4.1: Linear Inequalities in Two Variables

Properties of Inequalities

Substitution Property: The inequality formed by substituting one expression for an equal expression is equivalent to the original inequality:

$$\begin{aligned} 5x - 4x + 2 &< 6 \\ x &< 4 \quad \Rightarrow \text{The solution set is } \{x : x < 6\} \end{aligned}$$

Addition Property: The inequality formed by adding the same quantity to both sides of an inequality is equivalent to the original inequality:

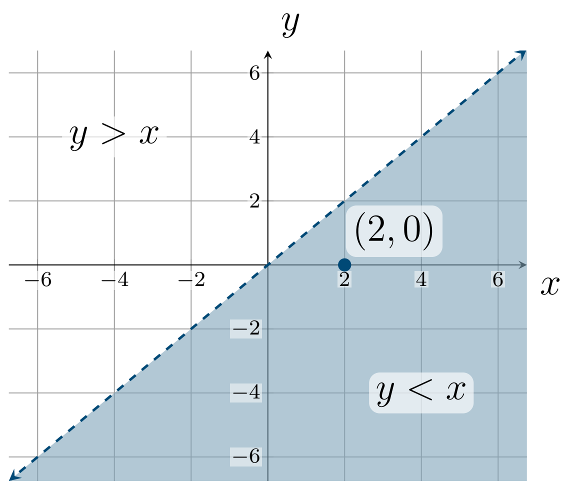
$$\begin{aligned} x - 4 &< 6 \\ x - 4 + 4 &< 6 + 4 \\ x &< 10 \end{aligned} \qquad\qquad \begin{aligned} x + 5 &\geq 12 \\ x + 5 + (-5) &\geq 12 + (-5) \\ x &\geq 7 \end{aligned}$$

Multiplication Property The inequality formed by multiplying both sides of an inequality by the same *positive* quantity is equivalent to the original inequality. The direction of the inequality is flipped when multiplying by a *negative* quantity:

$$\begin{aligned} \frac{1}{3}x &> 6 \\ 3\left(\frac{1}{3}x\right) &> 3(6) \\ x &> 18 \end{aligned} \qquad\qquad \begin{aligned} 5x - 5 + 5 &\leq 6x + 20 + 5 \\ \frac{-x}{-1} &\leq \frac{25}{-1} \\ x &\geq -25 \end{aligned}$$

One Linear Inequality in Two Variables

Definition.
Consider the inequality $y < x$:
The line created by this inequality divides the xy-plane into two **half-planes**. We can determine which half-plane is the solution region by selecting any point *not on the line* as a **test point**.



Example. Graph the inequality $3x - 2y \le 6$

$$-3x + 3x - 2y = 6 \quad -3x$$

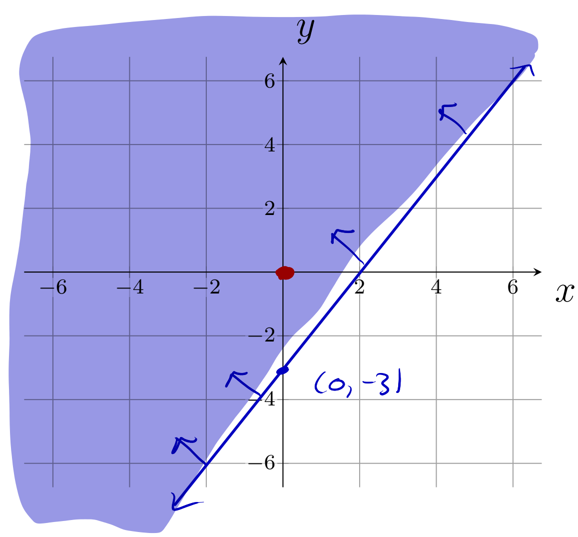
$$\frac{-2y}{-2} = \frac{-3x + 6}{-2}$$

$$y = \frac{3}{2}x - 3$$

Test point:

(x, y)	$\overbrace{3x - 2y}^{0} \le 6$
$(0, 0)$	$3(0) - 2(0)$ $= 0$

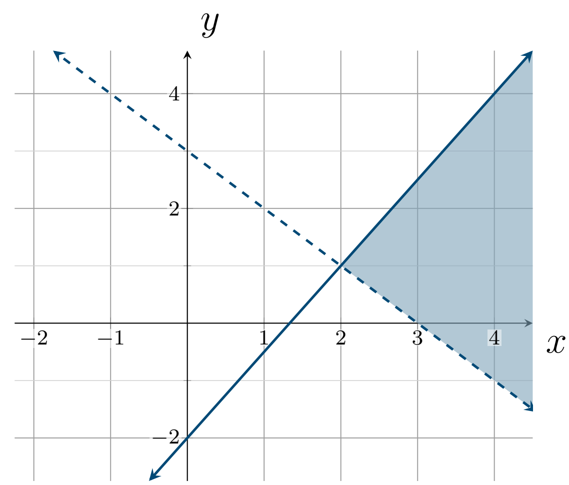
Since $0 \le 6$, we shade on the side of the test point



Definition.
A **system of inequalities** has two or more inequalities in two or more variables. The solution of the system is the intersection of the individual solution sets.

