# Image Restoration in the Presence of Noise

# Outline

- A model of the image degradation / restoration process
- Noise models
- Restoration in the presence of noise only – spatial filtering
- Periodic noise reduction by frequency domain filtering
- Linear, position-invariant degradations
- Estimating the degradation function
- Inverse filtering

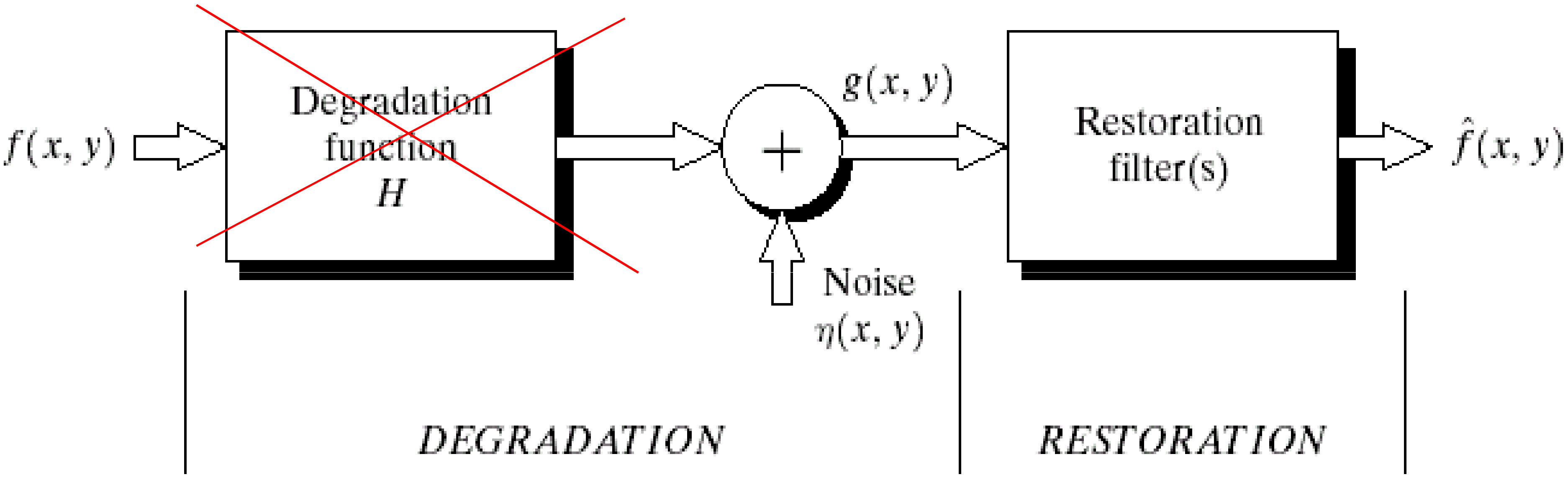
# Additive Noise only

- When the only degradation present in an image is noise, then

  $g(x,y)=f(x,y)+\eta(\mathbf{x,y})$

- G(u,v) = F(u,v) + N(u,v)
- Spatial filtering →additive noise

# Additive noise only




$$\begin{cases} g(x,y)=f(x,y)+\eta(x,y) \\ G(u,v)=F(u,v)+N(u,v) \end{cases}$$

# Spatial filters for de-noising additive noise

- Skills similar to image enhancement
- Mean filters
- Order-statistics filters
- Adaptive filters

# Mean filters

- Arithmetic mean

$$\hat{f}(x,y) = \frac{1}{mn} \sum_{(s,t)\in S_{xy}} g(s,t)$$

Window centered at (x,y)

•Let Sxy represent the set of coordinates in a rectangular window of size m*n
•Arithmetic mean computes the average value of the corrupted image g(x,y) in the area defined by Sxy
•The Arithmetic mean smoothes the image but it also blurs the image

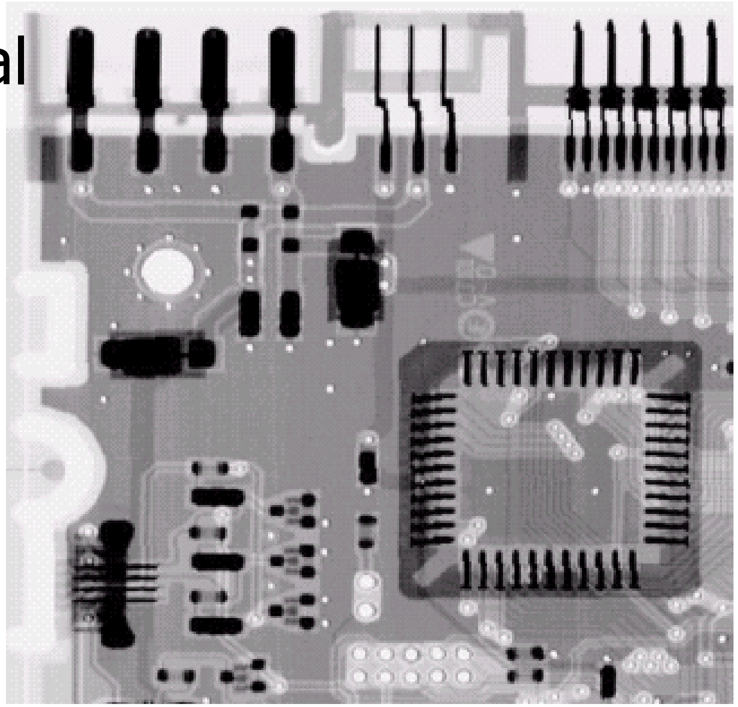
# Mean Filters

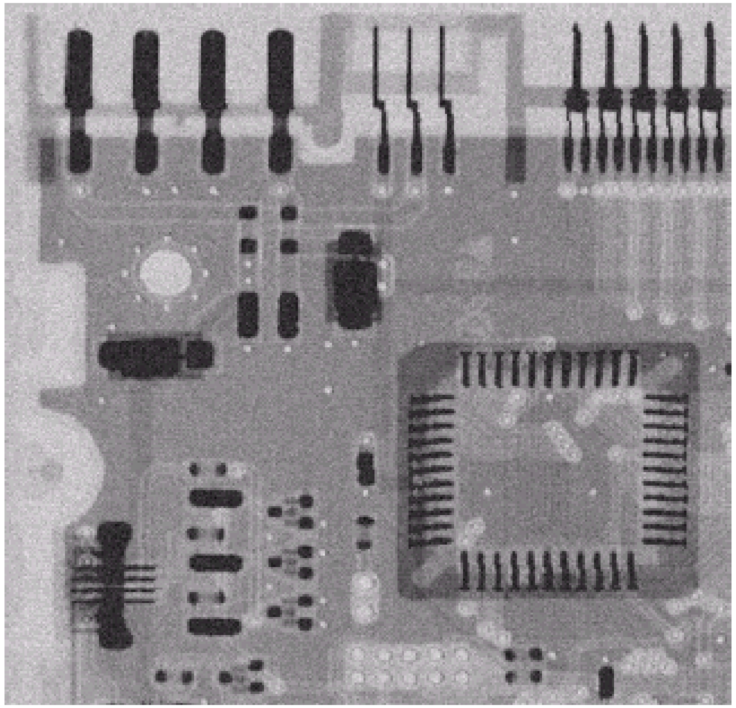
Geometric mean

$$\hat{f}(x,y) = \left[ \prod_{(s,t)\in S_{xy}} g(s,t) \right]^{1/mn}$$

- Each restored pixel is given by the product of the pixels in the subimage window raised to the power 1/mn
- Achieves smoothening than arithmetic mean filter
- There is a chance of loosing the image details.

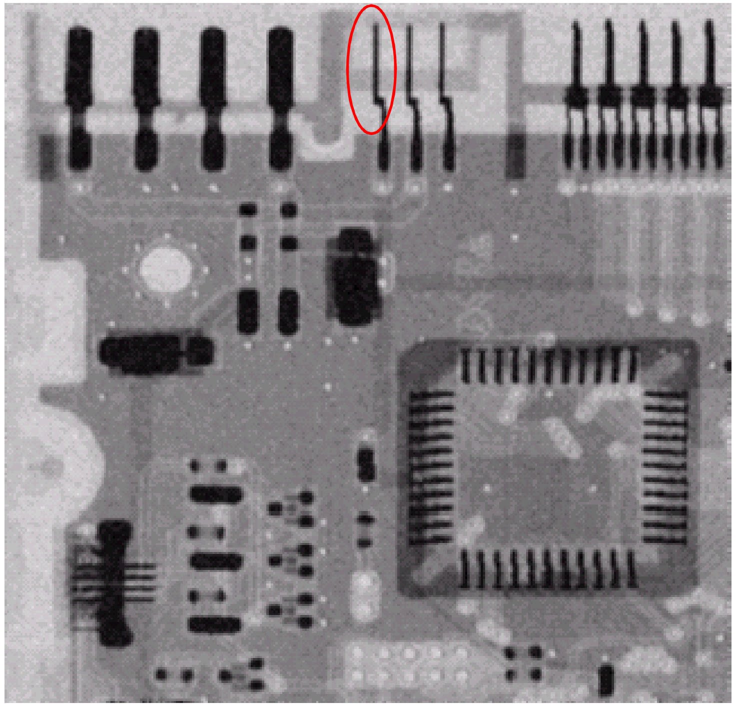
original

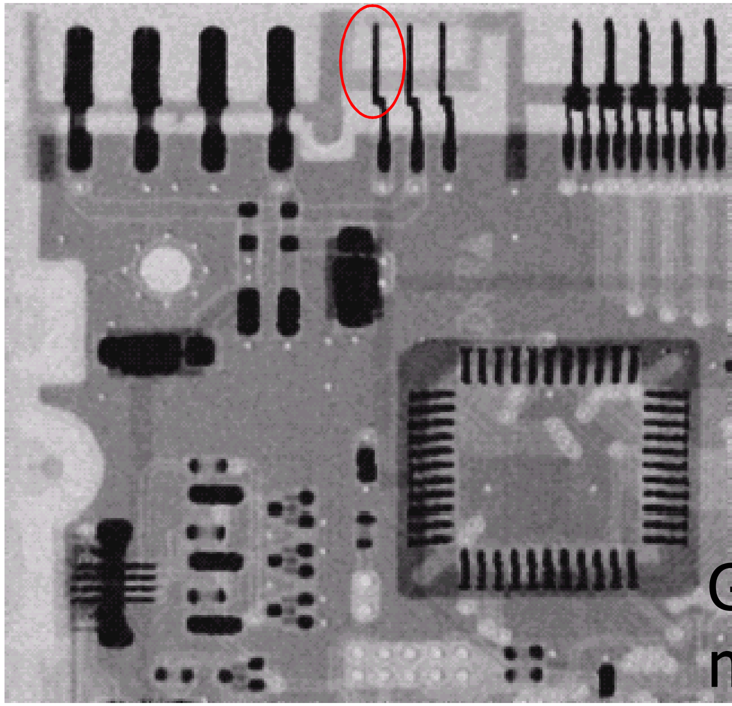




Noisy Gaussian
$\mu=0$
$\sigma=20$

Arith. mean





Geometric mean

# Mean Filters

Harmonic mean filter

$$\hat{f}(x,y) = \frac{mn}{\sum_{(s,t)\in S_{xy}} \frac{1}{g(s,t)}}$$

- The Harmonic mean does well for Gaussian noise and salt noise but fails for pepper noise.

