Moving Coil:

Mathematical Model and Control

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1 Moving Coil: mathematical model

The moving coil was already studied at the beginning of this document. Here we want to represent the moving coil model using a different approach, we would like to use a more mathematical (and less physical) approach to show a more formal representation. Most of the following material has been taken from the monumental work of [1].

We define two reference frames: one stationary $\mathscr{O}$ and one which lies on the moving coil $\mathscr{O}'$ and we define the following meaning of the electrical field $\vec{E}$ induced across the coil

- $\vec{E}'$ is the induced electrical field observed from the moving reference frame $\mathscr{O}'$.
- $\vec{E}$ is the induced electrical field observed from the stationary reference frame $\mathscr{O}$.

The Faraday's law applied to such that circuit can be written as follows

$$\oint_{\mathscr{C}} \vec{E}' \cdot d\vec{l} = -\frac{d}{dt}\int_{\mathscr{S}} \vec{B} \cdot \hat{n}\, da \tag{1.1}$$

where $\vec{E}'$ is the electrical field applied to the coil respect the moving reference frame.

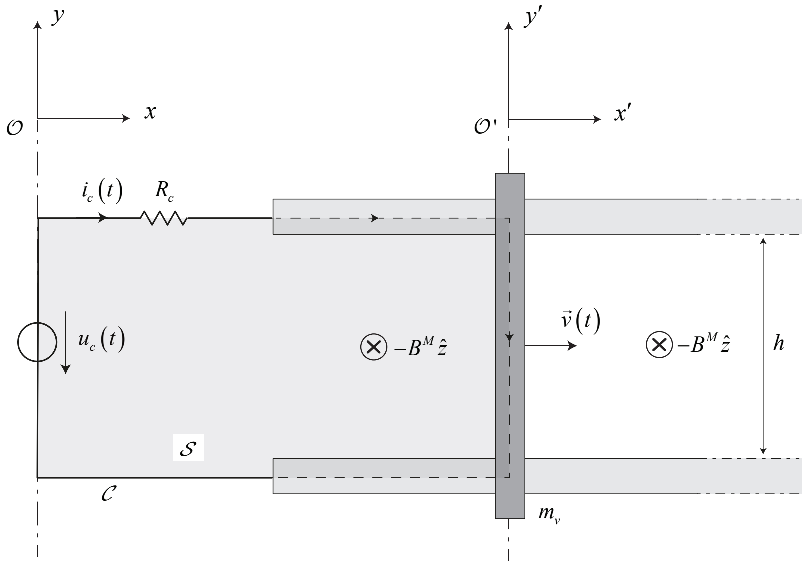


Figure 1: Moving coil actuator.

The circuit is moving at a certain speed $\vec{v}$, hence, the total time derivation of the linkage flux at the right hand side of Eq. 1.1 shall take into account that the linkage flux can change due to change of $\vec{B}$ and due to the change of the contour $\mathscr{C}$ (or $\mathscr{S}$) due to the movement of the coil.

The total time derivative* of the flux through the moving coil is (see [**?**])

$$\frac{d}{dt}\int_{\mathscr{S}} \vec{B} \cdot \hat{n}\, da = \int_{\mathscr{S}} \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}\, da + \oint_{\mathscr{C}} \left(\vec{B} \times \vec{v}\right) \cdot d\vec{l} \tag{1.2}$$

*The concept of the convective derivative can be used

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}$$

Hence Eq. 1.1 can now be written in the form

$$\oint_{\mathscr{C}} \left[\vec{E}' - \vec{v} \times \vec{B}\right] \cdot d\vec{l} = \int_{\mathscr{S}} \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}\, da \tag{1.3}$$

Eq. 1.3 can also be obtained as follows (see also Figure 2)

$$\oint_{\mathscr{C}} \vec{E}' \cdot d\vec{l} = -\frac{d}{dt}\int_{\mathscr{S}} \vec{B} \cdot \hat{n}\, da = -\frac{1}{dt}\left[\int_{\mathscr{S}(t+dt)} \vec{B}(t+dt) \cdot \hat{n}\, da - \int_{\mathscr{S}(t)} \vec{B}(t) \cdot \hat{n}\, da\right] \tag{1.4}$$

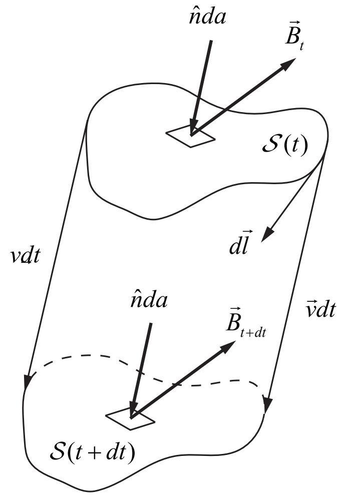


Figure 2: Evaluation of the $d\vec{B}/dt$.

where

$$\vec{B}(t+dt) = \vec{B}(t) + \frac{\partial \vec{B}}{\partial t} \tag{1.5}$$

The second term of Eq. 1.4 become

$$\begin{aligned} &-\frac{1}{dt}\left[\int_{\mathscr{S}(t+dt)} \left(\vec{B}(t) + \frac{\partial \vec{B}}{\partial t}\right) \cdot \hat{n}\, da - \int_{\mathscr{S}(t)} \vec{B}(t) \cdot \hat{n}\, da\right] = \\ &-\frac{1}{dt}\left[\int_{\mathscr{S}(t)} \vec{B}(t+dt) \cdot \hat{n}\, da - \int_{\mathscr{S}(t)} \vec{B}(t) \cdot \hat{n}\, da\right] - \int_{\mathscr{S}(t+dt)} \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}\, da \end{aligned} \tag{1.6}$$

