Assignment #1

Due date: October 11th, 2022 at 11:59pm

- Please write your name here Precious Udogu
- Please write your UID here 119343890
- Please write your solutions in the free space after problem statements. Hand in the hard copy of your solutions before the class begins.
- Your grades depend on the correctness and clarity of your answers.
- Write your answers with enough detail about your approach and concepts used, so that the grader will be able to understand it easily. You should ALWAYS prove the correctness of your algorithms either directly or by referring to a proof in the book.
- Write your answers in the spaces provided. If needed, attach other pages.
- You should scan and submit your solutions one week before the deadline to show your progress. Send your .pdf file to XXX@gmail.com with title "HW1-so-far".

(1)

1. A standard 52-card deck includes thirteen *ranks* of {2, 3, ... ,10, *J*, *Q*, *K*, *A*} in each of the four *suits* { , ♣, ♥, ♠}. In Poker, any subset of 5 cards is called a *hand*. A hand is called a *full house* if it includes exactly three cards of the same rank and two cards of another rank.

(a) Suppose that we draw 5 cards out of the 52 cards randomly. What is the probability that these 5 cards form a full house?

(b) Suppose that we draw 7 cards out of the 52 cards randomly. What is the probability that there exists at least one full-house hand within these 7 cards?

Solution :

a) P(5 cards form a full house) = (total ways to select 3 of 4 suits of that card) X (total ways to select one of the 13 ranks) X (total ways to select one of the remaining 12 ranks) X (total ways to select 2 of 4 suits of that rank)

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total ways to select 5 cards of 52 cards

$$= \frac{{}^{13}C_1 \times {}^{4}C_3 \times {}^{12}C_1 \times {}^{4}C_2}{{}^{52}C_5}$$

$$= \frac{3744}{2598960}$$

$$= 0.00144$$

b) P(there exists at least one full house within the 7 cards drawn) =

P(three of a kind, a pair, and two singletons) + P(three of a kind and two pairs) + P(two three of a kinds and a singleton)

$$P(\text{three of a kind, a pair, and two singletons}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}\binom{11}{2}\binom{4}{1}^2}{\binom{52}{7}}$$

$$P(\text{three of a kind and two pairs}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{2}^2}{\binom{52}{7}}$$

$$P(\text{two three of a kinds and a singleton}) = \frac{\binom{13}{2}\binom{4}{3}^2\binom{11}{1}\binom{4}{1}}{\binom{52}{7}}$$

P(there exists at least on full house within the 7 cards drawn) =

$$\frac{3294720 + 123552 + 54912}{\binom{52}{7}}$$

$$= 0.02596$$

2. Suppose that you play the following game against a house in Las Vegas. You pick a number between one and six, and then the house rolls three dices. The house pays you $1,500 if your number comes up on one dice, $2,000 if your number comes up on two dices, and $2.500 if your number comes up on all three dices. However, *you must pay the house $1,000* if your number does not show up at all. How much can you expect to win (or lose)?

Solution:

$P(\text{number appears on only one die}) = P(X = 1500)$

$$= \binom{3}{1}\left(\tfrac{1}{6}\right)^1\left(\tfrac{5}{6}\right)^2 = \tfrac{75}{216}$$

$P(\text{number appears on only two dice}) = P(X = 2000)$

$$= \binom{3}{2}\left(\tfrac{1}{6}\right)^2\left(\tfrac{5}{6}\right)^1 = \tfrac{15}{216}$$

$P(\text{number appears on three dice}) = P(X = 2500)$

$$= \binom{3}{3}\left(\tfrac{1}{6}\right)^3\left(\tfrac{5}{6}\right)^0 = \tfrac{1}{216}$$

$P(\text{number does not show up at all}) = P(X = -1000)$

$$= \binom{3}{0}\left(\tfrac{1}{6}\right)^0\left(\tfrac{5}{6}\right)^3 = \frac{125}{216}$$

$$EX = \sum x P(x)$$

$$= 1500\left(\tfrac{75}{216}\right) + 2000\left(\tfrac{15}{216}\right) + 2500\left(\tfrac{1}{216}\right) - 1000\left(\tfrac{125}{216}\right)$$

$$= 92.59$$

$\therefore$ One can expect to win \$92.59

(3)

3. You are looking for your hat in one of six drawers. There is a 10% chance that it is not in the drawers at all, but if it is in a drawer, it is equally likely to be in each. Suppose that we have opened the first two drawers and noticed that the hat is not in them.

