# Topics

- Probability
- Conditional Probability
- Bayes Rule
- HMM tagging
- Markov Chains
- Hidden Markov Models

# Introduction to Probability

- Experiment (trial)
  - Repeatable procedure with well-defined possible outcomes
- Sample Space (S)
    - the set of all possible outcomes
    - *finite or infinite*
  - Example
    - coin toss experiment
    - possible outcomes: S = {heads, tails}
  - Example
    - die toss experiment
    - possible outcomes: S = {1,2,3,4,5,6}

# Introduction to Probability

- Definition of sample space depends on what we are asking
  - Sample Space (S): the set of all possible outcomes
  - Example
    - die toss experiment for whether the number is even or odd
    - possible outcomes: {even,odd}
    - *not* {1,2,3,4,5,6}

# More definitions

- Events
  - an ***event*** is any subset of outcomes from the ***sample space***
- Example
  - die toss experiment
  - let A represent the event such that the outcome of the die toss experiment is divisible by 3
  - $A = \{3,6\}$
  - A is a subset of the sample space $S= \{1,2,3,4,5,6\}$

# Definition of Probability

- The probability law assigns to an event a nonnegative number
- Called P(A)
- Also called the probability A
- That encodes our knowledge or belief about the collective likelihood of all the elements of A
- Probability law must satisfy certain properties

# Probability Axioms

- Nonnegativity
  - P(A) >= 0, for every event A
- Additivity
  - If A and B are two disjoint events, then the probability of their union satisfies:
  - P(A U B) = P(A) + P(B)
- Normalization
  - The probability of the entire sample space S is equal to 1, i.e. P(S) = 1.

# An example

- An experiment involving a single coin toss

  There are two possible outcomes, H and T

  Sample space S is {H,T}

  If coin is fair, should assign equal probabilities to {H,T} outcomes

  Since they have to sum to 1

  $P(\{H\}) = 0.5$ and $P(\{T\}) = 0.5$

  $P(\{H,T\}) = P(\{H\})+P(\{T\}) = 1.0$

# Another example

- Experiment involving 3 coin tosses
- Outcome is a 3-long string of H or T
- S ={HHH,HHT,HTH,HTT,THH,THT,TTH,TTT}
- Assume each outcome is equiprobable
  - “Uniform distribution”
- What is probability of the event that exactly 2 heads occur?

  A = {HHT,HTH,THH} 3 events

  P(A) = P({HHT})+P({HTH})+P({THH}) union of the prob of individual events

  = 1/8 + 1/8 + 1/8

  = 3/8

# Probability definitions

- In summary:

$$P(E) = \frac{\text{number of outcomes corresponding to event E}}{\text{total number of outcomes}}$$

Probability of drawing a spade from 52 well-shuffled playing cards:

$$\frac{13}{52} = \frac{1}{4} = 0.25$$

# Probability and part of speech tags

- What's the probability of a random word (from a random dictionary page) being a verb?

$$P(drawingaverb) = \frac{ofwaystogetaverb}{allwords}$$

- How to compute each of these?

  All words = just count all the words in the dictionary

  # of ways to get a verb: # of words which are verbs!

  If a dictionary has 50,000 entries, and 10,000 are verbs….

  P(V) is 10000/50000 = 1/5 = .20

# Conditional Probability

- A way to reason about the outcome of an experiment based on partial information
  - In a word guessing game the first letter for the word is a “t”. What is the likelihood that the second letter is an “h”?
  - How likely is it that a person has a disease given that a medical test was negative?
  - A spot shows up on a radar screen. How likely is it that it corresponds to an aircraft?

# An intuition

- Let's say A is "it's raining".
- Let's say $P(A)$ in dry Florida is .01
- Let's say B is "it was sunny ten minutes ago"
- $P(A|B)$ means "what is the probability of it raining now if it was sunny 10 minutes ago"
- $P(A|B)$ is probably way less than $P(A)$
- Perhaps $P(A|B)$ is .0001
- Intuition: The knowledge about B should change our estimate of the probability of A.

