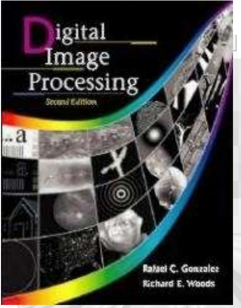


Chapter 4
Image Enhancement in the Frequency Domain

Background

- Any function that periodically repeats itself can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient (Fourier series).
- Even functions that are not periodic (but whose area under the curve is finite) can be expressed as the integral of sines and/or cosines multiplied by a weighting function (Fourier transform).

Background

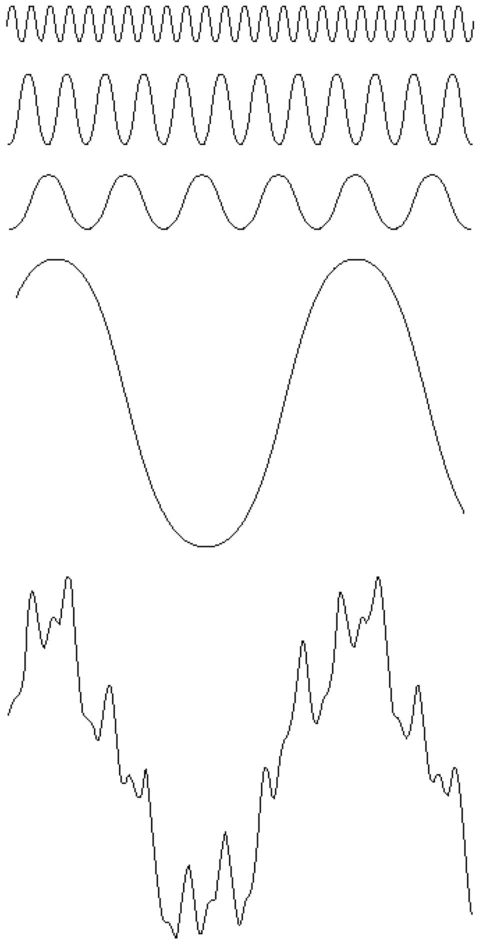


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

- The **frequency domain** refers to the plane of the two dimensional discrete Fourier transform of an image.
- The purpose of the Fourier transform is to represent a signal as a linear combination of sinusoidal signals of various frequencies.

Introduction to the Fourier Transform and the Frequency Domain

- The one-dimensional Fourier transform and its inverse
  - Fourier transform (continuous case)

$$F(u)=\int_{-\infty}^{\infty} f(x)\,e^{-j2\pi ux}\,dx \quad \text{where } j=\sqrt{-1}$$

  - Inverse Fourier transform:

$$\boxed{e^{j\theta}=\cos\theta + j\sin\theta}$$

$$f(x)=\int_{-\infty}^{\infty} F(u)\,e^{j2\pi ux}\,du$$

- The two-dimensional Fourier transform and its inverse
  - Fourier transform (continuous case)

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,e^{-j2\pi(ux+vy)}\,dx\,dy$$

  - Inverse Fourier transform:

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\,e^{j2\pi(ux+vy)}\,du\,dv$$

Introduction to the Fourier Transform and the Frequency Domain

- The one-dimensional Fourier transform and its inverse
  - Fourier transform (discrete case) DTC

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M} \quad \text{for } u = 0,1,2,\ldots,M-1$$

  - Inverse Fourier transform:

$$f(x) = \sum_{u=0}^{M-1} F(u)e^{j2\pi ux/M} \quad \text{for } x = 0,1,2,\ldots,M-1$$

Introduction to the Fourier Transform and the Frequency Domain

- Since $e^{j\theta} = \cos\theta + j\sin\theta$ and the fact $\cos(-\theta) = \cos\theta$ then discrete Fourier transform can be redefined

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\left[\cos 2\pi ux/M - j\sin 2\pi ux/M\right]$$

$$\text{for } u = 0,1,2,\ldots,M-1$$

  – Frequency (time) domain: the domain (values of u) over which the values of $F(u)$ range; because u determines the frequency of the components of the transform.
  – Frequency (time) component: each of the M terms of $F(u)$.

Introduction to the Fourier Transform and the Frequency Domain

- $F(u)$ can be expressed in polar coordinates:

$$F(u) = |F(u)| e^{j\varphi(u)}$$

$$\text{where } |F(u)| = \left[R^2(u) + I^2(u)\right]^{\frac{1}{2}} \quad (\text{magnitude or spectrum})$$

$$\varphi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right] \quad (\text{phase angle or phase spectrum})$$

  - $R(u)$: the real part of $F(u)$
  - $I(u)$: the imaginary part of $F(u)$

- Power spectrum:

$$P(u) = |F(u)|^2 = R^2(u) + I^2(u)$$

Introduction to the Fourier Transform and the Frequency Domain

- The two-dimensional Fourier transform and its inverse
  - Fourier transform (discrete case) DTC

$$F(u,v)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

$$\text{for } u=0,1,2,\ldots,M-1, v=0,1,2,\ldots,N-1$$

  - Inverse Fourier transform:

$$f(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$$

$$\text{for } x=0,1,2,\ldots,M-1, y=0,1,2,\ldots,N-1$$

    - u, v : the transform or frequency variables
    - x, y : the spatial or image variables

Introduction to the Fourier Transform and the Frequency Domain

- We define the Fourier spectrum, phase anble, and power spectrum as follows:

$$|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{\frac{1}{2}} \quad (\text{spectrum})$$

$$\varphi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right] \quad (\text{phase angle})$$

$$P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v) \quad (\text{power spectrum})$$

  - $R(u,v)$: the real part of $F(u,v)$
  - $I(u,v)$: the imaginary part of $F(u,v)$

Introduction to the Fourier Transform and the Frequency Domain

- Some properties of Fourier transform:

$$\Im\left[f(x,y)(-1)^{x+y}\right] = F\left(u-\frac{M}{2}, v-\frac{N}{2}\right) \quad (\text{shift})$$

$$F(0,0) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) \qquad (\text{average})$$

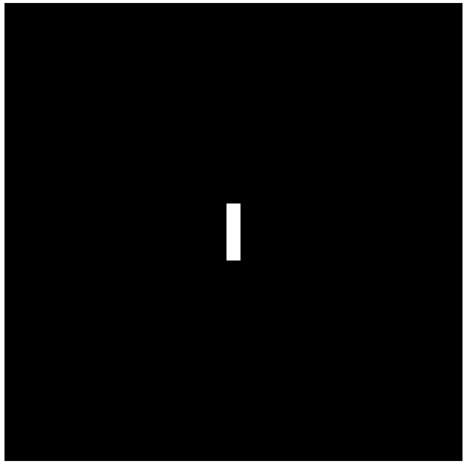
$$F(u,v) = F*(-u,-v) \qquad (\text{conujgate symmetric})$$

$$|F(u,v)| = |F(-u,-v)| \qquad (\text{symmetric})$$

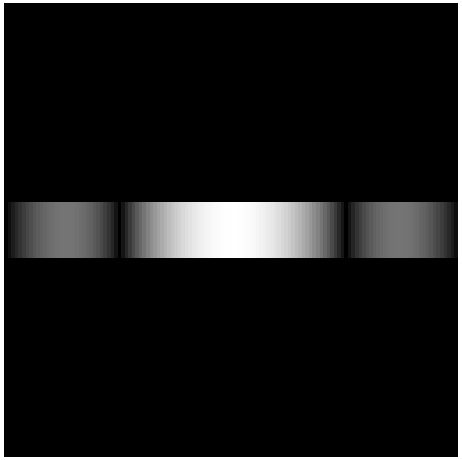
The Two-Dimensional DFT and Its Inverse

The 2D DFT **F(u,v)** can be obtained by

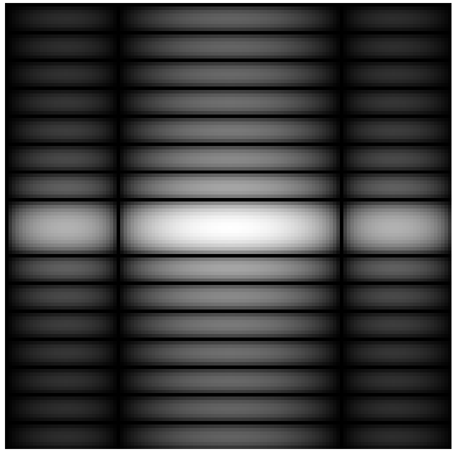
1. taking the 1D DFT of every row of image **f(x,y)**, **F(u,y)**,
2. taking the 1D DFT of every column of **F(u,y)**



(a)f(x,y)

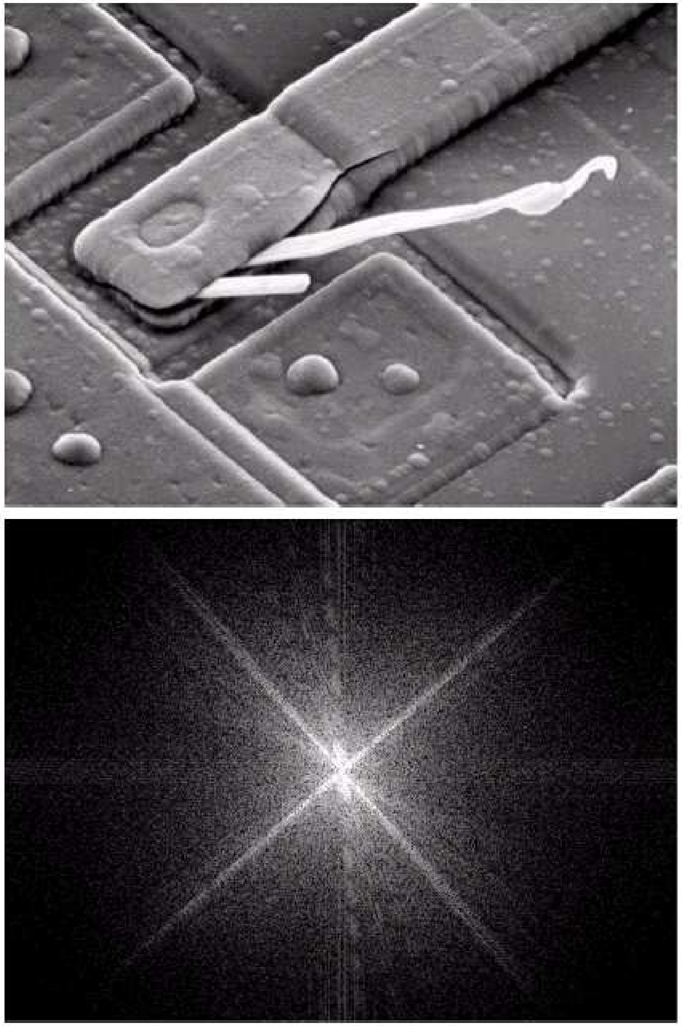


(b)F(u,y)



