Introduction to Noncommutative Geometry

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1 Day I: Banach Space, Hilbert Space and C-star Algebra

1.1 Banach Space and Hilbert Space

$$\begin{aligned} \text{Hilbert Space } \mathcal{H} &\longleftrightarrow \text{Banach Space } \mathcal{B} + \text{Norm from Inner Product} \\ &\longleftrightarrow \text{Vector Space} + \text{Inner Product} + \text{Complete Norm,} \end{aligned}$$

where one have unique inner product $\langle\ ,\ \rangle$ from the norm $\|\ \|$

$$\langle \phi, \psi \rangle =$$

1.2 C-star Algebra

Definition 1.1 (Algebra)**.** *An* ***algebra*** $\mathcal{A}$ *over a field* $\mathbb{F}$ *is a vector space over* $\mathbb{F}$ *equipped with a bilinear product* $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$*, denoted by* $a \cdot b$ *or simply* ab*.*

Definition 1.2 (Banach Algebra)**.** *A Banach algebra is an algebra* $\mathcal{A}$ *endowed with a norm* $\| \cdot \|$ *such that*

$$\|ab\| \leq \|a\| \|b\|, \quad \forall a, b \in \mathcal{A}.$$

Definition 1.3 ($\mathbb{C}^*$ Algebra)**.** $\mathbb{C}^*$ *algebra* $\longleftrightarrow$ *Banach algebra over* $\mathbb{C}$ + $\forall x \in \mathbb{C}^*$, $\phi : x \mapsto x^*$.

Example 1.1 (Matrix)**.**

Example 1.2 (Continues Linear Operator over Hilbert Space)**.**