Math 125 Class notes
Spring 2026

To accompany
Discrete Mathematics with Applications
by *Epp*

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Table Of Contents

1.1: Variables

> **Definition.**
> A **variable** is a placeholder for something which may or may not be unknown.

Example. Is there a number with the following property: doubling it and adding 3 gives the same result as squaring it?

$$\to \text{Is there a number } x \text{ with the property that } 2x + 3 = x^2?$$
$$\to \text{Is there a number } \square \text{ with the property that } 2 \cdot \square + 3 = \square^2?$$

Example. No matter what number might be chosen, if it is greater than 2, then its square is greater than 4.

$$\to \text{No matter what number } n \text{ might be chosen, if } n \text{ is greater than 2,}$$
$$\text{then } n^2 \text{ is greater than 4.}$$

Example. Use variables to rewrite the following sentences:

Are there numbers with the property that the sum of their squares equals the square of their sum?

Given any real number, its square is nonnegative.

Definition.

- A **universal statement** says that a certain property is true for all elements in a set.

- A **conditional statement** says that if one thing is true, then some other thing also has to be true.

- Given a property that may or may not be true, an **existential statement** says that there is at least one thing for which the property is true.

Definition.
A **universal conditional statement** is both universal and conditional:

> For every animal a, if a is a dog, then a is a mammal.

Conditional statements can be rewritten in ways that make them appear more to be purely universal or purely conditional:

> If a is a dog, then a is a mammal.
> All dogs are mammals

Example. Rewrite the following universal condition statement:

> For every real number x, if x is nonzero then x^2 is positive.

> If a real number is nonzero, then its square _____.

> For every nonzero real number x, _____.

> If x _____, then _____.

> The square of any nonzero real number is _____.

> All nonzero real numbers have _____.

Definition.

A **universal existence statement** is a statement that is universal because its first part says that a certain property is true for all objects of a given type, and it is existential because its second part asserts the existence of something:

Every real number has an additive inverse.

In the above example, note that the particular additive inverse depends on the given real number:

For every real number r, there is an additive inverse for r.

Example. Rewrite the following universal existence statement:

Every pot has a lid

All pots _____.

For ever pot P, there is _____.

For every pot P, there is a lid L such that _____.

> **Definition.**
> An **existential universal statement** is a statement that is existential because its first part asserts that a certain object exists and is universal because its second part says that the object satisfies a certain property for all things of a certain kind:
>
> There is a positive integer that is less than or equal to every positive integer.
>
> The number one satisfies the above statement, which can also be rewritten:
>
> There is a positive integer m that is less than or equal to every positive integer.

Example. Rewrite the following existence universal statement:

There is a person in my class who is at least as old as every person in my class.

Some _____ is at least as old as _____.

There is a person p in my class such that p is _____.

There is a person p in my class with the property that for every person q in my class, p is _____.

1.2: The Language of Sets

Definition.

- A **set** is a collection of objects.

- If S is a set, then we use

   − $x \in S$ to denote that the element x is in the set S.

   − $x \notin S$ to denote that the element x is *not* in the set S.

- The **set-roster notation** is used to denote all elements in a set between braces:

$$S = \{1, 2, \ldots, 100\}$$

   Here, we see that $67 \in S$, but $1337 \notin S$.

- The **axiom of extension** says that a set is completely determined by what its elements are – not the order in which they are listed.

Example.

Let $A = \{1, 2, 3\}$, $B = \{3, 1, 2\}$, and $C = \{1, 1, 2, 3, 3, 3\}$. What are the elements of A, B, and C? How are A, B, and C related?

Is $\{0\} = 0$?

How many elements are in the set $\{1, \{1\}\}$?

For each nonnegative integer n, let $U_n = \{n, -n\}$. Find U_1, U_2, and U_0.

Certain sets of numbers are so frequently referred to that they are given special names and symbols:

$$
\begin{array}{ll}
\mathbf{N} \text{ or } \mathbb{N} & \text{The set of all } \textbf{natural numbers} \\
\mathbf{Z} \text{ or } \mathbb{Z} & \text{The set of all } \textbf{integers} \\
\mathbf{Q} \text{ or } \mathbb{Q} & \text{The set of all } \textbf{rational numbers}, \text{ or quotient of integers} \\
\mathbf{R} \text{ or } \mathbb{R} & \text{The set of all } \textbf{real numbers}
\end{array}
$$

Note: We may additionally use superscripts to indicate further properties of these sets:

$$
\begin{array}{ll}
\mathbb{Z}^{+} \text{ or } \mathbb{Z}^{>0} & \text{The set of } \textit{positive} \text{ integers} \\
\mathbb{Q}^{-} \text{ or } \mathbb{Q}^{<0} & \text{The set of } \textit{negative} \text{ rational numbers} \\
\mathbb{R}^{nonneg} \text{ or } \mathbb{R}^{\geq 0} & \text{The set of } \textit{nonnegative} \text{ real numbers}
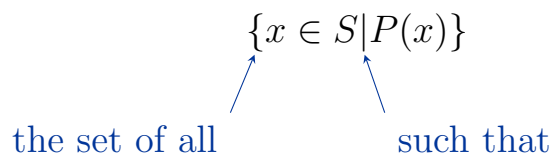\end{array}
$$

Note: Different sources denote the natural numbers $\mathbb{N}$ as $\mathbb{Z}^{+}$ *or* $\mathbb{Z}^{\geq 0}$.

