# Quantum Field Theory

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**Notations**:
- $X$: a smooth manifold, usually a compact manifold.
- $\mathcal{E}$: the space of fields, usually infinite dimensional.
- $\mathrm{Conn}(P, X)$: the space of connections on a principal bundle $P$ over $X$.
- $\mathrm{Maps}(\Sigma, X)$: the space of maps from $\Sigma$ to $X$.
- $\Omega^\bullet(X)$: the space of differential forms on $X$.
- $\Omega_c^\bullet(X)$: the space of differential forms with compact support on $X$.
- $\mathrm{Vect}(M)$: the space of smooth vector fields on a manifold $M$, which is Lie algebra of $\mathrm{Diff}(M)$.

## Day I: Overall Discussion and Mathematical Preliminaries

### Actions and Path Integrals

Action $S : \mathcal{E} \to \Bbbk$ where $\mathcal{E}$ always has infinite dimension, and $\mathbb{K}$ is a field (usually $\mathbb{R}$ or $\mathbb{C}$).

$$\text{QM in Imaginary Time} \xrightarrow{\text{Brownian Motion}} \text{Wiener Measure on Phase Space}$$

$$\text{Asymptotic Analysis} \longrightarrow \text{Perturbative Renormalisation Theory}$$

**Example** (*Some Examples of Classical Field Theories*)
- Scalar Field Theory $\mathcal{E} = C^\infty(X)$
- Gauge Theory $\mathcal{E} = \mathrm{Conn}(P, X)$
- $\sigma$ Model $\mathcal{E} = \mathrm{Maps}(\Sigma, X)$
- Gravity $\mathcal{E} = \mathrm{Metrics}(X)$ (Better descriptions does not depend on the background)

### Observables

Observables are functions on the space of fields, i.e. $\mathcal{O} \in C^\infty(\mathcal{E})$.

**Example** (*field theory*)
- Consider $X = \mathrm{pt}$, thus $\mathcal{E} = \mathbb{R}^n$ for example.
- $\dim X > 0$, the new algebraic structure arise form topological structures of $X$.

The Key Point is: Capture the data of open sets of $X \longrightarrow$ Consider the observables supported on open set $U$ of $X$ denoted by $\mathrm{Obs}(U)$ where $U$ is an open set of $X$.

Local data captures the open sets of $X$. The relations between open sets captures the global data of $X \longrightarrow$ The algebraic structure of the observables is a sheaf of $X$.

$$\bigsqcup_i U_i \longrightarrow \prod \mathrm{Obs}(U_i)$$

Which implies OPE in physics and factorization algebra in mathematics.

Higher product in QFT: The generalization of products of algebra ('products in any direction instead of left and right') e.g. QM gives only left and right module of an algebra; OPE has products in various directions.

Consider the $\dim X = 2$ case in detailed

**Example** (*Holomorphic/Chiral Field Theory*) Various angle of product $A(w)B(z)$ could be denoted by the time of $A(w)$ rotations around $B(z)$, which could be captured by the Fourier mode of $A(w)$. Thus, one can have

$$A(w)B(z) = \sum_{m\in\mathbb{Z}} \frac{\left(A_{(m)}B(z)\right)}{(z-w)^{m+1}},$$

which is the Chiral algebra due to Beilinson and Drinfeld and associated with the Dolbeault cohomology $H^0_{\bar{\partial}}(\Sigma^2 - \Delta) \cong \mathbb{C}((z^m))$, where $\Sigma^2$ is the complex surface and $\Delta$ is the diagonal of $\Sigma^2$. The higher structure could be captured by the higher cohomology $H^p_{\bar{\partial}}(\Sigma^2 - \Delta)$, which is the higher chiral algebra associated to the derived holomorphic section.

## de Rham Cohomology

Chain of differential forms $\Omega^\bullet(X)$

$$\Omega^\bullet(X) = \left(\xrightarrow{\mathrm{d}} \Omega^{n-1}(X) \xrightarrow{\mathrm{d}} \Omega^{n(X)} \xrightarrow{\mathrm{d}} \Omega^{n+1}(X) \xrightarrow{\mathrm{d}} \dots\right),$$

where d is the exterior derivative, and $\Omega^{n(X)}$ is the space of $n$-forms on $X$. The general construction of differential forms could be constructed over open set $U$ by

$$\Omega^n(U) = \bigoplus(1 \le i_1 \le \dots \le i_n \le n) C^\infty(U)\mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_n},$$

where one can prove that $\mathrm{d}^2 = 0$ and thus $(\Omega^\bullet(U), \mathrm{d})$ is a cochain complex. The cohomology of it is called the de Rham cohomology $H^\bullet(X)$.

**Theorem** (*de Rham cohomology is intrinsic*) The definition of de Rham cohomology does not depend on the choice of the open set $U$ and the choice of the coordinate system. This mean de Rham cohomology is intrinsic $\longrightarrow$ we can define the de Rham cochain complex on smooth manifold $X$.

**Definition** (*de Rham Cohomology on Compact Support*) Let $X$ be a smooth manifold, then the de Rham cohomology on compact support is defined as

$$H^\bullet_c(X) = H^\bullet(\Omega^\bullet_c(X), \mathrm{d})$$

where $\Omega^\bullet_c(X)$ is the space of differential forms with compact support.

**Theorem** (*Stokes' Theorem*) Let $X$ be a smooth manifold with boundary, then for any $\omega \in \Omega^n(X)$, we have

$$\int_X \mathrm{d}\omega = \int_{\partial X} \omega$$

which connects the local data $\mathrm{d}\Omega^\bullet_c(X)$ and the global data $\partial X$.

**Theorem** (*Poincaré Lemma*)

$$H^{p(\mathbb{R}^n)} = \begin{cases} \mathbb{R}, & p = 0 \\ 0, & p > 0 \end{cases}, \quad H^p_c(\mathbb{R}^n) = \begin{cases} 0, & p < n \\ \mathbb{R}, & p = n \end{cases}$$

Generator: $H^p(\mathbb{R}^n) \to$ constant function, $H^p_c(\mathbb{R}^n) \to$ a compact support function $\alpha = f(x)\mathrm{vol}_n$, and $\int_{\mathbb{R}^n} \alpha = 1$.

Important: An **Integration** arises from the de Rham cohomology!

(1) If $\alpha = \mathrm{d}\beta$ where $\beta \in \Omega_c^{n-1}(X)$, then $\int_X \alpha = 0$, thus the generator is $\alpha$ whose integral is non-zero. (2) **Dual Site**: Integration could be captured by the cohomology

$$\int_{\mathbb{R}^n} \Leftrightarrow H_c^n(\mathbb{R}^n) \cong \mathbb{R}$$

Path integral could be interpreted as the integration over $\mathcal{E}$, which leads to consider the cohomology of it.

