Math 125 Class notes
Spring 2026

To accompany
Discrete Mathematics with Applications
by *Epp*

Peter Westerbaan

Last updated: February 24, 2026

Table Of Contents

1.1: Variables

Definition.
A **variable** is a placeholder for something which may or may not be unknown.

Example. Is there a number with the following property: doubling it and adding 3 gives the same result as squaring it?

$\rightarrow$ Is there a number x with the property that $2x + 3 = x^2$?
$\rightarrow$ Is there a number $\Box$ with the property that $2 \cdot \Box + 3 = \Box^2$?

Example. No matter what number might be chosen, if it is greater than 2, then its square is greater than 4.

$\rightarrow$ No matter what number n might be chosen, if n is greater than 2, then n^2 is greater than 4.

Example. Use variables to rewrite the following sentences:

> Are there numbers with the property that the sum of their squares equals the square of their sum?

> Given any real number, its square is nonnegative.

Definition.

- A **universal statement** says that a certain property is true for all elements in a set.
- A **conditional statement** says that if one thing is true, then some other thing also has to be true.
- Given a property that may or may not be true, an **existential statement** says that there is at least one thing for which the property is true.

Definition.
A **universal conditional statement** is both universal and conditional:

For every animal a, if a is a dog, then a is a mammal.

Conditional statements can be rewritten in ways that make them appear more to be purely universal or purely conditional:

If a is a dog, then a is a mammal.
All dogs are mammals

Example. Rewrite the following universal condition statement:

For every real number x, if x is nonzero then x^2 is positive.

If a real number is nonzero, then its square ____________________.

For every nonzero real number x, ____________________.

If x ____________________, then ____________________.

The square of any nonzero real number is ____________________.

All nonzero real numbers have ____________________.

Definition.
A **universal existence statement** is a statement that is universal because its first part says that a certain property is true for all objects of a given type, and it is existential because its second part asserts the existence of something:

Every real number has an additive inverse.

In the above example, note that the particular additive inverse depends on the given real number:

For every real number r, there is an additive inverse for r.

Example. Rewrite the following universal existence statement:

Every pot has a lid

All pots ____________________.

For ever pot P, there is ____________________.

For every pot P, there is a lid L such that ____________________.

Definition.
An **existential universal statement** is a statement that is existential because its first part asserts that a certain object exists and is universal because its second part says that the object satisfies a certain property for all things of a certain kind:

> There is a positive integer that is less than or equal to every positive integer.

The number one satisfies the above statement, which can also be rewritten:

> There is a positive integer m that is less than or equal to every positive integer.

Example. Rewrite the following existence universal statement:

> There is a person in my class who is at least as old as every person in my class.

Some ____________________ is at least as old as ____________________.

There is a person p in my class such that p is ____________________.

There is a person p in my class with the property that for every person q in my class, p is ____________________.

1.2: The Language of Sets

Definition.

- A **set** is a collection of objects.
- If S is a set, then we use
  - $x \in S$ to denote that the element x is in the set S.
  - $x \notin S$ to denote that the element x is *not* in the set S.
- The **set-roster notation** is used to denote all elements in a set between braces:

$$S = \{1, 2, \ldots, 100\}$$

  Here, we see that $67 \in S$, but $1337 \notin S$.
- The **axiom of extension** says that a set is completely determined by what its elements are – not the order in which they are listed.

Example.

Let $A = \{1, 2, 3\}$, $B = \{3, 1, 2\}$, and $C = \{1, 1, 2, 3, 3, 3\}$. What are the elements of A, B, and C? How are A, B, and C related?

Is $\{0\} = 0$?

How many elements are in the set $\{1, \{1\}\}$?

For each nonnegative integer n, let $U_n = \{n, -n\}$. Find U_1, U_2, and U_0.

Certain sets of numbers are so frequently referred to that they are given special names and symbols:

N or $\mathbb{N}$	The set of all **natural numbers**
Z or $\mathbb{Z}$	The set of all **integers**
Q or $\mathbb{Q}$	The set of all **rational numbers**, or quotient of integers
R or $\mathbb{R}$	The set of all **real numbers**

Note: We may additionally use superscripts to indicate further properties of these sets:

$\mathbb{Z}^+$ or $\mathbb{Z}^{>0}$	The set of *positive* integers
$\mathbb{Q}^-$ or $\mathbb{Q}^{<0}$	The set of *negative* rational numbers
$\mathbb{R}^{nonneg}$ or $\mathbb{R}^{\geq 0}$	The set of *nonnegative* real numbers

Note: Different sources denote the natural numbers $\mathbb{N}$ as $\mathbb{Z}^+$ *or* $\mathbb{Z}^{\geq 0}$.

Definition. (Set-Builder Notation)
Let S be a set and let $P(x)$ be a property that elements of S may or may not satisfy. We may define a new set to be **the set of all elements x in S such that $P(x)$ is true**. We denote this set as follows:

$$\{x \in S | P(x)\}$$

the set of all such that

Example. Describe each of the following sets:

$\{x \in \mathbb{R} \mid -2 < x < 5\}$

$\{x \in \mathbb{Z} \mid -2 < x < 5\}$

$\{x \in \mathbb{Z}^+ \mid -2 < x < 5\}$

Definition.
If A and B are sets, then A is called a **subset** of B, written $A \subseteq B$, if, and only if, every element of A is also an element of B:

$$A \subseteq B \text{ means that for every element } x \text{, if } x \in A \text{, then } x \in B.$$
$$A \not\subseteq B \text{ means that there is at least one element } x \text{, such that } x \in A \text{ and } x \notin B.$$

A is a **proper subset** of B if, and only if, every element of A is in B, but there is at least one element of B that is not in A:

$$A \subsetneq B \text{ means that for every element } x \text{, if } x \in A \text{, then } x \in B,$$
$$\text{and there exists } x \in B \text{ such that } x \notin A.$$

Example. Let $A = \mathbb{Z}^+$, $B = \{n \in \mathbb{Z} \mid 0 \leq n \leq 100\}$, and $C = \{100, 200, 300, 400, 500\}$. Evaluate the truth and falsity of each of the following statements.

$B \subseteq A$

C is a proper subset of A

C and B have at least one element in common

$C \subseteq B$

$C \subseteq C$

Example. Determine which of the following statements are true:

$2 \in \{1, 2, 3\}$

$\{2\} \in \{1, 2, 3\}$

$2 \subseteq \{1, 2, 3\}$

$\{2\} \subseteq \{1, 2, 3\}$

$\{2\} \subseteq \{\{1\}, \{2\}\}$

$\{2\} \in \{\{1\}, \{2\}\}$

Definition.
Given elements a and b, the symbol (a, b) denotes the **ordered pair** consisting of a and b together with the specification that a is the first element of the pair, and b is the second element. Two ordered pairs (a, b) and (c, d) are equal if, and only if, $a = c$ and $b = d$:

$$(a, b) = (c, d) \text{ means that } a = c \text{ and } c = d.$$

Example.

Is $(1, 2) = (2, 1)$?

Is $\left(3, \dfrac{5}{10}\right) = \left(\sqrt{9}, \dfrac{1}{2}\right)$?

Definition.
Let $n \in \mathbb{N}$ and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered** n**-tuple**, $(x_1, x_2, \ldots, x_n)$, consists of $x_1, x_2, \ldots x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. and ordered 2-tuple is called an **ordered pair**, and ordered 3-tuple is called an **ordered triple**.
Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if, and only if, $x_1 = y_1$, $x_2 = y_2$, $\ldots$, and $x_n = y_n$:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \iff x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

Example.

