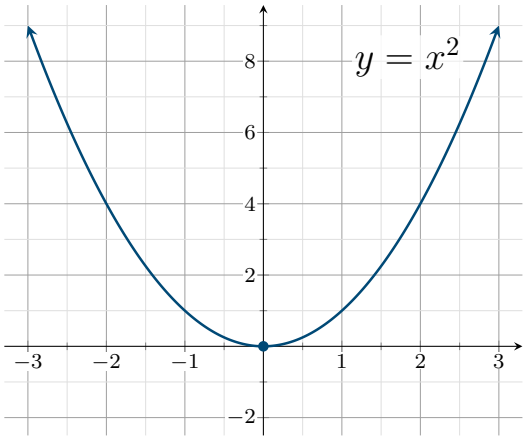
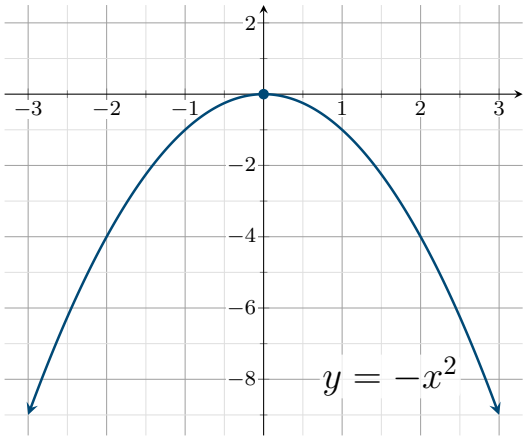
2.2: Quadratic Functions: Parabolas

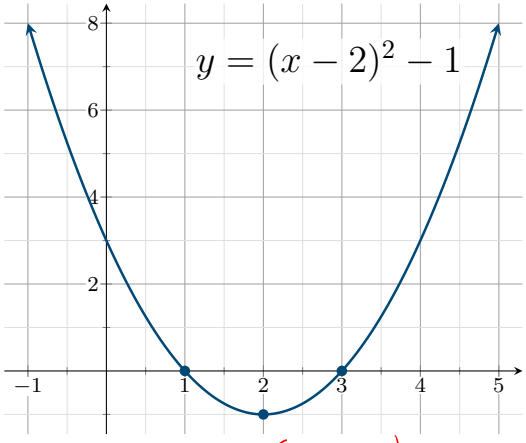
Definition.
A **quadratic function** has the form

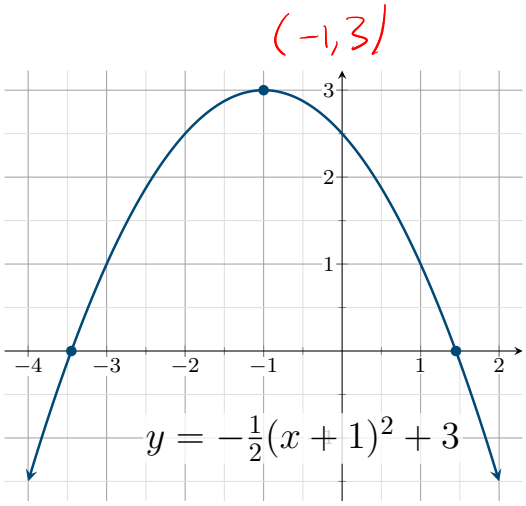
$$y = f(x) = ax^2 + bx + c \quad (a \neq 0)$$

where a, b, and c represent constants. A **parabola** is the shape of the graph of a quadratic function.









Definition.
The quadratic function $y = f(x) = ax^2 + bx + c$ has its **vertex** at

$$\left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right).$$

The optimal value occurs at the vertex of a parabola:

- A maximum if $a < 0$ ∩
- A minimum if $a > 0$ ∪

Example. Consider the function

$$2x + \frac{x^2}{2}$$

Is the vertex a maximum or minimum? Locate the vertex, x-intercepts, y-intercept, and then sketch the graph.