## Cartan Formula

Vector fields could act on smooth functions via

$$V(f) = V^i \frac{\partial f}{\partial x^i} = \frac{\mathrm{d}}{\mathrm{d}t} f\left(\varphi_{t(x)}\right) |_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \varphi_t^* f(x) |_{t=0}$$

Such an action could be extended to differential forms by

$$\mathrm{Vect}(M) \ni V : \alpha \mapsto \mathcal{L}_V \alpha = \frac{\mathrm{d}}{\mathrm{d}t} \varphi_t^* \alpha |_{t=0}$$

which has the property $\mathcal{L}_V(\alpha \wedge \beta) = \mathcal{L}_V \alpha \wedge \beta + \alpha \wedge \mathcal{L}_V \beta$, which implies that the Lie derivative is a derivation on the algebra of differential forms with degree 0. And we have contraction $\iota_V$ which is a derivation of degree $-1$ on the algebra of differential forms.

$$\mathcal{L}_V = \mathrm{d}\iota_V + \iota_V \mathrm{d}$$

Lie derivative is homotopy trivial i.e. chain homotopic.

### Proof of Poincaré Lemma

Use Cartan Formula, one can proof Poincaré Lemma.

*Proof.* (*Poincaré Lemma*). Rescaling invariance of $\mathbb{R}^n$ leads to the Euler vector field $E = x^i \frac{\partial}{\partial x^i}$. One can consider the associated diffeomorphism $\varphi_t$, where we assume $\varphi_0 = 1$ and thus $\varphi_{-\infty}^* \alpha = 0$. Thus, the closed form $\alpha$ could be rewritten as:

$$\begin{aligned} \alpha &= \varphi_0^* \alpha - \varphi_{-\infty}^* \alpha \\ &= \int_{-\infty}^0 \frac{\mathrm{d}}{\mathrm{d}t} \varphi_t^* \alpha \mathrm{d}t \\ &= \int_{-\infty}^0 \mathcal{L}_E(\varphi_t^* \alpha) \mathrm{d}t, \end{aligned}$$

using the Cartan formula and $\mathrm{d}\varphi^* = \varphi^* \mathrm{d}$, we have

$$\alpha = \mathrm{d} \int_{-\infty}^0 \varphi_t^* \iota_E \alpha \mathrm{d}t = \mathrm{d}\beta.$$

Thus, the closed form $\alpha$ is exact, which implies that the de Rham cohomology $H^{p(\mathbb{R}^n)}$ is trivial for $p > 0$. The same idea could be applied to the de Rham cohomology on compact support $H_c^p(\mathbb{R}^n)$. ∎

## Day II: Classical Field Theory

Assume $\mathcal{E} = \Gamma(E, X)$, i.e. a section of a bundle $E \to X$, where $X$ is an oriented manifold. The action is written as $S[\varphi] = \int_X \mathcal{L}[\varphi(x)]$, where $\varphi \ni \mathcal{E}$. Lagrangian $\mathcal{L}$ satisfies:

- Built up by jets of $\varphi$ (locality)
- Valued in $n$-form on $X$ (oriented)

The solution of Euler-Lagrange equation forms $\mathrm{Crit}(S)$, which denotes the critical locus of the action $S$.

## Examples

**Example** (*Phase Space Quantum Mechanics*) Consider $X = \mathbb{R}$, then $\mathcal{E} = \mathbb{R}^{2n}$, and the action is

$$S[\varphi] = \int_{\mathbb{R}^{2n}} p\,\mathrm{d}q - H(q,p), dt = \int [p\dot{q} - H]dt$$

where $H$ is the Hamiltonian. The Euler-Lagrange equation becomes $\mathrm{d}H = -\iota_{x_*\partial}\omega$, where $x : \mathbb{R} \to \mathcal{E}$.

**Example** (*Scalar Field Theory*) Consider $(X, g)$ a Riemannian manifold, then $\mathcal{E} = C^{\infty(X)}$, and the action is

$$S[\varphi] = \int_X \left[\frac{1}{2}|\nabla\varphi|^2 + V(\varphi)\right] d\mathrm{vol}_g$$

where $V(\varphi)$ is a potential function, and $d\mathrm{vol}_g = \sqrt{|g|}d^d x$. Assume $\partial X = \emptyset$, then the Euler-Lagrange equation is

$$\Delta\varphi = \frac{\partial V}{\partial\varphi}$$

where $\Delta f = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j f\right)$.

**Example** (*Chern-Simons Theory*) Consider $X$ a 3-manifold and $\mathfrak{g}$ a semisimple Lie algebra. Denote $P$ as a principal $\mathfrak{g}$-bundle over $X$, then the space of fields is $\mathcal{E} = \mathrm{Conn}(P, X)$. Assume $\mathfrak{g}$ is equipped with a non-degenerate invariant bilinear form $\langle\cdot,\cdot\rangle$ (Killing form), then the action is

$$\mathrm{CS}[A] = \int_X \frac{1}{2}\langle A, F_A\rangle + \frac{1}{6}\langle A, [A, A]\rangle$$

and the Euler-Lagrange equation is encoded by the flat connection $F_A = 0$.

## Symmetry (1)

### Global Symmetry and Noether's Theorem

Consider a classical action $S : \mathcal{E} \to \mathbb{R}$ with a group action $G \hookrightarrow \mathcal{E}$ such that $S[g(\varphi)] = S[\varphi]$. Then $G$ is a global symmetry of the action $S$.

For continuous symmetry, i.e. $G$ is a Lie group, the infinitesimal action of $G$ on $\mathcal{E}$ is given by a vector field $V \ni \mathrm{Vect}(\mathcal{E})$, which satisfies

$$\delta_{V^\alpha}\varphi = V^\alpha(\varphi)$$

thus the variation of the Lagrangian is

$$\delta_{V^\alpha}\mathcal{L} = dK_\alpha$$

where $K_\alpha$ is an $n-1$ form. Using the Euler-Lagrange equation and its boundary contribution, we obtain

$$\delta_{V^\alpha}\mathcal{L} \to (EL = 0)d\iota_{V^\alpha}\Theta = dK_\alpha$$

thus the Noether current is

$$J_\alpha = \iota_{V^\alpha}\Theta - K_\alpha, \quad dJ_\alpha + EL[\varphi]V_\alpha = 0$$

which is an $n-1$ form on $X$ and satisfies $dJ_\alpha|_{\mathrm{Crit}(S)} = 0$ when the Euler-Lagrange equation holds.

If $Y_1, Y_2 \subset X$ are codimension 1 (hyper) surfaces, homologous by $\Sigma$, then

$$\int_{Y_1} J_\alpha - \int_{Y_2} J_\alpha = \int_\Sigma dJ_\alpha = 0, \quad \varphi \ni \mathrm{Crit}(S)$$

and the integration over $J_\alpha$ is independent of the choice of hypersurface, so we can define the Noether charge as the integration over $J_\alpha$ on a hypersurface $Y$.

Alternatively, one can consider the 'gauged' symmetry, promoting $\varepsilon$ to a field $\varepsilon(x)$, and compute the variation of the action by integrating by parts:

$$\delta_{V^\alpha} S = \int_X -\varepsilon(x) d\hat{J}_\alpha$$

and $\hat{J}$ is the Noether current, satisfying the same equation as $J_\alpha$ up to an exact form.

## Day III: Breaking

## Day IV: Symmetry (2)

First, we will consider finite dimensional case. We consider $G$ as a finite dimensional Lie group, $\mathfrak{g}$ is the Lie algebra of $G$ and $W$ is finite dimensional representation of $G$.

