Non-degenerate Conditions in Proving Geometric Theorems

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This document is a collection of cases from the paper "Geometric Interpretation and Application of Non-degenerate Conditions in Proving Geometric Theorems".

Proof: Using the complex number method, using lowercase letters x to represent the complex number of a point X and using $\bar{x}$ the conjugate complex number of the complex number , the coordinates of point x X are $X=\left(x,\bar{x}\right)$. Solution: First calculate the coordinates of some points, then establish the identity, and finally interpret the geometric meaning of the identity.

Some cases provide multiple approaches.

Easy questions

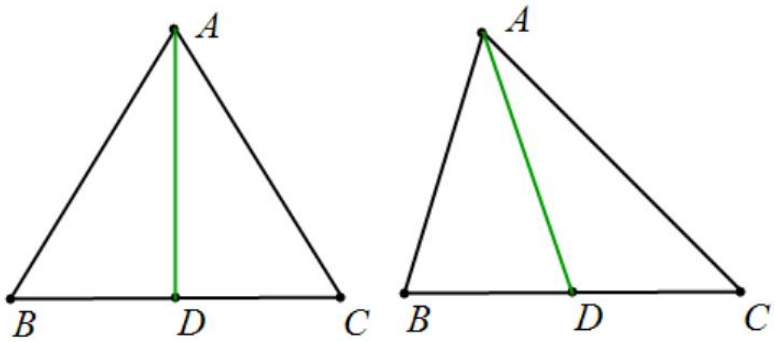


Figure 1

Example 1 : As shown in Figure 1, in triangle ABC , D is the midpoint *of* BC . If $AB = AC$, prove that: $AD \perp BC$.

the circumscribed circle of $d = \frac{b+c}{2}$ $\triangle$ *ABC be the unit circle,* , generating the identity:

$$|b-c|^4\left(\frac{\frac{b-a}{b-c}}{\frac{c-b}{c-a}}-\overline{\left(\frac{\frac{b-a}{b-c}}{\frac{c-b}{c-a}}\right)}\right)=\frac{(a-b)(a-c)(b-c)}{abc}|b-c|^2\left(\frac{a-d}{b-c}+\overline{\left(\frac{a-d}{b-c}\right)}\right),$$

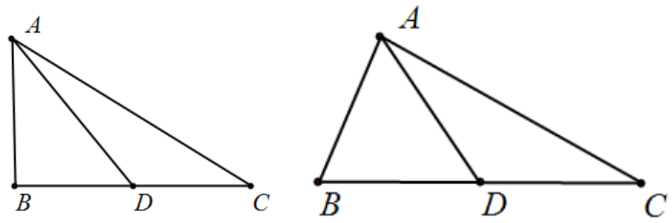
Extended proposition: In $\triangle$ ABC , D is the midpoint *of* BC , prove: $BC\sin(B-C)=4AD\sin A\cos\angle ADB$.

the circumscribed circle of $d = \frac{b+c}{2}$ $\triangle$ *ABC be the unit circle,* , generating the identity:

$$\frac{\frac{c-b}{c-a}+\overline{\left(\frac{c-b}{c-a}\right)}}{\frac{c-b}{c-a}-\overline{\left(\frac{c-b}{c-a}\right)}}-\frac{\frac{b-a}{b-c}+\overline{\left(\frac{b-a}{b-c}\right)}}{\frac{b-a}{b-c}-\overline{\left(\frac{b-a}{b-c}\right)}}=2\frac{\frac{c-b}{\frac{b+c}{2}-a}+\overline{\left(\frac{c-b}{\frac{b+c}{2}-a}\right)}}{\frac{c-b}{\frac{b+c}{2}-a}-\overline{\left(\frac{c-b}{\frac{b+c}{2}-a}\right)}},$$

Extended proposition: In $\triangle$ ABC , D is the midpoint *of* BC , prove: $\cot C-\cot B=2\cot\angle ADB$.

The two ideas are unified



The midpoint of side BC of △ ABC is D. If $\angle B = 90^\circ$, then $\cot\angle BAD = 2\cot\angle BAC$.

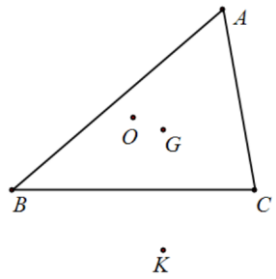
This conclusion is too trivial to be explored using the plural method.

Proof: Let $d = 0$, $b = -c$, find the identity:

$$\frac{\frac{a-d}{a-b}+\overline{\left(\frac{a-d}{a-b}\right)}}{\frac{a-d}{a-b}-\overline{\left(\frac{a-d}{a-b}\right)}}-2\frac{\frac{a-c}{a-b}+\overline{\left(\frac{a-c}{a-b}\right)}}{\frac{a-c}{a-b}-\overline{\left(\frac{a-c}{a-b}\right)}}=\frac{\frac{b-a}{b-c}+\overline{\left(\frac{b-a}{b-c}\right)}}{\frac{b-a}{b-c}-\overline{\left(\frac{b-a}{b-c}\right)}},$$

So we get the extended proposition:

△ ABC is D, then $\cot\angle BAD - 2\cot\angle BAC = \cot B$.



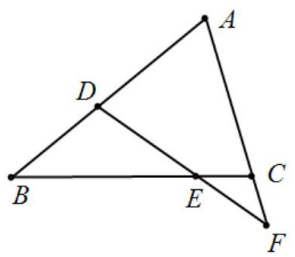
Example 2 : Given that the centroid G *of triangle* ABC *is symmetric* to the side BC at point K, prove that the necessary and sufficient condition for points A, B, K, and C to be cocircular is $AB^2+AC^2=2BC^2$: (The cocircularity of the four points is symmetric to the centroid)

Proof: Let $A=\left(a,\frac{1}{a}\right)$, $B=\left(b,\frac{1}{b}\right)$, $C=\left(c,\frac{1}{c}\right)$, $G=\left(\frac{a+b+c}{3},\frac{1/a+1/b+1/c}{3}\right)$,

$K=\left(\frac{2ab+2ac-bc}{3a},\frac{-a+2b+2c}{3bc}\right)$, the identity

$$\frac{2}{9}\left(2(c-b)\overline{(c-b)}-(a-b)\overline{(a-b)}-(a-c)\overline{(a-c)}\right)=1-k\bar{k},$$

Extended proposition: It is known that the symmetric point of the centroid G *of* $\triangle ABC$ about the side BC is K, prove: $\frac{2}{9}\left(2BC^2-AB^2-AC^2\right)=OA^2-OK^2$.



Example 3 : In triangle ABC , D *is on* line segment AB , E *is on* line segment BC , and F *is on the extension of* line segment AC . If D , E , and F are collinear, prove that:

$\frac{AD \cdot BE \cdot CF}{DB \cdot EC \cdot FA} = 1$(Extension of the Menelaos Theorem for Complex Number Identities)

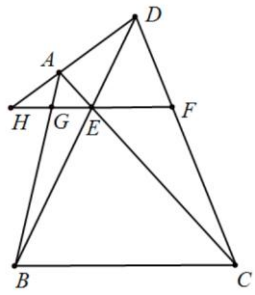
Proof: Assume $A = \left(a, \frac{1}{a}\right)$, $B = \left(b, \frac{1}{b}\right)$, $C = \left(c, \frac{1}{c}\right)$, $D = \left(d, \frac{a+b-d}{ab}\right)$,

$E = \left(e, \frac{b+c-e}{bc}\right)$, $F = \left(f, \frac{a+c-f}{ac}\right)$,

$(d-b)(e-c)(f-a)-(a-d)(b-e)(f-c) = abc[DEF]$,

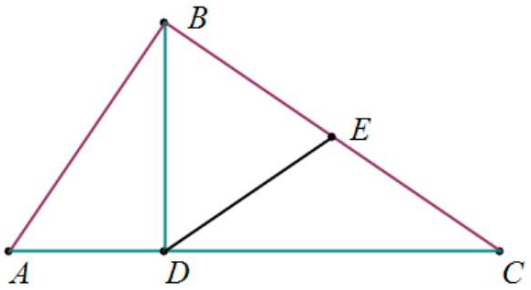
Extended proposition: In $\triangle ABC$, D *is on* line segment AB , E *is on* line segment BC , and F *is on the extension of* line segment AC . If D , E , and F are collinear, prove: $DB \cdot EC \cdot FA - AD \cdot BE \cdot CF = 4R \cdot S_{\triangle DEF}$, where R is the radius of the circumcircle of $\triangle ABC$.

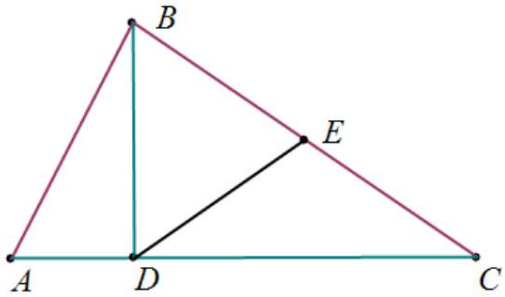
Example 4 : In quadrilateral *ABCD* , *AC* and *BD* intersect at *E*. Draw *HF* // *BC* through *E* , intersecting *CD* and *AB* at *F* and *G* , respectively, and the extension of *DA at H*. Prove that *EF* = 2 *EG* $\Leftrightarrow$ *EG* = *HG* . (Complex number identities extend the length of line segments .)



Proof: Identity $([ABC]-[DBC])((e-g)-(g-h))=[ABD](2(e-g)-(f-e))$,

As shown in the figure, in quadrilateral *ABCD* , *AC* and *BD* intersect at *E*. Draw *HF* // *BC* through *E* , *which intersects CD* and *AB* at *F* and *G* respectively , and intersects the extension of *DA at H*. Prove: $(S_{\triangle ABC}-S_{\triangle DBC})(EG-GH)=S_{\triangle ABD}(FE-2EG)$.





Example 5 : In triangle ABC , $\angle ABC = 90°$, $BD \perp AC$, D is the foot of the perpendicular, and E is the midpoint *of* BC . Prove that: $\angle EDC = \angle ABD$. (2016 Jiangsu Province College Entrance Examination Question)

Proof: Let DC be the real axis and DB be the imaginary axis, $A=(a,a)$, $B=(b,-b)$, $C=(c,c)$, $D=(0,0)$, $E=\left(\frac{b+c}{2},\frac{b-c}{2}\right)$, calculate

$$(b-a)\overline{(b-a)}\left(\frac{b-c}{b-a}+\overline{\left(\frac{b-c}{b-a}\right)}\right)=2\left(ac-b^2\right),$$

$$(d-c)(b-d)\overline{(d-c)(b-d)}\left(\frac{\frac{d-e}{d-c}}{\frac{b-d}{b-a}}-\overline{\left(\frac{\frac{d-e}{d-c}}{\frac{b-d}{b-a}}\right)}\right)=bc\left(ac-b^2\right),$$ generating the identity:

$$bc(b-a)\overline{(b-a)}\left(\frac{b-c}{b-a}+\overline{\left(\frac{b-c}{b-a}\right)}\right)=2(d-c)(b-d)\overline{(d-c)(b-d)}\left(\frac{\frac{d-e}{d-c}}{\frac{b-d}{b-a}}-\overline{\left(\frac{\frac{d-e}{d-c}}{\frac{b-d}{b-a}}\right)}\right),$$

$DB\cdot DC\cdot BA\cdot BC\cos B=2DE\cdot DC\cdot BD\cdot BA\sin\left(\angle EDC-\angle ABD\right)$, is simplified to $\cos\angle B=\sin\left(\angle EDC-\angle ABD\right)$, that is $90^\circ-\angle B=\angle EDC-\angle ABD$, that is $90^\circ-\angle EDC=\angle B-\angle ABD$, obviously.

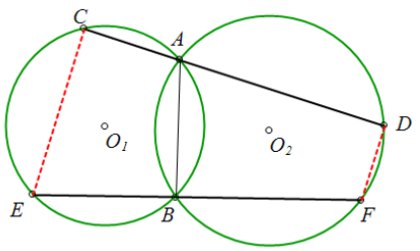
Example: In triangle ABC, BD is the height , E is the midpoint of BC, if BA⊥BC, prove: $\angle ABD=\angle EDC$.

Proof: Let DC be the real axis, DB be the imaginary axis, $d=0$, $e=\frac{b+c}{2}$,

$$\frac{\frac{b-d}{b-a}+\overline{\left(\frac{d-e}{d-c}\right)}}{\frac{b-d}{b-a}-\overline{\left(\frac{d-e}{d-c}\right)}}-\frac{\frac{d-e}{d-c}+\overline{\left(\frac{d-e}{d-c}\right)}}{\frac{d-e}{d-c}-\overline{\left(\frac{d-e}{d-c}\right)}}=\frac{a-c}{a}\frac{\frac{b-c}{b-a}+\overline{\left(\frac{b-c}{b-a}\right)}}{\frac{b-c}{b-a}-\overline{\left(\frac{b-c}{b-a}\right)}},$$

△ ABC, BD is the height , and E is the midpoint of BC. Prove that: $\cot\angle ABD - \cot\angle EDC = \frac{AC}{AD}\cot\angle ABC$.

Example 6 : Two circles intersect at points A and B. Draw straight lines through A and B *that intersect the two circles at points* C , D , F , and F *respectively* . Prove that: $CE \ // \ DF$.



Proof: Let the circumscribed circle of $\triangle ABF$ be the unit circle, $A=\left(a,\frac{1}{a}\right), B=\left(b,\frac{1}{b}\right), C=\left(c,\frac{a-c+d}{ad}\right), D=\left(d,\frac{1}{d}\right)$,

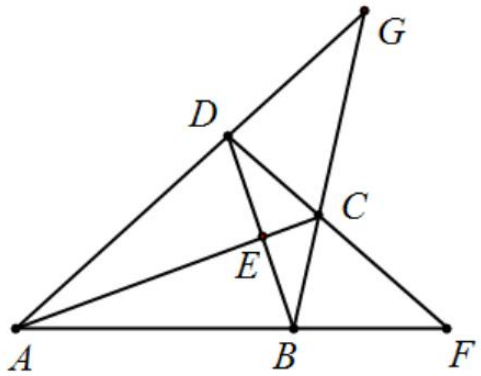
$E=\left(e,\frac{b-e+f}{bf}\right), F=\left(f,\frac{1}{f}\right)$,

$$(f-d)\overline{(f-d)}\left(\frac{e-c}{f-d}-\overline{\left(\frac{e-c}{f-d}\right)}\right)(a-b)(a-c)(b-e)=[ABEC]ab(d-f),$$

Extended proposition: For the inscribed quadrilateral $ABFD$, take point E *on the extension line of* FB and point C *on the extension line of* DA , prove:

$$2\left|S_{\triangle EFD}-S_{\triangle CFD}\right|=DF\cdot EC\left|\sin\left(\angle BAC+\angle BEC\right)\right|.$$

Note: Further simplifying it to , $\angle BAC+\angle BEC=\pi+\angle\left(FD,EC\right)$, goes around in a circle and returns to the starting point.



Example 7 : Given quadrilateral $ABCD$, no three points on its four vertices are collinear, AC intersects BD at E , AB intersects CD at F , and AD intersects BC at G. Prove that E , F , and G are not collinear .

Proof: Let $A=(0,0)$, $B=(1,1)$, $C=(c,\bar{c}), D=(d,\bar{d})$

$$E=\left(-\frac{c(d-\bar{d})}{-c+\bar{c}-\bar{c}d+c\bar{d}},\frac{\bar{c}(d-\bar{d})}{c-\bar{c}+\bar{c}d-c\bar{d}}\right) \quad , \quad F=\left(\frac{-\bar{c}d+c\bar{d}}{c-\bar{c}-d+\bar{d}},\frac{-\bar{c}d+c\bar{d}}{c-\bar{c}-d+\bar{d}}\right) \quad ,$$

$$G=\left(\frac{(c-\bar{c})d}{d-\bar{c}d-\bar{d}+c\bar{d}},\frac{(c-\bar{c})\bar{d}}{d-\bar{c}d-\bar{d}+c\bar{d}}\right),$$

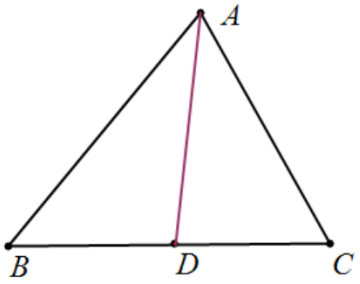
$$[EFG]([ABC]-[ABD])([BCD]-[BCA])([ACD]+[ABC])$$

$$=2[ABC][ABD][BCD][ACD].$$

Extended proposition: In quadrilateral $ABCD$, no three of its vertices are collinear. AC intersects BD at E , AB intersects CD at F , and AD intersects BC at G. Prove:

$$S_{\triangle EFG}(S_{\triangle ABC}-S_{\triangle ABD})(S_{\triangle BCD}-S_{\triangle BCA})(S_{\triangle ACD}+S_{\triangle ABC})$$

$$=2S_{\triangle ABC}S_{\triangle ABD}S_{\triangle BCD}S_{\triangle ACD}.$$



Example 8 : In $\triangle ABC$, AD is the angle bisector, then $\dfrac{AB}{AC}=\dfrac{BD}{DC}$.

Let BC be the real axis and point A be on the imaginary axis, generating the identity,

$$(b-c)^2(a-d)^2(a+d)^2\left(\frac{\frac{a-c}{a-d}}{\frac{a-d}{a-b}}-\overline{\left(\frac{\frac{a-c}{a-d}}{\frac{a-d}{a-b}}\right)}\right)$$

$$=[ABC]\big((a-c)(-a-c)(d-b)(d-b)-(a-b)(-a-b)(d-c)(d-c)\big).$$

Extended proposition: In $\triangle ABC$, *D is a point on* line segment BC , then

$$AD^2\cdot BC^2\sin(\angle CAD-\angle DAB)=\sin A\left(AB^2\cdot CD^2-AC^2\cdot BD^2\right).$$

The angle in the conclusion is a directed angle, and when AD is the exterior angle bisector, the same holds $\dfrac{AB}{AC}=\dfrac{BD}{DC}$.

Proof: Let $d=0$ DC be the real axis.

$$2\left((a-b)\overline{(a-b)}(c-d)\overline{(c-d)}-(a-c)\overline{(a-c)}(b-d)\overline{(b-d)}\right)$$

$$=[ABC](b-d)\overline{(d-c)}\left(\frac{\frac{a-c}{a-d}+\overline{\left(\frac{a-c}{a-d}\right)}}{\frac{a-c}{a-d}-\overline{\left(\frac{a-c}{a-d}\right)}}-\frac{\frac{a-d}{a-b}+\overline{\left(\frac{a-d}{a-b}\right)}}{\frac{a-d}{a-b}-\overline{\left(\frac{a-d}{a-b}\right)}}\right).$$

D on BC: $(AB\cdot CD-AC\cdot BD)(AB\cdot CD+AC\cdot BD)$

$$=2S_{\triangle ABC}\cdot BD\cdot DC(\cot\angle BAD-\cot\angle DAC).$$

D is on the extension line of BC: $(AB\cdot CD-AC\cdot BD)(AB\cdot CD+AC\cdot BD)$

$$=2S_{\triangle ABC}\cdot BD\cdot DC(\cot\angle BAD+\cot\angle DAC).$$

$$4(b-c)\overline{(b-c)}(d-a)\overline{(d-a)}\left((a-b)\overline{(a-b)}(c-d)\overline{(c-d)}-(a-c)\overline{(a-c)}(b-d)\overline{(b-d)}\right)$$

$$=[ABC]^2(d-b)\overline{(d-b)}(d-c)\overline{(d-c)}\left(\frac{\frac{a-d}{a-b}+\overline{\left(\frac{a-d}{a-b}\right)}}{\frac{a-d}{a-b}-\overline{\left(\frac{a-d}{a-b}\right)}}+\frac{\frac{a-c}{a-d}+\overline{\left(\frac{a-c}{a-d}\right)}}{\frac{a-c}{a-d}-\overline{\left(\frac{a-c}{a-d}\right)}}\right)\left(\frac{\frac{a-d}{a-b}+\overline{\left(\frac{a-d}{a-b}\right)}}{\frac{a-d}{a-b}-\overline{\left(\frac{a-d}{a-b}\right)}}-\frac{\frac{a-c}{a-d}+\overline{\left(\frac{a-c}{a-d}\right)}}{\frac{a-c}{a-d}-\overline{\left(\frac{a-c}{a-d}\right)}}\right).$$

D is on line BC,

$$BC^2\cdot DA^2\left(AB\cdot CD-AC\cdot BD\right)\left(AB\cdot CD+AC\cdot BD\right)$$

$$=4S_{\triangle ABC}{}^2\cdot BD^2\cdot DC^2\left(\cot\angle BAD+\cot\angle DAC\right)\left(\cot\angle BAD-\cot\angle DAC\right).$$

Example 15 : In $\triangle ABC$, AD is the angle bisector, then $AD^2 = AB \cdot AC - BD \cdot DC$.
Let BC be the real axis and point A be on the imaginary axis, generating the identity,

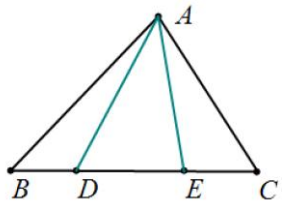
$$(a-d)^2(a+d)^2\left(\frac{\frac{a-c}{a-d}}{\frac{a-d}{a-b}}-\overline{\left(\frac{\frac{a-c}{a-d}}{\frac{a-d}{a-b}}\right)}\right)$$

$$=\frac{2a}{b+c}\left(|a-b|^2|a-c|^2-\left(|a-d|^2+(b-d)(d-c)\right)^2\right).$$

Extended proposition: In $\triangle ABC$, M is the midpoint, D *is a point on* the line segment BC , then

$$2AD^2 \cdot AB \cdot AC \sin(\angle CAD - \angle DAB) = \tan\angle AMC\left(\left(AD^2 + BD \cdot DC\right)^2 - AB^2 \cdot AC^2\right).$$

Example 16 : In $\triangle ABC$, points D and E are on BC , and $\angle CAE = \angle DAB$, prove $\frac{AB^2}{AC^2} = \frac{BD \cdot BE}{CD \cdot CE}$.



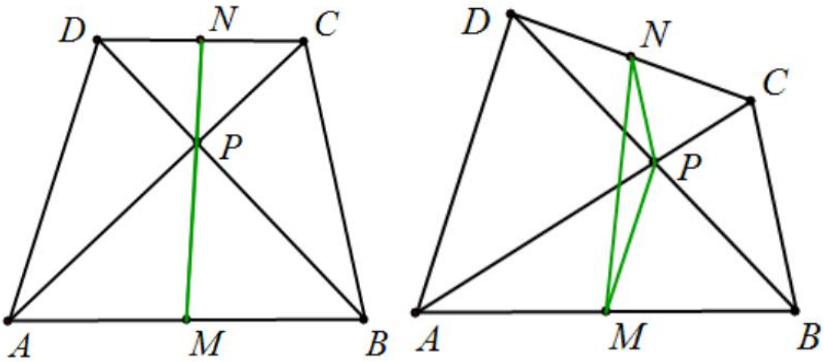
Proof: Let point A be on the imaginary axis, BC be the real axis,

$$(a-b)\overline{(a-b)}(c-d)(c-e)-(a-c)\overline{(a-c)}(b-d)(b-e)$$

$$+\frac{b-c}{2a}(a-e)\overline{(a-e)}(a-d)\overline{(a-d)}\left(\frac{a-c}{a-e}/\frac{a-d}{a-b}-\overline{\left(\frac{a-c}{a-e}/\frac{a-d}{a-b}\right)}\right)=0.$$

triangle ABC , points D and E are on BC . Prove that

$$\sin A\left(AB^2 \cdot CD \cdot CE - AC^2 \cdot BD \cdot BE\right) = BC^2 \cdot AD \cdot AE \sin\left(\angle CAE - \angle DAB\right).$$



Example 9 : Quadrilateral $ABCD$, AC intersects BD at P , M and N *are the midpoints of* AB and CD respectively . If $AB \ // \ DC$, prove that: M , N , and P are collinear .

Proof: Use the complex number method, using lowercase letters x to represent X the complex number of a point and using to represent $\bar{x}$ the conjugate complex number of the complex number . x

Let P be the origin, AC be the real axis, $\bar{d} = \frac{\bar{b}d}{b}$, $m = \frac{a+b}{2}$, $n = \frac{c+d}{2}$,

$[DAB]-[CAB] = -\frac{(b-\bar{b})(bc-ad)}{b}$, $[MPN] = -\frac{(b-\bar{b})(bc-ad)}{4b}$, so

$[DAB]-[CAB] = 4[MPN]$.

Extended proposition: Quadrilateral $ABCD$, AC intersects BD at P , M and N *are the midpoints of* AB and CD respectively , prove: $S_{\triangle DAB} - S_{\triangle CAB} = 4S_{\triangle MPN}$.

Example 1 0 : As shown in the figure, in $\triangle ABC$, D is a point on the line segment AB . Prove that: *The necessary and sufficient condition for* $(DA\cdot BC)^2+(DB\cdot AC)^2=(AB\cdot DC)^2$ $CA \perp CB$ *is* .

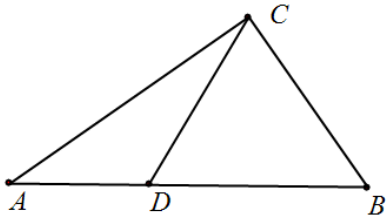


Figure 32

Explanation: When D is at the midpoint *of* AB , it is the Pythagorean theorem.

Proof: Let line AD be the real axis and point C be on the imaginary axis

$$(d-a)(d-a)(b-c)(b+c)+(d-b)(d-b)(a-c)(a+c)-(b-a)(b-a)(d-c)(d+c)$$

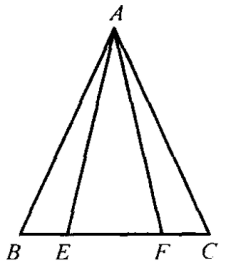
$$=(a-d)(b-d)(c-a)(-c-a)\left(\frac{c-b}{c-a}+\frac{-c-b}{-c-a}\right).$$

$$(DA\cdot BC)^2+(DB\cdot AC)^2-(AB\cdot DC)^2+2CA\cdot CB\cdot DA\cdot DB\cos\angle ACB=0.$$

When D is on line segment AB ,

$$(DA\cdot BC)^2+(DB\cdot AC)^2-(AB\cdot DC)^2=2CA\cdot CB\cdot DA\cdot DB\cos\angle ACB.$$

Example 12 : In triangle ABC , $AB = AC$, $\angle ACB = 90^\circ$, $\angle EAF = 45^\circ$, E and F are two points on the base BC (E is between B and F). Prove: $EF^2 = BE^2 + FC^2$.
triangle ABC , $AB = AC$, and E and F *are* two points on base BC (E is between B and F). Prove that $\angle EAF = \frac{\angle BAC}{2}$ the necessary and sufficient condition for is $EF^2 = BE^2 + FC^2 + 2BE \cdot FC \cos \angle BAC$.

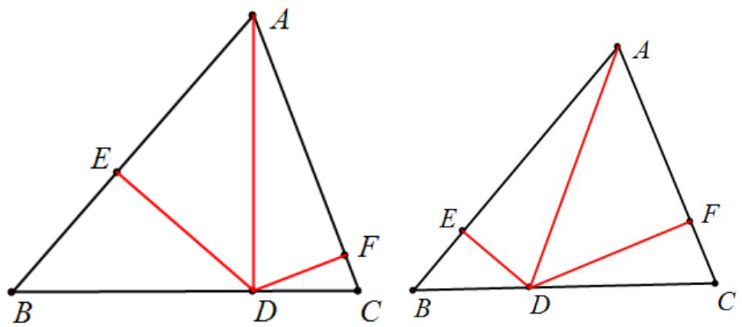


$$(e-f)^2 - (b-e)^2 - (f-c)^2 - 2(b-e)(f-c)\left(\frac{\frac{a-c}{a-b} + \frac{-a-c}{-a-b}}{2}\right)$$

$$= -\frac{2c}{a(a-c)(a+c)}(a-e)(-a-e)(a-c)(-a-c)\left(\frac{\frac{a-f}{a-e}}{\frac{a-c}{a-0}} - \overline{\left(\frac{\frac{a-f}{a-e}}{\frac{a-c}{a-0}}\right)}\right),$$

triangle ABC , $AB = AC$, and E and F *are* two points on base BC (E is between B and F). Prove:

$$EF^2 - \left(BE^2 + FC^2 + 2BE \cdot FC \cos \angle BAC\right) = 4AF \cdot AE \cos C \sin\left(\angle EAF - \frac{\angle BAC}{2}\right).$$



Example 13 : In triangle ABC , AD is the altitude, point D *is on* BC , draw $DE \perp AB$ *from* D at E , and draw $DF \perp AC$ at F.

Prove that points B , C , F , and E are concentric circles.

Proof: Let $d=0$, $\bar{b}=b$, $\bar{c}=c$, solve the equation $\frac{b-e}{b-a}=\overline{\left(\frac{b-e}{b-a}\right)}$ and $\frac{d-e}{b-a}=-\overline{\left(\frac{d-e}{b-a}\right)}$ get $e=-\frac{\left(a-\bar{a}\right)b}{2\left(\bar{a}-b\right)}$, $\bar{e}=\frac{\left(a-\bar{a}\right)b}{2\left(a-b\right)}$. Similarly $f=-\frac{\left(a-\bar{a}\right)c}{2\left(\bar{a}-c\right)}$, $\bar{f}=\frac{\left(a-\bar{a}\right)c}{2\left(a-c\right)}$.

Identity: $$\frac{\frac{f-c}{f-b}+\overline{\left(\frac{f-c}{f-b}\right)}}{\frac{f-c}{f-b}-\overline{\left(\frac{f-c}{f-b}\right)}}-\frac{\frac{e-c}{e-b}+\overline{\left(\frac{e-c}{e-b}\right)}}{\frac{e-c}{e-b}-\overline{\left(\frac{e-c}{e-b}\right)}}=\frac{\frac{d-a}{b-c}+\overline{\left(\frac{d-a}{b-c}\right)}}{\frac{d-a}{b-c}-\overline{\left(\frac{d-a}{b-c}\right)}},$$

Extended proposition: In acute angle $\triangle ABC$, from point D on line segment BC, draw DE $\perp$ AB *at* E , and draw $DF \perp AC$ at F.

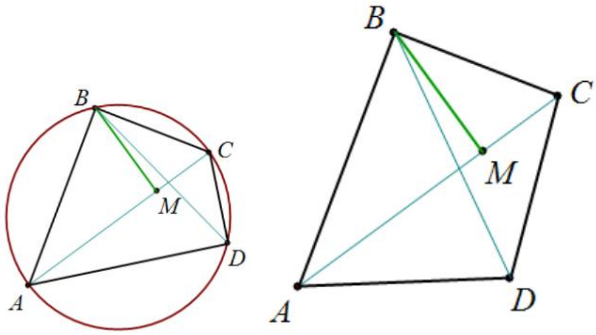
Verification: $\cot\angle BFC-\cot\angle BEC=\cot\angle ADC$.

Example 1 4 : As shown in Figure 1, in the inscribed quadrilateral *ABCD* , $AB = BD$, $BM \perp AC$, and the foot of the perpendicular is *M*. Prove: $AM = DC + CM$. (Archimedes' Broken Line Theorem)

Proof: Let circle O be the unit circle, $b=1$, $d=1/a$,

$$m=\frac{a+b+c-ac/b}{2}=\frac{1}{2}(1+a+c-ac) ,$$

$$|c-d|=\left|\frac{-1+ac}{a}\right|=|-1+ac|=|(a-m)-(m-c)| .$$



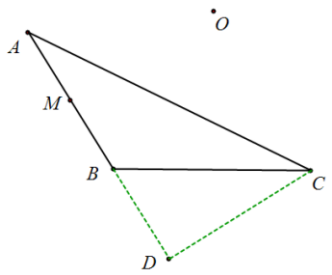
Assume $b=0$, $\bar{b}=0$, $\bar{a}=a$, $d=\frac{a^2}{\bar{d}}$, there is an identity

$$(b-a)(d-c)\overline{(b-a)(d-c)}\left(\frac{b-c}{b-a}-\overline{\left(\frac{b-c}{b-a}\right)}\right)\begin{vmatrix} a & \bar{a} & 1 \\ c & \bar{c} & 1 \\ d & \bar{d} & 1 \end{vmatrix}$$

$$=(a-d)(a-c)\overline{(a-c)(a-d)}\left((a-m-(m-c))\overline{(a-m-(m-c))}-(d-c)\overline{(d-c)}\right),$$

Extended Proposition: As shown in the figure, quadrilateral *ABCD* , $AB = BD$, $BM \perp AC$, and the foot of the perpendicular is *M*. Proof:

$$8S_{\triangle ACD}\cdot AB\cdot BC\cdot CD\sin(\angle ABC+\angle ADC)=AD\cdot AC^2\left((AM-MC)^2-DC^2\right).$$



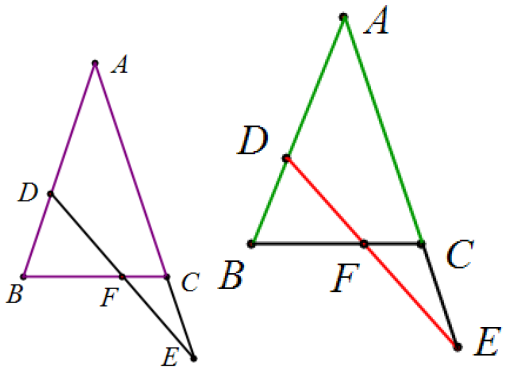
Example 17 : In $\triangle ABC$, O is the circumcenter, CD is the altitude, and M *is* the midpoint *of* AB . Prove that the necessary and sufficient condition for $DM = OA$ *is that* $\angle A$ and $\angle$B *differ* by 90°.

Proof: Let the circumcircle of $\triangle ABC$ be the unit circle,

$$\left(\frac{a+b}{2}-\frac{a+b+c-\frac{ab}{c}}{2}\right)\left(\frac{1/a+1/b}{2}-\frac{1/a+1/b+1/c-\frac{c}{ab}}{2}\right)-1$$

$$=-\frac{a^3b^3c^2}{4(a-b)^4(a-c)^2(b-c)^2}\left((a-b)\overline{(a-b)}(b-a)\overline{(b-a)}\left(\frac{\frac{a-c}{a-b}}{\frac{b-a}{b-c}}+\overline{\left(\frac{\frac{a-c}{a-b}}{\frac{b-a}{b-c}}\right)}\right)\right)^2,$$

In $\triangle ABC$, O is the circumcenter, CD is the altitude, and M *is* the midpoint *of* AB . Prove: $OA^2-DM^2=OA^2\cos^2(B-A)$.



Example 18 : In $AB = AC \triangle ABC$, take point D *on* AB, extend AC to E so that $CE = BD$, DE intersects BC at F. Prove: $FD = FE$.

Proof: Let A be the origin, AB be the real axis, $a = 0$, $e = \frac{c\bar{e}}{\bar{c}}$,

$$f = \frac{b\bar{c}d - bc\bar{e} - bd\bar{e} + cd\bar{e}}{\bar{c}d - b\bar{e}}, \left((d-b)^2 - (c-e)\overline{(c-e)}\right) - \left(b^2 - c\bar{c}\right)\left(1 - \frac{e}{c}\right)^2$$

$$= \left((d-f)\overline{(d-f)} - (f-e)\overline{(f-e)}\right)\frac{\left(\bar{c}d - b\bar{e}\right)^2}{\bar{c}^2 (d-e)\overline{(d-e)}}.$$

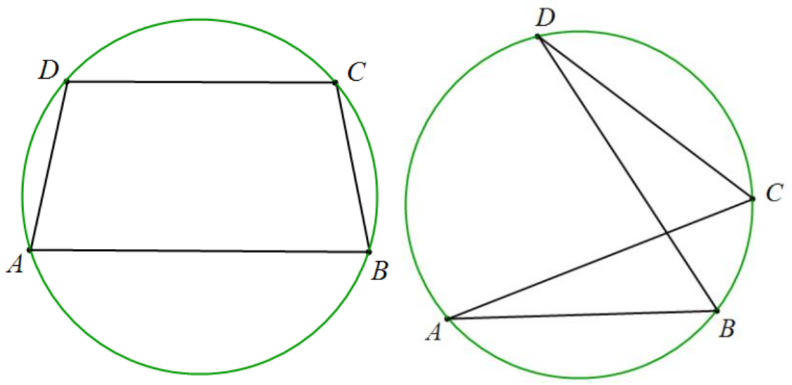
$\triangle ABC$, take point D *on* line AB, point E *on line* AC, and DE intersect BC at F. Prove that: $\left(BD^2 - CE^2\right) - \left(BA^2 - CA^2\right)\left(1 - \frac{AE}{AC}\right)^2 = (DF^2 - EF^2)\left(\frac{AC \cdot AD - AB \cdot AE}{AC \cdot DE}\right)^2$.

Note 1: Note that $AC \cdot AD - AB \cdot AE = 0$ F *does not exist* at this time .

Explanation 2 : Assume $AD = AB - DB$, $AE = AC + CE$, $DE = DF + FE$,

$$-\frac{2\left(AB \cdot CE + AC \cdot DB\right)\left(AB \cdot CE \cdot DF - AC \cdot DB \cdot FE\right)}{AC^2\left(DF + FE\right)} = 0, \text{ then } \frac{AC}{CE}\frac{EF}{FD}\frac{DB}{BA} = 1.$$

Its geometric meaning is that line BC intersects $\triangle ADE$, and Menelaos's theorem applies. The three conditions for the equality of the segments involved in the problem are combined in such a simple form. Even with some effort, a clever solution might emerge. Alternatively, the common side theorem can be used .



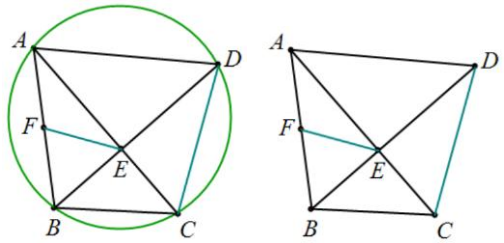
Example 19 : As shown in Figure 1, in the inscribed quadrilateral $ABCD$, if $AC = BD$, prove that $AB \ // \ CD$ or $AD \ // \ BC$. (Four points on a circle have equal diagonals and two sets of parallel lines have full and partial decompositions .)

Proof: Let the circle be the unit circle,

$$(a-b)(b-c)(c-d)(d-a)\left((a-c)\overline{(a-c)}-(b-d)\overline{(b-d)}\right)$$

$$= abcd\left([CDA]-[BDA]\right)\left([DAB]-[CAB]\right)$$

Inscribed in the circle is quadrilateral $ABCD$. Prove that: $AB \cdot BC \cdot CD \cdot DA \cdot (BD^2 - AC^2) = 16R^2 (S_{\triangle CDA} - S_{\triangle BDA}) \cdot (S_{\triangle DAB} - S_{\triangle CAB})$.



Example 20 : In a quadrilateral $ABCD$, $AC \perp BD$. AC intersects BD at E. F is the midpoint *of* AB . Prove that $EF \perp CD$. (Brahmagupta Theorem)

In the following, we assume that concircularity is discarded and $EF \perp CD$. Let E = 0, ED be the real axis, and EA be the imaginary axis.

$$e=0,\ \bar{b}=b,\ \bar{d}=d,\ \bar{a}=-a,,\ \bar{c}=-c,\ f=\frac{a+b}{2},\ \bar{f}=\frac{-a+b}{2},$$

Identity: $(a-b)\overline{(a-b)}(c-d)\overline{(c-d)}\left(\frac{a-d}{a-b}\frac{c-b}{c-d}-\overline{\left(\frac{a-d}{a-b}\frac{c-b}{c-d}\right)}\right)$

$$=-2(a-c)(b-d)\left((c-d)\overline{(c-d)}\left(\frac{e-f}{c-d}+\overline{\frac{e-f}{c-d}}\right)\right).$$

Geometric meaning: Quadrilateral $ABCD$, $AC \perp BD$, AC intersects BD at E , F is the midpoint of AB , prove that $AB\cdot BC\cdot CD\cdot DA\sin(\angle BAD+\angle BCD)=2AC\cdot BD\cdot CD\cdot EF\cos\alpha$, where α is the angle through which the straight line CD turns counterclockwise to EF .

$$ab\left(\frac{\frac{d-b}{d-a}+\overline{\left(\frac{d-b}{d-a}\right)}}{\frac{d-b}{d-a}-\overline{\left(\frac{d-b}{d-a}\right)}}-\frac{\frac{c-b}{c-a}+\overline{\left(\frac{c-b}{c-a}\right)}}{\frac{c-b}{c-a}-\overline{\left(\frac{c-b}{c-a}\right)}}\right)=(c-d)\overline{(c-d)}\left(\frac{e-f}{c-d}+\overline{\left(\frac{e-f}{c-d}\right)}\right).$$

Geometric meaning: Quadrilateral $ABCD$, $AC \perp BD$, AC intersects BD at E , F is the midpoint of AB , prove that $S_{\triangle EBC}(\cot\angle ACB-\cot ADB)=CD\cdot EF\cos\alpha$, where α is the angle through which the straight line CD turns counterclockwise to EF .

$$S_{\triangle EBC}(\cot\angle ACB-\cot ADB)=\frac{\mathrm{i}}{4}\begin{vmatrix}0 & 0 & 1\\ a & \bar{a} & 1\\ b & \bar{b} & 1\end{vmatrix}\left(\frac{\frac{c-b}{c-a}+\overline{\left(\frac{c-b}{c-a}\right)}}{\frac{c-b}{c-a}-\overline{\left(\frac{c-b}{c-a}\right)}}-\frac{\frac{d-b}{d-a}+\overline{\left(\frac{d-b}{d-a}\right)}}{\frac{d-b}{d-a}-\overline{\left(\frac{d-b}{d-a}\right)}}\right)\mathrm{i}$$

$$=-\frac{ab}{2}\left(-\frac{ac+bd}{ab}\right)=\frac{1}{2}(ac+bd),$$

$$CD \cdot EF \cos\alpha = |c-d||e-f| \frac{\frac{e-f}{c-d} + \overline{\left(\frac{e-f}{c-d}\right)}}{2\left|\frac{e-f}{c-d}\right|} = (c-d)\overline{(c-d)} \frac{\frac{e-f}{c-d} + \overline{\left(\frac{e-f}{c-d}\right)}}{2}$$

$$= \frac{1}{2}(ac+bd).$$

Example 21 : As shown in Figure 16 , in triangle ABC , $BC = 2\,AC$, D is the midpoint of BC , *and* E is the midpoint of DC . Prove that: $\angle BAD = \angle DAE$.

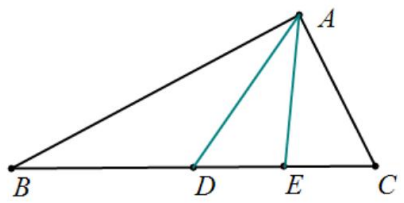


Figure 1 6

Proof: Let $b=0$, $d=2$, $e=3$, $c=4$,

Identity 1:

$$4(a-b)(a-e)\overline{(a-b)(a-e)}\left(\frac{\frac{a-d}{a-b}}{\frac{a-e}{a-d}}-\overline{\left(\frac{\frac{a-d}{a-b}}{\frac{a-e}{a-d}}\right)}\right)+\left(a-\bar{a}\right)\left((b-c)^2-4(a-c)\overline{(a-c)}\right)=0$$

Extension 1: In triangle ABC , D is the midpoint of BC , and E is the midpoint of DC . Prove that:

$$8AD^2\cdot AE\cdot AB\sin(\angle BAD-\angle DAE)=S_{\triangle ABC}(4CA^2-CB^2)\,.$$

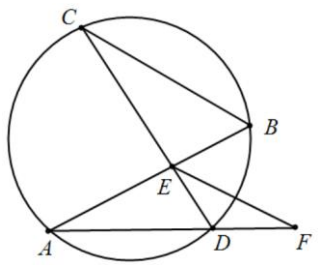
Identity 2 :

$$[ABC]\left(\frac{\frac{a-d}{a-b}+\overline{\left(\frac{a-d}{a-b}\right)}}{\frac{a-d}{a-b}-\overline{\left(\frac{a-d}{a-b}\right)}}-\frac{\frac{a-e}{a-d}+\overline{\left(\frac{a-e}{a-d}\right)}}{\frac{a-e}{a-d}-\overline{\left(\frac{a-e}{a-d}\right)}}\right)+\left((b-c)^2-4(a-c)\overline{(a-c)}\right)=0$$

Extension 2 : In triangle ABC , D is the midpoint of BC , and E is the midpoint of DC . Prove that:

$$4S_{\triangle ABC}\left(\cot\angle BAD-\cot\angle DAE\right)=BC^2-4AC^2\,.$$

Using different methods to generalize will yield different generalization results. In this case, the conclusion of generalization 2 is obviously more concise and beautiful.



Example 22 : Given that *E is the intersection of two chords AB* and *CD* in a circle , let the line *EF* // *CB* intersect the extension of *AD at F.* Prove that: $FE^2 = FA \cdot FD$. Complex number identities extend the product relationship between a quadrilateral inscribed in a circle and a line segment.

Proof: Let the circle be the unit circle, $O = (0,0)$,

$E = \left(\dfrac{abc + abd - acd - bcd}{ab - cd}, \dfrac{a + b - c - d}{ab - cd} \right)$, $F = \left(f, \dfrac{a + d - f}{ad} \right)$,

$$ad[ABC](b-c)\left[(e-f)\overline{(e-f)} - (f-a)\overline{(f-d)}\right]$$

$$= ([FBC] - [EBC])(a-b)(a-d)(c-e),$$

Extended Proposition: Given that *E* is the intersection of two chords *AB* and *CD in a circle* . *F* is a point on the extension of *AD* . Prove that:

$$\left(EF^2 - FA \cdot FD\right) \cdot S_{\triangle ABC} \cdot BC = \left(S_{\triangle FBC} - S_{\triangle EBC}\right) AB \cdot CE \cdot AD.$$

Note: When *F* runs $S_{\triangle FBC} - S_{\triangle EBC}$ from one side of the straight line *BC to the other side,*

it should be changed to $S_{\triangle FBC} + S_{\triangle EBC}$.

