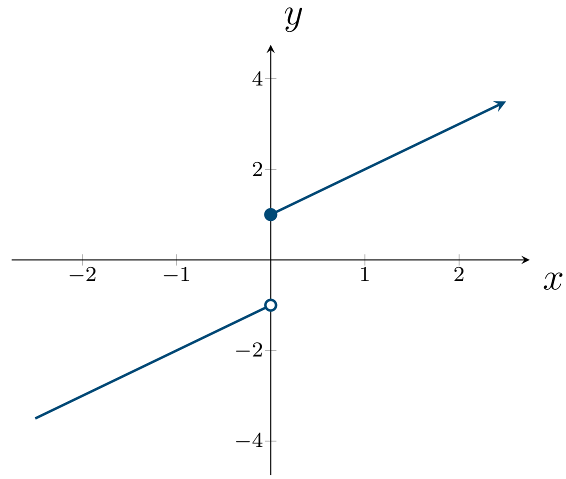
2.5: One-Sided Limits and Continuity

Consider the function

$$f(x) = \begin{cases} x - 1, & x < 0 \\ x + 1, & x \geq 0 \end{cases}$$



What is $\lim_{x \to 0} f(x)$?

$\lim_{x \to 0} f(x)$ DNE

Definition. (One-Sided Limits)
The function f has a **right-hand limit** L as x approaches a from the right, written

$$\lim_{x \to a^+} f(x) = L$$

if the values of $f(x)$ can be made as close to L as we please by taking x sufficiently close to (but not equal to) a and to the right of a.
The function f has a **left-hand limit** L as x approaches a from the left, written

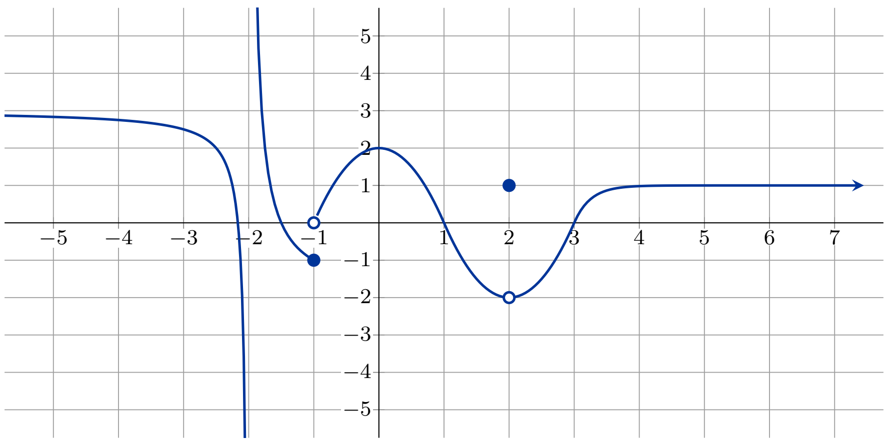
$$\lim_{x \to a^-} f(x) = M$$

if the values of $f(x)$ can be made as close to L as we please by taking x sufficiently close to (but not equal to) a and to the left of a.

Theorem 3
Let f be a function that is defined for all values of x close to $x = a$ with the possible exception of a itself. Then

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

Example. Using the graph below, evaluate the following limits:



$\lim_{x \to -2^-} f(x) = -\infty$

$\lim_{x \to -2^+} f(x) = \infty$

$\lim_{x \to -2} f(x)$ DNE

$\lim_{x \to -1^-} f(x) = -1$

$f(-1) = -1$

$\lim_{x \to -1^+} f(x) = 0$

$\lim_{x \to -1} f(x)$ DNE

$\lim_{x \to 1} f(x) = 0$

$\lim_{x \to 2} f(x) = -2$

$f(2) = 1$

$\lim_{x \to \infty} f(x) = 1$

Definition. (Continuity of a Function at a Number)
A function f is **continuous** at a if $\lim_{x \to a} f(x) = f(a)$.

