Math 123 Class notes
Fall 2025

To accompany
Applied Calculus
by *Tan*

Peter Westerbaan

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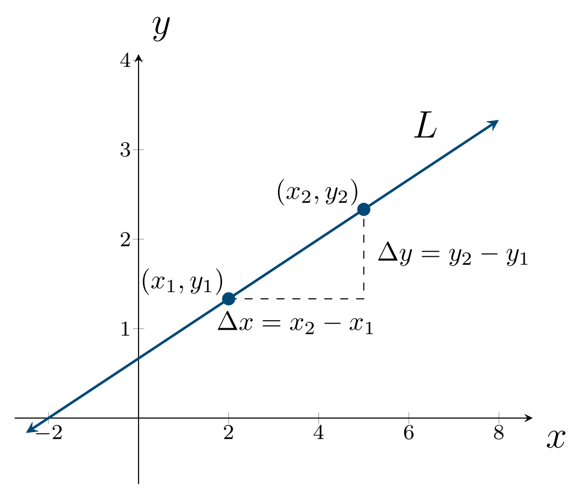
1.4: Straight Lines

Definition. (Slope of a Nonvertical Line)
If (x_1, y_1) and (x_2, y_2) are any two distinct points on a nonvertical line L, then the slope m of L is given by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



Example. Compute the slope of the line passing through the points

$(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (4, 2)$

$(x_1, y_1) = (3, 2)$ and $(x_2, y_2) = (-1, 2)$

$(x_1, y_1) = (4, 1)$ and $(x_2, y_2) = (4, 4)$

Definition. (Point-Slope Form of an Equation of a Line)
An equation of the line that has slope m and passes through the point (x_1, y_1) is given by

$$y - y_1 = m(x - x_1)$$

Example. Find the equation of the line going through the points

$(x_1, y_1) = (-2, 1)$ and $(x_2, y_2) = (3, -2)$

$(x_1, y_1) = (3, 4)$ and $(x_2, y_2) = (-1, 4)$

$(x_1, y_1) = (2, 0)$ and $(x_2, y_2) = (2, 1)$

Definition. (Slope-Intercept Form of an Equation of a Line)
An equation of the line that has slope m and intersects the y-axis at the point $(0, b)$ is given by

$$y = mx + b$$

Example. Rewrite the equations in the previous example in slope-intercept form.

Definition. (Parallel and Perpendicular lines)
Let L_1 and L_2 be lines with slopes m_1 and m_2 respectively. If L_1 and L_2 are *parallel*, then

$$m_1 = m_2.$$

If L_1 and L_2 are *perpendicular*, then

$$m_1 = -\frac{1}{m_2}.$$

Example.

Find the line *parallel* to $y = \frac{3}{2}x + 1$ that passes through the point $(-4, 10)$.

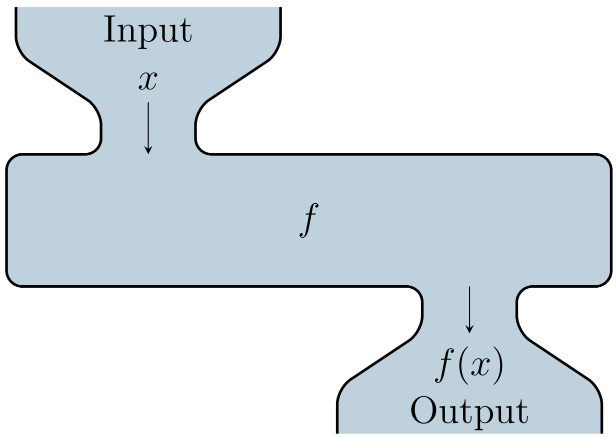
Find the line *perpendicular* to $y = \frac{3}{2}x + 1$ that passes through the point $(-3, 4)$.

