CONTROL OF AIRCRAFT

PROCESS MODELING

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CONTROL OF AIRCRAFT

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2. DUTCH ROLL MODE
3. ROLL DAMPING MODE
4. SPIRAL MODE
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LATERAL MODEL

We have considered here the general case where the inertia tensor of the aircraft is non diagonal (a plane with a vertical stabilizer for example does not have a symmetric mass repartition).

$$\mathcal{J} = \begin{pmatrix} I_{XX} & 0 & -I_{ZX} \\ 0 & I_{YY} & 0 \\ -I_{ZX} & 0 & I_{ZZ} \end{pmatrix}$$

The flight mechanics equations give:

$$\dot{v} = pw - ru + \left(\frac{F_{ay}}{m} + g \cos\theta \sin\varphi \right)$$

$$-I_{ZX}\dot{r} + q(I_{ZZ}r - I_{ZX}p) - I_{YY}qr + I_{XX}\dot{p} = M_{ax}$$

$$I_{ZZ}\dot{r} - q(-I_{ZX}r + I_{XX}p) + I_{YY}pq - I_{ZX}\dot{p} = M_{az}$$

$$\dot{\varphi} = p + q \sin\varphi \tan\theta + r \cos\varphi \tan\theta$$

By isolating $\dot{r}$ and $\dot{p}$ we obtain

$$\dot{v} = pw - ru + \left(\frac{F_{ay}}{m} + g\cos\theta\sin\varphi\right)$$

$$\dot{p} = -\frac{\left({I_{ZZ}}^2 - I_{YY}\,I_{ZZ} + {I_{ZX}}^2\right)\,q\,r + \left(I_{ZX}\,I_{ZZ} + \left(I_{XX} - I_{YY}\right)\,I_{ZX}\right)\,p\,q}{I_{XX}\,I_{ZZ} - {I_{ZX}}^2} - \frac{-I_{ZX}\,M_{az} - I_{ZZ}\,M_{ax}}{I_{XX}\,I_{ZZ} - {I_{ZX}}^2}$$

$$\dot{r} = \frac{\left(I_{ZX}\,I_{ZZ} + \left(I_{XX} - I_{YY}\right)\,I_{ZX}\right)\,q\,r + \left({I_{ZX}}^2 - I_{XX}\,I_{YY} + {I_{XX}}^2\right)\,p\,q}{I_{XX}\,I_{ZZ} - {I_{ZX}}^2} + \frac{I_{XX}\,M_{az} + I_{ZX}\,M_{ax}}{I_{XX}\,I_{ZZ} - {I_{ZX}}^2}$$

$$\dot{\varphi} = p + q\sin\varphi\tan\theta + r\cos\varphi\tan\theta$$

We take the following notation, for each variable of the state vector β, p, r, φ and the commands δ_l and δ_n (note that $\dot{v}V_{eq} = \dot{\beta}$):

$$Y_\beta = \frac{1}{mV_{eq}} \frac{\partial F_{ay}}{\partial \beta} = \frac{QSC_{y_\beta}}{mV_{eq}}$$

$$Y_p = \frac{1}{mV_{eq}} \frac{\partial F_{ay}}{\partial p} = \frac{QSl\frac{\ell_{ref}}{V_{eq}}C_{y_p}}{mV_{eq}}$$

$$Y_r = \frac{1}{mV_{eq}} \frac{\partial F_{ay}}{\partial r} = \frac{QSl\frac{\ell_{ref}}{V_{eq}}C_{y_r}}{mV_{eq}}$$

$$Y_{\delta_l} = \frac{1}{mV_{eq}} \frac{\partial F_{ay}}{\partial \delta_l} = \frac{QSC_{y_{\delta_l}}}{mV_{eq}}$$

$$Y_{\delta_n} = \frac{1}{mV_{eq}} \frac{\partial F_{ay}}{\partial \delta_n} = \frac{QSC_{y_{\delta_n}}}{mV_{eq}}$$

$$l_\beta = \frac{I_{zz}}{I_{xx}I_{zz} - {I_{zx}}^2}\frac{\partial M_{ax}}{\partial \beta} = \frac{I_{zz}}{I_{xx}I_{zz} - {I_{zx}}^2} QS\,\ell_{ref}\, C_{l_\beta}$$

