6.2: Integration by Substitution

Example. Find the derivative of the following functions:

$$f(x) = \frac{(2x+1)^4}{4}$$

$$f'(x) = \frac{4(2x+1)^3}{4} \cdot 2 = \boxed{2(2x+1)^3}$$

$$g(x) = \frac{1}{x+3} = (x+3)^{-1}$$

$$g'(x) = -1(x+3)^{-2} \cdot 1 = \boxed{\frac{-1}{(x+3)^2}}$$

Let $u = g(x)$, where g is differentiable on an interval, and let f be continuous on the corresponding range of g. On that interval,

$$\int f'\big(g(x)\big) \cdot g'(x)\,dx = \int f(u)\,du$$

Procedure: Substitution Rule (Change of Variables)

1. Let $u = g(x)$, where $g(x)$ is part of the integrand, usually the "inside function" of a composite function $f\big(g(x)\big)$.
2. Find $du = g'(x)\,dx$.
3. Use the substitution $u = g(x)$ and $du = g'(x)\,dx$ to convert the *entire* integral into one involving only u.
4. Find the resulting integral
5. Replace u by $g(x)$ to obtain the final solution as a function of x.

Example. Evaluate the following integrals:

$$\int 2x(x^2+3)^4\,dx = \int u^4\,du$$

$u = x^2+3$

$du = 2x\,dx$

$$= \frac{u^5}{5} + C$$

$$= \boxed{\frac{1}{5}(x^2+3)^5 + C}$$

$$\int (2x+1)^3\,dx = \int u^3 \cdot \frac{1}{2}\,du$$

$u = 2x+1$

$du = 2\,dx$

$\downarrow$

$\frac{1}{2}\,du = dx$

$$= \frac{u^4}{4}\cdot\frac{1}{2} + C$$

$$= \boxed{\frac{(2x+1)^4}{8} + C}$$

$$\int x^2\sqrt{x^3+1}\,dx = \int u^{1/2}\cdot\frac{1}{3}\,du$$

$u = x^3+1$

$du = 3x^2\,dx$

$\frac{1}{3}\,du = x^2\,dx$

$$= \frac{u^{3/2}}{3/2}\cdot\frac{1}{3}+C$$

$$= \boxed{\frac{2\left(x^3+1\right)^{3/2}}{9}+C}$$

$$\int t\sqrt[4]{1-t^2}\,dt = -\frac{1}{2}\int u^{1/2}\,du$$

$u = 1-t^2$

$du = -2t\,dt$

$-\frac{1}{2}\,du = t\,dt$

$$= -\frac{1}{2}\,\frac{u^{3/2}}{3/2}+C$$

$$= \boxed{-\frac{1}{3}\left(1-t^2\right)^{3/2}+C}$$

$$\int\sqrt{4-t}\,dt = -\int u^{1/2}\,du$$

$u = 4-t$

$du = -dt$

$$= -\frac{u^{3/2}}{3/2}+C$$

$$= \boxed{-\frac{2}{3}(4-t)^{3/2}+C}$$

$$\int(2-x)^6\,dx = -\int u^6\,dx$$

$u = 2-x$

$du = -dx$

$$= -\frac{u^7}{7}+C$$

$$= \boxed{\frac{-(2-x)^7}{7}+C}$$

$$\int e^{-3x}\,dx = -\frac{1}{3}\int e^{u}\,du$$

$u = -3x$

$du = -3\,dx$

$$= -\frac{1}{3}e^{u} + C$$

$$= \boxed{-\frac{1}{3}e^{-3x} + C}$$

$$\int \frac{t}{3t^2+1}\,dt = \frac{1}{6}\int \frac{1}{u}\,du$$

$u = 3t^2+1$

$du = 6t\,dt$

$$= \frac{1}{6}\ln|u| + C$$

$$= \boxed{\frac{1}{6}\ln(3t^2+1) + C}$$

$$\int \frac{(\ln(x))^2}{2x}\,dx = \frac{1}{2}\int (\ln(x))^2 \cdot \frac{1}{x}\,dx$$

$u = \ln(x)$

$du = \frac{1}{x}\,dx$

$$= \frac{1}{2}\int u^2\,du$$

$$= \frac{1}{2}\,\frac{u^3}{3} + C$$

$$= \boxed{\frac{1}{6}(\ln(x))^3 + C}$$

$$\int u^3\left(u^2+1\right)^{3/2}\,du = \int u^2(u^2+1)^{3/2}\,u\,du$$

$v = u^2+1$

$dv = 2u\,du$

$v-1 = u^2$

$$= \int (v-1)\,v^{3/2}\,dv$$

$$= \int v^{5/2} - v^{3/2}\,dv$$

$$= \frac{v^{7/2}}{7/2} - \frac{v^{5/2}}{5/2} + C$$

$$= \boxed{\frac{2(u^2+1)^{7/2}}{7} - \frac{2(u^2+1)^{5/2}}{5} + C}$$