Math 121 Class notes
Fall 2025

To accompany
Mathematical Applications for the Management Life and Social Sciences
by *Harshbarger* and *Reynolds*

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1.1: Solutions of Linear Equations and Inequalities in One Variable

Definition.
A **function** f is a special relation between x and y such that each input x results in *at most* one y. The symbol $f(x)$ is read "f of x" and is called the **value of** f **at** x

Example. Let $f(x) = 4x - 1$. Evaluate the following:

$f(1)$

$f\left(\frac{1}{2}\right)$

$f(-2)$

$f(0)$

$f(\text{☺})$

$f(f(x))$

Composite Functions:

Let f and g be functions of x. Then, the **composite functions** g of f (denoted $g \circ f$) and f of g (denoted $f \circ g$) are defined as:

$$(g \circ f)(x) = g(f(x))$$
$$(f \circ g)(x) = f(g(x))$$

Example. Let $g(x) = x - 1$. Find:

$(g \circ f)(x)$

$(f \circ g)(x)$

Operations with Functions:

Let f and g be functions of x and define the following:

Sum	$(f+g)(x) = f(x) + g(x)$
Difference	$(f-g)(x) = f(x) - g(x)$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$
Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ if $g(x) \neq 0$

Definition.

An **expression** is a meaningful string of numbers, variables and operations:

$$3x - 2$$

An **equation** is a statement that two quantities or algebraic expressions are equal:

$$3x - 2 = 7$$

A **solution** is a value of the variable that makes the equation true:

$$\begin{aligned} 3(3) - 2 &= 7 \\ 9 - 2 &= 7 \\ 7 &= 7 \end{aligned}$$

A **solution set** is the set of ALL possible solutions of an equation:

$3x - 2 = 7$ only has the solution $x = 3$,

$2(x-1) = 2x - 2$ is true for all possible values of x.

Properties of Equality:

Substitution Property: The equation formed by substituting one expression for an equal expression is equivalent to the original equation:

$$\begin{aligned} 3(x-3)-\frac{1}{2}(4x-18) &= 4 \\ 3x-9-2x+9 &= 4 \\ x &= 4 \end{aligned}$$

Addition Property: The equation formed by adding the same quantity to both sides of an equation is equivalent to the original equation:

$$\begin{aligned} x-4 &= 6 \\ x-4+4 &= 6+4 \\ x &= 10 \end{aligned} \qquad \begin{aligned} x+5 &= 12 \\ x+5+(-5) &= 12+(-5) \\ x &= 7 \end{aligned}$$

Multiplication Property: The equation formed by multiplying both sides of an equation by the same *nonzero* quantity is equivalent to the original equation:

$$\begin{aligned} \frac{1}{3}x &= 6 \\ 3\left(\frac{1}{3}x\right) &= 3(6) \\ x &= 18 \end{aligned} \qquad \begin{aligned} 5x &= 20 \\ \frac{5x}{5} &= \frac{20}{5} \\ x &= 4 \end{aligned}$$

Solving a linear equation:

Using the properties of equality above, we can solve any linear equation in 1 variable:

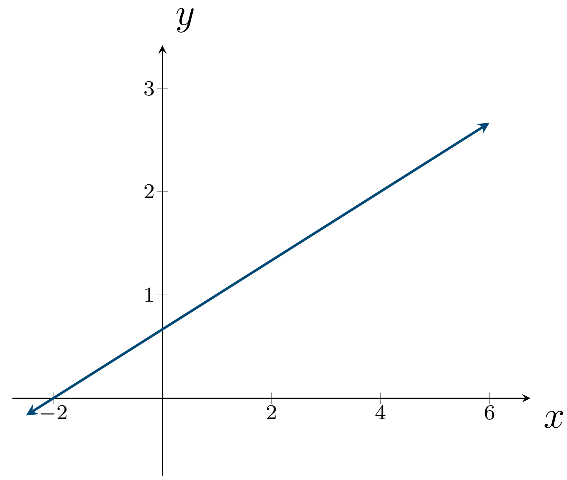
Example. Solve $\dfrac{3x}{4} + 3 = \dfrac{x-1}{3}$

1. Eliminate fractions: $$12\left(\frac{3x}{4} + 3\right) = 12\left(\frac{x-1}{3}\right)$$
2. Remove/evaluate parenthesis: $$9x + 36 = 4x - 4$$
3. Use addition property to isolate the variable to one side: $$9x + 36 - 36 - 4x = 4x - 4 - 36 - 4x$$
4. Use multiplication property to isolate variable: $$\frac{5x}{5} = \frac{-40}{5}$$
5. Verify solution via substitution: $$\underbrace{\frac{3(-8)}{4} + 3}_{-6+3=-3} \stackrel{?}{=} \underbrace{\frac{(-8)-1}{3}}_{\frac{-9}{3}=-3}$$

Example. Solve the following:

$$\frac{3x+1}{2} = \frac{x}{3} - 3 \qquad\qquad \frac{2x-1}{x-3} = 4 + \frac{5}{x-3}$$

Example. Solve $-2x + 6y = 4$ for y



Example. Suppose that the relationship between a firm's profit, P, and the number of items sold, x, can be described by the equation

$$5x - 4P = 1200$$

a) How many units must be produced and sold for the firm to make a profit of \$150?

b) Solve this equation for P in terms of x. Then, find the profit when 240 units are sold.

Definition.
An **inequality** is a statement that one quantity is greater than (or less than) another quantity.

Properties of Inequalities

Substitution Property: The inequality formed by substituting one expression for an equal expression is equivalent to the original inequality:

$$\begin{aligned} 5x - 4x + 2 &< 6 \\ x &< 4 \quad \Rightarrow \text{The solution set is } \{x : x < 6\} \end{aligned}$$

Addition Property: The inequality formed by adding the same quantity to both sides of an inequality is equivalent to the original inequality:

$$\begin{aligned} x - 4 &< 6 \\ x - 4 {\color{red}+4} &< 6 {\color{red}+4} \\ x &< 10 \end{aligned} \qquad \begin{aligned} x + 5 &\geq 12 \\ x + 5 {\color{red}+(-5)} &\geq 12 {\color{red}+(-5)} \\ x &\geq 7 \end{aligned}$$

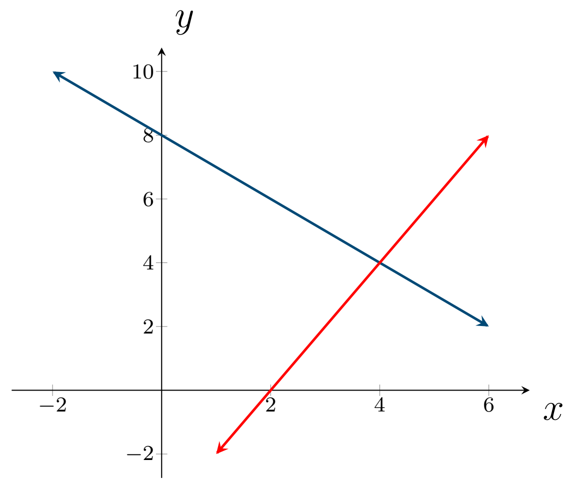
Multiplication Property The inequality formed by multiplying both sides of an inequality by the same *positive* quantity is equivalent to the original inequality. The direction of the inequality is flipped when multiplying by a *negative* quantity:

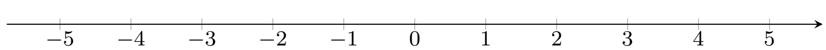
$$\begin{aligned} \frac{1}{3}x &> 6 \\ {\color{red}3}\left(\frac{1}{3}x\right) &> {\color{red}3}(6) \\ x &> 18 \end{aligned} \qquad \begin{aligned} 5x - 5 {\color{red}+5} &\leq 6x + 20 {\color{red}+5} \\ \frac{-x}{\color{red}-1} &\leq \frac{25}{\color{red}-1} \\ x &\geq -25 \end{aligned}$$

Example. Solve

$$-x + 8 \leq 2x - 4$$

first by gathering the x variable on the left, then again on the right. See that the multiplication property holds in both cases. Plot the solution set on a numberline.

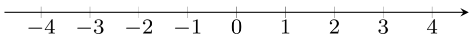


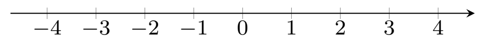


Example. Plot the following inequalities:

$x \leq 2$

$x > -3$





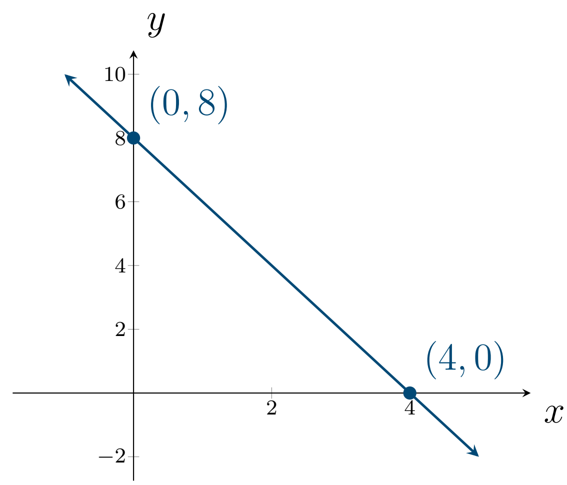
1.3: Linear Functions

Definition.
A **linear function** is a function of the form

$$y = f(x) = mx + b$$

where m and b are constants.

Example. $y = -2x + 8$



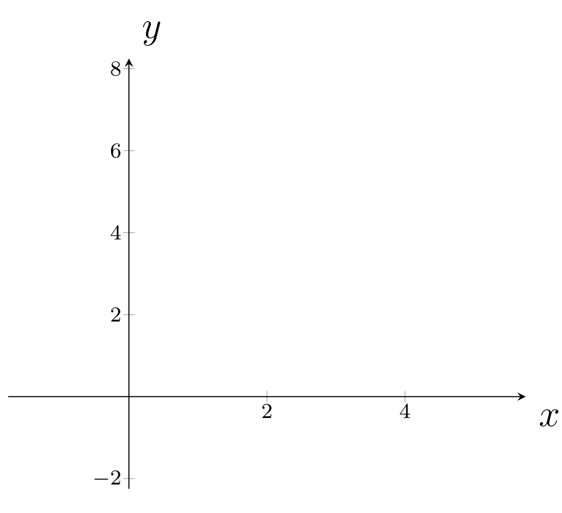
A linear function can be uniquely determined using only *two* distinct points.

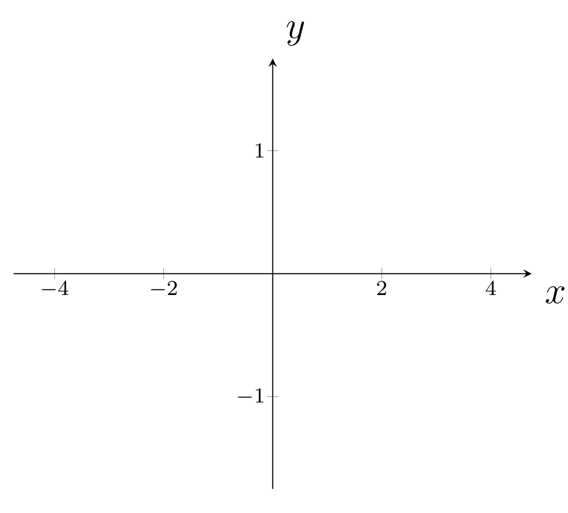
Definition.
The point(s) where a graph intersects the axes are called intercepts. The x-coordinate of the point where the function intersects the x-axis is called the **x-intercepts**. The y-coordinate of the point where the function intersects the x-axis is called the **y-intercepts**.

