

# Math 123 Class notes Fall 2025

To accompany  
*Applied Calculus*  
by *Tan*

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## 1.4: Straight Lines

### Definition. (Slope of a Nonvertical Line)

If  $(x_1, y_1)$  and  $(x_2, y_2)$  are any two distinct points on a nonvertical line  $L$ , then the slope  $m$  of  $L$  is given by

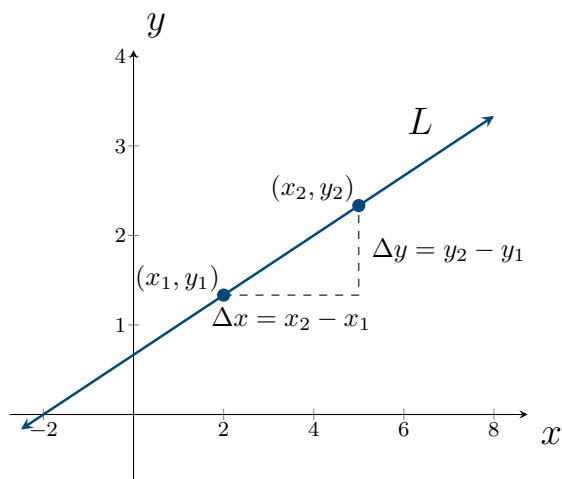
$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

**Example.** Compute the slope of the line passing through the points

$$(x_1, y_1) = (1, 1) \text{ and } (x_2, y_2) = (4, 2)$$

$$(x_1, y_1) = (3, 2) \text{ and } (x_2, y_2) = (-1, 2)$$

$$(x_1, y_1) = (4, 1) \text{ and } (x_2, y_2) = (4, 4)$$



**Definition. (Point-Slope Form of an Equation of a Line)**

An equation of the line that has slope  $m$  and passes through the point  $(x_1, y_1)$  is given by

$$y - y_1 = m(x - x_1)$$

**Example.** Find the equation of the line going through the points

$$(x_1, y_1) = (-2, 1) \text{ and } (x_2, y_2) = (3, -2)$$

$$(x_1, y_1) = (3, 4) \text{ and } (x_2, y_2) = (-1, 4)$$

$$(x_1, y_1) = (2, 0) \text{ and } (x_2, y_2) = (2, 1)$$

**Definition. (Slope-Intercept Form of an Equation of a Line)**

An equation of the line that has slope  $m$  and intersects the  $y$ -axis at the point  $(0, b)$  is given by

$$y = mx + b$$

**Example.** Rewrite the equations in the previous example in slope-intercept form.

**Definition. (Parallel and Perpendicular lines)**

Let  $L_1$  and  $L_2$  be lines with slopes  $m_1$  and  $m_2$  respectively. If  $L_1$  and  $L_2$  are *parallel*, then

$$m_1 = m_2.$$

If  $L_1$  and  $L_2$  are *perpendicular*, then

$$m_1 = -\frac{1}{m_2}.$$

**Example.**

Find the line *parallel* to  $y = \frac{3}{2}x + 1$  that passes through the point  $(-4, 10)$ .

Find the line *perpendicular* to  $y = \frac{3}{2}x + 1$  that passes through the point  $(-3, 4)$ .

## Forms of Linear Equations

General form:  $Ax + By = C$

Point-slope form:  $y - y_1 = m(x - x_1)$

Slope-intercept form:  $y = mx + b$

Vertical line:  $x = a$

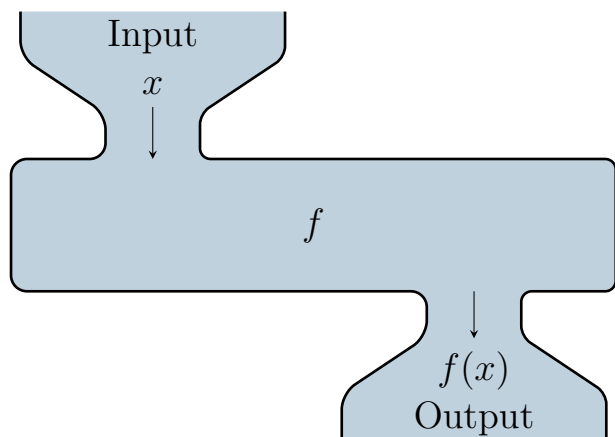
Horizontal line:  $y = b$

## 2.1: Functions and Their Graphs

### Definition.

A **function** is a rule that assigns to each element in a set  $A$  one and only one element in a set  $B$ .

In the context above, the set  $A$  is called the **domain**, and the set  $B$  is called the **range**.



**Example.** Let  $f(x) = 2x^2 - 2x + 1$ . Evaluate the following

$$f(1)$$

$$f(-2)$$

$$f(a)$$

$$f(a + h)$$



**Example.** Find the domain and range of the following functions:

$$f(x) = x$$

$$A = \pi r^2$$

$$y = \sqrt{x - 1}$$

$$y = \frac{1}{x^2 - 4}$$

**Example.** An open box is to be made from a rectangular piece of cardboard 16 inches long and 10 inches wide by cutting away identical squares ( $x$  inches by  $x$  inches) from each corner and folding up the resulting flaps. Find an expression that gives the volume  $V$  of the box as a function of  $x$ . What is the domain of the function?



