Math 125 Class notes
Spring 2026

To accompany
Discrete Mathematics with Applications
by *Epp*

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Math 125 Formula Sheet

1.1: Variables

Definition.
A **variable** is a placeholder for something which may or may not be unknown.

Example. Is there a number with the following property: doubling it and adding 3 gives the same result as squaring it?

$\to$ Is there a number x with the property that $2x + 3 = x^2$?
$\to$ Is there a number $\square$ with the property that $2 \cdot \square + 3 = \square^2$?

Example. No matter what number might be chosen, if it is greater than 2, then its square is greater than 4.

$\to$ No matter what number n might be chosen, if n is greater than 2, then n^2 is greater than 4.

Example. Use variables to rewrite the following sentences:

> Are there numbers with the property that the sum of their squares equals the square of their sum?

> Given any real number, its square is nonnegative.

Definition.

- A **universal statement** says that a certain property is true for all elements in a set.
- A **conditional statement** says that if one thing is true, then some other thing also has to be true.
- Given a property that may or may not be true, an **existential statement** says that there is at least one thing for which the property is true.

Definition.
A **universal conditional statement** is both universal and conditional:

> For every animal a, if a is a dog, then a is a mammal.

Conditional statements can be rewritten in ways that make them appear more to be purely universal or purely conditional:

> If a is a dog, then a is a mammal.
> All dogs are mammals

Example. Rewrite the following universal condition statement:

> For every real number x, if x is nonzero then x^2 is positive.

If a real number is nonzero, then its square ____________________.

For every nonzero real number x, ____________________.

If x ____________________, then ____________________.

The square of any nonzero real number is ____________________.

All nonzero real numbers have ____________________.

Definition.
A **universal existence statement** is a statement that is universal because its first part says that a certain property is true for all objects of a given type, and it is existential because its second part asserts the existence of something:

Every real number has an additive inverse.

In the above example, note that the particular additive inverse depends on the given real number:

For every real number r, there is an additive inverse for r.

Example. Rewrite the following universal existence statement:

Every pot has a lid

All pots ____________________.

For ever pot P, there is ____________________.

For every pot P, there is a lid L such that ____________________.

Definition.
An **existential universal statement** is a statement that is existential because its first part asserts that a certain object exists and is universal because its second part says that the object satisfies a certain property for all things of a certain kind:

> There is a positive integer that is less than or equal to every positive integer.

The number one satisfies the above statement, which can also be rewritten:

> There is a positive integer m that is less than or equal to every positive integer.

Example. Rewrite the following existence universal statement:

> There is a person in my class who is at least as old as every person in my class.

Some ____________________ is at least as old as ____________________.

There is a person p in my class such that p is ____________________.

There is a person p in my class with the property that for every person q in my class, p is ____________________.

1.2: The Language of Sets

Definition.

- A **set** is a collection of objects.
- If S is a set, then we use
  - $x \in S$ to denote that the element x is in the set S.
  - $x \notin S$ to denote that the element x is *not* in the set S.
- The **set-roster notation** is used to denote all elements in a set between braces:

$$S = \{1, 2, \ldots, 100\}$$

  Here, we see that $67 \in S$, but $1337 \notin S$.
- The **axiom of extension** says that a set is completely determined by what its elements are – not the order in which they are listed.

Example.

Let $A = \{1, 2, 3\}$, $B = \{3, 1, 2\}$, and $C = \{1, 1, 2, 3, 3, 3\}$. What are the elements of A, B, and C? How are A, B, and C related?

Is $\{0\} = 0$?

How many elements are in the set $\{1, \{1\}\}$?

For each nonnegative integer n, let $U_n = \{n, -n\}$. Find U_1, U_2, and U_0.

