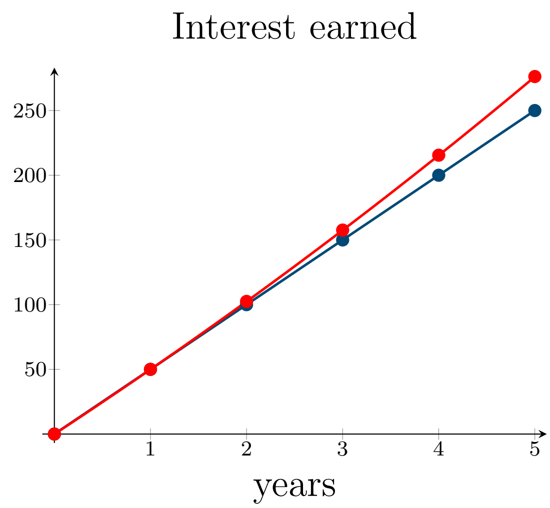
6.2: Compound Interest

Example. Suppose you invest \$1,000 at 5% annual interest. With simple interest, you can take 2 approaches:

1. Gain interest on *only* your initial investment
2. Reinvest the interest gained

Year	Simple interest	Simple interest reinvested
1	\$1,050.00	\$1,050.00
2	\$1,100.00	\$1,102.50
3	\$1,150.00	\$1,157.63
4	\$1,200.00	\$1,215.51
5	\$1,250.00	\$1,276.28



> **Definition.**
> **Compound interest** is a method where the interest for each period is added to the principal before interest is calculated for the next period.

Example. Using the example above, derive a formula for the future value of an investment compounded annually.

$1: \; 1000(1+0{,}5) = \$1050{,}00$

$2: \; 1050(1+0{,}5) = [1000(1+0{,}5)](1+0{,}5) = 1000(1+0{,}5)^2 = \$1102{,}50$

$3: \; 1102{,}50\,(1+0{,}5) = [1000(1+0{,}5)^2](1+0{,}5) = 1000(1+0{,}5)^3 = \$1157.63$

$\Rightarrow \; n: \; 1000(1+0{,}5)^n \;\Rightarrow\; \boxed{S = P(1+r)^n}$

Definition.
When interest is compounded multiple times a year (e.g. quarterly, monthly, etc.), the **nominal interest rate** is the interest rate *per year*.

If $\$P$ is invested for t years at a nominal interest rate r compounded m times per year, then the **total number of compounding periods** is

$$n = mt$$

the interest rate per compounding period (**periodic interest rate**) is

$$i = \frac{r}{m}$$

and the future value is

$$S = P(1+i)^n = P\Big(1 + \frac{r}{m}\Big)^{mt}$$

Example. If $\underbrace{\$3,000}_{P}$ is invested for $\underbrace{5}_{t}$ years at $\underbrace{9\%}_{r}$ compounded $\underbrace{4}_{m}$ times a year, how much interest is earned?

$$S = 3000\left(1 + \frac{0.09}{4}\right)^{4(5)}$$
$$= 3000(1.0225)^{20}$$
$$= 3000(1.5605\ldots)$$
$$= \$4681.53$$

$$S = P + I$$
$$4681.53 = 3000 + I \Rightarrow \boxed{I = \$1681.53}$$

Futurama

Example. For the following, identify the annual interest rate, the length in years, the periodic interest rate, and the number of periods:

Frequency	m
Annually	1
Semi-annually	2
Quarterly	4
Monthly	12
Weekly	52
Daily	365

$\underbrace{12\%}_{r=0.12}$ $\underbrace{\text{compounded monthly}}_{m=12}$ for $\underbrace{7 \text{ years}}_{t=7}$ $\quad i = \frac{r}{m} = \frac{12\%}{12} = 1\%$

$mt = 84$

$\underbrace{7.2\%}_{r=0.072}$ $\underbrace{\text{compounded quarterly}}_{m=4}$ for $\underbrace{11 \text{ quarters}}_{mt=11}$ $\quad i = \frac{r}{m} = \frac{7.2\%}{4} = 1.8\%$

$\Rightarrow t = \frac{11}{4} = 2.75$

Example. Ben and Taylor want to have \$200,000 in Arthur's college fund on his 18th birthday, and they want to know the impact on this goal of having \$10,000 invested at 9.8%, compounded quarterly, on his 1st birthday. To advise Ben and Taylor regarding this, find

the future value of the \$10,000 investment,

$P = 10,000$
$r = 9.8 = 0.098$
$t = 18-1 = 17 \text{ yrs}$
$m = 4$

$$S = 10,000\left(1+\frac{0.098}{4}\right)^{4(17)}$$
$$= 10,000(1.0245)^{68}$$
$$= 10,000(5.1858) = \boxed{\$51,857.73}$$

the amount of compound interest that the investment earns,

$S = P + I$

$51,857.73 = 10,000 + I \Rightarrow 41,857.73$

the impact this would have on their goal.

$$\frac{51,857.73}{200,000}(100\%) = 25.9\%$$

They'll earn about 25.9% of their goal.