Is $(1, 2, 3, 4) = (1, 2, 4, 3)$?

Is $\left(3, (-2)^2, \dfrac{1}{2}\right) = \left(\sqrt{9}, 4, \dfrac{3}{6}\right)$?

Definition.
Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$, denoted

$$A_1 \times A_2 \times \cdots \times A_n$$

is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1$, $a_2 \in A_2$, $\ldots$, $a_n \in A_n$:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in \mathcal{A}_n\}$$

Example. Let $A = \{x, y\}$, $B = \{1, 2, 3\}$, and $C = \{a, b\}$. Find the following:

$A \times B$

$B \times A$

$A \times A$

How many elements are in $A \times B$, $B \times A$, and $A \times A$?

$(A \times B) \times C$

$A \times B \times C$

Describe $\mathbb{R} \times \mathbb{R}$

Definition.
Let $n \in \mathbb{N}$. Given a finite set A, a **string of length n over A** is an ordered n-tuple of elements of A written without parentheses or commas. The elements of A are called the **characters** of the string. The **null string** over A is defined to be the "string" with no characters, often denoted λ, and is said to have length 0. If $A = \{0, 1\}$, then a string over A is called a **bit string**.

Example. Let $A = \{a, b\}$. List all strings of length 3 over A with at least two characters that are the same.

1.3: The Language of Relations and Functions

Definition.
Let A and B be sets. A **relation** R **from** A **to** B is a subset of $A \times B$. Given an ordered pair (x, y), x **is related to** y **by** R, written $x\,R\,y$, if, and only if, (x, y) is in R. The set A is called the **domain** of R and the set B is called its **co-domain**.
The notation for a relation R may be written symbolically as follows:

$$x\,R\,y \text{ means that } (x, y) \in R.$$

The notation $x \not R\, y$ means that x is not related to y by R:

$$x \not R\, y \text{ means that } (x, y) \notin R.$$

Example. Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ and define a relation R from A to B as follows; Given any $(x, y) \in A \times B$,

$$(x, y) \in R \text{ means that } \frac{x - y}{2} \text{ is an integer.}$$

State explicitly which ordered pairs are in $A \times B$ and which are in R

Is $1\,R\,3$?

Is $2\,R\,3$?

Is $2\,R\,2$?

What are the domain and co-domain of R?

Example. Define a relation C from $\mathbb{R}$ to $\mathbb{R}$ as follows: For any $(x, y) \in \mathbb{R} \times \mathbb{R}$,

$$(x, y) \in C \text{ means that } x^2 + y^2 = 1.$$

Is $(1, 0) \in C$? Is $(0, 0) \in C$? Is $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \in C$?

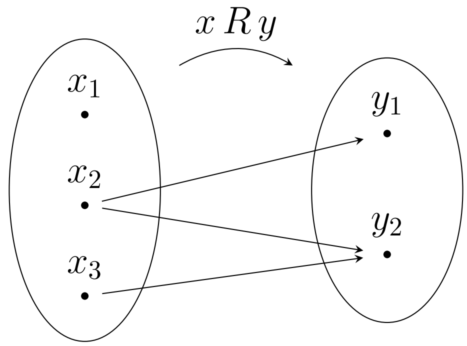
Is $-2\, C\, 0$? Is $0\, C\, (-1)$? Is $1\, C\, 1$?

What are the domain and co-domain of C?

Draw a graph for C by plotting the points of C in the Cartesian plane.

Definition.
Suppose R is a relation from set A to a set B. The **arrow diagram for** R is obtained as follows:

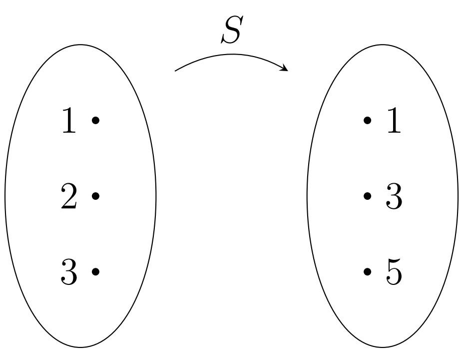
1. Represent the elements of A as points in one region and the elements of B as points in another region.
2. For each x in A and y in B, draw an arrow from x to y if, and only if, x is related to y by R.

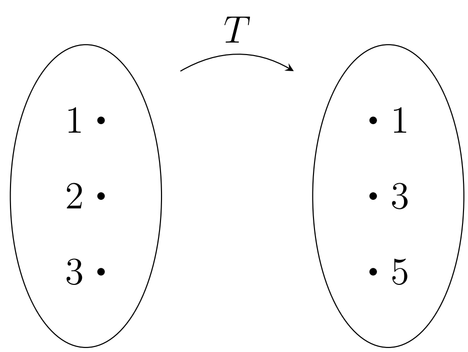


Example. Let $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$ and define relations S and T from A to B as follows: For every $(x, y) \in A \times B$,

$$(x, y) \in S \text{ means that } x < y$$
$$T = \{(2, 1), (2, 5)\}.$$

Draw arrow diagrams for S and T





Definition.
A **function** F **from a set** A **to a set** B is a relation with domain A and co-domain B that satisfies the following two properties:

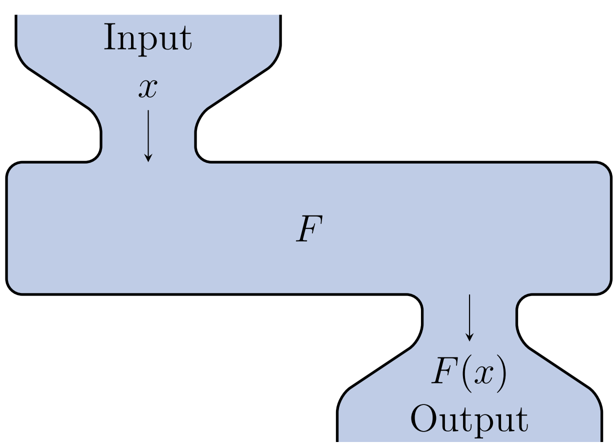
1. For every element x in A, there is an element y in B such that $(x, y) \in F$.
2. For all elements x in A and y and z in B,

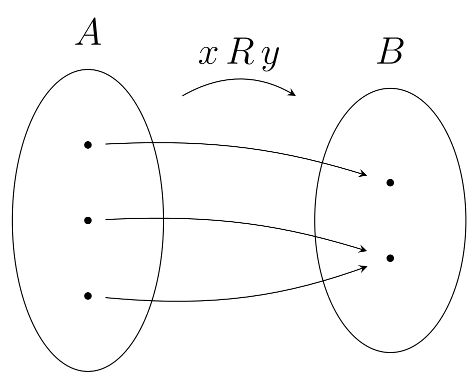
$$\text{if } (x, y) \in F \text{ and } (x, z) \in F, \text{ then } y = z.$$

Note: A relation from A to B is a function if, and only if,

1. Every element of A is the first element of an ordered pair of F
2. No two distinct ordered pairs in F have the same first element.

