6.1: Set Theory: Definitions and the Element Method of Proof

Element Argument: The Basic Method for Proving that One set is a Subset of Another
Let sets X and Y be given. To prove that $X \subseteq Y$,

1. **suppose** that x is a particular but arbitrarily chosen element of X,
2. **show** that x is an element of Y

Example. Define sets A and B as follows:

$$A = \{m \in \mathbb{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbb{Z}\} = \{\ldots -6, 0, 6, 12, \ldots\}$$
$$B = \{n \in \mathbb{Z} \mid n = 3s \text{ for some } s \in \mathbb{Z}\} = \{\ldots -9, -6, -3, 0, 3, 6, 9 \ldots\}$$

Prove that $A \subseteq B$

Let $m \in A$, then $m = 6r + 12$, $r \in \mathbb{Z}$

$$\begin{aligned} \Rightarrow m &= 6r + 12 \\ &= 3(2r+4) \\ &= 3s, \quad \text{where } s = 2r+4 \end{aligned}$$

Since $s = 2r + 4 \in \mathbb{Z}$, then $m \in B$.

Disprove that $B \subseteq A$

$3 \in B$, but $3 \notin A$

Definition.
Given sets A and B, A **equals** B, written $\mathbf{A} = \mathbf{B}$, if, and only if, every element of A is in B and every element of B is in A:

$$A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A.$$

Example. Define sets A and B as follows:

$$A = \{m \in \mathbb{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbb{Z}\}$$
$$B = \{n \in \mathbb{Z} \mid n = 3s \text{ for some } s \in \mathbb{Z}\}$$

Is $A = B$?

No, since $A \subseteq B$, but $B \nsubseteq A$.

Definition.
Given an integer n and a positive integer d, when n is divided by d, then

$$\begin{aligned} n \operatorname{div} d &= \text{ the integer quotient} \\ n \bmod d &= \text{ the nonnegative integer remainder} \end{aligned}$$

If n and d are integers and $d > 0$, then

$$n \operatorname{div} d = q \quad \text{and} \quad n \bmod d = r \quad \Leftrightarrow \quad n = dq + r$$

Example. Compute the following:

$32 \operatorname{div} 9, \ \ 32 \bmod 9$

$\frac{32}{9} = 3.\overline{5} = 3 R 5$ → $32 \operatorname{div} 9 = 3$; → $32 \bmod 9 = 5$

$365 \operatorname{div} 7, \ \ 365 \bmod 7$

$\frac{365}{7} = 52.\overline{142857} = 52 R 1$ → $365 \operatorname{div} 7 = 52$; → $365 \bmod 7 = 1$

Example. If it is currently 11:00, what time will it be in

51 hours?

$51 = 2(24) + 3 \longrightarrow$ 2:00

121 hours?

$121 = 5(24) + 1 \longrightarrow$ 12:00

11 hours?

$11 = 12 - 1 \longrightarrow$ 10:00

-1 hours?

10:00

Example. Let $A = \left\{4, \sqrt{16}, 19 \bmod 15\right\}$ and $B = \{12 \bmod 8\}$. Is $A \subseteq B$? Is $B \subseteq A$?

$A = \{4\}$ $\qquad$ $B = \{4\}$

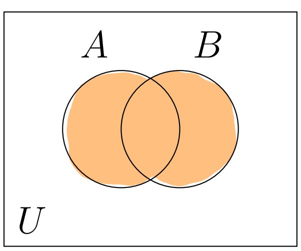
$$\left.\begin{aligned} 4 \in A \ \& \ 4 \in B &\rightarrow A \subseteq B \\ 4 \in B \ \& \ 4 \in A &\rightarrow B \subseteq A \end{aligned}\right\} \longrightarrow A = B$$

Definition.
Let A and B be subsets of a universal set U.

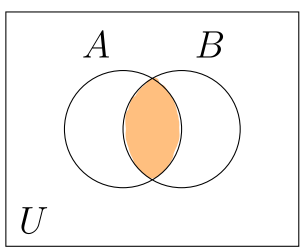
1. The **union** of A and B is the set of all elements that are in at least one of A or B.
$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$$
2. The **intersection** of A and B is the set of all elements that are common to both A and B.
$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$
3. The **difference** of A and B is the set of all elements that are in B and not A.
$$B - A = \{x \in U \mid x \in B \text{ and } x \notin A\}$$
4. The **complement** of A is the set of all elements in U that are not in A.
$$A^c = \{x \in U \mid x \notin A\}$$

Example. Represent the following sets using the Venn diagrams below:

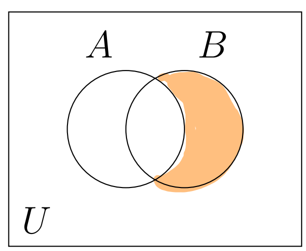
$A \cup B$



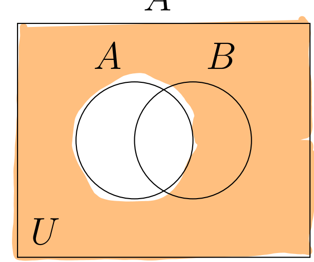
$A \cap B$



$B - A$



A^c



Example. Let the universal set be the set $U = \{a, b, c, d, e, f, g\}$, and let $A = \{a, c, e, g\}$ and $B = \{d, e, f, g\}$. Find

