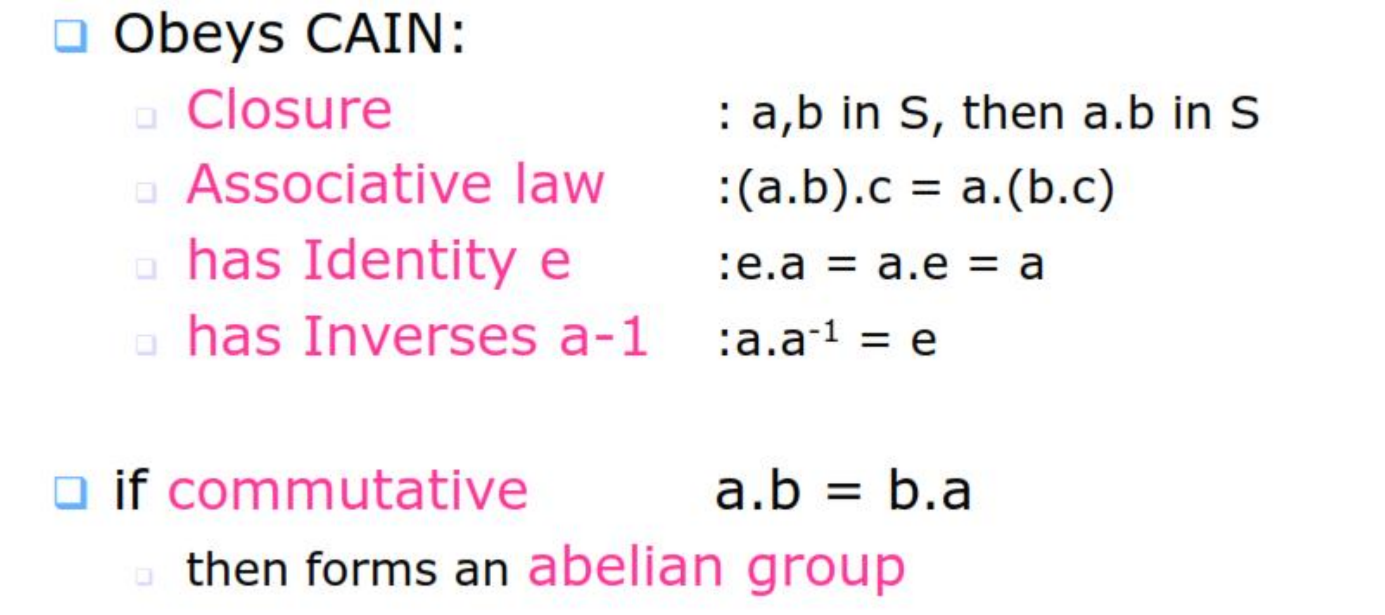
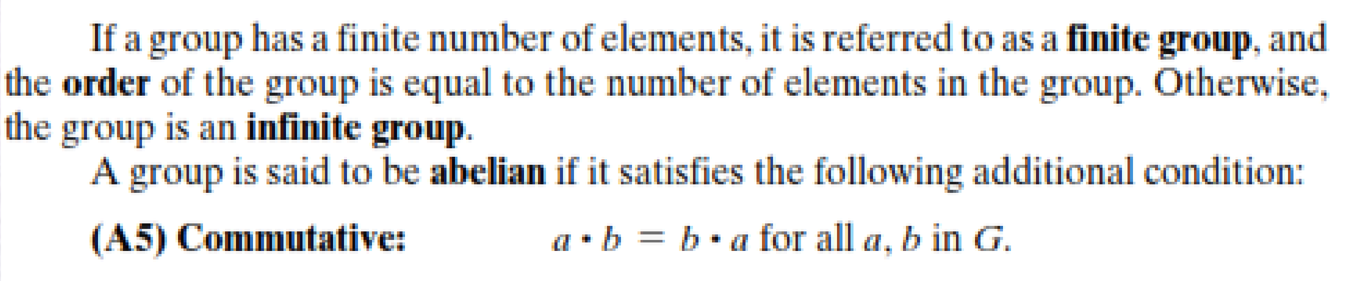
Groups

- A set of elements with a binary operation denoted by • that associates to each ordered pair (a,b) of elements in G an element $(a \bullet b)$ in G , such that the following axioms are obeyed:
  - (A1) Closure:
    - If a and b belong to G, then $a \bullet b$ is also in G
  - (A2) Associative:
    - $a \bullet (b \bullet c) = (a \bullet b) \bullet c$ for all a, b, c in G
  - (A3) Identity element:
    - There is an element e in G such that $a \bullet e = e \bullet a = a$ for all a in G
  - (A4) Inverse element:
    - For each a in G, there is an element a in G such that $a \bullet a = a \bullet a = e$
  - (A5) Commutative:
    - $a \bullet b = b \bullet a$ for all a, b in G



- Obeys CAIN:
  - Closure : a,b in S, then a.b in S
  - Associative law :(a.b).c = a.(b.c)
  - has Identity e :e.a = a.e = a
  - has Inverses a-1 :$a.a^{-1} = e$
- if commutative a.b = b.a
  - then forms an abelian group



If a group has a finite number of elements, it is referred to as a **finite group**, and the **order** of the group is equal to the number of elements in the group. Otherwise, the group is an **infinite group**.

A group is said to be **abelian** if it satisfies the following additional condition:

(A5) Commutative: $a \cdot b = b \cdot a$ for all a, b in G.

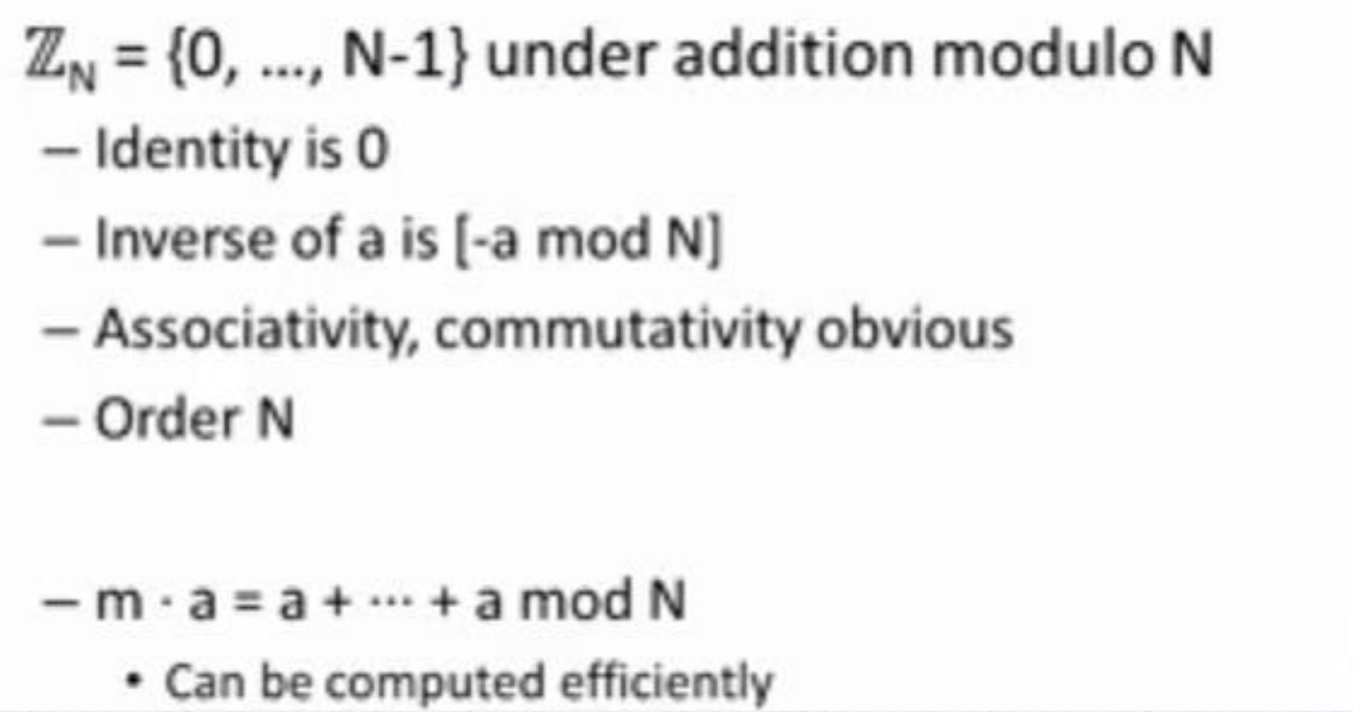
Example

- The set of integers (positive, negative, and 0) under addition is an abelian group.
- The set of nonzero real numbers under multiplication is an abelian group.

Cyclic Group

- Exponentiation is defined within a group as a repeated application of the group operator, so that $a^3 = a \bullet a \bullet a$
- We define $a^0 = e$ as the identity element, and $a^{-n} = (a')^n$, where a' is the inverse element of a within the group
- A group G is **cyclic** if every element of G is a power a^k (k is an integer) of a fixed element
- The element a is said to **generate** the group G or to be a **generator** of G
- A cyclic group is always abelian and may be finite or infinite

