Math 121 Class notes
Fall 2024

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1.1: Solutions of Linear Equations and Inequalities in One Variable

Definition.
A **function** f is a special relation between x and y such that each input x results in *at most* one y. The symbol $f(x)$ is read "f of x" and is called the **value of f at x**

Example. Let $f(x) = \frac{x^2}{2} + x$. Evaluate the following:

$f(1)$

$f\left(\frac{1}{2}\right)$

$f(-2)$

$f(0)$

$f(f(x))$

Composite Functions:

Let f and g be functions of x. Then, the **composite functions** g of f (denoted $g \circ f$) and f of g (denoted $f \circ g$) are defined as:

$$(g \circ f)(x) = g(f(x))$$
$$(f \circ g)(x) = f(g(x))$$

Example. Let $g(x) = x - 1$. Find:

$(g \circ f)(x)$

$(f \circ g)(x)$

Operations with Functions:

Let f and g be functions of x and define the following:

Sum	$(f+g)(x) = f(x) + g(x)$
Difference	$(f-g)(x) = f(x) - g(x)$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$
Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ if $g(x) \neq 0$

Definition.

An **expression** is a meaningful string of numbers, variables and operations:

$$3x - 2$$

An **equation** is a statement that two quantities or algebraic expressions are equal:

$$3x - 2 = 7$$

A **solution** is a value of the variable that makes the equation true:

$$\begin{aligned} 3(3) - 2 &= 7 \\ 9 - 2 &= 7 \\ 7 &= 7 \end{aligned}$$

A **solution set** is the set of ALL possible solutions of an equation:

$3x - 2 = 7$ only has the solution $x = 3$,

$2(x-1) = 2x - 2$ is true for all possible values of x.

Properties of Equality:

Substitution Property: The equation formed by substituting one expression for an equal expression is equivalent to the original equation:

$$\begin{aligned} 3(x-3)-\frac{1}{2}(4x-18)&=4\\ 3x-9-2x+9&=4\\ x&=4 \end{aligned}$$

Addition Property: The equation formed by adding the same quantity to both sides of an equation is equivalent to the original equation:

$$\begin{aligned} x-4&=6\\ x-4+4&=6+4\\ x&=10 \end{aligned} \qquad \begin{aligned} x+5&=12\\ x+5+(-5)&=12+(-5)\\ x&=7 \end{aligned}$$

Multiplication Property: The equation formed by multiplying both sides of an equation by the same *nonzero* quantity is equivalent to the original equation:

$$\begin{aligned} \frac{1}{3}x&=6\\ 3\left(\frac{1}{3}x\right)&=3(6)\\ x&=18 \end{aligned} \qquad \begin{aligned} 5x&=20\\ \frac{5x}{5}&=\frac{20}{5}\\ x&=4 \end{aligned}$$

Solving a linear equation:

Using the properties of equality above, we can solve any linear equation in 1 variable:

Example. Solve $\dfrac{3x}{4} + 3 = \dfrac{x-1}{3}$

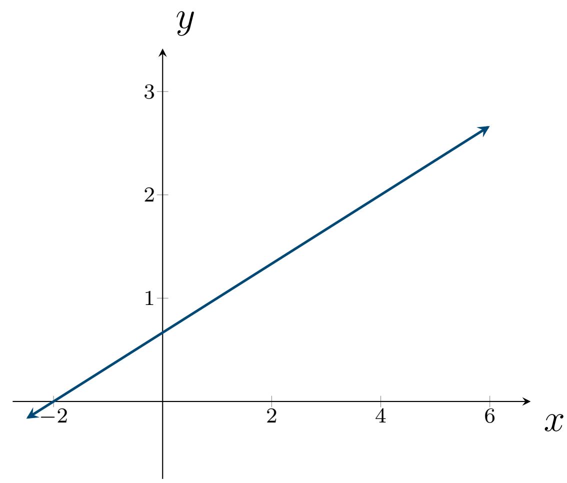
1. Eliminate fractions: $$12\left(\frac{3x}{4} + 3\right) = 12\left(\frac{x-1}{3}\right)$$
2. Remove/evaluate parenthesis: $$9x + 36 = 4x - 4$$
3. Use addition property to isolate the variable to one side: $$9x + 36 - 36 - 4x = 4x - 4 - 36 - 4x$$
4. Use multiplication property to isolate variable: $$\frac{5x}{5} = \frac{-40}{5}$$
5. Verify solution via substitution: $$\underbrace{\frac{3(-8)}{4} + 3}_{-6+3=-3} \stackrel{?}{=} \underbrace{\frac{(-8)-1}{3}}_{\frac{-9}{3}=-3}$$

