2.1: Logical Form and Logical Equivalence

Definition.
A **statement** (or **proposition**) is a sentence that is true or false, but not both.

Example. Determine which of the following are statements:

$2 + 2 = 4$

Is a statement

$2 + 2 = 5$

Is a statement

$x^2 + 2 = 11$

Is NOT a statement

Today is Saturday.

Is a statement

She is a computer science major.

Is NOT a statement

Jane is a computer science major.

Is a statement

Definition. (Compound Statements)
Let p and q be statement variables.

- The **negation** of p is "not p", and is denoted as $\sim p$ (or $\neg p$)
- The **conjunction** of p and q is "p and q", and is denoted at $p \wedge q$
- The **disjunction** of p and q is "p or q", and is denoted $p \vee q$.
- The **exclusive or** of p and q is "p x-or q", and is denoted $p \oplus q$ (or p XOR q)

The **order of operations** specifies that $\sim$ is performed first.

Example. Consider the following statements:

p : It is raining.
q : It is sunny.
r : It is cloudy.

Rewrite the following compound statements in words:

$\sim p$

It is not raining.

$p \vee q$

It is raining or it is sunny.

$q \wedge r$

It is sunny and it is cloudy.

$q \wedge \sim r$

It is sunny and it is not cloudy.

$p \wedge (q \vee r)$

It is raining and it is sunny or cloudy.

Alternatively:
It is raining and it is sunny or it is raining and it is cloudy.

$p \oplus q$

It is raining or it is sunny, but not both.

Alternatively:
It is either sunny or rainy.

Definition.
A **statement form** (or **propositional form**) is an expression made up of statement variables (e.g., p, q, and r), and logical connectives (e.g. $\sim$, $\wedge$, $\vee$, and $\oplus$).
The **truth table** for a given statement form displays the truth values that correspond to all possible combinations of truth values for its component statement variables.

Example. Let p and q be statement variables. Fill out the following truth tables:

p	$\sim p$
T	F
F	T

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$
T	T	T	T	F
T	F	F	T	T
F	T	F	T	T
F	F	F	F	F

This is equivalent to XOR

p	q	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$(p \vee q) \wedge \sim (p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Example. Construct a truth table for the statement form $(p \wedge q) \vee \sim r$.

p	q	r	$p \wedge q$	$\sim r$	$(p \wedge q) \vee \sim r$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	F	F	F
F	F	F	F	T	T

> **Definition.**
> Two *statement forms* are called **logically equivalent** if, and only if, they have identical true values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms P and Q is denoted $P \equiv Q$.

Example. Use truth tables to test if the following statement forms are equivalent:

p and $\sim(\sim p)$

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

$\sim(p \wedge q)$ and $\sim p \wedge \sim q$

p	q	$p \wedge q$	$\sim p$	$\sim q$	$\sim(p \wedge q)$	$\sim p \wedge \sim q$
T	T	T	F	F	F	F
T	F	F	F	T	T	F
F	T	F	T	F	T	F
F	F	F	T	T	T	T

Not equivalent

Definition. (De Morgan's Laws)
The negation of an *and* statement is logically equivalent to the *or* statement in which each component is negated.
The negation of an *or* statement is logically equivalent to the *and* statement in which each component is negated.

Example. Use truth tables to show that the following statement forms are equivalent:

$\sim (p \wedge q)$ and $\sim p \vee \sim q$

p	q	$p \wedge q$	$\sim p$	$\sim q$	$\sim (p \wedge q)$	$\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	F	T	T	T
F	T	F	T	F	T	T
F	F	F	T	T	T	T

$\sim (p \vee q)$ and $\sim p \wedge \sim q$

p	q	$p \vee q$	$\sim p$	$\sim q$	$\sim (p \vee q)$	$\sim p \wedge \sim q$
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	F	F
F	F	F	T	T	T	T

Example. Using De Morgan's law to write the negation of the following statements:

Jim is at least 6 feet tall and weighs at least 200 pounds.

Jim is less than 6 feet tall OR Jim weighs less than 200 pounds.

The bus was late or Tom's watch was slow

The bus was not late AND Tom's watch was not slow.

$-1 < x \leq 4 \longrightarrow (-1, 4]$

$x \leq -1$ or $x > 4 \longrightarrow (-\infty, -1] \cup (4, \infty)$

> **Definition.**
> A **tautology** is a statement form that is always true.
> A **contradiction** is a statement form that is always false.

Example. Complete the truth tables for $p \wedge \sim p$ and $p \vee \sim p$

p	$\sim p$	$p \wedge \sim p$	$p \vee \sim p$
T	F	F	T
F	T	F	T
		Contradiction	Tautology

Example. Let $\mathbf{t}$ be a tautology, and $\mathbf{c}$ be a contradiction. Show that $p \wedge \mathbf{t} \equiv p$ and $p \wedge \mathbf{c} \equiv \mathbf{c}$

p	t	c	$p \wedge t$	$p \wedge c$
T	T	F	T	F
F	T	F	F	F

Theorem 2.1.1 Logical Equivalences (p 49)
Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold:

1. Commutative laws:
   $p \wedge q \equiv q \wedge p$ $\quad$ $p \vee q \equiv q \vee p$
2. Associative laws:
   $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $\quad$ $(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. Distributive laws:
   $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $\quad$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. Identity laws:
   $p \wedge \mathbf{t} \equiv p$ $\quad$ $p \vee \mathbf{c} \equiv p$
5. Negation laws:
   $p \vee \sim p \equiv \mathbf{t}$ $\quad$ $p \wedge \sim p \equiv \mathbf{c}$
6. Double negative law:
   $\sim (\sim p) \equiv p$
7. Idempotent laws:
   $p \wedge p \equiv p$ $\quad$ $p \vee p \equiv p$
8. Universal bound laws:
   $p \vee \mathbf{t} \equiv \mathbf{t}$ $\quad$ $p \wedge \mathbf{c} \equiv \mathbf{c}$
9. De Morgan's laws:
   $\sim (p \wedge q) \equiv \sim p \vee \sim q$ $\quad$ $\sim (p \vee q) \equiv \sim p \wedge \sim q$
10. Absorption laws:
    $p \wedge (p \vee q) \equiv p$ $\quad$ $p \wedge (p \vee q) \equiv p$
11. Negations of $\mathbf{t}$ and $\mathbf{c}$:
    $\sim \mathbf{t} \equiv \mathbf{c}$ $\quad$ $\sim \mathbf{c} \equiv \mathbf{t}$