Example



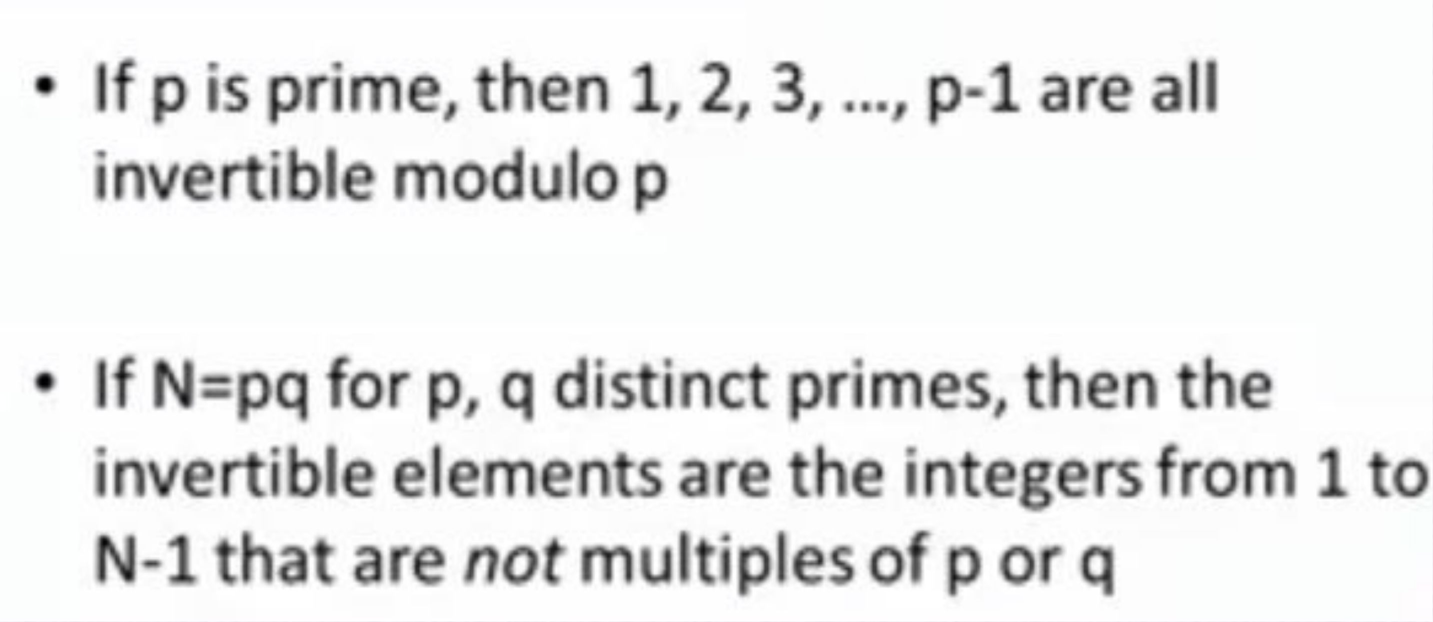
$\mathbb{Z}_N$ = {0, …, N-1} under addition modulo N

- Identity is 0
- Inverse of a is [-a mod N]
- Associativity, commutativity obvious
- Order N
- m · a = a + ⋯ + a mod N
  - Can be computed efficiently

Contd…

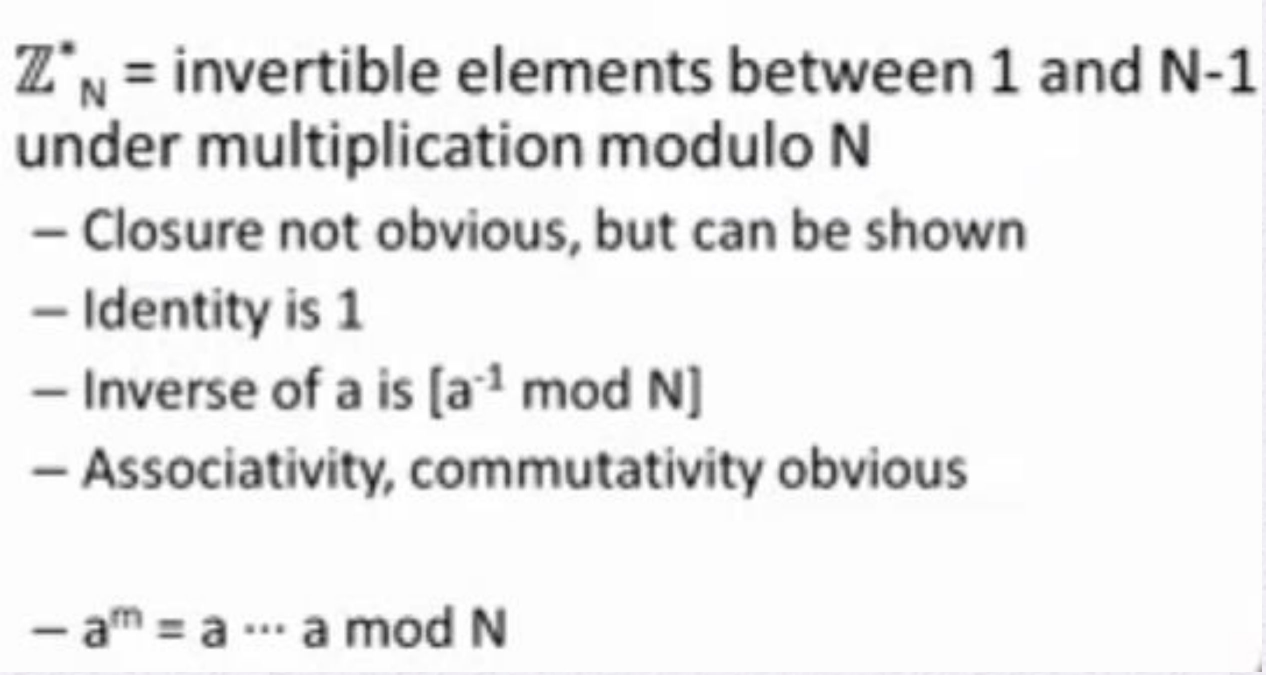
- Modular Inverses uses gcd, inverse of b mod N
- Gcd(b,N) =1.

Contd…



- If p is prime, then 1, 2, 3, …, p-1 are all invertible modulo p

- If N=pq for p, q distinct primes, then the invertible elements are the integers from 1 to N-1 that are *not* multiples of p or q

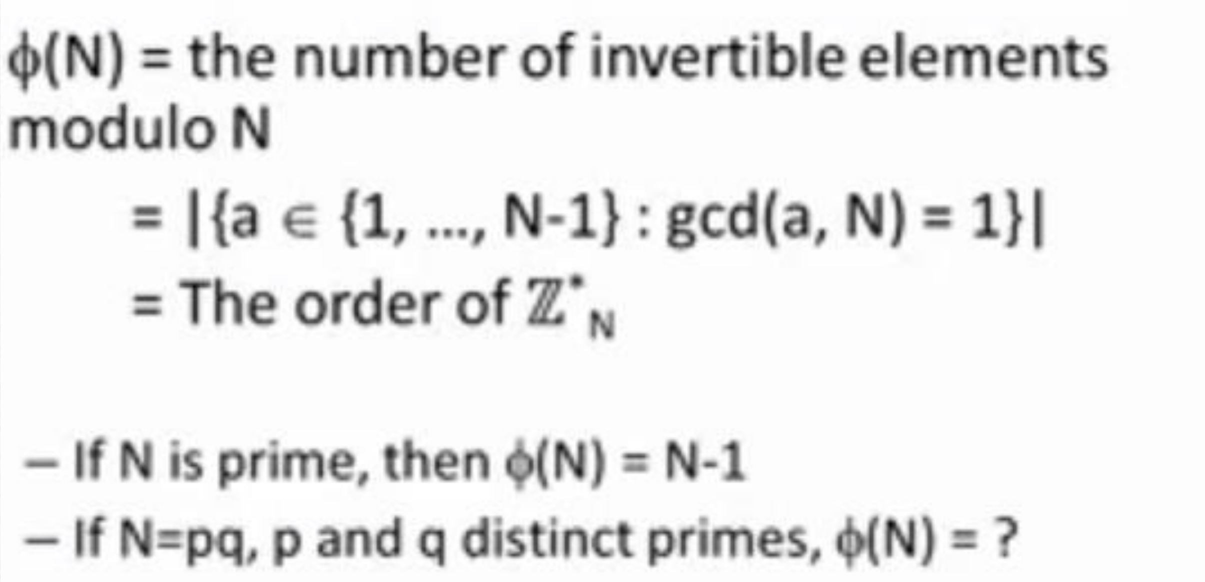


$\mathbb{Z}^{*}_{N}$ = invertible elements between 1 and N-1 under multiplication modulo N

- Closure not obvious, but can be shown
- Identity is 1
- Inverse of a is [a^{-1} mod N]
- Associativity, commutativity obvious

- a^m = a ⋯ a mod N

Contd…



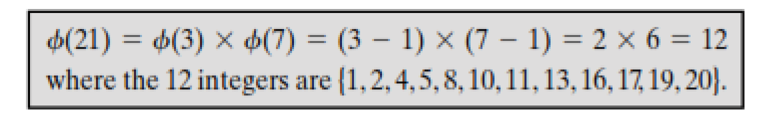
ϕ(N) = the number of invertible elements modulo N

= |{a ∈ {1, …, N-1} : gcd(a, N) = 1}|

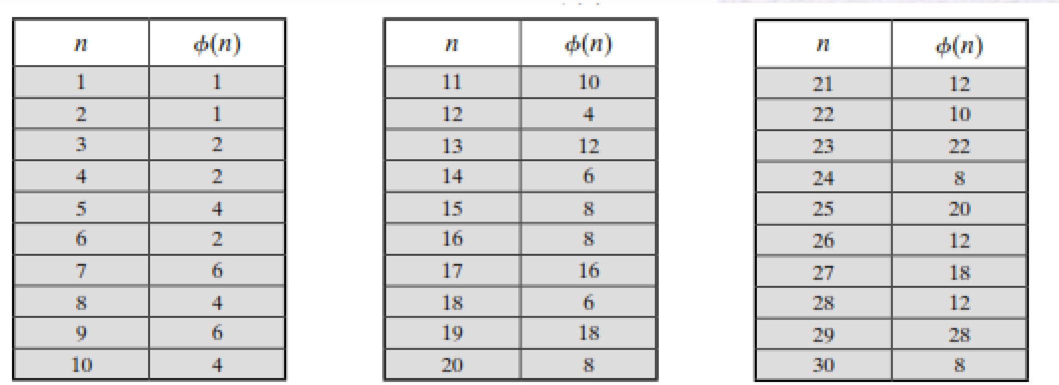
= The order of $\mathbb{Z}^*_N$

– If N is prime, then ϕ(N) = N-1

– If N=pq, p and q distinct primes, ϕ(N) = ?