- To solve for the y-intercept:
  - Set $x = 0$,
  - Solve for y.
- To solve for the x-intercept:
  - Set $y = 0$,
  - Solve for x.

Example. Find the intercepts and graph the following lines:

$3x + 2y = 12$ $\qquad\qquad$ $x = 4y$





Definition.
If a nonvertical line passes through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, its **slope**, denoted by m, is found using

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

Δy is "delta y", and represents the change in y
Δx is "delta x", and represents the change in x

Note: The slope of a vertical line is undefined.

Example. Find the slope of the line passing through the points $(-2, 1)$ and $(5, 3)$.

Note:

- Two distinct nonvertical lines are *parallel* if and only if their slopes are *equal*.
- Two distinct nonvertical lines are *perpendicular* if and only if their slopes are *negative reciprocals*:
  e.g. If ℓ_1 has a nonzero slope m, then ℓ_2 is perpendicular if its slope is $-1/m$.

Point-slope form

Definition.
The equation of the line passing through the point (x_1, y_1) with slope m can be written in the point-slope form:
$$y - y_1 = m(x - x_1)$$

Example. Find the equation of each line that passes through the point $(-3, 4)$ and has

a slope of $m = \frac{1}{4}$

the point $(-2, 1)$ on the line

a slope of zero (horizontal)

an undefined slope (vertical)

Slope-intercept form

Definition.
The slope-intercept form of the equation of a line with slope m and y-intercept b is

$$y = mx + b$$

Example (Example 7, p.82)**.** The population of U.S. males, y (in thousands), projected from 2015 to 2060 can be modeled by

$$y = 1125.9x + 142,960$$

where x is the number of years after 2000.

- Find the slope and y-intercept of the graph of this function.

- What does the y-intercept tell us about the population of U.S. males?

- Interpret the slope as a rate of change.

Example. Each day, a young person should sleep 8 hours plus $\frac{1}{4}$ hour for each year the person is under 18 years of age. Assuming that the relation is linear, write the equation relating hours of sleep y and age x

Forms of Linear Equations

$$\begin{aligned} \text{General form: } & Ax + By = C \\ \text{Point-slope form: } & y - y_1 = m(x - x_1) \\ \text{Slope-intercept form: } & y = mx + b \\ \text{Vertical line: } & x = a \\ \text{Horizontal line: } & y = b \end{aligned}$$

1.4: Graphs and Graphing Utilities

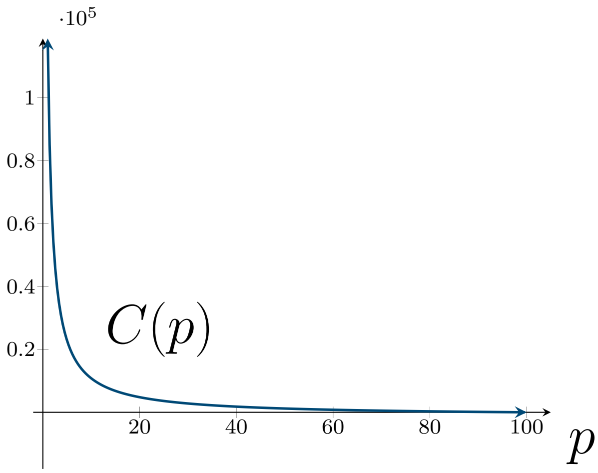
As graphing calculators are *not* required for this course, we will use Desmos:

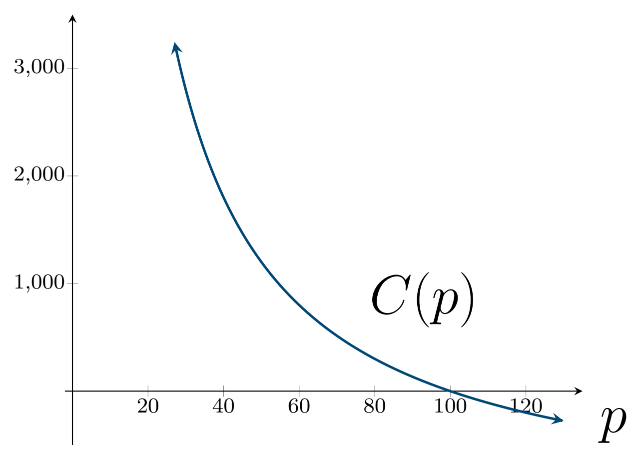
desmos.com/calculator

Example. For a certain city, the cost C of obtaining drinking water with p percent impurities (by volume) is given by

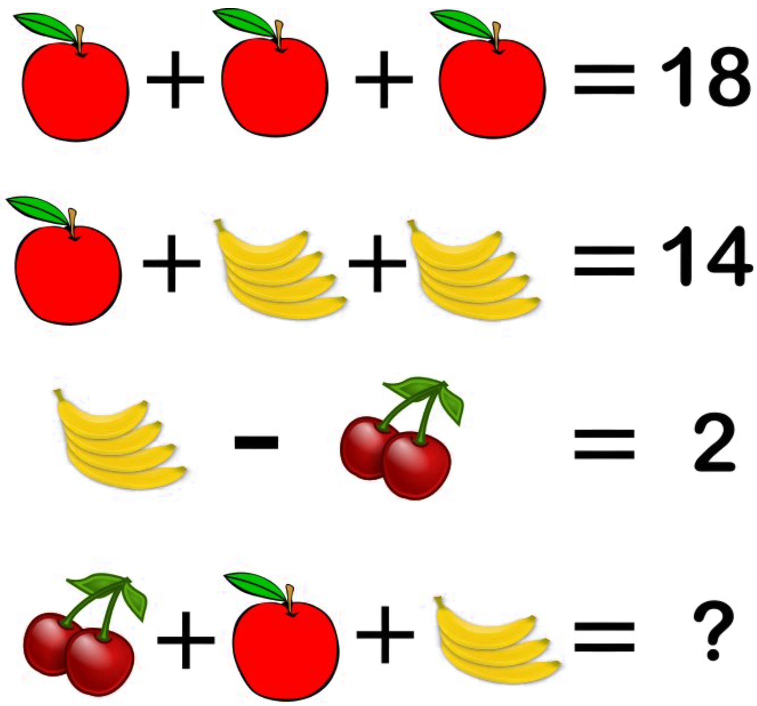
$$C = \frac{120,000}{p} - 1200$$

The equation for C requires that $p \neq 0$, and because p is the percent impurities, we know $0 < p \leq 100$. Use the restriction on p and a graphing calculator to obtain an accurate graph of the equation.





1.5: Solutions of Systems of Linear Equations



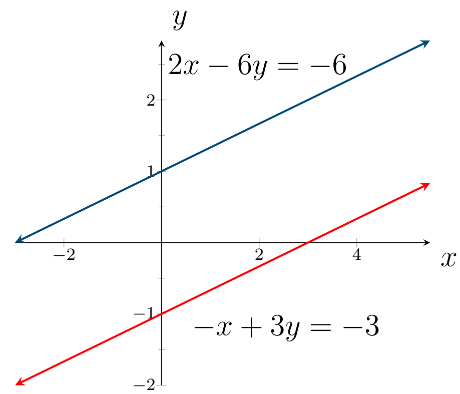
Definition.
A **system of equations** is 2 (or more) equations. The ordered pairs (x, y) that satisfies *all* equations in the system are the **solutions** of the system.

When solving a system of linear equations, there are three possible outcomes:

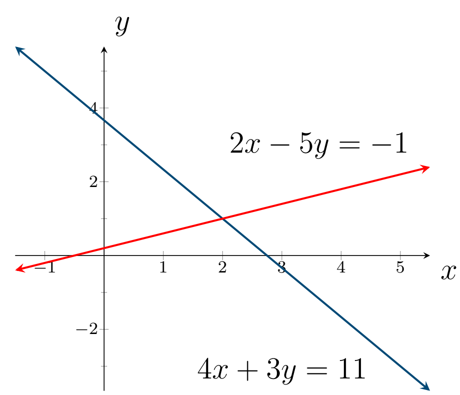
1. No solution (*Inconsistent*),
2. Exactly one solution,
3. Infinitely many solutions (*Dependent*).

Example. Use graphing to find the solutions to the following systems

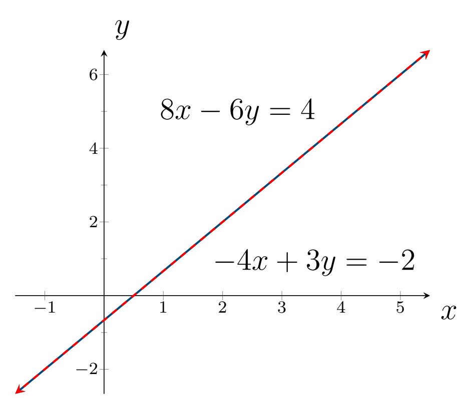
$$\begin{aligned} 2x - 6y &= -6 \\ -x + 3y &= -3 \end{aligned}$$



$$\begin{aligned} 4x + 3y &= 11 \\ 2x - 5y &= -1 \end{aligned}$$



$$\begin{aligned} -4x + 3y &= -2 \\ 8x - 6y &= 4 \end{aligned}$$



Equivalent systems result when

1. One expression is replaced by an equivalent expression.
2. Two equations are interchanged.
3. A multiple of one equation is added to another equation.
4. An equation is multiplied by a nonzero constant.

Substitution Method

Example. Solve the system $\begin{cases} 2x+3y=4 \\ x-2y=3 \end{cases}$

1. Solve one equation for either one of the variables in terms of the other.
   $$x = 2y + 3$$
2. Substitute this expression into the other equation to give the equation in one unknown.
   $$2(2y+3) + 3y = 4$$
3. Solve this equation for the unknown.
   $$4y + 6 + 3y = 4$$
   $$7y = -2 \Rightarrow y = -\frac{2}{7}$$
4. Substitute solution into the equation in Step 1.
   $$x = 2\left(-\frac{2}{7}\right) + 3 \Rightarrow x = \frac{17}{7}$$
5. Check the solution (x, y).
   $$2\left(\frac{17}{7}\right) + 3\left(-\frac{2}{7}\right) = 4$$
   $$\left(\frac{17}{7}\right) - 2\left(-\frac{2}{7}\right) = 3$$

Example. Use the substitution method to solve the system

$$
\begin{aligned}
4x + 5y &= 18 \qquad (1)\\
3x - 9y &= -12 \qquad (2)
\end{aligned}
$$

Elimination Method

Example. Solve the system $\begin{cases} 2x - 5y = 4 \\ x + 2y = 3 \end{cases}$

1. Multiply one or both equations by a nonzero number so the coefficients of one of the variables may cancel.

$$\Rightarrow \begin{cases} 2x - 5y = 4 \\ -2x - 4y = -6 \end{cases}$$

2. Add or subtract the equations to eliminate one of the variables.

$$0x - 9y = -2$$

3. Solve for the remaining variable.

$$\Rightarrow y = \frac{2}{9}$$

4. Substitute solution in one of the original equations and solve for the other variable.

$$2x - 5\left(\frac{2}{9}\right) = 4 \Rightarrow x = \frac{23}{9}$$

5. Check the solution (x, y)

$$\begin{aligned} 2\left(\frac{23}{9}\right) - 5\left(\frac{2}{9}\right) &= 4 \\ \left(\frac{23}{9}\right) + 2\left(\frac{2}{9}\right) &= 3 \end{aligned}$$

Example. Use the elimination method to solve the following systems:

$$\begin{aligned} 2x - 6y &= -6 \\ -x + 3y &= -3 \end{aligned}$$

$$\begin{aligned} 4x + 3y &= 11 \\ 2x - 5y &= -1 \end{aligned}$$

$$\begin{aligned} -4x + 3y &= -2 \\ 8x - 6y &= 4 \end{aligned}$$

Example. A woman has \$500,000 invested in two rental properties. One yields an annual return of 10% on her investment, and the other returns 12% per year on her investment. Her total annual return from the two investments is \$53,000. Let x represent the amount of the 10% investment and y represent the amount of the 12% investment.

