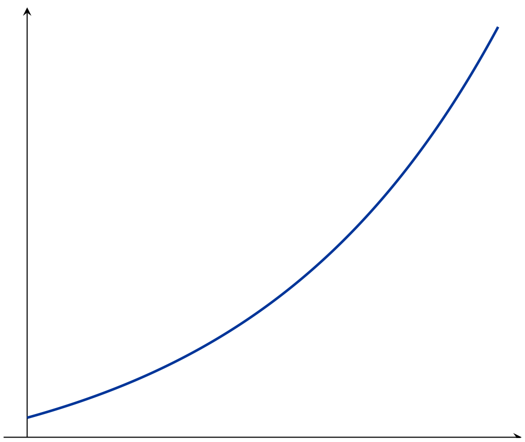
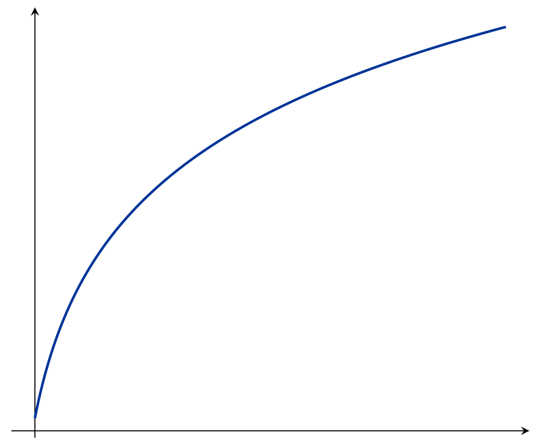
4.2: Applications of the Second Derivative

Definition.
Consider any differentiable function $f(x)$ on the interval (a, b). We say f is

concave up if $f'(x)$ is increasing



concave down if $f'(x)$ is decreasing

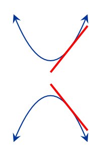


Thus, for every value of x on the interval (a, b), if

- $f''(x) > 0$, then f' is increasing, and f is concave *up* on (a, b).
- $f''(x) < 0$, then f' is decreasing, and f is concave *down* on (a, b).
- If f is continuous at c and f changes concavity at c, then f has an **inflection point** at c.

Note: $f(x)$ is

- concave up if its tangent lines lie below the curve
- concave down if its tangent lines lie above the curve



Determining the Intervals of Concavity of the Graph of f

1. Determine the values of x for which f'' is zero or undefined.
2. Determine the sign of $f''(x)$ to the left and right of each point from above:
   Let c be a convenient test point on the interval of interest. Then,
   a) if $f''(c) > 0$, then f is concave up on that interval. 
   b) if $f''(c) < 0$, then f is concave down on that interval. 

Example. Find the intervals where the following functions are concave up and concave down:

$$f(x) = x^3 - 3x^2 - 24x + 32$$

Graph

$$g(x) = (x+1)^{2/3}$$

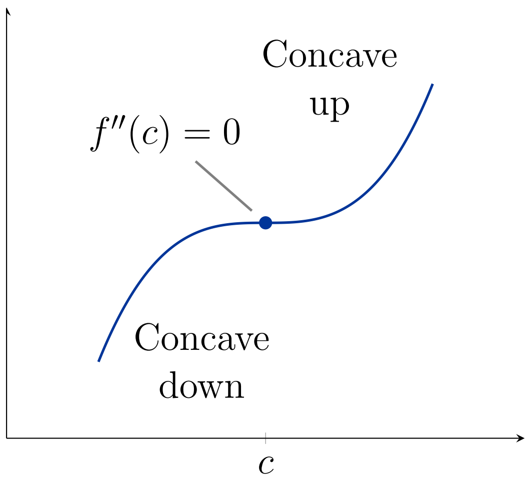
$$h(x) = x + \frac{1}{x}$$

$$j(x) = \frac{x^2}{1 - x^2}$$

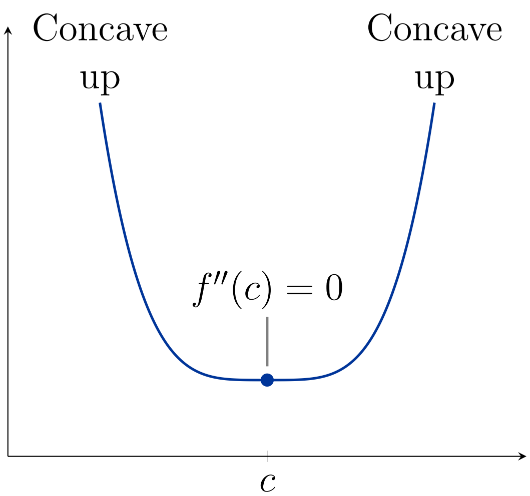
Finding inflection points

1. Compute $f''(x)$.
2. Locate where $f''(x) = 0$ or $f''(x)$ does not exist.
3. Determine if the sign of $f''(x)$ changes at the points found above.