(c)F(u,v)

Filtering in the Frequency Domain



a
b

FIGURE 4.4 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

Basics of Filtering in the Frequency Domain

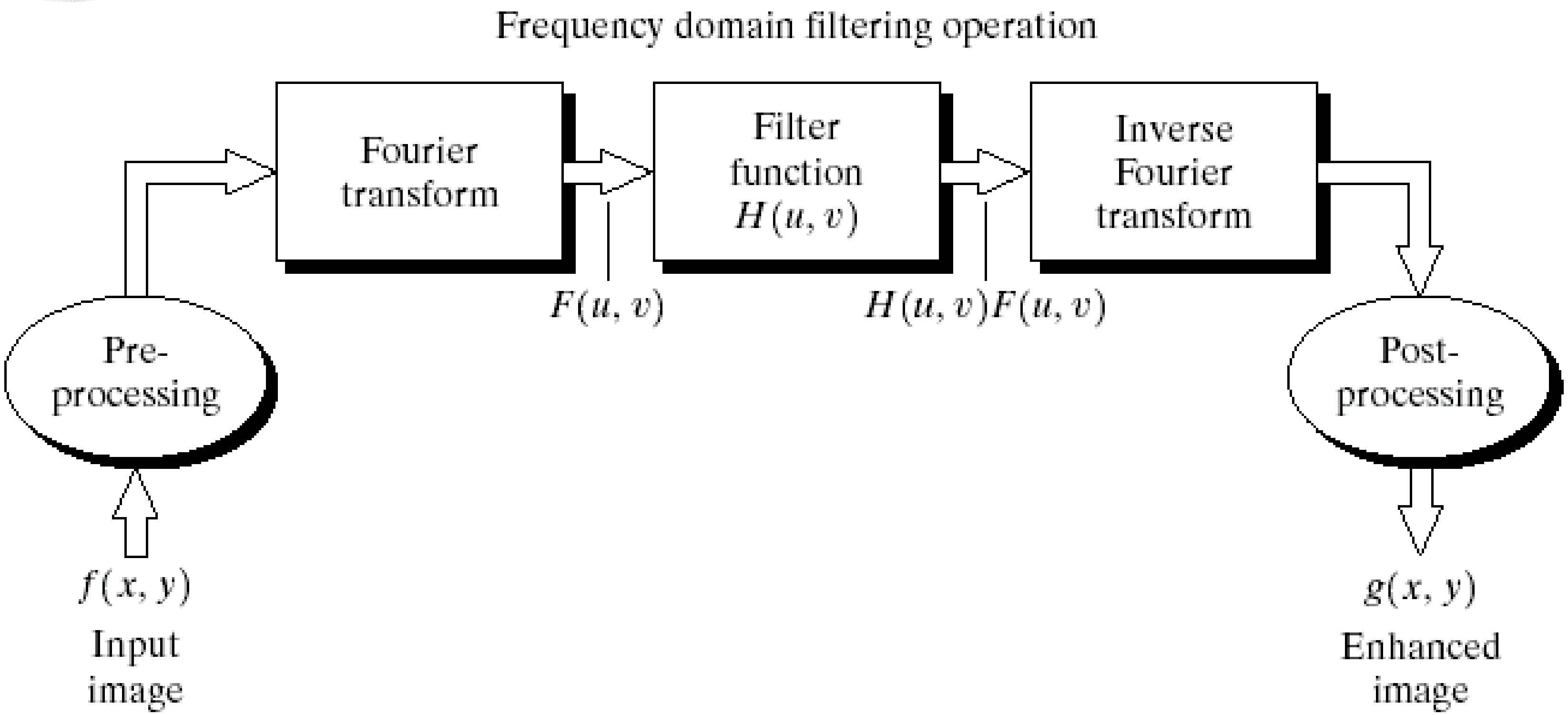


FIGURE 4.5 Basic steps for filtering in the frequency domain.

Some Basic Filters and Their Functions

- Multiply all values of $F(u,v)$ by the filter function (notch filter):

$$H(u,v)=\begin{cases} 0 & \text{if } (u,v)=(M/2,N/2) \\ 1 & \text{otherwise.} \end{cases}$$

  – All this filter would do is set $F(0,0)$ to zero (force the average value of an image to zero) and leave all frequency components of the Fourier transform untouched.

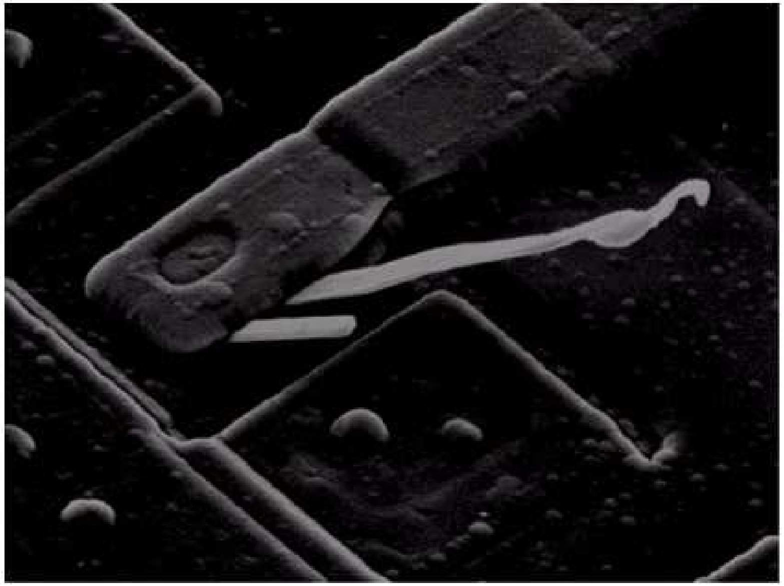
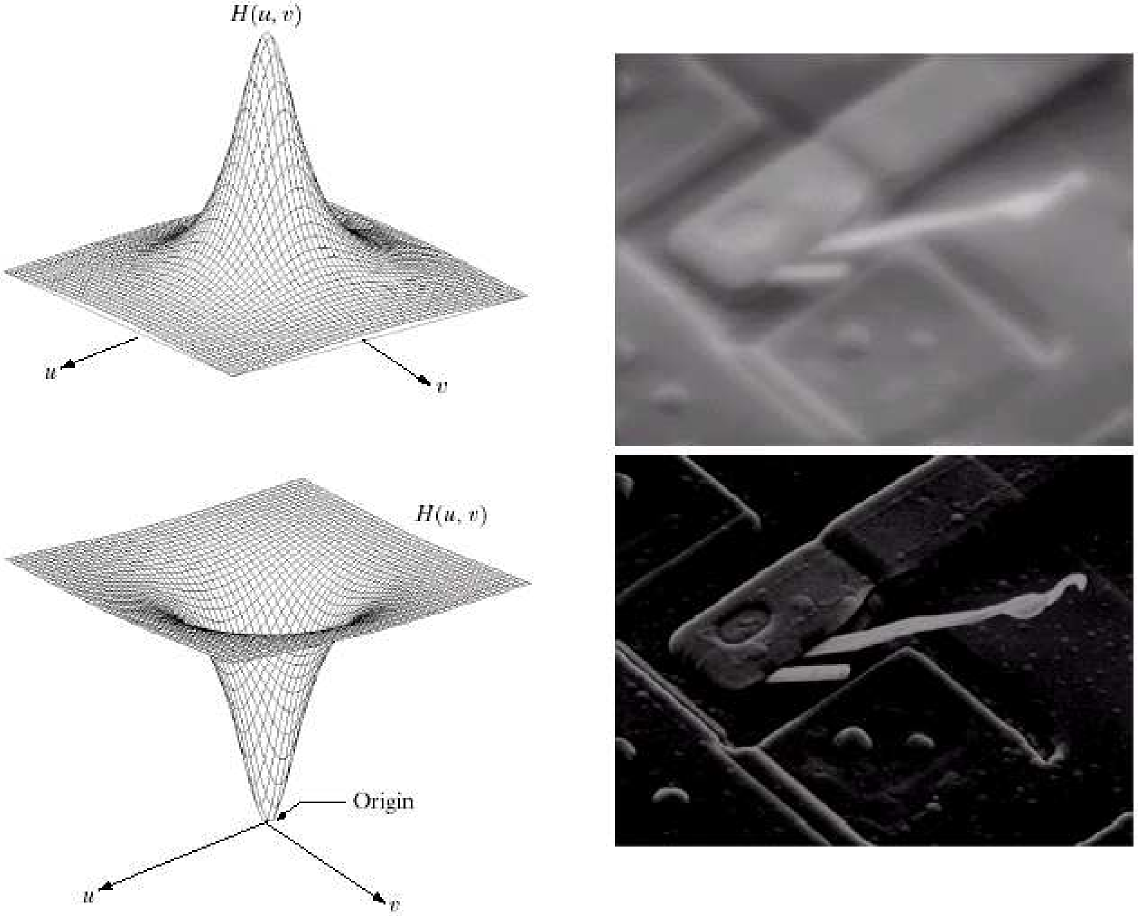


FIGURE 4.6 Result of filtering the image in Fig. 4.4(a) with a notch filter that set to 0 the $F(0, 0)$ term in the Fourier transform.

