2.5: Application: Number Systems and Circuits for Addition

Recall our how we write numbers in base 10:

$$\begin{aligned} 5,049 &= 5 \cdot 1000 + 0 \cdot 100 + 4 \cdot 10 + 9 \cdot 1 \\ &= 5 \cdot 10^3 + 0 \cdot 10^2 + 4 \cdot 10^1 + 9 \cdot 10^0 \end{aligned}$$

Definition.
Any integer $b > 1$ can be used as a base for a numbering system. A numbering system of base b has the digits $0, 1, \ldots, b-1$.
A **base** 2 **notation** or **binary notation**, uses the digits $0, 1$. In binary, every integer is represented as sum of products of the form

$$d \cdot 2^n$$

where $n \in \mathbb{Z}$ and $d \in \{0, 1\}$.

Example. Below is the binary representation for the integers 1 to 9:

$$\begin{array}{rcccccccccr}
1_{10} & = & & & & & & & 1 \cdot 2^0 & = & 1_2 \\
2_{10} & = & & & & & 1 \cdot 2^1 & + & 0 \cdot 2^0 & = & 10_2 \\
3_{10} & = & & & & & 1 \cdot 2^1 & + & 1 \cdot 2^0 & = & 11_2 \\
4_{10} & = & & & 1 \cdot 2^2 & + & 0 \cdot 2^1 & + & 0 \cdot 2^0 & = & 100_2 \\
5_{10} & = & & & 1 \cdot 2^2 & + & 0 \cdot 2^1 & + & 1 \cdot 2^0 & = & 101_2 \\
6_{10} & = & & & 1 \cdot 2^2 & + & 1 \cdot 2^1 & + & 0 \cdot 2^0 & = & 110_2 \\
7_{10} & = & & & 1 \cdot 2^2 & + & 1 \cdot 2^1 & + & 1 \cdot 2^0 & = & 111_2 \\
8_{10} & = & 1 \cdot 2^3 & + & 0 \cdot 2^2 & + & 0 \cdot 2^1 & + & 0 \cdot 2^0 & = & 1000_2 \\
9_{10} & = & 1 \cdot 2^3 & + & 0 \cdot 2^2 & + & 0 \cdot 2^1 & + & 1 \cdot 2^0 & = & 1001_2
\end{array}$$

Converting binary → decimal:
To convert from binary to decimal, multiply each digit by its corresponding power of 2 and sum the results.

Example. Represent the following in decimal notation (base-10):

$110_2 = 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0$

$= 1 \cdot 4 + 1 \cdot 2 + 0 \cdot 1$

$= 4 + 2$

$= \boxed{6_{10}}$

$1011_2 = 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$

$= 1 \cdot 8 + 0 \cdot 4 + 1 \cdot 2 + 1 \cdot 0$

$= 8 + 2$

$= \boxed{10_{10}}$

11110_2

$= 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0$

$= 1 \cdot 16 + 1 \cdot 8 + 1 \cdot 4 + 1 \cdot 2 + 0 \cdot 1$

$= 16 + 8 + 4 + 2$

$= \boxed{30}$

101011_2

$= 1 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$

$= 32 + 8 + 2 + 1$

$= \boxed{43}$

Converting decimal $\to$ binary:
To convert from decimal to binary, we repeated divide by 2, and record the remainders.

