5.5: Differentiation of Logarithmic Functions

Definition. (Natural log)
The inverse of the exponential function is the logarithm:

$$e^{y} = x \longleftrightarrow \ln(x) = y, \qquad x > 0$$

We may also denote this as $\log_e(x)$

Example. Solve the following exponential equations [Graphs]

$y = e^{-3}$

$\boxed{y \approx 0.0498}$

$e^{x^2 - x} = e^2$

Same base $\Rightarrow$ $x^2 - x = 2$

$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$\boxed{x = -1}$ $\boxed{x = 2}$

$e^x = 10$

$\ln(10) = \boxed{x \approx 2.303}$

$4e^{1-x^2} = 6$

$e^{1-x^2} = \frac{3}{2}$

$\ln\left(\frac{3}{2}\right) = 1 - x^2$

$x^2 = 1 - \ln\left(\frac{3}{2}\right)$

$x = \pm\sqrt{1 - \ln\left(\frac{3}{2}\right)} \approx \boxed{\pm 0.771}$

Rule 3: Derivative of $\ln(x)$

$$\frac{d}{dx}\Big[\ln|x|\Big] = \frac{1}{x}, \qquad x \neq 0$$

Rule 4: Derivative of $\ln(f(x))$

$$\frac{d}{dx}\Big[\ln(f(x))\Big] = \frac{f'(x)}{f(x)}$$

Example. Find the derivative of the following functions

$y = \ln(\sqrt[3]{x}) = \ln(x^{1/3})$

$y' = \frac{1}{x^{1/3}} \cdot \frac{1}{3}x^{-2/3} = \boxed{\frac{1}{3x}}$

$f(x) = \sqrt{x}\ln(x) = x^{1/2}\ln(x)$

$f'(x) = \frac{1}{2}x^{-1/2}\ln(x) + x^{1/2}\frac{1}{x}$

$= \boxed{\frac{\ln(x)}{2\sqrt{x}} + \sqrt{x}}$

$g(x) = \ln\left(x^2+1\right)$

$g'(x) = \boxed{\frac{2x}{x^2+1}}$

$h(t) = \dfrac{1+\ln(t)}{1-\ln(t)}$

$h'(t) = \dfrac{(1-\ln(t))\frac{1}{t} - (1+\ln(t))(-\frac{1}{t})}{(1-\ln(t))^2}$

$= \boxed{\frac{2}{t(1-\ln(t))^2}}$

Additional properties of logarithms

$$e^{y} = x \qquad \ln(x) = y$$

$$e^{1} = e \qquad \ln(e) = 1$$

$$e^{0} = 1 \qquad \ln(1) = 0$$

$$e^{x}e^{y} = e^{x+y} \qquad \ln(xy) = \ln(x) + \ln(y)$$

$$\frac{e^{x}}{e^{y}} = e^{x-y} \qquad \ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$

$$e^{xy} = (e^{x})^{y} \qquad \ln(x^{y}) = y\ln(x)$$

$$e^{\ln(x)} = x \qquad \ln(e^{x}) = x$$

Definition. (Other logarithms)
Logarithms are defined for any base $a > 0$:

$$a^{y} = x \quad \longleftrightarrow \quad \log_{a}(x) = y, \qquad x > 0$$

Example. Find the derivative of the following functions

$y = \ln(\sqrt[3]{x}) = \ln(x^{1/3}) = \frac{1}{3}\ln(x)$

$y' = \frac{1}{3} \cdot \frac{1}{x} = \boxed{\frac{1}{3x}}$

$f(x) = \ln\left((x^2+1)(x^3+2)^6\right)$

$= \ln(x^2+1) + 6\ln(x^3+2)$

$f'(x) = \boxed{\frac{2x}{x^2+1} + \frac{6 \cdot 3x^2}{x^3+2}}$

$g(x) = \ln\left(\frac{1-x}{1+x}\right) = \ln(1-x) - \ln(1+x)$

$g'(x) = \boxed{\frac{-1}{1-x} + \frac{1}{1+x}}$

$h(x) = \ln\left(x + \sqrt{x^2-1}\right)$

$= \ln\left(x + (x^2-1)^{1/2}\right)$

$h'(x) = \frac{1 + \frac{1}{2}(x^2-1)^{-1/2} \cdot (2x)}{x + (x^2-1)^{1/2}}$

$= \boxed{\frac{1 + \frac{x}{\sqrt{x^2-1}}}{x + \sqrt{x^2-1}}}$

$j(t) = 2(\ln(5t))^{3/2}$

$= 2(\ln(5) + \ln(t))^{3/2}$

$j'(t) = \boxed{3(\ln(5) + \ln(t))^{1/2} \cdot \frac{1}{t}}$

$k(u) = \ln\left(\sqrt{u^2-4}\right)$

$= \ln\left((u^2-4)^{1/2}\right)$

$k'(u) = \frac{1}{2}\,\frac{1}{u^2-4} \cdot 2u = \boxed{\frac{u}{u^2-4}}$

Logarithmic Differentiation
Using properties of logs, we can transform a function before taking it's derivative:

1. Take the natural log of both sides rewriting products and quotients as sums and differences.
2. Differentiate both sides.
3. Solve for $\frac{dy}{dx}$

Example. Differentiate

$$y = x^2(x-1)\left(x^2+4\right)^3$$

$$\ln(y) = 2\ln(x) + \ln(x-1) + 3\ln(x^2+4)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{1}{x-1} + \frac{3\cdot 2x}{x^2+4}$$

$$\frac{dy}{dx} = y\left(\frac{2}{x} + \frac{1}{x-1} + \frac{6x}{x^2+4}\right)$$

$$\boxed{\frac{dy}{dx} = x^2(x-1)(x^2+4)^3\left(\frac{2}{x} + \frac{1}{x-1} + \frac{6x}{x^2+4}\right)}$$