Continuity Checklist:
In order for f to be continuous at a, the following three conditions must hold:

1. $f(a)$ is defined (a is in the domain of f),
2. $\lim_{x \to a} f(x)$ exists,
3. $\lim_{x \to a} f(x) = f(a)$ (the value of f equals the limit of f at a).

Example. Determine the values of x for which the following functions are continuous:

$f(x) = 3x^3 + 2x^2 - x + 10$

Continuous on $(-\infty, \infty)$

$$g(x) = \frac{8x^{10} - 4x + 1}{x^2 + 1}$$

$x^2 + 1 \neq 0$

$x^2 \neq -1$ $\Rightarrow$ Continuous on $(-\infty, \infty)$

$$h(x) = \frac{4x^3 - 3x^2 + 1}{x^2 - 3x + 1}$$

$x^2 - 3x + 1 \neq 0$

$$x \neq \frac{3 \pm \sqrt{(-3)^2 - 4(1)(1)}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

$\Rightarrow$ Continuous on $\left(-\infty, \frac{3-\sqrt{5}}{2}\right) \cup \left(\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right) \cup \left(\frac{3+\sqrt{5}}{2}, \infty\right)$

Example. Determine whether the following are continuous at a:

$$f(x) = x^2 + \sqrt{7-x},\ a = 4 \quad \checkmark$$

$f(4) = 16 + \sqrt{3}$

$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x) = \lim_{x \to 4} f(x) = 16 + \sqrt{3}$

$$g(x) = \frac{1}{x-3},\ a = 3 \quad \times$$

$g(3)$ DNE

$$h(x) = \begin{cases} \dfrac{x^2+x}{x+1}, & x \neq -1 \\ 0, & x = -1 \end{cases},\ a = -1 \quad \times$$

$\lim_{x \to -1} h(x) = \lim_{x \to -1} \frac{x^2+x}{x+1}$

$= \lim_{x \to -1} \frac{x(x+1)}{x+1}$

$= \lim_{x \to -1} x = -1 \neq h(-1) = 0$

$$j(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases},\ a = 0 \quad \checkmark$$

$j(0) = 0$

$\lim_{x \to 0^-} j(x) = \lim_{x \to 0} -x = 0$

$\lim_{x \to 0^+} j(x) = \lim_{x \to 0} x = 0$

$\Rightarrow \lim_{x \to 0} j(x) = 0$

$$k(x) = \begin{cases} \dfrac{x^2+x-6}{x^2-x}, & x \neq 2 \\ -1, & x = 2 \end{cases},\ a = 2 \quad \times$$

$k(2) = -1$

$\lim_{x \to 2} \frac{x^2+x-6}{x^2-x} = \frac{0}{2} = 0 \neq k(2) = -1$

Properties of Continuous Functions

1. The constant function $f(x) = c$ is continuous everywhere.
2. The identify function $f(x) = x$ is continuous everywhere.

If f and g are continuous at $x = a$, then

$[f(x)]^n$, where n is a real number, is continuous at $x = a$ whenever it is defined at that number

$f \pm g$ is continuous at $x = a$

fg is continuous at $x = a$

f/g is continuous at $x = a$ provided that $g(a) \neq 0$

Polynomial and Rational Functions

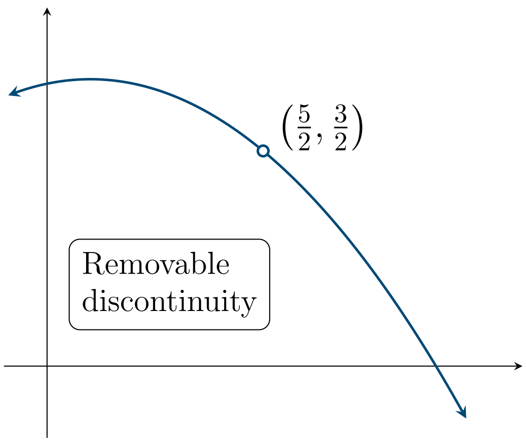
1. A polynomial function is continuous for all x.
2. A rational function (a function of the form $\frac{p}{q}$, where p and q are polynomials) is continuous for all x for which $q(x) \neq 0$.

