PHYS 512 - Assignment 2

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Problem 1

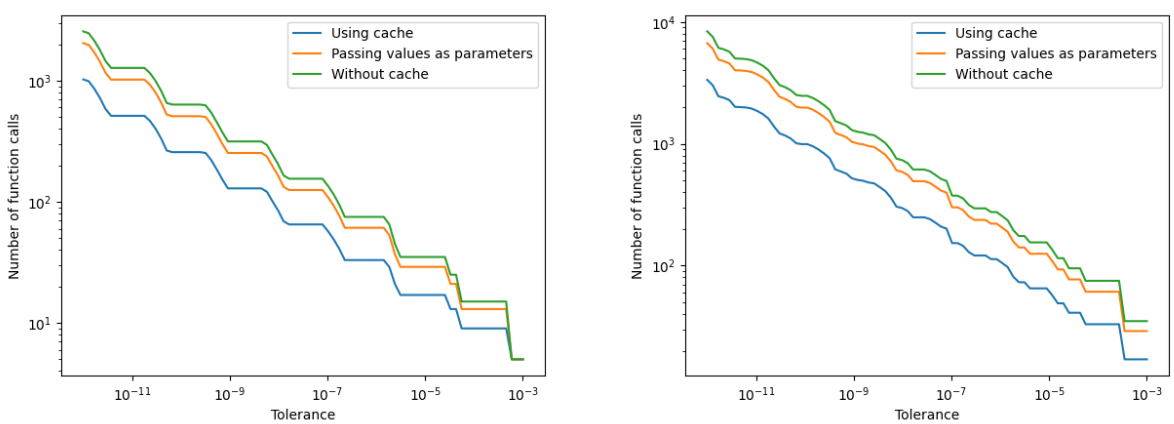
For this problem, I have thought of two ways of saving processing time by reducing the number of calls to $f(x)$:

1) Reuse values computed for boundary values

2) Build and use a cache as calculations are executed

The big advantage of the first method is that it comes with almost no memory cost and can easily save precious computing time. However, it does not guarantee that $f(x)$ won't be computed more than once for the same x. On the other hand, the second method does guarantee this by building and using a mapping $x \to f(x)$ for each value of x. This, however, comes with a memory cost, but is only of the order of kilobytes even for a tolerance of 10^{-11}. Since insertion and access for a hashmap is of the order of $\mathcal{O}(1)$, the cache itself does not add any computing overhead.

Fig. 1 compares these two methods with the simple bare method for a set of tolerances. As expected, method 2 requires less function calls than method 1, which itself requires less function calls than the bare method. In the case of Fig. 1b, method 2 requires, on average, 300 function calls less than the bare method, and method 1 requires, on average, 600 function calls less than method 1.



(a)

(b)

Figure 1: Number of function calls required for integration at various tolerances, comparing the bare method with two optimization methods. (a) Integration of e^x in the range $[0, 1]$. (b) Integration of $1/(1 + x^2)$ in the range $[-1, 1]$.

Problem 2

To model $\log_2(x)$ in the range $[0.5, 1]$, we build the Chebyshev matrix and uses truncated versions of it to find what's the minimum order for which the accuracy is less than 10^{-6}. Doing so yields the results shown in Fig. 2.

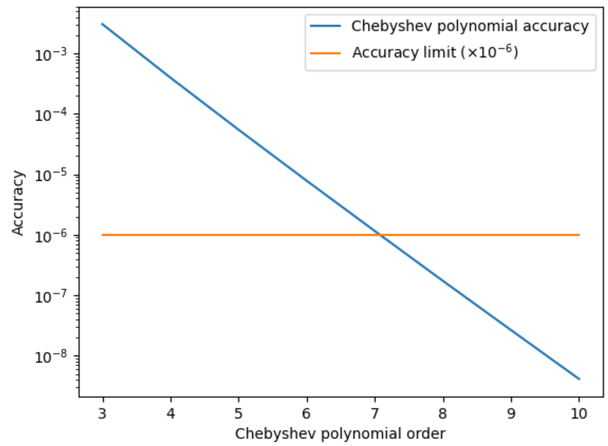


Figure 2: Chebyshev fit accuracy for various polynomial orders. Note that the lowest order for which the accuracy is less than 10^{-6} is 8. The accuracy for order 7 is around 1.15×10^{-6}.

Note that to make use of the full range for which Chebyshev polynomials have their useful properties, we compute $y = f(x)$ for $x \in [0.5, 1]$, but then rescale the x array so that it is in the range $[-1, 1]$. With these results, we find that the lowest order for which the accuracy is less than 10^{-6} is 8. Equipped with this information, we may now perform Legendre polynomial fitting using the same points and order then compare the results.

Fig. 3 shows the residuals for Chebyshev and Legendre polynomials. The RMS errors are 1.94×10^{-7} for Chebyshev and 1.84×10^{-7} for Legendre. The maximum errors are 3.2×10^{-7} for Chebyshev and 8.1×10^{-7} for Legendre. Notice that Chebyshev has higher RMS error while Legendre has larger maximum error.