# More precisely

- Given an experiment, a corresponding sample space S, and a probability law
- Suppose we know that the outcome is some event B
- We want to quantify the likelihood that the outcome also belongs to some other event A
- We need a new probability law that gives us the conditional probability of A given B
- $P(A|B)$

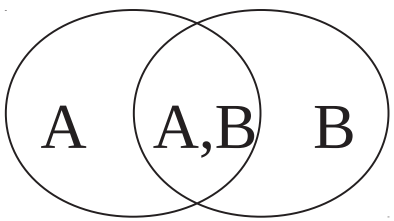
# Conditional Probability

- let A and B be events in the sample space
- $P(A|B)$ = the conditional *probability* of event A *occurring given* some fixed event B *occurring*
- *definition:* $P(A|B) = P(A \cap B) / P(B)$

# Conditional probability

- $P(A|B) = P(A \cap B)/P(B)$
- Or

$$P(A|B) = \frac{P(A,B)}{P(B)}$$




*Note:* $P(A,B) = P(A|B) \cdot P(B)$

*Also:* $P(A,B) = P(B,A)$

# Independence

- What is $P(A,B)$ if A and B are independent?
- $P(A,B)=P(A) \cdot P(B)$ iff A,B independent.

  $P(heads,tails) = P(heads) \cdot P(tails) = 0.5 \cdot 0.5 = 0.25$

  *Note: $P(A|B)=P(A)$ iff A,B independent*

  *Also: $P(B|A)=P(B)$ iff A,B independent*

# Bayes Theorem

$$P\left(B\middle|A\right)=\frac{P\left(A\middle|B\right)P\left(B\right)}{P\left(A\right)}$$

- Idea: The probability of an A conditional on another event B is generally different from the probability of B conditional on A. There is a definite relationship between the two.

# Deriving Bayes Rule

The probability of event A given event B is

$$P(A|B) = \frac{P(A,B)}{P(B)}$$

# Deriving Bayes Rule

The probability of event B given event A is

$$P(B|A) = \frac{P(A,B)}{P(A)}$$

# Deriving Bayes Rule

$$P(A|B)=\frac{P(A,B)}{P(B)}$$

$$P(B|A)=\frac{P(A,B)}{P(A)}$$

$$P(A|B)P(B)=P(A,B)$$

$$P(B|A)P(A)=P(A,B)$$

# Deriving Bayes Rule

$$P(A|B)=\frac{P(A,B)}{P(B)} \qquad P(B|A)=\frac{P(A,B)}{P(A)}$$

$$P(A|B)P(B)=P(A,B) \qquad P(B|A)P(A)=P(A,B)$$

$$P(A|B)P(B)=P(B|A)P(A)$$

$$P(A|B)=\frac{P(B|A)P(A)}{P(B)}$$

# Bayes Rule

$$P(A|B)=\frac{P(B|A)P(A)}{P(B)}$$

the theorem may be paraphrased as:

Conditional/Posterior probability =

(LIKELIHOOD multiplied by PRIOR) divided by NORMALIZING CONSTANT

# Hidden Markov Model (HMM) Tagging

- Using an HMM to do POS tagging
- HMM is a special case of Bayesian inference
- It is also related to the “noisy channel” model in ASR (Automatic Speech Recognition)

# POS tagging as a sequence classification task

- Given a sentence (an “observation” or “sequence of observations”)
  - *Secretariat is expected to race tomorrow*
  - sequence of n words w1…wn.
- What is the best sequence of tags which corresponds to this sequence of observations?
- Probabilistic/Bayesian view:
  - Consider all possible sequences of tags
  - Out of this universe of sequences, choose the tag sequence which is most probable given the observation sequence of n words w1…wn.

# Getting to HMM

- Let $T = t_1, t_2, \ldots, t_n$
- Let $W = w_1, w_2, \ldots, w_n$
- Goal: Out of all sequences of tags $t_1 \ldots t_n$, get the the most probable sequence of POS tags T underlying the observed sequence of words $w_1, w_2, \ldots, w_n$

$$\hat{t}_1^n = \underset{t_1^n}{\operatorname{argmax}} P(t_1^n | w_1^n)$$

- Hat ^ means "our estimate of the best = the most probable tag sequence"
- $\text{Argmax}_x f(x)$ means "the x such that f(x) is maximized"

  it maximizes our estimate of the best tag sequence

# Getting to HMM

- This equation is guaranteed to give us the best tag sequence

$$\hat{t}_1^n = \underset{t_1^n}{\operatorname{argmax}} P(t_1^n | w_1^n)$$

- But how do we make it operational? How do we compute this value?
- Intuition of Bayesian classification:
  - Use Bayes rule to transform it into a set of other probabilities that are easier to compute
  - Thomas Bayes: British mathematician (1702-1761)

