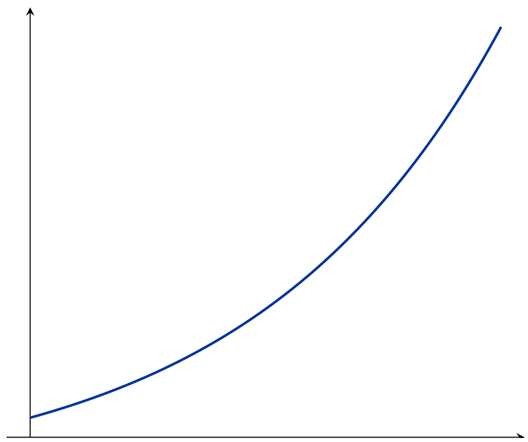
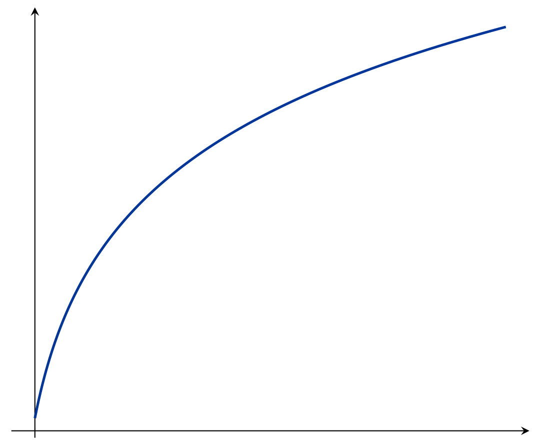
## 4.2: Applications of the Second Derivative

**Definition.**
Consider any differentiable function $f(x)$ on the interval $(a, b)$. We say $f$ is

concave up if $f'(x)$ is increasing



concave down if $f'(x)$ is decreasing



Thus, for every value of $x$ on the interval $(a, b)$, if

– $f''(x) > 0$, then $f'$ is increasing, and $f$ is concave *up* on $(a, b)$.

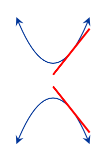
– $f''(x) < 0$, then $f'$ is decreasing, and $f$ is concave *down* on $(a, b)$.

– If $f$ is continuous at $c$ and $f$ changes concavity at $c$, then $f$ has an **inflection point** at $c$.

*Note*: $f(x)$ is

– concave up if its tangent lines lie below the curve

– concave down if its tangent lines lie above the curve



**Determining the Intervals of Concavity of the Graph of $f$**

1. Determine the values of $x$ for which $f''$ is zero or undefined.
2. Determine the sign of $f''(x)$ to the left and right of each point from above: Let $c$ be a convenient test point on the interval of interest. Then,
   a) if $f''(c) > 0$, then $f$ is concave up on that interval.
   b) if $f''(c) < 0$, then $f$ is concave down on that interval.

**Example.** Find the intervals where the following functions are concave up and concave down:

$f(x) = x^3 - 3x^2 - 24x + 32$ Graph

$f'(x) = 3x^2 - 6x - 24$

$f''(x) = 6x - 6$

$f''(x) = 0$ $f''(x)$ DNE

$6x - 6 = 0$ —

$x = 1$

6

| | 0 | 10 |
|---|---|---|
| $6x-6$ | $-$ | $+$ |
| | $\cap$ | $\cup$ |

$\boxed{\begin{array}{ll} \text{Concave down:} & (-\infty, 6) \\ \text{Concave up:} & (6, \infty) \end{array}}$

$g(x) = (x + 1)^{2/3}$

$g'(x) = \frac{2}{3}(x+1)^{-1/3}$

$g''(x) = \frac{-2}{9}(x+1)^{-4/3} = \frac{-2}{3(x+1)^{4/3}}$

$g''(x) = 0$ $g''(x)$ DNE

— $3(x+1)^{4/3} \neq 0$

$x \neq -1$

$-1$

| | $-10$ | $0$ |
|---|---|---|
| $-2$ | $-$ | $-$ |
| $3$ | $+$ | $+$ |
| $(x+1)^{4/3}$ | $+$ | $+$ |
| | $-$ | $-$ |
| | $\cap$ | $\cap$ |

$\boxed{\text{Concave down: } (-\infty, -1) \cup (-1, \infty)}$

$$h(x) = x + \frac{1}{x} = x + x^{-1}$$

$$h'(x) = 1 - x^{-2}$$

$$h''(x) = 2x^{-3} = \frac{2}{x^3}$$

Solve $h''(x) = 0$ & $h''(x)$ DNE

$x \neq 0$

| | $-1$ | $0$ | $1$ |
|---|---|---|---|
| $\frac{2}{x^3}$ | $-$ | | $+$ |
| | $\cap$ | | $\cup$ |

