Infinite sequences and series

- **Numerical sequences** or simply a **sequence** is a set of ordered numbers. A sequence is an **unending succession** of numbers called **terms**, which have a definite order. A sequence we can consider as a **function** whose **domain** is the **set of all natural numbers** in ascending (increasing) order. We write a sequence as $a_1, a_2, a_3, \ldots, a_n$
- a_1 is the first term ; a_2 is the second term ; a_3 is the third term a_n is called the general term.

Example 5, 8, 11, 14, …, 3n + 2 a_1 = 5 $3 \cdot 1 + 2 = 5$ a_2 = 8 $3 \cdot 2 + 2 = 8$ a_n = 3n + 2

- As a function it is f(x) = 3x + 2 where x∈ N N = $\{1, 2, 3, \ \ldots . \}$
- The set of numbers 1, 2, 3, 5, 8, 13, 21, is called Fibonacci sequence. The order is: the term after is the sum of the two previous numbers.
- The set of numbers 3, 7, 11, 15,…,4n - 1 is called **arithmetic progression**. The order is: the term after is the sum of the previous term and a constant number (in our case constant number is 4). Also, we can say that the difference between term after and term previous is a constant number.
- The set of numbers 3, 9, 27, 81, …,3^n is called **geometric progression**. The order is: the after term is the previous multiplied by 3 , or the ratio between term after and term previous is a constant number (multiplicative factor)
- In general, all these infinite sequences we called **series**. We are interested in, the sum of these numbers, The sum of series

Infinite sequences and series

- The arithmetic progression we can write as

a, a + d, (a + d) + d, (a + 2d) + d,…, a + (n - 1)d ("d" is the difference) d = a_n - a_{n-1}, a_n = a_1 + (n-1)d

S =($\frac{a_1+a_n}{2}$)n

- Find the sum 2, 7, 12, 17, 22, 27, 32, 37 a_1= 2, n = 8, a_n= 37 S =($\frac{a_1+a_n}{2}$)n = (2 + 37)(8/2) = 156
- The geometric progression we can write as:

a, ar, (ar)r, (ar^2)r,…, ar^{n-1} "r" is the ratio r = $\frac{a_n}{a_{n-1}}$ a_n= $a_1 r^{n-1}$ S = $\frac{a(r^n - 1)}{r - 1}$

- In economy we are interested in the geometric progression and the sum of the consecutive terms.

<u>Example</u>. If we want to find the sum of the six first terms of the geometric progression 2, 4, 8, 16, 32, 64, 128, …, 2^n we can add six terms together 2 + 4 + 8 + 16 + 32 + 64 = 126

Infinite sequences and series

- To find the sum of such geometric series exist a special formula S = $\frac{a(r^n - 1)}{r - 1}$ where "a" is the first term, "r" is

geometric ratio, r > 1, and "n" is the number of terms. Formula for r < 1 is S = $\frac{a(1 - r^n)}{1 - r}$

- The following sequences are arithmetic progression or geometric progression?

5, 10, 15, 20, … 5, 10, 20, 40, … 1, - 3, 9, - 27, … 36, 30, 24, 18, … 36, 18, 9, 4.5,… 1000, - 100, 10, - 1,…

d = 5 r = 2 r = - 3 d = - 6 r = 0.5 r = - 0.1

- Practice problem 1 page 231
- Practice problem 2 page 232
- Find the sum 100$(1.002)^1$+ 100$(1.002)^2$ + 100$(1.002)^3$ + … + 100$(1.002)^{14}$

  S = 100$(1.002)^1 \frac{(1.002^{14} - 1)}{1.002 - 1}$ = 100.2(1.0283 – 1)/0.002 = 2.8356/0.002 = 1417.8

Infinite sequences and series

- Exercise. 1. Find the value of the geometric series
- 1000 + 1000(1.03) + $1000(1.03)^2$ + … + $1000(1.03)^9$
- Apply the formula S = $\frac{a(r^n-1)}{r-1}$ = $\frac{1000[(1.03)^{10}-1]}{1.03-1}$ = $\frac{1000(1.3439-1)}{0.03}$ = $1000 \cdot 11.463879 = 11463.879$

Infinite sequences and series

- There are two particular of geometric series involving saving and loans. Example page 233
- Practice problem 3 page 234
- **Example.** A person saves \$100 in a bank account at the beginning of each month. The bank offers a return of 12% compound interest in one year, or 1% every month.

a) Determine the total amount saved after 12 months.

b) After how many months does the amount saved first exceed \$2000.