The first term at the right side of the equation represents the variation of the flux of $\vec{B}$ due to the only change of the surface, and we can write

$$-\frac{1}{dt}\left[\int_{\mathscr{S}(t)} \vec{B}(t+dt) \cdot \hat{n}\, da - \int_{\mathscr{S}(t)} \vec{B}(t) \cdot \hat{n}\, da\right] = \oint_{\mathscr{C}} \left(\vec{v} \times \vec{B}\right) \cdot d\vec{l} \tag{1.7}$$

we obtain that

$$\frac{d\vec{B}}{dt} = \frac{\partial \vec{B}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right)\vec{B} = \frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \left(\vec{B} \times \vec{v}\right) + \vec{v}\left(\vec{\nabla} \cdot \vec{B}\right)$$

where $\vec{v}$ is considered a constant vector. Applying Stokes theorem we obtain Eq. 1.2

while the second term represent the variation of the flux due to the change of the vector $\vec{B}$

$$\int_{\mathscr{S}(t+dt)} \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}\, da = \int_{\mathscr{S}(t)} \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}\, da \qquad dt \to 0 \tag{1.8}$$

and we obtain the Eq. 1.3.

Now we apply the Faraday's law to the circuit respect the stationary reference frame $\mathscr{O}$ and by the fact that the circuit $\mathscr{C}$ and the surface $\mathscr{S}$ are fixed respect to the same stationary reference frame $\mathscr{O}$ the Faraday's law can be written in the form

$$\oint_{\mathscr{C}} \vec{E} \cdot d\vec{l} = -\int_{\mathscr{S}} \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}\, da \tag{1.9}$$

Galilean invariance Eq. 1.3 and Eq. 1.9 must be equal, hence results that the electrical field of the coil measured respect to $\mathscr{O}$ can be expressed as (see also section **??**)

$$\boxed{\vec{E} = \vec{E}' - \vec{v} \times \vec{B}} \tag{1.10}$$

Hence the Faraday's law applied to the moving coil

$$\boxed{\oint_{\mathscr{C}} \vec{E}' \cdot d\vec{l} = -\frac{d}{dt} \int_{\mathscr{S}} \vec{B} \cdot \hat{n}\, da} \tag{1.11}$$

where the circuit $\mathscr{C}$ and the surface $\mathscr{S}$ are not considered fixed, can be written in the form

$$\boxed{\oint_{\mathscr{C}} \left[\vec{E}' - \vec{v} \times \vec{B}\right] \cdot d\vec{l} = \int_{\mathscr{S}} \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}\, da} \tag{1.12}$$

where the circuit $\mathscr{C}$ and the surface $\mathscr{S}$ can be considered fixed but where we already account the speed of the coil.

From another, but equivalent, point of view, we can suppose to freeze, at a certain instant, the circuit and to define the contour $\mathscr{C}$ and its surface $\mathscr{S}$ as fixed respect to the stationary reference frame. Even if the contour $\mathscr{C}$ and the surface $\mathscr{S}$ are fixed in the space, the charge present into the coil is subjected to a vector speed $\vec{v}$ and hence is subjected to a force $\vec{f} = q(\vec{E} + \vec{v} \times \vec{B})$ respect to the reference frame $\mathscr{O}$, from Galilean invariance we can impose that $\vec{f} = \vec{f}' = q\vec{E}'$ which results in $\vec{E}' = \vec{E} + \vec{v} \times \vec{B}$. That means, the induced electrical field along the coil seen from a stationary reference frame is given by

$$\vec{E} = \vec{E}' - \vec{v} \times \vec{B} \tag{1.13}$$

For ideal conductive coil we can write

$$\vec{E} = -\vec{v} \times \vec{B}.$$

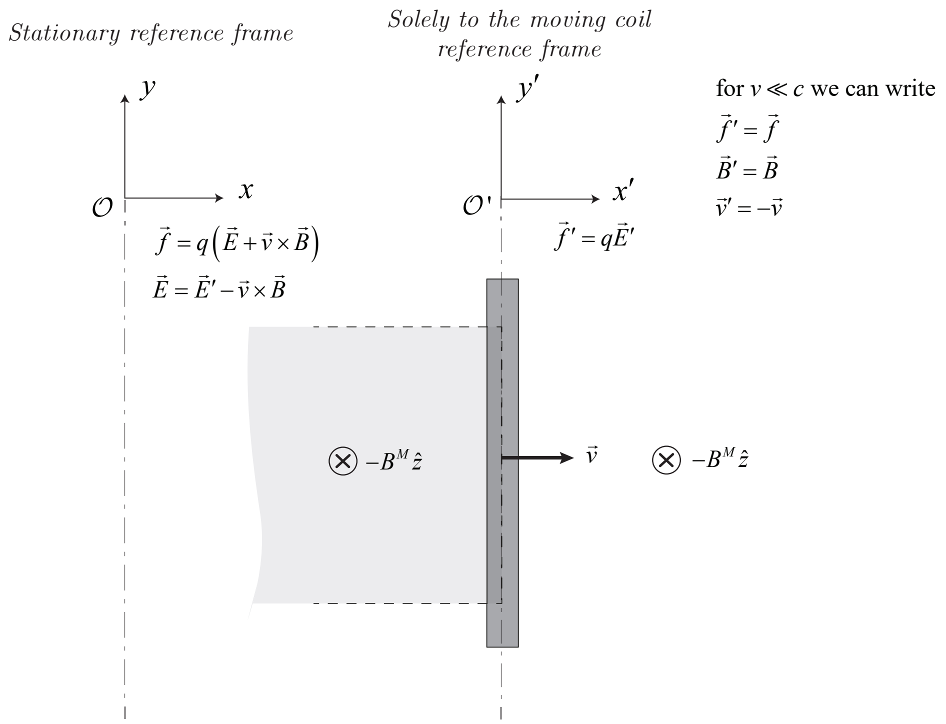


Figure 3: Lorentz force applied to the coil charge due to effect of the speed.