Certain sets of numbers are so frequently referred to that they are given special names and symbols:

N or $\mathbb{N}$	The set of all **natural numbers**
Z or $\mathbb{Z}$	The set of all **integers**
Q or $\mathbb{Q}$	The set of all **rational numbers**, or quotient of integers
R or $\mathbb{R}$	The set of all **real numbers**

Note: We may additionally use superscripts to indicate further properties of these sets:

$\mathbb{Z}^+$ or $\mathbb{Z}^{>0}$	The set of *positive* integers
$\mathbb{Q}^-$ or $\mathbb{Q}^{<0}$	The set of *negative* rational numbers
$\mathbb{R}^{nonneg}$ or $\mathbb{R}^{\geq 0}$	The set of *nonnegative* real numbers

Note: Different sources denote the natural numbers $\mathbb{N}$ as $\mathbb{Z}^+$ *or* $\mathbb{Z}^{\geq 0}$.

Definition. (Set-Builder Notation)
Let S be a set and let $P(x)$ be a property that elements of S may or may not satisfy. We may define a new set to be **the set of all elements x in S such that $P(x)$ is true**. We denote this set as follows:

$$\{x \in S | P(x)\}$$

the set of all such that

Example. Describe each of the following sets:

$\{x \in \mathbb{R} \mid -2 < x < 5\}$

$\{x \in \mathbb{Z} \mid -2 < x < 5\}$

$\{x \in \mathbb{Z}^+ \mid -2 < x < 5\}$

Definition.
If A and B are sets, then A is called a **subset** of B, written $A \subseteq B$, if, and only if, every element of A is also an element of B:

$$A \subseteq B \text{ means that for every element } x, \text{ if } x \in A, \text{ then } x \in B.$$
$$A \nsubseteq B \text{ means that there is at least one element } x, \text{ such that } x \in A \text{ and } x \notin B.$$

A is a **proper subset** of B if, and only if, every element of A is in B, but there is at least one element of B that is not in A:

$$A \subsetneq B \text{ means that for every element } x, \text{ if } x \in A, \text{ then } x \in B,$$
$$\text{and there exists } x \in B \text{ such that } x \notin A.$$

Example. Let $A = \mathbb{Z}^+$, $B = \{n \in \mathbb{Z} \mid 0 \leq n \leq 100\}$, and $C = \{100, 200, 300, 400, 500\}$. Evaluate the truth and falsity of each of the following statements.

$B \subseteq A$

C is a proper subset of A

C and B have at least one element in common

$C \subseteq B$

$C \subseteq C$

Example. Determine which of the following statements are true:

$2 \in \{1, 2, 3\}$

$\{2\} \in \{1, 2, 3\}$

$2 \subseteq \{1, 2, 3\}$

$\{2\} \subseteq \{1, 2, 3\}$

$\{2\} \subseteq \{\{1\}, \{2\}\}$

$\{2\} \in \{\{1\}, \{2\}\}$

Definition.
Given elements a and b, the symbol (a,b) denotes the **ordered pair** consisting of a and b together with the specification that a is the first element of the pair, and b is the second element. Two ordered pairs (a,b) and (c,d) are equal if, and only if, $a=c$ and $b=d$:

$$(a,b)=(c,d) \text{ means that } a=c \text{ and } c=d.$$

Example.

Is $(1,2)=(2,1)$?

Is $\left(3,\dfrac{5}{10}\right)=\left(\sqrt{9},\dfrac{1}{2}\right)$?

Definition.
Let $n\in\mathbb{N}$ and let $x_1,x_2,\ldots,x_n$ be (not necessarily distinct) elements. The **ordered n-tuple**, $(x_1,x_2,\ldots,x_n)$, consists of $x_1,x_2,\ldots x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. and ordered 2-tuple is called an **ordered pair**, and ordered 3-tuple is called an **ordered triple**.
Two ordered n-tuples $(x_1,x_2,\ldots,x_n)$ and $(y_1,y_2,\ldots,y_n)$ are **equal** if, and only if, $x_1=y_1$, $x_2=y_2$, $\ldots$, and $x_n=y_n$:

$$(x_1,x_2,\ldots,x_n)=(y_1,y_2,\ldots,y_n) \iff x_1=y_1,x_2=y_2,\ldots,x_n=y_n.$$

Example.