$\phi(21) = \phi(3) \times \phi(7) = (3 - 1) \times (7 - 1) = 2 \times 6 = 12$
where the 12 integers are {1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20}.



n	$\phi(n)$
1	1
2	1
3	2
4	2
5	4
6	2
7	6
8	4
9	6
10	4
11	10
12	4
13	12
14	6
15	8
16	8
17	16
18	6
19	18
20	8
21	12
22	10
23	22
24	8
25	20
26	12
27	18
28	12
29	28
30	8

Rings

- A **ring** *R* , sometimes denoted by {R , + , * }, is a set of elements with two binary operations, called *addition* and *multiplication*, such that for all *a* , *b* , *c* in *R* the following axioms are obeyed:

(A1–A5)

R is an abelian group with respect to addition; that is, *R* satisfies axioms A1 through A5. For the case of an additive group, we denote the identity element as 0 and the inverse of *a* as –*a*

(M1) Closure under multiplication:

If *a* and *b* belong to *R* , then *ab* is also in *R*

(M2) Associativity of multiplication:

a (*bc*) = (*ab*)*c* for all *a* , *b* , *c* in R

(M3) Distributive laws:

a (*b* + *c*) = *ab* + *ac* for all *a* , *b* , *c* in *R*

(*a* + *b*)*c* = *ac* + *bc* for all *a* , *b* , *c* in *R*

- In essence, a ring is a set in which we can do addition, subtraction [*a* - *b* = *a* + (-*b*)], and multiplication without leaving the set

Rings (cont.)

- A ring is said to be commutative if it satisfies the following additional condition:

  (M4) Commutativity of multiplication:

  ab = ba for all a, b in R

- An ***integral domain*** is a commutative ring that obeys the following axioms.

  (M5) Multiplicative identity:

  There is an element 1 in R such that $a\,1 = 1a = a$ for all a in R

  (M6) No zero divisors:

  If a, b in R and $ab = 0$, then either $a = 0$ or $b = 0$

Fields

- A **field** *F* , sometimes denoted by {F, +,* }, is a set of elements with two binary operations, called *addition* and *multiplication*, such that for all *a, b, c* in *F* the following axioms are obeyed:

  (A1–M6)

  F is an integral domain; that is, *F* satisfies axioms A1 through A5 and M1 through M6

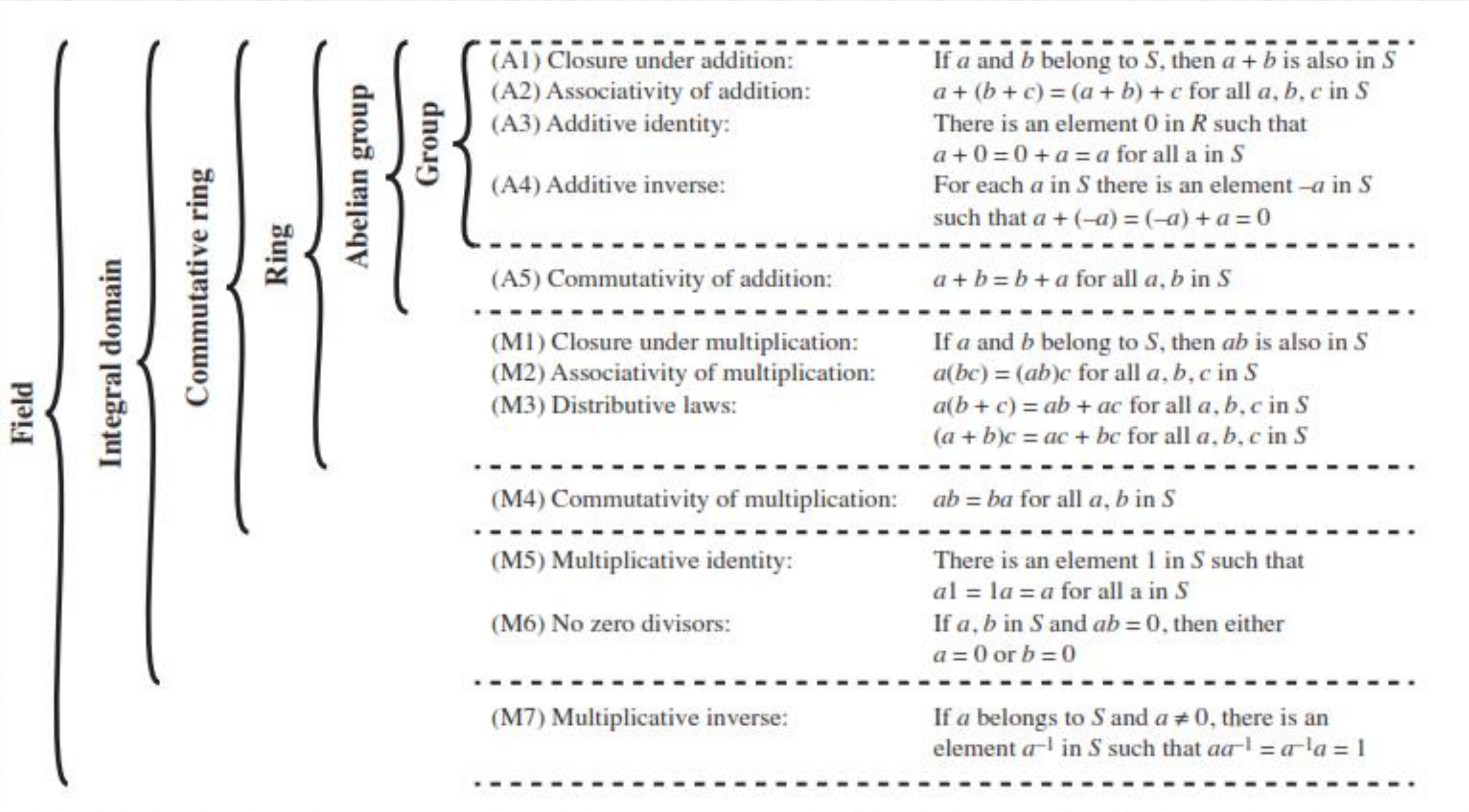
  (M7) Multiplicative inverse:

  For each *a* in *F*, except 0, there is an element a^{-1} in *F* such that $aa^{-1} = (a^{-1})a = 1$

- In essence, a field is a set in which we can do addition, subtraction, multiplication, and division without leaving the set. Division is defined with the following rule: $a/b = a(b^{-1})$

Familiar examples of fields are the rational numbers, the real numbers, and the complex numbers. Note that the set of all integers is not a field, because not every element of the set has a multiplicative inverse.

Contd…



Field

Integral domain

Commutative ring

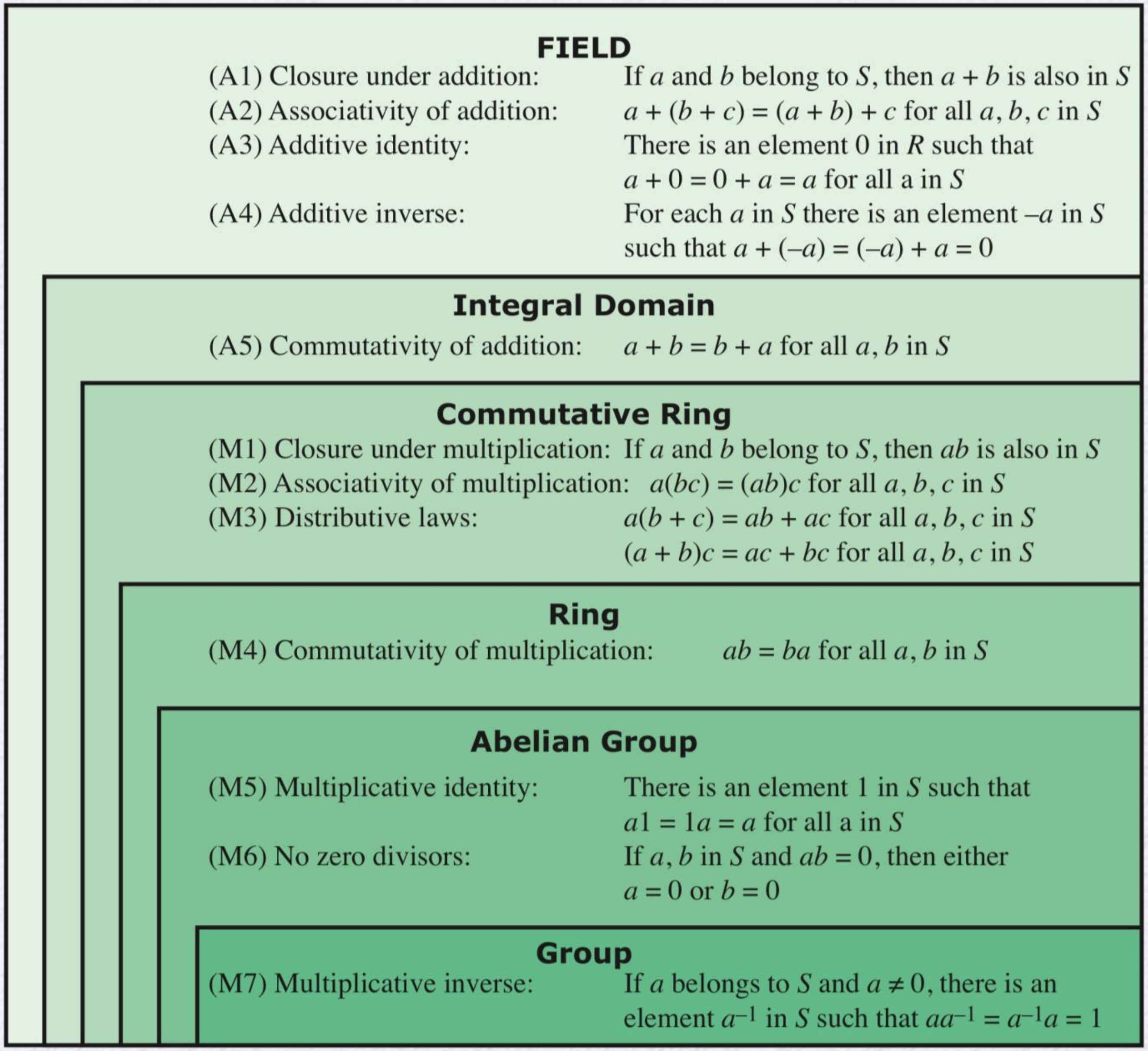
Ring

Abelian group

Group

(A1) Closure under addition:	If a and b belong to S, then $a + b$ is also in S
(A2) Associativity of addition:	$a + (b + c) = (a + b) + c$ for all a, b, c in S
(A3) Additive identity:	There is an element 0 in R such that $a + 0 = 0 + a = a$ for all a in S
(A4) Additive inverse:	For each a in S there is an element $-a$ in S such that $a + (-a) = (-a) + a = 0$
(A5) Commutativity of addition:	$a + b = b + a$ for all a, b in S
(M1) Closure under multiplication:	If a and b belong to S, then ab is also in S
(M2) Associativity of multiplication:	$a(bc) = (ab)c$ for all a, b, c in S
(M3) Distributive laws:	$a(b + c) = ab + ac$ for all a, b, c in S $(a + b)c = ac + bc$ for all a, b, c in S
(M4) Commutativity of multiplication:	$ab = ba$ for all a, b in S
(M5) Multiplicative identity:	There is an element 1 in S such that $a1 = 1a = a$ for all a in S
(M6) No zero divisors:	If a, b in S and $ab = 0$, then either $a = 0$ or $b = 0$
(M7) Multiplicative inverse:	If a belongs to S and $a \neq 0$, there is an element a^{-1} in S such that $aa^{-1} = a^{-1}a = 1$