# Mean filters (cont.)

- Contra-harmonic mean filter

$$\hat{f}(x,y) = \frac{\sum_{(s,t)\in S_{xy}} g(s,t)^{Q+1}}{\sum_{(s,t)\in S_{xy}} g(s,t)^{Q}}$$

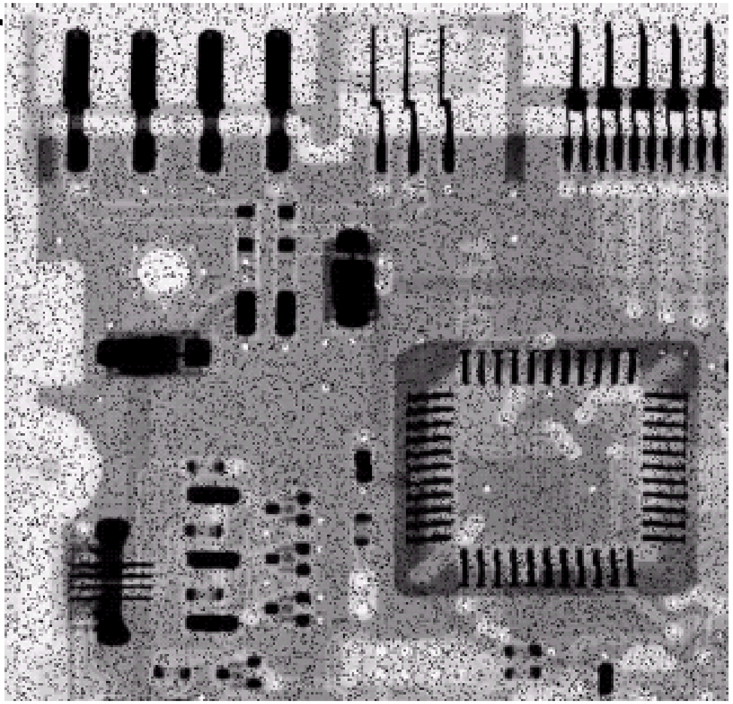
Q=-1, harmonic

Q=0, airth. mean

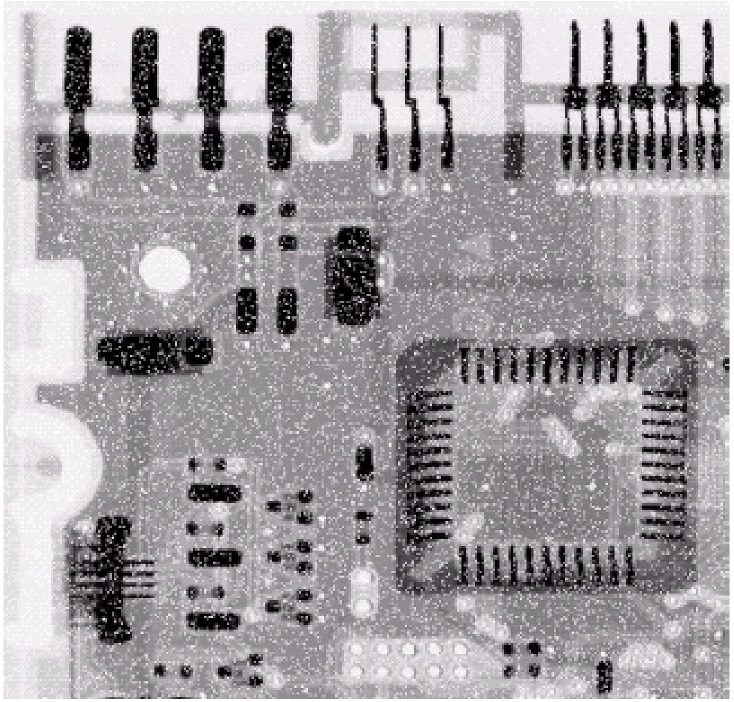
Q=+, ?

•In contraharmonic mean, positive Q eliminates pepper noise and negative Q eliminates salt noise
•Arithmetic filter if Q=0
•Harmonic Filter Q=-1
•Well suitated for Impluse noise

