

5.5: Differentiation of Logarithmic Functions

Definition. (Natural log)

The inverse of the exponential function is the logarithm:

$$e^y = x \longleftrightarrow \ln(x) = y, \quad x > 0$$

We may also denote this as $\log_e(x)$

Example. Solve the following exponential equations

[Graphs](#)

$$y = e^{-3}$$

$$y \approx 0.0498$$

$$e^{x^2-x} = e^2$$

$$\text{Same base} \Rightarrow x^2 - x = 2$$

$$x^2 - x - 2 = 0$$

$$(x+1)(x-2) = 0$$

$$x = -1 \quad x = 2$$

$$e^x = 10$$

$$\ln(10) = x \approx 2.303$$

$$4e^{1-x^2} = 6$$

$$e^{1-x^2} = \frac{3}{2}$$

$$\ln\left(\frac{3}{2}\right) = 1 - x^2$$

$$x^2 = 1 - \ln\left(\frac{3}{2}\right)$$

$$x = \pm \sqrt{1 - \ln\left(\frac{3}{2}\right)} \approx \pm 0.771$$

Rule 3: Derivative of $\ln(x)$

$$\frac{d}{dx} [\ln|x|] = \frac{1}{x}, \quad x \neq 0$$

Rule 4: Derivative of $\ln(f(x))$

$$\frac{d}{dx} [\ln(f(x))] = \frac{f'(x)}{f(x)}$$

Example. Find the derivative of the following functions

$$y = \ln(\sqrt[3]{x}) = \ln(x^{1/3})$$

$$y' = \frac{1}{x^{1/3}} \cdot \frac{1}{3} x^{-2/3} = \boxed{\frac{1}{3x}}$$

$$f(x) = \sqrt{x} \ln(x) = x^{1/2} \ln(x)$$

$$f'(x) = \frac{1}{2} x^{-1/2} \ln(x) + x^{1/2} \cdot \frac{1}{x}$$

$$= \boxed{\frac{\ln(x)}{2\sqrt{x}} + \sqrt{x}}$$

$$g(x) = \ln(x^2 + 1)$$

$$g'(x) = \boxed{\frac{2x}{x^2 + 1}}$$

$$h(t) = \frac{1 + \ln(t)}{1 - \ln(t)}$$

$$h'(t) = \frac{(1 - \ln(t)) \cdot \frac{1}{t} - (1 + \ln(t)) \cdot \left(-\frac{1}{t}\right)}{(1 - \ln(t))^2}$$

$$= \boxed{\frac{2}{t(1 - \ln(t))^2}}$$

Additional properties of logarithms

$$e^y = x$$

$$\ln(x) = y$$

$$e^1 = a$$

$$\ln(a) = 1$$

$$e^0 = 1$$

$$\ln(1) = 0$$

$$e^x e^y = e^{x+y}$$

$$\ln(xy) = \ln(x) + \ln(y)$$

$$\frac{e^x}{e^y} = e^{x-y}$$

$$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$

$$e^{xy} = (e^x)^y$$

$$\ln(x^y) = y \ln(x)$$

$$e^{\ln(x)} = x$$

$$\ln(e^x) = x$$

Definition. (Other logarithms)

Logarithms are defined for any base $a > 0$:

$$a^y = x \iff \log_a(x) = y, \quad x > 0$$

Example. Find the derivative of the following functions

$$y = \ln(\sqrt[3]{x}) = \ln(x^{1/3}) = \frac{1}{3} \ln(x)$$

$$y' = \frac{1}{3} \cdot \frac{1}{x} = \boxed{\frac{1}{3x}}$$

$$f(x) = \ln((x^2 + 1)(x^3 + 2)^6)$$

$$= \ln(x^2 + 1) + 6 \ln(x^3 + 2)$$

$$f'(x) = \boxed{\frac{2x}{x^2 + 1} + \frac{6 \cdot 3x^2}{x^3 + 2}}$$

$$g(x) = \ln\left(\frac{1-x}{1+x}\right) = \ln(1-x) - \ln(1+x)$$

$$g'(x) = \boxed{\frac{-1}{1-x} - \frac{1}{1+x}}$$

$$h(x) = \ln(x + \sqrt{x^2 - 1})$$

$$= \ln(x + (x^2 - 1)^{1/2})$$

$$h'(x) = \frac{1 + \frac{1}{2}(x^2 - 1)^{-1/2} \cdot (2x)}{x + (x^2 - 1)^{1/2}}$$

$$= \boxed{\frac{1 + \frac{x}{\sqrt{x^2 - 1}}}{x + \sqrt{x^2 - 1}}}$$

$$j(t) = 2(\ln(5t))^{3/2}$$

$$= 2(\ln(5) + \ln(t))^{3/2}$$

$$j'(t) = \boxed{3(\ln(5) + \ln(t))^{1/2} \cdot \frac{1}{t}}$$

$$k(u) = \ln(\sqrt{u^2 - 4})$$

$$= \ln((u^2 - 4)^{1/2})$$

$$k'(u) = \frac{1}{2} \cdot \frac{1}{u^2 - 4} \cdot 2u = \boxed{\frac{u}{u^2 - 4}}$$

Logarithmic Differentiation

Using properties of logs, we can transform a function before taking its derivative:

1. Take the natural log of both sides rewriting products and quotients as sums and differences.
2. Differentiate both sides.
3. Solve for $\frac{dy}{dx}$

Example. Differentiate

$$y = x^2(x-1)(x^2+4)^3$$

$$\ln(y) = 2 \ln(x) + \ln(x-1) + 3 \ln(x^2+4)$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{2}{x} + \frac{1}{x-1} + \frac{3 \cdot 2x}{x^2+4}$$

$$\frac{dy}{dx} = y \left(\frac{2}{x} + \frac{1}{x-1} + \frac{6x}{x^2+4} \right)$$

$$\frac{dy}{dx} = x^2(x-1)(x^2+4)^3 \left(\frac{2}{x} + \frac{1}{x-1} + \frac{6x}{x^2+4} \right)$$