Quantum Field Theory

Lectured by Prof. Si Li and Noted by Xinyu Xiang

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Warning: Lots of possible typos!!!!!!!!!!!! **Notations**:

- X: a smooth manifold, usually a compact manifold.
- $\mathcal{E}$: the space of fields, usually infinite dimensional.
- $\mathrm{Conn}(P, X)$: the space of connections on a principal bundle P over X.
- $\mathrm{Maps}(\Sigma, X)$: the space of maps from Σ to X.
- $\Omega^\bullet(X)$: the space of differential forms on X.
- $\Omega_c^\bullet(X)$: the space of differential forms with compact support on X.
- $\mathrm{Vect}(M)$: the space of smooth vector fields on a manifold M, which is Lie algebra of $\mathrm{Diff}(M)$.

1 Day I: Overall Discussion and Mathematical Preliminaries

1.1 Actions and Path Integrals

Action $S : \mathcal{E} \to \mathbf{k}$ where $\mathcal{E}$ always has infinite dimension, and is a field (usually $\mathbb{R}$ or $\mathbb{C}$).

$$\text{QM in Imaginary Time} \xrightarrow{\text{Brownian Motion}} \text{Wiener Measure on Phase Space}$$

$$\text{Asymptotic Analysis} \longrightarrow \text{Perturbative Renormalisation Theory}$$

Example 1.1. *Some Examples of Classical Field Theories*

(a) Scalar Field Theory $\mathcal{E} = C^\infty(X)$

(b) Gauge Theory $\mathcal{E} = \mathrm{Conn}(P, X)$

(c) σ *Model* $\mathcal{E} = Maps(\Sigma, X)$

(d) Gravity $\mathcal{E} = Metrics(X)$ *(More better descriptions does not depends on the background)*

1.2 Observables

Observables are functions on the space of fields, i.e. $\mathcal{O} \in C^\infty(\mathcal{E})$.

Example 1.2 (field theory)**.** *(a) Consider* $X = pt$*, thus* $\mathcal{E} = \mathbb{R}^n$ *for example.*

(b) $\dim X > 0$*, the new algebraic structure arise form topological structures of* X*.*

The Key Point is: Capture the data of open sets of $X \longrightarrow$ Consider the observables supported on open set U of X denoted by $\mathrm{Obs}(U)$ where U is an open set of X.

Local data captures the open sets of X. The relations between open sets captures the global data of $X \longrightarrow$ The algebraic structure of the observables is a sheaf of X.

$$\bigsqcup_i U_i \longrightarrow \bigotimes_i \mathrm{Obs}(U_i)$$

Which implies OPE in physics and factorization algebra in mathematics.

Higher product in QFT: The generalization of products of algebra ('products in any direction instead of left and right') e.g. QM gives only left and right module of an algebra; OPE has products in various directions.

Consider the $\dim X = 2$ case in detailed

Example 1.3 (Holomorphic/Chiral Field Theory). *Various angle of product $A(w)B(z)$ could be denoted by the time of $A(w)$ rotations around $B(z)$, which could be captured by the Fourier mode of $A(w)$, thus one can have*

$$A(w)B(z) = \sum_{m\in\mathbb{Z}} \frac{(A_{(m)B(z)})}{(z-w)^{m+1}}$$

which is the Chiral algebra due to Beilinson and Drinfeld and associated with the Doubult cohomology $H^0_{\bar{\partial}}(\Sigma^2 - \Delta) \cong \mathbb{C}((z^m))$, where Σ^2 is the complex surface and Δ is the diagonal of Σ^2. The higher structure could be captured by the higher cohomology $H^p_{\bar{\partial}}(\Sigma^2 - \Delta)$, which is the higher chiral algebra associated to the derived holomorphic section.