$a = \frac{1}{2} > 0 \Rightarrow$ Vertex is a minimum

Vertex: $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) = \left(-\frac{2}{2(1/2)}, f\left(-\frac{2}{2(1/2)}\right)\right) = (-2, -2)$

x-intercepts $(y = 0)$

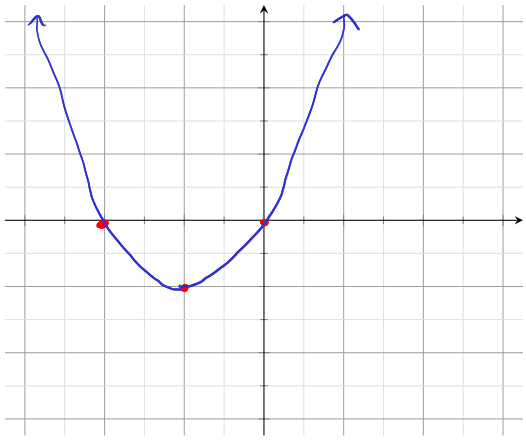
$0 = 2x + \frac{x^2}{2} = \frac{1}{2}x(4 + x)$

$x = 0$ $\quad$ $x = -4$

$(0,0)$ $\quad$ $(-4,0)$

y-intercept $(x = 0)$

$y = 2(0) + \frac{0^2}{2} = 0 \Rightarrow \boxed{(0,0)}$



Example. Consider the function

$$x^2 + 5 - 4x$$

Is the vertex a maximum or minimum? Locate the vertex, x-intercepts, y-intercept, and then sketch the graph.

$a = 1 > 0 \Rightarrow$ Vertex is a minimum

Vertex: $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) = \left(-\frac{-4}{2(1)}, f\left(-\frac{-4}{2(1)}\right)\right) = (2, 1)$

x-intercepts ($y = 0$)

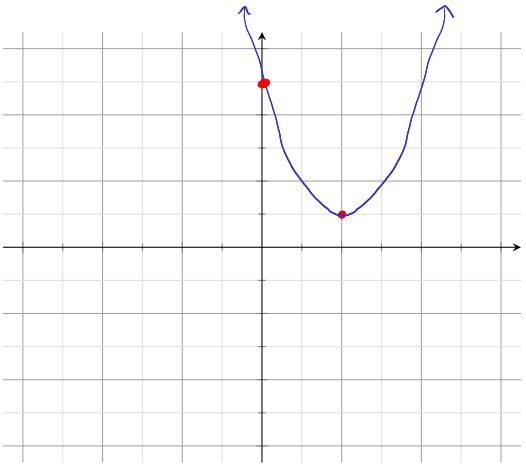
$0 = \underset{a}{1}x^2 \underset{b}{-4}x \underset{c}{+5}$

$x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$

Determinant:
$(-4)^2 - 4(1)(5) = -4 < 0 \Rightarrow$ No solutions!
$\Rightarrow$ No x-intercepts

y-intercept ($x = 0$)

$y = 0^2 - 4(0) + 5 = 5 \rightarrow \boxed{(0, 5)}$



Example. Ace Cruises offers a sunset cruise to a group of 50 people for a price of \$30 per person, but it reduces the price per person by \$0.50 for each additional person above 50. Find the revenue function. What price maximizes the revenue? What is this maximal value?

Number of people	Price per person	Total Revenue
50	\$30.00	\$1500.00
51	\$29.50	\$1504.50
52	\$29.00	\$1508.00
⋮		
$50+x$	$30-0.5x$	$R(x)=(50+x)(30-0.5x)$

$$R(x)=(50+x)(30-0.5x)$$
$$= \underbrace{1500}_{c} \underbrace{+5x}_{b} \underbrace{-0.5x^2}_{a}$$

$a<0 \Rightarrow$ Vertex is a maximum.

Vertex $\left(-\frac{b}{2a}, R\left(-\frac{b}{2a}\right)\right)$

$$-\frac{b}{2a} = -\frac{5}{2(-0.5)} = 5$$

$$R(5) = 1500+5(5)-0.5(5)^2$$
$$= 1500+25-\frac{25}{2}$$
$$= \boxed{\$1512.50}$$