# Mathematical Modeling and Scientific Computation

May 31, 2018

## 1 Simple Dynamics: Systems of Differential Equations

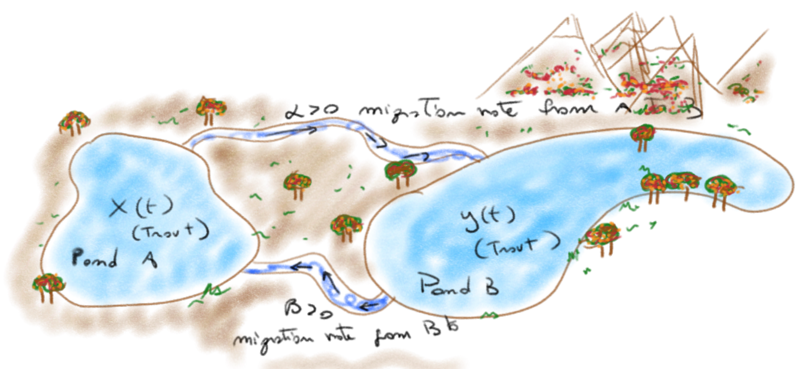


Figure 1: A simple model of trout dynamics.

Every spring in two ponds in Alto Adige (an Italian region) it is possible to witness some interesting migratory patterns of the local Trout populations (see figure 1). There are two small ponds which we shall call $A$ and $B$ respectively which contain some trout. At some time $t$, we will denote by $a(t)$ the number of trout in pond $A$ and by $b(t)$ the number of trout in in pond $B$. The two ponds are connected by two small streams which usually have very little water in them. In spring, however, because of the melting Alpine glaciers, the rivers swell allowing the trout to swim freely in them. The values $\alpha, \beta \in [0, 1]$ represent the proportion of trout swimming from pond $A$ to pond $B$ and from pond $B$ to pond $A$ respectively. We can represent the dynamics as follows:

$$\begin{aligned}\frac{dx}{dt} &= -\alpha x(t) + \beta y(t) \\ \frac{dy}{dt} &= \alpha x(t) - \beta y(t).\end{aligned}$$

**Your task:** Implement this model in Python and study the dynamics by choosing some initial conditions for the system. (You may also find an analytical solution by hand and compare it to your empirical solutions.)

# 2 A more complicated model: Fixing a Dam

In 2015, the Kariba Dam at the Zambezi River between Zambia and Zimbabwe was found to be in danger of failing, and it was advised that action be taken fast. However, there are a number of ways in which to resolve the problem:

- doing repair work on the existing dam
- rebuilding a dam at the site of the existing one
- removing the existing dam and replacing it with ten to twenty smaller dams along the river.

**Your Task:** Pick one of the ways to resolve the issues with the Kariba Dam, implement the model of section 2.1 and investigate the problem by means of simulations. (This is an open ended problem which was featured in the 2017 Mathematical Contest in Modeling[1].)

## 2.1 Model of Kariba Dam

Here is a simplified model of the existing Kariba Dam (see figure 2) and the flow of a simplified river. To simplify your task, we have created a simplified model of the dam using available data on Kariba Dam[2].
The model is defines the following characteristics:

- $w$ as the width of the dam in $m$
- $l$ as the length of the water basin in $m$
- $H_{Op}$ as the maximum operating height in $m$

[1] see `http://www.comap.com/undergraduate/contests/mcm/contests/2017/problems/` for the official problem description.
[2] Lagassé, P. (2000). Columbia Encyclopedia

- $H(t)$ as the current water height in $m$
- $A_i(t)$ as the surface area of the dam's water basin in $m^2$
- $A_o$ as the surface area of the spillway openings in $m^2$
- $V(t)$ as the volume of the water currently contained by the dam in $m^3$
- $v_1$ as the velocity of the water exiting the spillway in $m/s^2$
- $v_2$ as the velocity of the water travelling from the lake to the spillway in $m/s^2$
- $F_i$ as the flow rate of the water into the dam's water basin in $m^3/s$
- $F_o(t)$ as the flow rate of the water out of the dam's spillways in $m^3/s$
- $g$ as the gravitation constant — $9.81 m/s^2$

The volume of the Kariba Dam is calculated in the following way:

$$\begin{aligned} A_i &= w * l \\ V(t) &= A_i H(t). \end{aligned}$$

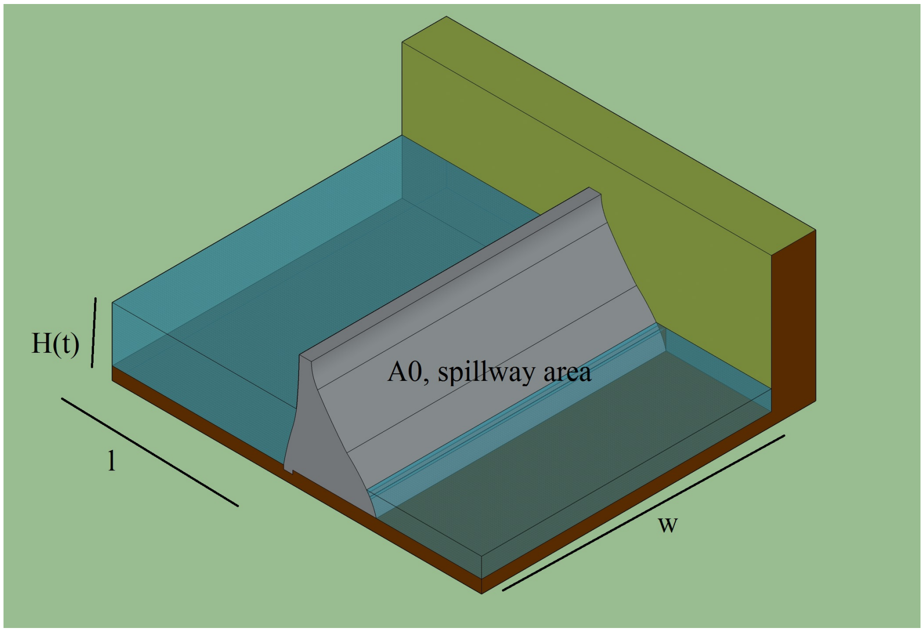


Figure 2: A simplified dam

Given the current volume of water, we need to determine the flow through the spillway $F_o$. This is done by way of Bernoulli's incompressible flow equation. We have

$$\begin{aligned} \frac{{v_1}^2}{2g} + h_0 &= \frac{{v_2}^2}{2g} + H(t) \Rightarrow \\ {v_1}^2 - {v_2}^2 &= 2g(H(t) - h_0) \end{aligned}$$

and by Bernoulli's law, we have $A_1 v_1 = A_2 v_2$ hence:

$$ {v_2}^2 = \frac{2g(H(t) - h_0)}{1 - {A_1}^2/{A_2}^2}. $$

Thus we find the volumetric flow rate through the dam's spillway to be:

$$ F_o(t) = A_1 v_1 = A_1 \sqrt{\frac{2g(H(t) - h_0)}{1 - A_1^2/A_2^2}}. \tag{1} $$

However, we must note that $A_1^2/A_2^2 \approx 0$ since $A_1 << A_2$ and thus we may simplify equation 1 to be:

$$ F_o(t) = A_1 \sqrt{2g(H(t) - h_0)} \tag{2} $$

we can then use this outward flow rate alter the current volume of water in the dam over time using the following simple equation:

$$ \dot{V}(t) = V(t) + F_i - F_o(t) \tag{3} $$

This allows us to model the dam adapting to different weather conditions over the course of a year.