1.3 de Rham Cohomology

Chain of differential forms $\Omega^\bullet(X)$

$$\Omega^\bullet(X) = \left(\cdots \xrightarrow{\mathrm{d}} \Omega^{n-1}(X) \xrightarrow{\mathrm{d}} \Omega^n(X) \xrightarrow{\mathrm{d}} \Omega^{n+1}(X) \xrightarrow{\mathrm{d}} \cdots\right) \tag{1.1}$$

where d is the exterior derivative, and $\Omega^n(X)$ is the space of n-forms on X. The general construction of differential forms could be constructed over open set U by

$$\Omega^n(U) = \bigoplus_{1\le i_1\le\cdots\le i_n\le n} C^\infty(U)\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_n}$$

where one can prove that $\mathrm{d}^2 = 0$ and thus $(\Omega^\bullet(U), \mathrm{d})$ is a cochain complex. The cohomology of it is called the de Rham cohomology $H^\bullet(X)$.

Proposition 1.1. *The definition of de Rham cohomology does not depend on the choice of the open set U and the choice of the coordinate system i.e. it is intrinsic $\longrightarrow$ we can define the de Rham cochain complex on smooth manifold X.*

Proof. Consider □

Definition 1.1 (de Rham Cohomology on Compact Support). *Let X be a smooth manifold, then the de Rham cohomology on compact support is defined as*

$$H^\bullet_c(X) = H^\bullet(\Omega^\bullet_c(X), \mathrm{d}) \tag{1.2}$$

where $\Omega^\bullet_c(X)$ is the space of differential forms with compact support.

Theorem 1.2 (Stokes' Theorem). *Let X be a smooth manifold with boundary, then for any $\omega \in \Omega^n(X)$, we have*

$$\int_X \mathrm{d}\omega = \int_{\partial X} \omega$$

which connects the local data $\mathrm{d}\Omega^\bullet(X)$ and the global data ∂X.

Theorem 1.3 (Poincaré Lemma)**.**

$$H^p(\mathbb{R}^n) = \begin{cases} \mathbb{R} & p = 0 \\ 0 & p > 0 \end{cases}, \quad H_c^p(\mathbb{R}^n) = \begin{cases} 0 & p < 0 \\ \mathbb{R} & p = n \end{cases}$$

Generator: $H^p(\mathbb{R}^n) \to$ *constant function,* $H_c^p(\mathbb{R}^n) \to$ *a compact support function* $\alpha = f(x)\mathrm{vol}_n$*, and* $\int_{\mathbb{R}^n} \alpha = 1$.

Proof. □

Important: An *Integration* arises from the de Rham cohomology!

Observation. (1) if $\alpha = \mathrm{d}\beta$ where $\beta \in \Omega_c^{n-1}(X)$, then $\int_X \alpha = 0$, thus the generator is α whose integral is non-zero.

(2) **Dual Site**: Integration could be captured by the cohomology

$$\int_{\mathbb{R}^n} \leftrightarrow H_c^n(\mathbb{R}^n) \cong \mathbb{R}$$

Path integral could be interpreted as the integration over $\mathcal{E}$, which leads to consider the cohomology of it.

□

1.4 Cartan Formula

Vector fields could acts on smooth functions via

$$V(f) = V^i \frac{\partial f}{\partial x^i} = \frac{\mathrm{d}}{\mathrm{d}t} f(\varphi_t(x))\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \varphi_t^* f(x)\bigg|_{t=0}$$

Such an action could be extended to differential forms by

$$\mathrm{Vect}(M) \ni V : \alpha \mapsto \mathcal{L}_V \alpha = \frac{\mathrm{d}}{\mathrm{d}t} \varphi_t^* \alpha \bigg|_{t=0}$$

which has the property $\mathcal{L}_V(\alpha \wedge \beta) = \mathcal{L}_V \alpha \wedge \beta + \alpha \wedge \mathcal{L}_V \beta$, which implies that the Lie derivative is a derivation on the algebra of differential forms with degree 0. And we have contraction ι_V which is a derivation of degree -1 on the algebra of differential forms.

$$\mathcal{L}_V = \mathrm{d}\iota_V + \iota_V \mathrm{d}$$

Lie derivative is homotopy trivial i.e. chain homotopic.