- The first payment is invested for 12 months, so its value is $100(1.01)^{12}$
- The second payment is invested for 11 months, so its value is $100(1.01)^{11}$
- The third payment is invested for 10 months, so its value is $100(1.01)^{10}$ and so on.
- The last payment is invested for one month, so its value is $100(1.01)^{1}$
- The total value of saving at the end of 12 months is

$$100(1.01)^{12} + 100(1.01)^{11} + 100(1.01)^{10} + \ldots + 100(1.01)^{1} = 100(1.01)^{1} + 100(1.01)^{2} + \ldots + 100(1.01)^{11} + 100(1.01)^{12}$$

- This total value is the sum of the first 12 terms of a geometric progression in which the first term 100(1.01) and the geometric ratio is 1.01. applying the formula obtain

$$S = \frac{a(r^n - 1)}{r - 1} = 100(1.01)\,\frac{(1.01^{12} - 1)}{1.01 - 1} = 101\frac{1.126 - 1}{0.01} = 101\frac{0.1268}{0.01} = 101 \cdot 12.682 = 1280.932$$

Infinite sequences and series

- To answer the second question, we must use the same formula
- $100(1.01)\,\frac{(1.01^n - 1)}{1.01 - 1} = 2000$

$101(1.01^n\ - 1) = 2000$ or $(1.01)^n - 1 = 0.198$ $(1.01)^n = 1.198$ now take logs of both sides and find

$n\log(1.01) = \log(1.198)$ $n = \frac{\log(1.198)}{\log(1.01)} = \frac{0.07845}{0.00432} = 18.2$. after $\cong$19 months savings exceed \$2000.

Infinite sequences and series

<u>Example</u>. Determine the monthly repayments needed to repay a \$100000 loan which is pay back over 25 years when the interest rate is 8% compounded annually. The time interval between two consecutive repayments is one month, whereas the period which interest is charge is one year.

- The amount of interest is 100000·8% = \$8000. If denote \$x each instalment the debt after the first year is 100000 + 8000 – 12x = 108000 – 12x = 100000(1.08) – 12x.
- The second year start with this debt and the interest charged is [100000(1.08) – 12x](1.08) = 100000$(1.08)^{2}$ - 12x(1.08). During the second year we repay [10000$(1.08)^2$ - 12x(1.08)](1.08) – 12x.
- So, each year we multiply by 1.08 and subtract 12x.
- So, at the end of the third year our debt is 100000$(1.08)^3$ - 12x$(1.08)^2$ –12x(1.08)- 12x. Continuing this pattern we see that after 25 years the amount owed is 100000$(1.08)^{25}$ - 12x$(1.08)^{24}$ – 12x $(1.08)^{23}$ - … - 12x(1.08)- 12x

or 100000$(1.08)^{25}$ – 12x [1 + 1.08 +$(1.08)^2$ + …+ $(1.08)^{24}$] = 0

Infinite sequences and series

- At the end of the first year, the dept is 100000 $(1.08)^1$ – 12x
- At the end of the second year, the dept is 100000$(1.08)^2$ - 12x(1.08)
- At the end of the third year, the dept is 100000$(1.08)^3$ - 12x$(1.08)^2$ –12x(1.08)- 12x
- After 25 years the dept is
- 100000$(1.08)^{25}$ - 12x$(1.08)^{24}$ –12x$(1.08)^{23}$ - 12x$(1.08)^{22}$ - 12x$(1.08)^{21}$ -… - 12x(1.08)- 12x = 0 or
- 100000$(1.08)^{25}$ - 12x [1 + 1.08 +$(1.08)^2$ + …+ $(1.08)^{24}$] = 0 the dept is completely cleared
- 1 + 1.08 +$(1.08)^2$ + …+ $(1.08)^{24}$= 1 $\frac{(1.08^{25}-1)}{1.08-1}$ = 73.106 100000$(1.08)^{25}$ = 684847.520
- 684847.520 – 12x(73.106) = 0 or x = 780.66. Total payment is 780.66(12)(25) = 234198
- The debt at the end of the first year is 108000 – 9367.92 = 98632.08
- The debt at the end of the second year is 100000$(1.08)^2$ – 9367.92(1.08) = 116640 – 19485.2736
- = 97154.7264

Infinite sequences and series

- the geometric series inside the square brackets can be computed using the formula
- S = $\frac{a(r^n - 1)}{r - 1} = \frac{1.08^{25} - 1}{1.08 - 1}$ = 73.106
- the amount owed at the end of 25 years is
- 684847.520 – 877.272x = 0   x = \$780.66. The monthly repayment on 25-year loan of \$100000 is \$780.66, assuming
- that the interest rate remains fixed at 8% throughout this period. The total repayment is
- $25 \cdot 12 \cdot 780.66 = 300 \times 780.66$ = \$234198
-