Example 23 : In triangle ABC , $\angle B = \angle C$. Prove that AB = AC.

Extension 1:

Let the circumcircle of $\triangle ABC$ be the unit circle, then

$$\frac{abc}{(a-b)(b-c)(a-c)}(c-a)(b-a)\overline{(c-a)}\,\overline{(b-a)}\left(\frac{\frac{c-b}{c-a}}{\frac{b-a}{b-c}}-\overline{\left(\frac{\frac{c-b}{c-a}}{\frac{b-a}{b-c}}\right)}\right)$$

$$=(a-b)\overline{(a-b)}-(a-c)\overline{(a-c)},$$

Geometric meaning $2R\cdot BC\sin(C-B)=AB^2-AC^2$,

The corresponding identity $\sin(C+B)\sin(C-B)=\sin^2 C-\sin^2 B$.

Extension 2 : $(c-a)\overline{(c-a)}\left(\frac{c-b}{c-a}-\overline{\left(\frac{c-b}{c-a}\right)}\right)=(b-c)\overline{(b-c)}\left(\frac{b-a}{b-c}-\overline{\left(\frac{b-a}{b-c}\right)}\right)$,

geometric meaning $AC\cdot\sin\angle ACB=AB\cdot\sin\angle ABC$. Law of Sines

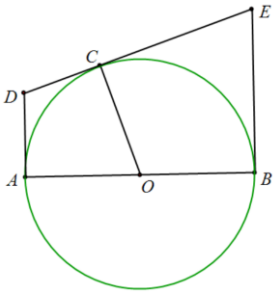
Extension 3 :

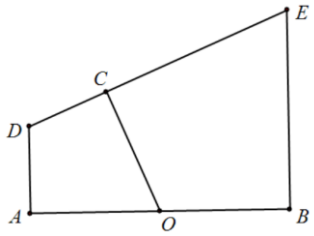
$$\frac{\frac{b-a}{b-c}+\overline{\left(\frac{b-a}{b-c}\right)}}{\frac{b-a}{b-c}-\overline{\left(\frac{b-a}{b-c}\right)}}-\frac{\frac{c-b}{c-a}+\overline{\left(\frac{c-b}{c-a}\right)}}{\frac{c-b}{c-a}-\overline{\left(\frac{c-b}{c-a}\right)}}=(a-b)\overline{(a-b)}-(a-c)\overline{(a-c)}$$

, geometric meaning $2S_{\triangle ABC}(\cot B-\cot C)=AB^2-AC^2$.

$$\cot B=\frac{\cos B}{\sin B}=\frac{(a^2+c^2-b^2)/(2ac)}{2S/(ac)}=\frac{a^2+c^2-b^2}{4S_{\triangle ABC}},$$

$$2S_{\triangle ABC}(\cot B-\cot C)=2S_{\triangle ABC}\left(\frac{a^2+c^2-b^2}{4S_{\triangle ABC}}-\frac{a^2+b^2-c^2}{4S_{\triangle ABC}}\right)=AB^2-AC^2.$$





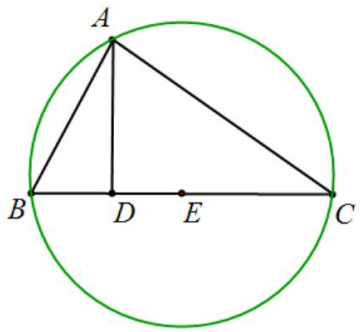
Example 24 : C is any point on the circle O *with* AB as diameter ; the tangent through point C intersects the tangent through points A and B *at points* D and E *respectively* . Prove $CO^2 = CD \cdot CE$.

Note: The original problem is very simple. We have already proven that $OD \perp OE$, which can be obtained from the Projection Theorem. However, after generalization, it becomes somewhat more difficult.

Proof: Let $o = 0$ line AB be the real axis, $a = -b$, $\bar{d} = 2a - d$, $\bar{e} = 2b - e$, solve the equation $\frac{c}{c-d} = -\overline{\left(\frac{c}{c-d}\right)}$, $\frac{c-d}{c-e} = \overline{\left(\frac{c-d}{c-e}\right)}$, and get $c = \frac{b(d+e)}{4b+d-e}$, $\bar{c} = \frac{b(d+e)}{d-e}$,

$$2(a-b)^2\left(c\bar{c} - (d-c)\overline{(c-e)}\right) = (d-e)\overline{(d-e)}(c-a)\overline{(c-a)}\left(\frac{c-b}{c-a} + \overline{\left(\frac{c-b}{c-a}\right)}\right).$$

In quadrilateral $ABED$, $AD \perp AB$, $BE \perp AB$, O is the midpoint of AB , $OC \perp DE$ at C , and C *is on* line segment DE . Prove $AB^2(CO^2 - CD \cdot CE) = DE^2 \cdot CA \cdot CB \cdot \cos\angle ACB$.

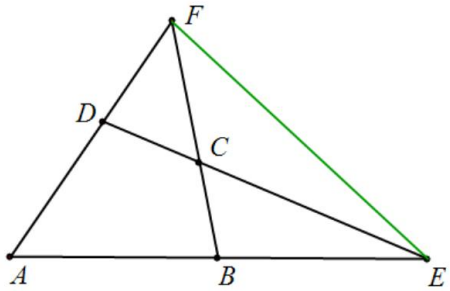


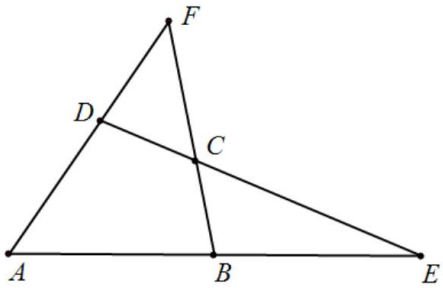
Example 25 : $\triangle ABC$ In the equation , , $\angle B = 2\angle C$ E *is* the midpoint of BC , and AD $\perp$ BC intersects BC at D. Prove that: $AB = 2DE$.

$$(a-b)\overline{(a-b)} - 4(d-e)\overline{(d-e)} = 2\frac{b(a-b)(a-c)^3}{a^2(b-c)^3}\left(\frac{\left(\frac{c-b}{c-a}\right)^2}{\frac{b-a}{b-c}} - \overline{\left(\frac{\left(\frac{c-b}{c-a}\right)^2}{\frac{b-a}{b-c}}\right)}\right),$$

in Figure 1 $\triangle ABC$, E is the midpoint of BC , AD $\perp$ BC intersects BC at D , prove that :

$\left(AB^2 - 4DE^2\right)\sin B = CA^2 \sin\left(2C - B\right)$.





Example 26 : As shown in Figure 1, in quadrilateral $ABCD$, AB intersects DC at E , and AD intersects BC at F. If points $EF^2 = EA\cdot EB + FA\cdot FDA$, B , C , and D *are concentric, prove that:* . (Complex number identities extend the relationship between concentricity of four points and the side lengths of a complete quadrilateral)

Proof: Assume $A=(0,0)$, $B=(1,1)$, $C=(c,\bar{c})$, $D=(d,\bar{d})$,

$$E=\left(\frac{-\bar{c}d+c\bar{d}}{c-\bar{c}-d+\bar{d}},\frac{-\bar{c}d+c\bar{d}}{c-\bar{c}-d+\bar{d}}\right),\quad F=\left(\frac{(c-\bar{c})d}{d-\bar{c}d-\bar{d}+c\bar{d}},\frac{(c-\bar{c})\bar{d}}{d-\bar{c}d-\bar{d}+c\bar{d}}\right),$$

Generating identities: $-[ABCD][ABD]$

$$=([BCD]-[BCA])([DAB]-[CAB])\left((e-f)\overline{(e-f)}-(e-a)\overline{(e-b)}-(f-a)\overline{(f-d)}\right)$$

, Expand the proposition: As shown in Figure 1, quadrilateral $ABCD$, AB intersects DC at E , AD intersects BC at F , prove: $\sin E\sin F\left(EF^2-EA\cdot EB-FA\cdot FD\right)=2S_{\triangle ABD}\sin(B+D)$.

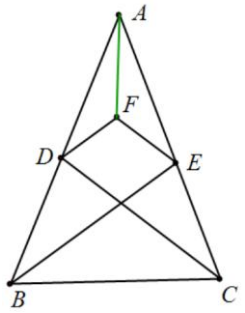
Example 27 : There is a point P inside rectangle $ABCD$. Prove that: $PA^2+PC^2=PB^2+PD^2$.

Proof: Let P be the origin, $a\bar{a}+c\bar{c}-b\bar{b}-(a+c-b)\overline{(a+c-b)}$

$$=(a-b)\overline{(a-b)}\left(\frac{a-(a+c-b)}{a-b}+\overline{\left(\frac{a-(a+c-b)}{a-b}\right)}\right),$$

P inside parallelogram $ABCD$. Prove: $PA^2+PC^2-PB^2-PD^2=2AB\cdot AD\cos A$.

Example 28 : As shown in Figure 1, in triangle ABC , D and E *are on* AB and AC respectively , $DF \ // \ BE$, $EF \ // \ CD$, if $BD = CE$, prove that AF *is* the angle bisector of $\angle A$. *(* Complex number identities extend to equal-length line segments and angle bisectors)



Proof: Suppose $A=\left(a,\frac{1}{a}\right)$, $B=\left(b,\frac{1}{b}\right)$, $C=\left(c,\frac{1}{c}\right)$, , $D=\left(d,\frac{a+b-d}{ab}\right)$,

$E=\left(e,\frac{a+c-e}{ac}\right)$, can be solved $F=\left(f,\overline{f}\right)$.

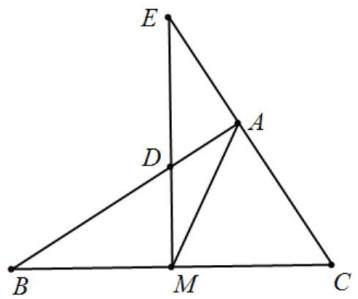
$$[AEF][ADF][ABC]\left((b-d)\overline{(b-d)}-(c-e)\overline{(c-e)}\right)$$

$$=[BDE][CDE]|a-f|^4\left(\frac{\frac{a-c}{a-f}}{\frac{a-f}{a-b}}-\overline{\left(\frac{\frac{a-c}{a-f}}{\frac{a-f}{a-b}}\right)}\right),$$

Extended proposition: As shown in Figure 1, in $\triangle ABC$, D and E *are on* AB and AC respectively , $DF \ // \ BE$, $EF \ // \ CD$, prove: $S_{\triangle AEF}\cdot S_{\triangle ADF}\sin A\left(BD^2-CE^2\right)=S_{\triangle BDE}\cdot S_{\triangle CDE}\cdot AF^2\sin\left(\angle CAF-\angle BAF\right)$.

Note: When D *is on the extension of* line segment AB and E *is on* line segment AC , AF is the exterior angle bisector of $\angle A$.

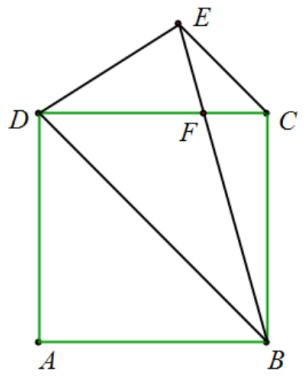
Example 29 : As shown in Figure 1, in triangle ABC , M is the midpoint of BC , $MD \perp BC$, angle AB intersects the extended line CA at D , *and* if $AB \perp AC$, prove: $MD \cdot ME = MA^2$ (Complex number identities extend the perpendicular relationship between line segments and perpendicularity)



Proof: Assume $M=(0,0)$, $C=(c,c)$, $B=-C$, $D=(d,-d)$, $E=(e,-e)$, we can find $A=\left(\frac{-cd+ce+2de}{d+e},-\frac{cd-ce+2de}{d+e}\right)$,

$$8de\left(d(-e)-a\bar{a}\right)=(d-e)^2(a-b)\overline{(a-b)}\left(\frac{a-c}{a-b}+\overline{\left(\frac{a-c}{a-b}\right)}\right),$$

Extended proposition: As shown in Figure 1, in $\triangle ABC$, M is the midpoint of BC , $MD \perp BC$ intersects AB at D , and intersects the extended line *of CA at E.* Prove: $4MD \cdot ME\left(MD \cdot ME - MA^2\right) = DE^2 \cdot AB \cdot AC \cos A$.

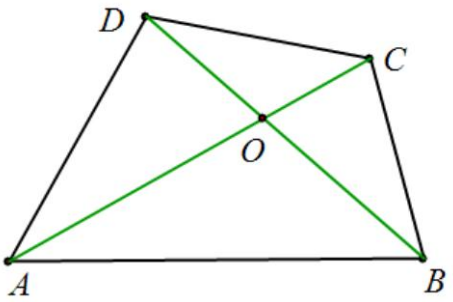


Example 30 : Square $ABCD$, $CE \ // \ BD$, and $BD = BE$, F is the intersection of lines BE and CD , prove that: $DE = DF$.

Proof: Let C be the origin, $b = -\mathrm{i}$, $d = -1$, solve for $e = \dfrac{(1-\mathrm{i})f}{1+f}$, $\bar{e} = \dfrac{(1+\mathrm{i})f}{1+f}$, and get the identity:

$$d^2\left((d-f)\overline{(d-f)}-(d-e)\overline{(d-e)}\right) = f^2\left((b-d)\overline{(b-d)}-(b-e)\overline{(b-e)}\right),$$

Extended proposition: Square $ABCD$, $CE \ // \ BD$, F is the intersection of straight lines BE and CD , prove: $CD^2\left(DF^2 - DE^2\right) = CF^2\left(BD^2 - BE^2\right)$.



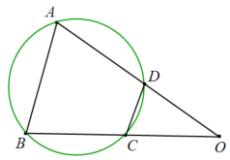
Example 3 1 : Quadrilateral $ABCD$, AC intersects BD at O. If $S_{\triangle OAB}S_{\triangle OCD} = S_{\triangle OBC}{}^2$, prove: $AB \ // \ DC$. (Area relationship and parallelism of quadrilaterals)

Proof: Let O be the origin, OC be the real axis, $\bar{d} = \dfrac{d\bar{b}}{b}$,

$$[OAB][OCD]-[OBC]^2 = ([DAB]-[CAB])[OBC],$$

Extended proposition: Quadrilateral $ABCD$, AC intersects BD at O , prove: $S_{\triangle OAB}S_{\triangle OCD} - S_{\triangle OBC}{}^2 = (S_{\triangle DAB} - S_{\triangle CAB})S_{\triangle OBC}$.

Example 32 : Quadrilateral $ABCD$, line AD intersects BC at O. Prove that $OA\cdot OD = OB\cdot OC$ the necessary and sufficient condition for this is that the four points A , B , C , and D are cocircular.



Proof: Let O be the origin and line OB be the real axis.

(1) Since points O , A , and D are collinear, we have $\frac{a}{d} = \overline{\left(\frac{a}{d}\right)}$, which is $\bar{d} = \frac{\bar{a}d}{a}$.

Substituting this into the following two equations, we find that there is a common factor $bc - \bar{a}d = 0$.

(2) $OA\cdot OD = OB\cdot OC$, that is $bc - \bar{a}d = 0$.

(3) Points $\dfrac{\frac{a-c}{a-b}}{\frac{d-c}{d-b}} - \overline{\left(\dfrac{\frac{a-c}{a-b}}{\frac{d-c}{d-b}}\right)} = \dfrac{(a-\bar{a})(b-c)(a-d)(bc-\bar{a}d)}{(a-b)(b-\bar{a})(c-d)(ac-\bar{a}d)} = 0$ A , B , C , and D *are concentric circles,* i.e.

$$(a-b)(\bar{a}-b)(c-d)\left(c-\frac{\bar{a}d}{a}\right)\left(\frac{\frac{a-c}{a-b}}{\frac{d-c}{d-b}} - \overline{\left(\frac{\frac{a-c}{a-b}}{\frac{d-c}{d-b}}\right)}\right) + \frac{(a-\bar{a})(b-c)(a-d)}{a}(bc-\bar{a}d) = 0$$

, so we get a new proposition: quadrilateral $ABCD$, line AD intersects BC at O , prove: $AB\cdot BC\cdot CD\cdot DA\sin(\angle BAD + \angle BCD) = 2\left(S_{\triangle ABC} - S_{\triangle DBC}\right)\left(OA\cdot OD - OB\cdot OC\right)$.

Note: This conclusion is a generalization of the circle power theorem, which quantifies the circle $OA\cdot OD = OB\cdot OC \Leftrightarrow$ power theorem. From this conclusion, we can know that points A , B , C , and D are concentric circles.

This is a beautiful conclusion. The equation is rich in geometric elements , involving segment lengths, angles, areas, collinearity, and cocircularity . The left side of the conclusion $AB\cdot BC\cdot CD\cdot DA\sin(\angle BAD + \angle BCD)$ multiplies the product of the four sides of the quadrilateral by a trigonometric function, $\sin(\angle BAD + \angle BCD)$ the values of which determine whether the four points are cocircular. The right side of the conclusion $2\left(S_{\triangle ABC} - S_{\triangle DBC}\right)\left(OA\cdot OD - OB\cdot OC\right)$ multiplies the difference in the areas of the two triangles by the difference in the products of the line segments, which

$S_{\triangle ABC} - S_{\triangle DBC}$ determines whether lines *AD* and *BC* are parallel or intersecting. The known condition that lines *AD* intersect *BC* at *O* was previously overlooked. However, it is an essential condition for the circle power theorem to hold and should be reflected in the conclusion.

The ultimate ideal is to generate an identity that allows us to see at a glance:

are five points *ABCDO* on a plane , *BCO and ADO* are collinear, and points $OA \cdot OD = OB \cdot OC$ *A* , *B* , *C* and *D are cocircular* . If you know any three of the four conditions, you can deduce the fourth one.

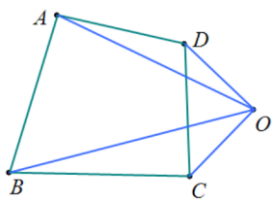
five points *A* , *B* , *C* , *D* , and *O* on a plane . Points *O* , *A* , and *D are* collinear. Points *O* , *B* , and *C* are collinear . Points *A* , *B* , *C* , and *D* are cocircular. $OA \cdot OD = OB \cdot OC$ Can we establish an identity and study the relationship between these conditions?

$$2(a-b)\overline{(a-b)}(c-d)\overline{(c-d)}\left(\frac{a-d}{a-b}\frac{c-b}{c-d}-\overline{\left(\frac{a-d}{a-b}\frac{c-b}{c-d}\right)}\right)$$

$$+\left(\begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix}-\begin{vmatrix} d & \bar{d} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix}\right)\left((o-a)\overline{(o-a)}\left(\frac{o-d}{o-a}+\overline{\left(\frac{o-d}{o-a}\right)}\right)-(o-b)\overline{(o-b)}\left(\frac{o-c}{o-b}+\overline{\left(\frac{o-c}{o-b}\right)}\right)\right)$$

$$-\begin{vmatrix} o & \bar{o} & 1 \\ d & \bar{d} & 1 \\ a & \bar{a} & 1 \end{vmatrix}\left((b-c)\overline{(b-c)}\left(\frac{b-a}{b-c}+\overline{\left(\frac{b-a}{b-c}\right)}\right)-(c-d)\overline{(c-d)}\left(\frac{c-b}{c-d}+\overline{\left(\frac{c-b}{c-d}\right)}\right)\right)$$

$$-\begin{vmatrix} o & \bar{o} & 1 \\ c & \bar{c} & 1 \\ b & \bar{b} & 1 \end{vmatrix}\left((a-b)\overline{(a-b)}\left(\frac{a-d}{a-b}+\overline{\left(\frac{a-d}{a-b}\right)}\right)-(d-a)\overline{(d-a)}\left(\frac{d-c}{d-a}+\overline{\left(\frac{d-c}{d-a}\right)}\right)\right)=0$$



Translating complex number identities into ordinary geometric quantities, we consider that directed line segments, directed angles, and directed areas differ from ordinary line segments, angles, and areas by a sign, and the choice of sign depends on the relative positions of the points. For example, in this problem, the positions that can be formed by the five free points are very complex. The following relationship holds true under the conditions shown in Figure 3.

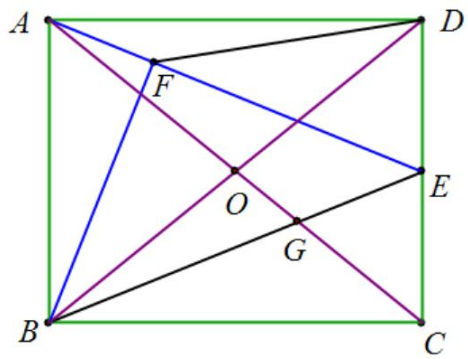
$$\frac{1}{2}AB\cdot BC\cdot CD\cdot DA\sin\left(\angle BAD+\angle BCD\right)$$

$$-\left(S_{\triangle ABC}-S_{\triangle DBC}\right)(OA\cdot OD\cos\angle AOD-OB\cdot OC\cos\angle BOC)$$

$$-S_{\triangle ODA}(BA\cdot BC\cos\angle ABC-CB\cdot CD\cos\angle BCD)$$

$$-S_{\triangle OCB}(AD\cdot AB\cos\angle BAD-DC\cdot DA\cos\angle CDA)=0.$$

Example 33 : Square ABCD, E is the midpoint of CD, BF ⊥ AE, prove that: ∠EFD = 45 °.



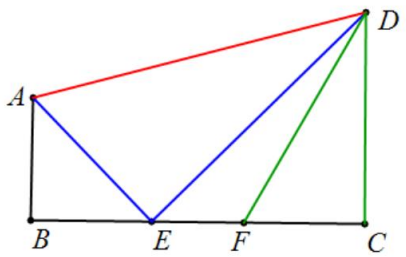
Proof: Assume $\bar{a}=-a$ that $b=0$, $\bar{c}=c$, $d=a+c$, $e=\frac{c+d}{2}=\frac{a+2c}{2}$,

$f=\frac{\left(-a+\bar{a}\right)c}{\bar{a}-2c}=\frac{2ac}{a+2c}$, Discover

$$3\frac{\frac{b-e}{a-c}+\overline{\left(\frac{b-e}{a-c}\right)}\left(\frac{f-d}{f-e}\right)^2+\overline{\left(\frac{f-d}{f-e}\right)^2}}{\frac{b-e}{a-c}-\overline{\left(\frac{b-e}{a-c}\right)}\left(\frac{f-d}{f-e}\right)^2-\overline{\left(\frac{f-d}{f-e}\right)^2}}=4\frac{\frac{b-e}{a-e}+\overline{\left(\frac{b-e}{a-e}\right)}\frac{b-d}{a-c}+\overline{\left(\frac{b-d}{a-c}\right)}}{\frac{b-e}{a-e}-\overline{\left(\frac{b-e}{a-e}\right)}\frac{b-d}{a-c}-\overline{\left(\frac{b-d}{a-c}\right)}}.$$

Extended proposition: Rectangle ABCD, E is the midpoint of CD, BF ⊥ AE, AC intersects BD and BE at O and G respectively. Prove: $3\cot\angle CGE\cot\angle 2EFD=4\cot\angle AEB\cot\angle COD$.

Example 34 : Quadrilateral ABCD, AB $\perp$ BC, CD $\perp$ BC, E and F are two points on BC, BE=CF , and $\angle$ADE = $\angle$CDF . Prove that $\angle$AED =90 °.



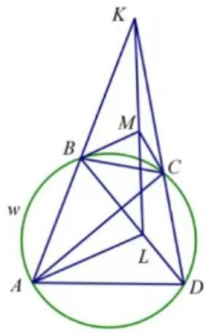
Proof: Assume $\bar{a}=-a$ that $b=0$, $\bar{c}=c$, $\bar{d}=2c-d$, $\bar{e}=e$, $f=b+c-e$, $\bar{f}=f$,

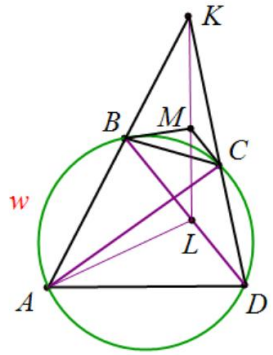
$$\frac{\frac{d-c}{d-f}+\overline{\left(\frac{d-c}{d-f}\right)}}{\frac{d-c}{d-f}-\overline{\left(\frac{d-c}{d-f}\right)}}-\frac{\frac{d-e}{d-a}+\overline{\left(\frac{d-e}{d-a}\right)}}{\frac{d-e}{d-a}-\overline{\left(\frac{d-e}{d-a}\right)}}=\frac{c}{e}\frac{\frac{e-a}{e-d}+\overline{\left(\frac{e-a}{e-d}\right)}}{\frac{e-a}{e-d}-\overline{\left(\frac{e-a}{e-d}\right)}}.$$

Extended proposition: In quadrilateral ABCD, AB $\perp$ BC, CD $\perp$ BC, E and F are two points on BC, BE=CF , prove

$$\cot\angle FDC-\cot\angle ADE=\frac{BC}{BE}\cot\angle ADE .$$

Example 35 : As shown in the figure, convex quadrilateral ABCD is inscribed in circle w. Lines AB and CD intersect at point K. Take point L on diagonal BD such that $\angle$ DAL = $\angle$CAB . Construct CM / /DB through point C, intersecting KL at point M. Prove that line BM is a tangent to circle w. (2025 National High School Mathematics League Fujian Preliminary)





Proof: Assume that $k=0$, $c=1$, $\bar{d}=d$, Since A, B, C, and D are concentric, we have $b=\frac{cd}{\bar{a}}$; Since B, D, and L are collinear, we have $\bar{l}=\frac{-ad+\bar{a}d-\bar{a}l+a\bar{a}l}{a\left(-1+\bar{a}\right)}$;

Since **CM / / DB , we have** $m=\frac{l}{d}$, $\bar{m}=\frac{-ad+\bar{a}d-\bar{a}l+a\bar{a}l}{a\left(-1+\bar{a}\right)d}$,

$$[ABD]\left(\frac{\frac{a-l}{a-d}+\overline{\left(\frac{a-l}{a-d}\right)}}{\frac{a-l}{a-d}-\overline{\left(\frac{a-l}{a-d}\right)}}-\frac{\frac{a-b}{a-c}+\overline{\left(\frac{a-b}{a-c}\right)}}{\frac{a-b}{a-c}-\overline{\left(\frac{a-b}{a-c}\right)}}\right)=[ACD]\left(\frac{\frac{b-m}{b-c}+\overline{\left(\frac{b-m}{b-c}\right)}}{\frac{b-m}{b-c}-\overline{\left(\frac{b-m}{b-c}\right)}}-\frac{\frac{a-b}{a-c}+\overline{\left(\frac{a-b}{a-c}\right)}}{\frac{a-b}{a-c}-\overline{\left(\frac{a-b}{a-c}\right)}}\right)$$

Extended Proposition: As shown in the figure, convex quadrilateral ABCD is inscribed in circle w. Lines AB and CD intersect at point K. Take point L on diagonal BD and make CM / / DB through point C. It intersects KL at point M. Prove :

$S_{\triangle ABD}\left(\cot\angle DAL-\cot\angle CAB\right)=S_{\triangle ACD}\left(\cot\angle CBM-\cot\angle CAB\right)$.

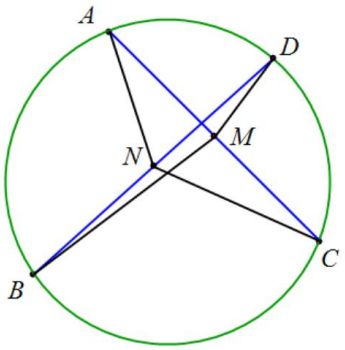
Example 36 : In a quadrilateral ABCD , M is the midpoint of AC and N is the midpoint of BD. If $\angle AMB = \angle AMD$, prove that $\angle ANB = \angle BNC$. (2019 Danube Mathematical Competition)

Proof: Let the circle be the unit circle, $m=\frac{a+c}{2}, n=\frac{b+d}{2}$,

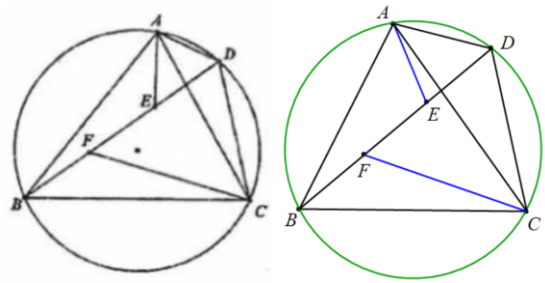
$$\frac{\frac{n-b}{n-a}+\overline{\left(\frac{n-b}{n-a}\right)}}{\frac{n-b}{n-a}-\overline{\left(\frac{n-b}{n-a}\right)}}-\frac{\frac{n-c}{n-b}+\overline{\left(\frac{n-c}{n-b}\right)}}{\frac{n-c}{n-b}-\overline{\left(\frac{n-c}{n-b}\right)}}=\frac{\frac{m-b}{m-a}+\overline{\left(\frac{m-b}{m-a}\right)}}{\frac{m-b}{m-a}-\overline{\left(\frac{m-b}{m-a}\right)}}-\frac{\frac{m-a}{m-d}+\overline{\left(\frac{m-a}{m-d}\right)}}{\frac{m-a}{m-d}-\overline{\left(\frac{m-a}{m-d}\right)}},$$

Extended proposition: In the inscribed quadrilateral ABCD, M is the midpoint of AC and N is the midpoint of BD. Prove

$\cot\angle AMB-\cot\angle AMD=\cot\angle ANB-\cot\angle BNC$.



Example 37 : As shown in the figure, in the inscribed quadrilateral ABCD, points E and F are on the diagonal BD, satisfying DE=BF . If $\angle DAE = \angle CAB$, prove that $\angle DCF = \angle ACB$.



Proof: Let the circle be the unit circle, $\bar{e} = \dfrac{b+d-e}{bd}$, $f = b+d-e$,

$$\frac{\dfrac{a-d}{a-e}+\overline{\left(\dfrac{a-d}{a-e}\right)}}{\dfrac{a-d}{a-e}-\overline{\left(\dfrac{a-d}{a-e}\right)}}-\frac{\dfrac{a-c}{a-b}+\overline{\left(\dfrac{a-d}{a-e}\right)}}{\dfrac{a-c}{a-b}-\overline{\left(\dfrac{a-d}{a-e}\right)}}=\frac{b-e}{e-d}\left(\frac{\dfrac{c-b}{c-a}+\overline{\left(\dfrac{a-d}{a-e}\right)}}{\dfrac{c-b}{c-a}-\overline{\left(\dfrac{a-d}{a-e}\right)}}-\frac{\dfrac{c-f}{c-d}+\overline{\left(\dfrac{a-d}{a-e}\right)}}{\dfrac{c-f}{c-d}-\overline{\left(\dfrac{a-d}{a-e}\right)}}\right),$$

As shown in the figure, in the inscribed quadrilateral ABCD, points E and F are on the diagonal BD, satisfying DE=BF . Prove

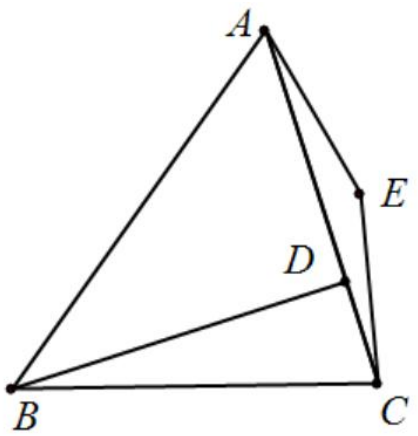
$ED\left(\cot\angle DAE-\cot\angle CAB\right)=BE\left(\cot\angle ACB-\cot\angle DCF\right)$,

Example 38 : Convex quadrilateral ABCD is inscribed in circle O, and point 0 is not on any diagonal . If the circumcenter of O_1 triangle AOC lies on line BD, prove that the circumcenter of triangle BOD O_2 lies on line AC.

Proof: Let circle O be the unit circle, then $o_1 = \frac{ac}{a+c}$, $o_2 = \frac{bd}{b+d}$, $[BDO][ACO_1] = [ACO][BDO_2]$.

Convex quadrilateral ABCD is inscribed in circle O, the circumcenter of △ AOC O_1, and the circumcenter of △ BOD . O_2 Prove $S_{\triangle BDO} \cdot S_{\triangle ACO_1} = S_{\triangle ACO} \cdot S_{\triangle BDO_2}$.

Example 39 : In triangle ABC , D is a point on AC . Draw a perpendicular line to AB through the midpoint of AD , which intersects the perpendicular line to BC through the midpoint of DB at point E. Prove: EA = EC if and only if BD $\perp$ AC .



Proof: Assume $d=0$, $a=1$, Solve the equation $\dfrac{e-\dfrac{a+d}{2}}{a-b}=-\overline{\left(\dfrac{e-\dfrac{a+d}{2}}{a-b}\right)}$,

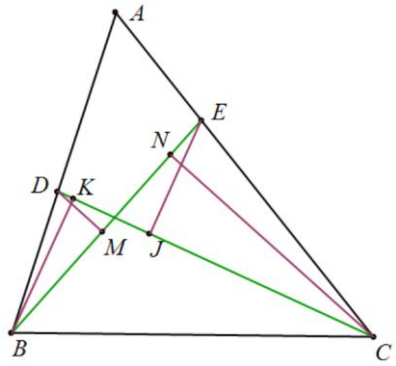
$\dfrac{e-\dfrac{c+d}{2}}{c-b}=-\overline{\left(\dfrac{e-\dfrac{c+d}{2}}{c-b}\right)}$, and get $e=-\dfrac{-2b+b^2+b\bar{b}+2c-2bc}{2\left(b-\bar{b}\right)}$,

$\bar{e}=\dfrac{-2\bar{b}+b\bar{b}+\bar{b}^2+2c-2\bar{b}c}{2\left(b-\bar{b}\right)}$,

$$\frac{[ABC]}{2}\frac{\dfrac{d-c}{d-b}+\overline{\left(\dfrac{d-c}{d-b}\right)}}{\dfrac{d-c}{d-b}-\overline{\left(\dfrac{d-c}{d-b}\right)}}=(a-e)\overline{(a-e)}-(c-e)\overline{(c-e)},$$

$\triangle$ ABC , D is a point on AC . Draw a perpendicular line to AB through the midpoint of AD , which intersects the perpendicular line to BC through the midpoint of DB at point E. Prove $2S_{\triangle ABC}\cot\angle BDC=CE^2-AE^2$:

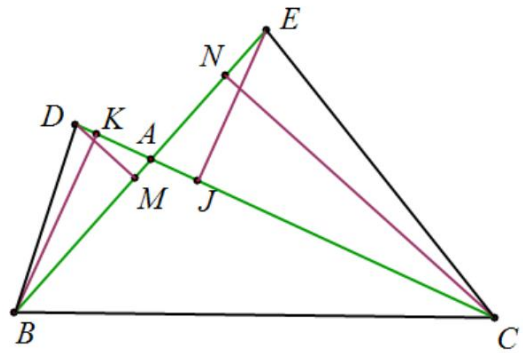
Example 40 : On sides AB and AC of triangle ABC , there are points D and E respectively . The feet of the perpendiculars of points E and B on CD are J and K respectively , and the feet of the perpendiculars of points D and C on BE are M and N respectively . If the four points B , C , E , and D are concentric , prove that the four points J, K, M, and N are concentric. (2025 Pan-African Mathematical Olympiad)



Proof: Let the intersection of BE and CD be the origin, $b=1$, $\bar{d}=\frac{\bar{c}d}{c}$, $k=\frac{c+\bar{c}}{2\bar{c}}$, $j=\frac{\left(c+\bar{c}\right)e}{2\bar{c}}$, $m=\frac{\left(c+\bar{c}\right)d}{2c}$, $n=\frac{c+\bar{c}}{2}$, Find the identity:

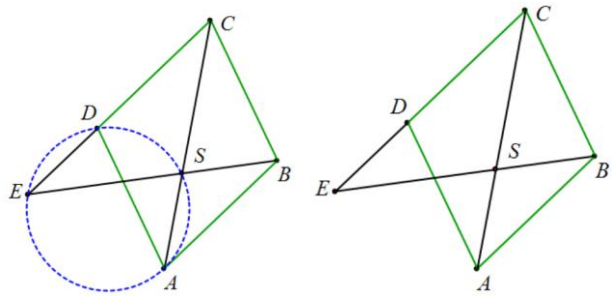
$$\frac{\frac{e-c}{e-b}+\overline{\left(\frac{e-c}{e-b}\right)}}{\frac{e-c}{e-b}-\overline{\left(\frac{e-c}{e-b}\right)}}-\frac{\frac{d-c}{d-b}+\overline{\left(\frac{d-c}{d-b}\right)}}{\frac{d-c}{d-b}-\overline{\left(\frac{d-c}{d-b}\right)}}=\frac{de}{c}\left(\frac{\frac{k-j}{k-m}+\overline{\left(\frac{k-j}{k-m}\right)}}{\frac{k-j}{k-m}-\overline{\left(\frac{k-j}{k-m}\right)}}-\frac{\frac{n-j}{n-m}+\overline{\left(\frac{k-j}{k-m}\right)}}{\frac{n-j}{n-m}-\overline{\left(\frac{k-j}{k-m}\right)}}\right).$$

On sides AC and AB of triangle ABC , there are points D and E respectively . The feet of the perpendiculars of points E and B on CD are J and K respectively . The feet of the perpendiculars of points D and C on BE are M and N respectively . Prove $AB\cdot AC\left(\cot\angle BEC-\cot\angle BDC\right)=AD\cdot AE\left(\cot\angle MKJ-\cot\angle MNJ\right)$:



Special case: Four points B , C , E , and D are cocircular ⇔ Four points J, K, M, and N are cocircular.

Example 41 : In parallelogram ABCD, E is a point on the extension of CD, and AD = AE. BE intersects AC at S. If AB = AD, then A, S, D, and E are cocircular.



Proof: Let $a=0$, $d=1$, $b=a+c-d$, $e=-\dfrac{1-c}{1-\bar{c}}$, and find the identity:

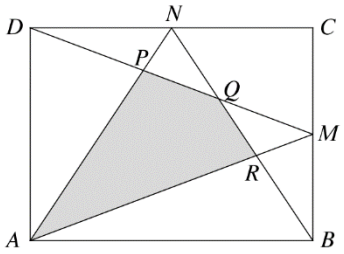
$$(b-e)(d-a)\overline{(b-e)(d-a)}\left(\frac{\frac{c-a}{b-e}}{\frac{d-a}{d-e}}-\overline{\left(\frac{\frac{c-a}{b-e}}{\frac{d-a}{d-e}}\right)}\right)=-\left(b\bar{b}-1\right)(a-c)\overline{(a-c)}\left(\frac{d-e}{a-c}-\overline{\left(\frac{d-e}{a-c}\right)}\right)$$

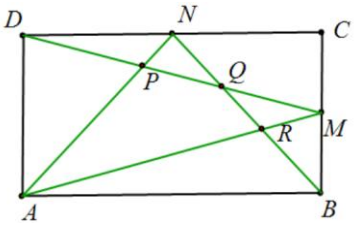
, parallelogram ABCD, E is the point on the extension of CD, and AD = AE, BE intersects AC at S, prove

$BE\cdot DA\sin\left(\angle EDA-\angle ESA\right)=\left(AB^2-AD^2\right)\sin\angle CAB$.

Example 42 : In parallelogram $ABCD$, M and N *are the midpoints of* BC and CD respectively , AM intersects BN at R , DM intersects BN at Q , and AN intersects DM at P. Prove that the necessary and sufficient condition for A , P , Q , and R to be cocircular is *that* $BA \perp BC$.

Proof: Let $\dfrac{\dfrac{A-\dfrac{A+C-B+C}{2}}{A-\dfrac{B+C}{2}}}{\dfrac{A+C-B-\dfrac{B+C}{2}}{\dfrac{A+C-B+C}{2}-B}}=t$, $\left(\dfrac{B-C}{B-A}\right)^2=s$, $1-4t-4s+ts=0$,



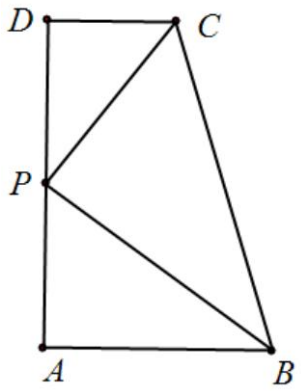


In parallelogram $ABCD$, M and N *are the midpoints of* BC and CD respectively , AM intersects BN at R , DM intersects BN at Q , and AN intersects DM at P. Prove: $\dfrac{10}{3}\cot\angle BAD=\cot\angle MAN-\cot\angle BQM$.

Proof: Let A be the origin, AB be the real axis, $d=a+c-b$, $m=\dfrac{b+c}{2}$, $n=\dfrac{d+c}{2}$,

$$\frac{10}{3}\frac{\dfrac{a-d}{a-b}+\overline{\left(\dfrac{a-d}{a-b}\right)}}{\dfrac{a-d}{a-b}-\overline{\left(\dfrac{a-d}{a-b}\right)}}=\frac{\dfrac{a-n}{a-m}+\overline{\left(\dfrac{a-n}{a-m}\right)}}{\dfrac{a-n}{a-m}-\overline{\left(\dfrac{a-n}{a-m}\right)}}-\frac{\dfrac{d-m}{n-b}+\overline{\left(\dfrac{d-m}{n-b}\right)}}{\dfrac{d-m}{n-b}-\overline{\left(\dfrac{d-m}{n-b}\right)}}=-\frac{10\left(2b-c-\bar{c}\right)}{3\left(c-\bar{c}\right)},$$

Example 43 : In trapezoid ABCD, AB//DC, P is the midpoint of AD , AB+CD=BC, prove: P B $\perp$PC .

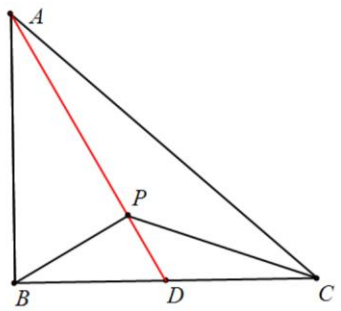


Proof: Let AB be the real axis, AD be the imaginary axis, $a=0$, $p=\frac{d}{2}$,

$$(b-c)\overline{(b-c)}-(b-a+c-d)\overline{(b-a+c-d)}=2[PBC]\frac{\frac{p-c}{p-b}+\overline{\left(\frac{p-c}{p-b}\right)}}{\frac{p-c}{p-b}-\overline{\left(\frac{p-c}{p-b}\right)}}.$$

In trapezoid ABCD, AB//DC, P is the midpoint of AD , prove: $(AB+CD)^2-BC^2=4PB\cdot PC\cos\angle BPC$.

$$(A-B+D-C)^2-(B-C)^2=4\left(\frac{A+D}{2}-B\right)\cdot\left(\frac{A+D}{2}-C\right)$$



Example 45 : In △ ABC, BA ⊥ BC, AD is the median, BP ⊥ AD, prove: $\angle ACD = \angle CPD$.

Proof: Let BC be the real axis, BA be the imaginary axis, $b = 0$, $d = \frac{c}{2}$,

$$\overline{p} = \frac{2ac - 2ap - cp}{2a - c}, \quad \frac{\frac{c-d}{c-a} + \overline{\left(\frac{c-d}{c-a}\right)}}{\frac{c-d}{c-a} - \overline{\left(\frac{c-d}{c-a}\right)}} - \frac{\frac{p-c}{p-d} + \overline{\left(\frac{p-c}{p-d}\right)}}{\frac{p-c}{p-d} - \overline{\left(\frac{p-c}{p-d}\right)}} = \frac{\frac{p-b}{p-a} + \overline{\left(\frac{p-b}{p-a}\right)}}{\frac{p-b}{p-a} - \overline{\left(\frac{p-b}{p-a}\right)}}.$$

△ ABC, BA ⊥ BC, AD is the median, prove: $\cot\angle ACD - \cot\angle CPD = \cot\angle BPA$.

Example 46 : *P* and *Q are the midpoints of* the diagonals *AC* and *BD of the* inscribed quadrilateral *ABCD* . If $\angle BPA = \angle DPA$, prove that $\angle AQB = \angle CQB$. (2011 National High School Mathematics League)

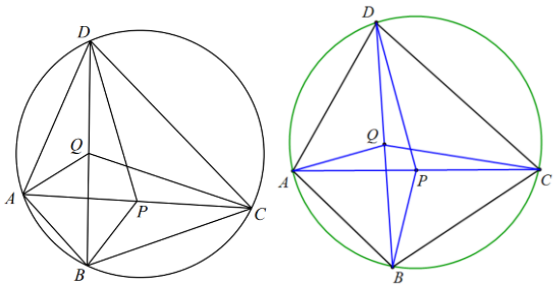


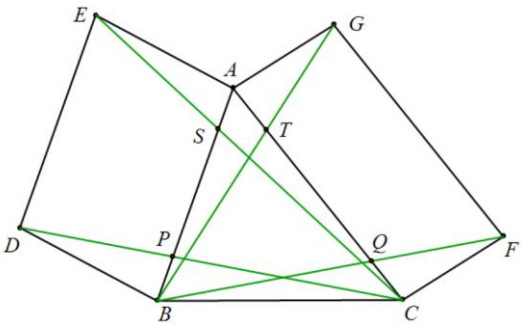
Figure 21

Proof: Let the circle be the unit circle, $p=\frac{a+c}{2}$, $q=\frac{b+d}{2}$, the identity:

$$\frac{\frac{p-a}{p-d}+\overline{\left(\frac{p-a}{p-d}\right)}}{\frac{p-a}{p-d}-\overline{\left(\frac{p-a}{p-d}\right)}}-\frac{\frac{p-b}{p-a}+\overline{\left(\frac{p-b}{p-a}\right)}}{\frac{p-b}{p-a}-\overline{\left(\frac{p-b}{p-a}\right)}}=\frac{\frac{q-c}{q-b}+\overline{\left(\frac{q-c}{q-b}\right)}}{\frac{q-c}{q-b}-\overline{\left(\frac{q-c}{q-b}\right)}}-\frac{\frac{q-b}{q-a}+\overline{\left(\frac{q-b}{q-a}\right)}}{\frac{q-b}{q-a}-\overline{\left(\frac{q-b}{q-a}\right)}}$$

$$=\frac{(ac-bd)(2ac+2bd-ab-bc-cd-da)}{(a-b)(b-c)(c-d)(d-a)},$$

shown in the figure , *P* and *Q are* the midpoints of the diagonals *AC* and *BD of the* inscribed quadrilateral *ABCD* . Prove that : $\cot\angle DPA-\cot\angle APB=\cot\angle BQC-\cot\angle AQB$.



