Sequence and Series

Chapter 11
MATH182A

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November 30, 2016

The Root Test

Given $\sum a_n$, perform the following:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

- If $L > 1$, series diverges.
- If $L < 1$, series converges.
- If $L = 1$, test fails.

Example

$$\sum(\frac{n^2+1}{2n^2+1})^n \text{ , thus, } a_n = (\frac{n^2+1}{2n^2+1})^n$$

$$\lim_{n\to\infty} \sqrt[n]{(\frac{n^2+1}{2n^2+1})^n}$$

$$\lim_{n\to\infty} (\frac{n^2+1}{2n^2+1})^n * \frac{n^2}{n^2}$$

$$\lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{2+\frac{1}{n^2}}$$

$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$

$$\lim_{n\to\infty} \frac{1+0}{2+0} = \frac{1}{2}$$

$$L = \frac{1}{2}, \frac{1}{2} < 1, a_n \text{ is convergent.}$$

Example

$$\sum(1+\frac{1}{n})^{n^2}, \text{ thus, } a_n = (\frac{n^2+1}{2n^2+1})^n$$

$$\lim_{n\to\infty} \sqrt[n]{(1+\frac{1}{n})^{n^2}}$$

$$\lim_{n\to\infty}(1+\frac{1}{n})^n$$

$$\text{Theorem: } \lim_{n\to\infty}(1+\frac{1}{n})^n = e$$

$$L = e, e > 1, a_n \text{ is divergent.}$$

Example

$$\sum(\frac{-2n}{n+2})^{7n}, \text{ thus, } a_n = (\frac{-2n}{n+2})^{7n}$$

$$\lim_{n\to\infty} \sqrt[n]{(\frac{-2n}{n+2})^{7n}}$$

$$\lim_{n\to\infty}(\frac{2n}{n+2})^7$$

$$\lim_{n\to\infty}(\frac{-2n}{n+2} * \frac{n}{n})^7$$

$$\lim_{n\to\infty}(\frac{2}{1+\frac{2}{n}})^7$$

$$\lim_{n\to\infty}\frac{2}{n} = 0, \lim_{n\to\infty} = (\frac{2}{1+0})^7 = 2^7$$

$$L = 2^7, 2^7 > 1, a_n \text{ is divergent.}$$

Example

$$1 - \frac{1*3}{3!} + \frac{1*3*5}{5!} - \frac{1*3*5*7}{7!}\dots$$

$$nth \text{ term: } a_n = (-1)^{n-1}\frac{1*3*5*\dots*(2n-1)}{(2n-1)!}$$

$$a_{n+1} = (-1)^n\frac{1*3*5*\dots*(2n-1)*(2n+1)}{(2n+1)!}$$

Appply ratio test...

$$|\frac{a_{n+1}}{a_n}| = \frac{(-1)^n \frac{1*3*5*...*(2n-1)*(2n+1)}{(2n+1)!}}{(-1)^{n-1} \frac{1*3*5*...*(2n-1)}{(2n-1)!}} * \frac{(2n-1)!}{(2n-1)!}$$

$$|\frac{a_{n+1}}{a_n}| = \frac{(2n-1)! * (2n+1)}{(2n+1)!}$$

$$= \frac{(2n-1)! * (2n+1)}{(2n+1)(2n)(2n-1)!}$$

$$= \frac{1}{2n}$$

$$\lim_{n \to \infty} \frac{1}{2n} = 0$$

$L = 0, 0 < 1, a_n$ is convergent.

P Series

$$\sum \frac{1}{n^\rho}$$

- If $\rho \geq 1$, series converges.
- If $\rho < 1$, series diverges.

Power Series

$$\sum_{n=0}^{\infty} C_n(x^n) = C_0 + C_1(x) + C_2(x^2) + \ldots + C_n(x^n)$$

Three possibilities for convergence:

- Convergent when $x = a$ (converges only for a single value of x)
- Convergent when x exists in a list of possible x values
- Convergent for all x

Example

$$\sum_{n=9}^{\infty} \frac{x^n}{\sqrt{n}},\ a_n = \frac{x^n}{\sqrt{n}}$$

Goal is to find the range of x-values in which the function converges.

Apply ratio test...

$$|\frac{a_{n+1}}{a_n}| = |\frac{\frac{x^{n+1}}{\sqrt{n+1}}}{\frac{x^n}{\sqrt{n}}}|$$

$$= |x|(\frac{\sqrt{n+1}}{\sqrt{n}})$$

$$= |x|(1+\frac{1}{n})^{\frac{1}{2}}$$

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = |x|$$

In order to converge absolutely, L must be < 1.

Thus, a_n converges for $-1 < x < 1$

Example

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}, a_n = (-1)^n \frac{x^n}{n+1}$$

$$|\frac{a_{n+1}}{a_n}| = |\frac{(-1)^{n+1}\frac{x^{n+1}}{n+2}}{(-1)^n \frac{x^n}{n+1}}|$$

$$= \lim_{n\to\infty} |x| \frac{1+\frac{1}{n}}{1+\frac{2}{n}}$$

$$= \lim_{n\to\infty} |x|$$

In order to converge absolutely, L must be < 1.

Thus, a_n converges for $-1 < x < 1$

Example

$$\sum \frac{x^n}{n!},\ a_n = \frac{x^n}{n!}$$

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}$$

$$\lim_{n\to\infty} = |x|\frac{n!}{(n+1)!}$$

$$\lim_{n\to\infty} = |x|\frac{n!}{(n+1)(n!)}$$

$$\lim_{n\to\infty} = \frac{|x|}{(n+1)} = 0$$

Thus, L always equals zero, regardless of x, so a_n converges for all x.

Example

$$\sum (-2)^n \frac{x^n}{\sqrt[4]{n}},\ a_n = (-2)^n \frac{x^n}{\sqrt[4]{n}}$$

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{(-2)^{n+1}\frac{x^{n+1}}{\sqrt[4]{n+1}}}{(-2)^n\frac{x^n}{\sqrt[4]{n}}}$$

$$\lim_{n\to\infty} = 2|x|(1+\frac{1}{n})^{\frac{1}{4}}$$

$$\lim_{n\to\infty} = 2|x|$$

To converge, x must fall in the range $\frac{-1}{2} < x < \frac{1}{2}$ so that $L < 1$.