

3.3: The Chain Rule

Example. Let $f(x) = (x^3 + x + 1)^2$. Find $f'(x)$

using the product rule

$$f(x) = (x^3 + x + 1)(x^3 + x + 1)$$

$$\Rightarrow f'(x) = (x^3 + x + 1) \frac{d}{dx}[x^3 + x + 1] + \frac{d}{dx}[x^3 + x + 1] (x^3 + x + 1)$$

This is the
chain rule
answer

$$= (x^3 + x + 1)(3x^2 + 1) + (3x^2 + 1)(x^3 + x + 1)$$

$$\rightarrow = 2(x^3 + x + 1)(3x^2 + 1)$$

$$= 3x^2 \cdot 2(x^3 + x + 1) + 1 \cdot 2(x^3 + x + 1)$$

$$= 6x^5 + 6x^3 + 6x^2 + 2x^3 + 2x + 2$$

$$= \boxed{6x^5 + 8x^3 + 6x^2 + 2x + 2}$$

by expanding

$$f(x) = (x^3 + x + 1)(x^3 + x + 1) = x^3(x^3 + x + 1) + x(x^3 + x + 1) + (x^3 + x + 1)$$

$$= (x^6 + x^4 + x^3) + (x^4 + x^2 + x) + (x^3 + x + 1)$$

$$= x^6 + 2x^4 + 2x^3 + x^2 + 2x + 1$$

$$\Rightarrow f'(x) = \boxed{6x^5 + 8x^3 + 6x^2 + 2x + 2}$$

What about $\frac{d}{dx}[(x^3 + x + 1)^{100}]$?

Composite Functions:

Let f and g be functions of x . Then, the **composite functions** g of f (denoted $g \circ f$) and f of g (denoted $f \circ g$) are defined as:

$$(g \circ f)(x) = g(f(x))$$

$$(f \circ g)(x) = f(g(x))$$

Example. 'Break-down' the following composite functions:

$$\frac{1}{x+3}$$

$$f(x) = \frac{1}{x}$$

$$g(x) = x+3$$

$$\Rightarrow f(g(x)) = \frac{1}{g(x)} = \frac{1}{x+3}$$

$$(x^4 + 3x - 8)^3$$

$$f(x) = x^3$$

$$g(x) = x^4 + 3x - 8$$

$$f(g(x)) = (g(x))^3 = (x^4 + 3x - 8)^3$$

$$\left(\frac{1-x}{x^3+1}\right)^4$$

$$f(x) = x^4$$

$$g(x) = \frac{1-x}{x^3+1}$$

$$f(g(x)) = [g(x)]^4$$

$$= \left(\frac{1-x}{x^3+1}\right)^4$$

$$\frac{3}{\sqrt{(x+1)^2 - 1}}$$

$$f(x) = \frac{3}{x}$$

$$g(x) = \sqrt{(x+1)^2 - 1} \leftarrow$$

This is also
a composite
function!!

$$f(g(x)) = \frac{3}{g(x)}$$

$$= \frac{3}{\sqrt{(x+1)^2 - 1}}$$

Rule 7: The Chain Rule

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Note:

$$\frac{d}{dx} \left[f \left(g \left(h \left(j(x) \right) \right) \right) \right] = f' \left(g \left(h \left(j(x) \right) \right) \right) \cdot g' \left(h \left(j(x) \right) \right) \cdot h' \left(j(x) \right) \cdot j'(x)$$

The General Power Rule

$$\frac{d}{dx}[(f(x))^n] = n(f(x))^{n-1}f'(x)$$

Example. Use the chain rule to show

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\frac{d}{dx} \left[\frac{f}{g} \right] = \frac{d}{dx} \left[f \cdot [g]^{-1} \right] = f \frac{d}{dx} [g^{-1}] + \frac{d}{dx} [f] g^{-1}$$

$$= f(x) g^{-2} (-1)g' + f' g^{-1} = -\frac{f g'}{g^2} + \frac{f'}{g} \left(\frac{g}{g} \right) = \frac{f'g - fg'}{g^2}$$