Group, Ring, and Field



FIELD

(A1) Closure under addition: If a and b belong to S, then $a + b$ is also in S

(A2) Associativity of addition: $a + (b + c) = (a + b) + c$ for all a, b, c in S

(A3) Additive identity: There is an element 0 in R such that
$a + 0 = 0 + a = a$ for all a in S

(A4) Additive inverse: For each a in S there is an element $–a$ in S
such that $a + (–a) = (–a) + a = 0$

Integral Domain

(A5) Commutativity of addition: $a + b = b + a$ for all a, b in S

Commutative Ring

(M1) Closure under multiplication: If a and b belong to S, then ab is also in S

(M2) Associativity of multiplication: $a(bc) = (ab)c$ for all a, b, c in S

(M3) Distributive laws: $a(b + c) = ab + ac$ for all a, b, c in S
$(a + b)c = ac + bc$ for all a, b, c in S

Ring

(M4) Commutativity of multiplication: $ab = ba$ for all a, b in S

Abelian Group

(M5) Multiplicative identity: There is an element 1 in S such that
$a1 = 1a = a$ for all a in S

(M6) No zero divisors: If a, b in S and $ab = 0$, then either
$a = 0$ or $b = 0$

Group

(M7) Multiplicative inverse: If a belongs to S and $a \neq 0$, there is an
element a^{-1} in S such that $aa^{-1} = a^{-1}a = 1$

Figure 4.2 Group, Ring, and Field

Contd…

- Familiar examples of fields are the rational numbers, the real numbers, and the complex numbers. Note that the set of all integers is not a field, because not every element of the set has a multiplicative inverse; in fact, only the elements 1 and -1 have multiplicative inverses in the integers.

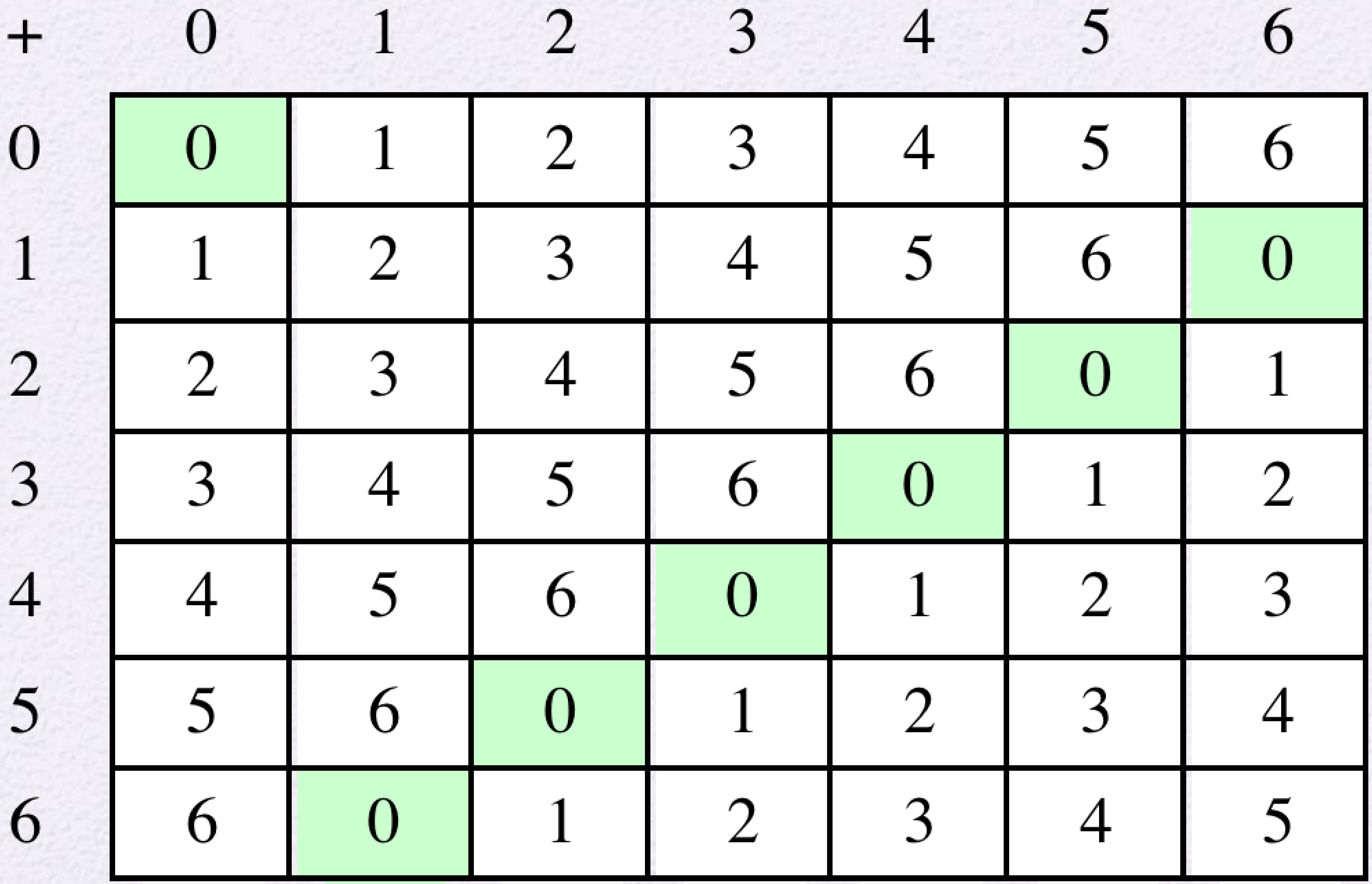
Finite Fields of the Form GF(*p*)

- Finite fields play a crucial role in many cryptographic algorithms

- It can be shown that the order of a finite field must be a power of a prime p^n, where n is a positive integer
  - The only positive integers that are divisors of p are p and 1

- The finite field of order p^n is generally written GF(p^n)
  - GF stands for Galois field, in honor of the mathematician who first studied finite fields

Table 4.5(a)
Arithmetic in GF(7)



+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

(a) Addition modulo 7

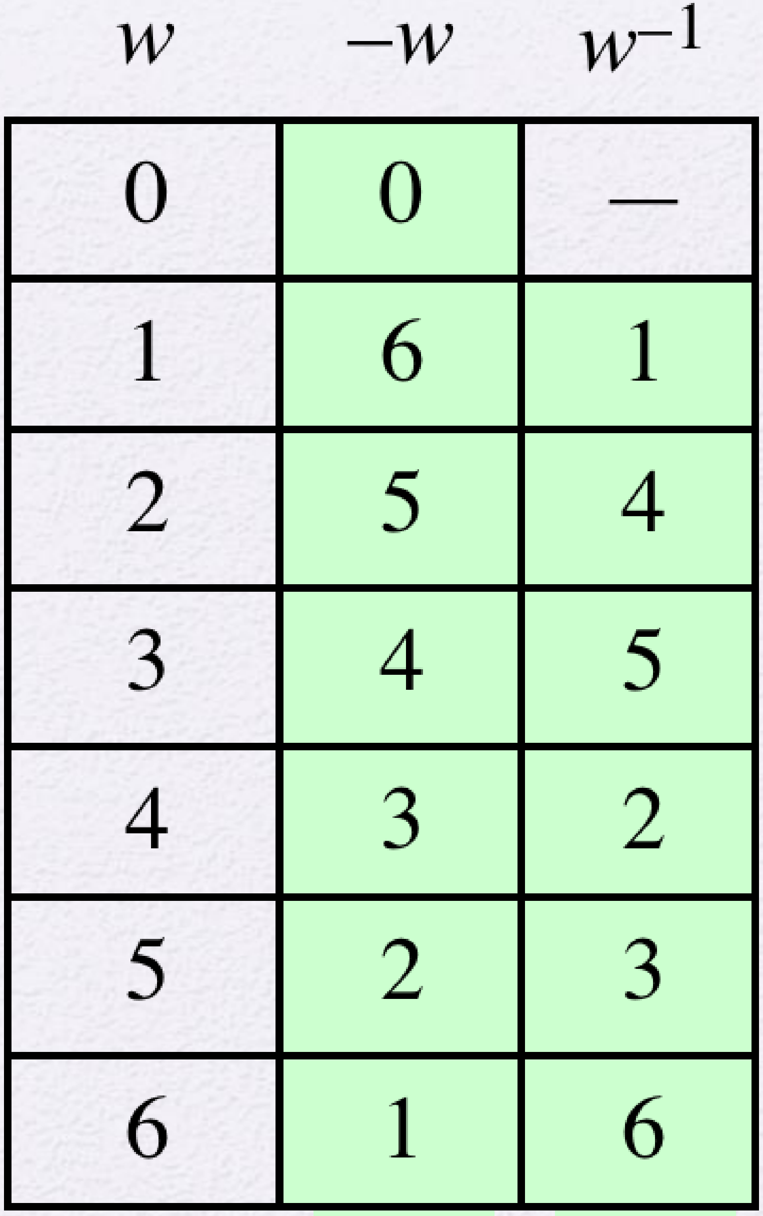
Table 4.5(b)
Arithmetic in GF(7)

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

(b) Multiplication modulo 7

Table 4.5(c)

Arithmetic in GF(7)



w	$-w$	w^{-1}
0	0	—
1	6	1
2	5	4
3	4	5
4	3	2
5	2	3
6	1	6

(c) Additive and multiplicative inverses modulo 7

In this section, we have shown how to construct a finite field of order *p*, where *p* is prime.