Now we can write the vector $\vec{B}$ as a superposition of two fields: the $\vec{B}_i$ due to the current $i(t)$ and the external field $\vec{B}^M$ which is supposed to be constant and isotropic,

$$\vec{B}(t) = \vec{B}_i(t) + \vec{B}^M$$

Therefore, the last term in Eq. 1.12 can be rewritten as

$$\int_{\mathcal{S}} \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}\, da = \int_{\mathcal{S}} \frac{\partial \vec{B}_i}{\partial t} \cdot \hat{n}\, da + \int_{\mathcal{S}} \frac{\partial \vec{B}^M}{\partial t} \cdot \hat{n}\, da \tag{1.14}$$

where

$$\frac{\partial \vec{B}^M}{\partial t} = 0 \tag{1.15}$$

$$\frac{\partial \vec{B}_i}{\partial t} = \frac{\partial \vec{B}_i(i)}{\partial i} \frac{di(t)}{dt} \tag{1.16}$$

Hence Eq. 1.14 can be written as

$$\int_{\mathcal{S}} \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}\, da = \int_{\mathcal{S}} \left[\frac{\partial \vec{B}_i(i)}{\partial i} \frac{di(t)}{dt}\right] \cdot \hat{n}\, da = \frac{\partial \phi(i)}{\partial i} \frac{di(t)}{dt} = L \frac{di(t)}{dt} \tag{1.17}$$

where $\mathcal{S}$ is considered constant and $\partial\phi(i)/\partial i = L$ is the definition of inductance.

Now we apply Eq. 1.12 in clockwise around the contour $\mathscr{C}$ of the Figure 1 and we consider the case that the coil is composed of N_c turns

$$\begin{aligned}\psi(t) &= N_c\phi(t) = N_c \int_S \vec{B}\cdot d\vec{s} \\ L_c &= N_c\, L\end{aligned} \tag{1.18}$$

we obtain

$$\underbrace{\int_{(-)}^{(+)} \vec{E}\cdot d\vec{l}}_{\text{source}} + \underbrace{\int \vec{E}\cdot d\vec{l}}_{\text{resistor}} + \underbrace{\int \vec{E}'\cdot d\vec{l}}_{\text{coil}} = -\frac{d}{dt}\int_S \vec{B}\cdot\hat{n}\, da. \tag{1.19}$$

or

$$\underbrace{\int_{(-)}^{(+)} \vec{E}\cdot d\vec{l}}_{\text{source}} + \underbrace{\int \vec{E}\cdot d\vec{l}}_{\text{resistor}} + \underbrace{\int \left[\vec{E}' - \vec{v}_v\times\vec{B}^M N_c\right]\cdot d\vec{l}}_{\text{coil}} = -L_c\frac{di_c(t)}{dt}. \tag{1.20}$$

The integrals over the fixed components (source and resistor) follow from Equations

$$-u_c = \underbrace{\int_{(-)}^{(+)} \vec{E}\cdot d\vec{l}}_{\text{source}} \tag{1.21}$$

$$i_c\, R_c = \underbrace{\int \vec{E}\cdot d\vec{l}}_{\text{resistor}} \tag{1.22}$$

Concerning the coil we consider the case where the whole resistance is accounted in R_c and we set $\vec{E}' = \vec{J}/\sigma = 0$, we obtain

$$u_c(t) - i_c(t)\, R_c - L_c\frac{di_c}{dt} + \int_{\text{coil}} \left(\vec{v}_v\times\vec{B}^M N_c\right)\cdot d\vec{l} = 0 \tag{1.23}$$

We are considering the case where $\vec{v}_v = v_v\hat{x}$, $\vec{B}^{\,M} = -B^M\,\hat{z}$ and $d\vec{l} = dy\hat{y}$. Therefore ($\hat{x}\times\hat{z} = -\hat{y}$),

$$\int_{\text{coil}} \left(\vec{v}_v\times\vec{B}^M N_c\right)\cdot d\vec{l} = \int_h^0 v_v\, B^M N_c dy = -N_c B^M h v_v(t) \tag{1.24}$$

we obtain

$$\boxed{u_c(t) - i_c(t)R_c - L_c\frac{di_c(t)}{dt} - N_c B^M h v_v(t) = 0} \tag{1.25}$$

To complete the linear pm-actuator model we must add the mechanical equations using the Newton's equation. To apply the Newton's law we first must calculate the corresponding force actuated by the iteration between the current i_c and the magnetic field $\vec{B}^M$.

