

2.4: Limits

Example. Suppose that the position function of a maglev train (in feet) is given by

$$s(t) = 4t^2, \quad (0 \leq t \leq 30)$$

Using the position function, compute the *average* velocity of the train

on the interval $[t, 2]$

| | | | | | |
|-----|-----|-----|------|-------|--------|
| t | 1.5 | 1.9 | 1.99 | 1.999 | 1.9999 |
|-----|-----|-----|------|-------|--------|

on the interval $[2, t]$

| | | | | | |
|-----|-----|-----|------|-------|--------|
| t | 2.5 | 2.1 | 2.01 | 2.001 | 2.0001 |
|-----|-----|-----|------|-------|--------|

What do the tables above suggest about *instantaneous* velocity of the train at $t = 2$?

Definition. (Limit of a Function)

The function f has the **limit** L as x approaches a , written

$$\lim_{x \rightarrow a} f(x) = L$$

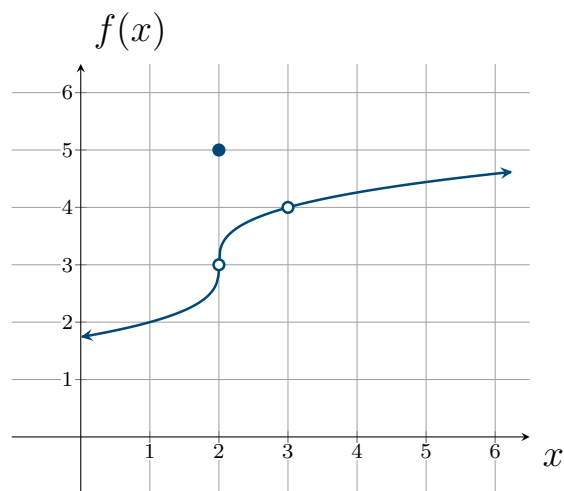
if the value of $f(x)$ can be made as close to the number L as we please by taking x sufficiently close to (but not equal to) a .

Example. Using the graph of f , determine the following values:

$$f(1) \text{ and } \lim_{x \rightarrow 1} f(x)$$

$$f(2) \text{ and } \lim_{x \rightarrow 2} f(x)$$

$$f(3) \text{ and } \lim_{x \rightarrow 3} f(x)$$



Example. Find the limit of the following functions at the value specified:

[Graphs](#)

$$f(x) = x^3 \quad \text{at } x = 2$$

$$g(x) = \begin{cases} x + 2, & x \neq 1 \\ 1, & x = 1 \end{cases} \quad \text{at } x = 1$$

$$h(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases} \quad \text{at } x = 0$$

$$j(x) = \frac{1}{(x-1)^2} \quad \text{at } x = 1$$

$$k(x) = 4 \quad \text{at } x = 0$$

Theorem 1: Properties of Limits

Suppose

$$\lim_{x \rightarrow a} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = M$$

Then

1. $\lim_{x \rightarrow a} [f(x)]^r = \left[\lim_{x \rightarrow a} f(x) \right]^r$ where r is a positive constant
2. $\lim_{x \rightarrow a} cf(x) = c \lim_{x \rightarrow a} f(x)$ where c is a real number
3. $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = L \pm M$
4. $\lim_{x \rightarrow a} [f(x)g(x)] = \left[\lim_{x \rightarrow a} f(x) \right] \left[\lim_{x \rightarrow a} g(x) \right] = LM$
5. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{M}$ provided $M \neq 0$

Example. Use the above theorem to evaluate the following limits:

$$\lim_{x \rightarrow 1} (5x^{3/2} - 2)$$

$$\lim_{x \rightarrow 3} \frac{2x^3 \sqrt{x^2 + 7}}{x + 1}$$

Suppose that $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$. Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

has an **indeterminate form** of $\frac{0}{0}$. To evaluate such a limit, we replace the given function with a function that's equivalent everywhere except at $x = a$, and then evaluate the limit.

Example. Evaluate the following

$$\lim_{t \rightarrow 2} \frac{4t^2 - 16}{t - 2}$$

$$\lim_{h \rightarrow 0} \frac{\sqrt{4 + h} - 2}{h}$$

Limit of a Function at Infinity

The function f has the limit L as x increases without bound, written

$$\lim_{x \rightarrow \infty} f(x) = L$$

if $f(x)$ can be made arbitrarily close to L by taking x large enough.

The function f has the limit M as x decreases without bound, written

$$\lim_{x \rightarrow -\infty} f(x) = M$$

if $f(x)$ can be made arbitrarily close to M by taking x to be negative and sufficiently large enough in absolute value.

When the above limits exist, the equations $y = L$ and/or $y = M$ are called **horizontal asymptotes**.

Example. Evaluate the following infinite limits

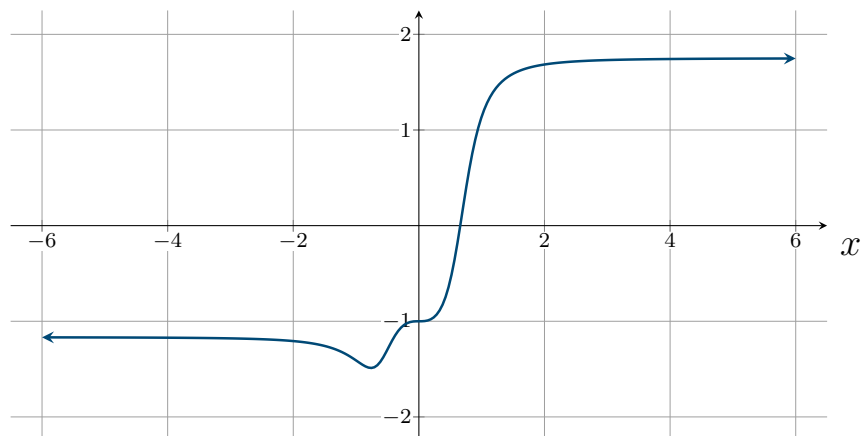
$$\lim_{x \rightarrow \infty} \frac{2x^2 + 3x - 4}{x^2 - 7x + 1}$$

$$\lim_{x \rightarrow -\infty} \frac{3x^2 + 4}{2x^3}$$

$$\lim_{x \rightarrow \pm\infty} \frac{3x^5 + 2x^3 - 4}{x^4 + 4x^2 - 1}$$

Graph

$f(x)$



$$\lim_{x \rightarrow \infty} \frac{7x^3 - 2}{-x^3 + \sqrt{25x^6 - 4}}$$

$$\lim_{x \rightarrow -\infty} \frac{7x^3 - 2}{-x^3 + \sqrt{25x^6 - 4}}$$

Example. The company *Custom Office* makes a line of executive desks. It is estimated that the total cost of making x *Senior Executive Model* desks is

$$C(x) = 100x + 200,000$$

dollars per year. The average cost of making x desks is given by

$$\overline{C}(x) = \frac{C(x)}{x}$$

Compute $\lim_{x \rightarrow \infty} \overline{C}(x)$ and interpret the result.

Theorem 2

For all $n > 0$,

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x^n} = 0$$

provided that $\frac{1}{x^n}$ is defined.