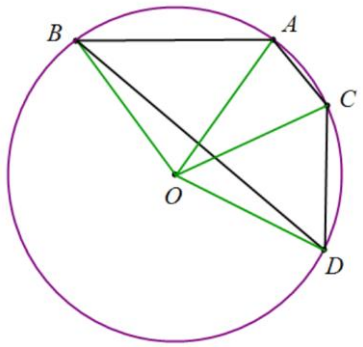
Example 47 : Construct parallelograms ABDE and ACFG outside sides AB and AC of triangle ABC, with S △ ABG = S △ ACE . Connect CD and CE to intersect AB at P and S, respectively. Connect BF and BG to intersect AC at Q and T, respectively . Prove: ST//PQ . (Proposition by Ye Zhonghao)

Proof: Let $a=0$, $b=1$, $e=a+d-b$, , $g=a+f-c$, $s=\dfrac{-c+\bar{c}-\bar{c}d+c\bar{d}}{c-\bar{c}-d+\bar{d}}$,

$p=\dfrac{-\bar{c}d+c\bar{d}}{c-\bar{c}-d+\bar{d}}$, $t=\dfrac{c\left(c-\bar{c}-f+\bar{f}\right)}{-c+\bar{c}-\bar{c}f+c\bar{f}}$, $q=-\dfrac{c\left(f-\bar{f}\right)}{-c+\bar{c}-\bar{c}f+c\bar{f}}$, find

$[EDC][BFG]([SPQ]-[TPQ])=[ABC]^2([ABG]-[AEC])$.

Construct parallelograms ABDE and ACFG outside sides AB and AC of triangle ABC. Connect CD and CE to intersect AB at P and S respectively. Connect BF and BG to intersect AC at Q and T respectively. Prove: $S_{\triangle EDC}S_{\triangle BFG}\left(S_{\triangle SPQ}-S_{\triangle TPQ}\right)=S_{\triangle ABC}{}^2\left(S_{\triangle ABG}-S_{\triangle AEC}\right)$.



Example 48 : There are four points A, B, C, and D on a circle O. AB$\perp$CD. Prove that: $S_{\triangle OCA} = S_{\triangle ODB}$.

Proof: Let circle O be the unit circle, the identity

$$\frac{([OAC]-[OBD])}{(b-a)\overline{(b-a)}\left(\frac{d-c}{b-a}+\overline{\left(\frac{d-c}{b-a}\right)}\right)([ABC]-[DBC])}=\frac{abcd}{(a-b)(b-c)(a-d)(c-d)},$$

There are four points A, B, C, and D on circle O. Prove that: $2(S_{\triangle OCA}-S_{\triangle ODB})\sin\frac{\angle BOC}{2}\sin\frac{\angle AOD}{2}=S_{ABDC}\cos\alpha$, where α is the angle between lines AB and CD.

Note: When D crosses the straight line OB, $S_{\triangle OCA}-S_{\triangle ODB}$ it should be changed to $S_{\triangle OCA}+S_{\triangle ODB}$.

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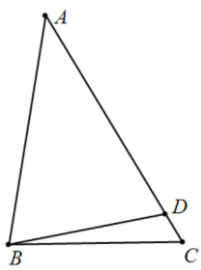


Example 49 : In △ABC, D is a point on AC, AB = AD, prove: $2\angle DBC+\angle ACB=\angle ABC$.

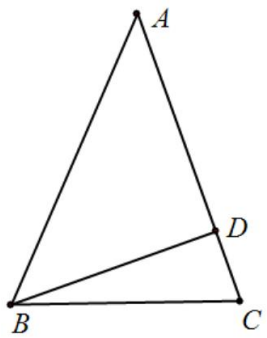
Identity: $\dfrac{\left(\dfrac{b-d}{b-c}\right)^2\dfrac{c-b}{c-a}}{\dfrac{b-a}{b-c}}\dfrac{\dfrac{b-a}{b-d}}{\dfrac{d-b}{d-a}}\dfrac{a-c}{a-d}=1$.

Proof: Let the circumcircle of △ABC be the unit circle, then $\overline{d}=\dfrac{a+c-d}{ac}$,

$$\frac{(a-b)\overline{(a-b)}-(a-d)\overline{(a-d)}}{(b-c)^2\overline{(b-c)}^2(b-a)\overline{(b-a)}(c-a)\overline{(c-a)}\left(\dfrac{\left(\dfrac{b-d}{b-c}\right)^2\dfrac{c-b}{c-a}}{\dfrac{b-a}{b-c}}-\overline{\left(\dfrac{\left(\dfrac{b-d}{b-c}\right)^2\dfrac{c-b}{c-a}}{\dfrac{b-a}{b-c}}\right)}\right)}$$

$$=\frac{ab^3c^3}{(a-b)(a-c)(b-c)^5},$$

In △ABC, D is a point on AC. Prove that: $\left(AB^2-AD^2\right)\sin A=BD^2\sin\left(2\angle DBC+\angle ACB-\angle ABC\right)$.



Example 50 : In △ABC, AB = AC, BD ⊥ AC, prove it $\frac{AD}{DC}+1=2\frac{AB^2}{CB^2}$.

Proof: Let the circumcircle of △ABC be the unit circle, $d=\frac{ab+b^2-ac+bc}{2b}$, identity:

$$\frac{\frac{a-d}{d-c}+1-2\frac{(a-b)\overline{(a-b)}}{(c-b)\overline{(c-b)}}}{(a-b)\overline{(a-b)}-(a-c)\overline{(a-c)}}$$

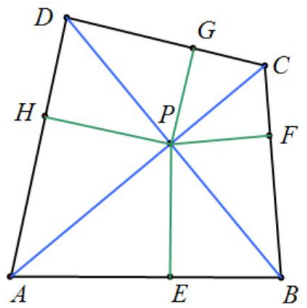
$$=\frac{2ab^2c^2}{(a+b)(a-c)(b-c)^3}\left((a-b)\overline{(a-b)}-(b-c)\overline{(b-c)}\right),$$

In △ABC, BD ⊥ AC, prove that:

$$BC^2\cdot AC\cdot CD\left(\frac{AD}{DC}+1-2\frac{AB^2}{CB^2}\right)=\left(AB^2-AC^2\right)\left(BA^2-BC^2\right).$$

Note: Through the identity, it is found that $\frac{AD}{DC}+1-2\frac{AB^2}{CB^2}\Rightarrow AB=AC$ it is obviously not true.

Example 51 : In quadrilateral ABCD , AC $\perp$ BD, and AC and BD intersect at P, PE $\perp$ AB, PF $\perp$ BC, PG $\perp$ CD, PH $\perp$ AD, and E, F, G, and H are perpendicular feet. Prove that E, F, G, and H are concentric circles.



Let P be the origin, PC be the real axis, $\bar{d}=\frac{\bar{b}d}{b}$, $e=\frac{a\left(b-\bar{b}\right)}{2\left(a-\bar{b}\right)}$, $f=\frac{\left(\bar{b}-b\right)c}{2\left(\bar{b}-c\right)}$,

$g=\frac{\left(b-\bar{b}\right)cd}{2\left(bc-\bar{b}d\right)}$, $h=\frac{a\left(b-\bar{b}\right)d}{2\left(ab-\bar{b}d\right)}$, identity

$$(a-c)\overline{(a-c)}[EHP][FGP]\frac{bc\bar{c}\left(\frac{d}{c}+\overline{\left(\frac{d}{c}\right)}\right)}{acd},$$

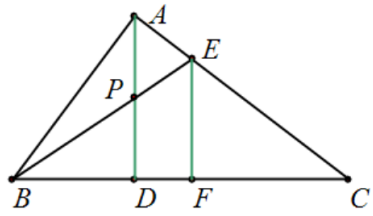
$$=[BCP](f-e)\overline{(f-e)}(h-g)\overline{(h-g)}\left(\frac{\frac{e-h}{e-f}}{\frac{g-h}{g-f}}-\overline{\left(\frac{\frac{e-h}{e-f}}{\frac{g-h}{g-f}}\right)}\right),$$

Given that in quadrilateral ABCD , AC and BD intersect at P, PE $\perp$ AB, PF $\perp$ BC, PG $\perp$ CD, PH $\perp$ AD, and E, F, G, and H are the feet of perpendiculars. Prove that:

$$4S_{\triangle EHP}\cdot S_{\triangle FGP}\cdot AC^2\cdot PB\cos\angle DPC$$

$$=S_{\triangle BCP}\cdot PA\cdot EF\cdot FG\cdot GH\cdot HE\sin\left(\angle HEF+\angle HGF\right).$$

Example 52 : In triangle ABC , AD is the altitude, AB $\perp$ AC , P is the midpoint of AD, BP intersects AC at E, draw a perpendicular line to BC through E, and F is the foot of the perpendicular. Prove: $FE^2 = AE \cdot EC$.



Proof: Let DC be the real axis, DA be the imaginary axis, $p = \frac{a}{2}$, $e = \frac{ab - ac + bc}{2b - c}$,

$f = \frac{bc}{2b - c}$, identities $2\left((f-e)\overline{(f-e)} - (a-e)\overline{(e-c)}\right) = -[CEF]\frac{\frac{a-c}{a-b} + \overline{\left(\frac{a-c}{a-b}\right)}}{\frac{a-c}{a-b} - \overline{\left(\frac{a-c}{a-b}\right)}}$,

acute angle $\triangle$ABC , AD is the altitude, P is the midpoint of AD, BP intersects AC at E, and a perpendicular to BC is drawn through E, with F being the foot of the perpendicular.

Prove: $FE^2 - AE \cdot EC = 2S_{\triangle AEF} \cot A$.

triangle ABC , AD is the altitude, P is the midpoint of AD, BP intersects AC at E, and a perpendicular line is drawn through E to BC, with F being the foot of the perpendicular.
Prove: $FE^2 = AE \cdot EC \Leftrightarrow AB \perp AC$.

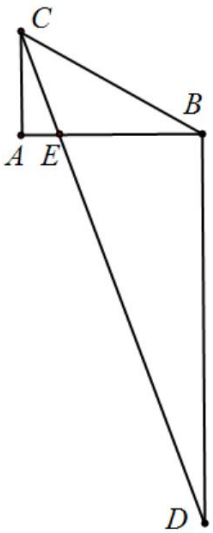
Proof: Let DC be the real axis, DA be the imaginary axis, $d = 0$, $p = \frac{a}{2}$,

Solve the equation $\frac{b-e}{b-p} = \overline{\left(\frac{b-e}{b-p}\right)}$, $\frac{a-e}{a-c} = \overline{\left(\frac{a-e}{a-c}\right)}$, and we get $e = \frac{ab - ac + bc}{2b - c}$;

$f = \frac{e + \bar{e}}{2} = \frac{bc}{2b - c}$, $AB \perp AC$,Right now $\frac{a-c}{a-b} + \overline{\left(\frac{a-c}{a-b}\right)} = \frac{2\left(a^2 - bc\right)}{(a-b)(a+b)} = 0$,

$FE^2 - AE \cdot EC = (f-e)\overline{(f-e)} - (a-e)\overline{(e-c)} = \frac{c(b-c)\left(a^2 - bc\right)}{(2b-c)^2}$,

so $FE^2 = AE \cdot EC \Leftrightarrow AB \perp AC$.



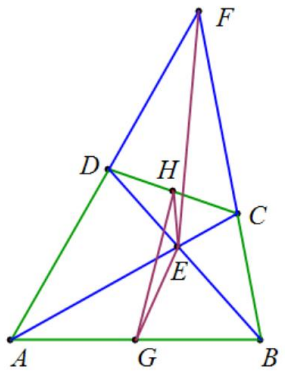
Example 53 : In triangle ABC, ∠A=90°. Draw a straight line CED intersecting AB at E and the perpendicular line BD of AB at D, and making ∠BCE=2∠ACE, then DE=2BC.

Proof: Let AB be the real axis and AC be the imaginary axis, $\overline{d}=2b-d$, $e=\dfrac{bc}{b+c-d}$, identity

$$\frac{(d-e)\overline{(d-e)}-4(c-b)\overline{(c-b)}}{(c-b)\overline{(c-b)}(c-a)^2\overline{(c-a)}^2\left(\dfrac{\left(\dfrac{c-e}{c-a}\right)^2}{\dfrac{c-b}{c-e}}-\overline{\left(\dfrac{\left(\dfrac{c-e}{c-a}\right)^2}{\dfrac{c-b}{c-e}}\right)}\right)}=\frac{(b+c-d)(b+2c-d)}{2bc^5},$$

triangle ABC, ∠A=90°. Draw a straight line CED that intersects AB at E and the perpendicular line BD to AB at D. Prove that:

$$AC^3\cdot AB\left(4BC^2-DE^2\right)=CE^3\cdot CB(AC+BD)(2AC+BD)\sin(2\angle ACE-\angle BCE).$$



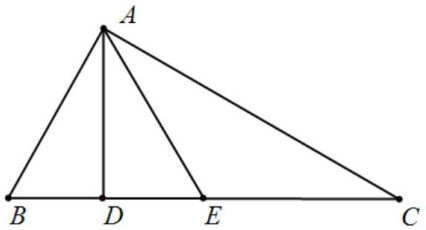
Example 54 : In a circle-inscribed quadrilateral ABCD, AC intersects BD at E. Extending AD and BC intersects at F. G and H are the midpoints of AB and CD, respectively. Prove that EF is a tangent to the circumcircle of triangle EGH. (Preliminary question for the 50th IMO in 2009)

Proof: Let $e=0$ ED be the real axis, $\bar{b}=b$, $\bar{d}=d$, $\bar{c}=\dfrac{\bar{a}c}{a}$, $h=\dfrac{c+d}{2}$, $g=\dfrac{a+b}{2}$,

the solution is $f=\dfrac{abc+abd-acd-bcd}{ab-cd}$, $\bar{f}=\dfrac{\bar{a}bc+abd-\bar{a}cd-bcd}{ab-cd}$,

Identity $$\frac{\dfrac{a-d}{a-e}+\overline{\left(\dfrac{a-d}{a-e}\right)}}{\dfrac{a-d}{a-e}-\overline{\left(\dfrac{a-d}{a-e}\right)}}-\frac{\dfrac{b-e}{b-c}+\overline{\left(\dfrac{b-e}{b-c}\right)}}{\dfrac{b-e}{b-c}-\overline{\left(\dfrac{b-e}{b-c}\right)}}=\frac{\dfrac{g-h}{g-e}+\overline{\left(\dfrac{g-h}{g-e}\right)}}{\dfrac{g-h}{g-e}-\overline{\left(\dfrac{g-h}{g-e}\right)}}-\frac{\dfrac{e-h}{e-f}+\overline{\left(\dfrac{e-h}{e-f}\right)}}{\dfrac{e-h}{e-f}-\overline{\left(\dfrac{e-h}{e-f}\right)}}.$$

In quadrilateral ABCD, AC intersects BD at E. Extending AD and BC intersects at F. G and H are the midpoints of AB and CD respectively. Prove that: $\cot\angle DAC-\cot\angle DBC=\cot\angle HGE-\cot\angle HEF$.



Example 55 : In triangle ABC, BA $\perp$ AC, AD $\perp$ BC at point D, and E is the midpoint of BC. Prove that: $\angle$BAD = $\angle$CAE .

Proof: Let DC be the real axis and DA be the imaginary axis, $d=0$, $\bar{a}=-a$, $e=\dfrac{b+c}{2}$,

Identity: $$\frac{\dfrac{a-d}{a-b}+\overline{\left(\dfrac{a-d}{a-b}\right)}}{\dfrac{a-d}{a-b}-\overline{\left(\dfrac{a-d}{a-b}\right)}}-\frac{\dfrac{a-c}{a-e}+\overline{\left(\dfrac{a-c}{a-e}\right)}}{\dfrac{a-c}{a-e}-\overline{\left(\dfrac{a-c}{a-e}\right)}}=-\frac{b+c}{b}\frac{\dfrac{a-c}{a-b}+\overline{\left(\dfrac{a-c}{a-b}\right)}}{\dfrac{a-c}{a-b}-\overline{\left(\dfrac{a-c}{a-b}\right)}}.$$

triangle ABC, AD $\perp$ BC at point D, and E is the midpoint of BC. Prove that : $\cot\angle BAD-\cot\angle EAC=2\dfrac{DE}{DB}\cot\angle BAC$.

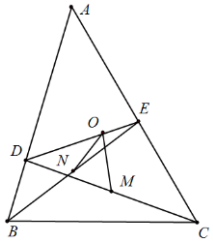


Figure 33

Example 56 : As shown in Figure 33, let △O be the circumcenter of ABC, let D and E lie on AB and AC, respectively, and let O lie on DE. Let M and N be the midpoints of CD and BE, respectively. Prove that ∠MON = ∠BAC. (Question from the 2004 Iranian Contest)

Proof: Let circle O be the unit circle and line DE be the real axis, then $d=\overline{d}$, solve the equation $\frac{a-d}{a-b}=\overline{\left(\frac{a-d}{a-b}\right)}$ to get $d=\frac{a+b}{1+ab}$, then $m=\frac{c+d}{2}=\frac{a+b+c+abc}{2(1+ab)}$,

similarly $n=\frac{a+b+c+abc}{2(1+ac)}$, verify $\frac{\frac{m}{n}}{\frac{a-c}{a-b}}=\overline{\left(\frac{\frac{m}{n}}{\frac{a-c}{a-b}}\right)}=\frac{(a-b)(1+ac)}{(1+ab)(a-c)}$.

Note: Setting the unit circle as the normal operation is a routine operation, and setting the line DE as the real axis is a stroke of genius.

Proof: Let circle O be the unit circle, $\overline{d}=\frac{a+b-d}{ab}$, $\overline{e}=\frac{a+c-e}{ac}$, identity

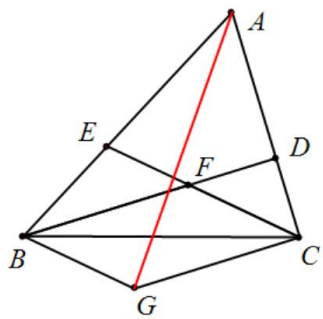
$$(a-b)\overline{(a-b)}m\overline{m}\left(\frac{\frac{a-c}{a-b}}{\frac{m}{n}}-\overline{\left(\frac{\frac{a-c}{a-b}}{\frac{m}{n}}\right)}\right)=\frac{b+c}{4(b-c)}[ABC][DOE],$$

Extended Proposition: Let △O be the circumcenter of ABC, let D and E lie on AB and AC respectively, and let M and N be the midpoints of CD and BE respectively. Proof: $OM\cdot ON\sin(A-\angle MON)=S_{\triangle DOE}\cos A$.

Proof: Let circle O be the unit circle, $\overline{d}=\frac{a+b-d}{ab}$, $\overline{e}=\frac{a+c-e}{ac}$, identity

$$2[MON]\left(\frac{\frac{a-c}{a-b}+\overline{\left(\frac{a-c}{a-b}\right)}}{\frac{a-c}{a-b}-\overline{\left(\frac{a-c}{a-b}\right)}}-\frac{\frac{m}{n}+\overline{\left(\frac{m}{n}\right)}}{\frac{m}{n}-\overline{\left(\frac{m}{n}\right)}}\right)=[DOE]\frac{\frac{a-c}{a-b}+\overline{\left(\frac{a-c}{a-b}\right)}}{\frac{a-c}{a-b}-\overline{\left(\frac{a-c}{a-b}\right)}},$$

Extended Proposition: Let △O be the circumcenter of ABC, let D and E lie on AB and AC respectively, and let M and N be the midpoints of CD and BE respectively. Proof: $2S_{\triangle MON}\left(\cot\angle MON-\cot A\right)=S_{\triangle ODE}\cot A$.



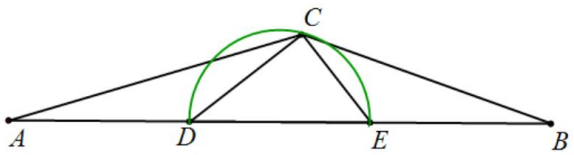
Example 57 : Points D and E are on sides AC and AB of triangle ABC respectively, and BE=CD. BD and CE intersect at point F. The midpoint of side BC is M , and the point symmetric to F about M is G. Prove that AG bisects $\angle$BAC .

Proof: Let the circumcircle of △ABC be the unit circle, $\bar{d}=\frac{a+c-d}{ac}$, $\bar{e}=\frac{a+b-e}{ab}$,,

$$[ABC]^3(c-d)\overline{(c-d)}-(b-e)\overline{(b-e)}$$

$$=\left([ABC]+[ADE]\right)^2(a-g)^2\overline{(a-g)}^2\left(\frac{\frac{a-c}{a-g}}{\frac{a-g}{a-b}}-\overline{\left(\frac{\frac{a-c}{a-g}}{\frac{a-g}{a-b}}\right)}\right),$$

Points D and E are on sides AC and AB of triangle ABC respectively. BD and CE intersect at point F. The midpoint of side BC is M, and the point symmetric of F about M is G. Prove $S_{\triangle ABC}{}^2(CD^2-BE^2)\sin A=(S_{\triangle ABC}+S_{\triangle ADE})^2AG^2\sin(\angle CAG-\angle GAB)$.



Example 58 : Given that D and E are the points dividing AB into three equal parts, draw a semicircle with E as the diameter , and select a point on C the semicircle to verify $\tan\angle ACD\tan\angle BCE=\frac{1}{4}$.

prove: $S_{\triangle CAD}=\frac{1}{2}CA\cdot CD\sin\angle ACD$,

$S_{\triangle CAD}+S_{\triangle CDE}=\frac{1}{2}CA\cdot CE\sin\left(90^{\circ}+\angle ACD\right)=\frac{1}{2}CA\cdot CE\cos\angle ACD$,

Depend on $S_{\triangle CAD}=S_{\triangle CDE}$, divide the two $\frac{1}{2}=\frac{CD}{CE}\tan\angle ACD$, similarly $\frac{1}{2}=\frac{CE}{CD}\tan\angle BCE$,

Multiply the two formulas together to get $\tan\angle ACD\tan\angle BCE=\frac{1}{4}$.

Next, we explore the possibility of abandoning the condition CD⊥CE.

Let line AB be the real axis, $d=0$, $e=1$, $a=-1$, $b=2$,

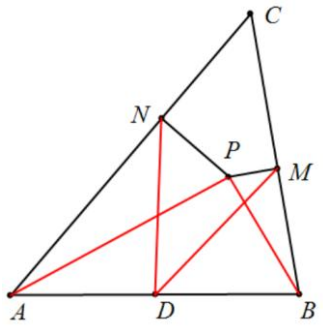
$$\frac{4\left(\frac{\frac{c-d}{c-a}-\overline{\left(\frac{c-d}{c-a}\right)}}{\frac{c-d}{c-a}+\overline{\left(\frac{c-d}{c-a}\right)}}\right)\left(\frac{\frac{c-b}{c-e}-\overline{\left(\frac{c-b}{c-e}\right)}}{\frac{c-b}{c-e}+\overline{\left(\frac{c-b}{c-e}\right)}}\right)+1}{(c-d)\overline{(c-d)}\left(\frac{c-e}{c-d}+\overline{\left(\frac{c-e}{c-d}\right)}\right)}$$

$$=\frac{(c-e)\overline{(c-e)}\left(\frac{c-d}{c-e}+\overline{\left(\frac{c-d}{c-e}\right)}\right)-b\bar{b}}{(c-a)\overline{(c-a)}\left(\frac{c-d}{c-a}+\overline{\left(\frac{c-d}{c-a}\right)}\right)(c-e)\overline{(c-e)}\left(\frac{c-b}{c-e}+\overline{\left(\frac{c-b}{c-e}\right)}\right)},$$

Extended proposition: Given that D and E are the points that divide AB into three equal parts, and C is an arbitrary point, prove that $2CA\cdot CB\cdot\cos\angle ACD\cos\angle BCE\left(1-4\tan\angle ACD\tan\angle BCE\right)$

$=\left(CD^{2}+CE^{2}-5DE^{2}\right)\cos\angle DCE$.

Example 59 : As shown in Figure 1, select point P in triangle ABC such that $\angle PAC = \angle PBC$. Draw perpendiculars PM and PN from point P to sides BC and CA, respectively . Let D be the midpoint of side AB. Prove that DN = DM .



Proof: Let the circumcircle of △ABC be the unit circle, $d=\dfrac{a+b}{2}$,

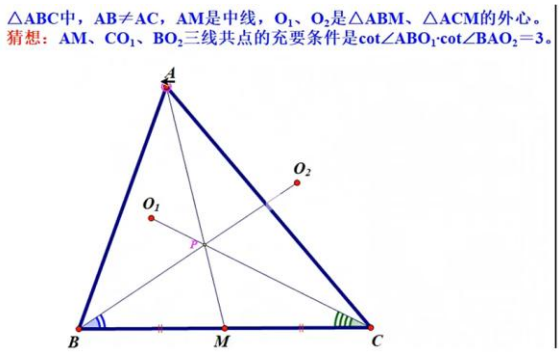
$m=\dfrac{b+c+p-bc\overline{p}}{2}$, $n=\dfrac{a+c+p-ac\overline{p}}{2}$, Identity:

$$\frac{(a-b)c}{4(a-c)(b-c)}(a-p)\overline{(a-p)}(b-p)\overline{(b-p)}\left(\frac{\frac{b-c}{b-p}}{\frac{a-p}{a-c}}-\overline{\left(\frac{\frac{b-c}{b-p}}{\frac{a-p}{a-c}}\right)}\right)$$

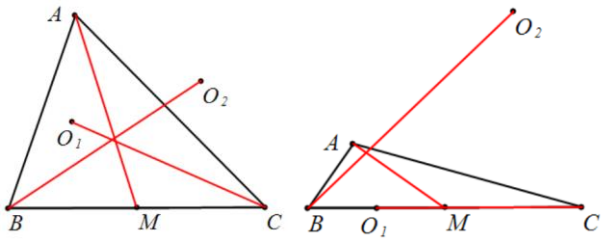
$$=(d-m)\overline{(d-m)}-(d-n)\overline{(d-n)} \quad .$$

As shown in Figure 1, take point P in △ ABC and draw perpendiculars PM and PN from point P to sides BC and CA respectively . Let D be the midpoint of side AB.

Prove that: $DN^2-DM^2=PA\cdot PB\sin\angle CBP\sin\left(\angle PAC-\angle PBC\right)$.



Example 60 : $\triangle ABC$ In , $AB \neq AC$, AM is the median, O_1 , O_2 is the circumcenter of $\triangle ABM$, $\triangle ACM$. Conjecture: AM, CO_1, BO_2 The necessary and sufficient condition for the three lines to have a common point is $\cot\angle BCO_1 \cdot \cot\angle CBO_2 = 3$.



The computer outputs the identity:

Extended proposition: $\triangle ABC$ In the middle, $AB \neq AC$, AM is the median, O_1 and , O_2 is the circumcenter $\triangle ABM$ of $\triangle ACM$. Prove:

$$2S_{\triangle BMO_1} \cdot S_{\triangle BMO_2} \cdot MA\left(3 - \cot\angle BCO_1 \cot\angle CBO_2\right)\cos\angle AMB$$

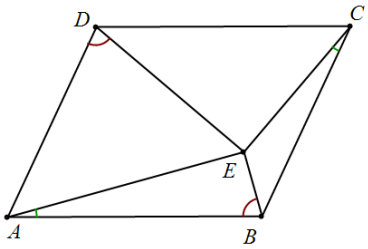
$$= 3MC \cdot \left(S_{\triangle ABO_2} \cdot S_{\triangle MCO_1} - S_{\triangle MBO_2} \cdot S_{\triangle ACO_1}\right).$$

The AM, CO_1, BO_2 three lines with a common point are equivalent to

$$S_{\triangle ABO_2} \cdot S_{\triangle MCO_1} - S_{\triangle MBO_2} \cdot S_{\triangle ACO_1} = 0.$$

Note: The original question specifies this $AB \neq AC$, so it's ruled out $\cos\angle AMB = 0$. To rule it out $S_{\triangle BMO_1} \cdot S_{\triangle BMO_2} = 0$, $\angle$BAM or $\angle$MAC = 90 °. This point doesn't need to be emphasized, as the existence of the cotangent guarantees it.

Example 6 1 : E is a point inside parallelogram $ABCD$. Prove that: $\angle BAE = \angle BCE \Leftrightarrow \angle EBA = \angle EDA$.



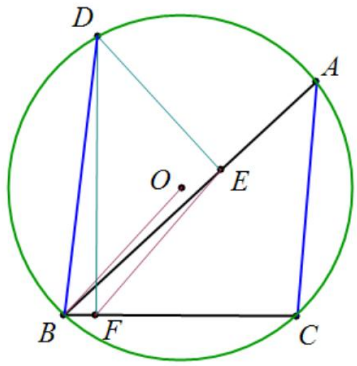
Proof: Let $a=0$, $b=1$, $d=a+c-b=c-1$,

$$(c-b)\overline{(c-b)}\left(\frac{c-e}{c-b}-\overline{\left(\frac{c-e}{c-b}\right)}\right)\left(\frac{\frac{a-e}{a-b}+\overline{\left(\frac{a-e}{a-b}\right)}}{\frac{a-e}{a-b}-\overline{\left(\frac{a-e}{a-b}\right)}}-\frac{\frac{c-b}{c-e}+\overline{\left(\frac{c-b}{c-e}\right)}}{\frac{c-b}{c-e}-\overline{\left(\frac{c-b}{c-e}\right)}}\right)$$

$$=(d-e)\overline{(d-e)}\left(\frac{d-a}{d-e}-\overline{\left(\frac{d-a}{d-e}\right)}\right)\left(\frac{\frac{b-a}{b-e}+\overline{\left(\frac{b-a}{b-e}\right)}}{\frac{b-a}{b-e}-\overline{\left(\frac{b-a}{b-e}\right)}}-\frac{\frac{d-e}{d-a}+\overline{\left(\frac{d-e}{d-a}\right)}}{\frac{d-e}{d-a}-\overline{\left(\frac{d-e}{d-a}\right)}}\right)$$

$$=-\frac{2\left(ce-c\bar{c}e-e^2+\bar{c}e^2-\bar{c}\bar{e}+c\bar{c}\bar{e}+\bar{e}^2-c\bar{e}^2\right)}{e-\bar{e}},$$

$$S_{\triangle AED}\left(\cot\angle ABE-\cot\angle EDA\right)=S_{\triangle BCE}\left(\cot\angle BAE-\cot\angle BCE\right),$$



Example 62 : Point D is a point on the circumcircle O of triangle ABC. Through D, draw DE⊥AB and DF⊥BC. Prove: If AC∥DB, then EF∥OB.

Proof: Let the circumscribed circle of △ABC be the unit circle,

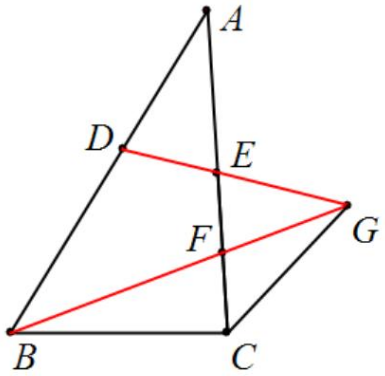
$e=\frac{1}{2}\left(a+b+d-ab\overline{d}\right)$, $f=\frac{1}{2}\left(b+c+d-bc\overline{d}\right)$, found

$$\frac{(d-b)\overline{(d-b)}\left(\frac{a-c}{d-b}-\overline{\left(\frac{a-c}{d-b}\right)}\right)}{(o-b)\overline{(o-b)}\left(\frac{e-f}{o-b}-\overline{\left(\frac{e-f}{o-b}\right)}\right)}=2$$ that the geometric meaning is:

$[DBC]-[ADC]=2([OBF]-[EOF])$;

$$\frac{\frac{a-c}{d-b}}{\frac{e-f}{o-b}}=\overline{\left(\frac{\frac{a-c}{d-b}}{\frac{e-f}{o-b}}\right)}$$, geometric meaning is: the angle between the lines AC and DB is equal to the angle between the lines EF and OB;

Note: Point D does not need to be on the circumcircle of △ABC.



Example 63 : In △ABC, there are points D and E on AB and AC, BC=BD, AD=AE, F is the midpoint of CE, DE intersects BF at G. Prove: $\angle BDC = \angle DGC$.

Proof: Let DE be the real axis, A be on the imaginary axis, $d=-e$, $\bar{a}=-a$,

$\bar{b}=-\dfrac{ab+2ae-be}{a+e}$, $\bar{c}=\dfrac{-ac+2ae-ce}{a-e}$, $f=\dfrac{c+e}{2}$,

$g=\dfrac{e\left(2ab+ac-2bc+ae-ce-e^2\right)}{2ab-ac+3ae-2be-ce-e^2}$,

$$(b-d)\overline{(b-d)}-(b-c)\overline{(b-c)}$$

$$=\frac{2a^2bc-a^2c^2-2a^2be+4a^2ce-2ac^2e+a^2e^2-4abe^2+2bce^2-c^2e^2-2ae^3+2be^3+e^4}{(-a+e)(a+e)}$$

$$,\frac{\dfrac{d-c}{d-b}}{\dfrac{g-c}{g-d}}-\overline{\left(\frac{\dfrac{d-c}{d-b}}{\dfrac{g-c}{g-d}}\right)}$$

$$=\frac{4ae(-2a+c+e)\left(2a^2bc-a^2c^2-2a^2be+4a^2ce-2ac^2e+a^2e^2-4abe^2+2bce^2-c^2e^2-2ae^3+2be^3+e^4\right)}{(a-e)(e-c)\left(2ab-ac+ae-ce-e^2\right)\left(-2a^2b+a^2c-5a^2e+2abe+2ace+ce^2+e^3\right)}$$

, so the proposition is established.

The above approach is very stupid.

Computers pieced together the identity:

$$\begin{vmatrix} f & \bar{f} & 1 \\ b & \bar{b} & 1 \\ d & \bar{d} & 1 \end{vmatrix}(d-b)\overline{(d-b)}(g-c)\overline{(g-c)}\left(\frac{\dfrac{d-c}{d-b}}{\dfrac{g-c}{g-d}}-\overline{\left(\frac{\dfrac{d-c}{d-b}}{\dfrac{g-c}{g-d}}\right)}\right)$$

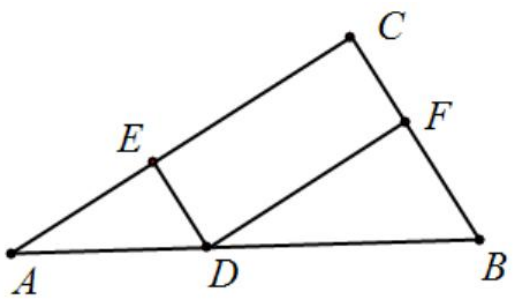
$$=\begin{vmatrix} a & \overline{a} & 1 \\ d & \overline{d} & 1 \\ g & \overline{g} & 1 \end{vmatrix}^2 \frac{(b+e)(e-c)}{2(a-e)(a+e)}\left((b-d)\overline{(b-d)}-(b-c)\overline{(b-c)}\right),$$

So there is an extended proposition: In △ABC, there are two points D and E on AB and AC, AD=AE, F is the midpoint of CE, DE intersects BF at G, prove:

$$4S_{\triangle FBD}\sin\left(\angle BDC-\angle DGC\right)=\sin\angle DCG\sin\angle ADE\left(BC^2-BD^2\right).$$

The new proposition not only includes: $\angle BDC=\angle DGC \Leftrightarrow BC=BD$, but also quantifies these geometric quantities.

Obviously, the original question is $BC=BD \Rightarrow \angle BDC=\angle DGC$. Now we can integrate the two into a single identity. By doing this multiple times, we can integrate all the conditions in the question into the identity.



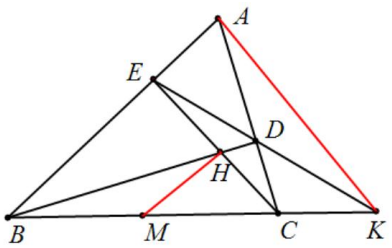
Example 6 4 : In triangle ABC, D, E, and F are on BA, AC, and CB respectively, and quadrilateral DFCE is a parallelogram. If ∠C = 90 °, prove that:
$AE \cdot EC + BF \cdot FC = AD \cdot DB$.

Proof: Let $a = 0$, $b = 1$, $e = dc$, $f = c + d - e$,

$$(a-d)\overline{(d-b)} - (a-e)\overline{(e-c)} - (b-f)\overline{(f-c)} = \left(\frac{\frac{c-b}{c-a} + \overline{\left(\frac{c-b}{c-a}\right)}}{\frac{c-b}{c-a} - \overline{\left(\frac{c-b}{c-a}\right)}} \right) [ADF],$$

In △ABC, D, E, and F are on BA, AC, and CB respectively, and quadrilateral DFCE is a parallelogram. Prove that:
$AE \cdot EC + BF \cdot FC - AD \cdot DB = 2CE \cdot CF \cot \angle ACB$.

$$AE \cdot EC + BF \cdot FC - AD \cdot DB = 4S_{\triangle ADF} \cot \angle ACB = CE \cdot CF \cot \angle ACB,$$



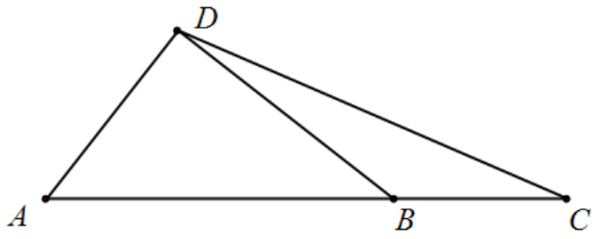
Example 6 5 : In △ABC, BD and CE are altitudes, H is the orthocenter, M is the midpoint of BC, and K is the intersection of lines ED and BC. Prove that: MH⊥AK.

Proof: Let the circumscribed circle of △ABC be the unit circle,

$d=\dfrac{ab+b^2-ac+bc}{2b}$, $e=\dfrac{-ab+ac+bc+c^2}{2c}$, $m=\dfrac{b+c}{2}$, $h=a+b+c$,

$\bar{k}=\dfrac{b+c-k}{bc}$, found $2[EDK]=[BCO](a-k)\overline{(a-k)}\left(\dfrac{m-h}{a-k}+\overline{\left(\dfrac{m-h}{a-k}\right)}\right)$ that the

geometric meaning is: $2S_{\triangle EDK}=\overrightarrow{AK}\cdot\overrightarrow{MH}\sin 2A$;

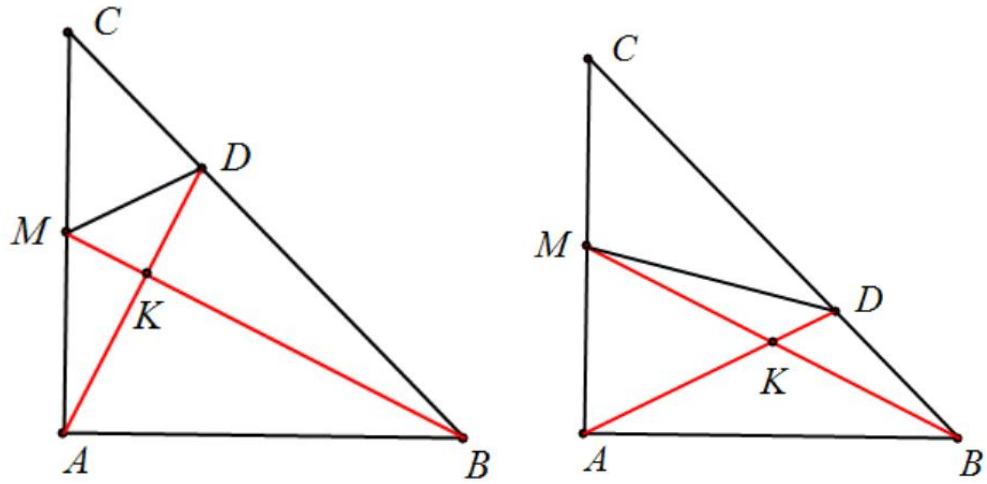


Example 6 6 : C is on the extension line of AB, and 2 BC=AB, if DA $\perp$DB . Prove: $\tan\angle ADC+3\tan A=0$.

Proof: Let AB be the real axis, $a=0$, $b=2$, $c=3$,

$$\frac{\frac{d-c}{d-a}-\overline{\left(\frac{d-c}{d-a}\right)}}{\frac{d-c}{d-a}+\overline{\left(\frac{d-c}{d-a}\right)}}+3\frac{\frac{a-d}{a-b}-\overline{\left(\frac{a-d}{a-b}\right)}}{\frac{a-d}{a-b}+\overline{\left(\frac{a-d}{a-b}\right)}}=2\frac{\frac{d-b}{d-a}+\overline{\left(\frac{d-b}{d-a}\right)}}{\frac{d-b}{d-a}-\overline{\left(\frac{d-b}{d-a}\right)}}\frac{\frac{d-c}{d-a}-\overline{\left(\frac{d-c}{d-a}\right)}}{\frac{d-c}{d-a}+\overline{\left(\frac{d-c}{d-a}\right)}}\frac{\frac{a-d}{a-b}-\overline{\left(\frac{a-d}{a-b}\right)}}{\frac{a-d}{a-b}+\overline{\left(\frac{a-d}{a-b}\right)}}$$

C is on the extension of AB, and 2 BC=AB. D is any point. Prove: $\tan\angle ADC+3\tan A=2\cot\angle ADB\tan\angle ADC\tan A$.



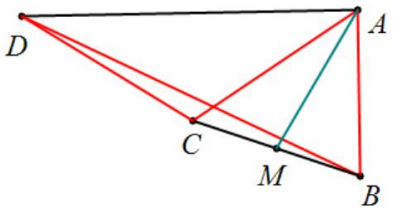
Example 6 7 : △ABC, AB = AC, AB ⊥ AC, M is the midpoint of AC, D is a point on BC, AD ⊥ MB, prove: $\angle DMC = \angle AMB$.

Proof: Let $a=0$, $b=1$, $c=\mathrm{i}$, $m=\frac{\mathrm{i}}{2}$, since BCD are collinear, we can get

$$\bar{d}=\mathrm{i}(d-1-\mathrm{i}),\quad \frac{\frac{\mathrm{i}+d}{2}-1}{\mathrm{i}-d}\,\frac{\frac{0-d}{\frac{\mathrm{i}}{2}-1}+\frac{0-\bar{d}}{-\frac{\mathrm{i}}{2}-1}}{\frac{0-d}{\frac{\mathrm{i}}{2}-1}-\frac{0-\bar{d}}{-\frac{\mathrm{i}}{2}-1}}=\frac{\frac{\frac{\mathrm{i}}{2}-1}{\frac{\mathrm{i}}{2}-0}+\frac{-\frac{\mathrm{i}}{2}-1}{-\frac{\mathrm{i}}{2}-0}}{\frac{\frac{\mathrm{i}}{2}-1}{\frac{\mathrm{i}}{2}-0}-\frac{-\frac{\mathrm{i}}{2}-1}{-\frac{\mathrm{i}}{2}-0}}-\frac{\frac{\frac{\mathrm{i}}{2}-\mathrm{i}}{\frac{\mathrm{i}}{2}-d}+\frac{-\frac{\mathrm{i}}{2}+\mathrm{i}}{-\frac{\mathrm{i}}{2}-\bar{d}}}{\frac{\frac{\mathrm{i}}{2}-\mathrm{i}}{\frac{\mathrm{i}}{2}-d}-\frac{-\frac{\mathrm{i}}{2}+\mathrm{i}}{-\frac{\mathrm{i}}{2}-\bar{d}}}$$

In △ABC, AB = AC, AB ⊥ AC, M is the midpoint of AC, D is a point on BC, AD intersects MB at K. Prove: $\frac{DB+CB}{2CD}\cot\angle AKB=\cot\angle DMC-\cot\angle AMB$.

Special case: △ABC, AB = AC, AB ⊥ AC, M is the midpoint of AC, D is a point on BC, and CD = 2 DB, AD intersects MB at K. Prove: $\cot\angle AKB=\cot\angle DMC-\cot\angle AMB$.



Example 6 8 : Convex quadrilateral ABCD, AB$\perp$AD, M is the midpoint of BC, if $\angle ADC = \angle BAM$, prove: $\angle ADB = \angle MAC$.
(2016 Russian Mathematical Olympiad)

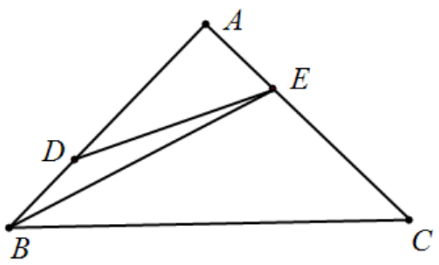
Proof: Let AD be the real axis, DB be the imaginary axis, $a = 0$, $m = \dfrac{b+c}{2}$,

$$(d-c)(a-b)\overline{(d-c)(a-b)}\left(\frac{\frac{d-a}{d-c}}{\frac{a-b}{a-m}} - \overline{\left(\frac{\frac{d-a}{d-c}}{\frac{a-b}{a-m}}\right)}\right)$$

$$=(d-b)(a-m)\overline{(d-b)(a-m)}\left(\frac{\frac{d-a}{d-b}}{\frac{a-m}{a-c}} - \overline{\left(\frac{\frac{d-a}{d-b}}{\frac{a-m}{a-c}}\right)}\right).$$

Convex quadrilateral ABCD, AB$\perp$AD, M is the midpoint of BC, prove:

$AB \cdot CD \sin(\angle ADC - \angle BAM) = AC \cdot BD \sin(\angle ADB - \angle MAC)$.