GF(*p*) is defined with the following properties:

- 1. GF(*p*) consists of *p* elements

- 2. The binary operations + and * are defined over the set. The operations of addition, subtraction, multiplication, and division can be performed without leaving the set. Each element of the set other than 0 has a multiplicative inverse

- We have shown that the elements of GF(*p*) are the integers {0, 1, . . . , *p* – 1} and that the arithmetic operations are addition and multiplication mod *p*

Polynomial Arithmetic

- We can distinguish three classes of polynomial arithmetic:

- Ordinary polynomial arithmetic, using the basic rules of algebra

- Polynomial arithmetic in which the arithmetic on the coefficients is performed modulo *p*; that is, the coefficients are in GF(*p*)

- Polynomial arithmetic in which the coefficients are in GF(*p*), and the polynomials are defined modulo a polynomial *m* (*x*) whose highest power is some integer *n*

Ordinary Polynomial Arithmetic Example

As an example:

let $f(x) = x^3 + x^2 + 2$ and $g(x) = x^2 - x + 1$, where S is the set of integers

Then:

$$f(x) + g(x) = x^3 + 2x^2 - x + 3$$

$$f(x) - g(x) = x^3 + x + 1$$

$$f(x) * g(x) = x^5 + 3x^2 - 2x + 2$$

Figures 4.3a through 4.3c show the manual calculations

$$\begin{array}{r}
x^3 + x^2 \qquad + 2 \\
+ \;\; (x^2 - x + 1) \\
\hline
x^3 + 2x^2 - x + 3
\end{array}$$

(a) Addition

$$\begin{array}{r}
x^3 + x^2 \qquad + 2 \\
- \;\; (x^2 - x + 1) \\
\hline
x^3 \qquad + x + 1
\end{array}$$

(b) Subtraction

$$\begin{array}{r}
x^3 + x^2 \qquad + 2 \\
\times \;\; (x^2 - x + 1) \\
\hline
x^3 + x^2 \qquad + 2 \\
- x^4 - x^3 \qquad - 2x \qquad \\
x^5 + x^4 \qquad + 2x^2 \qquad\qquad \\
\hline
x^5 \qquad\qquad + 3x^2 - 2x + 2
\end{array}$$

(c) Multiplication

$$\begin{array}{rr}
 & x + 2 \qquad\qquad \\
\cline{2-2}
x^2 - x + 1 \,\big) & x^3 + x^2 \qquad + 2 \\
 & x^3 - x^2 + x \qquad \\
\cline{2-2}
 & 2x^2 - x + 2 \\
 & 2x^2 - 2x + 2 \\
\cline{2-2}
 & x \qquad
\end{array}$$

(d) Division

Figure 4.3 Examples of Polynomial Arithmetic

Polynomial Arithmetic With Coefficients in Z_p

- If each distinct polynomial is considered to be an element of the set, then that set is a ring

- When polynomial arithmetic is performed on polynomials over a field, then division is possible
  - Note: this does not mean that *exact division* is possible

- If we attempt to perform polynomial division over a coefficient set that is not a field, we find that division is not always defined
  - Even if the coefficient set is a field, polynomial division is not necessarily exact
  - With the understanding that remainders are allowed, we can say that polynomial division is possible if the coefficient set is a field

Polynomial Division

- We can write any polynomial in the form:

  $$f(x) = q(x)\, g(x) + r(x)$$

  - *r(x)* can be interpreted as being a remainder
  - *So r(x) = f(x)* mod *g(x)*

- If there is no remainder we can say *g(x)* **divides** *f(x)*
  - Written as g(*x*) | *f(x)*
  - We can say that g(*x*) is a **factor** of *f(x)*
  - *Or* g(*x*) is a **divisor** of *f(x)*

- A polynomial *f(x)* over a field *F* is called **irreducible** if and only if *f(x)* cannot be expressed as a product of two polynomials, both over *F*, and both of degree lower than that of *f(x)*
  - An irreducible polynomial is also called a **prime polynomial**

- The greatest common divisor of two integers is the largest positive integer that exactly __________ both integers.
- A) multiplies B) exponentially multiplies
- C) squares D) divides

- Two integers are ___________ if their only common positive integer factor is 1.
- A) relatively prime B) congruent modulo
- C) polynomials D) residual

- The ___________ of two numbers is the largest integer that divides both numbers.

- A) greatest common divisor B) prime polynomial

- C) lowest common divisor D) integral divisor

- A ring is said to be __________ if it satisfies the condition ab = ba for all a, b in R.
- A) cyclic B) commutative
- C) abelian D) infinite

- . A __________ is a set of elements on which two arithmetic operations have been defined and which has the properties of ordinary arithmetic, such as closure, associativity, commutativity, distributivity, and having both additive and multiplicative inverses.
- A) field B) modulus
- C) group D) ring

- A __________ is a field with a finite number of elements.
- A) finite group B) finite order
- C) finite field D) finite ring

- If $b|a$, we say that b is a ____________ of a.
- A) residue B) group
- C) divisor D) modulus

- For given integers a and b, the extended ___________ algorithm not only calculates the greatest common divisor d but also two additional integers x and y.
- A) modular B) Euclidean
- C) associative D) cyclic

- A group is said to be __________ if it satisfies the condition $a * b = b * a$ for all a, b in G.
- A) abelian B) infinite
- C) cyclic D) commutative

- With the understanding that remainders are allowed, we can say that polynomial division is possible if the coefficient set is a ___________ .

- A) ring B) field
- C) factor D) divisor

- By analogy to integers, an irreducible polynomial is also called a ___________ .
- A) constant polynomial B) monic polynomial
- C) polynomial ring D) prime polynomial

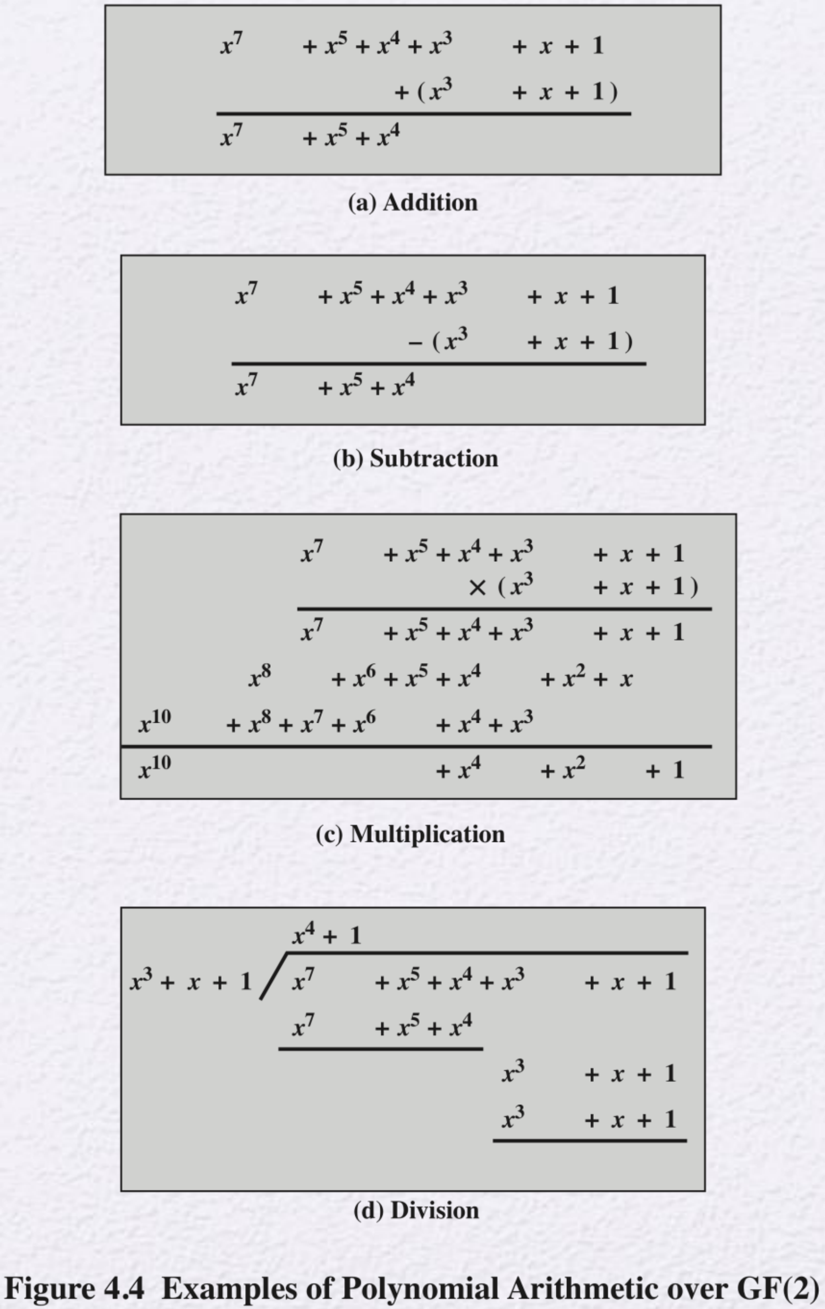
- The congruence relation is used to define ____________ .
- A) finite groups B) greatest common divisor
- C) lowest common divisor D) residue classes

- As a __________ relation, mod expresses that two arguments have the same remainder with respect to a given modulus.
- A) finite B) monic
- C) congruence D) cyclic

- . The order of a finite field must be of the form pn where p is a prime and n is a __ .
- A) identity element B) positive integer
- C) commutative ring D) associative

Example of Polynomial Arithmetic Over GF(2)

(Figure 4.4 can be found on page 110 in the textbook)



$$\begin{array}{r} x^7 + x^5 + x^4 + x^3 \quad + x + 1 \\ +\,(x^3 \quad + x + 1) \\ \hline x^7 + x^5 + x^4 \qquad\qquad\qquad\quad \end{array}$$

(a) Addition

$$\begin{array}{r} x^7 + x^5 + x^4 + x^3 \quad + x + 1 \\ -\,(x^3 \quad + x + 1) \\ \hline x^7 + x^5 + x^4 \qquad\qquad\qquad\quad \end{array}$$

(b) Subtraction

$$\begin{array}{r} x^7 + x^5 + x^4 + x^3 \quad + x + 1 \\ \times\,(x^3 \quad + x + 1) \\ \cline{1-1} x^7 + x^5 + x^4 + x^3 \quad + x + 1 \\ x^8 + x^6 + x^5 + x^4 \quad + x^2 + x \quad \\ x^{10} + x^8 + x^7 + x^6 \quad + x^4 + x^3 \qquad\qquad\quad \\ \hline x^{10} \qquad\qquad\qquad + x^4 \quad + x^2 \quad + 1 \end{array}$$