$$l_p = \frac{I_{zz}}{I_{xx}I_{zz} - {I_{zx}}^2}\frac{\partial M_{ax}}{\partial p} = \frac{I_{zz}}{I_{xx}I_{zz} - {I_{zx}}^2} QS\,\ell_{ref}\,\frac{\ell_{ref}}{V_{eq}}\, C_{l_p}$$

$$l_r = \frac{I_{zz}}{I_{xx}I_{zz} - {I_{zx}}^2}\frac{\partial M_{ax}}{\partial r} = \frac{I_{zz}}{I_{xx}I_{zz} - {I_{zx}}^2} QS\,\ell_{ref}\,\frac{\ell_{ref}}{V_{eq}}\, C_{l_r}$$

$$l_{\delta_l} = \frac{I_{zz}}{I_{xx}I_{zz} - {I_{zx}}^2}\frac{\partial M_{ax}}{\partial \delta_l} = \frac{I_{zz}}{I_{xx}I_{zz} - {I_{zx}}^2} QS\,\ell_{ref}\, C_{l_{\delta_l}}$$

$$l_{\delta_n} = \frac{I_{zz}}{I_{xx}I_{zz} - {I_{zx}}^2}\frac{\partial M_{ax}}{\partial \delta_n} = \frac{I_{zz}}{I_{xx}I_{zz} - {I_{zx}}^2} QS\,\ell_{ref}\, C_{l_{\delta_n}}$$

$$n_\beta = \frac{I_{XX}}{I_{XX} I_{ZZ} - I_{ZX}^2} \frac{\partial M_{az}}{\partial \beta} = \frac{I_{XX}}{I_{XX} I_{ZZ} - I_{ZX}^2} Q S \ell_{ref} C_{n_\beta}$$

$$n_p = \frac{I_{XX}}{I_{XX} I_{ZZ} - I_{ZX}^2} \frac{\partial M_{az}}{\partial p} = \frac{I_{XX}}{I_{XX} I_{ZZ} - I_{ZX}^2} Q S \ell_{ref} \frac{\ell_{ref}}{V_{eq}} C_{n_p}$$

$$n_r = \frac{I_{XX}}{I_{XX} I_{ZZ} - I_{ZX}^2} \frac{\partial M_{az}}{\partial r} = \frac{I_{XX}}{I_{XX} I_{ZZ} - I_{ZX}^2} Q S \ell_{ref} \frac{\ell_{ref}}{V_{eq}} C_{n_r}$$

$$n_{\delta_l} = \frac{I_{XX}}{I_{XX} I_{ZZ} - I_{ZX}^2} \frac{\partial M_{az}}{\partial \delta_l} = \frac{I_{XX}}{I_{XX} I_{ZZ} - I_{ZX}^2} Q S \ell_{ref} C_{n_{\delta_l}}$$

$$n_{\delta_n} = \frac{I_{XX}}{I_{XX} I_{ZZ} - I_{ZX}^2} \frac{\partial M_{az}}{\partial \delta_n} = \frac{I_{XX}}{I_{XX} I_{ZZ} - I_{ZX}^2} Q S \ell_{ref} C_{n_{\delta_n}}$$

Signs of the gradients of aerodynamic coefficients

$Cy_{\beta} < 0$

$Cy_{\delta_n} > 0$

$Cn_{\beta} > 0$ in the case of a stable aircraft

$Cn_{\delta_n} < 0$ in the case of tail fins

$C_{l_{\beta}} < 0$ generally

$C_{l_p} < 0$

$C_{l_r} > 0$

C_{n_p} ? (no general rule, depends on the airframe)

$C_{n_r} < 0$

We deduce the following state space model, after linearization and supposing that at equilibrium $\varphi_{eq} = 0$. At the equilibrium, we have $q_{eq} = 0$, $w_{eq} = 0$. Moreover, the terms of order greater than 2 are neglected.