- Write an equation that states that the sum of investments is \$500,000.

- What is the annual return on the 10% investment? What about the 12% investment?

- Write an equation that states the sum of the annual return is \$53,000.

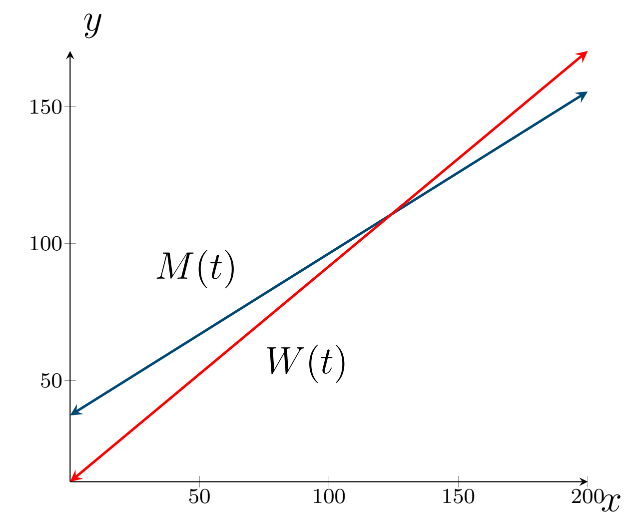
- Solve these two equations simultaneously to find how much is invested in each property.

Example. A nurse has two solutions that contain different concentrations of a certain medication. One is a 12.5% concentration, and the other is a 5% concentration. How many cubic centimeters of each should she mix to obtain 20 cubic centimeters of an 8% concentration?

Example. Using U.S. Bureau of Labor Statistics data for selected years from 1950 and projected to 2050, the number of men M and women W in the workforce (both in millions) can be modeled by the functions

$$M(t) = 0.591t + 37.3 \qquad \text{and} \qquad W(t) = 0.786t + 13.1$$

where t is the number of years after 1940. Find the year these functions predict that there will be equal numbers of men and women in the U.S. workforce.



1.6: Applications of Functions in Business and Economics

Definition.
Profit is the difference between the revenue and total cost:

$$P(x) = R(x) - C(x)$$

where

$$\begin{aligned} P(x) &= \text{profit from sale of } x \text{ units,} \\ R(x) &= \text{total revenue from sale of } x \text{ units,} \\ C(x) &= \text{total cost from production and sale of } x \text{ units.} \end{aligned}$$

Note: In general, the symbols used in economics are π, TR and TC respectively.

In general, **total revenue** is

$$\text{Revenue} = (\text{price per unit})(\text{number of units})$$

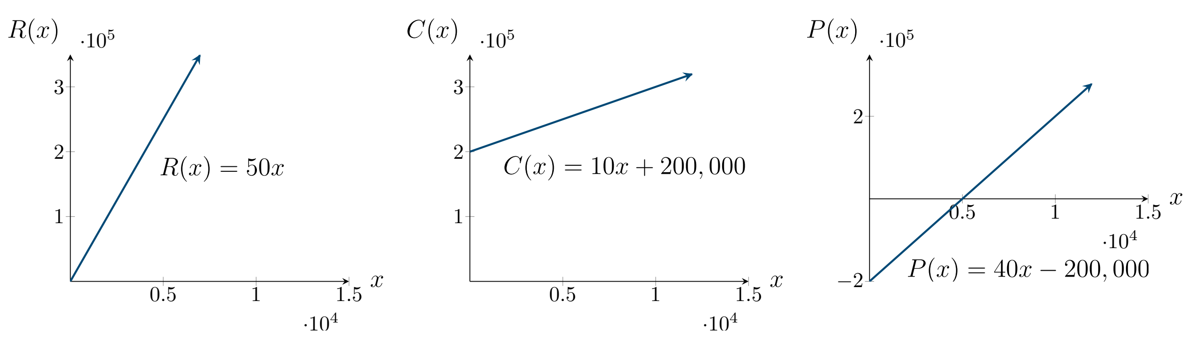
The **total cost** is composed of fixed cost and variable cost:

- **Fixed costs** (FC) remain constant regardless of the number of units produced.
- **Variable costs** (VC) are directly related to the number of units produced.

The total cost is given by

$$\text{Cost} = \text{variable costs} + \text{fixed costs}$$

Example. Suppose a firm manufactures MP3 players and sells them for \$50 each. The costs incurred in the production and sale of the MP3 players are \$200,000 plus \$10 for each player produced and sold. Write the profit function for the production and sale of x players.



Example. The ABC company produces widgets which sell at \$25 each. ABC can produce 30 widgets at a total cost of \$2,050, and 180 widgets at a total cost of \$4,300. Find the revenue, cost, and profit functions.

Definition. (Marginals)
The

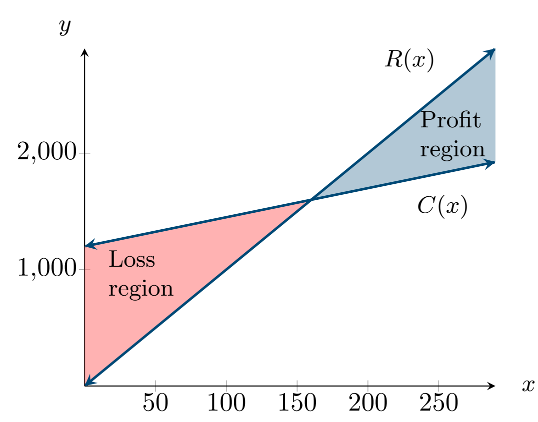
- **marginal profit** ($\overline{MP}$) is the rate of change in profit...
- **marginal cost** ($\overline{MC}$) is the rate of change in costs...
- **marginal revenue** ($\overline{MR}$) is the rate of change in revenue...

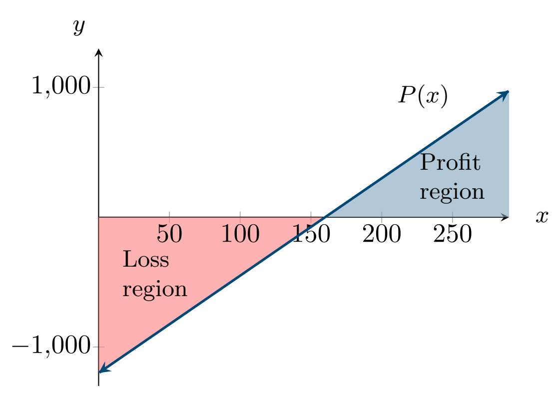
with respect to the number of units produced and sold. When these functions are linear, the marginals are given by the slope of their respective function.

Example. A manufacturer sells widgets for \$10 per unit. The manufacturer's variable costs are \$2.50 per unit, and the total cost of 100 units is \$1,450.

- Find the profit function. What are the marginal revenue, cost and profit?

- Find the **break-even point** (where $R(x) = C(x)$ and $P(x) = 0$). What happens if we sell more or less than the break-even point?

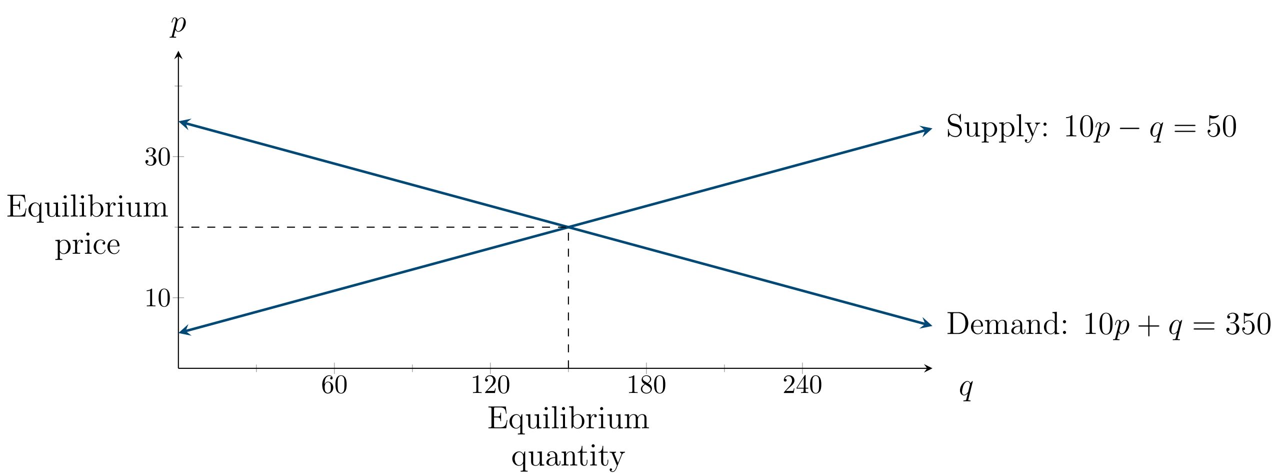




Definition.

- **Market equilibrium** occurs when the quantity of a commodity demanded is equal to the quantity supplied.
- The **law of demand** states that the quantity demanded will decrease as the price increases
- The **law of supply** states that the quantity supplied will increase as the price increases

Example. Below is a graph containing a supply and demand curve. Find the market equilibrium.



Example. Find the market equilibrium point for the following demand and supply functions:

$$\begin{aligned} \text{Supply:} \quad & p = 2q + 170 \\ \text{Demand:} \quad & p = -5q + 450 \end{aligned}$$

Example. Using the supply and demand functions above, modify the supply function to include a \$14 tax per unit sold, then find the new market equilibrium point.

Example. Retailers will buy 45 Wi-Fi routers from a wholesaler if the price is \$10 each but only 20 if the price is \$60. The wholesaler will supply 56 routers at \$42 each and 70 at \$50 each. Assuming that the supply and demand functions are linear, find the market equilibrium point.