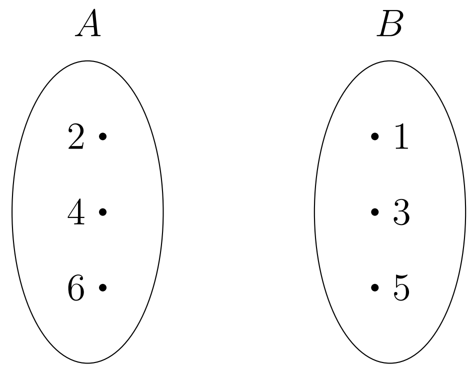
Note: If A and B are sets and F is a function from A to B, then given any element x in A, the unique element in B that is related to x by F is denoted $F(x)$, which is read "F **of** x".



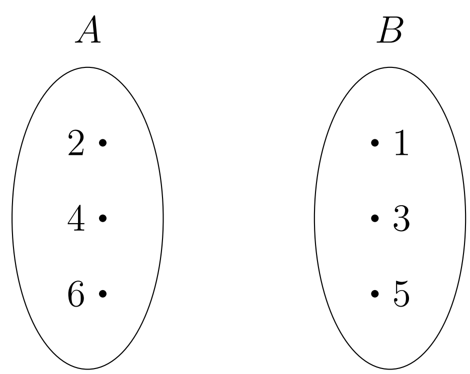


Example. Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$. Which of the relations R, S, and T defined below are functions from A to B?

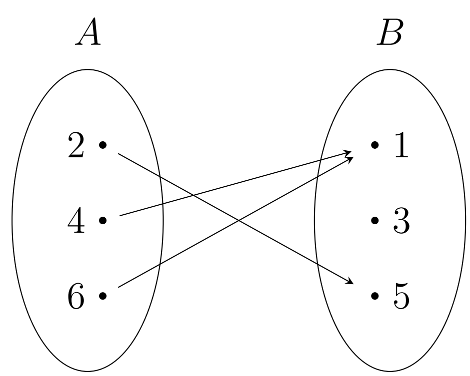
$R = \{(2,5), (4,1), (4,3), (6,5)\}$



For every $(x, y) \in A \times B$, $(x, y) \in S$ means that $y = x + 1$.



T is defined by the arrow diagram



2.1: Logical Form and Logical Equivalence

Definition.
A **statement** (or **proposition**) is a sentence that is true or false, but not both.

Example. Determine which of the following are statements:

$2 + 2 = 4$

$2 + 2 = 5$

$x^2 + 2 = 11$

Today is Saturday.

She is a computer science major.

Jane is a computer science major.

Definition. (Compound Statements)
Let p and q be statement variables.

- The **negation** of p is "not p", and is denoted as $\sim p$ (or $\neg p$)
- The **conjunction** of p and q is "p and q", and is denoted at $p \wedge q$
- The **disjunction** of p and q is "p or q", and is denoted $p \vee q$.
- The **exclusive or** of p and q is "p x-or q", and is denoted $p \oplus q$ (or p XOR q)

The **order of operations** specifies that $\sim$ is performed first.

Example. Consider the following statements:

p : It is raining.
q : It is sunny.
r : It is cloudy.

Rewrite the following compound statements in words:

$\sim p$

$p \vee q$

$q \wedge r$

$q \wedge \sim r$

$p \wedge (q \vee r)$

$p \oplus q$

Definition.
A **statement form** (or **propositional form**) is an expression made up of statement variables (e.g., p, q, and r), and logical connectives (e.g. $\sim$, $\wedge$, $\vee$, and $\oplus$).
The **truth table** for a given statement form displays the truth values that correspond to all possible combinations of truth values for its component statement variables.

Example. Let p and q be statement variables. Fill out the following truth tables:

p	$\sim p$
T	
F	

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$
T	T			
T	F			
F	T			
F	F			

p	q	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$(p \vee q) \wedge \sim (p \wedge q)$
T	T				
T	F				
F	T				
F	F				

Example. Construct a truth table for the statement form $(p \wedge q) \vee \sim r$.

Definition.
Two *statement forms* are called **logically equivalent** if, and only if, they have identical true values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms P and Q is denoted $P \equiv Q$.

Example. Use truth tables to test if the following statement forms are equivalent:

p and $\sim (\sim p)$

$\sim (p \wedge q)$ and $\sim p \wedge \sim q$

Definition. (De Morgan's Laws)
The negation of an *and* statement is logically equivalent to the *or* statement in which each component is negated.
The negation of an *or* statement is logically equivalent to the *and* statement in which each component is negated.

Example. Use truth tables to show that the following statement forms are equivalent:

$\sim (p \wedge q)$ and $\sim p \vee \sim q$

$\sim (p \vee q)$ and $\sim p \wedge \sim q$

Example. Using De Morgan's law to write the negation of the following statements:

Jim is at least 6 feet tall and weighs at least 200 pounds.

The bus was late or Tom's watch was slow.

$-1 < x \leq 4$

Definition.
A **tautology** is a statement form that is always true.
A **contradiction** is a statement form that is always false.

Example. Complete the truth tables for $p \wedge \sim p$ and $p \vee \sim p$

Example. Let $\mathbf{t}$ be a tautology, and $\mathbf{c}$ be a contradiction. Show that $p \wedge \mathbf{t} \equiv p$ and $p \wedge \mathbf{c} \equiv \mathbf{c}$

Theorem 2.1.1 Logical Equivalences (p 49)
Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold:

1. Commutative laws:
   $p \wedge q \equiv q \wedge p$ $\qquad$ $p \vee q \equiv q \vee p$
2. Associative laws:
   $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $\qquad$ $(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. Distributive laws:
   $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $\qquad$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. Identity laws:
   $p \wedge \mathbf{t} \equiv p$ $\qquad$ $p \vee \mathbf{c} \equiv p$
5. Negation laws:
   $p \vee \sim p \equiv \mathbf{t}$ $\qquad$ $p \wedge \sim p \equiv \mathbf{c}$
6. Double negative law:
   $\sim (\sim p) \equiv p$
7. Idempotent laws:
   $p \wedge p \equiv p$ $\qquad$ $p \vee p \equiv p$
8. Universal bound laws:
   $p \vee \mathbf{t} \equiv \mathbf{t}$ $\qquad$ $p \wedge \mathbf{c} \equiv \mathbf{c}$
9. De Morgan's laws:
   $\sim (p \wedge q) \equiv \sim p \vee \sim q$ $\qquad$ $\sim (p \vee q) \equiv \sim p \wedge \sim q$
10. Absorption laws:
    $p \wedge (p \vee q) \equiv p$ $\qquad$ $p \wedge (p \vee q) \equiv p$
11. Negations of $\mathbf{t}$ and $\mathbf{c}$:
    $\sim \mathbf{t} \equiv \mathbf{c}$ $\qquad$ $\sim \mathbf{c} \equiv \mathbf{t}$

2.2: Conditional Statements

Definition.
If p and q are statement variables, the **conditional** of q by p is "If p then q", or "p implies q" and is denoted by $p \to q$. It is false when p is true and q is false; otherwise it is true. We call p the **hypothesis** (or **antecedent**) of the conditional and q the **conclusion** (or **consequent**).
A conditional statement that is always true because the hypothesis is false is called **vacuously true**.

$$\text{If } \underbrace{4,686 \text{ is divisible by } 6,}_{\text{hypothesis}} \text{ then } \underbrace{4,686 \text{ is divisible by } 3}_{\text{conclusion}}$$

Example. Consider the following statement:

If Lander is open, then we will have class.