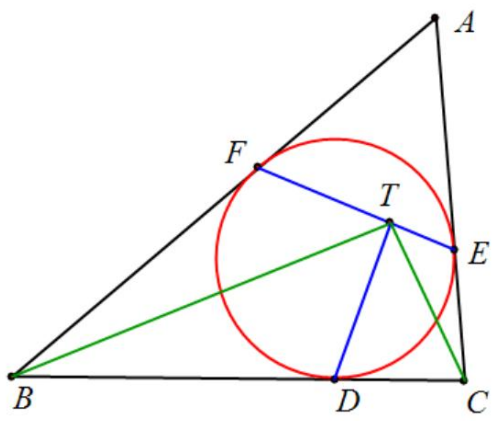


Example 69: As shown in Figure 1, in $\triangle ABC$, $AB = AC$, $AB \perp AC$, take $BD = AE = \frac{AB}{3}$ on AB and AC, prove that: $\angle ADE = \angle EBC$.

Proof: Let BC be the real axis, point A is on the imaginary axis, $d = \frac{a+2b}{3}$, $e = \frac{2a+b}{3}$,

$$\frac{\frac{a-e}{a-b}+\overline{\left(\frac{a-e}{a-b}\right)}}{\frac{a-e}{a-b}-\overline{\left(\frac{a-e}{a-b}\right)}} = \frac{\frac{d-a}{d-e}+\overline{\left(\frac{d-a}{d-e}\right)}}{\frac{d-a}{d-e}-\overline{\left(\frac{d-a}{d-e}\right)}} - \frac{\frac{b-e}{b-c}+\overline{\left(\frac{b-e}{b-c}\right)}}{\frac{b-e}{b-c}-\overline{\left(\frac{b-e}{b-c}\right)}},$$

$\triangle ABC$, $AB = AC$. On AB and AC, take $BD = AE = \frac{AB}{3}$. Prove: $\cot\angle A = \cot\angle ADE - \cot\angle CBE$.



Example 7 0 : The incircle of △ABC cuts BC , CA, AB at D, E, and F respectively, DT⊥EF at T, prove that: ∠BTD = ∠CTD.

Proof: Let the inscribed circle be the unit circle, $\bar{d}=\frac{1}{d}$, $\bar{e}=\frac{1}{e}$, $\bar{f}=\frac{1}{f}$, $a=\frac{2ef}{e+f}$,

$b=\frac{2df}{d+f}$, $c=\frac{2de}{d+e}$,

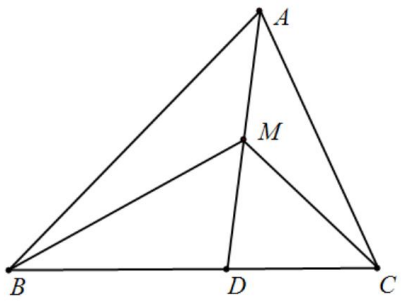
Since E, T, and F are collinear, $\begin{vmatrix} e & \bar{e} & 1 \\ f & \bar{f} & 1 \\ t & \bar{t} & 1 \end{vmatrix}=0$ the solution is $\bar{t}=\frac{e+f-t}{ef}$:

Verify $$2\frac{\frac{f-e}{t-d}+\overline{\left(\frac{f-e}{t-d}\right)}}{\frac{f-e}{t-d}-\overline{\left(\frac{f-e}{t-d}\right)}}=\frac{\frac{t-c}{t-d}+\overline{\left(\frac{t-d}{t-b}\right)}}{\frac{t-c}{t-d}-\overline{\left(\frac{t-d}{t-b}\right)}}-\frac{\frac{t-d}{t-b}+\overline{\left(\frac{t-d}{t-b}\right)}}{\frac{t-d}{t-b}-\overline{\left(\frac{t-d}{t-b}\right)}}$$, therefore

$2\cot\angle DTE=\cot\angle DTC-\cot\angle BTD$.

Extension: The incircle of △ABC cuts BC, CA, AB at D, E, F respectively. T is a point on line segment EF. Prove that: $2\cot\angle DTE=\cot\angle DTC-\cot\angle BTD$.

Example 71 : In triangle ABC, D is a point on BC, M is the midpoint of AD, and MD bisects $\angle$ BMC . If $\angle$ BAC=90 °, prove that: $\angle$ ADB=2 $\angle$ BAD .



Proof: Let $d=0$, $c=1$, $a=2m$, solve the equation $\dfrac{\frac{m-c}{m-d}}{\frac{m-d}{m-b}}=\overline{\left(\dfrac{\frac{m-c}{m-d}}{\frac{m-d}{m-b}}\right)}$ and get

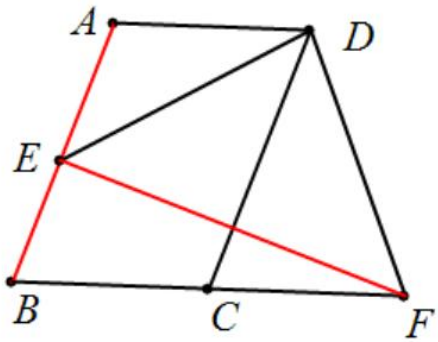
$b=\dfrac{m\overline{m}}{m+\overline{m}-m\overline{m}}$.

$$2\frac{\frac{a-c}{a-b}+\overline{\left(\frac{a-c}{a-b}\right)}}{\frac{a-c}{a-b}-\overline{\left(\frac{a-c}{a-b}\right)}}\frac{\frac{m-c}{m-d}+\overline{\left(\frac{m-c}{m-d}\right)}}{\frac{m-c}{m-d}-\overline{\left(\frac{m-c}{m-d}\right)}}=\frac{\frac{a-d}{a-b}+\overline{\left(\frac{a-d}{a-b}\right)}}{\frac{a-d}{a-b}-\overline{\left(\frac{a-d}{a-b}\right)}}\left(\frac{\left(\frac{a-d}{a-b}\right)^2+\overline{\left(\frac{a-d}{a-b}\right)^2}}{\left(\frac{a-d}{a-b}\right)^2-\overline{\left(\frac{a-d}{a-b}\right)^2}}-\frac{\frac{d-b}{d-a}+\overline{\left(\frac{d-b}{d-a}\right)}}{\frac{d-b}{d-a}-\overline{\left(\frac{d-b}{d-a}\right)}}\right)$$

$\triangle$ ABC, D is a point on BC, M is the midpoint of AD, and MD bisects $\angle$ BMC .

Prove that: $2\cot\angle DMC\cot\angle BAC=\cot\angle BAD\left(\cot\angle 2BAD-\cot\angle ADB\right)$.

Example 72 : In a parallelogram ABCD, E is a point on AB. Extend BC to F so that BC = CF. If FE $\perp$ AB, prove that: $\angle$ E DC = $\angle$ FDC .

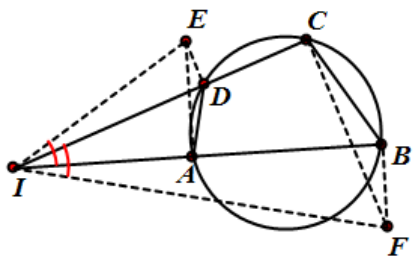


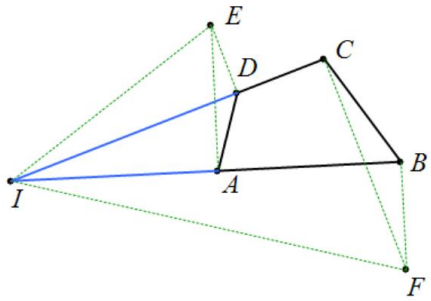
Proof: Assume $b=0$, $c=1$, $f=2$, $a=b+d-c$, $\bar{e}=\dfrac{e\bar{a}}{a}$,

$$\frac{\dfrac{d-f}{d-c}+\overline{\left(\dfrac{d-f}{d-c}\right)}}{\dfrac{d-f}{d-c}-\overline{\left(\dfrac{d-f}{d-c}\right)}}-\frac{\dfrac{d-c}{d-e}+\overline{\left(\dfrac{d-c}{d-e}\right)}}{\dfrac{d-c}{d-e}-\overline{\left(\dfrac{d-c}{d-e}\right)}}=2\frac{\dfrac{e-f}{a-b}+\overline{\left(\dfrac{e-f}{a-b}\right)}}{\dfrac{e-f}{a-b}-\overline{\left(\dfrac{e-f}{a-b}\right)}}$$

In a parallelogram ABCD, E is a point on AB. Extend BC to F so that BC = CF. Prove: $\cot\angle CDF-\cot\angle CDE=2\cot\angle BEF$.

Example 73 : A quadrilateral $ABCD$ is inscribed in a circle, $I = AB \cap CD$, and the straight lines passing through A, D and perpendicular to AB, AD intersect at E, and the straight lines passing through B, C and perpendicular to CD, CB intersect at point F, then $\angle EIA = \angle FIC$.





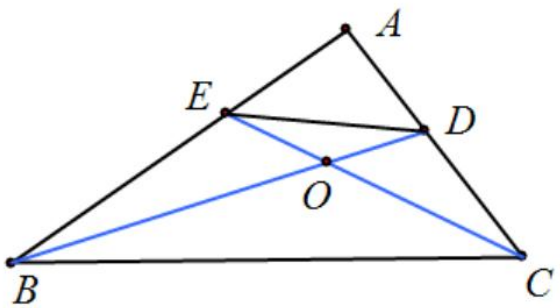
Proof: Let IA be the real axis, $i = 0$, $a = 1$, $\bar{b} = b$, $\bar{d} = \frac{d\bar{c}}{c}$, , $e = \frac{2\left(c - \bar{c}d\right)}{c - \bar{c}}$,

$$f = \frac{2c\left(b - \bar{c}\right)}{c - \bar{c}}, \quad \left([ACD] - [BCD]\right)^2 [ABF][CDE] \left(\frac{\frac{i-e}{i-d} + \overline{\left(\frac{i-e}{i-d}\right)}}{\frac{i-e}{i-d} - \overline{\left(\frac{i-e}{i-d}\right)}} - \frac{\frac{i-a}{i-f} + \overline{\left(\frac{i-a}{i-f}\right)}}{\frac{i-a}{i-f} - \overline{\left(\frac{i-a}{i-f}\right)}} \right)$$

$$= 8|a-b|^4 |c-d|^4 \left(\frac{a-d}{a-b}\frac{c-b}{c-d} - \overline{\left(\frac{a-d}{a-b}\frac{c-b}{c-d}\right)} \right),$$

Quadrilateral $ABCD$, $I = AB \cap CD$, respectively through A, D and perpendicular to AB, AD the straight line intersect at E, respectively through B, C and perpendicular to CD, CB the straight line intersect at point F, then

$$4\left(S_{\triangle ACD} - S_{\triangle BCD}\right)^2 BF \cdot DE \left(\cot \angle FIA - \cot \angle DIE\right) = AB^2 \cdot BC \cdot CD^2 \cdot DA \sin\left(A + C\right) .$$



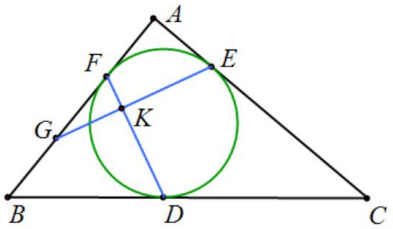
Example 74 : △ABC, the angle bisectors BC and CE of ∠B and ∠C intersect at O. If ∠A = 90 °, prove: $2S_{\triangle BCO} = S_{BCDE}$.

Proof: Assume $o = 0$ that $b = 1$, solve the equation $\frac{e}{c} = \overline{\left(\frac{e}{c}\right)}$, $\frac{\frac{b-e}{b-o}}{\frac{b-o}{b-c}} = \overline{\left(\frac{\frac{b-e}{b-o}}{\frac{b-o}{b-c}}\right)}$, and

we get $e = \frac{c}{c+\bar{c}-1}$, the same thing $d = \frac{c\bar{c}}{c+\bar{c}-c\bar{c}}$. $2[BCO] - S[BCE] - S[CDE]$

$$= -\frac{c+\bar{c}}{c\bar{c}}[CDO][BOE]\frac{\frac{d-c}{e-b} + \overline{\left(\frac{d-c}{e-b}\right)}}{\frac{d-c}{e-b} - \overline{\left(\frac{d-c}{e-b}\right)}}.$$

The angle bisectors BC and CE of △ABC, ∠B and ∠C intersect at O. Prove that: $2S_{\triangle BCO} - S_{BCDE} = 2S_{\triangle ODE}\sin(2\angle BOC)\cot A$.



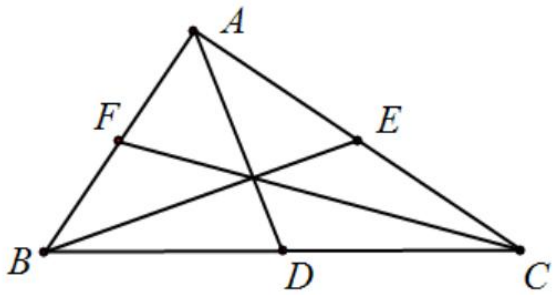
Example 75 : As shown in the figure, △ABC has an incircle that cuts BC, CA, and AB at points D, E, and F respectively. Take point G on AB, satisfying BG = FA, and DF intersects GE at K. If ∠A = 90 °, prove that: ∠DKE = 90 °.

Proof: Let the inscribed circle be the unit circle, $a=\frac{2ef}{e+f}$, $b=\frac{2df}{d+f}$, $g=a+b-f$,

there is an identity $\frac{\frac{a-e}{a-f}+\overline{\left(\frac{a-e}{a-f}\right)}}{\frac{a-e}{a-f}-\overline{\left(\frac{a-e}{a-f}\right)}}+\frac{[GDF]+[DEF]}{[DEF]}\frac{\frac{d-f}{g-e}+\overline{\left(\frac{d-f}{g-e}\right)}}{\frac{d-f}{g-e}-\overline{\left(\frac{d-f}{g-e}\right)}}=0$,

As shown in the figure, △ABC has an incircle that cuts BC, CA, and AB at points D, E, and F respectively. Take point G on AB that satisfies BG = FA. DF intersects GE at K. Prove: $\cot A=\frac{S_{GDEF}}{S_{\triangle DEF}}\cot\angle DKE$.

Example 76 : *In* triangle ABC , D , E , and F are the midpoints of sides BC , CA , and AB , respectively . Prove that $\angle\ DAC = \angle\ ABE$ if and only if $\angle\ AFC = \angle\ ADB$.
1995 British Mathematical Olympiad exam question
Proof: Let G be the centroid of △ABC, and let $\angle DAC = \angle ABE = \angle FDG \Leftrightarrow$ B, D, G, and F be cocircular $\Leftrightarrow \angle AFC = \angle ADB$.
The above is just for a special triangle.



Proof: Assume $b=0$, $c=1$, $d=\frac{c}{2}$, $e=\frac{a+c}{2}$, $f=\frac{a}{2}$,

$$\frac{\frac{b-a}{b-e}+\overline{\left(\frac{b-a}{b-e}\right)}}{\frac{b-a}{b-e}-\overline{\left(\frac{b-a}{b-e}\right)}}-\frac{\frac{a-c}{a-d}+\overline{\left(\frac{a-c}{a-d}\right)}}{\frac{a-c}{a-d}-\overline{\left(\frac{a-c}{a-d}\right)}}-2\left(\frac{\frac{b-a}{f-c}+\overline{\left(\frac{b-a}{f-c}\right)}}{\frac{b-a}{f-c}-\overline{\left(\frac{b-a}{f-c}\right)}}-\frac{\frac{c-b}{d-a}+\overline{\left(\frac{c-b}{d-a}\right)}}{\frac{c-b}{d-a}-\overline{\left(\frac{c-b}{d-a}\right)}}\right),$$

In triangle ABC , D , E , and F *are* the midpoints of sides BC , CA , and AB , respectively .
Prove that:

$$\cot\angle ABE-\cot\angle CAD=2\left(\cot\angle AFC-\cot\angle ADB\right),$$

Example 77 : As shown in Figure 26, in square ABCD, E and F are the midpoints of DC and EC respectively. Prove that: 2∠DAE=∠FAB.

Proof: Assume $A=0$, $B=1$, $D=\mathrm{i}$, $\dfrac{\frac{3}{4}+\mathrm{i}}{1}=\dfrac{25}{16}\left(\dfrac{\mathrm{i}}{\frac{1}{2}+\mathrm{i}}\right)^2$.

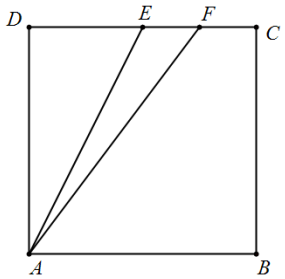


Figure 26

Proof: Assume $a=0$ that $b=1$, $c=b+d$, $\overline{d}=\dfrac{1}{d}$, $e=\dfrac{c+d}{2}$, $f=\dfrac{c+e}{2}$,

$$-\frac{4}{3}(a-d)\overline{(a-d)}\left(\frac{a-e}{a-d}+\overline{\left(\frac{a-e}{a-d}\right)}\right)\left(\frac{\left(\frac{a-d}{a-e}\right)^2+\overline{\left(\frac{a-e}{a-d}\right)}^2}{\left(\frac{a-d}{a-e}\right)^2-\overline{\left(\frac{a-e}{a-d}\right)}^2}-\frac{\frac{a-f}{a-b}+\overline{\left(\frac{a-f}{a-b}\right)}}{\frac{a-f}{a-b}-\overline{\left(\frac{a-f}{a-b}\right)}}\right)$$

$$=\frac{\frac{a-d}{a-b}+\overline{\left(\frac{a-d}{a-b}\right)}}{\frac{a-d}{a-b}-\overline{\left(\frac{a-d}{a-b}\right)}}\text{,but }\frac{8}{3}\cos\angle DAE\left(\cot\angle BAF-\cot\angle 2DAE\right)=\frac{AD}{AE}\cot\angle DAB,$$

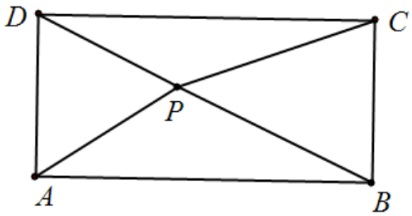
$$4(a-e)^2\overline{(a-e)}^2(a-f)\overline{(a-f)}\left(\frac{\left(\frac{a-d}{a-e}\right)^2}{\frac{a-f}{a-b}}-\frac{\overline{\left(\frac{a-d}{a-e}\right)}^2}{\overline{\left(\frac{a-f}{a-b}\right)}}\right)$$

$$=[ADF](a-b)\overline{(a-b)}\left(\frac{a-d}{a-b}+\overline{\left(\frac{a-d}{a-b}\right)}\right)\text{,but}$$

$$16AE^2\cdot AF\sin\left(2\angle DAE-\angle BAF\right)=3AB^3\sin 2\angle DAB.$$

As shown in Figure 26, in rhombus ABCD, E and F are the midpoints of DC and EC respectively. Prove that: $\dfrac{8}{3}\cos\angle DAE\left(\cot\angle BAF-\cot\angle 2DAE\right)=\dfrac{AD}{AE}\cot\angle DAB$, $16AE^2\cdot AF\sin\left(2\angle DAE-\angle BAF\right)=3AB^3\sin 2\angle DAB$.

Example 78 : In rectangle ABCD , there is a point P that satisfies ∠APD + ∠BPC = 180°. Prove that ∠DAP+∠BCP=90° .

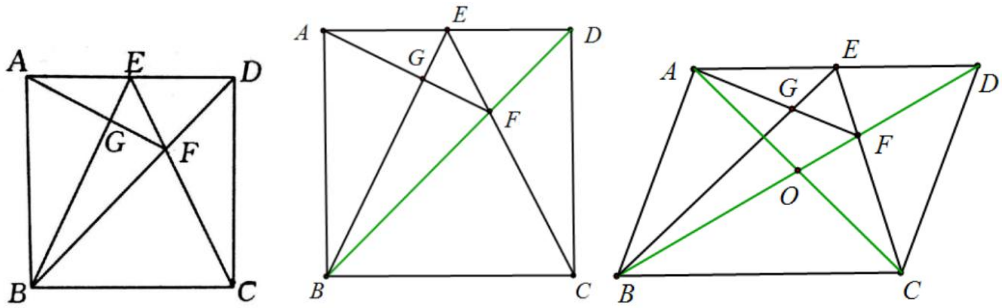


Proof: Let $a=0$, $b=1$, find the identity:

$$\frac{d(p-b)(p-d)\overline{(p-b)(p-d)}\left(\frac{p-a}{p-d}\frac{p-c}{p-b}-\overline{\left(\frac{p-a}{p-d}\frac{p-c}{p-b}\right)}\right)}{b(a-p)(c-p)\overline{(a-p)(c-p)}\left(\frac{a-d}{a-p}\frac{c-b}{c-p}+\overline{\left(\frac{p-a}{p-d}\frac{p-c}{p-b}\right)}\right)}=1,$$

Extended proposition: There is a point P inside rectangle ABCD . Prove $PB\cdot PD\sin(\angle APD+\angle BPC)=-AB\cdot CB\cos(\angle DAP+\angle BCP)$.

Example 79 : In square ABCD, E is the midpoint of side AD, BD and CE intersect at point F. Prove that : AF⊥BE.



Proof of the original question:

$\angle GAE+\angle GEA=\angle ECD+\angle GEA=\angle EBA+\angle GEA=90^{\circ}$.

rhombus ABCD, E is the midpoint of side AD, and BD and CE intersect at point F. Prove that : $3\cot\angle ABC=5\cot\angle EGF$.

Proof: Let $b=0$, $c=1$, $d=a+c$, $e=\dfrac{a+d}{2}$, $f=\dfrac{2d}{3}$, $\bar{a}=\dfrac{1}{a}$,

$$3\frac{\dfrac{b-a}{b-c}+\overline{\left(\dfrac{b-a}{b-c}\right)}}{\dfrac{b-a}{b-c}-\overline{\left(\dfrac{b-a}{b-c}\right)}}=5\frac{\dfrac{b-e}{a-f}+\overline{\left(\dfrac{b-e}{a-f}\right)}}{\dfrac{b-e}{a-f}-\overline{\left(\dfrac{b-e}{a-f}\right)}}.$$

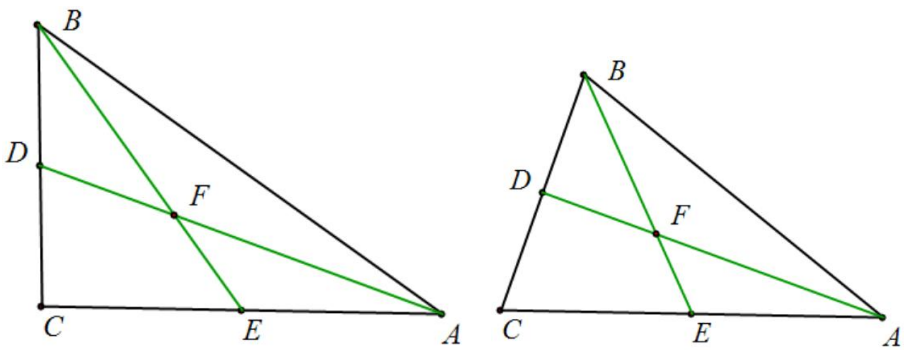
Step-by-step derivation

parallelogram ABCD, E is the midpoint of side AD, BD and CE intersect at F, and AC intersects BD at O. Prove that: $3\cot\angle ABC+4\cot\angle DOC=5\cot\angle EGF$.

Proof: Let $b=0$, $c=1$, $d=a+c$, $e=\dfrac{a+d}{2}$, $f=\dfrac{2d}{3}$,

$$3\frac{\dfrac{b-a}{b-c}+\overline{\left(\dfrac{b-a}{b-c}\right)}}{\dfrac{b-a}{b-c}-\overline{\left(\dfrac{b-a}{b-c}\right)}}+4\frac{\dfrac{b-d}{a-c}+\overline{\left(\dfrac{b-d}{a-c}\right)}}{\dfrac{b-d}{a-c}-\overline{\left(\dfrac{b-d}{a-c}\right)}}=5\frac{\dfrac{b-e}{a-f}+\overline{\left(\dfrac{b-e}{a-f}\right)}}{\dfrac{b-e}{a-f}-\overline{\left(\dfrac{b-e}{a-f}\right)}}.$$

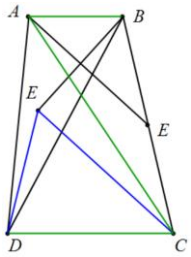
Example 80 : In triangle ABC, ∠ C = 90°, and the midlines AD and BE intersect at F. Prove $S_{\triangle ABC}=\frac{AB^2}{3}\tan\angle EFA$.



Proof: Assume $c=0$ that $\bar{a}=a$, $e=\frac{a}{2}$, $d=\frac{b}{2}$,

$$3[ABC]+4(a-b)\overline{(a-b)}\frac{\frac{a-d}{e-b}-\overline{\left(\frac{a-d}{e-b}\right)}}{\frac{a-d}{e-b}+\overline{\left(\frac{a-d}{e-b}\right)}}+[ABC]\frac{\frac{b-e}{a-d}-\overline{\left(\frac{a-d}{e-b}\right)}}{\frac{b-e}{a-d}+\overline{\left(\frac{a-d}{e-b}\right)}}\frac{\frac{c-b}{c-a}+\overline{\left(\frac{a-d}{e-b}\right)}}{\frac{c-b}{c-a}-\overline{\left(\frac{a-d}{e-b}\right)}}=0$$

Extended proposition: In △ **ABC, the medians AD and BE intersect at F. Prove it** $AB^2\tan\angle EFA-3S_{\triangle ABC}=(\cot\angle ACB)(\tan\angle EFA)S_{\triangle ABC}$.



Example 81 : In trapezoid ABCD , M is the midpoint of BC, and B is the symmetry of E about line AM. If AC = AB + CD, prove that ∠CDB = ∠DEC .

Proof: Let $b=1$, $e=-1$, $\overline{a}=-a$, $\overline{m}=-m$, $c=2m-b$, solve the equation

$\frac{a-b}{d-c}=\overline{\left(\frac{a-b}{d-c}\right)}$ to get $\overline{d}=\frac{4m-2a-d-ad}{a-1}$,

$$\frac{[ADC]}{2}\left(\frac{\frac{d-b}{d-c}+\overline{\left(\frac{d-b}{d-c}\right)}}{\frac{d-b}{d-c}-\overline{\left(\frac{d-b}{d-c}\right)}}-\frac{\frac{e-c}{e-d}+\overline{\left(\frac{e-c}{e-d}\right)}}{\frac{e-c}{e-d}-\overline{\left(\frac{e-c}{e-d}\right)}}\right)=(a-c)\overline{(a-c)}-(b-a+c-d)\overline{(b-a+c-d)}$$

Extended proposition: In trapezoid ABCD , M is the midpoint of BC, and B is the symmetry of E about line AM. Prove it

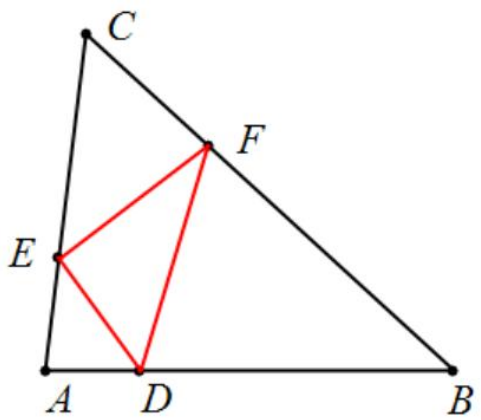
$2S_{\triangle ADC}\left(\cot\angle CDB-\cot\angle DEC\right)=AC^2-\left(BA+CD\right)^2$.

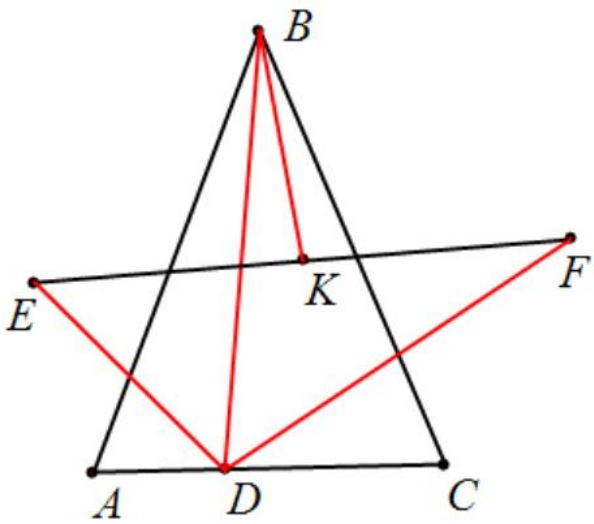
Example 82 : **As shown in the figure, points D, E, and F are on sides AB, AC, and BC of triangle ABC, respectively . BF = 2 CF, CE = 2 AE. If ∠DEF = 90 ° , prove that ∠ADE = ∠EDF . (2005 St. Petersburg Mathematical Olympiad)**

Proof: Let the circumscribed circle of △ABC be the unit circle, $\overline{d}=\dfrac{a+b-d}{ab}$,

$$e=\frac{2a+c}{3},\quad f=\frac{2c+b}{3},,\quad \frac{\dfrac{d-e}{d-a}+\overline{\left(\dfrac{d-e}{d-a}\right)}}{\dfrac{d-e}{d-a}-\overline{\left(\dfrac{d-e}{d-a}\right)}}-\frac{\dfrac{d-f}{d-e}+\overline{\left(\dfrac{d-f}{d-e}\right)}}{\dfrac{d-f}{d-e}-\overline{\left(\dfrac{d-f}{d-e}\right)}}=2\frac{\dfrac{e-d}{e-f}+\overline{\left(\dfrac{e-d}{e-f}\right)}}{\dfrac{e-d}{e-f}-\overline{\left(\dfrac{e-d}{e-f}\right)}}.$$

As shown in the figure, points D, E, and F are on sides AB, AC, and BC of △ABC respectively . BF = 2 CF, CE = 2 AE. Prove $\cot\angle ADE-\cot\angle EDF=2\cot\angle FED$.





Example 83 : In the figure, in triangle ABC, BA = BC, point D is on line segment AC, the midpoint of DE is on BA, the midpoint of DF is on BC, and K is the midpoint of EF and is inside triangle ABC $\angle EDA = \angle FDC$ **. If , prove ...** $\angle ABD = \angle KBC$ **(2000 St. Petersburg Mathematical Olympiad)**

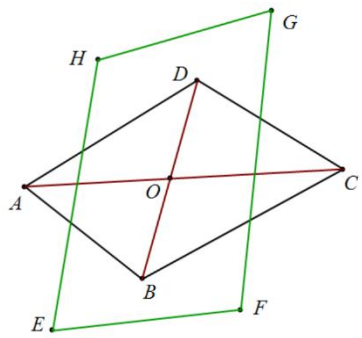
Proof: Assume $c=1$, $a=-1$, $\overline{d}=d$, $\overline{b}=-b$, According to **the midpoint of DE is on BA, we can get** $\overline{e}=\dfrac{-4b-2bd+e-be}{1+b}$, Similarly

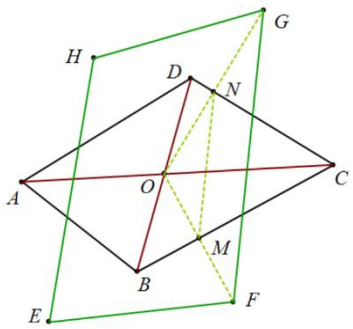
$\overline{f}=\dfrac{-4b+2bd+f+bf}{1-b}$, $k=\dfrac{e+f}{2}$,

$$(d-e)(d-f)\overline{(d-e)}\overline{(d-f)}\left(\frac{\frac{d-a}{d-e}}{\frac{d-f}{d-c}}-\overline{\left(\frac{\frac{d-a}{d-e}}{\frac{d-f}{d-c}}\right)}\right)$$

$$=\frac{4(d-a)(d-c)}{(b-a)(b-c)}(b-a)(b-c)\overline{(b-a)}\overline{(b-c)}\left(\frac{\frac{b-d}{b-a}}{\frac{b-c}{b-k}}-\overline{\left(\frac{\frac{b-d}{b-a}}{\frac{b-c}{b-k}}\right)}\right).$$

As shown in the figure, in △ABC, BA = BC, point D is on line segment AC, the midpoint of DE is on BA, the midpoint of DF is on BC, K is the midpoint of EF, and is inside △ABC. Prove it $DE\cdot DF\sin(\angle EDA-\angle FDC)=4BD\cdot BK\sin(\angle ABD-\angle KBC)$.





Example 84 : Convex quadrilateral ABCD, diagonal AC intersects BD at O, the symmetric points of point O with respect to AB, BC, CD, and DA are E, F, G, and H. Prove that the necessary and sufficient condition for AC⊥BD is that the four points E, F, G, and H are cocircular.

Proof: Suppose $o=0$, $\bar{a}=a$, $c=1$, $\bar{d}=d\bar{b}/b$, $e=\dfrac{a\left(b-\bar{b}\right)}{\left(a-\bar{b}\right)}$, $f=\dfrac{b-\bar{b}}{\left(1-\bar{b}\right)}$,

$$g=\frac{\left(b-\bar{b}\right)d}{\left(b-\bar{b}d\right)}\ ,\quad h=\frac{a\left(b-\bar{b}\right)d}{\left(ab-\bar{b}d\right)}\ ,\quad \frac{\left(\dfrac{o-c}{o-b}\right)^2-\overline{\left(\dfrac{o-c}{o-b}\right)}^2}{\left(\dfrac{o-c}{o-b}\right)^2+\overline{\left(\dfrac{o-c}{o-b}\right)}^2}=\frac{\dfrac{f-e}{f-g}\dfrac{h-g}{h-e}-\overline{\left(\dfrac{f-e}{f-g}\dfrac{h-g}{h-e}\right)}}{\dfrac{f-e}{f-g}\dfrac{h-g}{h-e}+\overline{\left(\dfrac{f-e}{f-g}\dfrac{h-g}{h-e}\right)}}\ ,$$

that is $\tan\left(\angle EFG+\angle EHG\right)=\tan\left(2\angle BOC\right)$. Combining the geometric properties, we have $\angle EFG+\angle EHG=2\angle BOC$.

Proof: Let OM⊥BC at M, ON⊥CD at N, then O, M, C, and N are concentric; since MN is the median of △OFG, then FG∥MN;

$\angle OFG=\angle OMN=\angle OCD$ Similarly $\angle OFE=\angle OBA$, $\angle OHG=\angle ODC$, $\angle OHE=\angle OAB$,

$\angle EFG+\angle EHG=\angle OFG+\angle OFE+\angle OHG+\angle OHE$

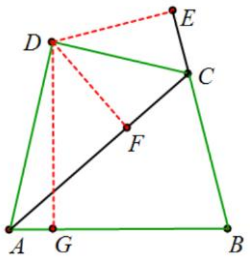
$=\angle OCD+\angle OBA+\angle ODC+\angle OAB=\left(\angle OCD+\angle ODC\right)+\left(\angle OBA+\angle OAB\right)$

$=\left(180^\circ-\angle COD\right)+\left(180^\circ-\angle AOB\right)=2\angle BOC$,

So $\angle EFG+\angle EHG=2\angle BOC$, therefore $\angle EFG+\angle EHG=180^\circ \Leftrightarrow \angle BOC=90^\circ$.

problem

Example 1 : Simson's Theorem



Solve the equation $\frac{c-e}{c-b}=\overline{\left(\frac{c-e}{c-b}\right)}$ and $\frac{d-e}{c-b}=-\overline{\left(\frac{d-e}{c-b}\right)}$ get

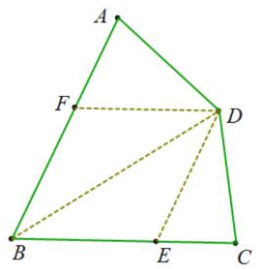
$\left\{e,\bar{e}\right\}=\left\{\frac{\bar{b}c-b\bar{c}+\bar{b}d-\bar{c}d+b\bar{d}-c\bar{d}}{2\left(\bar{b}-\bar{c}\right)},\frac{-\bar{b}c+b\bar{c}+\bar{b}d-\bar{c}d+b\bar{d}-c\bar{d}}{2\left(b-c\right)}\right\}$, and similarly we get $\left\{f,\bar{f}\right\}$, $\left\{g,\bar{g}\right\}$,

$$[A,B,C]^2(b-c)\overline{(b-c)}(a-d)\overline{(a-d)}\left(\frac{d-c}{d-a}\frac{b-a}{b-c}-\overline{\left(\frac{d-c}{d-a}\frac{b-a}{b-c}\right)}\right)$$

$$=-4(a-b)\overline{(a-b)}(b-c)\overline{(b-c)}(c-a)\overline{(c-a)}[E,F,G].$$

Extended proposition: Convex quadrilateral $ABCD$, through D draw $DE\perp BC$, $DF\perp CA$, $DG\perp AB$, R is the radius of the circumscribed circle of $\triangle ABC$, prove:

$$AB\cdot BC\cdot CD\cdot DA\sin\left(\angle ADC+\angle ABC\right)=8R^2\cdot S_{\triangle EFG}.$$



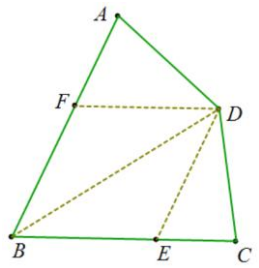
Example 2 : Convex quadrilateral $ABCD$, take points E and F *on* BC and BA to form parallelogram $BEDF$. Prove that the necessary and sufficient condition for points A , B , C , and D to be cocircular is $BD^2 = BC \cdot BE + BA \cdot BF$.

Proof: Let B be the origin and BC be the real axis, then $\overline{f} = \dfrac{\overline{a} f}{a}$, $d = (e + f)$,

$$(a-b)\overline{(a-b)}(c-d)\overline{(c-d)}\left(\frac{a-d}{a-b}\frac{c-b}{c-d} - \overline{\left(\frac{a-d}{a-b}\frac{c-b}{c-d}\right)}\right) = -[A,B,C]\left(ec + f\overline{a} - d\overline{d}\right).$$

Extended proposition: Convex quadrilateral $ABCD$, take points E and F *on* BC and BA to form parallelogram $BEDF$, prove:

$$AB \cdot BC \cdot CD \cdot DA \sin(A + C) = 2S_{\triangle ABC}\left(BC \cdot BE + BA \cdot BF - BD^2\right).$$



Example 3 : Convex quadrilateral $ABCD$, take points E and F *on* BC and BA to form parallelogram $BEDF$. Prove that the necessary and sufficient condition for points A , B , C , and D to be cocircular is $\cos B = \frac{1}{2}\left(\frac{EC}{BF} + \frac{FA}{BE}\right)$.

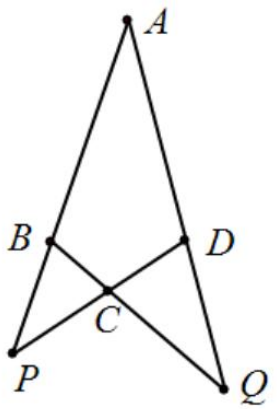
Proof: $\cos B = \frac{1}{2}\left(\frac{EC}{BF} + \frac{FA}{BE}\right)$ Equivalent to $2BE \cdot BF \cdot \cos B = BE \cdot EC + BF \cdot FA$,

Let B be the origin and BC be the real axis, then $\overline{f} = \frac{\overline{a} f}{a}$, $d = (e + f)$,

$$(a-b)\overline{(a-b)}(c-d)\overline{(c-d)}\left(\frac{a-d}{a-b}\frac{c-b}{c-d} - \overline{\left(\frac{a-d}{a-b}\frac{c-b}{c-d}\right)}\right)$$

$$= -[A,B,C]\left(e(c-e) + f\overline{(a-f)} - ee\left(\frac{b-f}{b-e} + \overline{\left(\frac{b-f}{b-e}\right)}\right)\right).$$

Extended proposition: Convex quadrilateral $ABCD$, take points E and F *on* BC and BA to form parallelogram $BEDF$, prove: $AB \cdot BC \cdot CD \cdot DA \sin(A + C) = 2S_{\triangle ABC}(BE \cdot EC + BF \cdot FA - 2BE \cdot BF \cdot \cos B)$.



Example 4 : Convex quadrilateral $ABCD$, AB intersects CD at P , BC intersects AD at Q. Prove that the necessary and sufficient condition for points A , B , C , and D *to be cocircular is* . $BD^2 = AB^2 \cdot AP \cdot \frac{DQ}{AQ} + AD^2 \cdot \frac{BP}{AP}$

prove: $S_1 = |b-d|^2 (a-p)\overline{(a-q)} - |a-b|^2 (a-p)\overline{(d-q)} - |a-d|^2 (b-p)\overline{(a-q)}$,

$S_2 = (a-b)(c-d)\overline{(a-b)(c-d)}\left(\frac{a-d}{a-b}\frac{c-b}{c-d} - \overline{\left(\frac{a-d}{a-b}\frac{c-b}{c-d}\right)}\right)$,

$([ABC]-[DBC])([BCD]-[ACD])S_1 = -[ABD](a-b)\overline{(a-d)}S_2$,

Extended proposition: Convex quadrilateral $ABCD$, take points E and F *on* BC and BA to form parallelogram $BEDF$. Prove that:

$$(S_{\triangle ABC} - S_{\triangle DBC})(S_{\triangle BCD} - S_{\triangle ACD})(BD^2 \cdot AP \cdot AQ - AB^2 \cdot AP \cdot DQ - AD^2 \cdot BP \cdot AQ)$$

$$= AB^3 \cdot AD^3 \cdot BC \cdot CD \sin A \sin(A+C)$$

Example 5 : Ptolemy's theorem $AC\cdot BD=AB\cdot CD+BC\cdot AD$.
Considering that it is inconvenient to use complex numbers to represent line segments, we will $AB\cdot CD+BC\cdot AD-AC\cdot BD=0$ find a dual formula $AB\cdot CD+BC\cdot AD+AC\cdot BD=0$ and multiply the two formulas to get $AB^2\cdot CD^2+BC^2\cdot AD^2+2AB\cdot BC\cdot CD\cdot DA-AC^2\cdot BD^2=0$.

$$(a-b)(c-d)\overline{(a-b)(c-d)}+(b-c)(d-a)\overline{(b-c)(d-a)}$$

$$-2(a-b)(c-d)\overline{(a-b)(c-d)}\frac{\frac{a-d}{a-b}\frac{c-b}{c-d}+\overline{\left(\frac{a-d}{a-b}\frac{c-b}{c-d}\right)}}{2}-(a-c)(b-d)\overline{(a-c)(b-d)}=0$$

.

$$AB^2\cdot CD^2+BC^2\cdot AD^2-2AB\cdot BC\cdot CD\cdot DA\cos(A+C)-AC^2\cdot BD^2=0,$$

$A+C=\pi$ At that time, $\cos(A+C)=-1$, so there was $AC\cdot BD=AB\cdot CD+BC\cdot AD$.

$A+C=\frac{\pi}{2}$ At that time, $\cos(A+C)=0$, so there was $AB^2\cdot CD^2+BC^2\cdot AD^2=AC^2\cdot BD^2$.

$$AB^2\cdot CD^2+BC^2\cdot AD^2-2AB\cdot BC\cdot CD\cdot DA\left(2\cos^2\frac{A+C}{2}-1\right)-AC^2\cdot BD^2=0,$$

$$AB^2\cdot CD^2+BC^2\cdot AD^2+2AB\cdot BC\cdot CD\cdot DA-AC^2\cdot BD^2=4AB\cdot BC\cdot CD\cdot DA\cos^2\frac{A+C}{2}$$

$$(AB\cdot CD+BC\cdot AD-AC\cdot BD)(AB\cdot CD+BC\cdot AD+AC\cdot BD)=4AB\cdot BC\cdot CD\cdot DA\cos^2\frac{A+C}{2}$$

.

