3.1: Predicates and Quantified Statements I

Definition.
A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.
The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

Example. Let $P(x)$ be the predicate "$x^2 > x$" with domain the set $\mathbb{R}$. Write $P(2)$, $P\big(\frac{1}{2}\big)$, and $P\big(-\frac{1}{2}\big)$, and indicate which of these statements are true and which are false.

Definition.
If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x. The truth set of $P(x)$ is denoted

$$\{x \in D \mid P(x)\}$$

Example. Let $Q(n)$ be the predicate "n is a factor of 8". Find the truth set of $Q(n)$ if

the domain of n is $\mathbb{Z}^+$ | the domain of n is $\mathbb{Z}$

Definition.
Let $Q(x)$ be a predicate and D the domain of x.

- **Quantifiers** are words that refer to quantities such as "some" or "all" and tell for how many elements a given predicate is true.
- The **universal quantifier** is represented by the symbol "$\forall$".
- A **universal statement** is a statement of the form "$\forall x \in D$, $Q(x)$".
  - It is defined to be true if, and only if, $Q(x)$ is true for *each* individual x in D.
  - It is defined to be false if, and only if, $Q(x)$ is false for *at least one* x in D.
- A value for x for which $Q(x)$ is false is called a **counterexample** to the universal statement.

Example. Let $D = \{1, 2, 3, 4, 5\}$, and consider the statement

$$\forall x \in D, x^2 \geq x.$$

Write one way to read this statement out loud, and show that it is true.

The above example uses the **method of exhaustion**.

Example. Consider the statement

$$\forall x \in \mathbb{R}, x^2 \geq x.$$

Find a counter example to show that this statement is false.

Definition.
Let $Q(x)$ be a predicate and D the domain of x.

- The **existential quantifier** is represented by the symbol "$\exists$".
- An **existential statement** is a statement of the form "$\exists x \in D$ such that $Q(x)$".
  - It is defined to be true if, and only if, $Q(x)$ is true for *at least one* x in D.
  - It is false if, and only if, $Q(x)$ is false *for all* x in D.

Example. Consider the statement

$$\exists m \in \mathbb{Z}^+ \text{ such that } m^2 = m.$$

Write one way to read this statement out loud, and show that it is true.

Example. Let $E = \{5, 6, 7, 8\}$ and consider the statement

$$\exists m \in E \text{ such that } m^2 = m.$$

Show that this statement is false.

Example. Rewrite the following statements formally using quantifiers and variables:

All triangles have three sides.

No dogs have wings.

Some programs are structured.

Definition.
A **universal conditional statement** is of the form:

$$\forall x,\ \text{if } P(x) \text{ then } Q(x).$$

Example. Rewrite each of the following statements in the form

$$\forall \underline{\hspace{2cm}},\ \text{if } \underline{\hspace{2cm}} \text{ then } \underline{\hspace{2cm}}$$

If a real number is an integer, then it is a rational number.

All bytes have eight bits.

No fire trucks are green.