5.2: Logarithmic Functions and Their Properties

Definition.

For $a > 0$ and $a \neq 1$, the **logarithmic function**

$$y = \log_a(x) \qquad \text{(logarithmic form)}$$

has domain $x > 0$, base a, and is defined by

$$a^y = x \qquad \text{(exponential form)}$$

Example. Rewrite the following in exponential form

$4 = \log_2(16)$

$2^4 = 16$

$5 = \log_{10}(100,000)$

$10^5 = 100,000$

$\frac{1}{2} = \log_{16}(4)$

$16^{1/2} = 4$

$\boxed{x^{1/m} = \sqrt[m]{x}}$

$-4 = \log_3\left(\frac{1}{81}\right)$

$3^{-4} = \frac{1}{81}$

$\boxed{x^{-m} = \frac{1}{x^m}}$

$-\frac{1}{4} = \log_{625}\left(\frac{1}{5}\right)$

$625^{-1/4} = \frac{1}{5}$

$-\frac{5}{3} = \log_{\frac{1}{8}}(32)$

$\left(\frac{1}{8}\right)^{-5/3} = 32$

Example. Simplify the following:

$y = \log_3(9)$

$3^y = 9 \Rightarrow y = 2$

$\Rightarrow \boxed{\log_3(9) = 2}$

$y = \log_4(2)$

$4^y = 2 \Rightarrow y = 1/2$

$\Rightarrow \boxed{\log_4(2) = 1/2}$

Example. Solve the following:

$\log_5(x) = 4$

$x = 5^4$

$\boxed{x = 625}$

$\log_8(x) = 1$

$x = 8^1$

$\boxed{x = 8}$

$\log_{81}(x) = -\frac{1}{4}$

$x = 81^{-1/4}$

$= \dfrac{1}{81^{1/4}}$

$= \dfrac{1}{\sqrt[4]{81}}$

$\boxed{x = 1/3}$

$\log_{10}(x + 4) = 3$

$x + 4 = 10^3$

$-4 + x + 4 = 1000 - 4$

$\boxed{x = 996}$

Common logarithms:	$\log(x) = \log_{10}(x)$
Natural logarithms:	$\ln(x) = \log_e(x)$

Example (The Rule of 70). If \$$P$ is invested for t years at interest rate r, compounded continuously, then the future value of the investment is given by

$$S = Pe^{rt}.$$

Find the value of t when the investment $\underbrace{\text{doubles}}_{S=2P}$.

$$\frac{2P}{P} = \frac{Pe^{rt}}{P}$$

$$2 = e^{rt} \longrightarrow \log_e(2) = rt$$

$$\frac{\ln(2)}{r} = \frac{rt}{r}$$

$$\Rightarrow \quad t = \frac{\ln(2)}{r} \approx \frac{0.69314\ldots}{r}$$

If we represent the interest rate as a percentage (e.g. 5% instead of 0.05), then we have

$$t \approx \frac{70}{r}$$

Suppose r=5%:

$$\boxed{t \approx \frac{70}{5} = 14} \qquad \boxed{t = \frac{\ln(2)}{0.05} \approx 13.863}$$

Change of base formula:
If $a > 0$, $b > 0$ with $a \neq 1$ and $b \neq 1$, then

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}.$$

Note: This works for any valid base!

Base e : $\log_b(x) = \dfrac{\ln(x)}{\ln(b)}$ Base 10 : $\log_b(x) = \dfrac{\log(x)}{\log(b)}$

Example. Solve the following

$3^x = 10$

$$x = \log_3(10)$$
$$= \frac{\ln(10)}{\ln(3)}$$
$$\boxed{x \approx 2.0959}$$

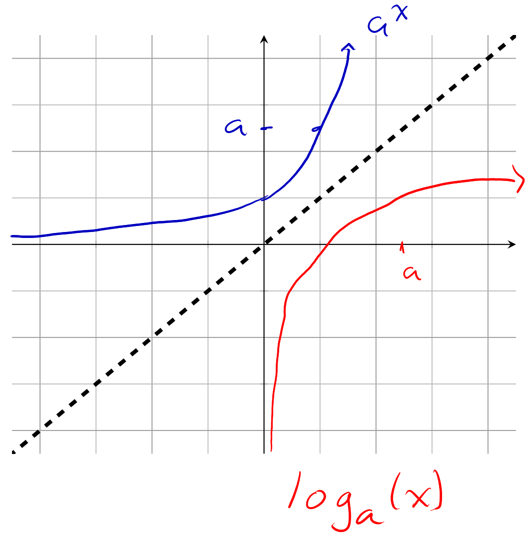
$6.5^x = 5$

$$x = \log_{6.5}(5)$$
$$= \frac{\ln(5)}{\ln(6.5)}$$
$$\boxed{x \approx 0.8598}$$

Example. Fill in the tables below and graph a^x and $\log_a(x)$ on the same axes.