That is, at the beginning, in order to express the common circle information, it is used $\cos(A+C)=-1$, and after the identity equation is generated, it is re-understood to obtain a new proposition.

$$\frac{1}{2}(a-b)(a-b)\frac{\frac{a-d}{a-b}-\frac{a-\bar{d}}{a-b}}{2}+\frac{1}{2}(c-b)(\bar{c}-b)\frac{\frac{c-d}{c-b}-\frac{\bar{c}-\bar{d}}{\bar{c}-b}}{2}$$

$$=\frac{1}{4}\left(-bc+b\bar{c}-ad+2bd-\bar{c}d+a\bar{d}-2b\bar{d}+c\bar{d}\right),$$

$$\left((a-b)(a-b)+(b-c)(b-\bar{c})+(c-d)(\bar{c}-\bar{d})+(d-a)(\bar{d}-a)\right)^2$$

$$-8(a-b)(a-b)(c-b)(\bar{c}-b)\frac{\frac{a-d}{a-b}\frac{c-d}{c-b}+\frac{a-\bar{d}}{a-b}\frac{\bar{c}-\bar{d}}{\bar{c}-b}}{2}$$

$$-2\left((a-b)^2(a-b)^2+(b-c)^2(b-\bar{c})^2+(c-d)^2(\bar{c}-\bar{d})^2+(d-a)^2(\bar{d}-a)^2\right)$$

$$=-\left(bc-b\bar{c}+ad-2bd+\bar{c}d-a\bar{d}+2b\bar{d}-c\bar{d}\right)^2,$$

For any quadrilateral, $\left(AB^2+BC^2+CD^2+DA^2\right)^2-8AB\cdot BC\cdot CD\cdot DA\cos(A+C)$

$$-2\left(AB^4+BC^4+CD^4+DA^4\right)=16{S_{ABCD}}^2.$$

$$\left(AB^2+BC^2+CD^2+DA^2\right)^2-8AB\cdot BC\cdot CD\cdot DA\left(2\cos^2\frac{A+C}{2}-1\right)$$

$$-2\left(AB^4+BC^4+CD^4+DA^4\right)=16{S_{ABCD}}^2,$$

$$\left(AB^2+BC^2+CD^2+DA^2\right)^2+8AB\cdot BC\cdot CD\cdot DA$$

$$-2\left(AB^4+BC^4+CD^4+DA^4\right)-16AB\cdot BC\cdot CD\cdot DA\cos^2\frac{A+C}{2}=16{S_{ABCD}}^2,$$

$$S_{ABCD}=\sqrt{(-AB+BC+CD+DA)(AB-BC+CD+DA)(AB+BC-CD+DA)(AB+BC+CD-DA)-AB\cdot BC\cdot CD\cdot DA\cos^2\frac{A+C}{2}}$$

$$S_{ABCD}=\sqrt{\begin{array}{l}(-AB+BC+CD+DA)(AB-BC+CD+DA)(AB+BC-CD+DA)(AB+BC+CD-DA)\\-\frac{1}{4}(AB\cdot CD+BC\cdot AD-AC\cdot BD)(AB\cdot CD+BC\cdot AD+AC\cdot BD)\end{array}}$$

Example 6 : Let A , B , C , and D be four points on a circle, and O be any point. Prove that:

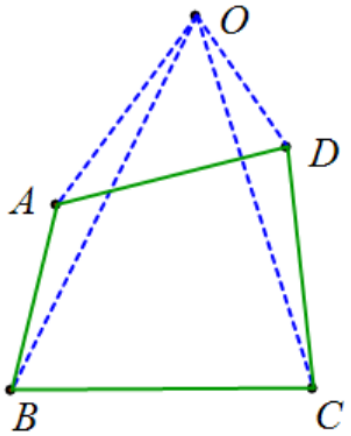
$$OA^2 \cdot S_{\triangle BCD} - OB^2 \cdot S_{\triangle CDA} + OC^2 \cdot S_{\triangle DAB} - OD^2 \cdot S_{\triangle ABC} = 0 .$$

Proof 1: With O as the origin, since points **A , B , C , and D are concentric,**

$$\begin{vmatrix} x_A^2 + y_A^2 & x_A & y_A & 1 \\ x_B^2 + y_B^2 & x_B & y_B & 1 \\ x_C^2 + y_C^2 & x_C & y_C & 1 \\ x_D^2 + y_D^2 & x_D & y_D & 1 \end{vmatrix} = 0$$ we can expand the first column and note $x_A^2 + y_A^2 = OA^2$

the determinant formula for the area. The proposition is proven.

The above proof is from Professor Shan Xuan's book "Techniques of Analytic Geometry." Mr. Shan commented: "This problem seems difficult, and indeed, it is very tricky without using analytic geometry."



,

Identity: $$a\bar{a}\begin{vmatrix} b & \bar{b} & 1 \\ c & \bar{c} & 1 \\ d & \bar{d} & 1 \end{vmatrix} - b\bar{b}\begin{vmatrix} c & \bar{c} & 1 \\ d & \bar{d} & 1 \\ a & \bar{a} & 1 \end{vmatrix} + c\bar{c}\begin{vmatrix} d & \bar{d} & 1 \\ a & \bar{a} & 1 \\ b & \bar{b} & 1 \end{vmatrix} - d\bar{d}\begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix}$$

$$+(a-b)\overline{(a-b)}(c-d)\overline{(c-d)}\left(\frac{a-d}{a-b}\frac{c-b}{c-d} - \overline{\frac{a-d}{a-b}\frac{c-b}{c-d}}\right) = 0 .$$

Extended Proposition: Given quadrilateral $ABCD$ and any point O , prove that:

$$OA^2 \cdot S_{\triangle BCD} - OB^2 \cdot S_{\triangle CDA} + OC^2 \cdot S_{\triangle DAB} - OD^2 \cdot S_{\triangle ABC}$$

$$= \frac{1}{2} AB \cdot BC \cdot CD \cdot DA \sin(\angle BAD + \angle BCD) .$$

Proof: Use the complex number method, using lowercase letters x to represent X the complex number of a point and using to represent $\bar{x}$ the conjugate complex number of the complex number . x

$$\frac{1}{2}AB\cdot BC\cdot CD\cdot DA\sin(\angle BAD+\angle BCD)$$

$$=\frac{1}{2}\left|(a-b)(b-c)(c-d)(d-a)\right|\frac{\frac{a-d}{a-b}\frac{c-b}{c-d}-\overline{\left(\frac{a-d}{a-b}\frac{c-b}{c-d}\right)}}{2\left|\frac{a-d}{a-b}\frac{c-b}{c-d}\right|\mathrm{i}}$$

$$=\frac{1}{4}\left|(a-b)\overline{(a-b)}(c-d)\overline{(c-d)}\right|\frac{\frac{a-d}{a-b}\frac{c-b}{c-d}-\overline{\left(\frac{a-d}{a-b}\frac{c-b}{c-d}\right)}}{\mathrm{i}}$$

$$=\frac{\mathrm{i}}{4}\left(\overline{(a-d)(c-b)}(a-b)(c-d)-\overline{(a-b)(c-d)}(a-d)(c-b)\right);$$

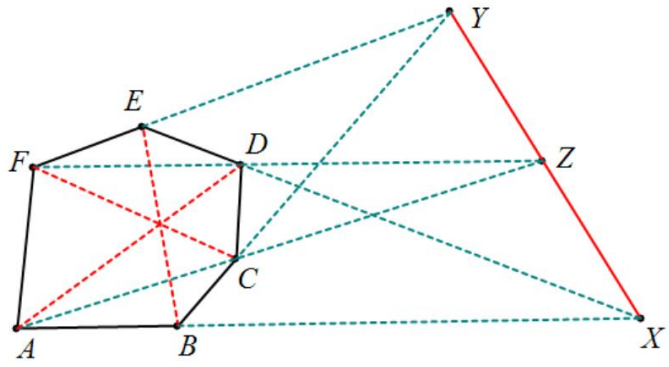
Let O be the origin, $OA^2\cdot S_{\triangle BCD}-OB^2\cdot S_{\triangle CDA}+OC^2\cdot S_{\triangle DAB}-OD^2\cdot S_{\triangle ABC}$

$$=\frac{\mathrm{i}}{4}\left(a\bar{a}\begin{vmatrix} b & \bar{b} & 1\\ c & \bar{c} & 1\\ d & \bar{d} & 1\end{vmatrix}-b\bar{b}\begin{vmatrix} c & \bar{c} & 1\\ d & \bar{d} & 1\\ a & \bar{a} & 1\end{vmatrix}+c\bar{c}\begin{vmatrix} d & \bar{d} & 1\\ a & \bar{a} & 1\\ b & \bar{b} & 1\end{vmatrix}-d\bar{d}\begin{vmatrix} a & \bar{a} & 1\\ b & \bar{b} & 1\\ c & \bar{c} & 1\end{vmatrix}\right)$$

$$=\frac{\mathrm{i}}{4}(a\bar{a}\left(b\bar{c}+d\bar{b}+c\bar{d}-d\bar{c}-b\bar{d}-c\bar{b}\right)-b\bar{b}\left(c\bar{d}+a\bar{c}+d\bar{a}-a\bar{d}-c\bar{a}-d\bar{c}\right)$$

$$+c\bar{c}\left(d\bar{a}+b\bar{d}+a\bar{b}-b\bar{a}-d\bar{b}-a\bar{d}\right)-d\bar{d}\left(a\bar{b}+b\bar{a}+b\bar{c}-c\bar{b}-a\bar{c}-b\bar{a}\right))$$

$$=\frac{\mathrm{i}}{4}\left(\overline{(a-d)(c-b)}(a-b)(c-d)-\overline{(a-b)(c-d)}(a-d)(c-b)\right);$$



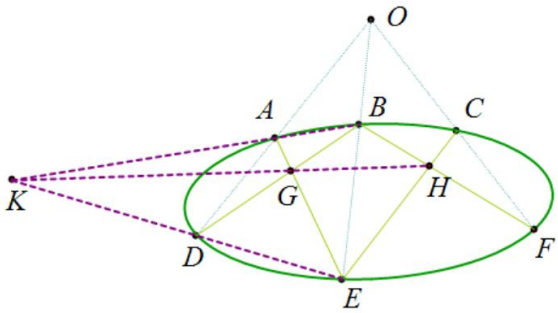
Example 7 : Hexagon $ABCDEF$, line AB intersects DE at X , line BC intersects EF at Y , and line AC intersects DF at Z. Prove that the necessary and sufficient condition for lines AD , BE , and CF *to intersect at a point is that points* X , Y , and Z are collinear.

Identity: $[ABC][DEF]([FAD][CBE]-[FBE][CAD])$

$$=([EAB]-[DAB])([FAC]-[DAC])([FBC]-[EBC])[XYZ].$$

Extended proposition: In hexagon $ABCDEF$, line AB intersects *line* DE at X , line BC intersects *line* EF at Y , and line AC intersects line DF at Z. Prove that: $[ABC][DEF]([FAD][CBE]-[FBE][CAD])$

$$=([EAB]-[DAB])([FAC]-[DAC])([FBC]-[EBC])[XYZ].$$



Example 8 : Given six points A , B , C , D , E , and F on a quadratic curve , straight lines DA , EB , and FC intersect at a point O , straight line AE intersects BD at G , and straight line CE intersects BF at H. Prove that straight lines AB , GH , and DE intersect at a point.

Extended proposition: Given six points A , B , C , D , E , and F , straight lines DA , EB , and FC intersect at point O , straight line AE intersects BD at G , and straight line CE intersects BF at H. Prove that:

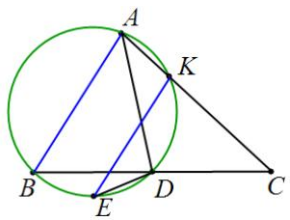
$$S_{\triangle ACF}{}^{2} \cdot S_{\triangle ABD}{}^{2} \cdot S_{\triangle BCE} \cdot S_{\triangle ACE} \left(S_{\triangle GBA} \cdot S_{\triangle HDE} - S_{\triangle HBA} \cdot S_{\triangle GDE} \right)$$

$$= S_{\triangle ABE} \cdot S_{\triangle ABG} \cdot S_{\triangle ACH} \cdot S_{\triangle ACD} \left(S_{\triangle ACD} \cdot S_{\triangle BCF} \cdot S_{\triangle AEF} \cdot S_{\triangle BDE} - S_{\triangle BEF} \cdot S_{\triangle ACF} \cdot S_{\triangle BCD} \cdot S_{\triangle ADE} \right).$$

The six-point conic section is equivalent to $S_{\triangle ACD} \cdot S_{\triangle BCF} \cdot S_{\triangle AEF} \cdot S_{\triangle BDE} = S_{\triangle BEF} \cdot S_{\triangle ACF} \cdot S_{\triangle BCD} \cdot S_{\triangle ADE}$.

Straight lines AB , GH , and DE intersect at one point, which is equivalent to $S_{\triangle GBA} \cdot S_{\triangle HDE} = S_{\triangle HBA} \cdot S_{\triangle GDE}$.

Straight lines DA , EB , and FC intersect at one point, which is equivalent to $S_{\triangle BAD} \cdot S_{\triangle ECF} = S_{\triangle BCF} \cdot S_{\triangle EAD}$.



Example 9 : As shown in the figure, D is the midpoint of side BC *in triangle* ABC , K is the intersection of AC and the circumcircle O of triangle ABD , and EK is parallel to AB and intersects circle O at E. If $AD = DE$, prove that $AB + AK = KC$. (2019 Jiangsu High School Mathematics Competition)

$c = 2d - b$, solve the equation $\frac{a-k}{a-c} = \overline{\left(\frac{a-k}{a-c}\right)}$, and get $k = \frac{bd(a+b-2d)}{2ab-ad-bd}$, $e = \frac{ab}{k}$,

$$(a-b)\overline{(a-b)} - (k-c-(a-k))\overline{(k-c-(a-k))}$$

$$= \frac{\left(\frac{a-b}{a-d}\right)^2}{\frac{b}{d}}\left((d-a)\overline{(d-a)} - (d-e)\overline{(d-e)}\right),$$

Extended Proposition: As shown in the figure, D is the midpoint of side BC *in triangle* ABC , K is the intersection of AC and the circumcircle O of triangle ABD , and EK is parallel to AB and intersects circle O at E. Prove:

$$AB^2 - (CK - KA)^2 = \left(\frac{AB}{AD}\right)^2 \cdot \left(AD^2 - ED^2\right).$$

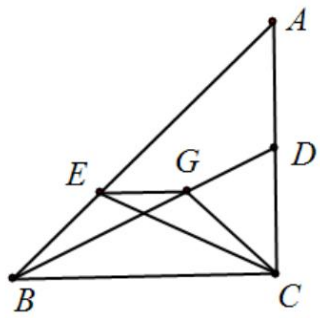
General complex number solution: $(a-b)\overline{(a-b)} - (k-c-(a-k))\overline{(k-c-(a-k))}$

$$= \frac{2(a-b)^2(b-d)\left(2a^2b - a^2d + abd - ad^2 + bd^2 - 2d^3\right)}{abd(a+b-2d)(2ab-ad-bd)};$$

$$(d-a)\overline{(d-a)} - (d-e)\overline{(d-e)}$$

$$= \frac{2(a-d)^2(b-d)\left(2a^2b - a^2d + abd - ad^2 + bd^2 - 2d^3\right)}{ad^2(a+b-2d)(2ab-ad-bd)};$$ This means that $AD = DE$ and $AB + AK = KC$ are equivalent $2a^2b - a^2d + abd - ad^2 + bd^2 - 2d^3 = 0$, so it is proved that:

$AD = DE \Leftrightarrow AB + AK = KC$.



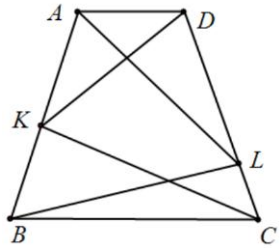
Example 10 : Given G as the centroid of triangle ABC , D as the midpoint of side CA , *and* a line through G and parallel to base BC *intersecting* AB at point E. Prove: $\angle AEC = \angle DGC$ *if and only if* $\angle ACB = 90°$. (2010 British Mathematical Olympiad)

Proof: Let $A=\left(a,\bar{a}\right)$, $B=\left(b,b\right)$, $C=\left(0,0\right)$, $D=\left(\frac{a}{2},\frac{\bar{a}}{2}\right)$, $G=\left(\frac{a+b}{3},\frac{\bar{a}+b}{3}\right)$,

$e=\left(\frac{a+2b}{3},\frac{\bar{a}+2b}{3}\right)$, identity $\dfrac{\frac{g-d}{g-c}+\overline{\left(\frac{g-d}{g-c}\right)}}{\frac{g-d}{g-c}-\overline{\left(\frac{g-d}{g-c}\right)}}-\dfrac{\frac{e-a}{e-c}+\overline{\left(\frac{e-a}{e-c}\right)}}{\frac{e-a}{e-c}-\overline{\left(\frac{e-a}{e-c}\right)}}=\frac{2}{3}\dfrac{\frac{c-b}{c-a}+\overline{\left(\frac{c-b}{c-a}\right)}}{\frac{c-b}{c-a}-\overline{\left(\frac{c-b}{c-a}\right)}}$,

Extended proposition: Given G as the centroid of triangle ABC , D as the midpoint of side CA , a line passing through G and parallel to base BC *intersects* AB at point E.

Proof: $\cot\angle AEC-\cot\angle DGC=\frac{2}{3}\cot\angle ACB$.



Example 11 : As shown in Figure 1, in trapezoid $ABCD$, it is known *that* $AD \parallel BC$. K and L are points on AB and CD *respectively, satisfying* $\angle BAL = \angle CDK$. Proof: $\angle BAL = \angle CKD$. (2008 Singapore Mathematical Olympiad)

$$A=\left(a,\bar{a}\right),\quad B=(0,0),\quad C=(1,1),\quad D=\left(d,d-\left(a-\bar{a}\right)\right),\quad K=\left(k,\frac{k\bar{a}}{a}\right),$$

$$L=\left(l,\frac{a-\bar{a}-l-al+\bar{a}l+dl}{-1+d}\right),$$

$$\text{Equation 1}\left([AKC]-[LKC]\right)(a-b)\overline{(a-b)}(d-c)\overline{(d-c)}\left(\frac{\frac{a-l}{a-b}}{\frac{d-c}{d-k}}-\overline{\left(\frac{\frac{a-l}{a-b}}{\frac{d-c}{d-k}}\right)}\right)$$

$$=[ABD](l-a)\overline{(l-a)}(k-d)\overline{(k-d)}\left(\frac{\frac{l-b}{l-a}}{\frac{k-d}{k-c}}-\overline{\left(\frac{\frac{l-b}{l-a}}{\frac{k-d}{k-c}}\right)}\right),$$

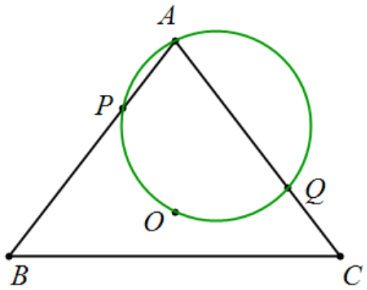
Extended Proposition 1: As shown in Figure 1, in trapezoid $ABCD$, *AD is known* $\parallel$ BC. K and L *are* points on AB and CD respectively. Prove that: $\left(S_{\triangle AKC}-S_{\triangle LKC}\right)AB\cdot DC\sin\left(\angle BAL-\angle CDK\right)=S_{\triangle ABD}\cdot LB\cdot KC\sin\left(\angle BLA-\angle CKD\right)$.

Identity 2

$$\left([LCK]-[ACK]\right)\left(\frac{\frac{a-l}{a-b}+\overline{\left(\frac{a-l}{a-b}\right)}}{\frac{a-l}{a-b}-\overline{\left(\frac{a-l}{a-b}\right)}}-\frac{\frac{d-c}{d-k}+\overline{\left(\frac{d-c}{d-k}\right)}}{\frac{d-c}{d-k}-\overline{\left(\frac{d-c}{d-k}\right)}}\right)=[ABD]\left(\frac{\frac{l-b}{l-a}+\overline{\left(\frac{l-b}{l-a}\right)}}{\frac{l-b}{l-a}-\overline{\left(\frac{l-b}{l-a}\right)}}-\frac{\frac{k-d}{k-c}+\overline{\left(\frac{k-d}{k-c}\right)}}{\frac{k-d}{k-c}-\overline{\left(\frac{k-d}{k-c}\right)}}\right)$$

,

Extended Proposition 2 : As shown in Figure 1, in trapezoid $ABCD$, it is known *that* $AD \parallel BC$. K and L are points on line segments AB and CD *respectively.* Prove that: $\left(S_{\triangle AKC}-S_{\triangle LKC}\right)\left(\cot\angle BAL-\cot\angle CDK\right)=S_{\triangle ABD}\left(\cot\angle BLA-\cot\angle CKD\right)$.



Example 12 : In triangle ABC , given that $AB = AC$, points P and Q lie on sides AB and AC , *respectively, and do not* coincide with vertices of the triangle . Prove that the circumcircle of triangle *APQ passes through the circumcenter of ABC* if and only if $AP = CQ$. (2012 Estonian National Team Selection Exam)

Proof: Assume $A=(1,1)$ that $B=\left(b,\frac{1}{b}\right)$, $C\left(\frac{1}{b},b\right)$, $P=\left(p,\frac{a+b-p}{ab}\right)$,

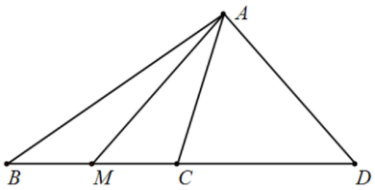
$Q=\left(q,\frac{a+c-q}{ac}\right)$, $O=(0,0)$, $[APOQ]([APC]+[BCQ])$

$$=\left(\frac{c-b}{c-a}+\overline{\left(\frac{c-b}{c-a}\right)}\right)\overline{(a-p)}(a-b)\overline{(a-q)}(a-c)\left((c-q)\overline{(c-q)}-(a-p)\overline{(a-p)}\right),$$

Extended proposition: In triangle ABC , it is known that $AB = AC$, points P and Q are on sides AB and AC *respectively and do not* coincide with the vertices of the triangle . Proof:

$$4\left(S_{\triangle APC}+S_{\triangle BCQ}\right)\cdot OP\cdot OQ\sin\left(\angle A+\angle POQ\right)=\left(CQ^2-AP^2\right)\cdot CB\cdot CA\cos C\,.$$

Example 13 : In triangle $2\angle ACB = 180^\circ + \angle CBA$ ABC , M *is the midpoint of* side BC . Draw a circle with vertex A *as center that intersects line* BC at points M and D. Prove: $MD = AB$. (2010 Croatian Mathematical Competition Final)



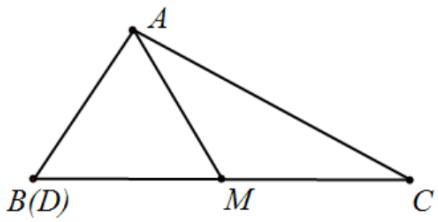
Proof: Assume $A=(a,-a)$, $B=(b,b), C=(c,c)$, $M=\dfrac{B+C}{2}$, $D=-M$,

$$(c-a)^2(c+a)^2(b-a)(b+a)\left(\frac{\left(\frac{c-b}{c-a}\right)^2}{\frac{b-a}{b-c}}-\frac{\left(\frac{c-b}{c+a}\right)^2}{\frac{b+a}{b-c}}\right)$$

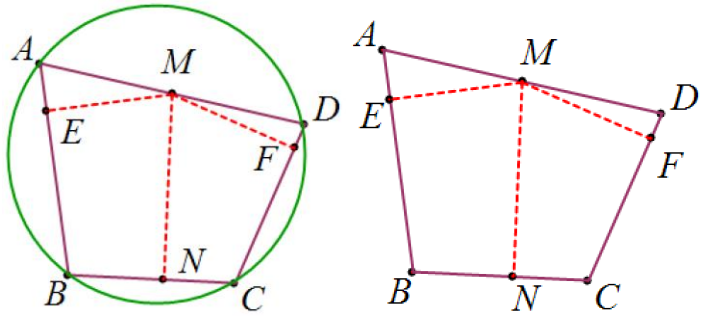
$$=2a(b-c)^3\left((m-d)(m-d)-(a-b)(-a-b)\right) ,$$

Extended proposition: In $\triangle ABC$, M is the midpoint of side BC . Draw a circle with vertex A *as* the center that intersects the line BC *at points* M and D. Proof: $-CA^2\sin(2\angle ACB-\angle CBA)=(MD^2-AB^2)\sin\angle ABC$.

Note: This identity also corresponds to the classic figure, $\angle A = 90^\circ$, $\angle B = 60^\circ$, $\angle C = 30^\circ$, in which case B and D coincide.



Example 14 : In the inscribed quadrilateral $ABCD$, from the midpoint M of AD, draw $MN \perp BC$, $ME \perp AB$, $MF \perp CD$, with N , E , and F as the feet of perpendiculars. Prove: MN passes through the midpoint of segment EF . (2018 Shanxi High School Mathematics Competition)



Abandoning the cocircularity of points A , B , C , and D , *explore* the relationship between the cocircularity of points A, B, C, and D and MN passing through the midpoint of line segment EF. Among them, *the* cocircularity *of* points A , B , C , *and* D *is* equivalent *to* , *and* MN $\angle A+\angle C=180^{\circ}$ passing *through* the midpoint of line segment $S_{\triangle MEN}=S_{\triangle MFN}$ EF is equivalent to .

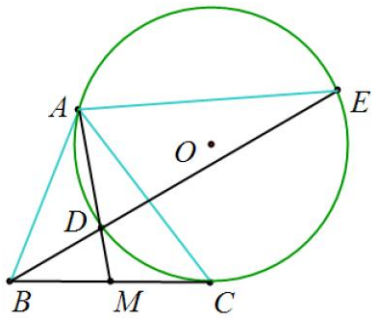
Establish the identity: $$\frac{\begin{vmatrix} m & \overline{m} & 1 \\ e & \overline{e} & 1 \\ n & \overline{n} & 1 \end{vmatrix}+\begin{vmatrix} m & \overline{m} & 1 \\ f & \overline{f} & 1 \\ n & \overline{n} & 1 \end{vmatrix}}{(a-b)\overline{(a-b)}\left(\dfrac{\dfrac{a-d}{a-b}}{\dfrac{c-d}{b-c}}-\overline{\left(\dfrac{\dfrac{a-d}{a-b}}{\dfrac{c-d}{b-c}}\right)}\right)}$$

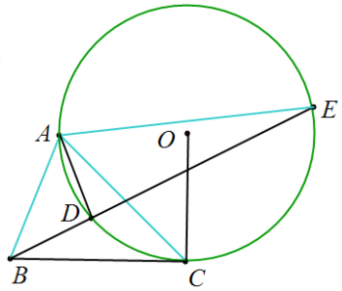
$$=\frac{\begin{vmatrix} m & \overline{m} & 1 \\ e & \overline{e} & 1 \\ n & \overline{n} & 1 \end{vmatrix}\left(\begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}-\begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ d & \overline{d} & 1 \end{vmatrix}\right)}{\begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}\begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ d & \overline{d} & 1 \end{vmatrix}},$$

Extended proposition: As shown in the figure, in the convex quadrilateral $ABCD$, from the midpoint M of AD , draw $MN \perp BC$, $ME \perp AB$, $MF \perp CD$, and N , E , and F are the feet of perpendiculars. Proof:

$$S_{\triangle MEN}\cdot(S_{\triangle ABC}-S_{\triangle ABD})\cdot AB\cdot BC\cdot DA\sin(\angle A+\angle C)=2CD\cdot S_{\triangle ABC}\cdot S_{\triangle ABD}\cdot\left(S_{\triangle MEN}-S_{\triangle MFN}\right)$$

.





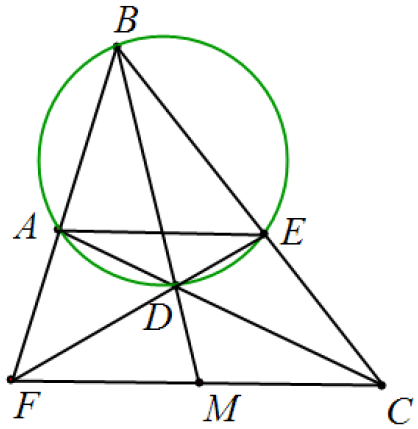
Example 15 : As shown in the figure, in acute angle $\triangle ABC$, M is the midpoint of BC, circle O passes through point A and is tangent to line BC *at point* C, line AM *intersects* circle O at another point D, and line BD *intersects* circle O at another point E. Prove: $\angle EAC = \angle BAC$. (2018 Shaanxi High School Mathematics League)

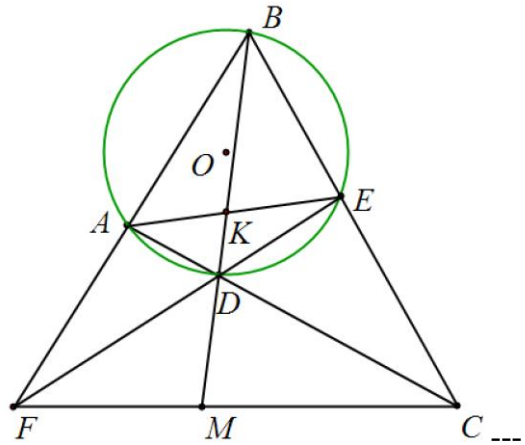
Proof: Let the circumscribed circle of $\triangle$ $b = \dfrac{cc(d+e)-de(c+c)}{cc-de}$ *ADC be the unit circle,* ,

$$\frac{\dfrac{a-e}{a-c}}{\dfrac{a-c}{a-b}} - \overline{\left(\frac{\dfrac{a-e}{a-c}}{\dfrac{a-c}{a-b}}\right)} = \frac{acd(a-e)(c-e)}{e(a-c)^2(a-d)(c-d)}\left(\begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ d & \bar{d} & 1 \end{vmatrix} + \begin{vmatrix} a & \bar{a} & 1 \\ c & \bar{c} & 1 \\ d & \bar{d} & 1 \end{vmatrix}\right),$$

Extended Proposition: As shown in the figure, in the acute angle $\triangle ABC$, circle O passes through point A *and* is tangent to line BC *at point C. The line* passing through B *intersects* circle O at *points* D and E. Proof: $2CE \cdot (S_{\triangle ABD} - S_{\triangle ACD}) = AB \cdot AD \cdot CD \sin(\angle CAB - \angle EAC)$.

Example 16 : A circle passes through vertices A and B *of* triangle ABC , intersecting line segments AC and BC at points D and E *respectively* . Lines BA and ED intersect at point F , and lines BD and CF intersect at point M. Proof: $MF = MC \Leftrightarrow MB \cdot MD = MC^2$ (2003 U.S. Mathematical Olympiad test question)





, $$f = \frac{ab(d+e) - de(a+b)}{ab - de}$$, $$c = \frac{ad(b+e) - be(a+d)}{ad - be}$$,

$$m = \frac{-ab^2d - abd^2 + 2b^2d^2 + ab^2e - b^2de + ad^2e - bd^2e}{-2abd + b^2d + bd^2 + abe + ade - 2bde}, \quad k = \frac{ae(b+d) - bd(a+e)}{ae - bd},$$

$$\frac{(m-c)\overline{(m-c)} - (m-d)\overline{(m-b)}}{(m-c)\overline{(m-c)} - (m-f)\overline{(m-f)}} k\bar{k} \frac{\dfrac{a-b}{a-e}}{\dfrac{d-b}{d-e}} \left(\frac{\begin{vmatrix} a & \bar{a} & 1 \\ e & \bar{e} & 1 \\ b & \bar{b} & 1 \end{vmatrix} - \begin{vmatrix} a & \bar{a} & 1 \\ e & \bar{e} & 1 \\ d & \bar{d} & 1 \end{vmatrix}}{\begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ d & \bar{d} & 1 \end{vmatrix} - \begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ e & \bar{e} & 1 \end{vmatrix}} \right)^2 = 1.$$

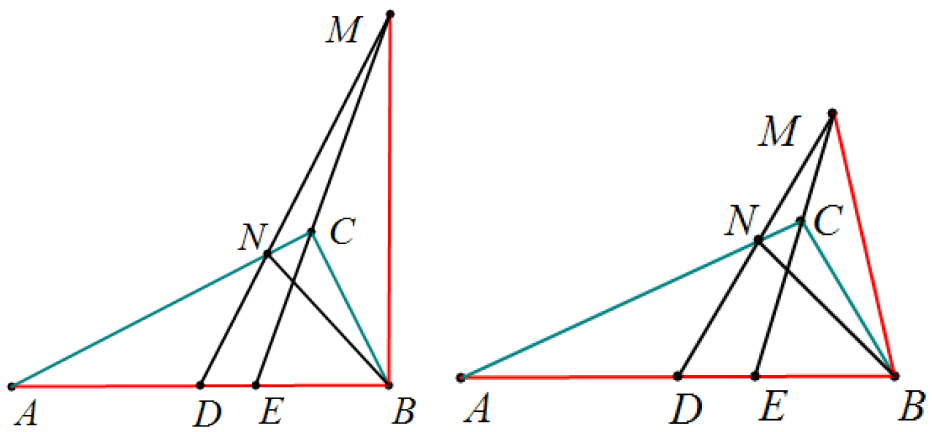
Extended Proposition: In the circle O , *quadrilateral* $ADEB$ *is inscribed.* Lines BE and AD intersect at point C , lines BA and ED intersect at point F , lines AE and BD intersect at point K , and lines BD and CF intersect at point M. Proof:

$$\left(MC^2 - MB \cdot MD\right) \cdot OK^2 \cdot AB \cdot DE \cdot \left(\frac{S_{ABCD}}{S_{\triangle ABD} - S_{\triangle ABE}} \right)^2 = AE \cdot DB \cdot R^2 \left(MC^2 - MF^2\right).$$

Interpreting diversity, equivalence

$$\frac{a-e}{f-c} - \frac{1/a - 1/e}{f1 - c1} = \frac{(ad-be)(ab-de)\left(abd + a^2e - 2abe - 2ade + bde + ae^2\right)}{a(b-d)e(a-b-d+e)(-abd + abe + ade - bde)},$$

$$\begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ d & \bar{d} & 1 \end{vmatrix} + \begin{vmatrix} e & \bar{e} & 1 \\ b & \bar{b} & 1 \\ d & \bar{d} & 1 \end{vmatrix} = -\frac{(b-d)\left(abd + a^2e - 2abe - 2ade + bde + ae^2\right)}{abde},$$



Example 17 : In the right triangle ABC, point D is the midpoint of the hypotenuse AB. $MB \perp AB$, MD intersects AC at N, and the extension of MC *intersects* AB at E. Prove: $\angle DBN = \angle BCE$. (2007 China Southeast Mathematical Olympiad)

Proof: Let D be the origin, DB be the real axis, $d=0$, $a=-b$, $\bar{n}=\dfrac{\overline{mn}}{m}$,

$$\bar{c}=\frac{bcm-bmn+b\overline{mn}+c\overline{mn}}{m(b+n)},$$

Identity: $$\begin{vmatrix} d & \bar{d} & 1 \\ b & \bar{b} & 1 \\ m & \overline{m} & 1 \end{vmatrix} n(c-n)(b-m)\overline{(b-m)}\left(\frac{b-d}{b-m}+\overline{\left(\frac{b-d}{b-m}\right)}\right)$$

$$-\frac{1}{2}\left(\begin{vmatrix} d & \bar{d} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix}-\begin{vmatrix} d & \bar{d} & 1 \\ b & \bar{b} & 1 \\ m & \overline{m} & 1 \end{vmatrix}\right) m(b+n)^2\overline{(c-a)}\left(\frac{c-b}{c-a}+\overline{\left(\frac{c-b}{c-a}\right)}\right)$$

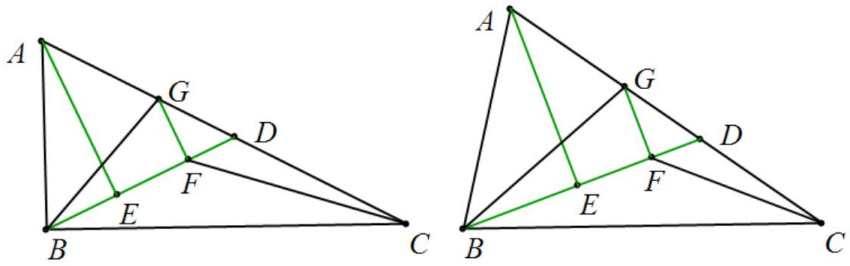
$$+m(b+n)(b-n)\overline{(b-n)}(c-b)\overline{(c-b)}\left(\frac{\frac{b-d}{b-n}}{\frac{c-b}{m-c}}-\overline{\left(\frac{\frac{b-d}{b-n}}{\frac{c-b}{m-c}}\right)}\right)=0.$$

Extended Proposition: As shown in the figure, in triangle ABC, point D is the midpoint of the hypotenuse AB, MD intersects AC at N; the extension of MC *intersects* AB at E. Proof:

$$4S_{\triangle DBM}\cdot CN\cdot DN\cdot BM\cdot BD\cos\angle MBD=2\left(S_{\triangle DBC}-S_{\triangle DBM}\right)\cdot AN^2\cdot DM\cdot CB\cos\angle ACB$$

$$+DM\cdot AN\cdot BD\cdot BN\cdot CB\cdot MC\sin\left(\angle DBN-\angle BCE\right).$$

Example 18 : As shown in the figure , in triangle ABC , $\angle ABC = 90°$. D and G are two points on side CA. Connect BD and BG . *Draw* perpendiculars to BD *through* points A and G , with their feet at E and F , respectively . Connect CF. Given $BE = EF$, prove that $\angle ABG = \angle DFC$. (2006 China Southeast Mathematical Olympiad)



Proof: Let the circumcircle of $\triangle ABC$ be the unit circle, solve the equation $\frac{a-d}{a-c}=\overline{\left(\frac{a-d}{a-c}\right)}$, and we get $\overline{d}=\frac{a+c-d}{ac}$; Similarly $\overline{g}=\frac{a+c-g}{ac}$, establish the identity: $abc(d-g)(b-c)\overline{(b-c)}\left(\frac{b-a}{b-c}+\overline{\left(\frac{b-a}{b-c}\right)}\right)$

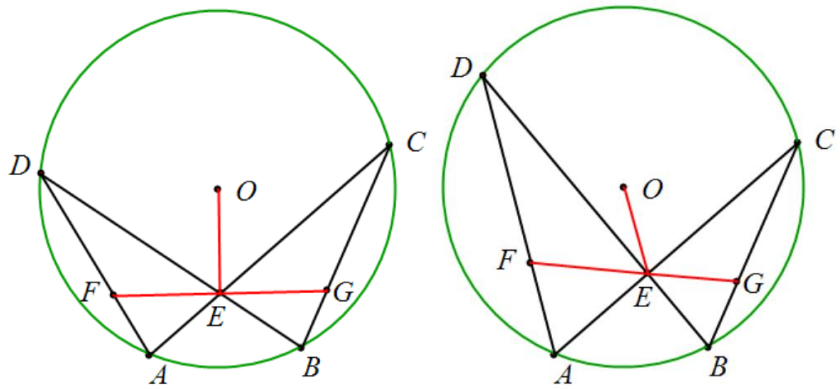
$$-\frac{2(a-b)(a-c)(b-c)(c-g)}{\left(\begin{vmatrix} a & \overline{a} & 1 \\ c & \overline{c} & 1 \\ b & \overline{b} & 1 \end{vmatrix}-\begin{vmatrix} a & \overline{a} & 1 \\ c & \overline{c} & 1 \\ f & \overline{f} & 1 \end{vmatrix}\right)}\left((e-b)\overline{(e-b)}-(e-f)\overline{(e-f)}\right)$$

$$+\frac{2a^2b^2c^2}{(a-b)(b-c)}(b-g)(b-d)\overline{(b-g)}\overline{(b-d)}\left(\frac{\frac{b-a}{b-g}}{\frac{b-d}{f-c}}-\overline{\left(\frac{\frac{b-a}{b-g}}{\frac{b-d}{f-c}}\right)}\right)=0.$$

Extended Proposition: As shown in the figure , in triangle ABC , D and G are two points on side CA. Connect BD and BG . Draw perpendicular lines to BD *through points* A and G , with their feet at E and F *respectively* . Connect CF. Prove:

$$\frac{S_{\triangle ACB}\cdot CG}{(S_{\triangle ACB}-S_{\triangle ACF})}\left(EB^2-EF^2\right)-DG\cdot BC\cdot BA\cos\angle ABC$$

$$=BG\cdot BD\cdot FC\frac{\sin(\angle ABG-\angle DFC)}{\sin\angle BAC}.$$



Example 19 : In the circle O, inscribe quadrilateral $ABCD$. AC intersects BD at E. Draw a line through E *that intersects* AD at F and BC at G. If $OE \perp FG$, prove that $EF = EG$. (Butterfly Theorem)

Proof: Let O be the unit circle, $e = \dfrac{ac(b+d)-bd(a+c)}{ac-bd}$, solve the equation $\dfrac{a-d}{a-f} = \overline{\left(\dfrac{a-d}{a-f}\right)}$, and we get $\overline{f} = \dfrac{a+d-f}{ad}$. Solve the equation $\dfrac{e-f}{e-g} = \overline{\left(\dfrac{e-f}{e-g}\right)}$, $\dfrac{b-g}{b-c} = \overline{\left(\dfrac{b-g}{b-c}\right)}$, and we get

$$g = \frac{a^2b^2c^2 - 2a^2b^2cd + a^2b^2d^2 - a^2c^2d^2 + 2abc^2d^2 - b^2c^2d^2 - ab^2c^2f + 2ab^2cdf + a^2c^2df - 2abc^2df + b^2c^2df - ab^2d^2f}{a^2bc^2 - 2a^2bcd + a^2bd^2 - a^2cd^2 + 2abcd^2 - b^2cd^2 - abc^2f + a^2cdf + b^2cdf - abd^2f}$$

the sum

$$\overline{g} = \frac{a^2c^2 - 2a^2cd + 2abd^2 - b^2d^2 - ac^2f + a^2df - 2abdf + b^2df + 2acdf - ad^2f}{a^2bc^2 - 2a^2bcd + a^2bd^2 - a^2cd^2 + 2abcd^2 - b^2cd^2 - abc^2f + a^2cdf + b^2cdf - abd^2f}.$$

Identity: $(a-b)(b-c)(a-d)(c-d)\left((e-f)\overline{(e-f)} - (e-g)\overline{(e-g)}\right)$

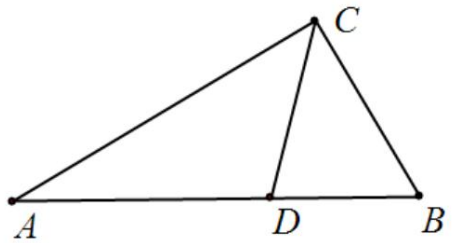
$$= a^2b^2c^2d^2\overline{(e-f)(e-g)}\left(\frac{f-g}{\overline{f-g}}\right)\left(\overline{\frac{b-d}{a-c}}\left(\frac{a-c}{b-d} - \overline{\left(\frac{a-c}{b-d}\right)}\right)\right)^2 e\bar{e}\left(\frac{g-f}{e-o} + \overline{\left(\frac{g-f}{e-o}\right)}\right)$$

.

the circle O, inscribe the quadrilateral $ABCD$. AC intersects BD at E. Draw a straight line through E *that intersects* AD at F and BC at G. R is the radius of the circle. Prove this

$$AB \cdot BC \cdot CD \cdot DA \cdot (EF^2 - EG^2) = 8R^2 \cdot EF \cdot EG \cdot OE \cdot FG \sin^2 \angle AEB \cos \angle OEF \quad .$$

(Extended version of the Butterfly Theorem)



Example 20 : In $\triangle ABC$, $\angle ACB = 90^\circ$, point D is on AB , and $(CD \cdot AB)^2 = 2CA \cdot AD \cdot DB \cdot BC$, prove that CD bisects $\angle$ ACB . (Mathematics Bulletin Call for Solutions 2639)

Proof: Let AB be the real axis and C be on the imaginary axis, then $b = \frac{c^2}{a}$,

$$\left((a-b)(a-b)(d-c)(d+c)\right)^2 - 4(c-a)(-c-a)(d-a)(d-a)(d-b)(d-b)(b-c)(b+c)$$

$$= \frac{(b-a)^2}{4c^2}\left((c-d)^2(c+d)^2\left(\frac{\frac{c-b}{c-d}}{\frac{c-d}{c-a}} - \overline{\left(\frac{\frac{c-b}{c-d}}{\frac{c-d}{c-a}}\right)}\right)\right)^2 .$$

$\triangle ABC$ In the equation, $\angle ACB = 90^\circ$, point D is on AB . Prove that: $(CD \cdot AB)^2 = 2CA \cdot AD \cdot DB \cdot BC$ the necessary and sufficient condition for is $\angle BCD = \angle ACD$.

$\triangle ABC$ In the equation, $\angle ACB = 90^\circ$, point D is on AB . Prove that: $(AB \cdot CD)^4 - 4CA^2 \cdot AD^2 \cdot DB^2 \cdot BC^2 = AB^4 \cdot CD^4 \sin^2(\angle BCD - \angle ACD)$.

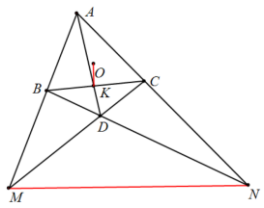
Proof: The original formula is equivalent to

$4CA^2 \cdot AD^2 \cdot DB^2 \cdot BC^2 = AB^4 \cdot CD^4 \cos^2(\angle BCD - \angle ACD)$.

$2CA \cdot AD \cdot DB \cdot BC = AB^2 \cdot CD^2 \cos(\angle BCD - \angle ACD)$.

$2\frac{\sin A \sin BAD \cdot DB}{CD^2} = \sin 2\angle ACD$.

$2\sin ACD \cdot \sin BCD = \sin 2\angle ACD$.



Example 2-1 : The circumcenter of acute angle ABC is O , K is a point on side BC (not *its* midpoint), D is a point on the extension of line segment AK , lines BD and AC intersect at point N , and lines CD *and* AB intersect at point M. Prove: If $OK \perp MN$, then points A , B , D , and C are concentric . (2010 National High School Mathematics League)

Proof: Let the circumscribed circle of $\triangle ABC$ be the unit circle,

Identity: $$[A,B,D][A,C,D][A,D,O][B,D,O][C,D,O]|m-n|^2\left(\frac{o-k}{m-n}+\overline{\left(\frac{o-k}{m-n}\right)}\right)$$

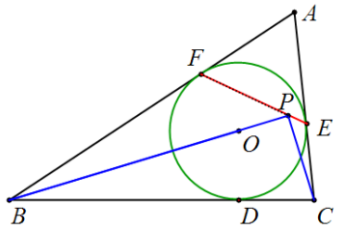
$$=-\left([A,B,D]-[A,D,C]\right)[A,K,O][D,N,O]|a-b|^2|c-d|^2\left(\frac{a-c}{a-b}\frac{d-b}{d-c}-\overline{\left(\frac{a-c}{a-b}\frac{d-b}{d-c}\right)}\right)$$

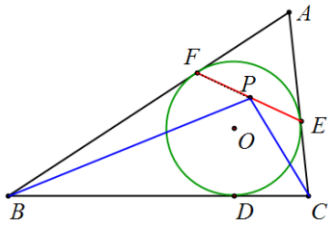
acute angle ABC is O , K is a point on side BC (not *its* midpoint), D is a point on the extension of line segment AK , BD intersects AC at point N , and CD *intersects* AB at point M. Let the angle between OK *and* MN *be* , α prove :

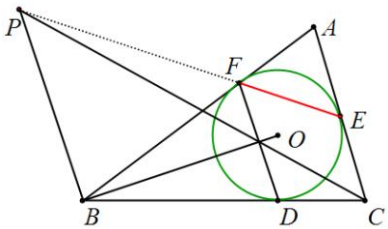
$$4S_{\triangle ABD}\cdot S_{\triangle ACD}\cdot S_{\triangle ADO}\cdot S_{\triangle BDO}\cdot S_{\triangle CDO}\cdot MN\cdot OK\left|\cos\alpha\right|$$

$$=\left|S_{\triangle ABD}-S_{\triangle ADC}\right|\cdot S_{\triangle AKO}\cdot S_{\triangle DMO}\cdot S_{\triangle DNO}\cdot AB\cdot BD\cdot DC\cdot CA\left|\sin\left(A+D\right)\right|.$$

In the original question, it is emphasized that K *is not* the midpoint of side BC , which is clearly reflected in the identity. $S_{\triangle ABD}=S_{\triangle ADC}$ This corresponds to K *being* the midpoint of side BC .