Some Basic Filters and Their Functions

Lowpass filter

Highpass filter

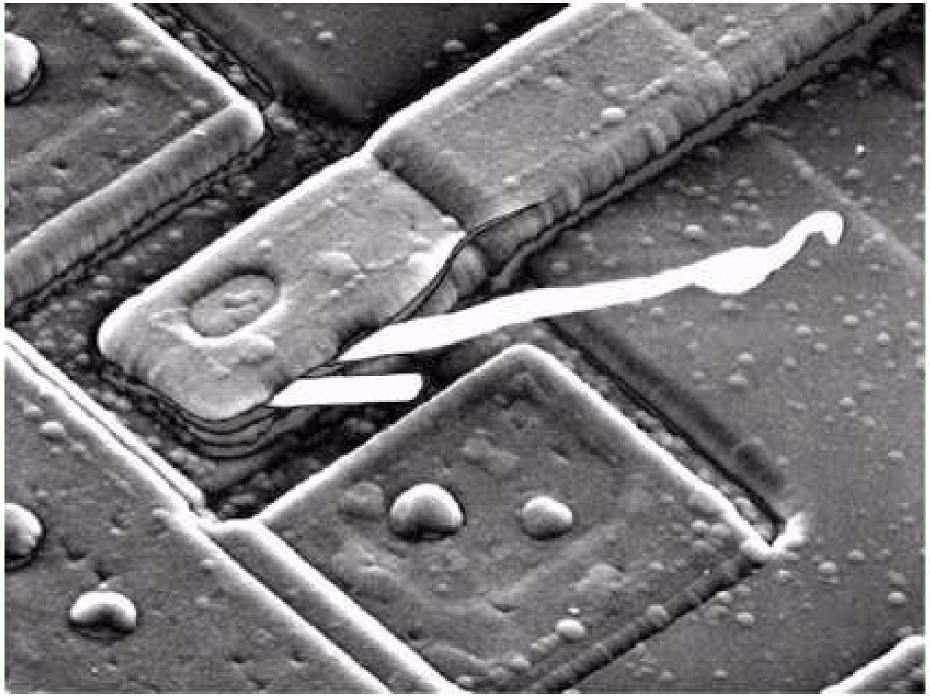


a b
c d

FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

Some Basic Filters and Their Functions

FIGURE 4.8 Result of highpass filtering the image in Fig. 4.4(a) with the filter in Fig. 4.7(c), modified by adding a constant of one-half the filter height to the filter function. Compare with Fig. 4.4(a).



Correspondence between Filtering in the Spatial and Frequency Domain

- Convolution theorem:
  - The discrete convolution of two functions $f(x,y)$ and $h(x,y)$ of size MXN is defined as

$$f(x,y)*h(x,y)=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)h(x-m,y-n)$$

  - Let $F(u,v)$ and $H(u,v)$ denote the Fourier transforms of $f(x,y)$ and $h(x,y)$, then

$$f(x,y)*h(x,y)\Leftrightarrow F(u,v)H(u,v) \qquad \text{Eq. (4.2-31)}$$

$$f(x,y)h(x,y)\Leftrightarrow F(u,v)*H(u,v) \qquad \text{Eq. (4.2-32)}$$

Smoothing Frequency-Domain Filters

- The basic model for filtering in the frequency domain

$$G(u,v) = H(u,v)F(u,v)$$

where $F(u,v)$: the Fourier transform of the image to be smoothed

$H(u,v)$: a filter transfer function

- Smoothing is fundamentally a lowpass operation in the frequency domain.
- There are several standard forms of lowpass filters (LPF).
  - Ideal lowpass filter
  - Butterworth lowpass filter
  - Gaussian lowpass filter

Ideal Lowpass Filters (ILPFs)

- The simplest lowpass filter is a filter that “cuts off” all high-frequency components of the Fourier transform that are at a distance greater than a specified distance D_0 from the origin of the transform.
- The transfer function of an ideal lowpass filter

$$H(u,v)=\begin{cases}1 & \text{if } D(u,v)\le D_0\\ 0 & \text{if } D(u,v)>D_0\end{cases}$$

where $D(u,v)$: the distance from point (u,v) to the center of ther frequency rectangle

$$D(u,v)=\left[(u-M/2)^2+(v-N/2)^2\right]^{\frac{1}{2}}$$

Ideal Lowpass Filters (ILPFs)

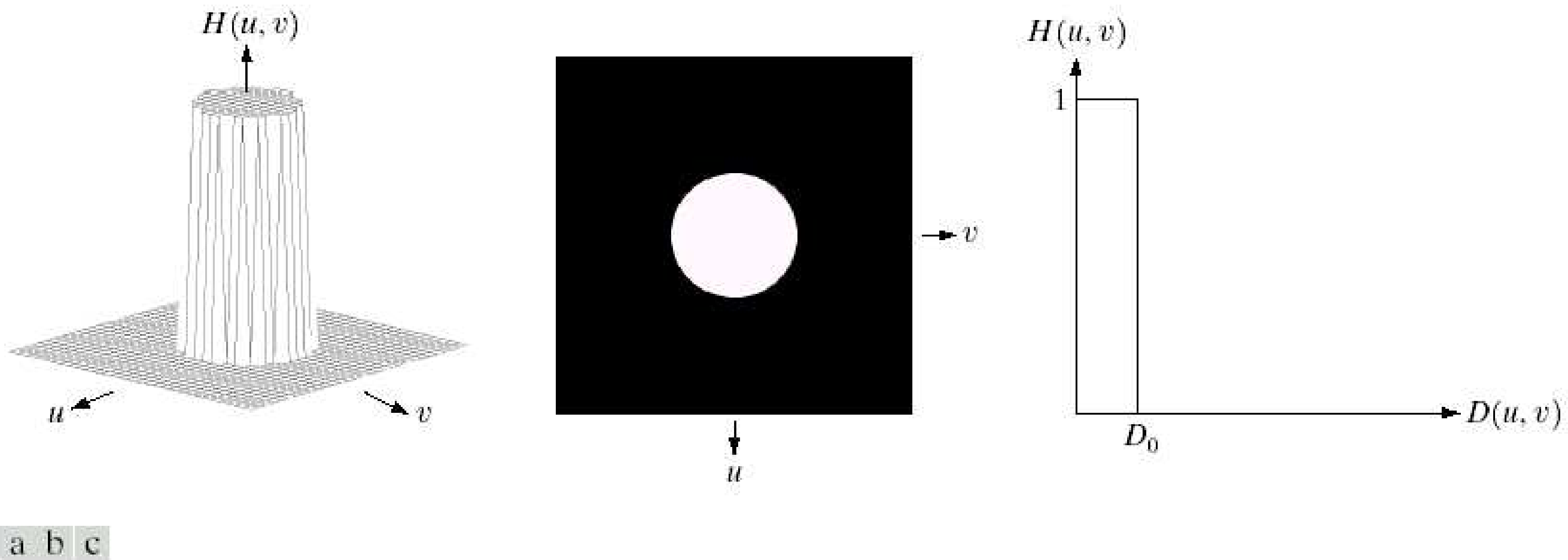
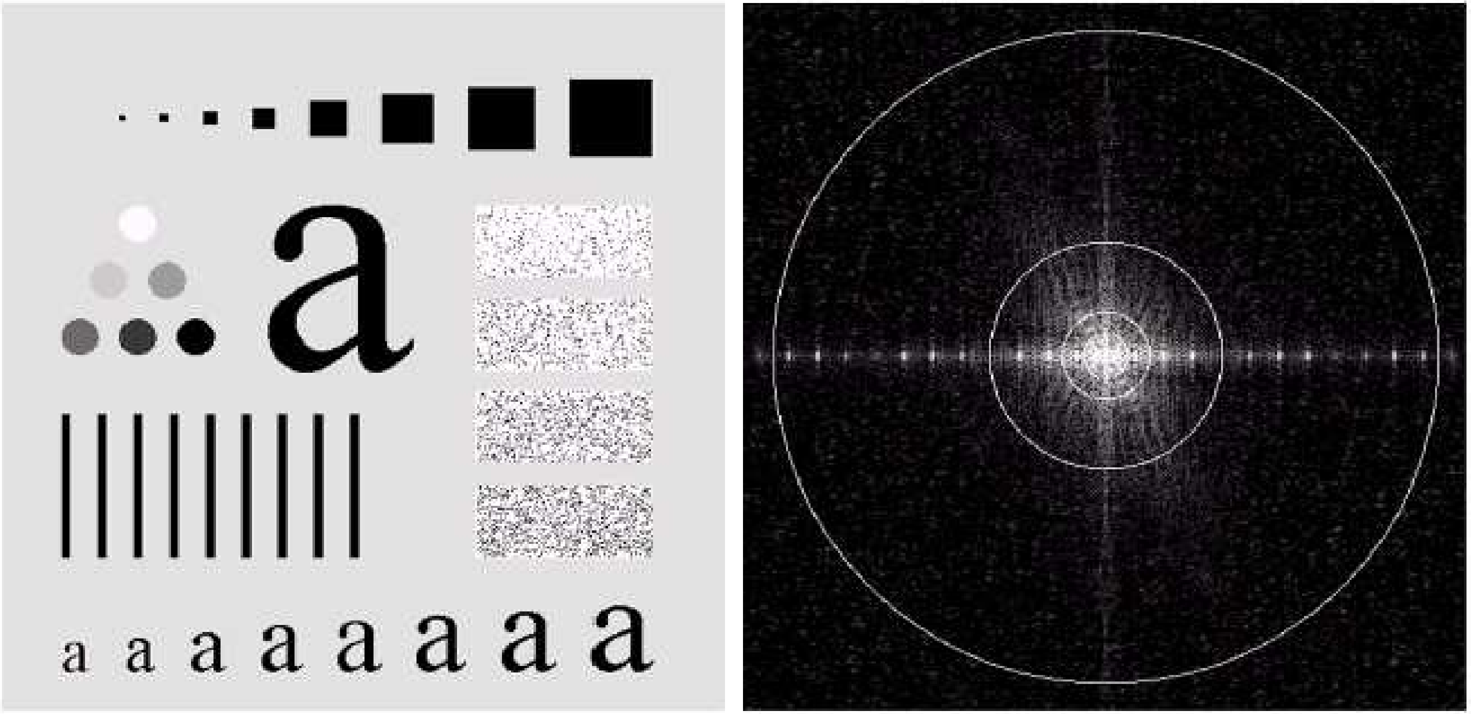