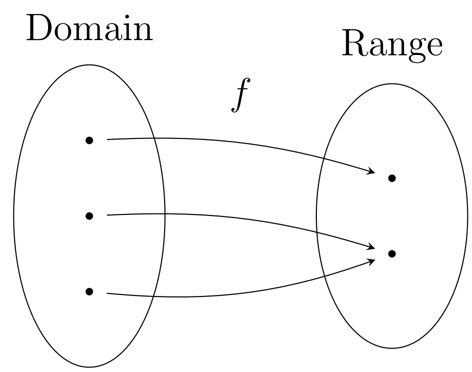
Forms of Linear Equations

$$\begin{aligned} \text{General form: } & Ax + By = C \\ \text{Point-slope form: } & y - y_1 = m(x - x_1) \\ \text{Slope-intercept form: } & y = mx + b \\ \text{Vertical line: } & x = a \\ \text{Horizontal line: } & y = b \end{aligned}$$

2.1: Functions and Their Graphs

Definition.
A **function** is a rule that assigns to each element in a set A one and only one element in a set B.
In the context above, the set A is called the **domain**, and the set B is called the **range**.





Example. Let $f(x) = 2x^2 - 2x + 1$. Evaluate the following

$f(1)$

$f(-2)$

$f(a)$

$f(a+h)$

Example. Find the domain and range of the following functions:

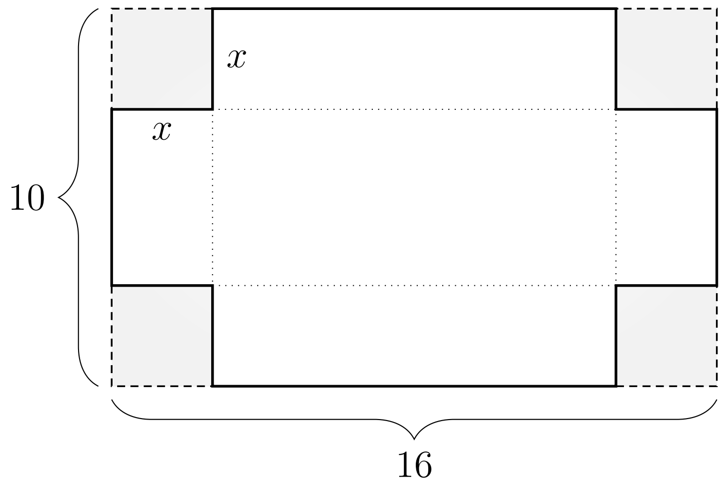
$f(x) = x$

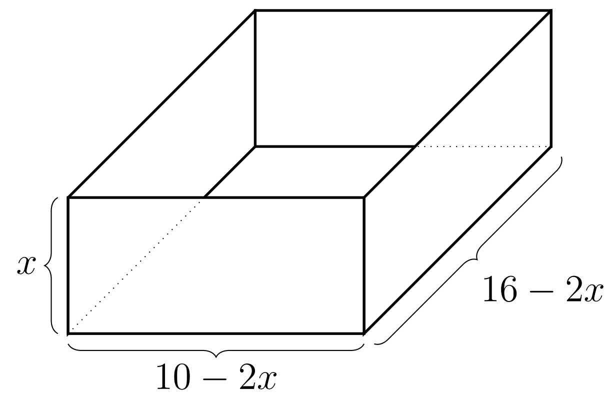
$A = \pi r^2$

$y = \sqrt{x-1}$

$y = \dfrac{1}{x^2 - 4}$

Example. An open box is to be made from a rectangular piece of cardboard 16 inches long and 10 inches wide by cutting away identical squares (x inches by x inches) from each corner and folding up the resulting flaps. Find an expression that gives the volume V of the box as a function of x. What is the domain of the function?





Definition.
A **piecewise** function is a function with different definitions for different portions of the domain.

