3.1: Predicates and Quantified Statements I

Definition.
A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.
The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

Example. Let $P(x)$ be the predicate "$x^2 > x$" with domain the set $\mathbb{R}$. Write $P(2)$, $P\big(\frac{1}{2}\big)$, and $P\big(-\frac{1}{2}\big)$, and indicate which of these statements are true and which are false.

$P(2)$: $2^2 > 2$
$4 > 2$
True

$P(\frac{1}{2})$: $(\frac{1}{2})^2 > \frac{1}{2}$
$\frac{1}{4} > \frac{1}{2}$
False

$P(-\frac{1}{2})$: $(-\frac{1}{2})^2 > -\frac{1}{2}$
$\frac{1}{4} > -\frac{1}{2}$
True

Definition.
If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x. The truth set of $P(x)$ is denoted

$$\{x \in D \mid P(x)\}$$

Example. Let $Q(n)$ be the predicate "n is a factor of 8". Find the truth set of $Q(n)$ if

the domain of n is $\mathbb{Z}^+$

$\{1, 2, 3 \ldots\}$

Truth set:

$\{1, 2, 4, 8\}$

the domain of n is $\mathbb{Z}$

$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Truth set:

$\{-8, -4, -2, -1, 1, 2, 4, 8\}$

Definition.
Let $Q(x)$ be a predicate and D the domain of x.

- **Quantifiers** are words that refer to quantities such as "some" or "all" and tell for how many elements a given predicate is true.
- The **universal quantifier** is represented by the symbol "$\forall$".
- A **universal statement** is a statement of the form "$\forall x \in D,\ Q(x)$".
  - It is defined to be true if, and only if, $Q(x)$ is true for *each* individual x in D.
  - It is defined to be false if, and only if, $Q(x)$ is false for *at least one* x in D.
- A value for x for which $Q(x)$ is false is called a **counterexample** to the universal statement.

Example. Let $D = \{1, 2, 3, 4, 5\}$, and consider the statement

$$\forall x \in D, x^2 \geq x.$$

Write one way to read this statement out loud, and show that it is true.

The square of every element in D is greater than or equal to that element

$1^2 \geq 1$ $\quad$ $2^2 \geq 2$ $\quad$ $3^2 \geq 2$ $\quad$ $4^2 \geq 4$ $\quad$ $5^2 \geq 5$

$1 \geq 1$ $\quad$ $4 \geq 2$ $\quad$ $9 \geq 2$ $\quad$ $16 \geq 4$ $\quad$ $25 \geq 5$

The above example uses the **method of exhaustion**.

Example. Consider the statement

$$\forall x \in \mathbb{R}, x^2 \geq x.$$

Find a counter example to show that this statement is false.

Let $x = -1 \Rightarrow x^2 = (-1)^2 = 1$ so $x^2 \not\geq x$

Definition.
Let $Q(x)$ be a predicate and D the domain of x.

- The **existential quantifier** is represented by the symbol "$\exists$".
- An **existential statement** is a statement of the form "$\exists x \in D$ such that $Q(x)$".
  - It is defined to be true if, and only if, $Q(x)$ is true for *at least one* x in D.
  - It is false if, and only if, $Q(x)$ is false *for all* x in D.

Example. Consider the statement

$$\exists m \in \mathbb{Z}^+ \text{ such that } m^2 = m.$$

Write one way to read this statement out loud, and show that it is true.

There is a positive integer equal to its square.

Let $m = 1 \Rightarrow m^2 = 1 \Rightarrow m^2 = m$

Example. Let $E = \{5, 6, 7, 8\}$ and consider the statement

$$\exists m \in E \text{ such that } m^2 = m.$$

Show that this statement is false.

$$\left.\begin{array}{l} 5^2 = 25 \neq 5 \\ 6^2 = 36 \neq 6 \\ 7^2 = 49 \neq 7 \\ 8^2 = 64 \neq 8 \end{array}\right\}$$

False because it is false
for ALL elements in E

Example. Rewrite the following statements formally using quantifiers and variables:

All triangles have three sides.

$\forall$ triangles T, T has 3 sides.

No dogs have wings.

$\forall$ dogs D, D does not have wings.

Some programs are structured.

$\exists$ a program P such that P is structured.

> **Definition.**
> A **universal conditional statement** is of the form:
>
> $$\forall x, \text{ if } P(x) \text{ then } Q(x).$$

Example. Rewrite each of the following statements in the form

$$\forall \underline{\hspace{2cm}}, \text{ if } \underline{\hspace{2cm}} \text{ then } \underline{\hspace{2cm}}$$

If a real number is an integer, then it is a rational number.

$\forall x \in \mathbb{R}$, if $x \in \mathbb{Z}$, then $x \in \mathbb{Q}$

Set of numbers written as fractions (p 7)

All bytes have eight bits.

$\forall x$, if x is a byte, then x has 8 bits

No fire trucks are green.

$\forall x$, if x is a fire truck, then x is NOT green.