

Introduction to Index Theory

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1 Motivation

Consider $T \in \mathcal{B}(\mathcal{H})$ which is a bounded operator in Hilbert space. The decomposition $H = \ker(T) \oplus H_0$ and $H = \text{im}(T) \oplus H_1$ leads to an isomorphism (which is finite dimensional)

$$T|_{H_0} : H_0 \rightarrow \text{im}(T).$$

where the index of T is

$$\text{ind}(T) = \dim \ker(T) - \dim \text{coker}(T).$$

If index of T is 0, there would be no obstruction to extend an operator to an isomorphic one, i.e. index captured the obstruction to extend to an isomorphism.

Consider an continuous transformation family of T which would be written as $(T_t)_{t \in [0,1]}$. Which could be written as

$$\text{ind}(T_1) = \text{ind}(T_0).$$

Thus, one could expect the index of an operator is homotopy invariant.

Example 1.1 (The Topology of Harday Space). Consider Harday space

$$H^2(\mathbb{S}^1) = \left\{ \sum_{n \in \mathbb{N}} a_n z^n, \sum_{n \in \mathbb{N}} |a_n|^2 < \infty \right\}$$

There would be an projection operator $p : L^2(\mathbb{S}^1) \rightarrow H^2(\mathbb{S}^1)$, $f \in C(\mathbb{S}^1)$ and there is a multiplication operator $M_f : L^2(\mathbb{S}^1) \rightarrow L^2(\mathbb{S}^1)$. Finally, one could consider the topological operator $pM_f p : H^2(\mathbb{S}^1) \rightarrow H^2(\mathbb{S}^1)$

Now consider $f(z) = z$ and $f(z) = z^{-1}$, the shift operator $T_f : H^2(\mathbb{S}^1) \rightarrow H^2(\mathbb{S}^1)$ is defined as

$$(a_0, a_1, a_2, \dots) \mapsto (0, a_0, a_1, \dots), \quad \text{and} \quad (a_0, a_1, a_2, \dots) \mapsto (a_1, a_2, \dots).$$

Theorem 1.1 (Noether, Fuly 1935). Suppose $f \neq 0$ for all $z \in \mathbb{S}^1$ (i.e. f is a loop on $\mathbb{C} - 0$), we have $\text{ind}(T_f) = -\text{winding}(f)$.

Proof. The index is depends on on homotopy class of $f : \mathbb{S}^1 \rightarrow \mathbb{C} - \{0\}$ which could be characterized by $\pi_1(\mathbb{C} - \{0\}) = \mathbb{Z}$. So that $[f] = [z^k]$ where k is the dimension of cokernel i.e. the index of f , which is associated with the winding number of f , thus one have

$$\text{ind}(T_f) = -\text{winding}(f)$$

which proved the theorem. □

2 A-S Index Theorem

Consider Elliptic differential operator $\mathcal{D} = a_\alpha(x)\partial^\alpha \xrightarrow{\sigma} (x, \zeta) \equiv a_\alpha \zeta^\alpha$ which could be defined by Fourier transformation, where ζ^α for $\zeta = (\zeta_1, \dots, \zeta_n)$ could be defined $\zeta^\alpha = (\zeta_1^\alpha, \dots, \zeta_n^\alpha)$ and a_α is matrix take value in $C_c^\infty(\mathbb{R}^n)$. Elliptic means $\sigma \in GL_n(\mathbb{C})$.

$\mathcal{D} = \gamma_i \partial_i$ is Dirac operator where

$$\begin{cases} \gamma_i^2 = 1, & \text{for } i = 1, \dots, n, \\ \gamma_i \gamma_j + \gamma_j \gamma_i = 0, & \text{for } i \neq j. \end{cases}$$

Now we want to define the Dirac operator \mathcal{D} on a manifold M with a Riemannian metric g . $\mathcal{D}^2 = \Delta$ is always well defined on M . If one want to extend the definition of Dirac operator, one need to introduce a spin structure on M i.e. taking an universal cover over $SO(n)$ which is the automorphism group of Δ . The result would lead to spin group $Spin(n)$ which is a double cover of $SO(n)$, and the spinor bundle S is defined as the associated vector bundle of $Spin(n)$. A manifold with a spin structure is called a spin manifold, where is the Dirac operator lives in.