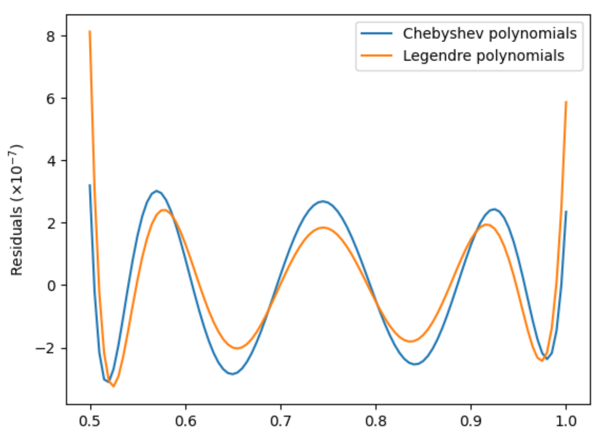


Figure 3: Residuals for fitting $\log_2(x)$ in the range $[0.5, 1]$ using Chebyshev and Legendre polynomials.

Problem 3

a) The problem is set up by writing a system of 15 differential equations, each representing the time evolution of a decay product (or U-238). Since the half-lives differ greatly, we opted for the implicit Runge-Kutta method (i.e., `Radau` in scipy) which should be able to handle stiff equations better. The results are shown in Fig. 4.

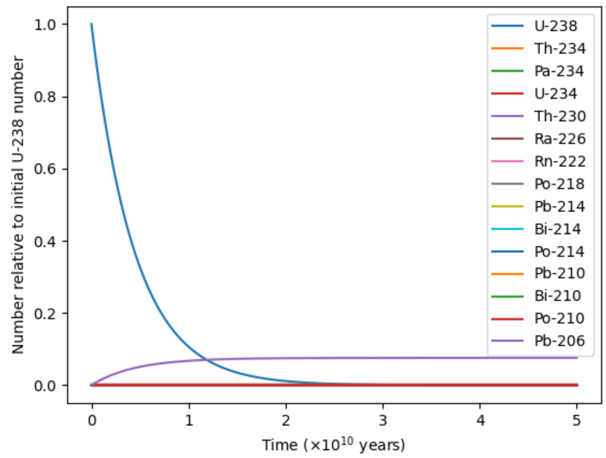
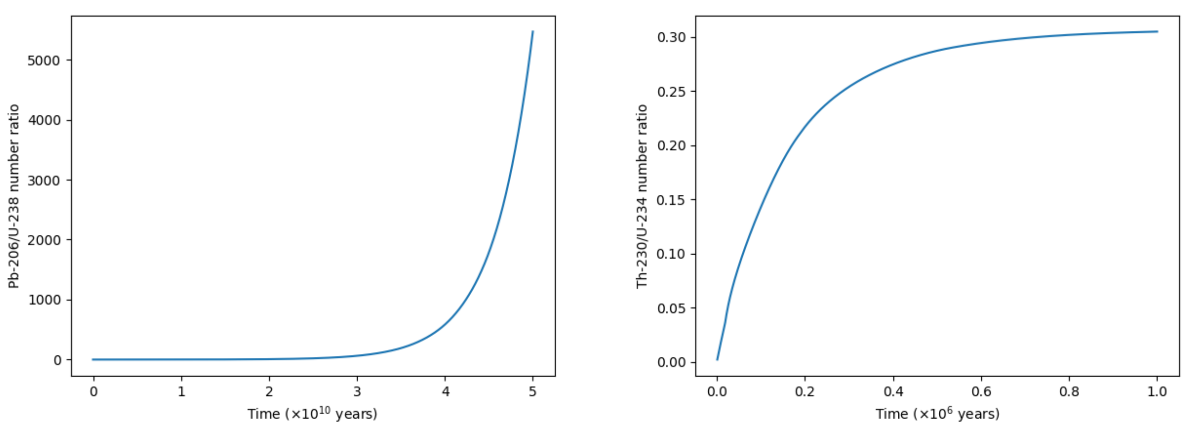


Figure 4: U-238 decay plot showing all products of the decay chain assuming starting from a sample of pure U-238.

b) Plots of the ratio of Pb-206 to U-238 and Th-230 to U-234 are shown in Fig. 5. The results do make sense analytically. For Fig. 5a, we can approximate U-238 decaying directly into Pb-206 with an effective half-time, which would result in an exponentially increasing amount of Pb-206, as shown in the figure. For Fig. 5b, since Th-230 comes directly after U-234 in the decay chain, we also expected an increasing exponential behavior, but notice how the rate is decreasing since the half-life of Th-230 is smaller than U-234. Not shown in this plot is that eventually everything will decay into the final product, Pb-206.



(a)

(b)

Figure 5: Ratios of decay products over time. (a) Ratio of Pb-206 to U-238, the end product and initial sample, respectively. (b) Ratio of Th-230 to U-234, two consecutive decay products.