**Definition.**

A **piecewise** function is a function with different definitions for different portions of the domain.

**Example.** Rewrite the following as piecewise functions:

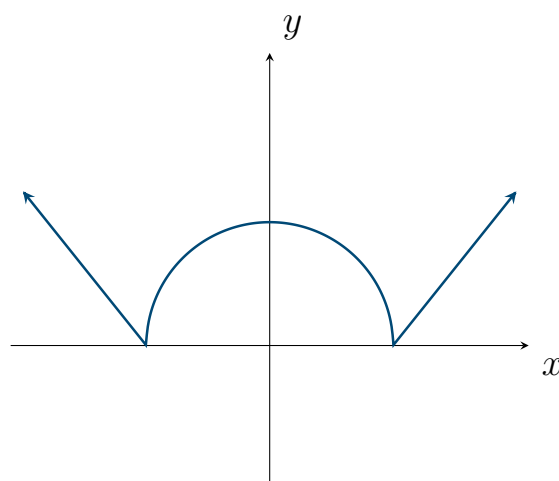
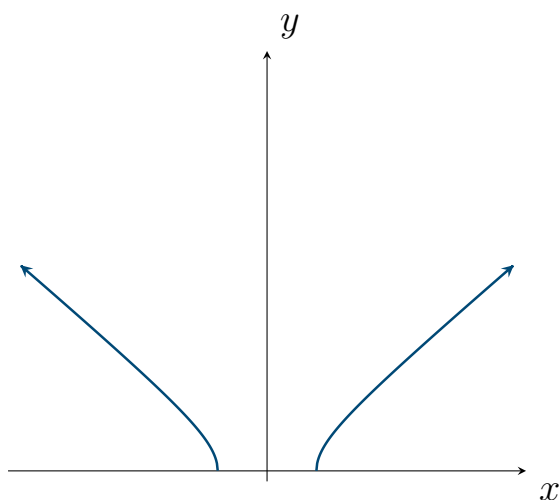
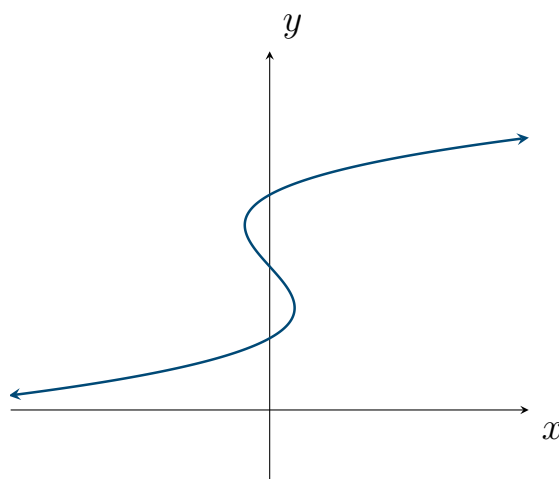
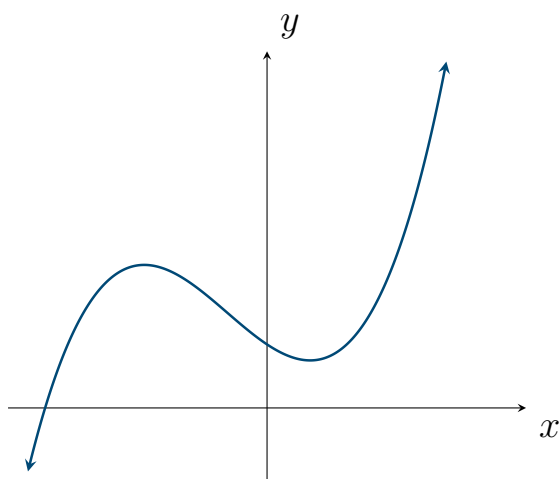
$$|x| = \qquad \qquad \qquad \frac{x}{|x|} =$$

$$|x - 1| + |4 - x| =$$

**Definition. (Vertical Line Test)**

A curve in the  $xy$ -plane is the graph of a function  $y = f(x)$  (an explicit function) if and only if each vertical line intersects it in at most one point

**Example.** Use the vertical line test on the following graphs to determine which graphs may represent an explicit function:



## 2.2: The Algebra of Functions

### Definition.

Let  $f$  and  $g$  be functions with domains  $A$  and  $B$ , respectively. Then the **sum**  $f + g$ , **difference**  $f - g$ , and **product**  $fg$  of  $f$  and  $g$  are functions with domain  $A \cap B$ .