Is $(1,2,3,4)=(1,2,4,3)$?

Is $\left(3,(-2)^2,\dfrac{1}{2}\right)=\left(\sqrt{9},4,\dfrac{3}{6}\right)$?

Definition.
Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$, denoted

$$A_1 \times A_2 \times \cdots \times A_n$$

is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1$, $a_2 \in A_2$, $\ldots$, $a_n \in A_n$:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in \mathcal{A}_n\}$$

Example. Let $A = \{x, y\}$, $B = \{1, 2, 3\}$, and $C = \{a, b\}$. Find the following:

$A \times B$

$B \times A$

$A \times A$

How many elements are in $A \times B$, $B \times A$, and $A \times A$?

$(A \times B) \times C$

$A \times B \times C$

Describe $\mathbb{R} \times \mathbb{R}$

Definition.
Let $n \in \mathbb{N}$. Given a finite set A, a **string of length** n **over** A is an ordered n-tuple of elements of A written without parentheses or commas. The elements of A are called the **characters** of the string. The **null string** over A is defined to be the "string" with no characters, often denoted λ, and is said to have length 0. If $A = \{0, 1\}$, then a string over A is called a **bit string**.

Example. Let $A = \{a, b\}$. List all strings of length 3 over A with at least two characters that are the same.

1.3: The Language of Relations and Functions

Definition.
Let A and B be sets. A **relation** R **from** A **to** B is a subset of $A \times B$. Given an ordered pair (x, y), x **is related to** y **by** R, written $x\,R\,y$, if, and only if, (x, y) is in R. The set A is called the **domain** of R and the set B is called its **co-domain**.
The notation for a relation R may be written symbolically as follows:

$$x\,R\,y \text{ means that } (x, y) \in R.$$

The notation $x \not R\, y$ means that x is not related to y by R:

$$x \not R\, y \text{ means that } (x, y) \notin R.$$

Example. Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ and define a relation R from A to B as follows; Given any $(x, y) \in A \times B$,

$$(x, y) \in R \text{ means that } \frac{x - y}{2} \text{ is an integer.}$$

State explicitly which ordered pairs are in $A \times B$ and which are in R

Is $1\,R\,3$? Is $2\,R\,3$? Is $2\,R\,2$?

What are the domain and co-domain of R?

Example. Define a relation C from $\mathbb{R}$ to $\mathbb{R}$ as follows: For any $(x, y) \in \mathbb{R} \times \mathbb{R}$,

$$(x, y) \in C \text{ means that } x^2 + y^2 = 1.$$

Is $(1, 0) \in C$? Is $(0, 0) \in C$? Is $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \in C$?

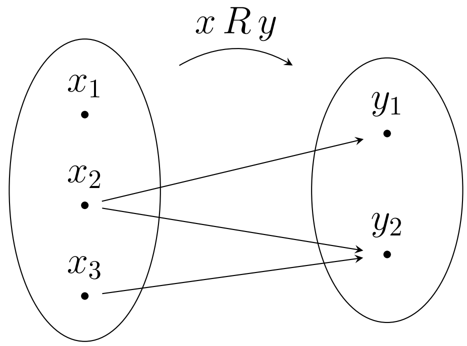
Is $-2\,C\,0$? Is $0\,C\,(-1)$? Is $1\,C\,1$?

What are the domain and co-domain of C?

Draw a graph for C by plotting the points of C in the Cartesian plane.

Definition.
Suppose R is a relation from set A to a set B. The **arrow diagram for** R is obtained as follows:

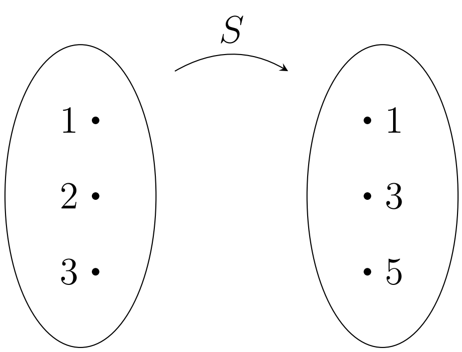
1. Represent the elements of A as points in one region and the elements of B as points in another region.
2. For each x in A and y in B, draw an arrow from x to y if, and only if, x is related to y by R.

