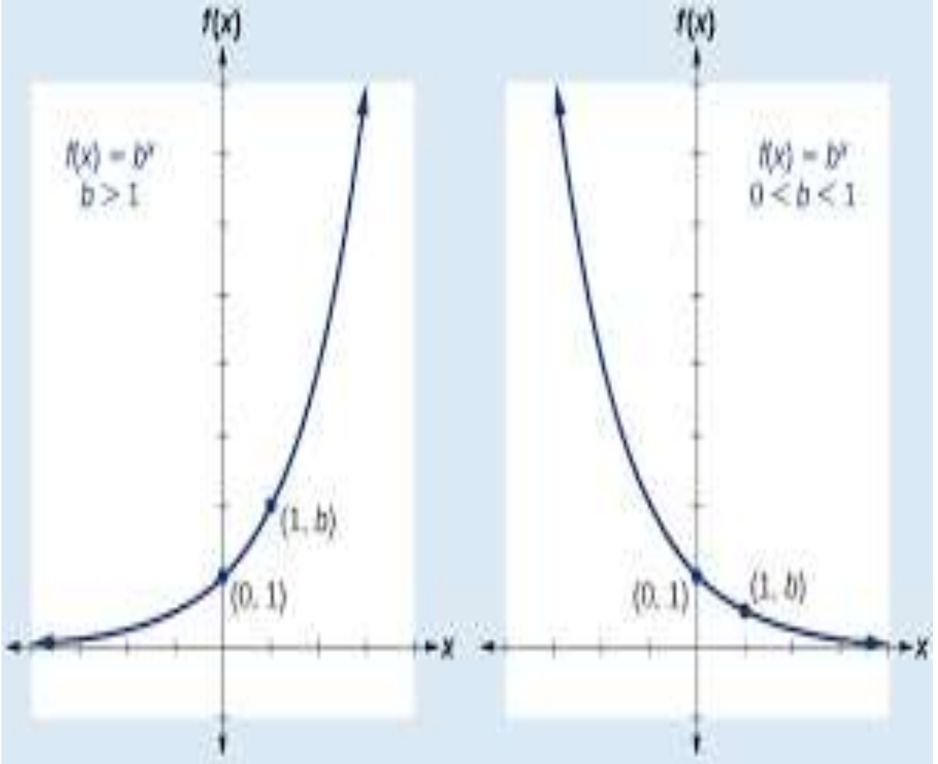
Exponential and logarithmic function

Expressions   f(x) = a^x or   y = a^x where "a" is a real number (a > 1 or a < 1) are called exponential function

The general form of its graph →



Properties of this function

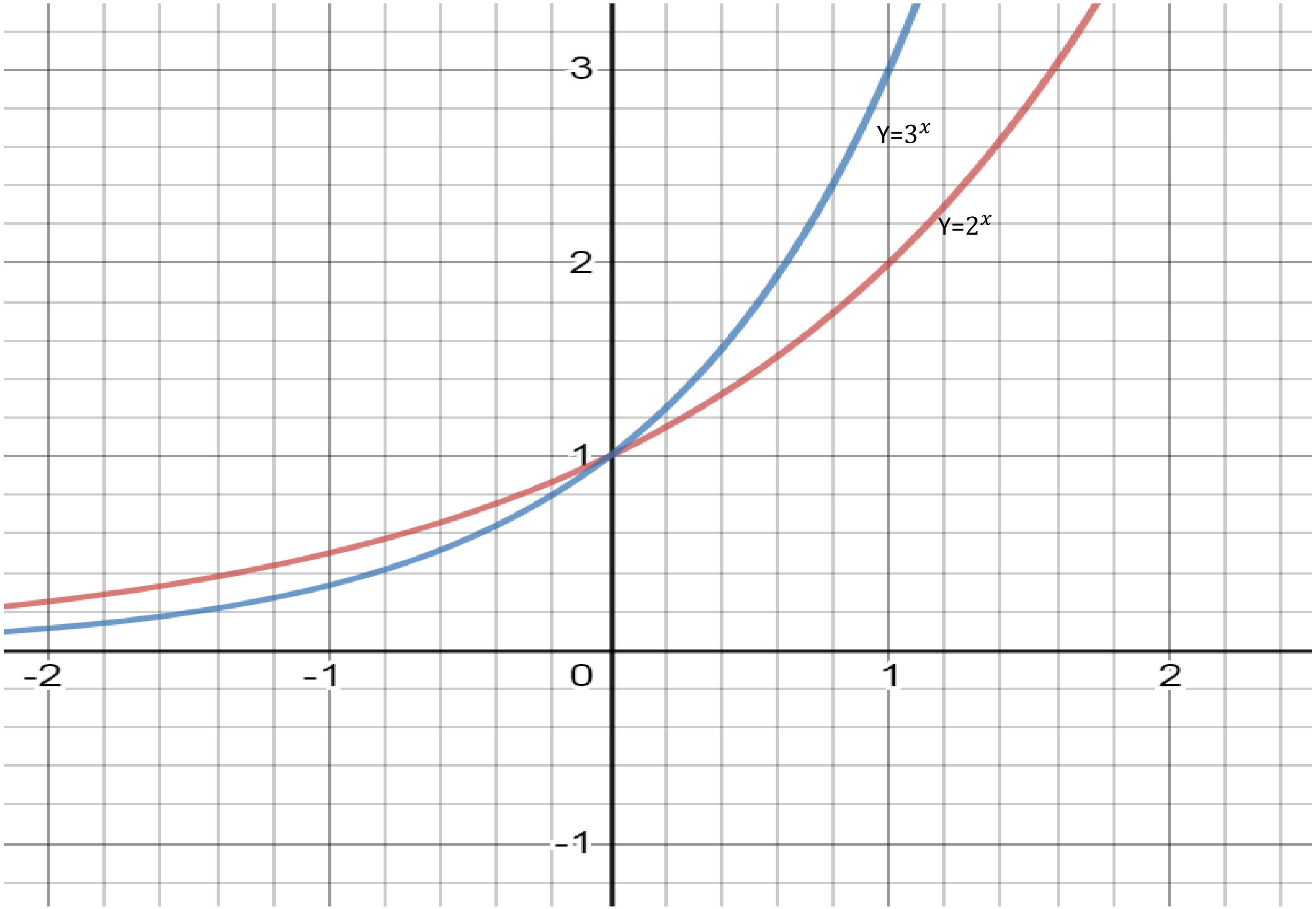
- increased function for a > 1
- decreased function for a < 1
- all the graphs pass through point (0, 1)
- this function is determined and continuous for every real number "x"
- for increased function, the values of "y" approach zero as "x" approach - ∞ and they are + ∞ as "x" goes +∞
- for decreased function, the values of "y" goes + ∞ as "x" goes - ∞ and approach zero as "x" goes +∞