Example. Solve the following:

$$\frac{3x+1}{2} = \frac{x}{3} - 3$$

$$\frac{2x-1}{x-3} = 4 + \frac{5}{x-3}$$

Example. Solve $-2x + 6y = 4$ for y



Example. Suppose that the relationship between a firm's profit, P, and the number of items sold, x, can be described by the equation

$$5x - 4P = 1200$$

a) How many units must be produced and sold for the firm to make a profit of \$150?

b) Solve this equation for P in terms of x. Then, find the profit when 240 units are sold.

Definition.
An **inequality** is a statement that one quantity is greater than (or less than) another quantity.

Properties of Inequalities

Substitution Property: The inequality formed by substituting one expression for an equal expression is equivalent to the original inequality:

$$\begin{aligned} 5x - 4x + 2 &< 6 \\ x &< 4 \quad \Rightarrow \text{The solution set is } \{x : x < 6\} \end{aligned}$$

Addition Property: The inequality formed by adding the same quantity to both sides of an inequality is equivalent to the original inequality:

$$\begin{aligned} x - 4 &< 6 \\ x - 4 + 4 &< 6 + 4 \\ x &< 10 \end{aligned} \qquad \begin{aligned} x + 5 &\geq 12 \\ x + 5 + (-5) &\geq 12 + (-5) \\ x &\geq 7 \end{aligned}$$

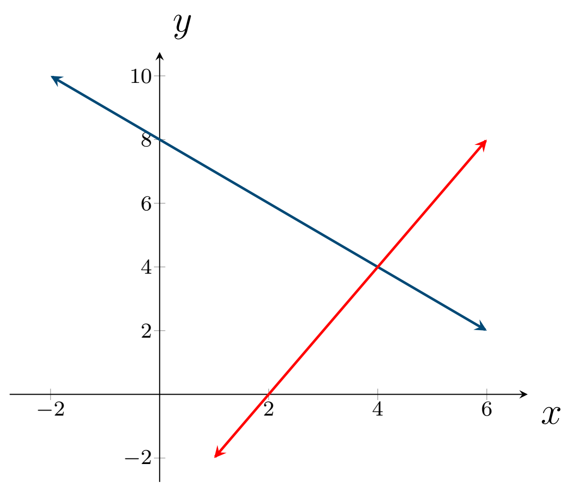
Multiplication Property The inequality formed by multiplying both sides of an inequality by the same *positive* quantity is equivalent to the original inequality. The direction of the inequality is flipped when multiplying by a *negative* quantity:

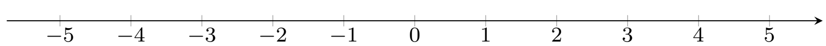
$$\begin{aligned} \frac{1}{3}x &> 6 \\ 3\left(\frac{1}{3}x\right) &> 3(6) \\ x &> 18 \end{aligned} \qquad \begin{aligned} 5x - 5 &\leq 6x + 20 \\ -x &\leq 25 \\ x &\geq -25 \end{aligned}$$

Example. Solve

$$-x + 8 \leq 2x - 4$$

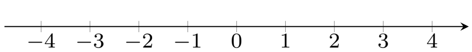
first by gathering the x variable on the left, then again on the right. See that the multiplication property holds in both cases. Plot the solution set on a numberline.

