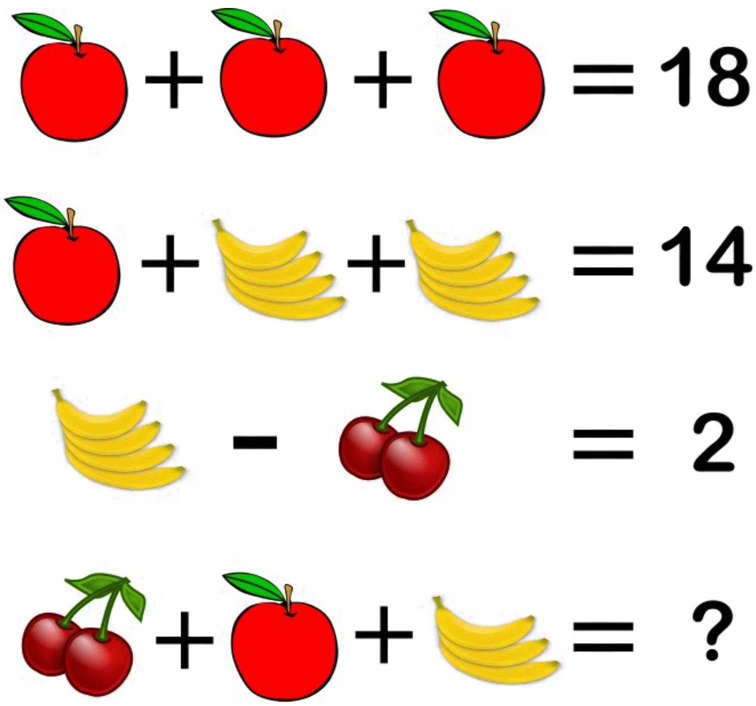
1.5: Solutions of Systems of Linear Equations



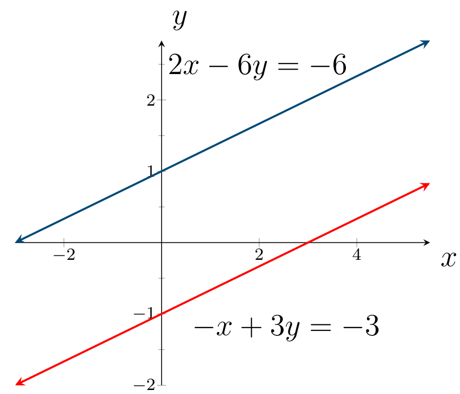
> **Definition.**
> A **system of equations** is 2 (or more) equations. The ordered pairs (x, y) that satisfies *all* equations in the system are the **solutions** of the system.

When solving a system of linear equations, there are three possible outcomes:

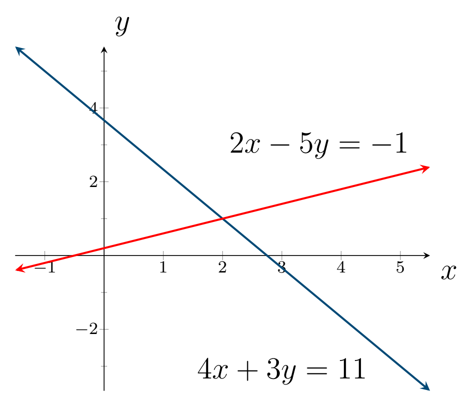
1. No solution (*Inconsistent*),
2. Exactly one solution,
3. Infinitely many solutions (*Dependent*).

Example. Use graphing to find the solutions to the following systems

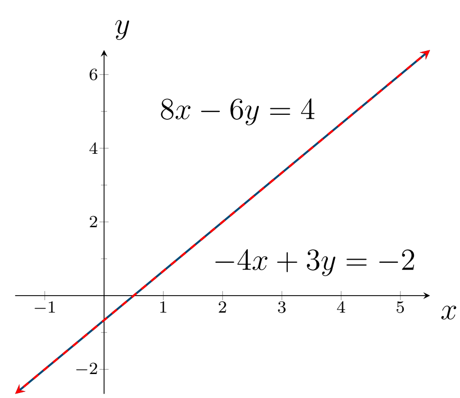
$$\begin{aligned} 2x - 6y &= -6 \\ -x + 3y &= -3 \end{aligned}$$



$$\begin{aligned} 4x + 3y &= 11 \\ 2x - 5y &= -1 \end{aligned}$$



$$\begin{aligned} -4x + 3y &= -2 \\ 8x - 6y &= 4 \end{aligned}$$



Equivalent systems result when

1. One expression is replaced by an equivalent expression.
2. Two equations are interchanged.
3. A multiple of one equation is added to another equation.
4. An equation is multiplied by a nonzero constant.