Definition. (Set-Builder Notation)
Let S be a set and let $P(x)$ be a property that elements of S may or may not satisfy.
We may define a new set to be **the set of all elements x in S such that $P(x)$ is
true**. We denote this set as follows:

$$\{x \in S | P(x)\}$$

the set of all   such that

Example. Describe each of the following sets:

$$\{x \in \mathbb{R} \mid -2 < x < 5\}$$

$$\{x \in \mathbb{Z} \mid -2 < x < 5\}$$

$$\{x \in \mathbb{Z}^+ \mid -2 < x < 5\}$$

Definition.

If A and B are sets, then A is called a **subset** of B, written $A \subseteq B$, if, and only if, every element of A is also an element of B:

$A \subseteq B$ means that for every element x, if $x \in A$, then $x \in B$.

$A \nsubseteq B$ means that there is at least one element x, such that $x \in A$ and $x \notin B$.

A is a **proper subset** of B if, and only if, every element of A is in B, but there is at least one element of B that is not in A:

$A \subsetneq B$ means that for every element x, if $x \in A$, then $x \in B$,

and there exists $x \in B$ such that $x \notin A$.

Example. Let $A = \mathbb{Z}^+$, $B = \{n \in \mathbb{Z} \mid 0 \leq n \leq 100\}$, and $C = \{100, 200, 300, 400, 500\}$. Evaluate the truth and falsity of each of the following statements.

$B \subseteq A$

C is a proper subset of A

C and B have at least one element in common

$C \subseteq B$

$C \subseteq C$

Example. Determine which of the following statements are true:

$2 \in \{1, 2, 3\}$ $\qquad\qquad\qquad\qquad\qquad$ $\{2\} \in \{1, 2, 3\}$

$2 \subseteq \{1, 2, 3\}$ $\qquad\qquad\qquad\qquad\qquad$ $\{2\} \subseteq \{1, 2, 3\}$

$\{2\} \subseteq \{\{1\}, \{2\}\}$ $\qquad\qquad\qquad\qquad\qquad$ $\{2\} \in \{\{1\}, \{2\}\}$

Definition.

Given elements a and b, the symbol (a, b) denotes the **ordered pair** consisting of a and b together with the specification that a is the first element of the pair, and b is the second element. Two ordered pairs (a, b) and (c, d) are equal if, and only if, $a = c$ and $b = d$:

$$(a, b) = (c, d) \text{ means that } a = c \text{ and } c = d.$$

Example.

Is $(1, 2) = (2, 1)$?

Is $\left(3, \dfrac{5}{10}\right) = \left(\sqrt{9}, \dfrac{1}{2}\right)$?

Definition.

Let $n \in \mathbb{N}$ and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered n-tuple**, $(x_1, x_2, \ldots, x_n)$, consists of $x_1, x_2, \ldots x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. and ordered 2-tuple is called an **ordered pair**, and ordered 3-tuple is called an **ordered triple**.

Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if, and only if, $x_1 = y_1$, $x_2 = y_2$, $\ldots$, and $x_n = y_n$:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \iff x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

Example.

Is $(1, 2, 3, 4) = (1, 2, 4, 3)$?

Is $\left(3, (-2)^2, \dfrac{1}{2}\right) = \left(\sqrt{9}, 4, \dfrac{3}{6}\right)$?

Definition.
Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$, denoted

$$A_1 \times A_2 \times \cdots \times A_n$$

is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1$, $a_2 \in A_2$, $\ldots$, $a_n \in A_n$:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in \mathcal{A}_n\}$$

Example. Let $A = \{x, y\}$, $B = \{1, 2, 3\}$, and $C = \{a, b\}$. Find the following:

$A \times B$

$B \times A$

$A \times A$

How many elements are in $A \times B$, $B \times A$, and $A \times A$?

$(A \times B) \times C$

$A \times B \times C$

Describe $\mathbb{R} \times \mathbb{R}$

Definition.
Let $n \in \mathbb{N}$. Given a finite set A, a **string of length** n **over** A is an ordered n-tuple of elements of A written without parentheses or commas. The elements of A are called the **characters** of the string. The **null string** over A is defined to be the "string" with no characters, often denoted λ, and is said to have length 0. If $A = \{0, 1\}$, then a string over A is called a **bit string**.

Example. Let $A = \{a, b\}$. List all strings of length 3 over A with at least two characters that are the same.

1.3: The Language of Relations and Functions

Definition.
Let A and B be sets. A **relation** R **from** A **to** B is a subset of $A \times B$. Given an ordered pair (x, y), x **is related to** y **by** R, written $x \, R \, y$, if, and only if, (x, y) is in R. The set A is called the **domain** of R and the set B is called its **co-domain**. The notation for a relation R may be written symbolically as follows:

$$x \, R \, y \text{ means that } (x, y) \in R.$$

The notation $x \, \not\!R \, y$ means that x is not related to y by R:

$$x \, \not\!R \, y \text{ means that } (x, y) \notin R.$$

Example. Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ and define a relation R from A to B as follows; Given any $(x, y) \in A \times B$,

$$(x, y) \in R \text{ means that } \frac{x - y}{2} \text{ is an integer.}$$

State explicitly which ordered pairs are in $A \times B$ and which are in R

Is $1 \, R \, 3$?   Is $2 \, R \, 3$?   Is $2 \, R \, 2$?

What are the domain and co-domain of R?

Example. Define a relation C from $\mathbb{R}$ to $\mathbb{R}$ as follows: For any $(x, y) \in \mathbb{R} \times \mathbb{R}$,

$$(x, y) \in C \text{ means that } x^2 + y^2 = 1.$$

Is $(1, 0) \in C$?   Is $(0, 0) \in C$?   Is $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \in C$?

Is $-2\,C\,0$?   Is $0\,C\,(-1)$?   Is $1\,C\,1$?

