

## 1.5: Solutions of Systems of Linear Equations

$$\text{🍏} + \text{🍏} + \text{🍏} = 18$$

$$\text{🍏} + \text{🍌} + \text{🍌} = 14$$

$$\text{🍌} - \text{🍒} = 2$$

$$\text{🍒} + \text{🍏} + \text{🍌} = ?$$

### Definition.

A **system of equations** is 2 (or more) equations. The ordered pairs  $(x, y)$  that satisfies *all* equations in the system are the **solutions** of the system.

When solving a system of linear equations, there are three possible outcomes:

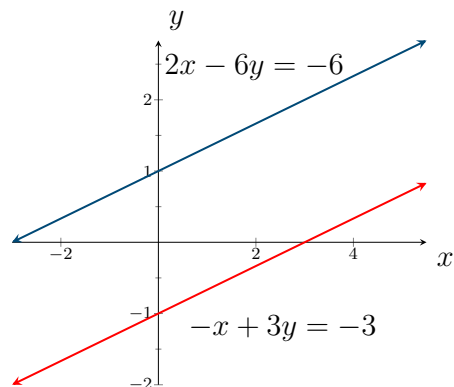
1. No solution (*Inconsistent*),
2. Exactly one solution,
3. Infinitely many solutions (*Dependent*).

**Example.** Use graphing to find the solutions to the following systems

$$2x - 6y = -6$$

$$-x + 3y = -3$$

*No solution*



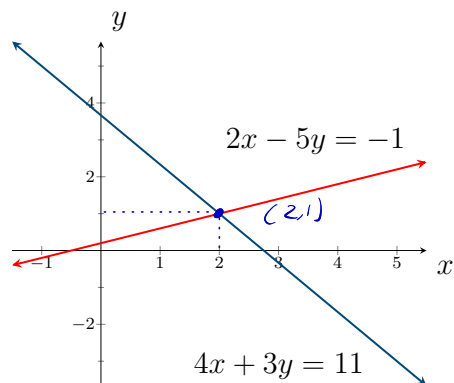
$$4x + 3y = 11$$

$$2x - 5y = -1$$

$$(x, y) = (2, 1)$$

$$4(2) + 3(1) = 8 + 3 = 11 \quad \checkmark$$

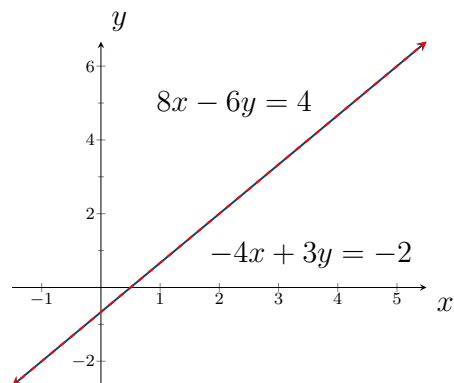
$$2(2) - 5(1) = 4 - 5 = -1 \quad \checkmark$$



$$-4x + 3y = -2$$

$$8x - 6y = 4$$

*Infinitely many solutions*



**Equivalent systems** result when

1. One expression is replaced by an equivalent expression.
2. Two equations are interchanged.
3. A multiple of one equation is added to another equation.
4. An equation is multiplied by a nonzero constant.

### Substitution Method

**Example.** Solve the system  $\begin{cases} 2x + 3y = 4 \\ x - 2y = 3 \end{cases}$

1. Solve one equation for either one of the variables in terms of the other.

$$x = 2y + 3$$

2. Substitute this expression into the other equation to give the equation in one unknown.

$$2(\textcolor{red}{2}y + \textcolor{red}{3}) + 3y = 4$$

3. Solve this equation for the unknown.

$$4y + 6 + 3y = 4$$

$$7y = -2 \Rightarrow y = -\frac{2}{7}$$

4. Substitute solution into the equation in Step 1.

$$x = 2\left(-\frac{\textcolor{red}{2}}{\textcolor{red}{7}}\right) + 3 \Rightarrow x = \frac{17}{7}$$

