5.5: Differentiation of Logarithmic Functions

Definition. (Natural log)

The inverse of the exponential function is the logarithm:

$$e^y = x \iff \ln(x) = y, \qquad x > 0$$

We may also denote this as $\log_e(x)$

Example. Solve the following exponential equations

$$y = e^{-3}$$

$$e^{x^2 - x} = e^2$$

$$e^x = 10$$

$$4e^{1-x^2} = 6$$

Rule 3: Derivative of $\ln(x)$

$$\frac{d}{dx}\left[\ln|x|\right] = \frac{1}{x}, \qquad x \neq 0$$

Rule 4: Derivative of $\ln\left(f(x)\right)$

$$\frac{d}{dx}\left[\ln\left(f(x)\right)\right] = \frac{f'(x)}{f(x)}$$

Example. Find the derivative of the following functions

$$y = \ln\left(\sqrt[3]{x}\right)$$

$$f(x) = \sqrt{x}\ln(x)$$

$$g(x) = \ln\left(x^2 + 1\right)$$

$$h(t) = \frac{1 + \ln(t)}{1 - \ln(t)}$$

Additional properties of logarithms

$$e^y = x \qquad\qquad \ln(x) = y$$

$$e^1 = e \qquad\qquad \ln(e) = 1$$

$$e^0 = 1 \qquad\qquad \ln(1) = 0$$

$$e^x e^y = e^{x+y} \qquad\qquad \ln(xy) = \ln(x) + \ln(y)$$

$$\frac{e^x}{e^y} = e^{x-y} \qquad\qquad \ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$

$$e^{xy} = (e^x)^y \qquad\qquad \ln(x^y) = y\ln(x)$$

$$e^{\ln(x)} = x \qquad\qquad \ln(e^x) = x$$

Definition. (Other logarithms)
Logarithms are defined for any base $a > 0$:

$$a^y = x \quad \longleftrightarrow \quad \log_a(x) = y, \qquad x > 0$$

Example. Find the derivative of the following functions

$$y = \ln\left(\sqrt[3]{x}\right) \qquad\qquad\qquad\qquad f(x) = \ln\left(\left(x^2 + 1\right)\left(x^3 + 2\right)^6\right)$$

$$g(x) = \ln\left(\frac{1 - x}{1 + x}\right) \qquad\qquad\qquad\qquad h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$$

$$j(t) = 2\left(\ln(5t)\right)^{3/2} \qquad\qquad\qquad\qquad k(u) = \ln\left(\sqrt{u^2 - 4}\right)$$

Logarithmic Differentiation

Using properties of logs, we can transform a function before taking it's derivative:

1. Take the natural log of both sides rewriting products and quotients as sums and differences.

2. Differentiate both sides.

3. Solve for $\frac{dy}{dx}$

Example. Differentiate

$$y = x^2(x-1)\left(x^2+4\right)^3$$