Create the truth table for $p \to q$

p	q	$p \to q$
T	T	
T	F	
F	T	
F	F	

Note: The **order of operations** states that $\to$ is performed last

Example. Create the truth table for $p \vee \sim q \to \sim p$.

p	q	$\sim q$	$p \vee \sim q$	$\sim p$	$p \vee \sim q \to \sim p$
T	T				
T	F				
F	T				
F	F				

Example. Use a truth table to show that $p \vee q \to r \equiv (p \to r) \wedge (q \to r)$

p	q	r	$p \vee q$	$p \vee q \to r$	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

Definition.
The **negation** of "if p then q" is logically equivalent to "p and not q":

$$\sim (p \to q) \equiv p \wedge \sim q$$

Example. Write negations for each of the following statements:

If my car is in the repair shop, then I cannot get to class.

If Sara lives in Athens, then she lives in Greece.

Definition.
The **contrapositive** of a conditional statement of the form "If p then q" is

$$\text{If } \sim q \text{ then } \sim p : \quad \sim q \to \sim p$$

A conditional statement is logically equivalent to its contrapositive.

Example. Write each of the following statements in its equivalent contrapositive form:

If Howard can swim across the lake, then Howard can swim to the island.

If today is Easter, then tomorrow is Monday.

Definition.
Suppose a conditional statement of the form "If p then q" is given.

- The **converse** is "If q then p": $q \to p$
- The **inverse** is "If $\sim p$ then $\sim q$": $\sim p \to \sim q$

Example. Write the converse and inverse of each of the following statements:

If Howard can swim across the lake, then Howard can swim to the island.
Converse:

Inverse:

If today is Easter, then tomorrow is Monday.
Converse:

Inverse:

Note:

1. A conditional statement and its converse are *not* logically equivalent.
2. A conditional statement and its inverse are *not* logically equivalent.
3. The converse and the inverse of a conditional statement are logically equivalent to each other.

Definition.
If p and q are statements, p **only if** q means "if not q then not p":

$$\sim q \to \sim p \equiv p \to q$$

Example. Rewrite the following statement in if-then form in two ways, one of which is the contrapositive of the other:

> John will break the world's record for the mile run only if he runs the mile in under four minutes.

$\sim q \to \sim p$

$p \to q$

Note:

1. "p only if q" does *not* mean p if q
2. It is possible for "p only if q" to be true at the same time that "p if q" is false.

   e.g.: If John runs a mile in under four minutes, he still might not be fast enough to break the record.

Definition.
Given statement variables p and q, the **biconditional of** p **and** q is "p if, and only if, q" and is denoted $p \leftrightarrow q$. It is true if both p and q have the same truth values and is false otherwise. The words *if and only if* are sometimes abbreviated **iff**.

Note: The **order of operations** states that $\leftrightarrow$ is coequal with $\rightarrow$

Example. Create the truth table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	
T	F	
F	T	
F	F	

Order of Operations for Logical Operators

$\sim$	Evaluate negations first
$\wedge, \vee$	Evaluate $\wedge$ and $\vee$ second. When both present, parentheses may be needed.
$\rightarrow, \leftrightarrow$	Evaluate $\rightarrow$ and $\leftrightarrow$ third. When both present, parentheses may be needed.

Definition.
If r and s are statements:

1. r is a **sufficient condition** for s means "if r then s". $\quad r \to s$

2. r is a **necessary condition** for s means "if not r then not s". $\quad \sim r \to \sim s$

By property of the contrapositive:

3. r is a *necessary and sufficient condition* for s means "r if, and only if s."
$$r \leftrightarrow s$$

Example. Rewrite the following statement in the form "If A then B":

> Having two $45°$ angles is a sufficient condition for this triangle to be a right triangle.

Example. Use the contrapositive to rewrite the following statement in two ways:

> George's attaining age 35 is a necessary condition for his being president of the United States.

2.5: Application: Number Systems and Circuits for Addition

Recall our how we write numbers in base 10:

$$\begin{aligned} 5,049 &= 5 \cdot 1000 + 0 \cdot 100 + 4 \cdot 10 + 9 \cdot 1 \\ &= 5 \cdot 10^3 + 0 \cdot 10^2 + 4 \cdot 10^1 + 9 \cdot 10^0 \end{aligned}$$

Definition.
Any integer $b > 1$ can be used as a base for a numbering system. A numbering system of base b has the digits $0, 1, \ldots, b-1$.
A **base** 2 **notation** or **binary notation**, uses the digits $0, 1$. In binary, every integer is represented as sum of products of the form

$$d \cdot 2^n$$

where $n \in \mathbb{Z}$ and $d \in \{0, 1\}$.

Example. Below is the binary representation for the integers 1 to 9:

$$\begin{array}{ccccccccccccc}
1_{10} & = & & & & & & & 1 \cdot 2^0 & = & 1_2 \\
2_{10} & = & & & & & 1 \cdot 2^1 & + & 0 \cdot 2^0 & = & 10_2 \\
3_{10} & = & & & & & 1 \cdot 2^1 & + & 1 \cdot 2^0 & = & 11_2 \\
4_{10} & = & & & 1 \cdot 2^2 & + & 0 \cdot 2^1 & + & 0 \cdot 2^0 & = & 100_2 \\
5_{10} & = & & & 1 \cdot 2^2 & + & 0 \cdot 2^1 & + & 1 \cdot 2^0 & = & 101_2 \\
6_{10} & = & & & 1 \cdot 2^2 & + & 1 \cdot 2^1 & + & 0 \cdot 2^0 & = & 110_2 \\
7_{10} & = & & & 1 \cdot 2^2 & + & 1 \cdot 2^1 & + & 1 \cdot 2^0 & = & 111_2 \\
8_{10} & = & 1 \cdot 2^3 & + & 0 \cdot 2^2 & + & 0 \cdot 2^1 & + & 0 \cdot 2^0 & = & 1000_2 \\
9_{10} & = & 1 \cdot 2^3 & + & 0 \cdot 2^2 & + & 0 \cdot 2^1 & + & 1 \cdot 2^0 & = & 1001_2
\end{array}$$

Converting binary → decimal:
To convert from binary to decimal, multiply each digit by its corresponding power of 2 and sum the results.

Example. Represent the following in decimal notation (base-10):

110_2

1011_2

11110_2

101011_2

Converting decimal → binary:
To convert from decimal to binary, we repeated divide by 2, and record the remainders.

Example.

$$\begin{aligned}27_{10} &= 16+8+2+1\\ &= 1\cdot 2^4 + 1\cdot 2^3 + 0\cdot 2^2 + 1\cdot 2^1 + 1\cdot 2^0\\ &= 11011_2\end{aligned}$$

Example. Represent the following in binary notation:

243_{10} 587_{10}

990_{10} 531_{10}

Binary arithmetic:
In binary arithmetic, 10_2 behaves similarly to 10 in decimal arithmetic.

Example. Add 1101_2 and 111_2 using binary notation.

Example. Subtract 1011_2 from 11000_2 using binary notation.

Definition.
The 8-bit two's complement for an integer a between -128 and 127 is the 8-bit binary representation for

$$\begin{cases} a, & \text{if } a \geq 0 \\ 2^8 - |a|, & \text{if } a < 0. \end{cases}$$

Two's complement allows maximum representation for 2^8 integers with 8 binary digits.