1.4.1 Proof of Poincaré Lemma

Use Cartan Formula, one can proof Poincaré Lemma.

Proof. Rescaling invariance of $\mathbb{R}^n$ leads to the Euler vector field $E = x^i \frac{\partial}{\partial x^i}$. One can consider the associated diffeomorphism φ_t, where we assume $\varphi_0 = 1$ and thus $\varphi_{-\infty}^* \alpha = 0$, thus the closed form α could be rewritten as

$$\begin{aligned} \alpha &= \varphi_0^* \alpha - \varphi_{-\infty}^* \alpha \\ &= \int_{-\infty}^0 \frac{\mathrm{d}}{\mathrm{d}t} \varphi_t^* \alpha \mathrm{d}t \\ &= \int_{-\infty}^0 \mathcal{L}_E(\varphi_t^* \alpha) \mathrm{d}t \end{aligned}$$

using the Cartan formula and $\mathrm{d}\varphi^* = \varphi^*\mathrm{d}$, we have

$$\alpha = \mathrm{d}\int_{-\infty}^{0} \varphi_t^* \iota_E \alpha \, \mathrm{d}t = \mathrm{d}\beta,$$

thus, the closed form α is exact, which implies that the de Rham cohomology $H^p(\mathbb{R}^n)$ is trivial for $p > 0$. The same idea could be applied to the de Rham cohomology on compact support $H_c^p(\mathbb{R}^n)$. $\square$

2 Day II: Classical Field Theory

Assume $\mathcal{E} = \Gamma(E, X)$ i.e. a section of a bundle $E \to X$, where X is oriented manifold. And the action would be written as $S[\phi] = \int_X \mathcal{L}[\phi(x)]$ where $\phi \in \mathcal{E}$. Lagrangian $\mathcal{L}$ satisfies:

(a) built up by jets of ϕ (locality);

(b) valued in n form on X (oriented).

The solution of Euler-Lagrange equation forms $\mathrm{Crit}(S)$, which denotes the critical of the action S.

2.1 Examples

Example 2.1 (Phase Space Quantum Mechanics)**.** *Consider $X = \mathbb{R}$, then $\mathcal{E} = \mathbb{R}^{2n}$, and the action is*

$$S[\phi] = \int_{\mathbb{R}^{2n}} p\mathrm{d}q - H(q,p)\mathrm{d}t = \int [p\dot{q} - H]\,\mathrm{d}t$$

where H is the Hamiltonian. The Euler-Lagrange equation would become $\mathrm{d}H = -\iota_{x_*\partial}\omega$, where $x : \mathbb{R} \to \mathcal{E}$.

Example 2.2 (Scalar Field Theory)**.** *Consider (X, g) a Riemann Manifold, then $\mathcal{E} = C^\infty(X)$, and the action is*

$$S[\phi] = \int_X \left[\frac{1}{2}|\nabla\phi|^2 + V(\phi)\right] \mathrm{dvol}_g$$

where $V(\phi)$ is a potential function, and $\mathrm{dvol}_g = \sqrt{|g|}\mathrm{d}^d x$. Assume $\partial X =$, then the Euler-Lagrange equation is

$$\Delta\phi = \frac{\partial V}{\partial \phi}$$

where $\Delta f = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j f\right)$.

Example 2.3 (Chern-Simons Theory)**.** *Consider X a 3-manifold and $\mathfrak{g}$ a semi-simple Lie algebra. Denote P is a principal $\mathfrak{g}$-bundle over X, then the space of fields is $\mathcal{E} = \mathrm{Conn}(P, X)$. Assume $\mathfrak{g}$ is equipped with a non-degenerate invariant bilinear form $\langle\cdot,\cdot\rangle$ (Killing form), then the action is*

$$\mathrm{CS}[A] = \int_X \frac{1}{2}\langle A, F_A\rangle + \frac{1}{6}\langle A, [A, A]\rangle,$$

and the Euler-Lagrange equation encoded by the flat connection $F_A = 0$.