4.1: Linear Inequalities in Two Variables

Properties of Inequalities

Substitution Property: The inequality formed by substituting one expression for an equal expression is equivalent to the original inequality:

$$\begin{aligned} 5x - 4x + 2 &< 6 \\ x &< 4 \quad \Rightarrow \text{The solution set is } \{x : x < 6\} \end{aligned}$$

Addition Property: The inequality formed by adding the same quantity to both sides of an inequality is equivalent to the original inequality:

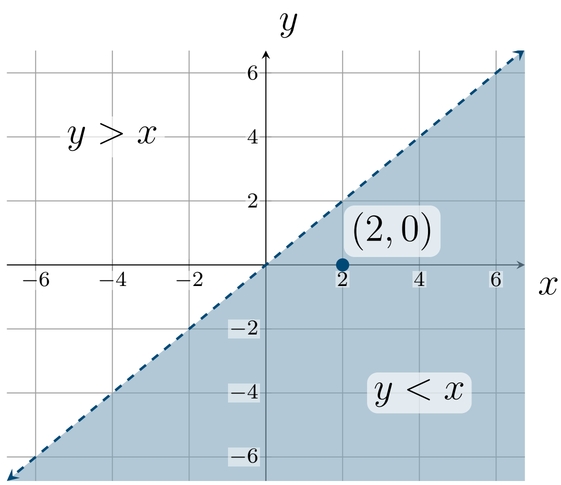
$$\begin{aligned} x - 4 &< 6 \\ x - 4 + 4 &< 6 + 4 \\ x &< 10 \end{aligned} \qquad \begin{aligned} x + 5 &\geq 12 \\ x + 5 + (-5) &\geq 12 + (-5) \\ x &\geq 7 \end{aligned}$$

Multiplication Property The inequality formed by multiplying both sides of an inequality by the same *positive* quantity is equivalent to the original inequality. The direction of the inequality is flipped when multiplying by a *negative* quantity:

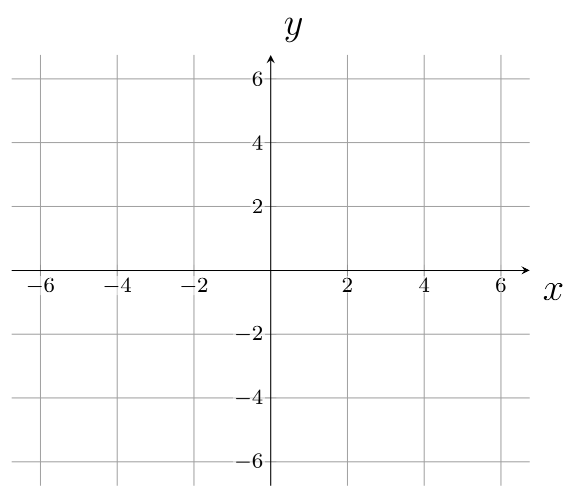
$$\begin{aligned} \frac{1}{3}x &> 6 \\ 3\left(\frac{1}{3}x\right) &> 3(6) \\ x &> 18 \end{aligned} \qquad \begin{aligned} 5x - 5 + 5 &\leq 6x + 20 + 5 \\ \frac{-x}{-1} &\leq \frac{25}{-1} \\ x &\geq -25 \end{aligned}$$

One Linear Inequality in Two Variables

Definition.
Consider the inequality $y < x$:
The line created by this inequality divides the xy-plane into two **half-planes**. We can determine which half-plane is the solution region by selecting any point *not on the line* as a **test point**.



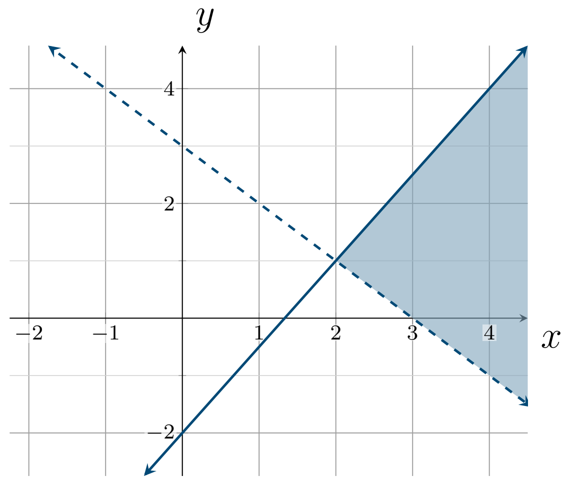
Example. Graph the inequality $3x - 2y \leq 6$



Definition.
A **system of inequalities** has two or more inequalities in two or more variables. The solution of the system is the intersection of the individual solution sets.

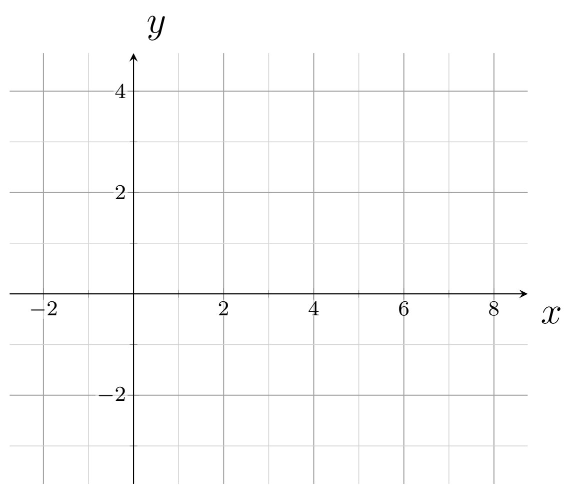
Example.

$$\begin{aligned} 3x - 2y &\geq 4 \\ x + \ y - 3 &> 0 \end{aligned}$$



Example. Graph the solution of the system

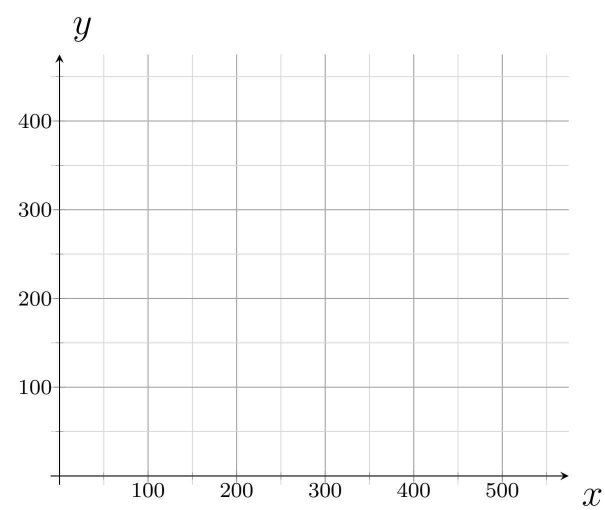
$$\begin{aligned} 3x - 4y &\leq \ 12 \\ 2x + 5y &> \ 10 \\ x - 8y &\geq -16 \end{aligned}$$



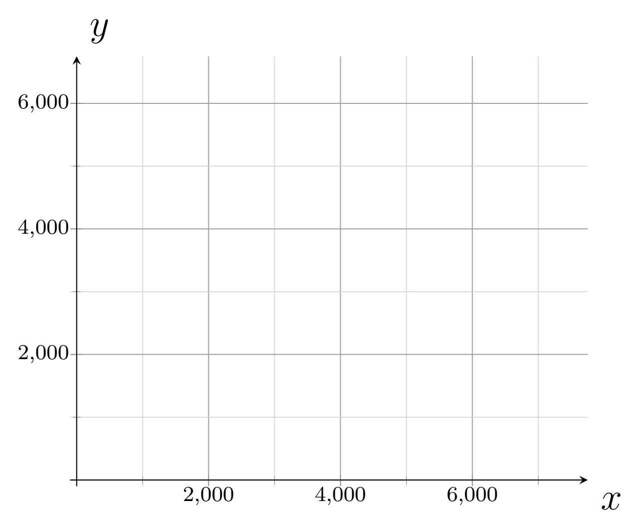
Example. CDF Appliances has assembly plants in Atlanta and Fort Worth, where the company produces a variety of kitchen appliances, including a 12-cup coffee maker and a cappuccino machine. The following table shows each factory's assembly capabilities for the two products and the numbers needed to fill orders.

	Atlanta	**Fort Worth**	**Needed**
Coffee maker	160/hr	800/hr	At least 64,000
Cappucino machine	200/hr	200/hr	At least 40,000

Write the system of inequalities that describes the number of assembly hours needed at each plant to fill the orders and graph the solution region for the system



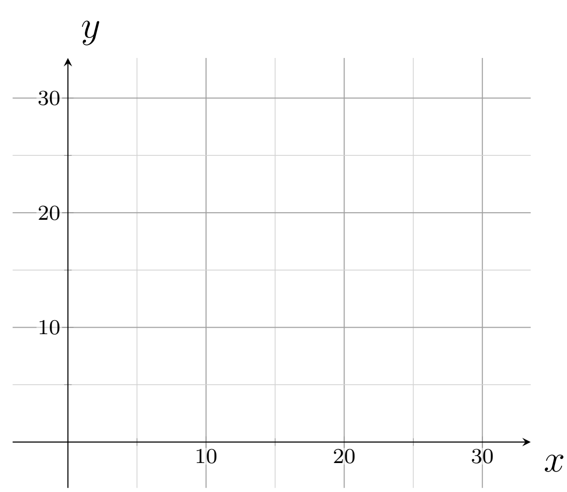
Example. A farm co-op has 6000 acres available to plant with corn and soybeans. Each acre of corn requires 9 gallons of fertilizer/herbicide and 3/4 hour of labor to harvest. Each acre of soybeans requires 3 gallons of fertilizer/herbicide and 1 hour of labor to harvest. The co-op has available at most 40,500 gallons of fertilizer/herbicide and at most 5250 hours of labor for harvesting. The number of acres of each crop is limited (constrained) by the available resources: land, fertilizer/herbicide, and labor for harvesting. Write the system of inequalities that describes the constraints and graph the solution region for the system.



Example. Graph the solution region for the system

$$\begin{aligned} 5x+2y &\leq 54 \\ 2x+4y &\leq 60 \\ x \geq 0,\ y &\geq\ 0 \end{aligned}$$

Then compute the corners of this region.

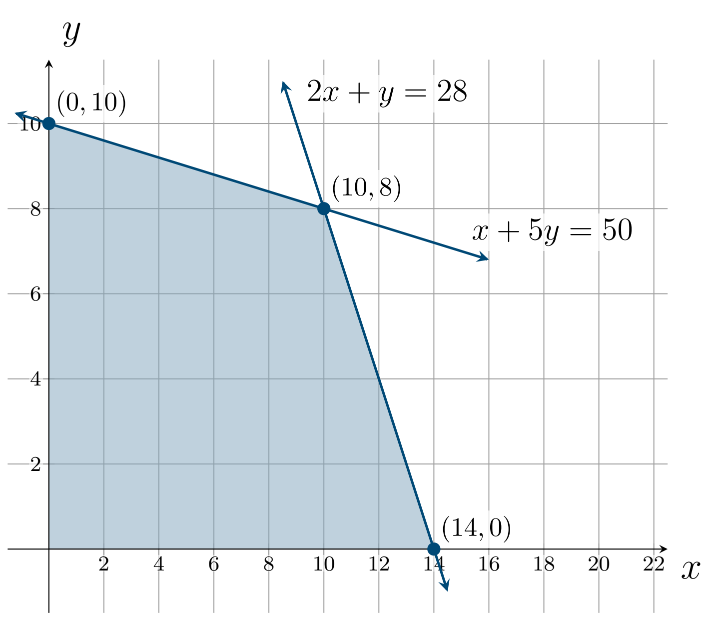


4.2: Linear Programming: Graphical Methods

Definition.
Linear programming is an optimization technique that can be used to solve linearly constrained problems:

$$
\begin{aligned}
\max F &= 3x + y \\
\text{subject to} \quad x + 5y &\leq 50 \\
2x + y &\leq 28 \\
x \geq 0, y &\geq 0
\end{aligned}
$$

The **constraints** of a linear program (LP) may be limitations or requirements of the variables. The **objective function** is the function that we wish to optimize (e.g. maximize profit *or* minimize cost).