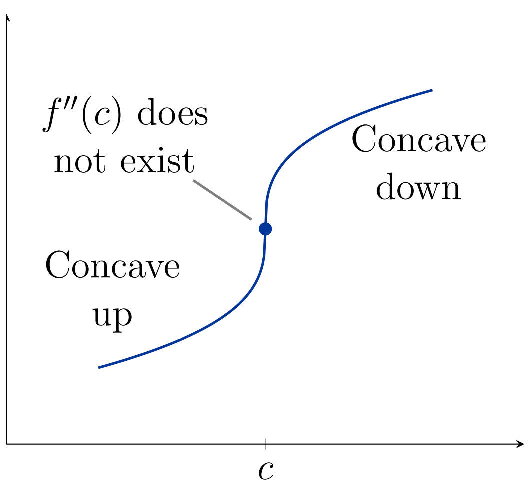
Inflection point at $x = c$



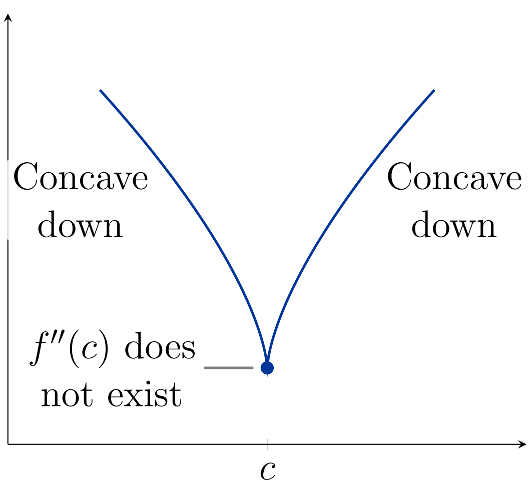
No inflection point at $x = c$



Inflection point at $x = c$



No inflection point at $x = c$



Example. For the following functions, determine the intervals of concavity and find any inflection points.

$f(x) = (x-1)^{5/3}$ Graph

$$g(x) = \frac{1}{x^2+1}$$

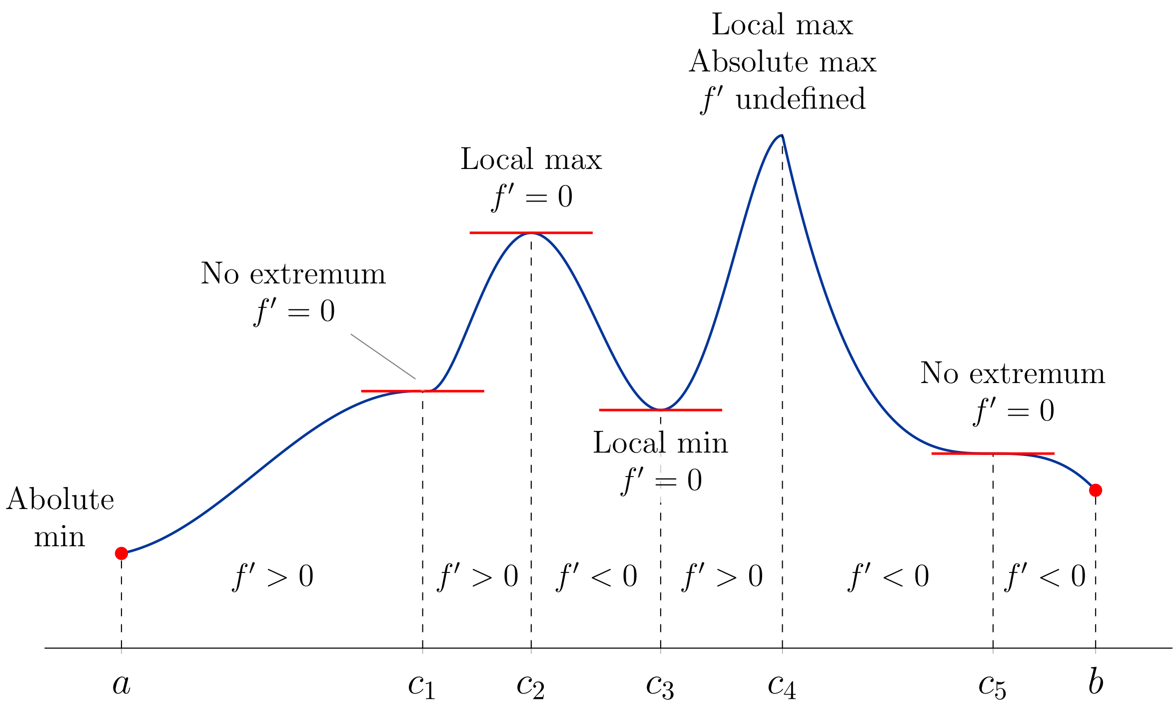
Second Derivative Test for Local Extrema
Suppose f'' is continuous on an open interval containing c with $f'(c) = 0$.

- If $f''(c) > 0$, then f has a local minimum at c.
- If $f''(c) < 0$, then f has a local maximum at c.
- If $f''(c) = 0$, then the test is inconclusive; f may have a local maximum, local minimum, or neither at c.

Example. Find the relative extrema of

$$f(x) = x^3 - 3x^2 - 24x + 32$$

Graph



$f(x)$	$f'(x)$	$f''(x)$
increasing	positive	—
decreasing	negative	—
max/min	crit. pt. & changes sign	—
concave up	increasing	positive
concave down	decreasing	negative
Inflection point	max/min	crit. pt. & changes sign