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)g(x)$$

The **quotient**  $f/g$  of  $f$  and  $g$  has domain  $A \cap B$  excluding all numbers  $x$  such that  $g(x) = 0$  and rule given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

**Example.** Let  $f(x) = \sqrt{x+1}$  and  $g(x) = 4 - x$ . Find the domain of the following:

$$f(x) + g(x) =$$

$$f(x) - g(x) =$$

$$f(x)g(x) =$$

$$\frac{f(x)}{g(x)} =$$

**Definition. (The Composition of Two Functions)**

Let  $f$  and  $g$  be functions. Then the composition of  $g$  and  $f$  is the function  $g \circ f$  defined by

$$(g \circ f)(x) = g(f(x))$$

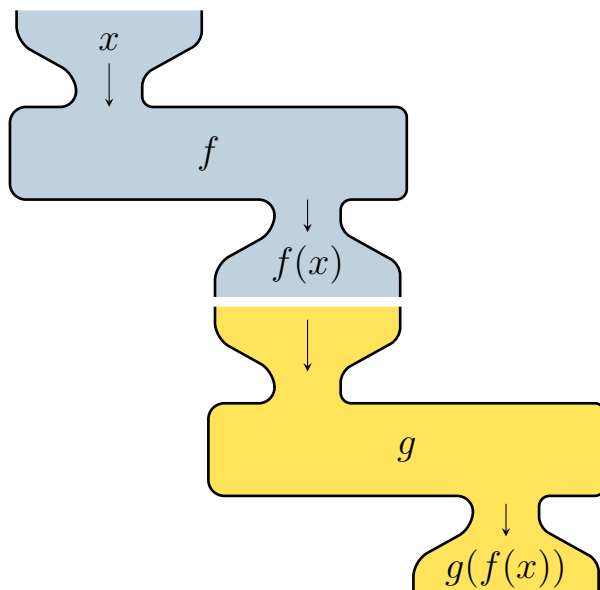
The domain of  $g \circ f$  is the set of all  $x$  is the domain of  $f$  such that  $f(x)$  lies in the domain of  $g$ .

**Example.** Let  $f(x) = \sqrt{x+1}$  and  $g(x) = 4 - x$ . Find the domain of the following:

$$g(f(x)) =$$

$$f(g(x)) =$$

$$f(f(x)) =$$



## 2.4: Limits

**Example.** Suppose that the position function of a maglev train (in feet) is given by

$$s(t) = 4t^2, \quad (0 \leq t \leq 30)$$

Using the position function, compute the *average* velocity of the train

on the interval  $[t, 2]$

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$t$	1.5	1.9	1.99	1.999	1.9999
-----	-----	-----	------	-------	--------

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on the interval  $[2, t]$

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$t$	2.5	2.1	2.01	2.001	2.0001
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What do the tables above suggest about *instantaneous* velocity of the train at  $t = 2$ ?

**Definition. (Limit of a Function)**

The function  $f$  has the **limit**  $L$  as  $x$  approaches  $a$ , written

$$\lim_{x \rightarrow a} f(x) = L$$

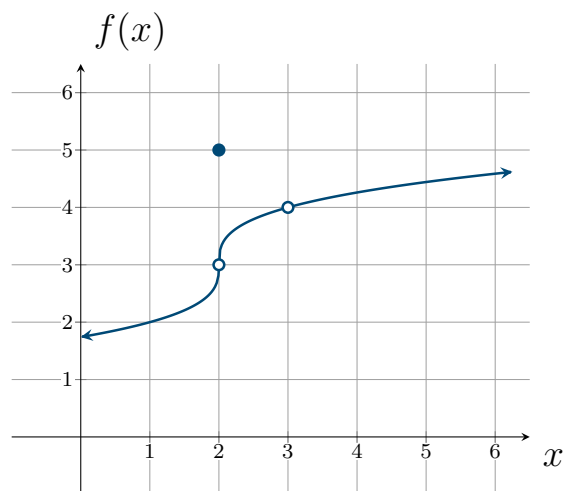
if the value of  $f(x)$  can be made as close to the number  $L$  as we please by taking  $x$  sufficiently close to (but not equal to)  $a$ .

**Example.** Using the graph of  $f$ , determine the following values:

$$f(1) \text{ and } \lim_{x \rightarrow 1} f(x)$$

$$f(2) \text{ and } \lim_{x \rightarrow 2} f(x)$$

$$f(3) \text{ and } \lim_{x \rightarrow 3} f(x)$$





**Example.** Find the limit of the following functions at the value specified:

[Graphs](#)

$$f(x) = x^3 \quad \text{at } x = 2$$

$$g(x) = \begin{cases} x + 2, & x \neq 1 \\ 1, & x = 1 \end{cases} \quad \text{at } x = 1$$

$$h(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases} \quad \text{at } x = 0$$

$$j(x) = \frac{1}{(x-1)^2} \quad \text{at } x = 1$$

$$k(x) = 4 \quad \text{at } x = 0$$

### Theorem 1: Properties of Limits

Suppose

$$\lim_{x \rightarrow a} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = M$$

Then

1.  $\lim_{x \rightarrow a} [f(x)]^r = \left[ \lim_{x \rightarrow a} f(x) \right]^r$  where  $r$  is a positive constant
2.  $\lim_{x \rightarrow a} cf(x) = c \lim_{x \rightarrow a} f(x)$  where  $c$  is a real number
3.  $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = L \pm M$
4.  $\lim_{x \rightarrow a} [f(x)g(x)] = \left[ \lim_{x \rightarrow a} f(x) \right] \left[ \lim_{x \rightarrow a} g(x) \right] = LM$
5.  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{M}$  provided  $M \neq 0$

**Example.** Use the above theorem to evaluate the following limits:

$$\lim_{x \rightarrow 1} (5x^{3/2} - 2)$$

$$\lim_{x \rightarrow 3} \frac{2x^3 \sqrt{x^2 + 7}}{x + 1}$$

Suppose that  $\lim_{x \rightarrow a} f(x) = 0$  and  $\lim_{x \rightarrow a} g(x) = 0$ . Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

has an **indeterminate form** of  $\frac{0}{0}$ . To evaluate such a limit, we replace the given function with a function that's equivalent everywhere except at  $x = a$ , and then evaluate the limit.

