

## 1.2: The Language of Sets

### Definition.

- A **set** is a collection of objects.
- If  $S$  is a set, then we use
  - $x \in S$  to denote that the element  $x$  is in the set  $S$ .
  - $x \notin S$  to denote that the element  $x$  is *not* in the set  $S$ .
- The **set-roster notation** is used to denote all elements in a set between braces:

$$S = \{1, 2, \dots, 100\}$$

Here, we see that  $67 \in S$ , but  $1337 \notin S$ .

- The **axiom of extension** says that a set is completely determined by what its elements are – not the order in which they are listed.

### Example.

Let  $A = \{1, 2, 3\}$ ,  $B = \{3, 1, 2\}$ , and  $C = \{1, 1, 2, 3, 3, 3\}$ . What are the elements of  $A$ ,  $B$ , and  $C$ ? How are  $A$ ,  $B$ , and  $C$  related?

Is  $\{0\} = 0$ ?

How many elements are in the set  $\{1, \{1\}\}$ ?

For each nonnegative integer  $n$ , let  $U_n = \{n, -n\}$ . Find  $U_1$ ,  $U_2$ , and  $U_0$ .

Certain sets of numbers are so frequently referred to that they are given special names and symbols:

**N** or  $\mathbb{N}$  The set of all **natural numbers**

**Z** or  $\mathbb{Z}$  The set of all **integers**

**Q** or  $\mathbb{Q}$  The set of all **rational numbers**, or quotient of integers

**R** or  $\mathbb{R}$  The set of all **real numbers**

*Note:* We may additionally use superscripts to indicate further properties of these sets:

$\mathbb{Z}^+$  or  $\mathbb{Z}^{>0}$  The set of *positive* integers

$\mathbb{Q}^-$  or  $\mathbb{Q}^{<0}$  The set of *negative* rational numbers

$\mathbb{R}^{nonneg}$  or  $\mathbb{R}^{\geq 0}$  The set of *nonnegative* real numbers

*Note:* Different sources denote the natural numbers  $\mathbb{N}$  as  $\mathbb{Z}^+$  or  $\mathbb{Z}^{\geq 0}$ .

### Definition. (Set-Builder Notation)

Let  $S$  be a set and let  $P(x)$  be a property that elements of  $S$  may or may not satisfy. We may define a new set to be **the set of all elements  $x$  in  $S$  such that  $P(x)$  is true**. We denote this set as follows:

$$\{x \in S \mid P(x)\}$$

↑                      ↑  
the set of all            such that

**Example.** Describe each of the following sets:

$$\{x \in \mathbb{R} \mid -2 < x < 5\}$$

$$\{x \in \mathbb{Z} \mid -2 < x < 5\}$$

$$\{x \in \mathbb{Z}^+ \mid -2 < x < 5\}$$

### Definition.

If  $A$  and  $B$  are sets, then  $A$  is called a **subset** of  $B$ , written  $A \subseteq B$ , if, and only if, every element of  $A$  is also an element of  $B$ :

$A \subseteq B$  means that for every element  $x$ , if  $x \in A$ , then  $x \in B$ .

$A \not\subseteq B$  means that there is at least one element  $x$ , such that  $x \in A$  and  $x \notin B$ .

$A$  is a **proper subset** of  $B$  if, and only if, every element of  $A$  is in  $B$ , but there is at least one element of  $B$  that is not in  $A$ :

$A \subsetneq B$  means that for every element  $x$ , if  $x \in A$ , then  $x \in B$ ,  
and there exists  $x \in B$  such that  $x \notin A$ .

**Example.** Let  $A = \mathbb{Z}^+$ ,  $B = \{n \in \mathbb{Z} \mid 0 \leq n \leq 100\}$ , and  $C = \{100, 200, 300, 400, 500\}$ . Evaluate the truth and falsity of each of the following statements.

$$B \subseteq A$$

$$C \text{ is a proper subset of } A$$

$$C \text{ and } B \text{ have at least one element in common}$$

$$C \subseteq B$$

$$C \subseteq C$$

**Example.** Determine which of the following statements are true:

$$2 \in \{1, 2, 3\}$$

$$\{2\} \in \{1, 2, 3\}$$

$$2 \subseteq \{1, 2, 3\}$$

$$\{2\} \subseteq \{1, 2, 3\}$$

$$\{2\} \subseteq \{\{1\}, \{2\}\}$$

$$\{2\} \in \{\{1\}, \{2\}\}$$

### Definition.

Given elements  $a$  and  $b$ , the symbol  $(a, b)$  denotes the **ordered pair** consisting of  $a$  and  $b$  together with the specification that  $a$  is the first element of the pair, and  $b$  is the second element. Two ordered pairs  $(a, b)$  and  $(c, d)$  are equal if, and only if,  $a = c$  and  $b = d$ :

$$(a, b) = (c, d) \text{ means that } a = c \text{ and } b = d.$$

### Example.

Is  $(1, 2) = (2, 1)$ ?

Is  $\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right)$ ?

### Definition.

Let  $n \in \mathbb{N}$  and let  $x_1, x_2, \dots, x_n$  be (not necessarily distinct) elements. The **ordered  $n$ -tuple**,  $(x_1, x_2, \dots, x_n)$ , consists of  $x_1, x_2, \dots, x_n$  together with the ordering: first  $x_1$ , then  $x_2$ , and so forth up to  $x_n$ . An ordered 2-tuple is called an **ordered pair**, and an ordered 3-tuple is called an **ordered triple**.

Two ordered  $n$ -tuples  $(x_1, x_2, \dots, x_n)$  and  $(y_1, y_2, \dots, y_n)$  are **equal** if, and only if,  $x_1 = y_1, x_2 = y_2, \dots$ , and  $x_n = y_n$ :

$$(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n) \iff x_1 = y_1, x_2 = y_2, \dots, x_n = y_n.$$

### Example.

Is  $(1, 2, 3, 4) = (1, 2, 4, 3)$ ?

Is  $\left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right)$ ?

**Definition.**

Given sets  $A_1, A_2, \dots, A_n$ , the **Cartesian product** of  $A_1, A_2, \dots, A_n$ , denoted

$$A_1 \times A_2 \times \cdots \times A_n$$

is the set of all ordered  $n$ -tuples  $(a_1, a_2, \dots, a_n)$  where  $a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n$ :

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n\}$$

**Example.** Let  $A = \{x, y\}$ ,  $B = \{1, 2, 3\}$ , and  $C = \{a, b\}$ . Find the following:

$$A \times B$$

$$B \times A$$

$$A \times A$$

How many elements are in  $A \times B$ ,  $B \times A$ , and  $A \times A$ ?

$$(A \times B) \times C$$

$$A \times B \times C$$

Describe  $\mathbb{R} \times \mathbb{R}$

### Definition.

Let  $n \in \mathbb{N}$ . Given a finite set  $A$ , a **string of length  $n$  over  $A$**  is an ordered  $n$ -tuple of elements of  $A$  written without parentheses or commas. The elements of  $A$  are called the **characters** of the string. The **null string** over  $A$  is defined to be the “string” with no characters, often denoted  $\lambda$ , and is said to have length 0. If  $A = \{0, 1\}$ , then a string over  $A$  is called a **bit string**.

**Example.** Let  $A = \{a, b\}$ . List all strings of length 3 over  $A$  with at least two characters that are the same.