Algebraic Curve

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1 Day 0: Preliminary

2 Day I

Definition 2.1 (Polynomial). *The collection of polynomials would denoted by* $\mathbb{K}[x_1, \cdots, x_n]$, *whose elements are of the form*

$$f = \sum_{i_1, \cdots, i_n} a_{i_1, \cdots, i_n} x_1^{i_1} \cdots x_n^{i_n},$$

where $a_{i_1, \cdots, i_n} \in \mathbb{K}$, *and* $i_1, \cdots, i_n$ *are non-negative integers.*

Definition 2.2 (Algebraic Closed Field). *If*

Remark 2.1. *Finite field is not algebraic closed: Consider* $f = (x - a_1) \cdots (x - a_n) + 1$ *which has no zero point.*

Definition 2.3 (Unique Factorization Domain (UFD)).

Proposition 2.1. *(1)* $\mathbb{K}[x_1, \cdots, x_n]$ *is a commutative ring with unity called the polynomial ring in n variables over* $\mathbb{K}$.

(2) If R is UFD, then $R[X]$ *is a UFD, which means that every non-zero polynomial can be factored uniquely into irreducible polynomials, up to order and units.*

From here on, we assume that $\mathbb{K}$ is an algebraic closed field.

Definition 2.4 (Affine Variety). *An affine variety is a subset of* $\mathbb{K}^n$ *defined by the vanishing of a set of polynomials, i.e., it is the solution set of a system of polynomial equations.*

Formally, given a set of polynomials $f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_n]$, *the affine variety* $V(f_1, \ldots, f_m)$ *is defined as:*

$$V(f_1, \ldots, f_m) = \{(a_1, \ldots, a_n) \in \mathbb{K}^n; f_i(a_1, \ldots, a_n) = 0 \text{ for all } i = 1, \ldots, m\}.$$

Proposition 2.2 (Zariski Topology). *Consider* $f, g \in \mathbb{K}[x, y]$

(1) $V(fg) = V(f) \cup V(g)$,

(2) $V(f, g) = V(f) \cap V(g)$, $V(f_\lambda)_{\lambda \in \Lambda} = \bigcap_{\lambda \in \Lambda} V(f_\lambda)$,

(3) $V(0) = \mathbb{A}^2_{\mathbb{K}}$.

Definition 2.5 (Affine Curve). *Consider* $f \in \mathbb{K}[x, y]$, $V(f)$ *denotes affine curve.*

(1) $\deg V(f) = \deg f$,

- *(a)* $\deg = 1$: *Line,*
- *(b)* $\deg = 2$: *conic curve (non-degenerate),*

(2) $F = F_1^{n_1} F_2^{n_2} \cdots F_m^{n_m}$, *where* F_i *irreducible.*

Example 2.1. $(x+y)^2$ *is irreducible,* xy *is reducible.*

Example 2.2. $y^2 - x^3 + x$ *is irreducible (left as exercise).*

Definition 2.6 (Field of Fractions). *The field of fractions of a UFD* R *is the smallest field in which* R *can be embedded, denoted by* $K(R)$*. It consists of elements of the form* $\frac{a}{b}$ *where* $a, b \in R$ *and* $b \neq 0 \in R$*.*

Formally, if R *is a UFD, then the field of fractions* $K(R)$ *is defined as:*

$$Q_{\text{uot}}(R) = \left\{ \frac{a}{b} \mid a, b \in R, b \neq 0 \right\},$$

which is indeed a field.

Lemma 2.3. *Consider* $f \in \mathbb{K}[x,y]$ *and* $\deg f > 0$*, then*

(1) $V(f)$ *has infinitely many points,*

(2) $\mathbb{A}^2_{\mathbb{K}} - V(f)$ *has infinitely many points.*

Theorem 2.4 (Simple Bezout Theorem). *If* $F, G \in \mathbb{K}[x,y] \subset \mathbb{K}(x)[y]$ *has no common component, then* $V(F,G)$ *is a finite set* $\Leftrightarrow$ $F = 0$, $G = 0$ *have finite solutions in* $\mathbb{K}^2$*.*

Proof. (1) Assume there is an element α such that $F = \alpha F'$ and $G = \alpha G'$, where we consider the ring $\mathbb{K}(x)[y]$, then

$$\begin{cases} aF = HF' \\ bG = HG', \end{cases}$$

where $a \in \mathbb{K}[x]$ and $H \in \mathbb{K}[x,y]$.