Theorem 2.1. *Let \hat{A} be the A-S characteristic class of a spin manifold M with dimension n , we have*

$$\int_M \hat{A}(M) \in \mathbb{Z}.$$

Here, $\hat{A}(M)$ a characteristic class of a spin manifold M with dimension n and curvature R , which is defined as

$$\hat{A}(M) = \left(\frac{R}{\sinh R} \right)^{1/2} \in H^\bullet(M).$$

2.1 Interlude: Euler Characteristic and Ball

It is well known that $\chi(M) = 2 - 2g$ for a closed orientable surface M with genus g . Use this consideration to $M = \mathbb{S}^2$, one could consider the vector field over \mathbb{S}^2 , where $\chi(\mathbb{S}^2) = 2$.

Consider the vector field $V : M \rightarrow TM$, which could be interpreted as a section of the tangent bundle TM , which could be also be written alternatively as the diagonal of embedding $M \rightarrow M \times M$ thus one have

$$\chi(\mathbb{S}^2) = \#(M, M),$$

which implies that the number of zeroes of a vector field on M is equal to the Euler characteristic of M .

2.2 Thom Class

The characteristic class could be viewed as the degree of 'non linearly independence' of the frame of the vector bundle $E \rightarrow M$.

Generalize this idea to the complex vector bundle $E \rightarrow M$ of rank k , then the obstruction of the existence of linear independent frame is equivalent to the existence of characteristic class of the vector bundle E which is $c_i \in H^{2i}(M)$. And the Pontryagin class would become $p_i(E) = c_{2i}(E \otimes \mathbb{C})$.

Theorem 2.2 (Atiyah-Singer Index Theorem). *Let M be a closed spin manifold of dimension n , and \mathcal{D} be an elliptic operator on M with index $\text{ind}(\mathcal{D})$. Then*

$$\text{ind}(\mathcal{D}) = \int_{TM} \text{Todd}(TM \otimes \mathbb{C}) \text{ch}(\sigma_{\mathcal{D}}),$$

where E is the vector bundle associated with the elliptic operator \mathcal{D} , $\sigma_{\mathcal{D}} \in K(TM)$.

2.3 Relation to K Theory

3 Application

Gauss-Bonnet theorem $D = d + d^*$; Riemann-Roch theorem: $D = \bar{\partial} + \bar{\partial}^*$

3.1 Signature Theorem

Consider the intersection number $\#(V, W) = \int_M P_V \wedge P_W$ where P_V is the Poincaré dual of the vector field V . Let M be an closed oriented manifold, then the intersection number $H^{2k}(M) \times H^{2k}(M) \rightarrow \mathbb{R}$ could be written as

$$(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta,$$

which is a quadratic form η on $H^{2k}(M)$, and then one could define the signature of M as the signature of the quadratic form η on $H^{2k}(M)$, which is defined as

$$\sigma(M) = \text{Signature}(M) = (\# > 0 \text{ Eigenvalues}) - (\# < 0 \text{ Eigenvalues}).$$

Theorem 3.1 (Signature Theorem). *Let M be a closed oriented manifold of dimension n , then*

$$\sigma(M) = \int_M L(M).$$

Proof. We have an algebra homomorphism: $\sigma : \Omega_{4k}^{SO} \otimes \mathbb{Q} \rightarrow \mathbb{Q}$ where Ω_{4k}^{SO} is the cobordism group of oriented manifolds of dimension $4k$. However, we have an isomorphism

$$\Omega_{4k}^{SO} \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{P}^2(\mathbb{C}), \mathbb{P}^2(\mathbb{C}), \dots],$$

which is the polynomial algebra generated by the oriented cobordism classes $\mathbb{P}^{2j}[\mathbb{C}]$, it's enough to verify

$$\sigma(\mathbb{P}^{2i}) = 1 = \langle L_i(p_1(\mathbb{P}^{2i}), \dots, p_n(\mathbb{P}^{2i})), [\mathbb{P}^{2i}] \rangle.$$

□

4 NCG Approach