The expression of the force due to the current i_c in the magnetic field $\vec{B}^M$ is evaluated as follows

$$\vec{f}(t) = i(t)\int_{\text{wire}} d\vec{l}\times\vec{B}^{\text{ext}} \tag{1.26}$$

where $\vec{B}^{\text{ext}}$ is an external field. Applying the integration of Eq. 1.26 it results in the following equation (**integration path follows the current direction**)

$$f_v^m(t) = N_c\, i_c(t) \int_h^0 d\vec{l} \times \vec{B}^M = N_c\, B^M\, h\, i_c(t) \quad \text{positive x-direction} \tag{1.27}$$

where N_c by the number of turns linkage to the magnetic field. The complete set of system equations becomes

$$\begin{cases} \dfrac{dx_v(t)}{dt} = v_v(t) \\ \dfrac{dv_v(t)}{dt} = N_c \dfrac{B^M h}{m_v} i_c(t) - \dfrac{b_v}{m_v} v_v(t) - \dfrac{1}{m_v} f_v^e(t) \\ \dfrac{di_c(t)}{dt} = -\dfrac{R_c}{L_c} i_c(t) - \dfrac{1}{L_c} N_c B^M h v_v(t) + \dfrac{1}{L_c} u_c(t) \end{cases} \tag{1.28}$$

the first two equations represent the dynamic motion of the bar where the force $f_v^m(t) = N_c B^M h\, i(t)$ is generated by the interaction of the current $i_c(t)$ and the magnetic field $\vec{B}^M$. The third equation represents the Kirchhoff's voltage law of the electrical circuit which generates the current $i_c(t)$ by applying the voltage $u_c(t)$. The equivalent auto-inductance L_c and the back-emf term, $N_c B^M h v_v(t)$ represent the effect of the Faraday's law.

2 Moving Coil: control system design

In the following project we are going to consider a linear actuator consisting of a conductive bar of mass m in sliding contact with a pair of stationary conducting rails as shown in Figure 4. The rails are connected to a voltage source $u(t)$. The bar is moving trough a constant uniform $\vec{B}^M$ magnetic field with a time dependent velocity $v(t)$ relative to the rails. We can consider the rails an inertial reference frame. This model could be useful for modeling a plotter or in general a linear actuator.

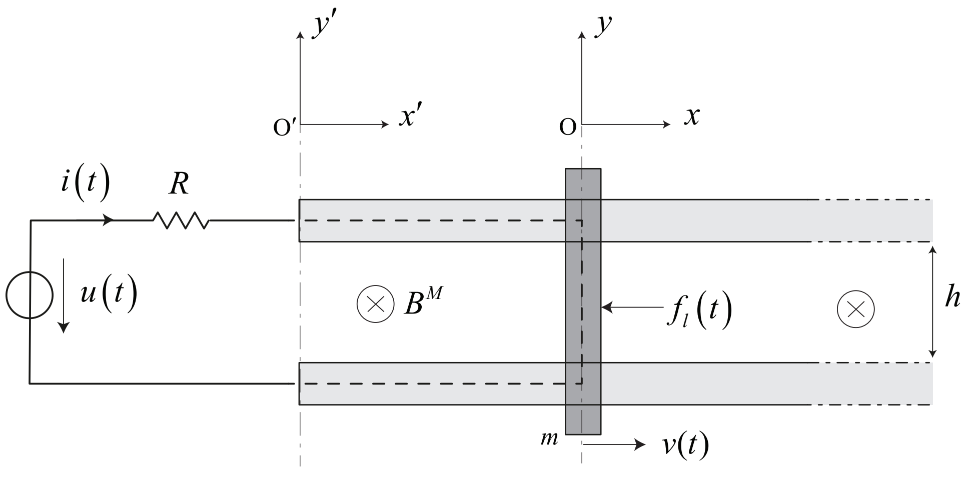


Figure 4: Linear actuator.

2.1 Model Derivation

Consider

$$\psi(t) = \int_S \vec{B} \cdot d\vec{s} \tag{2.1}$$

the flux linkage to the circuit. At this point we have to point out that the magnetic field $\vec{B}$ include two main contributes: one constant term which comes from the external permanent magnet $\vec{B}^M$ and one which is generated by the flowing current i, that means $\vec{B} = \vec{B}_i + \vec{B}^M$.

Applying the general Kirchhoff's law we obtain

$$u(t) - Ri(t) + \frac{d\psi(t)}{dt} = 0 \tag{2.2}$$

The term $d\psi(t)/dt$ represents the voltage induced in the circuit by a time rate of change of magnetic flux through the circuit. The change in flux can be due to a change in circuit current and/or the movement of the bar. Hence, the magnetic flux can be written as function of space and current $\psi(i, x)$

$$\psi(t) = \int_S \vec{B} \cdot d\vec{s} = \int_S \left(\vec{B}_i + \vec{B}^M\right) \cdot d\vec{s} = \psi(i, x) \tag{2.3}$$

Hence the total time derivative of $\psi(i, x)$ can be written as follows

$$\frac{d\psi(i,x)}{dt} = \frac{\partial\psi(i,x)}{\partial i(t)}\frac{di(t)}{dt} + \frac{\partial\psi(i,x)}{\partial x(t)}\frac{dx(t)}{dt} \tag{2.4}$$

where h (length of the bar) and B^M (magnitude of the permanent magnet magnetic field) are considered constant parameters

$$\frac{\partial\psi(i,x)}{\partial x(t)}\frac{dx(t)}{dt} = \frac{\partial Bxh}{\partial x}v(t) \approx B^M hv(t) \tag{2.5}$$

and

$$\frac{\partial\psi(i,x)}{\partial i(t)}\frac{di(t)}{dt} = L(x)\frac{di}{dt} \approx L\frac{di}{dt} \tag{2.6}$$

Eq. 2.2 becomes

$$u(t) - Ri(t) - L\frac{di(t)}{dt} - B^M hv(t) = 0 \tag{2.7}$$

The expression of the force due to the current i in the magnetic field $\vec{B}^M$ is given as follows (**integration path follows the current direction**)

$$f_m = i(t)\int_h^0 d\vec{l} \times \vec{B}^M = B^M h\, i(t) \quad \text{positive x-direction} \tag{2.8}$$

The complete set of system equations becomes

$$\begin{cases} \dfrac{dx(t)}{dt} = v(t) \\ \dfrac{dv(t)}{dt} = \dfrac{B^M h}{m} i(t) - \dfrac{b}{m} v(t) - \dfrac{1}{m} f_l(t) \\ \dfrac{di(t)}{dt} = -\dfrac{R}{L} i(t) - \dfrac{1}{L} B^M h v(t) + \dfrac{1}{L} u(t) \end{cases} \tag{2.9}$$

the first two equations represent the dynamic motion of the bar where the force $f_m(t) = B^M h\, i(t)$ is generated by the interaction of the current $i(t)$ and the magnetic field $\vec{B}^M$). The third equation represents the dynamic of the electrical circuit which generates the current $i(t)$ by applying the voltage $u(t)$. The equivalent auto-inductance L represents the effect of the Faraday's law.

