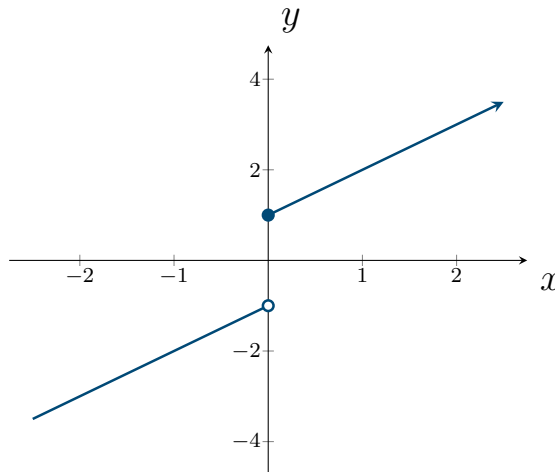


## 2.5: One-Sided Limits and Continuity

Consider the function

$$f(x) = \begin{cases} x - 1, & x < 0 \\ x + 1, & x \geq 0 \end{cases}$$

What is  $\lim_{x \rightarrow 0} f(x)$ ?



### Definition. (One-Sided Limits)

The function  $f$  has a **right-hand limit**  $L$  as  $x$  approaches  $a$  from the right, written

$$\lim_{x \rightarrow a^+} f(x) = L$$

if the values of  $f(x)$  can be made as close to  $L$  as we please by taking  $x$  sufficiently close to (but not equal to)  $a$  and to the right of  $a$ .

The function  $f$  has a **left-hand limit**  $L$  as  $x$  approaches  $a$  from the left, written

$$\lim_{x \rightarrow a^-} f(x) = L$$

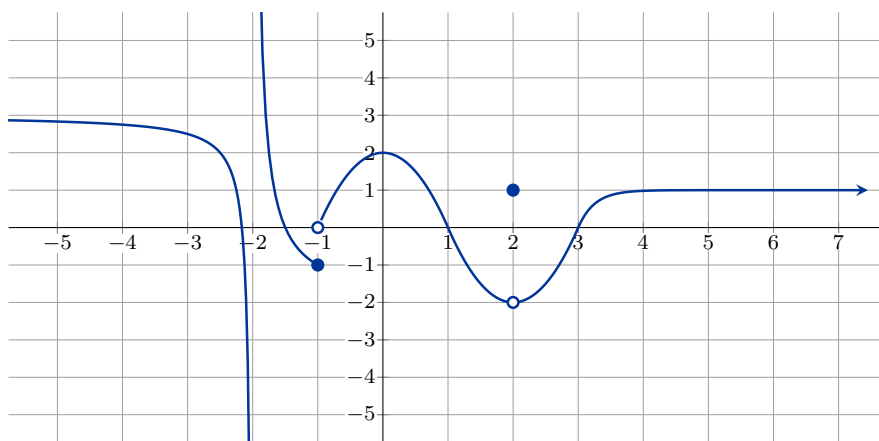
if the values of  $f(x)$  can be made as close to  $L$  as we please by taking  $x$  sufficiently close to (but not equal to)  $a$  and to the left of  $a$ .

**Theorem 3**

Let  $f$  be a function that is defined for all values of  $x$  close to  $x = a$  with the possible exception of  $a$  itself. Then

$$\lim_{x \rightarrow a} f(x) = L \quad \text{if and only if} \quad \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L$$