What are the domain and co-domain of C?

Draw a graph for C by plotting the points of C in the Cartesian plane.

Definition.

Suppose R is a relation from set A to a set B. The **arrow diagram for** R is obtained as follows:

1. Represent the elements of A as points in one region and the elements of B as points in another region.

2. For each x in A and y in B, draw an arrow from x to y if, and only if, x is related to y by R.
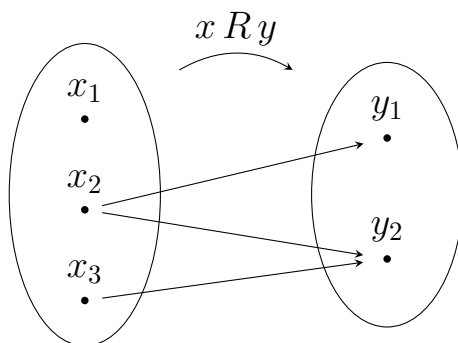
Example. Let $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$ and define relations S and T from A to B as follows: For every $(x, y) \in A \times B$,

$$(x, y) \in S \text{ means that } x < y$$
$$T = \{(2, 1), (2, 5)\}.$$

Draw arrow diagrams for S and T

Definition.

A **function** F **from a set** A **to a set** B is a relation with domain A and co-domain B that satisfies the following two properties:

1. For every element x in A, there is an element y in B such that $(x, y) \in F$.

2. For all elements x in A and y and z in B,

$$\text{if } (x, y) \in F \text{ and } (x, z) \in F, \text{ then } y = z.$$

*Note*: A relation from A to B is a function if, and only if,

1. Every element of A is the first element of an ordered pair of F

2. No two distinct ordered pairs in F have the same first element.

Note: If A and B are sets and F is a function from A to B, then given any element x in A, the unique element in B that is related to x by F is denoted $F(x)$, which is read "F **of** x".
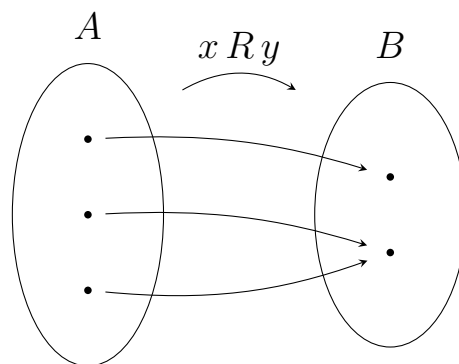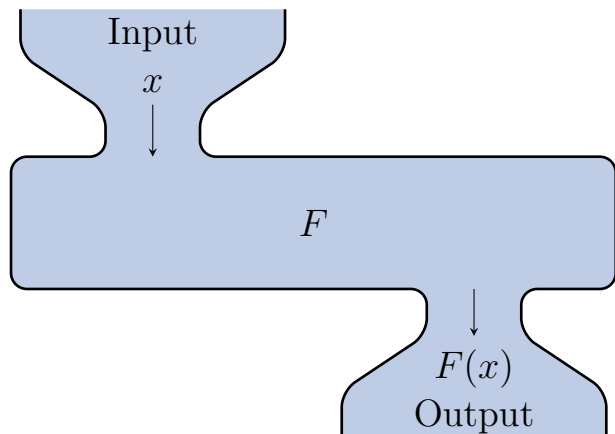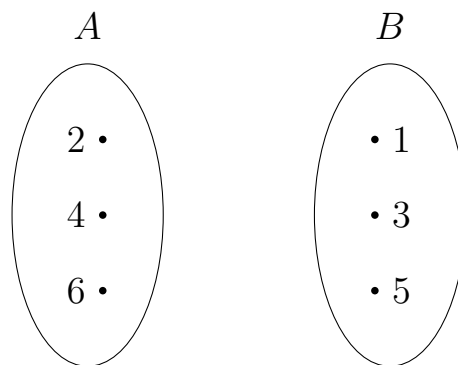
Example. Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$. Which of the relations R, S, and T defined below are functions from A to B?

$R = \{(2, 5), (4, 1), (4, 3), (6, 5)\}$
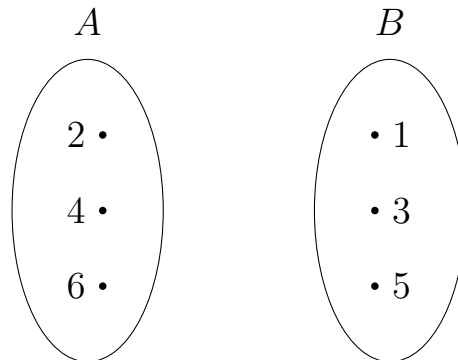
For every $(x, y) \in A \times B$, $(x, y) \in S$ means that $y = x + 1$.

T is defined by the arrow diagram
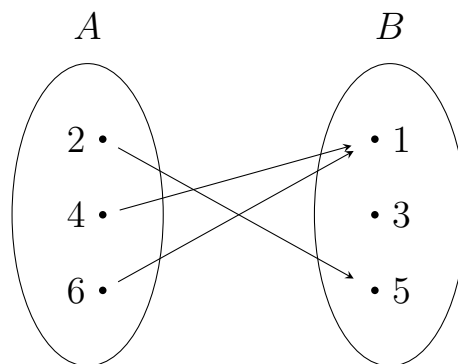
2.1: Logical Form and Logical Equivalence

> **Definition.**
> A **statement** (or **proposition**) is a sentence that is true or false, but not both.

Example. Determine which of the following are statements:

$2 + 2 = 4$ $2 + 2 = 5$

$x^2 + 2 = 11$ Today is Saturday.

