5.5: Differentiation of Logarithmic Functions

Definition. (Natural log)
The inverse of the exponential function is the logarithm:

$$e^{y} = x \;\longleftrightarrow\; \ln(x) = y, \qquad x > 0$$

We may also denote this as $\log_e(x)$

Example. Solve the following exponential equations Graphs

$y = e^{-3}$

$e^{x^2-x} = e^2$

$e^x = 10$

$4e^{1-x^2} = 6$

Definition. (Other logarithms)
Logarithms are defined for any base $a > 0$:

$$a^{y} = x \quad\longleftrightarrow\quad \log_a(x) = y, \qquad x > 0$$

Rule 3: Derivative of $\ln(x)$

$$\frac{d}{dx}\Big[\ln|x|\Big] = \frac{1}{x}, \qquad x \neq 0$$

Rule 4: Derivative of $\ln(f(x))$

$$\frac{d}{dx}\Big[\ln(f(x))\Big] = \frac{f'(x)}{f(x)}$$

Example. Find the derivative of the following functions

$y = \ln(\sqrt[3]{x})$

$f(x) = \sqrt{x}\ln(x)$

$g(x) = \ln\left(x^2 + 1\right)$

$h(t) = \dfrac{1 + \ln(t)}{1 - \ln(t)}$

Additional properties of logarithms

$e^y = x$	$\ln(x) = y$
$e^1 = a$	$\ln(a) = 1$
$e^0 = 1$	$\ln(1) = 0$
$e^x e^y = e^{x+y}$	$\ln(xy) = \ln(x) + \ln(y)$
$\dfrac{e^x}{e^y} = e^{x-y}$	$\ln\left(\dfrac{x}{y}\right) = \ln(x) - \ln(y)$
$e^{xy} = (e^x)^y$	$\ln(x^y) = y\ln(x)$
$e^{\ln(x)} = x$	$\ln(e^x) = x$

Example. Find the derivative of the following functions

$y = \ln\left(\sqrt[3]{x}\right)$

$f(x) = \ln\left(\left(x^2+1\right)\left(x^3+2\right)^6\right)$

$g(x) = \ln\left(\dfrac{1-x}{1+x}\right)$

$h(x) = \ln\left(x + \sqrt{x^2-1}\right)$

$j(t) = 2(\ln(5t))^{3/2}$

$k(u) = \ln\left(\sqrt{u^2-4}\right)$

Logarithmic Differentiation
Using properties of logs, we can transform a function before taking it's derivative:

1. Take the natural log of both sides rewriting products and quotients as sums and differences.
2. Differentiate both sides.
3. Solve for $\frac{dy}{dx}$

Example. Differentiate

$$y = x^2(x-1)\left(x^2+4\right)^3$$