

1.1: Solutions of Linear Equations and Inequalities in One Variable

Definition.

A **function** f is a special relation between x and y such that each input x results in *at most* one y . The symbol $f(x)$ is read “ f of x ” and is called the **value of f at x** .

Example. Let $f(x) = 4x - 1$. Evaluate the following:

$$\begin{aligned} f(1) &= 4(1) - 1 \\ &= 4 - 1 \\ &= \boxed{3} \end{aligned}$$

$$\begin{aligned} f\left(\frac{1}{2}\right) &= 4\left(\frac{1}{2}\right) - 1 \\ &= \frac{4}{2} - 1 \\ &= 2 - 1 = \boxed{1} \end{aligned}$$

$$\begin{aligned} f(-2) &= 4(-2) - 1 \\ &= -8 - 1 \\ &= \boxed{-9} \end{aligned}$$

$$\begin{aligned} f(0) &= 4(0) - 1 \\ &= 0 - 1 \\ &= \boxed{-1} \end{aligned}$$

$$f(\textcircled{S}) = \boxed{4\textcircled{S} - 1}$$

$$\begin{aligned} f(f(x)) &= 4(f(x)) - 1 \\ &= 4(\overbrace{4x - 1}) - 1 \\ &= 16x - 4 - 1 = \boxed{16x - 5} \end{aligned}$$

Composite Functions:

Let f and g be functions of x . Then, the **composite functions** g of f (denoted $g \circ f$) and f of g (denoted $f \circ g$) are defined as:

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ (f \circ g)(x) &= f(g(x)) \end{aligned}$$

Example. Let $g(x) = x - 1$. Find:

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= (f(x)) - 1 \\ &= (4x - 1) - 1 \\ &= \boxed{4x - 2} \end{aligned}$$

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= 4(g(x)) - 1 \\ &= 4(\overbrace{x - 1}) - 1 \\ &= 4x - 4 - 1 \\ &= \boxed{4x - 5} \end{aligned}$$

Operations with Functions:

Let f and g be functions of x and define the following:

Sum	$(f + g)(x) = f(x) + g(x)$
Difference	$(f - g)(x) = f(x) - g(x)$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$
Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ if $g(x) \neq 0$

Definition.

An **expression** is a meaningful string of numbers, variables and operations:

$$3x - 2$$

An **equation** is a statement that two quantities or algebraic expressions are equal:

$$3x - 2 = 7$$

A **solution** is a value of the variable that makes the equation true:

$$\begin{aligned} 3(3) - 2 &= 7 \\ 9 - 2 &= 7 \\ 7 &= 7 \end{aligned}$$

A **solution set** is the set of ALL possible solutions of an equation:

$3x - 2 = 7$ only has the solution $x = 3$,

$2(x - 1) = 2x - 2$ is true for all possible values of x .

Properties of Equality:

Substitution Property: The equation formed by substituting one expression for an equal expression is equivalent to the original equation:

$$\begin{aligned}3(x - 3) - \frac{1}{2}(4x - 18) &= 4 \\3x - 9 - 2x + 9 &= 4 \\x &= 4\end{aligned}$$

Addition Property: The equation formed by adding the same quantity to both sides of an equation is equivalent to the original equation:

$$\begin{array}{ll}x - 4 = 6 & x + 5 = 12 \\x - 4 + 4 = 6 + 4 & x + 5 + (-5) = 12 + (-5) \\x = 10 & x = 7\end{array}$$

Multiplication Property: The equation formed by multiplying both sides of an equation by the same *nonzero* quantity is equivalent to the original equation:

$$\begin{array}{ll}\frac{1}{3}x = 6 & 5x = 20 \\3\left(\frac{1}{3}x\right) = 3(6) & \frac{5x}{5} = \frac{20}{5} \\x = 18 & x = 4\end{array}$$

Solving a linear equation:

Using the properties of equality above, we can solve any linear equation in 1 variable:

