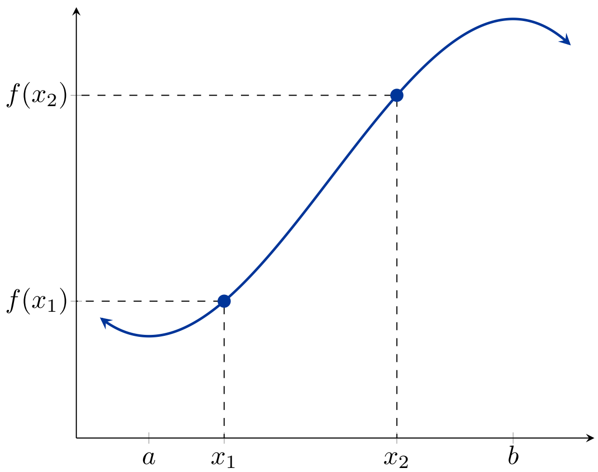
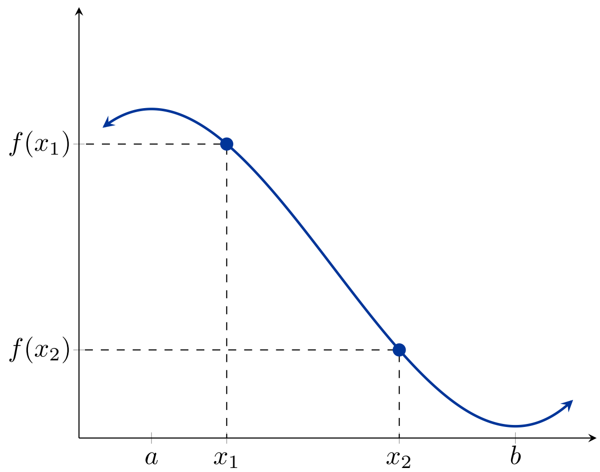
4.1: Applications of the First Derivative

Definition.
Consider the function $f(x)$ on the interval (a, b). Given *any* two numbers x_1 and x_2 in (a, b) where $x_1 < x_2$, we say f is

increasing if $f(x_1) < f(x_2)$



decreasing if $f(x_1) > f(x_2)$



Thus, for every value of x on the interval (a, b), if

- $f'(x) > 0$, then f is increasing on (a, b).
- $f'(x) < 0$, then f is decreasing on (a, b).
- $f'(x) = 0$, then f is constant on (a, b).

Example. Find the intervals where $f(x) = x^2$ is increasing and decreasing.

Determining intervals where a function is increasing or decreasing.

1. Find all values of x such that $f'(x) = 0$ or $f'(x)$ is undefined.
2. Determine the sign of $f'(x)$ on each open interval.

Example. Suppose that f is continuous everywhere and

$$f'(x) = \frac{(x-1)(x+2)}{(x-4)^2(x+5)}.$$

We see that $f'(-2) = f(1) = 0$ and $f(-5)$ and $f(4)$ are undefined. Complete a sign chart to show where $f(x)$ is increasing and decreasing.

Example. Find the intervals where the following functions are increasing and decreasing:

$f(x) = x^3 - 3x^2 - 24x + 32$ Graph

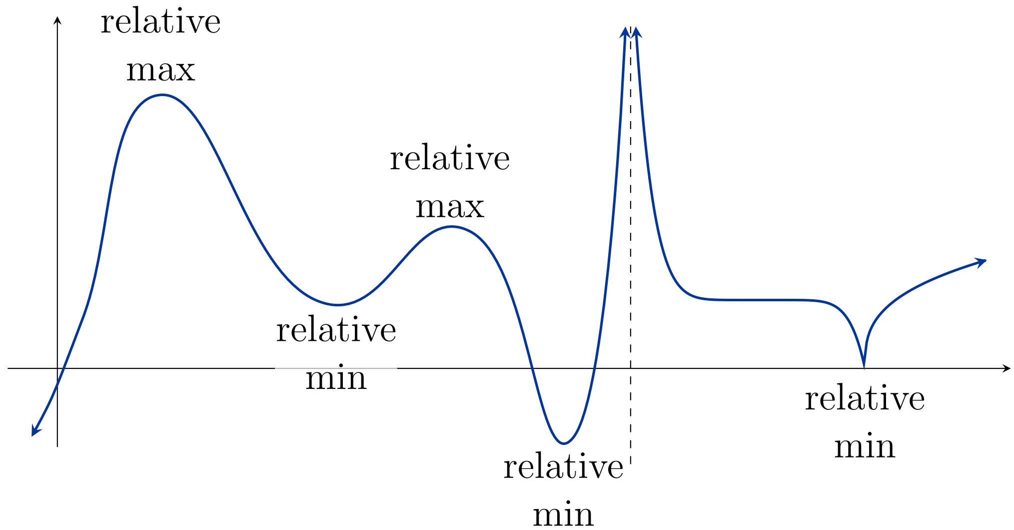
$g(x) = (x + 1)^{2/3}$

$$h(x) = x + \frac{1}{x}$$

$$j(x) = \frac{x^2}{1 - x^2}$$

Definition. (Relative Extrema)
A function f has a

- **relative maximum** at $x = c$ if $f(c) \geq f(x)$ for every x in (a, b)
- **relative minimum** at $x = c$ if $f(c) \leq f(x)$ for every x in (a, b)



Definition.
A **critical point** of a function f is any number x in the domain of f such that $f'(x) = 0$ or $f'(x)$ does not exist.

Procedure for Finding the Relative Extrema of a Continuous Function f
The First Derivative Test:

1. Determine the critical points of f.
2. Determine the sign change of $f'(x)$ to the left and right of each critical point:
   If, at $x = c$, $f'(x)$ ...

   a) changes sign from *positive* to *negative*, then f has a *relative maximum*

   b) changes sign from *negative* to *positive*, then f has a *relative minimum*

   c) does not change sign, then f does *not* have a relative extremum

   at $x = c$.

Example. Consider the function $f(x) = 6x - x^3$. Graph

Use $f'(x)$ to find the intervals on which the function is increasing and decreasing.

Identify the function's local extreme values (e.g. "local max of __ at $x =$ __")

Example. Find the relative maximums/relative minimums of the following:

$f(x) = x^3 - 3x^2 - 24x + 32$ Graph

$g(x) = (x + 1)^{2/3}$

$$h(x) = x + \frac{1}{x}$$

$$j(x) = \frac{x^2}{1 - x^2}$$