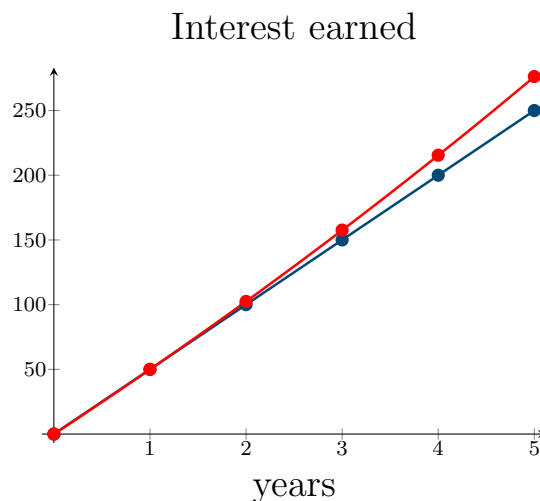


6.2: Compound Interest

Example. Suppose you invest \$1,000 at 5% annual interest. With simple interest, you can take 2 approaches:

1. Gain interest on *only* your initial investment
2. Reinvest the interest gained

Year	Simple interest	Simple interest reinvested
1	\$1,050.00	\$1,050.00
2	\$1,100.00	\$1,102.50
3	\$1,150.00	\$1,157.63
4	\$1,200.00	\$1,215.51
5	\$1,250.00	\$1,276.28



Definition.

Compound interest is a method where the interest for each period is added to the principal before interest is calculated for the next period.

Example. Using the example above, derive a formula for the future value of an investment compounded annually.

$$1: 1000(1+0.05) = \$1050.00$$

$$2: 1050(1+0.05) = [1000(1+0.05)](1+0.05) = 1000(1+0.05)^2 = \$1102.50$$

$$3: 1102.50(1+0.05) = [1000(1+0.05)^2](1+0.05) = 1000(1+0.05)^3 = \$1157.63$$

$$\Rightarrow n: 1000(1+0.05)^n \Rightarrow \boxed{S = P(1+r)^n}$$

Definition.

When interest is compounded multiple times a year (e.g. quarterly, monthly, etc.), the **nominal interest rate** is the interest rate *per year*.

If \$ P is invested for t years at a nominal interest rate r compounded m times per year, then the **total number of compounding periods** is

$$n = mt$$

the interest rate per compounding period (**periodic interest rate**) is

$$i = \frac{r}{m}$$

and the future value is

$$S = P(1 + i)^n = P\left(1 + \frac{r}{m}\right)^{mt}$$

Example. If \$3,000 is invested for 5 years at 9% compounded 4 times a year, how much interest is earned? P t r m

$$\begin{aligned} S &= 3000 \left(1 + \frac{0.09}{4}\right)^{4(5)} \\ &= 3000(1.0225)^{20} \\ &= 3000(1.9494) \\ &= \$5848.18 \end{aligned}$$

$$\begin{aligned} S &= P + I \\ 5848.18 &= 3000 + I \Rightarrow I = \$2848.18 \end{aligned}$$

Example. For the following, identify the annual interest rate, the length in years, the periodic interest rate, and the number of periods:

12% compounded monthly for 7 years
 $r = 0.12$ $m = 12$ $t = 7$ $mt = 84$

7.2% compounded quarterly for 11 quarters
 $r = 0.072$ $m = 4$ $mt = 11 \Rightarrow t = \frac{11}{4} = 2.75$

Frequency	m
Annually	1
Semi-annually	2
Quarterly	4
Monthly	12
Weekly	52
Daily	365

Example. Ben and Taylor want to have \$200,000 in Arthur's college fund on his 18th birthday, and they want to know the impact on this goal of having \$10,000 invested at 9.8%, compounded quarterly, on his 1st birthday. To advise Ben and Taylor regarding this, find

the future value of the \$10,000 investment,

$P = 10,000$
 $r = 9.8\% = 0.098$
 $t = 18 - 1 = 17 \text{ yrs}$
 $m = 4$

$S = 10,000 \left(1 + \frac{0.098}{4} \right)^{4(17)}$
 $= 10,000 (1.0245)^{68}$
 $= 10,000 (5.1858) = \boxed{\$51,857.73}$

the amount of compound interest that the investment earns,

$$S = P + I$$

$$51,857.73 = 10,000 + I \Rightarrow I = 41,857.73$$

the impact this would have on their goal.

$$\frac{51,857.73}{200,000} (100\%) = 25.9\%$$

They'll earn about 25.9% of their goal.

