6.4: Present Values of Annuities

Example. Suppose we wish to invest a lump sum of money, A_n, into an annuity that earns interest at a rate of 10% per year, so that we may receive payments of \$100 for 5 years. What is the amount of the lump sum?

1. $100 = P(1+0.10)^1 \implies P = \dfrac{100}{(1+0.10)^1} = 90.91$

. $100 = P(1+0.10)^2 \implies P = \dfrac{100}{(1+0.10)^2} = 82.64$

. $100 = P(1+0.10)^3 \implies P = \dfrac{100}{(1+0.10)^3} = 75.13$

. $100 = P(1+0.10)^4 \implies P = \dfrac{100}{(1+0.10)^4} = 68.30$

. $100 = P(1+0.10)^5 \implies P = \dfrac{100}{(1+0.10)^5} = 62.09$

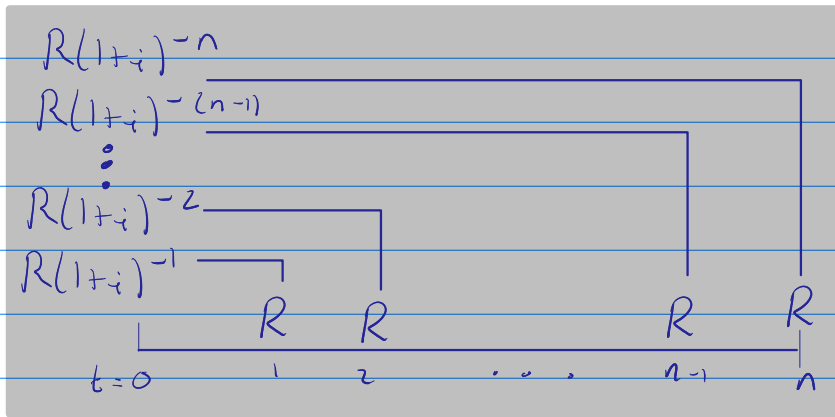$+$ ─────────

\$ 379.08

Definition.
If a payment of \$$R$ is to be withdrawn at the *end of each period* for n periods from an account that earns interest at a rate of i per period, then the account is an **ordinary annuity** and the **present value** is

$$A_n = R \cdot a_{\overline{n}|i} = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

The notation $a_{\overline{n}|i}$ represents the present value of an ordinary annuity of \$1 per period for n periods with an interest rate of i per period.

Derivation

$$A_n = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$R(1+i)^{-n}$$
$$R(1+i)^{-(n-1)}$$
$$\vdots$$
$$R(1+i)^{-2}$$
$$R(1+i)^{-1}$$

$$R \quad R \quad \cdots \quad R \quad R$$
$$t=0 \quad 1 \quad 2 \quad \cdots \quad n-1 \quad n$$

$$(1+i)^n A_n = \left(R(1+i)^{-1} + R(1+i)^{-2} + \ldots R(1+i)^{-(n-1)} + R(1+i)^{-n}\right)(1+i)^n$$

$$(1+i)^n A_n = R(1+i)^{n-1} + R(1+i)^{n-2} + \ldots R(1+i)^{1} + R$$

$$(1+i)^n A_n = R\left[\frac{(1+i)^n - 1}{i}\right] \qquad \leftarrow \text{From section 6.3}$$

$$A_n = R\left[\frac{(1+i)^n - 1}{i}\right]\frac{1}{(1+i)^n}$$

$$\boxed{A_n = R\left[\frac{1-(1+i)^{-n}}{i}\right]}$$

Example. Find the lump sum that one must invest in an annuity to receive $1000 at the end of each month for the next 16 years, if the annuity pays 9% compounded monthly.

