8.1: Relations on Sets

Definition.
A relation R from A to B is called a **binary relation** because it is a subset of a Cartesian product of two sets.

Example. Define a relation L from $\mathbb{R}$ to $\mathbb{R}$:

$$\forall x, y \in \mathbb{R},\ x\, L\, y \Leftrightarrow x < y.$$

Is $57\, L\, 53$?

No

$57 \not< 53$

Is $(-17)\, L\, (-14)$?

Yes

$-17 < -14$

Is $143\, L\, 143$?

No

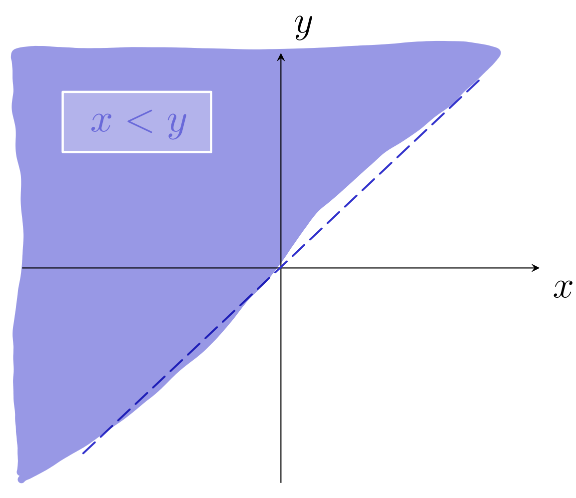
$143 \not< 143$

Is $(-35)\, L\, 1$?

Yes

$-35 < 1$

Draw the graph of L as a subset of the Cartesian plane $\mathbb{R} \times \mathbb{R}$.



Definition.
Two integers m and n are **congruent modulo** 2 if, and only if, $m \bmod 2 = n \bmod 2$.

Example. Define a relation E from $\mathbb{Z}$ to $\mathbb{Z}$:

$$\forall (m,n) \in \mathbb{Z} \times \mathbb{Z},\ m\,E\,n \Leftrightarrow m - n \text{ is even.}$$

Is $4\,E\,0$? Is $2\,E\,6$? Is $3\,E\,(-3)$? Is $5\,E\,2$?

Yes Yes Yes No

$5-2=3$ is NOT even

Prove that if n is any odd integer, then $n\,E\,1$.

n odd $\Rightarrow n = 2k+1,\ k \in \mathbb{Z}$

$\Rightarrow (n-1) = (2k+1) - 1 = 2k$ is even

$\Rightarrow n\,E\,1$

Example. Let $X = \{a, b, c\}$. Define a relation $\mathbf{S}$ from $\mathscr{P}(X)$ to $\mathscr{P}(X)$ as follows:

$$\forall A, B \in \mathscr{P}(X),\ A\,\mathbf{S}\,B \Leftrightarrow A \text{ has at least as many elements as } B.$$

Is $\{a,b\}\,\mathbf{S}\,\{b,c\}$?

2 elements 2 elements

Yes

Is $\{a\}\,\mathbf{S}\,\varnothing$?

1 element 0 elements

Yes

Is $\{b,c\}\,\mathbf{S}\,\{a,b,c\}$?

2 elements 3 elements

No

$(2 \not\geq 3)$

Is $\{c\}\,\mathbf{S}\,\{a\}$?

1 element 1 element

Yes

Definition.
Let R be a relation from A to B. Define the inverse relation R^{-1} from B to A as follows:

$$R^{-1} = \{(y, x) \in B \times A \,|\, (x, y) \in R\}.$$

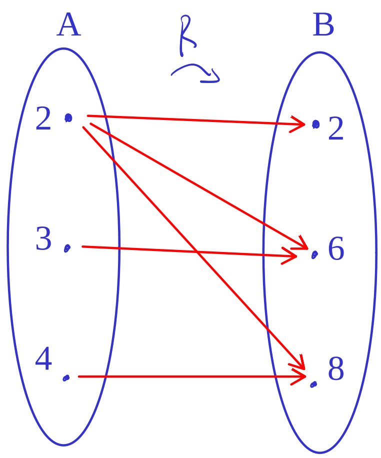
Example. Let $A = \{2, 3, 4\}$ and $B = \{2, 6, 8\}$, and let R be the "divides" relation from A to B:

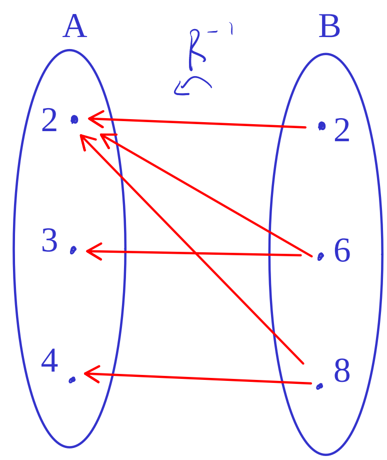
$$\forall (x, y) \in A \times B,\ x\,R\,y \Leftrightarrow x \mid y$$

Explicitly state which ordered pairs are in R and R^{-1}. Draw arrow diagrams for both.

