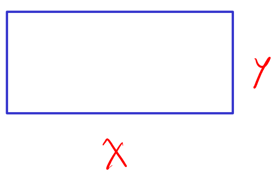
4.5: Optimization II

Guidelines for Solving Optimization Problems

1. Assign a letter to each variable mentioned in the problem. If appropriate, draw and label a figure.
2. Find an expression for the quantity to be optimized.
3. Use the conditions given in the problem to write the quantity to be optimized as a function f of *one* variable. Note any restrictions to be placed on the domain of f from physical considerations of the problem.
4. Optimize the function f over its domain.

Example. Show among all rectangles with an 8 meter perimeter, the one with the *largest* area is a square.



Perimeter

$p = 2x + 2y$

$8 = 2x + 2y$

solve for y

$\frac{8 - 2x}{2} = \frac{2y}{2}$

$4 - x = y$

Area

$A = xy$

Subsitute expression for y

$A = \underbrace{x}_{\geq 0}\underbrace{(4-x)}_{\geq 0} = 4x - x^2$

$\rightarrow 0 \leq x \leq 4$

Solve $A' = 0$

$A' = 4 - 2x$

$4 - 2x = 0$

$x = 2$

Evaluate A at critical points and end points

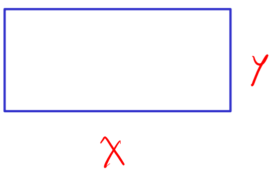
x	A
0	0
2	$4\,m^2$
4	0

Crit. Pt. $\rightarrow$ 2

$y = 4 - x$
$= 4 - 2$
$= 2$

$\boxed{x = y = 2\,m \Rightarrow \text{square}}$

Example. A man wishes to have a rectangular-shaped garden in his backyard. He has 50 feet of fencing with which to enclose his garden. Find the dimensions for the largest garden he can have if he uses all of his fencing.



Perimeter

$p = 2x + 2y$

$50 = 2x + 2y$

solve for y

$\frac{50 - 2x}{2} = \frac{2y}{2}$

$25 - x = y$

Area

$A = xy$

Subsitute expression for y

$A = \underbrace{x}_{\geq 0}\underbrace{(25 - x)}_{\geq 0} = 25x - x^2$

$0 \leq x \leq 25$

Solve $A' = 0$

$A' = 25 - 2x$

$25 - 2x = 0$

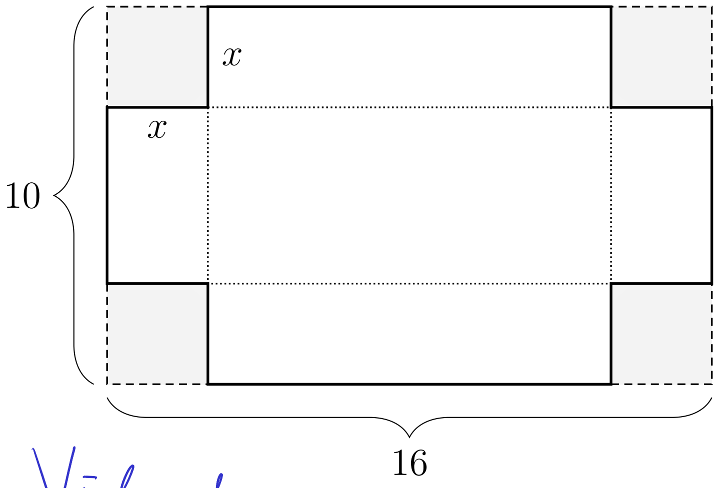
$x = \frac{25}{2} = 12.5$

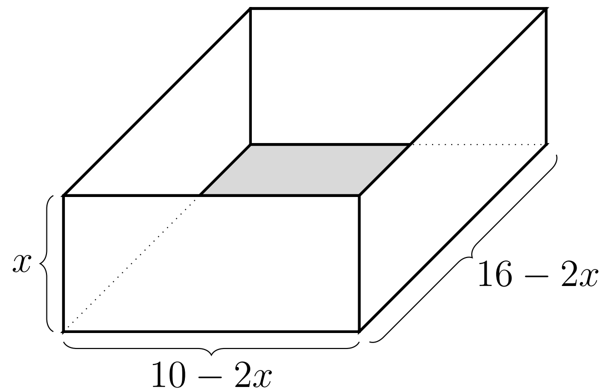
Evaluate A at critical points and end points