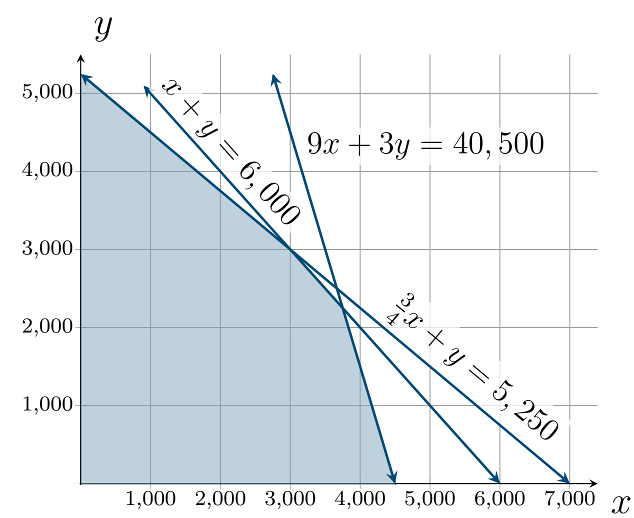
Linear programming (graphical method)

1. Write the objective function and constraint inequalities from the problem.
2. Graph the solution of the constraint system.
3. Find the corners of the resulting feasible region.
4. Evaluate the objective function at each corner.
5. If two corners give the optimal value, then the entire boundary joining these two points optimizes the function.

Example. A farm co-op has 6000 acres available on which to plant corn and soybeans. The following table summarizes each crop's requirement for fertilizer/herbicide, harvesting labor hours, and the available amounts of these resources.

	Corn	**Soybeans**	**Available**
Fertilizer/herbicide	9 gal/acre	3 gal/acre	40,500 gal
Harvesting labor	3/4 hr/acre	1 hr/acre	5,250 hr

Setup the system of inequalities that represents the constraints.

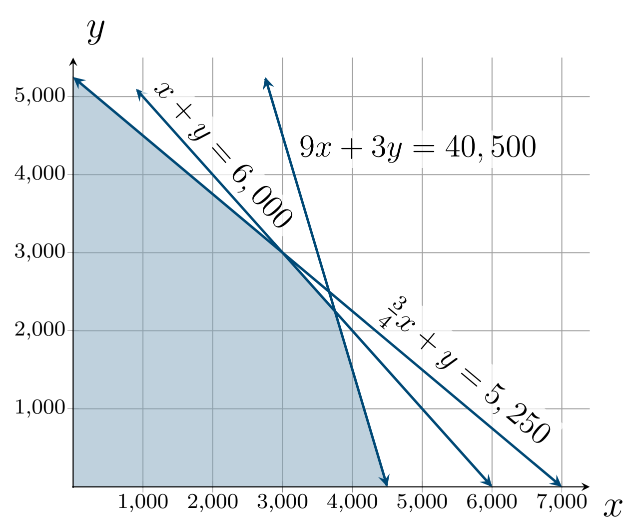


Example. Using the linear constraints from above, suppose the co-ops profits per acre are \$240 for corn and \$160 for soybeans. This gives us the following linear program:

$$\begin{aligned} \max P = 240x + 160y & \\ \text{subject to} \quad x + y &\le 6,000 \\ 9x + 3y &\le 40,500 \\ \frac{3}{4}x + y &\le 5,250 \\ x \ge 0, y &\ge 0 \end{aligned}$$

1. Find the "corners" of the feasible region
2. Evaluate the profit function at the corners

(x, y)	$P = 240x + 160y$
$(0, 0)$	\$0
$(0, 5250)$	\$840,000
$(3000, 3000)$	\$1,200,000
$(3750, 2250)$	\$1,260,000
$(4500, 0)$	\$1,080,000

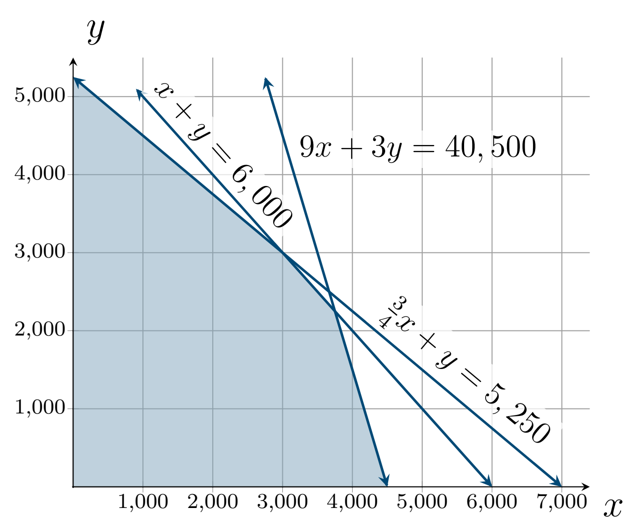


Example. Suppose the profits per acre are instead \$300 for corn, and \$100 for soybeans. This gives us the following linear program:

$$\begin{aligned} \max P = 300x + 100y & \\ \text{subject to} \qquad x + \quad y &\leq 6,000 \\ 9x + 3y &\leq 40,500 \\ \frac{3}{4}x + \quad y &\leq 5,250 \\ x \geq 0, y &\geq 0 \end{aligned}$$

Evaluate the profit function at the corners. What combination of corn and soy-beans maximizes the profit?

(x, y)	$P = 300x + 100y$
$(0, 0)$	\$0
$(0, 5250)$	\$525,000
$(3000, 3000)$	\$1,200,000
$(3750, 2250)$	\$1,350,000
$(4500, 0)$	\$1,350,000



Example. Two chemical plants, one at Macon and one at Jonesboro, produce three types of fertilizer: low phosphorus (LP), medium phosphorus (MP), and high phosphorus (HP). At each plant, the fertilizer is produced in a single production run, so the three types are produced in fixed proportions. The Macon plant produces 1 ton of LP, 2 tons of MP, and 3 tons of HP in a single operation and charges \$600 for what is produced in one operation. On the other hand, one operation of the Jonesboro plant produces 1 ton of LP, 5 tons of MP, and 1 ton of HP, and it charges \$1,000 for what it produces in one operation. If a customer needs 100 tons of LP, 260 tons of MP, and 180 tons of HP, how many production runs should be ordered from each plant to minimize costs?

Organize the information from the problem in the following table:

	Macon	Jonesboro	Requirements
Units of LP			
Units of MP			
Units of HP			

What is the objective function?

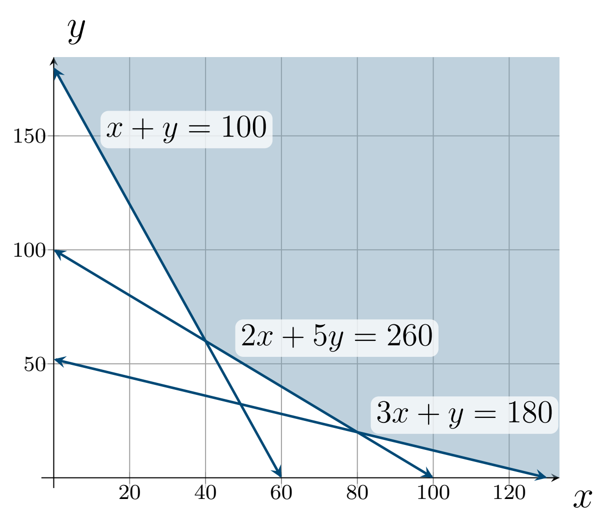
Write the linear program that we aim to solve below:

Example. From above, we get the following linear program

$$\begin{aligned} \min C = 600x + 1000y \\ \text{subject to} \quad x + \quad y &\geq 100 \\ 2x + \quad 5y &\geq 260 \\ 3x + \quad y &\geq 180 \\ x \geq 0, y &\geq 0 \end{aligned}$$

Graph the solution region of the constraints and evaluate the objective function at the corners of the feasible region above.

(x, y)	$P = 600x + 1,000y$
$(0, 180)$	$\$180,000$
$(40, 60)$	$\$84,000$
$(80, 20)$	$\$68,000$
$(130, 0)$	$\$78,000$



5.1: Exponential Functions

Properties of Exponents:

- If m is a positive integer, then $x^m = \underbrace{x \cdot x \ldots x}_{m \text{ times}}$:

$$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81, \quad (-5)^3 = (-5) \cdot (-5) \cdot (-5) = -125$$

- Additivity — If the bases are the same, then $x^a \cdot x^b = x^{a+b}$ and $\dfrac{x^a}{x^b} = x^{a-b}$:

$$4^3 \cdot 4^2 = 4^{3+2} = 4^5 = 1024, \qquad \frac{3^{17}}{3^{12}} = 3^{17-12} = 3^5 = 243$$

- If $x \neq 0$, then $x^0 = 1$:

$$4^0 = 1, \qquad (-7)^0 = 1, \qquad 2025^0 = 1$$

- Distributive — $(ab)^m = a^m b^m$ and $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$:

$$(3 \cdot 4)^2 = 3^2 \cdot 4^2 = 9 \cdot 16 = 144, \qquad \left(\frac{4}{5}\right)^2 = \frac{4^2}{5^2} = \frac{16}{25}$$

- If $m \neq 0$, then $x^{-m} = \dfrac{1}{x^m}$ and $x^m = \dfrac{1}{x^{-m}}$:

$$2^{-4} = \frac{1}{2^4} = \frac{1}{16}, \qquad \left(\frac{1}{3}\right)^{-2} = 3^2 = 9$$

- Multiplicity — $(x^a)^b = x^{ab}$:

$$\left(3^2\right)^4 = 3^8 = 6561$$

- Fractional exponents — $x^{1/m} = \sqrt[m]{x}$:

$$8^{1/3} = \sqrt[3]{8} = 2, \qquad 16^{3/2} = \left(16^{1/2}\right)^3 = \left(\sqrt{16}\right)^3 = 4^3 = 64$$

Definition.
An **exponential function** is of the form

$$f(x) = a^x$$

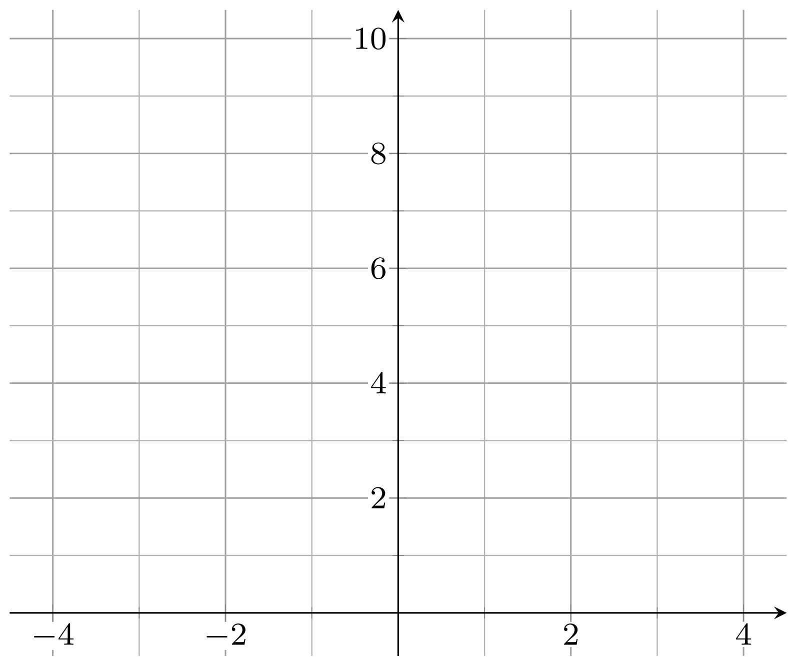
where $a > 0$ and $a \neq 1$. *Note:* The variable is in the exponent (e.g. 2^x vs x^2)

