

Quantum Field Theory

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Warning: Lots of possible typos!!!!!!!!!!!!!! **Notations:**

- X : a smooth manifold, usually a compact manifold.
- \mathcal{E} : the space of fields, usually infinite dimensional.
- \mathbf{k} : a field, usually \mathbb{R} or \mathbb{C} .
- $\text{Conn}(P, X)$: the space of connections on a principal bundle P over X .
- $\text{Maps}(\Sigma, X)$: the space of maps from a surface Σ to X .
- $\Omega^\bullet(X)$: the space of differential forms on X .
- $\Omega_c^\bullet(X)$: the space of differential forms with compact support on X .
- $\text{Vect}(M)$: the space of smooth vector fields on a manifold M , which is Lie algebra of $\text{Diff}(M)$.

1 Day I: Overall Discussion and Mathematical Preliminaries

1.1 Actions and Path Integrals

Action $S : \mathcal{E} \rightarrow \mathbf{k}$ where \mathcal{E} always has infinite dimension, and \mathbf{k} is a field (usually \mathbb{R} or \mathbb{C}).

QM in Imaginary Time $\xrightarrow{\text{Brownian Motion}}$ Wiener Measure on Phase Space

Asymptotic Analysis \longrightarrow Perturbative Renormalisation Theory

Example 1.1. *Some Examples of Classical Field Theories*

- (a) *Scalar Field Theory* $\mathcal{E} = C^\infty(X)$
- (b) *Gauge Theory* $\mathcal{E} = \text{Conn}(P, X)$
- (c) *σ Model* $\mathcal{E} = \text{Maps}(\Sigma, X)$
- (d) *Gravity* $\mathcal{E} = \text{Metrics}(X)$ (More better descriptions does not depends on the background)

1.2 Observables

Observables are functions on the space of fields, i.e. $\mathcal{O} \in C^\infty(\mathcal{E})$.

Example 1.2 (field theory). (a) Consider $X = pt$, thus $\mathcal{E} = \mathbb{R}^n$ for example.

(b) $\dim X > 0$, the new algebraic structure arise from topological structures of X .

The Key Point is: Capture the data of open sets of $X \rightarrow$ Consider the observables supported on open set U of X denoted by $\text{Obs}(U)$ where U is an open set of X .

Local data captures the open sets of X . The relations between open sets captures the global data of $X \rightarrow$ The algebraic structure of the observables is a sheaf of X .

$$\bigsqcup_i U_i \longrightarrow \bigotimes_i \text{Obs}(U_i)$$

Which implies OPE in physics and factorization algebra in mathematics.

Higher product in QFT: The generalization of products of algebra ('products in any direction instead of left and right') e.g. QM gives only left and right module of an algebra; OPE has products in various directions.

Consider the $\dim X = 2$ case in detailed

Example 1.3 (Holomorphic/Chiral Field Theory). Various angle of product $A(w)B(z)$ could be denoted by the time of $A(w)$ rotations around $B(z)$, which could be captured by the Fourier mode of $A(w)$, thus one can have

$$A(w)B(z) = \sum_{m \in \mathbb{Z}} \frac{(A_{(m)}B(z))}{(z-w)^{m+1}}$$

which is the Chiral algebra due to Beilinson and Drinfeld.

1.3 de Rham Cohomology

Chain of differential forms $\Omega^\bullet(X)$

$$\Omega^\bullet(X) = \left(\dots \xrightarrow{d} \Omega^{n-1}(X) \xrightarrow{d} \Omega^n(X) \xrightarrow{d} \Omega^{n+1}(X) \xrightarrow{d} \dots \right) \quad (1.1)$$

where d is the exterior derivative, and $\Omega^n(X)$ is the space of n -forms on X . The general construction of differential forms could be constructed over open set U by

$$\Omega^n(U) = \bigoplus_{1 \leq i_1 \leq \dots \leq i_n \leq n} C^\infty(U) dx^{i_1} \wedge \dots \wedge dx^{i_n}$$

where one can prove that $d^2 = 0$ and thus $(\Omega^\bullet(U), d)$ is a cochain complex. The cohomology of it is called the de Rham cohomology $H^\bullet(X)$.

Proof. TBD. □

Proposition 1.1. The definition of de Rham cohomology does not depend on the choice of the open set U and the choice of the coordinate system i.e. it is intrinsic \rightarrow we can define the de Rham cochain complex on smooth manifold X .

Proof. TBD. □

Definition 1.1 (de Rham Cohomology on Compact Support). Let X be a smooth manifold, then the de Rham cohomology on compact support is defined as

$$H_c^\bullet(X) = H^\bullet(\Omega_c^\bullet(X), d) \quad (1.2)$$

where $\Omega_c^\bullet(X)$ is the space of differential forms with compact support.

Theorem 1.2 (Stokes' Theorem). *Let X be a smooth manifold with boundary, then for any $\omega \in \Omega^n(X)$, we have*

$$\int_X d\omega = \int_{\partial X} \omega$$

which connects the local data $d\Omega^\bullet(X)$ and the global data ∂X .

Theorem 1.3 (Poincaré Lemma).

$$H^p(\mathbb{R}^n) = \begin{cases} \mathbb{R} & p = 0 \\ 0 & p > 0 \end{cases}, \quad H_c^p(\mathbb{R}^n) = \begin{cases} 0 & p < 0 \\ \mathbb{R} & p = n \end{cases}$$

Generator: $H^p(\mathbb{R}^n) \rightarrow$ constant function, $H_c^p(\mathbb{R}^n) \rightarrow$ a compact support function $\alpha = f(x)\text{vol}_n$, and $\int_{\mathbb{R}^n} \alpha = 1$.

Proof. □

Important: An *Integration* arises from the de Rham cohomology!

Observation. (1) if $\alpha = d\beta$ where $\beta \in \Omega_c^{n-1}(X)$, then $\int_X \alpha = 0$, thus the generator is α whose integral is non-zero.

(2) **Dual Site:** Integration could be captured by the cohomology

$$\int_{\mathbb{R}^n} \leftrightarrow H_c^n(\mathbb{R}^n) \cong \mathbb{R}$$

Path integral could be interpreted as the integration over \mathcal{E} , which leads to consider the cohomology of it. □

1.4 Cartan Formula

Vector fields could acts on smooth functions via

$$V(f) = V^i \frac{\partial f}{\partial x^i} = \left. \frac{d}{dt} f(\varphi_t(x)) \right|_{t=0} = \left. \frac{d}{dt} \varphi_t^* f(x) \right|_{t=0}$$

Such an action could be extended to differential forms by

$$\text{Vect}(M) \ni V : \alpha \mapsto \mathcal{L}_V \alpha = \left. \frac{d}{dt} \varphi_t^* \alpha \right|_{t=0}$$

which has the property $\mathcal{L}_V(\alpha \wedge \beta) = \mathcal{L}_V \alpha \wedge \beta + \alpha \wedge \mathcal{L}_V \beta$, which implies that the Lie derivative is a derivation on the algebra of differential forms with degree 0. And we have contraction ι_V which is a derivation of degree -1 on the algebra of differential forms.

$$\mathcal{L}_V = d\iota_V + \iota_V d$$

Lie derivative is homotopy trivial i.e. chain homotopic.

Rescaling invariance of \mathbb{R}^n leads to the Euler vector field $E = x^i \frac{\partial}{\partial x^i}$.