## 0.1 Recursive formulation

We start with the equation

$$(\rho + m\delta(x - L))\partial_{tt}\psi = \partial_{xx}\psi, \tag{1}$$

which after the substitutions

$$x \to x' = \frac{m}{\rho}x \tag{2}$$

$$t \to t' = \frac{m}{\sqrt{2}\rho}t \tag{3}$$

$$L' = \frac{\rho L}{m} \tag{4}$$

becomes

$$(1 + \delta(x - L))\partial_{tt}\psi = \frac{1}{2}\partial_{xx}\psi, \tag{5}$$

with primes omitted, so the problem has one parameter (which can be also shown using dimensional analysis). Futher, the most general ansatz for the escaping wave to the right of the point mass is

$$\psi(x,t) = \begin{cases} f(x-t) + g(x+t) & x \in [0, L] \\ h(x-t) & x \geq L, \end{cases} \tag{6}$$

with conditions

$$\psi(0,t) = 0 \quad \text{zero at the boundary} \tag{7}$$

$$\psi(L-,t) = \psi(L+,t) \quad \text{continuity.} \tag{8}$$

which become

$$\psi(x,t) = \begin{cases} -g(t-x) + g(x+t) & x \in [0, L] \\ h(x-t) & x \geq L. \end{cases} \tag{9}$$

$$-g(t-L) + g(L+t) = h(L-t) \quad \text{continuity} \tag{10}$$

Integrating (5) we obtain

$$\partial_{tt}\psi(L,t) = \frac{1}{2}(\partial_x\psi(L+,t) - \partial_x\psi(L-,t)). \tag{11}$$

which glues together the solutions from different intervals along with the continuity condition (8).

The preceding statements give rise to the following equation for $g$

$$\begin{aligned} -g''(t-L) + g''(t+L) &= \frac{1}{2}(h'(L-t) - g'(t-L) - g'(t+L)) \\ &= \frac{1}{2}(g'(t-L) - g'(t+L) - g'(t-L) - g'(t+L)) \\ &= -g'(t+L), \end{aligned} \tag{12}$$

and after the shift in $t$ and with $T := 2L$ we get

$$g''(t+T) + g'(t+T) = g''(t) \tag{13}$$

integrating gives us

$$g'(t+T) + g(t+T) = g'(t) + (g'(T) - g'(0) + g(T)). \tag{14}$$

Adding a constant to $g$ does not change the equations, so we can add such constant in order to cancel the $(g'(T) - g'(0) + g(T))$ term, and we are then given the final form of equation for $g$

$$g'(t+T) + g(t+T) = g'(t), \tag{15}$$

which is a kind of delayed differential equation, and for which one can iteratively find a solution for arbitrary large times given the function values in an interval $[a, a+T]$ where $a$ is an arbitrary constant for convinience we shall fix $a = 0$. Now we shall introduce a family of functions $g_n \in \mathcal{C}([0,T])$ defined as

$$g(t+nT) = g_n(t) \quad t \in [0,T] \tag{16}$$

so the equation (17) becomes

$$\begin{aligned} & g'_{n+1}(t) + g_{n+1}(t) = g'_n(t), \\ & g_{n+1}(0) = g_n(T) \quad \text{continuity condition.} \\ & \text{with} \quad g_0(t) \quad \text{initial conditions} \end{aligned} \tag{17}$$

The above equation can be solved by iteration

$$g_{n+1}(t) = (Lg_n)(t) = g_n(t) - e^{-t} \int_0^t e^s g_n(s) ds + e^{-t}(g_n(T) - g_n(0)). \tag{18}$$

With $L : \mathcal{C}([0,T]) \to \mathcal{C}([0,T])$. The eigenvectors of $L$ are the soultions of the equation (17) with $g_{n+1} = \lambda g_n$

$$\lambda(g'_\lambda(t) + g_\lambda(t)) = g'_\lambda(t), \tag{19}$$

$$\lambda g_\lambda(0) = g_\lambda(T) \quad \text{quantization.} \tag{20}$$

from which we obtain

$$g_\lambda(t) = e^{\frac{\lambda}{\lambda+1}t}, \tag{21}$$

$$\lambda = e^{\frac{\lambda}{\lambda+1}T} \quad \text{quantization.} \tag{22}$$

The above quantization for $\lambda$ gives rise to the quasi-normal modes of equation (5).

