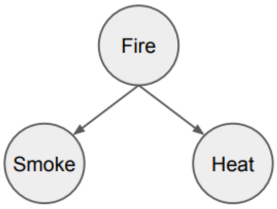
Problem 2

Consider the following Baysian network with 3 variables: {F=Fire,S=Smoke,H=Heat}.



Variable F represents the existence of a fire (F = 1), S = 1 if we see smoke; H = 1 if we observe heat and vice versa:

- F = Fire (1 = Fire, 0 = No fire)
- S = Smoke (1 = Observed smoke, 0 = No observed smoke)
- H = Heat (1 = Observed heat, 0 = No observed heat)

We know the following probabilities:

- P(F = 1) = 0.1 => P(F = 0) = 0.9
- P(S = 1|F = 1) = 0.9 => P(S = 0|F = 1) = 0.1
- P(S = 1|F = 0) = 0.001 => P(S = 0|F = 0) = 0.999
- P(H = 1|F = 1) = 0.99 => P(H = 0|F = 1) = 0.01
- P(H = 1|F = 0) = 0.0001 = P(H = 0|F = 0) = 0.9999

(A) Write down the joint distribution of the Bayesian network P(F, S, H) using the conditionally independent effects. Are Smoke and Heat somehow independent?

a. $P(F, S, H) = \mathrm{p(F)} * \mathrm{p(S|F)} * \mathrm{p(H|F)}$

b. Smoke and Heat should not be independent, because they share the same parent node Fire. They are only conditionally independent, conditioned on Fire.

(B) Before we observe any data, what is the prior probability that there is no fire?

a. P(F = 0) = 1 - P(F = 1) = 0.9

(C) Now suppose that we observe smoke. Use Bayes' theorem to evaluate the posterior probability that there is no fire given the observation of smoke and compare it to prior probability that there is no fire.

a. $P(F = 0|S = 1) = \frac{P(F=0,S=1)}{P(S=1)} = \frac{P(S = 1|F = 0) * P(F=0)}{P(S=1)} =$ $\frac{P(S = 1|F = 0) * P(F=0)}{P(S=1|F=1)*P(F=1)+P(S=1|F=0)*P(F=0)} = \frac{0.001*0.9}{0.9*0.1+0.001*0.9} \approx 0.00990$

b. The Probability of no fire with 0.9 is magnitudes higher than the probability of no fire given that we observed smoke.

(D) Next suppose that we also check the temperature and find that it is very hot. We have now observed the states of both smoke and heat. Compute the posterior probability that there is fire given the observations of both heat and smoke and compare it to the posterior probability of P(F = 0|S = 1).

a. $P(F = 1|S = 1, H = 1) = \frac{P(S = 1, H = 1|F = 1) * P(F=1)}{P(S = 1, H = 1|F = 1)*P(F=1)+P(S=1,H=1|F=0)*P(F=0)} =$

b. $\frac{P(S = 1|F = 1)*P(H = 1|F = 1)* P(F=1)}{P(S = 1|F = 1)*P(H = 1|F = 1)*P(F=1)+P(S = 1|F = 0)*P(H = 1|F = 0)*P(F=0)} =$

c. $\frac{0.9*0.99* 0.1}{0.9*0.99* 0.1+0.001*0.0001*0.9} \approx 0.9999989899$

d. Observing both Heat and Smoke, the Fire is almost guaranteed with a probability very close to 1. Heat, Smoke and Fire being 1 are highly correlated, therefore the probability of P(F = 0|S = 1) was very low and the probability of $P(F = 1|S = 1, H = 1)$ is really high.

Problem 3

Consider a Bayesian network with 2 variables and the following structure: A → B. We observed 4 samples for these variables:

A	B
1	1
0	0
1	1
0	0

(A) Calculate the maximum likelihood (ML) estimator of the parameter θ = P(B = 1|A). Recall $\hat{\theta}_{ML} = \max_{\theta} \mathrm{P(D|G)}$, where D =Data, G =Graph(Model).

a. $\theta_{0B} = \mathrm{P(B = 1|A = 0)}$
b. $\theta_{0B} = \mathrm{P(B = 1|A = 0)}$
c. $\theta_A = \mathrm{P(A = 1)}$
d. $\mathrm{P(D|G,\theta_A)} = \binom{4}{2} * \theta_A^2 * (1-\theta_A)^{4-2} => 6 * \theta_A^2 * (1-\theta_A)^2 = 0$
    i. $\theta_A = 0 \vee \theta_A = 1$
e. $\mathrm{P(D|G,\theta_{0B})} = \binom{2}{0} * \theta_{0B}^0 * (1-\theta_{0B})^2 => (1-\theta_{0B})^2 = 0$
    i. $\theta_{0B} = 1$
f. $\mathrm{P(D|G,\theta_{1B})} = \binom{2}{2} * \theta_{1B}^2 * (1-\theta_{1B})^0 => \theta_{1B}^2 = 0$
    i. $\theta_{1B} = 0$