Exponential and logarithmic function

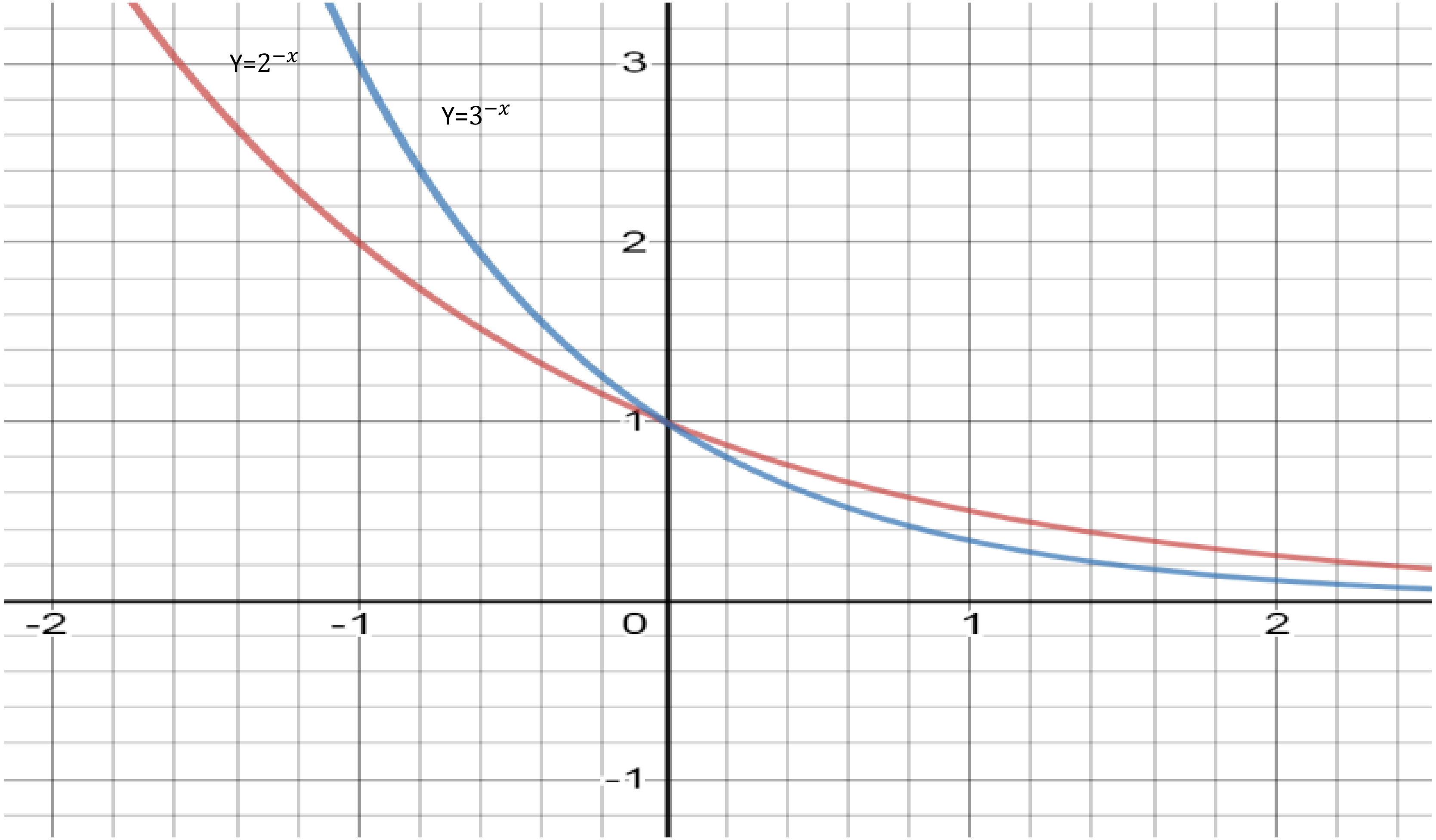
- **Example**. Find the value of "y" for the given exponential functions y = 2^x; y = 2^{-x}; y = 3^x; y = 3^{-x}
- Sketch the graph of these function.