**Example.** Evaluate the following

$$\lim_{t \rightarrow 2} \frac{4t^2 - 16}{t - 2}$$

$$\lim_{h \rightarrow 0} \frac{\sqrt{4 + h} - 2}{h}$$

Suppose that  $\lim_{x \rightarrow a} f(x) = L$  with  $L \neq 0$  and  $\lim_{x \rightarrow a} g(x) = 0$ . Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

does not exist. We can further specify if this limit tends towards  $-\infty$  or  $\infty$ .

**Example.** Evaluate the following

[Graphs](#)

$$\lim_{x \rightarrow 1} \frac{x}{x - 1}$$

$$\lim_{x \rightarrow 3} \frac{1}{(x - 3)^2}$$

$$\lim_{x \rightarrow -2} \frac{x - 2}{x^2 - 4}$$

$$\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 4}$$

### Limit of a Function at Infinity

The function  $f$  has the limit  $L$  as  $x$  increases without bound, written

$$\lim_{x \rightarrow \infty} f(x) = L$$

if  $f(x)$  can be made arbitrarily close to  $L$  by taking  $x$  large enough.

The function  $f$  has the limit  $M$  as  $x$  decreases without bound, written

$$\lim_{x \rightarrow -\infty} f(x) = M$$

if  $f(x)$  can be made arbitrarily close to  $M$  by taking  $x$  to be negative and sufficiently large enough in absolute value.

When the above limits exist, the equations  $y = L$  and/or  $y = M$  are called **horizontal asymptotes**.

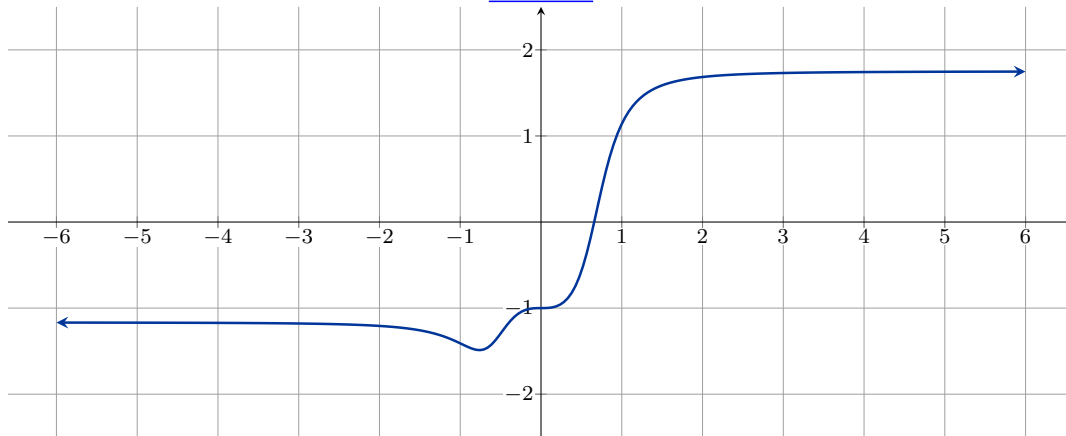
**Example.** Evaluate the following infinite limits

$$\lim_{x \rightarrow \infty} \frac{2x^2 + 3x - 4}{x^2 - 7x + 1}$$

$$\lim_{x \rightarrow -\infty} \frac{3x^2 + 4}{2x^3}$$

$$\lim_{x \rightarrow \pm\infty} \frac{3x^5 + 2x^3 - 4}{x^4 + 4x^2 - 1}$$

### Graph



$$\lim_{x \rightarrow -\infty} \frac{7x^3 - 2}{-x^3 + \sqrt{25x^6 - 4}}$$

$$\lim_{x \rightarrow \infty} \frac{7x^3 - 2}{-x^3 + \sqrt{25x^6 - 4}}$$

**Example.** The company *Custom Office* makes a line of executive desks. It is estimated that the total cost of making  $x$  *Senior Executive Model* desks is

$$C(x) = 100x + 200,000$$

dollars per year. The average cost of making  $x$  desks is given by

$$\overline{C}(x) = \frac{C(x)}{x}$$

Compute  $\lim_{x \rightarrow \infty} \overline{C}(x)$  and interpret the result.

**Theorem 2**

For all  $n > 0$ ,

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x^n} = 0$$

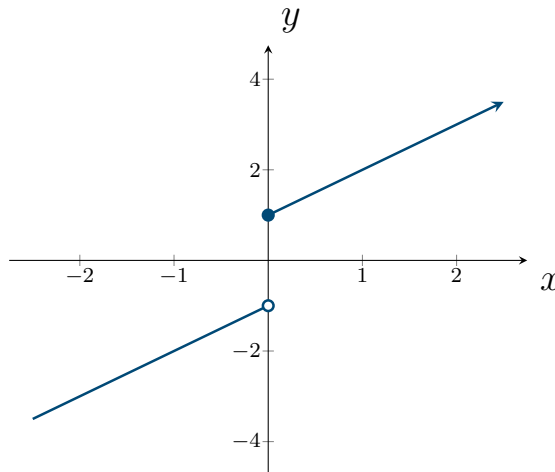
provided that  $\frac{1}{x^n}$  is defined.