### Chevalley-Eilenberg Cohomology

Consider $\mathfrak{g}^* \equiv \mathrm{Hom}\ (\mathfrak{g}, \mathbb{K})$. Consider the exterior algebra

$$\bigwedge \mathfrak{g}^* = \bigoplus_{p=0}^{\infty} \bigwedge^p \mathfrak{g}^*.$$

Assume the basis of $\mathfrak{g}$ is $\{e_1, \cdots, e_n\}$ and of $\mathfrak{g}^*$ is $\{c^1, \cdots, c^n\}$, which satisfies $c_\alpha c_\beta = -c_\beta c_\alpha$. Thus, one could identify the algebra above as a free object in the category of differential graded algebra, which is a ring equipped with anti-commute generators

$$\bigwedge \mathfrak{g}^* = \mathbb{K}[c^1, \cdots, c^n].$$

Consider the Lie algebra over $\mathfrak{g}$, which equipped with commutator $[\cdot, \cdot] : \wedge^2\ \mathfrak{g} \to \mathfrak{g}$. One the dual side, one would introduce a differential operator $\mathrm{d} : \mathfrak{g}^* \to \mathfrak{g}^*$, and we can extend it to the exterior algebra $\bigwedge \mathfrak{g}^*$ by

(1) Under the level of generators, we have $\mathrm{d}_{\mathrm{CE}} : \mathfrak{g}^* \to \wedge^2\ \mathfrak{g}^*$;

(2) Using the Leibniz rule, we can extend it to the exterior algebra $\bigwedge \mathfrak{g}^*$ by

$$\mathrm{d}_{\mathrm{CE}} : a \wedge b \mapsto \mathrm{d}_{\mathrm{CE}} a \wedge b + (-1)^{\deg a} a \wedge \mathrm{d}_{\mathrm{CE}} b.$$

And thus we have a differential graded algebra $\left(\bigwedge \mathfrak{g}^*, \mathrm{d}_{\mathrm{CE}}\right)$, which is called the Chevalley-Eilenberg complex.

Under the choice of basis above, we have $\left[e_\alpha, e_\beta\right] = f_{\alpha\beta}^\gamma e_\gamma$, which would lead to the derivation on the dual side

$$\mathrm{d}_{\mathrm{CE}} c^\alpha = \frac{1}{2} f_{\beta\gamma}^\alpha c^\beta \wedge c^\gamma \equiv \frac{1}{2} f_{\beta\gamma}^\alpha c^\beta c^\gamma.$$

Using the Leibniz rule, we can extend it to the exterior algebra $\bigwedge \mathfrak{g}^*$. Using the Jacobi identity, one can prove that $\mathrm{d}_{\mathrm{CE}}^2 = 0$ (left as exercise), thus we have a cochain complex $\left(\bigwedge \mathfrak{g}^*, \mathrm{d}_{\mathrm{CE}}\right)$ which is a differential graded algebra (DGA), where the generator $c^\alpha$ is called the 'ghost field' in physics, the degree is 'ghost number' and $\mathrm{d}_{\mathrm{CE}}$ is BRST operator.

*Proof.* Consider $\mathrm{d}_{\mathrm{CE}}^2$ acts on $c^\alpha$, the higher structure could be derived from Leibniz's rule.

$$\begin{aligned}
\mathrm{d}_{\mathrm{CE}}^2 c^\alpha &= \frac{1}{2} f_{\beta\gamma}^\alpha \left[\frac{1}{2} f_{\rho\lambda}^\beta c^\rho c^\lambda c^\gamma - \frac{1}{2} f_{\rho\lambda}^\gamma c^\beta c^\rho c^\lambda\right] \\
&= -\frac{1}{2} f_{\gamma\beta}^\alpha f_{\rho\lambda}^\beta c^\rho c^\lambda c^\lambda \\
&= \frac{1}{12} f_{\beta[\gamma}^\alpha f_{\rho\lambda]}^\beta c^\rho c^\lambda c^\lambda \\
&= 0.
\end{aligned}$$

Extend this result through Leibniz's rule, such a result would be valid in any degree. ■

Let $M$ be a $\mathfrak{g}$ representation where $\rho : \mathfrak{g} \to \mathrm{End}\ (W)$ satisfies

$$\rho(a)\rho(b)m - \rho(b)\rho(a)m = \rho([a, b])m, \quad a, b \in \mathfrak{g}, m \in M.$$

Consider the free $\bigwedge^\bullet \mathfrak{g}^*$-module generated by $M$:

$$\bigwedge^{\bullet} \mathfrak{g}^* \otimes M,$$

there is a natural extension of the Chevalley-Eilenberg differential $\mathrm{d}_{\mathrm{CE}}$ on it, which is defined by

(1) $\mathrm{d}_{\mathrm{CE}} : M \to g^* \otimes M$ is dual of $g \otimes M \xrightarrow{\rho} M$;

(2) $\mathrm{d}_{\mathrm{CE}}(a \otimes m) : \mathrm{d}_{\mathrm{CE}}(a) \otimes m + (-1)^{|a|} a \wedge \mathrm{d}_{\mathrm{CE}} m$

Where we can prove that $\mathrm{d}_{\mathrm{CE}}^2 = 0$, and thus we have a cochain complex $\bigwedge^{\bullet} \mathfrak{g}^* \otimes M$.

We denote $\wedge^p \mathfrak{g}^* \otimes M$ be $C^p(\mathfrak{g}^*, M)$, then we would find that it is $C^p(\mathfrak{g}^*)$-module, i.e.

$$C^p(\mathfrak{g}^*) \otimes C^q(\mathfrak{g}^*, M) \ni a \otimes v \mapsto a \wedge v \in C^{p+q}(\mathfrak{g}^*, M),$$

which is compatible with derivation

$$\mathrm{d}_{\mathrm{CE}}(a \wedge v) = \mathrm{d}_{\mathrm{CE}} a \wedge v + (-1)^{|a|} a \wedge \mathrm{d}_{\mathrm{CE}} v,$$

where $m \in M$ and $a \in \wedge^{\bullet} \mathfrak{g}^*$. The derivation could be written explicitly with basis $a_k$ of $M$, and it's dual basis $b^k$:

$$\mathrm{d}_{\mathrm{CE}} = (\rho_\alpha)^k_i b^i a_k c^\alpha + \frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma a_\alpha,$$

which could be easily verified that $\mathrm{d}_{\mathrm{CE}}^2 = 0$.

*Proof.* There is a general way to prove $\mathrm{d}_{\mathrm{CE}}^2 = 0$, which is to note that, under the dual transformation, one have identity $\langle \mathrm{d}_{\mathrm{CE}} \varphi_1, c_1 m_1 \rangle = \langle \varphi_1, \rho(c_1) m_1 \rangle$, $\langle \mathrm{d}_{\mathrm{CE}} \varphi_2, c_1 \wedge c_2 \rangle = \langle \varphi_2, [c_1, c_2] \rangle$ and Leibniz's law, so that:

$$\langle \partial_{\mathrm{CE}}^2 \varphi, c_1 \wedge c_2 \otimes m_1 \rangle = \langle \varphi, \rho([c_1, c_2]) m + (-1)(\rho(c_1)\rho(c_2) m + \rho(c_2)\rho(c_1) m) \rangle \xrightarrow{\mathfrak{g}\ \text{rep.}} 0,$$

$$\langle \mathrm{d}_{\mathrm{CE}}^2 \varphi, c_1 \wedge c_2 \wedge c_3 \rangle \xrightarrow{\text{Jacobian}} 0,$$

where we note that the dual of $c_1 \wedge c_2$ has degree 1 graded. ■

## Differential Graded Lie Algebra

We define a $\mathbb{Z}$-graded vector space

$$W = \bigoplus_{n \in \mathbb{Z}} W_n,$$

where $W_n$ is degree of $n$ component.

