

1.1: Solutions of Linear Equations and Inequalities in One Variable

Definition.

A **function** f is a special relation between x and y such that each input x results in *at most* one y . The symbol $f(x)$ is read “ f of x ” and is called the **value of f at x**

Example. Let $f(x) = \frac{x^2}{2} + x$. Evaluate the following:

$$\begin{aligned}
 f(1) &= \frac{1^2}{2} + 1\left(\frac{2}{2}\right) = \frac{1}{2} + \frac{2}{2} \\
 &= \frac{1+2}{2} = \boxed{\frac{3}{2}} \\
 f\left(\frac{1}{2}\right) &= \frac{\left(\frac{1}{2}\right)^2}{2} + \left(\frac{1}{2}\right) = \frac{\frac{1}{4}}{2} + \frac{1}{2}\left(\frac{4}{4}\right) \\
 &= \frac{1}{8} + \frac{4}{8} = \frac{1+4}{8} = \boxed{\frac{5}{8}} \\
 f(-2) &= \frac{(-2)^2}{2} + (-2) = \frac{4}{2} - 2 \\
 &= 2 - 2 = \boxed{0} \\
 f(f(x)) &= \frac{\left(\frac{x^2}{2} + x\right)^2}{2} + \left(\frac{x^2}{2} + x\right) = \frac{\left(\frac{x^2}{2} + x\right)^2}{2} + \left(\frac{x^2}{2} + x\right) = \frac{\frac{x^4}{4} + x^3 + x^2}{2} + \frac{x^2}{2} + x \\
 &= \frac{x^4}{8} + \frac{x^3}{2} + \frac{x^2}{2} + \frac{x^2}{2} + x = \boxed{\frac{x^4}{8} + \frac{x^3}{2} + x^2 + x}
 \end{aligned}$$

Composite Functions:

Let f and g be functions of x . Then, the **composite functions** g of f (denoted $g \circ f$) and f of g (denoted $f \circ g$) are defined as:

$$(g \circ f)(x) = g(f(x))$$

$$(f \circ g)(x) = f(g(x))$$

Example. Let $g(x) = x - 1$. Find:

$$\begin{aligned}
 (g \circ f)(x) &= g\left(\frac{x^2}{2} + x\right) \\
 &= \left(\frac{x^2}{2} + x\right) - 1 \\
 &= \left(\frac{x^2}{2} + x\right) - 1 \\
 &= \boxed{\frac{x^2}{2} + x - 1}
 \end{aligned}$$

$$\begin{aligned}
 (f \circ g)(x) &= f(g(x)) = \frac{(g(x))^2}{2} + (g(x)) \\
 &= \frac{(x-1)^2}{2} + (x-1) \\
 &= \frac{x^2 - 2x + 1}{2} + x - 1 \\
 &= \frac{x^2}{2} - x + \frac{1}{2} + x - 1 = \boxed{\frac{x^2}{2} - \frac{1}{2}}
 \end{aligned}$$

Operations with Functions:

Let f and g be functions of x and define the following:

Sum	$(f + g)(x) = f(x) + g(x)$
Difference	$(f - g)(x) = f(x) - g(x)$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$
Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ if $g(x) \neq 0$

Definition.

An **expression** is a meaningful string of numbers, variables and operations:

$$3x - 2$$

An **equation** is a statement that two quantities or algebraic expressions are equal:

$$3x - 2 = 7$$

A **solution** is a value of the variable that makes the equation true:

$$3(3) - 2 = 7$$

$$9 - 2 = 7$$

$$7 = 7$$

A **solution set** is the set of ALL possible solutions of an equation:

$3x - 2 = 7$ only has the solution $x = 3$,

$2(x - 1) = 2x - 2$ is true for all possible values of x .

