1.3: The Language of Relations and Functions

Definition.
Let A and B be sets. A **relation** R **from** A **to** B is a subset of $A \times B$. Given an ordered pair (x, y), x **is related to** y **by** R, written $x \, R \, y$, if, and only if, (x, y) is in R. The set A is called the **domain** of R and the set B is called its **co-domain**.
The notation for a relation R may be written symbolically as follows:

$$x \, R \, y \text{ means that } (x, y) \in R.$$

The notation $x \not R \, y$ means that x is not related to y by R:

$$x \not R \, y \text{ means that } (x, y) \notin R.$$

Example. Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ and define a relation R from A to B as follows; Given any $(x, y) \in A \times B$,

$$(x, y) \in R \text{ means that } \frac{x - y}{2} \text{ is an integer.}$$

State explicitly which ordered pairs are in $A \times B$ and which are in R

Is $1 \, R \, 3$? Is $2 \, R \, 3$? Is $2 \, R \, 2$?

What are the domain and co-domain of R?

Example. Define a relation C from $\mathbb{R}$ to $\mathbb{R}$ as follows: For any $(x, y) \in \mathbb{R} \times \mathbb{R}$,

$$(x, y) \in C \text{ means that } x^2 + y^2 = 1.$$

Is $(1, 0) \in C$? Is $(0, 0) \in C$? Is $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \in C$?

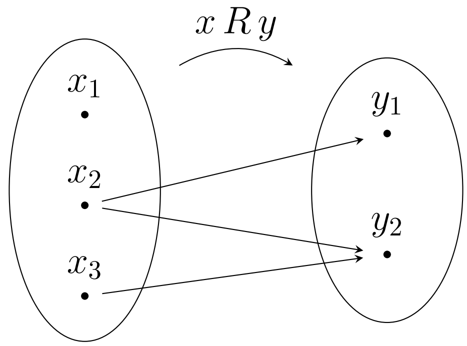
Is $-2\, C\, 0$? Is $0\, C\, (-1)$? Is $1\, C\, 1$?

What are the domain and co-domain of C?

Draw a graph for C by plotting the points of C in the Cartesian plane.

Definition.
Suppose R is a relation from set A to a set B. The **arrow diagram for** R is obtained as follows:

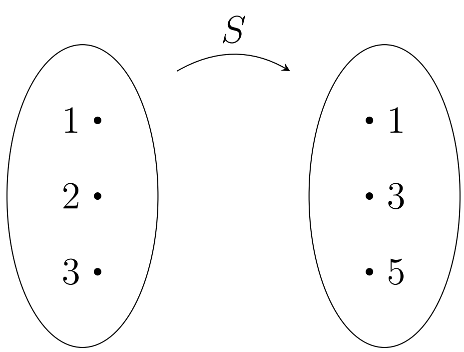
1. Represent the elements of A as points in one region and the elements of B as points in another region.
2. For each x in A and y in B, draw an arrow from x to y if, and only if, x is related to y by R.

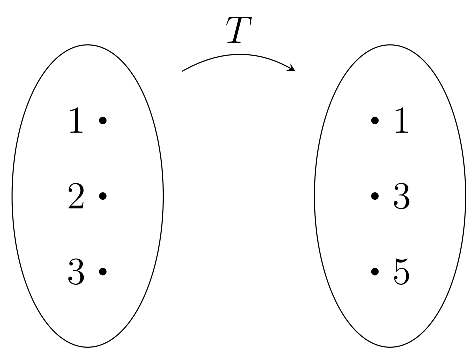


Example. Let $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$ and define relations S and T from A to B as follows: For every $(x, y) \in A \times B$,

$$(x, y) \in S \text{ means that } x < y$$
$$T = \{(2, 1), (2, 5)\}.$$

Draw arrow diagrams for S and T





Definition.
A **function** F **from a set** A **to a set** B is a relation with domain A and co-domain B that satisfies the following two properties:

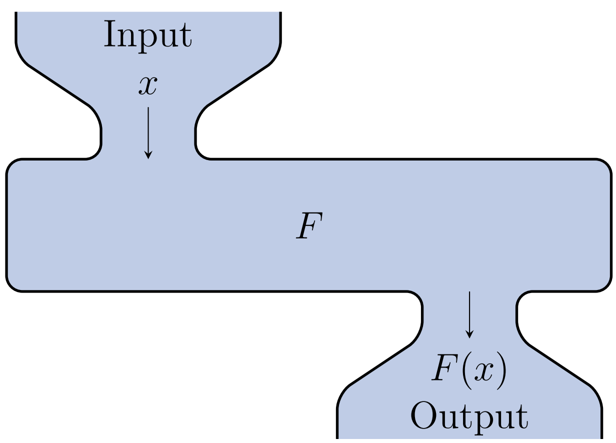
1. For every element x in A, there is an element y in B such that $(x, y) \in F$.
2. For all elements x in A and y and z in B,

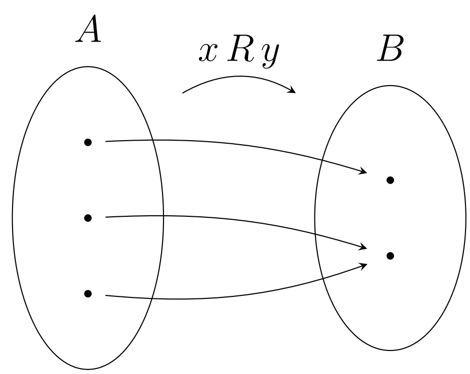
$$\text{if } (x, y) \in F \text{ and } (x, z) \in F, \text{ then } y = z.$$

Note: A relation from A to B is a function if, and only if,

1. Every element of A is the first element of an ordered pair of F
2. No two distinct ordered pairs in F have the same first element.

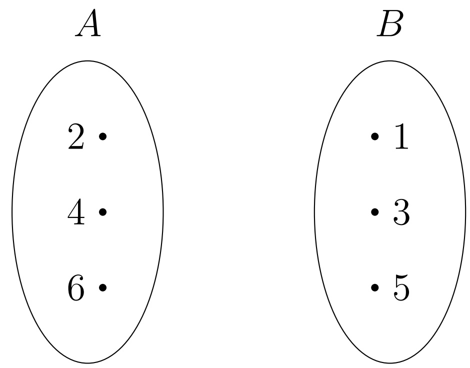
Note: If A and B are sets and F is a function from A to B, then given any element x in A, the unique element in B that is related to x by F is denoted $F(x)$, which is read "F **of** x".



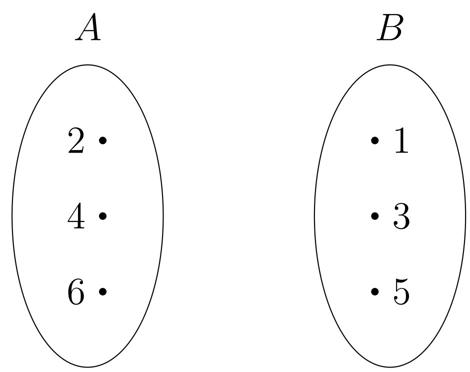


Example. Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$. Which of the relations R, S, and T defined below are functions from A to B?

$R = \{(2, 5), (4, 1), (4, 3), (6, 5)\}$



For every $(x, y) \in A \times B$, $(x, y) \in S$ means that $y = x + 1$.



T is defined by the arrow diagram