Example. Find the derivative of the following functions

$$F(x) = (x^3 + x + 1)^{100}$$

$$F'(x) = 100 \underbrace{(x^3 + x + 1)^{99}}_{\text{chain rule}} (3x^2 + 1)$$

$$G(t) = (3x + 1)^2$$

$$G'(t) = 2 \underbrace{(3x + 1)}_{\text{chain rule}} \cdot 3 = \boxed{6(3x + 1)}$$

$$H(u) = \sqrt{u^2 + 1} - 3 = (u^2 + 1)^{1/2} - 3$$

$$H'(u) = \frac{1}{2} \underbrace{(u^2 + 1)^{-1/2}}_{\text{chain rule}} \cdot 2u = \frac{u}{(u^2 + 1)^{1/2}} = \boxed{\frac{u}{\sqrt{u^2 + 1}}}$$

$$J(v) = v^2(2v + 3)^5$$

$$\begin{aligned} J'(v) &= \frac{d}{dv} [v^2] (2v + 3)^5 + v^2 \frac{d}{dv} [(2v + 3)^5] \\ &= 2v(2v + 3)^5 + v^2 \underbrace{5(2v + 3)^4 \cdot 2}_{\text{chain rule}} \\ &= \boxed{2v(2v + 3)^5 + 10v^2(2v + 3)^4} \end{aligned}$$

$$\kappa(x) = (2x^2 + 3)^4 (3x - 1)^5$$

$$\kappa'(x) = \frac{d}{dx} [(2x^2 + 3)^4] (3x - 1)^5 + (2x^2 + 3)^4 \frac{d}{dx} [(3x - 1)^5]$$

$$= 4(2x^2 + 3) \cdot 2(3x - 1)^5 + (2x^2 + 3)^4 5(3x - 1)^4 \cdot 3$$

$$= 8(2x^2 + 3)(3x - 1)^5 + 15(2x^2 + 3)^4(3x - 1)^4$$

$$\tau(x) = \frac{1}{(4x^2 - 7)^2} = (4x^2 - 7)^{-2}$$

$$\tau'(x) = -2(4x^2 - 7)^{-3} \cdot 8x = \frac{-16x}{(4x^2 - 7)^3}$$

Example. Find the equation of the line tangent to $f(x)$ at $\left(0, \frac{1}{8}\right)$

$$f(x) = \left(\frac{2x+1}{3x+2}\right)^3$$

$$f'(x) = 3\left(\frac{2x+1}{3x+2}\right)^2 \frac{d}{dx} \left[\frac{2x+1}{3x+2}\right] = 3\left(\frac{2x+1}{3x+2}\right)^2 \frac{(3x+2) \cdot 2 - (2x+1) \cdot 3}{(3x+2)^2}$$

$$= 3\left(\frac{2x+1}{3x+2}\right)^2 \frac{1}{(3x+2)^2} = \frac{3(2x+1)^2}{(3x+2)^4}$$

$$y - y_1 = m(x - x_1)$$

\uparrow \uparrow \uparrow
 x_1 y_1 m

$f(0) = \frac{1}{8}$ $f'(0) = \frac{3}{16}$ 0

$$y - \frac{1}{8} = \frac{3}{16}(x - 0)$$

$$y = \frac{3}{16}x + \frac{1}{8}$$

Example. The membership of The Fitness Center, which opened a few years ago, is approximated by the function

$$N(t) = 100(64 + 4t)^{2/3} \quad (0 \leq t \leq 52)$$

where $N(t)$ gives the number of members at the beginning of week t .

Find $N'(t)$

$$N'(t) = \frac{200}{3} (64 + 4t)^{-1/3} \cdot 4 = \frac{800}{3(64 + 4t)^{1/3}}$$

How fast was the center's membership increasing initially ($t = 0$)?

$$N'(0) = \frac{800}{3(64 + 4(0))^{1/3}} = \frac{800}{12} = \frac{200}{3} \approx 66.6$$

About 67 new members per week

How fast was the membership increasing at the beginning of the 40th week?

$$N'(40) = \frac{800}{3(64 + 4(40))^{1/3}} \approx 50.876$$

About 51 new members per week

What was the membership when the center first opened? At the beginning of the 40th week?

$$N(0) = 100 (64 + 4(0))^{2/3} = 100 \cdot 16 = 1600$$

$$N(40) = 100 (64 + 4(40))^{2/3} \approx 3688.349$$

Rule 1: Derivative of a Constant

$$\frac{d}{dx}[c] = 0$$

Rule 2: The Power Rule

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

Rule 3: Derivative of a Constant Multiple of a Function

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

Rule 4: The Sum Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

Rule 5: The Product Rule

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Rule 6: The Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Rule 7: The Chain Rule

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$