She is a computer science major.   Jane is a computer science major.

> **Definition. (Compound Statements)**
> Let p and q be statement variables.
>
> - The **negation** of p is "not p", and is denoted as $\sim p$ (or $\neg p$)
>
> - The **conjunction** of p and q is "p and q", and is denoted at $p \wedge q$
>
> - The **disjunction** of p and q is "p or q", and is denoted $p \vee q$.
>
> - The **exclusive or** of p and q is "p x-or q", and is denoted $p \oplus q$ (or p XOR q)
>
> The **order of operations** specifies that $\sim$ is performed first.

Example. Consider the following statements:

$$
\begin{aligned}
p &: \quad \text{It is raining.} \\
q &: \quad \text{It is sunny.} \\
r &: \quad \text{It is cloudy.}
\end{aligned}
$$

Rewrite the following compound statements in words:

$\sim p$ $\qquad\qquad\qquad\qquad\qquad\qquad p \vee q$

$q \wedge r$ $\qquad\qquad\qquad\qquad\qquad\qquad q \wedge \sim r$

$p \wedge (q \vee r)$ $\qquad\qquad\qquad\qquad\qquad\qquad p \oplus q$

Definition.
A **statement form** (or **propositional form**) is an expression made up of statement variables (e.g., p, q, and r), and logical connectives (e.g. $\sim$, $\wedge$, $\vee$, and $\oplus$).
The **truth table** for a given statement form displays the truth values that correspond to all possible combinations of truth values for its component statement variables.

Example. Let p and q be statement variables. Fill out the following truth tables:

p	$\sim p$
T	
F	

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$
T	T			
T	F			
F	T			
F	F			

p	q	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$(p \vee q) \wedge \sim (p \wedge q)$
T	T				
T	F				
F	T				
F	F				

Example. Construct a truth table for the statement form $(p \wedge q) \vee \sim r$.

Definition.

Two *statement forms* are called **logically equivalent** if, and only if, they have identical true values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms P and Q is denoted $P \equiv Q$.

Example. Use truth tables to test if the following statement forms are equivalent:

p and $\sim (\sim p)$

$\sim (p \wedge q)$ and $\sim p \wedge \sim q$

Definition. (De Morgan's Laws)
The negation of an *and* statement is logically equivalent to the *or* statement in which each component is negated.
The negation of an *or* statement is logically equivalent to the *and* statement in which each component is negated.

Example. Use truth tables to show that the following statement forms are equivalent:

$\sim (p \wedge q)$ and $\sim p \vee \sim q$

$\sim (p \vee q)$ and $\sim p \wedge \sim q$

Example. Using De Morgan's law to write the negation of the following statements:

Jim is at least 6 feet tall and weighs at least 200 pounds.

The bus was late or Tom's watch was slow.

$-1 < x \leq 4$

Definition.
A **tautology** is a statement form that is always true.
A **contradiction** is a statement form that is always false.

Example. Complete the truth tables for $p \wedge \sim p$ and $p \vee \sim p$

Example. Let $\mathbf{t}$ be a tautology, and $\mathbf{c}$ be a contradiction. Show that $p \wedge \mathbf{t} \equiv p$ and $p \wedge \mathbf{c} \equiv \mathbf{c}$

Theorem 2.1.1 Logical Equivalences (p 49)

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold:

1. Commutative laws:
$$p \wedge q \equiv q \wedge p \qquad\qquad\qquad p \vee q \equiv q \vee p$$

2. Associative laws:
$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r) \qquad\qquad (p \vee q) \vee r \equiv p \vee (q \vee r)$$

3. Distributive laws:
$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \qquad\qquad p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

4. Identity laws:
$$p \wedge \mathbf{t} \equiv p \qquad\qquad\qquad p \vee \mathbf{c} \equiv p$$

5. Negation laws:
$$p \vee \sim p \equiv \mathbf{t} \qquad\qquad\qquad p \wedge \sim p \equiv \mathbf{c}$$

6. Double negative law:
$$\sim (\sim p) \equiv p$$

7. Idempotent laws:
$$p \wedge p \equiv p \qquad\qquad\qquad p \vee p \equiv p$$

8. Universal bound laws:
$$p \vee \mathbf{t} \equiv \mathbf{t} \qquad\qquad\qquad p \wedge \mathbf{c} \equiv \mathbf{c}$$

9. De Morgan's laws:
$$\sim (p \wedge q) \equiv \sim p \vee \sim q \qquad\qquad \sim (p \vee q) \equiv \sim p \wedge \sim q$$

10. Absorption laws:
$$p \wedge (p \vee q) \equiv p \qquad\qquad\qquad p \wedge (p \vee q) \equiv p$$

11. Negations of $\mathbf{t}$ and $\mathbf{c}$:
$$\sim \mathbf{t} \equiv \mathbf{c} \qquad\qquad\qquad \sim \mathbf{c} \equiv \mathbf{t}$$

2.2: Conditional Statements

Definition.
If p and q are statement variables, the **conditional** of q by p is "If p then q", or "p implies q" and is denoted by $p \to q$. It is false when p is true and q is false; otherwise it is true. We call p the **hypothesis** (or **antecedent**) of the conditional and q the **conclusion** (or **consequent**).
A conditional statement that is always true because the hypothesis is false is called **vacuously true**.

$$\text{If } \underbrace{4,686 \text{ is divisible by } 6}_{\text{hypothesis}}, \text{ then } \underbrace{4,686 \text{ is divisible by } 3}_{\text{conclusion}}$$

Example. Consider the following statement:

If Lander is open, then we will have class.

