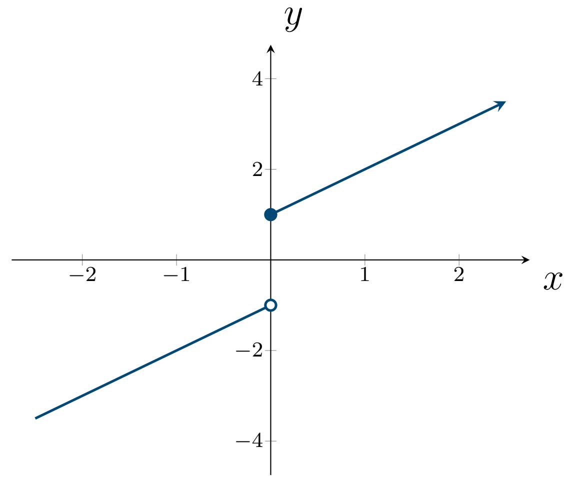
2.5: One-Sided Limits and Continuity

Consider the function

$$f(x) = \begin{cases} x-1, & x < 0 \\ x+1, & x \geq 0 \end{cases}$$

What is $\lim_{x \to 0} f(x)$?



Definition. (One-Sided Limits)
The function f has a **right-hand limit** L as x approaches a from the right, written

$$\lim_{x \to a^+} f(x) = L$$

if the values of $f(x)$ can be made as close to L as we please by taking x sufficiently close to (but not equal to) a and to the right of a.
The function f has a **left-hand limit** L as x approaches a from the left, written

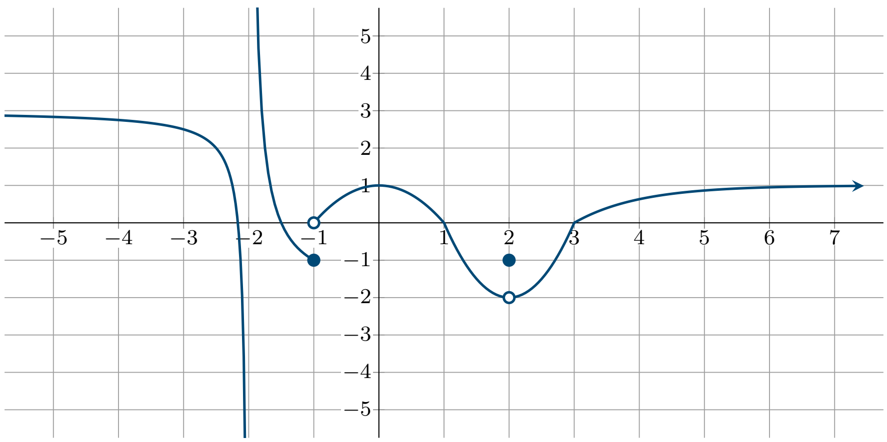
$$\lim_{x \to a^-} f(x) = M$$

if the values of $f(x)$ can be made as close to L as we please by taking x sufficiently close to (but not equal to) a and to the left of a.

Theorem 3
Let f be a function that is defined for all values of x close to $x = a$ with the possible exception of a itself. Then

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

Example. Using the graph below, evaluate the following limits:



$\lim_{x\to -2^-} f(x)$ $\qquad$ $\lim_{x\to -2^+} f(x)$ $\qquad$ $\lim_{x\to -2} f(x)$

$\lim_{x\to -1^-} f(x)$ $\qquad$ $\lim_{x\to -1^+} f(x)$ $\qquad$ $\lim_{x\to -1} f(x)$

$\lim_{x\to 1} f(x)$ $\qquad$ $\lim_{x\to 2} f(x)$ $\qquad$ $\lim_{x\to \infty} f(x)$

Definition. (Continuity of a Function at a Number)
A function f is **continuous** at a if $\lim_{x \to a} f(x) = f(a)$.

Continuity Checklist:
In order for f to be continuous at a, the following three conditions must hold:

1. $f(a)$ is defined (a is in the domain of f),
2. $\lim_{x \to a} f(x)$ exists,
3. $\lim_{x \to a} f(x) = f(a)$ (the value of f equals the limit of f at a).

