2.1: Quadratic Equations

Definition.
A **quadratic equation** in one variable is an equation of second degree that can be written in the *general form* as

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

where a, b, and c represent constants.

The **zero product property** states that for real numbers a and b, $ab = 0$ if and only if $a = 0$ or $b = 0$ or both.

Example. Solve the following for x:

$$x\underbrace{(x+3)}=0$$

$\boxed{x=0}$

$x+3=0$

$\boxed{x=-3}$

$$\underbrace{(x-4)}\underbrace{(3x+1)}=0$$

$x-4=0$

$\boxed{x=4}$

$3x+1=0$

$3x=-1$

$\boxed{x=-1/3}$

Solving quadratic equations via factoring:

Example. Solve $2x^2 + x = 3x + 12$

1. Rewrite the equation in the general form: $2x^2 - 2x - 12 = 0$
2. Rewrite bx using factors of ac: $2x^2 - 6x + 4x - 12 = 0$
3. Factor out like terms: $2x(x-3) + 4(x-3) = 0$
4. Factor by grouping: $(x-3)(2x+4) = 0$
5. Solve for the roots: $x = 3$ and $x = -2$

Example. Solve the following for x via factoring:

$$(x+3)(x-1) = 5$$

$$x(x-1) + 3(x-1) = 5$$

$$x^2 - x + 3x - 3 = 5$$

$$x^2 + 2x - 8 = 0$$

Sign of larger number

Factors have diff signs

$1 \cdot 8$

$2 \cdot 4$

$$x^2 \underbrace{+4x - 2x}_{} - 8 = 0$$

Order doesn't matter!

$$x(x+4) - 2(x+4) = 0$$

These should match!

$$(x-2)(x+4) = 0 \Rightarrow \boxed{x = 2,\ x = -4}$$

$$(-1)(-4x^2 + 8x - 3) = 0$$

$$4x^2 - 8x + 3 = 0$$

Sign of larger number

Factors have same signs

$4 \cdot 3 = 1 \cdot 12$

$2 \cdot 6$

$3 \cdot 4$

$$4x^2 - 2x - 6x + 3 = 0$$

$$2x(2x-1) - 3(2x-1) = 0$$

$$(2x-3)(2x-1) = 0$$

$$\Rightarrow \boxed{x = 3/2,\ x = 1/2}$$

$$\boxed{\text{Solutions to } x^2 = C \text{ are } x = \pm\sqrt{C}}$$

Example. Solve the following:

$\sqrt{(x-1)^2} = \sqrt{9}$

$x - 1 = \pm 3$

$x = 1 \pm 3$

$x = 1 - 3$

$\boxed{x = -2}$

$x = 1 + 3$

$\boxed{x = 4}$

Verify

$((-2) - 1)^2 = (-3)^2 = 9$ ✓

$((4) - 1)^2 = (3)^2 = 9$ ✓

$4x^2 - 1 = 0$

$\sqrt{4x^2} = \sqrt{1}$

$2x = \pm 1$

$\boxed{x = \pm \frac{1}{2}}$

Verify

$4\left(-\frac{1}{2}\right)^2 - 1 = 4\left(\frac{1}{4}\right) - 1 = 0$ ✓

$4\left(\frac{1}{2}\right)^2 - 1 = 4\left(\frac{1}{4}\right) - 1 = 0$ ✓

Definition.
The **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

gives the solutions to $ax^2 + bx + c = 0$.

Quadratic equations can have one, two, or no solutions. The **discriminant** is $b^2 - 4ac$:

- $b^2 - 4ac > 0$: The equation has *exactly* two distinct *real* solutions.
- $b^2 - 4ac = 0$: The equation has *exactly* one *real* solution.
- $b^2 - 4ac < 0$: The equation has no *real* solutions.

Example. Suppose some hooligans kick a ball up in the air off the roof of the library. Assuming the height, in ft, of the ball t seconds after kicking it is given by

$$h(t) = -32t^2 + 64t + 40$$

Solve for t when

the ball is 80 feet off of the ground

$$-80 + 80 = -32t^2 + 64t + 40 - 80$$

$$0 = -32t^2 + 64t - 40$$

$$t = \frac{-64 \pm \sqrt{64^2 - 4(-32)(-40)}}{2(-32)}$$

$$= \frac{-64 \pm \sqrt{\boxed{-1024}}}{-64}$$

$b^2 - 4ac < 0$

$\boxed{\Rightarrow \text{No solutions!}}$

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the ball is 72 feet off of the ground

$$-72 + 72 = -32t^2 + 64t + 40 - 72$$

$$0 = -32t^2 + 64t - 32$$

$$t = \frac{-64 \pm \sqrt{64^2 - 4(-32)(-\ \ )}}{2(-32)}$$

$$= \frac{-64 \pm \sqrt{\boxed{0}}}{-64}$$

$$= \boxed{1}$$

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the ball is 40 feet off of the ground

$$-40 + 40 = -32t^2 + 64t + 40 - 40$$

$$0 = -32t^2 + 64t$$

$$t = \frac{-64 \pm \sqrt{64^2 - 4(-32)(0)}}{2(-32)}$$

$$= \frac{-64 \pm \sqrt{\boxed{64^2}}}{2(-32)}$$

$b^2 - 4ac > 0$

$\Rightarrow$ Two solutions!

$$= \frac{-64 \pm 64}{-64} \rightarrow \boxed{\begin{array}{l} t = 0 \\ t = 2 \end{array}}$$

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the ball hits the ground

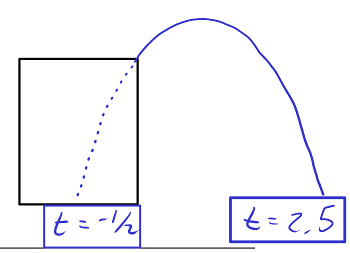
$$0 = -32t^2 + 64t + 40$$

$$t = \frac{-64 \pm \sqrt{64^2 - 4(-32)(40)}}{2(-32)}$$

$$= \frac{-64 \pm \sqrt{9216}}{2(-32)} = \frac{-64 \pm 96}{-64}$$

$$t = \frac{-64 - 96}{-64} = 2.5$$

$$t = \frac{-64 + 96}{-64} = -1/2$$



Example. The Social Security Trust Fund balance B, in billions of dollars, can be described by the function $B = -7.97t^2 + 312t - 356$ where t is the number of years past the year 1995. For planning purposes, it is important to know when the trust fund balance will be 0. Solve

$$0 = \underbrace{-7.97t^2}_{a} \underbrace{+\,312t}_{b} \underbrace{-\,356}_{c}.$$

$$t = \frac{-312 \pm \sqrt{(312)^2 - 4(-7.97)(-356)}}{2(-7.97)}$$

$$= \frac{-312 \pm \sqrt{85994.72}}{-15.97}$$

$\rightarrow t \approx 37.97$

$\rightarrow t \approx 1.78$