Create the truth table for $p \to q$

p	q	$p \to q$
T	T	
T	F	
F	T	
F	F	

Note: The **order of operations** states that $\rightarrow$ is performed last

Example. Create the truth table for $p\lor \sim q \rightarrow\sim p$.

p	q	$\sim q$	$p\lor \sim q$	$\sim p$	$p\lor \sim q \rightarrow\sim p$
T	T				
T	F				
F	T				
F	F				

Example. Use a truth table to show that $p \vee q \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$

p	q	r	$p \vee q$	$p \vee q \rightarrow r$	$p \rightarrow r$	$q \rightarrow r$	$(p \rightarrow r) \wedge (q \rightarrow r)$
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

Definition.
The **negation** of "if p then q" is logically equivalent to "p and not q":

$$\sim (p \to q) \equiv p \wedge \sim q$$

Example. Write negations for each of the following statements:

If my car is in the repair shop, then I cannot get to class.

If Sara lives in Athens, then she lives in Greece.

Definition.
The **contrapositive** of a conditional statement of the form "If p then q" is

$$\text{If } \sim q \text{ then } \sim p : \quad \sim q \to \sim p$$

A conditional statement is logically equivalent to its contrapositive.

Example. Write each of the following statements in its equivalent contrapositive form:

If Howard can swim across the lake, then Howard can swim to the island.

If today is Easter, then tomorrow is Monday.

Definition.

Suppose a conditional statement of the form "If p then q" is given.

- The **converse** is "If q then p": $q \to p$

- The **inverse** is "If $\sim p$ then $\sim q$": $\sim p \to \sim q$

Example. Write the converse and inverse of each of the following statements:

If Howard can swim across the lake, then Howard can swim to the island.

Converse:

Inverse:

If today is Easter, then tomorrow is Monday.

Converse:

Inverse:

Note:

1. A conditional statement and its converse are *not* logically equivalent.

2. A conditional statement and its inverse are *not* logically equivalent.

3. The converse and the inverse of a conditional statement are logically equivalent to each other.

Definition.
If p and q are statements, p **only if** q means "if not q then not p":

$$\sim q \to \sim p \equiv p \to q$$

Example. Rewrite the following statement in if-then form in two ways, one of which is the contrapositive of the other:

John will break the world's record for the mile run only if he runs the mile in under four minutes.

$\sim q \to \sim p$

$p \to q$

Note:

1. "p only if q" does *not* mean p if q

2. It is possible for "p only if q" to be true at the same time that "p if q" is false.

   e.g.: If John runs a mile in under four minutes, he still might not be fast enough to break the record.

Definition.
Given statement variables p and q, the **biconditional of p and q** is "p if, and only if, q" and is denoted $p \leftrightarrow q$. It is true if both p and q have the same truth values and is false otherwise. The words *if and only if* are sometimes abbreviated **iff**.

Note: The **order of operations** states that $\leftrightarrow$ is coequal with $\rightarrow$

Example. Create the truth table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	
T	F	
F	T	
F	F	

Order of Operations for Logical Operators

$\sim$	Evaluate negations first
$\wedge, \vee$	Evaluate $\wedge$ and $\vee$ second. When both present, parentheses may be needed.
$\rightarrow, \leftrightarrow$	Evaluate $\rightarrow$ and $\leftrightarrow$ third. When both present, parentheses may be needed.

Definition.

If r and s are statements:

1. r is a **sufficient condition** for s means "if r then s". $r \to s$

2. r is a **necessary condition** for s means "if not r then not s". $\sim r \to \sim s$

By property of the contrapositive:

3. r is a *necessary and sufficient condition* for s means "r if, and only if s."

$$r \leftrightarrow s$$

Example. Rewrite the following statement in the form "If A then B":

Having two 45° angles is a sufficient condition for this triangle to be a right triangle.

Example. Use the contrapositive to rewrite the following statement in two ways:

George's attaining age 35 is a necessary condition for his being president of the United States.

2.5: Application: Number Systems and Circuits for Addition

Recall our how we write numbers in base 10:

$$5,049 = 5 \cdot 1000 + 0 \cdot 100 + 4 \cdot 10 + 9 \cdot 1$$
$$= 5 \cdot 10^3 + 0 \cdot 10^2 + 4 \cdot 10^1 + 9 \cdot 10^0$$

Definition.

Any integer $b > 1$ can be used as a base for a numbering system. A numbering system of base b has the digits $0, 1, \ldots, b-1$.

A **base 2 notation** or **binary notation**, uses the digits $0, 1$. In binary, every integer is represented as sum of products of the form

$$d \cdot 2^n$$

where $n \in \mathbb{Z}$ and $d \in \{0, 1\}$.

Example. Below is the binary representation for the integers 1 to 9:

$$
\begin{aligned}
1_{10} &= & & & & & & & 1 \cdot 2^0 &= & 1_2 \\
2_{10} &= & & & & & 1 \cdot 2^1 &+ & 0 \cdot 2^0 &= & 10_2 \\
3_{10} &= & & & & & 1 \cdot 2^1 &+ & 1 \cdot 2^0 &= & 11_2 \\
4_{10} &= & & & 1 \cdot 2^2 &+ & 0 \cdot 2^1 &+ & 0 \cdot 2^0 &= & 100_2 \\
5_{10} &= & & & 1 \cdot 2^2 &+ & 0 \cdot 2^1 &+ & 1 \cdot 2^0 &= & 101_2 \\
6_{10} &= & & & 1 \cdot 2^2 &+ & 1 \cdot 2^1 &+ & 0 \cdot 2^0 &= & 110_2 \\
7_{10} &= & & & 1 \cdot 2^2 &+ & 1 \cdot 2^1 &+ & 1 \cdot 2^0 &= & 111_2 \\
8_{10} &= & 1 \cdot 2^3 &+ & 0 \cdot 2^2 &+ & 0 \cdot 2^1 &+ & 0 \cdot 2^0 &= & 1000_2 \\
9_{10} &= & 1 \cdot 2^3 &+ & 0 \cdot 2^2 &+ & 0 \cdot 2^1 &+ & 1 \cdot 2^0 &= & 1001_2
\end{aligned}
$$

Converting binary → decimal:
To convert from binary to decimal, multiply each digit by its corresponding power of 2 and sum the results.