$$\dot{X} = A_l X + B_l u$$

with

$$X = \begin{pmatrix} \beta \\ p \\ r \\ \varphi \end{pmatrix}$$

$$u = \begin{pmatrix} \delta_n \\ \delta_l \end{pmatrix}$$

State space representation

$$A_l = \begin{pmatrix} Y_\beta & Y_p & Y_r - 1 & \frac{g}{V_{eq}} \cos\theta_0 \\ l_\beta + \frac{I_{zx}}{I_{xx}} n_\beta & l_p + \frac{I_{zx}}{I_{xx}} n_p & l_r + \frac{I_{zx}}{I_{xx}} n_r & 0 \\ n_\beta + \frac{I_{zx}}{I_{zz}} l_\beta & n_p + \frac{I_{zx}}{I_{zz}} l_p & n_r + \frac{I_{zx}}{I_{zz}} l_r & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{pmatrix}$$

$$B_l = \begin{pmatrix} Y_{\delta_l} & Y_{\delta_n} \\ l_{\delta_l} + \frac{I_{zx}}{I_{xx}} n_{\delta_l} & l_{\delta_l} + \frac{I_{zx}}{I_{xx}} n_{\delta_n} \\ n_{\delta_l} + \frac{I_{zx}}{I_{zz}} l_{\delta_l} & n_{\delta_n} + \frac{I_{zx}}{I_{zz}} n_{\delta_l} \\ 0 & 0 \end{pmatrix}$$

Numerical application: B747 lateral model

Mass	m	288,773 kg
Altitude	z	12,192 m
Mach	M	0.8
Speed at equilibrium	V_{eq}	236 m/s
Air density	ρ	0.303 kg/m^3
Dynamic pressure	Q	8,439 Pa
Roll inertia	I_{XX}	$2.47 \cdot 10^7$ $kg \cdot m^2$
Yaw inertia	I_{ZZ}	$6.74 \cdot 10^7$ $kg \cdot m^2$
Roll-yaw inertia	I_{ZX}	$-2.12 \cdot 10^6$ $kg \cdot m^2$

Reference aerodynamic surface	S	511 m^2
Reference aerodynamic length	ℓ_{ref}	59.6 m

Aerodynamic coefficients gradients

Cy_β	-0.880 1/rad
Cy_p	0 s/rad
Cy_r	0 s/rad
Cy_{δ_l}	0 1/rad
Cy_{δ_n}	0.116 1/rad
Cl_β	-0.164 1/rad
Cl_p	-0.450$\times$2=-0.9 s/rad
Cl_r	0.3$\times$2=0.6 s/rad
Cl_{δ_l}	0.0137 1/rad
Cl_{δ_n}	0.007 1/rad
Cn_β	0.195 1/rad
Cn_p	-0.042$\times$2=-0.084 s/rad
Cn_r	-0.327$\times$2=0.654 s/rad
Cn_{δ_l}	0.0002 1/rad
Cn_{δ_n}	-0.126 1/rad

The coefficient 2 for Cl_p, Cl_r, Cn_p and Cn_r have been added because the original model was provided with coefficient l_p, l_r, n_p, n_r defined with a factor $\frac{\ell_{ref}}{2V_eq}$ instead of $\frac{\ell_{ref}}{V_eq}$.

```
In [80]: np.set_printoptions(formatter={'float': '{: 0.4f}'.format})

In [81]: print(Al)
[[-0.0557  0.0000 -1.0000  0.0416]
 [-1.7781 -0.5925  0.4097  0.0000]
 [ 0.8002 -0.0014 -0.1706  0.0000]
 [ 0.0000  1.0000  0.0000  0.0000]]

In [82]: print(Bl)
[[ 0.0000 -0.0366]
 [ 0.1431 -0.5719]
 [-0.0037  2.4152]
 [ 0.0000  0.0000]]

In [83]: Cl=np.eye(6)

In [84]: Dl=np.zeros((6,2))

In [85]: sysl=ss(Al,Bl,Cl,Dl)
```