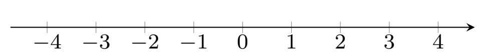




Example. Plot the following inequalities:

$$x \leq 2 \qquad\qquad x > -3$$





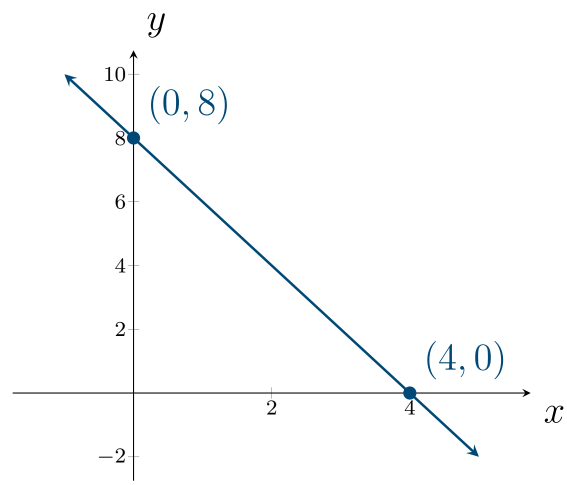
1.3: Linear Functions

Definition.
A **linear function** is a function of the form

$$y = f(x) = ax + b$$

where a and b are constants.

Example. $y = -2x + 8$



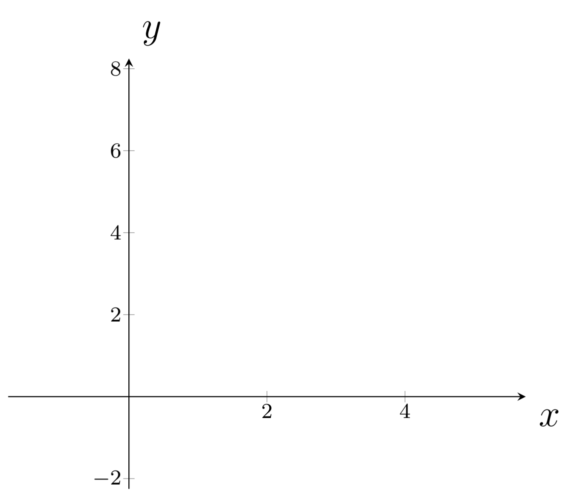
A linear function can be uniquely determined using only *two* distinct points.

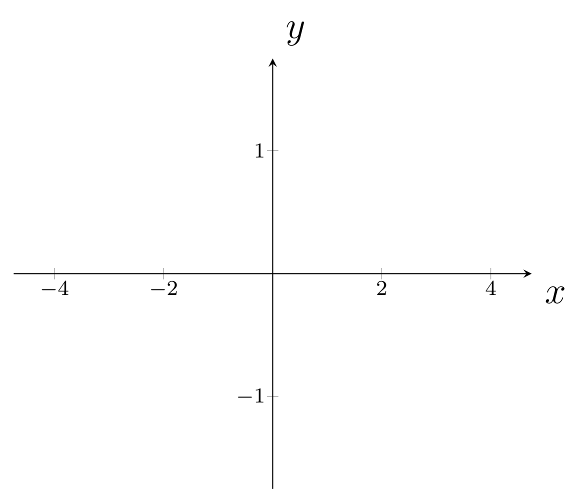
Definition.
The point(s) where a graph intersects the axes are called intercepts. The x-coordinate of the point where the function intersects the x-axis is called the **x-intercepts**. The y-coordinate of the point where the function intersects the x-axis is called the **y-intercepts**.

- To solve for the y-intercept:
  - Set $x = 0$,
  - Solve for y.
- To solve for the x-intercept:
  - Set $y = 0$,
  - Solve for x.

Example. Find the intercepts and graph the following lines:

$3x + 2y = 12$ $\qquad\qquad$ $x = 4y$





Definition.
If a nonvertical line passes through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, its **slope**, denoted by m, is found using

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

Δy is "delta y", and represents the change in y
Δx is "delta x", and represents the change in x

Note: The slope of a vertical line is undefined.