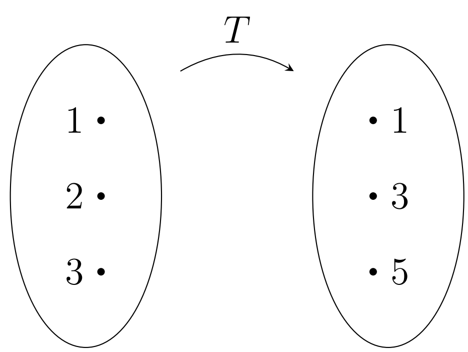


Example. Let $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$ and define relations S and T from A to B as follows: For every $(x, y) \in A \times B$,

$$(x, y) \in S \text{ means that } x < y$$
$$T = \{(2, 1), (2, 5)\}.$$

Draw arrow diagrams for S and T





Definition.
A **function** F **from a set** A **to a set** B is a relation with domain A and co-domain B that satisfies the following two properties:

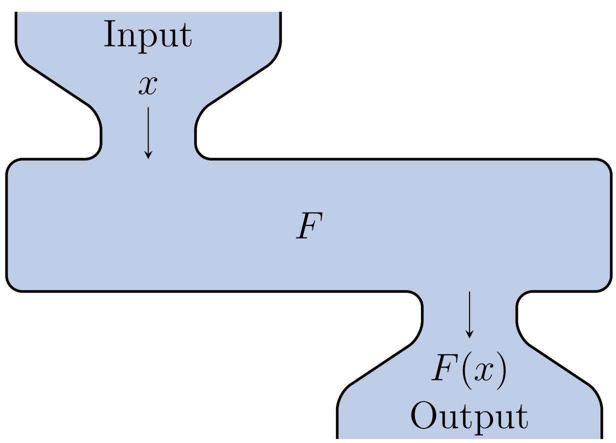
1. For every element x in A, there is an element y in B such that $(x, y) \in F$.
2. For all elements x in A and y and z in B,

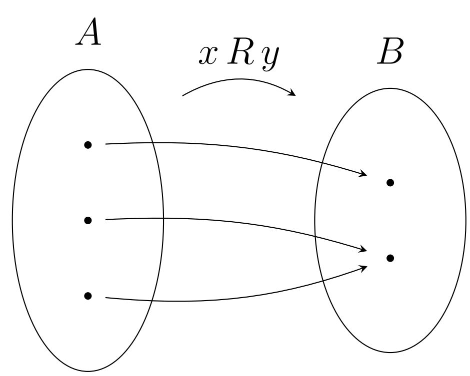
$$\text{if } (x, y) \in F \text{ and } (x, z) \in F, \text{ then } y = z.$$

Note: A relation from A to B is a function if, and only if,

1. Every element of A is the first element of an ordered pair of F
2. No two distinct ordered pairs in F have the same first element.

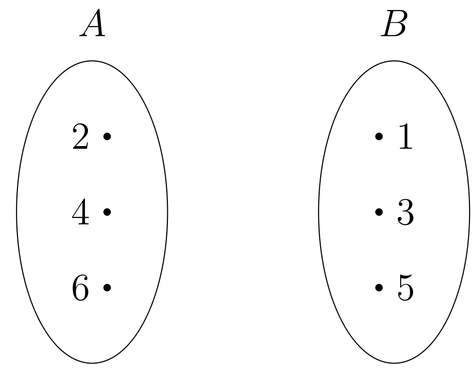
Note: If A and B are sets and F is a function from A to B, then given any element x in A, the unique element in B that is related to x by F is denoted $F(x)$, which is read "F **of** x".



