Quantum Field Theory

Lectured by Prof. Si Li and Noted by Xinyu Xiang

Jun. 2025

Warning: Lots of possible typos!!!!!!!!!!!! **Notations**:

- X: a smooth manifold, usually a compact manifold.
- $\mathcal{E}$: the space of fields, usually infinite dimensional.
- $\mathrm{Conn}(P, X)$: the space of connections on a principal bundle P over X.
- $\mathrm{Maps}(\Sigma, X)$: the space of maps from Σ to X.
- $\Omega^\bullet(X)$: the space of differential forms on X.
- $\Omega_c^\bullet(X)$: the space of differential forms with compact support on X.
- $\mathrm{Vect}(M)$: the space of smooth vector fields on a manifold M, which is Lie algebra of $\mathrm{Diff}(M)$.

1 Day I: Overall Discussion and Mathematical Preliminaries

1.1 Actions and Path Integrals

Action $S : \mathcal{E} \to \mathbf{k}$ where $\mathcal{E}$ always has infinite dimension, and is a field (usually $\mathbb{R}$ or $\mathbb{C}$).

$$\text{QM in Imaginary Time} \xrightarrow{\text{Brownian Motion}} \text{Wiener Measure on Phase Space}$$

$$\text{Asymptotic Analysis} \longrightarrow \text{Perturbative Renormalisation Theory}$$

Example 1.1. *Some Examples of Classical Field Theories*

(a) Scalar Field Theory $\mathcal{E} = C^\infty(X)$

(b) Gauge Theory $\mathcal{E} = \mathrm{Conn}(P, X)$

(c) σ *Model* $\mathcal{E} = \mathit{Maps}(\Sigma, X)$

(d) Gravity $\mathcal{E} = \mathit{Metrics}(X)$ *(More better descriptions does not depends on the background)*

1.2 Observables

Observables are functions on the space of fields, i.e. $\mathcal{O} \in C^\infty(\mathcal{E})$.

Example 1.2 (field theory). *(a) Consider* $X = pt$*, thus* $\mathcal{E} = \mathbb{R}^n$ *for example.*

(b) $\dim X > 0$*, the new algebraic structure arise form topological structures of* X*.*

The Key Point is: Capture the data of open sets of $X \longrightarrow$ Consider the observables supported on open set U of X denoted by $\mathrm{Obs}(U)$ where U is an open set of X.

Local data captures the open sets of X. The relations between open sets captures the global data of $X \longrightarrow$ The algebraic structure of the observables is a sheaf of X.

$$\bigsqcup_i U_i \longrightarrow \bigotimes_i \mathrm{Obs}(U_i)$$

Which implies OPE in physics and factorization algebra in mathematics.

Higher product in QFT: The generalization of products of algebra ('products in any direction instead of left and right') e.g. QM gives only left and right module of an algebra; OPE has products in various directions.

Consider the $\dim X = 2$ case in detailed

Example 1.3 (Holomorphic/Chiral Field Theory)**.** *Various angle of product $A(w)B(z)$ could be denoted by the time of $A(w)$ rotations around $B(z)$, which could be captured by the Fourier mode of $A(w)$, thus one can have*

$$A(w)B(z) = \sum_{m\in\mathbb{Z}} \frac{(A_{(m)B(z)})}{(z-w)^{m+1}}$$

which is the Chiral algebra due to Beilinson and Drinfeld and associated with the Doubult cohomology $H^0_{\bar\partial}(\Sigma^2 - \Delta) \cong \mathbb{C}((z^m))$, where Σ^2 is the complex surface and Δ is the diagonal of Σ^2. The higher structure could be captured by the higher cohomology $H^p_{\bar\partial}(\Sigma^2 - \Delta)$, which is the higher chiral algebra associated to the derived holomorphic section.