Example. Suppose a culture of bacteria has the property that each minute, every microorganism splits into two new organisms. The number of microorganisms after x minutes is given by

$$y = 2^x.$$

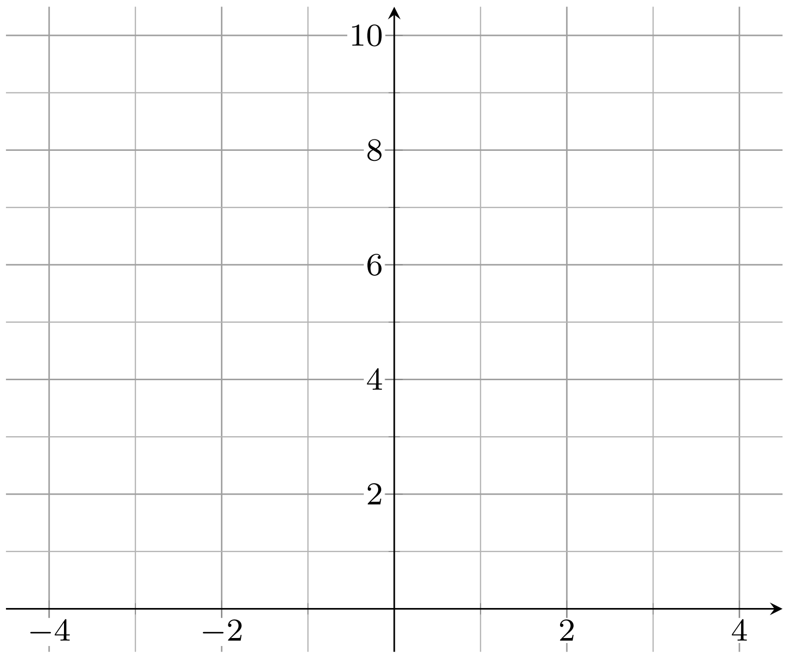
Fill out the table below, and graph this exponential function (include $x < 0$).

x	$y = 2^x$
0	
1	
2	
3	
4	



Example. Graph the exponential function

$$y = \left(\frac{1}{3}\right)^x$$

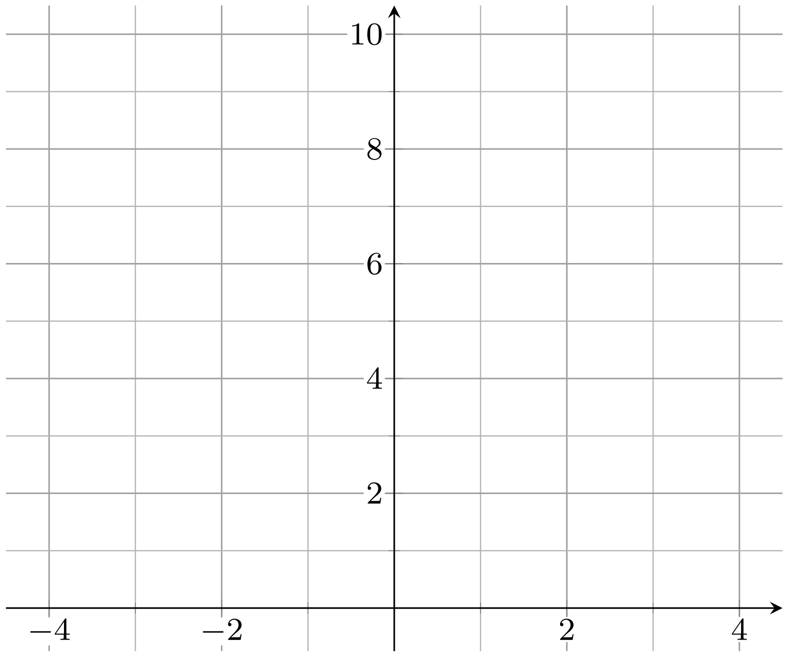


For an exponential function a^x, the function is

- increasing if $a > 1$, and
- decreasing if $0 < a < 1$.

Example. Graph the exponential function

$$y = \frac{1}{2}(2)^x$$



Example. Compound Interest:
If an initial principal P is invested at a rate r and compounded n times a year, the future value in t years is given by:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Suppose that \$800 dollars is invested, and is compounded quarterly at a rate of 6%:

$$A = 800(1.015)^{4t}$$

Find the future value after 10 years.

Example. Compounded continuously:
A special function that frequently occurs in the context of exponential functions is

$$y = e^x$$

where $e = 2.71828...$ (think irrational number like π). When an investment is compounded continuously, it's future value is given by

$$A = Pe^{rt}$$

> Suppose that we invest \$800, compounded continuously at 6%. Find the future value in 10 years.

5.2: Logarithmic Functions and Their Properties

Definition.

For $a > 0$ and $a \neq 1$, the **logarithmic function**

$$y = \log_a(x) \qquad \text{(logarithmic form)}$$

has domain $x > 0$, base a, and is defined by

$$a^y = x \qquad \text{(exponential form)}$$

Example. Rewrite the following in exponential form

$4 = \log_2(16)$

$5 = \log_{10}(100,000)$

$\frac{1}{2} = \log_{16}(4)$

$-4 = \log_3\left(\frac{1}{81}\right)$

$-\frac{1}{4} = \log_{625}\left(\frac{1}{5}\right)$

$-\frac{5}{3} = \log_{\frac{1}{8}}(32)$

Example. Simplify the following:

$\log_3(9)$

$\log_4(2)$

Example. Solve the following:

$\log_5(x) = 4$

$\log_8(x) = 1$

$\log_{81}(x) = -\frac{1}{4}$

$\log_{10}(x + 4) = 3$

$$\boxed{\begin{array}{ll} \textbf{Common logarithms:} & \log(x) = \log_{10}(x) \\ \textbf{Natural logarithms:} & \ln(x) = \log_e(x) \end{array}}$$

Example (The Rule of 70)**.** If \$$P$ is invested for t years at interest rate r, compounded continuously, then the future value of the investment is given by

$$S = Pe^{rt}.$$

Find the value of t when the investment doubles.

Change of base formula:
If $a > 0$, $b > 0$ with $a \neq 1$ and $b \neq 1$, then

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}.$$

Note: This works for any valid base!

Base e : $\quad \log_b(x) = \dfrac{\ln(x)}{\ln(b)}$ $\qquad$ Base 10 : $\quad \log_b(x) = \dfrac{\log(x)}{\log(b)}$

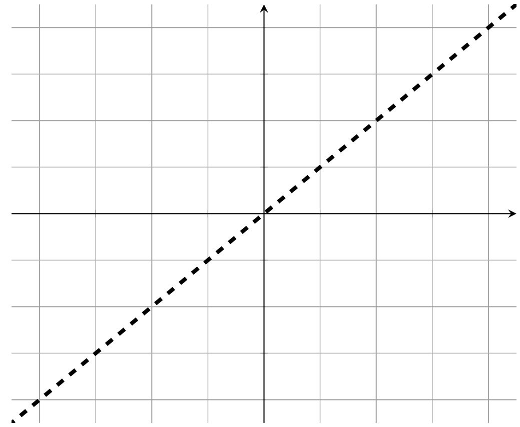
Example. Solve the following

$3^x = 10$ $\qquad$ $6.5^x = 5$

Example. Fill in the tables below and graph a^x and $\log_a(x)$ on the same axes.

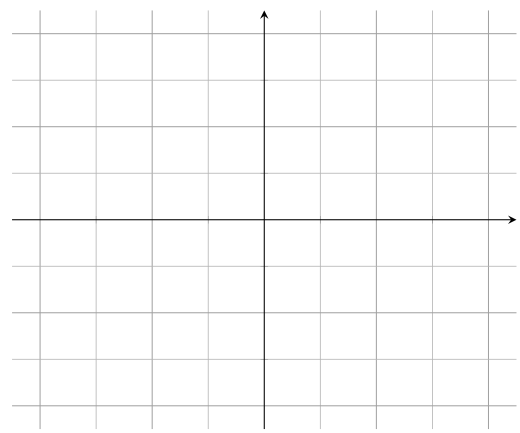
x	$y = a^x$
-2	
-1	
0	
1	
2	

x	$y = \log_a(x)$
	-2
	-1
	0
	1
	2

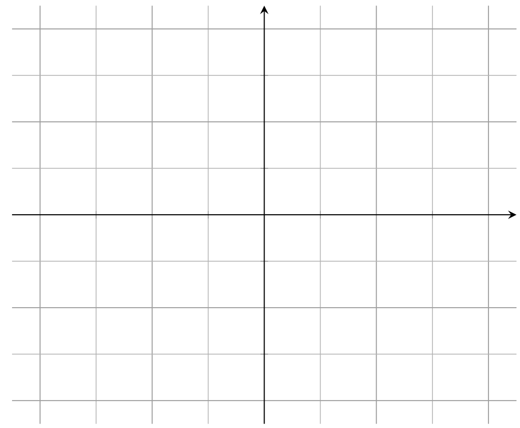


Try this!

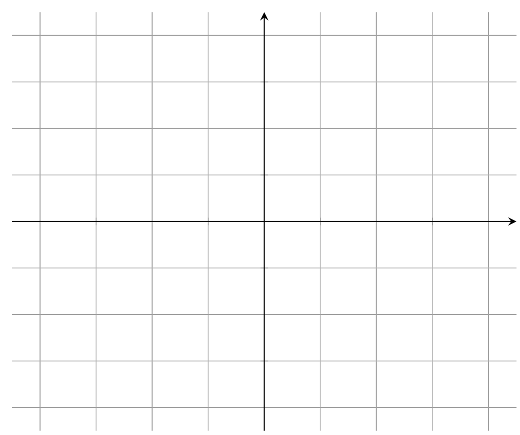
Example. Graph $\log(-x)$



Example. Graph $\ln(x)$



Example. Graph $-\log_2(-x)$



Example. Evaluate the following:

$$f(x) = \ln(x); \quad f\left(e^{-3x}\right) \qquad\qquad f(x) = 5^x; \quad f(\log_5(10))$$

Properties of exponents and logarithms: Assume $a > 0$:

$a^y = x$	$\log_a(x) = y$
$a^1 = a$	$\log_a(a) = 1$
$a^0 = 1$	$\log_a(1) = 0$
$a^x a^y = a^{x+y}$	$\log_a(xy) = \log_a(x) + \log_a(y)$
$\dfrac{a^x}{a^y} = a^{x-y}$	$\log_a\left(\dfrac{x}{y}\right) = \log_a(x) - \log_a(y)$
$a^{xy} = (a^x)^y$	$\log_a(x^y) = y\log_a(x)$
$a^{\log_a(x)} = x$	$\log_a(a^x) = x$

5.3: Equations and Applications with Exponential and Logarithmic Functions

Solving exponential equations:

Example. Solve $4\left(25^{2x}\right) = 312,500$

1. Isolate the exponential by rewriting the equation with a base raised to a power on one side: $$\frac{4\left(25^{2x}\right)}{4} = \frac{312,500}{4}$$
2. Take the logarithm of both sides: $$\ln\left(25^{2x}\right) = \ln(78,125)$$
3. Use a property of logarithms to remove the variable from the exponent: $$2x\ln(25) = \ln(78,125)$$
4. Solve for the variable: $$x = \frac{\ln(78,125)}{2\ln(25)} \approx 1.75$$

Example. Unless the exponential function uses base e or base 10, *it does not matter which logarithm we use.* Solve the following exponential equation first using base 10, then using base e:

$$6\left(4^{3x-2}\right) = 20$$

Example. Suppose the demand function for q thousand units of a certain commodity is given by

$$p = 30\left(3^{-q/2}\right)$$

At what price per unit will the demand equal 4000 units?