**Example.** Using the graph below, evaluate the following limits:



$$\lim_{x \rightarrow -2^-} f(x)$$

$$\lim_{x \rightarrow -2^+} f(x)$$

$$\lim_{x \rightarrow -2} f(x)$$

$$\lim_{x \rightarrow -1^-} f(x)$$

$$\lim_{x \rightarrow -1^+} f(x)$$

$$\lim_{x \rightarrow -1} f(x)$$

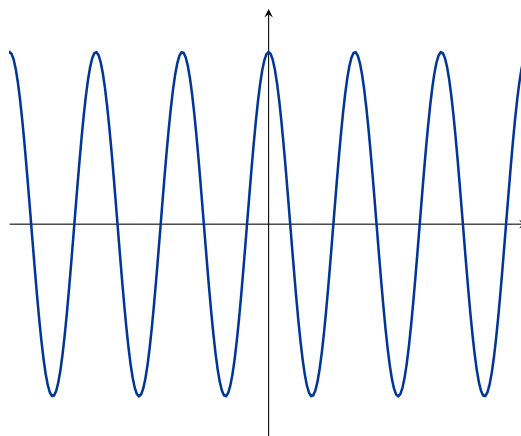
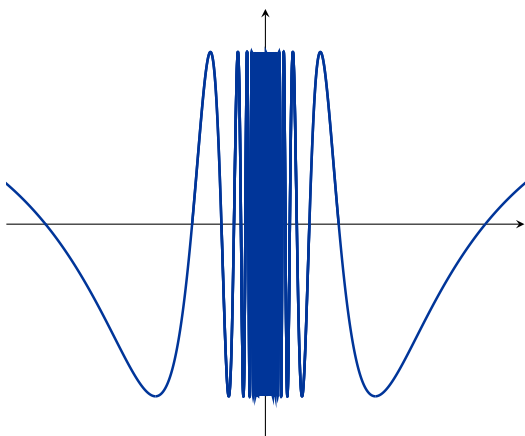
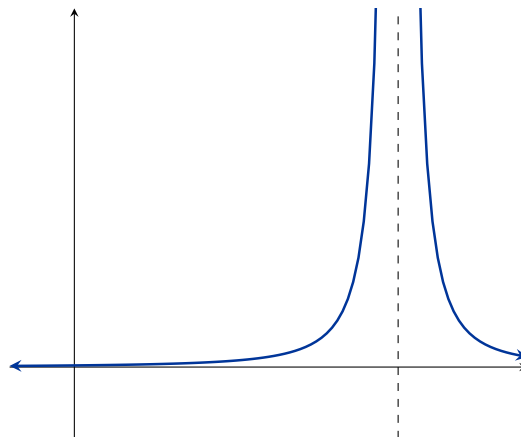
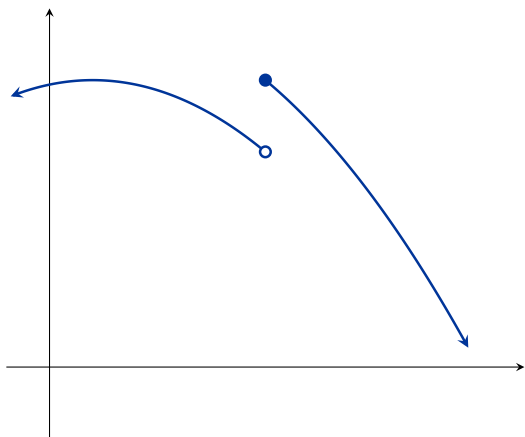
$$\lim_{x \rightarrow 1} f(x)$$

$$\lim_{x \rightarrow 2} f(x)$$

$$\lim_{x \rightarrow \infty} f(x)$$

Below are examples where the limit does not exist:

[Graph](#)



**Definition. (Continuity of a Function at a Number)**

A function  $f$  is **continuous** at  $a$  if  $\lim_{x \rightarrow a} f(x) = f(a)$ .

**Continuity Checklist:**

In order for  $f$  to be continuous at  $a$ , the following three conditions must hold:

1.  $f(a)$  is defined ( $a$  is in the domain of  $f$ ),
2.  $\lim_{x \rightarrow a} f(x)$  exists,
3.  $\lim_{x \rightarrow a} f(x) = f(a)$  (the value of  $f$  equals the limit of  $f$  at  $a$ ).

**Example.** Determine the values of  $x$  for which the following functions are continuous:

$$f(x) = 3x^3 + 2x^2 - x + 10$$

$$g(x) = \frac{8x^{10} - 4x + 1}{x^2 + 1}$$

$$h(x) = \frac{4x^3 - 3x^2 + 1}{x^2 - 3x + 1}$$

**Example.** Determine whether the following are continuous at  $a$ :

$$f(x) = x^2 + \sqrt{7-x}, \quad a = 4$$

$$g(x) = \frac{1}{x-3}, \quad a = 3$$

$$h(x) = \begin{cases} \frac{x^2+x}{x+1}, & x \neq -1 \\ 0, & x = -1 \end{cases}, \quad a = -1$$

$$j(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases}, \quad a = 0$$

$$k(x) = \begin{cases} \frac{x^2+x-6}{x^2-x}, & x \neq 2 \\ -1, & x = 2 \end{cases}, \quad a = 2$$

## Properties of Continuous Functions

1. The constant function  $f(x) = c$  is continuous everywhere.
2. The identity function  $f(x) = x$  is continuous everywhere.