2.2 Symmetry (1)

2.2.1 Global Symmetry and Noether's Theorem

Consider a classical action $S : \mathcal{E} \to \mathbb{R}$ with a group action $G \curvearrowright \mathcal{E}$ s.t. $S[g(\phi)] = S[\phi]$. Then G would become a global symmetry of the action S.

Consider the continuous symmetry i.e. G is a Lie group, then the infinitesimal action of G on $\mathcal{E}$ is given by a vector field $V \in \mathrm{Vect}(\mathcal{E})$, which satisfies

$$\delta_{V^\alpha}\phi = V^\alpha(\phi),$$

thus the variation of the Lagrangian is

$$\delta_{V^\alpha}\mathcal{L} = \mathrm{d}K_\alpha,$$

where K_α is a $n-1$ form. Furthermore, one can use the Euler-Lagrange equation and it's boundary contribution to obtain

$$\delta_{V^\alpha}\mathcal{L} \xrightarrow{\mathrm{EL}=0} \mathrm{d}\iota_{V^\alpha}\Theta = \mathrm{d}K_\alpha,$$

thus one have the Noether's current

$$J_\alpha = \iota_{V^\alpha}\Theta - K_\alpha, \quad \mathrm{d}J_\alpha + EL[\phi]V_\alpha = 0, \tag{2.1}$$

which is a $n-1$ form on X and satisfies $\mathrm{d}J_\alpha\big|_{\mathrm{Crit}(S)} = 0$ while the Euler-Lagrangian equation is satisfied. If we consider $Y_1, Y_2 \subset X$ is codimension 1 (hyper)surface, which are homologous by Σ, then we have

$$\int_{Y_1} J_\alpha - \int_{Y_2} J_\alpha = \int_\Sigma \mathrm{d}J_\alpha = 0, \quad \phi \in \mathrm{Crit}(S),$$

and the integration over J_α is independent of the choice of the hyper surface, thus we can define the Noether charge as the integration over J_α on a hyper surface Y [1].

Their is a alernative way to define the Noether current, which is more suitable for practical use. In brief, on can consider the 'gauged' symmetry which would promote ϵ to become a field $\epsilon(x)$, and the variation of the action could be compute by integrating by parts, finally one can obtain

$$\delta_{V^\alpha}S = \int_X -\epsilon(x)\mathrm{d}\hat{J}_\alpha,$$

and $\hat{J}$ would become the Noether current which satisfies (2.1) so that $\hat{J}_\alpha$ is identical to J_α up to an exact form.

3 Day III: Symmetry (2)

First, we will consider finite dimensional case. We consider G as a finite dimensional Lie group, $\mathfrak{g}$ is the Lie algebra of G and W is finite dimensional representation of G.

3.1 Chevalley-Eilenberg Cohomology

Consider $\mathfrak{g}^* \equiv \mathrm{Hom}(\mathfrak{g}, \mathbb{K})$. Consider the exterior algebra

$$\bigwedge \mathfrak{g}^* = \bigoplus_{p=0}^{\infty} \bigwedge^p \mathfrak{g}^*.$$

Assume the basis of $\mathfrak{g}$ is $\{e_1, \cdots, e_n\}$ and of $\mathfrak{g}^*$ is $\{c^1, \cdots, c^n\}$, which satisfies $c_\alpha c_\beta = -c_\beta c_\alpha$. Thus one shell identify the algebra above as a free object in the category of differential graded algebra, which is a ring equipped with anti-commute generators

$$\bigwedge \mathfrak{g}^* = \mathbb{K}[c^1, \cdots, c^n].$$

Consider the Lie algebra over $\mathfrak{g}$, which equipped with commutator $[\cdot,\cdot] : \wedge^2\mathfrak{g} \to \mathfrak{g}$. One the dual side, one shell introduce a differential operator $\mathrm{d} : \mathfrak{g}^* \to \mathfrak{g}^*$, and we can extend it to the exterior algebra $\bigwedge \mathfrak{g}^*$ by

[1] In physics, one always consider the Noether current which is the Hodge dual of J_α.