FIGURE 4.10 (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

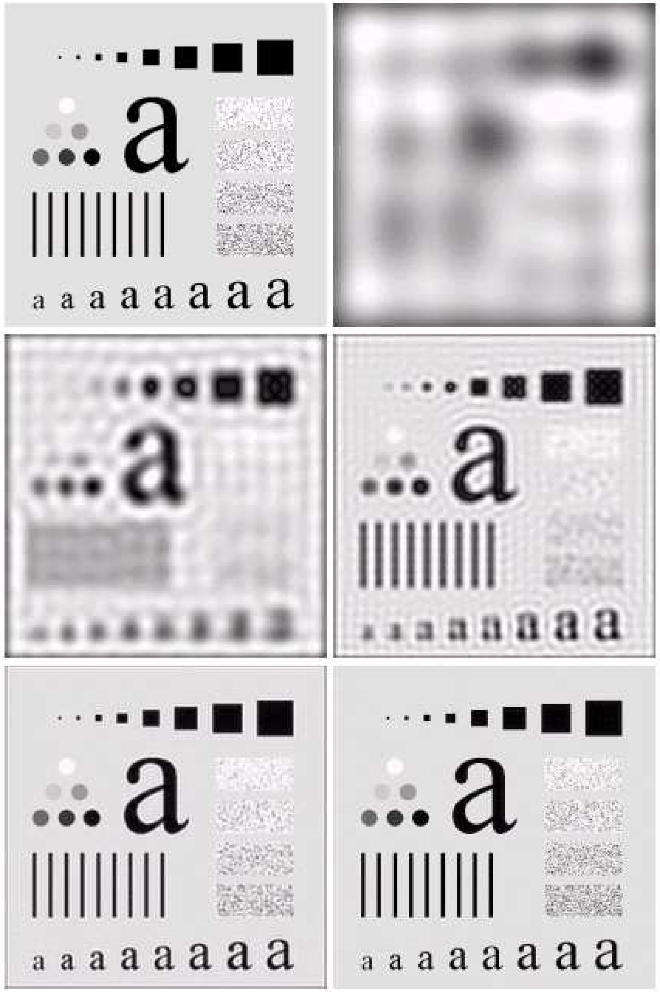
Ideal Lowpass Filters (ILPFs)



a b

FIGURE 4.11 (a) An image of size 500 × 500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.

Ideal Lowpass Filters

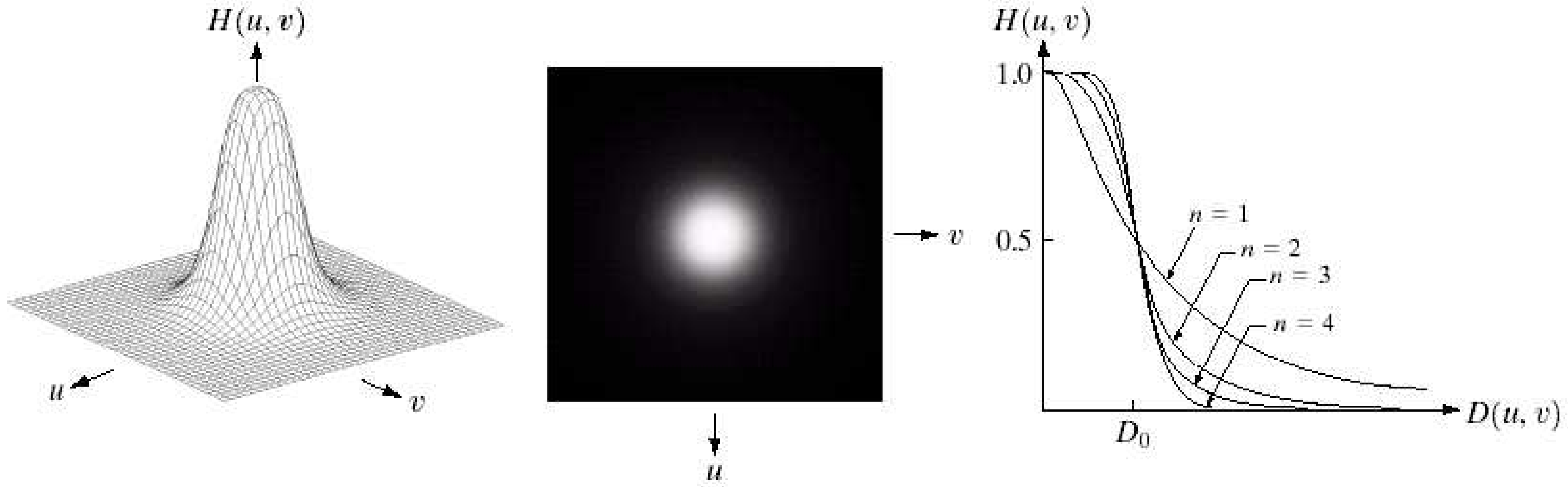


a	b
c	d
e	f

FIGURE 4.12 (a) Original image. (b)–(f) Results of ideal lowpass filtering with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). The power removed by these filters was 8, 5.4, 3.6, 2, and 0.5% of the total, respectively.

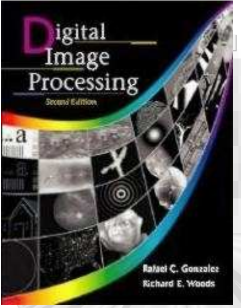
Butterworth Lowpass Filters (BLPFs) With order *n*

$$H(u,v)=\frac{1}{1+\left[D(u,v)/D_0\right]^{2n}}$$



a b c

FIGURE 4.14 (a) Perspective plot of a Butterworth lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.



Butterworth Lowpass Filters (BLPFs)

n=2

D_0=5,15,30,80,and 230

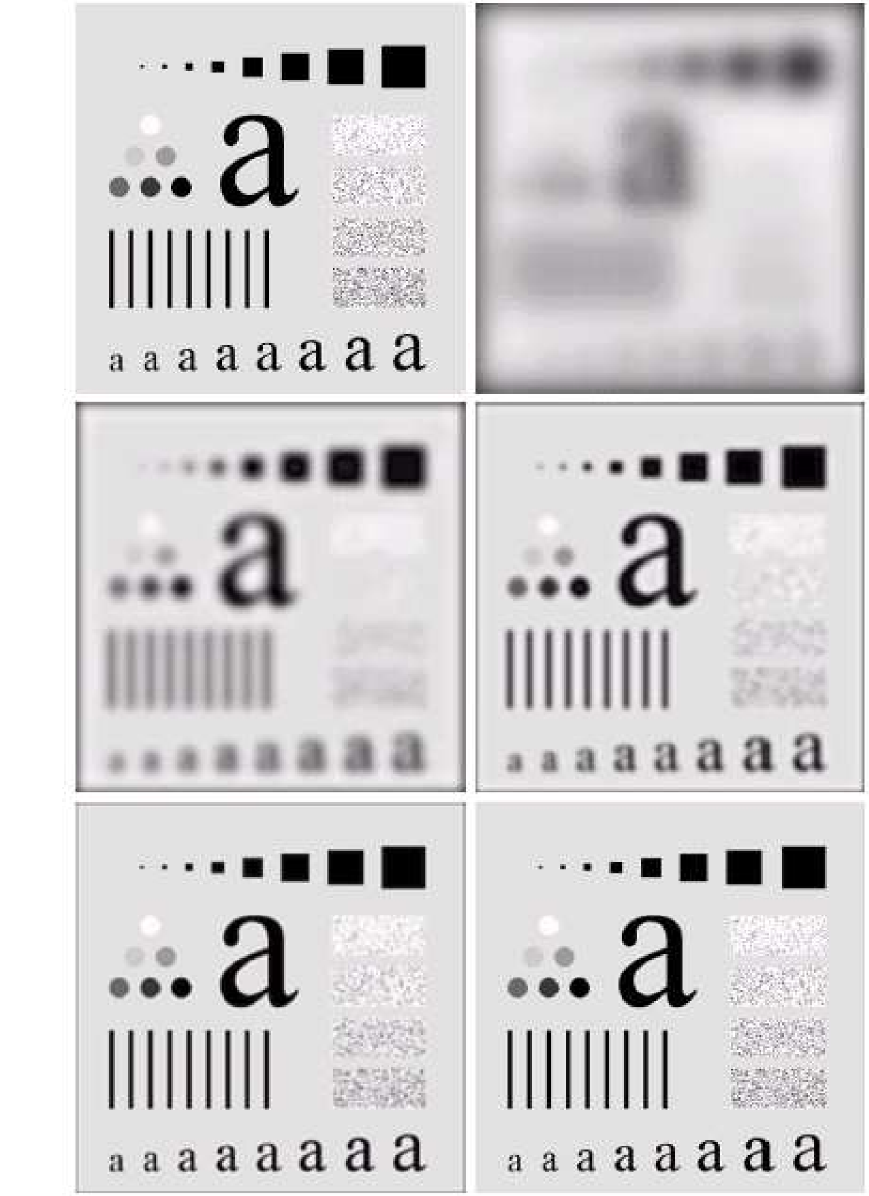


FIGURE 4.15 (a) Original image. (b)–(f) Results of filtering with BLPFs of order 2, with cutoff frequencies at radii of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Fig. 4.12.

