2.4: Limits

Example. Suppose that the position function of a maglev train (in feet) is given by

$$s(t) = 4t^2, \qquad (0 \le t \le 30)$$

Using the position function, compute the *average* velocity of the train

$$g(t) = \frac{s(t) - s(2)}{t-2} = \frac{4t^2 - 16}{t-2} = \frac{4(t^2-4)}{t-2} = \frac{4(t-2)(t+2)}{t-2}$$

$$= 4(t+2)$$

on the interval $[t, 2]$

t	1.5	1.9	1.99	1.999	1.9999
	14	15.6	15.96	15.996	15.9996

on the interval $[2, t]$

t	2.5	2.1	2.01	2.001	2.0001
	18	16.4	16.04	16.004	16.0004

What do the tables above suggest about *instantaneous* velocity of the train at $t = 2$?

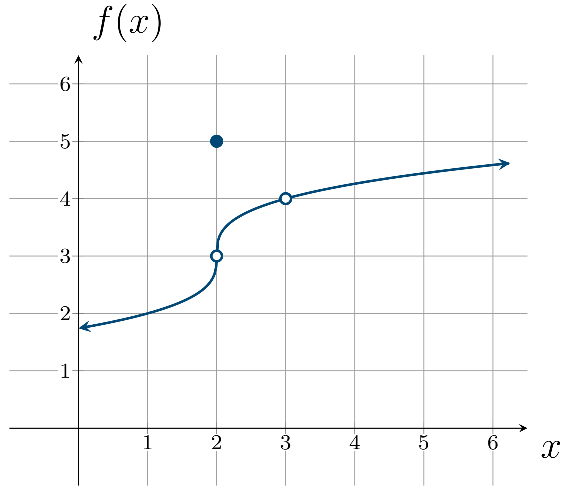
The train is going 16 ft/s a $t=2$ secs.

Definition. (Limit of a Function)
The function f has the **limit** L as x approaches a, written

$$\lim_{x\to a} f(x) = L$$

if the value of $f(x)$ can be made as close to the number L as we please by taking x sufficiently close to (but not equal to) a.

Example. Using the graph of f, determine the following values:



$f(1)$ and $\lim_{x\to 1} f(x)$

$f(1) = 2$

$\lim_{x\to 1} f(x) = 2$

$f(2)$ and $\lim_{x\to 2} f(x)$

$f(2) = 5$

$\lim_{x\to 2} f(x) = 3$

$f(3)$ and $\lim_{x\to 3} f(x)$

$f(3)$ DNE

$\lim_{x\to 3} f(x) = 4$

Example. Find the limit of the following functions at the value specified: Graphs

$f(x) = x^3 \quad \text{at } x = 2$

$\lim_{x \to 2} f(x) = 2^3 = 8$

$g(x) = \begin{cases} x+2, & x \neq 1 \\ 1, & x = 1 \end{cases} \quad \text{at } x = 1$

$\lim_{x \to 1} g(x) = 1 + 2 = 3$

$h(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases} \quad \text{at } x = 0$

$\lim_{x \to 0} h(x)$ DNE

$j(x) = \dfrac{1}{(x-1)^2} \quad \text{at } x = 1$

$\lim_{x \to 1} j(x) = \infty$

$k(x) = 4 \quad \text{at } x = 0$

$\lim_{x \to 0} k(x) = 4$

Theorem 1: Properties of Limits
Suppose

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M$$

Then

1. $\lim_{x\to a}[f(x)]^r = \left[\lim_{x\to a} f(x)\right]^r$ where r is a positive constant
2. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$ where c is a real number
3. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$
4. $\lim_{x\to a}[f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right] = LM$
5. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{M}$ provided $M \neq 0$

Example. Use the above theorem to evaluate the following limits:

$$\lim_{x\to 1}\left(5x^{3/2} - 2\right) = \lim_{x\to 1} 5x^{3/2} - \lim_{x\to 1} 2 = 5\lim_{x\to 1} x^{3/2} - 2$$

$$= 5(1) - 2 = \boxed{3}$$

$$\lim_{x\to 3}\frac{2x^3\sqrt{x^2+7}}{x+1} = \frac{\lim_{x\to 3} 2x^3\sqrt{x^2+7}}{\lim_{x\to 3} x+1} = \frac{2(3)^3 \lim_{x\to 3}\sqrt{x^2+7}}{3+1}$$

$$= \frac{54\sqrt{\lim_{x\to 3} x^2 + \lim_{x\to 3} 7}}{4} = \frac{54\sqrt{3^2+7}}{4} = \frac{54\sqrt{16}}{4} = \boxed{54}$$

Suppose that $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

has an **indeterminate form** of $\frac{0}{0}$. To evaluate such a limit, we replace the given function with a function that's equivalent everywhere except at $x = a$, and then evaluate the limit.