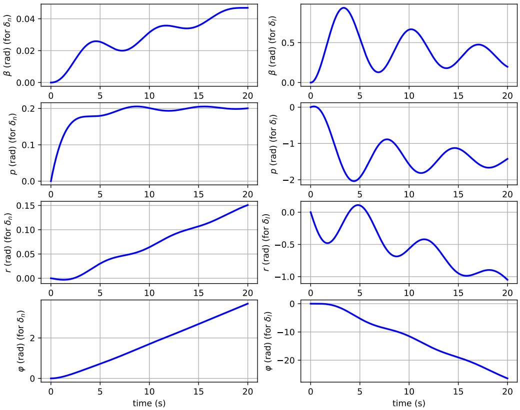
```
In [86]: control.matlab.damp(sysl)
_____Eigenvalue______ Damping___ Frequency_
  -0.07873    +0.9139j    0.08583      0.9173
  -0.07873    -0.9139j    0.08583      0.9173
   -0.6631                      1      0.6631
  0.001829                      1   -0.001829
```

The system, in this case, is unstable. It possesses a double pole weakly damped and two single poles, one very slow and the other less slow.
The double pole is the dutch roll mode.
The slowest single pole is the spiral mode (unstable here).
The fastest single pole is damping roll mode.

```
In [87]: eigenValues,eigenVectors=np.linalg.eig(asarray(Al))
    ...: print("Eigenvalues of Al")
    ...: print(eigenValues)
    ...: print("Eigenvectors of Al")
    ...: print(eigenVectors)
    ...:
Eigenvalues of Al
[-0.079+0.914j -0.079-0.914j -0.663+0.j     0.002+0.j   ]
Eigenvectors of Al
[[ 0.285-0.207j  0.285+0.207j -0.016+0.j     0.009+0.j   ]
 [-0.051+0.595j -0.051-0.595j -0.552+0.j     0.002+0.j   ]
 [-0.155-0.265j -0.155+0.265j  0.025+0.j     0.041+0.j   ]
 [ 0.651+0.j     0.651-0.j     0.833+0.j     0.999+0.j   ]]
```

The interpretation seems more difficult than for the longitudinal modes.

Step response to a roll and yaw fin deflection



If we examine the modulus of each of the components the eigenvectors (note that the two first eigenvectors are associated with the eigenvalues of the Dutch roll mode) we have

Variables of the state vector	dutch roll mode	roll damping mode	spiral mode
β	0.3521	0.0162	0.0088
p	0.5976	0.5524	0.0018
r	0.3074	0.0248	0.0410
φ	0.6515	0.8331	0.9991

Taking apart the influence on φ (coupling)

- the dutch roll has a dominating effect on β and p.
- the roll damping has a dominating effect on p.
- the spiral mode has dominating effect on r.

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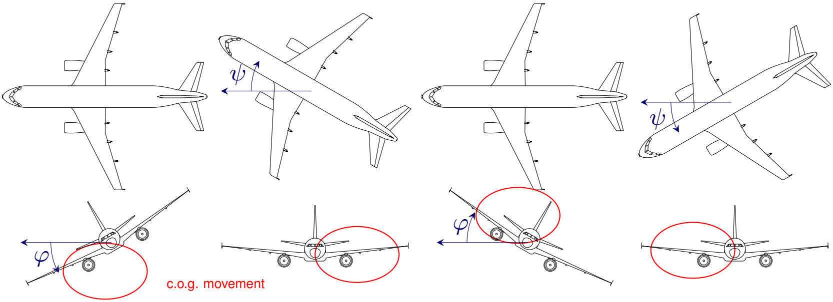
DUTCH ROLL MODE

The dutch roll mode consists in a coupling between yaw and roll axes. After an action on rudder for example, the aircraft oscillates around the yaw axis, with, in our example, a pulsation of around 0.92 rad/s and a weak damping of 0.086. It is caused by aircraft wing dihedral (the angle between the wings and the fuselage).

DUTCH ROLL MODE

- Yaw oscillations ψ, the yaw speed r increases.
- For $r > 0$, the speeds field is weaker on right wing so that its lift is lower than left wing's one and it creates a positive roll moment (because $l_r + \frac{I_{ZX}}{IXX} n_p > 0$), so p increases, so the roll φ increases.
- With the increase of roll, the coupling due to the term n_p, negative in our example, tends to decrease r on one hand, and the roll restoring mode ($l_p < 0$) tends to decrease p.
- We notices on the preceding graphs that the roll oscillations (p variable) have a phase shift of 90° with respect to β oscillations.
- In our example, the pole associated to this mode has a negative real part and a weak damping. The oscillations will diminish and will damp.