# Bayes Rule

$$P(x|y) = \frac{P(y|x)P(x)}{P(y)}$$

Breaks down any conditional probability $P(x|y)$ into three other probabilities

$P(x|y)$: The conditional probability of an event x assuming that y has occurred

# Bayes Rule

$$P(x|y) = \frac{P(y|x)P(x)}{P(y)}$$

$$\hat{t}_1^n = \operatorname*{argmax}_{t_1^n} \frac{P(w_1^n|t_1^n)P(t_1^n)}{P(w_1^n)}$$

We can drop the denominator: it does not change for each tag sequence; we are looking for the best tag sequence for the same observation, for the same fixed set of words

# Bayes Rule

$$P(x|y) = \frac{P(y|x)P(x)}{P(y)}$$

$$\hat{t}_1^n = \operatorname*{argmax}_{t_1^n} \frac{P(w_1^n|t_1^n)P(t_1^n)}{P(w_1^n)}$$

$$\hat{t}_1^n = \operatorname*{argmax}_{t_1^n} P(w_1^n|t_1^n)P(t_1^n)$$

# Likelihood and prior

$$\hat{t}_1^n = \operatorname*{argmax}_{t_1^n} \overbrace{P(w_1^n|t_1^n)}^{\text{likelihood}} \; \overbrace{P(t_1^n)}^{\text{prior}}$$

# Likelihood and prior

## Further Simplifications

1. the probability of a word appearing depends only on its own POS tag, i.e, independent of other words around it

$$P(w_1^n|t_1^n) \approx \prod_{i=1}^{n} P(w_i|t_i)$$

2. **BIGRAM** assumption: the probability of a tag appearing depends only on the previous tag

$$P(t_1^n) \approx \prod_{i=1}^{n} P(t_i|t_{i-1})$$

3. The most probable tag sequence estimated by the bigram tagger

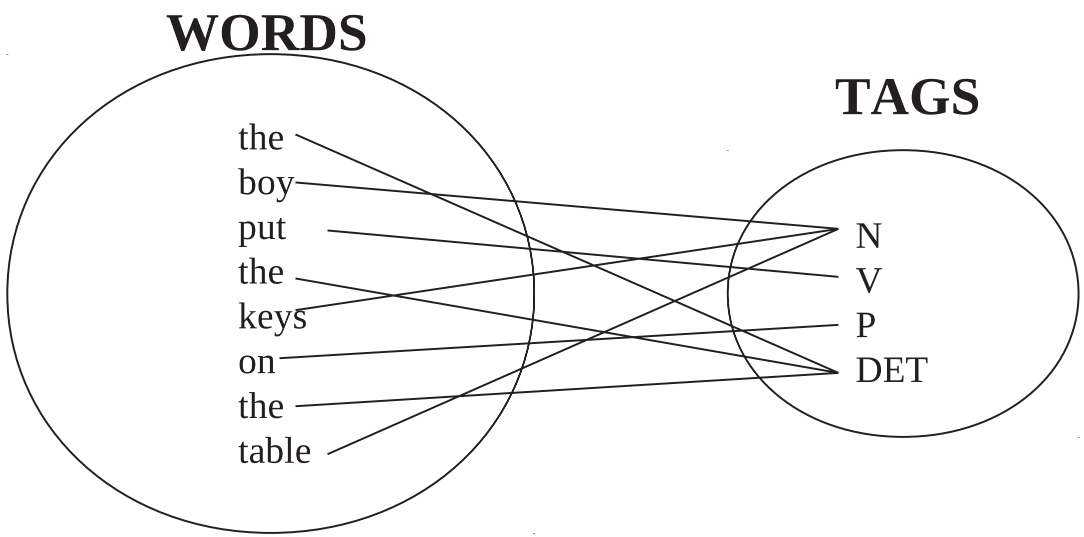
$$\hat{t}_1^n = \underset{t_1^n}{\operatorname{argmax}} P(t_1^n|w_1^n) \approx \underset{t_1^n}{\operatorname{argmax}} \prod_{i=1}^{n} P(w_i|t_i)P(t_i|t_{i-1})$$

# Likelihood ratio

## Further Simplifications

1. the probability of a word appearing depends only on its own POS tag, i.e, independent of other words around it

$$P(w_1^n|t_1^n) \approx \prod_{i=1}^{n} P(w_i|t_i)$$



# Prior probability
## Further Simplifications

2. **BIGRAM** assumption: the probability of a tag appearing depends only on the previous tag

$$P(t_1^n) \approx \prod_{i=1}^{n} P(t_i|t_{i-1})$$

Bigrams are groups of two written letters, two syllables, or two words; they are a special case of N-gram.