Example. Evaluate the following

$$\lim_{t \to 2} \frac{4t^2 - 16}{t - 2} = \lim_{t \to 2} \frac{4(t^2-4)}{t-2} = \lim_{t \to 2} \frac{4(t-2)(t+2)}{t-2}$$

$$= \lim_{t \to 2} 4(t+2)$$

$$= \boxed{16}$$

$$\lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} \left(\frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} \right) = \lim_{h \to 0} \frac{4+h-4}{h(\sqrt{4+h}+2)}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{4+h}+2)} = \lim_{h \to 0} \frac{1}{\sqrt{4+h}+2} = \boxed{\frac{1}{4}}$$

Suppose that $\lim_{x\to a} f(x) = L$ with $L \neq 0$ and $\lim_{x\to a} g(x) = 0$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

does not exist. We can further specify if this limit tends towards $-\infty$ or ∞.

Example. Evaluate the following Graphs

$\left(\frac{1}{0}\right)$ $\lim_{x\to 1} \frac{x}{x-1}$ DNE

x	$\frac{x}{x-1}$
0.75	-3
0.9	-9
0.99	-99
0.999	-999

x	$\frac{x}{x-1}$
1.25	5
1.1	11
1.01	101
1.001	1001

$\left(\frac{1}{0}\right)$ $\lim_{x\to 3} \frac{1}{(x-3)^2} = \infty$

x	$\frac{1}{(x-3)^2}$
2.9	100
2.99	10,000
2.999	1,000,000

x	$\frac{1}{(x-3)^2}$
3.1	100
3.01	10,000
3.001	1,000,000

$\left(\frac{1}{0}\right)$ $\lim_{x\to -2} \frac{x-2}{x^2-4}$ DNE

x	$\frac{x-2}{x^2-4}$
-2.25	-4
-2.1	-10
-2.01	-100
-2.001	-1000

x	$\frac{x-2}{x^2-4}$
-1.75	4
-1.9	10
-1.99	100
-1.999	1000

$\left(\frac{0}{0}\right)$ $\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)}$

$= \lim_{x\to 2} \frac{1}{x+2}$

$= \frac{1}{2+2}$

$= \boxed{\frac{1}{4}}$

> **Limit of a Function at Infinity**
> The function f has the limit L as x increases without bound, written
> $$\lim_{x\to\infty} f(x) = L$$
> if $f(x)$ can be made arbitrarily close to L by taking x large enough.
> The function f has the limit M as x decreases without bound, written
> $$\lim_{x\to-\infty} f(x) = M$$
> if $f(x)$ can be made arbitrarily close to M by taking x to be negative and sufficiently large enough in absolute value.
> When the above limits exist, the equations $y = L$ and/or $y = M$ are called **horizontal asymptotes**.

Example. Evaluate the following infinite limits

$$\lim_{x\to\infty} \frac{2x^2+3x-4}{x^2-7x+1} \left(\frac{1/x^2}{1/x^2}\right) = \lim_{x\to\infty} \frac{\frac{2x^2}{x^2}+\frac{3x}{x^2}-\frac{4}{x^2}}{\frac{x^2}{x^2}-\frac{7x}{x^2}+\frac{1}{x^2}} = \lim_{x\to\infty} \frac{2+\underbrace{\frac{3}{x}}_{0}-\underbrace{\frac{4}{x^2}}_{0}}{1-\underbrace{\frac{7}{x}}_{0}+\underbrace{\frac{1}{x^2}}_{0}} = \boxed{2}$$