Properties of Equality:

Substitution Property: The equation formed by substituting one expression for an equal expression is equivalent to the original equation:

$$\begin{aligned}3(x - 3) - \frac{1}{2}(4x - 18) &= 4 \\3x - 9 - 2x + 9 &= 4 \\x &= 4\end{aligned}$$

Addition Property: The equation formed by adding the same quantity to both sides of an equation is equivalent to the original equation:

$$\begin{array}{ll}x - 4 = 6 & x + 5 = 12 \\x - 4 + 4 = 6 + 4 & x + 5 + (-5) = 12 + (-5) \\x = 10 & x = 7\end{array}$$

Multiplication Property: The equation formed by multiplying both sides of an equation by the same *nonzero* quantity is equivalent to the original equation:

$$\begin{array}{ll}\frac{1}{3}x = 6 & 5x = 20 \\3\left(\frac{1}{3}x\right) = 3(6) & \frac{5x}{5} = \frac{20}{5} \\x = 18 & x = 4\end{array}$$

Solving a linear equation:

Using the properties of equality above, we can solve any linear equation in 1 variable:

Example. Solve $\frac{3x}{4} + 3 = \frac{x-1}{3}$

1. Eliminate fractions:
2. Remove/evaluate parenthesis:
3. Use addition property to isolate the variable to one side:
4. Use multiplication property to isolate variable:
5. Verify solution via substitution:

$$12\left(\frac{3x}{4} + 3\right) = 12\left(\frac{x-1}{3}\right)$$

$$9x + 36 = 4x - 4$$

$$9x + 36 - 36 - 4x = 4x - 4 - 36 - 4x$$

$$\frac{5x}{5} = \frac{-40}{5}$$

$$\underbrace{\frac{3(-8)}{4} + 3}_{-6 + 3 = -3} \stackrel{?}{=} \underbrace{\frac{(-8) - 1}{3}}_{\frac{-9}{3} = -3}$$

Example. Solve the following:

$$\begin{aligned} (2)(3)\left(\frac{3x+1}{2}\right) &= \left(\frac{x}{3} - 3\right)(2)(3) \\ 3(3x+1) &= 2x - (2)(3)3 \\ 9x + 3 - 2x &= 2x - 18 - 2x \\ 7x + 3 - 3 &= -18 - 3 \\ 7x &= -21 \\ \frac{7x}{7} &= \frac{-21}{7} \\ \boxed{x = -3} \end{aligned}$$

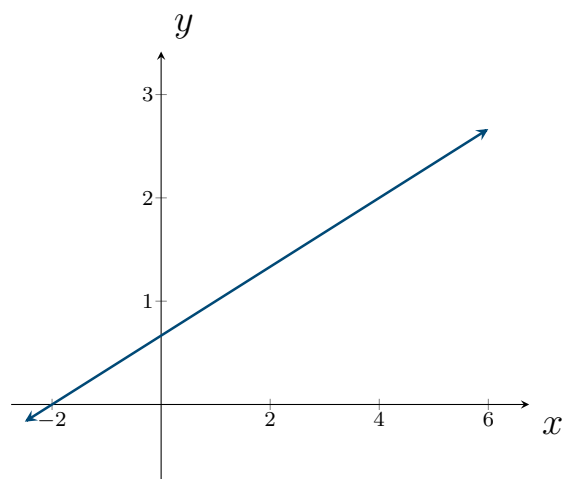
Verification of solution omitted

$$\begin{aligned} (x-3)\left(\frac{2x-1}{x-3}\right) &= \left(4 + \frac{5}{x-3}\right)(x-3) \\ 2x-1 &= 4(x-3) + 5 \\ 2x-1 - 4x &= 4x-12+5-4x \\ -2x - 1 + 1 &= -7 + 1 \\ \frac{-2x}{-2} &= \frac{-6}{-2} \\ \boxed{x = 3} \end{aligned}$$

When verifying this solution, we see that both sides are undefined at $x=3$.