Example. Determine the values of x for which the following functions are continuous:

$$f(x) = 3x^3 + 2x^2 - x + 10$$

$$g(x) = \frac{8x^{10} - 4x + 1}{x^2 + 1}$$

$$h(x) = \frac{4x^3 - 3x^2 + 1}{x^2 - 3x + 1}$$

Example. Determine whether the following are continuous at a:

$$f(x) = x^2 + \sqrt{7-x},\ a = 4$$

$$g(x) = \frac{1}{x-3},\ a = 3$$

$$h(x) = \begin{cases} \dfrac{x^2+x}{x+1}, & x \neq -1 \\ 0, & x = -1 \end{cases},\ a = -1$$

$$j(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases},\ a = 0$$

$$k(x) = \begin{cases} \dfrac{x^2+x-6}{x^2-x}, & x \neq 2 \\ -1, & x = 2 \end{cases},\ a = 2$$

Properties of Continuous Functions

1. The constant function $f(x) = c$ is continuous everywhere.
2. The identify function $f(x) = x$ is continuous everywhere.

If f and g are continuous at $x = a$, then

$[f(x)]^n$, where n is a real number, is continuous at $x = a$ whenever it is defined at that number

$f \pm g$ is continuous at $x = a$

fg is continuous at $x = a$

f/g is continuous at $x = a$ provided that $g(a) \neq 0$

Polynomial and Rational Functions

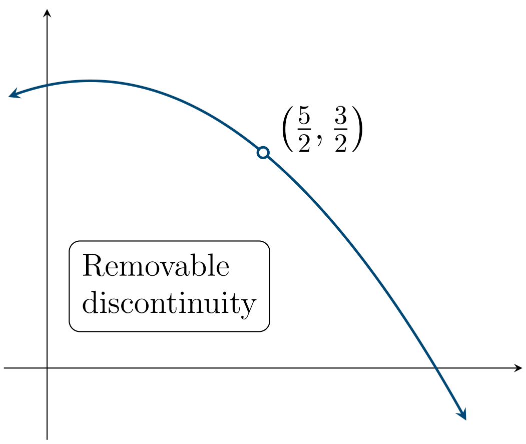
1. A polynomial function is continuous for all x.
2. A rational function (a function of the form $\frac{p}{q}$, where p and q are polynomials) is continuous for all x for which $q(x) \neq 0$.

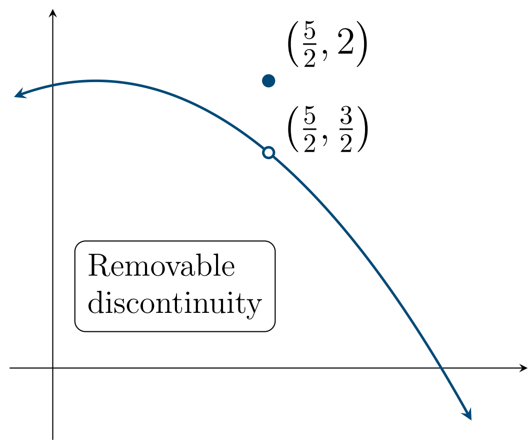
Definition.

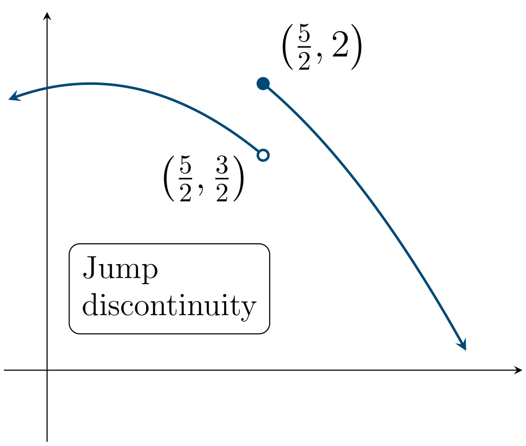
A **removable discontinuity** at $x = a$ is one that disappears when the function becomes continuous after defining $f(a) = \lim_{x \to a} f(x)$.