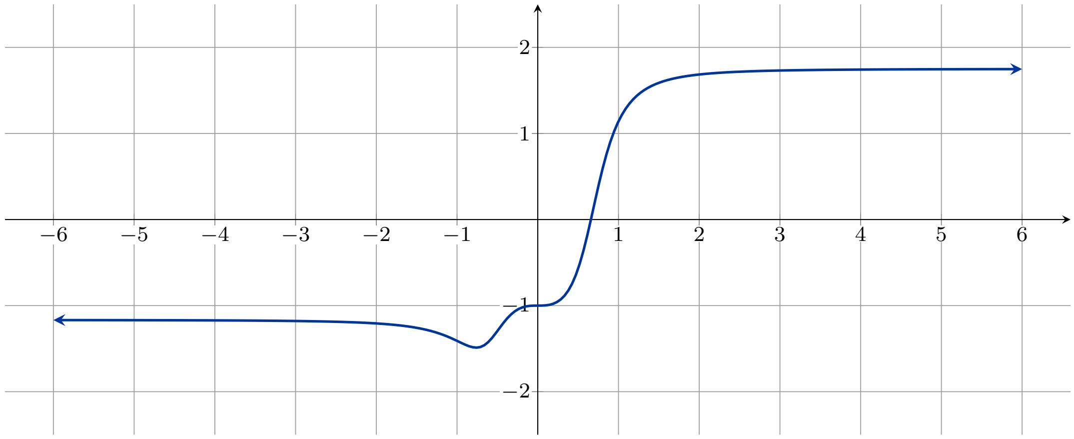
$$\lim_{x\to-\infty} \frac{3x^2+4}{2x^3} \left(\frac{1/x^3}{1/x^3}\right) = \lim_{x\to\infty} \frac{\frac{3x^2}{x^3}+\frac{4}{x^3}}{\frac{2x^3}{x^3}} = \lim_{x\to\infty} \frac{\frac{3}{x}+\underbrace{\frac{4}{x^3}}_{0}}{2} = \frac{0}{2} = \boxed{0}$$

$$\lim_{x\to\pm\infty} \frac{3x^5+2x^3-4}{x^4+4x^2-1} \left(\frac{1/x^4}{1/x^4}\right) = \lim_{x\to\infty} \frac{\frac{3x^5}{x^4}+\frac{2x^3}{x^4}-\frac{4}{x^4}}{\frac{x^4}{x^4}+\frac{4x^2}{x^4}-\frac{1}{x^4}} = \lim_{x\to\infty} \frac{3x+\underbrace{\frac{2}{x}}_{0}-\underbrace{\frac{4}{x^4}}_{0}}{1+\underbrace{\frac{4}{x^2}}_{0}-\underbrace{\frac{1}{x^4}}_{0}}$$

$$= \lim_{x\to\infty} 3x = \boxed{\infty}$$

$$\sqrt{x^6} = \begin{cases} x^3, & x \to \infty \\ -x^3, & x \to -\infty \end{cases}$$

Graph



$$\lim_{x\to-\infty} \frac{7x^3-2}{-x^3+\sqrt{25x^6-4}} \left(\frac{1/x^3}{1/x^3}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{7x^3}{x^3} - \frac{2}{x^3}}{-\frac{x^3}{x^3} + \frac{1}{x^3}\sqrt{25x^6-4}}$$

$$= \lim_{x\to\infty} \frac{7 - \underbrace{\frac{2}{x^3}}_{\to 0}}{-1 - \sqrt{\frac{1}{x^6}}\sqrt{25x^6-4}}$$

$$= \lim_{x\to\infty} \frac{7}{-1 - \sqrt{\frac{25x^6}{x^6} - \underbrace{\frac{4}{x^6}}_{\to 0}}}$$

$$= \lim_{x\to\infty} \frac{7}{-1 - \sqrt{25}} = \boxed{-\frac{7}{6}}$$

$$\lim_{x\to\infty} \frac{7x^3-2}{-x^3+\sqrt{25x^6-4}} \left(\frac{1/x^3}{1/x^3}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{7x^3}{x^3} - \frac{2}{x^3}}{-\frac{x^3}{x^3} + \frac{1}{x^3}\sqrt{25x^6-4}}$$

$$= \lim_{x\to\infty} \frac{7 - \underbrace{\frac{2}{x^3}}_{\to 0}}{-1 + \sqrt{\frac{1}{x^6}}\sqrt{25x^6-4}}$$

$$= \lim_{x\to\infty} \frac{7}{-1 + \sqrt{\frac{25x^6}{x^6} - \underbrace{\frac{4}{x^6}}_{\to 0}}}$$

$$= \lim_{x\to\infty} \frac{7}{-1 + \sqrt{25}} = \boxed{\frac{7}{4}}$$

Example. The company *Custom Office* makes a line of executive desks. It is estimated that the total cost of making x *Senior Executive Model* desks is

$$C(x) = 100x + 200,000$$

dollars per year. The average cost of making x desks is given by

$$\overline{C}(x) = \frac{C(x)}{x}$$

Compute $\lim_{x\to\infty} \overline{C}(x)$ and interpret the result.

$$\lim_{x\to\infty} \frac{100x+200,000}{x} = \lim_{x\to\infty} 100 + \underbrace{\frac{200,000}{x}}_{0} = 100$$

As the number of units, x, increases, the average cost per unit approaches (decreases) to $100 per unit

Theorem 2
For all $n > 0$,

$$\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$$

provided that $\frac{1}{x^n}$ is defined.