

6.3: Future Values of Annuities

Definition.

- An **annuity** is a financial plan characterized by regular payments (e.g. mortgages, student loans, etc.).
- The sum of all the payments and the interest earned is called the **future value of the annuity** or its **future value**.
- An **ordinary annuity** or (**annuity immediate**) is an annuity in which payments are made at the *end of each of the equal payment intervals*.

Example. Suppose that we invest \$100 at the end of each year for 5 years in an account that pays 10% compounded annually. How much money will you have at the end of the 5 years?

$$\begin{aligned} 1. \quad S &= 100 (1 + 0.10)^4 \\ 2. \quad S &= 100 (1 + 0.10)^3 \\ 3. \quad S &= 100 (1 + 0.10)^2 \\ 4. \quad S &= 100 (1 + 0.10)^1 \\ 5. \quad S &= 100 (1 + 0.10)^0 \\ \hline &\$610.51 \end{aligned}$$

"Geometric Series"

Definition.

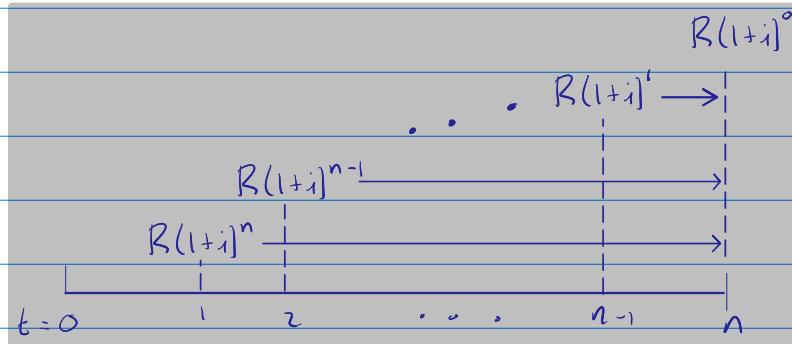
If $\$R$ is deposited at the *end of each period* for n periods in an annuity that earns interest at a rate of i per period, the **future value of the annuity** will be

$$S = R \cdot S_{\bar{n}|i} = R \left[\frac{(1 + i)^n - 1}{i} \right]$$

The notation $S_{\bar{n}|i}$ represents the future value of an ordinary annuity of \$1 per period for n periods with an interest rate of i per period.

Derivation

$$S = R \left[\frac{(1+i)^n - 1}{i} \right]$$



$$\begin{aligned} S &= R(1+i)^{n-1} + R(1+i)^{n-2} + \dots + R(1+i)^1 + R \\ - (1+i)S &= - \left(R(1+i)^n + R(1+i)^{n-1} + \dots + R(1+i)^2 + R(1+i)^1 \right) \end{aligned}$$

$$S - (1+i)S = R - R(1+i)^n$$

$$S(1 - (1+i)) = R[1 - (1+i)^n]$$

$$S = R \left[\frac{(1+i)^n - 1}{i} \right]$$

Example. Suppose a pair of twins take different steps to save for retirement. Both regularly make investments of \$2,000 into accounts that earn 10%, compounded annually. Starting at age 21:

Find the future value if twin A makes his payments for 8 years, and then lets his investment accrue compound interest every year for 36 years.

1. $S \leftarrow$ Find after 8 yrs

$$R: 2000$$

$$i: 10\% = 0.10$$

$$n: 8$$

$$S = 2000 \left[\frac{(1+0.10)^8 - 1}{0.10} \right]$$

$$= \$22,871.78$$

2. S is invested for 36 yrs.

$$S (1+0.10)^{36}$$

$$= \$707,027.91$$

Find the future value if twin B waits 8 years before making regular payments for the following 36 years.

$S \leftarrow$ Find after 36 yrs

$$R: 2000$$

$$i: 10\% = 0.10$$

$$n: 36$$

$$S = 2000 \left[\frac{(1+0.10)^{36} - 1}{0.10} \right]$$

$$= \$598,253.61$$

Example. Suppose that you wish to have \$50,000 saved up in 5 years. To do this, you want to make regularly monthly payments. What is the amount of the monthly payments if the interest rate is 5%? What if the interest rate is 15%?

$$S : \$50000$$

$R \leftarrow$ Find

$$i : \frac{r}{m} = \frac{5\%}{12} = 1.45\% = 0.0145$$

$$n : mt = 12(5) = 60$$

$$50000 = R \left[\frac{(1 + \frac{0.05}{12})^{60} - 1}{\frac{0.05}{12}} \right]$$

$$\Rightarrow R = \frac{50000}{\left[\frac{(1 + \frac{0.05}{12})^{60} - 1}{\frac{0.05}{12}} \right]} = \$735.23$$

$$r = 15\% \Rightarrow i = \frac{r}{m} = \frac{15\%}{12} = 1.25\% = 0.0125 \Rightarrow R = \frac{50000}{\left[\frac{(1 + \frac{0.15}{12})^{60} - 1}{\frac{0.15}{12}} \right]} = \$564.50$$

Example. A small business invests \$1,000 at the end of each month in an account that earns 6% compounded monthly. How long will it take until the business has \$100,000 toward the purchase of its own office building?

$$S : \$100,000$$

$$R : \$1000$$

$$\frac{r}{m} = i : \frac{6\%}{12} = 0.5\% = 0.005$$

$$mt = n : 12t \leftarrow \text{Find } t$$

$$\left(\frac{0.005}{1000} \right) 100,000 = 1000 \left[\frac{1.005^{12t} - 1}{0.005} \right] \left(\frac{0.005}{1000} \right)$$

$$0.5 + 1 = 1.005^{12t} - 1 + 1$$

$$\ln(1.5) = \ln(1.005^{12t})$$

$$\frac{\ln(1.5)}{12 \ln(1.005)} = \frac{12t \ln(1.005)}{12 \ln(1.005)}$$

$$t = \frac{\ln(1.5)}{12 \ln(1.005)} = 6.7746 \quad \begin{cases} \text{about 6 yrs.} \\ \text{and 10 months} \end{cases}$$

Definition.

An **annuity due** differs from an ordinary annuity in that the payments are made at the *beginning of each period*.

If $\$R$ is deposited at the *beginning of each period* for n periods in an annuity that earns interest at a rate of i per period, the **future value of the annuity** will be

$$S_{\text{due}} = R \cdot S_{\bar{n}|i} (1 + i) = R \left[\frac{(1 + i)^n - 1}{i} \right] (1 + i)$$

Example. Find the future value of an investment if \$150 is deposited at the *beginning of each month* for 9 years at an interest rate of 7.2% compounded monthly.

S : ← Find

R : 150

$$\frac{r}{m} = i : \frac{7.2\%}{12} = 0.6\% = 0.006$$

$$mt = n : 12(9) = 108$$

$$S = 150 \left[\frac{(1.006)^{108} - 1}{0.006} \right] (1.006) = \$22,836.59$$