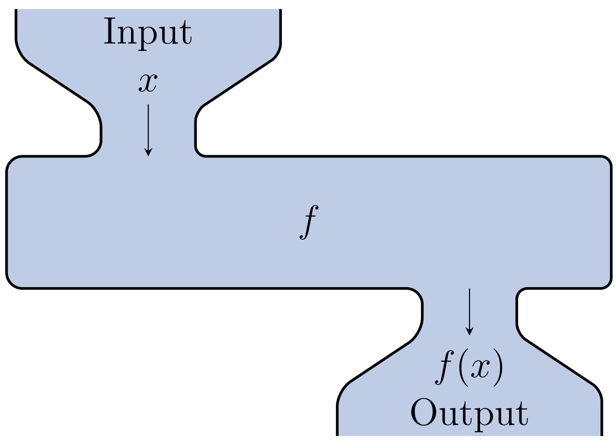
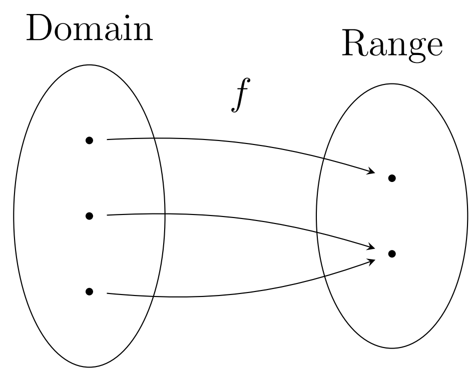
2.1: Functions and Their Graphs

Definition.
A **function** is a rule that assigns to each element in a set A one and only one element in a set B.
In the context above, the set A is called the **domain**, and the set B is called the **range**.





Example. Let $f(x) = 2x^2 - 2x + 1$. Evaluate the following

$$f(1) = 2(1)^2 - 2(1) + 1 = 2 - 2 + 1 = \boxed{1}$$

$$f(-2) = 2(-2)^2 - 2(-2) + 1 = 8 + 4 + 1 = \boxed{13}$$

$$f(a) = 2(a)^2 - 2(a) + 1 = \boxed{2a^2 - 2a + 1}$$

$$f(a+h) = 2(a+h)^2 - 2(a+h) + 1 = 2(a^2 + 2ah + h^2) - 2(a+h) + 1 = \boxed{2a^2 + 4ah + 2h^2 - 2a - 2h + 1}$$

Example. Find the domain and range of the following functions:

$f(x) = x$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

$A = \pi r^2$

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

$y = \sqrt{x-1}$

$x - 1 \geq 0$

$x \geq 1$

Domain: $[1, \infty)$

Range: $[0, \infty)$

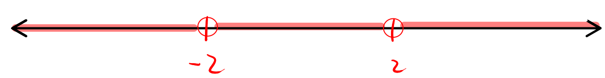
$y = \dfrac{1}{x^2 - 4}$

$x^2 - 4 \neq 0$

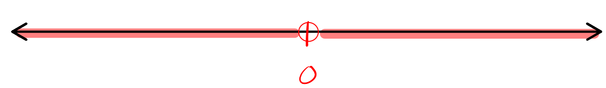
$x^2 \neq 4$

$x \neq \pm 2$

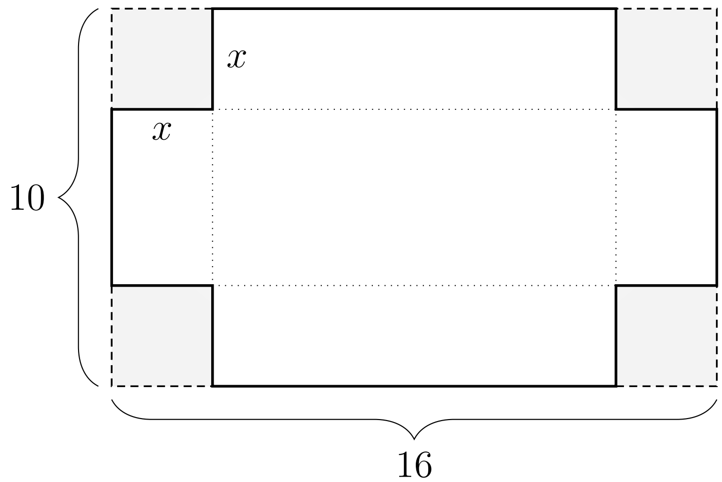
Domain: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