(c) Multiplication

$$\begin{array}{r l} & x^4 + 1 \\ \cline{2-2} x^3 + x + 1 \,\big) & x^7 + x^5 + x^4 + x^3 \quad + x + 1 \\ & x^7 + x^5 + x^4 \\ \cline{2-2} & \qquad\qquad\qquad x^3 \quad + x + 1 \\ & \qquad\qquad\qquad x^3 \quad + x + 1 \\ \cline{2-2} \end{array}$$

(d) Division

Figure 4.4 Examples of Polynomial Arithmetic over GF(2)

Polynomial GCD

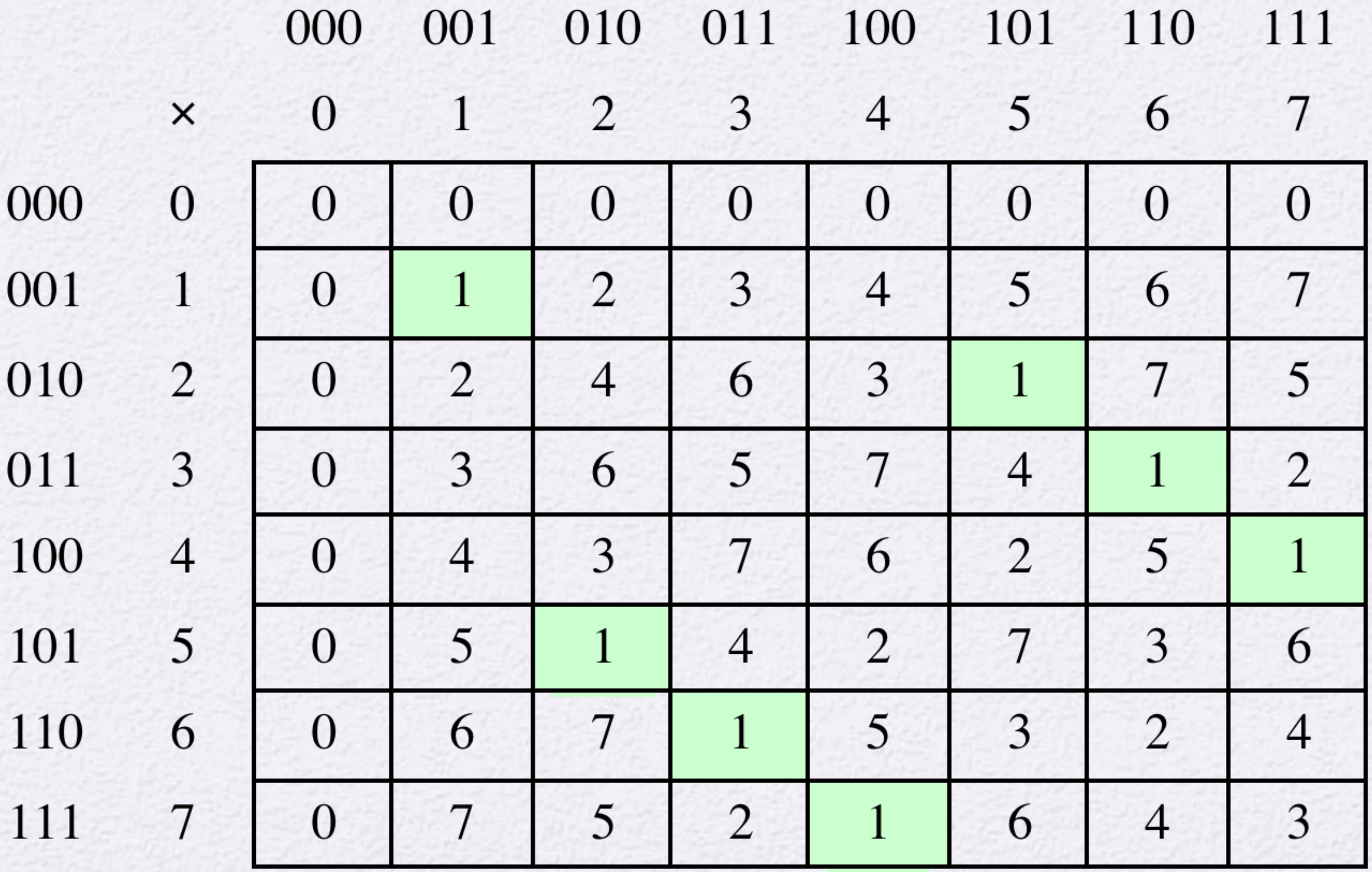
- The polynomial *c(x)* is said to be the greatest common divisor of *a(x)* and *b(x)* if the following are true:
  - c(x) divides both *a(x)* and *b(x)*
  - Any divisor of *a(x)* and *b(x)* is a divisor of *c(x)*
- An equivalent definition is:
  - gcd*[a(x), b(x)]* is the polynomial of maximum degree that divides both *a(x)* and *b(x)*
- The Euclidean algorithm can be extended to find the greatest common divisor of two polynomials whose coefficients are elements of a field

Table 4.6(a) Arithmetic in GF(2^3)

		000	001	010	011	100	101	110	111
	+	0	1	2	3	4	5	6	7
000	0	0	1	2	3	4	5	6	7
001	1	1	0	3	2	5	4	7	6
010	2	2	3	0	1	6	7	4	5
011	3	3	2	1	0	7	6	5	4
100	4	4	5	6	7	0	1	2	3
101	5	5	4	7	6	1	0	3	2
110	6	6	7	4	5	2	3	0	1
111	7	7	6	5	4	3	2	1	0

(a) Addition

Table 4.6(b)
Arithmetic in GF(2^3)

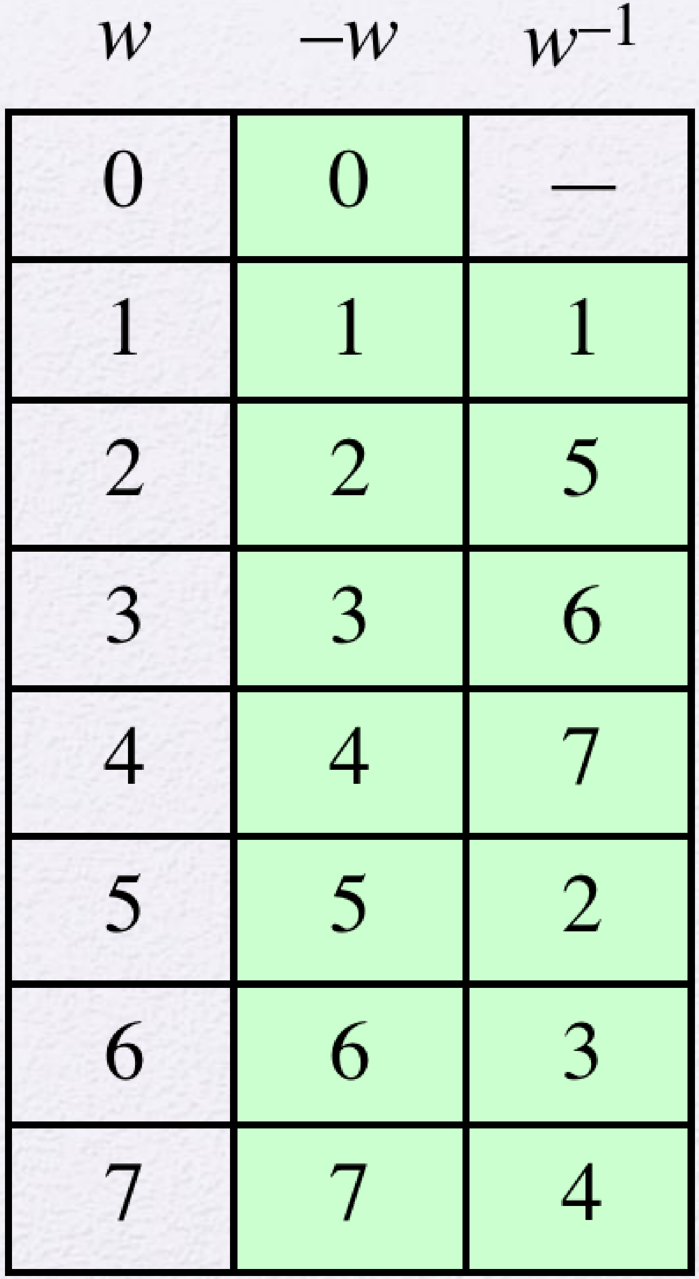


		000	001	010	011	100	101	110	111
	×	0	1	2	3	4	5	6	7
000	0	0	0	0	0	0	0	0	0
001	1	0	1	2	3	4	5	6	7
010	2	0	2	4	6	3	1	7	5
011	3	0	3	6	5	7	4	1	2
100	4	0	4	3	7	6	2	5	1
101	5	0	5	1	4	2	7	3	6
110	6	0	6	7	1	5	3	2	4
111	7	0	7	5	2	1	6	4	3

(b) Multiplication

Table 4.6(c)

Arithmetic in GF(2^3)