(1) **Degree Shift**: $W[n]_m \equiv W_{n+m}$;

(2) **Dual**: $W^*$ denote the linear dual of $W$

$$W_n^* = \mathrm{Hom}\ (W_{-n}, \mathbb{K});$$

(3) **Symmetry and Anti-Symmetry**: $\mathrm{Sym}^{\otimes n}(V) = V^{\otimes n} / \sim$ where $a \otimes b \sim (-)^{|a||b|} b \otimes a$, and $\bigwedge^V = V^{\otimes n} / \sim$ where $a \otimes b \sim (-1)^{|a||b|+1} b \otimes a$;

which has a natural isomorphism between $\bigwedge^m (V[1])$ and $\mathrm{Sym}^m(V)[m]$

**Proportion** Let $V$ be a DGA, then:

$$\bigwedge^m (V[1]) \cong \mathrm{Sym}^m(V)[m].$$

*Proof.* Consider the subspace generated by ideals

$$a \otimes b \sim (-1)^{(|a|+1)(|b|+1)+1} b \otimes a = (-1)^{|a||b|+|a|+|b|} b \otimes a, \quad a, b \in V[1],$$

$$a \otimes b \sim (-1)^{|a||b|} b \otimes a, \quad a, b \in V,$$

where $|a|$ is the degree of $a$ in $V$, thus the total degree in $V[1]$ is $|a| + |b| + 2$. The element in the left-hand side is

$$\frac{1}{n!}\left(a_1 a_2 \cdots a_n + (-1)^{(|a_1|+1)\left(\sum_{i=2}^{n}|a_i|+n-1\right)+n-1} a_2 \cdots a_n a_1 + \cdots\right) \in \bigwedge^n (V[1]),$$

and the element in the right-hand side is

$$\frac{1}{n!}\left(a_1 a_2 \cdots a_n + (-1)^{|a_1|\sum_{i=2}^{n}|a_i|} a_2 \cdots a_n a_1 + \cdots\right) \in \mathrm{Sym}^n(V)[n].$$

Consider the shuffle map

$$a_1 \otimes a_2 \otimes \cdots \otimes a_n \to a_n \otimes a_{n-1} \otimes \cdots \otimes a_1,$$

the overall sign in $\mathrm{Sym}^m(V)[m]$ and $\bigwedge^m(V[1])$ is the same, which is

$$(-1)^{\sum_{1\le i<j\le n}|a_i||a_j|} = (-1)^{\sum_{1\le i<j\le n}|a_i||a_j| + 2\sum_{i=1}^{n}|a_i|},$$

where the first term is the sign of the anti-symmetry monomials and the second term is the sign of the symmetry monomials. ∎

■

**Definition** (*Differential Graded Algebra*) A DGLA is a $\mathbb{Z}$-graded space

$$g = \bigoplus_{m\in\mathbb{Z}} g_m$$

together with bilinear map $[\cdot,\cdot] : g \otimes g \to g$ satisfying

(1) (graded bracket) $[g_\alpha, g_\beta] \subset g_{\alpha+\beta}$, $[\cdot,\cdot] \in \mathrm{Hom}\ (g \otimes g, g)$,

(2) (graded skew-symmetry) $[a,b] = -(-1)^{|a||b|}[b,a]$ $([\cdot,\cdot] : \wedge^2 g \to g)$,

(3) (graded Jacobi Identity) $[[a,b],c] = [a,[b,c]] - (-1)^{|a||b|}[b,[a,c]]$,

with a degree 1 map $d : g \to g$ (i.e., $d : g_\alpha \to g_{\alpha+1}$) satisfying $\mathrm{d}^2 = 0$ and

4. (graded Leibniz rule) $d[a,b] = [da,b] + (-1)^{|a|}[a,db]$.

**Example** (*Chern-Simons Theory*) Let $X$ be a manifold, $\mathfrak{g}$ a Lie algebra.

- $(\Omega^\bullet(X), d)$ de Rham complex,
- $(\Omega^\bullet(X) \otimes \mathfrak{g}, d, [\cdot,\cdot])$ is DGLA,
- $\Omega^p(X) \otimes \mathfrak{g}$ : degree $p$ component,
- $d : \Omega^p \otimes \mathfrak{g} \to \Omega^{p+1} \otimes \mathfrak{g}$ de Rham, $d(\alpha \otimes h) = d\alpha \otimes h$,
- $[\cdot,\cdot]$ induced from $\mathfrak{g}$,
- Let $\alpha_{1,2} \in \Omega^\bullet(X)$, $h_{1,2} \in \mathfrak{g}$, then $[\alpha_1 \otimes h_1, \alpha_2 \otimes h_2] = \alpha_1 \wedge \alpha_2 \otimes [h_1, h_2]$,

$\rightsquigarrow$ DGLA in Chern-Simons theory.

**Example** (*Kodaria-Spencer Gravity*) Let $X$ be a complex manifold. Let

- $\left(\Omega^{0,\bullet}(X), \overline{\partial}\right)$ Dolbeault Complex,
- $\left(\sum_{\bar{i}_1,\dots,\bar{i}_p} \varphi_{\bar{i}_1\dots\bar{i}_p} d\bar{z}^{\bar{i}_1} \wedge \dots \wedge d\bar{z}^{\bar{i}_p}\right)$ where $\overline{\partial} = d\bar{z}^{\bar{i}} \frac{\partial}{\partial \bar{z}^i}$,
- $T_X \otimes_{\mathbb{C}} \mathbb{C} = T_X^{1,0} \oplus T_X^{0,1}$, where we could choose the basis as

$$\operatorname{Span}\left\{\frac{\partial}{\partial z^i}\right\}, \quad \operatorname{Span}\left\{\frac{\partial}{\partial \bar{z}^i}\right\}.$$

Which leads that $\left(\Omega^{0,*}\left(X, T_X^{1,0}\right), \overline{\partial}, [\cdot,\cdot]\right)$ is a DGLA.