The current $i(t)$ and the position $x(t)$ are measured, hence, the state space representation can be represented as follows

$$\begin{cases} \dot{\vec{x}}(t) = \tilde{\mathbf{A}} \vec{x}(t) + \tilde{\mathbf{B}} u(t) + \tilde{\mathbf{E}} d(t) \\ \vec{y}(t) = \tilde{\mathbf{C}} \vec{x}(t) \end{cases}$$

$$\begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{dv(t)}{dt} \\ \frac{di(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{b}{m} & \frac{B^M h}{m} \\ 0 & -\frac{B^M h}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \\ i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ -\frac{1}{m} \\ 0 \end{bmatrix} f_l(t)$$

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{b}{m} & \frac{B^M h}{m} \\ 0 & -\frac{B^M h}{L} & -\frac{R}{L} \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix}$$

$$\tilde{\mathbf{E}} = \begin{bmatrix} 0 \\ -\frac{1}{m} \\ 0 \end{bmatrix}, \quad \tilde{\mathbf{C}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.2 Control Problem

Given the following data:

1. Rails friction $b = 20\,\mathrm{N\,s\,m^{-1}}$

2. Length of the bar $h = 0.2\,\mathrm{m}$
3. Mass of the bar $m = 0.03\,\mathrm{kg}$
4. Magnitude of the permanent magnet magnetic field $B^M = 1.2\,\mathrm{T}$
5. Equivalent circuit inductance $L = 1\,\mathrm{mH}$
6. Equivalent circuit resistance $R = 1\,\Omega$
7. $f_l(t) = b_2 v(t)$ is an unmodelized viscosity, where $b_2 = 0.1\,\mathrm{N\,s\,m^{-1}}$ for case (1) and $b_2 = 1\,\mathrm{N\,s\,m^{-1}}$ for case (2)

we want to design a controller which is able to position the bar (plotter head) from a initial point to a given final point without any residual error.

2.3 Control Design without Load Estimator

To meet the control requirements a possible implementation is a state feedback control with integrator. Obviously the full state is not measurable, hence a full state observer will be also implemented.

The selected control layout is reported in Figure 5. To implement the control structure we have to calculate the state feedback vector which includes the gain corresponding to the integral block $\begin{bmatrix}\mathbf{K}_x & k_i\end{bmatrix}$ and a state observer. For both purposes we have to calculate the corresponding $\mathbf{A}$ and $\mathbf{B}$ matrices in discrete time domain as follows

$$\mathbf{A} = \mathbf{I} + \tilde{\mathbf{A}}\ t_s \tag{2.10}$$

$$\mathbf{B} = \tilde{\mathbf{B}}\ t_s \tag{2.11}$$

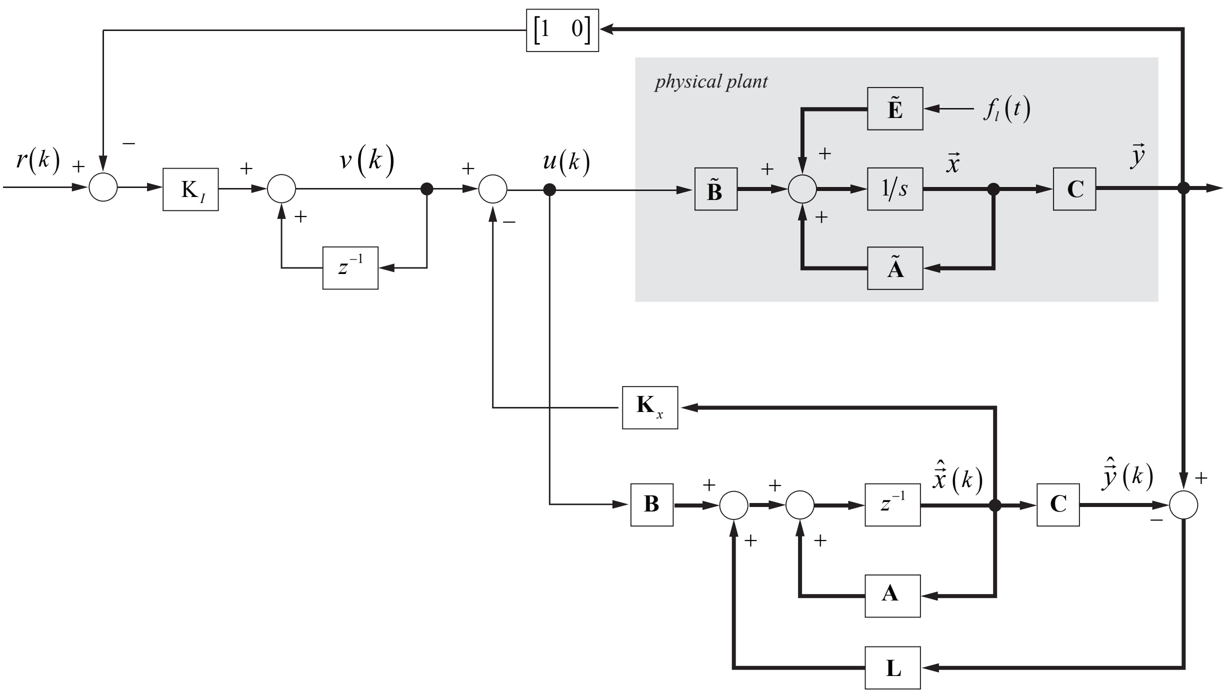


Figure 5: Control architecture.