5. Check the solution  $(x, y)$ .

$$2\left(\frac{\textcolor{red}{17}}{\textcolor{red}{7}}\right) + 3\left(-\frac{\textcolor{red}{2}}{\textcolor{red}{7}}\right) = 4$$

$$\left(\frac{\textcolor{red}{17}}{\textcolor{red}{7}}\right) - 2\left(-\frac{\textcolor{red}{2}}{\textcolor{red}{7}}\right) = 3$$

**Example.** Use the substitution method to solve the system

$$4x + 5y = 18 \quad (1)$$

$$3x - 9y = -12 \quad (2)$$

Pick one equation, and solve for one of the variables:

$$(2) \rightarrow +9y + 3x - 9y = -12 + 9y$$

$$\frac{3x}{3} = \frac{9y - 12}{3}$$

$$\boxed{x = 3y - 4} \quad (*)$$

Plug this expression back into the other equation and solve:

$$(1) \rightarrow 4(3y - 4) + 5y = 18$$

$$+16 + 12y - 16 + 5y = 18 + 16$$

$$\frac{17y}{17} = \frac{34}{17}$$

$$\boxed{y = 2}$$

Use the value found to solve for the remaining variable:

-Using expression found:

$$(*) \rightarrow x = 3(2) - 4$$

$$= 6 - 4$$

$$\boxed{x = 2}$$

-Using original equation:

$$(2) \rightarrow 3x - 9(2) = -12$$

$$+18 + 3x - 18 = -12 + 18$$

$$\frac{3x}{3} = \frac{6}{3}$$

$$\boxed{x = 2}$$

Solution:

$$(x, y) = (2, 2)$$

Verification omitted

## Elimination Method

**Example.** Solve the system  $\begin{cases} 2x - 5y = 4 \\ x + 2y = 3 \end{cases}$

1. Multiply one or both equations by a nonzero number so the coefficients of one of the variables may cancel.

$$\Rightarrow \begin{cases} 2x - 5y = 4 \\ -2x - 4y = -6 \end{cases}$$

2. Add or subtract the equations to eliminate one of the variables.

$$0x - 9y = -2$$

3. Solve for the remaining variable.

$$\Rightarrow y = \frac{2}{9}$$

4. Substitute solution in one of the original equations and solve for the other variable.

$$2x - 5\left(\frac{2}{9}\right) = 4 \Rightarrow x = \frac{23}{9}$$

5. Check the solution  $(x, y)$

$$\begin{aligned} 2\left(\frac{23}{9}\right) - 5\left(\frac{2}{9}\right) &= 4 \\ \left(\frac{23}{9}\right) + 2\left(\frac{2}{9}\right) &= 3 \end{aligned}$$

**Example.** Use the elimination method to solve the following systems:

$$\begin{array}{l}
 2x - 6y = -6 \\
 2(-x + 3y) = (-3) \cdot 2
 \end{array}
 \left. \vphantom{\begin{array}{l} 2x - 6y = -6 \\ 2(-x + 3y) = (-3) \cdot 2 \end{array}} \right\}
 \begin{array}{r}
 2x - 6y = -6 \\
 -2x + 6y = -6 \\
 \hline
 0x + 0y = -12 \\
 0 = -12 \longrightarrow
 \end{array}$$

This can't happen.  
This system is inconsistent  
There is no solution

$$\begin{array}{l}
 (5)(4x + 3y) = (11)(5) \\
 (3)(2x - 5y) = (-1)(3)
 \end{array}
 \left. \vphantom{\begin{array}{l} (5)(4x + 3y) = (11)(5) \\ (3)(2x - 5y) = (-1)(3) \end{array}} \right\}
 \begin{array}{r}
 20x + 15y = 55 \\
 6x - 15y = -3 \\
 \hline
 26x + 0y = 52 \\
 \hline
 26
 \end{array}$$

$$x = 2$$

$$\Rightarrow (x, y) = (2, 1)$$

Use either equation to solve for y:

$$\begin{array}{l}
 4(2) + 3y = 11 \\
 -8 + 8 + 3y = 11 - 8 \\
 3y = 3 \\
 \frac{3y}{3} = \frac{3}{3} \\
 y = 1
 \end{array}$$