Example 22 : $\triangle ABC$ has an incircle O , D , E , and F are the points of tangency between the circle and the three sides, and P is the intersection of the bisector of $\angle B$ *and* EF . Prove that: $PB \perp PC$.

Proof: Let O be the unit circle, $b=\dfrac{2df}{d+f}$, $c=\dfrac{2de}{d+e}$, $\bar{p}=\dfrac{e+f-p}{ef}$, if P is a point on line segment $\angle PBF-\angle PBC\ EF$, *explore the relationship* between $\angle BPC$ and .

Calculate $\dfrac{\dfrac{b-f}{\dfrac{b-p}{\dfrac{b-p}{b-c}}}-\overline{\left(\dfrac{\dfrac{b-f}{b-p}}{\dfrac{b-p}{b-c}}\right)}}{\dfrac{p-c}{p-b}+\overline{\left(\dfrac{p-c}{p-b}\right)}}$ and try to express the result in the form of geometric quantities, so there is the identity

$$2\frac{\begin{vmatrix} c & \bar{c} & 1 \\ 0 & \bar{0} & 1 \\ p & \bar{p} & 1 \end{vmatrix}}{\begin{vmatrix} b & \bar{b} & 1 \\ c & \bar{c} & 1 \\ 0 & \bar{0} & 1 \end{vmatrix}}(p-b)\overline{(p-b)}\left(\frac{\dfrac{b-f}{\dfrac{b-p}{\dfrac{b-p}{b-c}}}-\overline{\left(\dfrac{\dfrac{b-f}{b-p}}{\dfrac{b-p}{b-c}}\right)}}{}\right)=\left(\begin{vmatrix} d & \bar{d} & 1 \\ f & \bar{f} & 1 \\ b & \bar{b} & 1 \end{vmatrix}-\begin{vmatrix} d & \bar{d} & 1 \\ f & \bar{f} & 1 \\ p & \bar{p} & 1 \end{vmatrix}\right)\left(\frac{p-c}{p-b}+\overline{\left(\frac{p-c}{p-b}\right)}\right)$$

.

$\triangle ABC$ has an incircle O , D , E , and F are the points of tangency between the circle and the three sides, and P is a point on line segment $\dfrac{S_{\triangle COP}}{S_{\triangle BCO}}BF\cdot BC\cdot BP\sin\left(\angle PBF-\angle PBC\right)=2\left(S_{\triangle DFB}+S_{\triangle DFP}\right)PC\cos\angle BPC\ EF$. *Prove that:* .

Since traditional Euclidean geometry does not have the concept of "direction", when translating complex number identities into Euclidean geometry conclusions, it needs to be determined based on the specific graph, and sometimes there will be a difference of positive and negative signs.

point P *is on the extension line of* line segment EF , the conclusion changes to:

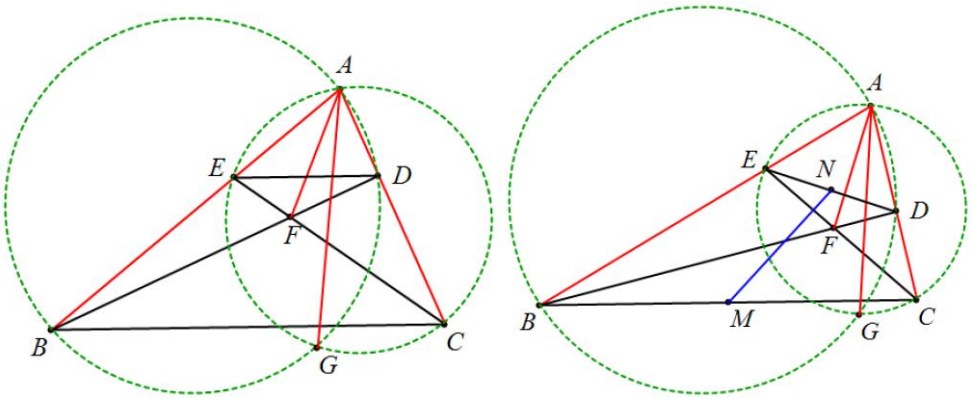
$$\frac{S_{\triangle COP}}{S_{\triangle BCO}} BF \cdot BC \cdot BP \sin\left(\angle PBF + \angle PBC\right) = 2\left(S_{\triangle DFB} - S_{\triangle DFP}\right) PC \cos \angle BPC .$$

If $S_{\triangle DFB} = S_{\triangle DFP}$, then $\angle PBF + \angle PBC = 180^{\circ}$, at this time $BP \ // \ DF$, P *is* on the exterior angle bisector of $\angle \ B$.

Example 23 : In triangle ABC , $DE \ // \ BC$, CE and BD intersect at point F , and the circumcircle of *triangle ADB* intersects the circumcircle of triangle AEC *at point* G , then $\angle \ CAG = \angle \ BAF$.

Proof: $a=0$, $c=kd$, $b=ke$, $f=\dfrac{b+kd}{1+k}$, solve the equation $\dfrac{\dfrac{a-g}{a-b}}{\dfrac{d-g}{d-b}}=\overline{\left(\dfrac{\dfrac{a-g}{a-b}}{\dfrac{d-g}{d-b}}\right)}$ and

$\dfrac{\dfrac{a-g}{a-c}}{\dfrac{e-g}{e-c}}=\overline{\left(\dfrac{\dfrac{a-g}{a-c}}{\dfrac{e-g}{e-c}}\right)}$ get $g=\dfrac{de(1+k)}{d+e}$, $\dfrac{\dfrac{f}{e}}{\dfrac{d}{g}}=k$.



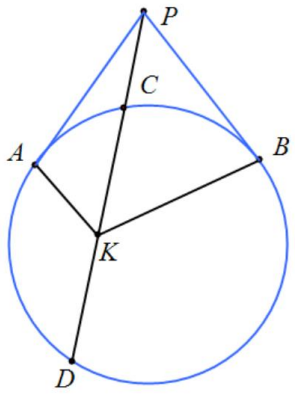
Assume $a=0$, $c=kd$, $b=te$, $f=\dfrac{dk+et-dkt-ekt}{1-kt}$, $g=\dfrac{de(1-kt)}{d+e-dk-et}$, then

$$4\left(\frac{c+b}{2}-\frac{d+e}{2}\right)\overline{\left(\frac{c+b}{2}-\frac{d+e}{2}\right)}\left(\frac{\dfrac{f}{e}}{\dfrac{d}{g}}-\overline{\left(\frac{\dfrac{f}{e}}{\dfrac{d}{g}}\right)}\right)=\left(d\bar{e}-\bar{d}e\right)(1-kt)(k-t),$$

triangle ABC , CE and BD intersect at point F , and the circumcircle of triangle ADB *intersects the* circumcircle of triangle AEC at point G. Prove that:

$$4MN^2\cdot AF\cdot AG\sin(BAF-CAG)=(AD\cdot AE-AB\cdot AC)(AC\cdot AE-AB\cdot AD)\sin A \text{ .in .}$$

$$\begin{vmatrix} a & \bar{a} & 1 \\ e & \bar{e} & 1 \\ d & \bar{d} & 1 \end{vmatrix}=\bar{d}e-d\bar{e}$$



Example 24 : There are four points A , C , B , and D *on a circle.* PA and PB are tangents, and K is the midpoint of CD . The necessary and sufficient condition to prove that P , C , and D are collinear is $\angle\ AKC = \angle\ BKC$.

Proof: Let the circle be the unit circle, $p = \dfrac{2ab}{a+b}$, $\overline{p} = \dfrac{2}{a+b}$,

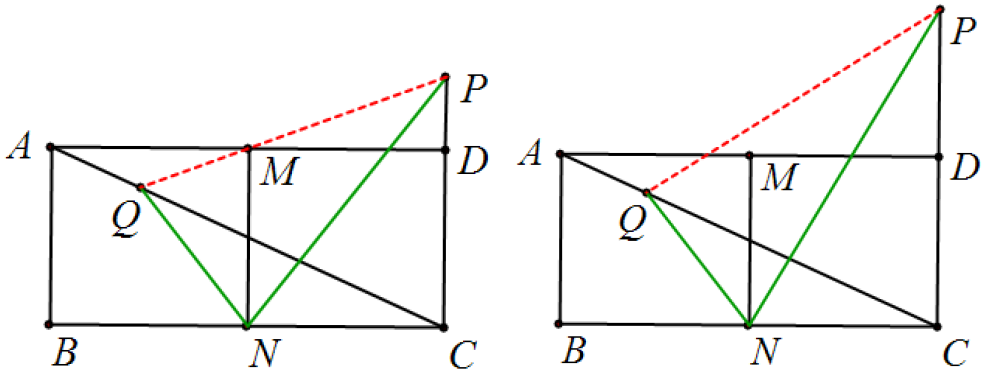
$$(d-c)^2\overline{(d-c)}^2\left(\frac{\frac{\frac{c+d}{2}-a}{\frac{d-c}{d-c}}}{\frac{\frac{c+d}{2}-b}{}}-\overline{\left(\frac{\frac{\frac{c+d}{2}-a}{\frac{d-c}{d-c}}}{\frac{\frac{c+d}{2}-b}{}}\right)}\right)+$$

$$\left(\frac{\frac{a+b}{2}}{a-b}\right)\left(\begin{vmatrix} a & \overline{a} & 1 \\ d & \overline{d} & 1 \\ c & \overline{c} & 1 \end{vmatrix}+\begin{vmatrix} b & \overline{b} & 1 \\ c & \overline{c} & 1 \\ d & \overline{d} & 1 \end{vmatrix}\right)\begin{vmatrix} p & \overline{p} & 1 \\ c & \overline{c} & 1 \\ d & \overline{d} & 1 \end{vmatrix}=0,$$

four points A , C , B , and D on a circle . PA and PB are tangents, and K is the midpoint of CD . Prove that:

$$AB\cdot KA\cdot KB\cdot DC^2\sin\left(\angle AKC-\angle CKB\right)=8OM\cdot S_{\triangle PCD}\cdot\left(S_{\triangle ADC}+S_{\triangle BCD}\right).$$

Example 25 : As shown in Figure 31 , in rectangle $ABCD$, M is the midpoint of AD , N is the midpoint of BC , P is on the extension of CD , *and* Q is a point on AC . *Connect* PN and QN . Prove that the necessary and sufficient condition for points P , M , and Q to be collinear is $\angle QNM = \angle PNM$.



Proof: Let NC be the real axis, NM be the imaginary axis, $n=0$, $d=c+m$,

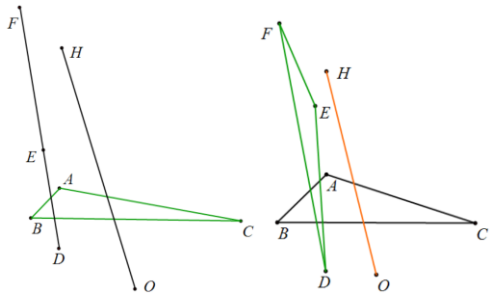
$\overline{d}=c-m$, $a=2m-d$, , $\overline{a}=2\overline{m}-\overline{d}$, $\overline{q}=-\dfrac{2cm-2cq-mq}{2c-m}$, $\overline{p}=2c-p$,

$$(m-q)\overline{(m-q)}\left(\frac{m-p}{m-q}-\overline{\left(\frac{m-p}{m-q}\right)}\right)=m^2\left(\frac{\frac{n-q}{n-m}}{\frac{n-m}{n-p}}-\overline{\left(\frac{\frac{n-q}{n-m}}{\frac{n-m}{n-p}}\right)}\right),$$

In rectangle $ABCD$, M is the midpoint of AD , N is the midpoint of BC , P is a point on CD , *and* Q is a point on AC . Connect PN and QN . Prove that

$2S_{\triangle PMQ}=NP\cdot NQ\left|\sin\left(\angle QNM-\angle PNM\right)\right|$:

Example 26 : In triangle ABC , let H be the orthocenter, O be the circumcenter, and R be the circumradius. Let D and A *be symmetric* about BC , E and B *be symmetric* about CA , and F and C *be symmetric* about AB . Prove that D , E, and F are collinear if and only if OH = 2R . (39th *IMO* - *Shortlist*)



Proof: Let the circumcircle of $\triangle ABC$ be the unit circle, $d = 2\dfrac{a+b+c-bc/a}{2} - a = b+c-bc/a$, $e = a+c-ca/b$, $f = a+b-ab/c$,

$$\frac{[D,E,F]}{[A,B,C]} = 4-(a+b+c)\overline{(a+b+c)},$$

triangle ABC , H is the orthocenter, O is the circumcenter, and R is the circumradius. Let D and A *be symmetric* about BC , E and B *be symmetric* about CA , and F and C *be symmetric* about AB . Proof: $\dfrac{S_{\triangle DEF}}{S_{\triangle ABC}} = 4-\left(\dfrac{OH}{R}\right)^2$.

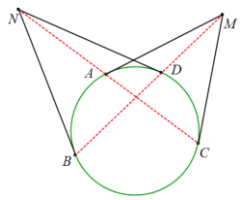


Figure 2 8

Example 27 : As shown in Figure 28 , in the inscribed quadrilateral $ABCD$, the tangents through A and C intersect at M , and the tangents through B and D intersect at N. *The necessary and sufficient condition for* N , A , and C to be collinear is that M , B , and D are collinear.

Proof: Let the circle be the unit circle, $m=\dfrac{2ac}{a+c}$, $n=\dfrac{2bd}{b+d}$,

$$\frac{n-a}{a-c}-\overline{\left(\frac{n-a}{a-c}\right)}=\frac{ab+bc+cd+da-2ac-2bd}{(c-a)(b+d)},$$

$$\frac{m-b}{b-d}-\overline{\left(\frac{m-b}{b-d}\right)}=\frac{ab+bc+cd+da-2ac-2bd}{(a+c)(d-b)}$$, so the necessary and sufficient condition for the three points N , A , and C to be collinear is that the three points M , B , and D are collinear.

Because A , B , C , and D are not equal, it is equivalent to say that points N , A , and C are collinear, and points M , B , and D are collinear $ab+bc+cd+da-2ac-2bd=0$. To prove the conclusion, we must ensure that the other terms in the numerator are $b-d\neq 0$. $a-c\neq 0$ As for the denominator, we generally don't care.

In fact, if we take a moment to study the denominator, we will have unexpected gains. $\begin{vmatrix} m & \overline{m} & 1 \\ b & \overline{b} & 1 \\ d & \overline{d} & 1 \end{vmatrix}=\dfrac{(b-d)(ab+bc+cd+da-2ac-2bd)}{bd(a+c)}$,

$\begin{vmatrix} n & \overline{n} & 1 \\ a & \overline{a} & 1 \\ c & \overline{c} & 1 \end{vmatrix}=\dfrac{(a-c)(ab+bc+cd+da-2ac-2bd)}{ac(b+d)}$, we can find

$$\frac{a^2-c^2}{ac}\begin{vmatrix} m & \overline{m} & 1 \\ b & \overline{b} & 1 \\ d & \overline{d} & 1 \end{vmatrix} = \frac{b^2-d^2}{bd}\begin{vmatrix} n & \overline{n} & 1 \\ a & \overline{a} & 1 \\ c & \overline{c} & 1 \end{vmatrix} = \frac{(a-c)(b-d)(ab+bc+cd+da-2ac-2bd)}{abcd}$$

that this is $\begin{vmatrix} 0 & \overline{0} & 1 \\ a & \overline{a} & 1 \\ c & \overline{c} & 1 \end{vmatrix}\begin{vmatrix} m & \overline{m} & 1 \\ b & \overline{b} & 1 \\ d & \overline{d} & 1 \end{vmatrix} = \begin{vmatrix} 0 & \overline{0} & 1 \\ b & \overline{b} & 1 \\ d & \overline{d} & 1 \end{vmatrix}\begin{vmatrix} n & \overline{n} & 1 \\ a & \overline{a} & 1 \\ c & \overline{c} & 1 \end{vmatrix}$.

Extended proposition: In the quadrilateral *ABCD* inscribed in circle O, the tangents through *A* and *C* intersect at M, and the tangents through *B* and *D* intersect at *N*.

Prove: $S_{\triangle OAC}S_{\triangle MBD} = S_{\triangle OBD}S_{\triangle NAC}$.

The existence of M requires that the chord AC cannot be a diameter, that is, the three points OAC are not collinear and the area is not 0. This is the implicit condition of this problem.

$[OAC] \neq 0, [OBD] \neq 0 \Rightarrow [MBD] = 0 \Leftrightarrow [NAC] = 0$.

Bold guess: $[OAC][MBD] = [OBD][NAC]$.

O and M correspond to each other, even if M moves to infinity. So the above formula is very symmetrical.

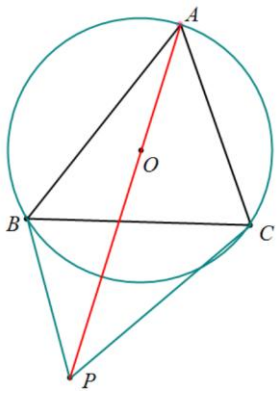


Figure 40

Example 28 : As shown in Figure 40 , in $\triangle ABC$, O is the circumcenter, $\angle ABP$ and $\angle ABC$ are complementary, $\angle ACP$ and $\angle ACB$ are complementary. Prove that: A , O , and P are collinear.

Proof: Let the circumscribed circle of $\triangle ABC$ be the unit circle, solve the equation

$$\frac{\frac{P-B}{B-A}}{\frac{B-A}{B-C}}=\frac{\frac{\overline{P}-1/B}{1/B-1/A}}{\frac{1/B-1/A}{1/B-1/C}} \text{ and } \frac{\frac{P-C}{C-A}}{\frac{C-A}{C-B}}=\frac{\frac{\overline{P}-1/C}{1/C-1/A}}{\frac{1/C-1/A}{1/C-1/B}} \text{ get } P=\frac{A^2+BC}{B+C}, \quad \overline{P}=\frac{A^2+BC}{A^2(B+C)},$$

so $\frac{P}{\overline{P}}=A^2=\frac{A}{\overline{A}}$.

Proof: Let the circumscribed circle of $\triangle ABC$ be the unit circle,

$$(a-b)^2 c\left(\frac{b-p}{b-a}\frac{b-c}{b-a}-\overline{\left(\frac{b-p}{b-a}\frac{b-c}{b-a}\right)}\right)+(c-a)^2 b\left(\frac{c-p}{c-a}\frac{c-b}{c-a}-\overline{\left(\frac{c-p}{c-a}\frac{c-b}{c-a}\right)}\right)$$

$$=(b-c)^2 a\left(\frac{p}{a}-\overline{\left(\frac{p}{a}\right)}\right),$$

$\triangle ABC$, O is the circumcenter, P is a point inside $\angle BAC$, prove:

$$\left|OC\cdot PB\sin(\angle ABP+\angle ABC)-OB\cdot PC\sin(\angle ACP+\angle ACB)\right|=OP\cdot BC\sin\angle AOP.$$

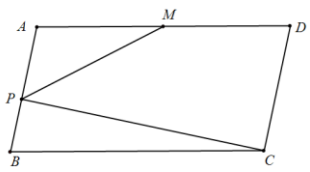


Figure 28

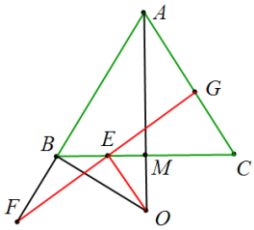
Example 29 : In parallelogram $ABCD$, the midpoint of AD *is* M , and the projection of C onto side PA is P. Prove that $BC = 2\ AB$ if and only if $\angle\ DAB = 2\ \angle\ APM$.

Proof: Assume $P = 0$,

$$\frac{\frac{a-d}{a-b}}{\left(\frac{p-a}{p-m}\right)^2} - \overline{\left(\frac{\frac{a-d}{a-b}}{\left(\frac{p-a}{p-m}\right)^2}\right)} = \frac{c}{2a^2(a-b)}\left((b-c)\overline{(b-c)} - 4(b-a)\overline{(b-a)}\right),$$

parallelogram $ABCD$, the midpoint of AD is M , and the projection of C onto side PA *is* P. Prove that: $CP \cdot \left(DA^2 - 4AB^2\right) = 4MP^2 \cdot DA\sin(2\angle APM - \angle BAD)$.

Example 30 : In triangle ABC, $AB = AC$. Draw line BO through line B, *intersecting the* extension of the median AM *at point O.* Draw line FG through point E on BC, intersecting lines AB and AC at points F and G, respectively. Prove that $OE \perp FG$ if and only if $FE = EG$. (Question from the 35th *IMO* exam, 1994)



Let MC be the real axis, MA be the imaginary axis, $b = -c$, $o = \dfrac{c^2}{a}$,

Solving the equation $\dfrac{a-f}{a-b} = \overline{\left(\dfrac{a-f}{a-b}\right)}$, we get $\overline{f} = -\dfrac{2ac + af - cf}{a+c}$;

Solve the $\dfrac{a-g}{a-c} = \overline{\left(\dfrac{a-g}{a-c}\right)}$ equation and $\dfrac{e-g}{e-f} = \overline{\left(\dfrac{e-g}{e-f}\right)}$ get

$$g = \frac{2ace - acf - c^2 f + aef - cef}{ac - c^2 + ae + ce - 2cf};\ \overline{g} = \frac{2ac^2 + acf - c^2 f - aef - cef}{ac - c^2 + ae + ce - 2cf}$$

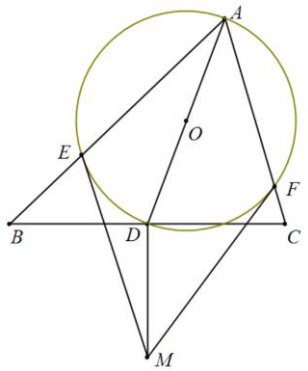
By calculation, $\dfrac{\dfrac{e-f}{e-o} + \overline{\left(\dfrac{e-f}{e-o}\right)}}{(e-f)\overline{(e-f)} - (e-g)\overline{(e-g)}}$ we can get the identity

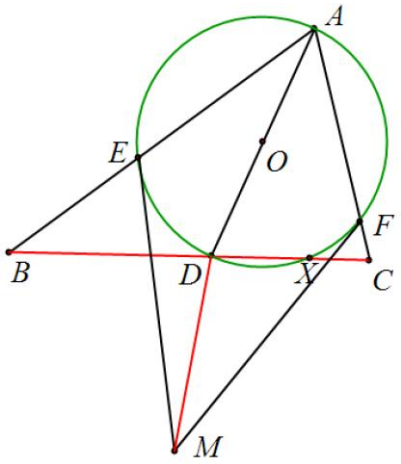
$$2\frac{a-f}{a-b}\frac{c-0}{e-b}(c-g)\overline{(c-g)}(e-f)\overline{(e-f)}(e-o)\overline{(e-o)}\left(\frac{e-f}{e-o} + \overline{\left(\frac{e-f}{e-o}\right)}\right)$$

$= (c-e)^2 (b-f)\overline{(b-f)}\left((e-f)\overline{(e-f)} - (e-g)\overline{(e-g)}\right)$, so we can get the proposition:

triangle ABC, $AB = AC$. Draw BO through B, intersecting *the extension* of the midline AM *at point O.* Draw line FG through point E on BC, intersecting lines AB and AC at points F and G *respectively*. Prove:

$$4\frac{AF}{AB}\cdot\frac{BM}{EB}\cdot CG^2 \cdot EF^3 \cdot EO\cos\angle OEF = CE^2 \cdot BF^2 \cdot \left(EF^2 - EG^2\right).$$





Example 3 1 : In $\triangle ABC$, *circle O* passes through points D , E , and F , $EO \perp EM$, $FO \perp FM$, prove that the necessary and sufficient condition for $BD = DC$ *is* $DM \perp BC$.

Proof: Let circle O be the unit circle, intersecting BC at points D and X. Let $a=-1$,

$$d=1 \quad , \quad , \quad b=\frac{ae(d+x)-dx(a+e)}{ae-dx} \quad , \quad c=\frac{af(d+x)-dx(a+f)}{af-dx} \quad , \quad m=\frac{2ef}{e+f} \quad ,$$

$$\frac{m-1}{b-1}+\overline{\left(\frac{m-1}{b-1}\right)}=\frac{(e+x)(-e-f+2ef-2x+ex+fx)}{2(-1+e)(e+f)x} \quad ,$$

$$b+c-2=\frac{2x(-e-f+2ef-2x+ex+fx)}{(e+x)(f+x)} .$$

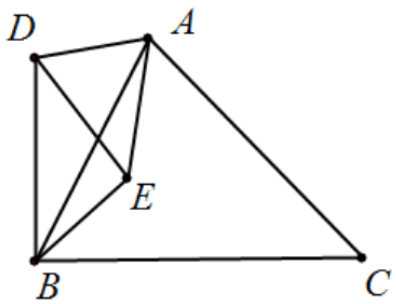
$$\frac{\frac{x-(-e)}{x-(-x)}}{\frac{f-(-e)}{f-(-x)}}\left((b-d)\overline{(b-d)}-(c-d)\overline{(c-d)}\right)=(b-c)\overline{(b-c)}\left(\frac{m-d}{b-c}+\overline{\left(\frac{m-d}{b-c}\right)}\right),$$

triangle ABC , circle O passes through points D , X , E , and F. $EO \perp EM$, $FO \perp FM$. Prove that: $\sin\angle OEX \dfrac{\sin\angle OXF}{\sin\angle OEF}\left(BD^2-CD^2\right)=2BC\cdot MD\cos\angle MDB$.

Note: $\dfrac{\frac{x-(-e)}{x-(-x)}}{\frac{f-(-e)}{f-(-x)}}$ -X and -E were introduced in the previous section . To reduce the number of new points, they were changed to trigonometric functions.

Example 3 2 : Given that $\triangle ABC \backsim \triangle ADE$ (similar to the original), $\angle DBC = 90°$.

Proof: $\dfrac{BE^2 - DE^2}{AC^2 - BC^2} = \dfrac{BD^2}{AB^2}$.



Proof: Assume $a = 0$, from $\dfrac{a-c}{a-b} = \dfrac{a-e}{a-d}$, we get $d = be/c$, $\bar{d} = \bar{b}\bar{e}/\bar{c}$,

$$(a-b)\overline{(a-b)}\left((b-e)\overline{(b-e)} - (d-e)\overline{(d-e)}\right) - (d-b)\overline{(d-b)}\left((a-c)\overline{(a-c)} - (b-c)\overline{(b-c)}\right)$$

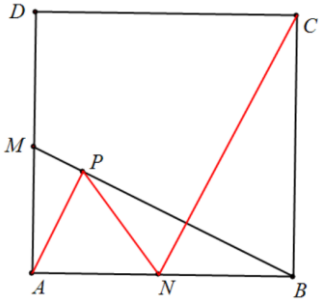
$$= b\bar{b}(b-c)\overline{(b-c)}\left(\frac{b-d}{b-c} + \overline{\left(\frac{b-d}{b-c}\right)}\right).$$

Extended proposition: Given that $\triangle ABC \backsim \triangle ADE$ (similar in sequence), prove: $AB^2(BE^2 - DE^2) - BD^2(AC^2 - BC^2) = 2AB^2 \cdot BC \cdot BD \cos\angle DBC$.

Example 3 3 : There is a point P in the convex quadrilateral $ABCD$, which satisfies $S_{\triangle ADP} \cdot S_{\triangle BPC} = S_{\triangle ABP} \cdot S_{\triangle DPC}$. Prove that: point P is on the diagonal of the quadrilateral.

Analysis: Its essence is to explore the relationship $[APD][BCP]-[ABP][CDP]$ between $[BPD]$ and $[APC]$. After calculation, we can get $[APD][BCP]-[ABP][CDP]+[APC][BPD]=0$, so we get the identity that all five points on the plane must satisfy.

Example 34 : Given that M and N *are on sides* DA and AB of square $ABCD$, and $AM = AN$, draw a perpendicular line through A to BM , with its foot at P. Prove that $\angle APN = \angle BNC$. (2000 I Love Math Junior High School Summer Camp Mathematics Competition)

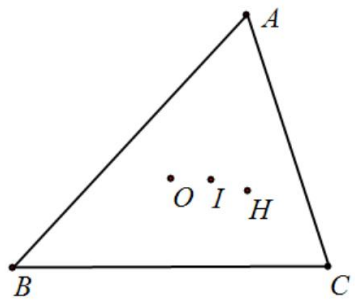


Proof: Suppose $A=(0,0)$, $B=(1,1)$, $C=(1+\mathrm{i},1-\mathrm{i})$, $D=(\mathrm{i},-\mathrm{i})$, $M=(\mathrm{i}m,-\mathrm{i}m)$, $N=(n,n)$, $P=\left(\frac{m}{-\mathrm{i}+m},\frac{m}{\mathrm{i}+m}\right)$, identity:

$$(p-a)(n-c)\overline{(p-a)(n-c)}\left(\frac{\frac{p-n}{p-a}}{\frac{n-c}{n-b}}-\overline{\left(\frac{\frac{p-n}{p-a}}{\frac{n-c}{n-b}}\right)}\right)(b-m\mathrm{i})(b+m\mathrm{i})$$

$$=-2\mathrm{i}m(m-n)(b-n)^2.$$

Extended proposition: Given that M and N *are on sides* DA and AB of square $ABCD$ respectively , draw a perpendicular line through A to BM , with its foot at P. Prove: $BM\cdot PN\cdot NC\sin(\angle APN-\angle BNC)=AB\cdot BN\cdot(AN-AM)$.



Example 35 : The circumcenter, incenter, and orthocenter of triangle ABC are points O, I, and H, *respectively*. Prove that $OI \,//\, BC$ is necessary and sufficient if $AI \perp IH$.

prove: $O=(0,0)$, $A=\left(a^2,\frac{1}{a^2}\right)$, $B=\left(b^2,\frac{1}{b^2}\right)$, $C=\left(c^2,\frac{1}{c^2}\right)$,

$H=\left(a^2+b^2+c^2,\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$, $I=\left(-ab-bc-ca,-\frac{1}{ab}-\frac{1}{bc}-\frac{1}{ca}\right)$,

Identity 1 : $$\frac{[OBC]-[IBC]}{(i-h)\overline{(i-h)}\left(\frac{i-a^2}{i-h}+\overline{\left(\frac{i-a^2}{i-h}\right)}\right)}\frac{[BCI]}{[OBC]}=-\frac{b^2-c^2}{b^2+c^2}.$$

Extended proposition: The circumcenter, incenter, and orthocenter of acute angle $\triangle ABC$ are points O, I, and H, *respectively*. Prove: $2\left(S_{\triangle IBC}-S_{\triangle OBC}\right)S_{\triangle IBC}=S_{\triangle OBC}IA\cdot IH\cdot\tan A\cos\angle AIH$.

Identity 2 : $$\frac{\frac{i-a^2}{i-h}+\overline{\left(\frac{i-a^2}{i-h}\right)}}{\frac{i-a^2}{i-h}-\overline{\left(\frac{i-a^2}{i-h}\right)}}=\frac{(b+c)^2}{b^2+c^2}\frac{\frac{o-i}{o-d}+\overline{\left(\frac{o-i}{o-d}\right)}}{\frac{o-i}{o-d}-\overline{\left(\frac{o-i}{o-d}\right)}}.$$

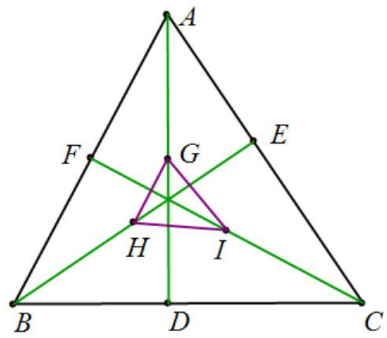
Extended proposition: The circumcenter, incenter, and orthocenter of acute angle $\triangle ABC$ *are points* O, I, and H, respectively. *Let the angle between lines IA and IO be* α. Prove: $\cot\angle AIH=\left(1-\frac{1}{\cos A}\right)\cot\alpha$.

$\frac{(b+c)^2}{b^2+c^2}=1-\frac{-bc}{\frac{b^2+c^2}{2}}=1-\frac{OK}{OD}=1-\frac{1}{\cos\angle BOD}=1-\frac{1}{\cos A}$. Let D be the midpoint of BC, and extend OD to intersect the circle at K.

Special case 1: $\angle AIH=90^\circ \Leftrightarrow \alpha=90^\circ$.

Special case 2 : When $\angle A=60^\circ$, $\cot\angle AIH=-\cot\alpha$, $\angle AIH$ and are α complementary .

Example 36 : Prove that the necessary and sufficient condition for the midpoints of the three altitudes of a triangle to be collinear is that the triangle is a right triangle. (54th Romanian Mathematical Olympiad Final, 2003)



Let the circumcircle of $\triangle ABC$ be the unit circle, , $d=\dfrac{b+c+a-\dfrac{bc}{a}}{2}$,

$e=\dfrac{c+a+b-\dfrac{ca}{b}}{2}$, $f=\dfrac{a+b+c-\dfrac{ab}{c}}{2}$, $g=\dfrac{a+d}{2}$, $h=\dfrac{b+e}{2}$, $i=\dfrac{c+f}{2}$,

$$[GHI]=-\frac{(a+b)(b+c)(c+a)(a-b)(b-c)(c-a)}{16a^2b^2c^2}$$

Generating identities $16[GHI][ABC]$

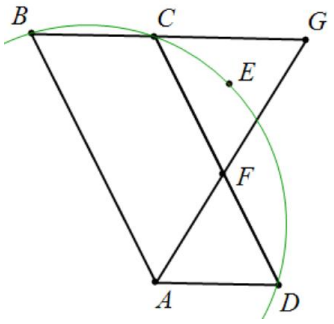
$$=|a-b|^2\left(\frac{a-c}{a-b}+\overline{\left(\frac{a-c}{a-b}\right)}\right)|b-c|^2\left(\frac{b-a}{b-c}+\overline{\left(\frac{b-a}{b-c}\right)}\right)|c-a|^2\left(\frac{c-b}{c-a}+\overline{\left(\frac{b-a}{b-c}\right)}\right),$$

Right now $16[GHI][ABC]=2AC\cdot AB\cos A\cdot 2BC\cdot BA\cos B\cdot 2CA\cdot CB\cos C$,

Right now $2[GHI]=[ABC]\cos A\cos B\cos C$.

Extended proposition: In $\triangle ABC$, *the midpoints of triangles* AD , BE , and CF *are* G , H , and I respectively . Prove: $2S_{\triangle GHI}=S_{\triangle ABC}\cos A\cos B\cos C$.

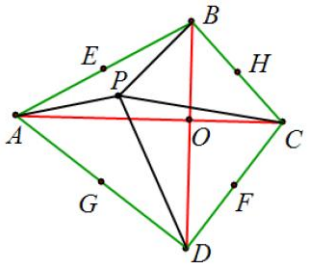
Example 37 : Suppose that , among the five points A , B , C , D , and E , $ABCD$ is a parallelogram, and $BCED$ is an inscribed quadrilateral. Let be l a line passing through point A , *intersecting line segment* DC at point F (F being an interior point of line segment l DC), *and intersecting* line BC at point G. If $EF = EG = EC$, prove that l is the angle bisector of $\angle$ DAB . (2007 *IMO* exam question)



Extended Proposition: Let $ABCD$ be a parallelogram l among the five points A , B , C , D , and E. *Let* be a line passing through point A , intersecting line segment DC *at point* F (F is an interior point of line segment l DC), *and intersecting* line BC at point G. If $EF = EG = EC$, prove:

$$8S_{\triangle BFG} \cdot CE \cdot ED \cdot DB \sin(\angle CED + \angle CBD) = CF^2 \cdot (BC + BG)(BG^2 - BA^2) .$$

Example 38 : In a convex quadrilateral $ABCD$, the two diagonals AC and BD are perpendicular, the two opposite sides AB and DC are not parallel, and point P is the intersection of the perpendicular bisectors of line segments AB and CD , and P is inside quadrilateral $ABCD$. Prove: The necessary and sufficient condition for $ABCD$ to be an inscribed quadrilateral is that the areas of $\triangle ABP$ and $\triangle CDP$ are equal . (1998 IMO *exam* question)



Proof: Let AC be the real axis and BD be the imaginary axis to build a coordinate system.

Solve the $(p-c)\overline{(p-c)}=(p-d)\overline{(p-d)}$ equation

$$\overline{p}=-\frac{a^2c+b^2c-ac^2-bc^2+a^2d+b^2d-ad^2-bd^2}{2(bc-ad)}$$ and

$(p-a)\overline{(p-a)}=\overline{(p-b)(p-b)}$ get $p=\dfrac{a^2c+b^2c-ac^2+bc^2-a^2d-b^2d-ad^2+bd^2}{2(bc-ad)}$.

A , B , C , and D are concentric circles, that is $OA\cdot OD=OB\cdot OC$, $ac+bd=0$.

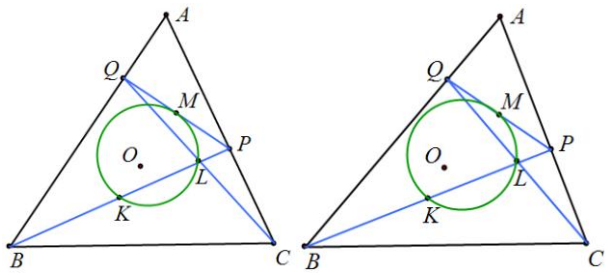
identities $$\frac{ad-bc}{4}\left(\begin{vmatrix} a & \overline{a} & 1 \\ p & \overline{p} & 1 \\ b & b & 1 \end{vmatrix}-\begin{vmatrix} p & \overline{p} & 1 \\ d & \overline{d} & 1 \\ c & c & 1 \end{vmatrix}\right)=\left(\frac{a+b}{2}-\frac{c+d}{2}\right)\left(\frac{a+d}{2}-\frac{b+c}{2}\right)ac+bd ,$$

Note: Through the identity equation, we can clearly see why it is agreed that " the two pairs of sides AB and DC are not parallel ".

Extended proposition: In a convex quadrilateral $ABCD$, the two diagonals AC and BD are perpendicular to each other, point P is the intersection of the perpendicular bisectors of line segments AB and CD , and P is inside the quadrilateral $ABCD$. Proof:

$$(OB\cdot OC-OA\cdot OD)\cdot(S_{\triangle APB}-S_{\triangle PDC})=EF\cdot GH\cdot(OA\cdot OC-OB\cdot OD).$$

Example 39 : Let O be the circumcenter of triangle ABC , and let points P and Q *be* points on sides CA and AB, *respectively. Let* K , L , and M be the midpoints of segments BP , CQ , and PQ, *respectively* . Let Γ be a circle passing through points K , L , and M. If line PQ is tangent to circle Γ, prove that $OP = OQ$. (2009 *IMO* exam question)



Proof: Let the circumcircle of $\triangle ABC$ be the unit circle, $o=0$, , $\bar{a}=\frac{1}{a}$, $\bar{b}=\frac{1}{b}$, $\bar{c}=\frac{1}{c}$,

$k=\frac{p+b}{2}$, $m=\frac{p+q}{2}$, $l=\frac{c+q}{2}$

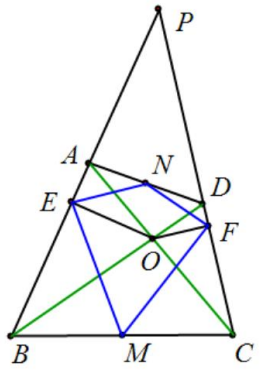
Solve the equation $\begin{vmatrix} a & \bar{a} & 1 \\ p & \bar{p} & 1 \\ c & \bar{c} & 1 \end{vmatrix}=0$ and get $\bar{p}=\frac{a+c-p}{ac}$; Solve the equation

$\begin{vmatrix} a & \bar{a} & 1 \\ q & \bar{q} & 1 \\ b & \bar{b} & 1 \end{vmatrix}=0$ and get $\bar{q}=\frac{a+b-q}{ab}$;

$$8abc(m-l)\overline{(m-l)}(k-m)\overline{(k-m)}\left(\frac{\frac{q-p}{m-l}}{\frac{k-m}{k-l}}-\overline{\left(\frac{\frac{q-p}{m-l}}{\frac{k-m}{k-l}}\right)}\right)=(b-c)(c-p)(b-q)\left(p\bar{p}-q\bar{q}\right).$$

Extended Proposition: Let O be the circumcenter of triangle ABC , and let points P and Q *be* points on sides CA and AB , *respectively. Let* K , L , and M be the midpoints of segments BP , CQ , and PQ , *respectively* . Prove

$16OA\cdot MK\cdot KL\cdot LM\cdot PQ\sin\left(\angle PML-\angle MKL\right)=BC\cdot BQ\cdot CP\cdot\left(OP^2-OQ^2\right)$:



Example 40 : Given an acute angle $\triangle PBC$, $PB \neq PC$. Let A and D *be* points on sides PB and PC, *respectively. Connect lines* AC and BD , intersecting at O. Through point O , *draw* lines $OE \perp AB$ and $OF \perp CD$, with their feet at E and F. *The midpoints of* segments BC and AD *are* M and N , respectively .

(1) If points $EM \cdot FN = EN \cdot FM$ A , B , C , and D *are concentric, prove that:* .

(2) If so , are points $EM \cdot FN = EN \cdot FM$ A , B , C , and D necessarily concentric? Prove your conclusion. (Question from the 24th China Mathematical Olympiad in 2009)

Proof: Let O be the origin, OA be the real axis, $m = \dfrac{b+c}{2}$, $n = \dfrac{a+d}{2}$, $\bar{d} = \dfrac{d\bar{b}}{b}$, and solve for

$$e = \frac{a\left(b-\bar{b}\right)}{2\left(a-\bar{b}\right)},\quad \bar{e} = -\frac{a\left(b-\bar{b}\right)}{2\left(a-b\right)},\quad f = \frac{cd\left(b-\bar{b}\right)}{2\left(bc-\bar{b}d\right)},\quad \bar{f} = -\frac{cd\left(b-\bar{b}\right)}{2b\left(c-d\right)},$$

$$16(a-b)\overline{(a-b)}(c-d)\overline{(c-d)}\Big((e-m)\overline{(e-m)}(f-n)\overline{(f-n)}-(e-n)\overline{(e-n)}(f-m)\overline{(f-m)}\Big)$$

$$+\left(\begin{vmatrix} b & \bar{b} & 1 \\ c & \bar{c} & 1 \\ d & \bar{d} & 1 \end{vmatrix} - \begin{vmatrix} a & \bar{a} & 1 \\ c & \bar{c} & 1 \\ d & \bar{d} & 1 \end{vmatrix}\right)\left(\begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix} - \begin{vmatrix} d & \bar{d} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix}\right)\left(ac-\bar{b}d\right)^2 = 0.$$

Extended proposition: $AB^2 \cdot CD^2 \cdot (EM^2 \cdot FN^2 - EN^2 \cdot FM^2)$

$$= (S_{\triangle BCD} - S_{\triangle ACD}) \cdot (S_{\triangle ABC} - S_{\triangle DBC}) \cdot (OA \cdot OC - OB \cdot OD)^2 .$$

Explanation: According to the identity, when points A , B , C , and D are cocircular, , we can infer that $EM \cdot FN = EN \cdot FM$. Otherwise, this is not necessarily the case. Since $S_{\triangle BCD} - S_{\triangle ACD} \neq 0$, , and $S_{\triangle ABC} - S_{\triangle DBC}$, may be 0, in which case AD is parallel to BC .

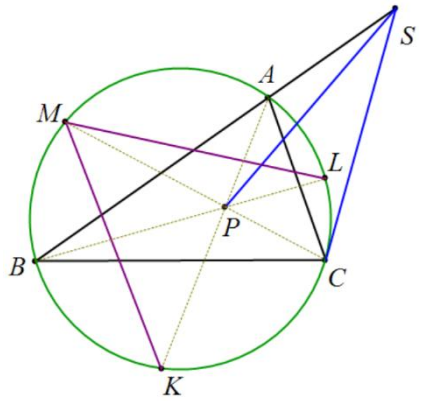


Figure 1

Example 4-1 : As shown in Figure 1, P is a point inside triangle ABC ($AC \neq BC$). AP , BP , and CP intersect the circumcircle O of triangle ABC *at points* K , L , and M , respectively . The tangent to circle O at point C *intersects line* AB at point S. Given that $SC = SP$, prove that $MK = ML$. (2010 *IMO* exam question)

Proof: Let O be the unit circle, P be on the real axis, $k = \dfrac{a-p}{ap-1}$, $l = \dfrac{b-p}{bp-1}$,

$m = \dfrac{c-p}{cp-1}$, $s = \dfrac{ab(c+c)-cc(a+b)}{ab-cc}$,

$$[A,B,C](c-p)\overline{(p-m)}\left((m-k)\overline{(m-k)}-(m-l)\overline{(m-l)}\right)$$

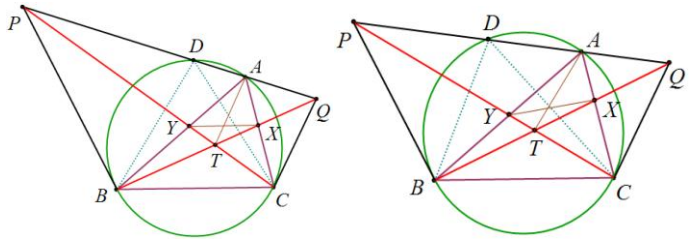
$$=[MKL]\left((c-a)\overline{(c-a)}-(c-b)\overline{(c-b)}\right)\left((s-p)\overline{(s-p)}-(s-c)\overline{(s-c)}\right),$$

Extended Proposition: As shown in Figure 1, P is a point inside triangle ABC ($AC \neq BC$), AP , BP , and CP *intersect the circumcircle* O of triangle ABC at K , L , and M respectively , and the tangent to circle O at point C *intersects line* AB at S. Proof:

$$S_{\triangle ABC} \cdot CP \cdot PM \cdot \left(MK^2 - ML^2\right) = S_{\triangle MKL} \cdot \left(CA^2 - CB^2\right)\left(SP^2 - SC^2\right).$$

Note: The identity reminds us to pay attention to $CA = CB$ the situation where S is at infinity.