If  $f$  and  $g$  are continuous at  $x = a$ , then

$[f(x)]^n$ , where  $n$  is a real number, is continuous at  $x = a$  whenever it is defined at that number

$f \pm g$  is continuous at  $x = a$

$fg$  is continuous at  $x = a$

$f/g$  is continuous at  $x = a$  provided that  $g(a) \neq 0$

## Polynomial and Rational Functions

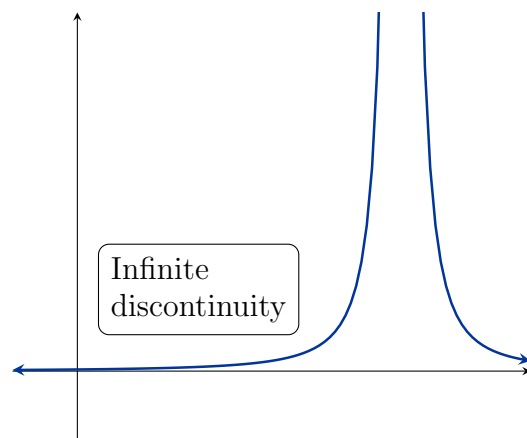
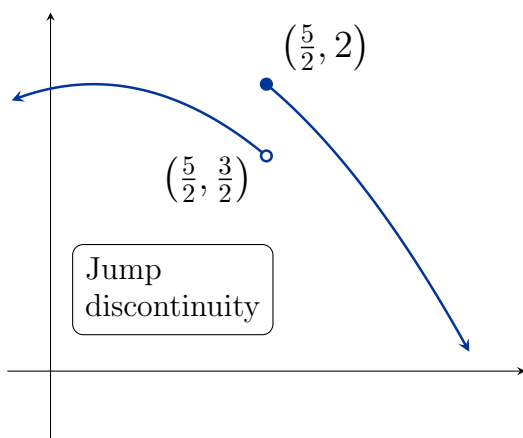
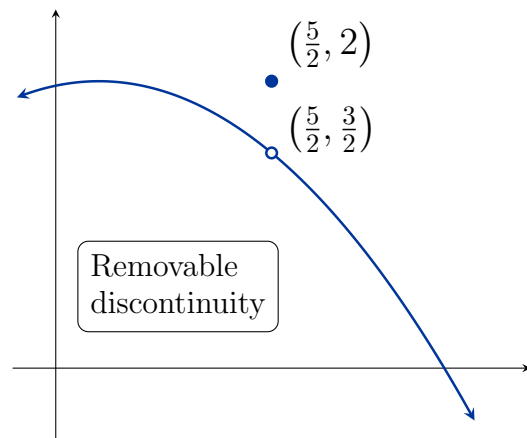
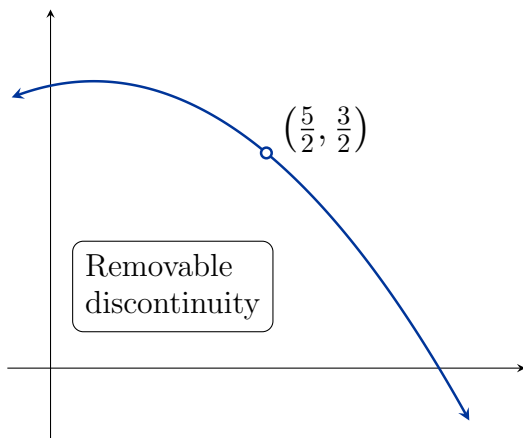
1. A polynomial function is continuous for all  $x$ .
2. A rational function (a function of the form  $\frac{p}{q}$ , where  $p$  and  $q$  are polynomials) is continuous for all  $x$  for which  $q(x) \neq 0$ .

## Definition.

A **removable discontinuity** at  $x = a$  is one that disappears when the function becomes continuous after defining  $f(a) = \lim_{x \rightarrow a} f(x)$ .

