## 0.1 Recursive formulation

We start with the equation

$$(\rho + m\delta(x-L))\partial_{tt}\psi = \partial_{xx}\psi, \tag{1}$$

which after the substitutions

$$x \to x' = \frac{m}{\rho}x \tag{2}$$

$$t \to t' = \frac{m}{\sqrt{2\rho}}t \tag{3}$$

$$L' = \frac{\rho L}{m} \tag{4}$$

becomes

$$(1 + \delta(x-L))\partial_{tt}\psi = \frac{1}{2}\partial_{xx}\psi, \tag{5}$$

with primes omitted, so the problem has one parameter (which can be also shown using dimensional analysis). Futher, the most general ansatz for the escaping wave to the right of the point mass is

$$\psi(x,t) = \begin{cases} f(x-t) + g(x+t) & x \in [0, L] \\ h(x-t) & x \geq L, \end{cases} \tag{6}$$

with conditions

$$\psi(0,t) = 0 \quad \text{zero at the boundary} \tag{7}$$

$$\psi(L-,t) = \psi(L+,t) \quad \text{continuity.} \tag{8}$$

which become

$$\psi(x,t) = \begin{cases} -g(t-x) + g(x+t) & x \in [0, L] \\ h(x-t) & x \geq L. \end{cases} \tag{9}$$

$$-g(t-L) + g(L+t) = h(L-t) \quad \text{continuity} \tag{10}$$

Integrating (1) we obtain

$$\partial_{tt}\psi(L,t) = \frac{1}{2}(\partial_x\psi(L+,t) - \partial_x\psi(L-,t)). \tag{11}$$

which glues together the solutions from different intervals along with the continuity condition (8).

The preceding statements give rise to the following equation for $g$

$$\begin{aligned} -g''(t-L) + g''(t+L) &= \frac{1}{2}(h'(L-t) - g'(t-L) - g'(t+L)) \\ &= \frac{1}{2}(g'(t-L) - g'(t+L) - g'(t-L) - g'(t+L)) \\ &= -g'(t+L), \end{aligned} \tag{12}$$

and after the shift in $t$ and with $T := 2L$ we get

$$g''(t+T) + g'(t+T) = g''(t) \tag{13}$$

integration gives us

$$g'(t+T) + g(t+T) = g'(t) + (g'(T) - g'(0) + g(T)). \tag{14}$$

Adding a constant to $g$ does not change the equations, so we can add such constant in order to cancel the $(g'(T) - g'(0) + g(T))$ term, and we are then given the final form of equation for $g$

$$g'(t+T) + g(t+T) = g'(t), \tag{15}$$

which is a kind of delayed differential equation, and for which one can iteratively find a solution for arbitrary large times given the function values in an interval $[a, a+T]$ where $a$ is an arbitrary constant for convinience we shall fix $a = 0$. Now we shall introduce a family of functions $g_n \in \mathcal{C}([0,T])$ defined as

$$g(t+nT) = g_n(t) \quad t \in [0,T] \tag{16}$$

so the equation (17) becomes

$$\begin{aligned} &g'_{n+1}(t) + g_{n+1}(t) = g'_n(t), \\ &g_{n+1}(0) = g_n(T) \quad \text{continuity condition.} \\ &\text{with} \quad g_0(t) \quad \text{initial conditions} \end{aligned} \tag{17}$$

The above equation can be solved by iteration

$$g_{n+1}(t) = (Lg_n)(t) = g_n(t) - e^{-t}\int_0^t e^s g_n(s)ds + e^{-t}(g_n(T) - g_n(0)). \tag{18}$$

With $L : \mathcal{C}([0,T]) \to \mathcal{C}([0,T])$. The eigenvectors of $L$ are the soultions of the equation (17) with $g_{n+1} = \lambda g_n = \lambda g_\lambda$

$$\lambda(g'_\lambda(t) + g_\lambda(t)) = g'_\lambda(t), \tag{19}$$

$$\lambda g_\lambda(0) = g_\lambda(T) \quad \text{quantization.} \tag{20}$$

from which we obtain

$$g_\lambda(t) = e^{\frac{\lambda}{\lambda+1}t}, \tag{21}$$

$$\lambda = e^{\frac{\lambda}{\lambda+1}T} \quad \text{quantization.} \tag{22}$$

The last term in (18) may look suspicious due to its form (explicit function times constant) but for eigenvectors it is simply zeros altogether with lower boundary of integration according to the quantization condition. The quantization for $\lambda$ gives rise to the quasi-normal modes of equation (1).

