

Algebraic Curve

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1 Day I

Definition 1.1 (Polynomial). The collection of polynomials would denoted by $\mathbb{K}[x_1, \dots, x_n]$, whose elements are of the form

$$f = \sum_{i_1, \dots, i_n} a_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n},$$

where $a_{i_1, \dots, i_n} \in \mathbb{K}$, and i_1, \dots, i_n are non-negative integers.

Definition 1.2 (Algebraic Closed Field). If

Remark 1.1. Finite field is not algebraic closed: Consider $f = (x - a_1) \cdots (x - a_n) + 1$ which has no zero point.

Definition 1.3 (Unique Factorization Domain (UFD)).

Proposition 1.1. (1) $\mathbb{K}[x_1, \dots, x_n]$ is a commutative ring with unity called the polynomial ring in n variables over \mathbb{K} .

(2) If R is UFD, then $R[X]$ is a UFD, which means that every non-zero polynomial can be factored uniquely into irreducible polynomials, up to order and units.

From here on, we assume that \mathbb{K} is an algebraic closed field.

Definition 1.4 (Affine Variety). An affine variety is a subset of \mathbb{K}^n defined by the vanishing of a set of polynomials, i.e., it is the solution set of a system of polynomial equations.

Formally, given a set of polynomials $f_1, \dots, f_m \in \mathbb{K}[x_1, \dots, x_n]$, the affine variety $V(f_1, \dots, f_m)$ is defined as:

$$V(f_1, \dots, f_m) = \{(a_1, \dots, a_n) \in \mathbb{K}^n; f_i(a_1, \dots, a_n) = 0 \text{ for all } i = 1, \dots, m\}.$$

Proposition 1.2 (Zariski Topology). Consider $f, g \in \mathbb{K}[x, y]$

- (1) $V(fg) = V(f) \cup V(g)$,
- (2) $V(f, g) = V(f) \cap V(g)$, $V(f_\lambda)_{\lambda \in \Lambda} = \bigcap_{\lambda \in \Lambda} V(f_\lambda)$,
- (3) $V(0) = \mathbb{A}_{\mathbb{K}}^2$.

Definition 1.5 (Affine Curve). Consider $f \in \mathbb{K}[x, y]$, $V(f)$ denotes affine curve.

- (1) $\deg V(f) = \deg f$,
 - (a) $\deg = 1$: Line,
 - (b) $\deg = 2$: conic curve (non-degenerate),
- (2) $F = F_1^{n_1} F_2^{n_2} \cdots F_m^{n_m}$, where F_i irreducible.

Example 1.1. $(x + y)^2$ is irreducible, xy is reducible.

Example 1.2. $y^2 - x^3 + x$ is irreducible (left as exercise).

Definition 1.6 (Field of Fractions). The field of fractions of a UFD R is the smallest field in which R can be embedded, denoted by $K(R)$. It consists of elements of the form $\frac{a}{b}$ where $a, b \in R$ and $b \neq 0 \in R$.

Formally, if R is a UFD, then the field of fractions $K(R)$ is defined as:

$$Q_{\text{quot}}(R) = \left\{ \frac{a}{b} \mid a, b \in R, b \neq 0 \right\},$$

which is indeed a field.

Lemma 1.3. Consider $f \in \mathbb{K}[x, y]$ and $\deg f > 0$, then

- (1) $V(f)$ has infinitely many points,
- (2) $\mathbb{A}_{\mathbb{K}}^2 - V(f)$ has infinitely many points.

Theorem 1.4 (Simple Bezout Theorem). If $F, G \in \mathbb{K}[x, y] \subset \mathbb{K}(x)[y]$ has no common component, then $V(F, G)$ is a finite set $\Leftrightarrow F = 0, G = 0$ have finite solutions in \mathbb{K}^2 .

Proof. (1) Assume there is an element α such that $F = \alpha F'$ and $G = \alpha G'$, where we consider the ring $\mathbb{K}(x)[y]$, then

$$\begin{cases} aF = HF' \\ bG = HG' \end{cases}$$

where $a \in \mathbb{K}[x]$ and $H \in \mathbb{K}[x, y]$.

(2) TBD

□

Theorem 1.5. Consider irreducible $F, G \in \mathbb{K}[x, y]$, $F|G \Leftrightarrow V(F) \subset V(G)$.