Infinite sequences and series

- Example. Total reserves of a non-renewable resource are 250 million tonnes. Annual consumption currently at 20 million tonnes per year, is expected to rise by 2% a year. After how many years will stocks be exhausted.
- The consumption in the first year is 20 million tonnes. The consumption in the second year is 20(1.02) million tonnes
- The consumption in the third year is $20(1.02)^2$ million tonnes. The total consumption (in million of tonnes) during the
- next "n" years will be
- 20 + 20(1.02) + $20(1.02)^2$ + … + $20(1.02)^{n-1}$. This is the sum of "n" terms of a geometric series with the first term
- a = 20 and ratio r = 1.02, so it is equal S = $\frac{a(r^n - 1)}{r - 1}$ = $\frac{20(1.02^n - 1)}{1.02 - 1}$ = 1000 $(1.02^n - 1)$ = 250 or 1.02^n = 1.25
- $\log 1.02^n$ = log(1.25) n = $\frac{\log(1.25)}{\log(1.02)}$ = $\frac{0.0969}{0.0086}$ = 11.27. So, the reserves will be completely exhausted after 12 years

Infinite sequences and series

- Exercise 2. An individual saves \$5000 in a bank account at the beginning of each year for ten years. No further saving or withdrawals are made from the account. Determine the total amount saved if the annual interest rate is 8% compounded.
- a) annually. b) semi-annually

Infinite sequences and series

- Solution.
- At the end of the first-year total amount is 5000(1.08)
- At the end of the second-year total amount is 5000(1.08) + $5000(1.08)^2$
- At the end of the third-year total amount is 5000(1.08) + $5000(1.08)^2$ + $5000(1.08)^3$
- At the end of the period of time total amount is
- S = 5000(1.08) + $5000(1.08)^2$ + $5000(1.08)^3$+ … + $5000(1.08)^{10}$ or
- S = 5000(1.08)[1 + 1.08 + $(1.08)^2$ + $(1.08)^3$ + … +$(1.08)^9$] = 5000(1.08) $\frac{1(1.08^{10}-1)}{1.08-1}$ = 5400$\frac{1.158924}{0.08}$ = 78227.43
-

Infinite sequences and series

- b) semi-annually. At the end of the first-year total amount is [5000(1.04)](1.04) = $5000(1.04)^2$
- At the end of the second-year total amount is $5000(1.04)^2$ + $5000(1.04)^2$ $(1.04)^2$ = $5000(1.04)^2$ + $5000(1.04)^4$
- At the end of the third-year total amount is $5000(1.04)^2$ + $5000(1.04)^4$ + $5000(1.04)^6$
- At the end of the period total amount is
- S = $5000(1.04)^2$ + $5000(1.04)^4$ + $5000(1.04)^6$+ … + $5000(1.04)^{20}$ or
- S = 5000[$(1.04)^2$ + $(1.04)^4$ + $(1.04)^6$ + … + $(1.04)^{20}$]
- the part of this sum is geometric progression where a = $(1.04)^2$ r = $(1.04)^2$
- and n = 10. Applying the formula, we obtain
- S = $5000(1.04)^2\frac{[(1.04)^2]^{10}-1}{(1.04)^2-1}$ = $5000\cdot 1.0816\frac{2.191123-1}{1.0816-1}$ = $5408\frac{1.191123}{0.0816}$ = 5408·14. 5970 = 78941.10

Infinite sequences and series

Exercise 3. A person invests \$5000 at the beginning of the year in a saving account that offers a return 4.5% compounded annually at the beginning of each subsequent year an additional \$1000 is invested in the account. How much will there be in account at the end of ten years.

- **<u>Solution</u>**. At the end of the first-year total sum is 5000·1.045
- At the end of the second-year total sum is (5000·1.045 + 1000)·1.045 = 5000· $(1.045)^2$+ 1000·1.045
- At the end of the third-year total sum is[5000· $(1.045)^2$+ 1000·1.045](1.045) = 5000· $(1.045)^3$+ 1000· $(1.045)^2$
- And the tenth-year total sum is 5000· $(1.045)^{10}$+ 1000(1.045) + 1000· $(1.045)^2$ +… + 1000· $(1.045)^9$ or
- 5000· $(1.045)^{10}$ + 1000(1.045)[1 + (1.045) + $(1.045)^2$+… + $(1.045)^8$]

Infinite sequences and series

- 5000· $(1.045)^{10}$ = 5000·1.5529694217 = 7764.8471086
- 1000(1.045)[1 + (1.045) + $(1.045)^2$+… + $(1.045)^8$] = 1045$\frac{1[(1.045)^9-1]}{1.045-1}$ = 1045$\frac{0.4860951404}{0.045}$ = 1045·10.8021142314 = 11288.209371842 the total sum is
- 7764.8471086 + 11288.209371842 = 19053.056480442

Infinite sequences and series

Exercise 4. A person wishes to save a regular amount at the beginning of each month to buy a car in 18 months 'time.