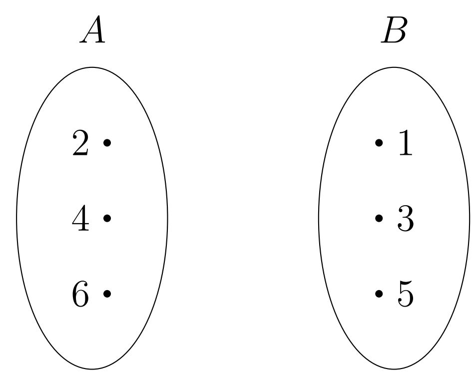


Example. Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$. Which of the relations R, S, and T defined below are functions from A to B?

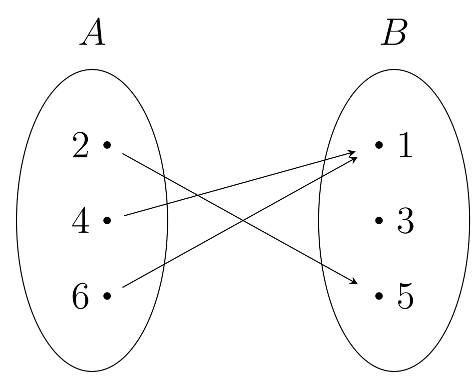
$R = \{(2, 5), (4, 1), (4, 3), (6, 5)\}$



For every $(x, y) \in A \times B$, $(x, y) \in S$ means that $y = x + 1$.



T is defined by the arrow diagram



2.1: Logical Form and Logical Equivalence

Definition.
A **statement** (or **proposition**) is a sentence that is true or false, but not both.

Example. Determine which of the following are statements:

$2 + 2 = 4$

$2 + 2 = 5$

$x^2 + 2 = 11$

Today is Saturday.

She is a computer science major.

Jane is a computer science major.

Definition. (Compound Statements)
Let p and q be statement variables.

- The **negation** of p is "not p", and is denoted as $\sim p$ (or $\neg p$)
- The **conjunction** of p and q is "p and q", and is denoted at $p \wedge q$
- The **disjunction** of p and q is "p or q", and is denoted $p \vee q$.
- The **exclusive or** of p and q is "p x-or q", and is denoted $p \oplus q$ (or p XOR q)

The **order of operations** specifies that $\sim$ is performed first.

Example. Consider the following statements:

$$\begin{array}{rl} p: & \text{It is raining.} \\ q: & \text{It is sunny.} \\ r: & \text{It is cloudy.} \end{array}$$

Rewrite the following compound statements in words:

$\sim p$ $\qquad$ $p \vee q$

$q \wedge r$ $\qquad$ $q \wedge \sim r$

$p \wedge (q \vee r)$ $\qquad$ $p \oplus q$

> **Definition.**
> A **statement form** (or **propositional form**) is an expression made up of statement variables (e.g., p, q, and r), and logical connectives (e.g. $\sim$, $\wedge$, $\vee$, and $\oplus$).
> The **truth table** for a given statement form displays the truth values that correspond to all possible combinations of truth values for its component statement variables.

Example. Let p and q be statement variables. Fill out the following truth tables:

p	$\sim p$
T	
F	

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$
T	T			
T	F			
F	T			
F	F			

p	q	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$(p \vee q) \wedge \sim (p \wedge q)$
T	T				
T	F				
F	T				
F	F				

Example. Construct a truth table for the statement form $(p \wedge q) \vee \sim r$.

Definition.
Two *statement forms* are called **logically equivalent** if, and only if, they have identical true values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms P and Q is denoted $P \equiv Q$.

Example. Use truth tables to test if the following statement forms are equivalent:

p and $\sim(\sim p)$

$\sim(p \wedge q)$ and $\sim p \wedge \sim q$

Definition. (De Morgan's Laws)
The negation of an *and* statement is logically equivalent to the *or* statement in which each component is negated.
The negation of an *or* statement is logically equivalent to the *and* statement in which each component is negated.

Example. Use truth tables to show that the following statement forms are equivalent:

$\sim(p \wedge q)$ and $\sim p \vee \sim q$

$\sim(p \vee q)$ and $\sim p \wedge \sim q$

Example. Using De Morgan's law to write the negation of the following statements:

Jim is at least 6 feet tall and weighs at least 200 pounds.