Bigrams are used as the basis for simple statistical analysis of text

The bigram assumption is related to the first-order Markov assumption

# Likelihood and prior

## Further Simplifications

3. The most probable tag sequence estimated by the bigram tagger

$$\hat{t}_1^n = \underset{t_1^n}{\operatorname{argmax}} P(t_1^n|w_1^n) \approx \underset{t_1^n}{\operatorname{argmax}} \prod_{i=1}^{n} P(w_i|t_i)P(t_i|t_{i-1})$$

$$\hat{t}_1^n = \underset{t_1^n}{\operatorname{argmax}} \overbrace{P(w_1^n|t_1^n)}^{\text{likelihood}} \overbrace{P(t_1^n)}^{\text{prior}}$$

$$P(w_1^n|t_1^n) \approx \prod_{i=1}^{n} P(w_i|t_i)$$

$$P(t_1^n) \approx \prod_{i=1}^{n} P(t_i|t_{i-1})$$

bigram assumption

# Two kinds of probabilities (1)

- Tag transition probabilities $p(t_i|t_{i-1})$
  - Determiners likely to precede adjs and nouns
    - That/DT flight/NN
    - The/DT yellow/JJ hat/NN
    - So we expect P(NN|DT) and P(JJ|DT) to be high
    - But P(DT|JJ) to be:?

# Two kinds of probabilities (1)

- Tag transition probabilities $p(t_i|t_{i-1})$
  - Compute P(NN|DT) by counting in a labeled corpus:

$$P(t_i|t_{i-1}) = \frac{C(t_{i-1}, t_i)}{C(t_{i-1})}$$

# of times DT is followed by NN

$$P(NN|DT) = \frac{C(DT, NN)}{C(DT)} = \frac{56,509}{116,454} = .49$$

# Two kinds of probabilities (2)

- Word likelihood probabilities $p(w_i|t_i)$
  - P(is|VBZ) = probability of VBZ (3sg pres verb) being "is"

    If we were expecting a third person singular verb, how likely is it that this verb would be *is*?

  - Compute P(is|VBZ) by counting in a labeled corpus:

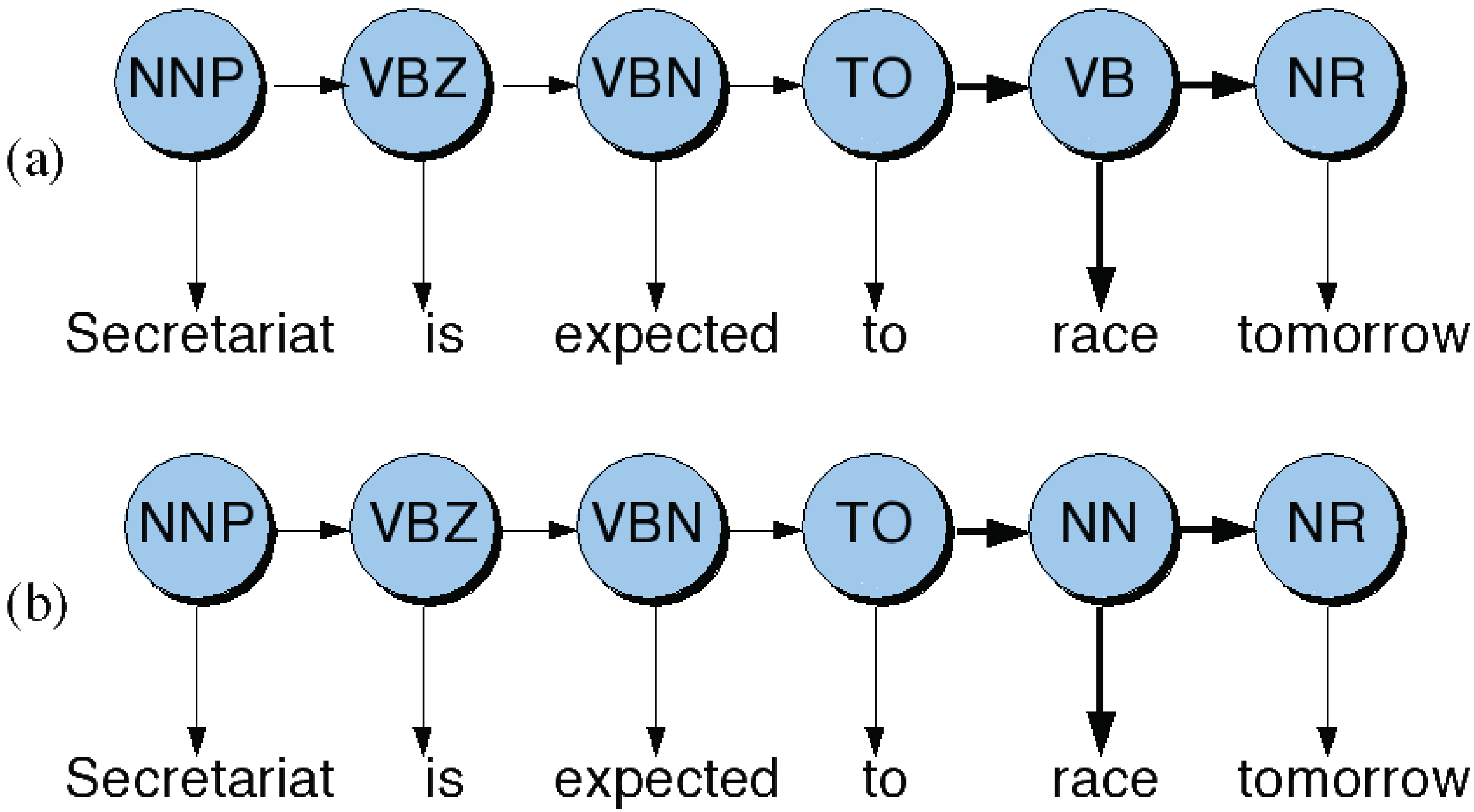
$$P(w_i|t_i) = \frac{C(t_i, w_i)}{C(t_i)}$$

$$P(is|VBZ) = \frac{C(VBZ, is)}{C(VBZ)} = \frac{10,073}{21,627} = .47$$

# An Example: the verb “race”

- Secretariat/NNP is/VBZ expected/VBN to/TO **race**/VB tomorrow/NR
- People/NNS continue/VB to/TO inquire/VB the/DT reason/NN for/IN the/DT **race**/NN for/IN outer/JJ space/NN
- How do we pick the right tag?

# Disambiguating “race”



# Disambiguating “race”

- $P(NN|TO) = .00047$

  $P(VB|TO) = .83$

  The tag transition probabilities $P(NN|TO)$ and $P(VB|TO)$:

  ‘How likely are we to expect verb/noun given the previous tag TO?’

- $P(race|NN) = .00057$

  $P(race|VB) = .00012$

  Lexical likelihoods from the Brown corpus for ‘race’ given a POS tag NN or VB.

# Disambiguating “race”

- P(NR|VB) = .0027

  P(NR|NN) = .0012

  tag sequence probability of an adverb occurring given the previous

  tag verb / noun

P(VB|TO)P(NR|VB)P(race|VB) = .83 x .0027 x .00012 = .00000027

P(NN|TO)P(NR|NN)P(race|NN)= .

00047x.0012x.00057=.00000000032

Multiply the lexical likelihoods with the tag sequence probabilities:

Hence the *race* is tagged as **verb** !

# Hidden Markov Models

- What we’ve described with these two kinds of probabilities is a Hidden Markov Model (HMM)
- Let’s just spend a bit of time tying this into the model
- In order to define HMM, we will first introduce the Markov Chain, or observable Markov Model.