Butterworth Lowpass Filters (BLPFs)
Spatial Representation

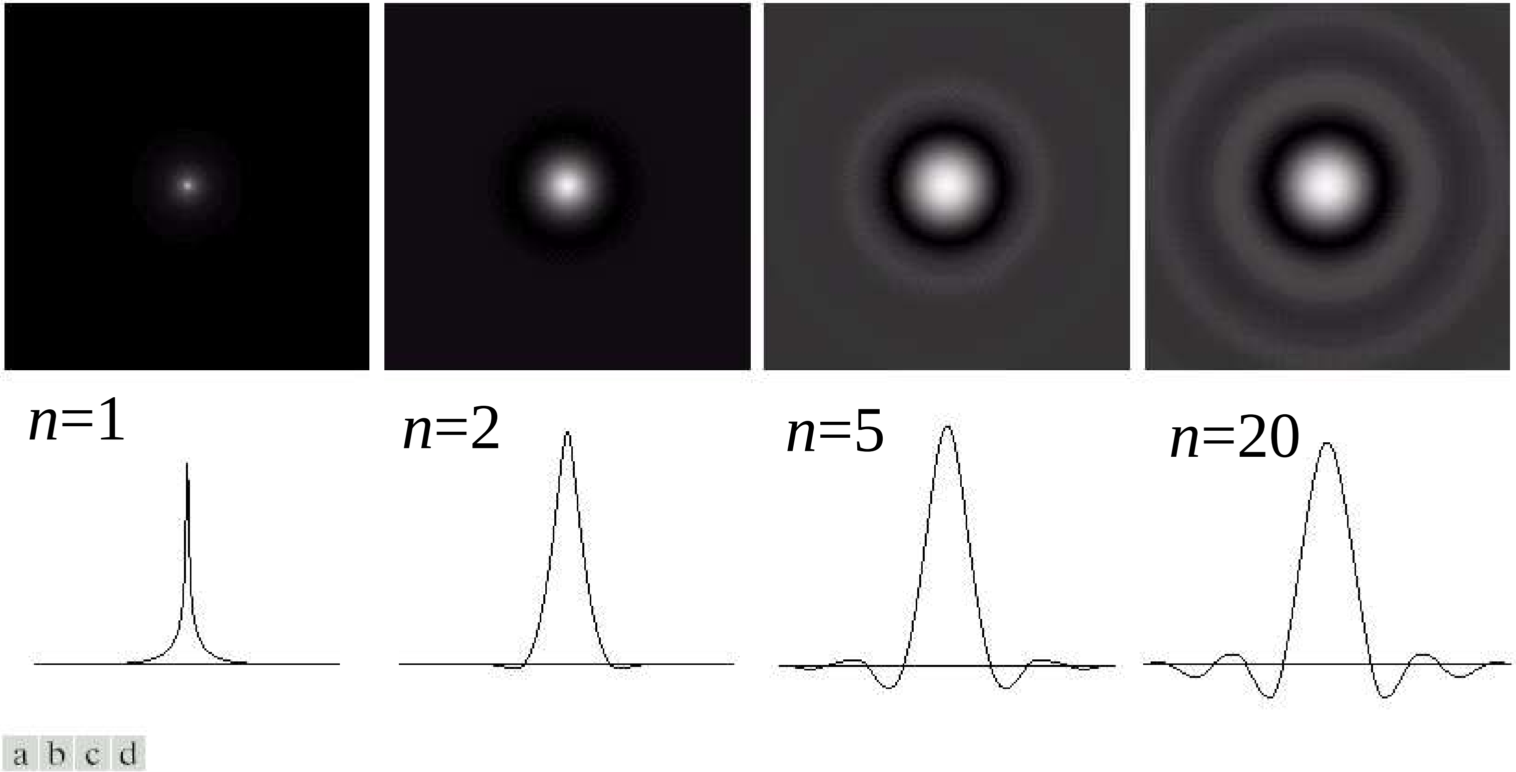
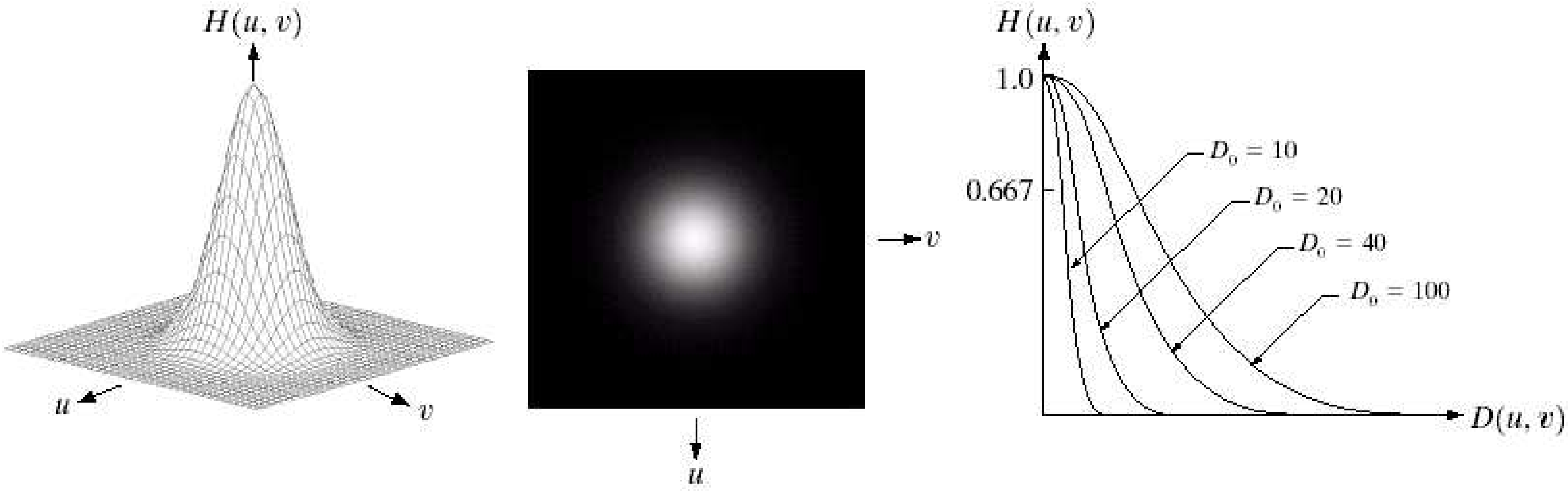


FIGURE 4.16 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding gray-level profiles through the center of the filters (all filters have a cutoff frequency of 5). Note that ringing increases as a function of filter order.

Gaussian Lowpass Filters (FLPFs)

$$H(u,v)=e^{-D^2(u,v)/2D_0^2}$$



a b c

FIGURE 4.17 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

Gaussian Lowpass Filters (FLPFs)

