3.6: Implicit Differentiation and Related Rates

Implicit Functions	Explicit Functions	Graph
$x^2y + y - x^2 + 1 = 0$	$y = \dfrac{x^2 - 1}{x^2 + 1}$	
$x^2 + y^2 = 4$	$y = \pm\sqrt{4 - x^2}$	
$y^3 + y^2 - xy + \dfrac{x^4}{4} = y$		

Implicit Differentiation:

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with ${}^{dy}/_{dx}$ on one side of the equation.
3. Solve for ${}^{dy}/_{dx}$.

Example. Find the derivatives of the following by rewriting each function explicitly before taking the derivative, and by using implicit differentiation. Compare the results.

$y^2 = x$

Explicitly:

$$y = \pm\sqrt{x} = \pm x^{1/2}$$

$$y' = \frac{dy}{dx} = \pm\frac{1}{2}x^{-1/2}$$

$$\frac{dy}{dx} = \boxed{\pm\frac{1}{2\sqrt{x}}}$$

← equivalent →

Implicitly:

$$\frac{d}{dx}\left[y^2\right] = \frac{d}{dx}\left[x\right]$$

$$2y^1\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \boxed{\frac{1}{2y}}$$

$\sqrt{x} + \sqrt{y} = 4$

Explicitly:

$$\sqrt{y} = 4 - \sqrt{x} \to y = (4-\sqrt{x})^2$$

$$\frac{dy}{dx} = 2(4-\sqrt{x})^1\left(-\frac{1}{2\sqrt{x}}\right)$$

$$= \boxed{-\frac{4-\sqrt{x}}{\sqrt{x}}}$$

← equivalent →

Implicitly:

$$\frac{d}{dx}\left[\sqrt{x} + \sqrt{y}\right] = \frac{d}{dx}\left[4\right]$$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \boxed{-\frac{\sqrt{y}}{\sqrt{x}}}$$

Example. Find $\frac{dy}{dx}$ given the equation

$$\frac{d}{dx}\left[y^3 - y + 2x^3 - x\right] = \frac{d}{dx}\left[8\right]$$

$$3y^2 \frac{dy}{dx} - 1\frac{dy}{dx} + 6x^2 - 1 = 0$$

$$\frac{dy}{dx}(3y^2 - 1) = 1 - 6x^2$$

$$\boxed{\frac{dy}{dx} = \frac{1-6x^2}{3y^2-1}}$$

Note: This derivative is also an implicit function

Example. Consider the equation $x^2 + y^2 = 4$.

Find $\dfrac{dy}{dx}$ by implicit differentiation.

$$\frac{d}{dx}\left[x^2+y^2\right] = \frac{d}{dx}\left[4\right]$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

Find the slope of the tangent line to the graph of the function $y = f(x)$ at the point $(1, \sqrt{3})$.

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,\sqrt{3})} = \boxed{\frac{-1}{\sqrt{3}}}$$

Find an equation of the tangent line.

$$y - y_1 = m\,(x - x_1)$$

$$y - \sqrt{3} = \left.\frac{dy}{dx}\right|_{(1,\sqrt{3})} (x-1)$$

$$y = -\frac{1}{\sqrt{3}}(x-1) + \sqrt{3} = \boxed{-\frac{\sqrt{3}}{3}x + \frac{4\sqrt{3}}{3}}$$

Related Rates:
Related rates are problems that use a mathematical relationship between two or more objects under specific constraints. From this, we can differentiate this relationship and examine how each variable changes with respect to time.

The volume of a cone with radius r and height h is given by

$$\frac{d}{dt}\left[V\right] = \frac{d}{dt}\left[\frac{1}{3}\pi r^2 h\right]$$

Find dV/dt when r and h are changing.

$$\frac{dV}{dt} = \frac{\pi}{3}\left(\frac{d}{dt}\left[r^2\right] h + r^2 \frac{d}{dt}\left[h\right]\right)$$

$$\boxed{\frac{dV}{dt} = \frac{\pi}{3}\left(2r\frac{dr}{dt} h + r^2 \frac{dh}{dt}\right)}$$

Find dV/dt when r is constant and h is changing.

$$\frac{dV}{dt} = \frac{\pi}{3} r^2 \frac{d}{dt}\left[h\right]$$

$$\boxed{\frac{dV}{dt} = \frac{\pi}{3} r^2 \frac{dh}{dt}}$$

Find dV/dt when r is changing and h is constant.

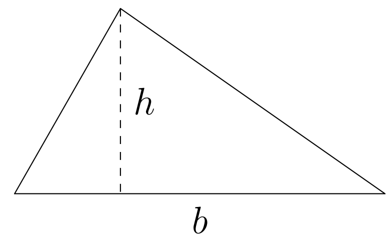
$$\frac{dV}{dt} = \frac{\pi}{3} \frac{d}{dt}\left[r^2\right] h$$

$$\boxed{\frac{dV}{dt} = \frac{\pi}{3} 2r \frac{dr}{dt} h}$$

Example. The altitude of a triangle is increasing at a rate of $1\,cm/min$ while the area of the triangle is increasing at a rate of $2\,cm^2/min$. How fast is the base of the triangle changing when the altitude is $10\,cm$ and the area is $100\,cm^2$.

$$\frac{dh}{dt} = 1\,\frac{cm}{min} \qquad \frac{dA}{dt} = 2\,\frac{cm^2}{min}$$

$$h = 10\,cm \qquad A = 100\,cm^2$$



$$A = \frac{1}{2} b h \longrightarrow 100 = \frac{1}{2} b \cdot 10 \Rightarrow \boxed{b = 20}$$

$$\frac{dA}{dt} = \frac{1}{2}\left[\frac{db}{dt} h + b \frac{dh}{dt}\right]$$

$$2 = \frac{1}{2}\left[10 \frac{db}{dt} + 20 \cdot 1\right]$$

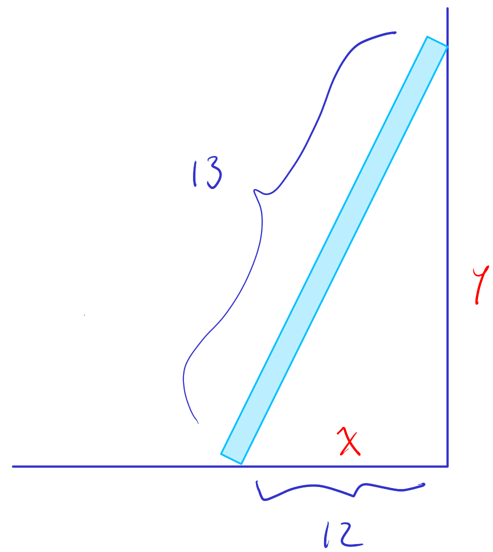
$$4 = 10 \frac{db}{dt} + 20$$

$$\boxed{-\frac{8}{5}\,\frac{cm}{min} = \frac{db}{dt}}$$

Example. The base of a 13-ft ladder leaning against a wall begins to slide away from the wall. At the instant of time when the base is 12 ft from the wall, the base is moving at a rate of 8 ft/sec. How fast is the top of the ladder sliding down the wall at that instant of time?

$x = 12 \qquad \frac{dx}{dt} = 8 \text{ ft/sec}$

Find $\frac{dy}{dt}$



$x^2 + y^2 = 13^2 \longrightarrow y = \sqrt{13^2 - 12^2} = 5$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{12}{5}\cdot 8 = -\frac{96}{5} = \boxed{-19.2 \text{ ft/sec}}$