

Problem 1.

(1)  $\frac{}{\{ \} \vdash \text{let rec } f = \text{fun } x \rightarrow \text{fun } n \rightarrow \text{if } n \leq 0 \text{ then } [] \text{ else } x :: (f \ x \ (n - 1)) \text{ in } (f \ 3 \ 2, f \ "a" \ 4) : \text{int list} * \text{string list}}$  (let rec rule)

(1):  $\{ f : \forall \alpha. \alpha \rightarrow \text{int} \rightarrow \alpha \text{ list} \} \vdash \text{fun } x \rightarrow \text{fun } n \rightarrow \text{if } n \leq 0 \text{ then } [] \text{ else } x :: (f \ x \ (n - 1)) : \forall \alpha. \alpha \rightarrow \text{int} \rightarrow \alpha \text{ list}$

let  $\Gamma_1 = \{ f : \forall \alpha. \alpha \rightarrow \text{int} \rightarrow \alpha \text{ list} \}$

prove:

$\frac{(3) \{ x : \tau, n : \text{int} \} + \Gamma_1 \vdash \text{if } n \leq 0 \text{ then } [] \text{ else } x :: (f \ x \ (n - 1)) : \tau \text{ list}}{\{ x : \tau \} + \Gamma_1 \vdash \text{fun } n \rightarrow \text{if } n \leq 0 \text{ then } [] \text{ else } x :: (f \ x \ (n - 1)) : \text{int} \rightarrow \tau \text{ list}}$  (fun rule)

$\Gamma_1 \vdash \text{fun } x \rightarrow \text{fun } n \rightarrow \text{if } n \leq 0 \text{ then } [] \text{ else } x :: (f \ x \ (n - 1)) : \forall \alpha. \alpha \rightarrow \text{int} \rightarrow \alpha \text{ list}$

(3):  $\{ x : \tau, n : \text{int}, f : \forall \alpha. \alpha \rightarrow \text{int} \rightarrow \alpha \text{ list} \} \vdash \text{if } n \leq 0 \text{ then } [] \text{ else } x :: (f \ x \ (n - 1)) : \tau \text{ list}$

let  $\Gamma_2 = \{ x : \tau, n : \text{int}, f : \forall \alpha. \alpha \rightarrow \text{int} \rightarrow \alpha \text{ list} \}$

prove:

$\frac{\frac{\text{Var rule}}{\Gamma_2 \vdash n : \text{int}} \quad \frac{\text{Const rule}}{\Gamma_2 \vdash 0 : \text{int}} \quad \frac{\text{relation rule}}{\Gamma_2 \vdash n \leq 0 : \text{bool}} \quad \frac{\text{[] rule}}{\Gamma_2 \vdash [] : \alpha \text{ list}} \quad \frac{\frac{\text{Var rule}}{\Gamma_2 \vdash x : \tau} \quad \frac{\text{:: rule}}{\Gamma_2 \vdash :: : \alpha \rightarrow \alpha \text{ list} \rightarrow \alpha \text{ list}}}{\Gamma_2 \vdash x :: (f \ x \ (n - 1)) : \tau \text{ list}} \quad \frac{(4) \Gamma_2 \vdash f \ x \ (n - 1) : \tau \text{ list}}{\Gamma_2 \vdash \text{if } n \leq 0 \text{ then } [] \text{ else } x :: (f \ x \ (n - 1)) : \tau \text{ list}}$  (if\_then\_else rule)

(4):  $\Gamma_2 \vdash f \ x \ (n - 1) : \tau \text{ list}$

Reminder:  $\Gamma_2 = \{ x : \tau, n : \text{int}, f : \forall \alpha. \alpha \rightarrow \text{int} \rightarrow \alpha \text{ list} \}$

prove:

$\frac{\frac{\text{Var rule}}{\Gamma_2 \vdash f : \forall \alpha. \alpha \rightarrow \text{int} \rightarrow \alpha \text{ list}} \quad \frac{\text{Var rule}}{\Gamma_2 \vdash x : \tau} \quad \frac{\frac{\text{Var rule}}{\Gamma_2 \vdash n : \text{int}} \quad \frac{\text{Const rule}}{\Gamma_2 \vdash 1 : \text{int}}}{\Gamma_2 \vdash n - 1 : \text{int}} \quad \frac{\text{Primitive operators rule}}{\Gamma_2 \vdash f \ x \ (n - 1) : \tau \text{ list}}$

(2)  $\{ f : \forall \alpha. \alpha \rightarrow \text{int} \rightarrow \alpha \text{ list} \} \vdash (f \ 3 \ 2, f \ "a" \ 4) : \text{int list} * \text{string list}$

Reminder:  $\Gamma_1 = \{ f : \forall \alpha. \alpha \rightarrow \text{int} \rightarrow \alpha \text{ list} \}$

prove:

$\frac{\frac{\text{Var rule}}{\Gamma_1 \vdash f : \forall \alpha. \alpha \rightarrow \text{int} \rightarrow \alpha \text{ list}} \quad \frac{\text{Const rule}}{\Gamma_1 \vdash 3 : \text{int}} \quad \frac{\text{Const rule}}{\Gamma_1 \vdash 2 : \text{int}}}{\Gamma_1 \vdash f \ 3 \ 2 : \text{int list}} \quad \frac{\frac{\text{Var rule}}{\Gamma_1 \vdash f : \forall \alpha. \alpha \rightarrow \text{int} \rightarrow \alpha \text{ list}} \quad \frac{\text{Const rule}}{\Gamma_1 \vdash "a" : \text{string}} \quad \frac{\text{Const rule}}{\Gamma_1 \vdash 4 : \text{int}}}{\Gamma_1 \vdash f \ "a" \ 4 : \text{string list}} \quad \frac{}{\Gamma_1 \vdash (f \ 3 \ 2, f \ "a" \ 4) : \text{int list} * \text{string list}}$  (tuple rule)

then, all assumptions are solved! DONE!