## 2.5: One-Sided Limits and Continuity

Consider the function

$$f(x) = \begin{cases} x - 1, & x < 0 \\ x + 1, & x \geq 0 \end{cases}$$

What is  $\lim_{x \rightarrow 0} f(x)$ ?



### Definition. (One-Sided Limits)

The function  $f$  has a **right-hand limit**  $L$  as  $x$  approaches  $a$  from the right, written

$$\lim_{x \rightarrow a^+} f(x) = L$$

if the values of  $f(x)$  can be made as close to  $L$  as we please by taking  $x$  sufficiently close to (but not equal to)  $a$  and to the right of  $a$ .

The function  $f$  has a **left-hand limit**  $L$  as  $x$  approaches  $a$  from the left, written

$$\lim_{x \rightarrow a^-} f(x) = L$$

if the values of  $f(x)$  can be made as close to  $L$  as we please by taking  $x$  sufficiently close to (but not equal to)  $a$  and to the left of  $a$ .

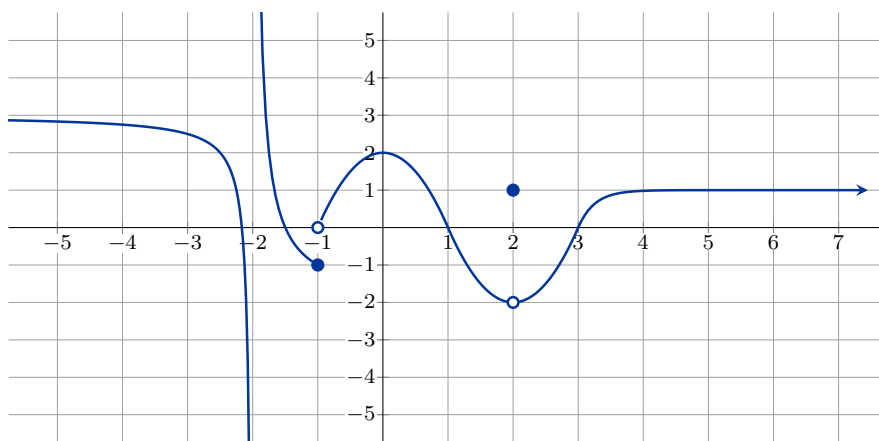


**Theorem 3**

Let  $f$  be a function that is defined for all values of  $x$  close to  $x = a$  with the possible exception of  $a$  itself. Then

$$\lim_{x \rightarrow a} f(x) = L \quad \text{if and only if} \quad \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L$$

**Example.** Using the graph below, evaluate the following limits:



$$\lim_{x \rightarrow -2^-} f(x)$$

$$\lim_{x \rightarrow -2^+} f(x)$$

$$\lim_{x \rightarrow -2} f(x)$$

$$\lim_{x \rightarrow -1^-} f(x)$$

$$\lim_{x \rightarrow -1^+} f(x)$$

$$\lim_{x \rightarrow -1} f(x)$$

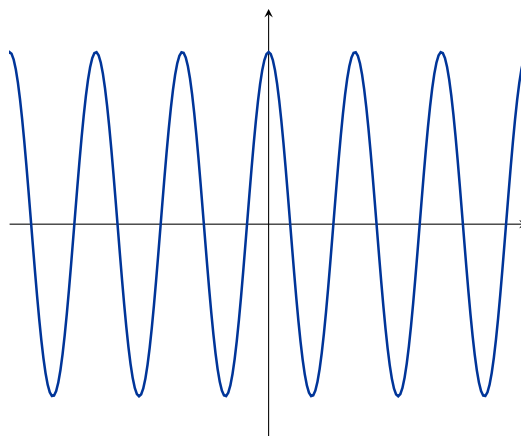
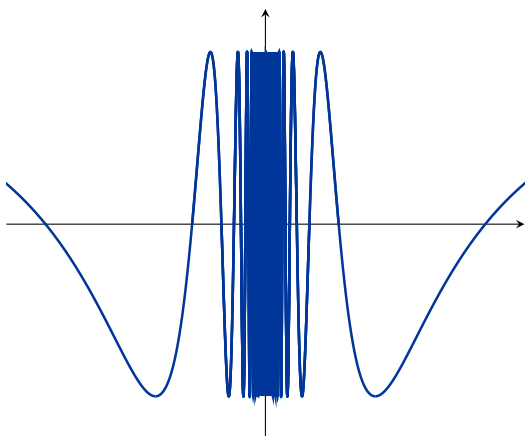
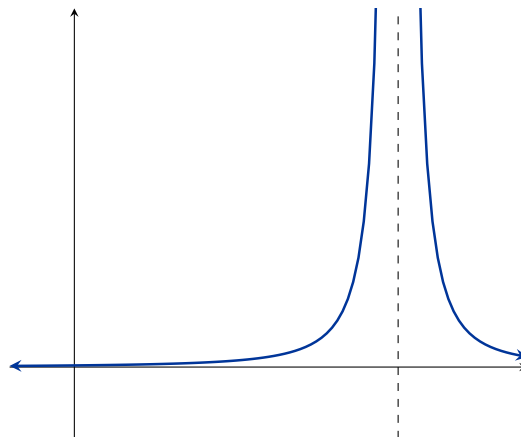
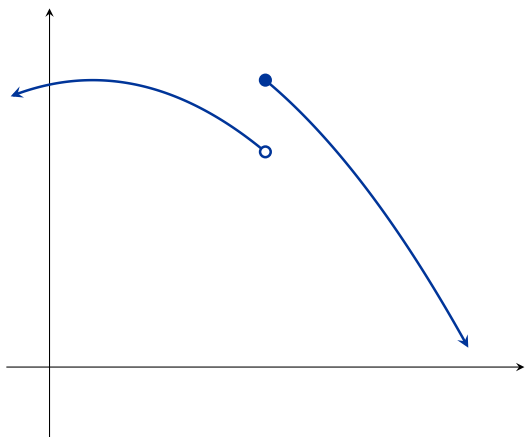
$$\lim_{x \rightarrow 1} f(x)$$