Example. Solve $-2x + 6y = 4$ for y

$$\begin{aligned} \underline{-2x} + 6y &= 4 \\ \underline{+2x} &+ 2x \\ 6y &= 2x + 4 \\ \underline{6} &\quad \underline{6} \\ y &= \frac{2x}{6} + \frac{4}{6} \\ y &= \frac{x}{3} + \frac{2}{3} \end{aligned}$$



Example. Suppose that the relationship between a firm's profit, P , and the number of items sold, x , can be described by the equation

$$5x - 4P = 1200$$

- a) How many units must be produced and sold for the firm to make a profit of \$150?

$$\begin{aligned} P &= \$150, \text{ Find } x \\ 5x - 4(150) &= 1200 \\ 5x - 600 &= 1200 \\ \underline{+600} &\quad \underline{+600} \\ 5x &= 1800 \\ \underline{5} &\quad \underline{5} \\ x &= 360 \end{aligned}$$

- b) Solve this equation for P in terms of x . Then, find the profit when 240 units are sold.

$$\begin{aligned} 5x - 4P &= 1200 \\ \underline{-5x} &\quad \underline{-5x} \\ -4P &= -5x + 1200 \\ \underline{-4} &\quad \underline{-4} \\ P &= \frac{-5x}{-4} + \frac{1200}{-4} \\ P &= \frac{5x}{4} - 300 \end{aligned}$$

Definition.

An **inequality** is a statement that one quantity is greater than (or less than) another quantity.

Properties of Inequalities

Substitution Property: The inequality formed by substituting one expression for an equal expression is equivalent to the original inequality:

$$5x - 4x + 2 < 6$$

$$x < 4 \Rightarrow \text{The solution set is } \{x : x < 6\}$$

Addition Property: The inequality formed by adding the same quantity to both sides of an inequality is equivalent to the original inequality:

$$x - 4 < 6$$

$$x - 4 + 4 < 6 + 4$$

$$x < 10$$

$$x + 5 \geq 12$$

$$x + 5 + (-5) \geq 12 + (-5)$$

$$x \geq 7$$

Multiplication Property The inequality formed by multiplying both sides of an inequality by the same *positive* quantity is equivalent to the original inequality. The direction of the inequality is flipped when multiplying by a *negative* quantity:

$$\frac{1}{3}x > 6$$

$$3\left(\frac{1}{3}x\right) > 3(6)$$

$$x > 18$$

$$5x - 5 \leq 6x + 20$$

$$-x \leq 25$$

$$x \geq -25$$

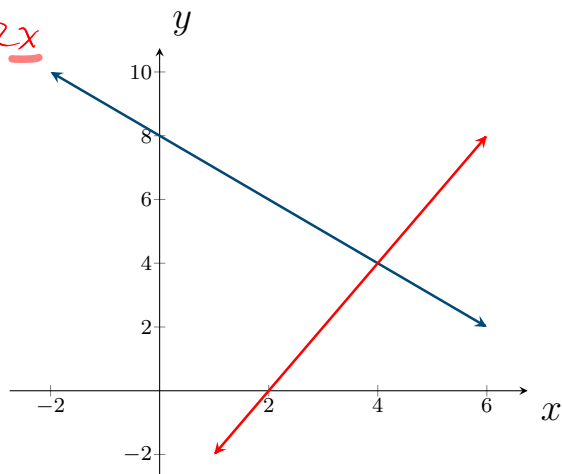
Example. Solve

$$-x + 8 \leq 2x - 4$$

first by gathering the x variable on the left, then again on the right. See that the multiplication property holds in both cases. Plot the solution set on a numberline.

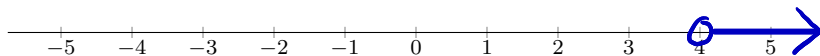
$$\begin{aligned} -x + 8 + x &\leq 2x - 4 + x \\ 8 + 4 &\leq 3x - 4 + 4 \\ \frac{12}{3} &\leq \frac{3x}{3} \\ 4 &\leq x \end{aligned}$$

$$\begin{aligned} -x + 8 - 2x &\leq 2x - 4 - 2x \\ -3x + 8 - 8 &\leq -4 - 8 \\ \frac{-3x}{-3} &\leq \frac{-12}{-3} \\ x &\geq 4 \end{aligned}$$



* Note:

- The inequality on the right changed direction in the last step because we divided by a negative number
- The answers above are equivalent. They are only formatted differently.



Example. Plot the following inequalities:

$$x \leq 2$$

$$x > -3$$

