1.2: The Language of Sets

Definition.

- A **set** is a collection of objects.
- If S is a set, then we use
  - $x \in S$ to denote that the element x is in the set S.
  - $x \notin S$ to denote that the element x is *not* in the set S.
- The **set-roster notation** is used to denote all elements in a set between braces:

  $$S = \{1, 2, \ldots, 100\}$$

  Here, we see that $67 \in S$, but $1337 \notin S$.
- The **axiom of extension** says that a set is completely determined by what its elements are – not the order in which they are listed.

Example.

Let $A = \{1, 2, 3\}$, $B = \{3, 1, 2\}$, and $C = \{1, 1, 2, 3, 3, 3\}$. What are the elements of A, B, and C? How are A, B, and C related?

A, B, and C all have the same elements. A, B, and C are equivalent sets

Is $\{0\} = 0$?

No

How many elements are in the set $\{1, \{1\}\}$?

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For each nonnegative integer n, let $U_n = \{n, -n\}$. Find U_1, U_2, and U_0.

$U_1 = \{1, -1\}$ $U_2 = \{2, -2\}$ $U_0 = \{0, -0\} = \{0\}$

Certain sets of numbers are so frequently referred to that they are given special names and symbols:

N or $\mathbb{N}$	The set of all **natural numbers**
Z or $\mathbb{Z}$	The set of all **integers**
Q or $\mathbb{Q}$	The set of all **rational numbers**, or quotient of integers
R or $\mathbb{R}$	The set of all **real numbers**

Note: We may additionally use superscripts to indicate further properties of these sets:

$\mathbb{Z}^+$ or $\mathbb{Z}^{>0}$	The set of *positive* integers
$\mathbb{Q}^-$ or $\mathbb{Q}^{<0}$	The set of *negative* rational numbers
$\mathbb{R}^{nonneg}$ or $\mathbb{R}^{\geq 0}$	The set of *nonnegative* real numbers

Note: Different sources denote the natural numbers $\mathbb{N}$ as $\mathbb{Z}^+$ *or* $\mathbb{Z}^{\geq 0}$.

$\mathbb{N} = \{1, 2, 3, \ldots\}$

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$\mathbb{Q} = \{x \mid x = \frac{a}{b}, a, b \in \mathbb{Z}, b \neq 0\}$

Definition. (Set-Builder Notation)
Let S be a set and let $P(x)$ be a property that elements of S may or may not satisfy. We may define a new set to be **the set of all elements x in S such that $P(x)$ is true**. We denote this set as follows:

$$\{x \in S | P(x)\}$$

the set of all such that

Example. Describe each of the following sets:

$\{x \in \mathbb{R} \mid -2 < x < 5\}$ All real numbers between -2 and 5, excluding -2 and 5.

$(-2, 5)$

$\{x \in \mathbb{Z} \mid -2 < x < 5\}$

$= \{-1, 0, 1, 2, 3, 4\}$

$\{x \in \mathbb{Z}^+ \mid -2 < x < 5\}$

$= \{1, 2, 3, 4\}$

Definition.
If A and B are sets, then A is called a **subset** of B, written $A \subseteq B$, if, and only if, every element of A is also an element of B:

$$A \subseteq B \text{ means that for every element } x, \text{ if } x \in A, \text{ then } x \in B.$$
$$A \not\subseteq B \text{ means that there is at least one element } x, \text{ such that } x \in A \text{ and } x \notin B.$$

A is a **proper subset** of B if, and only if, every element of A is in B, but there is at least one element of B that is not in A:

$$A \subsetneq B \text{ means that for every element } x, \text{ if } x \in A, \text{ then } x \in B,$$
$$\text{and there exists } x \in B \text{ such that } x \notin A.$$

Example. Let $A = \mathbb{Z}^+$, $B = \{n \in \mathbb{Z} \mid 0 \leq n \leq 100\}$, and $C = \{100, 200, 300, 400, 500\}$. Evaluate the truth and falsity of each of the following statements.

