are examples where the limit does not exist:

[Graph](#)

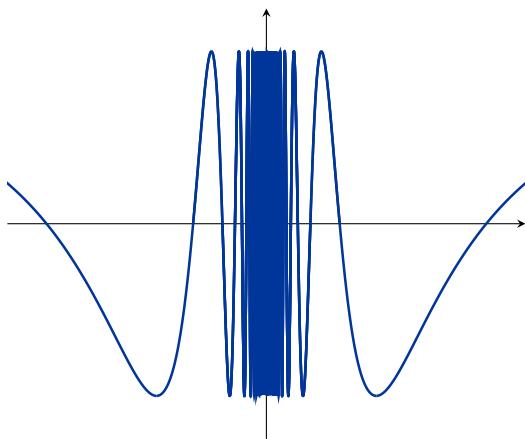
$$\lim_{x \rightarrow a} f(x)$$



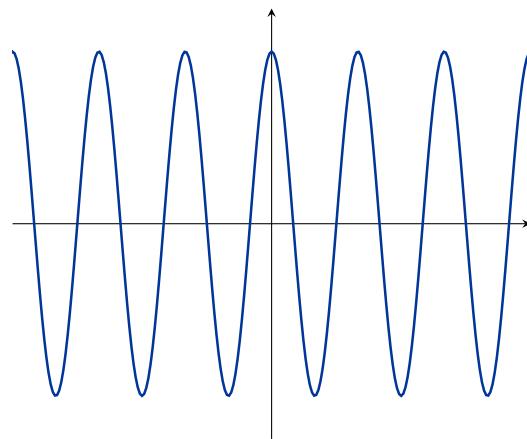
$$\lim_{x \rightarrow a} f(x)$$



$$\lim_{x \rightarrow 0} f(x)$$



$$\lim_{x \rightarrow \infty} f(x)$$



## Definition. (Continuity of a Function at a Number)

A function  $f$  is **continuous** at  $a$  if  $\lim_{x \rightarrow a} f(x) = f(a)$ .

## Continuity Checklist:

In order for  $f$  to be continuous at  $a$ , the following three conditions must hold:

1.  $f(a)$  is defined ( $a$  is in the domain of  $f$ ),
2.  $\lim_{x \rightarrow a} f(x)$  exists,
3.  $\lim_{x \rightarrow a} f(x) = f(a)$  (the value of  $f$  equals the limit of  $f$  at  $a$ ).

**Example.** Determine the values of  $x$  for which the following functions are continuous:

$$f(x) = 3x^3 + 2x^2 - x + 10 \quad \text{on } (-\infty, \infty)$$

$$g(x) = \frac{8x^{10} - 4x + 1}{x^2 + 1}$$

$x^2 + 1 \neq 0 \Rightarrow$  Continuous  
 $x^2 \neq -1 \quad \text{on } (-\infty, \infty)$

$$h(x) = \frac{4x^3 - 3x^2 + 1}{x^2 - 3x + 1}$$

$x^2 - 3x + 1 \neq 0$   
 $x \neq \frac{3 \pm \sqrt{(-3)^2 - 4(1)(1)}}{2} = \frac{3 \pm \sqrt{5}}{2}$

$\Rightarrow$  Continuous  
on  $(-\infty, \frac{3-\sqrt{5}}{2}) \cup (\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}) \cup (\frac{3+\sqrt{5}}{2}, \infty)$

**Example.** Determine whether the following are continuous at  $a$ :

$$f(x) = x^2 + \sqrt{7-x}, \quad a = 4 \quad \checkmark$$

$$f(4) = 16 + \sqrt{3}$$

$$g(x) = \frac{1}{x-3}, \quad a = 3 \quad \times$$

$$g(3) \text{ DNE}$$

$$\lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^+} f(x) = \lim_{x \rightarrow 4} f(x) = 16 + \sqrt{3}$$

$$h(x) = \begin{cases} \frac{x^2 + x}{x + 1}, & x \neq -1 \\ 0, & x = -1 \end{cases}, \quad a = -1 \quad \times$$

$$\begin{aligned} \lim_{x \rightarrow -1} h(x) &= \lim_{x \rightarrow -1} \frac{x^2 + x}{x + 1} \\ &= \lim_{x \rightarrow -1} \frac{x(x+1)}{x+1} \\ &= \lim_{x \rightarrow -1} x = -1 \neq h(-1) = 0 \end{aligned}$$

$$j(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}, \quad a = 0 \quad \checkmark$$

$$j(0) = 0$$

$$\lim_{x \rightarrow 0^-} j(x) = \lim_{x \rightarrow 0} -x = 0$$

$$\lim_{x \rightarrow 0^+} j(x) = \lim_{x \rightarrow 0} x = 0$$

$$\Rightarrow \lim_{x \rightarrow 0} j(x) = 0$$

$$k(x) = \begin{cases} \frac{x^2 + x - 6}{x^2 - x}, & x \neq 2 \\ -1, & x = 2 \end{cases}, \quad a = 2 \quad \times$$

$$k(2) = -1$$

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x^2 - x} = \frac{0}{2} = 0 \neq k(2) = -1$$

## Properties of Continuous Functions

1. The constant function  $f(x) = c$  is continuous everywhere.
2. The identity function  $f(x) = x$  is continuous everywhere.