## 0.2 Quasinormal Modes - asymptotic expansion

The transcendental equation for $\lambda$ can be written as

$$\frac{\beta}{\beta+1} = e^{\beta T} \tag{23}$$

where $\lambda = \frac{\beta}{\beta+1}$. We shall split $\beta$ into its real and imaginary part: $\beta = i\Omega - \Gamma$. Taking the absolute value of (23) we obtain the following equation

$$\frac{\Omega^2 + (\Gamma - 1)^2}{\Omega^2 + \Gamma^2} = e^{2\Gamma T} \tag{24}$$

Let's assume $\Gamma < 0$, then from the equality

$$e^{2\Gamma T} - 1 = \frac{1 - 2\Gamma}{\Omega^2 + \Gamma^2} \tag{25}$$

contradiction appears and so we deduce $\Gamma > 0$, which in turn leads to the fact that $\Gamma < \frac{1}{2}$. From the fact $\Gamma > 0$ one can conclude that $L$ is bounded, and thus continous on the space spanned by its eigenvectors because

$$|\lambda|^2 = e^{-2\Gamma T} < 1. \tag{26}$$

Equation (24) can be solved for $\Omega^2$

$$\begin{aligned} \Omega^2 &= \frac{1 - 2\Gamma - \Gamma^2(e^{2\Gamma T} - 1)}{e^{2\Gamma T} - 1} \\ &= \frac{1}{2\Gamma T} - \frac{2+T}{2T} + \Gamma\frac{6+T}{6} + \mathcal{O}(\Gamma^2), \\ \Omega &= \pm(\frac{1}{\sqrt{2T}}\frac{1}{\sqrt{\Gamma}} - \frac{2+T}{2\sqrt{2T}}\sqrt{\Gamma} + \mathcal{O}(\Gamma^{3/2})) \end{aligned} \tag{27}$$

We shall introduce a new parameter $\eta := \frac{\sqrt{T}}{2\sqrt{\pi\Gamma}}$ obtaining the parametrization $\beta(\eta)$ (the reason for such peculiar parametrization will become clear later.

$$\Gamma = \frac{T}{2\pi^2\eta^2}, \tag{28}$$

$$\Omega = \pm(\frac{\pi}{T}\eta - 2(2+T)\pi\frac{1}{\eta} + \mathcal{O}(\eta^{-3}), \tag{29}$$

$$\beta = \pm i(\frac{\pi}{T}\eta - 2(2+T)\pi\frac{1}{\eta}) - \frac{T}{2\pi^2\eta^2} + \mathcal{O}(\eta^{-3}) \tag{30}$$

Second equation for $\Omega$ and $\Gamma$ is given by

$$\Omega \cot \Omega T = \Gamma(\Gamma - 1) + \Omega^2 \tag{31}$$

the above equation can be solved for real $\Gamma$ when

$$\Delta = 1 - 4\Omega(\Omega - \cot(\Omega T)) \geq 0 \tag{32}$$

or equivalently for $\Omega \gg 1$

$$\cot(\Omega T) \geq \Omega \tag{33}$$

It can be easily seen (by assuming $\Omega T = n\pi + \epsilon_n$, $\epsilon_n \ll 1$) that (32) is fulfilled when $\Omega \in [n\pi/T, n\pi/T + \epsilon_n/T]$, where $\epsilon_n = (\sqrt{n^2\pi^2 + 4T} - n\pi)/2 \to T/n\pi$, hence $\Omega \to n\pi/T + \mathcal{O}(1/n)$. Substituting $\Omega T = n\pi + \epsilon$ to (32) we get

$$\Delta = 4\frac{n\pi}{T}(\frac{1}{\epsilon} - \frac{n\pi}{T}), \tag{34}$$

$$\Gamma_\pm = \frac{1}{2}(1 \pm \sqrt{\Delta}). \tag{35}$$

Up to this point we assumed $\Omega > 0$, but the same reasoning can be used for $\Omega < 0$ giving the same asymptotic relation (which is also a consequence of the fact, that if $\beta$ fulfills the (31) then also $\bar{\beta}$ does). Now the meaning of the parametrization in (28) becomes clear, $\beta(\eta)$ should approximate the solution of the (31) for $\eta \in \mathbb{Z}$.

Substituting $\Omega = 0$ to the (31) we get

$$\frac{\Gamma}{\Gamma - 1} = e^{-\Gamma T}, \tag{36}$$

which has no solution for real $\Gamma$, and so there is no real eigenvalue for $L$.

## 0.3 Iterations of $L$

We should now calculate $g(t) = (L^n h_0)(t - nT)$ for

$$h_0(t) = \sum_\lambda a_\lambda g_\lambda(t). \tag{37}$$

We have

$$(L^n h_0)(t) = \sum_\lambda \lambda^n a_\lambda g_\lambda(t). \tag{38}$$

Utilizing quantization condition for $\lambda$ we obtain

$$(L^n h_0)(t) = \sum_\lambda a_\lambda e^{\frac{\lambda}{\lambda+1}(t+nT)} = \tag{39}$$

$$\sum_\beta a_{\lambda(\beta)} e^{\beta(t+nT)} \tag{40}$$

Inserting the (28) to (39) yields

$$\begin{aligned}(L^n h_0)(t - nT) &= \sum_\eta a_{\lambda(\eta)} e^{\beta(\eta)t} \\ &\approx \int_{\eta_0}^\infty a_{\lambda(\eta)} \exp((ib\eta - \frac{ic}{\eta} - \frac{1}{\eta^2})t) d\eta + c.c.\end{aligned} \tag{41}$$

The remaining problem is the asymptotic form of $a_\lambda$ for reasonable initial data (which cannot be simply computed using scalar product, because eigenvectors are not orthogonal in their current form) and integration of the above integral to get energy leak in time (see (46)). The integration should be doable using the method of steepest descent or some other method for highly oscillatory integrands assuming $t \gg 1$. After this the asymptotic expansion in time should be easily obtained, and so the energy leak.