1.3 de Rham Cohomology

Chain of differential forms $\Omega^\bullet(X)$

$$\Omega^\bullet(X) = \left(\cdots \xrightarrow{\mathrm{d}} \Omega^{n-1}(X) \xrightarrow{\mathrm{d}} \Omega^n(X) \xrightarrow{\mathrm{d}} \Omega^{n+1}(X) \xrightarrow{\mathrm{d}} \cdots\right) \tag{1.1}$$

where d is the exterior derivative, and $\Omega^n(X)$ is the space of n-forms on X. The general construction of differential forms could be constructed over open set U by

$$\Omega^n(U) = \bigoplus_{1\leq i_1 \leq \cdots \leq i_n \leq n} C^\infty(U)\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_n}$$

where one can prove that $\mathrm{d}^2 = 0$ and thus $(\Omega^\bullet(U), \mathrm{d})$ is a cochain complex. The cohomology of it is called the de Rham cohomology $H^\bullet(X)$.

Proposition 1.1. *The definition of de Rham cohomology does not depend on the choice of the open set U and the choice of the coordinate system i.e. it is intrinsic $\longrightarrow$ we can define the de Rham cochain complex on smooth manifold X.*

Proof. Consider □

Definition 1.1 (de Rham Cohomology on Compact Support)**.** *Let X be a smooth manifold, then the de Rham cohomology on compact support is defined as*

$$H^\bullet_c(X) = H^\bullet(\Omega^\bullet_c(X), \mathrm{d}) \tag{1.2}$$

where $\Omega^\bullet_c(X)$ is the space of differential forms with compact support.

Theorem 1.2 (Stokes' Theorem)**.** *Let X be a smooth manifold with boundary, then for any $\omega \in \Omega^n(X)$, we have*

$$\int_X \mathrm{d}\omega = \int_{\partial X} \omega$$

which connects the local data $\mathrm{d}\Omega^\bullet(X)$ and the global data ∂X.

Theorem 1.3 (Poincaré Lemma)**.**

$$H^p(\mathbb{R}^n) = \begin{cases} \mathbb{R} & p = 0 \\ 0 & p > 0 \end{cases}, \quad H_c^p(\mathbb{R}^n) = \begin{cases} 0 & p < 0 \\ \mathbb{R} & p = n \end{cases}$$

Generator: $H^p(\mathbb{R}^n) \to$ *constant function,* $H_c^p(\mathbb{R}^n) \to$ *a compact support function* $\alpha = f(x)\mathrm{vol}_n$*, and* $\int_{\mathbb{R}^n} \alpha = 1$.

Proof. □

Important: An *Integration* arises from the de Rham cohomology!

Observation. (1) if $\alpha = \mathrm{d}\beta$ where $\beta \in \Omega_c^{n-1}(X)$, then $\int_X \alpha = 0$, thus the generator is α whose integral is non-zero.

(2) **Dual Site**: Integration could be captured by the cohomology

$$\int_{\mathbb{R}^n} \leftrightarrow H_c^n(\mathbb{R}^n) \cong \mathbb{R}$$

Path integral could be interpreted as the integration over $\mathcal{E}$, which leads to consider the cohomology of it.

□

1.4 Cartan Formula

Vector fields could acts on smooth functions via

$$V(f) = V^i \frac{\partial f}{\partial x^i} = \frac{\mathrm{d}}{\mathrm{d}t} f(\varphi_t(x))\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \varphi_t^* f(x)\bigg|_{t=0}$$

Such an action could be extended to differential forms by

$$\mathrm{Vect}(M) \ni V : \alpha \mapsto \mathcal{L}_V \alpha = \frac{\mathrm{d}}{\mathrm{d}t} \varphi_t^* \alpha\bigg|_{t=0}$$

which has the property $\mathcal{L}_V(\alpha \wedge \beta) = \mathcal{L}_V \alpha \wedge \beta + \alpha \wedge \mathcal{L}_V \beta$, which implies that the Lie derivative is a derivation on the algebra of differential forms with degree 0. And we have contraction ι_V which is a derivation of degree -1 on the algebra of differential forms.

$$\mathcal{L}_V = \mathrm{d}\iota_V + \iota_V \mathrm{d}$$

Lie derivative is homotopy trivial i.e. chain homotopic.