Concave down: $(-\infty, \ )$

Concave up: $(0, \infty)$

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$$j(x) = \frac{x^2}{1 - x^2}$$

$$j'(x) = \frac{(1-x^2)\,2x - x^2(-2x)}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}$$

$$j''(x) = \frac{(1-x^2)^2 \cdot 2 - 2x \cdot 2(1-x^2)\cdot(-2x)}{(1-x^2)^4} = \frac{6x^2+2}{(1-x^2)^3}$$

Solve $j''(x) = 0$ & $j''(x)$ DNE

$(1-x^2)^3 \neq 0$

$\pm 1 \neq x$

| | $-10$ | $-1$ | | $1$ | $10$ |
|---|---|---|---|---|---|
| $6x^2+2$ | $+$ | | $+$ | | $+$ |
| $(1-x^2)^3$ | $-$ | | $+$ | | $-$ |
| | $-$ | | $+$ | | $-$ |
| | $\cap$ | | $\cup$ | | $\cap$ |

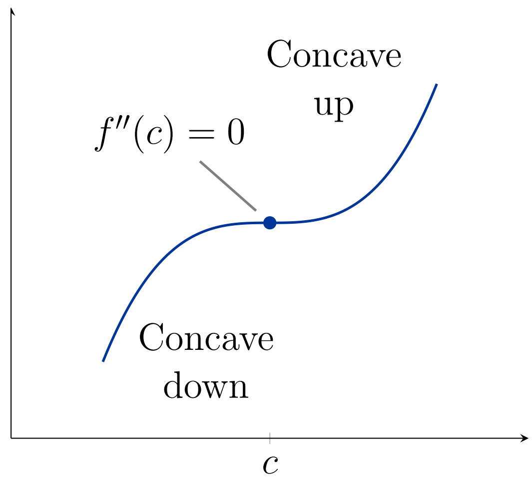
c. down: $(-\infty, -1) \cup (1, \infty)$

c. up $(-1, 1)$

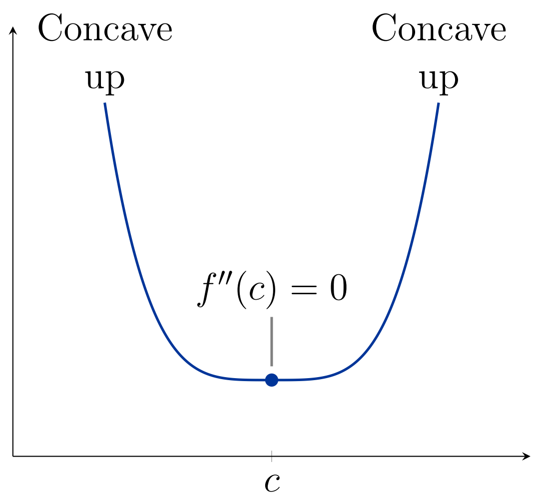
**Finding inflection points**

1. Compute $f''(x)$.
2. Locate where $f''(x) = 0$ or $f''(x)$ does not exist.
3. Determine if the sign of $f''(x)$ changes at the points found above.