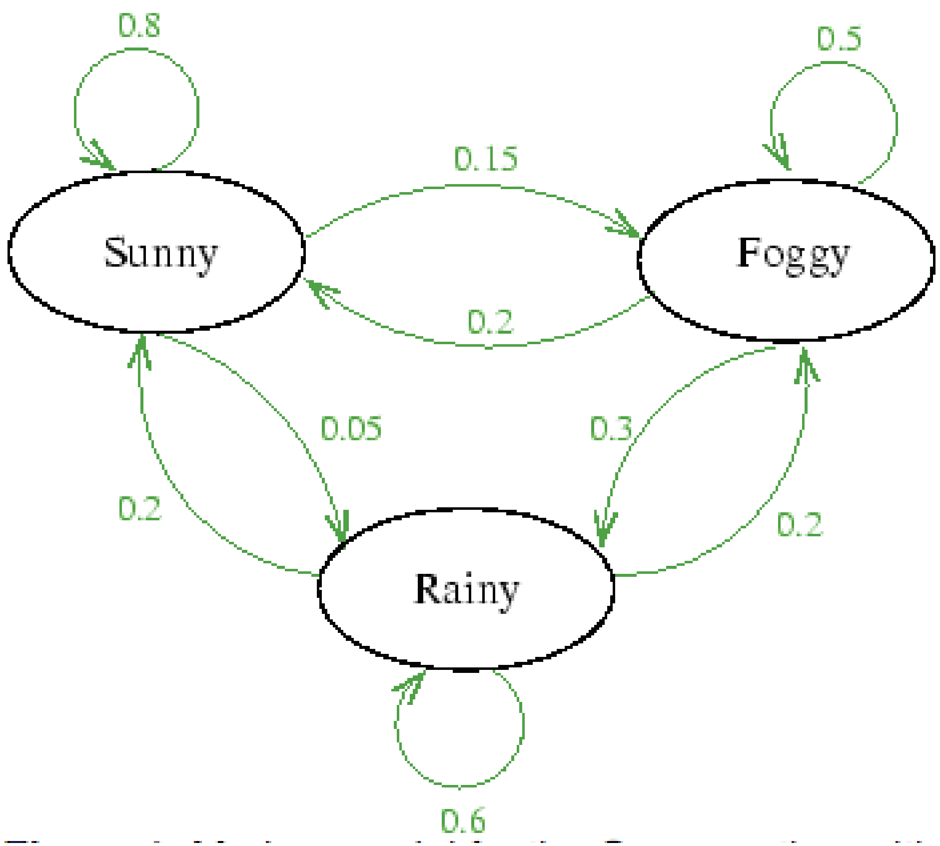
# Definitions

- A weighted finite-state automaton adds probabilities to the arcs
  - The sum of the probabilities leaving any arc must sum to one
- A Markov chain is a special case of a WFST in which the input sequence uniquely determines which states the automaton will go through
- Markov chains can't represent inherently ambiguous problems
  - Useful for assigning probabilities to unambiguous sequences

# Markov chain = “First-order observable Markov Model”

**Table 1:** Probabilities $p(q_{n+1}|q_n)$ of tomorrow's weather based on today's weather

| | Tomorrow's weather | | |
|---|---|---|---|
| Today's weather | | | |
| | 0.8 | 0.05 | 0.15 |
| | 0.2 | 0.6 | 0.2 |
| | 0.2 | 0.3 | 0.5 |



# Hidden Markov Model

- For Markov chains, the output symbols are the same as the states.
  - See **rainy** weather: we're in state **rainy**
- But in part-of-speech tagging (and other things)
  - The output symbols are **words**
  - But the hidden states are **part-of-speech tags**
- So we need an extension!
- A Hidden Markov Model is an extension of a Markov chain that allows both observed events (like words as input) and hidden events (like pos tags)

# Hidden Markov Model

- States $Q = q_1, q_2 \ldots q_N;$
- Observations $O = o_1, o_2 \ldots o_N;$
  - Each observation is a symbol from a vocabulary $V = \{v_1, v_2, \ldots v_V\}$
- Transition probabilities (prior)
  - Transition probability matrix $A = \{a_{ij}\}$. Probability of moving from state *i* to state *j*
- Observation likelihoods (likelihood)
  - probability matrix $B = \{b_i(o_t)\}$

    a sequence of observation likelihoods, each expressing the probability of an observation $o_t$ being generated from a state *i*.
- A special start and end state

