# 6.4: Boolean Algebras, Russell's Paradox, and the Halting Problem

**Definition.**

A **Boolean algebra** is a set $B$ together with two operations, generally denoted $+$ and $\cdot$, such that for all $a$ and $b$ in $B$ both $a + b$ and $a \cdot b$ are in $B$ and the following axioms are assumed to hold:

1. *Commutative Laws:* For all $a$ and $b$ in $B$,
$$a + b = b + a \text{ and } a \cdot b = b \cdot a$$

2. *Associative Laws:* For all $a$ and $b$ in $B$,
$$(a + b) + c = a + (b + c) \text{ and } (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

3. *Distributive Laws:* For all $a$ and $b$ in $B$,
$$a + (b \cdot c) = (a + b) \cdot (a + c) \text{ and } a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

4. *Identity Laws:* There exist distinct elements $0$ and $1$ in $B$ such that for each $a$ in $B$,
$$a + 0 = a \text{ and } a \cdot 1 = a$$

5. *Complement Laws:* For each $a$ in $B$, there exists an element in $B$, denoted $\overline{a}$ and called the **complement** or **negation** of $a$, such that
$$a + \overline{a} = 1 \text{ and } a \cdot \overline{a} = 0$$

**Properties of a Boolean Algebra**

Let $B$ be any Boolean algebra.

1. *Uniqueness of the Complement Laws:* For all $a$ and $x$ in $B$, if $a + x = 1$ and $a \cdot x = 0$, then $x = \overline{a}$.

2. *Uniqueness of $0$ and $1$:* If there exists $x$ in $B$ such that $a + x = a$ for every $a$ in $B$, then $x = 0$, and if there exists $y$ in $B$ such that $a \cdot y = a$ for every $a$ in $B$, then $y = 1$.

3. *Double Complement Law:* For every $a \in B$, $\overline{(\overline{a})} = a$.

4. *Idempotent Laws:* For every $a \in B$,

$$a + a = a \text{ and } a \cdot a = a.$$

5. *Universal Bound Laws:* For every $a \in B$,

$$a + 1 = 1 \text{ and } a \cdot 0 = 0.$$

6. *De Morgan's Laws:* For all $a$ and $b \in B$,

$$\overline{a + b} = \overline{a} \cdot \overline{b} \text{ and } \overline{a \cdot b} = \overline{a} + \overline{b}.$$

7. *Absorption Laws:* For all $a$ and $b \in B$,

$$(a + b) \cdot a = a \text{ and } (a \cdot b) + a = a.$$

8. *Complements of $0$ and $1$:*
$$\overline{0} = 1 \text{ and } \overline{1} = 0.$$

**Example.** Prove that for all elements $a$ in a Boolean algebra $B$:

$$\overline{(\overline{a})} = a.$$

$$a + a = a.$$

**Example.** Prove that for all elements $a$ in a Boolean algebra $B$:

$a \cdot a = a.$

$(a + b) \cdot a = a.$

**Russell's Paradox**

Define the following set $S$:

$$S = \{A \mid A \text{ is a set and } A \notin A\}.$$

Is $S$ an element of itself?

**The Barber Puzzle:** In a certain town, there is a male barber who shaves all those men, and only those men, who do not shave themselves.

Does the barber shave himself?

Is the sentence "The barber shaves himself" a statement?

**Example.** Determine whether each sentence is a statement:

If $1 + 1 = 3$, then $1 = 0$.

This sentence is false and $1 + 1 = 3$.

**The Halting Problem (Alan M. Turing)**

There is no computer algorithm that will accept any algorithm $X$ and data set $D$ as input and then will output "halts" or "loops forever" to indicate whether or not $X$ terminates in a finite number of steps when $X$ is run with data set $D$.