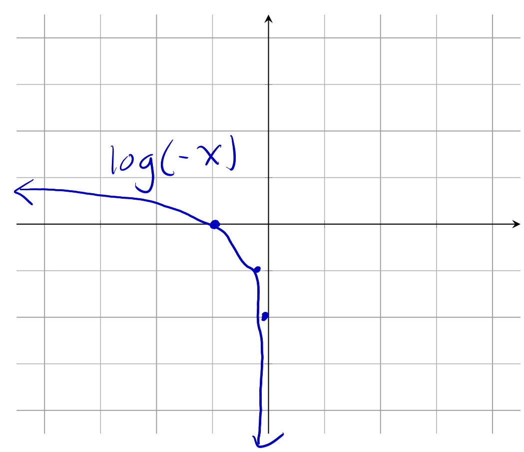
x	$y = a^x$
-2	$1/a^2$
-1	$1/a$
0	1
1	a
2	a^2

x	$y = \log_a(x)$
$1/a^2$	-2
$1/a$	-1
1	0
a	1
a^2	2



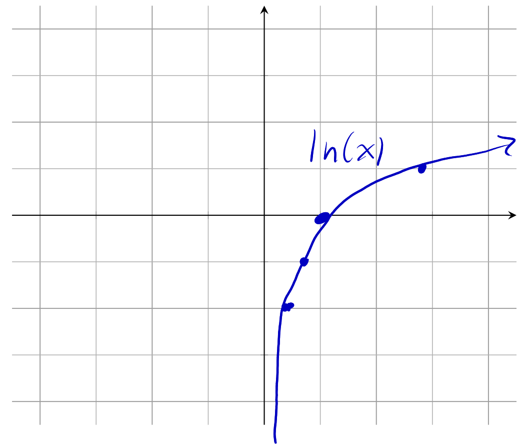
Example. Graph $\log(-x)$

x	$\log(-x)$
$-1/10^2$	-2
$-1/10$	-1
-1	0
-10	1
-100	2



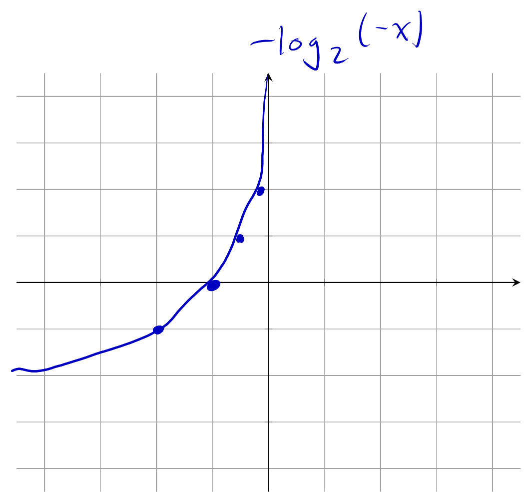
Example. Graph $\ln(x)$

x	$\ln(x)$
$0.368 \approx e^{-2}$	-2
$0.135 \approx e^{-1}$	-1
1	0
e	1
e^2	2



Example. Graph $-\log_2(-x)$

x	$-\log_2(-x)$
-2^2	-2
-2	-1
-1	0
$-1/2$	1
$-1/2^2$	2



Example. Evaluate the following:

$f(x) = \ln(x); \quad f\left(e^{-3x}\right)$

$f(x) = 5^x; \quad f(\log_5(10))$

$f(e^{-3x}) = \ln(e^{-3x})$

$= -3x$

$f(\log_5(10)) = 5^{\log_5(10)}$

$= 10$

Properties of exponents and logarithms: Assume $a > 0$:

$a^y = x$	$\log_a(x) = y$
$a^1 = a$	$\log_a(a) = 1$
$a^0 = 1$	$\log_a(1) = 0$
$a^x a^y = a^{x+y}$	$\log_a(xy) = \log_a(x) + \log_a(y)$
$\dfrac{a^x}{a^y} = a^{x-y}$	$\log_a\left(\dfrac{x}{y}\right) = \log_a(x) - \log_a(y)$
$a^{xy} = (a^x)^y$	$\log_a(x^y) = y\log_a(x)$
$a^{\log_a(x)} = x$	$\log_a(a^x) = x$