Substitution Method

Example. Solve the system $\begin{cases} 2x + 3y = 4 \\ x - 2y = 3 \end{cases}$

1. Solve one equation for either one of the variables in terms of the other.
$$x = 2y + 3$$
2. Substitute this expression into the other equation to give the equation in one unknown.
$$2(2y + 3) + 3y = 4$$
3. Solve this equation for the unknown.
$$4y + 6 + 3y = 4$$
$$7y = -2 \Rightarrow y = -\frac{2}{7}$$
4. Substitute solution into the equation in Step 1.
$$x = 2\left(-\frac{2}{7}\right) + 3 \Rightarrow x = \frac{17}{7}$$
5. Check the solution (x, y).
$$2\left(\frac{17}{7}\right) + 3\left(-\frac{2}{7}\right) = 4$$
$$\left(\frac{17}{7}\right) - 2\left(-\frac{2}{7}\right) = 3$$

Example. Use the substitution method to solve the system

$$\begin{aligned} 4x + 5y &= 18 \qquad (1) \\ 3x - 9y &= -12 \qquad (2) \end{aligned}$$

Elimination Method

Example. Solve the system $\begin{cases} 2x - 5y = 4 \\ x + 2y = 3 \end{cases}$

1. Multiply one or both equations by a nonzero number so the coefficients of one of the variables may cancel.

$$\Rightarrow \begin{cases} 2x - 5y = 4 \\ -2x - 4y = -6 \end{cases}$$

2. Add or subtract the equations to eliminate one of the variables.

$$0x - 9y = -2$$

3. Solve for the remaining variable.

$$\Rightarrow y = \frac{2}{9}$$

4. Substitute solution in one of the original equations and solve for the other variable.

$$2x - 5\left(\frac{2}{9}\right) = 4 \;\Rightarrow x = \frac{23}{9}$$

5. Check the solution (x, y)

$$2\left(\frac{23}{9}\right) - 5\left(\frac{2}{9}\right) = 4$$
$$\left(\frac{23}{9}\right) + 2\left(\frac{2}{9}\right) = 3$$

Example. Use the elimination method to solve the following systems:

$$\begin{aligned} 2x - 6y &= -6 \\ -x + 3y &= -3 \end{aligned}$$

$$\begin{aligned} 4x + 3y &= 11 \\ 2x - 5y &= -1 \end{aligned}$$

$$\begin{aligned} -4x + 3y &= -2 \\ 8x - 6y &= 4 \end{aligned}$$

Example. A woman has \$500,000 invested in two rental properties. One yields an annual return of 10% on her investment, and the other returns 12% per year on her investment. Her total annual return from the two investments is \$53,000. Let x represent the amount of the 10% investment and y represent the amount of the 12% investment.

- Write an equation that states that the sum of investments is \$500,000.

- What is the annual return on the 10% investment? What about the 12% investment?

- Write an equation that states the sum of the annual return is \$53,000.

- Solve these two equations simultaneously to find how much is invested in each property.

Example. A nurse has two solutions that contain different concentrations of a certain medication. One is a 12.5% concentration, and the other is a 5% concentration. How many cubic centimeters of each should she mix to obtain 20 cubic centimeters of an 8% concentration?

Example. Using U.S. Bureau of Labor Statistics data for selected years from 1950 and projected to 2050, the number of men M and women W in the workforce (both in millions) can be modeled by the functions

$$M(t) = 0.591t + 37.3 \qquad \text{and} \qquad W(t) = 0.786t + 13.1$$

where t is the number of years after 1940. Find the year these functions predict that there will be equal numbers of men and women in the U.S. workforce.