Example.

$$\begin{aligned} 3x-2y &\geq 4 \\ x+y-3 &> 0 \end{aligned}$$



Example. Graph the solution of the system

$$\begin{aligned} 3x-4y &\leq 12 \\ 2x+5y &> 10 \\ x-8y &\geq -16 \end{aligned}$$

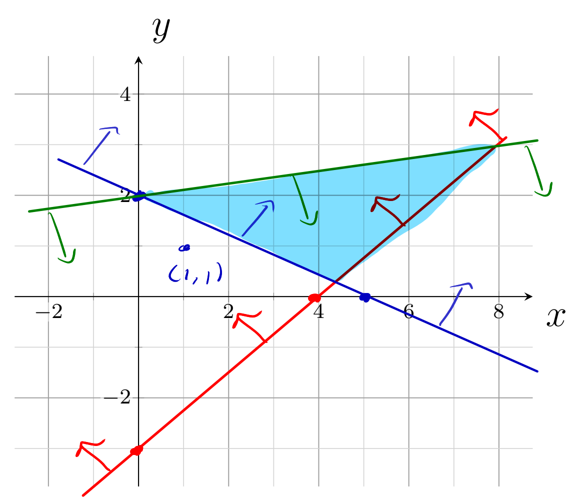
(0,−3) (4,0)

(0,2) (5,0)

$-x+x-8y \geq -16-x$

$\frac{-8y}{-8} \geq \frac{-x-16}{-8}$

$y \leq \frac{1}{8}x+2$



Test point:

(x, y)	$\overbrace{3x-4y\leq 12}^{-1}$	$\overbrace{2x+5y>10}^{7}$	$\overbrace{x-8y\geq -16}^{-7}$
(1, 1)	3(1)−4(1) = −1 ✓	2(1)+5(1) = 7 ×	(1)−8(1) = −7 ✓

Example. CDF Appliances has assembly plants in Atlanta and Fort Worth, where the company produces a variety of kitchen appliances, including a 12-cup coffee maker and a cappuccino machine. The following table shows each factory's assembly capabilities for the two products and the numbers needed to fill orders.

	Atlanta	**Fort Worth**	**Needed**
Coffee maker	160/hr	800/hr	At least 64,000
Cappucino machine	200/hr	200/hr	At least 40,000

Write the system of inequalities that describes the number of assembly hours needed at each plant to fill the orders and graph the solution region for the system

assembly hrs. in ATL → x; assembly hrs. in FTW → y

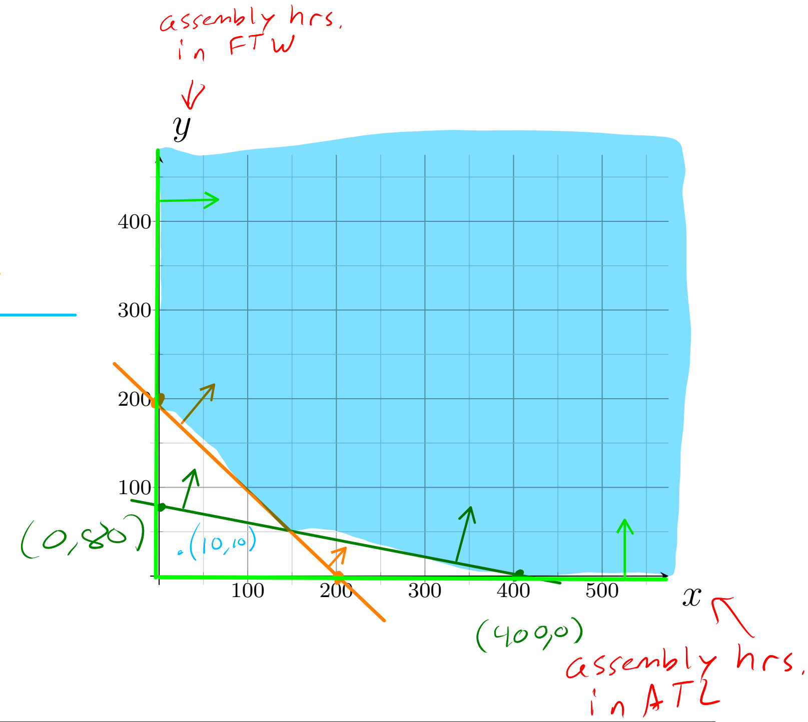
coffee makers → $160x + 800y \geq 64{,}000$ — $(0, 80)$, $(400, 0)$

cap. machines → $200x + 200y \geq 40{,}000$ — $(0, 200)$, $(200, 0)$

non-negative hours → $x \geq 0,\ y \geq 0$

Test point:

(x,y)	$160x + 800y \geq 64{,}000$	$200x + 200y \geq 40{,}000$
$(10,10)$	$160(10)+800(10)$ $=1600+8000$ $=9600$	$200(10)+200(10)$ $=2000+2000$ $=4000$



Example. A farm co-op has 6000 acres available to plant with corn (x) and soybeans (y). Each acre of corn requires 9 gallons of fertilizer/herbicide and 3/4 hour of labor to harvest. Each acre of soybeans requires 3 gallons of fertilizer/herbicide and 1 hour of labor to harvest. The co-op has available at most 40,500 gallons of fertilizer/herbicide and at most 5250 hours of labor for harvesting. The number of acres of each crop is limited (constrained) by the available resources: land, fertilizer/herbicide, and labor for harvesting. Write the system of inequalities that describes the constraints and graph the solution region for the system.

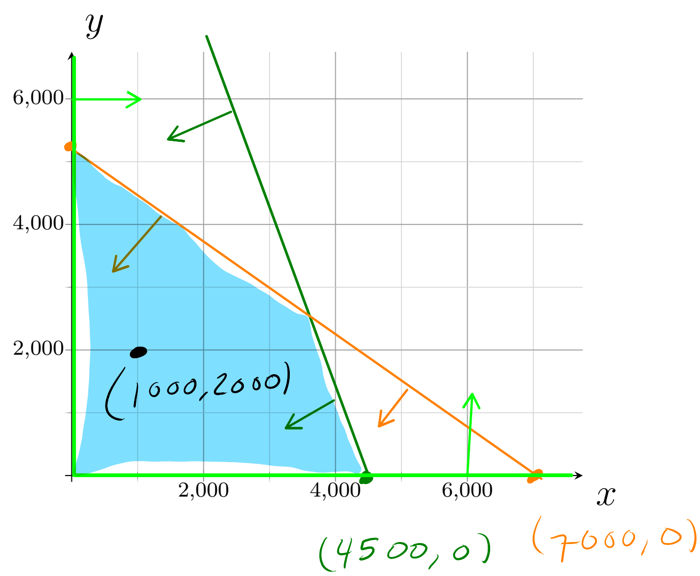
corn soybeans

fert./herb. $\longrightarrow 9x + 3y \le 40{,}500 \longrightarrow y \le -3x + 13500$

labor $\longrightarrow \frac{3}{4}x + 1y \le 5250 \longrightarrow y \le -\frac{3}{4}x + 5250$

non-negative # of crops $\longrightarrow x \ge 0,\ y \ge 0$

(x,y)	$\overbrace{9x + 3y}^{15000} \le 40{,}500$	$\overbrace{\frac{3}{4}x + y}^{2750} \le 5250$
$(1000,2000)$	$9(1000) + 3(2000)$	$\frac{3}{4}(1000) + (2000)$
	$= 9000 + 6000$	$= 750 + 2000$
	$= 15000$ ✓	$= 2750$ ✓



Example. Graph the solution region for the system

$$\begin{aligned} 5x+2y &\le 54 \\ 2x+4y &\le 60 \\ x \ge 0,\ y &\ge 0 \end{aligned}$$

$5x+2y \le 54$: $(0, 27)$, $(10.8, 0)$

$2x+4y \le 60$: $(0, 15)$, $(30, 0)$

Then compute the corners of this region.

$$\begin{aligned} 2(5x+2y &= 54) \\ -\ 1(2x+4y &= 60) \\ \hline 8x + 0y &= 48 \end{aligned}$$

$$\frac{8x}{8} = \frac{48}{8}$$

$$\boxed{x = 6}$$

$$2(6) + 4y = 60$$

$$12 + 4y = 60$$

$$4y = 48$$

$$\boxed{y = 12}$$

Test point:

(x, y)	$\underbrace{5x+2y}_{120} \le 54$	$\underbrace{2x+4y}_{80} \le 60$
$(20, 10)$	$5(20)+2(10)$ $=100+20$ $=120$ ✗	$2(20)+4(10)$ $=40+40$ $=80$ ✗

Corners:

$(0, 0)$

$(10.8, 0)$

$(6, 12)$

$(0, 15)$