(1) Under the level of generators, we have $\mathrm{d}_{\mathrm{CE}} : \mathfrak{g}^* \to \wedge^2\mathfrak{g}^*$;

(2) Using the Leibniz rule, we can extend it to the exterior algebra $\bigwedge \mathfrak{g}^*$ by

$$\mathrm{d}_{\mathrm{CE}} : a \wedge b \mapsto \mathrm{d}_{\mathrm{CE}} a \wedge b + (-1)^{\deg a} a \wedge \mathrm{d}_{\mathrm{CE}} b,$$

and thus we have a differential graded algebra $(\bigwedge \mathfrak{g}^*, \mathrm{d}_{\mathrm{CE}})$, which is called the Chevalley-Eilenberg complex.

Under the choice of basis above, we have $[e_\alpha, e_\beta] = f^\gamma_{\alpha\beta} e_\gamma$, which would lead to the derivation on the dual side

$$\mathrm{d}_{\mathrm{CE}} c^\alpha = \frac{1}{2} f^\alpha_{\beta\gamma} c^\beta \wedge c^\gamma \equiv \frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma.$$

Using the Leibniz rule, we can extend it to the exterior algebra $\bigwedge \mathfrak{g}^*$. Using the Jacobi identity, one can prove that $\mathrm{d}^2_{\mathrm{CE}} = 0$ (left as exercise), thus we have a cochain complex $(\bigwedge \mathfrak{g}^*, \mathrm{d}_{\mathrm{CE}})$ which is a differential graded algebra (dga), where the generator c^α is called the 'ghost field' in physics, the degree is 'ghost number' and d_{CE} is BRST operator.

Proof. Consider $\mathrm{d}^2_{\mathrm{CE}}$ acts on c^α, the higher structure could be derived from Leibniz's rule.

$$\begin{aligned}
\mathrm{d}^2_{\mathrm{CE}} c^\alpha =& \frac{1}{2} f^\alpha_{\beta\gamma} \left[\frac{1}{2} f^\beta_{\rho\lambda} c^\rho c^\lambda c^\gamma - \frac{1}{2} f^\gamma_{\rho\lambda} c^\beta c^\rho c^\lambda \right] \\
=& -\frac{1}{2} f^\alpha_{\gamma\beta} f^\beta_{\rho\lambda} c^\rho c^\lambda c^\lambda \\
=& \frac{1}{12} f^\alpha_{\beta[\gamma} f^\beta_{\rho\lambda]} c^\rho c^\lambda c^\lambda \\
=& 0
\end{aligned}$$

□

Let M be a $\mathfrak{g}$ representation where $\rho : \mathfrak{g} \to \mathrm{End}(W)$ satisfies

$$\rho(a)\rho(b)m - \rho(b)\rho(a)m = \rho([a,b])m, \quad a, b \in \mathfrak{g}, m \in M.$$

Consider the free $\bigwedge^\bullet \mathfrak{g}^*$-module generated by M:

$$\bigwedge\nolimits^\bullet \mathfrak{g}^* \otimes M,$$

there is a natural extension of the Chevalley-Eilenberg differential d_{CE} on it, which is defined by

(1) $\mathrm{d}_{\mathrm{CE}} : M \to g^* \otimes M$ is dual of $g \otimes M \xrightarrow{\rho} M$;

(2) $\mathrm{d}_{\mathrm{CE}}(a \otimes m) : \mathrm{d}_{\mathrm{CE}}(a) \otimes m + (-1)^{|a|} a \wedge \mathrm{d}_{\mathrm{CE}} m$

where we can prove that $\mathrm{d}^2_{\mathrm{CE}} = 0$ and thus we have a cochain complex $\bigwedge^\bullet \mathfrak{g}^* \otimes M$.