x	-4	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
2^{-x}	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
3^x	$\frac{1}{81}$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27
3^{-x}	81	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$

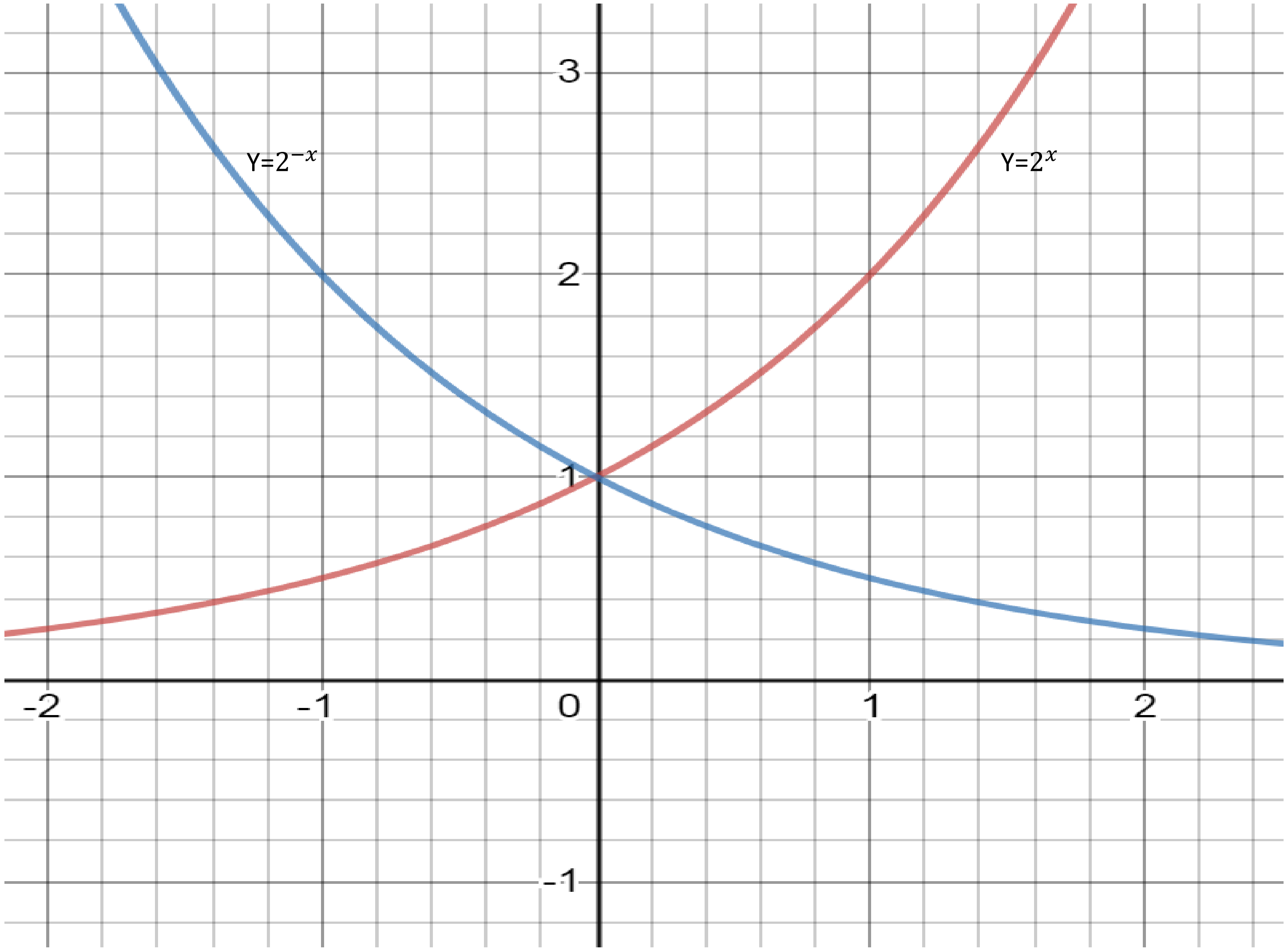
Exponential and logarithmic function



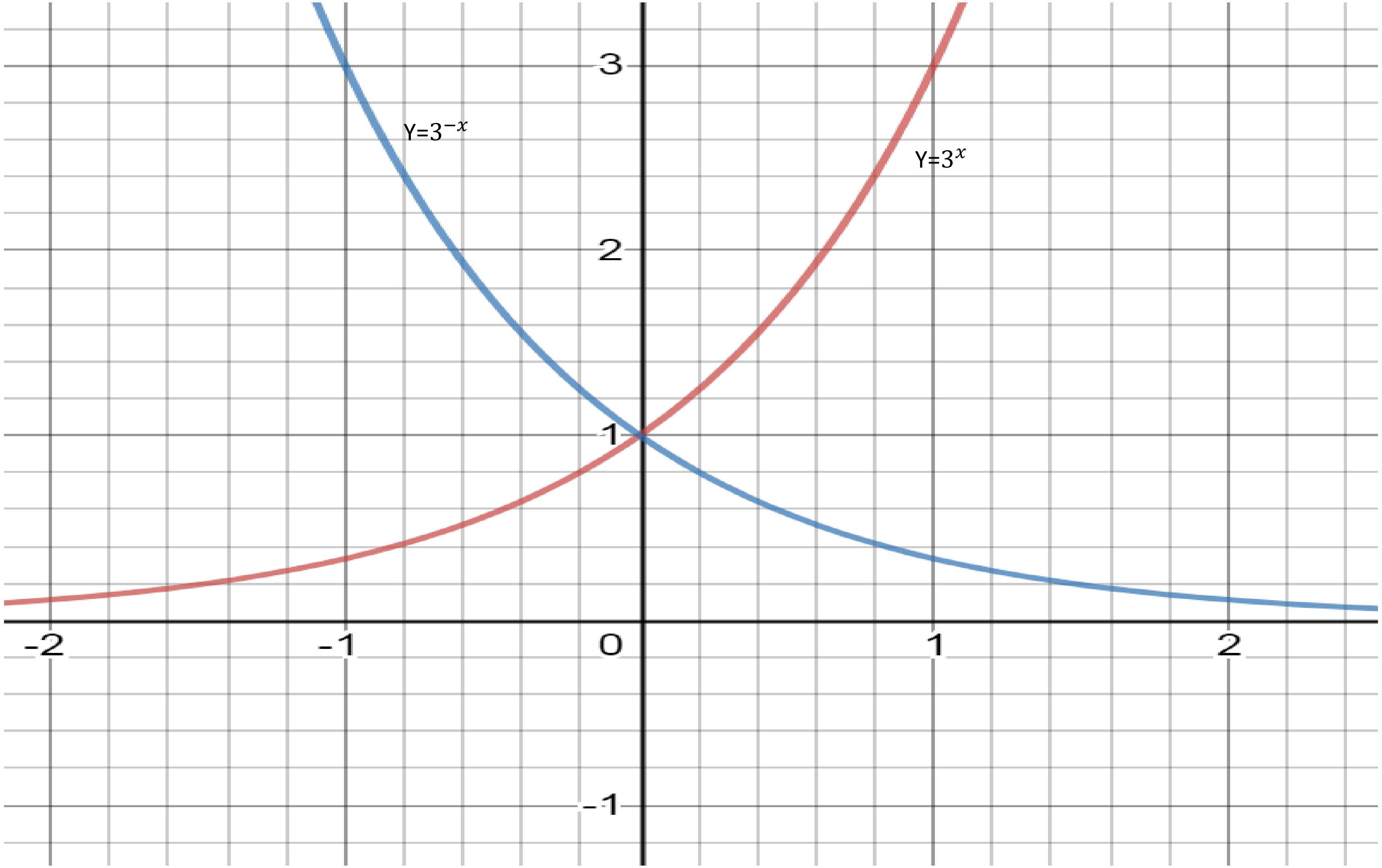
Exponential and logarithmic function



Exponential and logarithmic function



Exponential and logarithmic function



Exponential and logarithmic function

- **Example.** Is given the exponential function

y = $(1 + 1/x)^{x}$

x	1	10	100	1000
Y	2	1.1^{10} =2.593	1.01^{100} = 2.704	1.001^{1000} = 2.716

From the table we see that as “x” increase “y” also increased, but the rate of increase appears to be slowing down.