	x	A
	0	0
Crit. Pt. $\rightarrow$	$\boxed{\frac{25}{2}}$	$\boxed{156.25}$
	25	0

$$\boxed{\begin{array}{l} x = 12.5\text{ ft} \\ y = 12.5\text{ ft} \\ A = 156.25\text{ ft}^2 \end{array}}$$

Example. By cutting away identical squares from each corner of a rectangular piece of cardboard and folding up the resulting flaps, the cardboard may be turned into an open box. If the cardboard is 16" long and 10" wide, find the dimensions of the box that will yield the maximum volume.





$V = l \cdot w \cdot h$

$V(x) = \underbrace{(10-2x)}_{\geq 0}\underbrace{(16-2x)}_{\geq 0} \cdot \underbrace{x}_{\geq 0} \longrightarrow \left.\begin{array}{l} 0 \leq x \\ x \leq 5 \\ x \leq 8 \end{array}\right\} \quad 0 \leq x \leq 5$

$= 4x^3 - 52x^2 + 160x$

x	$V(x)$
0	0
2	144
5	0

$V'(x) = 12x^2 - 104x + 160$

Solve $V'(x) = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{104 \pm \sqrt{(-104)^2 - 4(12)(160)}}{2(12)} = \frac{104 \pm 56}{24} = \frac{13 \pm 7}{3}$$

$x = 2$ $\quad$ $\cancel{x = \frac{20}{3}}$

Outside [0,5]

max when $x = 2$

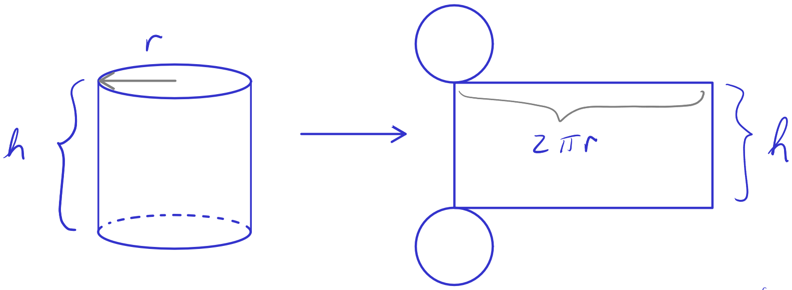
$\Rightarrow$ $l = 4$ in

$w = 12$ in

$h = 2$ in

$V = 144 \text{ in}^3$

Example. A cylindrical can is to be made to hold $1\,L$ $(1000\,cm^3)$ of oil. Find the dimensions of the can that will minimize the cost of the metal to manufacture the can.