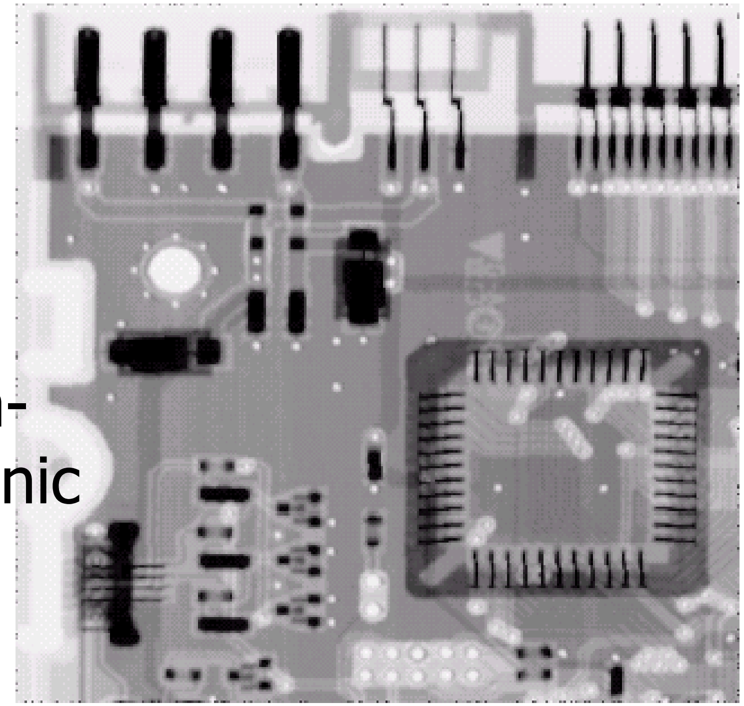
Pepper Noise



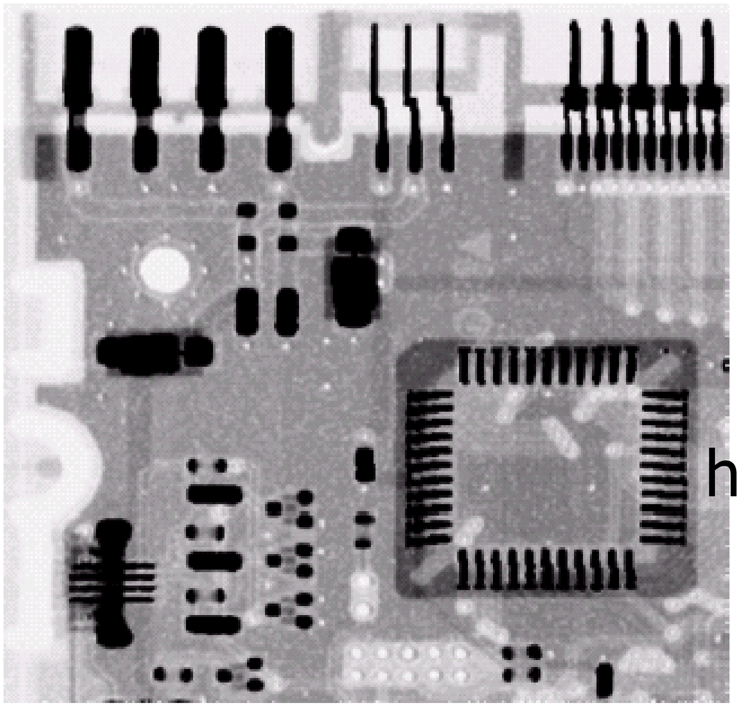
Salt Noise



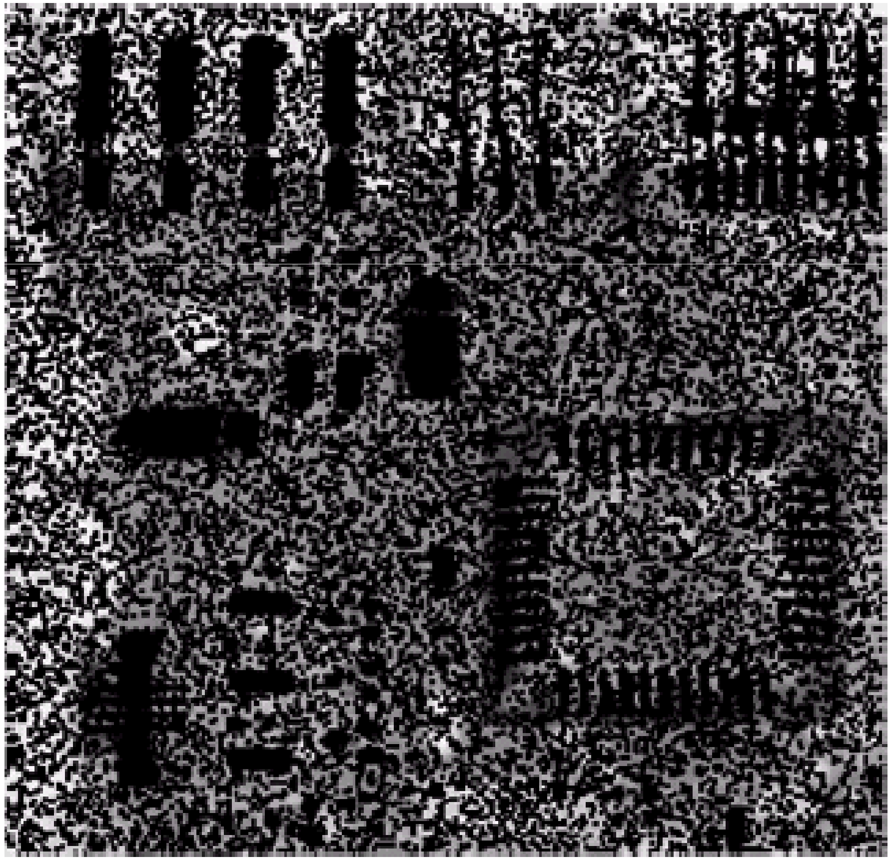
Contra-harmonic Q=1.5



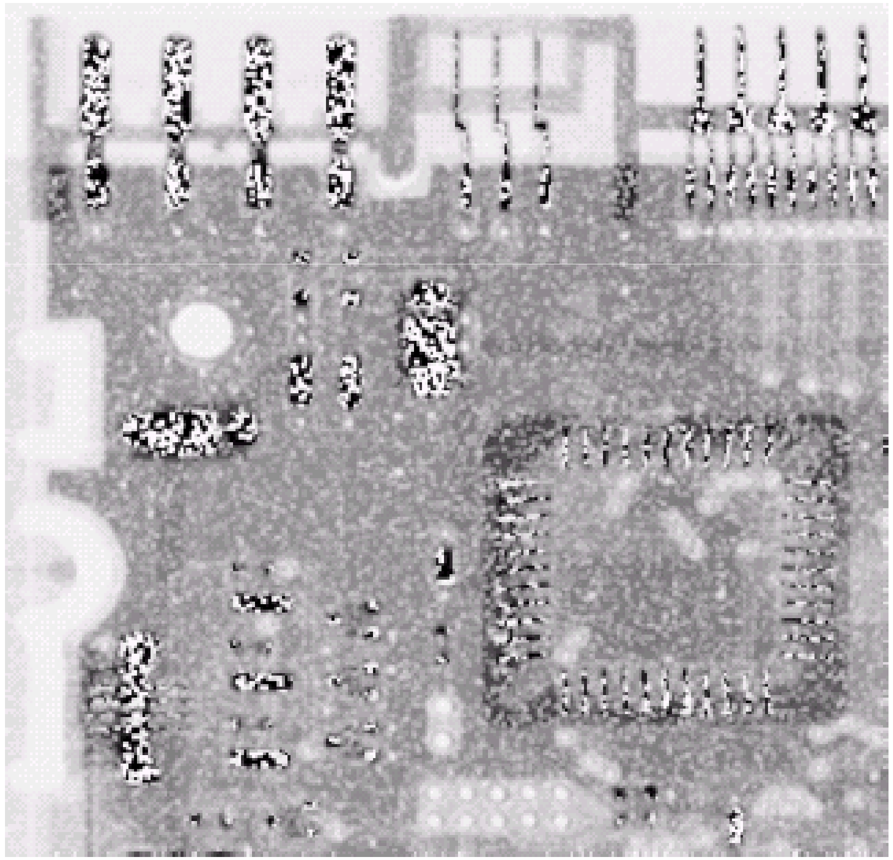
Contra-harmonic Q=-1.5



# Wrong sign in contra-harmonic filtering



Q=-1.5



Q=1.5

# Order-statistics filters

- Based on the ordering(ranking) of pixels
  - Suitable for unipolar or bipolar noise (salt and pepper noise)
- Median filters
- Max/min filters
- Midpoint filters
- Alpha-trimmed mean filters

# Order-statistics filters

- Median filter

$$\hat{f}(x,y) = \underset{(s,t)\in S_{xy}}{median}\{g(s,t)\}$$

- Max/min filters

$$\hat{f}(x,y) = \max_{(s,t)\in S_{xy}}\{g(s,t)\}$$

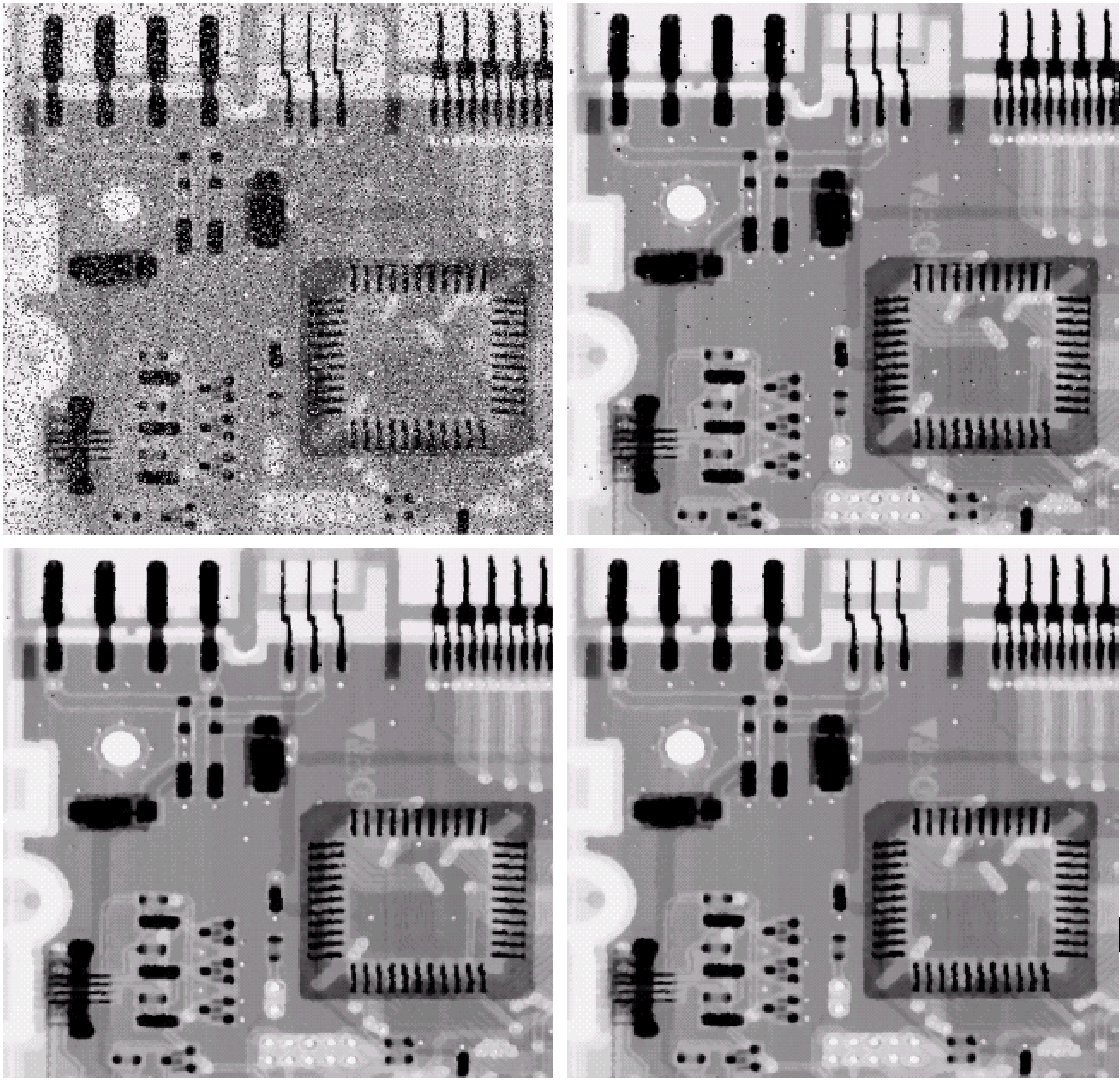
$$\hat{f}(x,y) = \min_{(s,t)\in S_{xy}}\{g(s,t)\}$$

# Median Filters

- Popular for random noise
- Excellent noise-reduction capabilities
- Less blurring
- Effective in the presence of both bipolar and unipolar impulse noise

# Max and Min Filters

- Max filter useful for finding the brightest point in the image
- Removes pepper noise
- Min filter useful for finding the darkest point in the image
- Reduces salt noise