Example. Rewrite the following as piecewise functions:

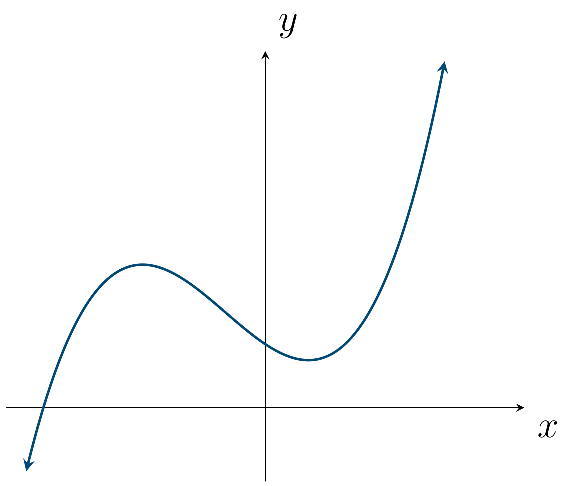
$|x| =$

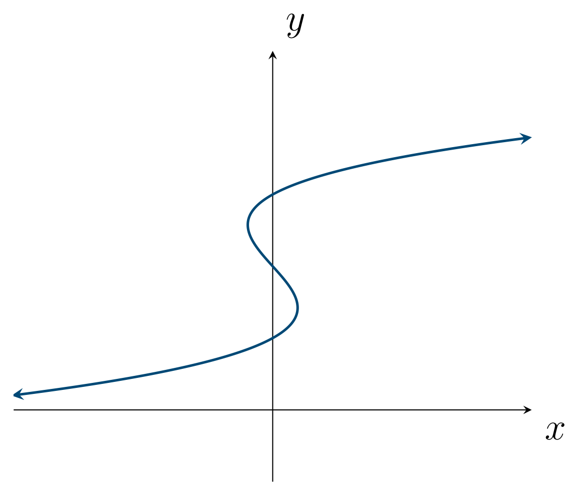
$\dfrac{x}{|x|} =$

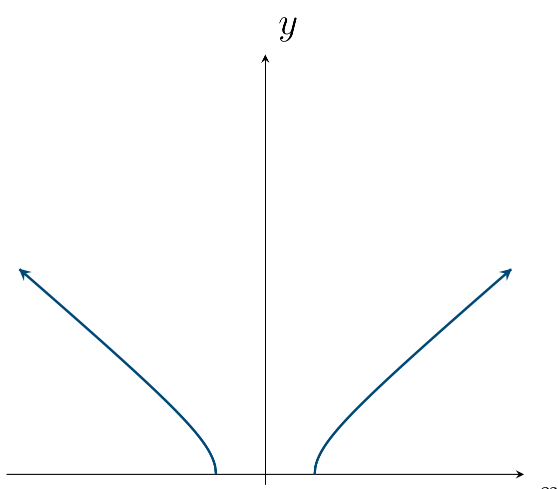
$|x-1| + |4-x| =$

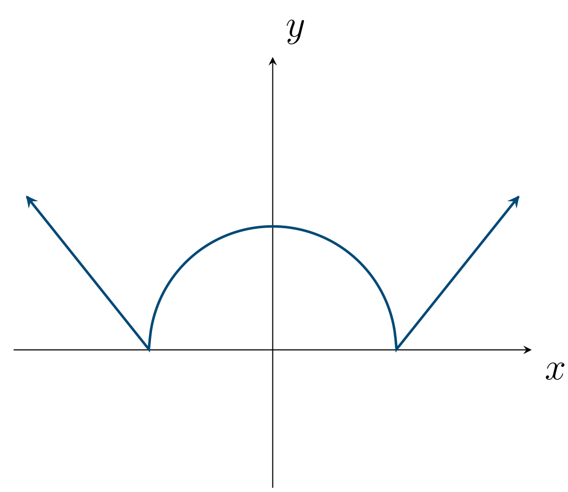
Definition. (Vertical Line Test)
A curve in the xy-plane is the graph of a function $y = f(x)$ (an explicit function) if and only if each vertical line intersects it in at most one point

Example. Use the vertical line test on the following graphs to determine which graphs may represent an explicit function:









2.2: The Algebra of Functions

Definition.
Let f and g be functions with domains A and B, respectively. Then the **sum** $f+g$, **difference** $f-g$, and **product** fg of f and g are functions with domain $A \cap B$.

