3.3: The Chain Rule

Example. Let $f(x) = \left(x^3 + x + 1\right)^2$. Find $f'(x)$

using the product rule

$$f(x) = (x^3+x+1)(x^3+x+1)$$

$$\Rightarrow f'(x) = (x^3+x+1)\frac{d}{dx}\left[x^3+x+1\right] + \frac{d}{dx}\left[x^3+x+1\right](x^3+x+1)$$

$$= (x^3+x+1)(3x^2+1) + (3x^2+1)(x^3+x+1)$$

This is the chain rule answer

$$\rightarrow = 2(x^3+x+1)(3x^2+1)$$

$$= 3x^2 \cdot 2(x^3+x+1) + 1 \cdot 2(x^3+x+1)$$

$$= 6x^5 + 6x^3 + 6x^2 + 2x^3 + 2x + 2$$

$$= \boxed{6x^5 + 8x^3 + 6x^2 + 2x + 2}$$

by expanding

$$f(x) = (x^3+x+1)(x^3+x+1) = x^3(x^3+x+1) + x(x^3+x+1) + (x^3+x+1)$$

$$= (x^6+x^4+x^3) + (x^4+x^2+x) + (x^3+x+1)$$

$$= x^6 + 2x^4 + 2x^3 + x^2 + 2x + 1$$

$$\Rightarrow f'(x) = \boxed{6x^5 + 8x^3 + 6x^2 + 2x + 2}$$

What about $\dfrac{d}{dx}\left[\left(x^3 + x + 1\right)^{100}\right]$?

Composite Functions:

Let f and g be functions of x. Then, the **composite functions** g of f (denoted $g \circ f$) and f of g (denoted $f \circ g$) are defined as:

$$(g \circ f)(x) = g(f(x))$$
$$(f \circ g)(x) = f(g(x))$$

Example. 'Break-down' the following composite functions:

$$\frac{1}{x+3}$$

$f(x) = \frac{1}{x}$

$g(x) = x+3$

$\Rightarrow f(g(x)) = \frac{1}{g(x)} = \frac{1}{x+3}$

$$\left(x^4 + 3x - 8\right)^3$$

$f(x) = x^3$

$g(x) = x^4 + 3x - 8$

$f(g(x)) = (g(x))^3 = (x^4 + 3x - 8)^3$

$$\left(\frac{1-x}{x^3+1}\right)^4$$

$f(x) = x^4$

$g(x) = \frac{1-x}{x^2+1}$

$f(g(x)) = [g(x)]^4$

$= \left(\frac{1-x}{x^3+1}\right)^4$

$$\frac{3}{\sqrt{(x+1)^2 - 1}}$$

$f(x) = \frac{3}{x}$

$g(x) = \sqrt{(x+1)^2 - 1}$ ←

This is also a composite function!!

$f(g(x)) = \frac{3}{g(x)}$

$= \frac{3}{\sqrt{(x+1)^2 - 1}}$

Rule 7: The Chain Rule

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Note:

$$\frac{d}{dx}\left[f\left(g\left(h\left(j(x)\right)\right)\right)\right] = f'\left(g\left(h\left(j(x)\right)\right)\right) \cdot g'\left(h\left(j(x)\right)\right) \cdot h'\left(j(x)\right) \cdot j'(x)$$

The General Power Rule

$$\frac{d}{dx}[(f(x))^n] = n(f(x))^{n-1}f'(x)$$

Example. Use the chain rule to show

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{d}{dx}\left[f \cdot [g]^{-1}\right] = f\frac{d}{dx}\left[g^{-1}\right] + \frac{d}{dx}\left[f\right]g^{-1}$$

$$= f(x)\,g^{-2}(-1)g' + f'g^{-1} = -\frac{fg'}{g^2} + \frac{f'}{g}\left(\frac{g}{g}\right) = \boxed{\frac{f'g - fg'}{g^2}}$$