bipolar Noise $P_a = 0.1$ $P_b = 0.1$

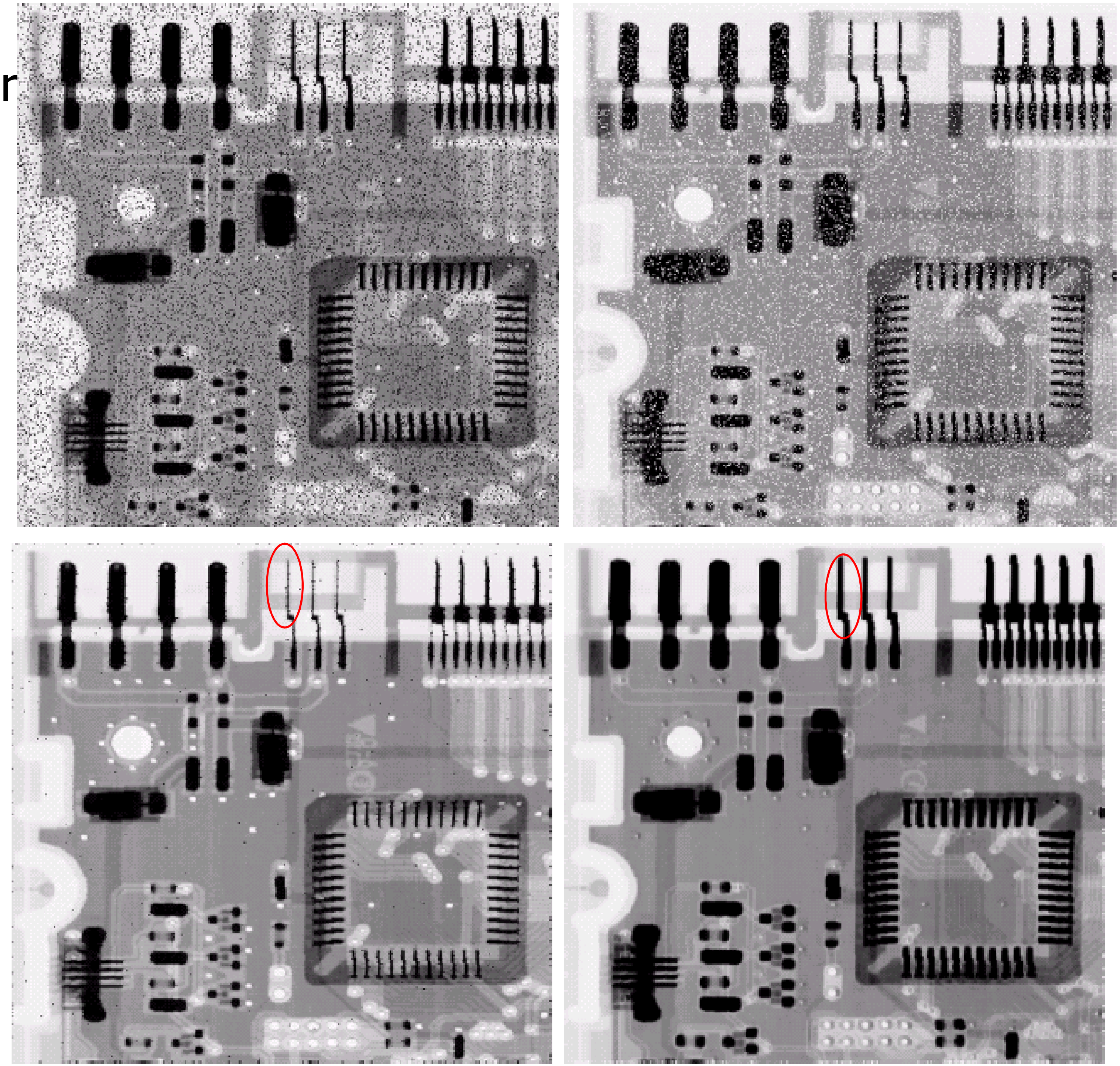
3x3 Median Filter Pass 1

3x3 Median Filter Pass 2

3x3 Median Filter Pass 3

Pepper noise

Salt noise



Max filter

Min filter

# Order-statistics filters (cont.)

- Midpoint filter

$$\hat{f}(x,y)=\frac{1}{2}\left[\max_{(s,t)\in S_{xy}}\{g(s,t)\}+\min_{(s,t)\in S_{xy}}\{g(s,t)\}\right]$$

- Alpha-trimmed mean filter
  - *Delete the d/2 lowest and d/2 highest gray-level pixels*

$$\hat{f}(x,y)=\frac{1}{mn-d}\sum_{(s,t)\in S_{xy}} g_r(s,t)$$

Middle (mn-d) pixels

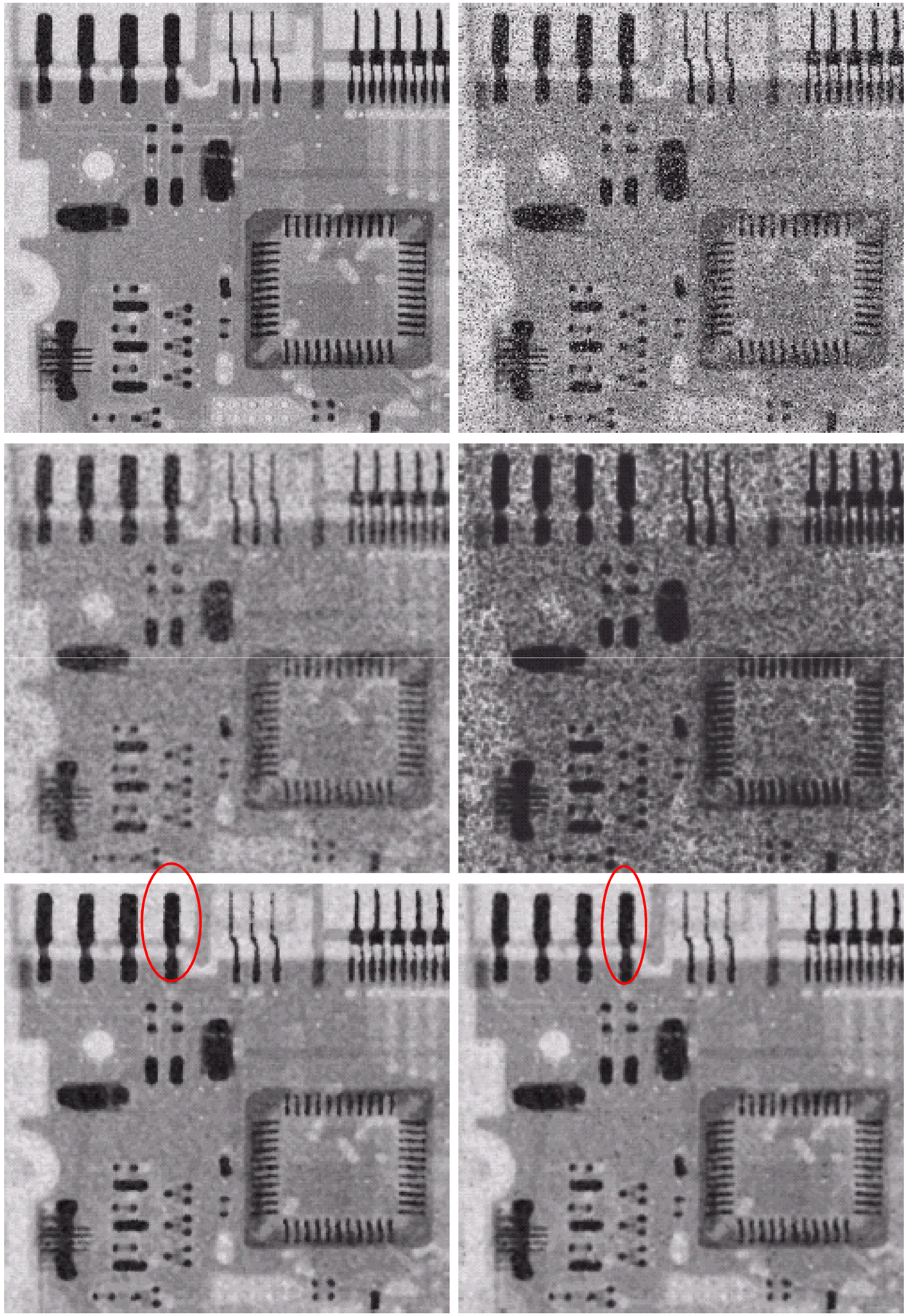
*Defaults to the arithmetic mean filter when d=0 and to the median filter when d=mn-1. Good for mixed short- and long-tailed noise*

Uniform noise

$\mu=0$

$\sigma^2=800$

Left +
Bipolar Noise
$P_a = 0.1$
$P_b = 0.1$



5x5
Arith. Mean
filter

5x5
Geometric
mean

5x5
Median
filter

5x5
Alpha-trim.
Filter
d=5

# Adaptive filters

- The filters discussed till now are non-adaptive filters.
  - whose coefficients are static, collectively forming the transfer function.
  - applied to an image regardless of how image characteristics vary from one point to another.
- Two adaptive filters are discussed.
  - whose behavior changes according to statistical characteristics of the image (local) inside the filter window.
  - whose performance is superior to that of non-adaptive filters.

# Adaptive filters

- Adapted to the behavior based on the statistical characteristics of the image inside the filter region Sxy
- Improved performance vs increased complexity
- Types:
  - (i) Adaptive local noise reduction filters
  - (ii) Adaptive median filter

# Adaptive local noise reduction filter

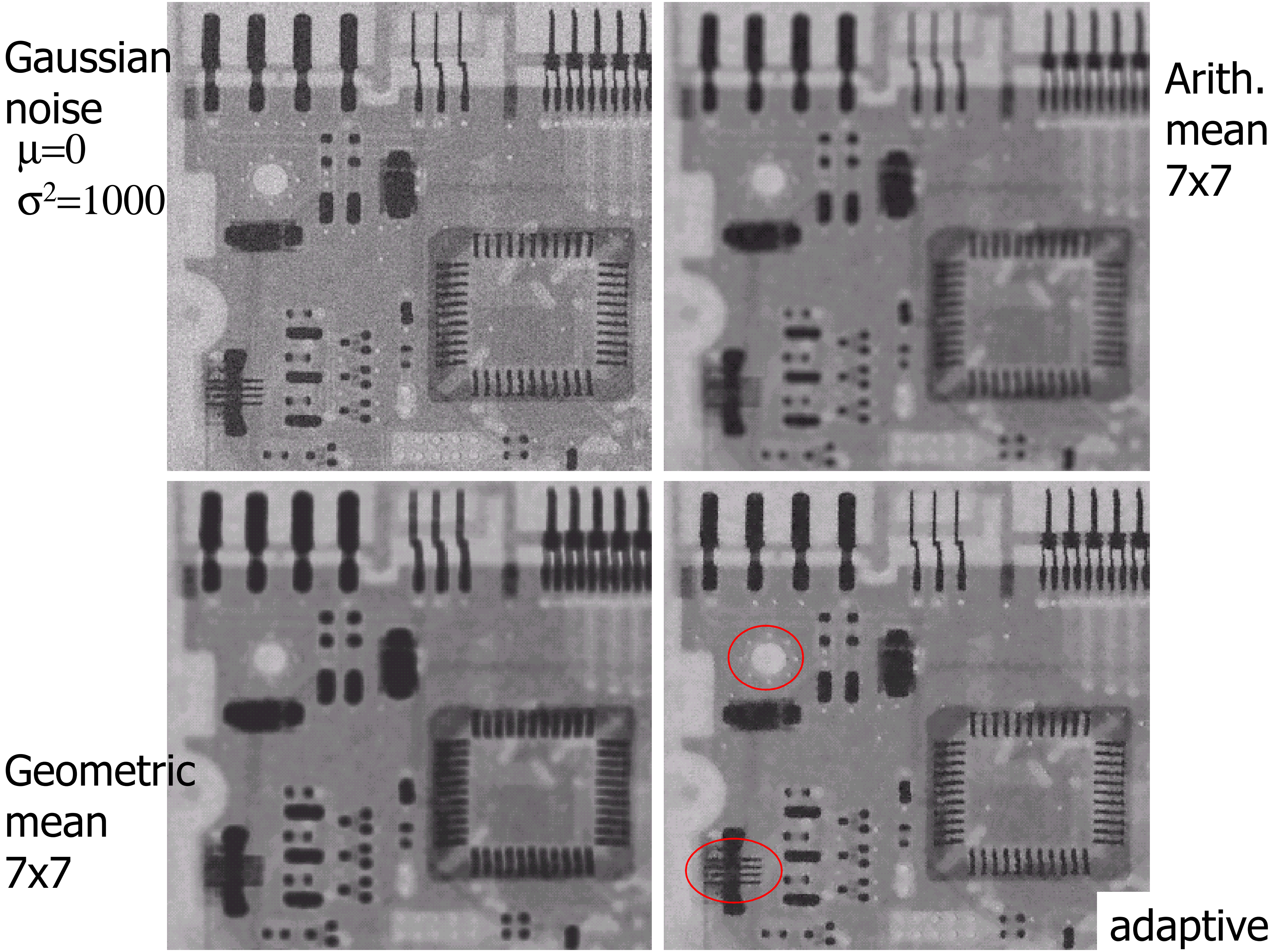
- Simplest statistical measurement
  - Mean and variance
- Known parameters on local region $S_{xy}$
  - $g(x,y)$: noisy image pixel value
  - $\sigma^2_\eta$ : noise variance (assume known a prior) in $S_{xy}$
  - $m_L$ : local mean
  - $\sigma^2_L$ : local variance

# Adaptive local noise reduction filter (cont.)

- Analysis: we want to do
  - If $\sigma^2_\eta$ is zero, return g(x,y)
  - If $\sigma^2_L > \sigma^2_\eta$, return value close to g(x,y)
  - If $\sigma^2_L = \sigma^2_\eta$, return the arithmetic mean $m_L$
- Formula

$$\hat{f}(x,y) = g(x,y) - \frac{\sigma_\eta^2}{\sigma_L^2}\left[g(x,y) - m_L\right]$$

- *only the variance of corrupting noise need to be known or estimated*
- • *Assume $\sigma^2_\eta \le \sigma^2_L$, otherwise, set the ratio =1*


Gaussian noise
μ=0
σ²=1000
Arith. mean 7x7
Geometric mean 7x7
adaptive

# Adaptive Median Filter

- The median filter performs relatively well on impulse noise as long as the spatial density of the impulse noise is not large ($P_a$, $P_b < 0.2$).

