

## 8.1: Relations on Sets

### Definition.

A relation  $R$  from  $A$  to  $B$  is called a **binary relation** because it is a subset of a Cartesian product of two sets.

**Example.** Define a relation  $L$  from  $\mathbb{R}$  to  $\mathbb{R}$ :

$$\forall x, y \in \mathbb{R}, x L y \Leftrightarrow x < y.$$

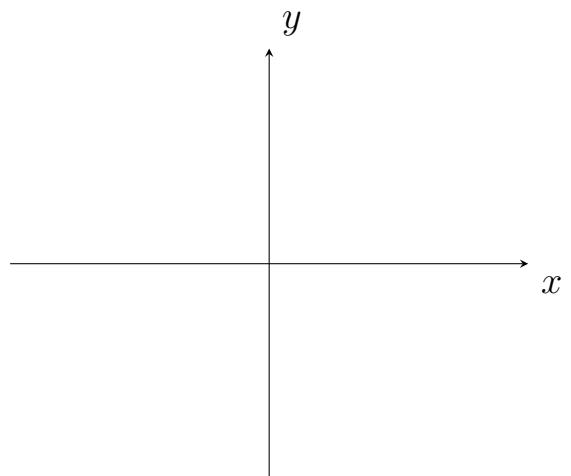
Is  $57 L 53$ ?

Is  $(-17) L (-14)$ ?

Is  $143 L 143$ ?

Is  $(-35) L 1$ ?

Draw the graph of  $L$  as a subset of the Cartesian plane  $\mathbb{R} \times \mathbb{R}$ .



**Definition.**

Two integers  $m$  and  $n$  are **congruent modulo 2** if, and only if,  $m \bmod 2 = n \bmod 2$ .

**Example.** Define a relation  $E$  from  $\mathbb{Z}$  to  $\mathbb{Z}$ :

$$\forall (m, n) \in \mathbb{Z} \times \mathbb{Z}, m E n \Leftrightarrow m - n \text{ is even.}$$

Is  $4 E 0$ ? Is  $2 E 6$ ? Is  $3 E (-3)$ ? Is  $5 E 2$ ?

Prove that if  $n$  is any odd integer, then  $n E 1$ .

**Example.** Let  $X = \{a, b, c\}$ . Define a relation  $\mathbf{S}$  from  $\mathcal{P}(X)$  to  $\mathcal{P}(X)$  as follows:

$$\forall A, B \in \mathcal{P}(X), A \mathbf{S} B \Leftrightarrow A \text{ has at least as many elements as } B.$$

Is  $\{a, b\} \mathbf{S} \{b, c\}$ ?

Is  $\{a\} \mathbf{S} \emptyset$ ?

Is  $\{b, c\} \mathbf{S} \{a, b, c\}$ ?

Is  $\{c\} \mathbf{S} \{a\}$ ?

**Definition.**

Let  $R$  be a relation from  $A$  to  $B$ . Define the inverse relation  $R^{-1}$  from  $B$  to  $A$  as follows:

$$R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}.$$

**Example.** Let  $A = \{2, 3, 4\}$  and  $B = \{2, 6, 8\}$ , and let  $R$  be the “divides” relation from  $A$  to  $B$ :

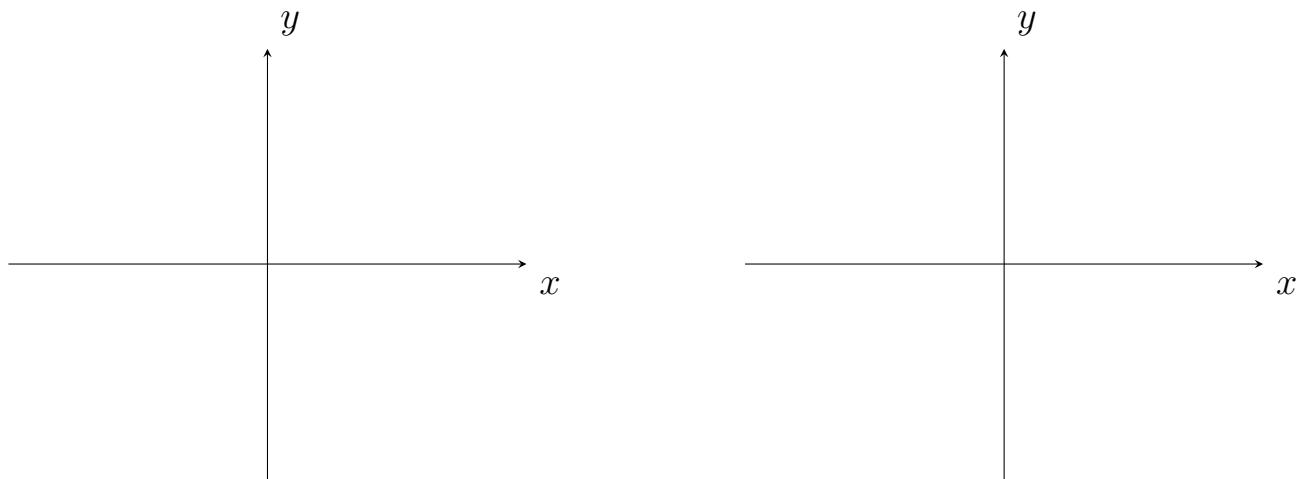
$$\forall (x, y) \in A \times B, x R y \Leftrightarrow x \mid y$$

Explicitly state which ordered pairs are in  $R$  and  $R^{-1}$ . Draw arrow diagrams for both.

**Example.** Define a relation  $R$  from  $\mathbb{R}$  to  $\mathbb{R}$  as follows:

$$\forall(x, y) \in \mathbb{R} \times \mathbb{R}, x R y \Leftrightarrow y = 2|x|.$$

Draw the graphs of  $R$  and  $R^{-1}$  in the Cartesian plane. Is  $R^{-1}$  a function?



**Definition.**

A **relation on a set**  $A$  is a relation from  $A$  to  $A$ .

A **graph**  $G$  consists of two finite sets:

- a nonempty set  $V(G)$  of **vertices** and
- a set  $E(G)$  of **edges**,

where each edge is associated with a set consisting of either one or two vertices called its endpoints.

A **directed graph** is a graph whose edges are directional.

**Example.** Let  $A = \{3, 4, 5, 6, 7, 8\}$  and define a relation  $R$  on  $A$  as follows

$$\forall x, y \in A, x R y \Leftrightarrow 2 \mid (x - y).$$

Draw the directed graph of  $R$ .

**Definition.**

Given sets  $A_1, A_2, \dots, A_n$ , an **n-ary relation**  $R$  on  $A_1 \times A_2 \times \dots \times A_n$  is a subset of  $A_1 \times A_2 \times \dots \times A_n$ . The special case of 2-ary, 3-ary, and 4-ary relations are called **binary**, **ternary**, and **quaternary relations**, respectively.