Example. Find the slope of the line passing through the points $(-2, 1)$ and $(5, 3)$.

Note:

- Two distinct nonvertical lines are *parallel* if and only if their slopes are *equal*.
- Two distinct nonvertical lines are *perpendicular* if and only if their slopes are *negative reciprocals*:
  e.g. If ℓ_1 has a nonzero slope m, then ℓ_2 is perpendicular if its slope is $-{}^{1}\!/_{m}$.

Point-slope form

Definition.
The equation of the line passing through the point (x_1, y_1) with slope m can be written in the point-slope form:

$$y - y_1 = m(x - x_1)$$

Example. Find the equation of each line that passes through the point $(-3, 4)$ and has

a slope of $m = \frac{1}{4}$

the point $(-2, 1)$ on the line

a slope of zero (horizontal)

an undefined slope (vertical)

Slope-intercept form

Definition.
The slope-intercept form of the equation of a line with slope m and y-intercept b is

$$y = mx + b$$

Example (Example 7, p.82)**.** The population of U.S. males, y (in thousands), projected from 2015 to 2060 can be modeled by

$$y = 1125.9x + 142,960$$

where x is the number of years after 2000.

- Find the slope and y-intercept of the graph of this function.

- What does the y-intercept tell us abut the population of U.S. males?

- Interpret the slope as a rate of change.

Forms of Linear Equations

$$\begin{aligned}
\text{General form: } & ax + by + c = 0 \\
\text{Point-slope form: } & y - y_1 = m(x - x_1) \\
\text{Slope-intercept form: } & y = mx + b \\
\text{Vertical line: } & x = a \\
\text{Horizontal line: } & y = b
\end{aligned}$$

1.4: Graphs and Graphing Utilities

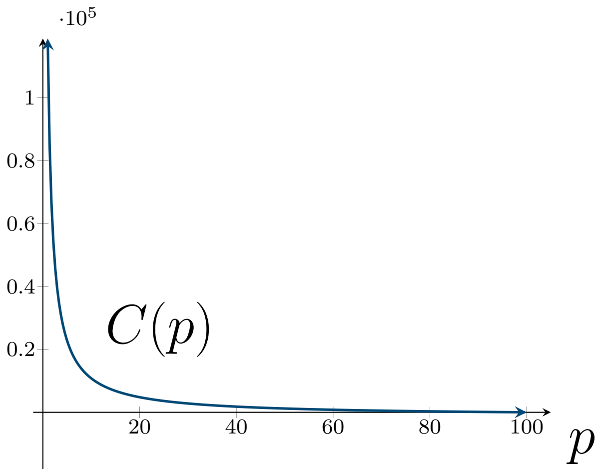
As graphing calculators are *not* required for this course, we will use Desmos:

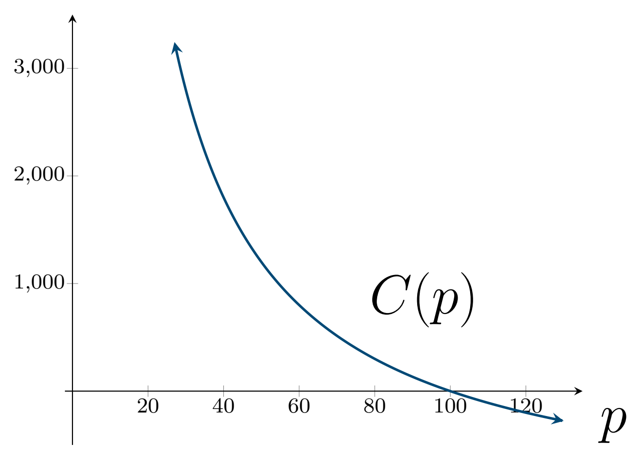
desmos.com/calculator

Example. For a certain city, the cost C of obtaining drinking water with p percent impurities (by volume) is given by