$$\lim_{x \rightarrow 2} f(x)$$

$$\lim_{x \rightarrow \infty} f(x)$$

Below are examples where the limit does not exist:

[Graph](#)



**Definition. (Continuity of a Function at a Number)**

A function  $f$  is **continuous** at  $a$  if  $\lim_{x \rightarrow a} f(x) = f(a)$ .

**Continuity Checklist:**

In order for  $f$  to be continuous at  $a$ , the following three conditions must hold:

1.  $f(a)$  is defined ( $a$  is in the domain of  $f$ ),
2.  $\lim_{x \rightarrow a} f(x)$  exists,
3.  $\lim_{x \rightarrow a} f(x) = f(a)$  (the value of  $f$  equals the limit of  $f$  at  $a$ ).

**Example.** Determine the values of  $x$  for which the following functions are continuous:

$$f(x) = 3x^3 + 2x^2 - x + 10$$

$$g(x) = \frac{8x^{10} - 4x + 1}{x^2 + 1}$$

$$h(x) = \frac{4x^3 - 3x^2 + 1}{x^2 - 3x + 1}$$

**Example.** Determine whether the following are continuous at  $a$ :

$$f(x) = x^2 + \sqrt{7-x}, \quad a = 4$$

$$g(x) = \frac{1}{x-3}, \quad a = 3$$

$$h(x) = \begin{cases} \frac{x^2+x}{x+1}, & x \neq -1 \\ 0, & x = -1 \end{cases}, \quad a = -1$$

$$j(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases}, \quad a = 0$$

$$k(x) = \begin{cases} \frac{x^2+x-6}{x^2-x}, & x \neq 2 \\ -1, & x = 2 \end{cases}, \quad a = 2$$

## Properties of Continuous Functions

1. The constant function  $f(x) = c$  is continuous everywhere.
2. The identity function  $f(x) = x$  is continuous everywhere.

If  $f$  and  $g$  are continuous at  $x = a$ , then

$[f(x)]^n$ , where  $n$  is a real number, is continuous at  $x = a$  whenever it is defined at that number

$f \pm g$  is continuous at  $x = a$

$fg$  is continuous at  $x = a$

$f/g$  is continuous at  $x = a$  provided that  $g(a) \neq 0$

## Polynomial and Rational Functions

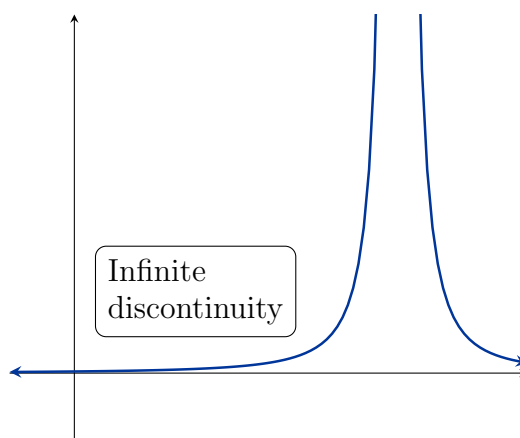
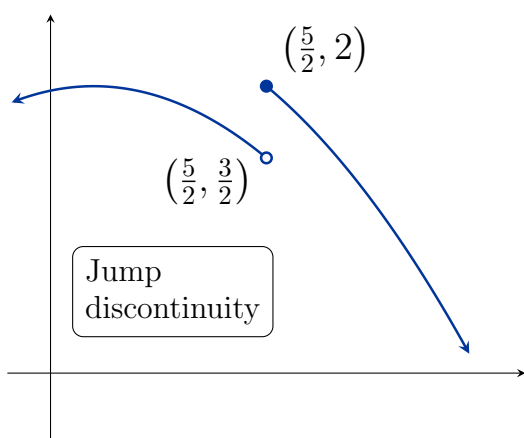
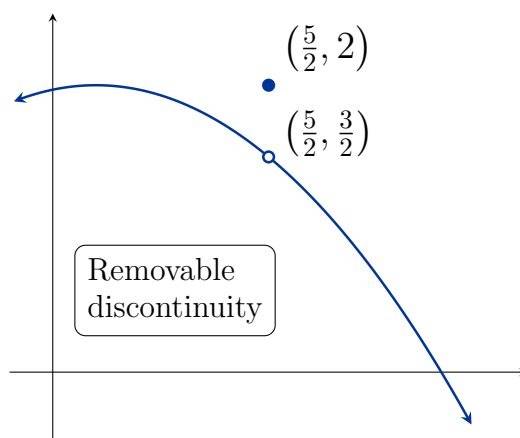
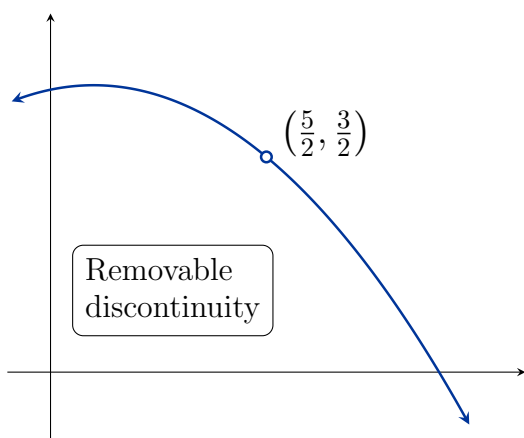
1. A polynomial function is continuous for all  $x$ .
2. A rational function (a function of the form  $\frac{p}{q}$ , where  $p$  and  $q$  are polynomials) is continuous for all  $x$  for which  $q(x) \neq 0$ .