$A \cup B = \{a, c, d, e, f, g\}$

$A \cap B = \{e, g\}$

$B - A = \{d, f\}$

$A^c = \{b, d, f\}$

> **Definition.**
> Given real numbers a and b with $a \leq b$:
>
> $$(a, b) = \{x \in \mathbb{R} \,|\, a < x < b\} \qquad (a, b] = \{x \in \mathbb{R} \,|\, a < x \leq b\}$$
> $$[a, b) = \{x \in \mathbb{R} \,|\, a \leq x < b\} \qquad [a, b] = \{x \in \mathbb{R} \,|\, a \leq x \leq b\}$$

Example. Let the universal set be $\mathbb{R}$, and let $A = (-1, 0]$ and $B = [0, 1)$. Find

$A \cup B = (-1, 0] \cup [0, 1)$

$= (-1, 1)$

$A \cap B = (-1, 0] \cap [0, 1)$

$= \{0\}$

$B - A = (0, 1)$

Exclude 0
since $0 \in A$

$A^c = (-\infty, -1] \cup (0, \infty)$

Definition.
Given sets $A_0, A_1, A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n,

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i, \text{ for at least one } i = 0, 1, 2, \ldots, n\}$$

$$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i, \text{ for every } i = 0, 1, 2, \ldots, n\}$$

Example. For each positive integer i, let $A_i = \left\{x \in \mathbb{R} \,\middle|\, -\frac{1}{i} < x < \frac{1}{i}\right\} = \left(-\frac{1}{i}, \frac{1}{i}\right)$. Find

$A_1 \cup A_2 \cup A_3$

$(-1,1) \cup (-\frac{1}{2}, \frac{1}{2}) \cup (-\frac{1}{3}, \frac{1}{3}) = (-1,1)$

Nested in $(-1,1)$

$A_1 \cap A_2 \cap A_3$

$(-1,1) \cap (-\frac{1}{2}, \frac{1}{2}) \cap (-\frac{1}{3}, \frac{1}{3}) = (-\frac{1}{3}, \frac{1}{3})$

$\bigcup_{i=1}^{\infty} A_i = (-1,1)$

$\bigcap_{i=1}^{\infty} A_i = \{0\}$

Only element common to all intervals

Definition.
The **empty set** (or **null set**), denoted $\varnothing$, is the set with no elements.

$$\{1,3\} \cap \{2,4\} = \varnothing$$

Two sets are called **disjoint** if, and only if, they have no elements in common:

$$A \cap B = \varnothing.$$

Sets $A_1, A_2, A_3, \ldots$ are **mutually disjoint** (or **pairwise disjoint**) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common:

$$A_i \cap A_j = \varnothing \text{ whenever } i \neq j.$$

Example.
Let $A_1 = \{3,5\}$, $A_2 = \{1,4,6\}$, and $A_3 = \{2\}$. Are A_1, A_2, and A_3 mutually disjoint?

Yes: $\left.\begin{aligned} A_1 \cap A_2 &= \varnothing \\ A_1 \cap A_3 &= \varnothing \\ A_2 \cap A_3 &= \varnothing \end{aligned}\right\}$ Exhaustively checked

Let $B_1 = \{2,4,6\}$, $B_2 = \{3,7\}$, and $B_3 = \{4,5\}$. Are B_1, B_2, B_3 mutually disjoint?

No: $B_1 \cap B_3 = \{4\}$

Definition.
A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, \ldots\}$ is a **partition** of a set A if, and only if,

1. A is the union of all the A_i;
2. the sets $A_1, A_2, A_3, \ldots$ are mutually disjoint.

Example.
Let $A = \{1, 2, 3, 4, 5, 6\}$, $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$, and $A_3 = \{5, 6\}$. Is $\{A_1, A_2, A_3\}$ a partition of A?

Yes: (1) $A_1 \cup A_2 \cup A_3 = A$

(2) $A_1 \cap A_2 = \emptyset$
$A_1 \cap A_3 = \emptyset$
$A_2 \cap A_3 = \emptyset$

Let $\mathbb{Z}$ be the set of all integers and let

$$T_i = \{n \in \mathbb{Z} \mid n = 3k + i, \text{ for some integer } k\}.$$

Is $\{T_0, T_1, T_2\}$ a partition of $\mathbb{Z}$?

$T_0 = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$

$T_1 = \{\ldots -8, -5, -2, 1, 4, 7, 10, \ldots\}$

$T_2 = \{\ldots -7, -4, -1, 2, 5, 8, 11, \ldots\}$

Yes: (1) $T_0 \cup T_1 \cup T_2 = \mathbb{Z}$

(2) $T_1 \cap T_2 = \emptyset$
$T_1 \cap T_3 = \emptyset$
$T_2 \cap T_3 = \emptyset$

Definition.
Given a set A, the **power set** of A, denoted $\mathscr{P}(A)$, is the set of all subsets of A.

Example. Find $\mathscr{P}(\{x, y\})$.

$$P(\{x, y\}) = \{\{\ \}, \{x\}, \{y\}, \{x, y\}\}$$