5.1: Exponential Functions

Properties of Exponents:

- If m is a positive integer, then $x^m = \underbrace{x \cdot x \ldots x}_{m \text{ times}}$:

$$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81, \quad (-5)^3 = (-5) \cdot (-5) \cdot (-5) = -125$$

- Additivity — If the bases are the same, then $x^a \cdot x^b = x^{a+b}$ and $\dfrac{x^a}{x^b} = x^{a-b}$:

$$4^3 \cdot 4^2 = 4^{3+2} = 4^5 = 1024, \qquad \frac{3^{17}}{3^{12}} = 3^{17-12} = 3^5 = 243$$

- If $x \neq 0$, then $x^0 = 1$: 0^0 undefined

$$4^0 = 1, \qquad (-7)^0 = 1, \qquad 2024^0 = 1$$

- Distributive — $(ab)^m = a^m b^m$ and $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$: $(a+b)^m \neq a^m + b^m$

$$(3 \cdot 4)^2 = 3^2 \cdot 4^2 = 9 \cdot 16 = 144, \qquad \left(\frac{4}{5}\right)^2 = \frac{4^2}{5^2} = \frac{16}{25}$$

- If $m \neq 0$, then $x^{-m} = \dfrac{1}{x^m}$ and $x^m = \dfrac{1}{x^{-m}}$:

$$2^{-4} = \frac{1}{2^4} = \frac{1}{16}, \qquad \left(\frac{1}{3}\right)^{-2} = 3^2 = 9$$

- Multiplicity — $(x^a)^b = x^{ab}$:

$$(3^2)^4 = 3^8 = 6561$$

$(x^3)^{1/3} = (8)^{1/3}$

$x = \sqrt[3]{8} = 2$

- Fractional exponents — $x^{1/m} = \sqrt[m]{x}$:

$$8^{1/3} = \sqrt[3]{8} = 2, \qquad 16^{3/2} = \left(16^{1/2}\right)^3 = \left(\sqrt{16}\right)^3 = 4^3 = 64$$

Definition.
An **exponential function** is of the form

$$f(x) = a^x$$

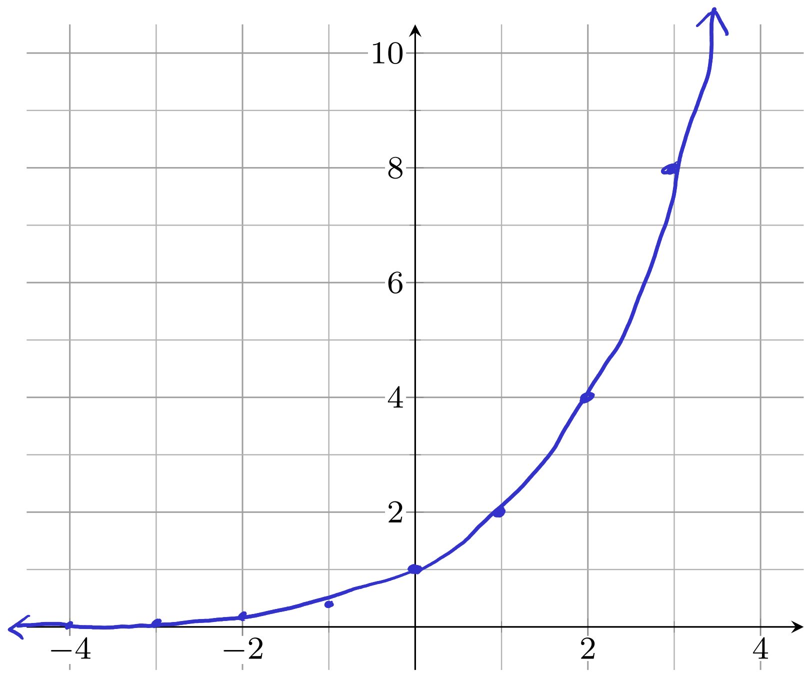
where $a > 0$ and $a \neq 1$. *Note:* The variable is in the exponent (e.g. 2^x vs x^2)

Example. Suppose a culture of bacteria has the property that each minute, every microorganism splits into two new organisms. The number of microorganisms after x minutes is given by

