## 5.2: Logarithmic Functions and Their Properties

**Definition.**

For $a > 0$ and $a \neq 1$, the **logarithmic function**
$$y = \log_a(x) \qquad \text{(logarithmic form)}$$
has domain $x > 0$, base $a$, and is defined by
$$a^y = x \qquad \text{(exponential form)}$$

**Example.** Rewrite the following in exponential form

$4 = \log_2(16)$

$2^4 = 16$

$5 = \log_{10}(100,000)$

$10^5 = 100,000$

$\frac{1}{2} = \log_{16}(4)$

$16^{1/2} = 4$

$\boxed{x^{1/m} = \sqrt[m]{x}}$

$-4 = \log_3\left(\frac{1}{81}\right)$

$3^{-4} = \frac{1}{81}$

$\boxed{x^{-m} = \frac{1}{x^m}}$

$-\frac{1}{4} = \log_{625}\left(\frac{1}{5}\right)$

$625^{-1/4} = \frac{1}{5}$

$-\frac{5}{3} = \log_{\frac{1}{8}}(32)$

$\left(\frac{1}{8}\right)^{-5/3} = 32$

**Example.** Simplify the following:

$y = \log_3(9)$

$3^y = 9 \Rightarrow y = 2$

$\Rightarrow \boxed{\log_3(9) = 2}$

$y = \log_4(2)$

$4^y = 2 \Rightarrow y = \frac{1}{2}$

$\Rightarrow \boxed{\log_4(2) = \frac{1}{2}}$

**Example.** Solve the following:

$\log_5(x) = 4$

$x = 5^4$

$\boxed{x = 625}$

$\log_8(x) = 1$

$x = 8^1$

$\boxed{x = 8}$

$\log_{81}(x) = -\frac{1}{4}$

$x = 81^{-1/4}$

$= \frac{1}{81^{1/4}}$

$= \frac{1}{\sqrt[4]{81}}$

$\boxed{x = \frac{1}{3}}$

$\log_{10}(x + 4) = 3$

$x + 4 = 10^3$

$-4 + x + 4 = 1000 - 4$

$\boxed{x = 996}$

$$\boxed{\begin{array}{ll} \textbf{Common logarithms:} & \log(x) = \log_{10}(x) \\ \textbf{Natural logarithms:} & \ln(x) = \log_e(x) \end{array}}$$

**Example** (The Rule of 70). If \$$P$ is invested for $t$ years at interest rate $r$, compounded continuously, then the future value of the investment is given by

$$S = Pe^{rt}.$$

Find the value of $t$ when the investment doubles.

$$S = 2P$$

$$\frac{2P}{P} = \frac{Pe^{rt}}{P}$$

$$2 = e^{rt} \longrightarrow \log_e(2) = rt$$

$$\frac{\ln(2)}{r} = \frac{rt}{r}$$

$$\Rightarrow \quad t = \frac{\ln(2)}{r} \approx \frac{0.69314\ldots}{r}$$

If we represent the interest rate as a percentage (e.g. 5% instead of 0.05), then we have

$$t \approx \frac{70}{r}$$

Suppose r=5%:

$$\boxed{t \approx \frac{70}{5} = 14}$$

$$\boxed{t = \frac{\ln(2)}{0.05} \approx 13.863}$$

**Change of base formula:**

If $a > 0$, $b > 0$ with $a \neq 1$ and $b \neq 1$, then

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}.$$

*Note*: This works for any valid base!

$$\text{Base } e: \quad \log_b(x) = \frac{\ln(x)}{\ln(b)} \qquad \text{Base } 10: \quad \log_b(x) = \frac{\log(x)}{\log(b)}$$

**Example.** Solve the following

$3^x = 10$

$$x = \log_3(10)$$

$$= \frac{\ln(10)}{\ln(3)}$$

$$\boxed{x \approx 2.0959}$$
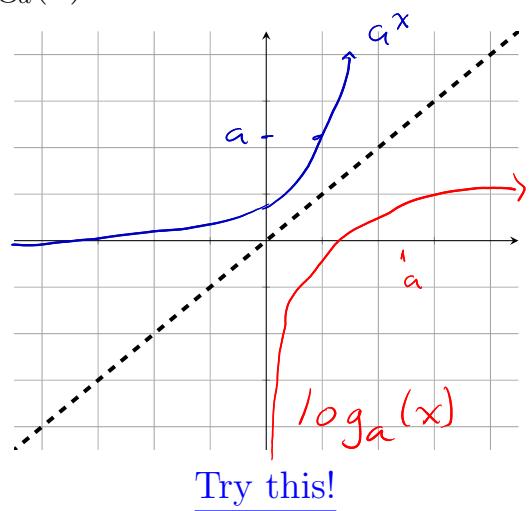
$6.5^x = 5$

