Algebraic Curve

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1 Day I

Definition 1.1 (Polynomial). *The collection of polynomials would denoted by $\mathbb{K}[x_1, \cdots, x_n]$, whose elements are of the form*

$$f = \sum_{i_1,\cdots,i_n} a_{i_1,\cdots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

where $a_{i_1,\cdots,i_n} \in \mathbb{K}$, and $i_1, \cdots, i_n$ are non-negative integers.

Definition 1.2 (Algebraic Closed Field). *If*

Remark 1.1. *Finite field is not algebraic closed: Consider $f = (x - a_1) \cdots (x - a_n) + 1$ which has no zero point.*

Definition 1.3 (Unique Factorization Domain (UFD)).

Proposition 1.1. *(1) $\mathbb{K}[x_1, \cdots, x_n]$ is a commutative ring with unity called the polynomial ring in n variables over $\mathbb{K}$.*

(2) If R is UFD, then $R[X]$ is a UFD, which means that every non-zero polynomial can be factored uniquely into irreducible polynomials, up to order and units.

From here on, we assume that $\mathbb{K}$ is an algebraic closed field.

Definition 1.4 (Affine Variety). *An affine variety is a subset of $\mathbb{K}^n$ defined by the vanishing of a set of polynomials, i.e., it is the solution set of a system of polynomial equations.*

Formally, given a set of polynomials $f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_n]$, the affine variety $V(f_1, \ldots, f_m)$ is defined as:

$$V(f_1, \ldots, f_m) = \{(a_1, \ldots, a_n) \in \mathbb{K}^n; f_i(a_1, \ldots, a_n) = 0 \text{ for all } i = 1, \ldots, m\}.$$

Proposition 1.2 (Zariski Topology). *Consider $f, g \in \mathbb{K}[x, y]$*

(1) $V(fg) = V(f) \cup V(g)$,

(2) $V(f, g) = V(f) \cap V(g)$, $V(f_\lambda)_{\lambda \in \Lambda} = \bigcap_{\lambda \in \Lambda} V(f_\lambda)$,

(3) $V(0) = \mathbb{A}^2_{\mathbb{K}}$.

Definition 1.5 (Affine Curve). *Consider $f \in \mathbb{K}[x, y]$, $V(f)$ denotes affine curve.*

(1) $\deg V(f) = \deg f$,

- *(a)* $\deg = 1$*: Line,*
- *(b)* $\deg = 2$*: conic curve (non-degenerate),*

(2) $F = F_1^{n_1} F_2^{n_2} \cdots F_m^{n_m}$*, where F_i irreducible.*

Example 1.1. *$(x+y)^2$ is irreducible, xy is reducible.*

Example 1.2. *$y^2 - x^3 + x$ is irreducible (left as exercise).*

Definition 1.6 (Field of Fractions)**.** *The field of fractions of a UFD R is the smallest field in which R can be embedded, denoted by $K(R)$. It consists of elements of the form $\frac{a}{b}$ where $a, b \in R$ and $b \neq 0 \in R$.*

Formally, if R is a UFD, then the field of fractions $K(R)$ is defined as:

$$Q_{\text{uot}}(R) = \left\{ \frac{a}{b} \mid a, b \in R, b \neq 0 \right\},$$

which is indeed a field.

Lemma 1.3. *Consider $f \in \mathbb{K}[x,y]$ and $\deg f > 0$, then*

(1) $V(f)$ has infinitely many points,

(2) $\mathbb{A}^2_{\mathbb{K}} - V(f)$ has infinitely many points.

Theorem 1.4 (Simple Bezout Theorem)**.** *If $F, G \in \mathbb{K}[x,y] \subset \mathbb{K}(x)[y]$ has no common component, then $V(F,G)$ is a finite set $\Leftrightarrow$ $F = 0$, $G = 0$ have finite solutions in $\mathbb{K}^2$.*

Proof. (1) Assume there is an element α such that $F = \alpha F'$ and $G = \alpha G'$, where we consider the ring $\mathbb{K}(x)[y]$, then

$$\begin{cases} aF = HF' \\ bG = HG', \end{cases}$$

where $a \in \mathbb{K}[x]$ and $H \in \mathbb{K}[x,y]$.

(2) TBD

□

Theorem 1.5. *Consider irreducible $F, G \in \mathbb{K}[x,y]$, $F|G \Leftrightarrow V(F) \subset V(G)$.*

Proof. (1) If $F|G$, then $G = FH$ for some $H \in \mathbb{K}[x,y]$, thus $V(F) \subset V(G)$.

(2) If $V(F) \subset V(G)$, by definition $F|G$.

□

2 Day II: Intersection Number (1)

Definition 2.1 (Localized Ring)**.** *Consider $\mathbb{K}[x,y]$ and a prime ideal $P \subset R$, the localized ring $\mathcal{O}_P$ is defined as:*

$$\mathcal{O}_P = \left\{ \frac{f}{g}; f, g \in \mathbb{K}[x,y], g(p) \neq 0 \right\},$$

the maximal ideal $\mathfrak{m}_P$ is defined as:

$$\mathfrak{m}_P = \left\{ \frac{f}{g}; f, g \in \mathbb{K}[x,y], g(p) \neq 0, f(p) = 0 \right\}.$$

which satisfies

$$0 \to \mathfrak{m}_P \to \mathcal{O}_P \to \mathbb{K}.$$