- The adaptive median filter can handle much more spatially dense impulse noise, and also performs some smoothing for non-impulse noise ($P_a$,$P_b$>0.2).

- The key insight in the adaptive median filter is that the filter size changes depending on the characteristics of the image.

- It also tries to preserve the details.

# Adaptive Median Filter

Remember that filtering looks at each original pixel image in turn and generates a new filtered pixel

- First examine the following notation:
- zmin = minimum gray level in Sxy
- zmax = maximum gray level in Sxy
- zmed = median of gray levels in Sxy
- zxy = gray level at coordinates (x, y)
- Smax = maximum allowed size of Sxy

# Adaptive Median Filter

Level A:

$A1 = zmed - zmin$

$A2 = zmed - zmax$

If $A1 > 0$ and $A2 < 0$, Go to level B

Else increase the window size

If window size ≤ repeat Smax level A

Else output zmed

Level B: $B1 = zxy - zmin$

$B2 = zxy - zmax$

If $B1 > 0$ and $B2 < 0$, output zxy

Else output zmed

# Adaptive Median Filter

The adaptive median filter has three purposes mainly :

- Remove impulse noise
- Provide smoothing of other noise
- Reduce distortion

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# Periodic noise reduction

- Periodic noise appears as concentrated bursts of energy in the Fourier transform at locations of the periodic interference
- This approach use selective filter to isolate the noise
- Three types of selective filters
  - Bandreject filter
  - Bandpass filter
  - Notch

# Periodic noise reduction

- Pure sine wave
  - Appear as a pair of impulse (conjugate) in the frequency domain

$$
\begin{cases}
f(x,y) = A\sin(u_0 x + v_0 y) \\
F(u,v) = -j\dfrac{A}{2}\left[\delta(u - \dfrac{u_0}{2\pi}, v - \dfrac{v_0}{2\pi}) - \delta(u + \dfrac{u_0}{2\pi}, v + \dfrac{v_0}{2\pi})\right]
\end{cases}
$$

# Periodic noise reduction (cont.)

- Bandreject filters
- Bandpass filters
- Notch filters
- Optimum notch filtering

# Band Reject Filters

- Removing periodic noise from an image involves removing a particular range of frequencies from that image.
- *Band reject filters can be used when the general location of the noise components is approximately known.*
- *Eg: image corrupted by additive periodic noise can be approximated as 2D sinusodial functions.*
- *Fourier transform of the sine consists of two impulses that are mirror images of each other about the origin*
- *The impulses form the imaginary part*

# Bandreject filters

* Reject an isotropic frequency

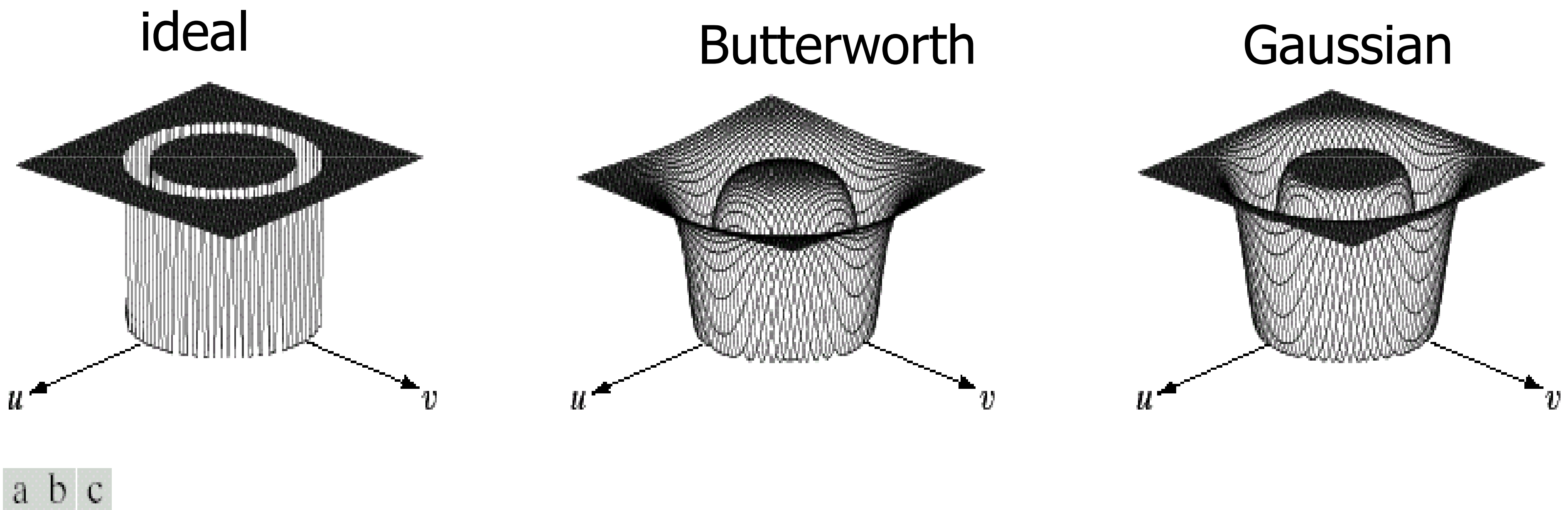
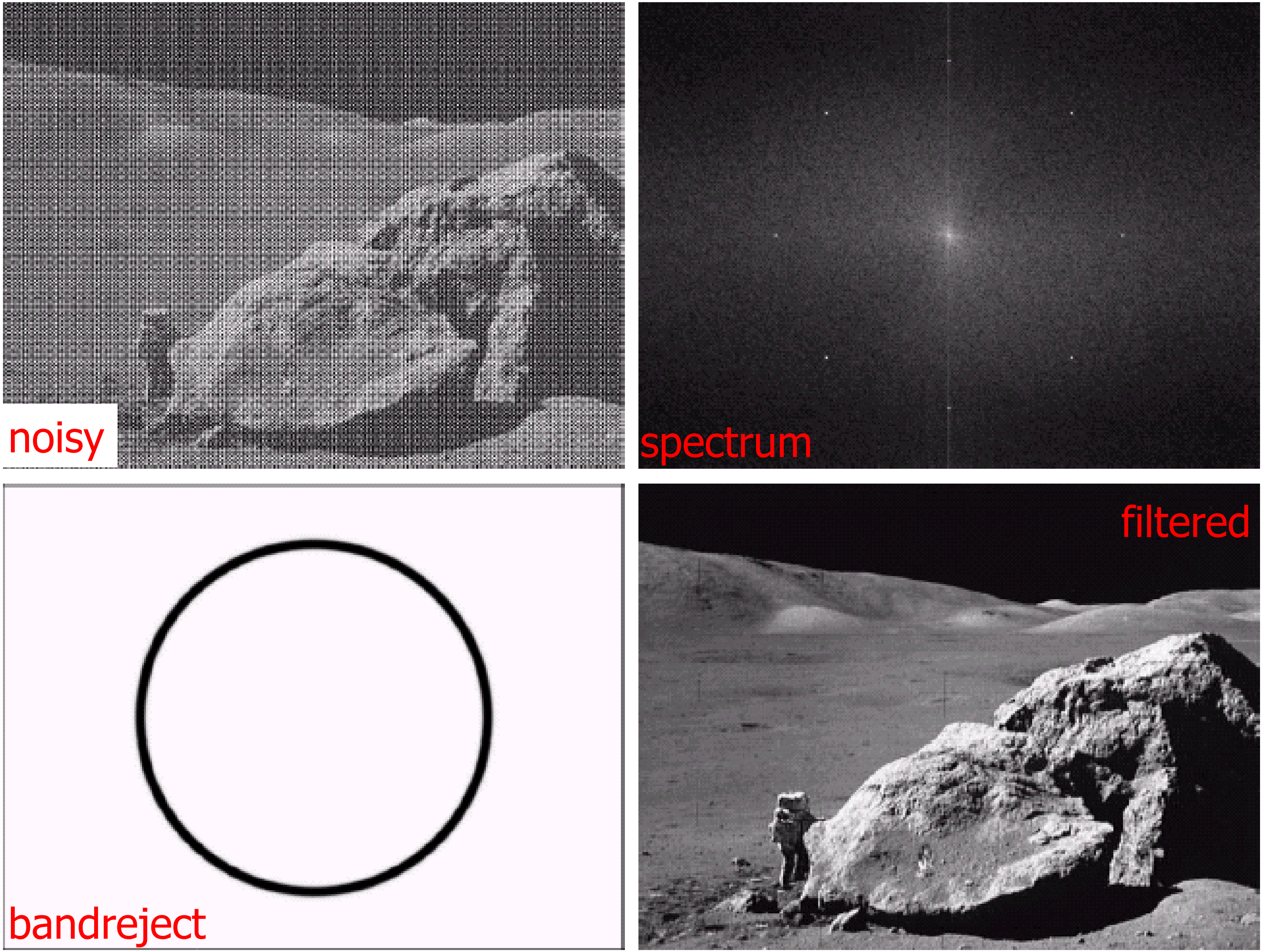



**FIGURE 5.15** From left to right, perspective plots of ideal, Butterworth (of order 1), and Gaussian bandreject filters.