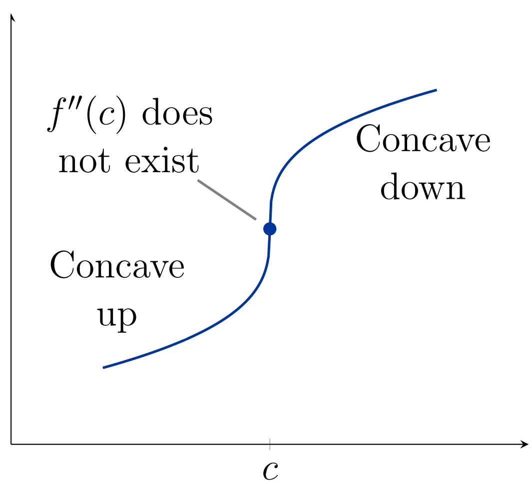
Inflection point at $x = c$



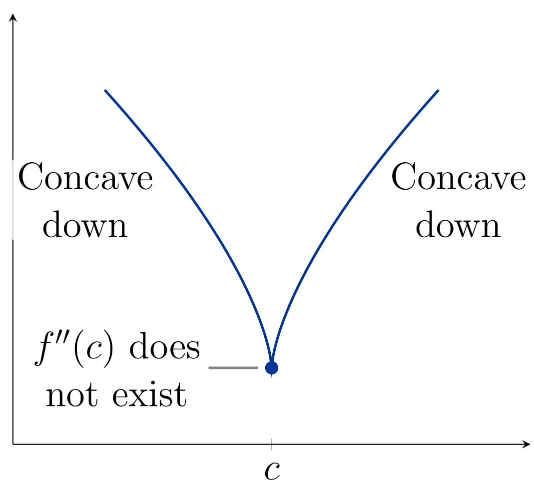

*No* inflection point at $x = c$




Inflection point at $x = c$




*No* inflection point at $x = c$



**Example.** For the following functions, determine the intervals of concavity and find any inflection points.

$f(x) = (x-1)^{5/3}$ [Graph]

$f'(x) = \frac{5}{3}(x-1)^{2/3}$

$f''(x) = \frac{10}{9}(x-1)^{-1/3} = \frac{10}{9(x-1)^{1/3}}$

$f''(x) = 0$ $\quad f''(x)$ DNE

— $\quad 9(x-1)^{1/3} \neq 0$

$x \neq 1$

| | 0 | 1 | 2 |
|---|---|---|---|
| $10$ | $+$ | | $+$ |
| $9$ | $+$ | | $+$ |
| $(x-1)^{1/3}$ | $-$ | | $+$ |
| | $-$ | | $+$ |
| | $\cap$ | | $\cup$ |

Concave down: $(-\infty, 1)$

Concave up: $(1, \infty)$

Inflection point: $(1, f(1))$

$\to (1, 0)$

$f(x)$ must be continuous in order for the inflection point to exist!!

$$g(x) = \frac{1}{x^2+1} = (x^2+1)^{-1}$$

$$g'(x) = -(x^2+1)^{-2} \cdot 2x = \frac{-2x}{(x^2+1)^2}$$

$$g''(x) = \frac{(x^2+1)^2(-2) - (-2x)\, 2(x^2+1)\cdot 2x}{(x^2+1)^4} = \frac{-2(x^2-4x+1)}{(x^2+1)^3}$$

$g''(x) = 0$

$x^2 - 4x + 1 = 0$

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(1)}}{2}$$

$$= \frac{4 \pm \sqrt{12}}{2}$$

$$= 2 \pm \sqrt{3}$$

$g''(x)$ DNE

$(x^2+1)^3 \neq 0$

| | $(-\infty, 2-\sqrt{3})$ | $(2-\sqrt{3}, 2+\sqrt{3})$ | $(2+\sqrt{3}, \infty)$ |
|---|---|---|---|
| | 0 | 2 | 10 |
| $-2$ | $-$ | $-$ | $-$ |
| $x^2-4x+1$ | $+$ | $-$ | $+$ |
| $(x^2+1)^3$ | $+$ | $+$ | $+$ |
| | $-$ | $+$ | $-$ |
| | $\cap$ | $\cup$ | $\cap$ |

Concave down: $(-\infty, 2-\sqrt{3}) \cup (2+\sqrt{3}, \infty)$

Concave up: $(2-\sqrt{3}, 2+\sqrt{3})$

Inflection points: $(2-\sqrt{3}, g(2-\sqrt{3}))$ $\quad$ $(2+\sqrt{3}, g(2+\sqrt{3}))$

$\rightarrow \left(2-\sqrt{3}, \frac{1}{8-4\sqrt{3}}\right)$ $\quad$ $\rightarrow \left(2+\sqrt{3}, \frac{1}{8+4\sqrt{3}}\right)$

**Second Derivative Test for Local Extrema**
Suppose $f''$ is continuous on an open interval containing $c$ with $f'(c) = 0$.

- If $f''(c) > 0$, then $f$ has a local minimum at $c$.
- If $f''(c) < 0$, then $f$ has a local maximum at $c$.
- If $f''(c) = 0$, then the test is inconclusive; $f$ may have a local maximum, local minimum, or neither at $c$.

**Example.** Find the relative extrema of

$$f(x) = x^3 - 3x^2 - 24x + 32$$

$f'(x) = 3x^2 - 6x - 24 = 3(x+2)(x-4)$

Graph

Quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(3)(-24)}}{2(3)} = \frac{6 \pm 18}{6} \to x = 4, \quad x = -2$$

$f''(x) = 6x - 6$

$f''(-2) = -18 \longrightarrow$ concave down $\frown$ $\Rightarrow$ Rel. max.

$f''(4) = 18 \longrightarrow$ concave up $\smile$ $\Rightarrow$ Rel. min.