6.3: Future Values of Annuities

Definition.

- An **annuity** is a financial plan characterized by regular payments (e.g. mortgages, student loans, etc.).
- The sum of all the payments and the interest earned is called the **future value of the annuity** or its **future value**.
- An **ordinary annuity** or (**annuity immediate**) is an annuity in which payments are made at the *end of each of the equal payment intervals.*

Example. Suppose that we invest \$100 at the end of each year for 5 years in an account that pays 10% compounded annually. How much money will you have at the end of the 5 years?

1. $S = 100(1+0.10)^4$
2. $S = 100(1+0.10)^3$
3. $S = 100(1+0.10)^2$
4. $S = 100(1+0.10)^1$
5. $+\ S = 100(1+0.10)^0$

"Geometric Series"

\$610.51

Definition.
If $\$R$ is deposited at the *end of each period* for n periods in an annuity that earns interest at a rate of i per period, the **future value of the annuity** will be

$$S = R \cdot S_{\overline{n}|i} = R\left[\frac{(1+i)^n - 1}{i}\right]$$

The notation $S_{\overline{n}|i}$ represents the future value of an ordinary annuity of \$1 per period for n periods with an interest rate of i per period.

Derivation

$$S = R\left[\frac{(1+i)^n - 1}{i}\right]$$

$$S = R(1+i)^{n-1} + R(1+i)^{n-2} + \ldots R(1+i)^1 + R$$

$$-(1+i)S = -\left(R(1+i)^n + R(1+i)^{n-1} + \ldots R(1+i)^2 + R(1+i)^1\right)$$

$$S - (1+i)S = R - R(1+i)^n$$

$$S\left(1-(1+i)\right) = R\left[1-(1+i)^n\right]$$

$$\boxed{S = R\left[\frac{(1+i)^n - 1}{i}\right]}$$

Example. Suppose a pair of twins take different steps to save for retirement. Both regularly make investments of $2,000 into accounts that earn 10%, compounded annually. Starting at age 21:

> Find the future value if twin A makes his payments for 8 years, and then lets his investment accrue compound interest every year for 36 years.

1. $S \leftarrow$ Find after 8 yrs

R: 2000

i: $10\% = 0.10$

n: 8

$$S = 2000\left[\frac{(1+0.10)^8 - 1}{0.10}\right]$$

$$= \boxed{\$22{,}871.78}$$

2. S in invested for 36 yrs.

$$S(1+0.10)^{36}$$

$$= \boxed{\$707{,}027.91}$$

> Find the future value if twin B waits 8 years before making regular payments for the following 36 years.

$S \leftarrow$ Find after 36 yrs

R: 2000

i: $10\% = 0.10$

n: 36

$$S = 2000\left[\frac{(1+0.10)^{36} - 1}{0.10}\right]$$

$$= \boxed{\$598{,}253.61}$$

Example. Suppose that you wish to have $50,000 saved up in 5 years. To do this, you want to make regularly monthly payments. What is the amount of the monthly payments if the interest rate is 5%? What if the interest rate is 15%?

$S:\ 50000$

$R \leftarrow$ Find

$i:\ \frac{r}{m} = \frac{5\%}{12} = 1.4\bar{6}\,\% = 0.014\bar{6}$

$n:\ mt = 12(5) = 60$

$$50000 = R\left[\frac{\left(1+\left(\frac{0.05}{12}\right)\right)^{60}-1}{\left(\frac{0.05}{12}\right)}\right]$$

$$\Rightarrow R = \frac{50\,000}{\left[\frac{\left(1+\left(\frac{0.05}{12}\right)\right)^{60}-1}{\left(\frac{0.05}{12}\right)}\right]} = \boxed{\$735.23}$$

$$r = 15\% \Rightarrow i = \frac{r}{m} = \frac{15\%}{12} = 1.25\% = 0.0125 \Rightarrow R = \frac{50\,000}{\left[\frac{\left(1+\left(\frac{0.15}{12}\right)\right)^{60}-1}{\left(\frac{0.15}{12}\right)}\right]} = \boxed{\$564.50}$$

Example. A small business invests $1,000 at the end of each month in an account that earns 6% compounded monthly. How long will it take until the business has $100,000 toward the purchase of its own office building?

$S:\ \$100{,}000$

$R:\ \$1{,}000$

$\frac{r}{m} = i:\ \frac{6\%}{12} = 0.5\% = 0.005$

$mt = n:\ 12t \leftarrow$ Find t

$$\left(\frac{0.005}{1000}\right)100{,}000 = 1{,}000\left[\frac{1.005^{12t}-1}{0.005}\right]\left(\frac{0.005}{1000}\right)$$

$$0.5+1 = 1.005^{12t} - 1 + 1$$

$$\ln(1.5) = \ln\left(1.005^{12t}\right)$$

$$\frac{\ln(1.5)}{12\ln(1.005)} = \frac{12t\ln(1.005)}{12\ln(1.005)}$$

$$t = \frac{\ln(1.5)}{12\ln(1.005)} = 6.7746$$

(about 6 yrs. and 10 months)

Definition.
An **annuity due** differs from an ordinary annuity in that the payments are made at the *beginning of each period.*

If $\$R$ is deposited at the *beginning of each period* for n periods in an annuity that earns interest at a rate of i per period, the **future value of the annuity** will be

$$S_{\text{due}} = R \cdot S_{\overline{n}|i}(1+i) = R\left[\frac{(1+i)^{n-1}}{i}\right](1+i)$$

Example. Find the future value of an investment if \$150 is deposited at the beginning of each month for 9 years at an interest rate of 7.2% compounded monthly.

S : $\leftarrow$ Find

R : 150

$\frac{r}{m} = i$: $\frac{7.2\%}{12} = 0.6\% = 0.006$

$mt = n$: $12(9) = 108$

$$S = 150\left[\frac{(1.006)^{108}-1}{0.006}\right](1.006) = \boxed{\$22{,}836.59}$$