8.3: Equivalence Relations

Definition.
Given a partition of a set A, the **relation induced by the partition**, R, is defined on A as follows:

$$\forall x, y \in A,\ x\,R\,y \ \Leftrightarrow\ \begin{array}{l}\text{there is a subset } A_i \text{ of the partition} \\ \text{such that both } x \text{ and } y \text{ are in } A_i.\end{array}$$

Example. Let $A = \{0, 1, 2, 3, 4\}$ and consider the following partition of A:

$$\{0, 3, 4\}, \{1\}, \{2\}.$$

Find the relation R induced by this partition

Let A be a set with a partition and let R be the relation induced by the partition. Then R is reflexive, symmetric, and transitive.

Definition.
Let A be a set and R a relation on A, R is an **equivalence relation** if, and only if, R is reflexive, symmetric, and transitive.

Example. Let X be the set of all nonempty subsets of $\{1, 2, 3\}$. Then

$$X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Define a relation $\mathbf{R}$ on X as follows:

$$\forall A, B \in X,\ A \,\mathbf{R}\, B \Leftrightarrow \text{ the least element of } A \text{ equals the least element of } B.$$

Prove that $\mathbf{R}$ is an equivalence relation on X.

Definition.
Suppose A is a set and R is an equivalence relation on A. For each element a in A, the **equivalence class of** a, denoted $[a]$ and called the **class of** a, is the set of all elements x in A such that x is related to a by R:

$$[a] = \{x \in A | x \, R \, a\}.$$

Example. Let $A = \{0, 1, 2, 3, 4\}$ and define a relation R on A as follows:

$$R = \{\{0,0\}, \{0,4\}, \{1,1\}, \{1,3\}, \{2,2\}, \{3,1\}, \{3,3\}, \{4,0\}, \{4,4\}\}.$$

Find the *distinct* equivalence classes of R.

Example. Recall the equivalence relation $\mathbf{R}$ defined on the nonempty subsets of $\{1, 2, 3\}$, X:

$$X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

List the equivalence classes of $\mathbf{R}$.

Example. Consider the equivalence relation R on a set A:

$$\forall x, y \in A \; x \, R \, y \Leftrightarrow x = y.$$

Describe the distinct equivalence classes of R.

Suppose A is a set, R is an equivalence relation on A and a and b are elements of A.

- If $a \, R \, b$, then $[a] = [b]$, and
- either $[a] \cap [b] = \varnothing$ or $[a] = [b]$.

Example. Let R be the relation of congruence modulo 3 on the set $\mathbb{Z}$:

$$m \, R \, n \Leftrightarrow 3 \mid (m - n).$$

Describe the distinct equivalence classes of R.

Definition.
Let $m, n \in \mathbb{Z}$ and let $d \in \mathbb{Z}^+$. We say m **is congruent to** n **modulo** d

$$m \equiv n \mod d$$

if, and only if,

$$d \mid (m - n).$$