Example.

$$\begin{aligned}27_{10} &= 16+8+2+1\\ &= 1\cdot 2^4 + 1\cdot 2^3 + 0\cdot 2^2 + 1\cdot 2^1 + 1\cdot 2^0\\ &= 11011_2\end{aligned}$$

Example. Represent the following in binary notation:

$243_{10} = 11110011_2$

$243 \div 2 = 121 \text{ r } 1$
$121 \div 2 = 60 \text{ r } 1$
$60 \div 2 = 30 \text{ r } 0$
$30 \div 2 = 15 \text{ r } 0$
$15 \div 2 = 7 \text{ r } 1$
$7 \div 2 = 3 \text{ r } 1$
$3 \div 2 = 1 \text{ r } 1$
$1 \div 2 = 0 \text{ r } 1$

$587_{10} = 1001001011_2$

$587 \div 2 = 293 \text{ r } 1$
$293 \div 2 = 146 \text{ r } 1$
$146 \div 2 = 73 \text{ r } 0$
$73 \div 2 = 36 \text{ r } 1$
$36 \div 2 = 18 \text{ r } 0$
$18 \div 2 = 9 \text{ r } 0$
$9 \div 2 = 4 \text{ r } 1$
$4 \div 2 = 2 \text{ r } 0$
$2 \div 2 = 1 \text{ r } 0$
$1 \div 2 = 0 \text{ r } 1$

$990_{10} = 1111011110_2$

$990 \div 2 = 495 \text{ r } 0$
$495 \div 2 = 247 \text{ r } 1$
$247 \div 2 = 123 \text{ r } 1$
$123 \div 2 = 61 \text{ r } 1$
$61 \div 2 = 30 \text{ r } 1$
$30 \div 2 = 15 \text{ r } 0$
$15 \div 2 = 7 \text{ r } 1$
$7 \div 2 = 3 \text{ r } 1$
$3 \div 2 = 1 \text{ r } 1$
$1 \div 2 = 0 \text{ r } 1$

$531_{10} = 1000010011_2$

$531 \div 2 = 265 \text{ r } 1$
$265 \div 2 = 132 \text{ r } 1$
$132 \div 2 = 66 \text{ r } 0$
$66 \div 2 = 33 \text{ r } 0$
$33 \div 2 = 16 \text{ r } 1$
$16 \div 2 = 8 \text{ r } 0$
$8 \div 2 = 4 \text{ r } 0$
$4 \div 2 = 2 \text{ r } 0$
$2 \div 2 = 1 \text{ r } 0$
$1 \div 2 = 0 \text{ r } 1$

Binary arithmetic:
In binary arithmetic, 10_2 behaves similarly to 10 in decimal arithmetic.

Example. Add 1101_2 and 111_2 using binary notation.

$$\begin{array}{r} {\scriptstyle 1\,1\,\,\,1\,\,\,} \\ 1101_2 \\ +\quad 111_2 \\ \hline 10100 \end{array}$$

Example. Subtract 1011_2 from 11000_2 using binary notation.

$$\begin{array}{rl} 1\cancel{1}\cancel{0}\cancel{0}\cancel{0}_2 & \leftarrow 24 \\ -\quad 1011_2 & \leftarrow 11 \\ \hline \boxed{1101_2} & \leftarrow 13 \end{array}$$

$$\boxed{\begin{array}{r} 10_2 \\ -\ 1_2 \\ \hline 1_2 \end{array}}$$

Definition.
The 8-bit two's complement for an integer a between -128 and 127 is the 8-bit binary representation for

$$\begin{cases} a, & \text{if } a \geq 0 \\ 2^8 - |a|, & \text{if } a < 0. \end{cases}$$

Two's complement allows maximum representation for 2^8 integers with 8 binary digits.

Example. Below are a few integers represented in binary using 8-bit two's complement:

$$-128 \to 2^8 - |-128| = 128_{10} = 10000000_2 \qquad 0 \to 0_{10} = 00000000_2$$

$$-127 \to 2^8 - |-127| = 129_{10} = 10000001_2 \qquad 1 \to 1_{10} = 00000001_2$$

$$\vdots \qquad 2 \to 2_{10} = 00000010_2$$

$$-2 \to 2^8 - |-2| = 254_{10} = 10000000_2 \qquad \vdots$$

$$-1 \to 2^8 - |-1| = 255_{10} = 11111111_2 \qquad 127 \to 127_{10} = 01111111_2$$

