

5.2: Logarithmic Functions and Their Properties

Definition.

For $a > 0$ and $a \neq 1$, the **logarithmic function**

$$y = \log_a(x) \quad (\text{logarithmic form})$$

has domain $x > 0$, base a , and is defined by

$$a^y = x \quad (\text{exponential form})$$

Example. Rewrite the following in exponential form

$$4 = \log_2(16)$$

$$5 = \log_{10}(100,000)$$

$$\frac{1}{2} = \log_{16}(4)$$

$$-4 = \log_3\left(\frac{1}{81}\right)$$

$$-\frac{1}{4} = \log_{625}\left(\frac{1}{5}\right)$$

$$-\frac{5}{3} = \log_{\frac{1}{8}}(32)$$

Example. Simplify the following:

$$\log_3(9)$$

$$\log_4(2)$$

Example. Solve the following:

$$\log_5(x) = 4$$

$$\log_8(x) = 1$$

$$\log_{81}(x) = -\frac{1}{4}$$

$$\log_{10}(x + 4) = 3$$

Common logarithms: $\log(x) = \log_{10}(x)$
Natural logarithms: $\ln(x) = \log_e(x)$

Example (The Rule of 70). If $\$P$ is invested for t years at interest rate r , compounded continuously, then the future value of the investment is given by

$$S = Pe^{rt}.$$

Find the value of t when the investment doubles.

Change of base formula:

If $a > 0$, $b > 0$ with $a \neq 1$ and $b \neq 1$, then

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}.$$

Note: This works for any valid base!

$$\text{Base } e : \quad \log_b(x) = \frac{\ln(x)}{\ln(b)} \qquad \text{Base } 10 : \quad \log_b(x) = \frac{\log(x)}{\log(b)}$$

Example. Solve the following

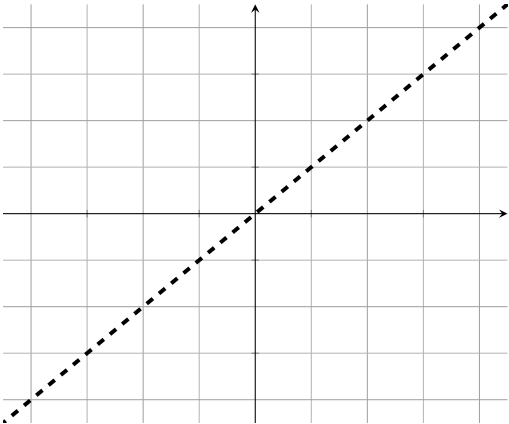
$$3^x = 10$$

$$6.5^x = 5$$

Example. Fill in the tables below and graph a^x and $\log_a(x)$ on the same axes.

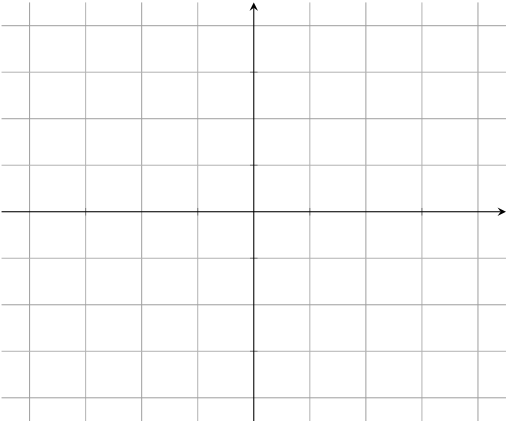
x	$y = a^x$
-2	
-1	
0	
1	
2	

x	$y = \log_a(x)$
-2	
-1	
0	
1	
2	

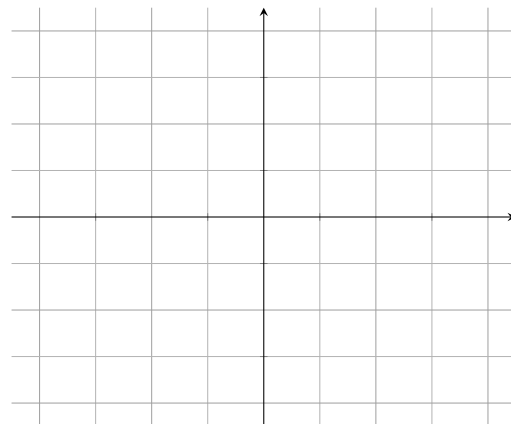


[Try this!](#)

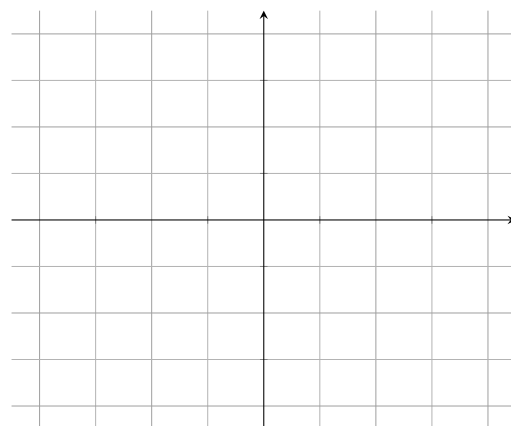
Example. Graph $\log(-x)$



Example. Graph $\ln(x)$



Example. Graph $-\log_2(-x)$



Example. Evaluate the following:

$$f(x) = \ln(x); \quad f(e^{-3x})$$

$$f(x) = 5^x; \quad f(\log_5(10))$$

Properties of exponents and logarithms: Assume $a > 0$:

$$a^y = x$$

$$\log_a(x) = y$$

$$a^1 = a$$

$$\log_a(a) = 1$$

$$a^0 = 1$$

$$\log_a(1) = 0$$

$$a^x a^y = a^{x+y}$$

$$\log_a(xy) = \log_a(x) + \log_a(y)$$

$$\frac{a^x}{a^y} = a^{x-y}$$

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

$$a^{xy} = (a^x)^y$$

$$\log_a(x^y) = y \log_a(x)$$

$$a^{\log_a(x)} = x$$

$$\log_a(a^x) = x$$