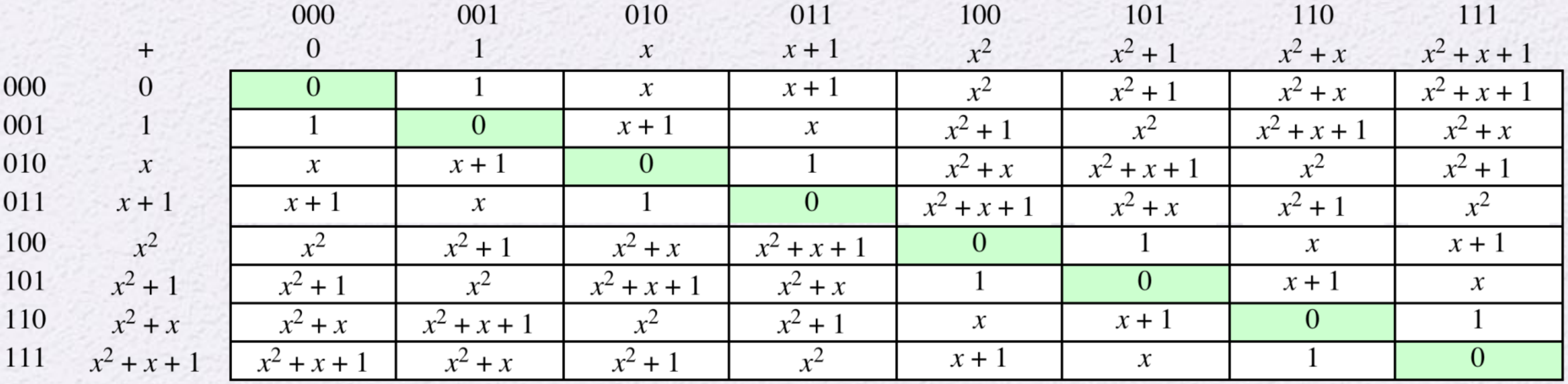


w	$-w$	w^{-1}
0	0	—
1	1	1
2	2	5
3	3	6
4	4	7
5	5	2
6	6	3
7	7	4

(c) Additive and multiplicative inverses

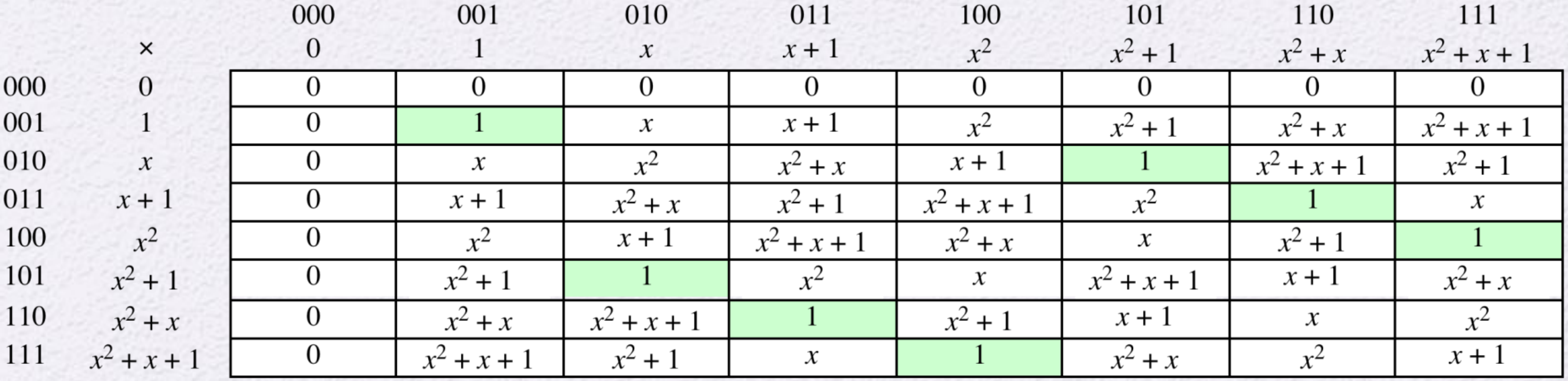
Table 4.7 (page 117 in textbook)

Polynomial Arithmetic Modulo ($x^3 + x + 1$)



		000	001	010	011	100	101	110	111
	+	0	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
000	0	0	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
001	1	1	0	$x+1$	x	x^2+1	x^2	x^2+x+1	x^2+x
010	x	x	$x+1$	0	1	x^2+x	x^2+x+1	x^2	x^2+1
011	$x+1$	$x+1$	x	1	0	x^2+x+1	x^2+x	x^2+1	x^2
100	x^2	x^2	x^2+1	x^2+x	x^2+x+1	0	1	x	$x+1$
101	x^2+1	x^2+1	x^2	x^2+x+1	x^2+x	1	0	$x+1$	x
110	x^2+x	x^2+x	x^2+x+1	x^2	x^2+1	x	$x+1$	0	1
111	x^2+x+1	x^2+x+1	x^2+x	x^2+1	x^2	$x+1$	x	1	0

(a) Addition

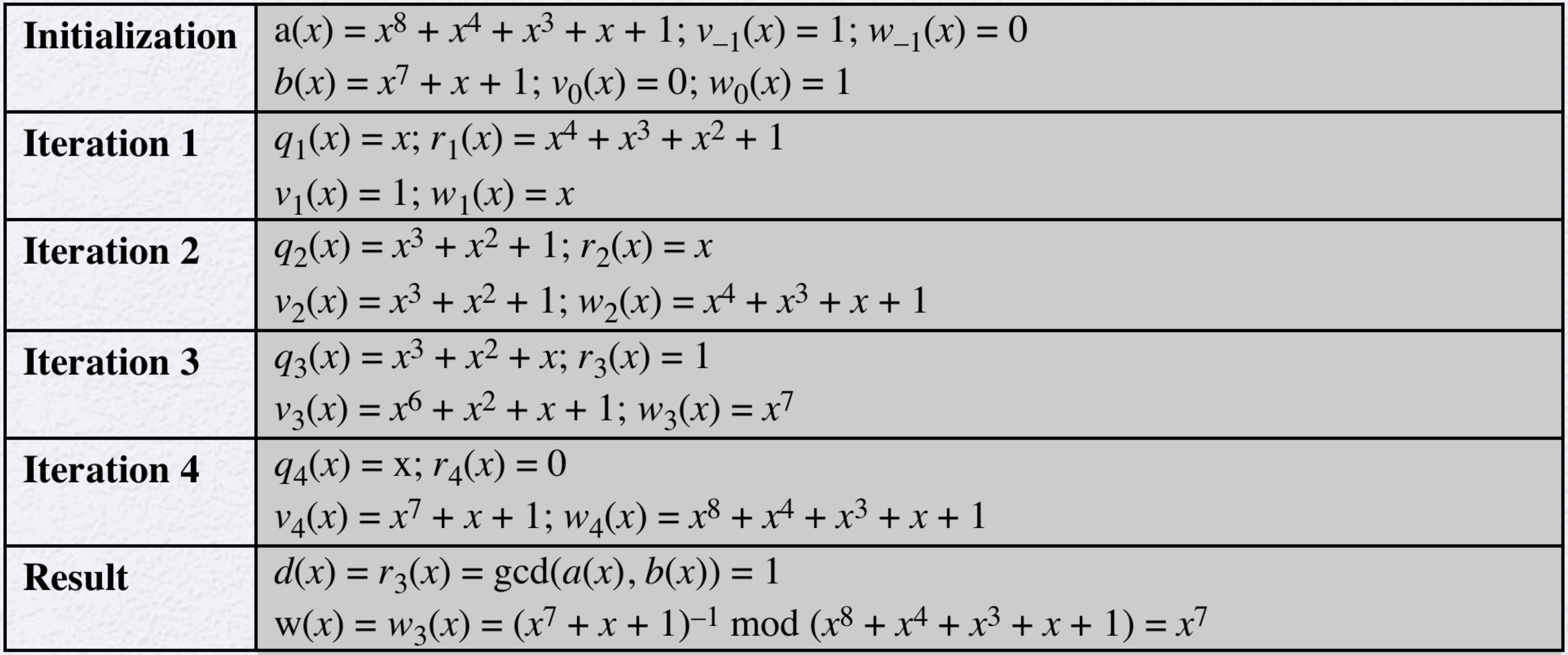


		000	001	010	011	100	101	110	111
	×	0	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
000	0	0	0	0	0	0	0	0	0
001	1	0	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
010	x	0	x	x^2	x^2+x	$x+1$	1	x^2+x+1	x^2+1
011	$x+1$	0	$x+1$	x^2+x	x^2+1	x^2+x+1	x^2	1	x
100	x^2	0	x^2	$x+1$	x^2+x+1	x^2+x	x	x^2+1	1
101	x^2+1	0	x^2+1	1	x^2	x	x^2+x+1	$x+1$	x^2+x
110	x^2+x	0	x^2+x	x^2+x+1	1	x^2+1	$x+1$	x	x^2
111	x^2+x+1	0	x^2+x+1	x^2+1	x	1	x^2+x	x^2	$x+1$

(b) Multiplication

Table 4.8

Extended Euclid [($x^8 + x^4 + x^3 + x + 1$), ($x^7 + x + 1$)]



Initialization	$\mathrm{a}(x) = x^8 + x^4 + x^3 + x + 1$; $v_{-1}(x) = 1$; $w_{-1}(x) = 0$ $b(x) = x^7 + x + 1$; $v_0(x) = 0$; $w_0(x) = 1$
Iteration 1	$q_1(x) = x$; $r_1(x) = x^4 + x^3 + x^2 + 1$ $v_1(x) = 1$; $w_1(x) = x$
Iteration 2	$q_2(x) = x^3 + x^2 + 1$; $r_2(x) = x$ $v_2(x) = x^3 + x^2 + 1$; $w_2(x) = x^4 + x^3 + x + 1$
Iteration 3	$q_3(x) = x^3 + x^2 + x$; $r_3(x) = 1$ $v_3(x) = x^6 + x^2 + x + 1$; $w_3(x) = x^7$
Iteration 4	$q_4(x) = \mathrm{x}$; $r_4(x) = 0$ $v_4(x) = x^7 + x + 1$; $w_4(x) = x^8 + x^4 + x^3 + x + 1$
Result	$d(x) = r_3(x) = \mathrm{gcd}(a(x), b(x)) = 1$ $\mathrm{w}(x) = w_3(x) = (x^7 + x + 1)^{-1} \bmod (x^8 + x^4 + x^3 + x + 1) = x^7$

(Table 4.8 can be found on page 118 in textbook)

