6.4: Present Values of Annuities

Example. Suppose we wish to invest a lump sum of money, A_n, into an annuity that earns interest at a rate of 10% per year, so that we may receive payments of \$100 for 5 years. What is the amount of the lump sum?

1. $100 = P(1+0.10)^1 \Rightarrow P = \frac{100}{(1+0.10)^1} = 90.91$

. $100 = P(1+0.10)^2 \Rightarrow P = \frac{100}{(1+0.10)^2} = 82.64$

. $100 = P(1+0.10)^3 \Rightarrow P = \frac{100}{(1+0.10)^3} = 75.13$

. $100 = P(1+0.10)^4 \Rightarrow P = \frac{100}{(1+0.10)^4} = 68.30$

. $100 = P(1+0.10)^5 \Rightarrow P = \frac{100}{(1+0.10)^5} = 62.09$

+

\$ 379.08

Definition.
If a payment of $\$R$ is to be withdrawn at the *end of each period* for n periods from an account that earns interest at a rate of i per period, then the account is an **ordinary annuity** and the **present value** is

$$A_n = R \cdot a_{\overline{n}|i} = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

The notation $a_{\overline{n}|i}$ represents the present value of an ordinary annuity of \$1 per period for n periods with an interest rate of i per period.

Example. Find the lump sum that one must invest in an annuity to receive \$1000 at the end of each month for the next 16 years, if the annuity pays 9% compounded monthly.

A_n: ← Find

R: 1000

$\frac{r}{m} = i$: $\frac{0.09}{12}$

$mt = n$: $12 \cdot 16 = 192$

$$A_n = 1000\left[\frac{1-\left(1+\frac{0.09}{12}\right)^{-192}}{\left(\frac{0.09}{12}\right)}\right]$$

$$= \boxed{\$101,572.77}$$

Example. Suppose that a couple plans to set up an ordinary annuity with a \$100,000 inheritance they received. What is the size of the quarterly payments they will receive for the next 6 years if the account pays 7% compounded quarterly?

A_n: 100 000

R: ← Find

$\frac{r}{m} = i$: $\frac{7\%}{4} = \frac{0.07}{4} = 0.0175$

$mt = n$: $4(6) = 24$

$$\frac{100,000}{\left[\frac{1-(1.0175)^{-24}}{0.0175}\right]} = \frac{R\left[\frac{1-(1.0175)^{-24}}{0.0175}\right]}{\left[\frac{1-(1.0175)^{-24}}{0.0175}\right]}$$

$$R = \frac{100,000}{\left[\frac{1-(1.0175)^{-24}}{0.0175}\right]}$$

$$= \boxed{\$5138.57}$$

Example. An inheritance of $250,000 is invested at 9% compounded monthly. If $2500 is withdrawn at the end of each month, how long will it be until the account balance is $0?

Use ln(...) to find unknown exponent

1. Isolate
2. ln(...)
3. expo
4. solve

A_n: 250 000

R: 2500

$\frac{r}{m} = i$: $\frac{9\%}{12} = \frac{0.09}{12} = 0.0075$

$mt = n$: $12t$ ← Find t or Find n

$$\left(\frac{0.0075}{2500}\right) 250{,}000 = 2500\left[\frac{1-(1.0075)^{-12t}}{0.0075}\right]\left(\frac{0.0075}{2500}\right)$$

$$0.75 - 1 = 1-(1.0075)^{-12t} - 1$$

$$(-1)(-0.25) = \left(-(1.0075)^{-12t}\right)(-1)$$

$$\ln(0.25) = \ln\left((1.0075)^{-12t}\right)$$

$$\frac{\ln(0.25)}{-12\ln(1.0075)} = \frac{-12t\ln(1.0075)}{-12\ln(1.0075)}$$

$$t = \frac{\ln(0.25)}{-12\ln(1.0075)} = 15.4610$$

$n = 12t$

$\Rightarrow n = 185.53$

There will be 185 regular withdrawals of $2,500, and the last withdrawal will be less

Definition.
If a payment of $\$R$ is to be withdrawn at the *beginning of each period* for n periods from an account that earns interest at a rate of i per period, then the account is an **annuity due** and the **present value** is

$$A_{(n,\text{due})} = R \cdot a_{\overline{n}|i}(1+i) = R\left[\frac{1-(1+i)^{-n}}{i}\right](1+i)$$

The notation $a_{\overline{n}|i}$ represents the present value of an ordinary annuity of $1 per period for n periods with an interest rate of i per period.

Example. Suppose that a court settlement results in a $750,000 award. If this is invested at 9% compounded semiannually, how much will it provide at the *beginning* of each half-year for a period of 7 years?

$A_{(n,\,due)}$: $\$750{,}000$

R : $\leftarrow$ Find

$\frac{r}{m} = i$: $\frac{9\%}{2} = 0.045$

$mt = n$: $2(7) = 14$

$$A_{(n,\,due)} = R\, a_{\overline{n}|i}(1+i)$$

$$\Rightarrow R = \frac{A_{(n,\,due)}}{a_{\overline{n}|i}(1+i)}$$

$$R = \frac{750{,}000}{\left[\frac{1-1.045^{-14}}{0.045}\right](1.045)}$$

$$= \boxed{\$70{,}205.97}$$