Explicitly, let $\{z^i\}$ be local holomorphic coordinates. $\alpha \in \Omega^{0,p}\left(X, T_X^{1,0}\right)$ takes the form

$$\alpha = \sum_{i,\bar{J}} \alpha^i_{\bar{J}} d\bar{z}^{\bar{J}} \otimes \frac{\partial}{\partial z^i}, \quad d\bar{z}^{\bar{J}} = d\bar{z}^{\bar{j}_1} \wedge \dots \wedge d\bar{z}^{\bar{j}_p},$$

$$\overline{\partial}\alpha = \sum_i \overline{\partial}\alpha^i_{\bar{J}} d\bar{z}^{\bar{J}} \otimes \frac{\partial}{\partial z^i} = \sum_i \frac{\partial \alpha^i_{\bar{J}}}{\partial \bar{z}^k} d\bar{z}^k \wedge d\bar{z}^{\bar{J}} \otimes \frac{\partial}{\partial z^i}.$$

Let $\alpha = \sum_i \alpha^i_{\bar{J}} d\bar{z}^{\bar{J}} \otimes \frac{\partial}{\partial z^i}$ and $\beta = \sum_m \beta^i_{\overline{M}} d\bar{z}^{\overline{M}} \otimes \frac{\partial}{\partial z^i}$. The Lie bracket is

$$[\alpha, \beta] = \sum_i \left(\alpha^j_{\bar{J}} \partial_j \beta^i_{\overline{M}} - \beta^j_{\overline{M}} \partial_j \alpha^i_{\bar{J}}\right) d\bar{z}^{\bar{J}} \wedge d\bar{z}^{\overline{M}} \otimes \frac{\partial}{\partial z^i}$$

On $\deg = 0$ component, this is the standard Lie bracket of $(1,0)$ vector fields. Finally, one can verify that $\left(\Omega^{0,*}\left(X, T_X^{1,0}\right), \overline{\partial}, [\cdot,\cdot]\right)$ is a DGLA. $\rightsquigarrow$ Mathematics: Deformation of complex structures $\leftrightarrow$ Physics: B-twisted topological string (Kodaria-Spencer gravity)

We can consider the Chevalley-Eilenberg complex for a DGLA $(\mathfrak{g}, \mathrm{d}, [\ ,\ ])$.

**Definition** (*Chevalley-Eilenberg Cohomology for DGLA*) For a DGLA $(\mathfrak{g}, \mathrm{d}, [\ ,\ ])$, the Chevalley-Eilenberg complex is defined as

$$C^\bullet(\mathfrak{g}) = \operatorname{Sym}^\bullet\left(\mathfrak{g}^*[-1]\right) = \bigwedge\nolimits^\bullet \mathfrak{g}^*[-\bullet],$$

equipped with the CE differential $\mathrm{d}_{\mathrm{CE}} = \mathrm{d}_1 + \mathrm{d}_2$, where

(1) $\mathrm{d}_1 : \mathfrak{g}^*[-1] \to \mathfrak{g}^*[-1]$ is the dual of $\mathrm{d} : \mathfrak{g} \to \mathfrak{g}$;

(2) $\mathrm{d}_2 : \mathfrak{g}^*[-1] \to \operatorname{Sym}^2\left(\mathfrak{g}^*[-1]\right) \cong \bigwedge^2 \mathfrak{g}^*[-2]$ is the dual of $[\ ,\ ] : \bigwedge^2 \to \mathfrak{g}$;

(3) (Graded Leibniz rule) The derivation extends to

$$\mathrm{d}_{\mathrm{CE}} : \operatorname{Sym}\left(\mathfrak{g}^*[-1]\right) \to \operatorname{Sym}(\mathfrak{g}[-1])$$

via the graded Leibniz rule

$$\mathrm{d}_{\mathrm{CE}}(ab) = \mathrm{d}_{\mathrm{CE}} a\, b + (-1)^{|a|} a\, \mathrm{d}_{\mathrm{CE}} b,$$

and satisfies $\mathrm{d}_{\mathrm{CE}}^2 = 0$.

**Remark** If $\mathfrak{g}$ degenerated to the ordinary Lie algebra, which would be 'bosonic' fields. However, the basic object to build CE complex for ordinary Lie algebra is 'fermionic' fields. So we need to impose $[-1]$ into the definition of CE complex of DGLA.

Another generalized consideration could be finished by this. Consider $L_\infty$ algebra, one can assume $\mathrm{d}_m : \mathfrak{g}^*[-1] \to \operatorname{Sym}^m(\mathfrak{g}^*[-1])[1] = \wedge^m \mathfrak{g}^*[m+1]$ is the dual of $l_m$, which is degree 1. Thus, we could shift by $[1]$, then obtain the dual side $l_m : \wedge^m \mathfrak{g} \to \mathfrak{g}^*[2-m]$. Which implied that, while we treat the dual side of $L_\infty$ algebra as $\mathfrak{g}^*[-1]$ embedded with a degree 1 derivation, the original $L_\infty$ algebra would have degree $2-m$ product $l_m$, which is compatible with our original consideration for ordinary $L_\infty$ algebra.

**Definition** (*DGLA Module*) Let $\mathfrak{g}$ be a DGLA. A $\mathfrak{g}$-module is a cochain complex $(M, d_M)$ with bilinear map

$$\mathfrak{g} \otimes M \to M$$

where $C^\bullet(\mathfrak{g}, M) = \mathrm{Sym}(\mathfrak{g}^*[-1]) \otimes N$ satisfying

(1) $\mathrm{d}_M$ is the dual of $\rho : \mathfrak{g}_n \otimes M_p \to M_{n+p}$,

(2) $\mathrm{d}_{\mathfrak{g}}$ is the dual of $[\ ,\ ] : \rho(a)\rho(b)m - (-1)^{|a||b|}\rho(b)\rho(a)m = \rho([a,b]) \cdot m$,

(3) (Chevalley-Eilenberg differential) $\mathrm{d}_{CE} = \mathrm{d}_M + \mathrm{d}_{\mathfrak{g}}$,

(4) (Leibniz's law) $\mathrm{d}_{\mathrm{CE}}(a \otimes m) = (\mathrm{d}_{\mathfrak{g}} a)m + (-1)^{|a|}\mathrm{ad}_M m$,

## Homotopic Lie Algebra ($L_\infty$ Algebra)

### Coderivation Side

The original definition could be viewed as a homotopic generalization of the Lie algebra, which is a DGLA $V$ with 'higher brackets' $\mu_n : V^{\otimes n} \to V$, where the first term at the chain level formed a (co)chain complex i.e. $\mu_1^2 = 0$. The higher brackets needed to satisfy some self-consistency conditions, which is so called 'homotopic Jacobian identity'. At some low level $n$, which could be written explicitly as

$$\mu_1\mu_2(a,b) = -\,\mu_2(\mu_1 a, b) - (-1)^{|a|}\,\mu_2(a, \mu_1 b),$$

$$\begin{aligned}\mu_1\mu_3(a,b,c) + \mu_3(\mu_1 a, b, c) + (-1)^{|a|}\,\mu_3(a, \mu_1 b, c) + (-1)^{|a|+|b|}\,\mu_3(a, b, \mu_1 c) \\ = -\,\mu_2\ (\ \mu_2(a,b), c\ ) - (-1)^{(|b|+|c|)|a|}\,\mu_2\ (\ \mu_2(b,c), a\ ) - (-1)^{(|a|+|b|)|c|}\,\mu_2\ (\ \mu_2(c,a), b\ ),\end{aligned}$$

where $a, b, c \in V$ is the element of $L_\infty$ algebra $V$.

