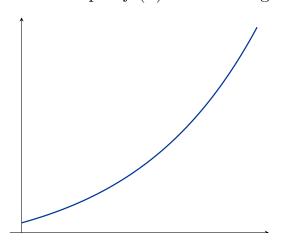
4.2: Applications of the Second Derivative

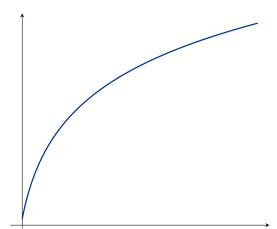
Definition.

Consider any differentiable function f(x) on the interval (a, b). We say f is

concave up if f'(x) is increasing

concave down if f'(x) is decreasing





Thus, for every value of x on the interval (a, b), if

- -f''(x) > 0, then f' is increasing, and f is concave up on (a, b).
- -f''(x) < 0, then f' is decreasing, and f is concave down on (a, b).
- If f is continuous at c and f changes concavity at c, then f has an **inflection** point at c.

Note: f(x) is

- concave up if its tangent lines lie below the curve
- concave down if its tangent lines lie above the curve



Determining the Intervals of Concavity of the Graph of f

- 1. Determine the values of x for which f'' is zero or undefined.
- 2. Determine the sign of f''(x) to the left and right of each point from above: Let c be a convenient test point on the interval of interest. Then,
 - a) if f''(c) > 0, then f is concave up on that interval.
 - b) if f''(c) < 0, then f is concave down on that interval.

Example. Find the intervals where the following functions are concave up and concave down:

$$f(x) = x^3 - 3x^2 - 24x + 32$$

Graph

$$g(x) = (x+1)^{2/3}$$

$$h(x) = x + \frac{1}{x}$$

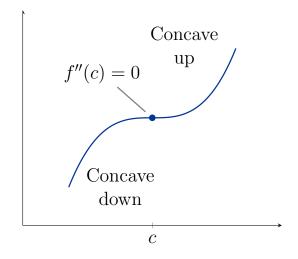
$$j(x) = \frac{x^2}{1 - x^2}$$

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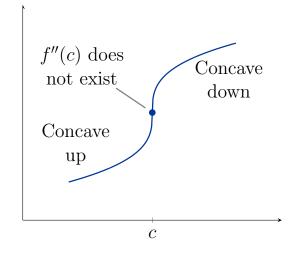
Finding inflection points

- 1. Compute f''(x).
- 2. Locate where f''(x) = 0 or f''(x) does not exist.
- 3. Determine if the sign of f''(x) changes at the points found above.

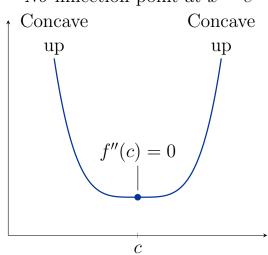
Inflection point at x = c



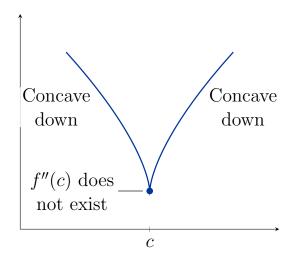
Inflection point at x = c



No inflection point at x = c



No inflection point at x = c



Example. For the following functions, determine the intervals of concavity and find any inflection points.

$$f(x) = (x-1)^{5/3}$$
 Graph

$$g(x) = \frac{1}{x^2 + 1}$$

Second Derivative Test for Local Extrema

Suppose f'' is continuous on an open interval containing c with f'(c) = 0.

- If f''(c) > 0, then f has a local minimum at c.
- If f''(c) < 0, then f has a local maximum at c.
- If f''(c) = 0, then the test is inconclusive; f may have a local maximum, local minimum, or neither at c.

Example. Find the relative extrema of

$$f(x) = x^3 - 3x^2 - 24x + 32$$

Graph

f(x)	f'(x)	f''(x)
increasing	positive	_
decreasing	negative	_
\max/\min	crit. pt. & changes sign	_
concave up	increasing	positive
concave down	decreasing	negative
Inflection point	\max/\min	crit. pt. & changes sign