Example. Represent the following in decimal notation (base-10):

110_2 $\qquad\qquad\qquad\qquad\qquad\qquad$ 1011_2

11110_2 $\qquad\qquad\qquad\qquad\qquad\qquad$ 101011_2

Converting decimal $\rightarrow$ binary:
To convert from decimal to binary, we repeated divide by 2, and record the remainders.

Example.

$$27_{10} = 16 + 8 + 2 + 1$$
$$= 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$$
$$= 11011_2$$

Example. Represent the following in binary notation:

243_{10} 587_{10}

990_{10} 531_{10}

Binary arithmetic:

In binary arithmetic, 10_2 behaves similarly to 10 in decimal arithmetic.

Example. Add 1101_2 and 111_2 using binary notation.

Example. Subtract 1011_2 from 11000_2 using binary notation.

Definition.
The 8-bit two's complement for an integer a between -128 and 127 is the 8-bit binary representation for
$$\begin{cases} a, & \text{if } a \geq 0 \\ 2^8 - |a|, & \text{if } a < 0. \end{cases}$$
Two's complement allows maximum representation for 2^8 integers with 8 binary digits.

Example. Below are a few integers represented in binary using 8-bit two's complement:

$-128 \to 2^8 - |-128| = 128_{10} = 10000000_2$ $\qquad\qquad$ $0 \to 0_{10} = 00000000_2$

$-127 \to 2^8 - |-127| = 129_{10} = 10000001_2$ $\qquad\qquad$ $1 \to 1_{10} = 00000001_2$

$\vdots$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $2 \to 2_{10} = 00000010_2$

$-2 \to 2^8 - |-2| = 254_{10} = 10000000_2$ $\qquad\qquad\qquad$ $\vdots$

$-1 \to 2^8 - |-1| = 255_{10} = 11111111_2$ $\qquad\qquad$ $127 \to 127_{10} = 01111111_2$

Example. Find the 8-bit two's complement for the following:

-46 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ 42

120 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ -82

Two's complement of a negative integer:

To find the decimal representation of the negative integer with a given 8-bit two's complement:

- Flip the bits

- Add 1

- Convert to base-10 and swap the sign

Example. Find the decimal representation of the integers with the following 8-bit two's complement:

11100101_2

11000000_2

Addition and Subtraction with Integers in Two's Complement Form:
When performing binary addition on integers written in Two's Complement form, we discard any "carry" bit.

Example. Perform binary addition using the Two's Complement form of the following:

83 and -55

-87 and -46

Definition.

Hexadecimal notation uses a **base** 16 **notation**. In hexadecimal, every integer is represented as sum of products of the form

$$d \cdot 16^n$$

where $n \in \mathbb{Z}$ and $d \in \{0, 1, \ldots, 9, A, B, C, D, E, F\}$.

Decimal	Hexadecimal	4-Bit Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
10	A	1010
11	B	1011
12	C	1100
13	D	1101
14	E	1110
15	F	1111

Example. Convert $3CF_{16}$ to decimal notation.

Example. Convert $B09F_{16}$ to binary notation.

Example. Convert 100110110101001_2 to hexadecimal notation.

3.1: Predicates and Quantified Statements I

> **Definition.**
> A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.
> The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

Example. Let $P(x)$ be the predicate "$x^2 > x$" with domain the set $\mathbb{R}$. Write $P(2)$, $P\left(\frac{1}{2}\right)$, and $P\left(-\frac{1}{2}\right)$, and indicate which of these statements are true and which are false.

> **Definition.**
> If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x. The truth set of $P(x)$ is denoted
> $$\{x \in D \mid P(x)\}$$

Example. Let $Q(n)$ be the predicate "n is a factor of 8". Find the truth set of $Q(n)$ if

the domain of n is $\mathbb{Z}^+$

the domain of n is $\mathbb{Z}$

Definition.
Let $Q(x)$ be a predicate and D the domain of x.

- **Quantifiers** are words that refer to quantities such as "some" or "all" and tell for how many elements a given predicate is true.

- The **universal quantifier** is represented by the symbol "$\forall$".

- A **universal statement** is a statement of the form "$\forall x \in D,\ Q(x)$".

  - It is defined to be true if, and only if, $Q(x)$ is true for *each* individual x in D.
  - It is defined to be false if, and only if, $Q(x)$ is false for *at least one* x in D.

- A value for x for which $Q(x)$ is false is called a **counterexample** to the universal statement.

Example. Let $D = \{1, 2, 3, 4, 5\}$, and consider the statement

$$\forall x \in D, x^2 \geq x.$$

Write one way to read this statement out loud, and show that it is true.

The above example uses the **method of exhaustion**.

Example. Consider the statement

$$\forall x \in \mathbb{R}, x^2 \geq x.$$

Find a counter example to show that this statement is false.

Definition.

Let $Q(x)$ be a predicate and D the domain of x.

- The **existential quantifier** is represented by the symbol "$\exists$".

- An **existential statement** is a statement of the form "$\exists x \in D$ such that $Q(x)$".