The infinite number of brackets could be rewritten into a more compact form via coalgebra, and it's coderivation. For this need, we introduce the graded algebra

$$S^c V = \bigoplus_{n=0}^{\infty} V^{\wedge n}[-n],$$

where we note that the monomial $a_1 a_2 \cdots a_n \in V^{\wedge n}$ satisfied

$$a_1 a_2 \cdots a_n = (-1)^{|a_i||a_{i+1}|} a_1 a_2 \cdots a_{i+1} a_i \cdots a_n.$$

We introduce the coproduct $\Delta : S^c V \to S^c V \otimes S^c V$, which is defined by

$$\Delta : a_1 \cdots a_n \mapsto \sum_{i=1}^{n} \sum_{\sigma \in \mathrm{Sh}(i,n)} (-1)^\sigma a_{\sigma(1)} \cdots a_{\sigma(i)} \otimes a_{\sigma(i+1)} \cdots a_{\sigma(n)},$$

where the shuffle map $\mathrm{Sh}(i,n)$ is the set of all possible ways of permutations which satisfies $\sigma(1) < \cdots \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$, and the sign $(-1)^\sigma$ is the sign of the permutation $\sigma$. The coproduct is coassociative, i.e.

$$(\Delta \otimes Id)\Delta = (Id \otimes \Delta)\Delta.$$

### Derivation Side

Day V: Perturbation Theory Consider a finite dimensional toy model for quantum field theory, where the path integral is defined as

$$Z = \sqrt{\det(Q)} \int_{\mathbb{R}^n} \prod_{i=1}^{n} \frac{\mathrm{d}x_i}{\sqrt{2\pi\hbar}} \mathrm{e}^{-\frac{1}{\hbar}S[x]},$$

where the action function is given by

$$S[x] = \frac{1}{2}Q(x) - \frac{1}{3!}\lambda I(x),$$

where $Q$ is a quadratic form, $I$ is a cubic form and $\lambda$ is a coupling constant.

There are two ways to compute the path integral, one is the non-perturbative way, which is to compute the path integral directly. While consider the definition of the path integral above, one would observe that the path integral would not be well-defined, since the integral would diverge. One way to make the path integral well-defined is to consider the embed the integration region $\mathbb{R}^n$ into $\mathbb{C}^n$. Thus, using the Cauchy integral, one could compute the path integral after changing the integration contour into $\Gamma \subset \mathbb{C}^n$, where the integration in $\Gamma$ would be convergent.

In physics, people usually consider another way to compute the path integral, which is called perturbation theory. In perturbation theory, one would consider the action as a perturbation of the free action, while $\lambda$ would be treated as a small parameter. Thus, the path integral could be computed by expanding the action in terms of $\lambda$

$$Z \to \sum_{m=0}^{\infty} \frac{\lambda^m}{m!\hbar^m} \left\langle \left( \frac{1}{3!} I(x) \right)^m \right\rangle,$$

where $\langle\ \rangle$ denotes the expectation value with respect to the free action, which is given by

$$\langle \mathcal{O} \rangle \to \sqrt{\det(Q)} \int_{\mathbb{R}^n} \prod_{i=1}^{n} \frac{\mathrm{d}x_i}{\sqrt{2\pi\hbar}} \mathcal{O}(x) \mathrm{e}^{-\frac{1}{\hbar} S_0[x]},$$

while $S_0 = \frac{1}{2} Q(x)$ is the free part i.e., quadratic part of the action. Here, each term in the expansion is well-defined and could be computed by Wick's contraction. Such a process could be explained by the Feynman diagram, where each term in the expansion corresponds to a multiplication of Feynman diagrams, and each Feynman diagram denotes a pattern of Wick's contraction.

**Definition** (*Graph*) By a graph $\Gamma$, we refer to the following data

(1) A set of vertices $V(\Gamma) = \{v_1, v_2, ..., v_n\}$, where each vertex $v_i$ is a point in the space-time $X$;

(2) A set of half-edges $HE(\Gamma)$;

(3) Inclusion maps from the set of half-edges to the set of vertices $\iota_\Gamma : HE(\Gamma) \to V(\Gamma)$, which assigns each half-edge to a vertex;

(4) A set of edges $E(\Gamma)$, which is a subset of the Cartesian product $HE(\Gamma) \times HE(\Gamma)$, where each edge connects two half-edges;

(5) For each $v \in V(\Gamma)$, $\#\{i_\Gamma^{-1}(v)\}$ is called the valency of $v$.

Using diagram to represent the process of Wick's contraction, one might meet the problem of over-counting. A proper way to avoid the over-counting is to consider the symmetry of the diagram i.e., the automorphism of a diagram and isomorphism between different diagrams.

**Definition** (*Graph Isomorphism*) A graph isomorphism between two graphs is a pair of bijective maps

$$\sigma_V : V(\Gamma_1) \to V(\Gamma_2), \quad \sigma_{HE} : HE(\Gamma_1) \to HE(\Gamma_2),$$

which are compatible with the inclusion maps:

$$\begin{tikzcd}
\mathrm{HE}(\Gamma) \arrow[r, "\sigma_{\mathrm{HE}}"] \arrow[d, "i_\Gamma"] & \mathrm{HE}(\Gamma') \arrow[d, "i_{\Gamma'}"] \\
\mathrm{V}(\Gamma) \arrow[r, "\sigma_V"] & \mathrm{V}(\Gamma')
\end{tikzcd}$$

and compatible with the edge set i.e., for any $a, b \in HE(\Gamma)$, we have

$$(a, b) \in E(\Gamma) \leftrightarrow (\sigma_{HE}(a), \sigma_{HE}(b)) \in E(\Gamma').$$

An automorphism of a graph is a graph isomorphism $\sigma_\Gamma : \Gamma \to \Gamma$, and we denote

$$\mathrm{Aut}(\Gamma) = \{\sigma_\Gamma : \Gamma \to \Gamma \mid \sigma_\Gamma \text{ is a graph isomorphism}\}.$$

Using the language above, we can describe the perturbation theory in terms of diagrams. Consider the theory in this section, given a graph $\Gamma$, we can associate a Feynman diagram to it, which is a collection of vertices and edges, where each vertex corresponds to an interaction term $I(x)$ and each edge corresponds to a contraction between two fields, which would correspond to $Q^{-1}$. After summing overall indices, we would obtain the contribution of this contraction pattern. Such an identification is called the Feynman rules, which is a set of rules to associate a Feynman diagram $\Gamma$ to a term in perturbative expansion $\omega_\Gamma$.