Example 42 : As shown in the figure, point D is the midpoint of arc BC *on* the circumcircle O *of acute angle* $\triangle ABC$. *Line* DA and the tangents to circle O through points B and C *intersect at points* P and Q , respectively . BQ intersects AC *at point* X , CP intersects AB at point Y , *and* BQ intersects CP at point T. *Prove* that AT bisects segment XY . (2017 National High School Mathematics League, Paper B)



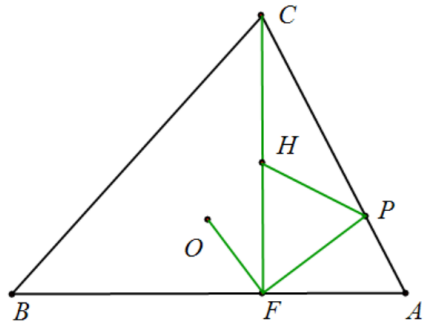
Point D *is the midpoint of* arc BC *on the circumcircle* O of acute angle $\triangle ABC$, which is equivalent to $DC = DB$; AT bisects segment XY , which is equivalent to $S_{\triangle AXT} = S_{\triangle AYT}$. Below, we discard the condition $DC = DB$, set circle O to be the unit circle, and solve the complex coordinates of P , Q , X , Y , and T *according to the constraints* to generate the identity

$$\begin{vmatrix} t & \bar{t} & 1 \\ x & \bar{x} & 1 \\ a & \bar{a} & 1 \end{vmatrix} - \begin{vmatrix} a & \bar{a} & 1 \\ y & \bar{y} & 1 \\ t & \bar{t} & 1 \end{vmatrix} = \begin{vmatrix} x & \bar{x} & 1 \\ y & \bar{y} & 1 \\ t & \bar{t} & 1 \end{vmatrix} \frac{\dfrac{b-0}{b-c}\begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix}}{\dfrac{c-b}{c-0}\begin{vmatrix} d & \bar{d} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix}} \left((d-b)\overline{(d-b)} - (d-c)\overline{(d-c)} \right),$$

Extended proposition: As shown in the figure, points A , B , C , and D *are concentric circles. Line* DA and the tangents of circle O through points B and C *intersect at points* P and Q respectively . The intersection of BQ and AC *is* X , the intersection of CP and AB is Y , and the intersection of BQ and CP is T. Prove:

$$S_{\triangle AXT} - S_{\triangle AYT} = \frac{S_{\triangle XYT}}{BC^2} \frac{S_{\triangle ABC}}{S_{\triangle DBC}} \left(DC^2 - DB^2 \right).$$

Example 43 : For an acute angle $\triangle ABC$, with $BC > CA$, 0 is its circumcenter, H is the orthocenter, *and* F is the foot of the height CH. Draw a perpendicular line OF *through* F, intersecting the side CA at P. Prove that $\angle \mathrm{FHP} = \angle \mathrm{BAC}$. (*IMO* 37 Preliminary)



Assume that the circumscribed circle of $\triangle ABC$ is the unit circle, $h = a + b + c$,

$f = \dfrac{a+b+c-ab/c}{2}$, $\overline{p} = \dfrac{a+c-p}{ac}$, identities:

$$4\left(\frac{a+c}{2}\right)\left(\frac{b+c}{2}\right)(f-p)\overline{(f-p)}\left(\frac{f-o}{f-p}+\overline{\left(\frac{f-o}{f-p}\right)}\right)$$

$$=c(a-b)(h-f)\overline{(h-f)}\left(\frac{\frac{h-p}{h-f}}{\frac{a-b}{a-c}}-\overline{\left(\frac{\frac{h-p}{h-f}}{\frac{a-b}{a-c}}\right)}\right),$$

Extended proposition: Suppose $\triangle ABC$ is an acute triangle with $BC > CA$, 0 is its circumcenter, H is the orthocenter, F is the foot of the altitude CH. P is a point on AC.

Prove: $4OM \cdot ON \cdot FP \cdot FO \cos\angle PFO = OC \cdot AC \cdot HP \cdot HF \sin(\angle PHF - \angle BAC)$.

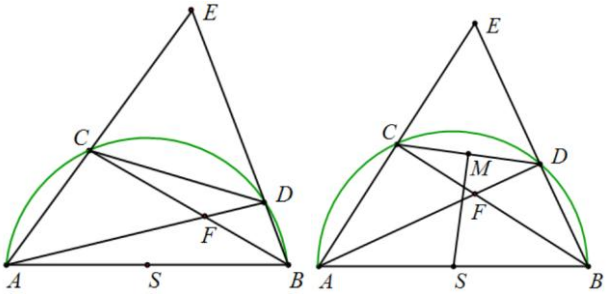
$$2[FPO]\frac{\frac{f-o}{f-p}+\overline{\left(\frac{f-o}{f-p}\right)}}{\frac{f-o}{f-p}-\overline{\left(\frac{f-o}{f-p}\right)}}=[FPC]\left(\frac{\frac{a-b}{a-c}+\overline{\left(\frac{a-b}{a-c}\right)}}{\frac{a-b}{a-c}-\overline{\left(\frac{a-b}{a-c}\right)}}-\frac{\frac{h-p}{h-f}+\overline{\left(\frac{h-p}{h-f}\right)}}{\frac{h-p}{h-f}-\overline{\left(\frac{h-p}{h-f}\right)}}\right),$$

Extended proposition: Suppose $\triangle ABC$ is an acute triangle with $BC > CA$, 0 is its circumcenter, H is the orthocenter, F is the foot of the altitude CH. P is a point on AC.

Prove: $2S_{\triangle FPO} \cot\angle PFO = S_{\triangle FPC}(\cot\angle CAB - \cot\angle FHP)$.

Note: The emphasis on $BC > CA$ is for guarantee $S_{\triangle FPC} \neq 0$.

Example 44 : Given two points C and D on a semicircle with diameter AB *and center* S , such that point C *lies on* arc AD and $\angle CSD = 90°$, and points E and F are the intersections of lines AC and BD , and AD and BC , *respectively* . Prove: $EF = AB$. (2007 Croatian Mathematical Competition)



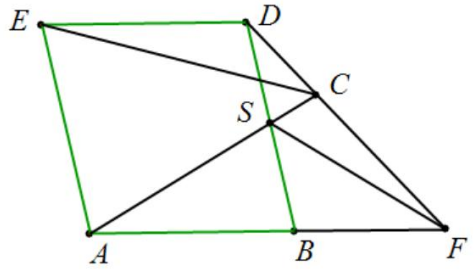
Proof: Assume $S=(0,0)$, $A=(-1,-1)$, $B=(1,1)$, $C=\left(c,\frac{1}{c}\right)$, $D=\left(d,\frac{1}{d}\right)$,

$$(e,\bar{e})=\left(\frac{c-d+2cd}{c+d},\frac{2-c+d}{c+d}\right) \quad , \quad (f,\bar{f})=\left(\frac{-c+d+2cd}{c+d},\frac{2+c-d}{c+d}\right) \quad ,$$

$$(s-d)\overline{(s-d)}\left(\frac{s-c}{s-d}+\overline{\left(\frac{s-c}{s-d}\right)}\right)=\frac{(c+d)^2}{8cd}\left(4-(e-f)\overline{(e-f)}\right),$$

Extended Proposition: As shown in the figure, there are two points C and D on the semicircle with AB as diameter and S *as center* , such that point C *is on* arc AD , E and F *are* the intersections of lines AC and BD , and AD and BC , *respectively* , and M is the midpoint of CD . Proof: $4SB^4\left|\cos\angle CSD\right| = SM^2\cdot\left|AB^2-EF^2\right|$.

Example 45 : Quadrilateral $ABDE$ is a parallelogram. AC and BD intersect at point S, and rays AB and DC intersect at point F. If points A, B, C, and D *are concentric, prove that* $\angle AFS = \angle ECD$. (2010 China-Europe Mathematical Olympiad)



Proof: Assume that $E = A + D - B$, according to the intersection relationship S, F,

$$(f-s)(c-e)\overline{(f-s)(c-e)}\left(\frac{\frac{f-a}{f-s}}{\frac{c-e}{c-d}}-\overline{\left(\frac{\frac{f-a}{f-s}}{\frac{c-e}{c-d}}\right)}\right)$$

$$=-\frac{[ACD]}{[ABC]+[ACD]}(a-b)(c-d)\overline{(a-b)(c-d)}\left(\frac{a-d}{a-b}\frac{c-b}{c-d}-\overline{\left(\frac{a-d}{a-b}\frac{c-b}{c-d}\right)}\right),$$

Extended Proposition: Convex quadrilateral $ABCD$, quadrilateral $ABDE$ is a parallelogram, AC and BD intersect at point S, and rays AB and DC intersect at point F. Proof: $(S_{\triangle ABC}+S_{\triangle ACD})\cdot FA\cdot FS\cdot CE\cdot CD\left|\sin(\angle AFS-\angle ECD)\right|$

$$=S_{\triangle ACD}\cdot AB\cdot BC\cdot CD\cdot DA\left|\sin(\angle DAB+\angle DCB)\right|.$$

Let AB be the real axis, $a=0$, $\bar{c}=\dfrac{c\bar{d}-cf+df-\bar{d}f}{d-f}$, $s=\dfrac{bc(d-f)}{bc-bf-cf+df}$,

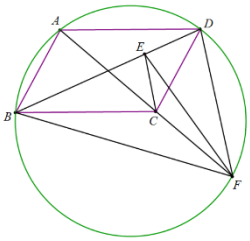
$e=a+d-b$,

$$\frac{\frac{b-a}{b-d}+\overline{\left(\frac{b-a}{b-d}\right)}}{\frac{b-a}{b-d}-\overline{\left(\frac{b-a}{b-d}\right)}}-\frac{\frac{c-a}{c-d}+\overline{\left(\frac{c-a}{c-d}\right)}}{\frac{c-a}{c-d}-\overline{\left(\frac{c-a}{c-d}\right)}}=\frac{b(c-f)}{f(d-c)}\left(\frac{\frac{c-e}{c-d}+\overline{\left(\frac{c-e}{c-d}\right)}}{\frac{c-e}{c-d}-\overline{\left(\frac{c-e}{c-d}\right)}}-\frac{\frac{f-a}{f-s}+\overline{\left(\frac{f-a}{f-s}\right)}}{\frac{f-a}{f-s}-\overline{\left(\frac{f-a}{f-s}\right)}}\right),$$

$$AF\cdot DC(\cot\angle DBA-\cot\angle DCA)=AB\cdot CF(\cot\angle DCE-\cot\angle SFA).$$

$E=A+D-B$, $E=A+D-B$, $E=A+D-B$, $E=A+D-B$,

Example 46 : In parallelogram $ABCD$, let E be a point on diagonal BD , satisfy $\angle ECD = \angle ACB$, and the extension of AC *intersects the circumcircle of* ABD at point F. Prove: $\angle DFE = \angle AFB$. (2014 National Junior High School League Problem)

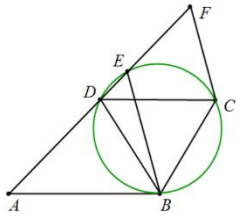


Proof: Let CF be the real axis, $c=1$, $a=-1$, $d=-b$, $\bar{e}=\bar{b}e/b$, $f=b\bar{d}/a=b\bar{b}$,

$$\frac{\frac{f-e}{f-d}+\overline{\left(\frac{f-e}{f-d}\right)}}{\frac{f-e}{f-d}-\overline{\left(\frac{f-e}{f-d}\right)}}-\frac{\frac{f-b}{f-a}+\overline{\left(\frac{f-b}{f-a}\right)}}{\frac{f-b}{f-a}-\overline{\left(\frac{f-b}{f-a}\right)}}=\frac{\frac{c-e}{c-d}+\overline{\left(\frac{c-e}{c-d}\right)}}{\frac{c-e}{c-d}-\overline{\left(\frac{c-e}{c-d}\right)}}-\frac{\frac{c-b}{c-a}+\overline{\left(\frac{c-b}{c-a}\right)}}{\frac{c-b}{c-a}-\overline{\left(\frac{c-b}{c-a}\right)}}$$

Extended proposition: In parallelogram $ABCD$, it is known that E *is a point on* the diagonal BD , and the extension of AC *intersects the circumcircle of* ABD at point F. Proof: $\cot\angle EFD-\cot\angle BFA=\cot\angle ECD-\cot\angle BCA$.

Example 47 : Let quadrilateral $ABCD$ be a trapezoid, $AB \parallel CD$, the circumcircle of triangle BCD *intersects line AD* at points D and E (D is between points A and E), and a line BE *parallel* to point C *intersects* AD at point F. *Prove* : $BC^2 = AD \cdot EF$. (2009-2010 Hungarian Mathematical Olympiad)



Proof: Let the circumscribed circle of $\triangle BCD$ be the unit circle,

$A=\left(a,\overline{a}\right)$, $B=\left(b,\frac{1}{b}\right)$, $C=\left(c,\frac{1}{c}\right)$, $D=\left(d,\frac{1}{d}\right)$, $E=\left(e,\frac{1}{e}\right)$, , $F=\left(f,\overline{f}\right)$,

$$-bcde\left((b-c)\overline{(b-c)}-(a-d)\overline{(e-f)}\right)=-abce+b^2de-bcde+c^2de+abcf-bcdf \quad ,$$

$$\frac{bcde}{b-e}\left([F,E,B]-[C,E,B]\right)=bcd-c^2d+bce-bde-bcf+cdf \ ,$$

$$\frac{bcde}{c-d}\left([D,A,B]-[C,A,B]\right)=abc-bcd-abe+b^2e-bce+cde,$$

Since A , D , and E are collinear, $\overline{a}=\frac{-a+d+e}{de}$; since F , D , and E are collinear,

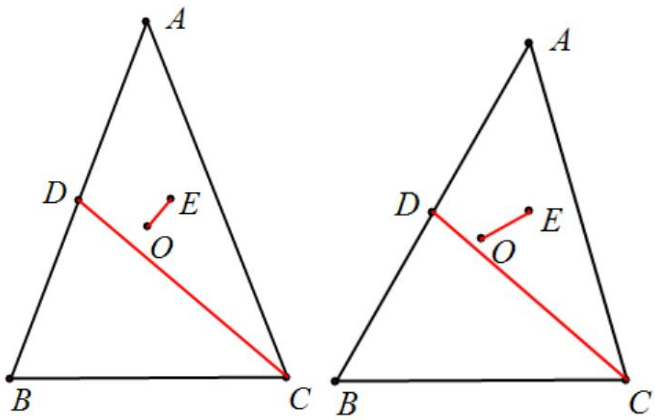
$$\overline{f}=\frac{d+e-f}{de}.$$

$$(b-d)\left((b-c)\overline{(b-c)}-(a-d)\overline{(e-f)}\right)-b\frac{a-d}{b-e}\left([F,E,B]-[C,E,B]\right)$$

$$+d\frac{b-c}{c-d}\left([D,A,B]-[C,A,B]\right)=0.$$

Extended Proposition: In the inscribed quadrilateral $DBCE$, point A *is on the extended* line segment ED , and point F *is on the extended* line segment DE . Proof:

$$\frac{\sin\angle BCD}{2}\left(BC^2-AD\cdot EF\right)=\frac{AD}{BD}\left(S_{\triangle CEB}-S_{\triangle FEB}\right)+\frac{BC}{CD}\left(S_{\triangle CAB}-S_{\triangle DAB}\right).$$



Example 48 : As shown in Figure 6 , let O be the circumcenter of triangle ABC , D be the midpoint of AB , E be the centroid of triangle ACD , and $AB = AC$. Prove that: $OE \perp CD$ (1983 British Olympiad Question)

Proof: Let the circumcircle of $\triangle ABC$ be the unit circle, $o=0$, $d=\frac{a+b}{2}$, $e=\frac{a+d+c}{3}$, $3e\bar{e}\left(\frac{c-d}{o-e}+\overline{\left(\frac{c-d}{o-e}\right)}\right)=(a-c)\overline{(a-c)}-(a-b)\overline{(a-b)}$,

Extended proposition: $6OE\cdot CD\cos\alpha = AC^2 - AB^2$, is α the angle formed by the straight line OE turning to CD .

Example 1 : As shown in Figure 6 , let O be the circumcenter of triangle ABC , D be the midpoint of AB , and E be the centroid of triangle ACD . Prove that $6OE\cdot CD|\cos\alpha| = |AC^2 - AB^2|$, α is the angle between lines OE and CD .

Example 1 : As shown in Figure 6, let O be the circumcenter of triangle ABC , D be the midpoint of AB , E be the centroid of triangle ACD , and $AB = AC$. Prove that: $OE \perp CD$ (1983 British Olympiad exam question)

prove :

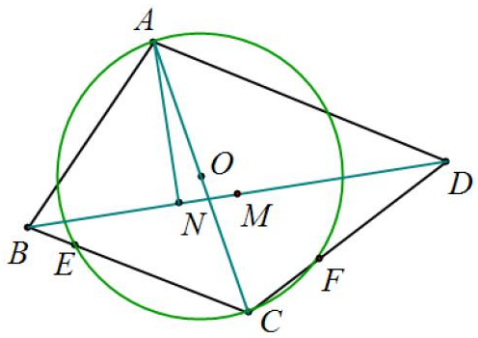
$$6(O-\frac{A+C+\frac{A+B}{2}}{3})(C-\frac{A+B}{2})+3(O-\frac{B+C}{2})(B-C)+3(O-\frac{C+A}{2})(A-C)$$
$$+[(A-B)^2-(A-C)^2]=0.$$

When $(O-\frac{B+C}{2})(B-C)=(O-\frac{C+A}{2})(A-C)=0$, that is $6OE\cdot CD|\cos\alpha| = |AC^2 - AB^2|$, α is the angle between straight lines OE and CD .

From the identity, we can derive a new proposition: Let O be the circumcenter of triangle ABC , D be the midpoint of AB , *and E be* the centroid of $OE \perp CD$ triangle ACD . Prove that $AB = AC$.

From a construction perspective, the equation " from " $AB = AC \Rightarrow OE \perp CD$ is natural . Conversely , constructing a triangle ABC without knowing $AB = AC$ is difficult to guarantee $OE \perp CD$. From a problem-solving perspective, we are more accustomed to using $AB = AC$, and isosceles triangles have many familiar properties. However $OE \perp CD$, the equation *" from"* is difficult to grasp and unsure how to apply. This is

reminiscent of *the Steiner- Lemmers Theorem :* "The two interior bisectors of an isosceles triangle are equal " is easy to prove, while " a triangle with two equal interior bisectors is an isosceles triangle" is more difficult. Therefore , properties like " $AB = AC \Leftarrow OE \perp CD$ from " are relatively rare in traditional geometry . However, using the identity method, such properties can be easily discovered.



Example 49 : As shown in the figure, in quadrilateral $ABCD$, $\angle ABC = \angle ADC < 90°$. The circle O with diameter AC intersects sides BC and CD *at points* E and F , *respectively* . M is the midpoint of BD , and $AN \perp BD$ at point N. Prove that points M , N , E , and F are concentric. (2020 Southeast China Mathematical Olympiad)

Proof: Let the circle be the unit circle, $c=-a$, $m=\dfrac{b+d}{2}$, $\bar{b}=\dfrac{-b+c+e}{ce}$,

$\bar{d}=\dfrac{c-d+f}{cf}$,

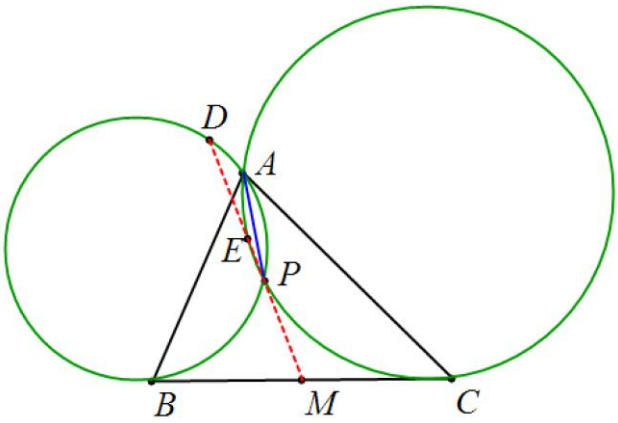
$$8(c-b)(c-d)(m-e)\overline{(m-e)}(n-f)\overline{(n-f)}\left(\frac{\frac{m-f}{m-e}}{\frac{n-f}{n-e}}-\overline{\left(\frac{\frac{m-f}{m-e}}{\frac{n-f}{n-e}}\right)}\right)$$

$$=(0-a)(e-f)\begin{vmatrix} m & \bar{m} & 1 \\ n & \bar{n} & 1 \\ c & \bar{c} & 1 \end{vmatrix}(b-c)\overline{(b-c)}(d-c)\overline{(d-c)}\left(\frac{\frac{b-a}{b-c}}{\frac{d-c}{d-a}}-\overline{\left(\frac{\frac{b-a}{b-c}}{\frac{d-c}{d-a}}\right)}\right),$$

Extended Proposition: As shown in the figure, in quadrilateral $ABCD$, the other intersection points of circle O *with* diameter AC and sides BC and CD *are* E and F respectively . M is the midpoint of BD , and $AN \perp BD$ at point N. Proof:

$$2OA\cdot MF\cdot ME\cdot NF\cdot NE\sin(\angle FME-\angle FNE)$$

$$=S_{\triangle MNC}\cdot EF\cdot AB\cdot DA\sin(\angle ABC-\angle ADC).$$



Example 50 : As shown in the figure, in acute angle $\triangle ABC$, M is the midpoint of side BC. *Point P is inside* $\triangle ABC$, such that AP bisects $CBAC$, and lines MP intersect the circumcircles of ABP and ACP at two points different from point P, D and E. Prove: If $DE = MP$, then $BC = 2\ BP$. (2019 National High School Mathematics Competition, Paper A)

Proof: With A as the origin and AP as the real axis, $m=\frac{b+c}{2}$, $\bar{c}=\frac{bc}{\bar{b}}$, solve the equation $\frac{m-e}{m-p}=\overline{\left(\frac{m-e}{m-p}\right)}$, , $\frac{\frac{a-c}{a-p}}{\frac{e-c}{e-p}}=\overline{\left(\frac{\frac{a-c}{a-p}}{\frac{e-c}{e-p}}\right)}$ and we get $e=\frac{\left(\bar{b}-c\right)\left(\bar{b}p-bc\right)}{\bar{b}^2+bc-2\bar{b}p}$;

$\bar{e}=-\frac{b\left(\bar{b}-c\right)(c-p)}{\bar{b}(b+c-2p)}$ similarly $d=\frac{b\left(\bar{b}-c\right)\left(\bar{b}-p\right)}{\bar{b}^2+bc-2\bar{b}p}$, $\bar{d}=\frac{\left(\bar{b}-c\right)(b-p)}{b+c-2p}$;

set up $p=1$,

$$(b-c)\overline{(b-c)}-4(b-p)\overline{(b-p)}=-\frac{4\bar{b}-4b\bar{b}-4\bar{b}^2+3b\bar{b}^2+b^2c+\bar{b}^2c-bc^2}{\bar{b}},$$

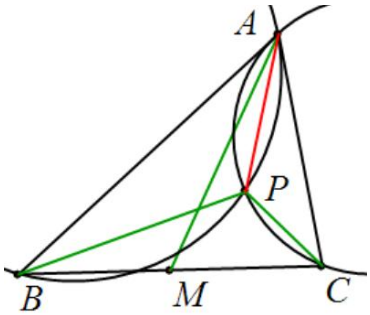
$$(e-d)-(m-p)=-\frac{4\bar{b}-4b\bar{b}-4\bar{b}^2+3b\bar{b}^2+b^2c+\bar{b}^2c-bc^2}{2\left(\bar{b}^2+bc-2\bar{b}\right)},$$

If it were the traditional complex number solution, it would end here. Not only has the original proposition been proved, but the inverse proposition has also been proved.
But from the perspective of identity, we can go one step further: $(b-c)\overline{(b-c)}-4(b-p)\overline{(b-p)}=\frac{2\left(\bar{b}^2+bc-2\bar{b}\right)}{\bar{b}}\left((e-d)-(m-p)\right)$, which can be transformed into

$(b-c)\overline{(b-c)}-4(b-p)\overline{(b-p)}=4\overline{\left(\frac{b+c}{2}-p\right)}((e-d)-(m-p))$, its geometric meaning is:

Extended proposition: As shown in the figure, in acute angle $\triangle ABC$, M is the midpoint of side BC . *Point P is inside* $\triangle ABC$, so that AP bisects $\angle BAC$, and the straight line MP intersects the circumcircles of ABP and ACP at two points *different from point* P , D and E *respectively* . Proof: $BC^2-4BP^2=4MP\cdot(ED-MP)$

Example 5 1 : In triangle ABC , AM is the median. Construct a circle through points A and B *that is tangent to* AC . Construct a circle through points A and C *that is tangent to* AB . The two circles intersect at P. Proof: $\angle CAP=\angle BAM$.



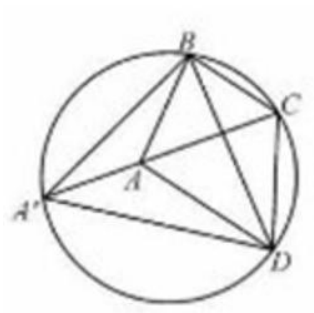
Identity: $2(a-p)\overline{(a-p)}\left(a-\frac{b+c}{2}\right)\overline{\left(a-\frac{b+c}{2}\right)}\left(\frac{\frac{a-c}{a-p}}{\frac{a-\frac{b+c}{2}}{a-b}}-\overline{\left(\frac{\frac{a-c}{a-p}}{\frac{a-\frac{b+c}{2}}{a-b}}\right)}\right)$

$$=(a-p)\overline{(a-p)}(c-a)\overline{(c-a)}\left(\frac{\frac{c-p}{c-a}}{\frac{a-p}{a-b}}-\overline{\left(\frac{\frac{c-p}{c-a}}{\frac{a-p}{a-b}}\right)}\right)$$

$$+(a-p)\overline{(a-p)}(b-a)\overline{(b-a)}\left(\frac{\frac{a-c}{a-p}}{\frac{b-a}{b-p}}-\overline{\left(\frac{\frac{a-c}{a-p}}{\frac{b-a}{b-p}}\right)}\right).$$

Extended proposition:

$2AM\sin(\angle PAC-\angle BAM)=CP\sin(\angle ACP-\angle BAP)+BP\sin(\angle PAC-\angle PBA)$.



Example 52 : In the convex quadrilateral $ABCD$, we have $AB \cdot CD = AD \cdot BC$. Prove: $\angle BAC + \angle CBD + \angle DCA + \angle ADB = 180^{\circ}$. (The 38th Russian Mathematical Olympiad, 2012)

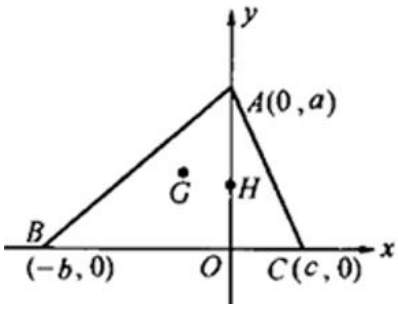
Identity

$$\left((a-c)\overline{(a-c)}(b-d)\overline{(b-d)}\right)^2\left(\frac{a-b}{a-c}\frac{b-c}{b-d}\frac{c-d}{c-a}\frac{d-a}{d-b}-\overline{\left(\frac{a-b}{a-c}\frac{b-c}{b-d}\frac{c-d}{c-a}\frac{d-a}{d-b}\right)}\right)$$

$$=\left(|a-b|^2|c-d|^2-|a-d|^2|b-c|^2\right)|a-b|^2|c-d|^2\left(\frac{a-d}{a-b}\frac{c-b}{c-d}-\overline{\left(\frac{a-d}{a-b}\frac{c-b}{c-d}\right)}\right).$$

Extended proposition: $AC^2 \cdot BD^2 \sin(\angle CAB + \angle DBC + \angle ACD + \angle BDA)$

$$=\left(DA^2 \cdot BC^2 - AB^2 \cdot CD^2\right)\sin(\angle BAD + \angle DCB)$$



Example 53 : As shown in $AB \neq AC, H, G$ Figure 54, the acute angles $\triangle ABC$ are the orthocenter and centroid of the triangle. Given $\frac{1}{S_{\triangle HAB}} + \frac{1}{S_{\triangle HAC}} = \frac{2}{S_{\triangle HBC}}$: . Prove: $\angle AGH = 90^{\circ}$. (2003 Chinese National Training Team Training Problem)

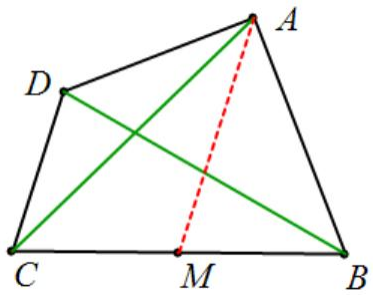
Proof: Let the circumscribed circle of $\triangle ABC$ be the unit circle, then $h = a + b + c$, $g = \frac{a+b+c}{3}, g = \frac{a+b+c}{3}$

$$\frac{1}{\begin{vmatrix} h & \bar{h} & 1 \\ a & \bar{a} & 1 \\ b & \bar{b} & 1 \end{vmatrix}} + \frac{1}{\begin{vmatrix} h & \bar{h} & 1 \\ c & \bar{c} & 1 \\ a & \bar{a} & 1 \end{vmatrix}} - \frac{2}{\begin{vmatrix} h & \bar{h} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix}}$$

$$= \frac{9a^2b^2c^2}{(a-b)(a+b)(a-c)(a+c)(b-c)(b+c)}(g-h)\overline{(g-h)}\left(\frac{g-a}{g-h} + \overline{\left(\frac{g-a}{g-h}\right)}\right),$$

Extended proposition:

$$72AB \cdot BC \cdot CA \cdot HA \cdot HB \cdot HC\left(\frac{1}{S_{\triangle HAB}} + \frac{1}{S_{\triangle HCA}} - \frac{2}{S_{\triangle HBC}}\right) = R^2 \cdot GA \cdot GH \cos\angle HGA.$$



Example 54 : In a convex quadrilateral $ABCD$, $\angle DAB = 90^\circ$, is M the midpoint of $\angle ADC = \angle BAM$ side BC , , prove: $\angle ADB = \angle CAM$. (The 42nd Russian Mathematical Olympiad, 2016)

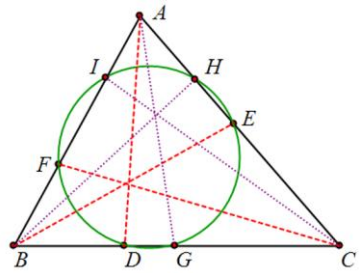
Proof: Let $a = 0$, $\bar{b} = -b$, $\bar{d} = d$,

$$(d-b)\overline{(d-b)}\left(a-\frac{b+c}{2}\right)\overline{\left(a-\frac{b+c}{2}\right)}\left(\frac{\frac{d-a}{d-b}}{\frac{a-\frac{b+c}{2}}{a-c}}-\overline{\left(\frac{\frac{d-a}{d-b}}{\frac{a-\frac{b+c}{2}}{a-c}}\right)}\right)$$

$$=(d-c)\overline{(d-c)}(a-b)\overline{(a-b)}\left(\frac{\frac{d-a}{d-c}}{\frac{a-b}{a-\frac{b+c}{2}}}-\overline{\left(\frac{\frac{d-a}{d-c}}{\frac{a-b}{a-\frac{b+c}{2}}}\right)}\right),$$

Extended proposition: In convex quadrilateral $ABCD$, $\angle DAB = 90^\circ$, M is the midpoint of side BC , prove:

$DB \cdot AC \sin(\angle ADB - \angle MAC) = DC \cdot AB \sin(\angle ADC - \angle BAM)$.



Example 55 : As shown in the figure, the three sides of triangle ABC intersect the circle at six points: D , E , F , G , H , and I. *It is known that* AD , BE , and CF intersect at one point. Prove that AG , BH , and CI intersect at one point.

Proof: Let the circle be the unit circle, then $a=\dfrac{fi(h+e)-he(f+i)}{fi-he}$,

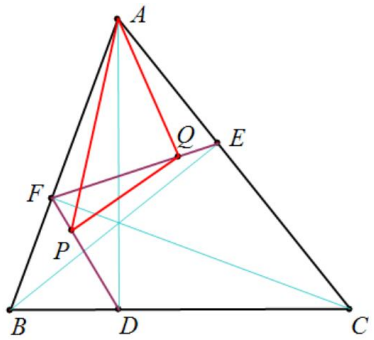
$\bar{a}=\dfrac{e+h-f-i}{eh-fi}$, and similarly $\{b,\bar{b}\}$, $\{c,\bar{c}\}$,

There is an identity

$$[GHI]([AFC][DBE]-[DFC][ABE])=[DEF]([GIC][ABH]-[AIC][GBH]),$$

Extended proposition: As shown in the figure, the three sides of $\triangle ABC$ *intersect the circle* at six points: D , E , F , G , H , and I. Prove: $S_{\triangle GHI}\left(S_{\triangle AFC}S_{\triangle DBE}-S_{\triangle DFC}S_{\triangle ABE}\right)=S_{\triangle DEF}\left(S_{\triangle GIC}S_{\triangle ABH}-S_{\triangle AIC}S_{\triangle GBH}\right)$.

Example 56 : Let AD , BE , and CF *be* the three altitudes of acute angle $\triangle ABC$, *and let* P and Q *lie on* line segments DE and EF , *respectively* . Prove that $\angle PAQ = \angle DAC$ if and only if AP bisects $\angle QPF$. (Extension of the 2006 German National Team Selection Exam)



the circumscribed circle of $d = \dfrac{a+b+c-bc/a}{2}$ $\triangle ABC$ *be the unit circle,* , $e = \dfrac{a+b+c-ac/b}{2}$, $f = \dfrac{a+b+c-ba/c}{2}$, solve the equation $\dfrac{d-p}{d-f} = \overline{\left(\dfrac{d-p}{d-f}\right)}$ and we get $\overline{p}$; solve the equation and $\dfrac{f-q}{e-f} = \overline{\left(\dfrac{f-q}{e-f}\right)}$ we get $\overline{q}$.

$$(a-b)^4(a-c)^2(p-a)^2\overline{(p-a)}^2\left(\frac{\frac{p-f}{p-a}}{\frac{p-a}{p-q}} - \overline{\left(\frac{\frac{p-f}{p-a}}{\frac{p-a}{p-q}}\right)}\right)$$

$$= 2a^3b^2c[PAB]^2(a-d)(a-q)\overline{(a-d)(a-q)}\left(\frac{\frac{a-c}{a-d}}{\frac{a-q}{a-p}} - \overline{\left(\frac{\frac{a-c}{a-d}}{\frac{a-q}{a-p}}\right)}\right),$$

Extended proposition: Let AD , BE , and CF *be* the three altitudes of acute angle $\triangle ABC$, *and let* P and Q *be on* line segments DE and EF *respectively* . Prove:

$$AB^4 \cdot AC \cdot PF \cdot PA \cdot PQ \sin(\angle FPA - \angle APQ) = 32 S_{\triangle PAB}{}^2 \cdot R^2 \cdot AD \cdot AQ \sin(\angle CAD - \angle QAP)$$

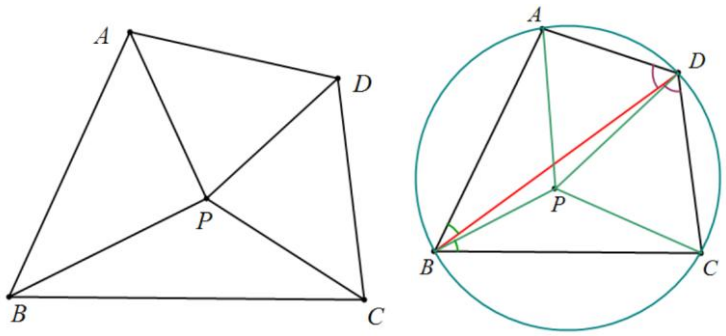


Figure 30

Example 57 : In convex quadrilateral $ABCD$, diagonal BD is neither a bisector of $\angle ABC$ nor a bisector of $\angle CDA$. *Point* P in quadrilateral $ABCD$ satisfies $\angle PBC = \angle DBA$ and $\angle PDC = \angle BDA$. Prove that $PA = PC$ *is the necessary and sufficient condition for points* A , B , C , and D to be concircular . (Question from the 45th *IMO exam, 2004*)

Proof: Because $(p-a)\overline{(p-a)}-(p-c)\overline{(p-c)}=\dfrac{(a-c)\left(p-ac\overline{p}\right)}{ac}=0$, so

$PA=PC \Leftrightarrow p-ac\overline{p}=0$.

First prove that points $\Rightarrow PA=PC$ A , B , C , and D *are concentric* .

From this $\dfrac{\dfrac{b-p}{b-c}}{\dfrac{b-a}{b-d}}=\dfrac{\dfrac{1/b-\overline{p}}{1/b-1/c}}{\dfrac{1/b-1/a}{1/b-1/d}}$, $\dfrac{\dfrac{d-c}{d-p}}{\dfrac{d-b}{d-a}}=\dfrac{\dfrac{1/d-1/c}{1/d-\overline{p}}}{\dfrac{1/d-1/b}{1/d-1/a}}$ we can get $p=\dfrac{ac+bd}{b+d}$,

$\overline{p}=\dfrac{ac+bd}{ac(b+d)}$,

Obviously satisfied $p-ac\overline{p}=0$.

Now prove that $\overline{d}=\dfrac{1}{d}$ points $\Leftarrow PA=PC$ A , B , C , and D *are concentric* . Assume that the circumcircle of $\triangle ABC$ is the unit circle .

Solve the system of equations $p-ac\overline{p}=0$, $\dfrac{\dfrac{b-p}{b-c}}{\dfrac{b-a}{b-d}}=\dfrac{\dfrac{1/b-\overline{p}}{1/b-1/c}}{\dfrac{1/b-1/a}{1/b-\overline{d}}}$, $\dfrac{\dfrac{d-c}{d-p}}{\dfrac{d-b}{d-a}}=\dfrac{\dfrac{\overline{d}-1/c}{\overline{d}-\overline{p}}}{\dfrac{\overline{d}-1/b}{\overline{d}-1/a}}$,

and we get $p=\dfrac{ac+bd}{b+d}$, $\overline{p}=\dfrac{ac+bd}{ac(b+d)}$, $\overline{d}=\dfrac{1}{d}$ (the other two solutions are $p=a$ or $p=c$, which do not match the question, so discard them), so $PA=PC \Rightarrow \overline{d}=\dfrac{1}{d}$, that is, the four points A , B , C , and D are concentric.

Proof: $A=(1,1)$, $B=(0,0)$, $C=(c,\bar{c})$, $D=(d,\bar{d})$, solve the equation

$$\frac{\frac{b-p}{b-c}}{\frac{b-a}{b-d}}=\overline{\left(\frac{\frac{b-p}{b-c}}{\frac{b-a}{b-d}}\right)},\quad \frac{\frac{d-c}{d-p}}{\frac{d-b}{d-a}}=\overline{\left(\frac{\frac{d-c}{d-p}}{\frac{d-b}{d-a}}\right)}$$, and get

$$P=\left(\begin{array}{l}\dfrac{c\left(-\bar{c}d^2+d^2\bar{d}+\bar{c}d^2\bar{d}+c\bar{d}^2-d\bar{d}^2-cd\bar{d}^2\right)}{d\left(-\bar{c}d-c\bar{c}d+\bar{c}d^2+c\bar{d}+c\bar{c}\bar{d}-c\bar{d}^2\right)},\\ \dfrac{\bar{c}\left(\bar{c}d^2-d^2\bar{d}-\bar{c}d^2\bar{d}-c\bar{d}^2+d\bar{d}^2+cd\bar{d}^2\right)}{\bar{d}\left(\bar{c}d+c\bar{c}d-\bar{c}d^2-c\bar{d}-c\bar{c}\bar{d}+c\bar{d}^2\right)}\end{array}\right),$$

Identity: $$[ABD]\left(|p-a|^2-|p-c|^2\right)|c-d|^2|b-d|^4\left(\frac{\frac{b-a}{b-d}}{\frac{b-d}{b-c}}-\overline{\left(\frac{\frac{b-a}{b-d}}{\frac{b-d}{b-c}}\right)}\right)$$

$$=[CDP][ABCD]\left|(b-c)\overline{(b-d)}-(b-d)\overline{(b-a)}\right|^2,$$

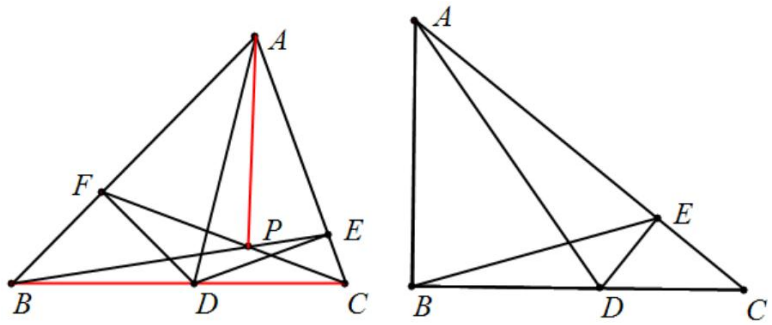
Extended Proposition: There is a point P *in* convex quadrilateral ABCD *such* that $\angle$ PBC = $\angle$ DBA *and* $\angle$ *PDC* = $\angle$ BDA . Proof :

$$S_{ABD}\cdot\left(PA^2-PC^2\right)\cdot CD\cdot\sin\left(\angle ABD-\angle DBC\right)$$

$$=S_{CDP}\cdot\left(BA^2+BC^2-2BA\cdot BC\cos\angle DBP\right)\cdot AD\cdot\sin\left(\angle BAD+\angle BCD\right).$$

Through the identity, we can clearly understand the intention of the question conditions. $\angle ABD-\angle DBC=BA^2+BC^2-2BA\cdot BC\cos\angle DBP=0$, can it be realized $PA^2-PC^2=0\Leftrightarrow\sin\left(\angle BAD+\angle BCD\right)=0$.

Example 58 : Let D *be a point on* side BC of a non-right triangle ABC . From point D, draw perpendiculars to AC and AB, with their feet at E and F *respectively* . *Lines* BF *and* CE *intersect* at point P. Prove that $AP \perp BC$ if and only if AD bisects $\angle BAC$ (2006 Romanian National Team Selection Exam Questions)



Proof: Let $a=0$, $d=1$, $\bar{c}=\dfrac{b-\bar{b}-c+\bar{b}c}{b-1}$, find the coordinates of E , F , and P ,

and the identity $[BDP]|a-d|^4\left(\dfrac{\frac{a-c}{a-d}}{\frac{a-d}{a-b}}-\overline{\left(\dfrac{\frac{a-c}{a-d}}{\frac{a-d}{a-b}}\right)}\right)$

$$=\frac{(c-1)}{2(b-c)}[BDA]^2|b-c|^2\left(\frac{p-a}{b-c}+\overline{\left(\frac{p-a}{b-c}\right)}\right),$$

Extended proposition: In $\triangle ABC$, D is a point on side BC . Draw perpendicular lines to AC and AB *from point* D , with feet of perpendiculars at E and F *respectively* . Straight *lines* BF and CE intersect at point P. Prove $BP\sin\angle PBC\sin(\angle CAD-\angle DAB)=PA\sin\angle CAD\sin\angle BAD\cos\alpha$.

Explanation: If $\angle B = 90°$ (Figure 2), B , F , and P converge into a point $[BDP]=0$. Therefore, the original question emphasizes that $\triangle ABC$ is not a right triangle.

Example 59 : As shown in Figure 1 , construct squares $ABDE$ and $CAFG$ from the two sides AB and AC *of* right triangle ABC . Squares BG intersect AC at H , and *square* CD intersects AB at I. Prove that $AH = AI$.

Proof: From the similarity of triangles $\frac{AH}{FG} = \frac{BA}{BF}$, we can get , $\frac{AI}{ED} = \frac{CA}{CE}$ so $AH = AI$.

Another proof: To prove $AH = AI$, we only need to prove $S_{\triangle HBF} = S_{\triangle IEC}$ (because the two triangles have the same base); and $S_{\triangle HBF} = S_{\triangle GBA} = S_{\triangle CBA}$, $S_{\triangle IEC} = S_{\triangle DAC} = S_{\triangle BAC}$, so $S_{\triangle HBF} = S_{\triangle IEC}$, $AH = AI$.

This proof may seem more complicated, but it requires less knowledge and is understandable to elementary school students.