An account offers a return of 4.8% compound annually. Work out the monthly savings

- If the total amount saved at the end of 18 months is $18000. Let's denote by "x" monthly savings
- The interest rate is 4.8/12 = 0.4. Month's savings are 1.004x
- . At the end of the period, the future value of the first month savings is 1.004^{18}x
- At the end of the period the future value of the second month savings is 1.004^{17}x
- At the end of the period the future value of the third month savings is 1.004^{16}x
- At the end of the period the future value of the last month savings is 1.004x .
- We know that this sum is 18000, so applying the formula we can write
- 18000 = 1.004x$\frac{(1.004)^{18}-1}{1.004-1}$ = 1.004x $\frac{1.0745-1}{0.004}$ = $\frac{0.0745x}{0.004}$ = 18.6995x x = 962.59258

Infinite sequences and series

- Exercise 5. A person borrows \$100000 at the beginning of the year and agrees to repay the loan in ten equal instalments at the end of each year. Interest is charged at a rate of 6% compounded annually.

A) Find the annual repayment.

B) Work out the total amount of interest paid and compare this with the total interest paid when repaying the loan in five equal annual instalments instead of ten

- A)The first year is $100000(1.06) – x$
- The second year is $[100000(1.06) – x](1.06) – x = 100000(1.06)^2 - x(1.06) – x$
- The third year is $[100000(1.06)^2 - (1.06)x – x](1.06) – x = 100000(1.06)^3 - x(1.06)^2 - x(1.06) – x$
- Year ten is $100000(1.06)^{10} - x(1.06)^9 - \ldots - x(1.06) – x = 0$
- $100000(1.06)^{10} – x[1 + (1.06) + (1.06)^2 + \ldots + (1.06)^9] = 0$
- $100000(1.06)^{10} = x[\frac{1[(1.06)^{10}-1]}{1.06-1}]$
- $100000(1.7908) = x\frac{0.7908}{0.06}$ $x = \frac{179080}{13.18} = 13587.2534$ $13587.2534 \times 10 = 135872.534$

Infinite sequences and series

- B)$100000(1.06)^5 – x[1 + (1.06) + (1.06)^2 + \ldots + (1.06)^5] = 0$
- $100000(1.06)^5 = x[\frac{1[(1.06)^5-1]}{1.06-1}]$
- $100000(1.3382) = x\frac{0.3382}{0.06}$ $x = \frac{133820}{5.6366} = 23742.5261$ $23742.5261 \times 5 = 118712.6305$

Infinite sequences and series

- Find the missing numbers at the table below

	Savings	Loans	Compound interest annually	Time	Monthly annually payment	Total sum
1	20000	no	4.2%	10 years	no	
2	no	20000	5.2%	8 years		
3	25000	no	6.4%		no	34000
4	no	25000	6.4%	5 years		
5	no	10000	6.8%	10 years		
6	no	10000	7.2%	8 years		

Infinite sequences and series

- Exercise 5 of the table
- $10000(1.068)^1 – 12x$
- $[10000(1.068)^1 – 12x](1.068) – 12x = 10000(1.068)^2 – 12x(1.068)^1 – 12x$
- $(10000(1.068)^2 – 12x(1.068)^1 – 12x)(1.068) – 12x =$
- $= 10000(1.068)^3 – 12x(1.068)^2 – 12x(1.068)^1 – 12x$
- The last year $10000(1.068)^{10} – 12x(1.068)^9 – 12x(1.068)^8 -\ldots - 12x(1.068) – 12x =$
- $= 10000(1.068)^{10} – 12x[(1.068)^9 + (1.068)^8 + \ldots + (1.068) + 1] =$
- $= 10000(1.9306) – 12x \frac{1[(1.068)^{10} - 1]}{1.068 - 1} = 0$
- $19306 – 12x \frac{0.9306}{0.068} = 0$ $\quad 164.2224x = 19306 \quad x = 117.5600$
- $S = 10 \cdot 12 \cdot 117.5600 = 14107.2$

Infinite sequences and series

- Key terms
- Arithmetic progression a_n - a_{n-1} = d
- Geometric progression $\frac{a_n}{a_{n-1}}$ = r
- Formula a_n = a_1 + (n – 1)d a_n= $a_1 r^{n-1}$ S = ($\frac{a_1+a_n}{2}$)n S = $\frac{a(r^n-1)}{r-1}$
- Geometric series
- Sum of "n" terms

- Note
- **For every key word ask the question "what is" and try to answer it**