2.1: Logical Form and Logical Equivalence

Definition.
A **statement** (or **proposition**) is a sentence that is true or false, but not both.

Example. Determine which of the following are statements:

$2 + 2 = 4$

$2 + 2 = 5$

$x^2 + 2 = 11$

Today is Saturday.

She is a computer science major.

Jane is a computer science major.

Definition. (Compound Statements)
Let p and q be statement variables.

- The **negation** of p is "not p", and is denoted as $\sim p$ (or $\neg p$)
- The **conjunction** of p and q is "p and q", and is denoted at $p \wedge q$
- The **disjunction** of p and q is "p or q", and is denoted $p \vee q$.
- The **exclusive or** of p and q is "p x-or q", and is denoted $p \oplus q$ (or p XOR q)

The **order of operations** specifies that $\sim$ is performed first.

Example. Consider the following statements:

$$\begin{aligned} p:&\quad \text{It is raining.} \\ q:&\quad \text{It is sunny.} \\ r:&\quad \text{It is cloudy.} \end{aligned}$$

Rewrite the following compound statements in words:

$\sim p$

$p \vee q$

$q \wedge r$

$q \wedge \sim r$

$p \wedge (q \vee r)$

$p \oplus q$

Definition.
A **statement form** (or **propositional form**) is an expression made up of statement variables (e.g., p, q, and r), and logical connectives (e.g. $\sim$, $\wedge$, $\vee$, and $\oplus$).
The **truth table** for a given statement form displays the truth values that correspond to all possible combinations of truth values for its component statement variables.

Example. Let p and q be statement variables. Fill out the following truth tables:

p	$\sim p$
T	
F	

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$
T	T			
T	F			
F	T			
F	F			

p	q	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$(p \vee q) \wedge \sim (p \wedge q)$
T	T				
T	F				
F	T				
F	F				

Example. Construct a truth table for the statement form $(p \wedge q) \vee \sim r$.

Definition.
Two *statement forms* are called **logically equivalent** if, and only if, they have identical true values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms P and Q is denoted $P \equiv Q$.

Example. Use truth tables to test if the following statement forms are equivalent:

p and $\sim(\sim p)$

$\sim(p \wedge q)$ and $\sim p \wedge \sim q$

Definition. (De Morgan's Laws)
The negation of an *and* statement is logically equivalent to the *or* statement in which each component is negated.
The negation of an *or* statement is logically equivalent to the *and* statement in which each component is negated.

Example. Use truth tables to show that the following statement forms are equivalent:

$\sim (p \wedge q)$ and $\sim p \vee \sim q$

$\sim (p \vee q)$ and $\sim p \wedge \sim q$

Example. Using De Morgan's law to write the negation of the following statements:

Jim is at least 6 feet tall and weighs at least 200 pounds.

The bus was late or Tom's watch was slow

$-1 < x \leq 4$

Definition.
A **tautology** is a statement form that is always true.
A **contradiction** is a statement form that is always false.

Example. Complete the truth tables for $p \wedge \sim p$ and $p \vee \sim p$

Example. Let $\mathbf{t}$ be a tautology, and $\mathbf{c}$ be a contradiction. Show that $p \wedge \mathbf{t} \equiv p$ and $p \wedge \mathbf{c} \equiv \mathbf{c}$

Theorem 2.1.1 Logical Equivalences (p 49)
Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold:

1. Commutative laws:
$p \wedge q \equiv q \wedge p$ $\qquad$ $p \vee q \equiv q \vee p$
2. Associative laws:
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $\qquad$ $(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. Distributive laws:
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $\qquad$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. Identity laws:
$p \wedge \mathbf{t} \equiv p$ $\qquad$ $p \vee \mathbf{c} \equiv p$
5. Negation laws:
$p \vee \sim p \equiv \mathbf{t}$ $\qquad$ $p \wedge \sim p \equiv \mathbf{c}$
6. Double negative law:
$\sim (\sim p) \equiv p$
7. Idempotent laws:
$p \wedge p \equiv p$ $\qquad$ $p \vee p \equiv p$
8. Universal bound laws:
$p \vee \mathbf{t} \equiv \mathbf{t}$ $\qquad$ $p \wedge \mathbf{c} \equiv \mathbf{c}$
9. De Morgan's laws:
$\sim (p \wedge q) \equiv \sim p \vee \sim q$ $\qquad$ $\sim (p \vee q) \equiv \sim p \wedge \sim q$
10. Absorption laws:
$p \wedge (p \vee q) \equiv p$ $\qquad$ $p \wedge (p \vee q) \equiv p$
11. Negations of $\mathbf{t}$ and $\mathbf{c}$:
$\sim \mathbf{t} \equiv \mathbf{c}$ $\qquad$ $\sim \mathbf{c} \equiv \mathbf{t}$