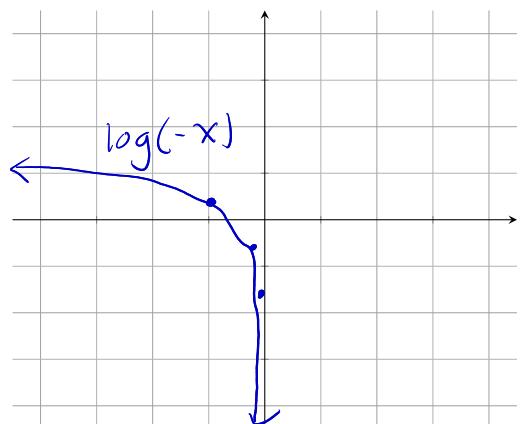$$x = \log_{6.5}(5)$$

$$= \frac{\ln(5)}{\ln(6.5)}$$

$$\boxed{x \approx 0.8598}$$

**Example.** Fill in the tables below and graph $a^x$ and $\log_a(x)$ on the same axes.

| $x$ | $y = a^x$ |
|-----|-----------|
| -2 | $1/a^2$ |
| -1 | $1/a$ |
| 0 | $1$ |
| 1 | $a$ |
| 2 | $a^2$ |

| $x$ | $y = \log_a(x)$ |
|-----|-----------------|
| $1/a^2$ | -2 |
| $1/a$ | -1 |
| $1$ | 0 |
| $a$ | 1 |
| $a^2$ | 2 |



Try this!

**Example.** Graph $\log(-x)$

| $x$ | $\log(-x)$ |
|-----|------------|
| $-1/10^2$ | $-2$ |
| $-1/10$ | $-1$ |
| $-1$ | $0$ |
| $-10$ | $1$ |
| $-100$ | $2$ |

**Example.** Graph $\ln(x)$

| $x$ | $\ln(x)$ |
|---|---|
| $0.368 \approx e^{-2}$ | $-2$ |
| $0.135 \approx e^{-1}$ | $-1$ |
| $1$ | $0$ |
| $e$ | $1$ |
| $e^2$ | $2$ |



**Example.** Graph $-\log_2(-x)$

| $x$ | $-\log_2(-x)$ |
|---|---|
| $-2^2$ | $-2$ |
| $-2$ | $-1$ |
| $-1$ | $0$ |
| $-\frac{1}{2}$ | $1$ |
| $-\frac{1}{2^2}$ | $2$ |

**Example.** Evaluate the following:

$$f(x) = \ln(x); \quad f\left(e^{-3x}\right) \qquad\qquad f(x) = 5^x; \quad f(\log_5(10))$$

$$f\left(e^{-3x}\right) = \ln\left(e^{-3x}\right) \qquad\qquad f(\log_5(10)) = 5^{\log_5(10)}$$

$$= -3x \qquad\qquad = 10$$
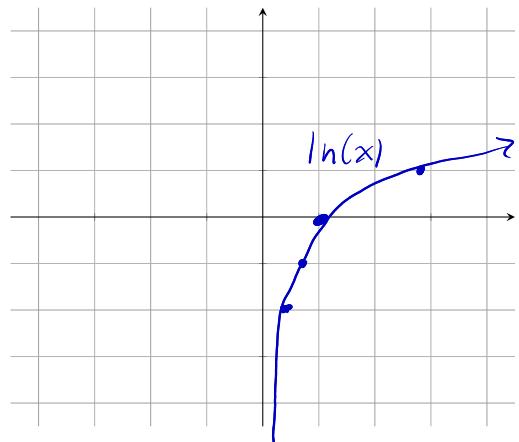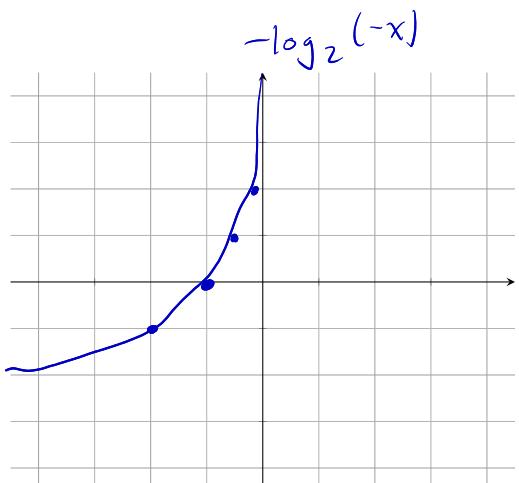
**Properties of exponents and logarithms:** Assume $a > 0$:

| | |
|---|---|
| $a^y = x$ | $\log_a(x) = y$ |
| $a^1 = a$ | $\log_a(a) = 1$ |
| $a^0 = 1$ | $\log_a(1) = 0$ |
| $a^x a^y = a^{x+y}$ | $\log_a(xy) = \log_a(x) + \log_a(y)$ |
| $\dfrac{a^x}{a^y} = a^{x-y}$ | $\log_a\left(\dfrac{x}{y}\right) = \log_a(x) - \log_a(y)$ |
| $a^{xy} = (a^x)^y$ | $\log_a(x^y) = y\log_a(x)$ |
| $a^{\log_a(x)} = x$ | $\log_a(a^x) = x$ |