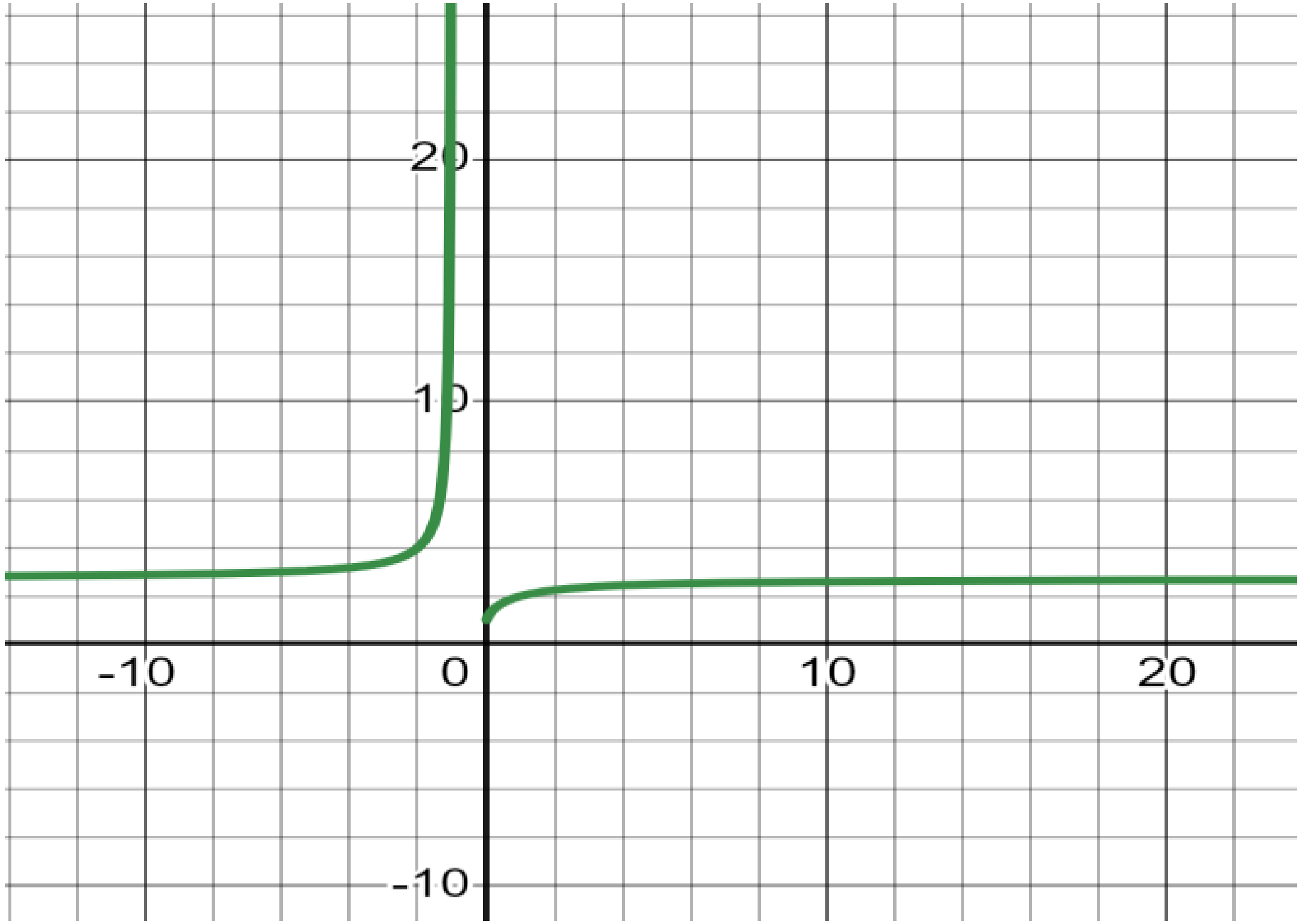
That means these values converge to a fixed value

2.704 – 2.593 = 0.111 and 2.716 – 2.704 = 0.012

- For values of “x” 10000; 100000; 1000000 by using calculator we obtain 2.71845 2.71826 2.718250
- We say these values converge (has limit) the value “e” as x goes to infinite.
- Write $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x}$ = e

Exponential and logarithmic function

- Graph of the function y = $(1+\frac{1}{x})^x$



Exponential and logarithmic function

1. The percentage, y, of households possessing refrigerators, t years they have been introduced in developed country, is

modelled by $y = 100 - 95e^{-0.15t}$

- Find the percentage of households that have refrigerators

a) at their launch $y(0) = 100 - 95e^{0} = 100 – 95 = 5\%$

b) after one year $y(1) = 100 - 95e^{-0.15} = 100 - 95 \cdot 0.860707 = 100 – 81.767257 = 18\%$

c) after 10 years $y(10) = 100 – 95e^{-1.5} = 100 - 95 \cdot 0.223130 = 100 – 21.197365 = 79\%$

d) after 20 years $y(20) = 100 - 95e^{-3} = 100 - 95 \cdot 0.049787 = 100 – 4.729771 = 95\%$

2. The number of items, N, produced each day by an assembly-line worker, t days after an initial training period is modelled by $N = 100 - 100e^{-0.4t}$.

- Calculate the number of items produced daily

a) 1 day after training period; $N(1) = 100 - 100e^{-0.4} = 100 - 100 \cdot 0.67032 = 100 – 67 = 33$

b) 2 day after training period; $N(2) = 100 - 100e^{-0.8} = 100 - 100 \cdot 0.44932 = 100 – 45 = 55$

c) 10 day after training period. $N(10) = 100 - 100e^{-4} = 100 - 100 \cdot 0.01831 = 100 – 2 = 98$

- What is the worker’s daily production in the long run?

