## 6.1: Set Theory: Definitions and the Element Method of Proof

**Element Argument: The Basic Method for Proving that One set is a Subset of Another**

Let sets $X$ and $Y$ be given. To prove that $X \subseteq Y$,

1. **suppose** that $x$ is a particular but arbitrarily chosen element of $X$,
2. **show** that $x$ is an element of $Y$

**Example.** Define sets $A$ and $B$ as follows:

$$A = \{m \in \mathbb{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbb{Z}\}$$
$$B = \{n \in \mathbb{Z} \mid n = 3s \text{ for some } s \in \mathbb{Z}\}$$

Prove that $A \subseteq B$

Disprove that $B \subseteq A$

**Definition.**
Given sets $A$ and $B$, $A$ **equals** $B$, written $\mathbf{A} = \mathbf{B}$, if, and only if, every element of $A$ is in $B$ and every element of $B$ is in $A$:

$$A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A.$$

**Example.** Define sets $A$ and $B$ as follows:

$$A = \{m \in \mathbb{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbb{Z}\}$$
$$B = \{n \in \mathbb{Z} \mid n = 3s \text{ for some } s \in \mathbb{Z}\}$$

Is $A = B$?

**Definition.**
Given an integer $n$ and a positive integer $d$, when $n$ is divided by $d$, then

$$\begin{aligned} n \operatorname{div} d = &\text{ the integer quotient} \\ n \bmod d = &\text{ the nonnegative integer remainder} \end{aligned}$$

If $n$ and $d$ are integers and $d > 0$, then

$$n \operatorname{div} d = q \quad \text{and} \quad n \bmod d = r \quad \Leftrightarrow \quad n = dq + r$$

**Example.** Compute the following:

$32 \operatorname{div} 9,\ \ 32 \bmod 9$

$365 \operatorname{div} 7,\ \ 365 \bmod 7$

**Example.** If it is currently 11:00, what time will it be in

51 hours?

121 hours?

11 hours?

$-1$ hours?

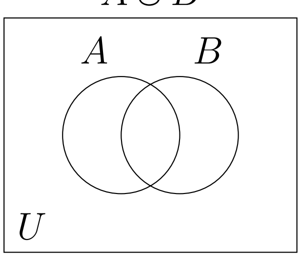
**Example.** Let $A = \left\{4, \sqrt{16}, 19 \bmod 15\right\}$ and $B = \{12 \bmod 8\}$. Is $A \subseteq B$? Is $B \subseteq A$?

**Definition.**
Let $A$ and $B$ be subsets of a universal set $U$.

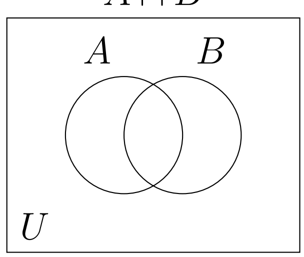
1. The **union** of $A$ and $B$ is the set of all elements that are in at least one of $A$ or $B$.
$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$$
2. The **intersection** of $A$ and $B$ is the set of all elements that are common to both $A$ and $B$.
$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$
3. The **difference** of $A$ and $B$ is the set of all elements that are in $B$ and not $A$.
$$B - A = \{x \in U \mid x \in B \text{ and } x \notin A\}$$
4. The **complement** of $A$ is the set of all elements in $U$ that are not in $A$.
$$A^c = \{x \in U \mid x \notin A\}$$

**Example.** Represent the following sets using the Venn diagrams below:

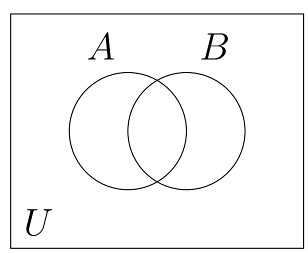
$A \cup B$



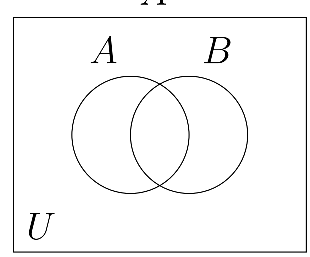

$A \cap B$




$B - A$




$A^c$



**Example.** Let the universal set be the set $U = \{a, b, c, d, e, f, g\}$, and let $A = \{a, c, e, g\}$ and $B = \{d, e, f, g\}$. Find

$A \cup B$ $\qquad$ $A \cap B$

$B - A$ $\qquad$ $A^c$

**Definition.**
Given real numbers $a$ and $b$ with $a \leq b$:

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\} \qquad (a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$
$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\} \qquad [a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

**Example.** Let the universal set be $\mathbb{R}$, and let $A = (-1, 0]$ and $B = [0, 1)$. Find

$A \cup B$ $\qquad$ $A \cap B$

$B - A$ $\qquad$ $A^c$

**Definition.**
Given sets $A_0, A_1, A_2, \ldots$ that are subsets of a universal set $U$ and given a nonnegative integer $n$,

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i, \text{ for at least one } i = 0, 1, 2, \ldots, n\}$$
$$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i, \text{ for every } i = 0, 1, 2, \ldots, n\}$$

**Example.** For each positive integer $i$, let $A_i = \left\{x \in \mathbb{R} \,\middle|\, -\frac{1}{i} < x < \frac{1}{i}\right\} = \left(-\frac{1}{i}, \frac{1}{i}\right)$. Find

$A_1 \cup A_2 \cup A_3$

$A_1 \cap A_2 \cap A_3$

$$\bigcup_{i=1}^{\infty} A_i$$

$$\bigcap_{i=1}^{\infty} A_i$$

**Definition.**
The **empty set** (or **null set**), denoted $\varnothing$, is the set with no elements.

$$\{1, 3\} \cap \{2, 4\} = \varnothing$$

Two sets are called **disjoint** if, and only if, they have no elements in common:

$$A \cap B = \varnothing.$$

Sets $A_1, A_2, A_3, \ldots$ are **mutually disjoint** (or **pairwise disjoint**) if, and only if, no two sets $A_i$ and $A_j$ with distinct subscripts have any elements in common:

$$A_i \cap A_j = \varnothing \text{ whenever } i \neq j.$$

**Example.**
Let $A_1 = \{3, 5\}$, $A_2 = \{1, 4, 6\}$, and $A_3 = \{2\}$. Are $A_1$, $A_2$, and $A_3$ mutually disjoint?

Let $B_1 = \{2, 4, 6\}$, $B_2 = \{3, 7\}$, and $B_3 = \{4, 5\}$. Are $B_1$, $B_2$, $B_3$ mutually disjoint?

**Definition.**
A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, \ldots\}$ is a **partition** of a set $A$ if, and only if,

1. $A$ is the union of all the $A_i$;
2. the sets $A_1, A_2, A_3, \ldots$ are mutually disjoint.

**Example.**
Let $A = \{1, 2, 3, 4, 5, 6\}$, $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$, and $A_3 = \{5, 6\}$. Is $\{A_1, A_2, A_3\}$ a partition of $A$?

Let $\mathbb{Z}$ be the set of all integers and let

$$T_i = \{n \in \mathbb{Z} \,|\, n = 3k + i, \text{ for some integer } k\}.$$

Is $\{T_0, T_1, T_2\}$ a partition of $\mathbb{Z}$?

**Definition.**
Given a set $A$, the **power set** of $A$, denoted $\mathscr{P}(A)$, is the set of all subsets of $A$.

**Example.** Find $\mathscr{P}(\{x, y\})$.