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$

The **quotient** f/g of f and g has domain $A \cap B$ excluding all numbers x such that $g(x) = 0$ and rule given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Example. Let $f(x) = \sqrt{x+1}$ and $g(x) = 4 - x$. Find the domain of the following:

$f(x) + g(x) =$

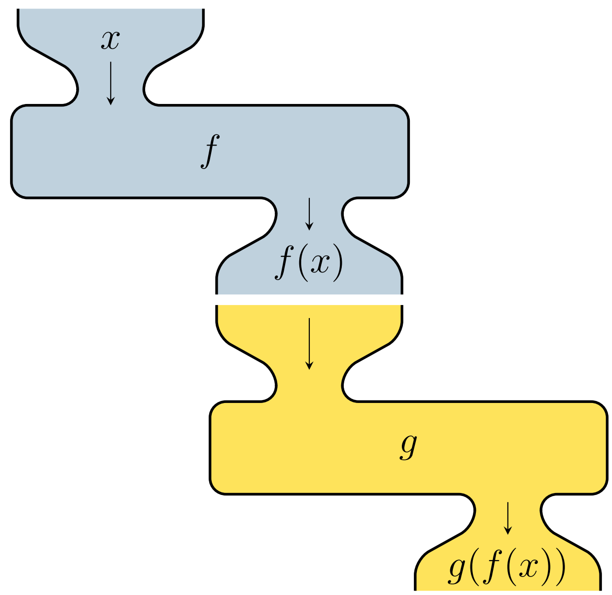
$f(x) - g(x) =$

$f(x)g(x) =$

$\dfrac{f(x)}{g(x)} =$

> **Definition. (The Composition of Two Functions)**
> Let f and g be functions. Then the composition of g and f is the function $g \circ f$ defined by
>
> $$(g \circ f)(x) = g(f(x))$$
>
> The domain of $g \circ f$ is the set of all x is the domain of f such that $f(x)$ lies in the domain of g.

Example. Let $f(x) = \sqrt{x+1}$ and $g(x) = 4 - x$. Find the domain of the following:



$g(f(x)) =$

$f(g(x)) =$

$f(f(x)) =$

2.4: Limits

Example. Suppose that the position function of a maglev train (in feet) is given by

$$s(t) = 4t^2, \qquad (0 \le t \le 30)$$

Using the position function, compute the *average* velocity of the train

on the interval $[t, 2]$

t	1.5	1.9	1.99	1.999	1.9999

on the interval $[2, t]$

t	2.5	2.1	2.01	2.001	2.0001

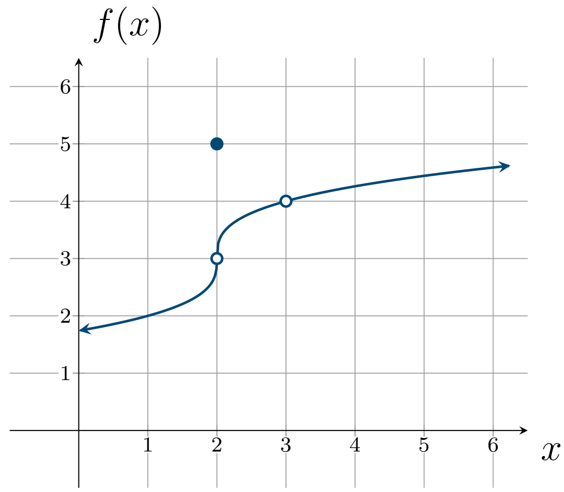
What do the tables above suggest about *instantaneous* velocity of the train at $t = 2$?

Definition. (Limit of a Function)
The function f has the **limit** L as x approaches a, written

$$\lim_{x \to a} f(x) = L$$

if the value of $f(x)$ can be made as close to the number L as we please by taking x sufficiently close to (but not equal to) a.

Example. Using the graph of f, determine the following values:



$f(1)$ and $\lim_{x \to 1} f(x)$

$f(2)$ and $\lim_{x \to 2} f(x)$

$f(3)$ and $\lim_{x \to 3} f(x)$

Example. Find the limit of the following functions at the value specified: Graphs

$f(x) = x^3 \quad \text{at } x = 2$

$g(x) = \begin{cases} x+2, & x \neq 1 \\ 1, & x = 1 \end{cases} \quad \text{at } x = 1$

$h(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases} \quad \text{at } x = 0$