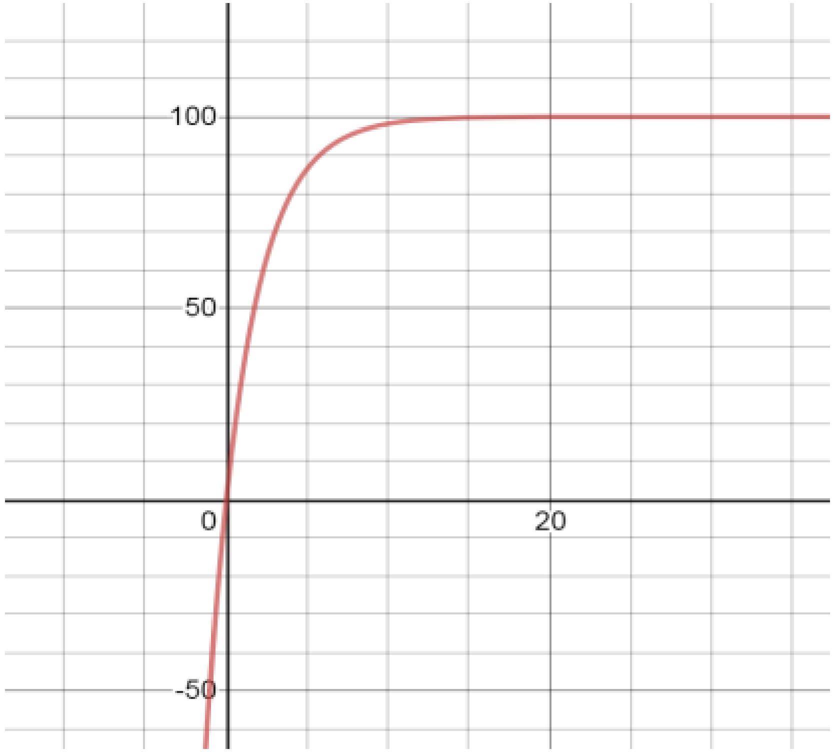
When time (t) is infinite the exponential function with base less than one goes to zero. So, the daily production is

$100 – 0 = 100$

Exponential and logarithmic function

Sketch the graph of N against t and explain why the general shape might have been expected.

3. The value of a second-hand car reduces exponentially with age, so that its value \$ y after t years can be modelled by the formula y = Ae^{-at}



- If the car was worth \$50000 when new and \$38000 after two years, find the values of A and a, correct to three decimal places.
- Use this model to predict the value of the car

a) when the car is five years old

b) in the long run

- If t = 0 (new car) y = Ae^{-at} =Ae^{0} = A = 50000

So, A = 50000. When t = 2 y = 38000 so 38000 = $50000e^{-2a}$

or e^{-2a} = 0.76 - 2alne = ln0.76 - 2a = - 0.274 so a = 0.137

- when t = 5 y = 50000 $e^{-5(0.137)}$ or y = 50000· $e^{-0.685}$ = 50000·0.504 = 25204. In the long run is 50000·0 = \$0

Exponential and logarithmic function

- On expression **y =** $\boldsymbol{a^x}$ we can find the exponent "x" if we know values of "y" and "a".

we write $\boldsymbol{log_a}$**y = x** this is a logarithmic function. This function is inverse function of the exponential function.

- On exponential function we know the base "a" and the values of independent variable "x", find "y"
- On logarithmic function we know the values of "a" and "y", find "x".
- In general form the logarithmic function is ***y =*** $\boldsymbol{log_a}$***x*** where the base "a" may be greater or less than one.
- Useful bases are numbers "10" and the number "e".
- We call logarithms to base "e" natural logarithms and notation **y =** $\boldsymbol{log_e}$**x = lnx**.

<u>Exercise.</u> Solve equations (round your answer to two decimal places)

- e^x = 5.9 xlne = ln5.9 x = 1.7749
- e^x = 0.45 e^x = 2 e^{3x} = 13.68 e^{-5x} = 0.34 $4e^{2x}$ = 7.98

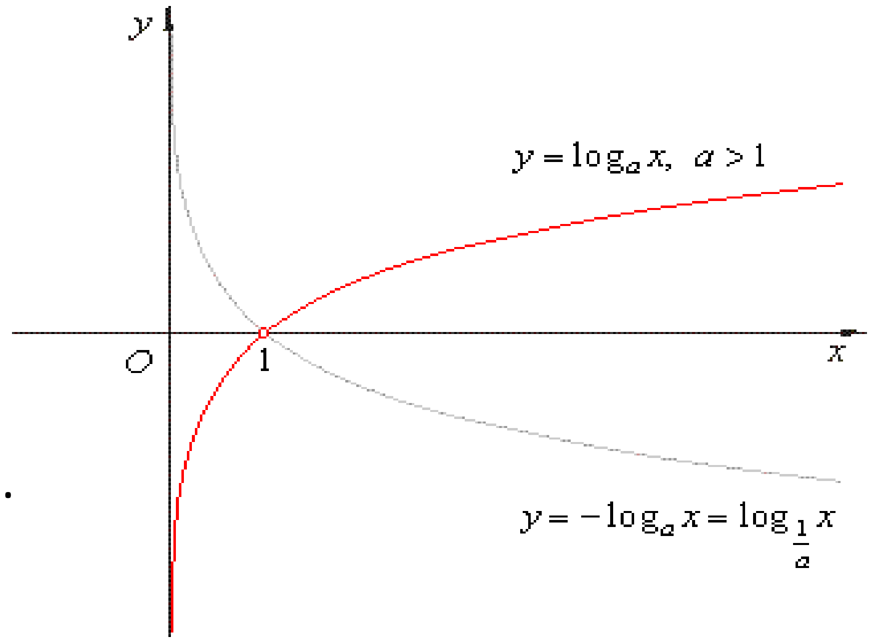
Exponential and logarithmic function

- The graph of the logarithmic function.
- Properties of the logarithmic function.
- As invers function of exponential function, its graph