Example. Find the derivative of the following functions

$F(x) = \left(x^3 + x + 1\right)^{100}$

$F'(x) = 100\left(x^3+x+1\right)^{99}(3x^2+1)$

$G(t) = (3t+1)^2$

$G'(t) = 2(3t+1)\cdot 3 = \boxed{6(3t+1)}$

$H(u) = \sqrt{u^2+1} - 3 = (u^2+1)^{1/2} - 3$

$H'(u) = \frac{1}{2}(u^2+1)^{-1/2}\cdot 2u = \frac{u}{(u^2+1)^{1/2}} = \boxed{\frac{u}{\sqrt{u^2+1}}}$

$J(\nu) = \nu^2(2\nu+3)^5$

$J'(\nu) = \frac{d}{d\nu}\left[\nu^2\right](2\nu+3)^5 + \nu^2\frac{d}{d\nu}\left[(2\nu+3)^5\right]$

$= 2\nu(2\nu+3)^5 + \nu^2 5(2\nu+3)^4 2$

$= \boxed{2\nu(2\nu+3)^5 + 10\nu^2(2\nu+3)^4}$

$$\kappa(x) = \left(2x^2+3\right)^4(3x-1)^5$$

$$K'(x) = \frac{d}{dx}\left[(2x^2+3)^4\right](3x-1)^5 + (2x^2+3)^4\,\frac{d}{dx}\left[(3x-1)^5\right]$$

$$= 4(2x^2+3)\cdot 2\,(3x-1)^5 + (2x^2+3)^4\,5(3x-1)^4\cdot 3$$

$$= \boxed{8\,(2x^2+3)(3x-1)^5 + 15\,(2x^2+3)^4(3x-1)^4}$$

$$\tau(x) = \frac{1}{\left(4x^2-7\right)^2} = (4x^2-7)^{-2}$$

$$\tau'(x) = -2(4x^2-7)^{-3}\cdot 8x = \boxed{\frac{-16x}{(4x^2-7)^3}}$$

Example. Find the equation of the line tangent to $f(x)$ at $\left(0, \dfrac{1}{8}\right)$

$$f(x) = \left(\frac{2x+1}{3x+2}\right)^3$$

$$f'(x) = 3\left(\frac{2x+1}{3x+2}\right)^2 \frac{d}{dx}\left[\frac{2x+1}{3x+2}\right] = 3\left(\frac{2x+1}{3x+2}\right)^2 \frac{(3x+2)2-(2x+1)3}{(3x+2)^2}$$

$$= 3\left(\frac{2x+1}{3x+2}\right)^2\frac{1}{(3x+2)^2} = \boxed{\frac{3(2x+1)^2}{(3x+2)^4}}$$

$$y - y_1 = m\,(x - x_1)$$

$f(0) = 1/8$ $\quad f'(0) = \frac{3}{16}$ $\quad 0$

$$y - 1/8 = \frac{3}{16}(x-0)$$

$$\boxed{y = \frac{3}{16}x + \frac{1}{8}}$$

Example. The membership of The Fitness Center, which opened a few years ago, is approximated by the function

$$N(t) = 100(64+4t)^{2/3} \qquad (0 \le t \le 52)$$

where $N(t)$ gives the number of members at the beginning of week t.

Find $N'(t)$

$$N'(t) = \frac{200}{3}(64+4t)^{-1/3} \cdot 4 = \frac{800}{3(64+4t)^{1/3}}$$

How fast was the center's membership increasing initially ($t = 0$)?

$$N'(0) = \frac{800}{3(64+4(0))^{1/3}} = \frac{800}{12} = \frac{200}{3} = 66.\bar{6}$$

About 67 new members per week

How fast was the membership increasing at the beginning of the 40th week?

$$N'(40) = \frac{800}{3(64+4(40))^{1/3}} \approx 50.876$$

About 51 new members per week

What was the membership when the center first opened? At the beginning of the 40th week?

$$N(0) = 100\,(64+4(0))^{2/3} = 100 \cdot 16 = 1600$$

$$N(40) = 100\,(64+4(40))^{2/3} \approx 3688.349$$

Rule 1: Derivative of a Constant

$$\frac{d}{dx}[c] = 0$$

Rule 2: The Power Rule

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

Rule 3: Derivative of a Constant Multiple of a Function

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

Rule 4: The Sum Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

Rule 5: The Product Rule

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Rule 6: The Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Rule 7: The Chain Rule

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$