1.4.1 Proof of Poincaré Lemma

Use Cartan Formula, one can proof Poincaré Lemma.

Proof. Rescaling invariance of $\mathbb{R}^n$ leads to the Euler vector field $E = x^i \frac{\partial}{\partial x^i}$. One can consider the associated diffeomorphism φ_t, where we assume $\varphi_0 = 1$ and thus $\varphi_{-\infty}^* \alpha = 0$, thus the closed form α could be rewritten as

$$\begin{aligned} \alpha &= \varphi_0^* \alpha - \varphi_{-\infty}^* \alpha \\ &= \int_{-\infty}^0 \frac{\mathrm{d}}{\mathrm{d}t} \varphi_t^* \alpha \mathrm{d}t \\ &= \int_{-\infty}^0 \mathcal{L}_E(\varphi_t^* \alpha) \mathrm{d}t \end{aligned}$$

using the Cartan formula and $\mathrm{d}\varphi^* = \varphi^*\mathrm{d}$, we have

$$\alpha = \mathrm{d}\int_{-\infty}^{0} \varphi_t^* \iota_E \alpha \,\mathrm{d}t = \mathrm{d}\beta,$$

thus, the closed form α is exact, which implies that the de Rham cohomology $H^p(\mathbb{R}^n)$ is trivial for $p > 0$. The same idea could be applied to the de Rham cohomology on compact support $H_c^p(\mathbb{R}^n)$. □

2 Day II: Classical Field Theory

Assume $\mathcal{E} = \Gamma(E, X)$ i.e. a section of a bundle $E \to X$, where X is oriented manifold. And the action would be written as $S[\phi] = \int_X \mathcal{L}[\phi(x)]$ where $\phi \in \mathcal{E}$. Lagrangian $\mathcal{L}$ satisfies:

(a) built up by jets of ϕ (locality);

(b) valued in n form on X (oriented).

The solution of Euler-Lagrange equation forms $\mathrm{Crit}(S)$, which denotes the critical of the action S.

2.1 Examples

Example 2.1 (Phase Space Quantum Mechanics). *Consider $X = \mathbb{R}$, then $\mathcal{E} = \mathbb{R}^{2n}$, and the action is*

$$S[\phi] = \int_{\mathbb{R}^{2n}} p\mathrm{d}q - H(q,p)\mathrm{d}t = \int [p\dot{q} - H]\,\mathrm{d}t$$

where H is the Hamiltonian. The Euler-Lagrange equation would become $\mathrm{d}H = -\iota_{x_*\partial}\omega$, where $x : \mathbb{R} \to \mathcal{E}$.

Example 2.2 (Scalar Field Theory). *Consider (X, g) a Riemann Manifold, then $\mathcal{E} = C^\infty(X)$, and the action is*

$$S[\phi] = \int_X \left[\frac{1}{2}|\nabla\phi|^2 + V(\phi)\right] \mathrm{dvol}_g$$

where $V(\phi)$ is a potential function, and $\mathrm{dvol}_g = \sqrt{|g|}\mathrm{d}^d x$. Assume $\partial X =$, then the Euler-Lagrange equation is

$$\Delta\phi = \frac{\partial V}{\partial \phi}$$

where $\Delta f = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j f\right)$.

Example 2.3 (Chern-Simons Theory). *Consider X a 3-manifold and $\mathfrak{g}$ a semi-simple Lie algebra. Denote P is a principal $\mathfrak{g}$-bundle over X, then the space of fields is $\mathcal{E} = \mathrm{Conn}(P, X)$. Assume $\mathfrak{g}$ is equipped with a non-degenerate invariant bilinear form $\langle\cdot,\cdot\rangle$ (Killing form), then the action is*

$$\mathrm{CS}[A] = \int_X \frac{1}{2}\langle A, F_A\rangle + \frac{1}{6}\langle A, [A, A]\rangle,$$

and the Euler-Lagrange equation encoded by the flat connection $F_A = 0$.