(2) TBD

□

Theorem 2.5. *Consider irreducible* $F, G \in \mathbb{K}[x,y]$, $F|G \Leftrightarrow V(F) \subset V(G)$*.*

Proof. (1) If $F|G$, then $G = FH$ for some $H \in \mathbb{K}[x,y]$, thus $V(F) \subset V(G)$.

(2) If $V(F) \subset V(G)$, by definition $F|G$.

□

3 Day II: Intersection Number (1)

Definition 3.1 (Localized Ring). *Consider* $\mathbb{K}[x,y]$ *and a prime ideal* $P \subset R$*, the localized ring* $\mathcal{O}_P$ *is defined as:*

$$\mathcal{O}_P = \left\{ \frac{f}{g}; f, g \in \mathbb{K}[x,y], g(p) \neq 0 \right\},$$

the maximal ideal $\mathfrak{m}_P$ *is defined as:*

$$\mathfrak{m}_P = \left\{ \frac{f}{g}; f, g \in \mathbb{K}[x,y], g(p) \neq 0, f(p) = 0 \right\}.$$

which satisfies

$$0 \to \mathfrak{m}_P \to \mathcal{O}_P \to \mathbb{K}.$$

3.1 Definition

Definition 3.2 (Intersection Number). *Consider $F, G \in \mathbb{K}[x, y]$ irreducible, $P = V(F) \cap V(G)$, then the intersection number $I_P(F, G)$ is defined as:*

$$\mu_P(F, G) = \dim_{\mathbb{K}} \mathcal{O}_P / \langle F, G \rangle,$$

where $\langle F, G \rangle$ is the ideal generated by F and G in the localized ring $\mathcal{O}_P$.

Proposition 3.1. *(1) $\mu_P(F, G) \in \mathbb{N} \cup \{\infty\}$;*

(2) $P \in F \cap G \Leftrightarrow \mu_P(F, G) \geq 1$, $\mu_P(F, G) = 1 \Leftrightarrow \langle F, G \rangle = \mu_P$

(3) $\mu_P(F, G) = \mu_P(G, F)$;

(4) $\mu_P(F, G + FH) = \mu_P(F, G)$;

(5) $\mu_P(FG, H) = \mu_P(F, H) + \mu_P(G, H)$;

(6) $I_{(0,0)}(x, y) = 1$,

Example 3.1. *Consider $F = y - x^2$ and $G = y$.*

Solution. **Use properties to compute**:

$$\begin{aligned}\mu_0(y, y - x^3) =& \mu_0(y, -x^3) \\ =& 2\mu_0(y, x) \\ =& 2\end{aligned}$$

where we used the property (4) to reduce the degree of the polynomial for the given variable, and use the fact that $\mu_0(x, y) = 1$. □

The most important part is to use the property (4) to reduce the degree of the polynomial for the given variable.

Example 3.2. *Consider $F = y^2 - x^3$ and $G = x^2 - y^3$.*

Solution.

$$\begin{aligned}\mu_0(y^2 - x^3, x^2 - y^3) =& \mu_0(y^2 - x^3 + x(x^2 - y^3), y^3 - x^2) \\ =& \mu_0(y^2 - xy^3, y^3 - x^2) \\ =& \mu_0(y^2, y^3 - x^2) + \mu_0(1 - xy, y^3 - x^2) \\ =& 2\mu_0(y, y^3 - x^2) + 0 \\ =& 2\mu_0(y, x^2) \\ =& 4\mu_0(y, x) \\ =& 4\end{aligned}$$

$\mu_0(1 - xy, y^3 - x^2)$ vanished since at $(0, 0)$, $1 - xy \neq 0$ and $y^3 - x^2 = 0$. □