Example. Solve $\frac{3x}{4} + 3 = \frac{x-1}{3}$

1. Eliminate fractions:

$$12\left(\frac{3x}{4} + 3\right) = 12\left(\frac{x-1}{3}\right)$$

$$9x + 36 = 4x - 4$$

2. Remove/evaluate parenthesis:

$$9x + 36 - 36 - 4x = 4x - 4 - 36 - 4x$$

3. Use addition property to isolate the variable to one side:

$$\frac{5x}{5} = \frac{-40}{5}$$

4. Use multiplication property to isolate variable:

$$\frac{3(-8)}{4} + 3 \stackrel{?}{=} \frac{(-8) - 1}{3}$$

$$\underbrace{-6 + 3}_{-3} = \underbrace{\frac{-9}{3}}_{-3} = -3$$

5. Verify solution via substitution:

Example. Solve the following:

$$(2)(3)\left(\frac{3x+1}{2}\right) = \left(\frac{x}{3} - 3\right)(2)(3)$$

$$3(3x+1) = 2x - (2)(3)3$$

$$9x + 3 - 2x = 2x - 18 - 2x$$

$$7x + 3 - 3 = -18 - 3$$

$$\frac{7x}{7} = \frac{-21}{7}$$

$$x = -3$$

$$(x-3)\left(\frac{2x-1}{x-3}\right) = \left(4 + \frac{5}{x-3}\right)(x-3)$$

$$2x-1 = 4(x-3) + 5$$

$$2x-1 - 4x = 4x - 12 + 5 - 4x$$

$$-2x - 1 + 1 = -7 + 1$$

$$\frac{-2x}{-2} = \frac{-6}{-2}$$

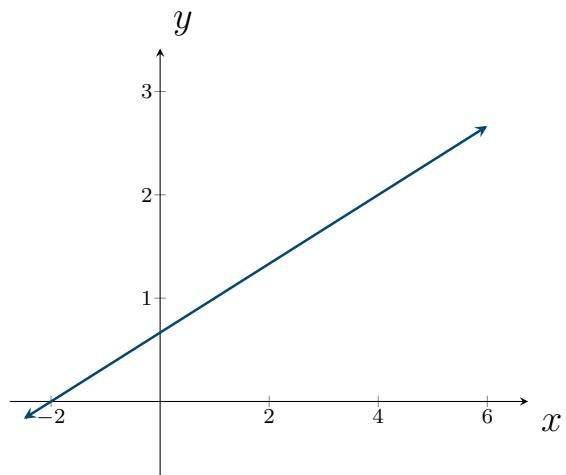
$$x = 3$$

Verification of solution omitted

When verifying this solution, we see that both sides are undefined at $x=3$.

Example. Solve $-2x + 6y = 4$ for y

$$\begin{aligned} -2x + 6y + 2x &= 4 + 2x \\ 6y &= 2x + 4 \\ \frac{6y}{6} &= \frac{2x}{6} + \frac{4}{6} \\ y &= \frac{2x}{6} + \frac{4}{6} \\ Y &= \frac{x}{3} + \frac{2}{3} \end{aligned}$$



Example. Suppose that the relationship between a firm's profit, P , and the number of items sold, x , can be described by the equation

$$5x - 4P = 1200$$

- a) How many units must be produced and sold for the firm to make a profit of \$150?

$$P = \$150, \text{ Find } x$$

$$\begin{aligned} 5x - 4(150) &= 1200 \\ 5x - 600 + 600 &= 1200 + 600 \\ \frac{5x}{5} &= \frac{1800}{5} \\ x &= 360 \end{aligned}$$

- b) Solve this equation for P in terms of x . Then, find the profit when 240 units are sold.

$$\begin{aligned} 5x - 4P - 5x &= 1200 - 5x \\ -4P &= \frac{-5x + 1200}{-4} \\ P &= \frac{-5x}{-4} + \frac{1200}{-4} \\ P &= \frac{5x}{4} - 300 \end{aligned}$$

$$\begin{aligned} x &= 240 \\ P &= \frac{5(240)}{4} - 300 \\ &= 5(60) - 300 \\ &= 300 - 300 \\ P &= 0 \end{aligned}$$

Break-even point

Example. Jill Ball has \$90,000 to invest. She has chosen one relatively safe investment fund that has an annual yield of 10% and another riskier one that has a 15% annual yield. How much should she invest in each fund if she would like to earn \$10,000 in one year from her investments?