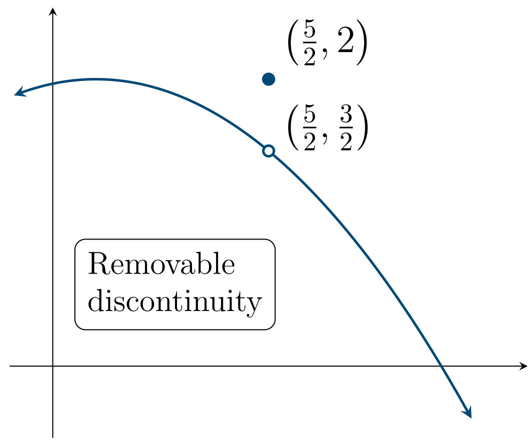
Definition.

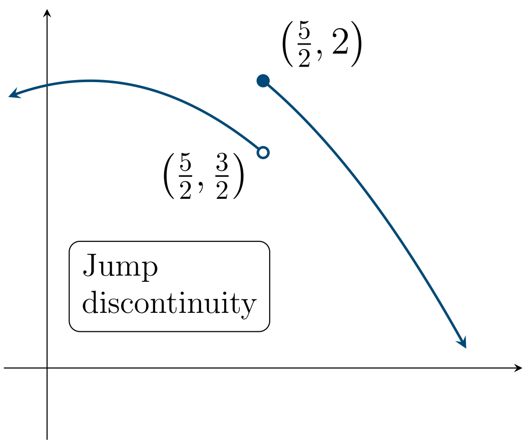
A **removable discontinuity** at $x = a$ is one that disappears when the function becomes continuous after defining $f(a) = \lim_{x \to a} f(x)$.

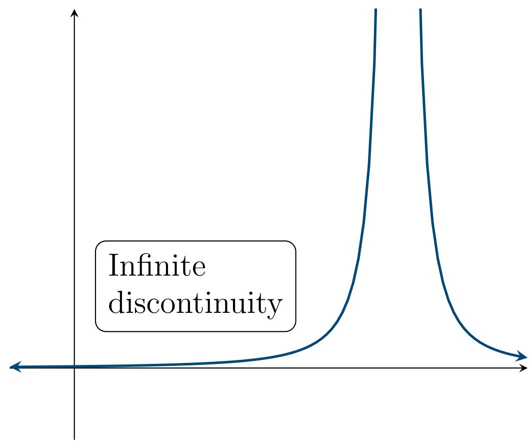
A **jump discontinuity** is one that occurs whenever $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$.

A **vertical discontinuity** occurs whenever $f(x)$ has a vertical asymptote.