Proof. (1) If $F|G$, then $G = FH$ for some $H \in \mathbb{K}[x, y]$, thus $V(F) \subset V(G)$.

(2) If $V(F) \subset V(G)$, by definition $F|G$.

□

2 Day II: Intersection Number (1)

Definition 2.1 (Localized Ring). Consider $\mathbb{K}[x, y]$ and a prime ideal $P \subset R$, the localized ring \mathcal{O}_P is defined as:

$$\mathcal{O}_P = \left\{ \frac{f}{g}; f, g \in \mathbb{K}[x, y], g(p) \neq 0 \right\},$$

the maximal ideal \mathfrak{m}_P is defined as:

$$\mathfrak{m}_P = \left\{ \frac{f}{g}; f, g \in \mathbb{K}[x, y], g(p) \neq 0, f(p) = 0 \right\}.$$

which satisfies

$$0 \rightarrow \mathfrak{m}_P \rightarrow \mathcal{O}_P \rightarrow \mathbb{K}.$$

Definition 2.2 (Intersection Number). Consider $F, G \in \mathbb{K}[x, y]$ irreducible, $P = V(F) \cap V(G)$, then the intersection number $I_P(F, G)$ is defined as:

$$I_P(F, G) = \dim_{\mathbb{K}} \mathcal{O}_P / \langle F, G \rangle,$$

where $\langle F, G \rangle$ is the ideal generated by F and G in the localized ring \mathcal{O}_P .

Proposition 2.1. (1) $I_P(F, G) \in \mathbb{N} \cup \{\infty\}$;

(2) $P \in F \cap G \Leftrightarrow I_P(F, G) \geq 1$, $I_P(F, G) = 1 \Leftrightarrow \langle F, G \rangle = I_P$

(3) $I_P(F, G) = I_P(G, F)$;

(4) $I_P(F, G + FH) = I_P(F, G)$;

(5) $I_P(FG, H) = I_P(F, H) + I_{P(G, H)}$;

(6) $I_{(0,0)}(x, y) = 1$,

Example 2.1. Consider $F = y - x^2$ and $G = y$.

Solution. Use properties to compute:

$$\begin{aligned} I_0(y, y - x^3) &= I_0(y, -x^3) \\ &= 2I_0(y, x) \\ &= 2 \end{aligned}$$

where we used the property (4) to reduce the degree of the polynomial for the given variable, and use the fact that $I_0(x, y) = 1$. \square

The most important part is to use the property (4) to reduce the degree of the polynomial for the given variable.

Example 2.2. Consider $F = y^2 - x^3$ and $G = x^2 - y^3$.

Solution.

$$\begin{aligned} I_0(y^2 - x^3, x^2 - y^3) &= I_0(y^2 - x^3 + x(x^2 - y^3), y^3 - x^2) \\ &= I_0(y^2 - xy^3, y^3 - x^2) \\ &= I_0(y^2, y^3 - x^2) + I_0(1 - xy, y^3 - x^2) \\ &= 2I_0(y, y^3 - x^2) + 0 \\ &= 2I_0(y, x^2) \\ &= 4I_0(y, x) \\ &= 4 \end{aligned}$$

$I_0(1 - xy, y^3 - x^2)$ vanished since at $(0, 0)$, $1 - xy \neq 0$ and $y^3 - x^2 = 0$. \square

Example 2.3. Consider $F = y - x - x^2$ and $G = y^2 - x^2 - 3x^2y$.

Solution.

$$\begin{aligned} I_0(y - x - x^2, y^2 - x^2 - 3x^2y) &= I_0(y - x - x^2, y^2 - x^2 - 3x^2y - (x + y)(y - x - x^2)) \\ &= I_0(y - x - x^2, -2x^2y + x^3) \\ &= I_0(y - x - x^2, x^2(x - 2y)) \\ &= 2I_0(y - x - x^2, x) + I_0(y - x - x^2, x - 2y) \\ &= 3. \end{aligned}$$

Another way to compute is to use definition of intersection number, where we plug the equation $y = x + x^2$ into the second equation, we have

$$I_0(y - x - x^2, y^2 - x^2 - 3x^2y) = m_0 \left((x + x^2)^2 - x^2 - 3x^2(x + x^2) \right) = m_0 \left(x^3(-1 - 2x) \right) = 0.$$

\square