

1.2: The Language of Sets

Definition.

- A **set** is a collection of objects.
- If S is a set, then we use
 - $x \in S$ to denote that the element x is in the set S .
 - $x \notin S$ to denote that the element x is *not* in the set S .
- The **set-roster notation** is used to denote all elements in a set between braces:

$$S = \{1, 2, \dots, 100\}$$

Here, we see that $67 \in S$, but $1337 \notin S$.

- The **axiom of extension** says that a set is completely determined by what its elements are – not the order in which they are listed.

Example.

Let $A = \{1, 2, 3\}$, $B = \{3, 1, 2\}$, and $C = \{1, 1, 2, 3, 3, 3\}$. What are the elements of A , B , and C ? How are A , B , and C related?

Is $\{0\} = 0$?

How many elements are in the set $\{1, \{1\}\}$?

For each nonnegative integer n , let $U_n = \{n, -n\}$. Find U_1 , U_2 , and U_0 .

Certain sets of numbers are so frequently referred to that they are given special names and symbols:

N or \mathbb{N}	The set of all natural numbers
Z or \mathbb{Z}	The set of all integers
Q or \mathbb{Q}	The set of all rational numbers , or quotient of integers
R or \mathbb{R}	The set of all real numbers

Note: We may additionally use superscripts to indicate further properties of these sets:

\mathbb{Z}^+ or $\mathbb{Z}^{>0}$	The set of <i>positive</i> integers
\mathbb{Q}^- or $\mathbb{Q}^{<0}$	The set of <i>negative</i> rational numbers
\mathbb{R}^{nonneg} or $\mathbb{R}^{\geq 0}$	The set of <i>nonnegative</i> real numbers

Note: Different sources denote the natural numbers \mathbb{N} as \mathbb{Z}^+ or $\mathbb{Z}^{\geq 0}$.

Definition. (Set-Builder Notation)

Let S be a set and let $P(x)$ be a property that elements of S may or may not satisfy. We may define a new set to be **the set of all elements x in S such that $P(x)$ is true**. We denote this set as follows:

$$\{x \in S \mid P(x)\}$$

the set of all such that

Example. Describe each of the following sets:

$$\{x \in \mathbb{R} \mid -2 < x < 5\}$$

$$\{x \in \mathbb{Z} \mid -2 < x < 5\}$$

$$\{x \in \mathbb{Z}^+ \mid -2 < x < 5\}$$

Definition.

If A and B are sets, then A is called a **subset** of B , written $A \subseteq B$, if, and only if, every element of A is also an element of B :

$A \subseteq B$ means that for every element x , if $x \in A$, then $x \in B$.

$A \not\subseteq B$ means that there is at least one element x , such that $x \in A$ and $x \notin B$.

A is a **proper subset** of B if, and only if, every element of A is in B , but there is at least one element of B that is not in A :

$A \subsetneq B$ means that for every element x , if $x \in A$, then $x \in B$,
and there exists $x \in B$ such that $x \notin A$.

Example. Let $A = \mathbb{Z}^+$, $B = \{n \in \mathbb{Z} \mid 0 \leq n \leq 100\}$, and $C = \{100, 200, 300, 400, 500\}$. Evaluate the truth and falsity of each of the following statements.

$$B \subseteq A$$

C is a proper subset of A

C and B have at least one element in common

$$C \subseteq B$$

$$C \subseteq C$$

Example. Determine which of the following statements are true:

$$2 \in \{1, 2, 3\}$$

$$\{2\} \in \{1, 2, 3\}$$

$$2 \subseteq \{1, 2, 3\}$$

$$\{2\} \subseteq \{1, 2, 3\}$$

$$\{2\} \subseteq \{\{1\}, \{2\}\}$$

$$\{2\} \in \{\{1\}, \{2\}\}$$

Definition.

Given elements a and b , the symbol (a, b) denotes the **ordered pair** consisting of a and b together with the specification that a is the first element of the pair, and b is the second element. Two ordered pairs (a, b) and (c, d) are equal if, and only if, $a = c$ and $b = d$:

$$(a, b) = (c, d) \text{ means that } a = c \text{ and } b = d.$$

Example.

$$\text{Is } (1, 2) = (2, 1)?$$

$$\text{Is } \left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right)?$$

Definition.

Let $n \in \mathbb{N}$ and let x_1, x_2, \dots, x_n be (not necessarily distinct) elements. The **ordered n -tuple**, (x_1, x_2, \dots, x_n) , consists of x_1, x_2, \dots, x_n together with the ordering: first x_1 , then x_2 , and so forth up to x_n . and ordered 2-tuple is called an **ordered pair**, and ordered 3-tuple is called an **ordered triple**.

Two ordered n -tuples (x_1, x_2, \dots, x_n) and (y_1, y_2, \dots, y_n) are **equal** if, and only if, $x_1 = y_1, x_2 = y_2, \dots$, and $x_n = y_n$:

$$(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n) \iff x_1 = y_1, x_2 = y_2, \dots, x_n = y_n.$$

Example.

$$\text{Is } (1, 2, 3, 4) = (1, 2, 4, 3)?$$

$$\text{Is } \left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right)?$$

Definition.

Given sets A_1, A_2, \dots, A_n , the **Cartesian product** of A_1, A_2, \dots, A_n , denoted

$$A_1 \times A_2 \times \cdots \times A_n$$

is the set of all ordered n -tuples (a_1, a_2, \dots, a_n) where $a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n$:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n\}$$

Example. Let $A = \{x, y\}$, $B = \{1, 2, 3\}$, and $C = \{a, b\}$. Find the following:

$$A \times B$$

$$B \times A$$

$$A \times A$$

How many elements are in $A \times B$, $B \times A$, and $A \times A$?

$$(A \times B) \times C$$

$$A \times B \times C$$

Describe $\mathbb{R} \times \mathbb{R}$

Definition.

Let $n \in \mathbb{N}$. Given a finite set A , a **string of length n over A** is an ordered n -tuple of elements of A written without parentheses or commas. The elements of A are called the **characters** of the string. The **null string** over A is defined to be the “string” with no characters, often denoted λ , and is said to have length 0. If $A = \{0, 1\}$, then a string over A is called a **bit string**.

Example. Let $A = \{a, b\}$. List all strings of length 3 over A with at least two characters that are the same.