

6.4: Present Values of Annuities

Example. Suppose we wish to invest a lump sum of money, A_n , into an annuity that earns interest at a rate of 10% per year, so that we may receive payments of \$100 for 5 years. What is the amount of the lump sum?

$$1. \quad 100 = P(1+0.10)^1 \Rightarrow P = \frac{100}{(1+0.10)^1} = 90.91$$

$$100 = P(1+0.10)^2 \Rightarrow P = \frac{100}{(1+0.10)^2} = 82.64$$

$$100 = P(1+0.10)^3 \Rightarrow P = \frac{100}{(1+0.10)^3} = 75.13$$

$$100 = P(1+0.10)^4 \Rightarrow P = \frac{100}{(1+0.10)^4} = 68.36$$

$$100 = P(1+0.10)^5 \Rightarrow P = \frac{100}{(1+0.10)^5} = 62.09$$

$$+ 379.08$$

Definition.

If a payment of $\$R$ is to be withdrawn at the *end of each period* for n periods from an account that earns interest at a rate of i per period, then the account is an **ordinary annuity** and the **present value** is

$$A_n = R \cdot a_{\overline{n}|i} = R \left[\frac{1 - (1+i)^{-n}}{i} \right]$$

The notation $a_{\overline{n}|i}$ represents the present value of an ordinary annuity of \$1 per period for n periods with an interest rate of i per period.

Example. Find the lump sum that one must invest in an annuity to receive \$1000 at the end of each month for the next 16 years, if the annuity pays 9% compounded monthly.

A_n : ← Find

R : 1000

$$\frac{r}{m} = i: \frac{0.09}{12}$$

$$mt = n: 12 \cdot 16 = 192$$

$$A_n = 1000 \left[\frac{1 - \left(1 + \frac{0.09}{12}\right)^{-192}}{\left(\frac{0.09}{12}\right)} \right]$$

$$= \boxed{\$101,572.77}$$

Example. Suppose that a couple plans to set up an ordinary annuity with a \$100,000 inheritance they received. What is the size of the quarterly payments they will receive for the next 6 years if the account pays 7% compounded quarterly?

A_n : 100 000

R : ← Find

$$\frac{r}{m} = i: \frac{7\%}{4} = \frac{0.07}{4} = 0.0175$$

$$mt = n: 4(6) = 24$$

$$\frac{100,000}{\left[\frac{1 - (1.0175)^{-24}}{0.0175} \right]} = R \left[\frac{1 - (1.0175)^{-24}}{0.0175} \right]$$

$$R = \frac{100,000}{\left[\frac{1 - (1.0175)^{-24}}{0.0175} \right]}$$

$$= \boxed{\$5138.57}$$

Example. An inheritance of \$250,000 is invested at 9% compounded monthly. If \$2500 is withdrawn at the end of each month, how long will it be until the account balance is \$0?

Use $\ln(\dots)$ to find unknown exponent

1. Isolate
2. $\ln(\dots)$
3. expo
4. solve

$$A_n: 250,000$$

$$R: 2500$$

$$\frac{r}{m} = i: \frac{9\%}{12} = \frac{0.09}{12} = 0.0075$$

$$mt = n: 12t \leftarrow \begin{array}{c} \text{Find } t \\ \text{or} \\ \text{Find } n \end{array}$$

$$\left(\frac{0.0075}{2500} \right) 250,000 = 2500 \left[\frac{1 - (1.0075)^{-12t}}{0.0075} \right] \left(\frac{0.0075}{2500} \right)$$

$$0.75 - 1 = 1 - (1.0075)^{-12t} - 1$$

$$(-1)(-0.25) = (-1)(1.0075)^{-12t}(-1)$$

$$\ln(0.25) = \ln((1.0075)^{-12t})$$

$$\frac{\ln(0.25)}{-12 \ln(1.0075)} = \frac{-12t \ln(1.0075)}{-12 \ln(1.0075)}$$

$$t = \frac{\ln(0.25)}{-12 \ln(1.0075)} = 15.4610$$

$$n = 12t$$

$$\Rightarrow n = 185.53$$

There will be 185 regular withdrawals of \$2,500, and the last withdrawal will be less

Definition.

If a payment of \$ R is to be withdrawn at the *beginning of each period* for n periods from an account that earns interest at a rate of i per period, then the account is an **annuity due** and the **present value** is

$$A_{(n, \text{due})} = R \cdot a_{\overline{n}|i}(1+i) = R \left[\frac{1 - (1+i)^{-n}}{i} \right] (1+i)$$

The notation $a_{\overline{n}|i}$ represents the present value of an ordinary annuity of \$1 per period for n periods with an interest rate of i per period.

Example. Suppose that a court settlement results in a \$750,000 award. If this is invested at 9% compounded semiannually, how much will it provide at the *beginning* of each half-year for a period of 7 years?

$$A_{(n, \text{due})} : \$750,000$$

$$R : \leftarrow \text{Find}$$

$$\frac{r}{m} = i : \frac{9\%}{2} = 0.045$$

$$mt = n : 2(7) = 14$$

$$A_{(n, \text{due})} = R a_{\overline{n}|i}(1+i)$$

$$\Rightarrow R = \frac{A_{(n, \text{due})}}{a_{\overline{n}|i}(1+i)}$$

$$R = \frac{750,000}{\left[\frac{1 - 1.045^{-14}}{0.045} \right] (1.045)}$$

$$= \$70,205.97$$

Comparing annuity calculations:

	Ordinary annuities	Annuity due	
Future Value	S	S_{due}	Regular payments Total <i>after</i> n periods
Present Value	A_n	$A_{(n,\text{due})}$	Regular withdrawals Total <i>before</i> n periods
	end of each period	beginning of each period	