D_0=5,15,30,80,and 230

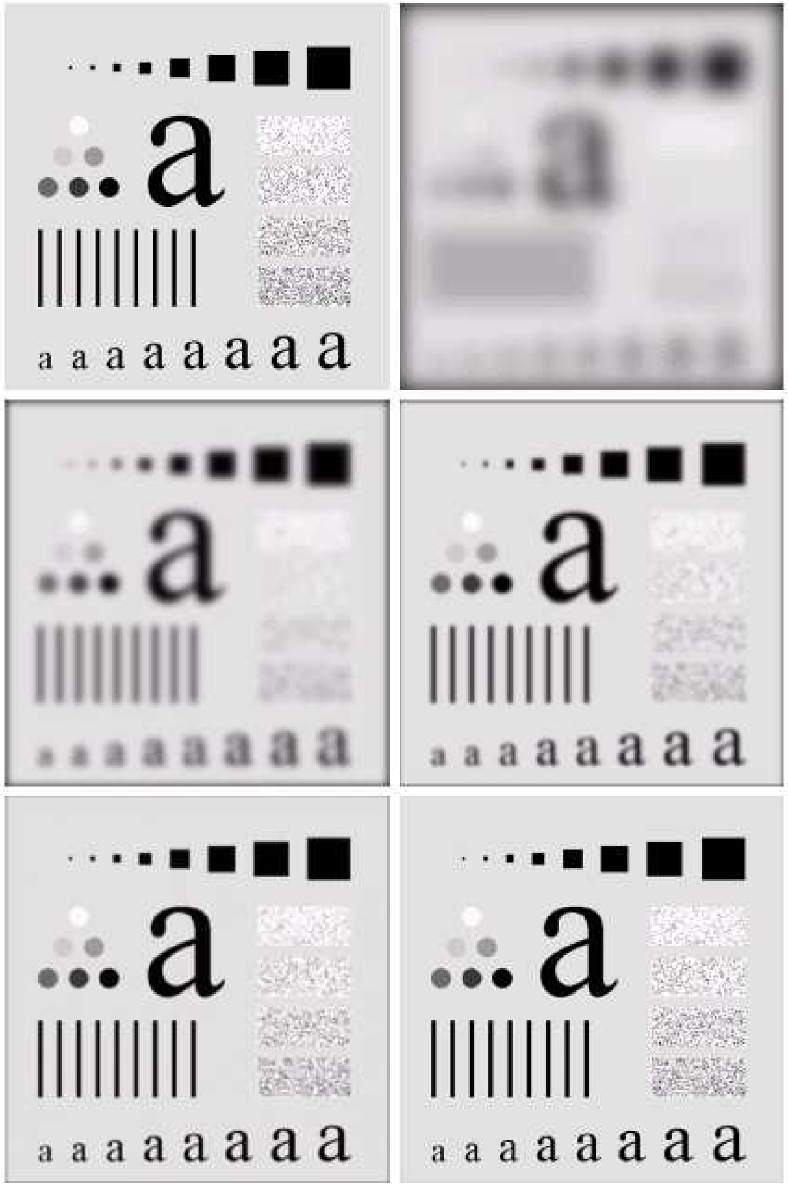


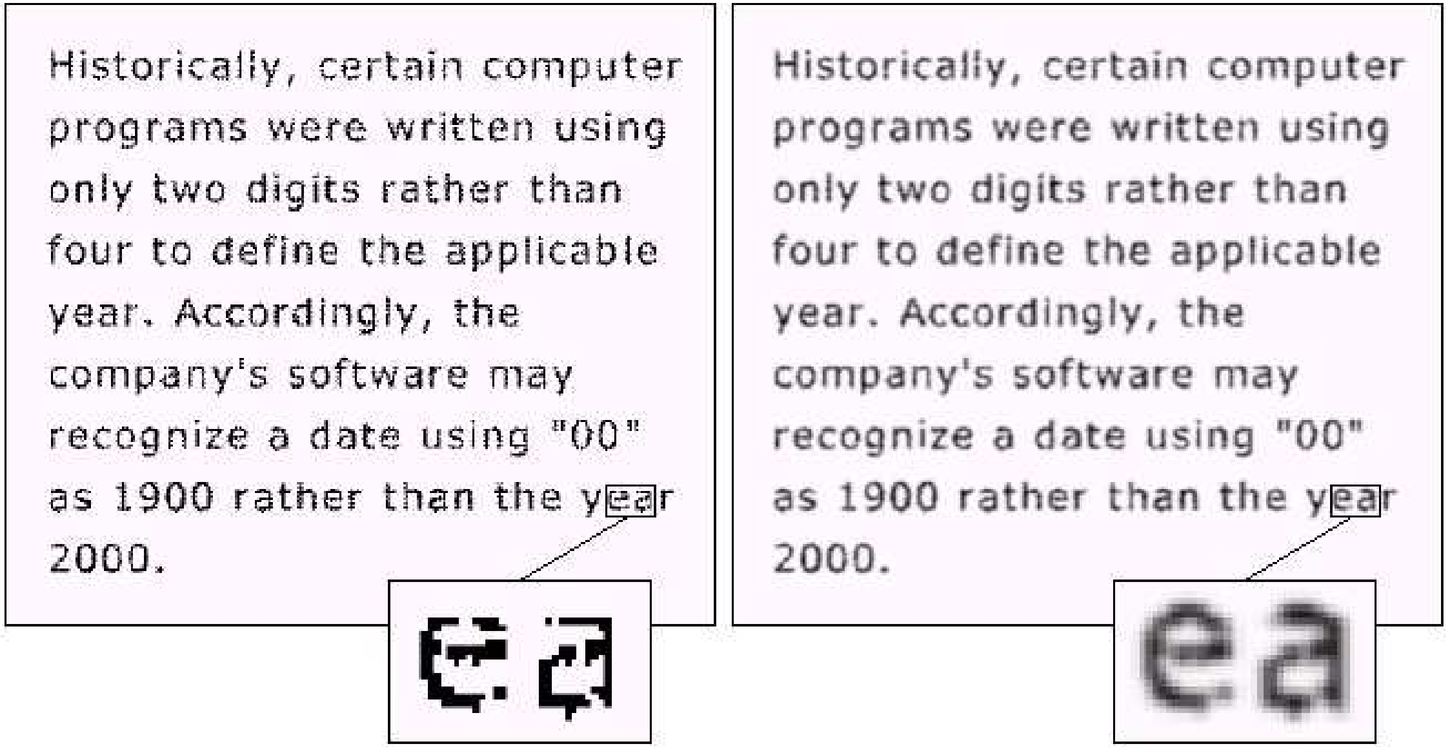
FIGURE 4.18 (a) Original image. (b)–(f) Results of filtering with Gaussian lowpass filters with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Figs. 4.12 and 4.15.

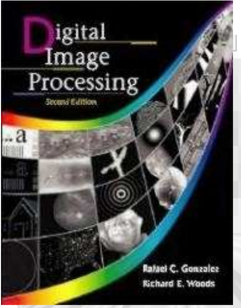
a b
c d
e f

Additional Examples of Lowpass Filtering

a b

FIGURE 4.19
(a) Sample text of poor resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).





Additional Examples of Lowpass Filtering

a b c

FIGURE 4.20 (a) Original image (1028×732 pixels). (b) Result of filtering with a GLPF with $D_0 = 100$. (c) Result of filtering with a GLPF with $D_0 = 80$. Note reduction in skin fine lines in the magnified sections of (b) and (c).

Sharpening Frequency Domain Filter

$$H_{hp}(u,v) = H_{lp}(u,v)$$

Ideal highpass filter

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

Butterworth highpass filter

$$H(u,v) = \frac{1}{1 + \left[D_0 / D(u,v)\right]^{2n}}$$

Gaussian highpass filter

$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$$

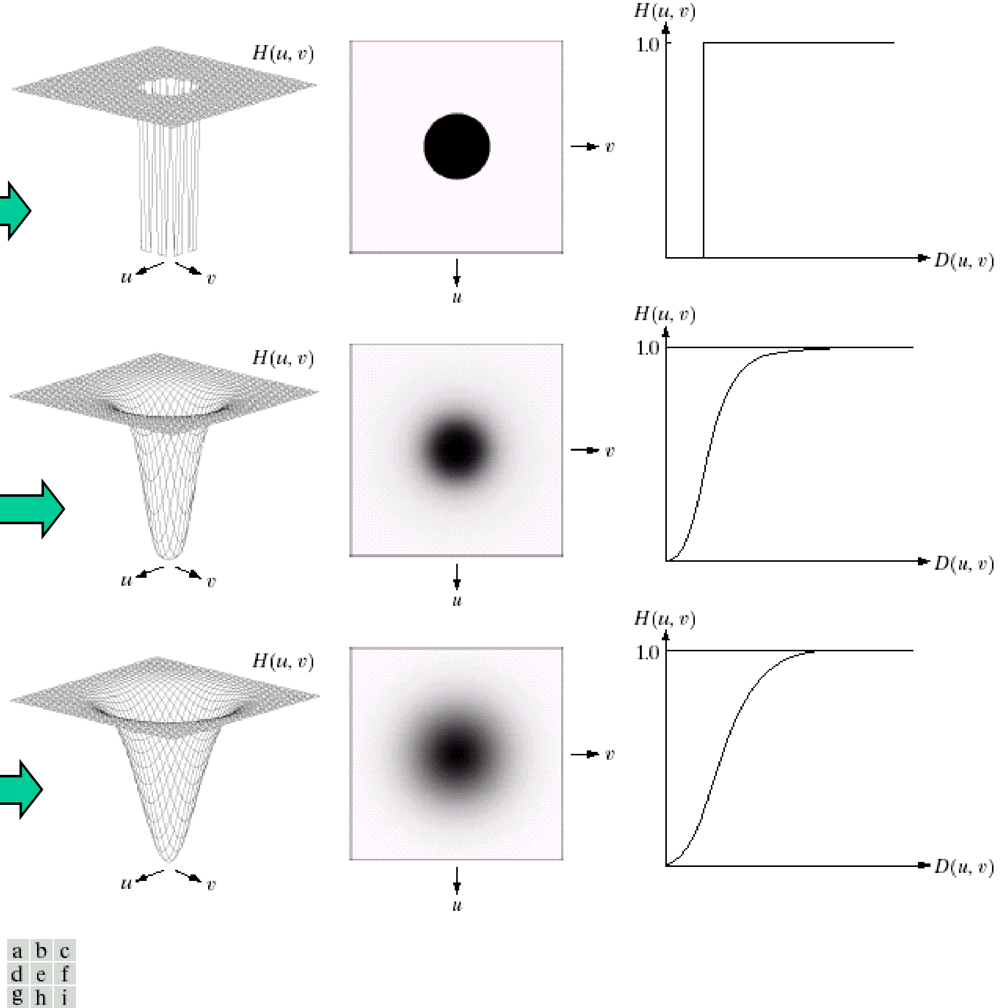
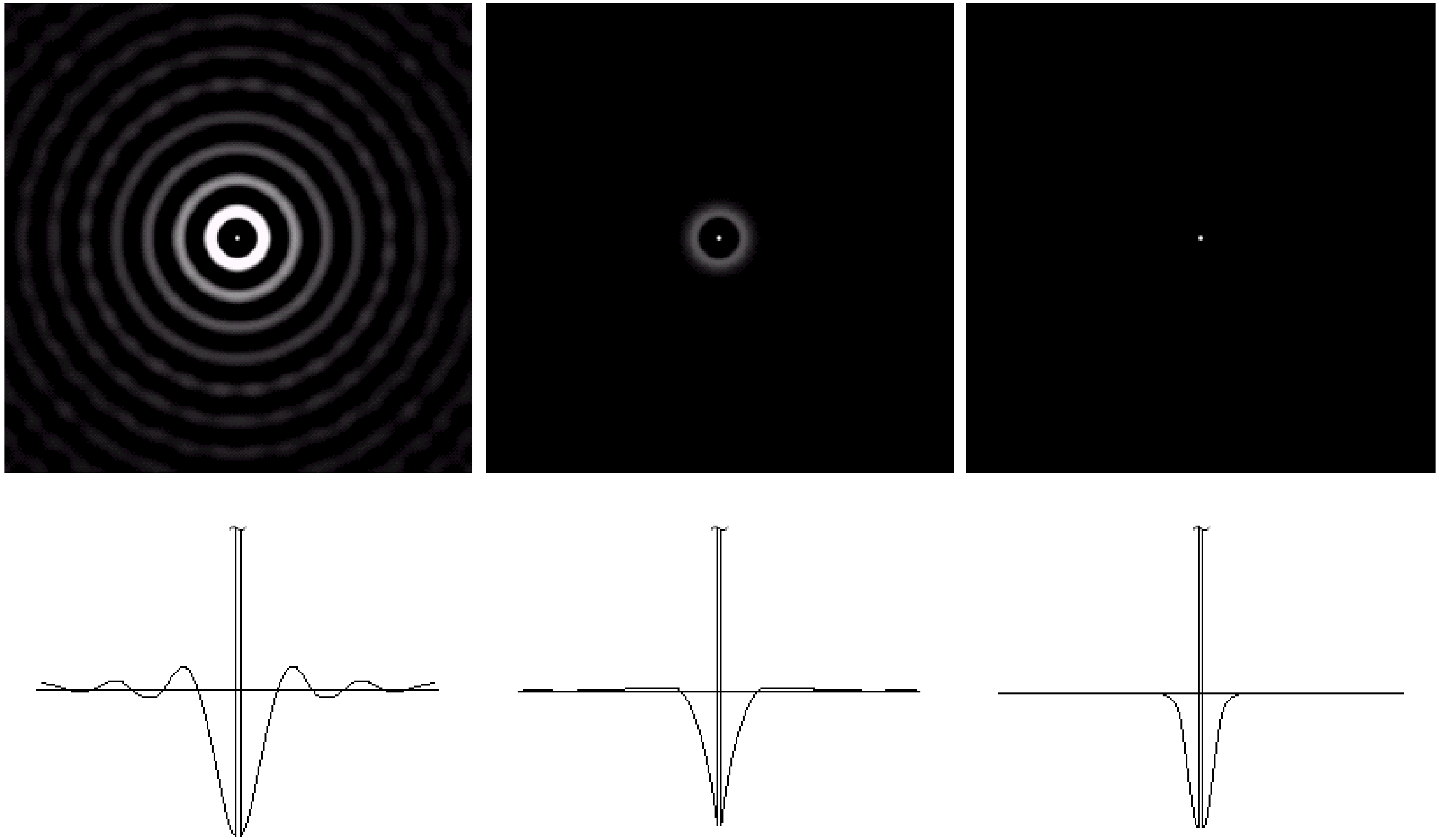