$R = \{(2,2), (2,6), (2,8), (3,6), (4,8)\}$

$R^{-1} = \{(2,2), (6,2), (8,2), (6,3), (8,4)\}$





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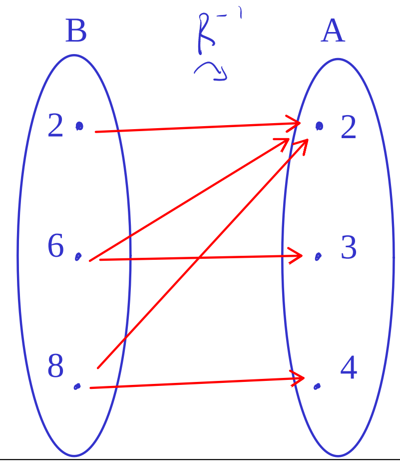
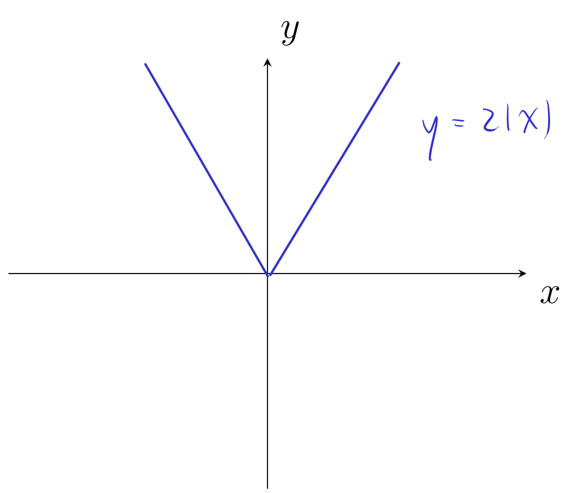
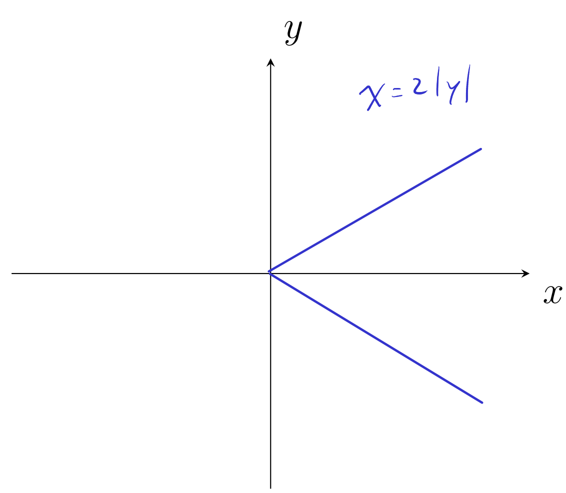


Example. Define a relation R from $\mathbb{R}$ to $\mathbb{R}$ as follows:

$$\forall (x, y) \in \mathbb{R} \times \mathbb{R},\ x\, R\, y \Leftrightarrow y = 2|x|.$$

Draw the graphs of R and R^{-1} in the Cartesian plane. Is R^{-1} a function?



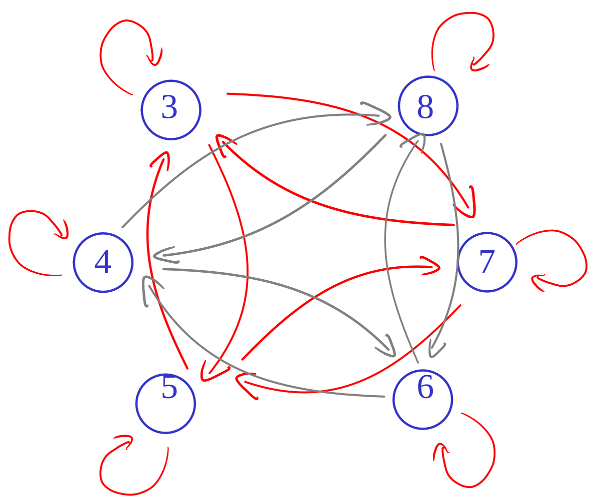


> **Definition.**
> A **relation on a set** A is a relation from A to A.
> A **graph** G consists of two finite sets:
>
> - a nonempty set $V(G)$ of **vertices** and
> - a set $E(G)$ of **edges**,
>
> where each edge is associated with a set consisting of either one or two vertices called its endpoints.
> A **directed graph** is a graph whose edges are directional.

Example. Let $A = \{3, 4, 5, 6, 7, 8\}$ and define a relation R on A as follows

$$\forall x, y \in A,\ \ x\,R\,y \Leftrightarrow 2 \mid (x - y).$$

Draw the directed graph of R.



Definition.
Given sets $A_1, A_2, \ldots, A_n$, an **n-ary relation** R on $A_1 \times A_2 \times \cdots \times A_n$ is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The special case of 2-ary, 3-ary, and 4-ary relations are called **binary**, **ternary**, and **quaternary relations**, respectively.