Once the matrices $\mathbf{A}$ and $\mathbf{B}$ are obtained we can proceed implementing the extended system to calculate $\begin{bmatrix} K_I & \mathbf{K}_x \end{bmatrix}$ as follows

$$\mathbf{A}' = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & 0 \end{bmatrix} = (n+1) \times (n+1) \text{ matrix}$$

$$\mathbf{B}' = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} = (n+1) \text{ column vector}$$

Obtained $\mathbf{A}'$ and $\mathbf{B}'$ we can proceed calculating the state feedback vector $\mathbf{K}'$ from the Ackermann's formula

$$\mathbf{K}' = \begin{bmatrix} 0 & 0 & ... & 0 & 1 \end{bmatrix} \mathbf{M}'^{-1} q(\mathbf{A}')$$

where $\mathbf{M}' = \begin{bmatrix} \mathbf{B}' & \mathbf{A}'\mathbf{B}' & ... & (\mathbf{A}')^{n+m-1}\,\mathbf{B}' \end{bmatrix}$. Ackermann's formula uses the **Cayley–Hamilton** theorem and $q(z)$ is the desired final polynomial characteristics.

The **Cayley-Hamilton** theorem states that $\mathbf{A}'$ satisfies its own characteristic equation, hence

$$q(\mathbf{A}') = \mathbf{A}'^{n} + \alpha_1 \mathbf{A}'^{n-1} + ... + \alpha_{n-1}\mathbf{A}' + \alpha_n \mathbf{I} = \mathbf{0} \tag{2.12}$$

Once obtained the matrix $\mathbf{K}'$ we can use the formula

$$\begin{bmatrix} \mathbf{K}_x & k_i \end{bmatrix} = \left\{ \mathbf{K}' + \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} \mathbf{A} - \mathbf{I}_4 & \mathbf{B} \\ \mathbf{CA} & \mathbf{CB} \end{bmatrix}^{-1} \tag{2.13}$$

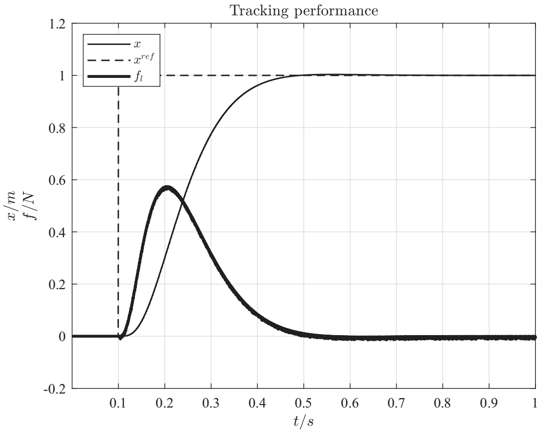
to obtain the state feedback gains $\begin{bmatrix} \mathbf{K}_x & k_i \end{bmatrix}$.

Regarding the state observer we have just to calculate the matrix $\mathbf{L}$. In this case the system is SIMO because the output vector $\vec{y}$ and obviously $\hat{\vec{y}}$ have dimension two and we have to use the Matlab command `place()`. The observer block becomes

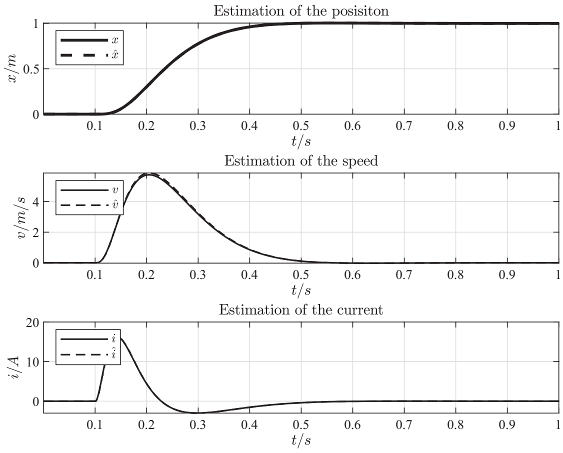
$$\begin{aligned} \hat{\vec{x}}(k+1) &= \mathbf{A}\,\hat{\vec{x}}(k) + \mathbf{B}\,u(k) + \mathbf{L}\left(\vec{y}(k) - \hat{\vec{y}}(k)\right) \\ \hat{\vec{y}}(k) &= \mathbf{C}\,\hat{\vec{x}}(k) \end{aligned} \tag{2.14}$$

as reported in Figure 5

2.3.1 Simulation results



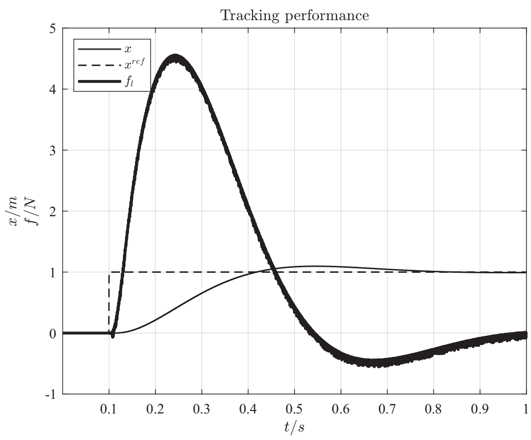
(a) Tracking performance for a given position set-point. The position reference x^{ref} is in dashed. Case (1).