$$V = \pi r^2 h$$

$$\rightarrow h = \frac{V}{\pi r^2} = \frac{1000}{\pi r^2}$$

$$\left.\begin{array}{l} r \neq 0 \\ h \geq 0 \end{array}\right\} r > 0$$

$$SA = 2(\text{Area of circle}) + \text{Area of rectangle}$$

$$= 2(\pi r^2) + 2\pi r \cdot h$$

$$= 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2}$$

$$= 2\pi r^2 + \frac{2000}{r}$$

$$S(r) = 2\pi r^2 + 2000 r^{-1}$$

$$S'(r) = 4\pi r - 2000 r^{-2}$$

Solve $S'(r) = 0$ & $S'(r)$ DNE $\rightarrow r \neq 0$

$$4\pi r - \frac{2000}{r^2} = 0$$

$$4\pi r = \frac{2000}{r^2}$$

$$r^3 = \frac{500}{\pi}$$

$$r = \sqrt[3]{\frac{500}{\pi}} \approx 5.419$$

r	$S(r)$
$\sqrt[3]{\frac{500}{\pi}}$	553.58 cm^3

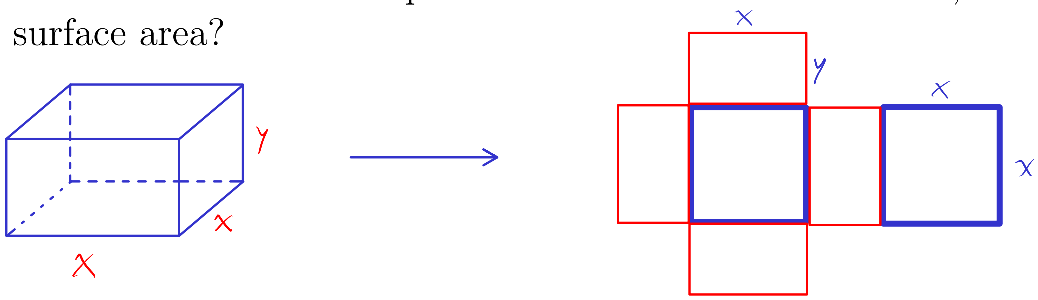
We can verify using the second derivative test:

$$S''(r) = 4\pi + 4000 r^{-3}$$

$$S''\left(\sqrt[3]{\frac{500}{\pi}}\right) > 0 \longrightarrow \text{Concave up} \Rightarrow \text{min}$$

$$\Rightarrow r = \sqrt[3]{\frac{500}{\pi}} \approx 5.419\text{ cm}, \quad h = \frac{1000}{\pi r} \approx 0.575\text{ cm}$$

Example. Of all boxes with a square base and a volume of $8\,\mathrm{m}^3$, which one has the minimum surface area?



$V = l \cdot w \cdot h$

$\Rightarrow 8 = x^2 \cdot y$

$\Rightarrow \frac{8}{x} = xy$

$x \neq 0$

$\underbrace{x \geq 0 \quad x \neq 0}_{x > 0}$

$SA = 4(\text{Area of side}) + 2(\text{Area of top/bottom})$

$= 4(xy) + 2(x^2)$

$= 4\left(\frac{8}{x}\right) + 2x^2$

$S(x) = 32x^{-1} + 2x^2$

$S'(x) = -32x^{-2} + 4x$

Solve

$S'(x) = 0$ & $S'(x)$ DNE

$x \neq 0$

$\frac{-32}{x^2} + 4x = 0$

$4x = \frac{32}{x^2}$

$x^3 = 8$

$x = \sqrt[3]{8} = 2$

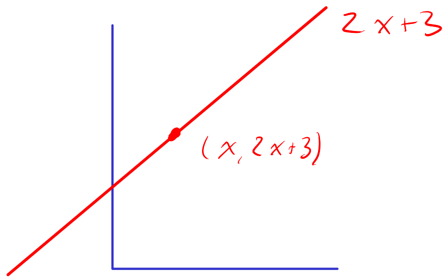
x	$S(x)$
2	24

We can verify using the second derivative test:

$S''(x) = 64x + 4$

$S''(2) > 0 \longrightarrow$ Concave up $\Rightarrow$ min

Example. Find the point on the line $y = 2x + 3$ that is closest to the origin.