	Investment Amt	Return
10%	$90,000 - x$	$0.10(90,000 - x) = 9,000 - 0.1x$
15%	x	$0.15x$

$$-9,000 + \underbrace{9,000 - 0.1x + 0.15x}_{0.05x} = 10,000 - 9,000$$

$$\frac{0.05x}{0.05} = \frac{1,000}{0.05} \Rightarrow x = 20,000$$

\$ 70,000 @ 10%
\$ 20,000 @ 15%

Example. A woman making \$2,000 per month has her salary reduced by 10% because of sluggish sales. One year later, after a dramatic improvement in sales, she is given a 20% raise over her reduced salary. Find her salary after the raise. What percent change is this from the \$2,000 per month?

$$\begin{array}{l} \text{Salary} \\ \text{After} \\ \text{Raise} \end{array} \quad 2000 \quad \begin{array}{l} \text{increase} \\ \cancel{(1-0.10)}(1+0.20) = 2000(0.9)(1.2) = \$2,160 \\ \text{10\%} \end{array}$$

decrease

$$\begin{array}{l} \text{Percent} \\ \text{change} \end{array} \quad \frac{2160 - 2000}{2000} = 0.08 = 8\% \quad \text{or} \quad 0.9(1.2) = \begin{array}{l} 1.08 \\ \text{100\% of original salary} \end{array} \quad \begin{array}{l} \text{8\% increase} \end{array}$$

Definition.

An **inequality** is a statement that one quantity is greater than (or less than) another quantity.

Properties of Inequalities

Substitution Property: The inequality formed by substituting one expression for an equal expression is equivalent to the original inequality:

$$5x - 4x + 2 < 6$$

$x < 4 \Rightarrow$ The solution set is $\{x : x < 6\}$

Addition Property: The inequality formed by adding the same quantity to both sides of an inequality is equivalent to the original inequality:

$$x - 4 < 6$$

$$x + 5 \geq 12$$

$$x - 4 + 4 < 6 + 4$$

$$x + 5 + (-5) \geq 12 + (-5)$$

$$x < 10$$

$$x \geq 7$$

Multiplication Property The inequality formed by multiplying both sides of an inequality by the same *positive* quantity is equivalent to the original inequality. The direction of the inequality is flipped when multiplying by a *negative* quantity:

$$\frac{1}{3}x > 6$$

$$5x - 5 + 5 \leq 6x + 20 + 5$$

$$3\left(\frac{1}{3}x\right) > 3(6)$$

$$\frac{-x}{-1} \leq \frac{25}{-1}$$

$$x > 18$$

$$x \geq -25$$

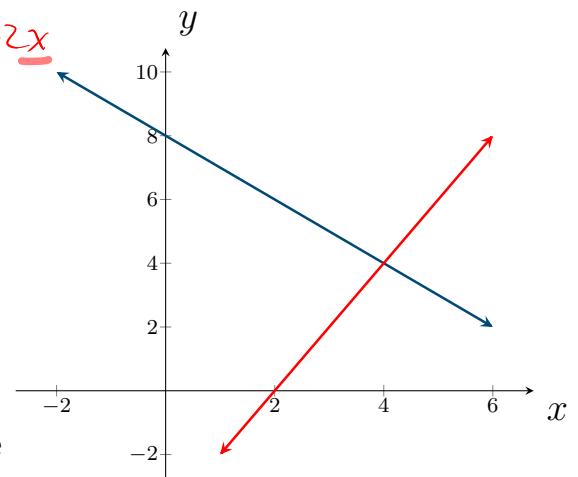
Example. Solve

$$-x + 8 \leq 2x - 4$$

first by gathering the x variable on the left, then again on the right. See that the multiplication property holds in both cases. Plot the solution set on a numberline.

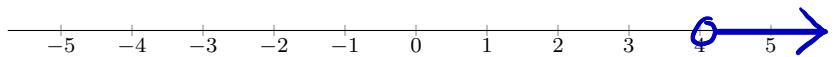
$$\begin{aligned} -x + 8 + x &\leq 2x - 4 + x \\ 8 + 4 &\leq 3x - 4 + 4 \\ 12 &\leq 3x \\ \frac{12}{3} &\leq \frac{3x}{3} \\ 4 &\leq x \end{aligned}$$

$$\begin{aligned} -x + 8 - 2x &\leq 2x - 4 - 2x \\ -3x + 8 - 8 &\leq -4 - 8 \\ -3x &\leq -12 \\ \frac{-3x}{-3} &\leq \frac{-12}{-3} \\ x &\geq 4 \end{aligned}$$



* Note:

- The inequality on the right changed direction in the last step because we divided by a negative number
- The answers above are equivalent. They are only formatted differently.



Example. Plot the following inequalities:

$$x \leq 2$$

$$x > -3$$

