2.5: Application: Number Systems and Circuits for Addition

Recall our how we write numbers in base 10:

$$
\begin{aligned}
5,049 &= 5 \cdot 1000 + 0 \cdot 100 + 4 \cdot 10 + 9 \cdot 1 \\
&= 5 \cdot 10^3 + 0 \cdot 10^2 + 4 \cdot 10^1 + 9 \cdot 10^0
\end{aligned}
$$

Definition.

Any integer $b > 1$ can be used as a base for a numbering system. A numbering system of base b has the digits $0, 1, \ldots, b-1$.

A **base** 2 **notation** or **binary notation**, uses the digits $0, 1$. In binary, every integer is represented as sum of products of the form

$$
d \cdot 2^n
$$

where $n \in \mathbb{Z}$ and $d \in \{0, 1\}$.

Example. Below is the binary representation for the integers 1 to 9:

$$
\begin{aligned}
1_{10} &= & & & & & & 1 \cdot 2^0 &=& 1_2 \\
2_{10} &= & & & & 1 \cdot 2^1 &+& 0 \cdot 2^0 &=& 10_2 \\
3_{10} &= & & & & 1 \cdot 2^1 &+& 1 \cdot 2^0 &=& 11_2 \\
4_{10} &= & & 1 \cdot 2^2 &+& 0 \cdot 2^1 &+& 0 \cdot 2^0 &=& 100_2 \\
5_{10} &= & & 1 \cdot 2^2 &+& 0 \cdot 2^1 &+& 1 \cdot 2^0 &=& 101_2 \\
6_{10} &= & & 1 \cdot 2^2 &+& 1 \cdot 2^1 &+& 0 \cdot 2^0 &=& 110_2 \\
7_{10} &= & & 1 \cdot 2^2 &+& 1 \cdot 2^1 &+& 1 \cdot 2^0 &=& 111_2 \\
8_{10} &= 1 \cdot 2^3 &+& 0 \cdot 2^2 &+& 0 \cdot 2^1 &+& 0 \cdot 2^0 &=& 1000_2 \\
9_{10} &= 1 \cdot 2^3 &+& 0 \cdot 2^2 &+& 0 \cdot 2^1 &+& 1 \cdot 2^0 &=& 1001_2
\end{aligned}
$$

Converting binary → decimal:
To convert from binary to decimal, multiply each digit by its corresponding power of 2 and sum the results.

Example. Represent the following in decimal notation (base-10):

110_2 $\qquad\qquad\qquad\qquad\qquad\qquad$ 1011_2

11110_2 $\qquad\qquad\qquad\qquad\qquad$ 101011_2

> **Converting decimal $\rightarrow$ binary:**
> To convert from decimal to binary, we repeated divide by 2, and record the remainders.

Example.

$$\begin{aligned}
27_{10} &= 16 + 8 + 2 + 1 \\
&= 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 \\
&= 11011_2
\end{aligned}$$

Example. Represent the following in binary notation:

243_{10} 587_{10}

990_{10} 531_{10}

Binary arithmetic:
In binary arithmetic, 10_2 behaves similarly to 10 in decimal arithmetic.

Example. Add 1101_2 and 111_2 using binary notation.

Example. Subtract 1011_2 from 11000_2 using binary notation.

Definition.
The 8-bit two's complement for an integer a between -128 and 127 is the 8-bit binary representation for
$$\begin{cases} a, & \text{if } a \geq 0 \\ 2^8 - |a|, & \text{if } a < 0. \end{cases}$$
Two's complement allows maximum representation for 2^8 integers with 8 binary digits.

Example. Below are a few integers represented in binary using 8-bit two's complement:

$-128 \rightarrow 2^8 - |-128| = 128_{10} = 10000000_2$

$-127 \rightarrow 2^8 - |-127| = 129_{10} = 10000001_2$

$$\vdots$$

$-2 \rightarrow 2^8 - |-2| = 254_{10} = 10000000_2$

$-1 \rightarrow 2^8 - |-1| = 255_{10} = 11111111_2$

$0 \rightarrow 0_{10} = 00000000_2$

$1 \rightarrow 1_{10} = 00000001_2$

$2 \rightarrow 2_{10} = 00000010_2$

$$\vdots$$

$127 \rightarrow 127_{10} = 01111111_2$

Example. Find the 8-bit two's complement for the following:

-46

42

120

-82

Two's complement of a negative integer:
To find the decimal representation of the negative integer with a given 8-bit two's complement:

- Flip the bits

- Add 1

- Convert to base-10 and swap the sign

Example. Find the decimal representation of the integers with the following 8-bit two's complement:

11100101_2

11000000_2

Addition and Subtraction with Integers in Two's Complement Form:
When performing binary addition on integers written in Two's Complement form, we discard any "carry" bit.

Example. Perform binary addition using the Two's Complement form of the following:

83 and -55

-87 and -46

Definition.
Hexadecimal notation uses a **base** 16 **notation**. In hexadecimal, every integer is represented as sum of products of the form

$$d \cdot 16^n$$

where $n \in \mathbb{Z}$ and $d \in \{0, 1, \ldots, 9, A, B, C, D, E, F\}$.

Decimal	Hexadecimal	4-Bit Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
10	A	1010
11	B	1011
12	C	1100
13	D	1101
14	E	1110
15	F	1111

Example. Convert $3CF_{16}$ to decimal notation.

Example. Convert $B09F_{16}$ to binary notation.

Example. Convert 100110110101001_2 to hexadecimal notation.