Computational Considerations

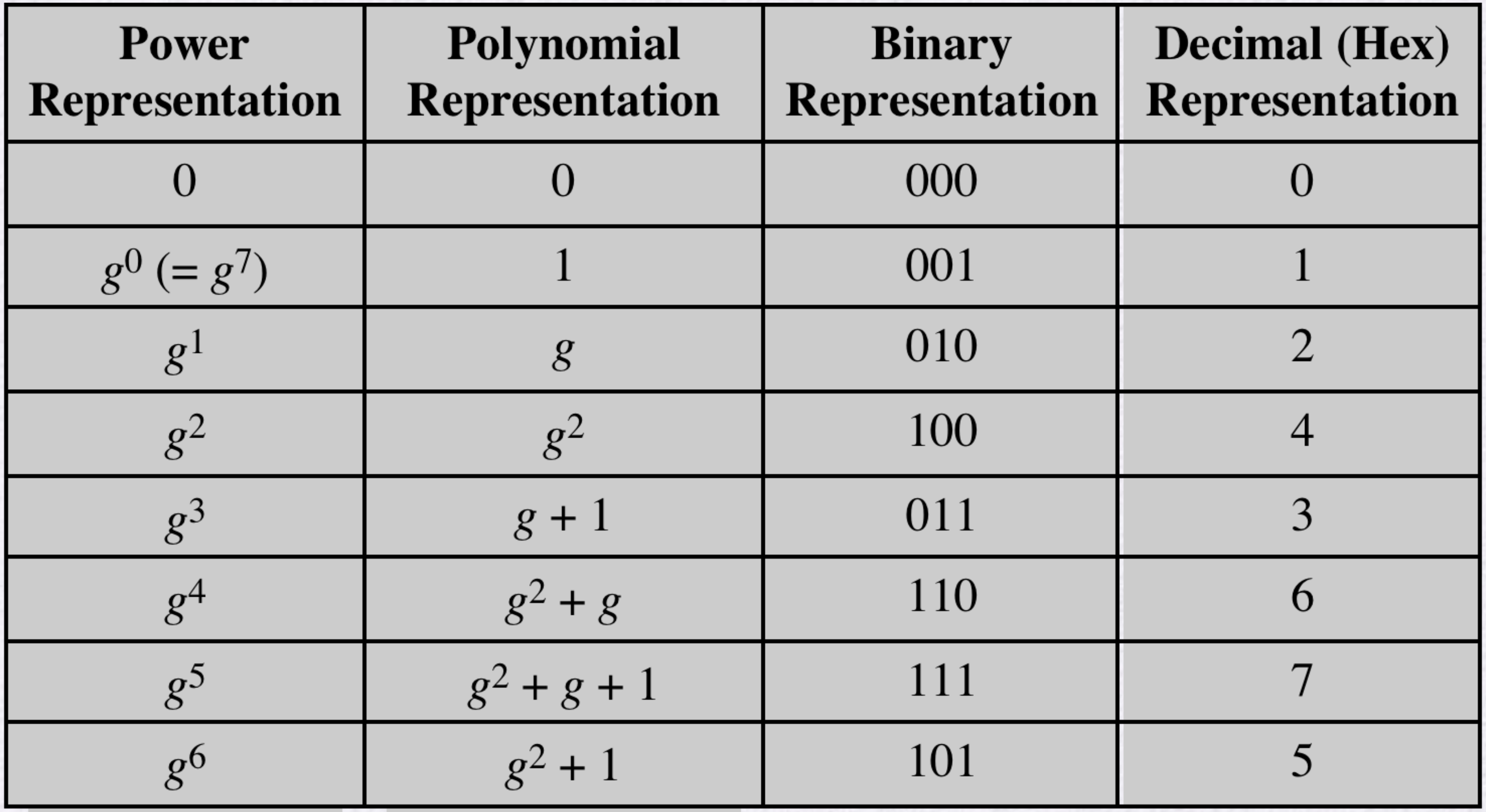
- Since coefficients are 0 or 1, they can represent any such polynomial as a bit string
- Addition becomes XOR of these bit strings
- Multiplication is shift and XOR
  - cf long-hand multiplication
- Modulo reduction is done by repeatedly substituting highest power with remainder of irreducible polynomial (also shift and XOR)

Using a Generator

- A **generator** g of a finite field F of order *q* (contains *q* elements) is an element whose first *q*-1 powers generate all the nonzero elements of F
  - The elements of F consist of 0, g^0, g^1, ..., g^{q-2}
- Consider a field F defined by a polynomial *fx*
  - An element b contained in F is called a **root** of the polynomial if *f(b) = 0*
- Finally, it can be shown that a root g of an irreducible polynomial is a generator of the finite field defined on that polynomial

Table 4.9

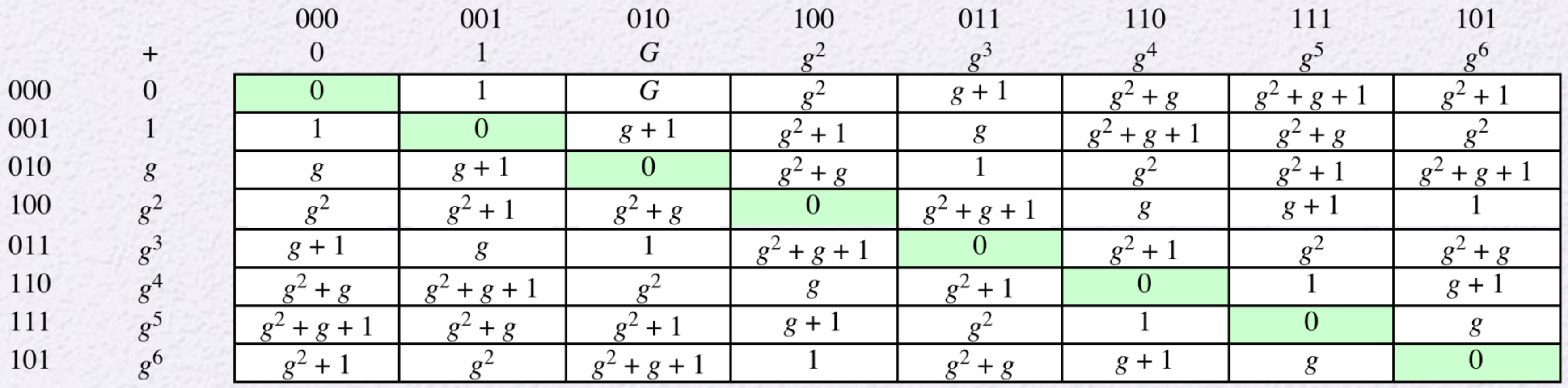
Generator for GF(2^3) using $x^3 + x + 1$



Power Representation	Polynomial Representation	Binary Representation	Decimal (Hex) Representation
0	0	000	0
g^0 (= g^7)	1	001	1
g^1	g	010	2
g^2	g^2	100	4
g^3	$g + 1$	011	3
g^4	$g^2 + g$	110	6
g^5	$g^2 + g + 1$	111	7
g^6	$g^2 + 1$	101	5

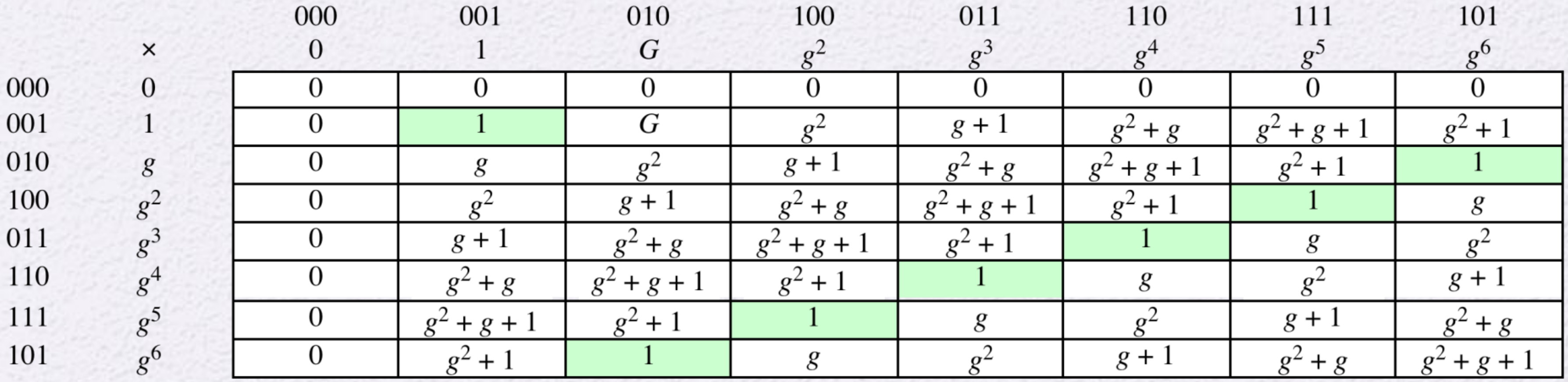
Table 4.10 (page 123 in textbook)

GF(2^3) Arithmetic Using Generator for the Polynomial ($x^3 + x + 1$)



		000	001	010	100	011	110	111	101
	+	0	1	G	g^2	g^3	g^4	g^5	g^6
000	0	0	1	G	g^2	$g + 1$	$g^2 + g$	$g^2 + g + 1$	$g^2 + 1$
001	1	1	0	$g + 1$	$g^2 + 1$	g	$g^2 + g + 1$	$g^2 + g$	g^2
010	g	g	$g + 1$	0	$g^2 + g$	1	g^2	$g^2 + 1$	$g^2 + g + 1$
100	g^2	g^2	$g^2 + 1$	$g^2 + g$	0	$g^2 + g + 1$	g	$g + 1$	1
011	g^3	$g + 1$	g	1	$g^2 + g + 1$	0	$g^2 + 1$	g^2	$g^2 + g$
110	g^4	$g^2 + g$	$g^2 + g + 1$	g^2	g	$g^2 + 1$	0	1	$g + 1$
111	g^5	$g^2 + g + 1$	$g^2 + g$	$g^2 + 1$	$g + 1$	g^2	1	0	g
101	g^6	$g^2 + 1$	g^2	$g^2 + g + 1$	1	$g^2 + g$	$g + 1$	g	0

(a) Addition



		000	001	010	100	011	110	111	101
	×	0	1	G	g^2	g^3	g^4	g^5	g^6
000	0	0	0	0	0	0	0	0	0
001	1	0	1	G	g^2	$g + 1$	$g^2 + g$	$g^2 + g + 1$	$g^2 + 1$
010	g	0	g	g^2	$g + 1$	$g^2 + g$	$g^2 + g + 1$	$g^2 + 1$	1
100	g^2	0	g^2	$g + 1$	$g^2 + g$	$g^2 + g + 1$	$g^2 + 1$	1	g
011	g^3	0	$g + 1$	$g^2 + g$	$g^2 + g + 1$	$g^2 + 1$	1	g	g^2
110	g^4	0	$g^2 + g$	$g^2 + g + 1$	$g^2 + 1$	1	g	g^2	$g + 1$
111	g^5	0	$g^2 + g + 1$	$g^2 + 1$	1	g	g^2	$g + 1$	$g^2 + g$
101	g^6	0	$g^2 + 1$	1	g	g^2	$g + 1$	$g^2 + g$	$g^2 + g + 1$

(b) Multiplication

Summary

- Divisibility and the division algorithm
- The Euclidean algorithm
- Modular arithmetic
- Groups, rings, and fields

- Finite fields of the form GF(p)
- Polynomial arithmetic
- Finite fields of the form GF(2^n)