Introduction to Index Theory

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1 Motivation

Consider $T \in \mathcal{B}(\mathcal{H})$ which is a bounded operator in Hilbert space. The decomposition $H = \ker(T) \oplus H_0$ and $H = \in (T) \oplus H_1$ leads to an isomorphism (which is finite dimensional)

$$T|_{H_0} : H_0 \to \operatorname{im}(T).$$

where the index of T is

$$\operatorname{ind}(T) = \dim\ker(T) - \dim\operatorname{coker}(T).$$

If index of T is 0, there would be no obstruction to extend an operator to an isomorphic one, i.e. index captured the obstruction to extend to an isomorphism.

Consider an continuous transformation family of T which would be written as $(T_t)_{t\in[0,1]}$. Which could be written as

$$\operatorname{ind}(T_1) = \operatorname{ind}(T_0).$$

Thus, one could expect the index of an operator is homotopy invariant.

Example 1.1 (The Topology of Harday Space)**.** *Consider Harday space*

$$H^2(\mathbb{S}^1) = \left\{ \sum_{n\in\mathbb{N}} a_n z^n, \sum_{n\in\mathbb{N}} |a_n|^2 < \infty \right\}$$

There would be an projection operator $p : L^2(\mathbb{S}^1) \to H^2(\mathbb{S}^1)$, $f \in C(\mathbb{S}^1)$ and there is a multplication operator $M_f : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$. Finally, one could consider the topological operator $pM_f p : H^2(\mathbb{S}^1) \to H^2(\mathbb{S}^1)$

Now consider $f(z) = z$ and $f(z) = z^{-1}$, the shift operator $T_f : H^2(\mathbb{S}^1) \to H^2(\mathbb{S}^1)$ is defined as

$$(a_0, a_1, a_2, \ldots) \mapsto (0, a_0, a_1, \ldots), \quad \textit{and } (a_0, a_1, a_2, \ldots) \mapsto (a_1, a_2, \ldots).$$

Theorem 1.1 (Noether, Fuly 1935)**.** *Suppose $f \neq 0$ for all $z \in \mathbb{S}^2$ (i.e. f is a loop on $\mathbb{C} - 0$), we have* $\operatorname{ind}(T_f) = -\operatorname{winding}(f)$.

Proof. The index is depends on on homotopy class of $f : \mathbb{S}^1 \to \mathbb{C} - \{0\}$ which could be characterized by $\pi_1(\mathbb{C} - \{0\}) = \mathbb{Z}$. So that $[f] = [z^k]$ where k is the dimension of cokernel i.e. the index of f, which is associated with the winding number of f, thus one have

$$\operatorname{ind}(T_f) = -\operatorname{winding}(f)$$

which proved the theorem. □

2 A-S Index Theorem

Consider Elliptic differential operator $\mathcal{D} = a_\alpha(x)\partial^\alpha \xrightarrow{\sigma} (x,\xi) \equiv a_\alpha \xi^\alpha$ which could be defined by Fourier transformation, where ξ^α for $\xi = (\xi_1, \cdots, \xi_n)$ could be defined $\xi^\alpha = (\xi_1^\alpha, \cdots, \xi_n^\alpha)$ and a_α is matrix take value in $C_c^\infty(\mathbb{R}^n)$. Elliptic means $\sigma \in GL_n(\mathbb{C})$.

$\not{\mathcal{D}} = \gamma_i \partial_i$ is Dirac operator where

$$\begin{cases} \gamma_i^2 = 1, & \text{for } i = 1, \cdots, n, \\ \gamma_i\gamma_j + \gamma_j\gamma_i = 0, & \text{for } i \neq j. \end{cases}$$

Now we want to define the Dirac operator $\not{\mathcal{D}}$ on a manifold M with a Riemannian metric g. $\not{\mathcal{D}}^2 = \Delta$ is always well defined on M. If one want to extend the definition of Dirac operator, one need to introduce a spin structure on M i.e. taking an universal cover over $SO(n)$ which is the automorphism group of Δ. The result would lead to spin group $Spin(n)$ which is a double cover of $SO(n)$, and the spinor bundle S is defined as the associated vector bundle of $Spin(n)$. A manifold with a spin structure is called a spin manifold, where is the Dirac operator lives in.

Theorem 2.1. *Let $\hat{A}$ be the A-S characteristic class of a spin manifold M with dimension n, we have*

$$\int_M \hat{A}(M) \in \mathbb{Z}.$$

Here, $\hat{A}(M)$ a characteristic class of a spin manifold M with dimension n and curvature R, which is defined as

$$\hat{A}(M) = \left(\frac{R}{\sinh R}\right)^{1/2} \in H^\bullet(M).$$

2.1 Interlude: Euler Characteristic and Ball

It is well known that $\chi(M) = 2 - 2g$ for a closed orientable surface M with genus g. Use this consideration to $M = \mathbb{S}^2$, one could consider the vector field over $\mathbb{S}^2$, where $\chi(\mathbb{S}^2) = 2$.

Consider the vector filed $V : M \to TM$, which could be interpreted as a section of the tangent bundle TM, which could be also be written alternatively as the diagnal of embadding $M \to M \times M$ thus one have

$$\chi(\mathbb{S}^2) = \#(M, M),$$

which implies that the number of zeroes of a vector field on M is equal to the Euler characteristic of M.

2.2 Thom Class

The characteristic class could be viewed as the degree of 'non linearly independence' of the frame of the vector bundle $E \to M$.

Generalize this idea to the complex vector bundle $E \to M$ of rank k, then the obstruction of the existence of linear independent frame is equivalent to the existence of characteristic class of the vector bundle E which is $c_i \in H^{2i}(M)$. And the Pontryagin class would become $p_i(E) = c_{2i}(E \otimes \mathbb{C})$.

Theorem 2.2 (Atiyah-Singer Index Theorem)**.** *Let M be a closed spin manifold of dimension n, and $\mathcal{D}$ be an elliptic operator on M with index* $\mathrm{ind}(\mathcal{D})$*. Then*

$$\mathrm{ind}(\mathcal{D}) = \int_{TM} \mathrm{Todd}(TM \otimes \mathbb{C})\, \mathrm{ch}(\sigma_D),$$

where E is the vector bundle associated with the elliptic operator $\mathcal{D}$, $\sigma_{\mathcal{D}} \in K(TM)$.

2.3 Relation to K Theory

3 Application

Gauss-Bonnet theorem $D = \mathrm{d} + \mathrm{d}^*$; Riemann-Roch theorem: $D = \bar{\partial} + \bar{\partial}^*$

3.1 Signature Theorem

Consider the intersection number $\#(V, W) = \int_M P_V \wedge P_W$ where P_V is the Poincaré dual of the vector field V. Let M be an closed oriented manifold, then the intersection number $H^{2k}(M) \times H^{2k}(M) \to \mathbb{R}$ could be written as

$$(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta,$$

which is a quadratic form η on $H^{2k}(M)$, and then one could define the signature of M as the signature of the quadratic form η on $H^{2k}(M)$, which is defined as

$$\sigma(M) = \mathrm{Signature}(M) = (\# > 0 \text{ Eigenvalues}) - (\# < 0 \text{ Eigenvalues}).$$

Theorem 3.1 (Signature Theorem)**.** *Let M be a closed oriented manifold of dimension n, then*

$$\sigma(M) = \int_M L(M).$$