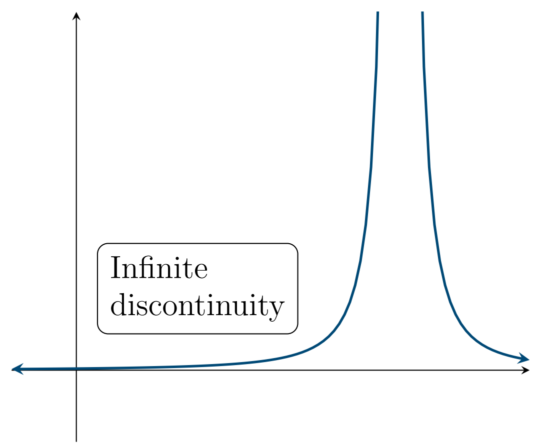
A **jump discontinuity** is one that occurs whenever $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$.

A **vertical discontinuity** occurs whenever $f(x)$ has a vertical asymptote.

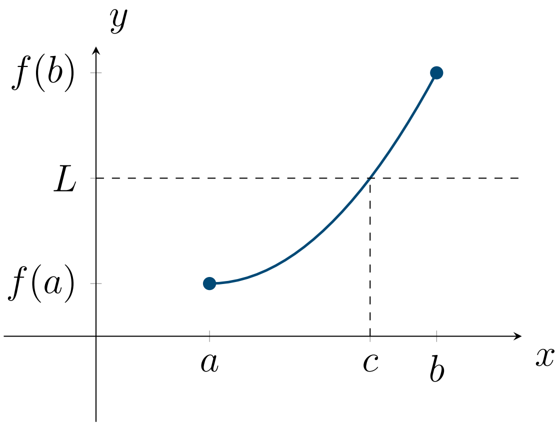


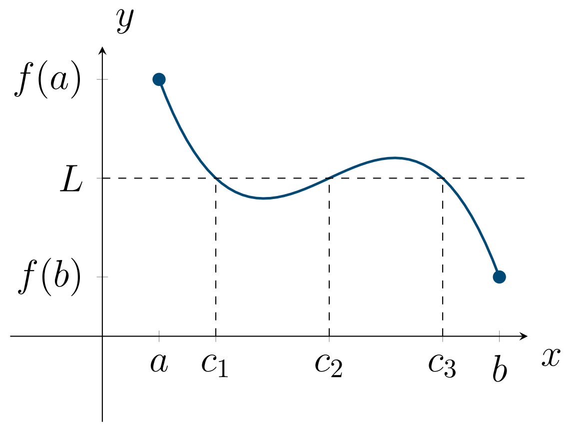




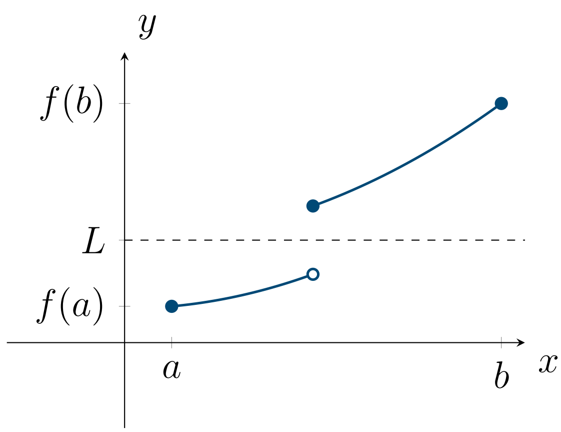


Theorem 4: Intermediate Value Theorem
Suppose f is continuous on the interval $[a, b]$ and L is a number strictly between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) satisfying $f(c) = L$.





Note: It is important that the function be continuous on the interval $[a, b]$:



Theorem 5: Existence of Zeros of a Continuous Function
If f is a continuous function on a closed interval $[a, b]$, and if $f(a)$ and $f(b)$ have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval (a, b).

Example. Check the conditions of the Intermediate Value Theorem to see if there exists a value c on the interval (a, b) such that the following equations hold: Graph

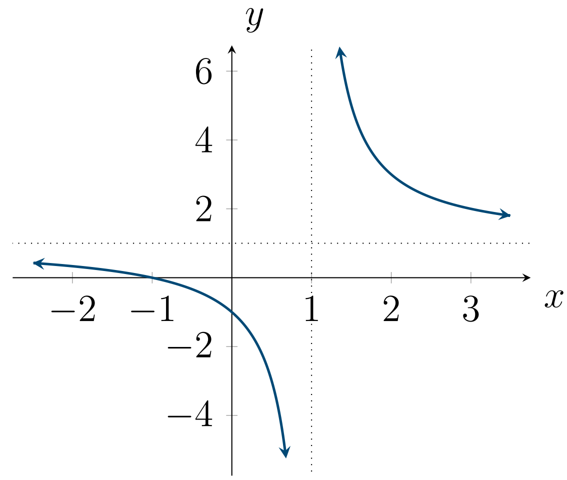
$x^x - x^2 = \frac{1}{2}$ on $[0, 2]$ $\sqrt{x^4 + 25x^3 + 10} = 5$ on $[0, 1]$