2.2 Symmetry (1)

2.2.1 Global Symmetry and Noether's Theorem

Consider a classical action $S : \mathcal{E} \to \mathbb{R}$ with a group action $G \curvearrowright \mathcal{E}$ s.t. $S[g(\phi)] = S[\phi]$. Then G would become a global symmetry of the action S.

Consider the continuous symmetry i.e. G is a Lie group, then the infinitesimal action of G on $\mathcal{E}$ is given by a vector field $V \in \mathrm{Vect}(\mathcal{E})$, which satisfies

$$\delta_{V^\alpha}\phi = V^\alpha(\phi),$$

thus the variation of the Lagrangian is

$$\delta_{V^\alpha}\mathcal{L} = \mathrm{d}K_\alpha,$$

where K_α is a $n-1$ form. Furthermore, one can use the Euler-Lagrange equation and it's boundary contribution to obtain

$$\delta_{V^\alpha}\mathcal{L} \xrightarrow{\mathrm{EL}=0} \mathrm{d}\iota_{V^\alpha}\Theta = \mathrm{d}K_\alpha,$$

thus one have the Noether's current

$$J_\alpha = \iota_{V^\alpha}\Theta - K_\alpha, \quad \mathrm{d}J_\alpha + EL[\phi]V_\alpha = 0, \tag{2.1}$$

which is a $n-1$ form on X and satisfies $\mathrm{d}J_\alpha\big|_{\mathrm{Crit}(S)} = 0$ while the Euler-Lagrangian equation is satisfied. If we consider $Y_1, Y_2 \subset X$ is codimension 1 (hyper)surface, which are homologous by Σ, then we have

$$\int_{Y_1} J_\alpha - \int_{Y_2} J_\alpha = \int_\Sigma \mathrm{d}J_\alpha = 0, \quad \phi \in \mathrm{Crit}(S),$$

and the integration over J_α is independent of the choice of the hyper surface, thus we can define the Noether charge as the integration over J_α on a hyper surface Y [1].

Their is a alernative way to define the Noether current, which is more suitable for practical use. In brief, on can consider the 'gauged' symmetry which would promote ϵ to become a field $\epsilon(x)$, and the variation of the action could be compute by integrating by parts, finally one can obtain

$$\delta_{V^\alpha}S = \int_X -\epsilon(x)\mathrm{d}\hat{J}_\alpha,$$

and $\hat{J}$ would become the Noether current which satisfies (2.1) so that $\hat{J}_\alpha$ is identical to J_α up to an exact form.

3 Day III: Symmetry (2)

First, we will consider finite dimensional case. We consider G as a finite dimensional Lie group, $\mathfrak{g}$ is the Lie algebra of G and W is finite dimensional representation of G.

3.1 Chevalley-Eilenberg Cohomology

Consider $\mathfrak{g}^* \equiv \mathrm{Hom}(\mathfrak{g}, \mathbb{K})$. Consider the exterior algebra

$$\bigwedge \mathfrak{g}^* = \bigoplus_{p=0}^{\infty} \bigwedge^p \mathfrak{g}^*.$$

Assume the basis of $\mathfrak{g}$ is $\{e_1, \cdots, e_n\}$ and of $\mathfrak{g}^*$ is $\{c^1, \cdots, c^n\}$, which satisfies $c_\alpha c_\beta = -c_\beta c_\alpha$. Thus one shell identify the algebra above as a free object in the category of differential graded algebra, which is a ring equipped with anti-commute generators

$$\bigwedge \mathfrak{g}^* = \mathbb{K}[c^1, \cdots, c^n].$$

Consider the Lie algebra over $\mathfrak{g}$, which equipped with commutator $[\cdot,\cdot] : \wedge^2\mathfrak{g} \to \mathfrak{g}$. One the dual side, one shell introduce a differential operator $\mathrm{d} : \mathfrak{g}^* \to \mathfrak{g}^*$, and we can extend it to the exterior algebra $\bigwedge \mathfrak{g}^*$ by

[1] In physics, one always consider the Noether current which is the Hodge dual of J_α.