  - It is defined to be true if, and only if, $Q(x)$ is true for *at least one x* in D.
  - It is false if, and only if, $Q(x)$ is false *for all x* in D.

Example. Consider the statement

$$\exists m \in \mathbb{Z}^+ \text{ such that } m^2 = m.$$

Write one way to read this statement out loud, and show that it is true.

Example. Let $E = \{5, 6, 7, 8\}$ and consider the statement

$$\exists m \in E \text{ such that } m^2 = m.$$

Show that this statement is false.

Example. Rewrite the following statements formally using quantifiers and variables:

All triangles have three sides.

No dogs have wings.

Some programs are structured.

Definition.
A **universal conditional statement** is of the form:

$$\forall x, \text{ if } P(x) \text{ then } Q(x).$$

Example. Rewrite each of the following statements in the form

$$\forall \underline{\hspace{2cm}}, \text{ if } \underline{\hspace{2cm}} \text{ then } \underline{\hspace{2cm}}$$

If a real number is an integer, then it is a rational number.

All bytes have eight bits.

No fire trucks are green.

3.2: Predicates and Quantified Statements II

Definition.

- The negation of a statement of the form

$$\forall x \text{ in } D, \ Q(x)$$

  is logically equivalent to a statement of the form

$$\exists x \text{ in } D \text{ such that } \sim Q(x).$$

$$\sim (\forall x \in D, \ Q(x)) \equiv \exists x \in D \text{ such that } \sim Q(x).$$

- The negation of a statement of the form

$$\exists x \text{ in } D \text{ such that } Q(x)$$

  is logically equivalent to a statement of the form

$$\forall x \text{ in } D, \ \sim Q(x).$$

$$\sim (\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \ \sim Q(x)$$

Example. Negate the following statements:

$\forall$ primes p, p is odd

$\exists$ a triangle T such that the sum of the angles of T equals $200°$

Example. Rewrite the following statements formally, then write the formal and informal negations.

No politicians are honest

The number $1,357$ is not divisible by any integer between 1 and 37.

Example. Write informal negations for the following statements:

All computer programs are finite.

Some computer hackers are over 40.

Negation of a Universal Conditional Statement

$$\sim (\forall x, \text{ if } P(x) \text{ then } Q(x)) \equiv \exists x \text{ such that } P(x) \text{ and } \sim Q(x)$$

Definition.
A statement of the form

$$\forall x \text{ in } D, \text{ if } P(x) \text{ then } Q(x)$$

is called **vacuously true** or **true by default** if, and only if, $P(x)$ is false for every x in D.

Example. The following statement is vacuously true since it's negation is false:

All kangaroos enrolled in my class are passing.

Definition.

Consider a statement of the form $\forall x \in D$, if $P(x)$ then $Q(x)$.

1. Its **contrapositive** is the statement $\qquad$ $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.

2. Its **converse** is the statement $\qquad$ $\forall x \in D$, if $Q(x)$ then $P(x)$.

3. Its **inverse** is the statement $\qquad$ $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

Example. Write a formal and informal contrapositive, converse, and inverse for the following statement:

If a real number is greater than 2, then its square is greater than 4.

Definition.

- "$\forall x, r(x)$ is a **sufficient condition** for $s(x)$" $\rightarrow$ "$\forall x$, if $r(x)$ then $s(x)$"

- "$\forall x, r(x)$ is a **necessary condition** for $s(x)$" $\rightarrow$ "$\forall x$, if $\sim r(x)$ then $\sim s(x)$"
  $\rightarrow$ "$\forall x$, if $s(x)$ then $r(x)$"

- "$\forall x, r(x)$ **only if** $s(x)$" $\rightarrow$ "$\forall x$, if $\sim s(x)$, then $\sim r(x)$"
  $\rightarrow$ "$\forall x$, if $r(x)$ then $s(x)$"

Example. Rewrite each of the following as a universal conditional statement, quantified either explicitly or implicitly. Do not use the word *necessary* or *sufficient*.

Squareness is a sufficient condition for rectangularity.

Being at least 35 years old is a necessary condition for being president of the United States.

Example. Rewrite the following as a universal conditional statement:

A product of two numbers is 0 only if one of the numbers is 0.

6.1: Set Theory: Definitions and the Element Method of Proof

Element Argument: The Basic Method for Proving that One set is a Subset of Another

Let sets X and Y be given. To prove that $X \subseteq Y$,

1. **suppose** that x is a particular but arbitrarily chosen element of X,

2. **show** that x is an element of Y

Example. Define sets A and B as follows:

$$A = \{m \in \mathbb{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbb{Z}\}$$
$$B = \{n \in \mathbb{Z} \mid n = 3s \text{ for some } s \in \mathbb{Z}\}$$

Outline a proof that $A \subseteq B$

Prove that $A \subseteq B$

Disprove that $B \subseteq A$

Definition.
Given sets A and B, A **equals** B, written $\mathbf{A} = \mathbf{B}$, if, and only if, every element of A is in B and every element of B is in A:

$$A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A.$$

Example. Define sets A and B as follows:

$$A = \{m \in \mathbb{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbb{Z}\}$$
$$B = \{n \in \mathbb{Z} \mid n = 3s \text{ for some } s \in \mathbb{Z}\}$$

Is $A = B$?