(B) Calculate the model evidence P(D|G). The prior distribution of θ is: P(θ) = P(θ|G) ~ β(3, 3).

a. $P(D|G) = \int P(D|\theta, G) * P(\theta|G)\, d\theta = \int P(D|\theta_A, G) * P(\theta_A|G)\, d\theta_A \int P(D|\theta_{0B}, G) * P(\theta_{0B}|G)\, d\theta_{0B} * \int P(D|\theta_{1B}, G) * P(\theta_{1B}|G)\, d\theta_{1B}$
b. For each factor we have the generic integral:
c. $\int \binom{n}{x} * \theta^x * (1-\theta)^{n-x} * \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} * \theta^{\alpha-1} * (1-\theta)^{\beta-1}\, d\theta$
d. $= \int \binom{n}{x} * \theta^{x+\alpha-1} * (1-\theta)^{n-x+\beta-1} * \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, d\theta$
e. $= \int \binom{n}{x} * \theta^{x+\alpha-1} * (1-\theta)^{n-x+\beta-1} * \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, d\theta$
f. $= \binom{n}{x} * \frac{\Gamma(x+\alpha)\Gamma(n-x+\beta)}{\Gamma(\alpha+n+\beta)} * \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$
g. And now we insert the parameter counts:
h. $P(D|G) = \binom{4}{2} * \frac{\Gamma(2+3)\Gamma(4-2+3)}{\Gamma(3+4+3)} * \frac{\Gamma(3+3)}{\Gamma(3)\Gamma(3)} * \binom{2}{0} * \frac{\Gamma(0+3)\Gamma(2-0+3)}{\Gamma(3+2+3)} * \frac{\Gamma(3+3)}{\Gamma(3)\Gamma(3)} * \binom{2}{2} * \frac{\Gamma(2+3)\Gamma(2-2+3)}{\Gamma(3+2+3)} * \frac{\Gamma(3+3)}{\Gamma(3)\Gamma(3)} = 6 * \left(\frac{\Gamma(3+3)}{\Gamma(3)\Gamma(3)}\right)^3 * \frac{\Gamma(5)\Gamma(5)}{\Gamma(10)} * \frac{\Gamma(3)\Gamma(5)}{\Gamma(8)} * \frac{\Gamma(5)\Gamma(3)}{\Gamma(8)} = 6 * \left(\frac{120}{4}\right)^3 * \frac{24^2}{362880} * \frac{48}{5040} * \frac{48}{5040} = \frac{2}{77175} \approx 0.00002$

(C) Calculate the posterior distribution P(θ|D, G) for both estimators (use the prior distribution from before). Again, estimate the parameter θ, but this time using the posterior distribution. Compare them to the ML estimators.

a. The posterior is proportional to $\theta^{x+\alpha-1} * (1-\theta)^{n-x+\beta-1}$
b. $P(\theta_A|G, D) \propto \theta^{2+3-1} * (1-\theta)^{4-2+3-1} => P(\theta_A|G, D) = \beta(5{,}5)$
c. $\mathrm{P}(\theta_{0\mathrm{B}}|\mathrm{G}, \mathrm{D}) \propto \theta^{0+3-1} * (1-\theta)^{2-0+3-1} => P(\theta_{0B}|G, D) = \beta(3{,}5)$
d. $\mathrm{P}(\theta_{1\mathrm{B}}|\mathrm{G}, \mathrm{D}) \propto \theta^{2+3-1} * (1-\theta)^{2-2+3-1} => P(\theta_{1B}|G, D) = \beta(5{,}3)$
e. We compare the estimators using the expected value of the beta distributions to the ML Estimators:
    i. $\theta_A => E[\beta(5{,}5)] = 10 \neq 0 \vee 1 = \hat{\theta}_A{}^{ML}$
    ii. $\theta_{0B} => E[\beta(3{,}5)] = 8 \neq 1 = \hat{\theta}_{0B}{}^{ML}$
    iii. $\theta_{1B} => E[\beta(5{,}3)] = 8 \neq 0 = \hat{\theta}_{1B}{}^{ML}$