The bus was late or Tom's watch was slow.

$-1 < x \leq 4$

Definition.
A **tautology** is a statement form that is always true.
A **contradiction** is a statement form that is always false.

Example. Complete the truth tables for $p \wedge \sim p$ and $p \vee \sim p$

Example. Let $\mathbf{t}$ be a tautology, and $\mathbf{c}$ be a contradiction. Show that $p \wedge \mathbf{t} \equiv p$ and $p \wedge \mathbf{c} \equiv \mathbf{c}$

Theorem 2.1.1 Logical Equivalences (p 49)
Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold:

1. Commutative laws:
   $p \wedge q \equiv q \wedge p$ $\qquad$ $p \vee q \equiv q \vee p$
2. Associative laws:
   $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $\qquad$ $(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. Distributive laws:
   $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $\qquad$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. Identity laws:
   $p \wedge \mathbf{t} \equiv p$ $\qquad$ $p \vee \mathbf{c} \equiv p$
5. Negation laws:
   $p \vee \sim p \equiv \mathbf{t}$ $\qquad$ $p \wedge \sim p \equiv \mathbf{c}$
6. Double negative law:
   $\sim (\sim p) \equiv p$
7. Idempotent laws:
   $p \wedge p \equiv p$ $\qquad$ $p \vee p \equiv p$
8. Universal bound laws:
   $p \vee \mathbf{t} \equiv \mathbf{t}$ $\qquad$ $p \wedge \mathbf{c} \equiv \mathbf{c}$
9. De Morgan's laws:
   $\sim (p \wedge q) \equiv \sim p \vee \sim q$ $\qquad$ $\sim (p \vee q) \equiv \sim p \wedge \sim q$
10. Absorption laws:
    $p \wedge (p \vee q) \equiv p$ $\qquad$ $p \wedge (p \vee q) \equiv p$
11. Negations of $\mathbf{t}$ and $\mathbf{c}$:
    $\sim \mathbf{t} \equiv \mathbf{c}$ $\qquad$ $\sim \mathbf{c} \equiv \mathbf{t}$

2.2: Conditional Statements

Definition.
If p and q are statement variables, the **conditional** of q by p is "If p then q", or "p implies q" and is denoted by $p \to q$. It is false when p is true and q is false; otherwise it is true. We call p the **hypothesis** (or **antecedent**) of the conditional and q the **conclusion** (or **consequent**).
A conditional statement that is always true because the hypothesis is false is called **vacuously true**.

$$\text{If } \underbrace{4,686 \text{ is divisible by } 6,}_{\text{hypothesis}} \text{ then } \underbrace{4,686 \text{ is divisible by } 3}_{\text{conclusion}}$$

Example. Consider the following statement:

If Lander is open, then we will have class.

Create the truth table for $p \to q$

p	q	$p \to q$
T	T	
T	F	
F	T	
F	F	

Note: The **order of operations** states that $\to$ is performed last

Example. Create the truth table for $p \vee \sim q \to \sim p$.

p	q	$\sim q$	$p \vee \sim q$	$\sim p$	$p \vee \sim q \to \sim p$
T	T				
T	F				
F	T				
F	F				

Example. Use a truth table to show that $p \vee q \to r \equiv (p \to r) \wedge (q \to r)$

p	q	r	$p \vee q$	$p \vee q \to r$	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

Definition.
The **negation** of "if p then q" is logically equivalent to "p and not q":

$$\sim (p \to q) \equiv p \wedge \sim q$$

Example. Write negations for each of the following statements:

If my car is in the repair shop, then I cannot get to class.

If Sara lives in Athens, then she lives in Greece.

Definition.
The **contrapositive** of a conditional statement of the form "If p then q" is

$$\text{If } \sim q \text{ then } \sim p: \quad \sim q \to \sim p$$

A conditional statement is logically equivalent to its contrapositive.

