3.3: The Chain Rule

Example. Let $f(x) = \left(x^3 + x + 1\right)^2$. Find $f'(x)$

using the product rule

by expanding

What about $\dfrac{d}{dx}\left[\left(x^3 + x + 1\right)^{100}\right]$?

Composite Functions:

Let f and g be functions of x. Then, the **composite functions** g of f (denoted $g \circ f$) and f of g (denoted $f \circ g$) are defined as:

$$(g \circ f)(x) = g(f(x))$$
$$(f \circ g)(x) = f(g(x))$$

Example. 'Break-down' the following composite functions:

$$\frac{1}{x+3} \qquad \left(x^4 + 3x - 8\right)^3$$

$$\left(\frac{1-x}{x^3+1}\right)^4 \qquad \frac{3}{\sqrt{(x+1)^2 - 1}}$$

Rule 7: The Chain Rule

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Note:

$$\frac{d}{dx}\left[f\Big(g\big(h(j(x))\big)\Big)\right] = f'\Big(g\big(h(j(x))\big)\Big) \cdot g'\big(h(j(x))\big) \cdot h'(j(x)) \cdot j'(x)$$

The General Power Rule

$$\frac{d}{dx}[(f(x))^n] = n(f(x))^{n-1}f'(x)$$

Example. Use the chain rule to show

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Example. Find the derivative of the following functions

$F(x) = \left(x^3 + x + 1\right)^{100}$

$G(t) = (3x + 1)^2$

$H(u) = \sqrt{u^2 + 1} - 3$

$J(\nu) = \nu^2(2\nu + 3)^5$

$$\kappa(x) = \left(2x^2 + 3\right)^4 (3x - 1)^5$$

$$\tau(x) = \frac{1}{(4x^2 - 7)^2}$$

Example. Find the equation of the line tangent to $f(x)$ at $\left(0, \frac{1}{8}\right)$

$$f(x) = \left(\frac{2x + 1}{3x + 2}\right)^3$$

Example. The membership of The Fitness Center, which opened a few years ago, is approximated by the function

$$N(t) = 100(64 + 4t)^{2/3} \qquad (0 \leq t \leq 52)$$

where $N(t)$ gives the number of members at the beginning of week t.

Find $N'(t)$

How fast was the center's membership increasing initially ($t = 0$)?

How fast was the membership increasing at the beginning of the 40th week?

What was the membership when the center first opened? At the beginning of the 40th week?

Rule 1: Derivative of a Constant

$$\frac{d}{dx}[c] = 0$$

Rule 2: The Power Rule

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

Rule 3: Derivative of a Constant Multiple of a Function

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

Rule 4: The Sum Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

Rule 5: The Product Rule

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Rule 6: The Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Rule 7: The Chain Rule

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$