5.2: Logarithmic Functions and Their Properties

Definition.

For $a > 0$ and $a \neq 1$, the **logarithmic function**

$$y = \log_a(x) \qquad \text{(logarithmic form)}$$

has domain $x > 0$, base a, and is defined by

$$a^y = x \qquad \text{(exponential form)}$$

Example. Rewrite the following in exponential form

$4 = \log_2(16)$

$5 = \log_{10}(100,000)$

$\frac{1}{2} = \log_{16}(4)$

$-4 = \log_3\left(\frac{1}{81}\right)$

$-\frac{1}{4} = \log_{625}\left(\frac{1}{5}\right)$

$-\frac{5}{3} = \log_{\frac{1}{8}}(32)$

Example. Simplify the following:

$\log_3(9)$

$\log_4(2)$

Example. Solve the following:

$\log_5(x) = 4$

$\log_8(x) = 1$

$\log_{81}(x) = -\frac{1}{4}$

$\log_{10}(x+4) = 3$

Common logarithms:	$\log(x) = \log_{10}(x)$
Natural logarithms:	$\ln(x) = \log_e(x)$

Example (The Rule of 70)**.** If $\$P$ is invested for t years at interest rate r, compounded continuously, then the future value of the investment is given by

$$S = Pe^{rt}.$$

Find the value of t when the investment doubles.

Change of base formula:
If $a > 0$, $b > 0$ with $a \neq 1$ and $b \neq 1$, then

$$\log_b(x) = \frac{\log_a(x)}{\log_a(x)}.$$

Note: This works for any valid base!

$$\text{Base } e: \quad \log_b(x) = \frac{\ln(x)}{\ln(b)} \qquad \text{Base } 10: \quad \log_b(x) = \frac{\log(x)}{\log(b)}$$

Example. Solve the following

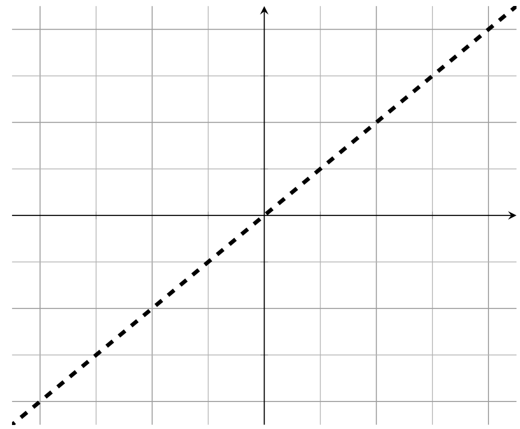
$3^x = 10$

$6.5^x = 5$

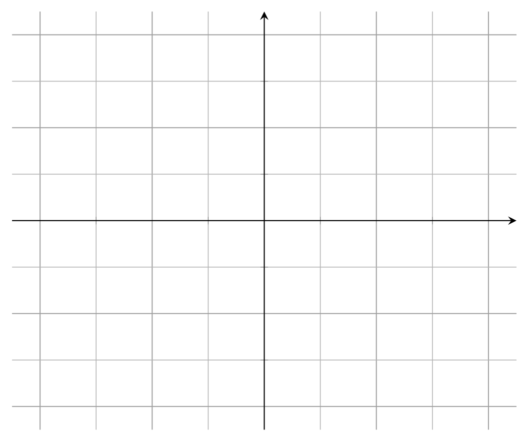
Example. Fill in the tables below and graph a^x and $\log_a(x)$ on the same axes.

x	$y = a^x$
-2	
-1	
0	
1	
2	

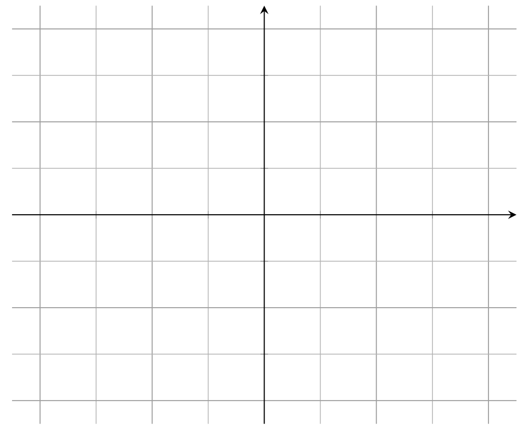
x	$y = \log_a(x)$
	-2
	-1
	0
	1
	2



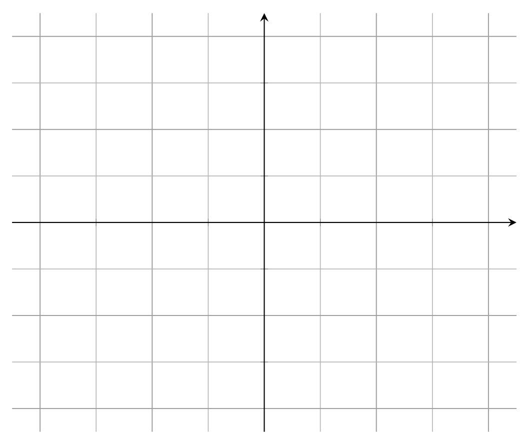
Example. Graph $\log(-x)$



Example. Graph $\ln(x)$



Example. Graph $-\log_2(-x)$



Example. Evaluate the following:

$f(x) = \ln(x); \quad f\left(e^{-3x}\right)$

$f(x) = 5^x; \quad f(\log_5(10))$

Properties of exponents and logarithms: Assume $a > 0$:

$$\boxed{\begin{array}{cc}
a^y = x & \log_a(x) = y \\[1em]
a^1 = a & \log_a(a) = 1 \\[1em]
a^0 = 1 & \log_a(1) = 0 \\[1em]
a^x a^y = a^{x+y} & \log_a(xy) = \log_a(x) + \log_a(y) \\[1em]
\dfrac{a^x}{a^y} = a^{x-y} & \log_a\left(\dfrac{x}{y}\right) = \log_a(x) - \log_a(y) \\[1em]
a^{xy} = (a^x)^y & \log_a(x^y) = y\log_a(x) \\[1em]
a^{\log_a(x)} = x & \log_a(a^x) = x
\end{array}}$$