6.1: Antiderivatives and the Rules of Integration

Definition. (Antiderivatives)
A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Example. Show that $F(x) = \frac{1}{3}x^3 - 2x^2 + x - 1$ is an antiderivative of $f(x) = x^2 - 4x + 1$.

$$F'(x) = 3\frac{1}{3}x^{3-1} - 2 \cdot 2x^{2-1} + x^{1-1} + 0$$

$$= x^2 - 4x + 1$$

$$= f(x)$$

Example. Let $F(x) = x$, $G(x) = x + 2$, and $H(x) = x + C$, where C is a constant. Show that F, G, and H are all antiderivatives of $f(x) = 1$.

$F(x) = x$ $\qquad$ $F'(x) = 1$

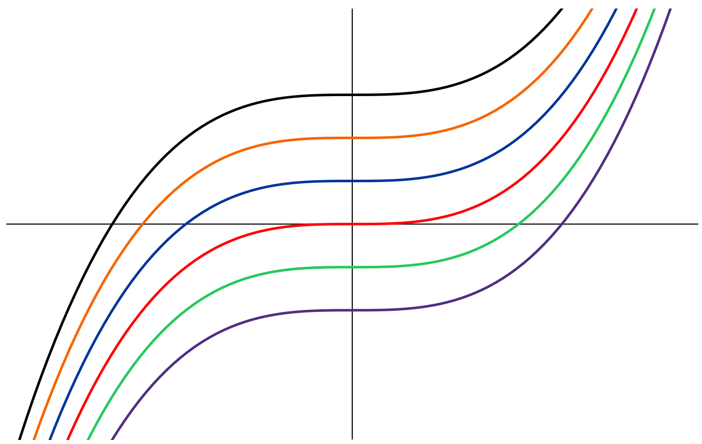
$G(x) = x + 2$ $\qquad$ $G'(x) = 1$

$H(x) = x + C$ $\qquad$ $H'(x) = 1$

$$\left.\begin{aligned} F'(x) &= 1 \\ G'(x) &= 1 \\ H'(x) &= 1 \end{aligned}\right\} = f(x)$$

Theorem 1
Let G be an antiderivative of a function f on an interval I. Then, every antiderivative of F of f on I must be of the form $F(x) = G(x) + C$, where C is a constant.

Example. If $f'(x) = x^2$, then $f(x) = \dfrac{x^3}{3} + C$ is the family of antiderivatives of $f'(x)$.



Definition. (Integration)
The process of finding the antiderivative is called **integration**:

$$\int f(x)\,dx = F(x) + C$$

The **indefinite integral** of f is the family of functions given by $F(x) + C$ where $F'(x) = f(x)$. The function to be integrated, f, is called the **integrand**. C is the **constant of integration**.

Rule 1: The Indefinite Integral of a Constant

$$\int k\,dx = kx + C \qquad (k,\ \text{a constant})$$

Rule 2: The Power Rule

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1} + C \qquad (n \neq -1)$$

Example. Find each of the following indefinite integrals

$$\int 2\,dx = \boxed{2x + C}$$

$$\int \pi^2\,dx = \boxed{\pi^2 x + C}$$

$$\int x^3\,dx = \boxed{\frac{x^4}{4} + C}$$

$$\int \frac{1}{x^{3/2}}\,dx = \int x^{-3/2}\,dx$$

$$= \frac{x^{-1/2}}{-1/2} + C$$

$$= \boxed{-2x^{-1/2} + C}$$

Rule 3: The Indefinite Integral of a Constant Multiple of a Function

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad (c,\ \text{a constant})$$

Rule 4: The Sum Rule

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Example. Find each of the following indefinite integrals

$$\int \frac{1}{5} - \frac{2}{t^3} + 2t\,dt = \frac{1}{5}\int dt - 2\int t^{-3}\,dt + 2\int t\,dt$$

$$= \frac{1}{5}t - 2\frac{t^{-2}}{-2} + 2\frac{t^2}{2} + C$$

$$= \boxed{\frac{t}{5} + \frac{1}{t^2} + t^2 + C}$$

$$\int 3x^5 + 4x^{3/2} - 2x^{-1/2}\,dx = 3\int x^5\,dx + 4\int x^{3/2}\,dx - 2\int x^{-1/2}\,dx$$

$$= \frac{3x^6}{6} + \frac{4x^{5/2}}{5/2} - \frac{2x^{1/2}}{1/2} + C$$

$$= \boxed{\frac{x^6}{2} + \frac{8x^{5/2}}{5} - 4x^{1/2} + C}$$

Rule 5: The Indefinite Integral of the Exponential Function

$$\int e^x \, dx = e^x + C$$

Rule 6: The Indefinite Integral of the Function $f(x) = x^{-1}$

$$\int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln|x| + c \qquad (x \neq 0)$$

Example. Find each of the following indefinite integrals

$$\int 2e^x - x^3 + x^e - e^e \, dx$$

$$= \boxed{2e^x - \frac{1}{4}x^4 + \frac{1}{e+1}x^{e+1} - e^e x + C}$$

$4x^{-2}$

$$\int 2x + \frac{3}{x} + \frac{4}{x^2} \, dx$$

$$= \boxed{x^2 + 3\ln|x| - \frac{4}{x} + C}$$

$$\int \frac{2}{\sqrt{x}} - \frac{2}{x} \, dx$$

$$= \int 2x^{-1/2} - \frac{2}{x} \, dx$$

$$= \boxed{4x^{1/2} - 2\ln|x| + C}$$

$$\int \frac{1}{4e^x} - \frac{4}{x} + e^x \, dx$$

$$= \int \frac{1}{4}e^{-x} - \frac{4}{x} + e^x \, dx$$

$$= \boxed{-\frac{1}{4}e^{-x} - 4\ln|x| + e^x + C}$$

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