Example. Below are a few integers represented in binary using 8-bit two's complement:

$$-128 \to 2^8 - |-128| = 128_{10} = 10000000_2$$

$$-127 \to 2^8 - |-127| = 129_{10} = 10000001_2$$

$$\vdots$$

$$-2 \to 2^8 - |-2| = 254_{10} = 10000000_2$$

$$-1 \to 2^8 - |-1| = 255_{10} = 11111111_2$$

$$0 \to 0_{10} = 00000000_2$$

$$1 \to 1_{10} = 00000001_2$$

$$2 \to 2_{10} = 00000010_2$$

$$\vdots$$

$$127 \to 127_{10} = 01111111_2$$

Example. Find the 8-bit two's complement for the following:

-46

42

120

-82

Two's complement of a negative integer:
To find the decimal representation of the negative integer with a given 8-bit two's complement:

- Flip the bits
- Add 1
- Convert to base-10 and swap the sign

Example. Find the decimal representation of the integers with the following 8-bit two's complement:

11100101_2

11000000_2

Addition and Subtraction with Integers in Two's Complement Form:
When performing binary addition on integers written in Two's Complement form, we discard any "carry" bit.

Example. Perform binary addition using the Two's Complement form of the following:

83 and -55

-87 and -46

Definition.
Hexadecimal notation uses a **base** 16 **notation**. In hexadecimal, every integer is represented as sum of products of the form

$$d \cdot 16^n$$

where $n \in \mathbb{Z}$ and $d \in \{0, 1, \ldots, 9, A, B, C, D, E, F\}$.

Decimal	Hexadecimal	4-Bit Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
10	A	1010
11	B	1011
12	C	1100
13	D	1101
14	E	1110
15	F	1111

Example. Convert $3CF_{16}$ to decimal notation.

Example. Convert $B09F_{16}$ to binary notation.

Example. Convert 100110110101001_2 to hexadecimal notation.

3.1: Predicates and Quantified Statements I

Definition.
A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.
The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

Example. Let $P(x)$ be the predicate "$x^2 > x$" with domain the set $\mathbb{R}$. Write $P(2)$, $P\big(\frac{1}{2}\big)$, and $P\big(-\frac{1}{2}\big)$, and indicate which of these statements are true and which are false.

Definition.
If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x. The truth set of $P(x)$ is denoted

$$\{x \in D \mid P(x)\}$$

Example. Let $Q(n)$ be the predicate "n is a factor of 8". Find the truth set of $Q(n)$ if

the domain of n is $\mathbb{Z}^+$

the domain of n is $\mathbb{Z}$

Definition.
Let $Q(x)$ be a predicate and D the domain of x.

- **Quantifiers** are words that refer to quantities such as "some" or "all" and tell for how many elements a given predicate is true.
- The **universal quantifier** is represented by the symbol "$\forall$".
- A **universal statement** is a statement of the form "$\forall x \in D$, $Q(x)$".
  - It is defined to be true if, and only if, $Q(x)$ is true for *each* individual x in D.
  - It is defined to be false if, and only if, $Q(x)$ is false for *at least one* x in D.
- A value for x for which $Q(x)$ is false is called a **counterexample** to the universal statement.

Example. Let $D = \{1, 2, 3, 4, 5\}$, and consider the statement

$$\forall x \in D, x^2 \geq x.$$

Write one way to read this statement out loud, and show that it is true.

The above example uses the **method of exhaustion**.

Example. Consider the statement

$$\forall x \in \mathbb{R}, x^2 \geq x.$$

Find a counter example to show that this statement is false.

Definition.
Let $Q(x)$ be a predicate and D the domain of x.

- The **existential quantifier** is represented by the symbol "$\exists$".
- An **existential statement** is a statement of the form "$\exists x \in D$ such that $Q(x)$".
  - It is defined to be true if, and only if, $Q(x)$ is true for *at least one* x in D.
  - It is false if, and only if, $Q(x)$ is false *for all* x in D.

Example. Consider the statement

$$\exists m \in \mathbb{Z}^+ \text{ such that } m^2 = m.$$

Write one way to read this statement out loud, and show that it is true.

Example. Let $E = \{5, 6, 7, 8\}$ and consider the statement

$$\exists m \in E \text{ such that } m^2 = m.$$

Show that this statement is false.

Example. Rewrite the following statements formally using quantifiers and variables:

All triangles have three sides.

No dogs have wings.

Some programs are structured.

Definition.
A **universal conditional statement** is of the form:

$$\forall x, \text{ if } P(x) \text{ then } Q(x).$$

Example. Rewrite each of the following statements in the form

$$\forall \underline{\hspace{2cm}}, \text{ if } \underline{\hspace{2cm}} \text{ then } \underline{\hspace{2cm}}$$

If a real number is an integer, then it is a rational number.

All bytes have eight bits.

No fire trucks are green.

3.2: Predicates and Quantified Statements II

Definition.

- The negation of a statement of the form

$$\forall x \text{ in } D,\ Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \text{ in } D \text{ such that } \sim Q(x).$$

$$\sim (\forall x \in D,\ Q(x)) \equiv \exists x \in D \text{ such that } \sim Q(x).$$

- The negation of a statement of the form

$$\exists x \text{ in } D \text{ such that } Q(x)$$

is logically equivalent to a statement of the form

$$\forall x \text{ in } D,\ \sim Q(x).$$

$$\sim (\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D,\ \sim Q(x)$$

Example. Negate the following statements:

$\forall$ primes p, p is odd

$\exists$ a triangle T such that the sum of the angles of T equals $200°$

Example. Rewrite the following statements formally, then write the formal and informal negations.

No politicians are honest

The number $1,357$ is not divisible by any integer between 1 and 37.

Example. Write informal negations for the following statements:

All computer programs are finite.

Some computer hackers are over 40.

Negation of a Universal Conditional Statement

$$\sim (\forall x,\ \text{if } P(x) \text{ then } Q(x)) \equiv \exists x \text{ such that } P(x) \text{ and } \sim Q(x)$$

Definition.
A statement of the form

$$\forall x \text{ in } D,\ \text{if } P(x) \text{ then } Q(x)$$

is called **vacuously true** or **true by default** if, and only if, $P(x)$ is false for every x in D.

Example. The following statement is vacuously true since it's negation is false:

All kangaroos enrolled in my class are passing.

Definition.
Consider a statement of the form $\forall x \in D$, if $P(x)$ then $Q(x)$.

1. Its **contrapositive** is the statement $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
2. Its **converse** is the statement $\forall x \in D$, if $Q(x)$ then $P(x)$.
3. Its **inverse** is the statement $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

Example. Write a formal and informal contrapositive, converse, and inverse for the following statement:

If a real number is greater than 2, then its square is greater than 4.

Definition.

- "$\forall x, r(x)$ is a **sufficient condition** for $s(x)$" $\rightarrow$ "$\forall x$, if $r(x)$ then $s(x)$"
- "$\forall x, r(x)$ is a **necessary condition** for $s(x)$" $\rightarrow$ "$\forall x$, if $\sim r(x)$ then $\sim s(x)$"
  $\rightarrow$ "$\forall x$, if $s(x)$ then $r(x)$"
- "$\forall x, r(x)$ **only if** $s(x)$" $\rightarrow$ "$\forall x$, if $\sim s(x)$, then $\sim r(x)$"
  $\rightarrow$ "$\forall x$, if $r(x)$ then $s(x)$"

Example. Rewrite each of the following as a universal conditional statement, quantified either explicitly or implicitly. Do not use the word *necessary* or *sufficient*.

Squareness is a sufficient condition for rectangularity.

Being at least 35 years old is a necessary condition for being president of the United States.

Example. Rewrite the following as a universal conditional statement:

A product of two numbers is 0 only if one of the numbers is 0.