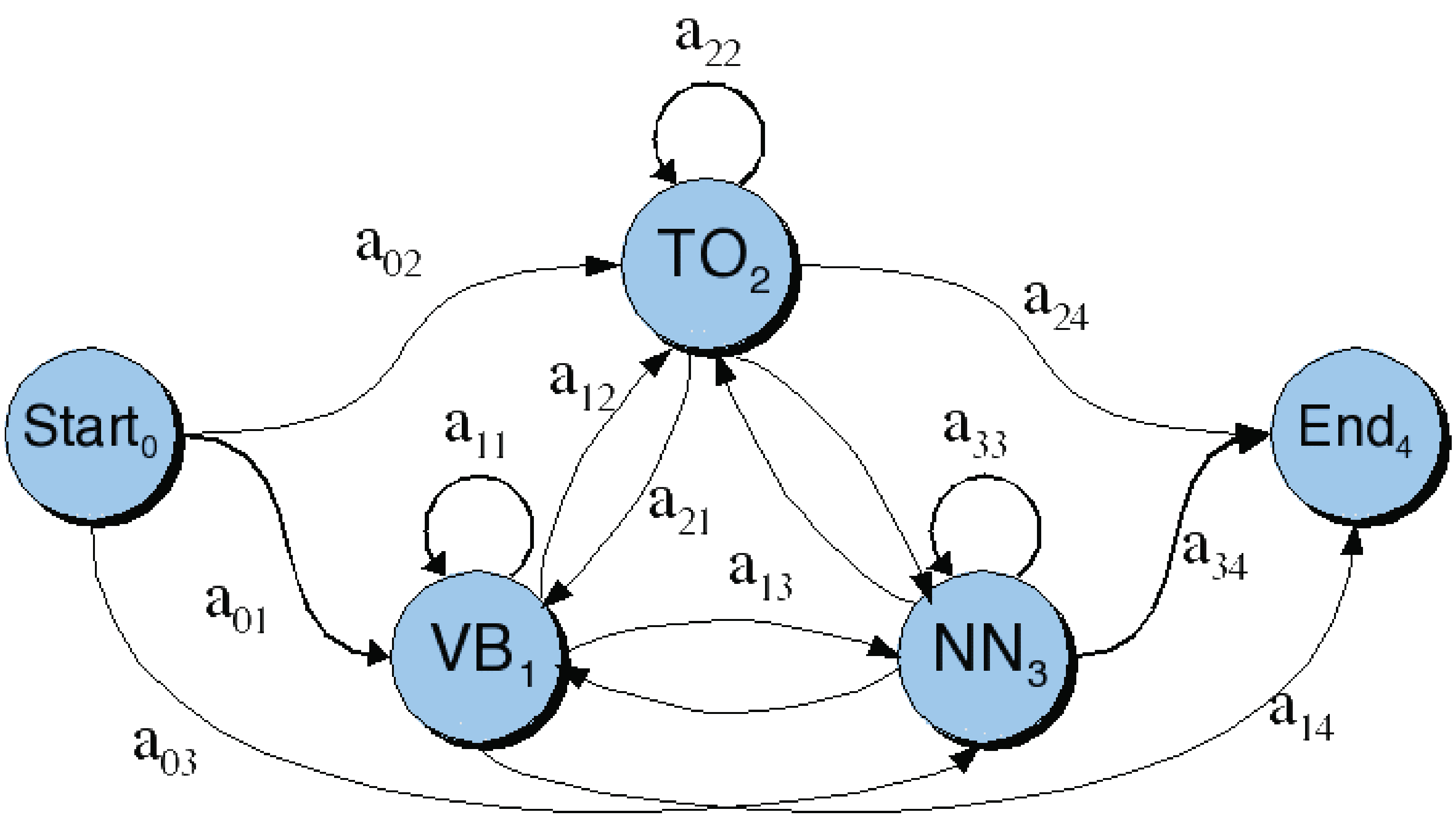
# HMM Taggers

- An HMM has two kinds of probabilities
  - A transition probabilities (PRIOR) (slide 35)
  - B observation likelihoods (LIKELIHOOD) (slide 35)
- HMM Taggers choose the tag sequence which maximizes the product of word likelihood and tag sequence probability

