2.2: Conditional Statements

Definition.
If p and q are statement variables, the **conditional** of q by p is "If p then q", or "p implies q" and is denoted by $p \to q$. It is false when p is true and q is false; otherwise it is true. We call p the **hypothesis** (or **antecedent**) of the conditional and q the **conclusion** (or **consequent**).
A conditional statement that is always true because the hypothesis is false is called **vacuously true**.

$$\text{If } \underbrace{4,686 \text{ is divisible by } 6,}_{\text{hypothesis}} \text{ then } \underbrace{4,686 \text{ is divisible by } 3}_{\text{conclusion}}$$

Example. Consider the following statement:

If Lander is open, then we will have class.

Create the truth table for $p \to q$

p	q	$p \to q$
T	T	
T	F	
F	T	
F	F	

Note: The **order of operations** states that $\to$ is performed last

Example. Create the truth table for $p \vee \sim q \to \sim p$.

p	q	$\sim q$	$p \vee \sim q$	$\sim p$	$p \vee \sim q \to \sim p$
T	T				
T	F				
F	T				
F	F				

Example. Use a truth table to show that $p \vee q \to r \equiv (p \to r) \wedge (q \to r)$

p	q	r	$p \vee q$	$p \vee q \to r$	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

Definition.
The **negation** of "if p then q" is logically equivalent to "p and not q":

$$\sim (p \to q) \equiv p \wedge \sim q$$

Example. Write negations for each of the following statements:

If my car is in the repair shop, then I cannot get to class.

If Sara lives in Athens, then she lives in Greece.

Definition.
The **contrapositive** of a conditional statement of the form "If p then q" is

$$\text{If } \sim q \text{ then } \sim p : \quad \sim q \to \sim p$$

A conditional statement is logically equivalent to its contrapositive.

Example. Write each of the following statements in its equivalent contrapositive form:

If Howard can swim across the lake, then Howard can swim to the island.

If today is Easter, then tomorrow is Monday.

Definition.
Suppose a conditional statement of the form "If p then q" is given.

- The **converse** is "If q then p": $q \to p$
- The **inverse** is "If $\sim p$ then $\sim q$": $\sim p \to \sim q$

Example. Write the converse and inverse of each of the following statements:

If Howard can swim across the lake, then Howard can swim to the island.
Converse:

Inverse:

If today is Easter, then tomorrow is Monday.
Converse:

Inverse:

Note:

1. A conditional statement and its converse are *not* logically equivalent.
2. A conditional statement and its inverse are *not* logically equivalent.
3. The converse and the inverse of a conditional statement are logically equivalent to each other.

Definition.
If p and q are statements, p **only if** q means "if not q then not p":

$$\sim q \to \sim p \equiv p \to q$$

Example. Rewrite the following statement in if-then form in two ways, one of which is the contrapositive of the other:

> John will break the world's record for the mile run only if he runs the mile in under four minutes.

$\sim q \to \sim p$

$p \to q$

Note:

1. "p only if q" does *not* mean p if q
2. It is possible for "p only if q" to be true at the same time that "p if q" is false.

   e.g.: If John runs a mile in under four minutes, he still might not be fast enough to break the record.

Definition.
Given statement variables p and q, the **biconditional of p and q** is "p if, and only if, q" and is denoted $p \leftrightarrow q$. It is true if both p and q have the same truth values and is false otherwise. The words *if and only if* are sometimes abbreviated **iff**.

Note: The **order of operations** states that $\leftrightarrow$ is coequal with $\rightarrow$

Example. Create the truth table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	
T	F	
F	T	
F	F	

Order of Operations for Logical Operators

$\sim$	Evaluate negations first
$\wedge, \vee$	Evaluate $\wedge$ and $\vee$ second. When both present, parentheses may be needed.
$\rightarrow, \leftrightarrow$	Evaluate $\rightarrow$ and $\leftrightarrow$ third. When both present, parentheses may be needed.

Definition.
If r and s are statements:

1. r is a **sufficient condition** for s means "if r then s". $\quad r \to s$

2. r is a **necessary condition** for s means "if not r then not s". $\quad \sim r \to \sim s$

By property of the contrapositive:

3. r is a *necessary and sufficient condition* for s means "r if, and only if s."
$r \leftrightarrow s$

Example. Rewrite the following statement in the form "If A then B":

> Having two $45°$ angles is a sufficient condition for this triangle to be a right triangle.

Example. Use the contrapositive to rewrite the following statement in two ways:

> George's attaining age 35 is a necessary condition for his being president of the United States.