6.1: Set Theory: Definitions and the Element Method of Proof

Element Argument: The Basic Method for Proving that One set is a Subset of Another
Let sets X and Y be given. To prove that $X \subseteq Y$,

1. **suppose** that x is a particular but arbitrarily chosen element of X,
2. **show** that x is an element of Y

Example. Define sets A and B as follows:

$$A = \{m \in \mathbb{Z} \,|\, m = 6r + 12 \text{ for some } r \in \mathbb{Z}\}$$
$$B = \{n \in \mathbb{Z} \,|\, n = 3s \text{ for some } s \in \mathbb{Z}\}$$

Prove that $A \subseteq B$

Disprove that $B \subseteq A$

Definition.
Given sets A and B, A **equals** B, written $\mathbf{A} = \mathbf{B}$, if, and only if, every element of A is in B and every element of B is in A:

$$A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A.$$

Example. Define sets A and B as follows:

$$A = \{m \in \mathbb{Z} \,|\, m = 6r + 12 \text{ for some } r \in \mathbb{Z}\}$$
$$B = \{n \in \mathbb{Z} \,|\, n = 3s \text{ for some } s \in \mathbb{Z}\}$$

Is $A = B$?

Definition.
Given an integer n and a positive integer d, when n is divided by d, then

$$\begin{aligned} n \operatorname{div} d = &\ \text{the integer quotient} \\ n \bmod d = &\ \text{the nonnegative integer remainder} \end{aligned}$$

If n and d are integers and $d > 0$, then

$$n \operatorname{div} d = q \quad \text{and} \quad n \bmod d = r \quad \Leftrightarrow \quad n = dq + r$$

Example. Compute the following:

$32 \operatorname{div} 9, \;\; 32 \bmod 9$

$365 \operatorname{div} 7, \;\; 365 \bmod 7$

Example. If it is currently 11:00, what time will it be in

51 hours?

121 hours?

11 hours?

-1 hours?

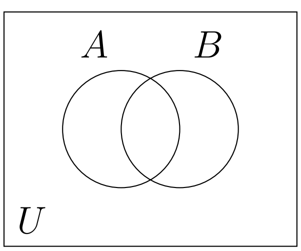
Example. Let $A = \left\{4, \sqrt{16}, 19 \bmod 15\right\}$ and $B = \{12 \bmod 8\}$. Is $A \subseteq B$? Is $B \subseteq A$?

Definition.
Let A and B be subsets of a universal set U.

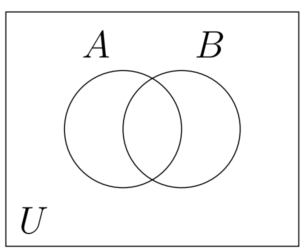
1. The **union** of A and B is the set of all elements that are in at least one of A or B.
$$A \cup B = \{x \in U \,|\, x \in A \text{ or } x \in B\}$$
2. The **intersection** of A and B is the set of all elements that are common to both A and B.
$$A \cap B = \{x \in U \,|\, x \in A \text{ and } x \in B\}$$
3. The **difference** of A and B is the set of all elements that are in B and not A.
$$B - A = \{x \in U \,|\, x \in B \text{ and } x \notin A\}$$
4. The **complement** of A is the set of all elements in U that are not in A.
$$A^c = \{x \in U \,|\, x \notin A\}$$

Example. Represent the following sets using the Venn diagrams below:

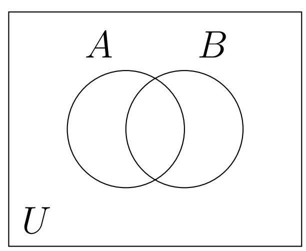
$A \cup B$



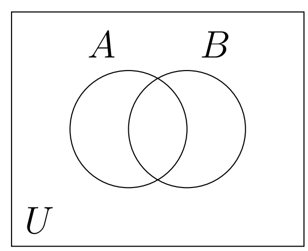
$A \cap B$



$B - A$



A^c



Example. Let the universal set be the set $U = \{a, b, c, d, e, f, g\}$, and let $A = \{a, c, e, g\}$ and $B = \{d, e, f, g\}$. Find

$A \cup B$

$A \cap B$

$B - A$

A^c

Definition.
Given real numbers a and b with $a \leq b$:

$$(a, b) = \{x \in \mathbb{R} \,|\, a < x < b\} \qquad (a, b] = \{x \in \mathbb{R} \,|\, a < x \leq b\}$$
$$[a, b) = \{x \in \mathbb{R} \,|\, a \leq x < b\} \qquad [a, b] = \{x \in \mathbb{R} \,|\, a \leq x \leq b\}$$

Example. Let the universal set be $\mathbb{R}$, and let $A = (-1, 0]$ and $B = [0, 1)$. Find

$A \cup B$

$A \cap B$

$B - A$

A^c

Definition.
Given sets $A_0, A_1, A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n,

$$\bigcup_{i=0}^{n} A_i = \{x \in U \,|\, x \in A_i, \text{ for at least one } i = 0, 1, 2, \ldots, n\}$$
$$\bigcap_{i=0}^{n} A_i = \{x \in U \,|\, x \in A_i, \text{ for every } i = 0, 1, 2, \ldots, n\}$$

Example. For each positive integer i, let $A_i = \left\{ x \in \mathbb{R} \,\middle|\, -\frac{1}{i} < x < \frac{1}{i} \right\} = \left(-\frac{1}{i}, \frac{1}{i} \right)$. Find

$A_1 \cup A_2 \cup A_3$

$A_1 \cap A_2 \cap A_3$

$\bigcup_{i=1}^{\infty} A_i$

$\bigcap_{i=1}^{\infty} A_i$

Definition.
The **empty set** (or **null set**), denoted $\varnothing$, is the set with no elements.

$$\{1,3\} \cap \{2,4\} = \varnothing$$

Two sets are called **disjoint** if, and only if, they have no elements in common:

$$A \cap B = \varnothing.$$

Sets $A_1, A_2, A_3, \dots$ are **mutually disjoint** (or **pairwise disjoint**) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common:

$$A_i \cap A_j = \varnothing \text{ whenever } i \neq j.$$

Example.
Let $A_1 = \{3,5\}$, $A_2 = \{1,4,6\}$, and $A_3 = \{2\}$. Are A_1, A_2, and A_3 mutually disjoint?

Let $B_1 = \{2,4,6\}$, $B_2 = \{3,7\}$, and $B_3 = \{4,5\}$. Are B_1, B_2, B_3 mutually disjoint?

Definition.
A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, \ldots\}$ is a **partition** of a set A if, and only if,

1. A is the union of all the A_i;
2. the sets $A_1, A_2, A_3, \ldots$ are mutually disjoint.

Example.
Let $A = \{1, 2, 3, 4, 5, 6\}$, $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$, and $A_3 = \{5, 6\}$. Is $\{A_1, A_2, A_3\}$ a partition of A?

Let $\mathbb{Z}$ be the set of all integers and let

$$T_i = \{n \in \mathbb{Z} \,|\, n = 3k + i, \text{ for some integer } k\}.$$

Is $\{T_0, T_1, T_2\}$ a partition of $\mathbb{Z}$?

Definition.
Given a set A, the **power set** of A, denoted $\mathscr{P}(A)$, is the set of all subsets of A.

Example. Find $\mathscr{P}(\{x, y\})$.