DUTCH ROLL MODE



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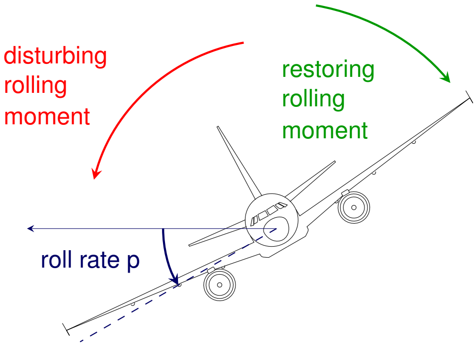
ROLL DAMPING MODE

- If the aircraft takes positive roll $\varphi > 0$, the right wing dives and the speeds field created makes that the right wing has globally a higher incidence than left wing, which creates a higher local lift on right wing than on the left wing.
- This creates a restoring moment, which tends to restore the equilibrium ($l_p < 0$).
- As this is a mode with a single pole after a perturbation, the return to equilibrium is done without oscillation (first order transfer function response). In our example, the mode pulsation is 0.66 rad/s.


disturbing
rolling
moment
restoring
rolling
moment
roll rate p

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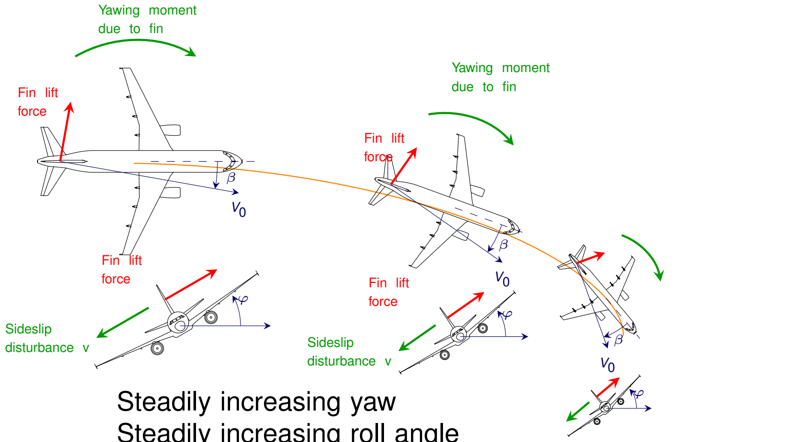
SPIRAL MODE

- Suppose that the aircraft is flying on a level flight and that a perturbation creates a roll angle φ and a small sideslip angle β.
- The lift of vertical stabilizer creates a positive moment around yaw axis.
- The coupling ($l_r > 0$) increases roll angle and sideslip angle β, which accentuates the phenomenon.
- Without any corrective action from the pilot, roll and yaw slowly diverge and altitude decreases: the aircraft describes a spiral towards the ground. In our example, this mode is slow with a pulsatio of $\omega_s = 0.002\ rad/s$, which corresponds to a time constant of about 3435 s, and this pole is unstable.
  To counter the effects of this mode, it is possible to modify dihedral or to modify the vertical stabilizer, by reducing its size.

SPIRAL MODE



Steadily increasing yaw
Steadily increasing roll angle

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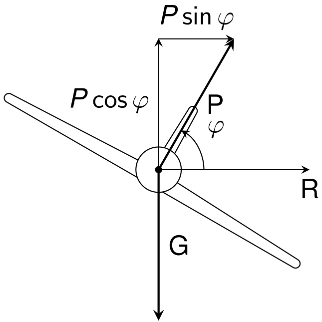
AIRCRAFT TURN



In a coordinated turn, the lift balances the lateral centrifugal force

$$P\sin\varphi = \frac{mV^2}{R}$$

and balances the weight in the vertical plane

$$P\cos\varphi = mg$$

From these 2 equations, we can deduce the roll φ needed to realize a balanced turn with a radius R.
We have

$$\tan\varphi = \frac{V^2}{Rg}$$