1.1: Solutions of Linear Equations and Inequalities in One Variable

Definition.
A **function** f is a special relation between x and y such that each input x results in *at most* one y. The symbol $f(x)$ is read "f of x" and is called the **value of f at x**

Example. Let $f(x) = \frac{x^2}{2} + x$. Evaluate the following:

$f(1) = \frac{1^2}{2} + 1\left(\frac{2}{2}\right) = \frac{1}{2} + \frac{2}{2} = \frac{1+2}{2} = \boxed{\frac{3}{2}}$

$f\left(\frac{1}{2}\right) = \frac{\left(1/2\right)^2}{2} + \left(1/2\right) = \frac{1/4}{2} + \frac{1}{2}\left(\frac{4}{4}\right) = \frac{1}{8} + \frac{4}{8} = \frac{1+4}{8} = \boxed{\frac{5}{8}}$

$f(-2) = \frac{(-2)^2}{2} + (-2) = \frac{4}{2} - 2 = 2 - 2 = \boxed{0}$

$f(0) = \frac{(0)^2}{2} + (0) = \boxed{0}$

$f(f(x)) = \frac{(f(x))^2}{2} + (f(x)) = \frac{\left(\frac{x^2}{2} + x\right)^2}{2} + \left(\frac{x^2}{2} + x\right) = \frac{\frac{x^4}{4} + x^3 + x^2}{2} + \frac{x^2}{2} + x$

$= \frac{x^4}{8} + \frac{x^3}{2} + \frac{x^2}{2} + \frac{x^2}{2} + x = \boxed{\frac{x^4}{8} + \frac{x^3}{2} + x^2 + x}$

Composite Functions:

Let f and g be functions of x. Then, the **composite functions** g of f (denoted $g \circ f$) and f of g (denoted $f \circ g$) are defined as:

$$(g \circ f)(x) = g(f(x))$$
$$(f \circ g)(x) = f(g(x))$$

Example. Let $g = x - 1$. Find:

$(g \circ f)(x) = g(f(x)) = (f(x)) - 1 = \left(\frac{x^2}{2} + x\right) - 1 = \boxed{\frac{x^2}{2} + x - 1}$

$(f \circ g)(x) = f(g(x)) = \frac{(g(x))^2}{2} + (g(x)) = \frac{(x-1)^2}{2} + (x-1) = \frac{x^2 - 2x + 1}{2} + x - 1 = \frac{x^2}{2} - x + \frac{1}{2} + x - 1 = \boxed{\frac{x^2}{2} - \frac{1}{2}}$

Operations with Functions:

Let f and g be functions of x and define the following:

Sum	$(f+g)(x) = f(x) + g(x)$
Difference	$(f-g)(x) = f(x) - g(x)$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$
Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ if $g(x) \neq 0$

Definition.

An **expression** is a meaningful string of numbers, variables and operations:

$$3x - 2$$

An **equation** is a statement that two quantities or algebraic expressions are equal:

$$3x - 2 = 7$$

A **solution** is a value of the variable that makes the equation true:

$$\begin{aligned} 3(3) - 2 &= 7 \\ 9 - 2 &= 7 \\ 7 &= 7 \end{aligned}$$

A **solution set** is the set of ALL possible solutions of an equation:

$3x - 2 = 7$ only has the solution $x = 3$,

$2(x-1) = 2x - 2$ is true for all possible values of x.

Properties of Equality:

Substitution Property: The equation formed by substituting one expression for an equal expression is equivalent to the original equation:

$$\begin{aligned} 3(x-3)-\frac{1}{2}(4x-18) &= 4 \\ 3x-9-2x+9 &= 4 \\ x &= 4 \end{aligned}$$

Addition Property: The equation formed by adding the same quantity to both sides of an equation is equivalent to the original equation:

$$\begin{aligned} x-4 &= 6 \\ x-4+4 &= 6+4 \\ x &= 10 \end{aligned} \qquad\qquad \begin{aligned} x+5 &= 12 \\ x+5+(-5) &= 12+(-5) \\ x &= 7 \end{aligned}$$

Multiplication Property: The equation formed by multiplying both sides of an equation by the same *nonzero* quantity is equivalent to the original equation:

$$\begin{aligned} \frac{1}{3}x &= 6 \\ 3\left(\frac{1}{3}x\right) &= 3(6) \\ x &= 18 \end{aligned} \qquad\qquad \begin{aligned} 5x &= 20 \\ \frac{5x}{5} &= \frac{20}{5} \\ x &= 4 \end{aligned}$$

Solving a linear equation:

Using the properties of equality above, we can solve any linear equation in 1 variable:

Example. Solve $\dfrac{3x}{4} + 3 = \dfrac{x-1}{3}$

1. Eliminate fractions: $$12\left(\frac{3x}{4} + 3\right) = 12\left(\frac{x-1}{3}\right)$$
2. Remove/evaluate parenthesis: $$9x + 36 = 4x - 4$$
3. Use addition property to isolate the variable to one side: $$9x + 36 - 36 - 4x = 4x - 4 - 36 - 4x$$
4. Use multiplication property to isolate variable: $$\frac{5x}{5} = \frac{-40}{5}$$
5. Verify solution via substitution: $$\underbrace{\frac{3(-8)}{4} + 3}_{-6+3=-3} \stackrel{?}{=} \underbrace{\frac{(-8)-1}{3}}_{\frac{-9}{3}=-3}$$