Verification omitted

$$\begin{array}{l}
 2(-4x + 3y) = (-2) \cdot 2 \\
 8x - 6y = 4
 \end{array}
 \left. \vphantom{\begin{array}{l} 2(-4x + 3y) = (-2) \cdot 2 \\ 8x - 6y = 4 \end{array}} \right\}
 \begin{array}{r}
 -8x + 6y = -4 \\
 8x - 6y = 4 \\
 \hline
 0x + 0y = 0
 \end{array}
 \longrightarrow$$

Any (x,y) pair will work.  
This solution has infinitely many solutions.

**Example.** A woman has \$500,000 invested in two rental properties. One yields an annual return of 10% on her investment, and the other returns 12% per year on her investment. Her total annual return from the two investments is \$53,000. Let  $x$  represent the amount of the 10% investment and  $y$  represent the amount of the 12% investment.

- Write an equation that states that the sum of investments is \$500,000.

$$x + y = 500,000$$

- What is the annual return on the 10% investment? What about the 12% investment?

$$0.10x$$

$$0.12y$$

- Write an equation that states the sum of the annual return is \$53,000.

$$0.10x + 0.12y = 53,000$$

- Solve these two equations simultaneously to find how much is invested in each property.

$$\begin{array}{r} x + y = 500,000 \\ 0.10x + 0.12y = 53,000 \end{array} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\}$$

$$\begin{array}{r} x + y = 500,000 \\ - (x + 0.12y = 53,000) \\ \hline -0.2y = -30,000 \\ \hline -0.2 \quad -0.2 \end{array}$$

Solve for  $x$  using the 1st equation:

$$x + 150,000 = 500,000$$

$$x = 350,000$$

$$y = 150,000$$

**Example.** A nurse has two solutions that contain different concentrations of a certain medication. One is a 12.5% concentration, and the other is a 5% concentration. How many cubic centimeters of each should she mix to obtain 20 cubic centimeters of an 8% concentration?

Let  $x$  and  $y$  represent the amount of each solution that we use.  
 Since we wish to obtain 20cc's in total, we have:

$$x + y = 20$$

Additionally, since we want the 20cc's to have an 8% concentration, we have:

$$0.125x + 0.05y = 0.08(20)$$

$$\Rightarrow \begin{array}{r} -0.05(x + y) = -0.05(20) \\ 0.125x + 0.05y = 0.08(20) \end{array}$$

$$\begin{array}{r} 0.075x \\ \hline 0.075 \end{array} \qquad \begin{array}{r} = 0.03(20) \\ \hline 0.075 \end{array}$$

$$\boxed{x = 8 \text{ cc}}$$

$$x + y = 20 \Rightarrow 8 + y = 20$$

$$\boxed{y = 12 \text{ cc}}$$

$$\Rightarrow \boxed{(x, y) = (8, 12)}$$

**Example.** Using U.S. Bureau of Labor Statistics data for selected years from 1950 and projected to 2050, the number of men  $M$  and women  $W$  in the workforce (both in millions) can be modeled by the functions

$$M(t) = 0.591t + 37.3 \quad \text{and} \quad W(t) = 0.786t + 13.1$$

where  $t$  is the number of years after 1940. Find the year these functions predict that there will be equal numbers of men and women in the U.S. workforce.

Since these functions will have the same value, we set them equal to each other:

$$\begin{aligned} M(t) &= W(t) \\ -0.786t + 0.591t + 37.3 &= 0.786t + 13.1 \quad -0.786t \\ -37.3 - 0.195t + 37.3 &= 13.1 \quad -37.3 \\ -0.195t &= -24.2 \\ \frac{-0.195t}{-0.195} &= \frac{-24.2}{-0.195} \\ t &\approx 124.10256 \end{aligned}$$

Since this is years after 1940, we round up and get:

$$1940 + 124.10256 = 2064.10256$$

$$\boxed{\text{year} = 2065}$$

This just after the end of 2064.  
This is part-way through Feb. 2065.