Example. Write each of the following statements in its equivalent contrapositive form:

If Howard can swim across the lake, then Howard can swim to the island.

If today is Easter, then tomorrow is Monday.

Definition.
Suppose a conditional statement of the form "If p then q" is given.

- The **converse** is "If q then p": $q \to p$
- The **inverse** is "If $\sim p$ then $\sim q$": $\sim p \to \sim q$

Example. Write the converse and inverse of each of the following statements:

If Howard can swim across the lake, then Howard can swim to the island.
Converse:

Inverse:

If today is Easter, then tomorrow is Monday.
Converse:

Inverse:

Note:

1. A conditional statement and its converse are *not* logically equivalent.
2. A conditional statement and its inverse are *not* logically equivalent.
3. The converse and the inverse of a conditional statement are logically equivalent to each other.

Definition.
If p and q are statements, p **only if** q means "if not q then not p":

$$\sim q \to \sim p \equiv p \to q$$

Example. Rewrite the following statement in if-then form in two ways, one of which is the contrapositive of the other:

> John will break the world's record for the mile run only if he runs the mile in under four minutes.

$\sim q \to \sim p$

$p \to q$

Note:

1. "p only if q" does *not* mean p if q
2. It is possible for "p only if q" to be true at the same time that "p if q" is false.

   e.g.: If John runs a mile in under four minutes, he still might not be fast enough to break the record.

Definition.
Given statement variables p and q, the **biconditional of** p **and** q is "p if, and only if, q" and is denoted $p \leftrightarrow q$. It is true if both p and q have the same truth values and is false otherwise. The words *if and only if* are sometimes abbreviated **iff**.

Note: The **order of operations** states that $\leftrightarrow$ is coequal with $\rightarrow$

Example. Create the truth table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	
T	F	
F	T	
F	F	

Order of Operations for Logical Operators

$\sim$	Evaluate negations first
$\wedge, \vee$	Evaluate $\wedge$ and $\vee$ second. When both present, parentheses may be needed.
$\rightarrow, \leftrightarrow$	Evaluate $\wedge$ and $\vee$ third. When both present, parentheses may be needed.

Definition.
If r and s are statements:

1. r is a **sufficient condition** for s means "if r then s". $\quad r \to s$

2. r is a **necessary condition** for s means "if not r then not s". $\quad \sim r \to \sim s$

By property of the contrapositive:

3. r is a *necessary and sufficient condition* for s means "r if, and only if s."
$$r \leftrightarrow s$$

Example. Rewrite the following statement in the form "If A then B":

> Having two 45° angles is a sufficient condition for this triangle to be a right triangle.

Example. Use the contrapositive to rewrite the following statement in two ways:

> George's attaining age 35 is a necessary condition for his being president of the United States.

2.5: Application: Number Systems and Circuits for Addition

Recall our how we write numbers in base 10:

$$\begin{aligned} 5,049 &= 5\cdot 1000 + 0\cdot 100 + 4\cdot 10 + 9\cdot 1 \\ &= 5\cdot 10^3 + 0\cdot 10^2 + 4\cdot 10^1 + 9\cdot 10^0 \end{aligned}$$

Definition.
Any integer $b > 1$ can be used as a base for a numbering system. A numbering system of base b has the digits $0, 1, \ldots, b-1$.
A **base 2 notation** or **binary notation**, uses the digits $0, 1$. In binary, every integer is represented as sum of products of the form

$$d \cdot 2^n$$

where $n \in \mathbb{Z}$ and $d \in \{0, 1\}$.