Example. What amount must be invested now to have $\overbrace{\$12,000}^S$ after $\overbrace{3}^t$ years with an interest rate of $\underbrace{6\%}_{r=0.06}$, compounded $\underbrace{\text{semi-annually}}_{m=2}$? Find P

$$S = P \left(1 + \frac{r}{m} \right)^{mt} \Rightarrow 12000 = P \left(1 + \frac{0.06}{2} \right)^{2(3)}$$

$$\frac{12000}{(1.03)^6} = \frac{P(1.03)^6}{(1.03)^6}$$

$$\frac{12000}{(1.03)^6} = P \approx \$10,049.81$$

Example. Three years after Google stock was first sold publicly, its share price had risen 500%. Google's 500% increase means that \$10,000 invested in Google stock at its initial public offering (I.P.O) was worth \$60,000 three years later. What interest rate compounded annually does this represent?

$$S = 60000$$

$$P = 10000$$

$$r \leftarrow \text{find}$$

$$m = 1$$

$$t = 3$$

$$\frac{60000}{10000} = \frac{10000 \left(1 + \frac{r}{1} \right)^{1(3)}}{10000}$$

$$(6)^{1/3} = ((1+r)^3)^{1/3}$$

$$-1 + 6^{1/3} = 1 + r - 1$$

$$6^{1/3} - 1 = r \approx 0.817$$

$$= 81.7\%$$

Example. Suppose we invest \$1 at a 100% interest rate for 1 year:

$$S = \left(1 + \frac{1.00}{m}\right)^m$$

Compute the future value

Annually

$$S = \left(1 + \frac{1.00}{1}\right)^1 = 2$$

Semi-annually

$$S = \left(1 + \frac{1.00}{2}\right)^2 = 2.25$$

Monthly

$$S = \left(1 + \frac{1.00}{12}\right)^{12} = 2.6130$$

Weekly

$$S = \left(1 + \frac{1.00}{52}\right)^{52} = 2.6926$$

Daily

$$S = \left(1 + \frac{1.00}{365}\right)^{365} = 2.7146$$

Each minute ($m = 525,600$)

$$S = \left(1 + \frac{1.00}{1}\right)^1 = 2.7183$$

Definition.

If \$ P is invested for t years at a nominal rate r compounded continuously, then the future value is given by the exponential function

$$S = Pe^{rt}$$

Example. Which investment strategy is worth more: \$3,000 for 8 years at

9%, compounded annually

$$\begin{aligned} S &= 3000 \left(1 + \frac{0.09}{1}\right)^{1(8)} \\ &= 3000 (1.09)^8 \\ &= 3000 (1.9926) \\ &= \$5977.69 \end{aligned}$$

8%, compounded continuously

$$\begin{aligned} S &= 3000 e^{0.08(8)} \\ &= 3000 e^{0.64} \\ &= 3000 (1.8965) \\ &= \$5689.44 \end{aligned}$$

Example. Suppose you invest \$900 at 11.5%, compounded continuously. How long will it take to gain \$700 in interest?

$$\begin{aligned} S &= P + I & \boxed{\text{Find } t} & \quad \frac{700}{900} = \frac{900(e^{0.115t} - 1)}{900} \\ \Rightarrow I &= S - P & & \quad 1 + \frac{7}{9} = e^{0.115t} - 1 + 1 \\ &= Pe^{rt} - P & & \quad \ln\left(\frac{16}{9}\right) = \ln(e^{0.115t}) \\ &= P[e^{rt} - 1] & & \quad \frac{\ln\left(\frac{16}{9}\right)}{0.115 \ln(e)} = \frac{0.115t \ln(e)}{0.115 \ln(e)} \\ & & & \quad \rightarrow t = \frac{\ln\left(\frac{16}{9}\right)}{0.115 \ln(e)} \\ & & & \quad \boxed{t \approx 5.0032} \\ & & & \quad \text{About 5 years} \end{aligned}$$

Definition.

Let r represent the annual (nominal) interest rate for an investment. Then the **annual percentage yield (APY)** is:

Periodic compounding:

$$\text{APY} = \left(1 + \frac{r}{m}\right)^m - 1$$

Continuous compounding:

$$\text{APY} = e^r - 1$$