$$y = 2^x.$$

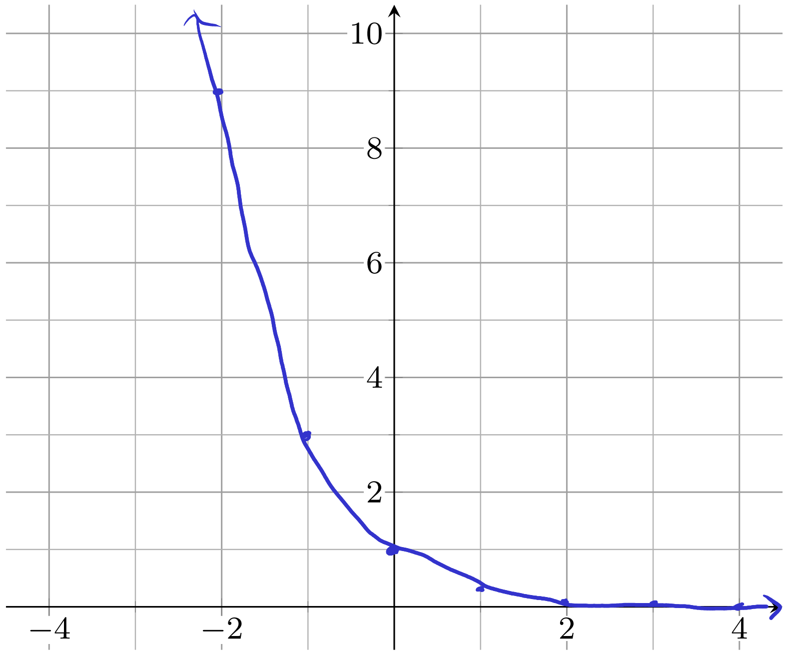
Fill out the table below, and graph this exponential function (include $x < 0$).

x	$y = 2^x$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$
4	$2^4 = 16$



Example. Graph the exponential function

$$y = \left(\frac{1}{3}\right)^x = \frac{1}{3^x} = 3^{-x}$$



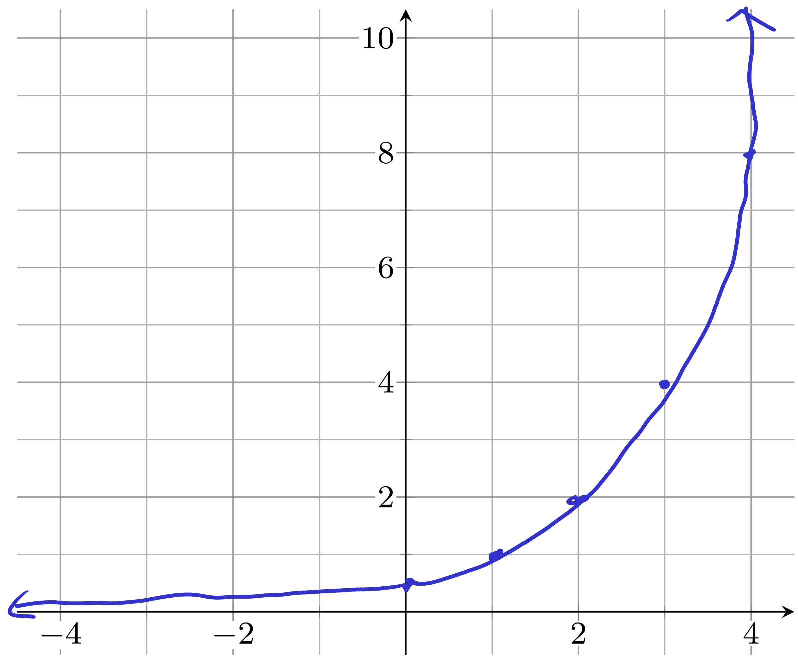
x	$\left(\frac{1}{3}\right)^x$
0	1
1	$1/3$
2	$1/9$
$\vdots$	
-1	3^1
-2	3^2

For an exponential function a^x, the function is

- increasing if $a > 1$, and
- decreasing if $0 < a < 1$.

Example. Graph the exponential function

$$y = \frac{1}{2}(2)^x = 2^{-1}(2^x) = 2^{-1+x} = 2^{x-1}$$



$2 > 1$ inc

x	$\frac{1}{2}(2^x)$
0	$\frac{1}{2}(1)$
1	$\frac{1}{2}(2)$
2	$\frac{1}{2}(4)$
3	$\frac{1}{2}(8)$

Example. Compound Interest:

If an initial principal P is invested at a rate r and compounded n times a year, the future value in t years is given by:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$n=4$

Suppose that \$800 dollars is invested, and is compounded quarterly at a rate of 6%:

$$A = 800(1.015)^{4t}$$

Find the future value after 10 years.

$$800(1.015)^{4(10)} = 800(1.015)^{40}$$
$$= 800(1.814\ldots)$$
$$= \boxed{\$1451.21}$$

Example. Compounded continuously:

A special function that frequently occurs in the context of exponential functions is

$$y = e^x$$

where $e = 2.71828...$ (think irrational number like π). When an investment is compounded continuously, it's future value is given by

$$A = Pe^{rt}$$

0.06

Suppose that we invest \$800, compounded continuously at 6%. Find the future value in 10 years.

$$800 \cdot e^{0.06(10)} = 800 \cdot e^{0.6}$$
$$= \boxed{\$1457.70}$$