(a) What is the probability that the hat is in the third drawer?

(b) What is the probability that the hat is not in any of the drawers?

Solution.

a) P(hat is in drawer) $= 1 - 0.1 = 0.9$

P(hat is in one of the 6 drawers) $= \frac{0.9}{6} = 0.15$

P(hat is in 3rd drawer | hat is not in first two drawers) $= \frac{0.15}{1-2(0.15)}$

$= 0.21429$

b) P(hat is not in any of the drawers | hat is not in first two drawers)

$= \frac{0.1}{1-2(0.15)}$

$= 0.14286$

4. Let S be the set of all sequences of three rolls of a dice.

- Let X be the sum of the number of dots on the three rolls. What is $E(X)$?
- Let Y be the product of the number of dots on the three rolls. What is $E(Y)$?

Solution

a) Let X = Sum of dots, and x = number of favourable outcomes

If there are 6 outcomes for one die,

Outcomes for 2 dice $= 6^2$

Outcomes for 3 dice $= 6^3 = 216$

$$P(x) = \frac{x}{6^3} \Rightarrow \frac{x}{216}$$

When $X = 3$, $P(X) = 1/216$

When $X = 4$, $P(X) = 3/216$

When $X = 5$, $P(X) = 6/216$

When $X = 6$, $P(X) = 10/216$

When $X = 7$, $P(X) = 15/216$

When $X = 8$, $P(X) = 21/216$

When $X = 9$, $P(X) = 25/216$

When $X = 10$, $P(X) = 27/216$

When $X = 11$, $P(X) = 27/216$

When $X = 12$, $P(X) = 25/216$

When $X = 13$, $P(X) = 21/216$

When $X = 14$, $P(X) = 15/216$

When $X = 15$, $P(X) = 10/216$

When $X = 16$, $P(X) = 6/216$

When $X = 17$, $P(X) = 3/216$

When $X = 18$, $P(X) = 1/216$

$$E(X) = \sum_{n=3}^{18} X \cdot P(X)$$

$$= 3\left(\tfrac{1}{216}\right) + 4\left(\tfrac{3}{216}\right) + 5\left(\tfrac{6}{216}\right) + 6\left(\tfrac{10}{216}\right) + 7\left(\tfrac{15}{216}\right) + 8\left(\tfrac{21}{216}\right)$$
$$+ 9\left(\tfrac{25}{216}\right) + 10\left(\tfrac{27}{216}\right) + 11\left(\tfrac{27}{216}\right) + 12\left(\tfrac{25}{216}\right) + 13\left(\tfrac{21}{216}\right)$$
$$+ 14\left(\tfrac{15}{216}\right) + 15\left(\tfrac{10}{216}\right) + 16\left(\tfrac{6}{216}\right) + 17\left(\tfrac{3}{216}\right) + 18\left(\tfrac{1}{216}\right)$$

$$= \frac{607}{54} \Rightarrow 11.2407$$

b) Let Y = product of number of dots, and y = number of favourable outcomes

$$P(Y) = y/6^3 \Rightarrow y/216$$

When $Y=1$, $P(Y) = 1/216$
When $Y=2$, $P(Y) = 1/216$
When $Y=3$, $P(Y) = 1/216$
When $Y=4$, $P(Y) = 2/216$
When $Y=5$, $P(Y) = 1/216$
When $Y=6$, $P(Y) = 2/216$
When $Y=8$, $P(Y) = 2/216$
When $Y=9$, $P(Y) = 1/216$
When $Y=10$, $P(Y) = 1/216$
When $Y=12$, $P(Y) = 3/216$
When $Y=15$, $P(Y) = 1/216$
When $Y=16$, $P(Y) = 2/216$
When $Y=18$, $P(Y) = 2/216$
When $Y=20$, $P(Y) = 2/216$
When $Y=24$, $P(Y) = 3/216$
When $Y=25$, $P(Y) = 1/216$
When $Y=27$, $P(Y) = 1/216$
When $Y=30$, $P(Y) = 2/216$
When $Y=32$, $P(Y) = 1/216$
When $Y=36$, $P(Y) = 3/216$
When $Y=40$, $P(Y) = 1/216$
When $Y=45$, $P(Y) = 1/216$
When $Y=48$, $P(Y) = 2/216$
When $Y=50$, $P(Y) = 1/216$
When $Y=54$, $P(Y) = 1/216$
When $Y=60$, $P(Y) = 2/216$
When $Y=64$, $P(Y) = 1/216$
When $Y=72$, $P(Y) = 2/216$
When $Y=75$, $P(Y) = 1/216$
When $Y=80$, $P(Y) = 1/216$
When $Y=90$, $P(Y) = 1/216$
When $Y=96$, $P(Y) = 1/216$
When $Y=100$, $P(Y) = 1/216$
When $Y=108$, $P(Y) = 1/216$
When $Y=120$, $P(Y) = 1/216$
When $Y=125$, $P(Y) = 1/216$
When $Y=144$, $P(Y) = 1/216$
When $Y=150$, $P(Y) = 1/216$
When $Y=180$, $P(Y) = 1/216$
When $Y=216$, $P(Y) = 1/216$