Example. What amount must be invested now to have $\overbrace{\$12,000}^{S}$ after $\overset{t}{\widetilde{3}}$ years with an interest rate of $\underset{r=0.06}{6\%}$, compounded $\underbrace{\text{semi-annually}}_{m=2}$?

Find P

$$S = P\left(1+\frac{r}{m}\right)^{mt} \Rightarrow 12000 = P\left(1+\frac{0.06}{2}\right)^{2(3)}$$

$$\frac{12000}{(1.03)^6} = \frac{P(1.03)^6}{(1.03)^6}$$

$$\frac{12000}{(1.03)^6} = \boxed{P \approx \$10{,}049.81}$$

Example. Three years after Google stock was first sold publicly, its share price had risen 500%. Google's 500% increase means that $10,000 invested in Google stock at its initial public offering (I.P.O.) was worth $60,000 three years later. What interest rate compounded annually does this represent?

$S = 60000$

$P = 10\,000$

$r \leftarrow$ find

$m = 1$

$t = 3$

$$\frac{60000}{10000} = \frac{10\,000\left(1+\frac{r}{1}\right)^{1(3)}}{10000}$$

$$(6)^{1/3} = \left((1+r)^3\right)^{1/3}$$

$$-1 + 6^{1/3} = 1 + r - 1$$

$$6^{1/3} - 1 = \boxed{\begin{aligned} r &\approx 0.817 \\ &= 81.7\% \end{aligned}}$$

Example. Suppose we invest \$1 at a 100% interest rate for 1 year:

$$S = \left(1 + \frac{1.00}{m}\right)^m$$

Compute the future value

Annually

$S = \left(1 + \frac{1.00}{1}\right)^1 = 2$

Semi-annually

$S = \left(1 + \frac{1.00}{2}\right)^2 = 2.25$

Monthly

$S = \left(1 + \frac{1.00}{12}\right)^{12} = 2.6130$

Weekly

$S = \left(1 + \frac{1.00}{52}\right)^{52} = 2.6926$

Daily

$S = \left(1 + \frac{1.00}{365}\right)^{365} = 2.7146$

Each minute ($m = 525,600$)

$S = \left(1 + \frac{1.00}{1}\right)^1 = 2.7183$

Definition.
If $\$P$ is invested for t years at a nominal rate r compounded continuously, then the future value is given by the exponential function

$$S = Pe^{rt}$$

Example. Which investment strategy is worth more: $\$3,000$ for 8 years at

9%, compounded annually

$$S = 3000\left(1+\frac{0.09}{1}\right)^{1(8)} = 3000(1.09)^8 = 3000(1.9926) = \$5977.69$$

8%, compounded continuously

$$S = 3000e^{0.08(8)} = 3000e^{0.64} = 3000(1.8965) = \$5689.44$$

Example. Suppose you invest $\$900$ at 11.5%, compounded continuously. How long will it take to gain $\$700$ in interest?

$$S = P + I$$

$$\Rightarrow I = S - P = Pe^{rt} - P = P\left[e^{rt} - 1\right]$$

$\boxed{\text{Find } t}$

$$\frac{700}{900} = \frac{900\left(e^{0.115t} - 1\right)}{900}$$

$$+1 + \frac{7}{9} = e^{0.115t} - 1 + 1$$

$$\ln\left(\frac{16}{9}\right) = \ln\left(e^{0.115t}\right)$$

$$\frac{\ln\left(\frac{16}{9}\right)}{0.115\ln(e)} = \frac{0.115t\ln(e)}{0.115\ln(e)}$$

$$\to t = \frac{\ln\left(\frac{16}{9}\right)}{0.115\ln(e)}$$

$$\boxed{t \approx 5.0032}$$

About 5 years

Definition.
Let r represent the annual (nominal) interest rate for an investment. Then the **annual percentage yield (APY)** is:

Periodic compounding:

$$\text{APY} = \left(1 + \frac{r}{m}\right)^m - 1$$

Continuous compounding:

$$\text{APY} = e^r - 1$$