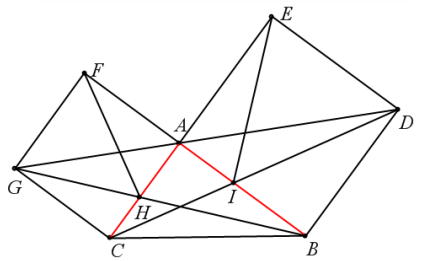


Figure 1

Proof: Let $a = 0$, $b = 1$, $e = \mathrm{i}$, $d = b + e$, $f = -c\mathrm{i}$, $g = c + f$, solution is

$$h = \frac{c(1+\mathrm{i})(c-\mathrm{i}\bar{c})}{\mathrm{i}c - \mathrm{i}\bar{c} + 2c\bar{c}} \quad , \quad i = \frac{(1-\mathrm{i})(c-\mathrm{i}\bar{c})}{-2\mathrm{i} + c - \bar{c}} \quad , \quad (i\bar{i} - h\bar{h})[BFG]^2[CED]^2$$

$$= \left((a-c)\overline{(a-c)}\left(\frac{a-b}{a-c} + \overline{\left(\frac{a-b}{a-c}\right)} \right) \right)^2 (b^2 - c\bar{c})[ABG]^2,$$

Extended proposition:

Figure 1 , draw squares $ABDE$ and $CAFG$ outward from the two sides AB and AC of triangle ABC . BG intersects AC at H , and CD intersects AB at I. Prove that $4(AH^2 - AI^2)S_{\triangle BFG}{}^2 S_{\triangle CDE}{}^2 = (AB \cdot AC \cos A)^2 (AB^2 - AC^2) S_{\triangle ABG}{}^2$:

Another proof: $S_{\triangle ADC} = S_{\triangle ADI} + S_{\triangle AIC}$, there is

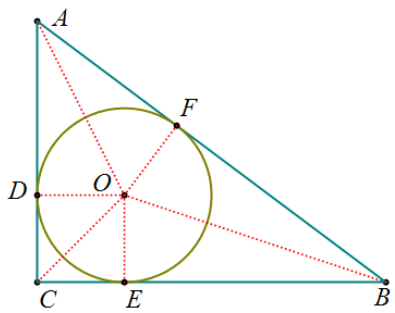
$$\frac{1}{2} AD \cdot AC \sin(45^\circ + A) = \frac{1}{2} AD \cdot AI \sin 45^\circ + \frac{1}{2} AI \cdot AC \sin A,$$

$$AI = \frac{\sqrt{2} AB \cdot AC \cdot \sin(45^\circ + A)}{AB + AC \cdot \sin A}, AH = \frac{\sqrt{2} AB \cdot AC \sin(45^\circ + A)}{AC + AB \sin A}$$

$AP = AQ$, which is equivalent to $AB + AC \cdot \sin\alpha = AC + AB \cdot \sin\alpha$, that is $(AB - AC)(1 - \sin A) = 0$, so $AB = AC$ or $\angle BAC = 90^\circ$.

Example 60 : In $\angle ACB = 90^\circ$ triangle ABC , , its incircle O is tangent to the three sides at D , E , and F. Prove that: $S_{\triangle ABC} = AF \cdot BF$.

Proof 1: Let $AD = AF = m$, $BE = BF = n$, $BC = a$, $AC = b$, then $a^2 + b^2 = (m+n)^2$, $m - n = b - a$, the solution is $mn = \frac{1}{2}ab$. Therefore $S_{\triangle ABC} = \frac{1}{2}ab = AF \cdot BF$.



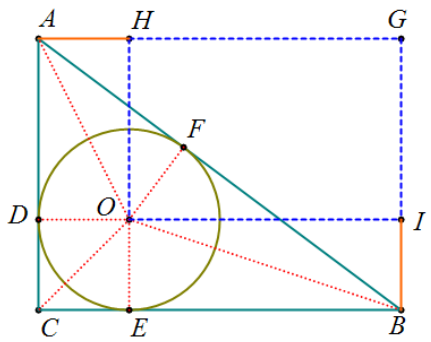
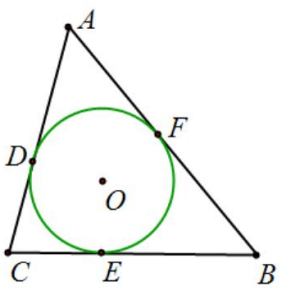


Figure 7 Figure 8

Proof 2: $S_{\triangle AOF} = S_{\triangle AOD} = S_{\triangle AOH}$, $S_{\triangle BOF} = S_{\triangle BOE} = S_{\triangle BOI}$, therefore $S_{\triangle ABC} + S_{OIGH} = S_{CBGA} = 2S_{\triangle ABC}$, therefore $S_{\triangle ABC} = S_{OIGH} = AD \cdot BE = AF \cdot BF$.

the two proofs , Proof 2 requires additional diagrams, but Proof 2 does have its significance. Because of this intuitive understanding, the conclusion can be seen at a glance.



Proof: Let the inscribed circle be the unit circle, then $\bar{d} = \frac{1}{d}$, $\bar{e} = \frac{1}{e}$, $\bar{f} = \frac{1}{f}$,

$a=\frac{2df}{d+f}$, $\bar{a}=\frac{2}{d+f}$, $b=\frac{2ef}{e+f}$, $\bar{b}=\frac{2}{e+f}$, $c=\frac{2ed}{e+d}$, $\bar{c}=\frac{2}{e+d}$, identity:

$$\frac{\frac{1}{4}\begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix}}{(a-f)\overline{(f-b)}}=\frac{d-e}{d+e},$$

$\triangle ABC$, its incircle O is tangent to the three sides at D , E , and F. Prove that: $S_{\triangle ABC}=AF\cdot FB\cot\frac{C}{2}$.

What if $\angle C$ is not 90 ° ? Will there be a conclusion $S_{\triangle ABC}=AF\cdot FB\sin C$? It turns out that the guess is wrong. Among the common special trigonometric function values, in addition to $\sin 90^\circ=1$, there are also $\tan 45^\circ=\cot 45^\circ=1$.

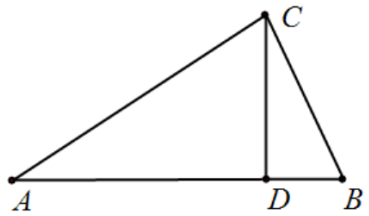
Controversial Issue

Example 1 : As shown in Figure 1, in triangle ABC , D is a point on line segment AB . Prove that:

(1) If $CA \perp CB$, $AD \cdot AB = AC^2$, then $DB \cdot DA = DC^2$.

(2) like $CA \perp CB$, $DB \cdot DA = DC^2$,but $AD \cdot AB = AC^2$.

(3) If $DB \cdot DA = DC^2$, $AD \cdot AB = AC^2$, then $CA \perp CB$. (Complex number identity extension projection theorem)



Proof: Let $A=(a,a)$, $B=(b,b)$, $C=(c,\bar{c})$, $D=(d,d)$,

$$(c-a)\overline{(c-a)}\left(\frac{c-b}{c-a}+\overline{\left(\frac{c-b}{c-a}\right)}\right)=2ab-ac-bc-a\bar{c}-b\bar{c}+2c\bar{c},$$

$$(a-c)\overline{(a-c)}-(a-d)(a-b)=ab-ac-a\bar{c}+c\bar{c}+ad-bd,$$

$$(d-c)\overline{(d-c)}-(a-d)(d-b)=ab+c\bar{c}-ad-bd-cd-\bar{c}d+2d^2,$$

First solve $2ab-ac-bc-a\bar{c}-b\bar{c}+2c\bar{c}=0$, we get $\bar{c}=\dfrac{2ab-ac-bc}{a+b-2c}$, so

$$\frac{(a-c)\overline{(a-c)}-(a-d)(a-b)}{(d-c)\overline{(d-c)}-(a-d)(d-b)}=-\frac{a-b}{a+b-2d}.$$

$$\frac{\dfrac{a+b-2d}{a-b}\left((a-c)\overline{(a-c)}-(a-d)(a-b)\right)+(d-c)\overline{(d-c)}-(a-d)(d-b)}{(c-a)\overline{(c-a)}\left(\dfrac{c-b}{c-a}+\overline{\left(\dfrac{c-b}{c-a}\right)}\right)}=\frac{a-d}{a-b},$$

Extended proposition: As shown in Figure 1, in $\triangle ABC$, D is a point on line segment

$$\frac{DA-DB}{AB}\left(AC^2-AD\cdot AB\right)+\left(DC^2-AD\cdot DB\right)=2\frac{AD}{AB}\left(CA\cdot CB\cos C\right)AB\text{. Prove that: .}$$

Explanation: Through the identity equation, we find that if $CA \perp CB$ $DB \cdot DA = DC^2$, then $AD \cdot AB = AC^2$ D *is* the midpoint *of* AB .

Example 2 : In triangle ABC , AD and AH are the median and height, respectively. Prove that: $\angle CAD = \angle HAB \Leftrightarrow AB \perp AC$. (2004 Turkish Mathematical Competition)

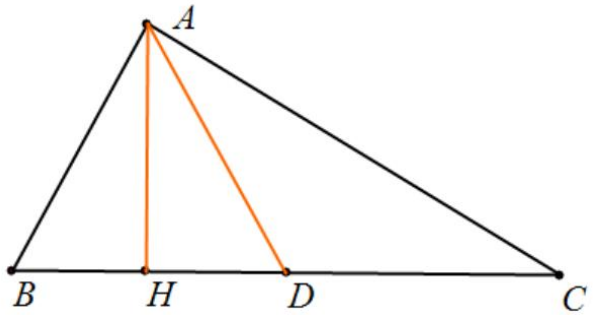


Figure 1

Proof: Let BC be the real axis, HA be the imaginary axis,

$$(a-d)(-a-d)(a-h)(-a-h)\left(\frac{\frac{a-c}{a-d}}{\frac{a-h}{a-b}}-\frac{\frac{-a-c}{-a-d}}{\frac{-a-h}{-a-b}}\right)$$

$$=\frac{a}{2(b-c)}(a-b)(-a-b)\left(\frac{a-c}{a-b}+\frac{-a-c}{-a-b}\right)\left((a-b)(-a-b)-(a-c)(-a-c)\right),$$

Note: $\angle CAD = \angle HAB \Leftrightarrow AB \perp AC$ Or $A = 90°$.
the acute angle ABC , AD and AH are the median and altitude respectively. Prove that

$$2BC \cdot AD \sin\left(\angle CAD - \angle HAB\right) = \left(AC^2 - AB^2\right)\cos A:$$

Why do we need to add the acute angle restriction? Because the conclusion is true in Figures 1 and 2, but not in Figure 3. Originally, H was on the right side of line AB , but in Figure 3 it is on the left. The order of the lines has changed, so the conclusion needs to be changed $2BC \cdot AD \sin\left(\angle CAD + \angle HAB\right) = \left(AC^2 - AB^2\right)\cos A$.

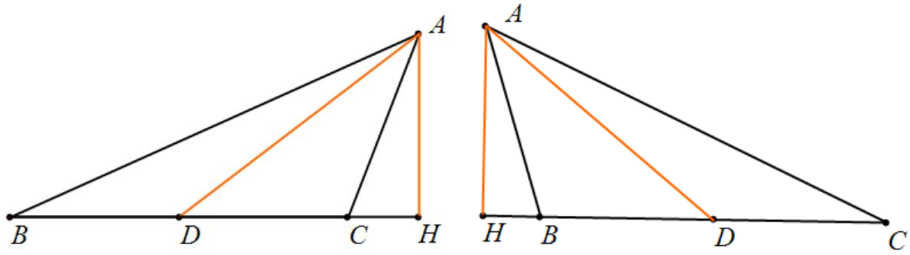


Figure 2 Figure 3

Example 3 : In triangle ABC , CH is the height. Given $\frac{BC^2}{AC^2}=\frac{BH}{AH}$: $\angle C=90°$ or $\angle A=\angle B$.

Proof: Let the circumcircle of $\triangle ABC$ be the unit circle, $h=\frac{a+b+c-ab/c}{2}$,

$$4\left((a-c)^2\overline{(a-c)}^2(b-h)\overline{(b-h)}-(b-c)^2\overline{(b-c)}^2(a-h)\overline{(a-h)}\right)$$

$$=\frac{(a-b)^5(a-c)}{a^3b^2(b-c)}\left(\frac{\frac{a-c}{a-b}}{\frac{b-a}{b-c}}-\overline{\left(\frac{\frac{a-c}{a-b}}{\frac{b-a}{b-c}}\right)}\right)\left(\frac{\frac{a-c}{a-b}}{\frac{b-a}{b-c}}+\overline{\left(\frac{\frac{a-c}{a-b}}{\frac{b-a}{b-c}}\right)}\right)\left(\frac{c-b}{c-a}+\overline{\left(\frac{c-b}{c-a}\right)}\right),$$

$\triangle ABC$, O is the circumcenter, CH is the height , prove: $(CA^2\cdot BH)^2-(CB^2\cdot AH)^2=2AB\cdot OC\cdot CA^2\cdot BC^2\sin(\angle A-\angle B)\cos(\angle A-\angle B)\cos C$.

Note: After establishing the identity, it's easy to spot a flaw in the original question. The corrected answer is: In triangle ABC , CH is the height. Given $\frac{BC^2}{AC^2}=\frac{BH}{AH}$: Prove: $\angle C=90°$ or $\angle A=\angle B$ or $\angle A$ and $\angle B$ differ by $90°$.

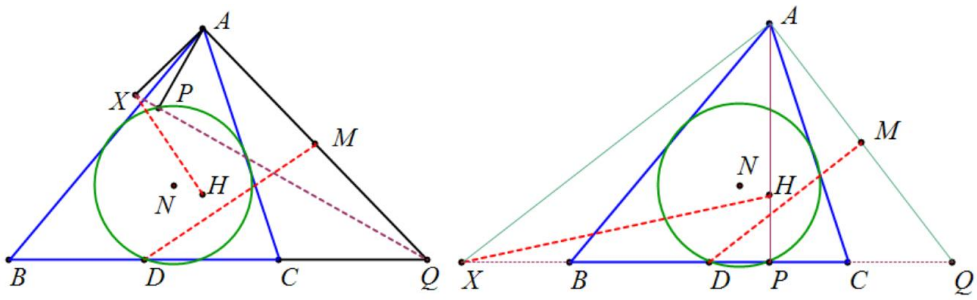


Figure 1 Figure 2

Example 4 : As shown in Figure 1 , P is a point on the nine-point circle of triangle ABC. Draw a perpendicular line through P *to* AP, intersecting line BC at *point* Q. Draw *a* perpendicular line through A to AQ , intersecting line PQ at point X. Let H *be* the orthocenter of triangle ABC , *and* D and M *be the midpoints of* BC and AQ , respectively . Prove that $HX \perp DM$. (Question from the 2025 Chinese National Team Selection Exam)

Suppose the circumscribed circle of $\triangle ABC$ is the unit circle, $O=0$, $H=A+B+C$, $N=\dfrac{H}{2}$, according to the problem, find the complex coordinates of the point and generate the identity equation

$$[PBC]|d-m|^2\left(\frac{h-x}{d-m}+\overline{\left(\frac{h-x}{d-m}\right)}\right)=2[ABC]\left(|p-n|^2-|d-n|^2\right).$$

Extended proposition:

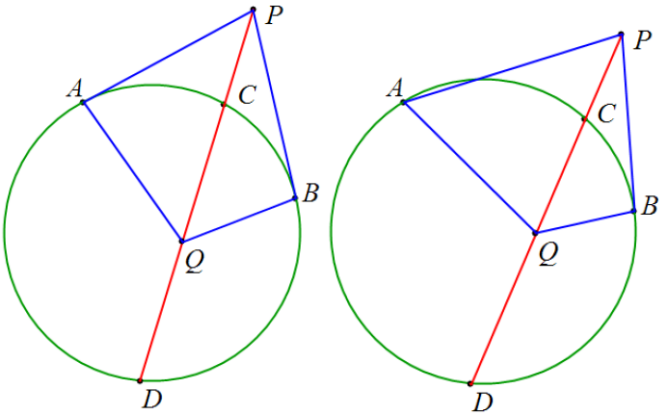
Figure 2 , in triangle ABC , a perpendicular line through P to AP intersects BC at Q , and a perpendicular line through A *to* AQ intersects PQ at X. Let H be the orthocenter of triangle ABC , N be the center of the nine-point circle of triangle ABC , and D and M *be the midpoints of* BC and AQ respectively : Prove $S_{\triangle PBC}\cdot DM\cdot HX\left|\cos\alpha\right|=S_{\triangle ABC}\cdot\left|NP^2-ND^2\right|$:

Generally speaking, candidates will not use this counterexample to overturn the question.

But what if a candidate gives a counterexample and proves that there is a problem with the question, will you give him points or not?

According to the computer's corrections, any situation where P falls on BC could be problematic. This also applies if P *coincides* with D. This is because the right side of the equation is 0, but there are two possible zero terms on the left side, which need to be eliminated.

Example 5 : As shown in the figure, two tangents and one secant are drawn through an external point P , *with the tangent points being* A and B. *The* secant intersects the circle at points C and D , *with* C being between P and D. Take a point Q on chord CD such that $\angle DAQ = \angle PBC$. Prove that $\angle DBQ = \angle PAC$. (Question from the 2003 National High School Mathematics League)



Proof: Let the circle be the unit circle, solve the equation $\frac{c-p}{c-d} = \overline{\left(\frac{c-p}{c-d}\right)}$, and get $\overline{p} = \frac{c+d-p}{cd}$. Similarly, we get . $\overline{q} = \frac{c+d-q}{cd}$

Generating identity: $\frac{\frac{b-c}{d-b}}{\frac{a-c}{a-d}} |b-q|^2 |a-p|^2 \left(\frac{\frac{b-d}{b-q}}{\frac{a-p}{a-c}} - \overline{\left(\frac{\frac{b-d}{b-q}}{\frac{a-p}{a-c}} \right)} \right)$

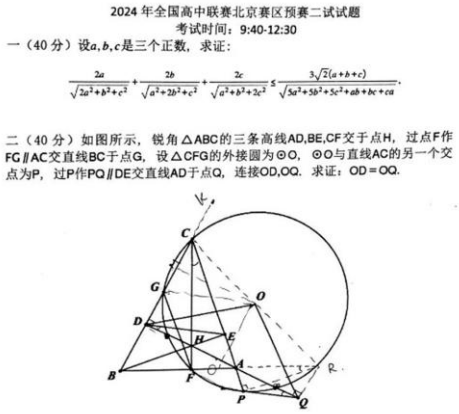
$$= |a-d|^2 |b-c|^2 \left(\frac{\frac{a-q}{a-d}}{\frac{b-c}{b-p}} - \overline{\left(\frac{\frac{a-q}{a-d}}{\frac{b-c}{b-p}} \right)} \right).$$

Extended proposition: It is known that points A , B , C , and D *are concentric,* Q *is on* line segment CD , and P is on the extended line $AQ \cdot BP \sin(\angle DAQ - \angle PBC) = AP \cdot BQ \sin(\angle DBQ - \angle PAC)$ of line segment CD .

Prove that: .

Research has found that the condition that PA and PB are tangent lines is redundant. This problem was tested in the 2003 National High School Mathematics League. As a professional competition, highly skilled testers create the questions, and afterward, dedicated testers assess the scientific nature and difficulty of the questions. Even after undergoing multiple rounds of scrutiny, redundant conditions still frequently occur. This stems from a lack of research on whether test conditions are redundant. Using a condition to prove a conclusion in a derivation does not necessarily mean that the

condition is necessary. A different approach might allow the conclusion to be derived without the condition. However, using the identity algorithm described in this article, by separating and then reassembling all the conditions, it is easy to identify which conditions are redundant.



2024 年全国高中联赛北京赛区预赛二试试题

考试时间：9:40-12:30

一（40 分）设a,b,c是三个正数，求证：

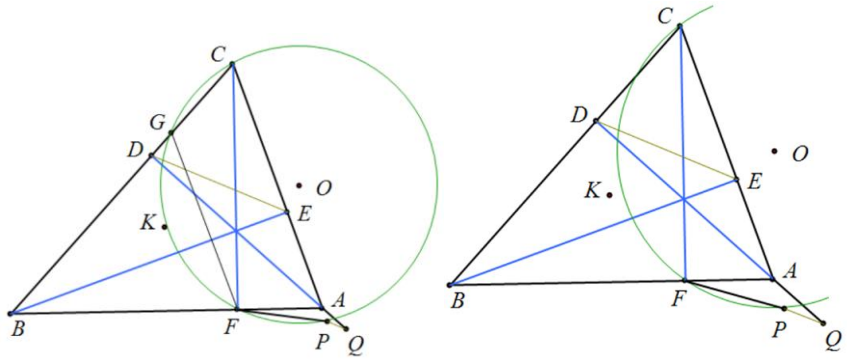
$$\frac{2a}{\sqrt{2a^2+b^2+c^2}}+\frac{2b}{\sqrt{a^2+2b^2+c^2}}+\frac{2c}{\sqrt{a^2+b^2+2c^2}}\le\frac{3\sqrt{2}(a+b+c)}{\sqrt{5a^2+5b^2+5c^2+ab+bc+ca}}.$$

二（40 分）如图所示，锐角△ABC的三条高线AD,BE,CF交于点H，过点F作FG∥AC交直线BC于点G，设△CFG的外接圆为⊙O，⊙O与直线AC的另一个交点为P，过P作PQ∥DE交直线AD于点Q，连接OD,OQ．求证：OD＝OQ.

AI provided by Professor Zhu Songchun's team at Peking University

arXiv preprint arXiv :2412.10673

Example 6 : As shown in the figure, the three altitudes of acute angle △ *ABC are AD* , *BE* , and *CF.* Draw *FG* // *AC through* point *F* , intersecting line *BC* at point *G.* Let the circumcircle *O of* △ *CFG* intersect line *AC* at another point at *P.* Draw *PQ* // *DE through P* , intersecting line *AD* at point *Q.* Connect *OD* and *OQ* . Prove that *OD* = *OQ* . (Question from the 2024 National High School League Beijing Division)



Proof: Let the circumcircle of △ *ABC* be the unit circle,

$$D=\left(\frac{a^2+ab+ac-bc}{2a},-\frac{a^2-ab-ac-bc}{2abc}\right),$$

$$E=\left(\frac{ab+b^2-ac+bc}{2b},\frac{ab-b^2+ac+bc}{2abc}\right),$$

$$F=\left(\frac{-ab+ac+bc+c^2}{2c},\frac{ab+ac+bc-c^2}{2abc}\right),\qquad P=\left(p,\frac{a+c-p}{ac}\right),$$

$$Q=\left(q,-\frac{a^2-bc-aq}{abc}\right) \quad , \quad O=\left(\frac{-ab+ac+bc+c^2+2bp}{2(b+c)},\frac{ab+3ac+bc+c^2-2cp}{2ac(b+c)}\right) \quad ,$$

$$[ABK][BCK]\left(|o-d|^2-|o-q|^2\right)=[BCQ]([QDE]-[PDE])$$

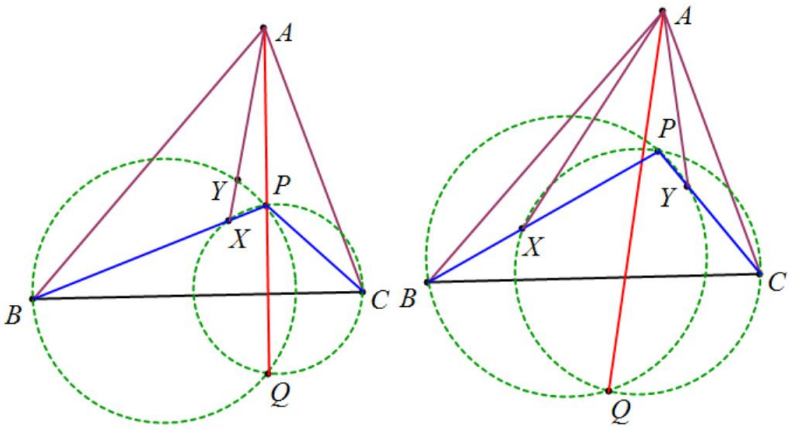
Extended proposition: As shown in the figure, the three altitudes of acute angle $\triangle ABC$ *are* AD , BE , and CF , and point P *is on the* extended line of line segment CA. Let the circumscribed circle O of $\triangle$ CF P *be* , and take point Q *on* line AD . Prove: $S_{\triangle ABK}\cdot S_{\triangle BCK}\left(OD^2-OQ^2\right)=S_{\triangle BCQ}\cdot KA^2\left(S_{\triangle PDE}-S_{\triangle QDE}\right)$.

Note: If we remove FG // AC , we can further remove G without affecting the conclusion.

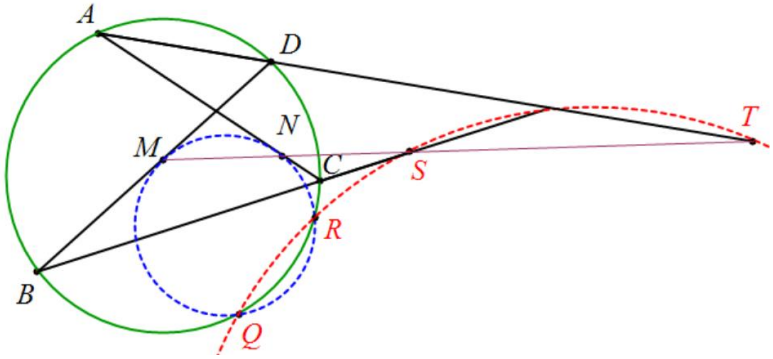
Example 7 : In triangle ABC , $AB > AC$. Two points X and Y in triangle ABC lie on the bisector of $\angle BAC$, *and* $\angle ABX = \angle ACY$. Let the extension of *BX intersect line segment CY* at point *P. Let the circumcircle* ω_1 of triangle *BPY intersect the circumcircle* ω_2 of triangle CPX *at* another point *Q*. Prove that points A , P , and Q are collinear. (2021 High School Mathematics League)

Extended Proposition: In triangle ABC , there are two points X and Y such that $\angle ABX = \angle ACY$. Let the extension of *BX intersect line segment CY* at point P , and let the circumcircle ω_1 of triangle *BPY intersect* the circumcircle ω_2 of triangle *CPX* at another point *Q*. Prove: $S_{\triangle APQ} \cdot S_{\triangle ABY} \cdot BC \sin(\angle CAY) = S_{\triangle CYQ} \cdot S_{\triangle ABP} \cdot AX \sin(\angle CAY - \angle XAB)$.

Note: A , X , and Y do not have to be collinear.



Example 8 : As shown in the figure, circle ω_1 is the circumcircle of quadrilateral $ABCD$. Lines AC and BD intersect at point E, and lines AD and BC intersect at point F. Circle ω_2 is tangent to line segments EB and EC at points M and N, respectively, and intersects circle ω_1 at points Q and R. Lines BC and AD *intersect line* MN at points S and T, respectively . Prove that points Q, R, S, and T are concentric. (The 16th Women's Mathematical Olympiad , 2017)



Proof: Let the circumscribed circle of $\triangle ABC$ be the unit circle, let $M=\left(m,\dfrac{b+d-m}{bd}\right)$,

$N=\left(n,\dfrac{a+c-n}{ac}\right)$, we can find $S=\left(s,\bar{s}\right)$ the identity , $T=\left(t,\bar{t}\right)$, $Q=\left(q,\dfrac{1}{q}\right)$, ,

$R=\left(r,\dfrac{1}{r}\right)$

$$\left([CMN]-[BMN]\right)\left([AMN]-[DMN]\right)^2(b-m)(a-n)|s-r|^2|q-t|^2\left(\frac{s-t}{s-r}\frac{q-r}{q-t}-\overline{\left(\frac{s-t}{s-r}\frac{q-r}{q-t}\right)}\right)$$

$$=\left([NMQ]+[NRQ]\right)[AMN][BMN](b-c)(d-a)|m-q|^2|r-n|^2\left(\frac{m-n}{m-q}\frac{r-q}{r-n}-\overline{\left(\frac{m-n}{m-q}\frac{r-q}{r-n}\right)}\right)$$

Extended Proposition: As shown in the figure, circle ω_1 is the circumcircle of quadrilateral $ABCD$. Lines AC and BD intersect at point E , and lines AD and BC intersect at point F. Circle ω_2 intersects line segments EB and EC at points M and N , respectively, and circle ω_1 at points Q and R. Lines BC and AD *intersect* line MN at points S and T , *respectively* . Prove

$$\left(S_{\triangle AMN}-S_{\triangle DMN}\right)^2\left(S_{\triangle CMN}-S_{\triangle BMN}\right)\cdot BM\cdot AN\cdot SR\cdot QT\cdot ST\sin\left(\angle TSR+\angle TQR\right)$$

$$=S_{\triangle AMN}S_{\triangle BMN}S_{\triangle ADS}BC\cdot AD\cdot MQ\cdot MN\cdot RN\sin\left(\angle NMQ+\angle NRQ\right):$$

Note: Tangency in the original question is a redundant condition.

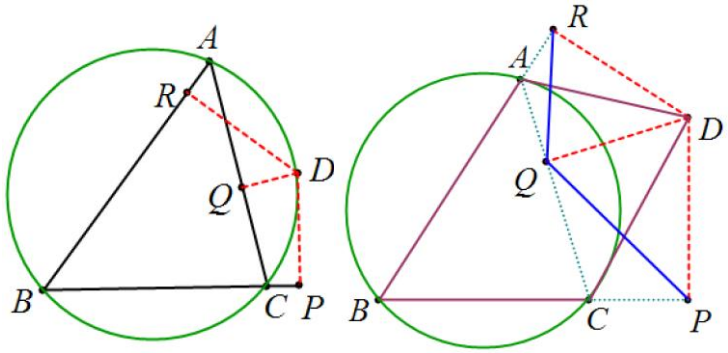
Example 9 : In a circle inscribed in quadrilateral $ABCD$, points P , Q , and R are projections from D onto lines BC , CA , and AB , *respectively* . Prove that $PQ = QR$ if and only if the angle bisectors of $\angle ABC$ and $\angle ADC$ *intersect on* AC . (Question from the 44th *IMO* exam)

Proof: Let the circumcircle of $\triangle ABC$ be the unit circle. According to the properties of the angle bisector, the intersection of the angle bisectors of $\angle ABC$ and $\angle ADC$ *on* AC is equivalent to $AB \cdot CD = AC \cdot BD$.

$$r = \frac{a+b+d-ab\overline{d}}{2}, \quad p = \frac{c+b+d-cb\overline{d}}{2}, \quad q = \frac{a+c+d-ac\overline{d}}{2},$$

$$4\left((q-p)\overline{(q-p)}-(q-r)\overline{(q-r)}\right)$$

$$=(a-b)\overline{(a-b)}(c-d)\overline{(c-d)}-(c-b)\overline{(c-b)}(a-d)\overline{(a-d)}.$$



The study found that: A , B , C , and D are cocircular, and the conditions are redundant. In quadrilateral $ABCD$, points P , Q , and R *are projections of* D onto lines BC , CA , and AB , respectively . Proof: $4R^2\left(QP^2 - QR^2\right) = \left(AB^2 \cdot CD^2 - BC^2 \cdot AD^2\right)$, where R is the radius of the circumcircle of triangle ABC .

12.如图，在凸四边形 $ABCD$ 中，M 为边 AB 的中点，且 $MC=MD$.分别过点 C、D 作边 BC、AD 的垂线，设两条垂线的交点为 P.过点 P 作 $PQ\perp AB$ 于 Q.求证:$\angle PQC=\angle PQD$.

证明：如图，联结 PA、PB，分别取 PA、PB 的中点 E、F，

联结 EM、ED、FM、FC.

则四边形 $PEMF$ 为平行四边形.....5 分

从而，$\angle PEM=\angle PFM$.

由 $ME=\frac{1}{2}BP=CF, MF=\frac{1}{2}AP=DE, MD=MC$

所以，$\triangle DEM\cong\triangle MFC$.....10 分

即，$\angle DEM=\angle MFC$.

所以，$\angle PED=\angle DEM-\angle PEM=\angle MFC-\angle PFM=\angle PFC$.

又，$\angle PED=2\angle PAD, \angle PFC=2\angle PBC$.

得 $\angle PAD=\angle PBC$.

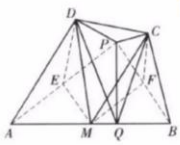
由于 $\angle PQA=\angle PDA=90^\circ$,

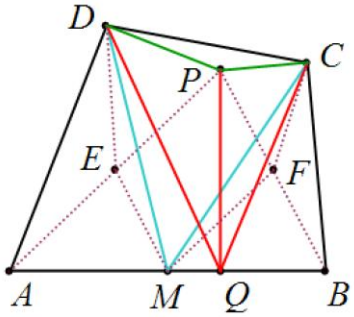
$\angle PQB=\angle PCB=90^\circ$,

则 P、Q、A、D 和 P、Q、B、C 分别四点共圆.....15 分

故 $\angle PQD=\angle PAD$，$\angle PQC=\angle PBC$.

所以，$\angle PQC=\angle PQD$.....20 分





Example 10 : As shown in the figure, in a convex quadrilateral $ABCD$, M is the midpoint of side AB , and $MC=MD$. *Draw perpendiculars* to sides BC and AD *through points* C and D , respectively . Let the intersection of these two perpendiculars be P. Draw PQ through point P *so that* $PQ\perp AB$ at Q. Prove that $\angle PQC=\angle PQD$. (2018 Hunan High School Mathematics Competition)

Proof: Let AB be the real axis, QP be the imaginary axis, $q=0$, $m=\frac{a+b}{2}$, solve the equation $\frac{p-d}{a-d}=\overline{\left(\frac{p-d}{a-d}\right)}$, and we get $\overline{p}=\frac{ad+a\overline{d}-2d\overline{d}}{d-\overline{d}}$. Solve the equation $\frac{p-c}{b-c}=-\overline{\left(\frac{p-c}{b-c}\right)}$, and we get $\overline{c}=-\frac{c\left(ad-bd+a\overline{d}+b\overline{d}-2d\overline{d}\right)}{-ad-bd+2cd-a\overline{d}+b\overline{d}-2c\overline{d}+2d\overline{d}}$.

$$\begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}\left((m-c)\overline{(m-c)}-(m-d)\overline{(m-d)}\right)$$

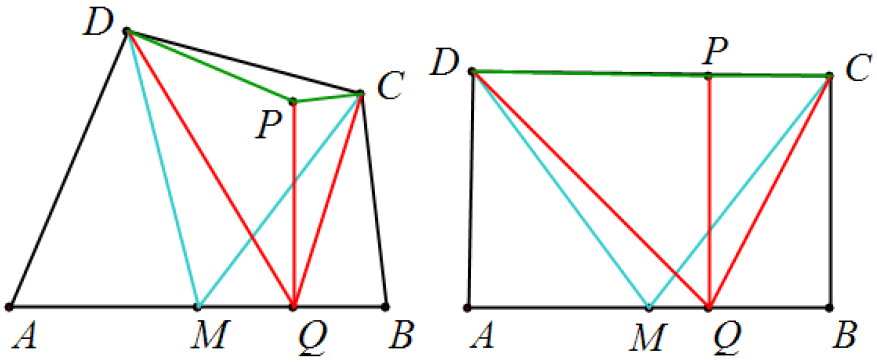
$$=\frac{a-b}{d-\overline{d}}\left(\begin{vmatrix} b & \overline{b} & 1 \\ c & \overline{c} & 1 \\ a & \overline{a} & 1 \end{vmatrix}-\begin{vmatrix} b & \overline{b} & 1 \\ c & \overline{c} & 1 \\ d & \overline{d} & 1 \end{vmatrix}\right)(q-p)\overline{(q-p)}\left(\frac{\frac{q-d}{q-p}}{\frac{q-p}{q-c}}-\overline{\left(\frac{\frac{q-d}{q-p}}{\frac{q-p}{q-c}}\right)}\right).$$

As shown in the figure, in a convex quadrilateral $ABCD$, M is the midpoint of side AB . *Draw perpendiculars* to sides BC and AD *through points* C and D , respectively . Let the intersection of these two perpendiculars be P. Draw PQ through point P *so that* $PQ\perp AB$ at Q. Prove:

$$S_{\triangle ABC}S_{\triangle ABD}\left(MC^2-MD^2\right)=\left(S_{\triangle BCA}-S_{\triangle BCD}\right)\cdot AB^2\cdot QC\cdot QD\sin\left(\angle PQC-\angle PQD\right).$$

Note: The identity reminds you of $S_{\triangle BCA}=S_{\triangle BCD}$ the situation you should pay attention

to. As shown in Figure 2, all the conditions of the problem are fully met, but at this time $S_{\triangle BCA}=S_{\triangle BCD}$, and $\angle PQC \neq \angle PQD$.



$$(m-c)\overline{(m-c)}-(m-d)\overline{(m-d)}$$

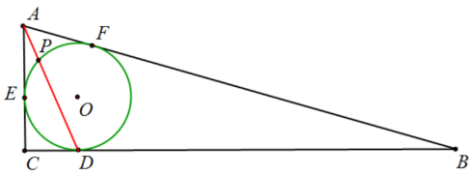
$$=\frac{\left(ad+bd-2cd+a\bar{d}+b\bar{d}-2d\bar{d}\right)\left(-2ac+ad+bd+a\bar{d}-b\bar{d}+2c\bar{d}-2d\bar{d}\right)}{2\left(ad+bd-2cd+a\bar{d}-b\bar{d}+2c\bar{d}-2d\bar{d}\right)},$$

$$\frac{\frac{q-d}{q-p}}{\frac{q-p}{q-c}}-\overline{\left(\frac{\frac{q-d}{q-p}}{\frac{q-p}{q-c}}\right)}=\frac{c\left(d-\bar{d}\right)^3\left(ad+bd-2cd+a\bar{d}+b\bar{d}-2d\bar{d}\right)}{\left(ad+a\bar{d}-2d\bar{d}\right)^2\left(ad+bd-2cd+a\bar{d}-b\bar{d}+2c\bar{d}-2d\bar{d}\right)},$$

Need to be excluded $S_{\triangle BCA}-S_{\triangle BCD}=\begin{vmatrix} b & \bar{b} & 1 \\ c & \bar{c} & 1 \\ a & \bar{a} & 1 \end{vmatrix}-\begin{vmatrix} b & \bar{b} & 1 \\ c & \bar{c} & 1 \\ d & \bar{d} & 1 \end{vmatrix}$

$=\dfrac{(c-b)\left(d-\bar{d}\right)\left(-2ac+ad+bd+a\bar{d}-b\bar{d}+2c\bar{d}-2d\bar{d}\right)}{ad+bd-2cd+a\bar{d}-b\bar{d}+2c\bar{d}-2d\bar{d}}$ AD // BC should be excluded .

x 5 x 10+(- x 6+ x 4) x 9!=0
4 x 6^2-4 x 4 x 6+4 x 5^2+ x 4^ 2!= 0
x 4^ 2!= 0
GB method deduce x 5 x 10 + (- x 6 + x 4) x 9 = 0
Wu Fafa did not find

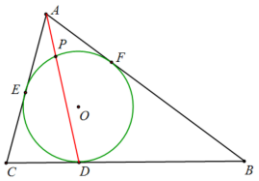


Example 11 : In right triangle ABC , $\angle ACB = 90°$. The incircle O *of triangle* ABC *is tangent* to sides BC , CA , and AB at points D , E , and F , respectively. Connect AD , which intersects the incircle O at point P. Connect BP and CP . If $\angle BPC = 90°$, prove that: $AE + AP = PD$. (2006 China Mathematical Olympiad)

Proof: Let circle O be the unit circle, $a = \dfrac{2ef}{e+f}$, $b = \dfrac{2df}{d+f}$, $c = \dfrac{2ed}{e+d}$, solve the equation $\dfrac{a-p}{a-d} = \overline{\left(\dfrac{a-p}{a-d}\right)}$ to get $p = \dfrac{de+df-2ef}{2d-e-f}$,

$$[F,E,O][D,E,F]\left((a-e)\overline{(a-e)}-((p-d)-(a-p))\left(\overline{(p-d)-(a-p)}\right)\right)$$

$$=[D,F,O][D,E,O](e-f)\overline{(e-f)}(p-c)\overline{(p-c)}\left(\frac{p-b}{p-c}+\overline{\left(\frac{p-b}{p-c}\right)}\right).$$

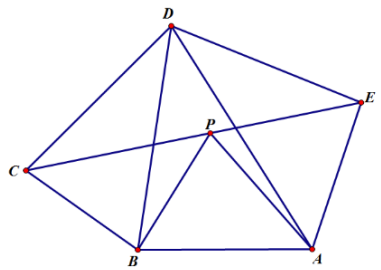


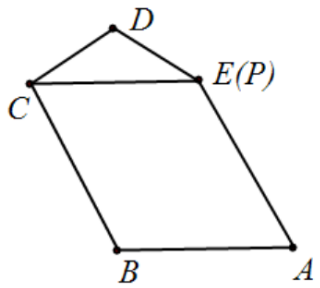
$\triangle ABC$, the inscribed circle O *of* $\triangle ABC$ *is tangent* to the sides BC , CA , and AB at points D , E , and F *respectively* . Connect AD , which intersects the inscribed circle O at point P. Connect BP and CP . R is the radius of the inscribed circle. Prove that $R^2 \cdot OE \cdot FD \cdot EF \cdot (AE^2 - (PD - AP)^2) = 2EF^2 \cdot OD \cdot EO \cdot DF \cdot PC \cdot PB \cos\angle BPC$:

Note: According to the identity equation, the original condition $\angle C = 90°$ is redundant.

Failure Question

Example 1 : As shown in the figure, in a convex pentagon $ABCDE$, $CD = DE$, $\angle EDC \neq 2\angle ADB$. Point P is inside the pentagon, satisfying $AP = AE$ and $BP = BC$. Prove: P is on the diagonal CE if and only if $S_{\triangle CBD} + S_{\triangle AED} = S_{\triangle BAD} + S_{\triangle BAP}$. (2019 IMO Pre - selection Question)





$D=(0,0)$, $K=(1,1)$, $P=(p,\overline{p})$, $C=(c,2-c)$, $E=(2-c,c)$,

$$A=\left(-\frac{2\overline{a}-2c-\overline{a}c+c^2-\overline{a}p+p\overline{p}}{c-\overline{p}},\overline{a}\right),\ B=\left(\frac{-2c+\overline{b}c+c^2-\overline{b}p+p\overline{p}}{-2+c+\overline{p}},\overline{b}\right),$$

identities $([CBD]+[AED]-[BAD]-[BAP])([BDC]-[BPD])(c-p)\overline{(c-p)}(e-c)$

$$=2[CPE][CPB](d-c)\overline{(d-c)}(d-a)\overline{(d-a)}\left(\frac{\frac{d-k}{d-c}}{\frac{d-a}{d-b}}-\overline{\left(\frac{\frac{d-k}{d-c}}{\frac{d-a}{d-b}}\right)}\right),$$

Extended Proposition: As shown in the figure, in the convex pentagon $ABCDE$, $CD = DE$, point P is inside the pentagon, and $AP = AE$, $BP = BC$. Prove:

$$2S_{\triangle CPB}\cdot DC\cdot DA\cdot DB\sin\left(\frac{\angle EDC}{2}-\angle ADB\right)\sin\angle PCE$$

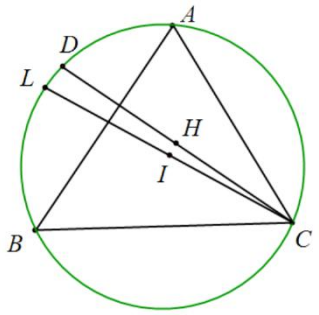
$=CP\cdot(S_{\triangle BPD}-S_{\triangle BDC})(S_{\triangle CBD}+S_{\triangle AED}-S_{\triangle BAD}-S_{\triangle BAP})$. (Note that it is directed area and directed angle)

This problem cannot find a non-degenerate condition $\angle EDC = 2\angle ADB$. This special case is rarely encountered.

According to the question, after manual addition, a complex number identity is generated.

Based on the identity, we construct a counterexample: quadrilateral $ABCD$ is a rhombus, $\angle A = 60°$, $DC = DE$, and P and E coincide. In this case , P *is* on diagonal CE , which is invalid . This counterexample is refuted by the author's argument that "

$S_{\triangle CBD}+S_{\triangle AED}=S_{\triangle BAD}+S_{\triangle BAP}$ point P is inside the pentagon ." So, this author is truly impeccable!



Example 2 : Given an acute angle $\triangle ABC$ with orthocenter H and incenter I, and $AC \neq BC$, extend CH and CI to intersect the circumcircle of $\triangle ABC$ *at* D and L , respectively . Prove that $\angle \mathrm{CIH} = 90°$ $\Leftrightarrow$ *and* $\angle IDL = 90°$.

the circumscribed circle of $A=a^2$ $\triangle ABC$ *be the unit circle,* , $B=b^2$, $C=c^2$,

$h=a^2+b^2+c^2$, $i=-ab-bc-ca$, $L=-ab$, $d=-\dfrac{a^2b^2}{c^2}$, the inner circle fails

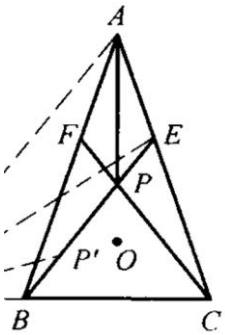
From $IC \perp IH$, we can get

$$\frac{i-h}{i-c^2}+\overline{\left(\frac{i-h}{i-c^2}\right)}=\frac{(a+b)\left(a^2b+ab^2+2abc+ac^2+bc^2\right)}{ab(a+c)(b+c)}=0$$

It is obtained $\dfrac{d-i}{d+ab}+\overline{\left(\dfrac{d-i}{d+ab}\right)}=\dfrac{a^2b+ab^2+2abc+ac^2+bc^2}{abc}=0$ from $DI \perp DL$.

Failed to generate complex identities and could not find the geometric meaning of the non-degenerate condition.

Example 3 : Let P be a point on the line bisector of $\angle A$ in ABC, and let lines BP and CP *intersect* AC and AB at E and F respectively . Proof: If $BE = CF$, then $\triangle ABC$ is an isosceles triangle .



Failed to generate complex identities and could not find the geometric meaning of the non-degenerate condition.