$$E(Y) = \sum_i Y \cdot P(Y)$$

$$= 1/216 + 2/216 + 3/216 + 8/216 + 5/216 + 12/216$$
$$+ 16/216 + 9/216 + 10/216 + 36/216 + 15/216 + 32/216$$
$$+ 36/216 + 40/216 + 72/216 + 25/216 + 27/216$$
$$+ 60/216 + 32/216 + 108/216 + 40/216 + 45/216$$
$$+ 96/216 + 50/216 + 54/216 + 120/216 + 64/216$$
$$+ 144/216 + 75/216 + 80/216 + 90/216 + 96/216$$
$$+ 100/216 + 108/216 + 120/216 + 125/216 + 144/216$$
$$+ 150/216 + 180/216 + 216/216 = \frac{2646}{216} \Rightarrow 12.25$$

(5)

5. Find x, y, and z such that

- $x \le 10$
- $x + y \le 17$
- $2x + 3z \le 25$
- $y + z \ge 11$
- $15x + 2y + z$ is maximized

Your solution:

- x: [ ]
- y: [ ]
- z: [ ]
- $15x + 2y + z$: [ ]

Solution

$x \le 10$

$x + y \le 17$

$2x + 3z \le 25$

$y + z \ge 11$

Let $p = 15x + 2y + z$

Using Simplex method,

$$\begin{array}{lllllllll}
x & & & + s_1 & & & & & = 10 \\
x & + y & & & + s_2 & & & & = 17 \\
2x & & + 3z & & & + s_3 & & & = 25 \\
 & y & + z & & & & + s_4 & & = 11 \\
-15x & - 2y & - z & & & & & + p & = 0
\end{array}$$

Tableau #1

x	y	z	s_1	s_2	s_3	s_4	p	
1	0	0	1	0	0	0	0	10
1	1	0	0	1	0	0	0	17
2	0	3	0	0	1	0	0	25
0	1	1	0	0	0	-1	0	11
-15	-2	-1	0	0	0	0	1	0

Tableau #2

x	y	z	s_1	s_2	s_3	s_4	p	
1	0	0	1	0	0	0	0	10
1	0	-1	0	1	0	0	0	6
2	0	3	0	0	1	0	0	25
0	1	1	0	0	0	-1	0	11
-15	0	1	0	0	0	-2	1	22

Tableau #3

x	y	z	s_1	s_2	s_3	s_4	p	
0	0	1	1	-1	0	-1	0	4
1	0	-1	0	1	0	1	0	6
0	0	5	0	-2	1	2	0	13
0	1	1	0	0	0	-1	0	11
0	0	-14	0	15	0	13	1	112

Tableau #4

	x	y	z	S_1	S_2	S_3	S_4	P	
S_1	0	0	0	1	-0.6	-0.2	-0.6	0	1.4
x	1	0	0	0	0.6	0.2	0.6	0	8.6
z	0	0	1	0	-0.4	0.2	-0.4	0	2.6
y	0	1	0	0	0.4	-0.2	-0.6	0	8.4
P	0	0	0	0	9.4	2.8	7.4	1	148.4

x : [8.6]

y : [8.4]

z : [2.6]

$15x + 2y + z$: [148.4]

(6)

6. We are given a graph G with vertex set $V(G)$ and edge set $E(G)$. Every edge e has a weight w_e. Write an LP whose optimal solution is equal to the length of the shortest path from a vertex s to a vertex t in this graph. You may write some constraints over the edge set or vertex set of the graph. For instance, you can write

$$\forall (u,v) \in E(G) \qquad w_u f_u + w_v f_v \leq 10.$$

In the above example, f's are the variables of your LP.