We denote $\wedge^p \mathfrak{g}^* \otimes M$ be $C^p(\mathfrak{g}^*, M)$, then we shell find that it is $C^p(\mathfrak{g}^*)$-module, i.e.

$$C^p(\mathfrak{g}^*) \otimes C^q(\mathfrak{g}^*, M) \ni a \otimes v \mapsto a \wedge v \in C^{p+q}(\mathfrak{g}^*, M),$$

which is compatible with derivation

$$\mathrm{d}_{\mathrm{CE}}(a \wedge v) = \mathrm{d}_{\mathrm{CE}} a \wedge v + (-1)^{|a|} a \wedge \mathrm{d}_{\mathrm{CE}} v,$$

where $m \in M$ and $a \in \wedge^\bullet \mathfrak{g}^*$. The derivation could be written explicitly with basis a_k of M and it's dual basis b^k:

$$\mathrm{d}_{\mathrm{CE}} = (\rho_\alpha)^k_i \, b^i a_k c^\alpha + \frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma a_\alpha,$$

which could be easily verified that $\mathrm{d}^2_{\mathrm{CE}} = 0$.

Proof. Consider the Chevalley-Eilenberg differential d_{CE} acts on $a \otimes m$, we have

$$\begin{aligned}\mathrm{d}_{\mathrm{CE}}(a \otimes m) =& \mathrm{d}_{\mathrm{CE}} a \otimes m + (-1)^{|a|} a \wedge \mathrm{d}_{\mathrm{CE}} m \\ =& \frac{1}{2} f^{\alpha}_{\beta\gamma} c^{\beta} c^{\gamma} a_{\alpha} m + (-1)^{|a|} a_{\beta} (\rho^i_j)_{\alpha} v^j m_i c^{\beta} c^{\alpha}, \\ \mathrm{d}^2_{\mathrm{CE}}(a \otimes m) =& \frac{1}{2} f^{\alpha}_{\beta\gamma} c^{\beta} c^{\gamma} a_{\alpha} \left(\rho^i_j\right)_{\rho} c^{\rho} + (-1)^{|a|}\end{aligned}$$

□

3.2 Differential Graded Lie Algebra

We define a $\mathbb{Z}$-graded vector space

$$W = \bigoplus_{n \in \mathbb{Z}} W_n,$$

where W_n is degree of n component.

1. **Degree Shift**: $W[n]_m \equiv W_{n+m}$;
2. **Dual**: W^* denote the linear dual of W

$$W^*_n = \mathrm{Hom}(W_{-n}, \mathbb{K});$$

3. **Symmetry and Anti-Symmetry**: $\mathrm{Sym}^{\otimes n}(V) = V^{\otimes n} / \sim$ where $a \otimes b \sim (-)^{|a||b|} b \otimes a$, and $\bigwedge^V = V^{\otimes n} / \sim$ where $a \otimes b \sim (-1)^{|a||b|+1} b \otimes a$;

which has a natural isomorphism between $\bigwedge^m (V[1])$ and $\mathrm{Sym}^m(V)[m]$

Proposition 3.1. *Let V be a dga, then:*

$$\bigwedge^m (V[1]) \cong \mathrm{Sym}^m(V)[m].$$

Proof. TBD. □

Definition 3.1 (Differential Graded Lie Algebra)**.** *A DGLA is a $\mathbb{Z}$-graded space*

$$g = \bigoplus_{m \in \mathbb{Z}} g_m$$

together with bilinear map $[\cdot, \cdot] : g \otimes g \to g$ satisfying

1. *(graded bracket) $[g_\alpha, g_\beta] \subset g_{\alpha+\beta}$, $[\cdot, \cdot] \in Hom(g \otimes g, g)$,*
2. *(graded skew-symmetry) $[a, b] = -(-1)^{|a||b|}[b, a] \quad \big([\cdot, \cdot] : \wedge^2 g \to g\big)$,*
3. *(graded Jacobi Identity) $[[a, b], c] = [a, [b, c]] - (-1)^{|a||b|}[b, [a, c]]$,*

with a degree 1 map $d : g \to g$ (i.e., $d : g_\alpha \to g_{\alpha+1}$) satisfying $\mathrm{d}^2 = 0$ and