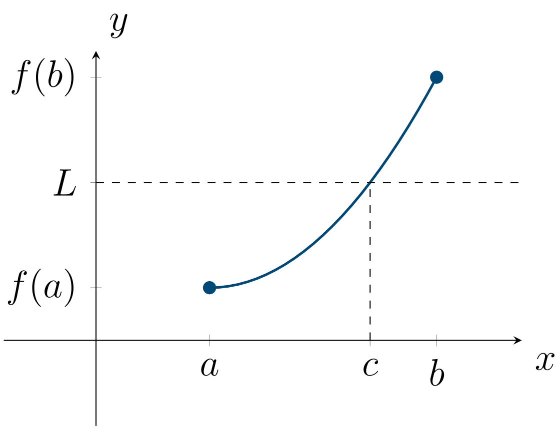


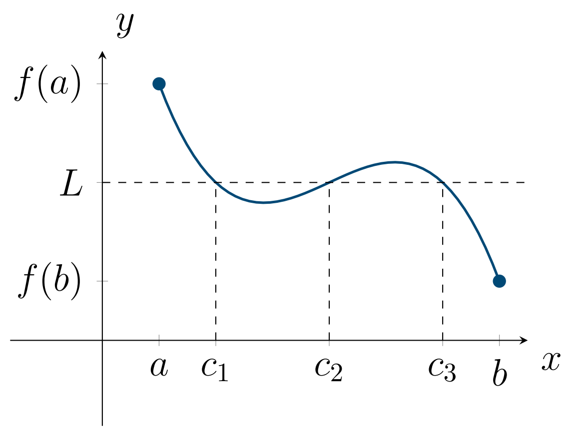




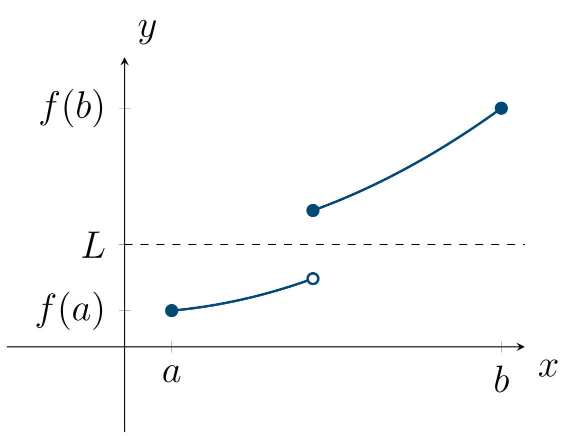


Theorem 4: Intermediate Value Theorem
Suppose f is continuous on the interval $[a, b]$ and L is a number strictly between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) satisfying $f(c) = L$.





Note: It is important that the function be continuous on the interval $[a, b]$:



Theorem 5: Existence of Zeros of a Continuous Function
If f is a continuous function on a closed interval $[a, b]$, and if $f(a)$ and $f(b)$ have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval (a, b).

Example. Check the conditions of the Intermediate Value Theorem to see if there exists a value c on the interval (a, b) such that the following equations hold: Graph

$x^x - x^2 = \frac{1}{2}$ on $[0, 2]$

$x=0$: $0^0 - 0^2 = 1$

$x=2$: $2^2 - 2^2 = 0$

$\boxed{\text{Since } 0 < \tfrac{1}{2} < 1 \text{, there exists } c \text{ such that } 0 < c < 2 \text{ and } f(c) = \tfrac{1}{2}}$

$\sqrt{x^4 + 25x^3 + 10} = 5$ on $[0, 1]$

$x=0$: $\sqrt{0+0+10} = \sqrt{10} \simeq 3.16$

$x=1$: $\sqrt{1+25+10} = \sqrt{36} = 6$

$\boxed{\text{Since } \sqrt{10} < 5 < 6 \text{, there exists } c \text{ such that } 0 < c < 1 \text{ and } f(c) = 5}$

$x + \sqrt{1 - x^2} = 0$ on $[-1, 0]$

$x=-1$: $-1 + \sqrt{1-1} = -1$

$x=0$: $0 + \sqrt{1-0} = 1$

$\boxed{\text{Since } -1 < 0 < 1 \text{, there exists } c \text{ such that } -1 < c < 0 \text{ and } f(c) = 0}$

$\frac{x^2}{x^2 + 1} = 0$ on $[-1, 1]$

$x=-1$: $\frac{(-1)^2}{(-1)^2+1} = \frac{1}{2}$

$x=1$: $\frac{(1)^2}{(1)^2+1} = \frac{1}{2}$

$\boxed{\text{Since the conditions of the IVT are not met, } c \text{ such that } -1 < c < 1 \text{ and } f(c) = 0 \text{ may not exist (not guaranteed)}}$

Note: $x = 0$ is a root.

Example. Consider the function

$$f(x) = \frac{x+1}{x-1}$$

on the interval $[0, 2]$. Does there exist a c on the interval $[0, 2]$ such that $f(c) = 1$?

$\left.\begin{array}{l} f(0) = -1 \\ f(2) = 3 \end{array}\right\} \quad -1 < 1 < 3$

Since $f(x)$ is discontinuous at $x = 1$, we cannot apply the IVT.