Distance: $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$$d = \sqrt{(x-0)^2 + (2x+3-0)^2}$$

$$= \sqrt{x^2 + 4x^2 + 12x + 9}$$

$$d(x) = \sqrt{5x^2 + 12x + 9} = (5x^2 + 12x + 9)^{1/2}$$

$$d'(x) = \frac{1}{2}(5x^2 + 12x + 9)^{-1/2}(10x + 12)$$

$$= \frac{5x + 6}{\sqrt{5x^2 + 12x + 9}}$$

x	$d(x)$
$-\frac{6}{5}$	1.3416

Solve $d'(x) = 0$ & $d'(x)$ DNE

$$\frac{5x+6}{\sqrt{5x^2+12x+9}} = 0$$

$$5x + 6 = 0$$

$$x = -\frac{6}{5}$$

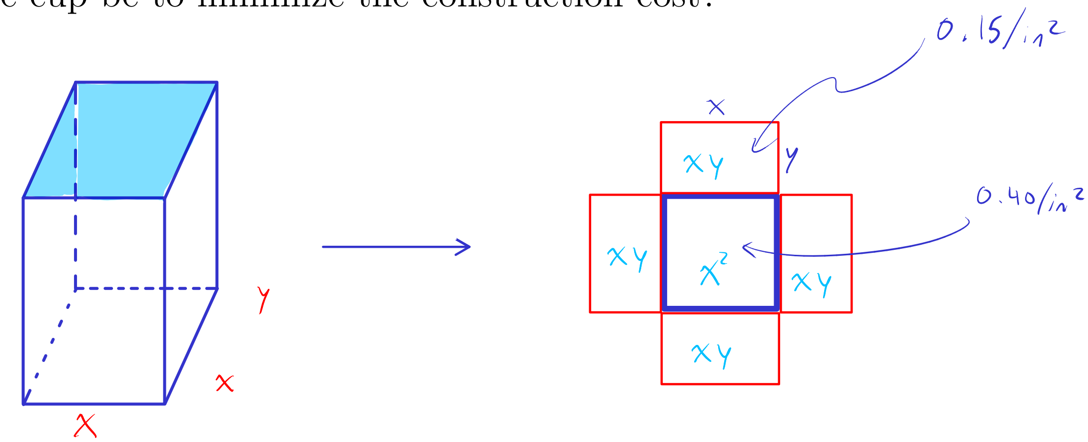
We can verify using the first derivative test:

	-10	$-\frac{6}{5}$	0
$5x+6$	$-$		$+$
$\sqrt{5x^2+12x+9}$	$+$		$+$
	$-$		$+$
	$\searrow$		$\nearrow$

Rel. min.

$$\boxed{(x, y) = \left(-\frac{6}{5}, \frac{3}{5}\right)}$$

Example. A pencil cup with a capacity of $36\,\text{in}^3$ (V) is to be constructed in the shape of a rectangular box with a square base and an open top. If the material for the sides cost \$0.15/in^2 and the material for the base costs \$0.40/in^2, what should the dimensions of the cup be to minimize the construction cost?



$\text{Cost} = (\text{Area of sides}) \cdot 0.15/\text{in}^2 + (\text{Area of base})\, 0.40/\text{in}^2$

$= 4(xy) \cdot 0.15 + (x^2)\, 0.40$

$V = l \cdot w \cdot h$

$= x^2 y$

$36 = x^2 y$

$\frac{36}{x} = xy$

$C(x) = 4\left(\frac{36}{x}\right) \cdot 0.15 + (x^2)\, 0.40 = 21.6\, x^{-1} + 0.40\, x^2$

$C'(x) = -21.6\, x^{-2} + 0.80\, x = \frac{-21.6}{x^2} + 0.80\, x$

$\left.\begin{array}{l} x \geq 0 \\ y \geq 0 \end{array}\right\} x > 0$

Solve $C'(x) = 0$ & $C'(x)$ DNE

$\frac{-21.6}{x^2} + 0.80\, x = 0$

$x^2 \neq 0$

$x = 0$

$0.80\, x = \frac{21.6}{x^2}$

$x^3 = 27$

$x = 3$

Verify (First deriv test)

	3	
	1	10
$C'(x)$	$-$	$+$

$\Rightarrow$ min at $x = 3$

$\boxed{C(3) = \$10.80}$