$j(x) = \dfrac{1}{(x-1)^2} \quad \text{at } x = 1$

$k(x) = 4 \quad \text{at } x = 0$

Theorem 1: Properties of Limits
Suppose

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M$$

Then

1. $\lim\limits_{x\to a}[f(x)]^r = \left[\lim\limits_{x\to a} f(x)\right]^r$ where r is a positive constant
2. $\lim\limits_{x\to a} cf(x) = c\lim\limits_{x\to a} f(x)$ where c is a real number
3. $\lim\limits_{x\to a}[f(x) \pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x) = L \pm M$
4. $\lim\limits_{x\to a}[f(x)g(x)] = \left[\lim\limits_{x\to a} f(x)\right]\left[\lim\limits_{x\to a} g(x)\right] = LM$
5. $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{L}{M}$ provided $M \neq 0$

Example. Use the above theorem to evaluate the following limits:

$$\lim_{x\to 1}\left(5x^{3/2} - 2\right)$$

$$\lim_{x\to 3} \frac{2x^3\sqrt{x^2+7}}{x+1}$$

Suppose that $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

has an **indeterminate form** of $\dfrac{0}{0}$. To evaluate such a limit, we replace the given function with a function that's equivalent everywhere except at $x = a$, and then evaluate the limit.

Example. Evaluate the following

$$\lim_{t\to 2} \frac{4t^2 - 16}{t - 2}$$

$$\lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h}$$

Limit of a Function at Infinity
The function f has the limit L as x increases without bound, written

$$\lim_{x\to\infty} f(x) = L$$

if $f(x)$ can be made arbitrarily close to L by taking x large enough.
The function f has the limit M as x decreases without bound, written

$$\lim_{x\to-\infty} f(x) = M$$

if $f(x)$ can be made arbitrarily close to M by taking x to be negative and sufficiently large enough in absolute value.
When the above limits exist, the equations $y = L$ and/or $y = M$ are called **horizontal asymptotes**.

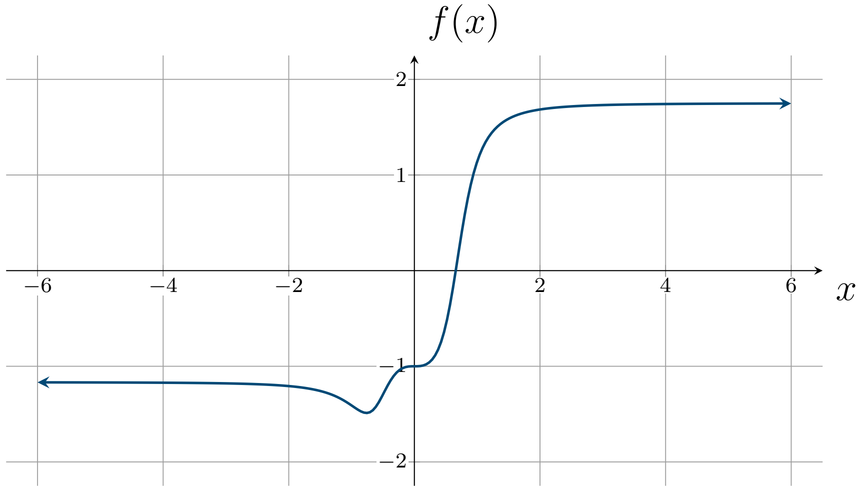
Example. Evaluate the following infinite limits

$$\lim_{x\to\infty} \frac{2x^2+3x-4}{x^2-7x+1}$$

$$\lim_{x\to-\infty} \frac{3x+4}{2x^2}$$

$$\lim_{x\to\pm\infty} \frac{3x^3+2x-4}{x^2+4x-1}$$

Graph



$$\lim_{x\to\infty} \frac{7x^3 - 2}{-x^3 + \sqrt{25x^6 - 4}}$$

$$\lim_{x\to-\infty} \frac{7x^3 - 2}{-x^3 + \sqrt{25x^6 - 4}}$$

Example. The company *Custom Office* makes a line of executive desks. It is estimated that the total cost of making x *Senior Executive Model* desks is

$$C(x) = 100x + 200,000$$

dollars per year. The average cost of making x desks is given by

$$\overline{C}(x) = \frac{C(x)}{x}$$

Compute $\lim_{x\to\infty} \overline{C}(x)$ and interpret the result.