## 0.4 Energy

The energy of the whole string is given by the functional

$$\begin{aligned}E(\psi) &= \frac{1}{2}\int_0^\infty dx(2(1 + \delta(x - L))\psi_t^2 + \psi_x^2) \\ &= \frac{1}{2}\int_0^L dx(2\psi_t^2 + \psi_x^2) + \frac{1}{2}\int_L^\infty dx(2\psi_t^2 + \psi_x^2) + \psi_t^2\Big|_{x=L} \\ &= E_{in}(\psi) + E_{esc}(\psi)\end{aligned} \tag{42}$$

with

$$E_{in}(\psi) = \frac{1}{2}\int_0^L dx(2\psi_t^2 + \psi_x^2) + \psi_t^2\Big|_{x=L} \tag{43}$$

$$E_{esc}(\psi) = \frac{1}{2}\int_L^\infty dx(2\psi_t^2 + \psi_x^2) \tag{44}$$

.

The energy is conserved, thus

$$\partial_t E_{in}(\psi) = -\partial_t E_{out}(\psi), \tag{45}$$

so it is enough to calculate only outgoing energy

$$\partial_t E_{in}(\psi) = -\partial_t E_{out}(\psi) = -\int_L^\infty dx(\partial_x(\psi_t\psi_x)) \tag{46}$$

$$= \psi_x\psi_t\Big|_{x=L+} \tag{47}$$

$$= -h'(L-t)^2 = -(g'(t-L) - g'(t+L))^2 = -g(t+L)^2. \tag{48}$$

After the cosmetic shift in time we get

$$\partial_t E_{in}(\psi) \approx -g(t)^2 \tag{49}$$

# 1 Notes

## 1.1 Scalar product

Scalar product is the standard scalar product for $L^2([0,T])$. Orthogonal Fourier basis is given by

$$\phi_n(x) = e^{ixn\pi/T} = e^{\omega_n x} \tag{50}$$

as well as the eigenfunctions of $L$.

$$g_n(x) = e^{\beta_n x} \tag{51}$$

The following quantities can be calculated

$$(e^{\beta x}, e^{\omega x}) = \begin{cases} T & \bar\beta + \omega = 0 \\ \frac{1}{\bar\beta+\omega}\left(e^{(\bar\beta+\omega)t} - 1\right) & \bar\beta + \omega \neq 0 \end{cases} \tag{52}$$

## 1.2 Asymptotic form of energy

The energy can be approximated as follows. First we define

$$I(t) := \int_a^\infty e^{a(x,t)} e^{itx} dx. \tag{53}$$

The above function can be expanded to finite series by repetitively applying integration by parts

$$I(t) = e^{ita}\sum_{n=0}^N \frac{d^n}{dx^n}\Big|_{x=a} e^{a(x,t)}\left(\frac{i}{t}\right)^{n+1} + \epsilon_{N+1}(t), \tag{54}$$

$$\epsilon_N(t) = \left(\frac{i}{t}\right)^N \int_a^\infty \frac{d^N}{dx^N} e^{a(x,t)} e^{itx} dx \tag{55}$$

N can be set to infinity if $\epsilon_N = O(x^{-N-1})$, which is true if $\frac{d^N}{dx^N}\Big|_{x=a} e^{a(x,t)} = O(x^{-1-\delta})$ for $\delta > 0$. If this is true we can write

$$I(t) = e^{ita} \sum_{n=0}^{\infty} \frac{d^n}{dx^n}\Big|_{x=a} e^{a(x,t)} \left(\frac{i}{t}\right)^{n+1}. \tag{56}$$

From now on we assume $\partial_x a(x,t) >> 1$. Using the identity for $n$-th derivative of function composition

$$\frac{d^n}{dx^n}\Big|_{x=x_0} e^{f(x)} = B_n(f'(x_0), ..., f^{(n)}(x_0)) e^{f(x)} \tag{57}$$

and the asymptotic form of Bell polynomials

$$B_n(x_1, ..., x_n) = x_1^n + \frac{n(n-1)}{2} x_1^{n-2} x_2 + O(x_i^{n-2}) \tag{58}$$

we can further simplify the formula for $I(t)$

$$I(t) = e^{ita} e^{a(a,t)} \frac{i}{t} \sum_{n=0}^{\infty} \left(\frac{ia'(a,t)}{t}\right)^n \tag{59}$$

$$= e^{ita} e^{a(a,t)} \frac{i}{t - ia'(a,t)}. \tag{60}$$

In the case of (41) after suitable change of variables we get

$$\log(|g(t)|^2) \sim A - Bt - 2\log t. \tag{61}$$

where $A$ is depends of initial value of $g(t)$, but $B$ does not (at least for large times).