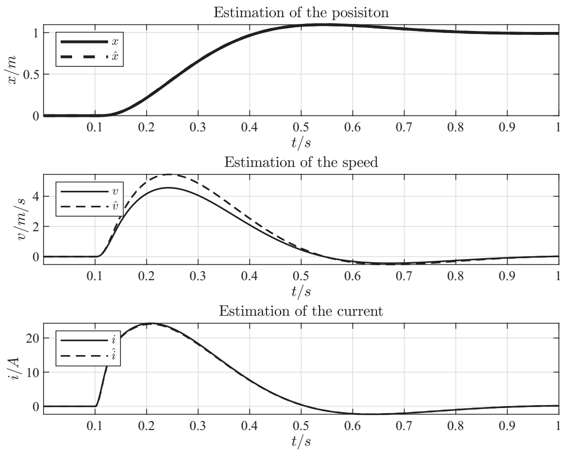
(b) Observer performance. Case (1).

Figure 6: Simulation results.

Limits of the model - The observer we have implemented presents some limits in particular in case (2) where unmodeled friction becomes relevant the estimation of the speed ($v(t)$) accumulate an error as function of the load as reported in Figure 7b where also the tracking performance are affected as shown in Figure 7a



(a) Tracking performance for a given position set-point. The position reference x^{ref} is in dashed. Case (2).



(b) Observer performance in case of high level of load force. Note the residual error in the estimation of the load force. Case (2).

Figure 7: Simulation results.

To avoid this degradation if the performance we would like to extend our state observer including an approximation of the dynamic of the load.

2.4 Control Design with Load Estimator

In this section we want to extend the observer including the dynamic of the load and try to increase the robustness against friction degradation of the rails. Let's start with a possible load model.

We can suppose a load of the form (the nature of the load force is friction, which means, a force which depends by speed):

$$f_l(t) = v(t)$$

where its dynamic can be expressed as

$$\dot{f}_l(t) = \dot{v}(t)$$

From Eq. 2.9 we can write

$$\dot{f}_l(t) = \dot{v}(t) = \frac{B^M h}{m} i(t) - \frac{b}{m} v(t) - \frac{1}{m} f_l(t)$$

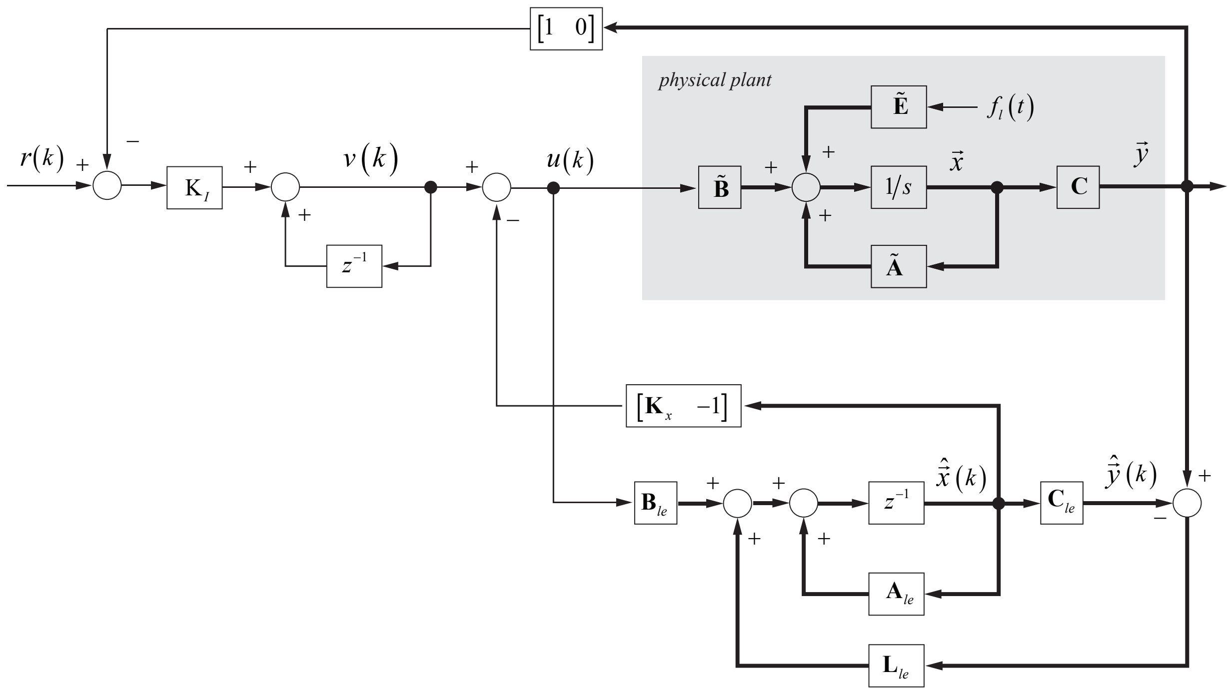


Figure 8: Control architecture with load estimator.