After using the Feynman rules, the perturbative component could be rephrased as

$$\frac{1}{m!\hbar^m}\langle I(x)^m\rangle = \sum_{\Gamma\in\mathcal{G}_m}\frac{1}{\mathrm{Aut}(\Gamma)}\lambda^{|V(\Gamma)|}\hbar^{l(\Gamma)-1}\omega_\Gamma,$$

where $\mathcal{G}_m$ is the set of trivalent graph with $m$ vertices and $l(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + 1$ is the loop number of $\Gamma$. Thus, the perturbative expansion could be rewritten as

$$\left\langle \mathrm{e}^{\frac{\lambda}{3!\hbar}I(x)}\right\rangle = \sum_\Gamma \frac{1}{\mathrm{Aut}(\Gamma)}\lambda^{|V(\Gamma)|}\hbar^{l(\Gamma)-1}\omega_\Gamma = \exp\left(\sum_{\Gamma\ \text{Connected}}\frac{1}{\mathrm{Aut}(\Gamma)}\lambda^{|V(\Gamma)|}\lambda^{l(\Gamma)-1}\right).$$

In general, while the interaction term is

$$I(x) = \frac{\lambda_3}{3!}I_3(x) + \frac{\lambda_4}{4!}I_4(x) + \cdots,$$

then the series expansion will have all possible graphs, which could be computed by the Feynman rules

$$\left\langle \mathrm{e}^{\frac{1}{\hbar}\sum_{m\geq 3}\frac{1}{m!}\lambda_m I_m(x)}\right\rangle = \exp\left(\sum_{\Gamma\ \text{Connected}}\left(\prod_{m\geq 3}\lambda_m^{|V(\Gamma)_m|}\right)\hbar^{l(\Gamma)-1}\frac{\omega_\Gamma}{|\mathrm{Aut}(\Gamma)|}\right),$$

where $|V(\Gamma)_m|$ is the number of vertices of type $I_m(x)$ in the graph $\Gamma$.

Day VI: UV Divergence

## Perturbative Quantum Field Theory

We would consider the perturbative theory of a scalar field theory, where $\mathcal{E} = C^\infty(X)$, $X = \mathbb{R}^d$ and the action is given by

$$S[\phi] = \int_X\left(\frac{1}{2}\phi\Box\phi + \frac{1}{2}m^2\phi^2\right)\mathrm{d}^d x,$$

where the observables could be defined as correlators, which could be defined as the expectation value of the product of fields

$$\langle\mathcal{O}\rangle = \frac{\int\mathcal{D}[\phi]\,\mathcal{O}\exp\left[-\frac{1}{\hbar}\int_X\mathrm{d}^d x\left(\frac{1}{2}\phi\Box\phi + \frac{1}{2}m^2\phi^2\right)\right]}{\int\mathcal{D}[\phi]\exp\left[-\frac{1}{\hbar}\int_X\mathrm{d}^d x\left(\frac{1}{2}\phi\Box\phi + \frac{1}{2}m^2\phi^2\right)\right]},$$

which could be computed by Wick's contraction and Green's function

$$(\Box + m^2)G(x,y) = \hbar\delta(x-y),$$

thus the observable $\langle\phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle$ could be computed by

$$\langle\phi(x_1)\phi(x_2)\cdots\phi(x_2 n)\rangle = \hbar^n\sum_{\sigma\in\mathbb{S}_{2n}}G\left(x_{\sigma(1)},x_{\sigma(2)}\right)G\left(x_{\sigma(3)},x_{\sigma(4)}\right)\cdots G\left(x_{\sigma(2n-1)},x_{\sigma(2n)}\right),$$

which has asymptotic expansion in the limit $x - y\to\infty$:

$$G(x,y)\sim\frac{1}{|x-y|^{d-2}},$$

for $d > 2$. Such an asymptotic expansion would lead to the divergence of the observable, which is called ultraviolet (UV) divergence.

Consider the interaction term

$$I_3(\phi) = \int_X\mathrm{d}^d x\,\frac{\lambda_3}{3!}\phi^3,\quad I_4(\phi) = \int_X\mathrm{d}^d x\,\frac{\lambda_4}{4!}\phi^4,$$

which would twist the observables to a new form which could be also computed by Feynman diagrams.

## Canonical Quantization

In classical mechanics, one would consider the phase space $(M, \omega)$, where $\omega$ is the symplectic form, which defined a symplectic structure on the phase space $M$

$$\omega(V_f, V_g) = \{f, g\},$$

where $\iota_{V_f}\omega = \mathrm{d}f$. The deformation quantization would lead to a non-commutative, where the commutative product in $C^\infty(M)$ is replaced by the Moyal product:

$$(\mathcal{A} = \mathbb{R}[x, y], \cdot, \{\ ,\ \}) \xrightarrow{\text{Deformation Quantization}} (\mathcal{A}[[\hbar]], \star, [\ ,\ ]),$$

where the Moyal product is defined by

$$f \star g = f(x,p)\exp\left(\frac{\mathrm{i}\hbar}{2}\left(\overleftarrow{\partial_x}\right)\vec{\partial}_p - \overleftarrow{\partial_p}\right)\vec{\partial}_x))g(x,p),$$

which is a non-commutative and associative product on the algebra $\mathcal{A}[[\hbar]]$.

*Proof. The Associativity of Moyal Product.* TBD. ∎

We can derive the Moyal product from the path integral, where the action is

$$S[x,p] = -\mathrm{i}\int_{\mathbb{R}} \mathbb{P}\mathrm{d}\mathbb{X},$$

which is called *topological quantum mechanics*, where the path integral is defined as:

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}[\mathbb{X}, \mathbb{P}]\, \mathcal{O} \exp\left(-\frac{1}{\hbar}S[\mathbb{X}, \mathbb{P}]\right)}{\int \mathcal{D}[\mathbb{X}, \mathbb{P}] \exp\left(-\frac{1}{\hbar}S[\mathbb{X}, \mathbb{P}]\right)},$$

where the green's function could be defined as:

$$G(t_1, t_2) = \frac{1}{2}\operatorname{sgn}(t_1 - t_2) = \begin{cases} \frac{1}{2}, & t_1 > t_2 \\ 0, & t_1 = t_2 \\ -\frac{1}{2}, & t_1 < t_2, \end{cases}$$

thus, $\langle \mathbb{X}(t_1)\mathbb{P}(t_2)\rangle = -\mathrm{i}\hbar G(t_1, t_2)$ and $\langle \mathbb{P}(t_1)\mathbb{X}(t_2)\rangle = \mathrm{i}\hbar G(t_1, t_2)$.

We can use the Moyal product to construct the algebra of observables, whose product could be interpreted as the OPE and derived from the path integral. Consider $f, g \in \mathbb{R}[x,p]$, thus, near the classical solution $(x,p)$ for each function, one can express the path integral as

$$\langle f(x + \mathbb{X}(t_1), p + \mathbb{P}(t_1))g(x + \mathbb{X}(t_2), p + \mathbb{P}(t_2))\rangle = f \star g.$$