Theorem 2
For all $n > 0$,

$$\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$$

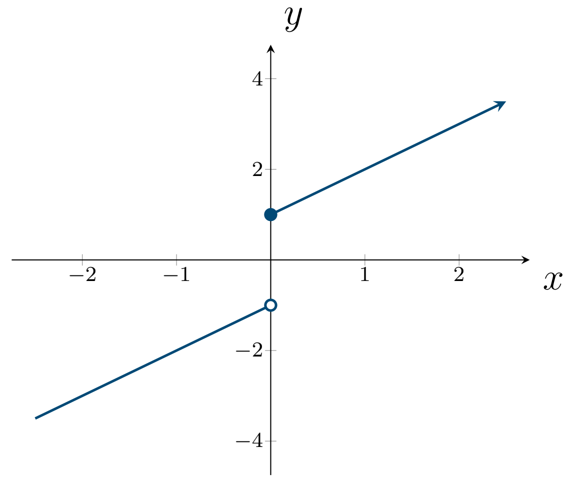
provided that $\frac{1}{x^n}$ is defined.

2.5: One-Sided Limits and Continuity

Consider the function

$$f(x) = \begin{cases} x-1, & x < 0 \\ x+1, & x \geq 0 \end{cases}$$

What is $\lim_{x \to 0} f(x)$?



Definition. (One-Sided Limits)
The function f has a **right-hand limit** L as x approaches a from the right, written

$$\lim_{x \to a^+} f(x) = L$$

if the values of $f(x)$ can be made as close to L as we please by taking x sufficiently close to (but not equal to) a and to the right of a.
The function f has a **left-hand limit** L as x approaches a from the left, written

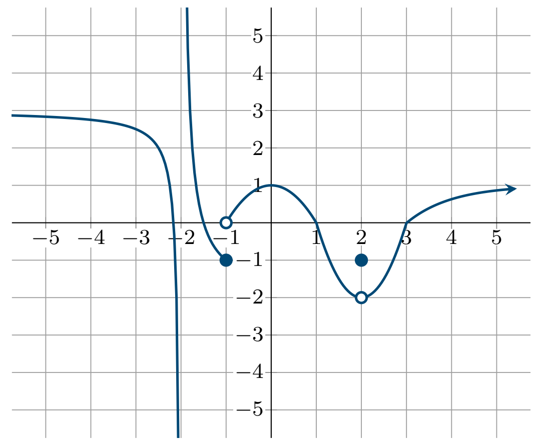
$$\lim_{x \to a^-} f(x) = M$$

if the values of $f(x)$ can be made as close to L as we please by taking x sufficiently close to (but not equal to) a and to the left of a.

Theorem 3
Let f be a function that is defined for all values of x close to $x = a$ with the possible exception of a itself. Then

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

Example. Using the graph to the right, evaluate the following limits:



$$\lim_{x\to -2} f(x)$$

$$\lim_{x\to -1} f(x)$$

$$\lim_{x\to 1} f(x)$$

$$\lim_{x\to 2} f(x)$$

$$\lim_{x\to \infty} f(x)$$

Definition. (Continuity of a Function at a Number)
A function f is **continuous** at a if $\lim_{x\to a} f(x) = f(a)$.

Continuity Checklist:
In order for f to be continuous at a, the following three conditions must hold:

1. $f(a)$ is defined (a is in the domain of f),
2. $\lim_{x\to a} f(x)$ exists,
3. $\lim_{x\to a} f(x) = f(a)$ (the value of f equals the limit of f at a).

