3.2: Predicates and Quantified Statements II

Definition.

- The negation of a statement of the form

$$\forall x \text{ in } D,\ Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \text{ in } D \text{ such that } \sim Q(x).$$

$$\sim (\forall x \in D,\ Q(x)) \equiv \exists x \in D \text{ such that } \sim Q(x).$$

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Example. Negate the following statements:

$\forall$ primes p, p is odd

$\exists$ prime p such that p is even

$\exists$ a triangle T such that the sum of the angles of T equals 200°

$\forall$ triangles T, the sum of the angles of T do NOT equal 200°.

Example. Rewrite the following statements formally, then write the formal and informal negations.

No politicians are honest

Formal: $\forall$ politicians P, P is not honest

Formal negation: $\exists$ a politician P such that P is honest

Informal negation: Some politicians are honest

The number $1,357$ is not divisible by any integer between 1 and 37.

Formal: $\forall$ integer n between 1 and 37, 1357 is NOT divisible by n

Formal negation: $\exists$ an integer n between 1 and 37, such that 1357 is divisible by n

Informal negation: The number 1357 is divisible by some integer between 1 and 37.

Example. Write informal negations for the following statements:

All computer programs are finite.

There is a computer program that is not finite.

Some computer hackers are over 40.

No computer hackers are over 40.

Negation of a Universal Conditional Statement

$$\sim (\forall x,\ \text{if } P(x) \text{ then } Q(x)) \equiv \exists x \text{ such that } P(x) \text{ and } \sim Q(x)$$

Definition.
A statement of the form

$$\forall x \text{ in } D,\ \text{if } P(x) \text{ then } Q(x)$$

is called **vacuously true** or **true by default** if, and only if, $P(x)$ is false for every x in D.

Example. The following statement is vacuously true since it's negation is false:

All kangaroos enrolled in my class are passing.

Definition.
Consider a statement of the form $\forall x \in D$, if $P(x)$ then $Q(x)$.

1. Its **contrapositive** is the statement $\quad \forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
2. Its **converse** is the statement $\quad \forall x \in D$, if $Q(x)$ then $P(x)$.
3. Its **inverse** is the statement $\quad \forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

Example. Write a formal and informal contrapositive, converse, and inverse for the following statement:

If a real number is greater than 2, then its square is greater than 4.

Formal: $\forall x \in \mathbb{R}$, if $x > 2$, then $x^2 > 4$

Formal contrapositive: $\forall x \in \mathbb{R}$, if $x^2 < 4$, then $x < 2$

Informal contrapositive: If the square of a real number is less than 4, then the number is less than 2.

Formal converse: $\forall x \in \mathbb{R}$, if $x^2 > 4$, then $x > 2$

Informal converse: If the square of a real number is greater than 4, then the number is greater than 2.

Formal inverse: $\forall x \in \mathbb{R}$, if $x \leq 2$, then $x^2 < 4$

Informal inverse: If a real number is less than or equal to 2, then the square of the number is less than or equal to 4.

Definition.

- "$\forall x, r(x)$ is a **sufficient condition** for $s(x)$" $\rightarrow$ "$\forall x$, if $r(x)$ then $s(x)$"
- "$\forall x, r(x)$ is a **necessary condition** for $s(x)$" $\rightarrow$ "$\forall x$, if $\sim r(x)$ then $\sim s(x)$"
  $\rightarrow$ "$\forall x$, if $s(x)$ then $r(x)$"
- "$\forall x, r(x)$ **only if** $s(x)$" $\rightarrow$ "$\forall x$, if $\sim s(x)$, then $\sim r(x)$"
  $\rightarrow$ "$\forall x$, if $r(x)$ then $s(x)$"

Example. Rewrite each of the following as a universal conditional statement, quantified either explicitly or implicitly. Do not use the word *necessary* or *sufficient.*

Squareness is a sufficient condition for rectangularity.

$\forall$ x, if x is square, then x is a rectangle

Being at least 35 years old is a necessary condition for being president of the United States.

$\forall$ person x, if x is younger than 35, then x cannot be POTUS

Example. Rewrite the following as a universal conditional statement:

A product of two numbers is 0 only if one of the numbers is 0.

$\forall x \in \mathbb{R}, y \in \mathbb{R}$, if $xy = 0$, then $x = 0$ or $y = 0$.