Definition.
Given an integer n and a positive integer d, when n is divided by d, then

$$n \operatorname{div} d = \text{ the integer quotient}$$
$$n \operatorname{mod} d = \text{ the nonnegative integer remainder}$$

If n and d are integers and $d > 0$, then

$$n \operatorname{div} d = q \quad \text{and} \quad n \operatorname{mod} d = r \quad \Leftrightarrow \quad n = dq + r$$

Example. Compute the following:

$$32 \operatorname{div} 9, \quad 32 \operatorname{mod} 9 \qquad\qquad\qquad 365 \operatorname{div} 7, \quad 365 \operatorname{mod} 7$$

Example. If it is currently 11:00 am, what time will it be in

37 hours? 51 hours?

11 hours? -1 hours?

Example. Let $A = \left\{ 4, \sqrt{16}, 19 \operatorname{mod} 15 \right\}$ and $B = \{12 \operatorname{mod} 8\}$. Is $A \subseteq B$? Is $B \subseteq A$?

Definition.
Let A and B be subsets of a universal set U.

1. The **union** of A and B is the set of all elements that are in at least one of A or B.
$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$$

2. The **intersection** of A and B is the set of all elements that are common to both A and B.
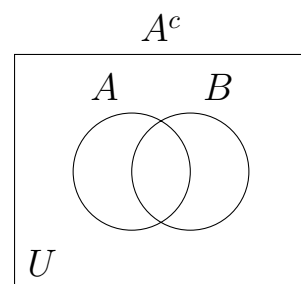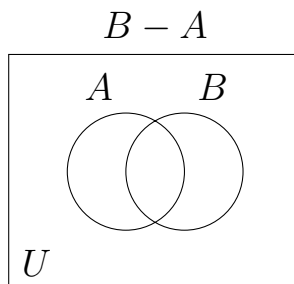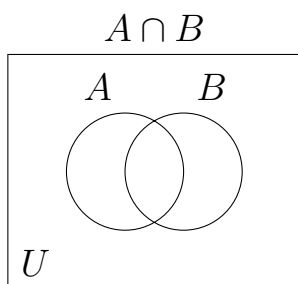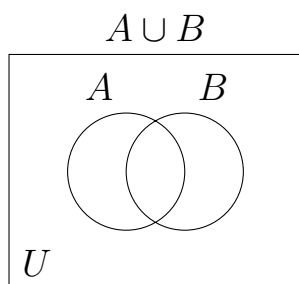$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$

3. The **difference** of A and B is the set of all elements that are in B and not A.
$$B - A = \{x \in U \mid x \in B \text{ and } x \notin A\}$$

4. The **complement** of A is the set of all elements in U that are not in A.
$$A^c = \{x \in U \mid x \notin A\}$$

Example. Represent the following sets using the Venn diagrams below:

$A \cup B$ $A \cap B$ $B - A$ A^c

Example. Let the universal set be the set $U = \{a, b, c, d, e, f, g\}$, and let $A = \{a, c, e, g\}$ and $B = \{d, e, f, g\}$. Find

$A \cup B$

$A \cap B$

$B - A$

A^c

Definition.

Given real numbers a and b with $a \leq b$:

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\} \qquad (a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$
$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\} \qquad [a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

Example. Let the universal set be $\mathbb{R}$, and let $A = (-1, 0]$ and $B = [0, 1)$. Find

$A \cup B$

$A \cap B$

$B - A$

A^c

Definition.
Given sets $A_0, A_1, A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n,

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i, \text{ for at least one } i = 0, 1, 2, \ldots, n\}$$

$$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i, \text{ for every } i = 0, 1, 2, \ldots, n\}$$

Example. For each positive integer i, let $A_i = \left\{ x \in \mathbb{R} \;\middle|\; -\frac{1}{i} < x < \frac{1}{i} \right\} = \left(-\frac{1}{i}, \frac{1}{i} \right)$.
Find

$A_1 \cup A_2 \cup A_3$ $\qquad\qquad\qquad$ $A_1 \cap A_2 \cap A_3$

$\displaystyle\bigcup_{i=1}^{\infty} A_i$ $\qquad\qquad\qquad$ $\displaystyle\bigcap_{i=1}^{\infty} A_i$

Definition.
The **empty set** (or **null set**), denoted $\varnothing$, is the set with no elements.

$$\{1,3\} \cap \{2,4\} = \varnothing$$

Two sets are called **disjoint** if, and only if, they have no elements in common:

$$A \cap B = \varnothing.$$

Sets $A_1, A_2, A_3, \ldots$ are **mutually disjoint** (or **pairwise disjoint**) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common:

$$A_i \cap A_j = \varnothing \text{ whenever } i \neq j.$$

Example.
Let $A_1 = \{3, 5\}$, $A_2 = \{1, 4, 6\}$, and $A_3 = \{2\}$. Are A_1, A_2, and A_3 mutually disjoint?

Let $B_1 = \{2, 4, 6\}$, $B_2 = \{3, 7\}$, and $B_3 = \{4, 5\}$. Are B_1, B_2, B_3 mutually disjoint?

> **Definition.**
> A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, \ldots\}$ is a **partition** of a set A if, and only if,
>
> 1. A is the union of all the A_i;
>
> 2. the sets $A_1, A_2, A_3, \ldots$ are mutually disjoint.

Example.
Let $A = \{1, 2, 3, 4, 5, 6\}$, $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$, and $A_3 = \{5, 6\}$. Is $\{A_1, A_2, A_3\}$ a partition of A?

Let $\mathbb{Z}$ be the set of all integers and let

$$T_i = \{n \in \mathbb{Z} \mid n = 3k + i, \text{ for some integer } k\}.$$

Is $\{T_0, T_1, T_3\}$ a partition of $\mathbb{Z}$?

Definition.
Given a set A, the **power set** of A, denoted $\mathscr{P}(A)$, is the set of all subsets of A.

Example. Find $\mathscr{P}(\{x, y\})$.