*Proof.* The process of Wick contraction could be realized by the operator $\left(\frac{\partial}{\partial P_0^L}\right)_{ij}$, which is defined by

$$\sum_{klpq} \mathrm{i}\hbar G_{kl}(t_i, t_j)\eta_{pq}\left(\frac{\partial}{\partial(\partial^k\phi_p \otimes \partial^l\phi_q)}\right)_{ij} \overset{f,g\in\mathbb{R}[x,p]}{\to} \mathrm{i}\hbar\sum_{pq} G(t_i, t_j)\omega_{pq}\left(\frac{\partial}{\partial(\phi_p \otimes \phi_q)}\right)_{ij},$$

where $\phi = (\mathbb{X}, \mathbb{P})$ and $\eta \equiv \omega$ gives the contribution of the correct sign in the Moyal product. The $n$ links contraction would lead to the operator

$$n \text{ Links Wick Contraction} \rightsquigarrow \frac{1}{n!}\left(\frac{\partial}{\partial P_0^L}\right)_{ij}^n,$$

thus the Feynman diagram could be rewritten as

$$\langle f(x+\mathbb{X}(t_1), p+\mathbb{P}(t_1)) g(x+\mathbb{X}(t_2), p+\mathbb{P}(t_2))\rangle = \mathrm{Mult}\left(\mathrm{e}^{\frac{\partial}{\partial P_0^L}}\left(f(\phi(t_1))\otimes g(\phi(t_2))\right)\right),$$

where the Mult is the multiplication of the algebra $\mathcal{A}[[\hbar]]$. Write the definition of the contraction operator, we obtain OPE

$$f(x,p)\star g(x,p)\sim f(x,p)\exp\left(\frac{\mathrm{i}\hbar}{2}\left(\overleftarrow{\partial_x}\right)\vec{\partial}_p-\overleftarrow{\partial_p}\right)\vec{\partial}_x))g(x,p)+\cdots,$$

where the $\cdots$ term is the correction term in $\hbar$ while $f, g$ is not polynomial. If $f, g\in\mathbb{R}[x,y]$ is polynomial, then the correction term is zero, and we have the Moyal product. ∎

## Counter Term in Perturbative $\phi^4$

If one consider the tree level Feynman diagram, we could proof that there is no UV divergence. However, if one consider the loop level Feynman diagram, some problems would occur. For example, consider the 1-loop Feynman diagram of the scalar field theory, which is given by

$$\hbar^2\lambda^2\int_{(\mathbb{R}^4)^2}\mathrm{d}x_1\mathrm{d}x_2\,\phi^2(x_1)\phi^2(x_2)\frac{1}{|x_1-x_2|^4},$$

where we assume $d=4$, such an integral would be divergent $\sim\log r$. The physical interpreter is considering the cutting off and set the coupling constants depending on the cutting of scaling.

After introducing the cutting off, the propagator would become:

$$G(x,y)=\int\mathrm{d}^4k\,\frac{\mathrm{e}^{\mathrm{i}k(x-y)}}{k^2}\xrightarrow{\text{Cutting Off}}\int_{\Lambda_0\le k\le\Lambda_1}\mathrm{d}^4k\,\frac{\mathrm{e}^{\mathrm{i}k(x-y)}}{k^2},$$

another way to interpret the cutting off is to consider the heat kernel cutting off, which is given by

$$P_\epsilon^L=\int_\epsilon^L\mathrm{d}t\,\mathrm{e}^{-t\square}=\int_\epsilon^L\frac{\mathrm{d}t}{(2\pi t)^2}\,\mathrm{e}^{-|x-y|^2/4t}.$$

Thus, using the heat kernel cutting off, we can rewrite the Feynman diagram with propagator as

$$\begin{aligned}&\hbar\lambda^2\int_{(\mathbb{R}^4)^2}\mathrm{d}^4x_1\mathrm{d}^4x_2\,\phi^2(x_1)\phi^2(x_2)\int_{[\epsilon,L]^2}\frac{\mathrm{d}t_1}{(2\pi t_1)^2}\frac{\mathrm{d}t_2}{(2\pi t_2)^2}\,\mathrm{e}^{-|x_1-x_2|^2/4t_1}\mathrm{e}^{-|x_1-x_2|^2/4t_2}\\&=-\hbar\lambda^2\frac{\ln\epsilon}{\pi^2}\int_{\mathbb{R}^4}\mathrm{d}^4x\,\phi^4(x)+\text{Smooth Terms}\ .\end{aligned}$$

To cancel the divergence, we need to introduce a counter term into action $S[\phi]$, which is given by

$$S\to S+I_1^{\mathrm{ct}},\quad I_1^{\mathrm{ct}}=\frac{\hbar\lambda^2}{4!\pi^2}\ln\epsilon\int_{\mathbb{R}^4}\mathrm{d}^4x\,\phi^4(x),$$

which could be interpreted as the renormalization of the coupling constant $\lambda$:

$$\lambda\to\lambda+\frac{\hbar\lambda^2}{\pi^2}\ln\epsilon,\quad\frac{\mathrm{d}\lambda}{\mathrm{d}\ln\epsilon}=\frac{\hbar\lambda^2}{\pi^2}.$$

## Day XI: Factorization Algebra

First, we would define the prefactorization algebra in the open set category $\mathrm{Ops}(M)$ whose objects are open subsets $U\subset M$ and morphisms are inclusions $U\subset V$.

Now, one could define the prefactorization algebra

**Definition** A prefactorization algebra $\mathcal{F}$ on $M$ would assign to each open subset $U$ with a vector space $\mathcal{F}$, together with a maps

$$m_V^{U_1,U_2,\cdots,U_n} : \mathcal{F}(U_1) \otimes \mathcal{F}(U_2) \otimes \cdots \otimes \mathcal{F}(U_n) \longrightarrow \mathcal{F}(V),$$

**Example** (*Prefactorization Algebra on* $\mathbb{R}$) Let $A$ be a unit associative $\mathbb{K}$ algebra, we can define a prefactorization algebra $A^{\text{fact}}$ on $\mathbb{R}$, which satisfies

- For an open set $U = (a, b)$, we have

$$A^{\text{fact}(U)} = A,$$

- For an open set $\mathcal{U} = \bigsqcup_{j \in J} I_j$, where $I_j$ be an open interval, we have

$$A^{\text{fact}(\mathcal{U})} = \underset{\text{Finite } K \subset J}{\operatorname{colim}} \left(\otimes_{j \in K} A\right).$$

- The structure maps $m_V^{U_1,\ldots,U_n}$ are multiplication maps, which could be defined as

$$A \otimes A \otimes A \longrightarrow A, \quad a_1 \otimes a_2 \otimes a_3 \mapsto a_1 a_2 a_3.$$

whose associativity is equavalent to the compability of this prefactorization algebra. Such a prefactorization algebra defined the algebraic structure of a topological quantum field theory on $\mathbb{R}$.

**Example** (*Prefactorization algebra on* $[0,1]$) Let $A$ be a $\mathbb{K}$ algebra over $X = [0,1]$, $M$, $N$ are the left and right module respectively, thus the prefactorization algebra $A_X^{\text{fact}}$ is defined as

- For an open set $U = (a, b)$, we have $A_X^{\text{fact}}(U) = A^{\text{fact}}$;
- For $U = U_1 \sqcup U_2$ while $U_1 \subset (0,1)$ and $U_2 \subset (0,1]$, we have $\mathcal{F}(U) = A^{\text{fact}}(U_1) \otimes M$;
- For $U = U_1 \sqcup U_2$ while $U_1 \subset [0,1)$ and $U_2 \subset (0,1]$, we have $\mathcal{F}(U) = N \otimes A^{\text{fact}}(U_2)$;
- For $U = U_1 \sqcup U_2 \sqcup U_3$, where $U_1 \subset [0,1)$, $U_2 \subset (0,1)$ and $U_3 \subset (0,1]$, we have

$$\mathcal{F}(U) = N \otimes A^{\text{fact}}(U_2) \otimes M,$$