$x + \sqrt{1 - x^2} = 0$ on $[-1, 0]$ $\dfrac{x^2}{x^2 + 1} = 0$ on $[-1, 1]$

Example. Consider the function

$$f(x) = \frac{x+1}{x-1}$$

on the interval $[0, 2]$. Does there exist a c on the interval $[0, 2]$ such that $f(c) = 1$?

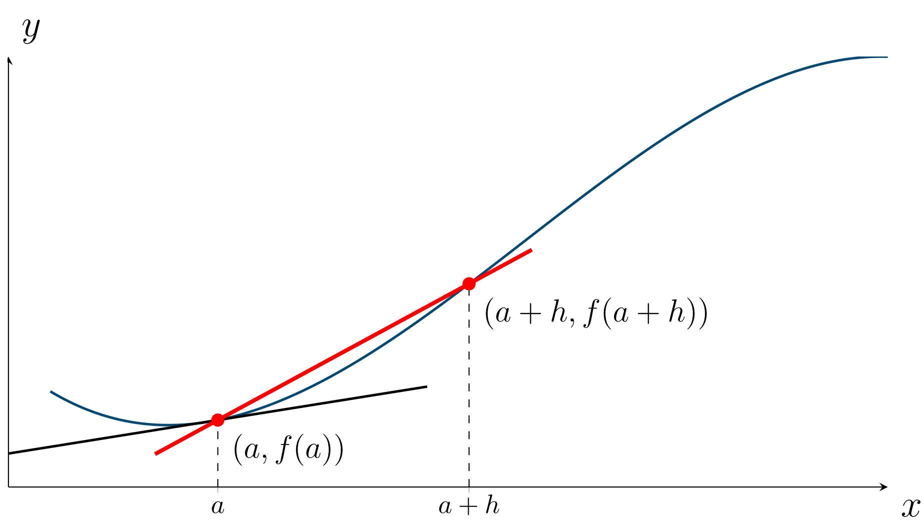


2.6: The Derivative

Definition.
Given a function $f(x)$:

- the **secant line** is the line that passes through two *distinct* points lying on the graph of $f(x)$,
- the **tangent line** is the line that intersects $f(x)$ in exactly one place (locally) and matches the slope of the graph at that point.

Graph



Definition. (Slope of a Tangent Line)
The slope of the tangent line to the graph of f at the point $P(x, f(x))$ is given by

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

if it exists.

Definition. (Average and Instantaneous Rates of Change)
The **average rate of change** of f over the interval $[x, x+h]$ or **slope of the secant line** to the graph of f through the points $(x, f(x))$ and $(x+h, f(x+h))$ is

$$\frac{f(x+h)-f(x)}{h}$$

The above fraction is referred to as the **difference quotient**.

The **instantaneous rate of change** of f at x or **slope of the tangent line** to the graph of f at $(x, f(x))$ is

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Definition. (Derivative of a Function)
The derivative of a function f with respect to x is the function f' (read "f prime"),

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

The domain of f' is the set of all x for which the limit exists.

Some other notations for the derivative are

$$D_x f(x) \qquad \frac{dy}{dx} \qquad y'$$

Example. Find the slope of the line tangent to the graph $f(x) = 3x + 5$ at any point $(x, f(x))$

Example. Let $f(x) = x^2$.

- Find $f'(x)$.
- Compute $f'(2)$ and interpret your result.

Example. Let $f(x) = x^2 - 4x$. Find the point on the graph where the tangent line is horizontal.

Example. Let $f(x) = \dfrac{1}{x}$. Find the equation of the tangent line at $x = 1$.

Differentiability and Continuity
If a function is differentiable at $x = a$, then it is continuous at $x = a$.

Example. For the graph below, identify each point where the derivative is undefined.