Solution

Let x_{uv} denote whether we use edge e or not, then the LPP is to

Minimize

$$\sum w_{uv} x_{uv}$$

Such that, (u,v) belongs to the edge set $E(G)$

$$\sum x_{uv} = \sum x_{vw}$$

for all v in $V(G) \setminus \{s,t\}$

and for (u,t) in edge set $E(G)$

$$\sum x_{ut} = 1$$

(7)

7. **Prove that the expected value of the binomial distribution with n draws and probability p is equal to np and its variance is $np(1-p)$**

Solution

$E[X] \to$ expected value

a) We begin by using the formula:

$$E[X] = \sum_{x=0}^{n} x\, C(n,x) p^x (1-p)^{n-x}$$

Since each term of the summation is multiplied by x, the value of the term corresponding to $x = 0$ will be 0, and so we can actually write:

$$E[X] = \sum_{x=1}^{n} x\, C(n,x) p^x (1-p)^{n-x}$$

By manipulating the factorials involved in the expression for $C(n,x)$ we can rewrite

$$x\, C(n,x) = n\, C(n-1, x-1)$$

This is true because:

$$x\, C(n,x) = \frac{x\, n!}{x!(n-x)!} = \frac{n!}{(x-1)!(n-x)!} = \frac{n(n-1)!}{(x-1)!((n-1)-(x-1))!}$$

$$= n\, C(n-1, x-1)$$

It follows that:

$$E[X] = \sum_{x=1}^{n} n\, C(n-1, x-1) p^x (1-p)^{n-x}$$

We factor out the n and one p from the above expression:

$$E[X] = np \sum_{x=1}^{n} C(n-1, x-1) p^{x-1} (1-p)^{(n-1)-(x-1)}$$

A change of variables $r = x-1$ gives us:

$$E[X] = np \sum_{r=0}^{n-1} C(n-1, r) p^r (1-p)^{(n-1)-r}$$

By the binomial formula, $(x+y)^k = \sum_{r=0}^{k} C(k,r) x^r y^{k-r}$ the summation above can be rewritten:

$$E[X] = (np)(p + (1-p))^{n-1} = np \quad \Big| \quad E[X] = np$$

b) Variance $\sigma^2 = E(x^2) - [E(x)]^2$

$$E(x^2) = \sum_{x=0}^{n} x^2 \cdot P(x)$$

$$E(x^2) = \sum_{x=0}^{n} [x + (x-1)x] \cdot P(x)$$

$$E(x^2) = \sum x \cdot P(x) + \sum (x-1)x \cdot P(x)$$

$$E(x^2) = np + \sum (x-1)x \cdot {}^nC_x \cdot p^x \cdot q^{n-x}$$

$$E(x^2) = np + \sum x(x-1) \cdot \frac{n!}{(n-x)! \cdot x!} \cdot p^x \cdot q^{n-x}$$

$$E(x^2) = np + \sum x(x-1) \cdot \frac{n!}{(n-x)! \cdot x \cdot (x-1) \cdot (x-2)!} \cdot p^x \cdot q^{n-x}$$

$$E(x^2) = np + \sum \frac{n \cdot (n-1) \cdot (n-2)!}{[(n-2)-(x-2)]! \cdot (x-2)!} \cdot p^2 \cdot p^{x-2} \cdot q^{(n-2)-(x-2)}$$

$$E(x^2) = np + n(n-1) \cdot p^2 \sum \frac{(n-2)!}{((n-2)-(x-2))! \cdot (x-2)!} \cdot p^{x-2} \cdot q^{(n-2)-(x-2)}$$

$$E(x^2) = np + n(n-1) \cdot p^2 \cdot (p+q)^{n-2}$$

$$E(x^2) = np + (n^2 \cdot p^2 - np^2) \cdot (1)^{n-2}$$

$$E(x^2) = np + n^2 \cdot p^2 - np^2 \quad , \quad E(x) = np$$

$$\therefore [E(x)]^2 = (np)^2$$

Variance $\sigma^2 = E(x^2) - [E(x)]^2$

$$\sigma^2 = (np + n^2 \cdot p^2 - np^2) - (np)^2$$

$$\sigma^2 = np + n^2 p^2 - np^2 - n^2 p^2$$

$$\sigma^2 = np - np^2$$

$$\sigma^2 = np(1-p)$$