noisy
spectrum
bandreject
filtered

# Band Reject Filters

$$H(u,v)=\begin{cases}1 & if\ D(u,v)<D_0-\frac{W}{2}\\ 0 & if\ D_0-\frac{W}{2}\le D(u,v)\le D_0+\frac{W}{2}\\ 1 & if\ D(u,v)>D_0+\frac{W}{2}\end{cases}$$

- D(u,v)=> it is the distance from the origin of the given frequency
- D0 is the radial center of the band
- W is the width of the band

# Band Reject Filters

- *Image heavily corrupted by sinusoidal noise of various frequencies*
- *The noise components are seen as symmetric pairs of bright spots in the Fourier spectrum*
- *Circularly symmetric band reject filters with order 4 is good choice*
- *The output has restored small details and texture effectively*

Butterworth and

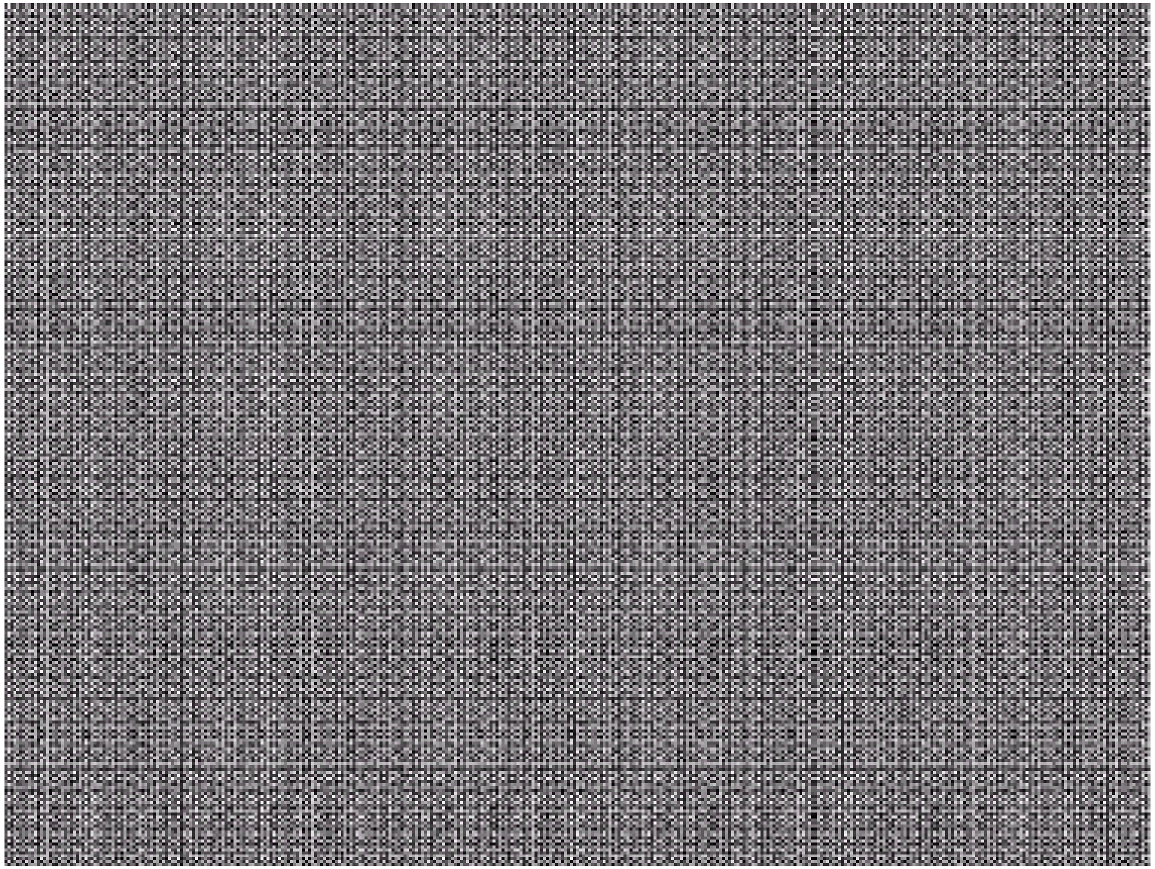
$$H(u,v) = 1 \Big/ \left[ 1 + \left( \frac{D(u,v)W}{D^2(u,v) - D_0^2} \right)^{2n} \right]$$

Gaussian band-reject filters:

$$H(u,v) = 1 - \exp\left[ -\frac{1}{2} \left( \frac{D^2(u,v) - D_0^2}{D(u,v)W} \right)^2 \right]$$

# Bandpass filters

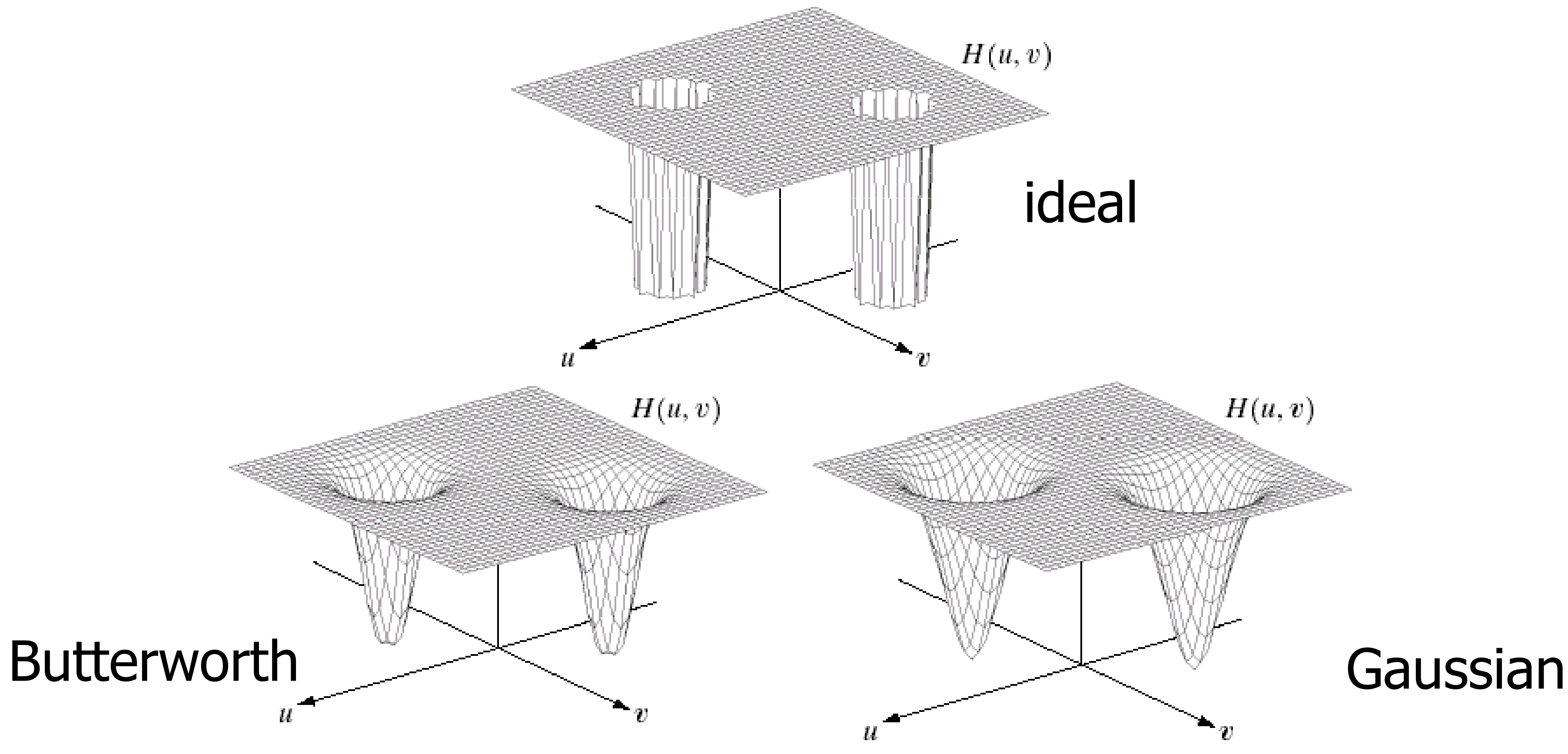
- $H_{bp}(u,v) = 1 - H_{br}(u,v)$



$$\Im^{-1}\{G(u,v)H_{bp}(u,v)\}$$

# Notch filters

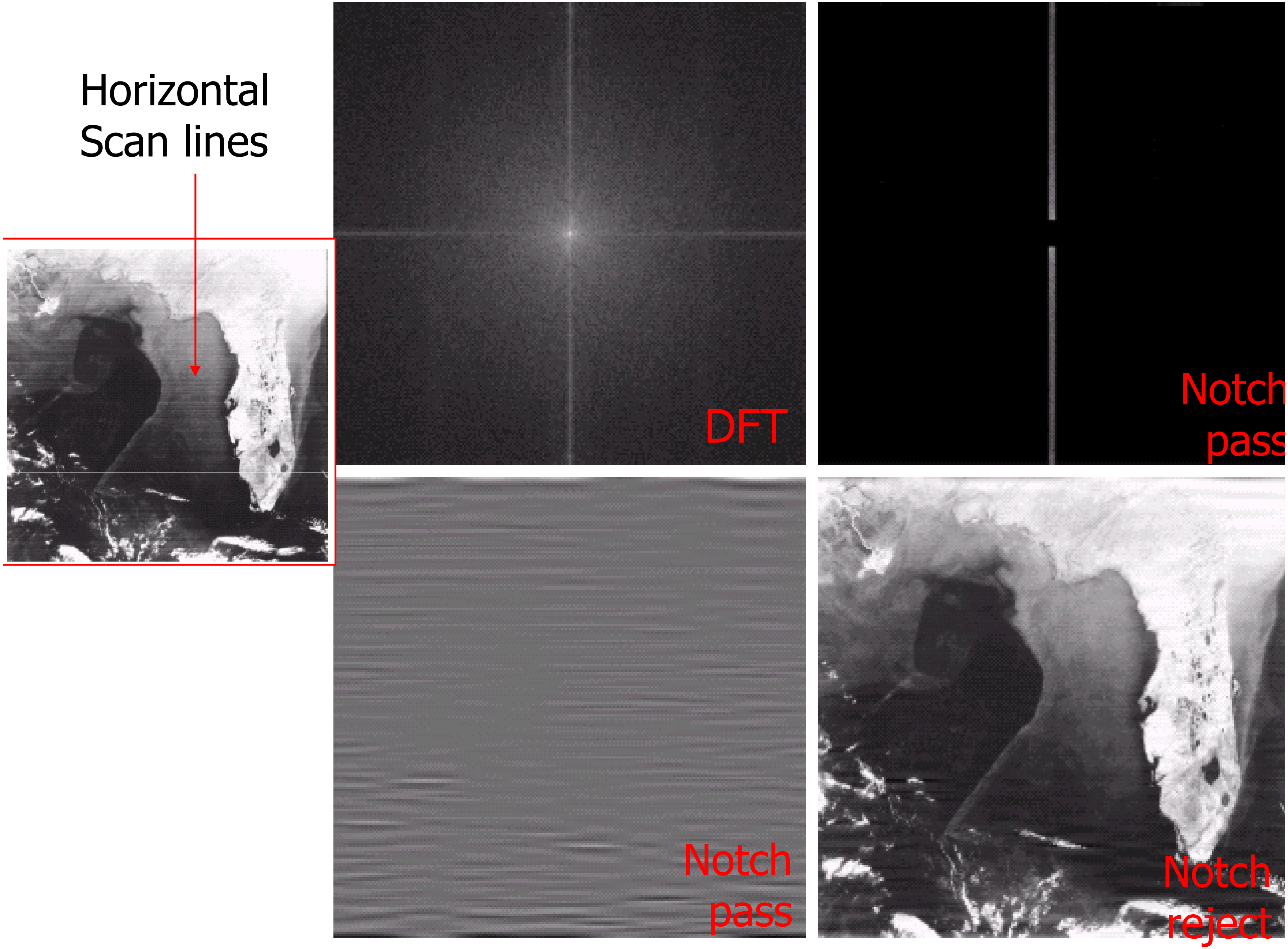
- *Reject(or pass) frequencies in predefined neighborhoods about a center frequency*