(1) Under the level of generators, we have $\mathrm{d}_{\mathrm{CE}} : \mathfrak{g}^* \to \wedge^2 \mathfrak{g}^*$;

(2) Using the Leibniz rule, we can extend it to the exterior algebra $\bigwedge \mathfrak{g}^*$ by

$$\mathrm{d}_{\mathrm{CE}} : a \wedge b \mapsto \mathrm{d}_{\mathrm{CE}} a \wedge b + (-1)^{\deg a} a \wedge \mathrm{d}_{\mathrm{CE}} b,$$

and thus we have a differential graded algebra $(\bigwedge \mathfrak{g}^*, \mathrm{d}_{\mathrm{CE}})$, which is called the Chevalley-Eilenberg complex.

Under the choice of basis above, we have $[e_\alpha, e_\beta] = f_{\alpha\beta}^{\gamma} e_\gamma$, which would lead to the derivation on the dual side

$$\mathrm{d}_{\mathrm{CE}} c^\alpha = \frac{1}{2} f^{\alpha}_{\beta\gamma} c^\beta \wedge c^\gamma \equiv \frac{1}{2} f^{\alpha}_{\beta\gamma} c^\beta c^\gamma.$$

Using the Leibniz rule, we can extend it to the exterior algebra $\bigwedge \mathfrak{g}^*$. Using the Jacobi identity, one can prove that $\mathrm{d}_{\mathrm{CE}}^2 = 0$, thus we have a cochain complex $(\bigwedge \mathfrak{g}^*, \mathrm{d}_{\mathrm{CE}})$ which is a differential graded algebra (dga), where the generator c^α is called the 'ghost field' in physics, the degree is 'ghost number' and d_{CE} is BRST operator.

Let M be a $\mathfrak{g}$ representation where $\rho : \mathfrak{g} \to \mathrm{End}(W)$ satisfies

$$\rho(a)\rho(b)m - \rho(b)\rho(a)m = \rho([a, b])m, \quad a, b \in \mathfrak{g}, m \in M.$$

Consider the free $\bigwedge^\bullet \mathfrak{g}^*$-module generated by M:

$$\bigwedge\nolimits^\bullet \mathfrak{g}^* \otimes M,$$

there is a natural extension of the Chevalley-Eilenberg differential d_{CE} on it, which is defined by

(1) $\mathrm{d}_{\mathrm{CE}} : M \to g^* \otimes M$ is dual of $g \otimes M \xrightarrow{\rho} M$;

(2) $\mathrm{d}_{\mathrm{CE}}(a \otimes m) : \mathrm{d}_{\mathrm{CE}}(a) \otimes m + (-1)^{|a|} a \wedge \mathrm{d}_{\mathrm{CE}} m$

where we can prove that $\mathrm{d}_{\mathrm{CE}}^2 = 0$ and thus we have a cochain complex $\bigwedge^\bullet \mathfrak{g}^* \otimes M$.

We denote $\wedge^p \mathfrak{g}^* \otimes M$ be $C^p(\mathfrak{g}^*, M)$, then we shell find that it is $C^p(\mathfrak{g}^*)$-module, i.e.

$$C^p(\mathfrak{g}^*) \otimes C^q(\mathfrak{g}^*, M) \ni a \otimes v \mapsto a \wedge v \in C^{p+q}(\mathfrak{g}^*, M),$$

which is compatible with derivation

$$\mathrm{d}_{\mathrm{CE}}(a \wedge v) = \mathrm{d}_{\mathrm{CE}} a \wedge v + (-1)^{|a|} a \wedge \mathrm{d}_{\mathrm{CE}} v,$$

where $m \in M$ and $a \in \wedge^\bullet \mathfrak{g}^*$.