6.4: Boolean Algebras, Russell's Paradox, and the Halting Problem

Definition.
A **Boolean algebra** is a set B together with two operations, generally denoted $+$ and $\cdot$, such that for all a and b in B both $a+b$ and $a \cdot b$ are in B and the following axioms are assumed to hold:

1. *Commutative Laws:* For all a and b in B,
$$a+b=b+a \text{ and } a \cdot b=b \cdot a$$
2. *Associative Laws:* For all a and b in B,
$$(a+b)+c=a+(b+c) \text{ and } (a \cdot b) \cdot c=a \cdot(b \cdot c)$$
3. *Distributive Laws:* For all a and b in B,
$$a+(b \cdot c)=(a+b) \cdot(a+c) \text{ and } a \cdot(b+c)=(a \cdot b)+(a \cdot c)$$
4. *Identity Laws:* There exist distinct elements 0 and 1 in B such that for each a in B,
$$a+0=a \text{ and } a \cdot 1=a$$
5. *Complement Laws:* For each a in B, there exists an element in B, denoted $\overline{a}$ and called the **complement** or **negation** of a, such that
$$a+\overline{a}=1 \text{ and } a \cdot \overline{a}=0$$

Properties of a Boolean Algebra

Let B be any Boolean algebra.

1. *Uniqueness of the Complement Laws:* For all a and x in B, if $a + x = 1$ and $a \cdot x = 0$, then $x = \overline{a}$.
2. *Uniqueness of* 0 *and* 1*:* If there exists x in B such that $a + x = a$ for every a in B, then $x = 0$, and if there exists y in B such that $a \cdot y = a$ for every a in B, then $y = 1$.
3. *Double Complement Law:* For every $a \in B$, $\overline{(\overline{a})} = a$.
4. *Idempotent Laws:* For every $a \in B$,
$$a + a = a \text{ and } a \cdot a = a.$$
5. *Universal Bound Laws:* For every $a \in B$,
$$a + 1 = 1 \text{ and } a \cdot 0 = 0.$$
6. *De Morgan's Laws:* For all a and $b \in B$,
$$\overline{a + b} = \overline{a} \cdot \overline{b} \text{ and } \overline{a \cdot b} = \overline{a} + \overline{b}.$$
7. *Absorption Laws:* For all a and $b \in B$,
$$(a + b) \cdot a = a \text{ and } (a \cdot b) + a = a.$$
8. *Complements of* 0 *and* 1*:*
$$\overline{0} = 1 \text{ and } \overline{1} = 0.$$

Example. Prove that for all elements a in a Boolean algebra B:

$\overline{(\overline{a})} = a.$

$a + a = a.$

Example. Prove that for all elements a in a Boolean algebra B:

$$a \cdot a = a.$$

$$(a + b) \cdot a = a.$$

Russell's Paradox
Define the following set S:

$$S = \{A \,|\, A \text{ is a set and } A \notin A\}.$$

Is S an element of itself?

The Barber Puzzle: In a certain town, there is a male barber who shaves all those men, and only those men, who do not shave themselves.
Does the barber shave himself?

Is the sentence "The barber shaves himself" a statement?

Example. Determine whether each sentence is a statement:

If $1 + 1 = 3$, then $1 = 0$.

This sentence is false and $1 + 1 = 3$.

The Halting Problem (Alan M. Turing)
There is no computer algorithm that will accept any algorithm X and data set D as input and then will output "halts" or "loops forever" to indicate whether or not X terminates in a finite number of steps when X is run with data set D.

Example boolean algebras:

Logical Equivalences For all statement variables p, q, and r:	**Set Properties** For all sets A, B, and C:
$p \vee q \equiv q \vee p$	$A \cup B = B \cup A$
$p \wedge q \equiv q \wedge p$	$A \cap B = B \cap A$
$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	$A \cap (B \cap C) = (A \cap B) \cap C$
$p \vee (q \vee r) \equiv (p \vee q) \vee r$	$A \cup (B \cup C) = (A \cup B) \cup C$
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$p \vee \mathbf{c} \equiv p$	$A \cup \varnothing = A$
$p \wedge \mathbf{t} \equiv p$	$A \cap U = A$
$p \vee \sim p \equiv \mathbf{t}$	$A \cup A^c = U$
$p \wedge \sim p \equiv \mathbf{c}$	$A \cap A^c = \varnothing$
$\sim (\sim p) \equiv p$	$(A^c)^c = A$
$p \wedge p \equiv p$	$A \cup A = A$
$p \vee p \equiv p$	$A \cap A = A$
$p \vee \mathbf{t} \equiv \mathbf{t}$	$A \cup U = U$
$p \wedge \mathbf{c} \equiv \mathbf{c}$	$A \cap \varnothing = \varnothing$
$\sim (p \vee q) \equiv \sim p \wedge \sim q$	$(A \cup B)^c = A^c \cap B^c$
$\sim (p \wedge q) \equiv \sim p \vee \sim q$	$(A \cap B)^c = A^c \cup B^c$
$p \vee (p \wedge q) \equiv p$	$A \cup (A \cap B) = A$
$p \wedge (p \vee q) \equiv p$	$A \cap (A \cup B) = A$
$\sim \mathbf{t} \equiv \mathbf{c}$	$U^c = \varnothing$
$\sim \mathbf{c} \equiv \mathbf{t}$	$\varnothing^c = U$

8.1: Relations on Sets

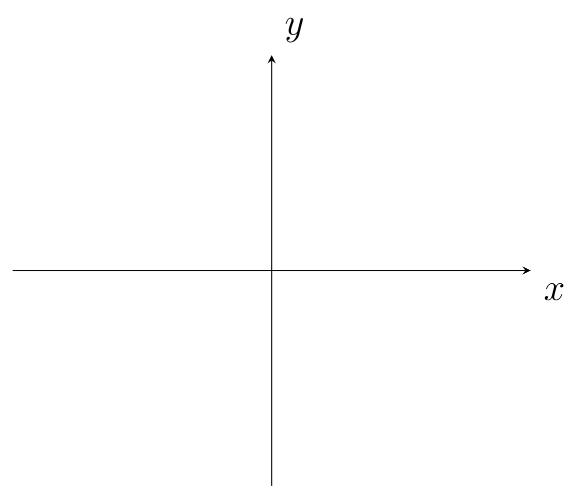
Definition.
A relation R from A to B is called a **binary relation** because it is a subset of a Cartesian product of two sets.

Example. Define a relation L from $\mathbb{R}$ to $\mathbb{R}$:

$$\forall x, y \in \mathbb{R},\ x \, L \, y \Leftrightarrow x < y.$$

Is $57 \, L \, 53$? Is $(-17) \, L \, (-14)$? Is $143 \, L \, 143$? Is $(-35) \, L \, 1$?

Draw the graph of L as a subset of the Cartesian plane $\mathbb{R} \times \mathbb{R}$.



Definition.
Two integers m and n are **congruent modulo** 2 if, and only if, $m \bmod 2 = n \bmod 2$.

Example. Define a relation E from $\mathbb{Z}$ to $\mathbb{Z}$:

$$\forall (m, n) \in \mathbb{Z} \times \mathbb{Z},\ m\,E\,n \Leftrightarrow m - n \text{ is even.}$$

Is $4\,E\,0$? Is $2\,E\,6$? Is $3\,E\,(-3)$? Is $5\,E\,2$?

Prove that if n is any odd integer, then $n\,E\,1$.