Example 3.3. *Consider $F = y - x - x^2$ and $G = y^2 - x^2 - 3x^2y$.*

Solution.

$$\begin{aligned}\mu_0(y - x - x^2, y^2 - x^2 - 3x^2y) =& \mu_0(y - x - x^2, y^2 - x^2 - 3x^2y - (x + y)(y - x - x^2)) \\ =& \mu_0(y - x - x^2, -2x^2y + x^3) \\ =& \mu_0(y - x - x^2, x^2(x - 2y)) \\ =& 2\mu_0(y - x - x^2, x) + \mu_0(y - x - x^2, x - 2y) \\ =& 3.\end{aligned}$$

Another way to compute is to use definition of intersection number, where we plug the equation $y = x + x^2$ into the second equation, we have

$$\mu_0(y - x - x^2, y^2 - x^2 - 3x^2 y) = \mathfrak{m}_0\left((x + x^2)^2 - x^2 - 3x^2(x + x^2)\right) = \mathfrak{m}_0\left(x^3(-1 - 2x)\right) = 0.$$

$\square$

Proposition 3.2. *If the lowest degree of F is x^n and the lowest degree of G is y^m, then the intersection number $I_{(0,0)}(F, G)$ is nm.*

Definition 3.3 (Short Exact Sequence)**.** *A short exact sequence of modules is a sequence of modules and homomorphisms*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

such that the image of f is equal to the kernel of g, i.e., $\mathrm{Im}(f) = \ker(g)$.

We would use the short exact sequence for linear space.

Definition 3.4. *Consider $P \in \mathbb{A}^2$ and $F, G, H \in \mathbb{K}[x, y]$, then*

(1) If F, G has no common component cross P, then

$$0 \to \mathcal{O}_P / \langle F, H \rangle \xrightarrow{\bullet G} \mathcal{O}_p / \langle F, GH \rangle \xrightarrow{\pi} \mathcal{O}_P / \langle F, G \rangle \to 0,$$

where π is the natural projection map.

(2) $\mu_P(F, GH) = \mu_P(F, G) + \mu_P(F, H)$.

Proof. (1) π is surjection;

(2) Consider an element acted by multiplication by G:

$$\bullet G : \frac{f}{g} + aF + bH \mapsto F(aG) + G\left(\frac{f}{g} + bH\right) \in \ker \pi,$$

where $a, b \in \mathbb{K}[x, y]$ and $g \in \mathcal{O}_P$. On the other side, consider $f/g \in \ker \pi$, thus $f/g = aF + bG \to b \in \mathcal{O}_p / \langle F, H \rangle$.

(3) $\bullet G$ is injection.

Note that all the vector spaces are finite dimensional, thus the dimension of the kernel is equal to the dimension of the image, and we can conclude that

$$\mu_P(F, GH) = \mu_P(F, G) + \mu_P(F, H),$$

which proofs the proposition (5). $\square$

3.2 The Algorithm to Compute Intersection Number

Consider $F(x, y) \in \mathbb{K}[x, y]$, in order to compute the insertion number $\mu_0(y, F(x, y))$, we can expand F as $F(x, y) = F(x, 0) + yH(x, y)$, thus

$$\mu_0(y, F(x, y)) = \mu_0(y, F(x, 0) + yH(x, y)) = \mu_0(y, F(x, 0)).$$

Assume $F(x, 0) = x^m f(x)$ where $f(x)$ is no vanishing at $x = 0$, thus

$$\mu_0(y, F(x, y)) = m.$$

Now we shell consider the linear (homogeneous 1 degree part). We denote $F \in \mathbb{K}[x, y]$ as

$$F = F_0 + F_1 + \cdots$$

where F_i is homogeneous degree i part. The F_1 part is important, because of the theorem below:

Theorem 3.3 (2.17 Intersection multiplicity 1). *If $F, G \in \mathbb{K}[x, y]$ pass through the origin, then*

$$\mu_0(F, G) = 1 \Leftrightarrow F, G \text{ Linear Independent}$$

Definition 3.5 (Tangents and multiplicities of points). *Let $F \in \mathbb{K}[x, y]$ be a curve, then*

(1) The smallest $m \in \mathbb{N}$ for which the homogeneous part F_m is non-zero is called the multiplicity $m_0(F)$ of F at the origin. Any linear factor of F_m (considered as a curve) is called a tangent to F at the origin.