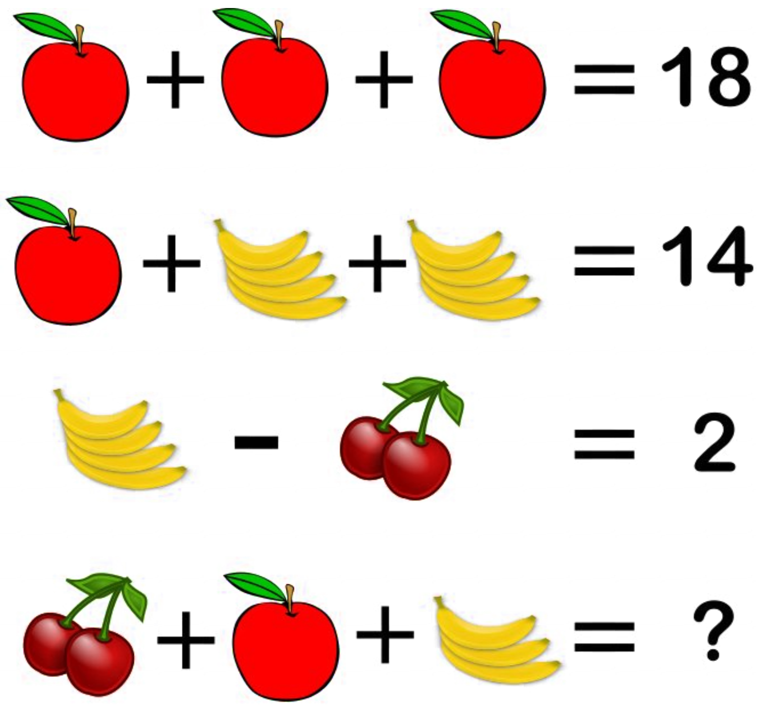
$$C = \frac{120,000}{p} - 1200$$

The equation for C requires that $p \neq 0$, and because p is the percent impurities, we know $0 < p \leq 100$. Use the restriction on p and a graphing calculator to obtain an accurate graph of the equation.





1.5 Solutions of Systems of Linear Equations



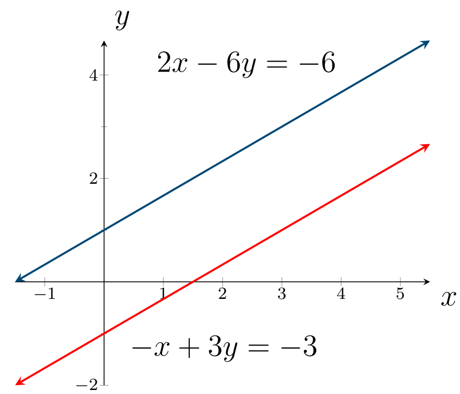
Definition.
A **system of equations** is 2 (or more) equations. The ordered pairs (x, y) that satisfies *all* equations in the system are the **solutions** of the system.

When solving a system of linear equations, there are three possible outcomes:

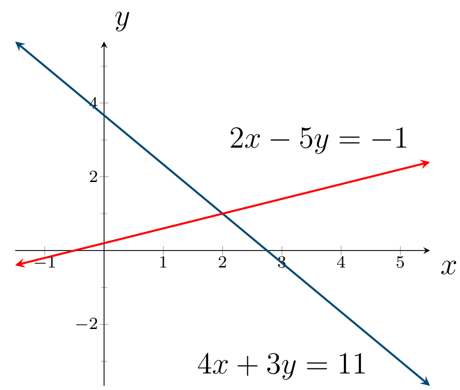
1. No solution (*Inconsistent*),
2. Exactly one solution,
3. Infinitely many solutions (*Dependent*).