## Definition.

A **removable discontinuity** at  $x = a$  is one that disappears when the function becomes continuous after defining  $f(a) = \lim_{x \rightarrow a} f(x)$ .

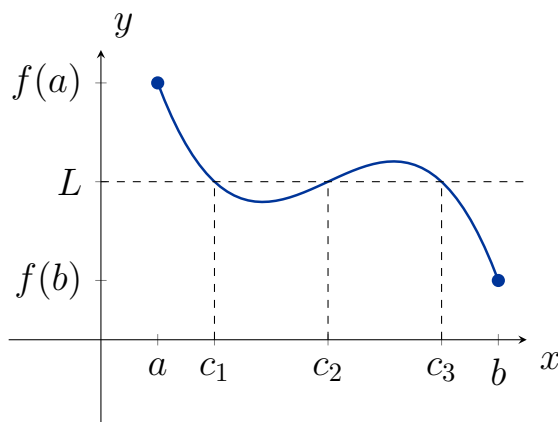
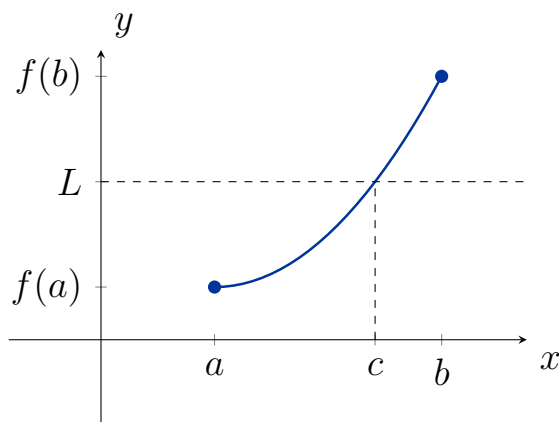
A **jump discontinuity** is one that occurs whenever  $\lim_{x \rightarrow a^-} f(x)$  and  $\lim_{x \rightarrow a^+} f(x)$  both exist, but  $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$ .

A **vertical discontinuity** occurs whenever  $f(x)$  has a vertical asymptote.

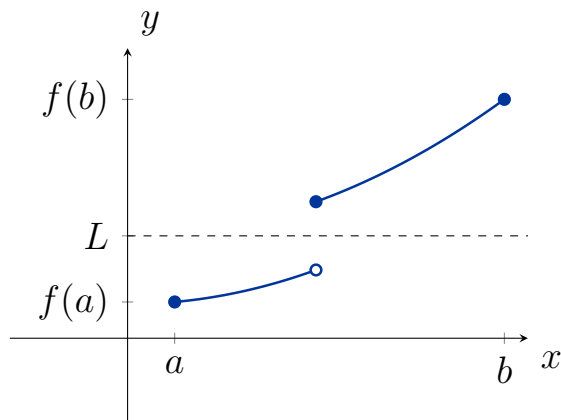


**Theorem 4: Intermediate Value Theorem**

Suppose  $f$  is continuous on the interval  $[a, b]$  and  $L$  is a number strictly between  $f(a)$  and  $f(b)$ . Then there exists at least one number  $c$  in  $(a, b)$  satisfying  $f(c) = L$ .



*Note:* It is important that the function be continuous on the interval  $[a, b]$ :

**Theorem 5: Existence of Zeros of a Continuous Function**

If  $f$  is a continuous function on a closed interval  $[a, b]$ , and if  $f(a)$  and  $f(b)$  have opposite signs, then there is at least one solution of the equation  $f(x) = 0$  in the interval  $(a, b)$ .