# Notch-Band Reject Filters

- *Due to symmetry of the Fourier transform notch filters appear in symmetric pairs about the origin .*
- *Notch filters pass rather than suppress the frequencies contained in the notch areas*
  - $Hnp(u,v)=1-Hnr(u,v)$

- *Hnp(u,v) is the transfer function of the notch pass filter corresponding to the notch reject filter with transfer functions Hnr(u,v)*


Horizontal
Scan lines
DFT
Notch
pass
Notch
pass
Notch
reject

# Bandpass filters

Let
$$D_1(u,v) = \sqrt{(u - M/2 - u_0)^2 + (v - N/2 - v_0)^2}$$
$$D_2(u,v) = \sqrt{(u - M/2 + u_0)^2 + (v - N/2 + v_0)^2}$$

Ideal, Butterworth, Gaussian **notch filters:**

$$H(u,v) = \begin{cases} 0 & \text{if} \quad D_1(u,v) \le D_0 \text{ or } D_2(u,v) \le D_0 \\ 1 & \text{otherwise} \end{cases}$$

$$H(u,v) = 1 \Bigg/ \left[ 1 + \left( \frac{D_0^2}{D_1(u,v) D_2(u,v)} \right)^n \right]$$

$$H(u,v) = 1 - \exp\left[ -\frac{1}{2} \left( \frac{D_1(u,v) D_2(u,v)}{D_0^2} \right) \right]$$

Note: for the filter coeff. to be real, notch areas must always be defined in symmetric pairs

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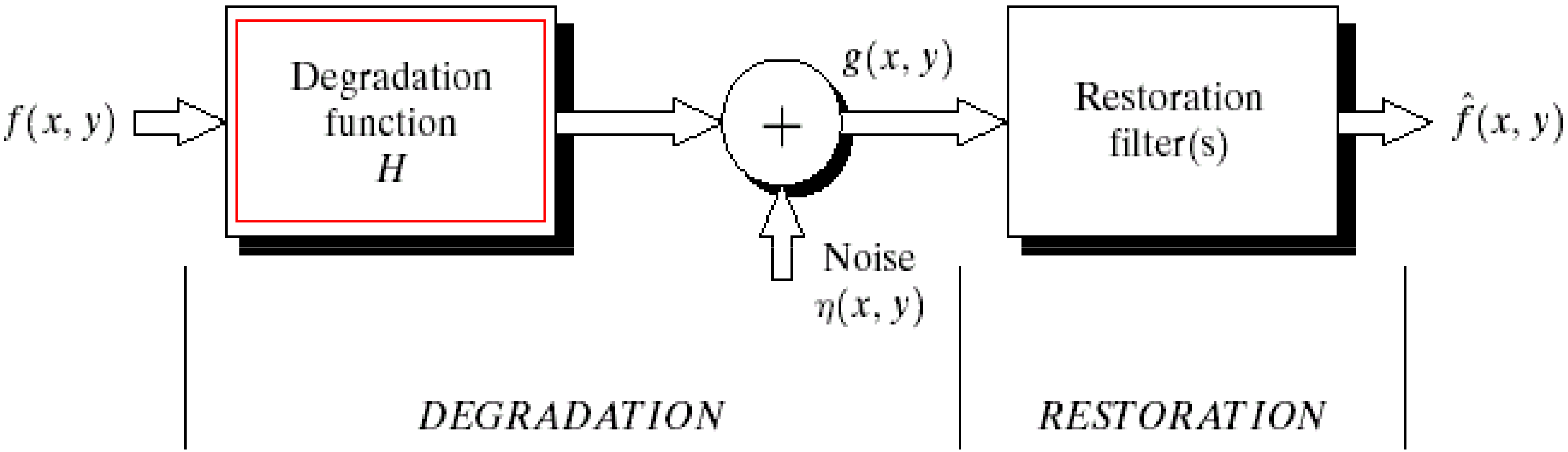
# A model of the image degradation /restoration process




$$\begin{cases} g(x,y)=f(x,y)*h(x,y)+\eta(x,y) \\ G(u,v)=F(u,v)H(u,v)+N(u,v) \end{cases}$$

If linear, position-invariant system

# Linear, position-invariant degradation

Properties of the degradation function H

- Linear system
  - $H[af_1(x,y)+bf_2(x,y)]=aH[f_1(x,y)]+bH[f_2(x,y)]$
- Position(space)-invariant system
  - $H[f(x,y)]=g(x,y)$
  - $\Leftrightarrow H[f(x-\alpha, y-\beta)]=g(x-\alpha, y-\beta)$
- c.f. 1-D signal
  - LTI (linear time-invariant system)

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\boxed{\delta(x-\alpha,y-\beta)}\,d\alpha\, d\beta$$

impulse

$$g(x,y)=H[f(x,y)]=H\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\delta(x-\alpha,y-\beta)\,d\alpha\, d\beta\right]$$

linear

$$g(x,y)=H[f(x,y)]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H\left[f(\alpha,\beta)\delta(x-\alpha,y-\beta)\right]d\alpha\, d\beta$$

$$g(x,y)=H[f(x,y)]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\boxed{H\left[\delta(x-\alpha,y-\beta)\right]}d\alpha\, d\beta$$

$$h(x,\alpha,y,\beta)=H\left[\delta(x-\alpha,y-\beta)\right]$$

$h(x,y)=H\left[\delta(x,y)\right]$

Impulse response (point spread function)

If position-invariant

$$H\left[\delta(x-\alpha,y-\beta)\right]=h(x-\alpha,y-\beta)$$

$$g(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)h(x-\alpha,y-\beta)\,d\alpha\, d\beta$$

$\eta(x,y)\neq 0$

$$g(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)h(x-\alpha,y-\beta)\,d\alpha\, d\beta+\eta(x,y)$$

# Linear, position-invariant degradation model

- Linear system theory is ready
- Non-linear, position-dependent system
  - May be general and more accurate
  - Difficult to solve compuatationally
- Image restoration: find H(u,v) and apply inverse process
  - Image deconvolution

# Estimating the degradation function

- Estimation by Image observation
- Estimation by experimentation
- Estimation by modeling

# Estimation by image observation

- *If H is unknown gather information from the image itself*
- *If the image is blurred, process the subimage by sharpening the subimage with sharpening filter*
- Estimate the original image in the window

$$H_s(u,v) = \frac{G_s(u,v)}{\hat{F}_s(u,v)}$$

Known (observed)

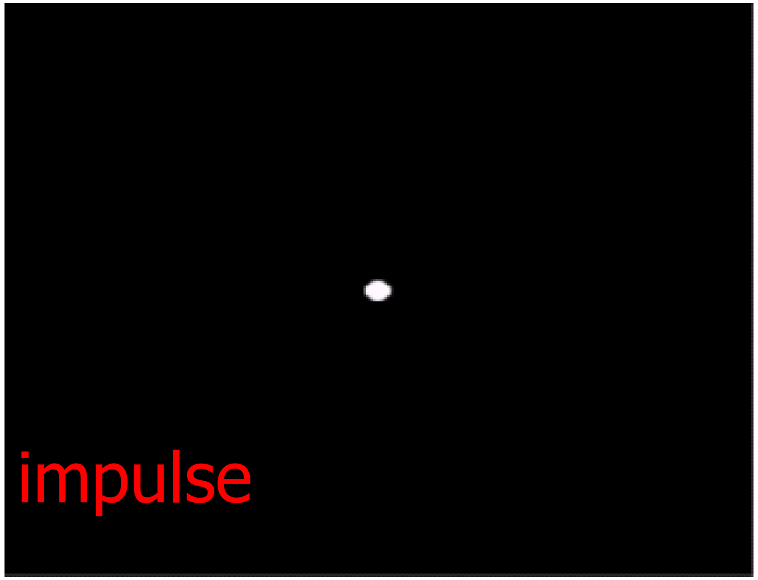
Estimate (processed)

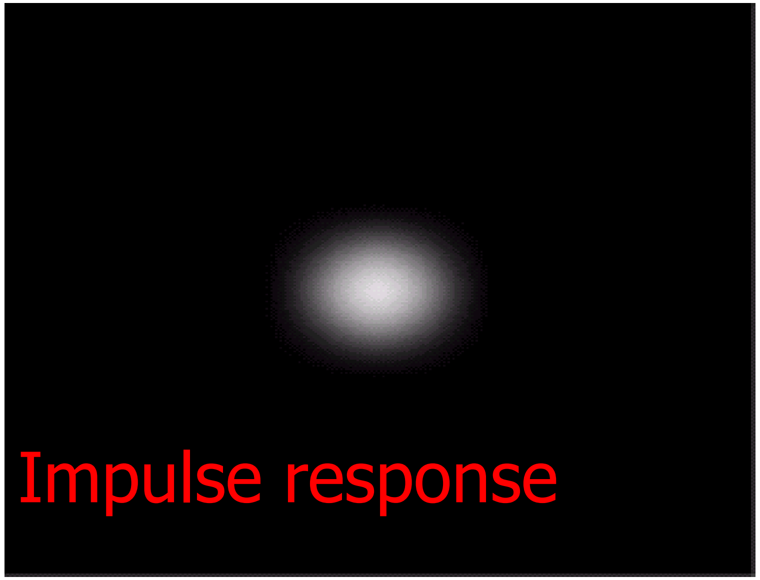
*-Obtain the complete degradation based on assumption of position invariance*