Example. Below is the binary representation for the integers 1 to 9:

$$\begin{array}{rcrcrcrcrcr}
1_{10} & = & & & & & & & 1\cdot 2^0 & = & 1_2 \\
2_{10} & = & & & & & 1\cdot 2^1 & + & 0\cdot 2^0 & = & 10_2 \\
3_{10} & = & & & & & 1\cdot 2^1 & + & 1\cdot 2^0 & = & 11_2 \\
4_{10} & = & & & 1\cdot 2^2 & + & 0\cdot 2^1 & + & 0\cdot 2^0 & = & 100_2 \\
5_{10} & = & & & 1\cdot 2^2 & + & 0\cdot 2^1 & + & 1\cdot 2^0 & = & 101_2 \\
6_{10} & = & & & 1\cdot 2^2 & + & 1\cdot 2^1 & + & 0\cdot 2^0 & = & 110_2 \\
7_{10} & = & & & 1\cdot 2^2 & + & 1\cdot 2^1 & + & 1\cdot 2^0 & = & 111_2 \\
8_{10} & = & 1\cdot 2^3 & + & 0\cdot 2^2 & + & 0\cdot 2^1 & + & 0\cdot 2^0 & = & 1000_2 \\
9_{10} & = & 1\cdot 2^3 & + & 0\cdot 2^2 & + & 0\cdot 2^1 & + & 1\cdot 2^0 & = & 1001_2
\end{array}$$

Converting binary → decimal:
To convert from binary to decimal, multiply each digit by its corresponding power of 2 and sum the results.

Example. Represent the following in decimal notation (base-10):

110_2

1011_2

11110_2

101011_2

Converting decimal → binary:
To convert from decimal to binary, we repeated divide by 2, and record the remainders.

Example.

$$\begin{aligned} 27_{10} &= 16 + 8 + 2 + 1 \\ &= 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 \\ &= 11011_2 \end{aligned}$$

Example. Represent the following in binary notation:

243_{10} 587_{10}

990_{10} 531_{10}

Binary arithmetic:
In binary arithmetic, 10_2 behaves similarly to 10 in decimal arithmetic.

Example. Add 1101_2 and 111_2 using binary notation.

Example. Subtract 1011_2 from 11000_2 using binary notation.

Definition.
The 8-bit two's complement for an integer a between -128 and 127 is the 8-bit binary representation for

$$\begin{cases} a, & \text{if } a \geq 0 \\ 2^8 - |a|, & \text{if } a < 0. \end{cases}$$

Two's complement allows maximum representation for 2^8 integers with 8 binary digits.

Example. Below are a few integers represented in binary using 8-bit two's complement:

$$-128 \to 2^8 - |-128| = 128_{10} = 10000000_2$$

$$-127 \to 2^8 - |-127| = 129_{10} = 10000001_2$$

$$\vdots$$

$$-2 \to 2^8 - |-2| = 254_{10} = 10000000_2$$

$$-1 \to 2^8 - |-1| = 255_{10} = 11111111_2$$

$$0 \to 0_{10} = 00000000_2$$

$$1 \to 1_{10} = 00000001_2$$

$$2 \to 2_{10} = 00000010_2$$

$$\vdots$$

$$127 \to 127_{10} = 01111111_2$$

Example. Find the 8-bit two's complement for the following:

-46

42

120

-82

Two's complement of a negative integer:
To find the decimal representation of the negative integer with a given 8-bit two's complement:

- Flip the bits
- Add 1
- Convert to base-10 and swap the sign

Example. Find the decimal representation of the integers with the following 8-bit two's complement:

11100101_2 $\qquad$ 11000000_2

Addition and Subtraction with Integers in Two's Complement Form:
When performing binary addition on integers written in Two's Complement form, we discard any "carry" bit.

Example. Perform binary addition using the Two's Complement form of the following:

-87 and -46 $\qquad$ 83 and -55

Definition.
Hexadecimal notation uses a **base** 16 **notation**. In hexadecimal, every integer is represented as sum of products of the form

$$d \cdot 16^n$$

where $n \in \mathbb{Z}$ and $d \in \{0, 1, \ldots, 9, A, B, C, D, E, F\}$.

Decimal	Hexadecimal	4-Bit Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
10	A	1010
11	B	1011
12	C	1100
13	D	1101
14	E	1110
15	F	1111

Example. Convert $3CF_{16}$ to decimal notation.

Example. Convert $B09F_{16}$ to binary notation.

Example. Convert 100110110101001_2 to hexadecimal notation.