If  $f$  and  $g$  are continuous at  $x = a$ , then

$[f(x)]^n$ , where  $n$  is a real number, is continuous at  $x = a$  whenever it is defined at that number

$f \pm g$  is continuous at  $x = a$

$fg$  is continuous at  $x = a$

$f/g$  is continuous at  $x = a$  provided that  $g(a) \neq 0$

## Polynomial and Rational Functions

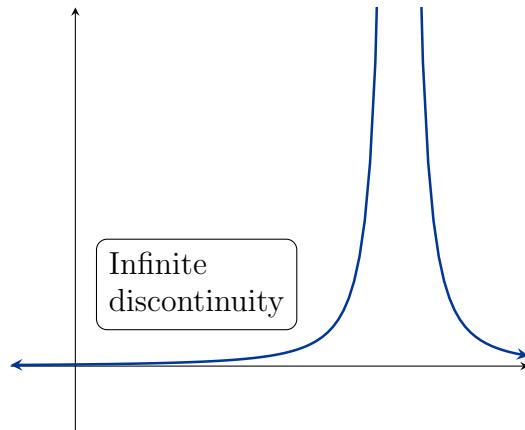
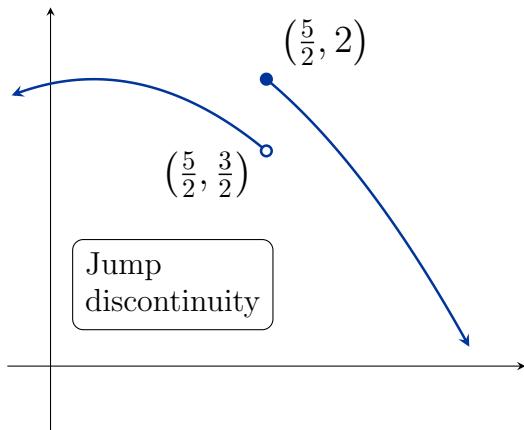
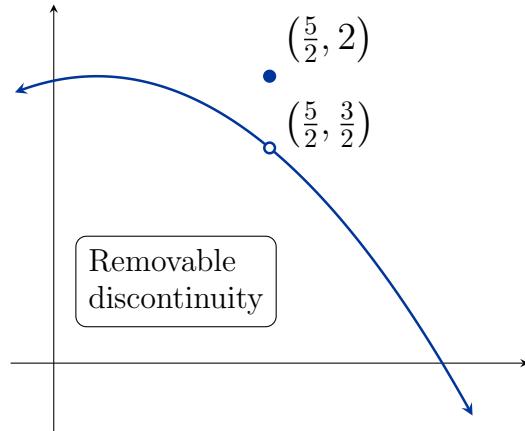
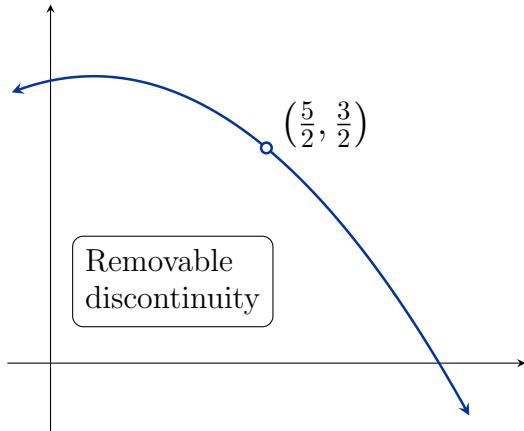
1. A polynomial function is continuous for all  $x$ .
2. A rational function (a function of the form  $\frac{p}{q}$ , where  $p$  and  $q$  are polynomials) is continuous for all  $x$  for which  $q(x) \neq 0$ .

## Definition.

A **removable discontinuity** at  $x = a$  is one that disappears when the function becomes continuous after defining  $f(a) = \lim_{x \rightarrow a} f(x)$ .

