Exercise Set 1

Graphs, Isomorphism, Paths and Trees

Answer all questions.

1. If G is a simple graph with at least two vertices, prove that G must contain two or more vertices of the same degree.
2. Prove that any two simple connected graphs with n vertices, all of degree two, are isomorphic.
3. Let G be a simple graph on n vertices. If G has k components, then the number m of edges of G satisfies
$$n - k \leq m \leq (n-k)(n-k+1)/2.$$
4. Prove that a simple graph with n vertices and more than $(n-1)(n-2)/2$ edges must be connected.
5. A graph G is said to be **bipartite** if the vertex set of G can be split into two disjoint sets A and B so that each edge of G joins a vertex of A and a vertex of B. Prove that every circuit in a bipartite graph has even length.
6. If G is a simple graph with vertex set V, its **complement** $\overline{G}$ is the simple graph with same vertex set V in which two vertices are adjacent if and only if they are **not** adjacent in G. Prove that a simple graph and its complement can not both be disconnected.
7. A simple graph that is isomorphic to its complement is **self-complementary**. It can be shown every self-complementary graph G has $4k$ or $4k+1$ vertices, where k is an integer. Find all self-complementary graphs with 4 and 5 vertices.
8. A round-robin tournament among n players (n being even) can be represented by a complete graph of n vertices. Discuss how you would schedule the tournament to finish in shortest possible time.
9. A set subset E' of edges of a graph G is **independent** if E' contains no circuit of G. Prove the following:
   (a) any subset of an independent set is independent;
   (b) if I and J are independent sets of edges with $|J| > |I|$, then there is an edge e that lies in J but not in I with the property that $I \cup \{e\}$ is independent.
10. The **line graph** $L(G)$ of a graph G is the graph whose vertices are in one-to-one correspondence with the edges of G such that two vertices of $L(G)$ being adjacent if and only if the corresponding edges of G are adjacent. Prove that if G is Eulerian graph then $L(G)$ is also Eulerian.
11. Prove that, if G is a bipartite graph with an odd number of vertices, then G is non-Hamiltonian.

12. Draw a graph in which an Euler line is also a Hamiltonian circuit. What can we say about such graphs in general ?

13. Is it possible, starting from any of the 64 squares of the chessboard, to move a knight such that it occupies every square exactly once and returns to the initial position ? If so, give one such tour.

14. It can be shown that there are only six different (non-isomorphic) trees of six vertices. Draw these six trees.

15. For a tree radius is defined as the eccentricity of the center(s). Also, diameter is defined as the length of the longest path in the tree. Show a tree in which its diameter is not equal to twice its radius. Under what condition does this inequality hold ? Elaborate.

16. Sketch all (unlabeled) binary trees with 6 pendant vertices. Find the path length of each.

17. Prove that a pendant edge (an edge whose one end vertex has degree 1) in a connected graph G is contained in every spanning tree of G.

18. Prove that any subgraph g of a connected graph G is contained in some spanning tree of G if and only if g contains no circuit.

19. Prove that any circuit in a graph G must have at least one edge common with a chord set.

20. Prove that a connected graph G is a tree of and only if adding an edge between any two vertices in G creates exactly one circuit.

21. Prove that the nullity of a graph does not change when you either insert a vertex in the middle of an edge, or remove a vertex of degree two by merging two edges incident on it.

22. Let T_1 and T_2 be two spanning trees of a connected graph G. If edge e is in T_1 but not in T_2, prove that there exists another edge f in T_2 but not in T_1 such that the subgraphs $(T_1 - e) \cup f$ and $(T_2 - f) \cup e$ are also spanning trees of G.

23. Construct a tree graph of a labeled complete graph of 4 vertices.