Example. Find the 8-bit two's complement for the following:

-46
$\longrightarrow 2^8 - |-46| = 210_{10}$
$= 11010010_2$

42
$\longrightarrow 42_{10} = 101010_2$

120
$\longrightarrow 120_{10} = 11010010_2$

-82
$\longrightarrow 2^8 - |-82| = 174_2 = 10101110_2$

Two's complement of a negative integer:
To find the decimal representation of the negative integer with a given 8-bit two's complement:

- Flip the bits
- Add 1
- Convert to base-10 and swap the sign

Example. Find the decimal representation of the integers with the following 8-bit two's complement:

$11100101_2 \longrightarrow$

Indicates negative

$$\begin{array}{r} 00011010_2 \\ +\quad\quad 1_2 \\ \hline 00011011_2 \end{array}$$

$= 27_{10} \longrightarrow \boxed{-27_{10}}$

$11000000_2 \longrightarrow$

$$\begin{array}{r} 00111111_2 \\ +\quad\quad 1_2 \\ \hline 1000000_2 \end{array}$$

$= 64_{10} \longrightarrow \boxed{-64_{10}}$

Addition and Subtraction with Integers in Two's Complement Form: When performing binary addition on integers written in Two's Complement form, we discard any "carry" bit.

Example. Perform binary addition using the Two's Complement form of the following:

83 and −55

$83 \longrightarrow 83_{10} \longrightarrow 01010011_2$

$-55 \longrightarrow 2^8 - |-55| = 201 \longrightarrow + \ 11001001_2$

Carry bit $\rightarrow$ $1\ 00011100_2$

$00011100_2 = 16 + 8 + 4 = \boxed{28_{10}}$

−87 and −46

$-87 \longrightarrow 2^8 - |-87| = 169_{10} \longrightarrow 10101001_2$

$-46 \longrightarrow 2^8 - |-46| = 210_{10} \longrightarrow + \ 11010010_2$

Carry bit $\rightarrow$ $1\ 01111011_2$

This 0 signals a positive result
and an overflow error

$1111011_2 \longrightarrow 64 + 32 + 16 + 8 + 2 + 1 = 123_{10} \neq -133_{10}$

Definition.
Hexadecimal notation uses a **base** 16 **notation**. In hexadecimal, every integer is represented as sum of products of the form

$$d \cdot 16^n$$

where $n \in \mathbb{Z}$ and $d \in \{0, 1, \ldots, 9, A, B, C, D, E, F\}$.

Decimal	Hexadecimal	4-Bit Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
10	A	1010
11	B	1011
12	C	1100
13	D	1101
14	E	1110
15	F	1111

Example. Convert $3CF_{16}$ to decimal notation.

12 15

$$3 \cdot 16^2 + 12 \cdot 16^1 + 15 \cdot 16^0 = 3 \cdot 256 + 12 \cdot 16 + 15 \cdot 1$$
$$= 768 + 192 + 15$$
$$= \boxed{975_{10}}$$

Example. Convert $B09F_{16}$ to binary notation.

B_{16} $\quad 0_{16}$ $\quad 9_{16}$ $\quad F_{16}$

11_{10} $\quad 0_{10}$ $\quad 9_{10}$ $\quad 15_{10}$

1011_2 $\quad 0000_2$ $\quad 0101_2$ $\quad 1111_2$

$$\rightarrow \quad \boxed{B09F_{16} = 1011\,0000\,0101\,1111_2}$$

Example. Convert $0100|1101|1010|1001_2$ to hexadecimal notation.

0100_2 $\quad 1101_2$ $\quad 1010_2$ $\quad 1001_2$

4_{10} $\quad 13_{10}$ $\quad 10_{10}$ $\quad 9_{10}$

4_{16} $\quad D_{16}$ $\quad A_{16}$ $\quad 9_{16}$

$$\boxed{0100110110101001_2 = 4DA9_{16}}$$