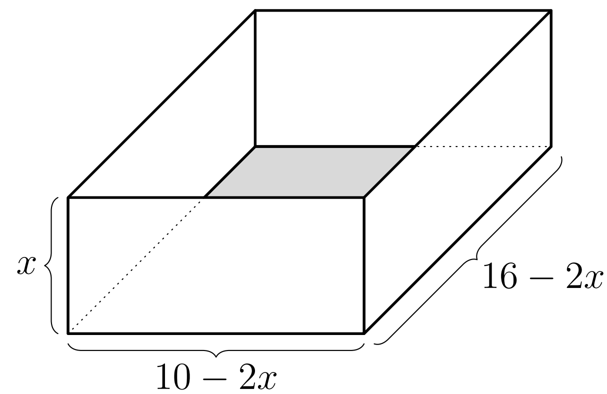


Range: $(-\infty, 0) \cup (0, \infty)$



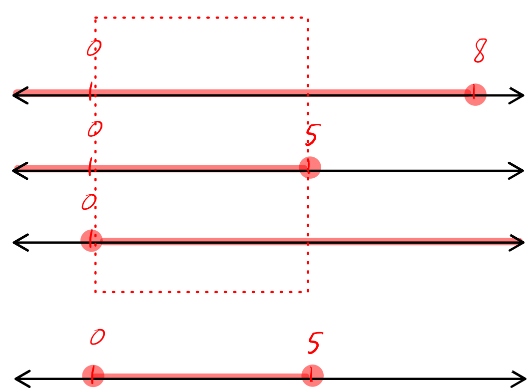
Example. An open box is to be made from a rectangular piece of cardboard 16 inches long and 10 inches wide by cutting away identical squares (x inches by x inches) from each corner and folding up the resulting flaps. Find an expression that gives the volume V of the box as a function of x. What is the domain of the function?





$$V = (\text{length})(\text{width})(\text{height})$$
$$= (16-2x)(10-2x)\,x$$
$$= x(160 - 52x + 4x^2)$$
$$= 4x^3 - 52x^2 + 160x$$

$$\begin{aligned} 16-2x &\geq 0 \\ 10-2x &\geq 0 \\ x &\geq 0 \end{aligned} \Rightarrow \begin{cases} x \leq 8 \\ x \leq 5 \\ x \geq 0 \end{cases}$$



$$\boxed{\text{Domain: } [0, 5]}$$

> **Definition.**
> A **piecewise** function is a function with different definitions for different portions of the domain.

Example. Rewrite the following as piecewise functions:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \qquad \frac{x}{|x|} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

$$|x-1| + |4-x| = \begin{cases} -(x-1) + (4-x), & x \leq 1 \\ (x-1) + (4-x), & 1 < x \leq 4 \\ (x-1) - (4-x), & 4 \leq x \end{cases} = \begin{cases} -2x+5, & x \leq 1 \\ 3, & 1 < x \leq 4 \\ 2x-5, & 4 \leq x \end{cases}$$

$$|x-1| = \begin{cases} -(x-1), & x \leq 1 \\ (x-1), & x > 1 \end{cases}$$

$$|4-x| = \begin{cases} -(4-x), & x \geq 4 \\ (4-x), & x < 4 \end{cases}$$

		1		4	
$\lvert x-1 \rvert$	$-(x-1)$		$(x-1)$		$(x-1)$
$\lvert 4-x \rvert$	$(4-x)$		$(4-x)$		$-(4-x)$

> **Definition. (Vertical Line Test)**
> A curve in the xy-plane is the graph of a function $y = f(x)$ (an explicit function) if and only if each vertical line intersects it in at most one point

Example. Use the vertical line test on the following graphs to determine which graphs may represent an explicit function: