

Quantum Field Theory

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Warning: Lots of possible typos!!!!!!!!!!!!!! **Notations:**

- X : a smooth manifold, usually a compact manifold.
- \mathcal{E} : the space of fields, usually infinite dimensional.
- $\text{Conn}(P, X)$: the space of connections on a principal bundle P over X .
- $\text{Maps}(\Sigma, X)$: the space of maps from Σ to X .
- $\Omega^\bullet(X)$: the space of differential forms on X .
- $\Omega_c^\bullet(X)$: the space of differential forms with compact support on X .
- $\text{Vect}(M)$: the space of smooth vector fields on a manifold M , which is Lie algebra of $\text{Diff}(M)$.

1 Day I: Overall Discussion and Mathematical Preliminaries

1.1 Actions and Path Integrals

Action $S : \mathcal{E} \rightarrow \mathbf{k}$ where \mathcal{E} always has infinite dimension, and \mathbf{k} is a field (usually \mathbb{R} or \mathbb{C}).

QM in Imaginary Time $\xrightarrow{\text{Brownian Motion}}$ Wiener Measure on Phase Space

Asymptotic Analysis \longrightarrow Perturbative Renormalisation Theory

Example 1.1. *Some Examples of Classical Field Theories*

- (a) *Scalar Field Theory* $\mathcal{E} = C^\infty(X)$
- (b) *Gauge Theory* $\mathcal{E} = \text{Conn}(P, X)$
- (c) *σ Model* $\mathcal{E} = \text{Maps}(\Sigma, X)$
- (d) *Gravity* $\mathcal{E} = \text{Metrics}(X)$ (More better descriptions does not depends on the background)

1.2 Observables

Observables are functions on the space of fields, i.e. $\mathcal{O} \in C^\infty(\mathcal{E})$.

Example 1.2 (field theory). (a) Consider $X = \text{pt}$, thus $\mathcal{E} = \mathbb{R}^n$ for example.

(b) $\dim X > 0$, the new algebraic structure arise from topological structures of X .

The Key Point is: Capture the data of open sets of $X \rightarrow$ Consider the observables supported on open set U of X denoted by $\text{Obs}(U)$ where U is an open set of X .

Local data captures the open sets of X . The relations between open sets captures the global data of $X \rightarrow$ The algebraic structure of the observables is a sheaf of X .

$$\bigsqcup_i U_i \rightarrow \bigotimes_i \text{Obs}(U_i)$$

Which implies OPE in physics and factorization algebra in mathematics.

Higher product in QFT: The generalization of products of algebra ('products in any direction instead of left and right') e.g. QM gives only left and right module of an algebra; OPE has products in various directions.

Consider the $\dim X = 2$ case in detailed

Example 1.3 (Holomorphic/Chiral Field Theory). Various angle of product $A(w)B(z)$ could be denoted by the time of $A(w)$ rotations around $B(z)$, which could be captured by the Fourier mode of $A(w)$, thus one can have

$$A(w)B(z) = \sum_{m \in \mathbb{Z}} \frac{(A_{(m)}B(z))}{(z-w)^{m+1}}$$

which is the Chiral algebra due to Beilinson and Drinfeld and associated with the Doublt cohomology $H^1_{\mathcal{D}}(\Sigma^2 - \Delta)$, where Σ^2 is the complex surface and Δ is the diagonal of Σ^2 . The higher structure could be captured by the higher cohomology $H^p_{\mathcal{D}}(\Sigma^2 - \Delta)$, which is the higher chiral algebra associated to the derived holomorphic section.

1.3 de Rham Cohomology

Chain of differential forms $\Omega^\bullet(X)$

$$\Omega^\bullet(X) = \left(\dots \xrightarrow{d} \Omega^{n-1}(X) \xrightarrow{d} \Omega^n(X) \xrightarrow{d} \Omega^{n+1}(X) \xrightarrow{d} \dots \right) \quad (1.1)$$

where d is the exterior derivative, and $\Omega^n(X)$ is the space of n -forms on X . The general construction of differential forms could be constructed over open set U by

$$\Omega^n(U) = \bigoplus_{1 \leq i_1 \leq \dots \leq i_n \leq n} C^\infty(U) dx^{i_1} \wedge \dots \wedge dx^{i_n}$$

where one can prove that $d^2 = 0$ and thus $(\Omega^\bullet(U), d)$ is a cochain complex. The cohomology of it is called the de Rham cohomology $H^\bullet(X)$.

Proposition 1.1. The definition of de Rham cohomology does not depend on the choice of the open set U and the choice of the coordinate system i.e. it is intrinsic \rightarrow we can define the de Rham cochain complex on smooth manifold X .

Proof. Consider □

Definition 1.1 (de Rham Cohomology on Compact Support). Let X be a smooth manifold, then the de Rham cohomology on compact support is defined as

$$H_c^\bullet(X) = H^\bullet(\Omega_c^\bullet(X), d) \quad (1.2)$$

where $\Omega_c^\bullet(X)$ is the space of differential forms with compact support.

Theorem 1.2 (Stokes' Theorem). Let X be a smooth manifold with boundary, then for any $\omega \in \Omega^n(X)$, we have

$$\int_X d\omega = \int_{\partial X} \omega$$

which connects the local data $d\Omega^\bullet(X)$ and the global data ∂X .

Theorem 1.3 (Poincaré Lemma).

$$H^p(\mathbb{R}^n) = \begin{cases} \mathbb{R} & p = 0 \\ 0 & p > 0 \end{cases}, \quad H_c^p(\mathbb{R}^n) = \begin{cases} 0 & p < 0 \\ \mathbb{R} & p = n \end{cases}$$

Generator: $H^p(\mathbb{R}^n) \rightarrow \text{constant function}$, $H_c^p(\mathbb{R}^n) \rightarrow \text{a compact support function } \alpha = f(x)\text{vol}_n$, and $\int_{\mathbb{R}^n} \alpha = 1$.

Proof. □

Important: An *Integration* arises from the de Rham cohomology!

Observation. (1) if $\alpha = d\beta$ where $\beta \in \Omega_c^{n-1}(X)$, then $\int_X \alpha = 0$, thus the generator is α whose integral is non-zero.

(2) **Dual Site:** Integration could be captured by the cohomology

$$\int_{\mathbb{R}^n} \leftrightarrow H_c^n(\mathbb{R}^n) \cong \mathbb{R}$$

Path integral could be interpreted as the integration over \mathcal{E} , which leads to consider the cohomology of it. □

1.4 Cartan Formula

Vector fields could acts on smooth functions via

$$V(f) = V^i \frac{\partial f}{\partial x^i} = \left. \frac{d}{dt} f(\varphi_t(x)) \right|_{t=0} = \left. \frac{d}{dt} \varphi_t^* f(x) \right|_{t=0}$$

Such an action could be extended to differential forms by

$$\text{Vect}(M) \ni V : \alpha \mapsto \mathcal{L}_V \alpha = \left. \frac{d}{dt} \varphi_t^* \alpha \right|_{t=0}$$

which has the property $\mathcal{L}_V(\alpha \wedge \beta) = \mathcal{L}_V \alpha \wedge \beta + \alpha \wedge \mathcal{L}_V \beta$, which implies that the Lie derivative is a derivation on the algebra of differential forms with degree 0. And we have contraction ι_V which is a derivation of degree -1 on the algebra of differential forms.

$$\mathcal{L}_V = d\iota_V + \iota_V d$$

Lie derivative is homotopy trivial i.e. chain homotopic.

1.4.1 Proof of Poincaré Lemma

Use Cartan Formula, one can proof Poincaré Lemma.

Proof. Rescaling invariance of \mathbb{R}^n leads to the Euler vector field $E = x^i \frac{\partial}{\partial x^i}$. One can consider the associated diffeomorphism φ_t , where we assume $\varphi_0 = 1$ and thus $\varphi_{-\infty}^* \alpha = 0$, thus the closed form α could be rewritten as

$$\begin{aligned} \alpha &= \varphi_0^* \alpha - \varphi_{-\infty}^* \alpha \\ &= \int_{-\infty}^0 \frac{d}{dt} \varphi_t^* \alpha dt \\ &= \int_{-\infty}^0 \mathcal{L}_E(\varphi_t^* \alpha) dt \end{aligned}$$

using the Cartan formula and $d\varphi^* = \varphi^*d$, we have

$$\alpha = d \int_{-\infty}^0 \varphi_t^* \iota_E \alpha dt = d\beta,$$

thus, the closed form α is exact, which implies that the de Rham cohomology $H^p(\mathbb{R}^n)$ is trivial for $p > 0$. The same idea could be applied to the de Rham cohomology on compact support $H_c^p(\mathbb{R}^n)$. \square

2 Day II: Classical Field Theory

Assume $\mathcal{E} = \Gamma(E, X)$ i.e. a section of a bundle $E \rightarrow X$, where X is oriented manifold. And the action would be written as $S[\phi] = \int_X \mathcal{L}[\phi(x)]$ where $\phi \in \mathcal{E}$. Lagrangian \mathcal{L} satisfies:

- (a) built up by jets of ϕ (locality);
- (b) valued in n form on X (oriented).