Example. Use graphing to find the solutions to the following systems

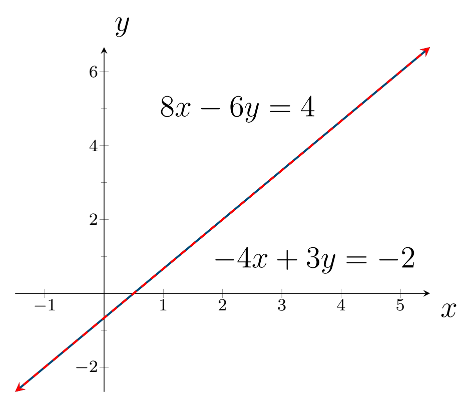
$$\begin{aligned} 2x - 6y &= -6 \\ -x + 3y &= -3 \end{aligned}$$



$$\begin{aligned} 4x + 3y &= 11 \\ 2x - 5y &= -1 \end{aligned}$$



$$\begin{aligned} -4x + 3y &= -2 \\ 8x - 6y &= 4 \end{aligned}$$



Equivalent systems result when

1. One expression is replaced by an equivalent expression.
2. Two equations are interchanged.
3. A multiple of one equation is added to another equation.
4. An equation is multiplied by a nonzero constant.

Substitution Method

Example. Solve the system $\begin{cases} 2x+3y=4 \\ x-2y=3 \end{cases}$

1. Solve one equation for either one of the variables in terms of the other.

$$x = 2y + 3$$

2. Substitute this expression into the other equation to give the equation in one unknown.

$$2(2y+3) + 3y = 4$$

3. Solve this equation for the unknown.

$$4y + 6 + 3y = 4$$
$$7y = -2 \Rightarrow y = -\frac{2}{7}$$

4. Substitute solution into the equation in Step 1.

$$x = 2\left(-\frac{2}{7}\right) + 3 \Rightarrow x = \frac{17}{7}$$

5. Check the solution (x, y).

$$2\left(\frac{17}{7}\right) + 3\left(-\frac{2}{7}\right) = 4$$
$$\left(\frac{17}{7}\right) - 2\left(-\frac{2}{7}\right) = 3$$

Example. Use the substitution method to solve the system

$$\begin{aligned} 4x + 5y &= \ \ 18 & (1) \\ 3x - 9y &= -12 & (2) \end{aligned}$$

Elimination Method

Example. Solve the system $\begin{cases} 2x - 5y = 4 \\ x + 2y = 3 \end{cases}$

1. Multiply one or both equations by a nonzero number so the coefficients of one of the variables may cancel.

$$\Rightarrow \begin{cases} 2x - 5y = 4 \\ -2x - 4y = -6 \end{cases}$$

2. Add or subtract the equations to eliminate one of the variables.

$$0x - 9y = -2$$

3. Solve for the remaining variable.

$$\Rightarrow y = \frac{2}{9}$$

4. Substitute solution in one of the original equations and solve for the other variable.

$$2x - 5\left(\frac{2}{9}\right) = 4 \ \Rightarrow x = \frac{23}{9}$$

5. Check the solution (x, y)

$$2\left(\frac{23}{9}\right) - 5\left(\frac{2}{9}\right) = 4$$
$$\left(\frac{23}{9}\right) + 2\left(\frac{2}{9}\right) = 3$$

Example. Use the elimination method to solve the following systems:

$$\begin{aligned} 2x - 6y &= -6 \\ -x + 3y &= -3 \end{aligned}$$

$$\begin{aligned} 4x + 3y &= 11 \\ 2x - 5y &= -1 \end{aligned}$$

$$\begin{aligned} -4x + 5y &= -2 \\ 8x - 6y &= 4 \end{aligned}$$

Example. A nurse has two solutions that contain different concentrations of a certain medication. One is a 12.5% concentration, and the other is a 5% concentration. How many cubic centimeters of each should she mix to obtain 20 cubic centimeters of an 8% concentration?

Example. Using U.S. Bureau of Labor Statistics data for selected years from 1950 and projected to 2050, the number of men M and women W in the workforce (both in millions) can be modeled by the functions

$$M(t) = 0.591t + 37.3 \qquad \text{and} \qquad W(t) = 0.786t + 13.1$$

where t is the number of years after 1940. Find the year these functions predict that there will be equal numbers of men and women in the U.S. workforce.