6.5: Evaluating Definite Integrals

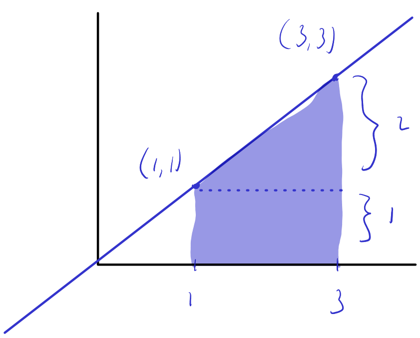
The Fundamental Theorem of Calculus
Let f be continuous on $[a,b]$. Then,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

where F is any antiderivative of f; that is, $F'(x) = f(x)$.

Example. Let R be the region under the graph of $f(x) = x$ on the interval $[1,3]$. Find the area of R Graph

using geometry



$A = \triangle + \square$

$= \frac{1}{2}(2)(2) + (1)\cdot 2$

$= \boxed{4}$

using the Fundamental Theorem of Calculus

$$A = \int_1^3 x\,dx = \frac{x^2}{2}\Big|_1^3 = \frac{(3)^2}{2} - \frac{(1)^2}{2} = \frac{9}{2} - \frac{1}{2} = \boxed{4}$$

Properties of the Definite Integral

Let f and g be integrable functions; then,

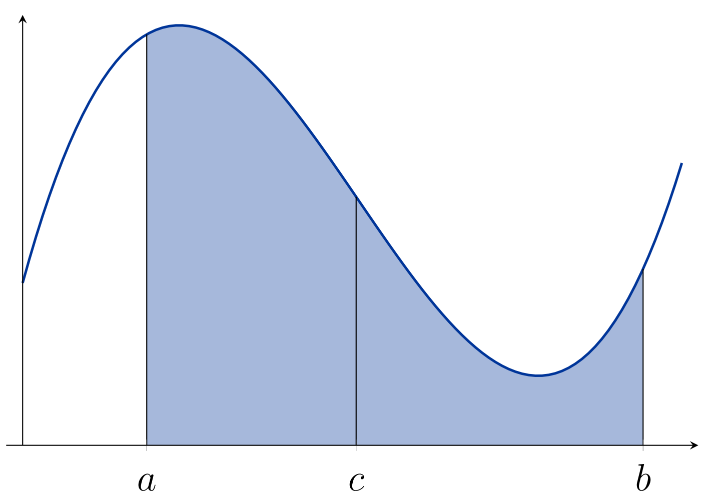
1. $\displaystyle\int_a^a f(x)\,dx = 0$

2. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ (c constant)

4. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ ($a < c < b$)



Example. Evaluate the following definite integrals

$$\int_0^4 x\sqrt{9+x^2}\,dx = \frac{1}{2}\int_9^{25} u^{1/2}\,du = \frac{1}{3}u^{3/2}\Big|_9^{25}$$

$u = 9 + x^2$

$du = 2x\,dx$

$\hookrightarrow \frac{1}{2}du = x\,dx$

$x = 4 \to u = 25$

$x = 0 \to u = 9$

$$= \frac{1}{3}(25)^{3/2} - \frac{1}{3}(9)^{3/2}$$

$$= \frac{125}{3} - \frac{27}{3}$$

$$= \boxed{\frac{98}{3}}$$

$$\int_0^2 xe^{2x^2}\,dx = \frac{1}{4}\int_0^8 e^u\,du = \frac{1}{4}e^u\Big|_0^8$$

$u = 2x^2$

$du = 4x\,dx$

$\hookrightarrow \frac{1}{4}du = x\,dx$

$x = 2 \to u = 8$

$x = 0 \to u = 0$

$$\boxed{\begin{aligned} &= \frac{1}{4}e^8 - \frac{1}{4}e^0 \\ &= \frac{1}{4}(e^8 - 1) \end{aligned}}$$

Example. Find the area of each region R described below: Graphs

Under $f(x) = \sqrt{x}$ from $x = 1$ to $x = 4$

$$A = \int_1^4 \sqrt{x}\, dx = \int_1^4 x^{1/2}\, dx = \frac{2}{3} x^{3/2} \Big|_1^4 = \frac{2}{3}(4)^{3/2} - \frac{2}{3}(1)^{3/2}$$

$$= \frac{2}{3}(8 - 1)$$

$$= \boxed{\frac{14}{3}}$$

Under $f(x) = e^{x/2}$ from $x = -1$ to $x = 1$

$$A = \int_{-1}^{1} e^{x/2}\, dx = 2\int_{-1/2}^{1/2} e^{u}\, du = 2e^{u}\Big|_{-1/2}^{1/2}$$

$$= \boxed{2\left(e^{1/2} - e^{-1/2}\right)}$$

$u = x/2$

$du = \frac{1}{2}\, dx$

$\hookrightarrow 2\, du = dx$

$x = 1, \quad u = 1/2$

$x = -1, \quad u = -1/2$