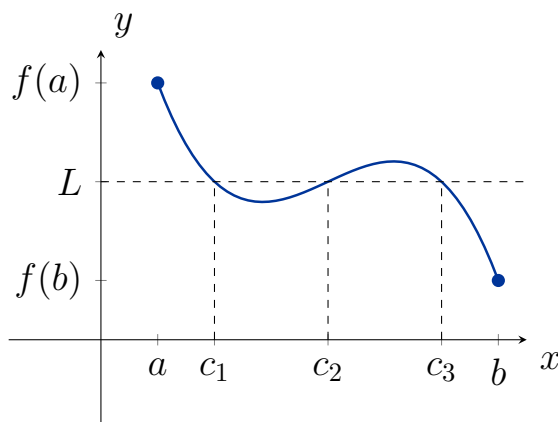
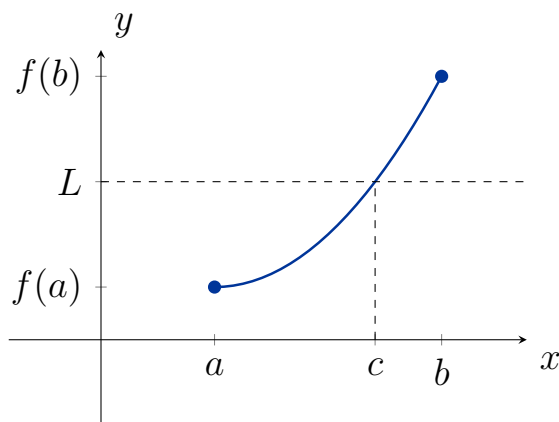
A **jump discontinuity** is one that occurs whenever  $\lim_{x \rightarrow a^-} f(x)$  and  $\lim_{x \rightarrow a^+} f(x)$  both exist, but  $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$ .

A **vertical discontinuity** occurs whenever  $f(x)$  has a vertical asymptote.

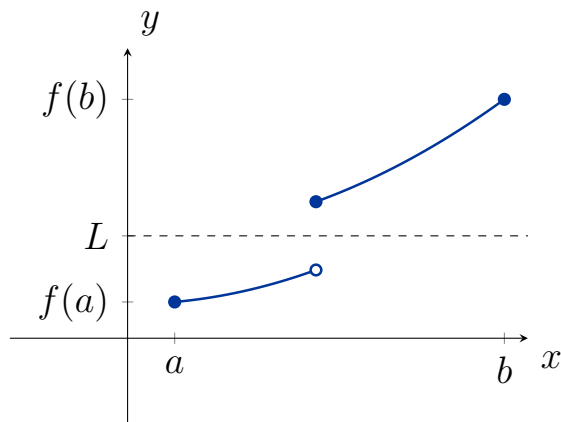


**Theorem 4: Intermediate Value Theorem**

Suppose  $f$  is continuous on the interval  $[a, b]$  and  $L$  is a number strictly between  $f(a)$  and  $f(b)$ . Then there exists at least one number  $c$  in  $(a, b)$  satisfying  $f(c) = L$ .



*Note:* It is important that the function be continuous on the interval  $[a, b]$ :

**Theorem 5: Existence of Zeros of a Continuous Function**

If  $f$  is a continuous function on a closed interval  $[a, b]$ , and if  $f(a)$  and  $f(b)$  have opposite signs, then there is at least one solution of the equation  $f(x) = 0$  in the interval  $(a, b)$ .



**Example.** Check the conditions of the Intermediate Value Theorem to see if there exists a value  $c$  on the interval  $(a, b)$  such that the following equations hold: [Graph](#)

$$x^x - x^2 = \frac{1}{2} \quad \text{on } [0, 2] \quad \sqrt{x^4 + 25x^3 + 10} = 5 \quad \text{on } [0, 1]$$

$$x + \sqrt{1 - x^2} = 0 \quad \text{on } [-1, 0] \quad \frac{x^2}{x^2 + 1} = 0 \quad \text{on } [-1, 1]$$

**Example.** Consider the function

$$f(x) = \frac{x+1}{x-1}$$

on the interval  $[0, 2]$ . Does there exist a  $c$  on the interval  $[0, 2]$  such that  $f(c) = 1$ ?

