# Algebraic Curve

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## 1 Day I

**Definition 1.1** (Polynomial). *The collection of polynomials would denoted by $\mathbb{K}[x_1, \cdots, x_n]$, whose elements are of the form*

$$f = \sum_{i_1,\cdots,i_n} a_{i_1,\cdots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

*where $a_{i_1,\cdots,i_n} \in \mathbb{K}$, and $i_1, \cdots, i_n$ are non-negative integers.*

**Definition 1.2** (Algebraic Closed Field). *If*

**Remark 1.1.** *Finite field is not algebraic closed: Consider $f = (x - a_1) \cdots (x - a_n) + 1$ which has no zero point.*

**Definition 1.3** (Unique Factorization Domain (UFD)).

**Proposition 1.1.** *(1) $\mathbb{K}[x_1, \cdots, x_n]$ is a commutative ring with unity called the polynomial ring in n variables over $\mathbb{K}$.*

*(2) If $R$ is UFD, then $R[X]$ is a UFD, which means that every non-zero polynomial can be factored uniquely into irreducible polynomials, up to order and units.*

From here on, we assume that $\mathbb{K}$ is an algebraic closed field.

**Definition 1.4** (Affine Variety). *An affine variety is a subset of $\mathbb{K}^n$ defined by the vanishing of a set of polynomials, i.e., it is the solution set of a system of polynomial equations.*

*Formally, given a set of polynomials $f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_n]$, the affine variety $V(f_1, \ldots, f_m)$ is defined as:*

$$V(f_1, \ldots, f_m) = \{(a_1, \ldots, a_n) \in \mathbb{K}^n; f_i(a_1, \ldots, a_n) = 0 \text{ for all } i = 1, \ldots, m\}.$$

**Proposition 1.2** (Zariski Topology). *Consider $f, g \in \mathbb{K}[x, y]$*

*(1)* $V(fg) = V(f) \cup V(g)$ ,

*(2)* $V(f, g) = V(f) \cap V(g)$, $V(f_\lambda)_{\lambda\in\Lambda} = \bigcap_{\lambda\in\Lambda} V(f_\lambda)$,

*(3)* $V(0) = \mathbb{A}^2_{\mathbb{K}}$.

**Definition 1.5** (Affine Curve). *Consider $f \in \mathbb{K}[x, y]$, $V(f)$ denotes affine curve.*

*(1)* $\deg V(f) = \deg f$,

- *(a)* $\deg = 1$*: Line,*
- *(b)* $\deg = 2$*: conic curve (non-degenerate),*

*(2)* $F = F_1^{n_1} F_2^{n_2} \cdots F_m^{n_m}$*, where $F_i$ irreducible.*

**Example 1.1.** *$(x+y)^2$ is irreducible, $xy$ is reducible.*

**Example 1.2.** *$y^2 - x^3 + x$ is irreducible (left as exercise).*

**Definition 1.6** (Field of Fractions)**.** *The field of fractions of a UFD $R$ is the smallest field in which $R$ can be embedded, denoted by $K(R)$. It consists of elements of the form $\frac{a}{b}$ where $a, b \in R$ and $b \neq 0 \in R$.*

*Formally, if $R$ is a UFD, then the field of fractions $K(R)$ is defined as:*

$$Q_{\text{uot}}(R) = \left\{ \frac{a}{b} \mid a, b \in R, b \neq 0 \right\},$$

*which is indeed a field.*

**Lemma 1.3.** *Consider $f \in \mathbb{K}[x,y]$ and $\deg f > 0$, then*

*(1) $V(f)$ has infinitely many points,*

*(2) $\mathbb{A}^2_{\mathbb{K}} - V(f)$ has infinitely many points.*

**Theorem 1.4** (Simple Bezout Theorem)**.** *If $F, G \in \mathbb{K}[x,y] \subset \mathbb{K}(x)[y]$ has no common component, then $V(F,G)$ is a finite set $\Leftrightarrow$ $F = 0$, $G = 0$ have finite solutions in $\mathbb{K}^2$.*

*Proof.* (1) Assume there is an element $\alpha$ such that $F = \alpha F'$ and $G = \alpha G'$, where we consider the ring $\mathbb{K}(x)[y]$, then

$$\begin{cases} aF = HF' \\ bG = HG', \end{cases}$$

where $a \in \mathbb{K}[x]$ and $H \in \mathbb{K}[x,y]$.

(2) TBD

□

**Theorem 1.5.** *Consider irreducible $F, G \in \mathbb{K}[x,y]$, $F|G \Leftrightarrow V(F) \subset V(G)$.*

*Proof.* (1) If $F|G$, then $G = FH$ for some $H \in \mathbb{K}[x,y]$, thus $V(F) \subset V(G)$.

(2) If $V(F) \subset V(G)$, by definition $F|G$.

□

## 2 Day II: Intersection Number (1)

**Definition 2.1** (Localized Ring)**.** *Consider $\mathbb{K}[x,y]$ and a prime ideal $P \subset R$, the localized ring $\mathcal{O}_P$ is defined as:*

$$\mathcal{O}_P = \left\{ \frac{f}{g}; f, g \in \mathbb{K}[x,y], g(p) \neq 0 \right\},$$

*the maximal ideal $\mathfrak{m}_P$ is defined as:*

$$\mathfrak{m}_P = \left\{ \frac{f}{g}; f, g \in \mathbb{K}[x,y], g(p) \neq 0, f(p) = 0 \right\}.$$

*which satisfies*

$$0 \to \mathfrak{m}_P \to \mathcal{O}_P \to \mathbb{K}.$$

## 2.1 Definition

**Definition 2.2** (Intersection Number). *Consider $F, G \in \mathbb{K}[x, y]$ irreducible, $P = V(F) \cap V(G)$, then the intersection number $I_P(F, G)$ is defined as:*

$$I_P(F,G) = \dim_{\mathbb{K}} \mathcal{O}_P / \langle F, G \rangle ,$$

*where $\langle F, G \rangle$ is the ideal generated by F and G in the localized ring $\mathcal{O}_P$.*

**Proposition 2.1.** *(1) $I_P(F,G) \in \mathbb{N} \cup \{\infty\}$;*

*(2) $P \in F \cap G \Leftrightarrow I_P(F,G) \geq 1$, $I_P(F,G) = 1 \Leftrightarrow \langle F, G \rangle = I_P$*

*(3) $I_P(F,G) = I_P(G,F)$;*

*(4) $I_P(F, G + FH) = I_P(F,G)$;*

*(5) $I_P(FG, H) = I_P(F,H) + I_P(G,H)$;*

*(6) $I_{(0,0)}(x,y) = 1$,*

**Example 2.1.** *Consider $F = y - x^2$ and $G = y$.*

*Solution.* **Use properties to compute**:

$$\begin{aligned} I_0(y, y - x^3) =& I_0(y, -x^3) \\ =& 2I_0(y,x) \\ =& 2 \end{aligned}$$

where we used the property (4) to reduce the degree of the polynomial for the given variable, and use the fact that $I_0(x,y) = 1$ . □

The most important part is to use the property (4) to reduce the degree of the polynomial for the given variable.

**Example 2.2.** *Consider $F = y^2 - x^3$ and $G = x^2 - y^3$.*

*Solution.*

$$\begin{aligned} I_0(y^2 - x^3, x^2 - y^3) =& I_0(y^2 - x^3 + x(x^2 - y^3), y^3 - x^2) \\ =& I_0(y^2 - xy^3, y^3 - x^2) \\ =& I_0(y^2, y^3 - x^2) + I_0(1 - xy, y^3 - x^2) \\ =& 2I_0(y, y^3 - x^2) + 0 \\ =& 2I_0(y, x^2) \\ =& 4I_0(y,x) \\ =& 4 \end{aligned}$$

$I_0(1 - xy, y^3 - x^2)$ vanished since at $(0,0)$, $1 - xy \neq 0$ and $y^3 - x^2 = 0$. □

**Example 2.3.** *Consider $F = y - x - x^2$ and $G = y^2 - x^2 - 3x^2y$.*

*Solution.*

$$\begin{aligned} I_0(y - x - x^2, y^2 - x^2 - 3x^2y) =& I_0(y - x - x^2, y^2 - x^2 - 3x^2y - (x+y)(y - x - x^2)) \\ =& I_0(y - x - x^2, -2x^2y + x^3) \\ =& I_0(y - x - x^2, x^2(x - 2y)) \\ =& 2I_0(y - x - x^2, x) + I_0(y - x - x^2, x - 2y) \\ =& 3. \end{aligned}$$

