5.3: Equations and Applications with Exponential and Logarithmic Functions

Solving exponential equations:

Example. Solve $4(25^{2x}) = 312,500$

1. Isolate the exponential by rewriting the equation with a base raised to a power on one side: $$\frac{4(25^{2x})}{4} = \frac{312,500}{4}$$
2. Take the logarithm of both sides: $$\ln\left(25^{2x}\right) = \ln(78,125)$$
3. Use a property of logarithms to remove the variable from the exponent: $$2x\ln(25) = \ln(78,125)$$
4. Solve for the variable: $$x = \frac{\ln(78,125)}{2\ln(25)} \approx 1.75$$

Example. Unless the exponential function uses base e or base 10, *it does not matter which logarithm we use.* Solve the following exponential equation first using base 10, then using base e:

$$6(4^{3x-2}) = 20$$

① $\dfrac{6\left(4^{3x-2}\right)}{6} = \dfrac{20}{6}$

② $\ln\left(4^{3x-2}\right) = \ln\left(\frac{10}{3}\right)$

③ $\ln\left(4^{3x-2}\right) = \ln\left(\frac{10}{3}\right)$

④ $\dfrac{(3x-2)\ln(4)}{\ln(4)} = \dfrac{\ln\left(\frac{10}{3}\right)}{\ln(4)}$

$+2 + 3x - 2 = \dfrac{\ln(10/3)}{\ln(4)} + 2$

$\dfrac{3x}{3} = \dfrac{\dfrac{\ln(10/3)}{\ln(4)} + 2}{3}$

$$\boxed{x = \frac{\dfrac{\ln(10/3)}{\ln(4)} + 2}{3} \approx 0.956}$$

Example. Suppose the demand function for q thousand units of a certain commodity is given by

$$p = 30\left(3^{-q/2}\right)$$

At what price per unit will the demand equal 4000 units?

$q = 4$, find p

$$p = 30\left(3^{-4/2}\right) = 30\left(3^{-2}\right) = 30\left(\frac{1}{3^2}\right) = \frac{30}{9} = \boxed{10.\overline{33}}$$

$$\boxed{p = \$10.33}$$

How many units, to the nearest thousand units, will be demanded if the price is \$17.31?

$p = \$17.31$, find q

$$\frac{17.31}{30} = \frac{30\left(3^{-q/2}\right)}{30}$$

$$\ln\left(\frac{17.31}{30}\right) = \ln\left(3^{-q/2}\right)$$

$$\left(\frac{-2}{\ln(3)}\right)\ln\left(\frac{17.31}{30}\right) = \frac{-q\ln(3)}{2}\left(\frac{-2}{\ln(3)}\right)$$

$$q = \frac{-2\ln\left(\frac{17.31}{30}\right)}{\ln(3)} \approx 1.001$$

$$\Rightarrow \boxed{q = 1 \text{ thousand units}}$$

Example. A company finds that its daily sales begin to fall after the end of an advertising campaign, and the decline is such that the number of sales is $S = 2000\left(2^{-0.1x}\right)$, where x is the number of days after the campaign ends.

How many sales will be made after $\underbrace{\text{10 days}}_{x}$ after the end of the campaign?

$$S = 2000\left(2^{-0.1(10)}\right) = 2000\left(2^{-1}\right) = 2000\left(\tfrac{1}{2}\right) = 1000$$

If the company does not want sales to drop below 350 per day, when should it start a new campaign?

$$\frac{35\cancel{0}}{200\cancel{0}} = \frac{2000\left(2^{-0.1q}\right)}{2000}$$

$$\ln\left(\frac{35}{200}\right) = \ln\left(2^{-0.1q}\right)$$

$$\frac{\ln\left(\frac{35}{200}\right)}{-0.1\ln(2)} = \frac{-0.1q\ln(2)}{-0.1\ln(2)}$$

$$\boxed{q = \frac{\ln\left(\frac{35}{200}\right)}{-0.1\ln(2)} \approx 25.146}$$

Example. The population of a certain city was 30,000 in 2000, and 40,500 in 2010. If the formula $P = P_0 e^{ht}$ applies to the growth of the city's population, what population is predicted for the year 2030?