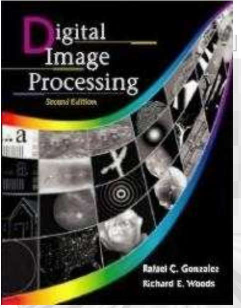
FIGURE 4.22 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

Highpass Filters
Spatial Representations



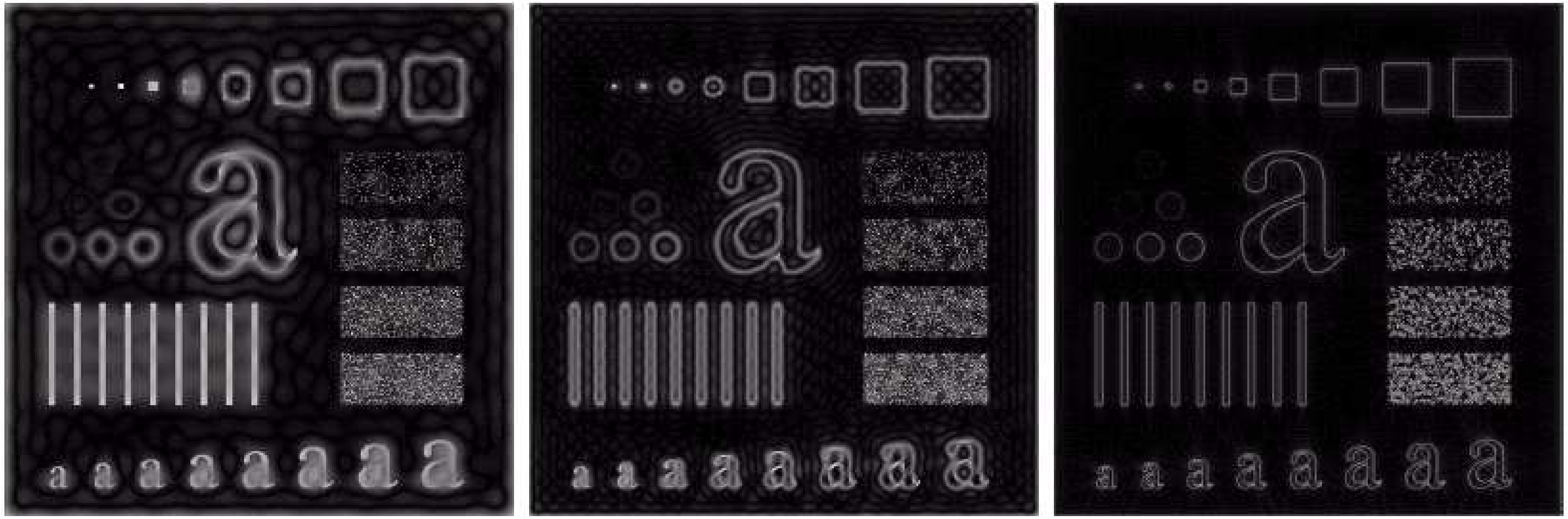
a b c

FIGURE 4.23 Spatial representations of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding gray-level profiles.



Ideal Highpass Filters

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

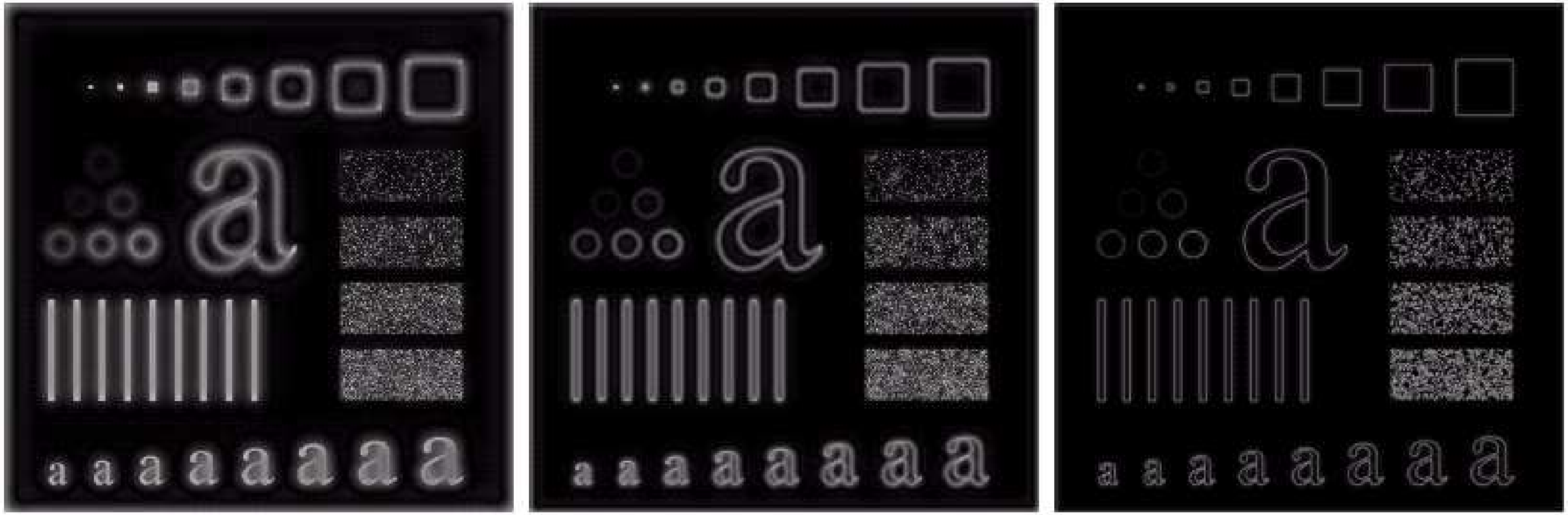


a b c

FIGURE 4.24 Results of ideal highpass filtering the image in Fig. 4.11(a) with $D_0 = 15$, 30, and 80, respectively. Problems with ringing are quite evident in (a) and (b).

Butterworth Highpass Filters

$$H(u,v) = \frac{1}{1 + \left[D_0 / D(u,v) \right]^{2n}}$$

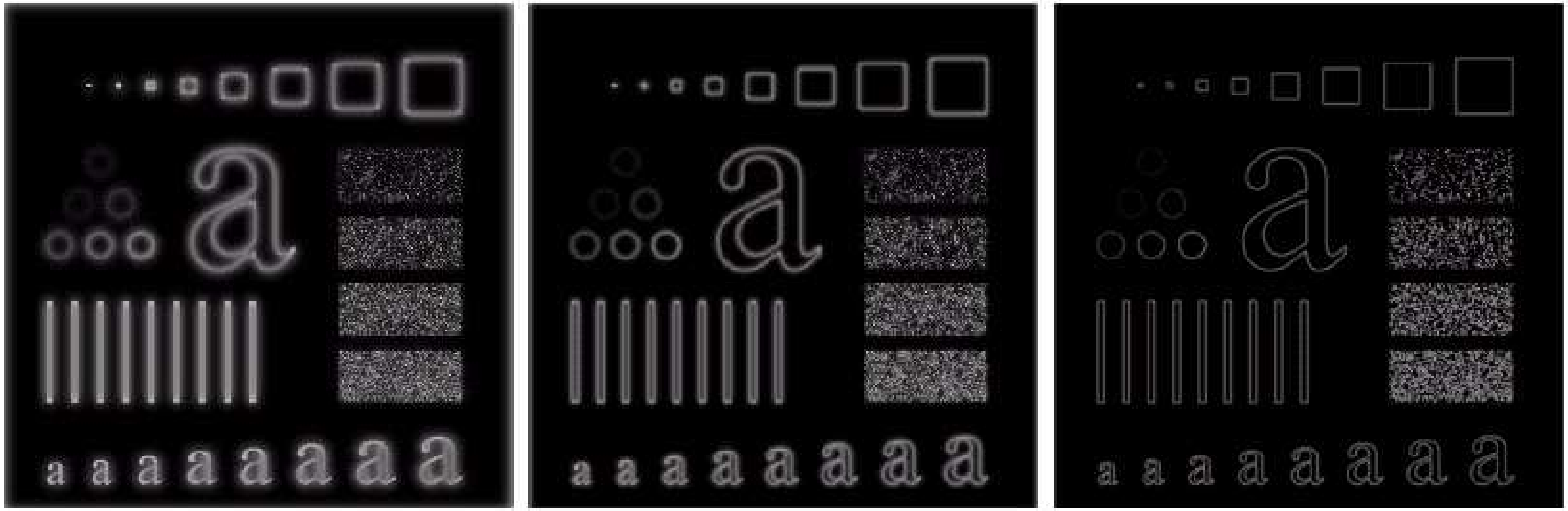


a b c

FIGURE 4.25 Results of highpass filtering the image in Fig. 4.11(a) using a BHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. These results are much smoother than those obtained with an ILPF.

Gaussian Highpass Filters

$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$$



a b c

FIGURE 4.26 Results of highpass filtering the image of Fig. 4.11(a) using a GHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. Compare with Figs. 4.24 and 4.25.