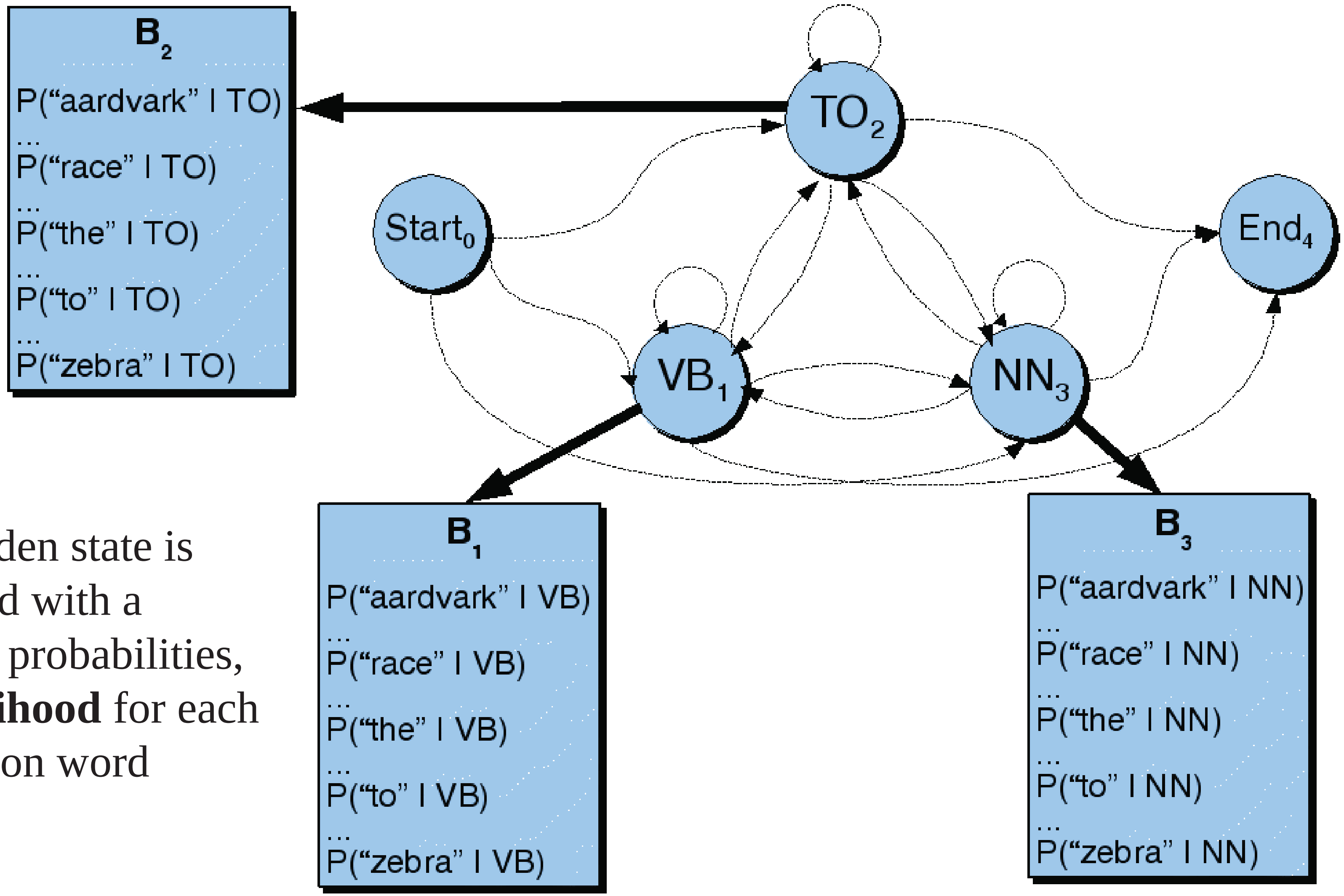
$$\hat{t}_1^n = \underset{t_1^n}{\operatorname{argmax}} \overbrace{P(w_1^n|t_1^n)}^{\text{likelihood}} \overbrace{P(t_1^n)}^{\text{prior}}$$

# Markov chain corresponding to hidden states of HMM, showing A probs

Transition probabilities are used to compute **prior probability**



# B observation likelihoods for HMM




Each hidden state is associated with a vector of probabilities, one **likelihood** for each observation word

# Next Time

- Transformation-Based Tagging