The extended (open loop) observer model can be described as follows

$$\begin{bmatrix} \frac{d\hat{x}(t)}{dt} \\ \frac{d\hat{v}(t)}{dt} \\ \frac{d\hat{i}(t)}{dt} \\ \frac{d\hat{f}_l(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{b}{m} & \frac{B^M h}{m} & -\frac{1}{m} \\ 0 & -\frac{B^M h}{L} & -\frac{R}{L} & 0 \\ 0 & -\frac{b}{m} & \frac{B^M h}{m} & -\frac{1}{m} \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \hat{v}(t) \\ \hat{i}(t) \\ \hat{f}_l(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \\ 0 \end{bmatrix} u(t) \tag{2.15}$$

where the closed loop form is

$$\begin{aligned}\dot{\hat{\vec{x}}}_{le}(t) &= \tilde{\mathbf{A}}_{le}\,\hat{\vec{x}}_{le}(t) + \tilde{\mathbf{B}}_{le}\,\tau_1(t) + \tilde{\mathbf{L}}_{le}\left(\vec{y}(t) - \hat{\vec{y}}(t)\right)\\ \hat{\vec{y}}(t) &= \mathbf{C}_{le}\,\hat{\vec{x}}_{le}(t)\end{aligned} \tag{2.16}$$

where $\vec{x}_{le} = \begin{bmatrix} x(t) & v(t) & i(t) & f_l(t)\end{bmatrix}^T$, $\mathbf{C}_{le} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$

$$\tilde{\mathbf{A}}_{le} = \begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & -\frac{b}{m} & \frac{B^M h}{m} & -\frac{1}{m}\\ 0 & -\frac{B^M h}{L} & -\frac{R}{L} & 0\\ 0 & -\frac{b}{m} & \frac{B^M h}{m} & -\frac{1}{m}\end{bmatrix} \tag{2.17}$$

The system $(\mathbf{A}_{le}, \mathbf{C}_{le})$ is fully observable. After the discretization

$$\begin{aligned}\tilde{\mathbf{A}}_{le} &\rightarrow \mathbf{A}_{le}\\ \tilde{\mathbf{B}}_{le} &\rightarrow \mathbf{B}_{le}\end{aligned} \tag{2.18}$$

$$\mathbf{A}_{le} = \mathbf{I} + \tilde{\mathbf{A}}_{le}\ t_s \tag{2.19}$$

or

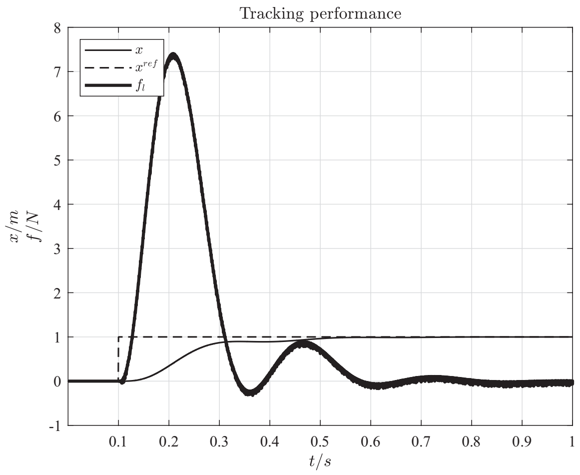
$$\mathbf{B}_{le} = \tilde{\mathbf{B}}_{le}\ t_s \tag{2.20}$$

we obtain the following state observer system

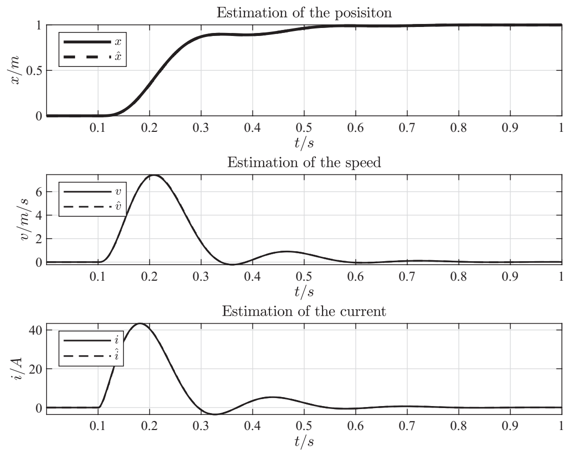
$$\begin{aligned}\hat{\vec{x}}_{le}(k+1) &= \mathbf{A}_{le}\,\hat{\vec{x}}_{le}(k) + \mathbf{B}_{le}\,\tau_1(t) + \mathbf{L}_{le}\left(\vec{y}(k) - \hat{\vec{y}}(k)\right)\\ \hat{\vec{y}}(k) &= \mathbf{C}_{le}\,\hat{\vec{x}}_{le}(k)\end{aligned} \tag{2.21}$$

Where matrix $\mathbf{L}_{le}$ is calculated applying `place()` formula considering the system $(\mathbf{A}_{le}, \mathbf{C}_{le})$.

2.4.1 Simulation Results



(a) Tracking performance for a given position set-point. The position reference x^{ref} is in dashed. Case (2).



(b) Observer performance in case of high level of load force. Note the residual error in the estimation of the load force. Case (2).

Figure 9: Simulation results.

Summary - In this example we have seen an application of the state feedback with integral control. This kind of approach is useful when it is necessary to achieve a zero error steady state tracking from a constant reference and the plant doesn't contain an integrator. In additional, we have implemented a state observer, at first glance, without including the dynamic of the load, which is here considered as a disturbance or plant parameter deviation (like ageing in the rails viscosity). We have seen that this plant deviation can degrade the performance of the servo and increasing the time to set, because the load enter in the system like a disturbance and the controller has to compensate it, spending "time"and "energy"which results in a modification of the global dynamics.

In the second part we tried to modelize the load dynamics supposing it comes from a not modelized viscosity of the rails. Increasing the observer (in terms of states elements), namely, including the dynamics of the load, we were able to estimate the load and to reduce, at least partially, the degradation of the tracking performance.

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