4. *(graded Leibniz rule) $d[a, b] = [da, b] + (-1)^{|a|}[a, db]$.*

Example 3.1 (de-Rham + Lie = DGLA)**.** *Let X be a manifold, $\mathfrak{g}$ a Lie algebra.*

- *$(\Omega^\bullet(X), d)$ de Rham complex,*
- *$(\Omega^\bullet(X) \otimes \mathfrak{g}, d, [\cdot, \cdot])$ is DGLA,*
- *$\Omega^p(X) \otimes \mathfrak{g}$: degree p component,*

- *$d : \Omega^p \otimes \mathfrak{g} \to \Omega^{p+1} \otimes \mathfrak{g}$ de Rham, $d(\alpha \otimes h) = d\alpha \otimes h$,*
- *$[\cdot, \cdot]$ induced from $\mathfrak{g}$,*
- *Let $\alpha_{1,2} \in \Omega^\bullet(X)$, $h_{1,2} \in \mathfrak{g}$, then $[\alpha_1 \otimes h_1, \alpha_2 \otimes h_2] = \alpha_1 \wedge \alpha_2 \otimes [h_1, h_2]$,*

$\rightsquigarrow$ DGLA in Chern-Simons theory.

Example 3.2 (Dolbeault + Lie = DGLA)**.** *Let X be a complex manifold. Let*

- *$(\Omega^{0,*}(X), \bar\partial)$ Dolbeault Complex,*
- *$(\sum_{\bar i_1,\ldots,\bar i_p} \varphi_{\bar i_1 \ldots \bar i_p} d\bar z^{\bar i_1} \wedge \ldots \wedge d\bar z^{\bar i_p})$ where $\bar\partial = d\bar z^{\bar i} \frac{\partial}{\partial \bar z^i}$,*
- *$T_X \otimes_{\mathbb{C}} \mathbb{C} = T_X^{1,0} \oplus T_X^{0,1}$, where we shell choose the basis as*

$$Span\{\frac{\partial}{\partial z^i}\}, \quad Span\{\frac{\partial}{\partial \bar z^i}\},$$

which leads that $(\Omega^{0,*}(X, T_X^{1,0}), \bar\partial, [\cdot, \cdot])$ is a DGLA.

Explicitly, let $\{z^i\}$ be local holomorphic coordinates. $\alpha \in \Omega^{0,p}(X, T_X^{1,0})$ takes the form

$$\alpha = \sum_{i,\bar J} \alpha^i_{\bar J} d\bar z^{\bar J} \otimes \frac{\partial}{\partial z^i}, \quad d\bar z^{\bar J} = d\bar z^{\bar j_1} \wedge \ldots \wedge d\bar z^{\bar j_p},$$

$$\bar\partial \alpha = \sum_i \bar\partial \alpha^i_{\bar J} d\bar z^{\bar J} \otimes \frac{\partial}{\partial z^i} = \sum_i \frac{\partial \alpha^i_{\bar J}}{\partial \bar z^k} d\bar z^k \wedge d\bar z^{\bar J} \otimes \frac{\partial}{\partial z^i}.$$

Let $\alpha = \sum_i \alpha^i_{\bar J} d\bar z^{\bar J} \otimes \frac{\partial}{\partial z^i}$ and $\beta = \sum_m \beta^i_{\bar M} d\bar z^{\bar M} \otimes \frac{\partial}{\partial z^i}$. The Lie bracket is

$$[\alpha, \beta] = \sum_i \left(\alpha^j_{\bar J} \partial_j \beta^i_{\bar M} - \beta^j_{\bar M} \partial_j \alpha^i_{\bar J} \right) d\bar z^{\bar J} \wedge d\bar z^{\bar M} \otimes \frac{\partial}{\partial z^i}$$

On deg $= 0$ *component, this is the standard Lie bracket of $(1,0)$ vector fields. Finally, one can verify that $(\Omega^{0,*}(X, T_X^{1,0}), \bar\partial, [\cdot, \cdot])$ is a DGLA. $\rightsquigarrow$ Mathematics: Deformation of complex structures $\longleftrightarrow$ Physics: B-twisted topological string (Kodaira-Spencer gravity)*

We can consider the Chevalley-Eilenberg complex for a DGLA $(\mathfrak{g}, \mathrm{d}, [\ ,\ ])$.

