5.6: Exponential Functions As Mathematical Models

Example. Consider the exponential function

$$Q(t) = Q_0 e^{kt}$$

What does Q_0 represent?

Q_0 is the initial value

$$Q(0) = Q_0 e^{k(0)} = Q_0 e^0 = Q_0$$

What does k represent?

k is the growth factor

Show that the rate of increase of $Q(t)$ is proportional to the quantity $Q(t)$.

$$Q'(t) = \underbrace{Q_0 e^{kt}}_{Q(t)} \cdot k = k\,Q(t)$$

Definition.
$Q(t)$ is said to exhibit **Exponential Growth**.

Example. Under ideal laboratory conditions, the number of bacteria in a culture grows in accordance with the law $Q(t) = Q_0 e^{kt}$, where Q_0 denotes the number of bacteria initially present in the culture, k is a constant determined by the strain of bacteria under consideration and other factors, and t is the elapsed time measured in hours. Suppose 10,000 bacteria are present initially in the culture and 60,000 are present 2 hours later.

How many bacteria will there be in the culture at the end of 4 hours?

$Q(0) = 10{,}000 \Rightarrow Q_0 = 10{,}000$

$Q(2) = 60{,}000 \Rightarrow \dfrac{60{,}000}{10{,}000} = \dfrac{10{,}000\, e^{k(2)}}{10000}$

$6 = e^{2k}$

$2k = \ln(6)$

$k = \dfrac{\ln(6)}{2}$

$\Rightarrow Q(4) = 10{,}000\, e^{\frac{\ln(6)}{2}(4)}$

$= 10{,}000\, e^{2\ln(6)}$

$= 10{,}000 \cdot 36$

$= \boxed{360{,}000}$

What is the $\underbrace{\text{rate of growth}}$ of the population after 4 hours?

$$Q'(t) = 10{,}000 e^{\frac{\ln(6)}{2}t}\left(\frac{\ln(6)}{2}\right) = \frac{\ln(6)}{2} Q(t)$$

$$\Rightarrow Q'(4) = \frac{\ln(6)}{2} Q(4) = \frac{\ln(6)}{2} \cdot 360{,}000 = 180{,}000 \ln(6)$$

$$\approx \boxed{322{,}516.7045}$$

Example. Radioactive substances decay exponentially. For example, the amount of radium present at any time t obeys the law $Q(t) = Q_0 e^{-kt}$, where Q_0 is the initial amount present and k is a specific positive constant. The **half-life of a radioactive substance** is the time required for a given amount to be reduced by one-half. It is known that the half-life of radium is approximately 1600 years. Suppose initially there are $\underbrace{200}_{Q_0}$ milligrams of pure radium.

What is the amount left after t years? What about 800 years?

$$Q(1600) = 100 \Rightarrow 100 = 200 e^{-k(1600)}$$

$$\frac{1}{2} = e^{-1600k}$$

$$-1600k = \ln\left(\tfrac{1}{2}\right)$$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{-1600} \Rightarrow \boxed{Q(t) = 200 e^{\frac{\ln\left(\frac{1}{2}\right)}{1600} t}}$$

$$\Rightarrow Q(8) = 200 e^{\frac{\ln\left(\frac{1}{2}\right)}{1600} 8} = 200 e^{\frac{\ln\left(\frac{1}{2}\right)}{2}} = 200\left(\tfrac{1}{2}\right)^{1/2} \approx \boxed{141.421}$$

How fast is the amount of radium decaying after t years? What about 800 years?

$$Q'(t) = 200 e^{\frac{\ln\left(\frac{1}{2}\right)}{1600} t} \left(\frac{\ln\left(\frac{1}{2}\right)}{1600}\right) = \frac{\ln\left(\frac{1}{2}\right)}{1600} Q(t)$$

$$\Rightarrow Q'(8) = \frac{\ln\left(\frac{1}{2}\right)}{1600} Q(8) = \frac{\ln\left(\frac{1}{2}\right)}{1600} \cdot 200\left(\tfrac{1}{2}\right)^{1/2} = \boxed{\frac{\sqrt{2}\ln\left(\frac{1}{2}\right)}{16}} \approx \boxed{-0.0613}$$

Example. Carbon 14, a radioactive isotope of carbon, has a half-life of 5730 years. What is its decay constant?

$$Q(t) = Q_0 e^{-kt}$$

$$Q(5730) = \frac{1}{2} Q_0 \implies \frac{1}{2} Q_0 = Q_0 e^{-k(5730)}$$

$$\frac{1}{2} = e^{-5730k}$$

$$-5730k = \ln\left(\frac{1}{2}\right)$$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{-5730} = \boxed{\frac{\ln(2)}{5730}} \approx \boxed{0.00012}$$

Example. The Camera Division of Eastman Optical produces a 35-mm single-lens reflex camera. Eastman's training department determines that after completing the basic training program, a new, previously inexperienced employee will be able to assemble

$$Q(t) = 50 - 30e^{-0.5t}$$

model F cameras per day t months after the employee starts work on the assembly line.

How many model F cameras can a new employee assemble per day after basic training?

$$Q(0) = 50 - 30e^{-0.5(0)} = 50 - 30e^{0} = 50 - 30 = \boxed{20}$$

How many model F cameras can an employee with 1 month of experience assemble per day? What about 2 months? 6 months?

$$Q(1) = 50 - 30e^{-0.5(1)} = 50 - 30e^{-0.5} \approx 50 - 18.196 = \boxed{31.804}$$

$$Q(2) = 50 - 30e^{-0.5(2)} = 50 - 30e^{-1} \approx 50 - 11.036 = \boxed{38.964}$$

$$Q(6) = 50 - 30e^{-0.5(6)} = 50 - 30e^{-3} \approx 50 - 1.496 = \boxed{48.506}$$

How many model F cameras can the average experienced employee ultimately be expected to assemble per day?

$$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} 50 - 30e^{-0.5t} = 50 - 30\underbrace{\lim_{t\to\infty} e^{-0.5t}}_{\to 0} = \boxed{50}$$

Example. The number of soldiers at Fort MacArthur who contracted influenza after t days during a flu epidemic is approximated by the *logistic model*

$$Q(t) = \frac{5000}{1 + 1249e^{-kt}}$$

If 40 soldiers contracted the flu by day 7, find how many soldiers contracted the flu by day 15.

$$Q(7) = 40 \Rightarrow 40 = \frac{5000}{1+1249e^{-k(7)}}$$

$$\longrightarrow 1+1249e^{-7k} = \frac{5000}{40}$$

$$1249e^{-7k} = 124$$

$$e^{-7k} = \frac{124}{1249}$$

$$\rightarrow -7k = \ln\left(\frac{124}{1249}\right)$$

$$k = \frac{\ln\left(\frac{124}{1249}\right)}{-7}$$

$$\Rightarrow Q(15) = \frac{5000}{1+1249e^{\frac{\ln\left(\frac{124}{1249}\right)}{7}(15)}} \approx \boxed{507.581}$$

At what rate is the number of soldiers contracting the flu changing on day 15?

$$Q(t) = \frac{5000}{1+1249e^{-kt}} = 5000\left(1+1249e^{-kt}\right)^{-1}$$

$$Q'(t) = -5000\left(1+1249e^{-kt}\right)^{-2}\left(1249e^{-kt}\right)\cdot(-k) = \frac{5000k\left(1249e^{-kt}\right)}{\left(1+1249e^{-kt}\right)^2}$$

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chain

$$Q'(15) = \frac{5000k\left(1249e^{-k15}\right)}{\left(1+1249e^{-k15}\right)^2} \approx \boxed{150.486}$$