is symmetric curve of exponential curve against the line

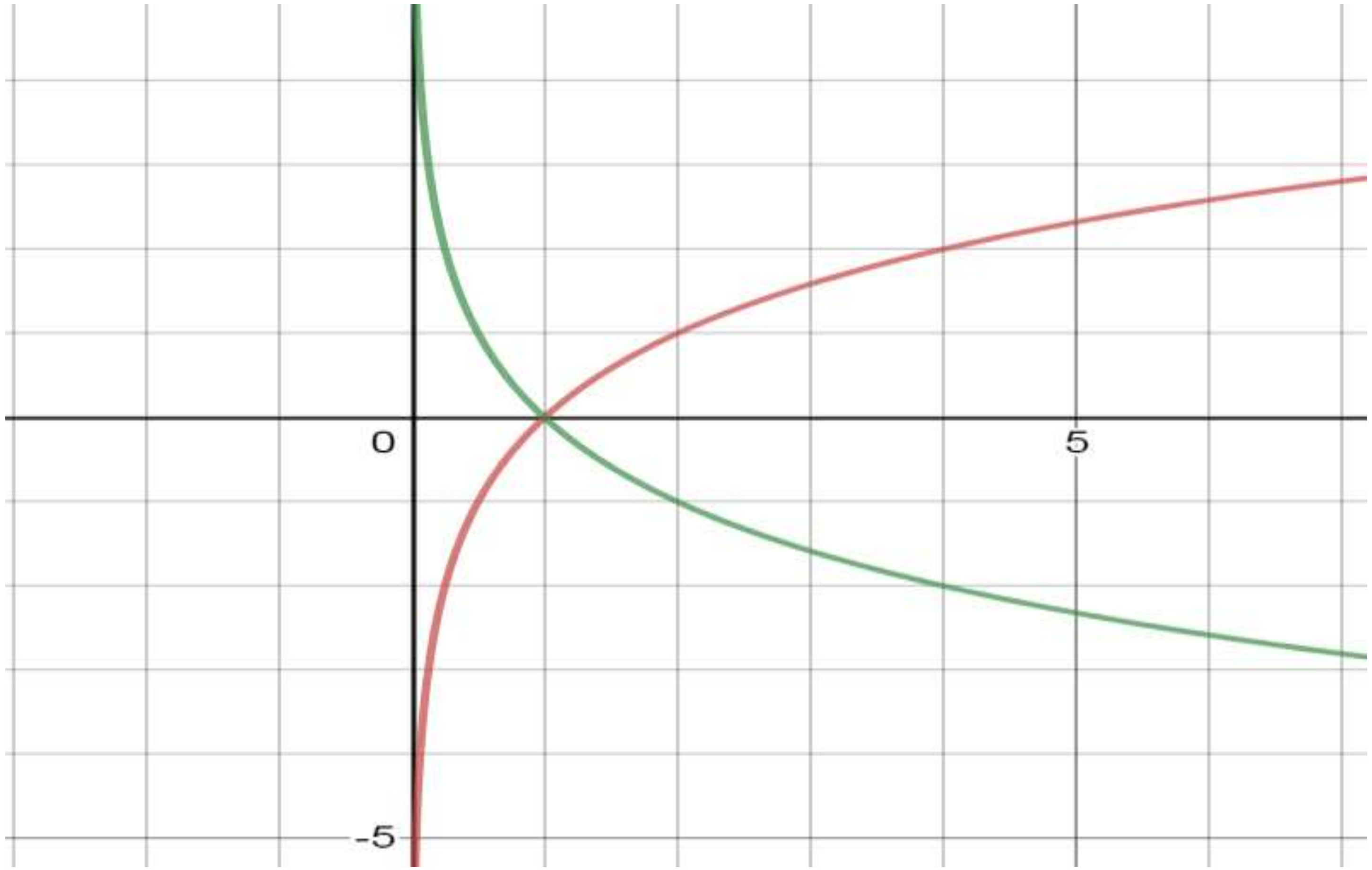
y = x

- It is increased function for bases a > 1
- It is decreased function for base a < 1
- It is determined and continuous for positive real numbers.
- All these graphs passes through point (1, 0)
- When base is greater than one "y"goes - ∞ as "x" goes zero
- When base is less than one "y"goes + ∞ as "x" goes zero