Year	Years after 2000	Pop
2000	0	30,000
2010	10	40,500
2030	30	?

$$30{,}000 = P_0 e^{h(0)} = P_0 e^0 = P_0 \Rightarrow \boxed{P_0 = 30000}$$

$$\frac{40{,}5\cancel{00}}{30{,}0\cancel{00}} = \frac{30{,}000\, e^{h(10)}}{30{,}000}$$

$$\ln\left(\frac{405}{300}\right) = \ln\left(e^{10h}\right)$$

$$\frac{\ln\left(\frac{405}{300}\right)}{10} = \frac{10h\ln(e)}{10} \qquad \ln(e) = 1$$

$$\boxed{h = \frac{\ln\left(\frac{405}{300}\right)}{10} \approx 0.03}$$

$$S = 30{,}000\, e^{\square(30)} = \boxed{73{,}811.25}$$

If you use $h = 0.03$

$$S = 30{,}000\, e^{0.03(30)} \approx \boxed{73{,}788.09}$$

Example. The Gompertz equation

$$N = 100(0.03)^{0.2^t}$$

predicts the size of a deer herd on a small island t decades from now.

What is the size of the deer population now ($t = 0$)?

$$N = 100\,(0.03)^{0.2^0} = 100\,(0.03)^1 = \boxed{3}$$

During what year will the deer population reach or exceed 70?

$$\frac{70}{100} = \frac{100\,(0.03)^{(0.2^t)}}{100}$$

$$\ln\left(\frac{70}{100}\right) = \ln\left((0.03)^{(0.2^t)}\right)$$

This is another exponential equation

$$\frac{\ln\left(\frac{7}{10}\right)}{\ln(0.03)} = \frac{(0.2^t)\ln(0.03)}{\ln(0.03)}$$

$$\ln\left(\frac{\ln\left(\frac{7}{10}\right)}{\ln(0.03)}\right) = \ln(0.2^t)$$

$$\frac{\ln\left(\frac{\ln\left(\frac{7}{10}\right)}{\ln(0.03)}\right)}{\ln(0.2)} = \frac{t\ln(0.2)}{\ln(0.2)}$$

$$\boxed{t = \frac{\ln\left(\frac{\ln\left(\frac{7}{10}\right)}{\ln(0.03)}\right)}{\ln(0.2)}} \approx 1.42$$

Example. One company's revenue from the sales of computer tablets from 2015 to 2020 can be modeled by the logistic function

$$y = \frac{9.46}{1 + 53.08e^{-1.28x}}$$

where x is the number of years past 2014 and y is in millions of dollars.

Estimate the sales revenue for $\underbrace{2020}_{x=6}$

$$y = \frac{9.46}{1+53.08e^{-1.28(6)}} \approx \boxed{9.234 \text{ million dollars}}$$

During what year will the sales revenue exceed \$4 million?

$$(1+53.08e^{-1.28x})\,4 = \frac{9.46}{1+53.08e^{-1.28x}}\,(1+53.08e^{-1.28x})$$

$$\frac{4(1+53.08e^{-1.28x})}{4} = \frac{9.46}{4}$$

$$-1 + 1+53.08e^{-1.28x} = \frac{9.46}{4} - 1$$

$$\frac{53.08e^{-1.28x}}{53.08} = \frac{\frac{9.46}{4} - 1}{53.08}$$

$$\ln\left(e^{-1.28x}\right) = \ln\left(\frac{\frac{9.46}{4} - 1}{53.08}\right)$$

$$\frac{-1.28x \underbrace{\ln(e)}_{1}}{-1.28} = \frac{\ln\left(\frac{\frac{9.46}{4} - 1}{53.08}\right)}{-1.28}$$

$$\boxed{x = \frac{\ln\left(\frac{\frac{9.46}{4} - 1}{53.08}\right)}{-1.28} \approx 2.860}$$

Example (Bonus)**.** Solve the following for x:

$$6^{x-2} = 2^{-3x}$$

$$\ln\left(6^{x-2}\right) = \ln\left(2^{-3x}\right)$$

$$(x-2)\ln(6) = -3x\ln(2)$$

$$-x\ln(6) + x\ln(6) - 2\ln(6) = -3x\ln(2) - x\ln(6)$$

$$-2\ln(6) = -3\underline{x}\ln(2) - \underline{x}\ln(6)$$

$$\frac{-2\ln(6)}{-3\ln(2)-\ln(6)} = \frac{x\left(-3\ln(2)-\ln(6)\right)}{-3\ln(2)-\ln(6)}$$

$$\boxed{\frac{-2\ln(6)}{-3\ln(2)-\ln(6)} = x \approx 0.926}$$