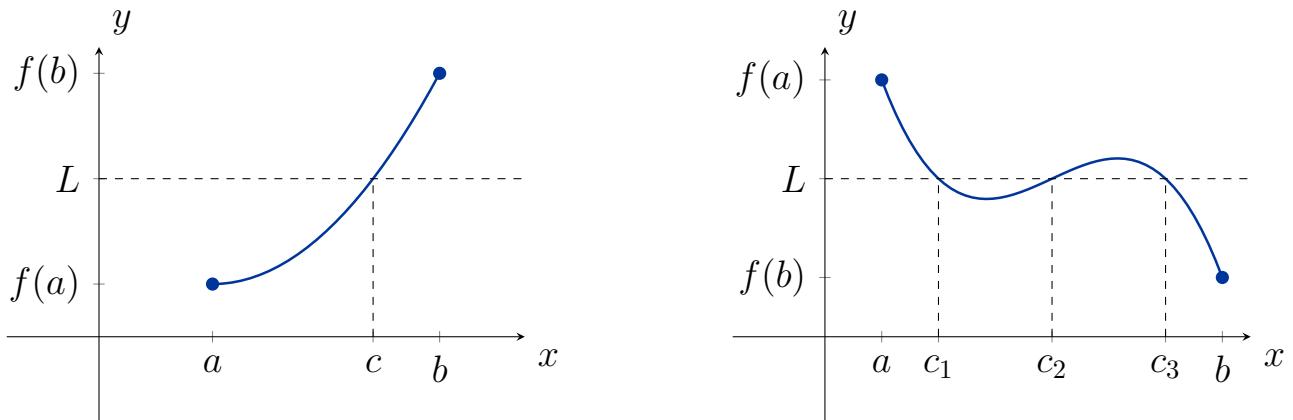
A **jump discontinuity** is one that occurs whenever  $\lim_{x \rightarrow a^-} f(x)$  and  $\lim_{x \rightarrow a^+} f(x)$  both exist, but  $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$ .

A **vertical discontinuity** occurs whenever  $f(x)$  has a vertical asymptote.

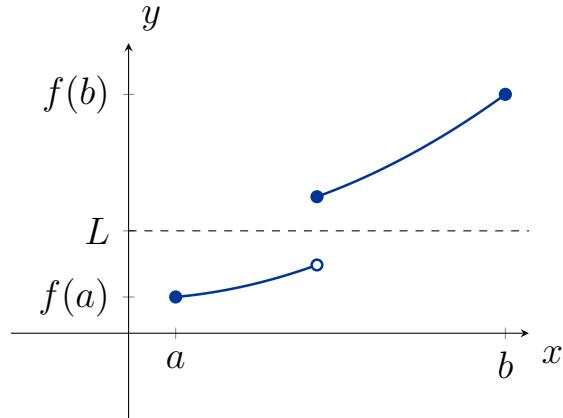


### Theorem 4: Intermediate Value Theorem

Suppose  $f$  is continuous on the interval  $[a, b]$  and  $L$  is a number strictly between  $f(a)$  and  $f(b)$ . Then there exists at least one number  $c$  in  $(a, b)$  satisfying  $f(c) = L$ .



Note: It is important that the function be continuous on the interval  $[a, b]$ :



### Theorem 5: Existence of Zeros of a Continuous Function

If  $f$  is a continuous function on a closed interval  $[a, b]$ , and if  $f(a)$  and  $f(b)$  have opposite signs, then there is at least one solution of the equation  $f(x) = 0$  in the interval  $(a, b)$ .

**Example.** Check the conditions of the Intermediate Value Theorem to see if there exists a value  $c$  on the interval  $(a, b)$  such that the following equations hold: [Graph](#)

$$x^x - x^2 = \frac{1}{2} \quad \text{on } [0, 2]$$

$$x=0 : 0^0 - 0^2 = 1$$

$$x=2 : 2^2 - 2^2 = 0$$

Since  $0 < \frac{1}{2} < 1$ , there exists  $c$  such that  $0 < c < 2$  and  $f(c) = \frac{1}{2}$

$$\sqrt{x^4 + 25x^3 + 10} = 5 \quad \text{on } [0, 1]$$

$$x=0 : \sqrt{0+0+10} = \sqrt{10} \approx 3.16$$

$$x=1 : \sqrt{1+25+10} = \sqrt{36} = 6$$

Since  $\sqrt{10} < 5 < 6$ , there exists  $c$  such that  $0 < c < 1$  and  $f(c) = 5$

$$x + \sqrt{1-x^2} = 0 \quad \text{on } [-1, 0]$$

$$x=-1 : -1 + \sqrt{1-1} = -1$$

$$x=0 : 0 + \sqrt{1-0} = 1$$

Since  $-1 < 0 < 1$ , there exists  $c$  such that  $-1 < c < 0$  and  $f(c) = 0$

$$\frac{x^2}{x^2 + 1} = 0 \quad \text{on } [-1, 1]$$

$$x=-1 : \frac{(-1)^2}{(-1)^2 + 1} = \frac{1}{2}$$

$$x=1 : \frac{(1)^2}{(1)^2 + 1} = \frac{1}{2}$$

Since the conditions of the I VT are not met,  $c$  such that  $-1 < c < 1$  and  $f(c) = 0$  may not exist (not guaranteed)

Note:  $x=0$  is a root.

**Example.** Consider the function

$$f(x) = \frac{x+1}{x-1}$$

on the interval  $[0, 2]$ . Does there exist a  $c$  on the interval  $[0, 2]$  such that  $f(c) = 1$ ?

$$\left. \begin{array}{l} f(0) = -1 \\ f(2) = 3 \end{array} \right\} \quad -1 < 1 < 3$$

Since  $f(x)$  is discontinuous  
at  $x=1$ , we cannot apply  
the IUT.

