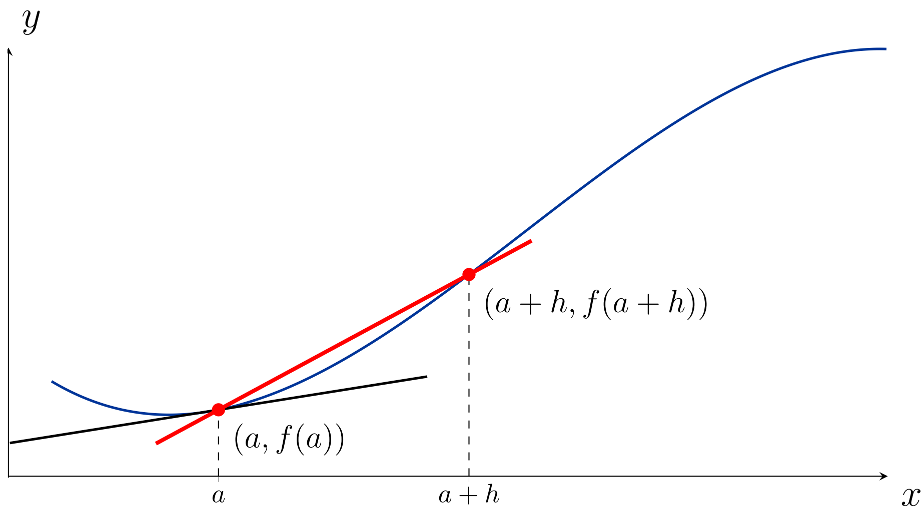
2.6: The Derivative

Definition.
Given a function $f(x)$:

- the **secant line** is the line that passes through two *distinct* points lying on the graph of $f(x)$,
- the **tangent line** is the line that intersects $f(x)$ in exactly one place (locally) and matches the slope of the graph at that point.

Graph



Definition. (Slope of a Tangent Line)
The slope of the tangent line to the graph of f at the point $P(x, f(x))$ is given by

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if it exists.

Definition. (Average and Instantaneous Rates of Change)
The **average rate of change** of f over the interval $[x, x+h]$ or **slope of the secant line** to the graph of f through the points $(x, f(x))$ and $(x+h, f(x+h))$ is

$$\frac{f(x+h)-f(x)}{h}$$

The above fraction is referred to as the **difference quotient**.

The **instantaneous rate of change** of f at x or **slope of the tangent line** to the graph of f at $(x, f(x))$ is

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Definition. (Derivative of a Function)
The derivative of a function f with respect to x is the function f' (read "f prime"),

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

The domain of f' is the set of all x for which the limit exists.

Some other notations for the derivative are

$$D_x f(x) \qquad \frac{dy}{dx} \qquad y'$$

Example. Find the slope of the line tangent to the graph $f(x) = 3x + 5$ at any point $(x, f(x))$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[3(x+h)+5]-[3x+5]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3x}+3h\cancel{+5}-\cancel{3x}\cancel{-5}}{h}$$

$$= \lim_{h\to 0} \frac{3h}{h}$$

$$= \lim_{h\to 0} 3$$

$$= \boxed{3}$$

Example. Let $f(x) = x^2$.

- Find $f'(x)$.
- Compute $f'(2)$ and interpret your result.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^2]-[x^2]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{h(2x+h)}{h}$$

$$= \lim_{h\to 0} 2x+h$$

$$= \boxed{2x} \quad \Rightarrow \quad f'(2) = 2(2)$$

The slope of the tangent line at x=2 is 2

Example. Let $f(x) = x^2 - 4x$. Find the point on the graph where the tangent line is horizontal.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^2 - 4(x+h)] - [x^2-4x]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{-4x} + 4h \cancel{-x^2} \cancel{-4x}}{h}$$

$$= \lim_{h\to 0} \frac{h(2x+h+4)}{h}$$

$$= \lim_{h\to 0} 2x+h+4$$

$$= \boxed{2x+4}$$

Solve $f'(x) = 0$

$$2x+4=0$$
$$2x=-4$$
$$\boxed{x=-2}$$

Example. Let $f(x) = \dfrac{1}{x}$. Find the equation of the tangent line at $x = 2$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h}\left(\frac{x}{x}\right) - \frac{1}{x}\left(\frac{x+h}{x+h}\right)}{h}$$

$$= \lim_{h\to 0} \frac{x-(x+h)}{h(x+h)x}$$

$$= \lim_{h\to 0} \frac{-h}{h(x+h)x}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h)x}$$

$$= \frac{-1}{(x+0)x}$$

$$= \boxed{-\frac{1}{x^2}}$$

$(2, f(2)) = (2, \frac{1}{2})$

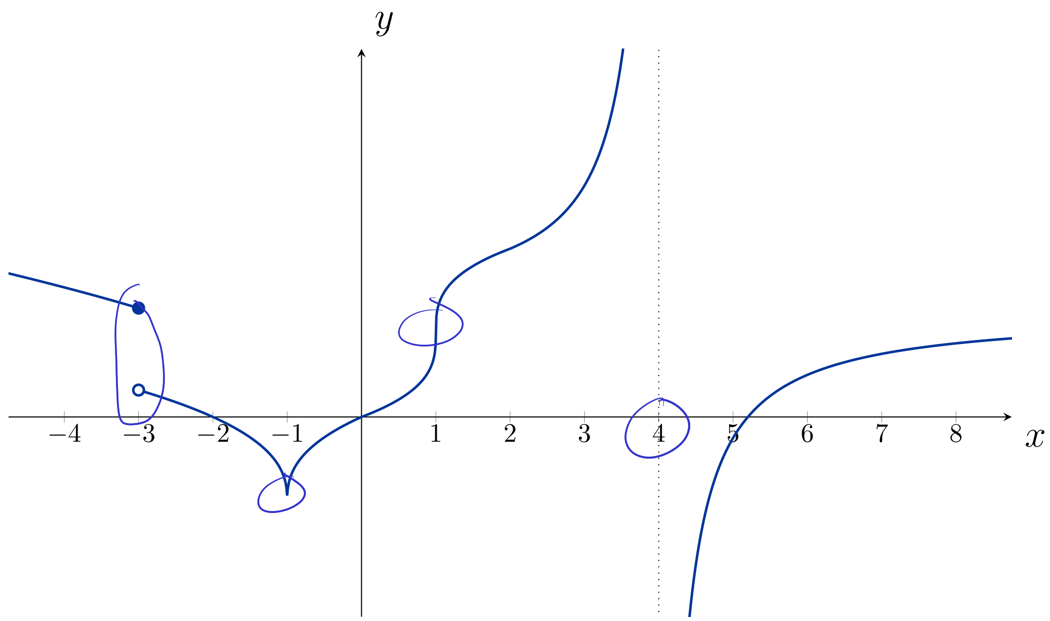
$m = f'(2) = -\frac{1}{2^2} = -\frac{1}{4}$

$\Rightarrow y - \frac{1}{2} = -\frac{1}{4}(x-2)$

$$\boxed{y = -\frac{1}{4}x + 1}$$

Differentiability and Continuity
If a function is differentiable at $x = a$, then it is continuous at $x = a$.

Example. For the graph below, identify each point where the derivative is undefined.



The derivative doesn't exist at....

x=-3: because f(x) is not continuous
x=-1: because f(x) has a sharp corner
x= 1: because f(x) has a vertical tangent line
x= 4: because f(x) is not continuous