Example. Determine the values of x for which the following functions are continuous:

$$f(x) = 3x^3 + 2x^2 - x + 10$$

$$g(x) = \frac{8x^{10} - 4x + 1}{x^2 + 1}$$

$$h(x) = \frac{4x^3 - 3x^2 + 1}{x^2 - 3x + 1}$$

Example. Determine whether the following are continuous at a:

$$f(x) = x^2 + \sqrt{7 - x},\ a = 4 \qquad g(x) = \frac{1}{x - 3},\ a = 3$$

$$h(x) = \begin{cases} \dfrac{x^2 + x}{x + 1}, & x \neq -1 \\ 0, & x = -1 \end{cases},\ a = -1 \qquad j(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases},\ a = 0$$

$$k(x) = \begin{cases} \dfrac{x^2 + x - 6}{x^2 - x}, & x \neq 2 \\ -1, & x = 2 \end{cases},\ a = 2$$

Properties of Continuous Functions

1. The constant function $f(x) = c$ is continuous everywhere.
2. The identify function $f(x) = x$ is continuous everywhere.

If f and g are continuous at $x = a$, then

$[f(x)]^n$, where n is a real number, is continuous at $x = a$ whenever it is defined at that number

$f \pm g$ is continuous at $x = a$

fg is continuous at $x = a$

f/g is continuous at $x = a$ provided that $g(a) \neq 0$

Polynomial and Rational Functions

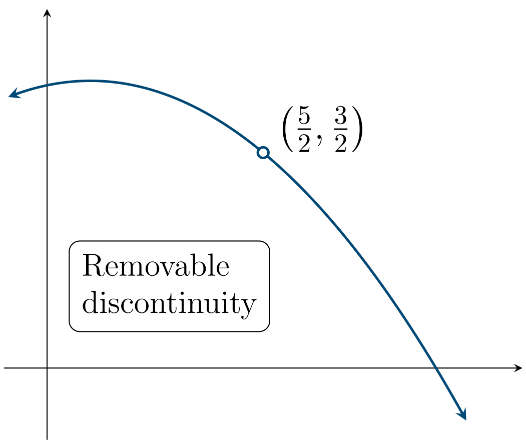
1. A polynomial function is continuous for all x.
2. A rational function (a function of the form $\frac{p}{q}$, where p and q are polynomials) is continuous for all x for which $q(x) \neq 0$.

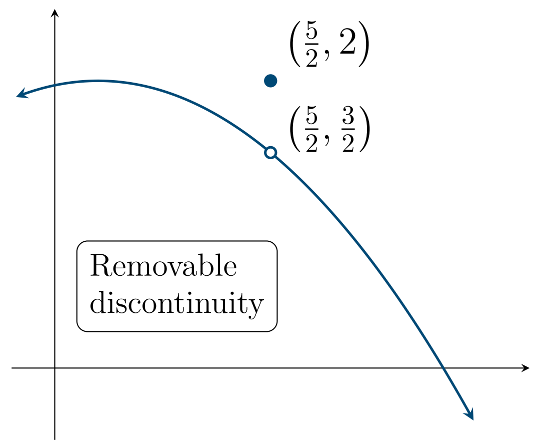
Definition.

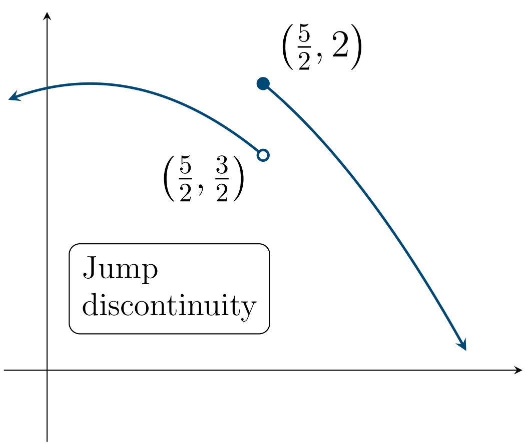
A **removable discontinuity** at $x = a$ is one that disappears when the function becomes continuous after defining $f(a) = \lim_{x \to a} f(x)$.

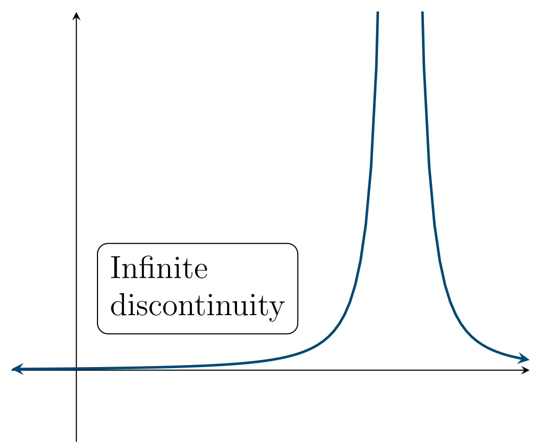
A **jump discontinuity** is one that occurs whenever $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$.