Another way to compute is to use definition of intersection number, where we plug the equation $y = x + x^2$ into the second equation, we have

$$I_0(y - x - x^2, y^2 - x^2 - 3x^2y) = \mathfrak{m}_0\left((x+x^2)^2 - x^2 - 3x^2(x+x^2)\right) = \mathfrak{m}_0\left(x^3(-1-2x)\right) = 0.$$

□

**Proposition 2.2.** *If the lowest degree of $F$ is $x^n$ and the lowest degree of $G$ is $y^m$, then the intersection number $I_{(0,0)}(F, G)$ is $nm$.*

**Definition 2.3** (Short Exact Sequence). *A short exact sequence of modules is a sequence of modules and homomorphisms*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

*such that the image of $f$ is equal to the kernel of $g$, i.e.,* $\mathrm{Im}(f) = \ker(g)$.

We would use the short exact sequence for linear space.

**Definition 2.4.** *Consider $P \in \mathbb{A}^2$ and $F, G, H \in \mathbb{K}[x, y]$, then*

*(1) If $F, G$ has no common component cross $P$, then*

$$0 \to \mathcal{O}_P / \langle F, H\rangle \xrightarrow{\bullet G} \mathcal{O}_p / \langle F, GH\rangle \xrightarrow{\pi} \mathcal{O}_P / \langle F, G\rangle \to 0,$$

*where $\pi$ is the natural projection map.*

*(2) $I_P(F, GH) = I_P(F, G) + I_P(F, H)$.*

*Proof.* (1) $\pi$ is surjection;

(2) Consider an element acted by multiplication by $G$:

$$\bullet G : \frac{f}{g} + aF + bH \mapsto F(aG) + G\left(\frac{f}{g} + bH\right) \in \ker \pi,$$

where $a, b \in \mathbb{K}[x, y]$ and $g \in \mathcal{O}_P$. On the other side, consider $f/g \in \ker \pi$, thus $f/g = aF + bG \to b \in \mathcal{O}_p / \langle F, H\rangle$.

(3) $\bullet G$ is injection.

Note that all the vector spaces are finite dimensional, thus the dimension of the kernel is equal to the dimension of the image, and we can conclude that

$$I_P(F, GH) = I_P(F, G) + I_P(F, H),$$

which proofs the proposition (5). □

## 2.2 The Algorithm to Compute Intersection Number

Consider $F(x, y) \in \mathbb{K}[x, y]$, in order to compute the insertion number $I_0(y, F(x, y))$, we can expand $F$ as $F(x, y) = F(x, 0) + yH(x, y)$, thus

$$I_0(y, F(x, y)) = I_0(y, F(x, 0) + yH(x, y)) = I_0(y, F(x, 0)).$$

Assume $F(x, 0) = x^m f(x)$ where $f(x)$ is no vanishing at $x = 0$, thus

$$I_0(y, F(x, y)) = m.$$

Now we shell consider the linear (homogeneous 1 degree part). We denote $F \in \mathbb{K}[x, y]$ as

$$F = F_0 + F_1 + \cdots$$

where $F_i$ is homogeneous degree $i$ part. The $F_1$ part is important, because of the theorem below:

**Theorem 2.3** (2.17 Intersection multiplicity 1). *If $F, G \in \mathbb{K}[x, y]$ pass through the origin, then*

$$I_0(F, G) = 1 \Leftrightarrow F, G \text{ Linear Independent}$$

**Definition 2.5** (Tangents and multiplicities of points). *Let $F \in \mathbb{K}[x, y]$ be a curve, then*

(1) *The smallest $m \in \mathbb{N}$ for which the homogeneous part $F_m$ is non-zero is called the multiplicity $m_0(F)$ of $F$ at the origin. Any linear factor of $F_m$ (considered as a curve) is called a tangent to $F$ at the origin.*

(2) *For a general point $P = (x_0, y_0) \in \mathbb{A}^2$, tangents at $P$ and the multiplicity $m_P(F)$ are defined by first shifting coordinates to $x' = x - x_0$ and $y' = y - y_0$, and then applying (a) to the origin $(x', y') = (0, 0)$.*