Implementation
Some Additional Properties of the 2D Fourier Transform

- Separability

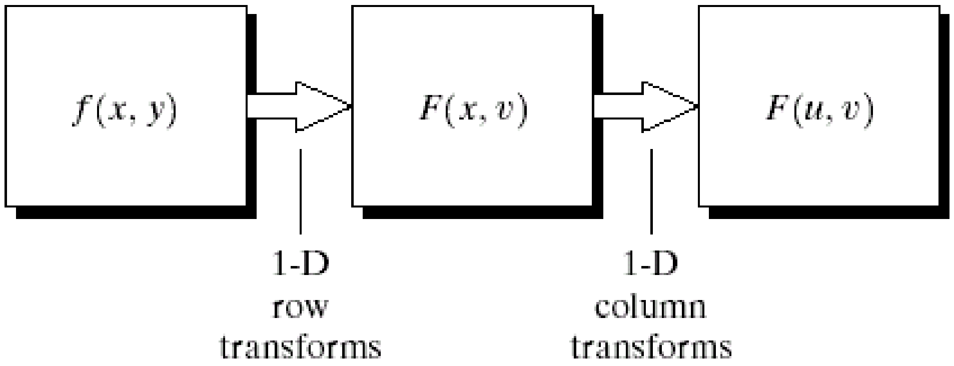
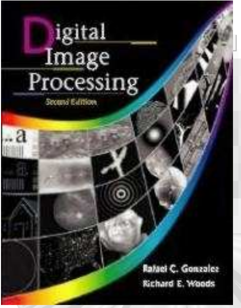


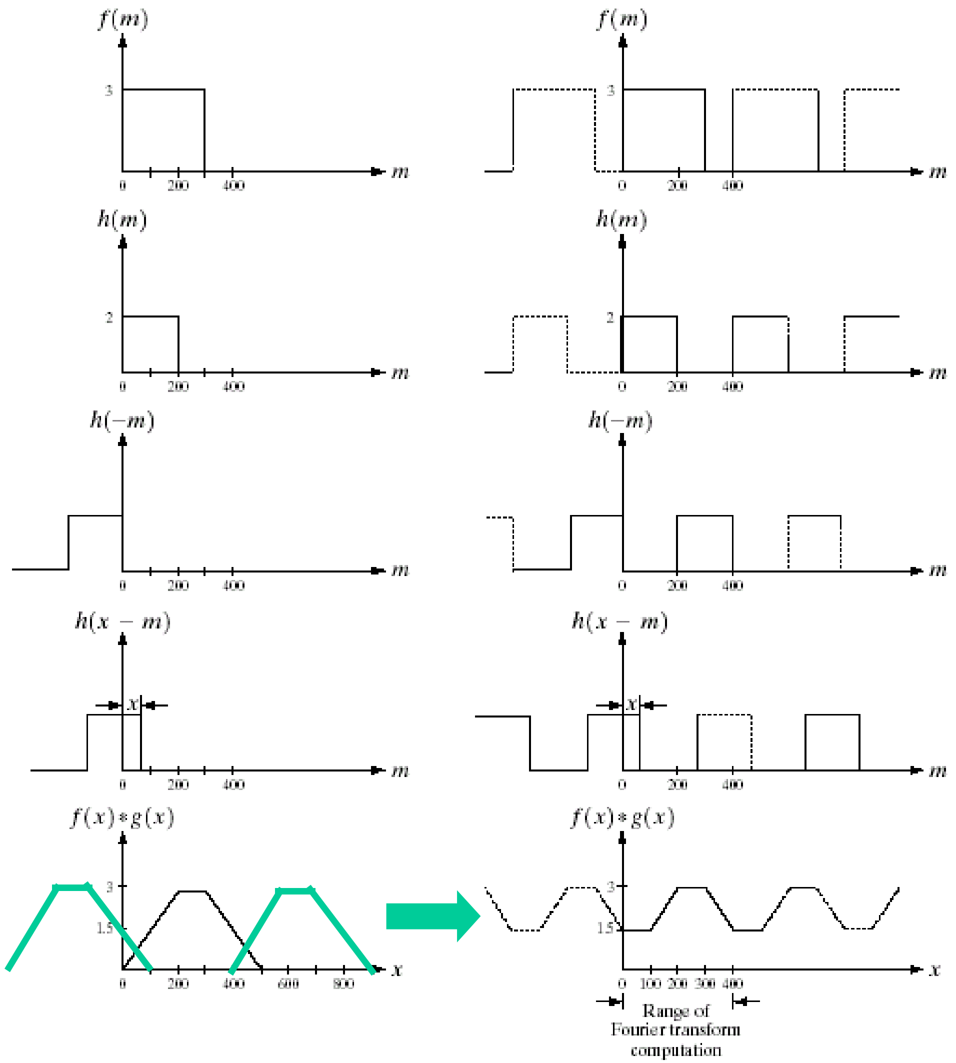
FIGURE 4.35 Computation of the 2-D Fourier transform as a series of 1-D transforms.



Implementation
More on Periodicity

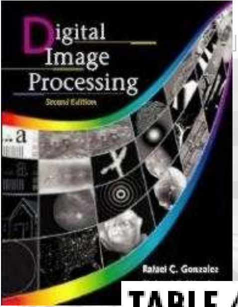
Convolution

$$f(x) * h(x) = \frac{1}{M} \sum_{m=0}^{M-1} f(m) h(x-m)$$



a f
b g
c h
d i
e j

FIGURE 4.36 Left: convolution of two discrete functions. Right: convolution of the same functions, taking into account the implied periodicity of the DFT. Note in (j) how data from adjacent periods corrupt the result of convolution.



Summary of Some Important Properties of the 2-D Fourier Transform

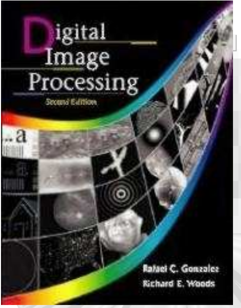
TABLE 4.1 Summary of some important properties of the 2-D Fourier transform.

Property	Expression(s)
Fourier transform	$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$
Inverse Fourier transform	$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$
Polar representation	$F(u,v) = \lvert F(u,v)\rvert e^{-j\phi(u,v)}$
Spectrum	$\lvert F(u,v)\rvert = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$, $R = \text{Real}(F)$ and $I = \text{Imag}(F)$
Phase angle	$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$
Power spectrum	$P(u,v) = \lvert F(u,v)\rvert^2$
Average value	$\bar{f}(x,y) = F(0,0) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$
Translation	$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$ $f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$ When $x_0 = u_0 = M/2$ and $y_0 = v_0 = N/2$, then $f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$ $f(x-M/2, y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$

Summary of Some Important Properties of the 2-D Fourier Transform

TABLE 4.1
(continued)

Conjugate symmetry	$F(u,v) = F^*(-u,-v)$ $\lvert F(u,v)\rvert = \lvert F(-u,-v)\rvert$
Differentiation	$\dfrac{\partial^n f(x,y)}{\partial x^n} \Leftrightarrow (ju)^n F(u,v)$ $(-jx)^n f(x,y) \Leftrightarrow \dfrac{\partial^n F(u,v)}{\partial u^n}$
Laplacian	$\nabla^2 f(x,y) \Leftrightarrow -(u^2+v^2)F(u,v)$
Distributivity	$\Im[f_1(x,y)+f_2(x,y)] = \Im[f_1(x,y)] + \Im[f_2(x,y)]$ $\Im[f_1(x,y)\cdot f_2(x,y)] \neq \Im[f_1(x,y)]\cdot\Im[f_2(x,y)]$
Scaling	$af(x,y) \Leftrightarrow aF(u,v),\ f(ax,by) \Leftrightarrow \dfrac{1}{\lvert ab\rvert}F(u/a,v/b)$
Rotation	$x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$ $f(r,\theta+\theta_0) \Leftrightarrow F(\omega,\varphi+\theta_0)$
Periodicity	$F(u,v) = F(u+M,v) = F(u,v+N) = F(u+M,v+N)$ $f(x,y) = f(x+M,y) = f(x,y+N) = f(x+M,y+N)$
Separability	See Eqs. (4.6-14) and (4.6-15). Separability implies that we can compute the 2-D transform of an image by first computing 1-D transforms along each row of the image, and then computing a 1-D transform along each column of this intermediate result. The reverse, columns and then rows, yields the same result.



Summary of Some Important Properties of the 2-D Fourier Transform

TABLE 4.1 *(continued)*

Property	Expression(s)
Computation of the inverse Fourier transform using a forward transform algorithm	$\frac{1}{MN} f^*(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F^*(u, v) e^{-j2\pi(ux/M + vy/N)}$ This equation indicates that inputting the function $F^*(u, v)$ into an algorithm designed to compute the forward transform (right side of the preceding equation) yields $f^*(x, y)/MN$. Taking the complex conjugate and multiplying this result by MN gives the desired inverse.
Convolution†	$f(x, y) * h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) h(x - m, y - n)$
Correlation†	$f(x, y) \circ h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n) h(x + m, y + n)$
Convolution theorem†	$f(x, y) * h(x, y) \Leftrightarrow F(u, v) H(u, v)$; $f(x, y) h(x, y) \Leftrightarrow F(u, v) * H(u, v)$
Correlation theorem†	$f(x, y) \circ h(x, y) \Leftrightarrow F^*(u, v) H(u, v)$; $f^*(x, y) h(x, y) \Leftrightarrow F(u, v) \circ H(u, v)$

Summary of Some Important Properties of the 2-D Fourier Transform

TABLE 4.1
(continued)

Some useful FT pairs:

Impulse	$\delta(x, y) \Leftrightarrow 1$
Gaussian	$A\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2(x^2+y^2)} \Leftrightarrow Ae^{-(u^2+v^2)/2\sigma^2}$
Rectangle	$\mathrm{rect}[a, b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
Cosine	$\cos(2\pi u_0x + 2\pi v_0y) \Leftrightarrow \dfrac{1}{2}[\delta(u + u_0, v + v_0) + \delta(u - u_0, v - v_0)]$
Sine	$\sin(2\pi u_0x + 2\pi v_0y) \Leftrightarrow j\dfrac{1}{2}[\delta(u + u_0, v + v_0) - \delta(u - u_0, v - v_0)]$

† Assumes that functions have been extended by zero padding.