Example. Let $X = \{a, b, c\}$. Define a relation $\mathbf{S}$ from $\mathscr{P}(X)$ to $\mathscr{P}(X)$ as follows:

$$\forall A, B \in \mathscr{P}(X),\ A\,\mathbf{S}\,B \Leftrightarrow A \text{ has at least as many elements as } B.$$

Is $\{a, b\}\,\mathbf{S}\,\{b, c\}$?

Is $\{a\}\,\mathbf{S}\,\varnothing$?

Is $\{b, c\}\,\mathbf{S}\,\{a, b, c\}$?

Is $\{c\}\,\mathbf{S}\,\{a\}$?

Definition.
Let R be a relation from A to B. Define the inverse relation R^{-1} from B to A as follows:

$$R^{-1} = \{(y, x) \in B \times A \,|\, (x, y) \in R\}.$$

Example. Let $A = \{2, 3, 4\}$ and $B = \{2, 6, 8\}$, and let R be the "divides" relation from A to B:

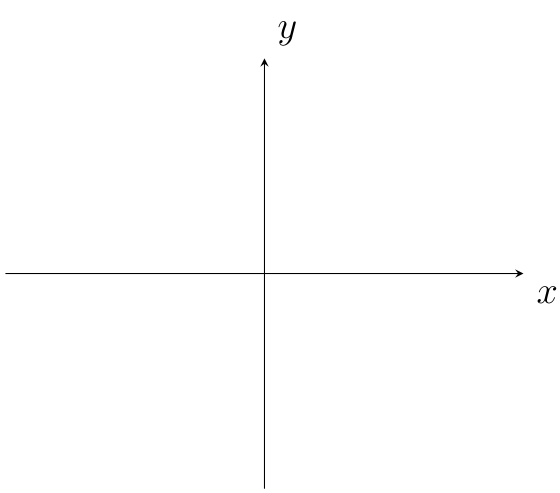
$$\forall (x, y) \in A \times B, \; x \, R \, y \Leftrightarrow x \mid y$$

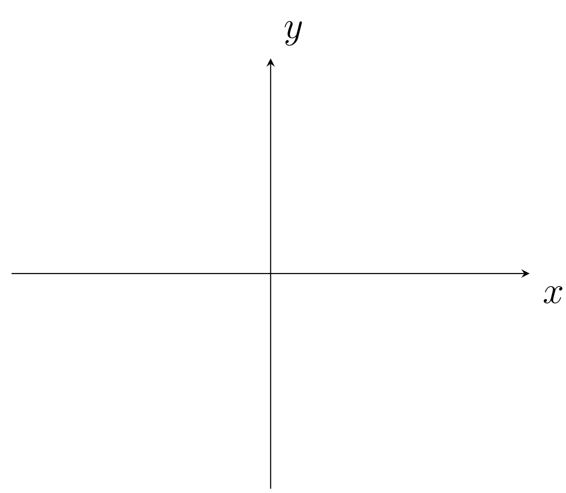
Explicitly state which ordered pairs are in R and R^{-1}. Draw arrow diagrams for both.

Example. Define a relation R from $\mathbb{R}$ to $\mathbb{R}$ as follows:

$$\forall (x, y) \in \mathbb{R} \times \mathbb{R},\ x\,R\,y \Leftrightarrow y = 2|x|.$$

Draw the graphs of R and R^{-1} in the Cartesian plane. Is R^{-1} a function?





Definition.
A **relation on a set** A is a relation from A to A.
A **graph** G consists of two finite sets:

- a nonempty set $V(G)$ of **vertices** and
- a set $E(G)$ of **edges**,

where each edge is associated with a set consisting of either one or two vertices called its endpoints.
A **directed graph** is a graph whose edges are directional.

Example. Let $A = \{3, 4, 5, 6, 7, 8\}$ and define a relation R on A as follows

$$\forall x, y \in A,\ x\, R\, y \Leftrightarrow 2 \mid (x - y).$$

Draw the directed graph of R.

Definition.
Given sets $A_1, A_2, \ldots, A_n$, an **n-ary relation** R on $A_1 \times A_2 \times \cdots \times A_n$ is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The special case of 2-ary, 3-ary, and 4-ary relations are called **binary**, **ternary**, and **quaternary relations**, respectively.

8.2: Reflexivity, Symmetry, and Transitivity

Definition.
Let R be a relation on a set A.

1. R is **reflexive** if, and only if, for every $x \in A$, $x\,R\,x$.

$$\forall x \in A, (x, x) \in R$$

2. R is **symmetric** if, and only if, for every $x, y \in A$, if $x\,R\,y$ then $y\,R\,x$.

$$\forall x, y \in A, \text{ if } (x, y) \in R \text{ then } (y, x) \in R$$

3. R is **transitive** if, and only if, for every $x, y, z \in A$, if $x\,R\,y$ and $y\,R\,z$, then $x\,R\,z$.

$$\forall x, y, z \in A, \text{ if } (x, y) \in R \text{ and } (y, z) \in R \text{ then } (x, z) \in R$$

Note: A relation R is

not reflexive $\Leftrightarrow \exists x \in A$ such that $x \not R\, x$
or $(x, x) \notin R$.

not symmetric $\Leftrightarrow \exists x, y \in A$ such that $x\,R\,y$ but $y \not R\, x$
or $(x, y) \in R$ but $(y, x) \notin R$.

not transitive $\Leftrightarrow \exists x, y, z \in A$ such that $x\,R\,y$ and $y\,R\,z$, but $x \not R\, z$
or $(x, y) \in R$ and $(y, z) \in R$, but $(x, z) \notin R$.

irreflexive $\Leftrightarrow \forall x \in A, x \not R\, x$

asymmetric $\Leftrightarrow \forall x, y \in A$, if $x\,R\,y$ then $y \not R\, x$

intransitive $\Leftrightarrow \forall x, y, z \in A$, if $x\,R\,y$ and $y\,R\,z$, then $x \not R\, z$

Example. Define a relation R on $\mathbb{R}$ as follows:

$$x \, R \, y \Leftrightarrow x = y.$$

Is R reflexive? Is R symmetric? Is R transitive?

Example. Define a relation R on $\mathbb{R}$ as follows:

$$x \, R \, y \Leftrightarrow x < y.$$

Is R reflexive? Is R symmetric? Is R transitive?

Example. Let $A = \{0, 1, 2, 3\}$ and define relations R, S, and T on A as follows:

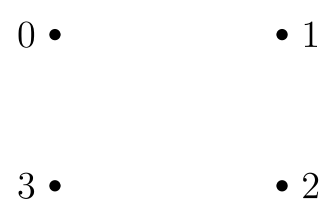
$$
\begin{aligned}
R &= \{(0,0),(0,1),(0,3),(1,0),(1,1),(2,2),(3,0),(3,3)\}\\
S &= \{(0,0),(0,2),(0,3),(2,3)\}\\
T &= \{(0,1),(2,3)\}
\end{aligned}
$$

For each relation, draw the directed graph, then identify if it is reflexive, symmetric, and/or transitive.

R

S

T



Example. Define a relation T on $\mathbb{Z}$ as follows:

$$\forall m, n \in \mathbb{Z},\ m\,T\,n \Leftrightarrow 3 \mid (m-n).$$

This relation is called **congruence modulo 3.**

Is T reflexive? Is T symmetric? Is T transitive?

Example. Define a relation S on $\mathbb{R}$ as follows:

$$\forall x, y \in \mathbb{R},\ x\,S\,y \Leftrightarrow |x| + |y| = 1.$$

Is S reflexive? Is S symmetric? Is S transitive?