How many units, to the nearest thousand units, will be demanded if the price is \$17.31?

Example. A company finds that its daily sales begin to fall after the end of an advertising campaign, and the decline is such that the number of sales is $S = 2000\left(2^{-0.1x}\right)$, where x is the number of days after the campaign ends.

How many sales will be made after 10 days after the end of the campaign?

If the company does not want sales to drop below 350 per day, when should it start a new campaign?

Example. The population of a certain city was 30,000 in 2000, and 40,500 in 2010. If the formula $P = P_0 e^{ht}$ applies to the growth of the city's population, what population is predicted for the year 2030?

Example. The Gompertz equation

$$N = 100(0.03)^{0.2^t}$$

predicts the size of a deer herd on a small island t decades from now.

What is the size of the deer population now ($t = 0$)?

During what year will the deer population reach or exceed 70?

Example. One company's revenue from the sales of computer tablets from 2015 to 2020 can be modeled by the logistic function

$$y = \frac{9.46}{1 + 53.08e^{-1.28x}}$$

where x is the number of years past 2014 and y is in millions of dollars.

Estimate the sales revenue for 2020

During what year will the sales revenue exceed \$4 million?

Example (Bonus)**.** Solve the following for x:

$$6^{x-2} = 2^{-3x}$$

6.1: Simple Interest

Definition.
The **simple interest** I is given by

$$I = Prt$$

where

$$\begin{aligned} I = &\ \text{interest (in dollars)} \\ P = &\ \text{principal (in dollars)} \\ r = &\ \text{annual interest rate (as a decimal)} \\ t = &\ \text{time (in years)} \end{aligned}$$

Note: The time measurements of r and t must agree

From this, the **future value** of simple interest is

$$S = P + I = P + Prt = P(1 + rt)$$

Example.

If \$8,000 is invested for 2 years at an annual rate of 9%, how much interest will be received at the end of the 2–year period? What will the future value be?

If \$4,000 is borrowed for 39 weeks at an annual interest rate of 15%, how much interest is due at the end of the 39 weeks?

An investor wants to have \$20,000 in 9 months. If the best available simple interest rate is 6.05% per year, how much must be invested now?

Definition.
The **return on investment** (ROI) is the ratio between the gain and cost of an investment:

$$ROI = \frac{\text{Gains on investment}}{\text{Cost of investment}}$$

The **earned (effective) interest rate** is the equivalent interest rate of the investment when all the fees and dividends are included.

Example. Mary Spaulding bought Wind-Gen Electric stock for $\$6,125.00$. After 6 months, the value of her shares had risen by $\$138.00$ and dividends totaling $\$144.14$ had been paid.

Find the return on investment on this investment.

Find the simple interest rate she earned on this investment if she sold the stock at the end of the 6 months.

Example.

To buy a Treasury bill (T-bill) that matures to $\$10,000$ in 6 months, you must pay $\$9,750$. What annual simple interest rate does this earn?

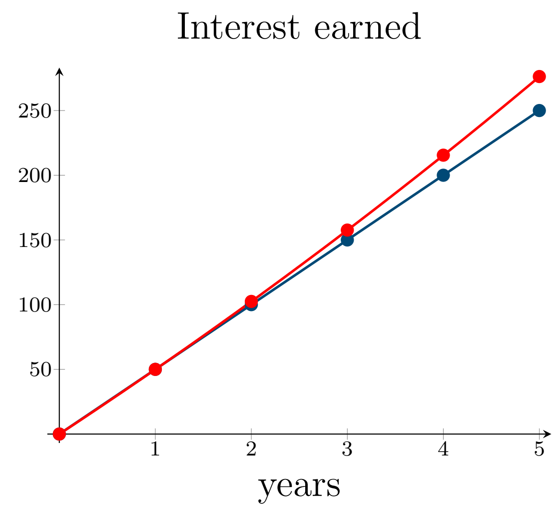
If the bank charges a fee of $\$40$ to buy a T-bill, what is the actual interest rate you earn?

6.2: Compound Interest

Example. Suppose you invest \$1,000 at 5% annual interest. With simple interest, you can take 2 approaches:

1. Gain interest on *only* your initial investment
2. Reinvest the interest gained

Year	Simple interest	Simple interest reinvested
1	\$1,050.00	\$1,050.00
2	\$1,100.00	\$1,102.50
3	\$1,150.00	\$1,157.63
4	\$1,200.00	\$1,215.51
5	\$1,250.00	\$1,276.28



Definition.
Compound interest is a method where the interest for each period is added to the principal before interest is calculated for the next period.

Example. Using the example above, derive a formula for the future value of an investment compounded annually.

Definition.
When interest is compounded multiple times a year (e.g. quarterly, monthly, etc.), the **nominal interest rate** is the interest rate *per year*.

If $\$P$ is invested for t years at a nominal interest rate r compounded m times per year, then the **total number of compounding periods** is

$$n = mt$$

the interest rate per compounding period (**periodic interest rate**) is

$$i = \frac{r}{m}$$

and the future value is

$$S = P(1+i)^n = P\left(1 + \frac{r}{m}\right)^{mt}$$

Example. If $\$3,000$ is invested for 5 years at 9% compounded 4 times a year, how much interest is earned?

Example. For the following, identify the annual interest rate, the length in years, the periodic interest rate, and the number of periods:

12% compounded monthly for 7 years

7.2% compounded quarterly for 11 quarters

Frequency	m
Annually	1
Semi-annually	2
Quarterly	4
Monthly	12
Weekly	52
Daily	365

Example. Ben and Taylor want to have $\$200,000$ in Arthur's college fund on his 18th birthday, and they want to know the impact on this goal of having $\$10,000$ invested at 9.8%, compounded quarterly, on his 1st birthday. To advise Ben and Taylor regarding this, find

the future value of the $\$10,000$ investment,

the amount of compound interest that the investment earns,

the impact this would have on their goal.

Example. What amount must be invested now to have \$12,000 after 3 years with an interest rate of 6%, compounded semi-annually?

Example. Three years after Google stock was first sold publicly, its share price had risen 500%. Google's 500% increase means that \$10,000 invested in Google stock at its initial public offering (I.P.O) was worth \$60,000 three years later. What interest rate compounded annually does this represent?

Example. Suppose we invest \$1 at a 100% interest rate for 1 year:

$$S = \left(1 + \frac{1.00}{m}\right)^m$$

Compute the future value

Annually

Semi-annually

Monthly

Weekly

Daily

Each minute ($m = 525,600$)

Definition.
If $\$P$ is invested for t years at a nominal rate r compounded continuously, then the future value is given by the exponential function

$$S = Pe^{rt}$$

Example. Which investment strategy is worth more: $\$3,000$ for 8 years at

9%, compounded annually

8%, compounded continuously

Example. Suppose you invest \$900 at 11.5%, compounded continuously. How long will it take to gain \$700 in interest?

Definition.

Let r represent the annual (nominal) interest rate for an investment. Then the **annual percentage yield (APY)** is:

Periodic compounding:

$$\text{APY} = \left(1 + \frac{r}{m}\right)^m - 1$$

Continuous compounding:

$$\text{APY} = e^r - 1$$

6.3: Future Values of Annuities

Definition.

- An **annuity** is a financial plan characterized by regular payments (e.g. mortgages, student loans, etc.).
- The sum of all the payments and the interest earned is called the **future value of the annuity** or its **future value**.
- An **ordinary annuity** or (**annuity immediate**) is an annuity in which payments are made at the *end of each of the equal payment intervals.*

Example. Suppose that we invest \$100 at the end of each year for 5 years in an account that pays 10% compounded annually. How much money will you have at the end of the 5 years?

Definition.
If $\$R$ is deposited at the *end of each period* for n periods in an annuity that earns interest at a rate of i per period, the **future value of the annuity** will be

$$S = R \cdot S_{\overline{n}|i} = R\left[\frac{(1+i)^n - 1}{i}\right]$$

The notation $S_{\overline{n}|i}$ represents the future value of an ordinary annuity of \$1 per period for n periods with an interest rate of i per period.

Example. Suppose a pair of twins take different steps to save for retirement. Both regularly make investments of $\$2,000$ into accounts that earn 10%, compounded annually. Starting at age 21:

> Find the future value if twin A makes his payments for 8 years, and then lets his investment accrue compound interest every year for 36 years.

> Find the future value if twin B waits 8 years before making regular payments for the following 36 years.

Example. Suppose that you wish to have \$50,000 saved up in 5 years. To do this, you want to make regularly monthly payments. What is the amount of the monthly payments if the interest rate is 5%? What if the interest rate is 15%?

Example. A small business invests \$1,000 at the end of each month in an account that earns 6% compounded monthly. How long will it take until the business has \$100,000 toward the purchase of its own office building?

Definition.
An **annuity due** differs from an ordinary annuity in that the payments are made at the *beginning of each period*.

If $\$R$ is deposited at the *beginning of each period* for n periods in an annuity that earns interest at a rate of i per period, the **future value of the annuity** will be

$$S_{\text{due}} = R \cdot S_{\overline{n}|i}(1+i) = R\left[\frac{(1+i)^n - 1}{i}\right](1+i)$$

Example. Find the future value of an investment if \$150 is deposited at the beginning of each month for 9 years at an interest rate of 7.2% compounded monthly.

6.4: Present Values of Annuities

Example. Suppose we wish to invest a lump sum of money, A_n, into an annuity that earns interest at a rate of 10% per year, so that we may receive payments of \$100 for 5 years. What is the amount of the lump sum?

Definition.
If a payment of \$$R$ is to be withdrawn at the *end of each period* for n periods from an account that earns interest at a rate of i per period, then the account is an **ordinary annuity** and the **present value** is

$$A_n = R \cdot a_{\overline{n}|i} = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

The notation $a_{\overline{n}|i}$ represents the present value of an ordinary annuity of \$1 per period for n periods with an interest rate of i per period.

Example. Find the lump sum that one must invest in an annuity to receive \$1000 at the end of each month for the next 16 years, if the annuity pays 9% compounded monthly.

Example. Suppose that a couple plans to set up an ordinary annuity with a $\$100,000$ inheritance they received. What is the size of the quarterly payments they will receive for the next 6 years if the account pays 7% compounded quarterly?

Example. An inheritance of $\$250,000$ is invested at 9% compounded monthly. If \$2500 is withdrawn at the end of each month, how long will it be until the account balance is \$0?

Definition.
If a payment of $\$R$ is to be withdrawn at the *beginning of each period* for n periods from an account that earns interest at a rate of i per period, then the account is an **annuity due** and the **present value** is

$$A_{(n,\text{due})} = R \cdot a_{\overline{n}|i}(1+i) = R\left[\frac{1-(1+i)^{-n}}{i}\right](1+i)$$

The notation $a_{\overline{n}|i}$ represents the present value of an ordinary annuity of \$1 per period for n periods with an interest rate of i per period.

