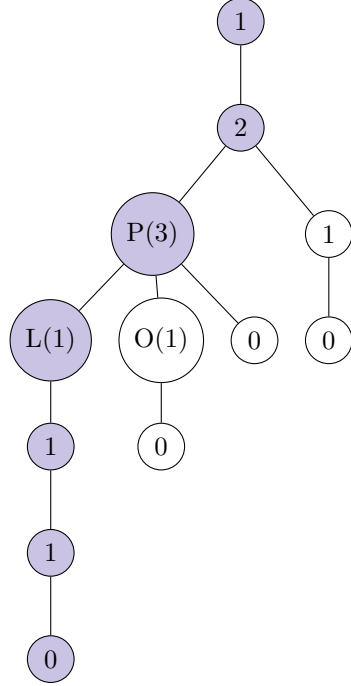


## Ordered Trees

Given an ordered tree  $T$ , let  $O$  be the first node in a preorder traversal of  $T$  that is not in the left-down path of  $T$ . Let  $P$  be  $O$ 's parent, and let  $L$  be  $P$ 's leftmost child (or, equivalently,  $O$ 's left sibling). The tree below gives an example illustrating these three nodes.

$$D = [1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0]$$

$T =$



Let  $D$  be the dyck word corresponding to  $T$  and let  $k$  be the index of the first 01 substring in  $D$ . Note that the node  $O$  within  $T$  corresponds to  $D_k$ .

The cool-lex rule for shifts in ordered trees can be broken down into 3 cases:

- Case 1:  $O$  has at least 1 child  
 Since  $O$  corresponds to  $D_k$ ,  $O$  having at least one child corresponds to the case where  $D_{k+1} = 1$   
 Shift  $L$  to be  $O$ 's first child.
- Case 2:  $O$  has no children and  $O$  is the child of the root  
 This case corresponds to the case where  $D_{k+1} = 0$  and the non-increasing prefix is tight (i.e., the non-increasing prefix has exactly as many ones as zeroes).  
 Shifts in this case are the same as in case 1.
- Case 3:  $O$  has no children and  $O$  is not child of the root

This case corresponds to the case where  $D_{k+1} = 0$  and the non-increasing prefix is not-tight (i.e., the non-increasing prefix has more ones than zeroes).

Shift L to be the first child of P's parent

Shift O to be the first child of the root.

Note: The order of these shifts matters. P cannot be the root, but if P's parent is the root, the O and L are both shifted to be P's first child. The shifting of O must be done second so that after both shifts are done O is the first child of the root.

Illustration of case 1 (shifting a 1):

$[1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0] \Rightarrow$   
 $[1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0]$

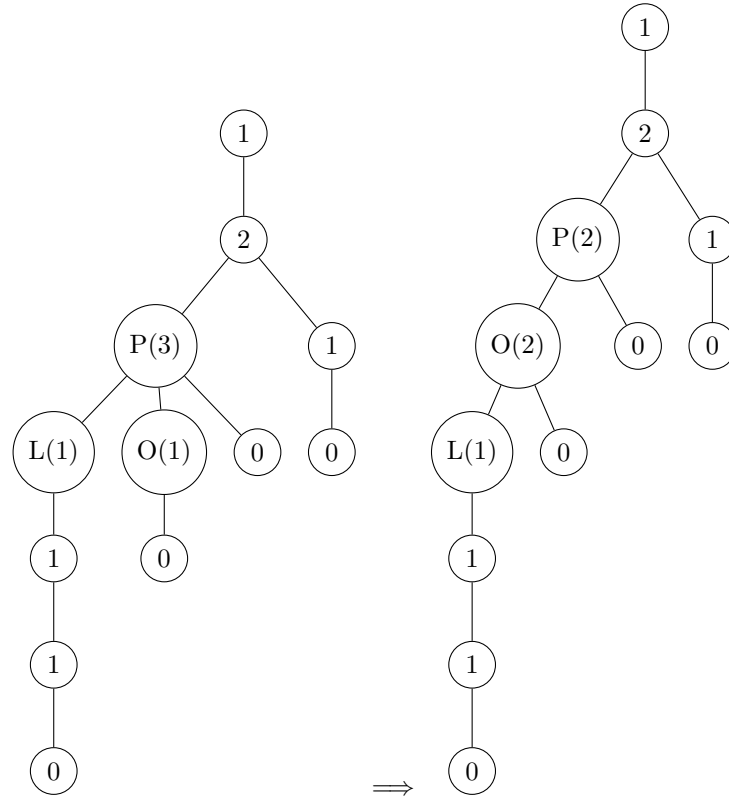


Illustration of case 3 (shifting a 0):

$[1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0] \Rightarrow$   
 $[1, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 1, 0, 0, 0]$

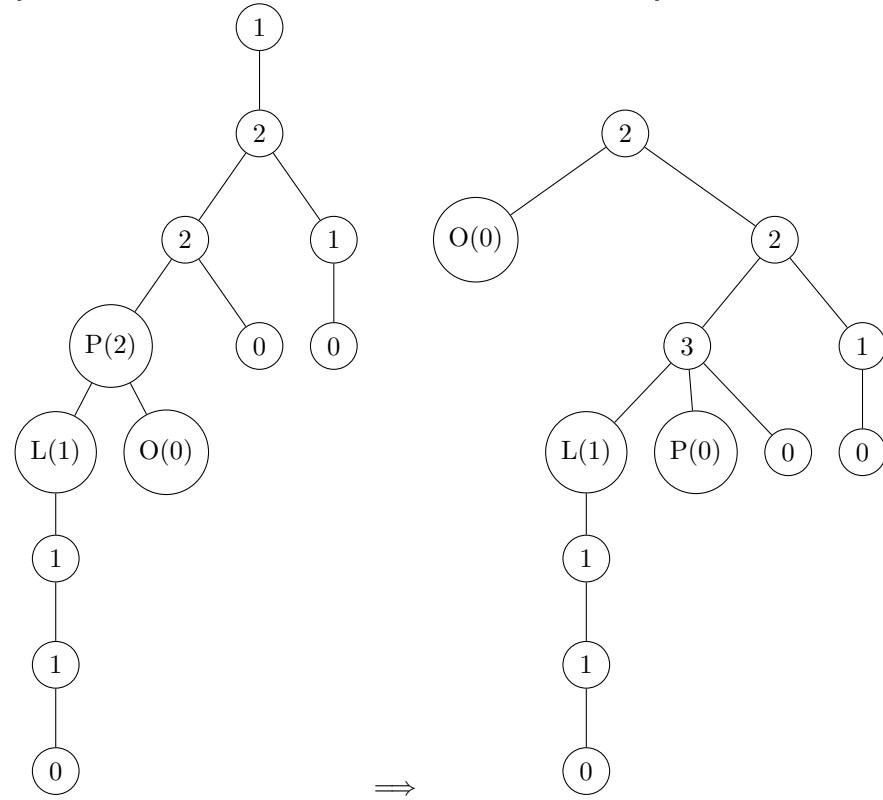


Illustration of case 2 ( $D_k = 0$ , prefix is tight, shift a 1)

$[1, 1, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0] \Rightarrow$   
 $[1, 1, 1, 1, 0, 0, 0, 0, 1, 0, 1, 0]$