$B \subseteq A$ False; $0 \in B$, but $0 \notin A$

C is a proper subset of A True

C and B have at least one element in common True

$C \subseteq B$ False; $200 \in C$, but $200 \notin B$

$C \subseteq C$ True

Example. Determine which of the following statements are true:

$2 \in \{1, 2, 3\}$

True

$\{2\} \in \{1, 2, 3\}$

False

$2 \subseteq \{1, 2, 3\}$

False

$\{2\} \subseteq \{1, 2, 3\}$

True

$\{2\} \subseteq \{\{1\}, \{2\}\}$

False

$\{2\} \in \{\{1\}, \{2\}\}$

True

Definition.
Given elements a and b, the symbol (a, b) denotes the **ordered pair** consisting of a and b together with the specification that a is the first element of the pair, and b is the second element. Two ordered pairs (a, b) and (c, d) are equal if, and only if, $a = c$ and $b = d$:

$$(a, b) = (c, d) \text{ means that } a = c \text{ and } c = d.$$

Example.

Is $(1, 2) = (2, 1)$?

No

Is $\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right)$?

Yes

Definition.
Let $n \in \mathbb{N}$ and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered n-tuple**, $(x_1, x_2, \ldots, x_n)$, consists of $x_1, x_2, \ldots x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. and ordered 2-tuple is called an **ordered pair**, and ordered 3-tuple is called an **ordered triple**.
Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if, and only if, $x_1 = y_1$, $x_2 = y_2$, $\ldots$, and $x_n = y_n$:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \iff x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

Example.

Is $(1, 2, 3, 4) = (1, 2, 4, 3)$?

No

Is $\left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right)$?

Yes

Definition.
Given sets $A_1, A_2, \dots, A_n$, the **Cartesian product** of $A_1, A_2, \dots, A_n$, denoted

$$A_1 \times A_2 \times \cdots \times A_n$$

is the set of all ordered n-tuples $(a_1, a_2, \dots, a_n)$ where $a_1 \in A_1$, $a_2 \in A_2$, $\dots$, $a_n \in A_n$:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \dots, a_n \in \mathcal{A}_n\}$$

Example. Let $A = \{x, y\}$, $B = \{1, 2, 3\}$, and $C = \{a, b\}$. Find the following:

$A \times B = \{(x,1), (x,2), (x,3), (y,1), (y,2), (y,3)\}$

$B \times A = \{(1,x), (2,x), (3,x), (1,y), (2,y), (3,y)\}$

$A \times A = \{(x,x), (x,y), (y,x), (y,y)\}$

How many elements are in $A \times B$, $B \times A$, and $A \times A$?
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$(A \times B) \times C = \{((x,1),a), ((x,2),a), ((x,3),a), ((y,1),a), ((y,2),a), ((y,3),a),$
$((x,1),b), ((x,2),b), ((x,3),b), ((y,1),b), ((y,2),b), ((y,3),b)\}$

$A \times B \times C = \{(x,1,a), (x,1,b), (x,2,a), (x,2,b), (x,3,a), (x,3,b),$
$(y,1,a), (y,1,b), (y,2,a), (y,2,b), (y,3,a), (y,3,b)\}$

Describe $\mathbb{R} \times \mathbb{R}$

The Cartesian plane we use for graphing

Definition.
Let $n \in \mathbb{N}$. Given a finite set A, a **string of length n over A** is an ordered n-tuple of elements of A written without parentheses or commas. The elements of A are called the **characters** of the string. The **null string** over A is defined to be the "string" with no characters, often denoted λ, and is said to have length 0. If $A = \{0, 1\}$, then a string over A is called a **bit string**.

Example. Let $A = \{a, b\}$. List all strings of length 3 over A with at least two characters that are the same.

aaa, aab, aba, abb, baa, bab, bba, bbb