# Estimation by experimentation

- If the image acquisition system is ready
- Obtain the impulse response

Suppose that equipment similar to the one used for acquisition be available; then it is possible to obtain an accurate estimation of the degradation by imaging an impulse using the same system settings. Then, $H(u,v) = G(u,v)/A$



# Estimation by modeling (1)

- Ex. Atmospheric model $H(u,v) = e^{-k(u^2+v^2)^{5/6}}$

original

k=0.0025

k=0.001

k=0.00025

# Estimation by modeling (2)

- Derive a mathematical model
- Ex. Motion of image

$$g(x,y)=\int_0^T f(x-x_0(t),y-y_0(t))dt$$

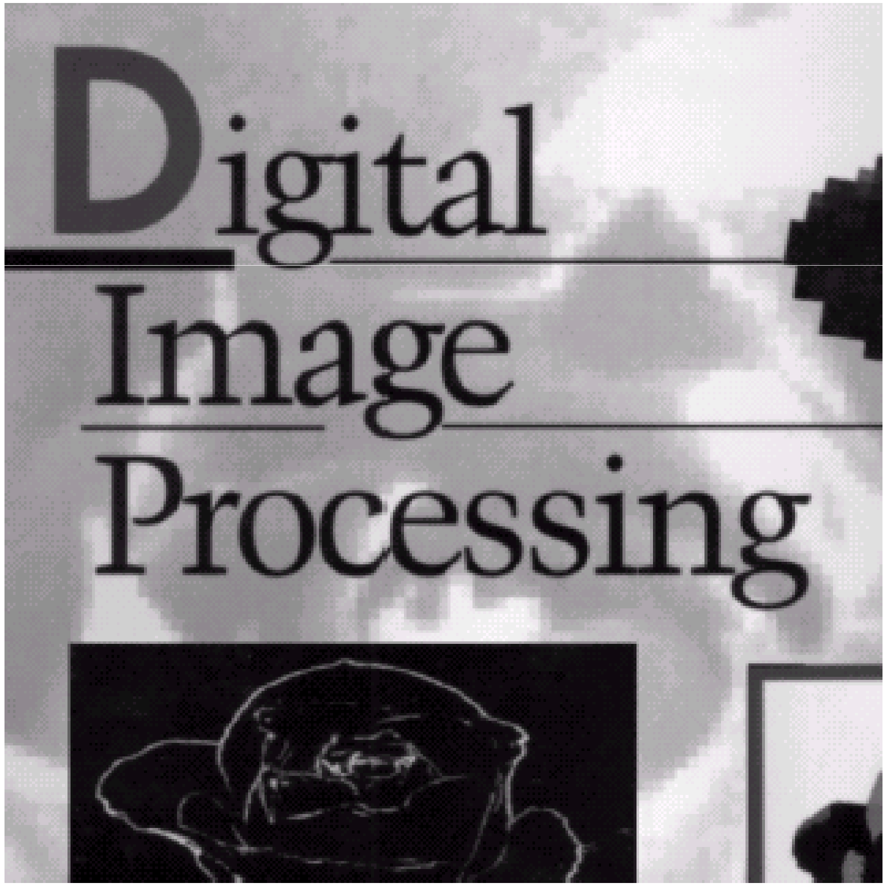
Planar motion

Fourier transform

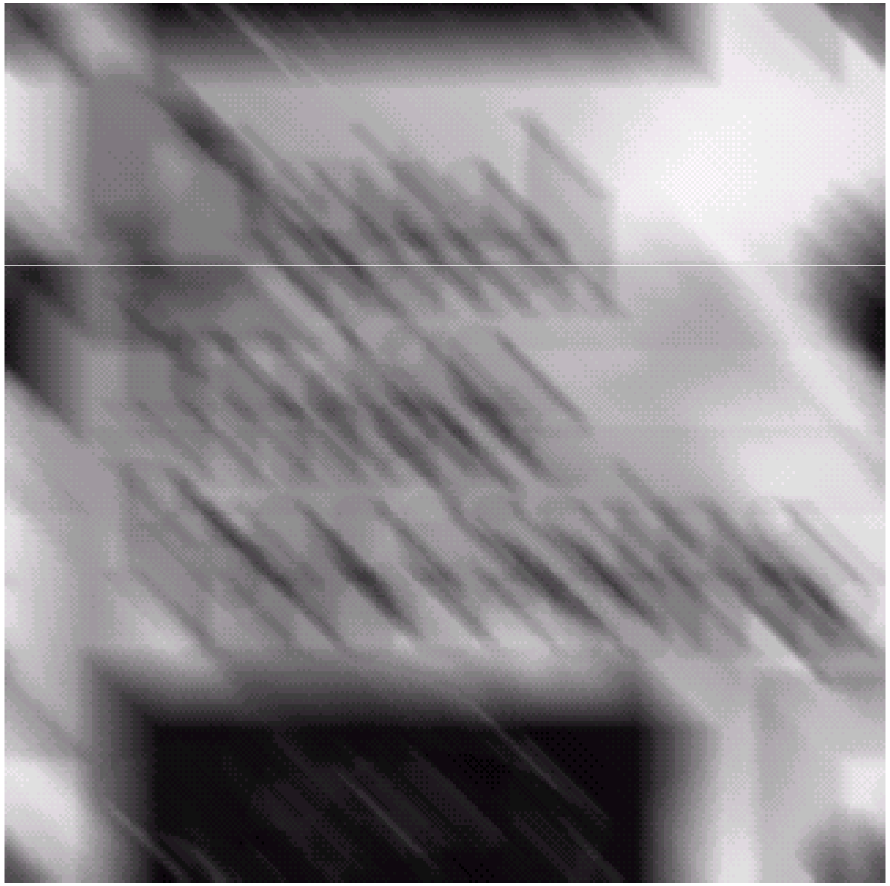
$$G(u,v)=F(u,v)\int_0^T e^{-j2\pi[ux_0(t)+vy_0(t)]}dt$$

# Estimation by modeling: example

original




Apply motion model



# Inverse filtering

- With the estimated degradation function H(u,v)

$$G(u,v)=F(u,v)H(u,v)+N(u,v)$$

Unknown noise

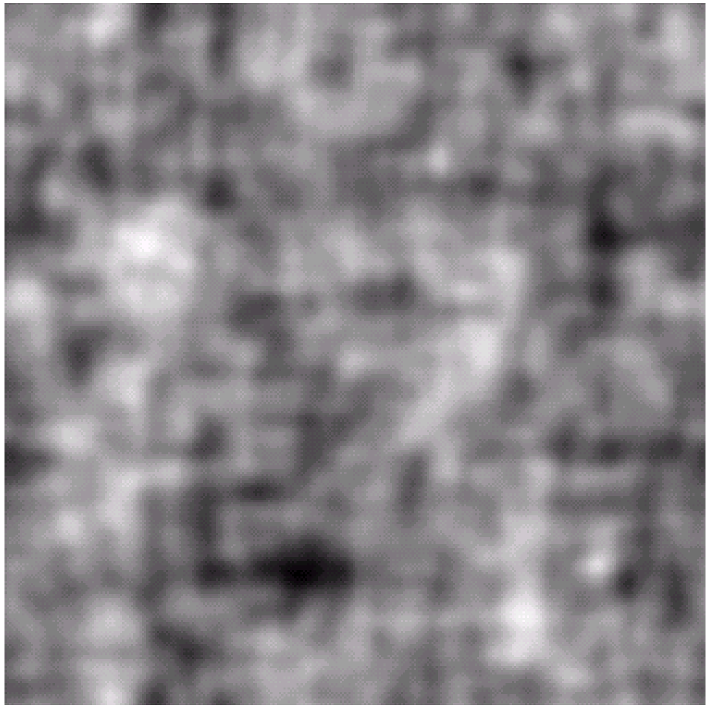
$$=> \quad \hat{F}(u,v) = \frac{G(u,v)}{H(u,v)} = F(u,v) + \frac{N(u,v)}{H(u,v)}$$

Estimate of original image

Problem: 0 or small values

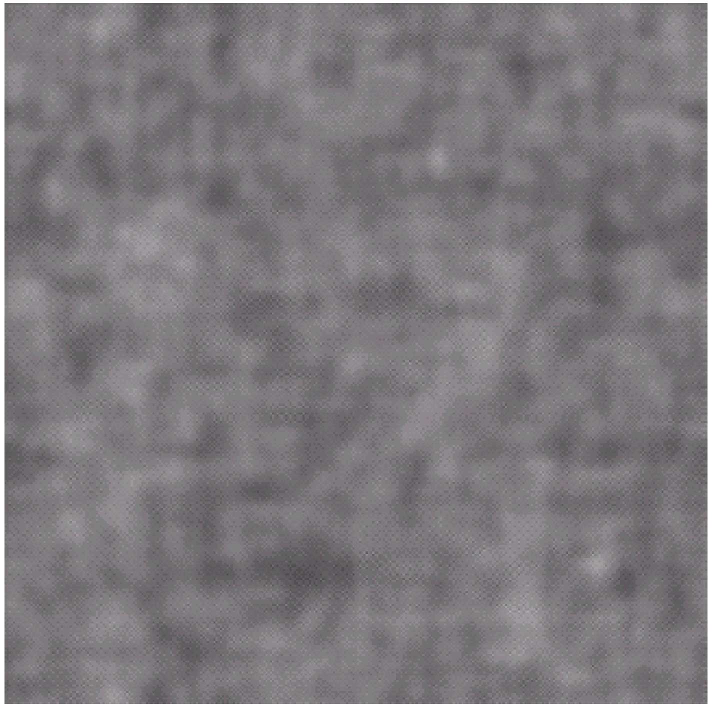
Sol: limit the frequency around the origin

Full inverse filter for k=0.0025



Cut Outside 40%

Cut Outside 70%



Cut Outside 85%