Definition 3.2 (Chevalley-Eilenberg Complex)**.** *For a DGLA $(\mathfrak{g}, \mathrm{d}, [\ ,\ ])$, the Chevalley-Eilenberg complex is defined as*

$$\mathrm{C}^\bullet(\mathfrak{g}) = \mathrm{Sym}^\bullet(g^*[-1]) = \bigwedge^\bullet \mathbf{g}^*[-\bullet],$$

equipped with the CE differential $\mathrm{d}_{\mathrm{CE}} = \mathrm{d}_1 + \mathrm{d}_2$, where

(1) $\mathrm{d}_1 : \mathfrak{g}^*[-1] \to \mathfrak{g}^*[-1]$ *is the dual of* $\mathrm{d} : \mathfrak{g} \to \mathfrak{g}$;

(2) $\mathrm{d}_2 : \mathfrak{g}^*[-1] \to \mathrm{Sym}^2(\mathfrak{g}^*[-1]) \cong \bigwedge^2 \mathfrak{g}^*[-2]$ *is the dual of* $[\ ,\ ] : \bigwedge^2 \to \mathfrak{g}$;

(3) (Graded Leibniz rule) The derivation extends to

$$\mathrm{d}_{\mathrm{CE}} : \mathrm{Sym}(\mathfrak{g}^*[-1]) \to \mathrm{Sym}(\mathfrak{g}[-1])$$

via the graded Leibniz rule

$$\mathrm{d}_{\mathrm{CE}}(ab) = \mathrm{d}_{\mathrm{CE}} a\, b + (-1)^{|a|} a\, \mathrm{d}_{\mathrm{CE}} b,$$

and satisfies $\mathrm{d}_{\mathrm{CE}}^2 = 0$.

Remark 3.1. *If $\mathfrak{g}$ degenerated to the ordinary Lie algebra, which would be 'bosonic' fields. However, the basic object to build CE complex for ordinary Lie algebra is 'fermionic' fields. So we need to impose $[-1]$ into the definition of CE complex of DGLA.*

Definition 3.3 (DGLA-module)**.** *Let $\mathfrak{g}$ be a DGLA. A $\mathfrak{g}$-module is a cochain complex (M, d_M) with bilinear map*

$$\mathfrak{g} \otimes M \to M$$

where $C^\bullet(\mathfrak{g}, M) = \mathrm{Sym}\,(\mathfrak{g}^*[-1]) \otimes N$ satisfying

(1) $\mathfrak{g}_n \otimes M_p \to M_{n+p}$,

(2) $a \cdot (b \cdot m) - (-1)^{|a||b|} b \cdot (a \cdot m) = [a, b] \cdot m$,

(3) $d_M(a \cdot m) = (da) \cdot m + (-1)^{|a|} a \cdot d_M m$,

(4) $\mathrm{d}_M^2 = 0$.

Equivalently, define the graded space

$$\mathfrak{g}_M = \mathfrak{g} \oplus M$$

equipped with

- *differential $d \oplus d_M$*
- $[-,-] : \mathfrak{g}_M \otimes \mathfrak{g}_M \to \mathfrak{g}_M$, *where*

$$\begin{aligned} &[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \quad \textit{Lie bracket on } \mathfrak{g} \\ &[-,-] : \mathfrak{g} \otimes M \to M \quad \mathfrak{g} \textit{ action on } M \\ &[-,-] : M \otimes M \to M \quad \textit{zero} \end{aligned}$$

Then $\mathfrak{g}_M$ is DGLA.

3.3 Homotopic Lie Algebra (L_∞ Algebra)