Example. Solve the following:

$$(2)(3)\left(\frac{3x+1}{2}\right) = \left(\frac{x}{3} - 3\right)(2)(3)$$

$$3(3x+1) = 2x - (2)(3)3$$

$$9x + 3 - 2x = 2x - 18 - 2x$$

$$7x + 3 - 3 = -18 - 3$$

$$\frac{7x}{7} = \frac{-21}{7}$$

$$\boxed{x = 21}$$

$$(x-3)\left(\frac{2x-1}{x-3}\right) = \left(4 + \frac{5}{x-3}\right)(x-3)$$

$$2x - 1 = 4(x-3) + 5$$

$$2x - 1 - 4x = 4x - 12 + 5 - 4x$$

$$-2x - 1 + 1 = -7 + 1$$

$$\frac{-2x}{-2} \quad \frac{-6}{-2}$$

$$\boxed{x = 3}$$

Verification of solutions omitted

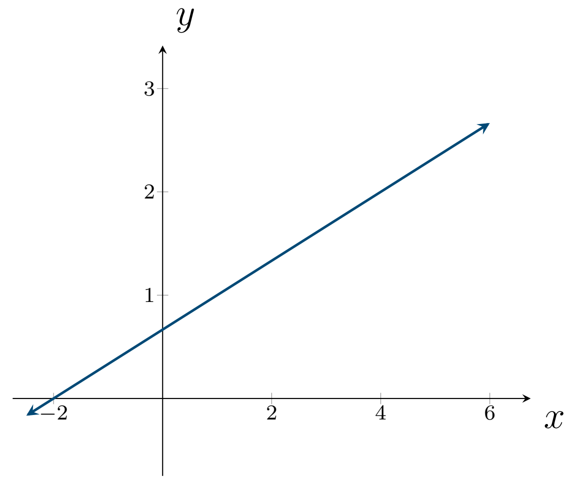
Example. Solve $-2x + 6y = 4$ for y

$-2x + 6y + 2x = 4 + 2x$

$\frac{6y}{6} = \frac{2x + 4}{6}$

$y = \frac{2x}{6} + \frac{4}{6}$

$\boxed{y = \frac{x}{3} + \frac{2}{3}}$



Example. Suppose that the relationship between a firm's profit, P, and the number of items sold, x, can be described by the equation

$$5x - 4P = 1200$$

a) How many units must be produced and sold for the firm to make a profit of \$150?

$P = \$150$, Find x

$5x - 4(150) = 1200$

$5x - 600 + 600 = 1200 + 600$

$\frac{5x}{5} = \frac{1800}{5}$

$\boxed{x = 360}$

b) Solve this equation for P in terms of x. Then, find the profit when 240 units are sold.

$5x - 4P - 5x = 1200 - 5x$

$\frac{-4P}{-4} = \frac{-5x + 1200}{-4}$

$P = \frac{-5x}{-4} + \frac{1200}{-4}$

$\boxed{P = \frac{5x}{4} - 300}$

Definition.
An **inequality** is a statement that one quantity is greater than (or less than) another quantity.

Properties of Inequalities

Substitution Property: The inequality formed by substituting one expression for an equal expression is equivalent to the original inequality:

$$\begin{aligned} 5x - 4x + 2 &< 6 \\ x &< 4 \quad \Rightarrow \text{The solution set is } \{x : x < 6\} \end{aligned}$$

Addition Property: The inequality formed by adding the same quantity to both sides of an inequality is equivalent to the original inequality:

$$\begin{aligned} x - 4 &< 6 \\ x - 4 + 4 &< 6 + 4 \\ x &< 10 \end{aligned} \qquad \begin{aligned} x + 5 &\geq 12 \\ x + 5 + (-5) &\geq 12 + (-5) \\ x &\geq 7 \end{aligned}$$

Multiplication Property The inequality formed by multiplying both sides of an inequality by the same *positive* quantity is equivalent to the original inequality. The direction of the inequality is flipped when multiplying by a *negative* quantity:

$$\begin{aligned} \frac{1}{3}x &> 6 \\ 3\left(\frac{1}{3}x\right) &> 3(6) \\ x &> 18 \end{aligned} \qquad \begin{aligned} 5x - 5 &\leq 6x + 20 \\ -x &\leq 25 \\ x &\geq -25 \end{aligned}$$

Example. Solve

$$-x + 8 \leq 2x - 4$$

first by gathering the x variable on the left, then again on the right. See that the multiplication property holds in both cases.

$$-x+8+x \le 2x-4+x$$

$$8+4 \le 3x-4+4$$

$$\frac{12}{3} \le \frac{3x}{3}$$

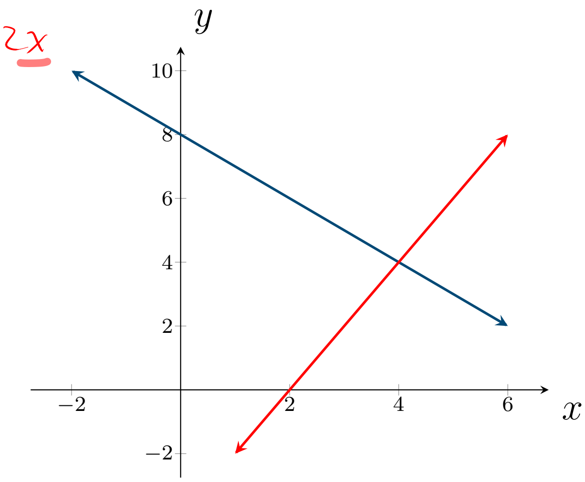
$$\boxed{4 \le x}$$

$$-x+8-2x \le 2x-4-2x$$

$$-3x+8-8 \le -4-8$$

$$\frac{-3x}{-3} \le \frac{-12}{-3}$$

$$\boxed{x \ge 4}$$



* Note:

-The inequality on the right changed direction in the last step because we divided by a negative number

-The answers above are equivalent. They are only formatted differently.