**Example.** Check the conditions of the Intermediate Value Theorem to see if there exists a value  $c$  on the interval  $(a, b)$  such that the following equations hold: [Graph](#)

$$x^x - x^2 = \frac{1}{2} \quad \text{on } [0, 2] \quad \sqrt{x^4 + 25x^3 + 10} = 5 \quad \text{on } [0, 1]$$

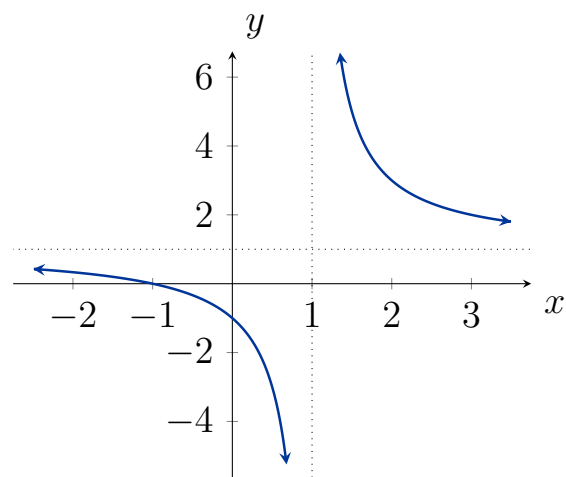
$$x + \sqrt{1 - x^2} = 0 \quad \text{on } [-1, 0] \quad \frac{x^2}{x^2 + 1} = 0 \quad \text{on } [-1, 1]$$



**Example.** Consider the function

$$f(x) = \frac{x+1}{x-1}$$

on the interval  $[0, 2]$ . Does there exist a  $c$  on the interval  $[0, 2]$  such that  $f(c) = 1$ ?



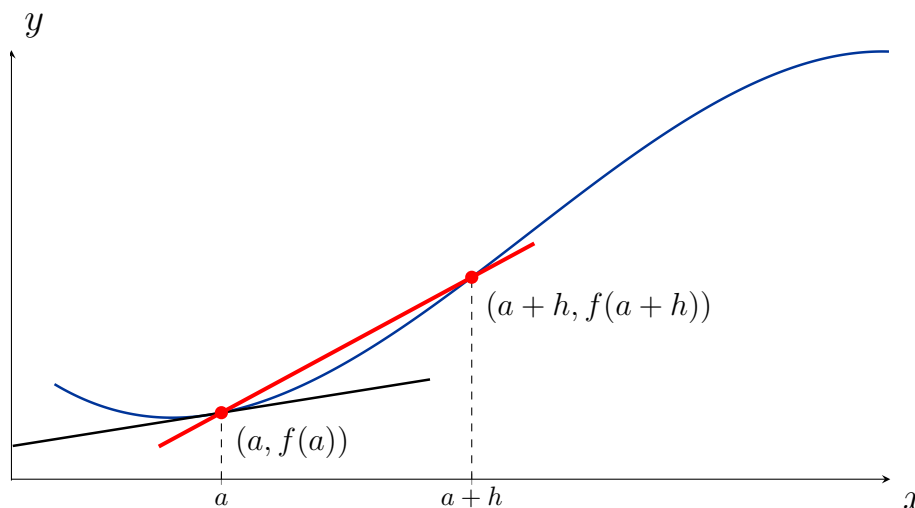
## 2.6: The Derivative

### Definition.

Given a function  $f(x)$ :

- the **secant line** is the line that passes through two *distinct* points lying on the graph of  $f(x)$ ,
- the **tangent line** is the line that intersects  $f(x)$  in exactly one place (locally) and matches the slope of the graph at that point.

[Graph](#)



### Definition. (Slope of a Tangent Line)

The slope of the tangent line to the graph of  $f$  at the point  $P(x, f(x))$  is given by

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

if it exists.

**Definition. (Average and Instantaneous Rates of Change)**

The **average rate of change** of  $f$  over the interval  $[x, x + h]$  or **slope of the secant line** to the graph of  $f$  through the points  $(x, f(x))$  and  $(x + h, f(x + h))$  is

$$\frac{f(x + h) - f(x)}{h}$$

The above fraction is referred to as the **difference quotient**.

The **instantaneous rate of change** of  $f$  at  $x$  or **slope of the tangent line** to the graph of  $f$  at  $(x, f(x))$  is

$$\lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

**Definition. (Derivative of a Function)**

The derivative of a function  $f$  with respect to  $x$  is the function  $f'$  (read “ $f$  prime”),

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

The domain of  $f'$  is the set of all  $x$  for which the limit exists.

Some other notations for the derivative are

$$D_x f(x) \qquad \frac{dy}{dx} \qquad y'$$

**Example.** Find the slope of the line tangent to the graph  $f(x) = 3x + 5$  at any point  $(x, f(x))$

**Example.** Let  $f(x) = x^2$ .

- Find  $f'(x)$ .
- Compute  $f'(2)$  and interpret your result.

**Example.** Let  $f(x) = x^2 - 4x$ . Find the point on the graph where the tangent line is horizontal.

**Example.** Let  $f(x) = \frac{1}{x}$ . Find the equation of the tangent line at  $x = 2$ .

## Differentiability and Continuity

If a function is differentiable at  $x = a$ , then it is continuous at  $x = a$ .

**Example.** For the graph below, identify each point where the derivative is undefined.