## 0.2 Quasinormal Modes - asymptotic expansion

The transcendental equation for $\lambda$ can be written as

$$\frac{\beta}{\beta - i} = e^{i\beta T} \tag{23}$$

where $\lambda = \frac{\beta}{\beta - i}$. We shall split $\beta$ into its real and imaginary part in the obvious way: $\beta = \Omega + i\Gamma$. Taking the absolute value of (23) gives rise to the following equation

$$\frac{\Omega^2 + (\Gamma - 1)^2}{\Omega^2 + \Gamma^2} = e^{2\Gamma T} \tag{24}$$

Let's assume $\Gamma < 0$, then from the equality

$$e^{2\Gamma T} - 1 = \frac{1 - 2\Gamma}{\Omega^2 + \Gamma^2} \tag{25}$$

contradiction appears and so we deduce $\Gamma > 0$, which in turn leads to the fact that $\Gamma < \frac{1}{2}$. From the fact $\Gamma > 0$ one can conclude that $L$ is bounded, and thus continous on the space spanned by its eigenvectors because

$$|\lambda|^2 = e^{-2\Gamma} < 1. \tag{26}$$

Equation (24) can be solved for $\Omega^2$

$$\begin{aligned} \Omega^2 &= \frac{1 - 2\Gamma - \Gamma^2(e^{2\Gamma T} - 1)}{e^{2\Gamma T} - 1} \\ &= \frac{1}{2\Gamma T} - \frac{2 + T}{2T} + \Gamma \frac{T + 6}{6} + \mathcal{O}(\Gamma^2). \end{aligned} \tag{27}$$

We shall introduce a new parameter $\eta := \frac{1}{\sqrt{\Gamma}}$ obtaining the parametrization $\beta(\eta)$

$$\Omega = \frac{1}{\sqrt{2T}}\eta - \frac{\sqrt{2T}(2 + T)}{4T}\frac{1}{\eta} + \mathcal{O}(\eta^{-3}), \tag{28}$$

$$\Gamma = \frac{1}{\eta^2}, \tag{29}$$

$$\beta = \frac{i}{\sqrt{2T}}\eta - \frac{i\sqrt{2T}(2 + T)}{4T}\frac{1}{\eta} - \frac{1}{\eta^2} + \mathcal{O}(\eta^{-3}) \tag{30}$$

We should now calculate $g(t) = (L^n h_0)(t - nT)$ for

$$h_0(t) = \sum_\lambda a_\lambda g_\lambda(t). \tag{31}$$

We have

$$(L^n h_0)(t) = \sum_\lambda \lambda^n a_\lambda g_\lambda(t). \tag{32}$$

Utilizing quantization condition for $\lambda$ we obtain

$$(L^n h_0)(t) = \sum_\lambda a_\lambda e^{\frac{\lambda}{\lambda+1}(t + nT)} \tag{33}$$

Inserting the (28) to (33) yields

$$\begin{aligned} (L^n h_0)(t - nT) &= \sum_\eta a_{\lambda(\eta)} e^{i\beta(\eta)t} \\ &\approx \int^\infty a_{\lambda(\eta)} \exp((ib\eta + \frac{ic}{\eta} - \frac{1}{\eta^2})t) d\eta \end{aligned} \tag{34}$$

The remaining problem is the asymptotic form of $a_\lambda$ for reasonable initial data (which cannot be simply computed using scalar product, because eigenvectors are not orthogonal in their current form) and integration of the above integral to get energy leak in time (see (38)). The integration should be doable using the method of steepest descent or some other method for highly oscillatory integrands assuming $t \gg 1$. After this the asymptotic expansion in time should be easily obtained, and so the energy leak.

## 0.3 Energy

The energy of the whole string is given by the functional

$$\begin{aligned} E(\psi) &= \frac{1}{2}\int_0^\infty dx((1+\delta(x-L))\psi_t^2 + \psi_x^2) \\ &= \frac{1}{2}\int_0^L dx(\psi_t^2+\psi_x^2) + \frac{1}{2}\int_L^\infty dx(\psi_t^2+\psi_x^2) + \frac{1}{2}\psi_t^2\Big|_{x=L} \\ &= E_{in}(\psi) + E_{esc}(\psi) \end{aligned} \tag{35}$$

$$E_{in}(\psi) = \frac{1}{2}\int_0^L dx(\psi_t^2+\psi_x^2) + \frac{1}{2}\psi_t^2\Big|_{x=L} \tag{36}$$

$$E_{esc}(\psi) = \frac{1}{2}\int_L^\infty dx(\psi_t^2+\psi_x^2) \tag{37}$$

$$\begin{aligned} \partial_t E_{in}(\psi) &= \int_0^L (\partial_x(\psi_t\psi_x)) + \psi_{tt}\psi_t\Big|_{x=L} = \\ &= (\psi_x + \psi_{tt})\psi_t\Big|_{x=L} = \\ &= (g'(t-L) + g'(t+L) - g''(t-L) + g''(t+L))(g'(t+L) - g'(t-L)) \\ &= (g'(t-L) + g'(t+L) - g'(t+L))(g'(t+L) - g'(t-L)) \\ &= -g'(t-L)g(t+L) \\ &= -(g'(t+L) + g(t+L))g(t+L) \\ &= -\frac{1}{2}\partial_t g(t+L)^2 - g(t+L)^2 \end{aligned} \tag{38}$$

So the highest order dependency on $t$ will be the $g(t+L)^2$ term (assuming $g \sim t^{-\alpha}$), so we can write that for large times we have

$$\partial_t E_{in}(\psi) \sim \quad -g^2(t+L) < 0 \tag{39}$$