Exponential and logarithmic function

- Graphs of functions y = $\log_2 x$ and
- y = $\log_{\frac{1}{2}} x$



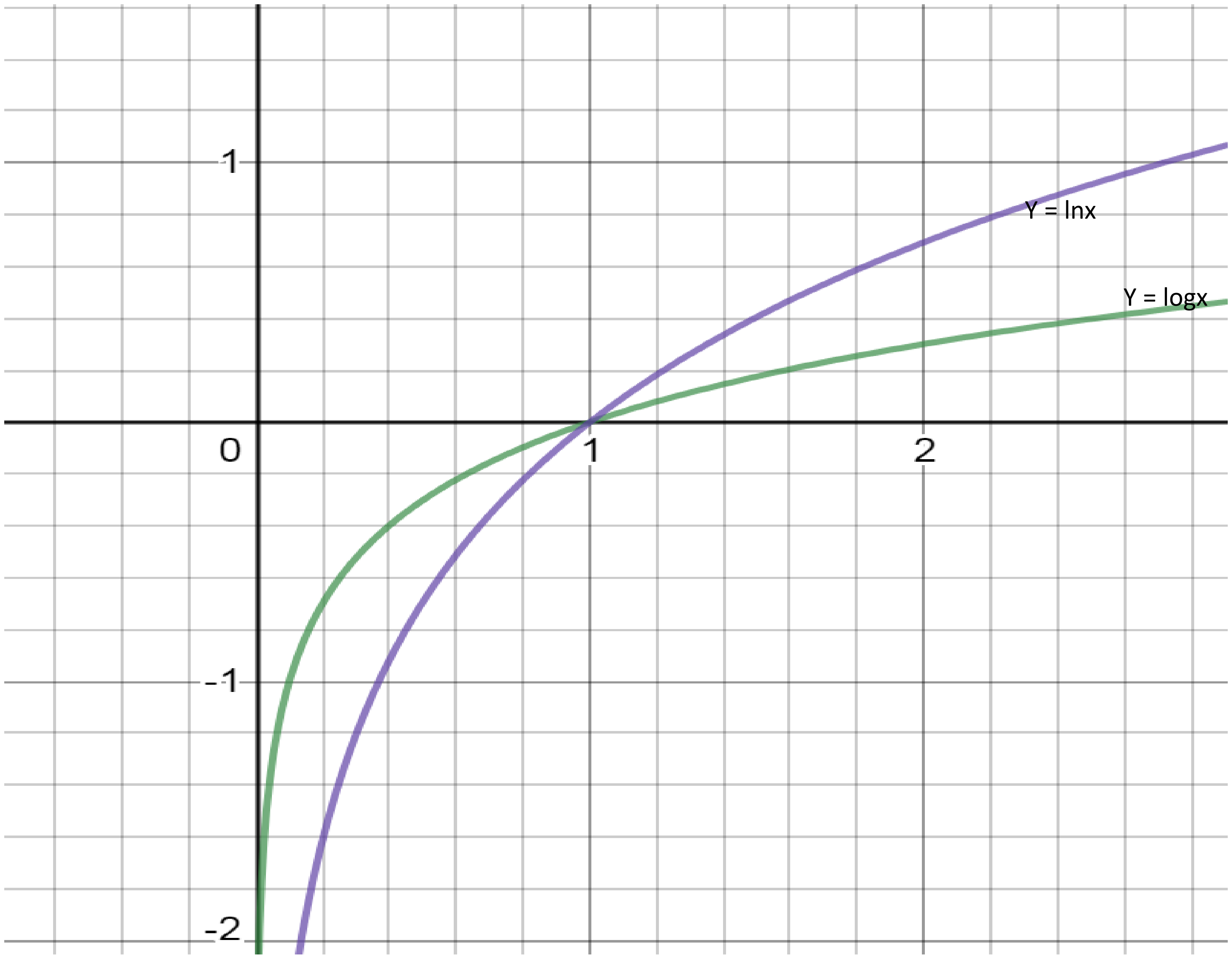
Exponential and logarithmic function

- Graph y = logx and y = lnx

log10 = 1 10^1 = 1

lne = 1 e^1 = 1

e = 2.7182818285



Exponential and logarithmic function

Example. A team of financial advisers guiding the launch of a national newspaper has modelled the future circulation of the newspaper by the equation N = c(1 - e^{-kt}) where N is the daily circulation after t days of publication, and c and k are positive constants.

- Transpose this formula to show that t = $\frac{1}{k}$ ln($\frac{c}{c-N}$)

c(1 - e^{-kt}) = N or (1 - e^{-kt}) = $\frac{N}{c}$ or - e^{-kt} = $\frac{N}{c}$ -1 e^{-kt} = 1 - $\frac{N}{c}$ -ktlne = ln($\frac{c-N}{c}$) or t = $\frac{1}{k}$ ln($\frac{c}{c-N}$)

when the paper is launched, audits show that c = 700000 and k = $\frac{1}{30}$ln2

a) Calculate the daily circulation after 30 days.

find at first - kt = - 30$\frac{1}{30}$ln2 = - ln2 = - 0.693 so N = 700000(1-$e^{-0.693}$) = 700000(1 – 0.5) = 350000

b) After how many days will the daily circulation first reach 525000.

find at first expression $\frac{c}{c-N} = \frac{700000}{700000-525000} = \frac{700000}{175000}$ = 4 $\frac{1}{k} = \frac{30}{ln2} = \frac{30}{0.693}$ = 43.29 t = 43.29ln4 = 43.29·1.386 = 60.01

- What advice can you give the newspaper proprietor if it is known that the paper will break even only the daily circulation exceeds 750000.
- If the daily circulation must be over 750000 then the owner has profit. But market saturation level is only 700000. For that reason the proprietor should sell the business .