Example. Suppose that a court settlement results in a $\$750,000$ award. If this is invested at 9% compounded semiannually, how much will it provide at the *beginning* of each half-year for a period of 7 years?

Comparing annuity calculations:

	Ordinary annuities	Annuity due	
Future Value	S	S_{due}	Regular payments Total *after* n periods
Present Value	A_n	$A_{(n,\text{due})}$	Regular withdrawals Total *before* n periods
	end of each period	beginning of each period	

2.1: Quadratic Equations

Definition.
A **quadratic equation** in one variable is an equation of second degree that can be written in the *general form* as

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

where a, b, and c represent constants.

The **zero product property** states that for real numbers a and b, $ab = 0$ if and only if $a = 0$ or $b = 0$ or both.

Example. Solve the following for x:

$x(x + 3) = 0$

$(x - 4)(3x + 1) = 0$

Solving quadratic equations via factoring:

Example. Solve $2x^2 + x = 3x + 12$

1. Rewrite the equation in the general form: $2x^2 - 2x - 12 = 0$

2. Rewrite bx using factors of ac: $2x^2 - 6x + 4x - 12 = 0$

3. Factor out like terms: $2x(x - 3) + 4(x - 3) = 0$

4. Factor by grouping: $(x - 3)(2x + 4) = 0$

5. Solve for the roots: $x = 3$ and $x = -2$

Example. Solve the following for x via factoring:

$(x + 3)(x - 1) = 5$ $\qquad$ $-4x^2 + 8x - 3 = 0$

$$\boxed{\text{Solutions to } x^2 = C \text{ are } x = \pm\sqrt{C}}$$

Example. Solve the following:

$(x-1)^2 = 9$

$4x^2 - 1 = 0$

Definition.
The **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

gives the solutions to $ax^2 + bx + c = 0$.

Quadratic equations can have one, two, or no solutions. The **discriminant** is $b^2 - 4ac$:

- $b^2 - 4ac > 0$: The equation has *exactly* two distinct *real* solutions.
- $b^2 - 4ac = 0$: The equation has *exactly* one *real* solution.
- $b^2 - 4ac < 0$: The equation has no *real* solutions.

Example. Suppose some hooligans kick a ball up in the air off the roof of the library. Assuming the height, in ft, of the ball t seconds after kicking it is given by

$$h(t) = -32t^2 + 64t + 40$$

Solve for t when

the ball is 80 feet off of the ground

the ball is 72 feet off of the ground

the ball is 40 feet off of the ground

the ball hits the ground

Example. The Social Security Trust Fund balance B, in billions of dollars, can be described by the function $B = -7.97t^2 + 312t - 356$ where t is the number of years past the year 1995. For planning purposes, it is important to know when the trust fund balance will be 0. Solve

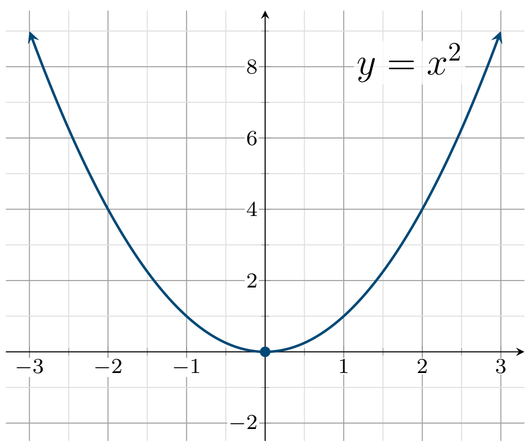
$$0 = -7.97t^2 + 312t - 356.$$

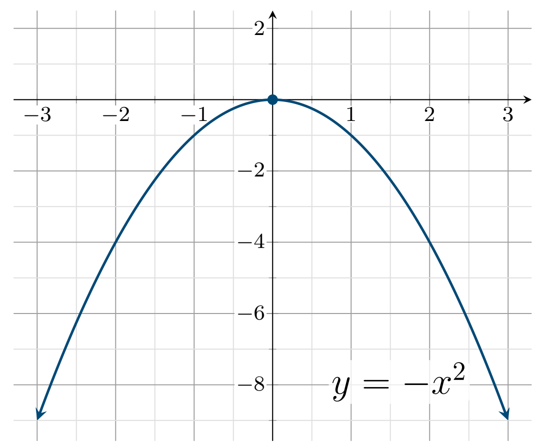
2.2: Quadratic Functions: Parabolas

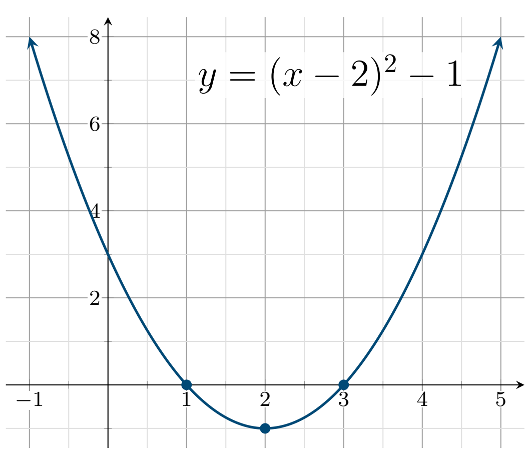
Definition.
A **quadratic function** has the form

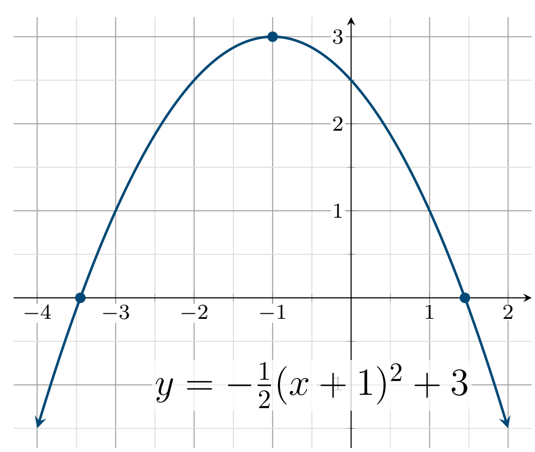
$$y = f(x) = ax^2 + bx + c \quad (a \neq 0)$$

where a, b, and c represent constants. A **parabola** is the shape of the graph of a quadratic function.









Definition.
The quadratic function $y = f(x) = ax^2 + bx + c$ has its **vertex** at

$$\left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right).$$

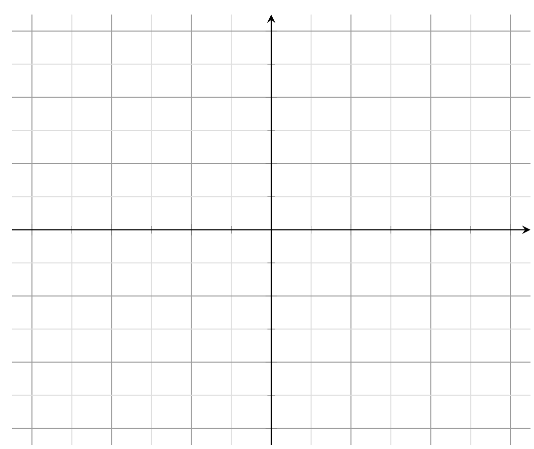
The optimal value occurs at the vertex of a parabola:

- A maximum if $a < 0$
- A minimum if $a > 0$

Example. Consider the function

$$2x + \frac{x^2}{2}$$

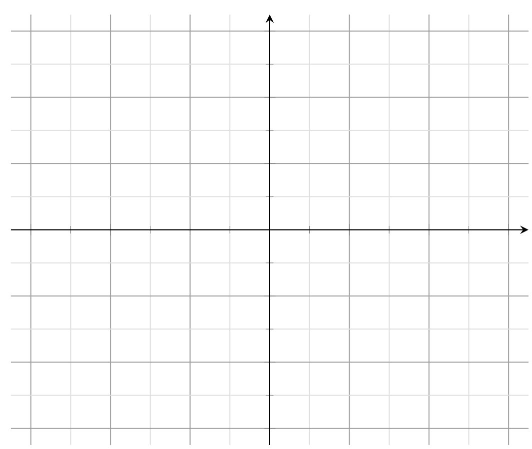
Is the vertex a maximum or minimum? Locate the vertex, x-intercepts, y-intercept, and then sketch the graph.



Example. Consider the function

$$x^2 + 5 - 4x$$

Is the vertex a maximum or minimum? Locate the vertex, x-intercepts, y-intercept, and then sketch the graph.



Example. Ace Cruises offers a sunset cruise to a group of 50 people for a price of \$30 per person, but it reduces the price per person by \$0.50 for each additional person above 50. Find the revenue function. What price maximizes the revenue? What is this maximal value?

2.3: Business Applications Using Quadratics

Recall the following:

Definition.
Profit is the difference between the revenue and total cost:

$$P(x) = R(x) - C(x)$$

where

$$\begin{aligned} P(x) &= \text{profit from sale of } x \text{ units,} \\ R(x) &= \text{total revenue from sale of } x \text{ units,} \\ C(x) &= \text{total cost from production and sale of } x \text{ units.} \end{aligned}$$

In general, **total revenue** is

$$\text{Revenue} = (\text{price per unit})(\text{number of units})$$

The **total cost** is composed of fixed cost and variable cost:

- **Fixed costs** (FC) remain constant regardless of the number of units produced.
- **Variable costs** (VC) are directly related to the number of units produced.

The total cost is given by

$$\text{Cost} = \text{variable costs} + \text{fixed costs}$$

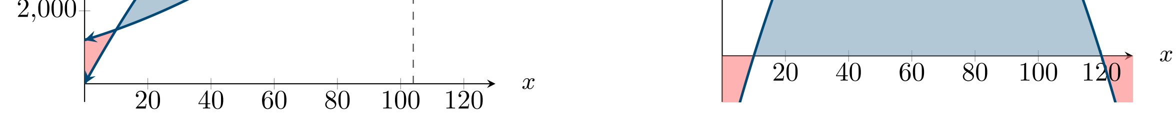
Example. Suppose that a company's cost include a fixed cost of $\$1,200$, and a variable cost per unit of $\frac{x}{4} + 18$ dollars, where x is the total number of units produced. If the selling price of their product is $(156 - \frac{3x}{4})$ dollars per unit, then

How many units should be sold to maximize the revenue?

Find the profit function.

How many units should be sold to maximize the profit?

Find the **break-even point** (e.g. where $R(x) = C(x)$ and $P(x) = 0$).



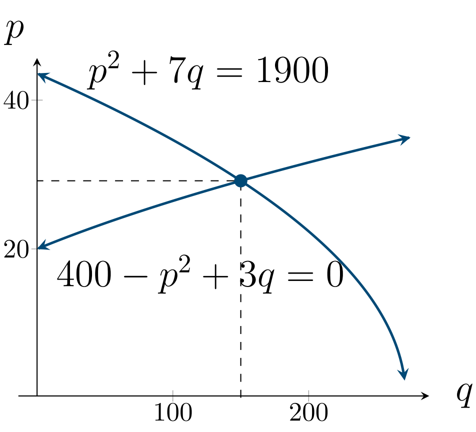
Example. Suppose that the demand function for a commodity is given by the equation

$$p^2 + 7q = 1900,$$

and the supply function is given by the equation

$$400 - p^2 + 3q = 0.$$

Find the **market equilibrium**



Example. If the supply and demand functions for a commodity are given by $p - q = 10$ and $q(2p - 10) = 2100$, what is the equilibrium price and what is the corresponding number of units supplied and demanded?