$$A_n: \leftarrow \text{Find}$$

$$R: 1000$$

$$\frac{r}{m} = i: \frac{0.09}{12}$$

$$mt = n: \quad 12 \cdot 16 = 192$$

$$A_n = 1000 \left[\frac{1 - \left(1 + \frac{0.09}{12}\right)^{-192}}{\left(\frac{0.09}{12}\right)} \right]$$

$$= \boxed{\$101,572.77}$$

Example. Suppose that a couple plans to set up an ordinary annuity with a $100,000 inheritance they received. What is the size of the quarterly payments they will receive for the next 6 years if the account pays 7% compounded quarterly?

$$A_n: \quad 100\,000$$

$$R: \leftarrow \text{Find}$$

$$\frac{r}{m} = i: \quad \frac{7\%}{4} = \frac{0.07}{4} = 0.0175$$

$$mt = n: \quad 4(6) = 24$$

$$\frac{100,000}{\left[\frac{1-(1.0175)^{-24}}{0.0175}\right]} = \frac{R\left[\frac{1-(1.0175)^{-24}}{0.0175}\right]}{\left[\frac{1-(1.0175)^{-24}}{0.0175}\right]}$$

$$R = \frac{100,000}{\left[\frac{1-(1.0175)^{-24}}{0.0175}\right]}$$

$$= \boxed{\$5138.57}$$

Example. An inheritance of $250,000 is invested at 9% compounded monthly. If $2500 is withdrawn at the end of each month, how long will it be until the account balance is $0?

Use ln(...) to find unknown exponent

1. Isolate
2. ln(...)
3. expo
4. solve

A_n: 250 000

R: 2500

$\frac{r}{m} = i$: $\frac{9\%}{12} = \frac{0.09}{12} = 0.0075$

$mt = n$: $12t$ ← Find t or Find n

$$\left(\frac{0.0075}{2500}\right) 250{,}000 = 2500 \left[\frac{1 - (1.0075)^{-12t}}{0.0075}\right] \left(\frac{0.0075}{2500}\right)$$

$$0.75 - 1 = 1 - (1.0075)^{-12t} - 1$$

$$(-1)(-0.25) = \left(-(1.0075)^{-12t}\right)(-1)$$

$$\ln(0.25) = \ln\left((1.0075)^{-12t}\right)$$

$$\ln(0.25) = -12t \ln(1.0075)$$

$$\frac{\ln(0.25)}{-12\ln(1.0075)} = \frac{-12t\ln(1.0075)}{-12\ln(1.0075)}$$

$$t = \frac{\ln(0.25)}{-12\ln(1.0075)} = 15.4610$$

$n = 12t$

$\Rightarrow n = 185.53$

There will be 185 regular withdrawals of $2,500, and the last withdrawal will be less

Definition.

If a payment of R is to be withdrawn at the *beginning of each period* for n periods from an account that earns interest at a rate of i per period, then the account is an **annuity due** and the **present value** is

$$A_{(n,\text{due})} = R \cdot a_{\overline{n}|i}(1+i) = R\left[\frac{1-(1+i)^{-n}}{i}\right](1+i)$$

The notation $a_{\overline{n}|i}$ represents the present value of an ordinary annuity of $1 per period for n periods with an interest rate of i per period.

Example. Suppose that a court settlement results in a $750,000$ award. If this is invested at 9% compounded semiannually, how much will it provide at the *beginning* of each half-year for a period of 7 years?

$A_{(n, due)} : \$750,000$

$R: \leftarrow \text{Find}$

$\frac{r}{m} = i: \frac{9\%}{2} = 0.045$

$mt = n: 2(7) = 14$

$A_{(n, due)} = R \, a_{\overline{n}|i} (1+i)$

$\Rightarrow R = \dfrac{A_{(n, due)}}{a_{\overline{n}|i}(1+i)}$

$R = \dfrac{750,000}{\left[\dfrac{1-1.045^{-14}}{0.045}\right](1.045)}$

$= \boxed{\$70,205.97}$

Comparing annuity calculations:

	Ordinary annuities	Annuity due	
Future Value	S	S_{due}	Regular payments Total *after* n periods
Present Value	A_n	$A_{(n,\text{due})}$	Regular withdrawals Total *before* n periods
	end of each period	beginning of each period	