The solution of Euler-Lagrange equation forms $\text{Crit}(S)$, which denotes the critical of the action S .

2.1 Examples

Example 2.1 (Phase Space Quantum Mechanics). Consider $X = \mathbb{R}$, then $\mathcal{E} = \mathbb{R}^{2n}$, and the action is

$$S[\phi] = \int_{\mathbb{R}^{2n}} p dq - H(q, p) dt = \int [p\dot{q} - H] dt$$

where H is the Hamiltonian. The Euler-Lagrange equation would become $dH = -\iota_{x_*} \partial \omega$, where $x : \mathbb{R} \rightarrow \mathcal{E}$.

Example 2.2 (Scalar Field Theory). Consider (X, g) a Riemann Manifold, then $\mathcal{E} = C^\infty(X)$, and the action is

$$S[\phi] = \int_X \left[\frac{1}{2} |\nabla \phi|^2 + V(\phi) \right] d\text{vol}_g$$

where $V(\phi)$ is a potential function, and $d\text{vol}_g = \sqrt{|g|} d^d x$. Assume $\partial X = \emptyset$, then the Euler-Lagrange equation is

$$\Delta \phi = \frac{\partial V}{\partial \phi}$$

where $\Delta f = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} g^{ij} \partial_j f)$.

Example 2.3 (Chern-Simons Theory). Consider X a 3-manifold and \mathfrak{g} a semi-simple Lie algebra. Denote P is a principal \mathfrak{g} -bundle over X , then the space of fields is $\mathcal{E} = \text{Conn}(P, X)$. Assume \mathfrak{g} is equipped with a non-degenerate invariant bilinear form $\langle \cdot, \cdot \rangle$ (Killing form), then the action is

$$\text{CS}[A] = \int_X \frac{1}{2} \langle A, F_A \rangle + \frac{1}{6} \langle A, [A, A] \rangle,$$

and the Euler-Lagrange equation encoded by the flat connection $F_A = 0$.

2.2 Symmetry

2.2.1 Global Symmetry and Noether's Theorem

Consider a classical action $S : \mathcal{E} \rightarrow \mathbb{R}$ with a group action $G \curvearrowright \mathcal{E}$ s.t. $S[g(\phi)] = S[\phi]$. Then G would become a global symmetry of the action S .

Consider the continuous symmetry i.e. G is a Lie group, then the infinitesimal action of G on \mathcal{E} is given by a vector field $V \in \text{Vect}(\mathcal{E})$, which satisfies

$$\delta_{V^\alpha} \phi = V^\alpha(\phi),$$

thus the variation of the Lagrangian is

$$\delta_{V^\alpha} \mathcal{L} = dK_\alpha,$$

where K_α is a $n - 1$ form. Furthermore, one can use the Euler-Lagrange equation and its boundary contribution to obtain

$$\delta_{V^\alpha} \mathcal{L} \xrightarrow{\text{EL}=0} d\iota_{V^\alpha} \Theta = dK_\alpha,$$

thus one have the Noether's current

$$J_\alpha = \iota_{V^\alpha} \Theta - K_\alpha, \quad dJ_\alpha + EL[\phi]V_\alpha = 0, \quad (2.1)$$

which is a $n - 1$ form on X and satisfies $dJ_\alpha|_{\text{Crit}(S)} = 0$ while the Euler-Lagrangian equation is satisfied. If we consider $Y_1, Y_2 \subset X$ is codimension 1 (hyper)surface, which are homologous by Σ , then we have

$$\int_{Y_1} J_\alpha - \int_{Y_2} J_\alpha = \int_{\Sigma} dJ_\alpha = 0, \quad \phi \in \text{Crit}(S),$$

and the integration over J_α is independent of the choice of the hyper surface, thus we can define the Noether charge as the integration over J_α on a hyper surface Y ¹.

There is an alternative way to define the Noether current, which is more suitable for practical use. In brief, one can consider the 'gauged' symmetry which would promote ϵ to become a field $\epsilon(x)$, and the variation of the action could be computed by integrating by parts, finally one can obtain

$$\delta_{V^\alpha} S = \int_X -\epsilon(x) d\hat{J}_\alpha,$$

and \hat{J} would become the Noether current which satisfies (2.1) so that \hat{J}_α is identical to J_α up to an exact form.

2.2.2 Gauge 'Symmetry'

¹In physics, one always consider the Noether current which is the Hodge dual of J_α .