A **vertical discontinuity** occurs whenever $f(x)$ has a vertical asymptote.

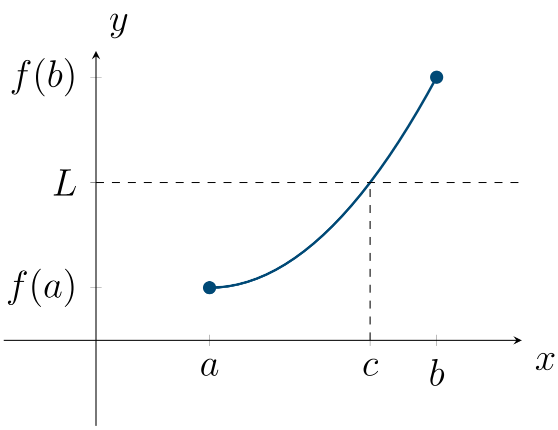


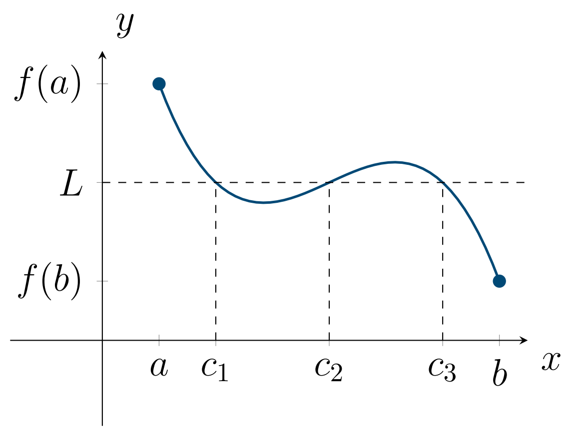




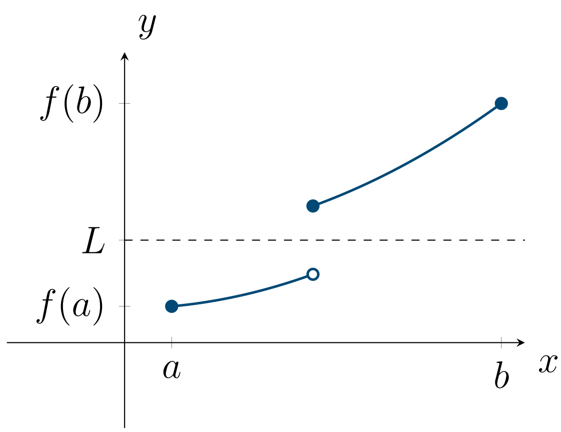


Theorem 4: Intermediate Value Theorem
Suppose f is continuous on the interval $[a, b]$ and L is a number strictly between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) satisfying $f(c) = L$.





Note: It is important that the function be continuous on the interval $[a, b]$:



Theorem 5: Existence of Zeros of a Continuous Function
If f is a continuous function on a closed interval $[a, b]$, and if $f(a)$ and $f(b)$ have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval (a, b).

Example. Check the conditions of the Intermediate Value Theorem to see if there exists a value c on the interval (a, b) such that the following equations hold: Graph

$x^x - \sqrt{x} = \frac{1}{2}$ on $[0, 1]$

$\sqrt{x^4 + 25x^3 + 10} = 5$ on $[0, 1]$

$x + \sqrt{1 - x^2} = 0$ on $[-1, 0]$

$\dfrac{x^2}{x^2 + 1} = 0$ on $[-1, 1]$

Example. Consider the function

$$f(x) = \frac{x+1}{x-1}$$

on the interval $[0, 2]$. Does there exist a c on the interval $[0, 2]$ such that $f(c) = 1$?