(2) For a general point $P = (x_0, y_0) \in \mathbb{A}^2$, tangents at P and the multiplicity $m_P(F)$ are defined by first shifting coordinates to $x' = x - x_0$ and $y' = y - y_0$, and then applying (a) to the origin $(x', y') = (0, 0)$.

4 Day III: Intersection Number (2)

Definition 4.1 (Cusps). *Let P be a point on an affine curve F. We say that P is a cusp if $m_P(F) = 2$, there is exactly one tangent L to F at P, and $\mu_P(F, L) = 3$.*

Definition 4.2 (Singular Curve and Non-singular Curve). *An affine curve $F \in \mathbb{K}[x, y]$ is called singular if it has a point P such that $\mu_P(F) > 1$. If F has no point P such that $\mu_P(F) > 1$, then F is called non-singular.*

where the multiplicity $\mu_P(F)$ is defined as the number of tangents at P.

Proposition 4.1 (Affine Jacobi Criterion). *Let $P = (x_0, y_0)$ be a point on an affine curve F.*

(a) P is a singular point of F if and only if

$$\frac{\partial F}{\partial x}(P) = \frac{\partial F}{\partial y}(P) = 0,$$

(b) If P is a smooth point of F, the tangent to F at P is given by

$$T_P F = \frac{\partial F}{\partial x}(P) \cdot (x - x_0) + \frac{\partial F}{\partial y}(P) \cdot (y - y_0).$$

Example 4.1. *Consider the tangent $T_P F$ of the curve $F \in \mathbb{K}[x, y]$, compute the intersection number $\mu_P(F, T_P F)$.*

Solution. First, one can consider some basic examples. For example, consider $F = y - x^2$, thus the tangent at $P = (0, 0)$ is $T_P F = y$, so that the intersection number is

$$\mu_0(y, y - x^2) = 2.$$

Moreover, one can prove that $\mu_P(T_P F, F) = 2$ for $F = y - x^2$. □

Theorem 4.2. *Let P be a smooth point on a curve F . Then for any two curves G and H that do not have a common component with F through P we have*

$$\langle F, G \rangle \subset \langle F, H \rangle \text{ in } \mathscr{O}_P \quad \Leftrightarrow \quad \mu_P(F, G) \geq \mu_P(F, H).$$

5 Day IV: Projective Curve

Definition 5.1 (Projective Space). *For $n \in \mathbb{N}$, we define the projective space $\mathbb{P}^n(\mathbb{K})$ as the set of equivalence classes of non-zero vectors in $\mathbb{K}^{n+1}$, where two vectors $(x_0, x_1, \ldots, x_n)$ and $(y_0, y_1, \ldots, y_n)$ are equivalent if there exists a non-zero scalar $\lambda \in \mathbb{K}$ such that*

$$\sim: (x_0, x_1, \ldots, x_n) = \lambda(y_0, y_1, \ldots, y_n).$$

The projective space could thus be defined as:

$$\mathbb{P}^n = \left\{ \mathbb{A}^{n+1}_{\mathbb{K}} - \{0\} \right\} / \sim .$$

Example 5.1. *Consider the projective space* $\mathbb{C}P^2 = \mathbb{C}^3 - \{0\} / \sim$*, one would induce the fiberation:*

$$\mathbb{S}^1 \to \mathbb{S}^5 \xrightarrow{\pi} \mathbb{C}P^2.$$

Example 5.2. *Consider the curve* $F = y - x^2$*, in* $\mathbb{P}^2$ *we can introduce the homogeneous coordinate* $[x : y : z]$*, thus the curve can be written as:*

$$F = yz - x^2,$$

while $z = 0$ *(the point at infinity), we have* $[0 : 1 : 0]$*, which is the point at infinity of the curve* F*.*